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A Comparison of Some Existence Tests

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We compare some tests to each other which guarantee the zero of a mapping from R^n to R^n.

1 Introduction

We use the term existence test to mean a (computer-assisted) method which guarantees that a mapping $f : R^n \to R^n$ has a zero x^* within some $D \subseteq R^n$. For $n = 1$ a well-known existence test is the application of the intermediate-value theorem.

EXAMPLE 1. Let

$$y = f(x) = 3x - 1, \quad x \in R.$$

Since any computer cannot represent $\frac{1}{3}$ exactly as a decimal number, any computer is unable to represent the zero of f exactly as a decimal number. However, the interval $I = [\underline{x}, \overline{x}] = [0.33, 0.34]$ fulfills

$$f(\underline{x}) < 0, \quad f(\overline{x}) > 0. \tag{1}$$

Since f is continuous, the intermediate-value theorem guarantees the existence of some $x^* \in I$ satisfying $f(x^*) = 0$. The calculation of (1) can be verified by a computer. Hence, (1) describes an existence test for $n = 1$.

We want to point out that the existence tests shall be performed by a computer; i.e., the test (1) shall be done by a computer. But as we will see in the following example, this is not always possible.

EXAMPLE 2. Let

$$y = f(x) = x - \sqrt{2}, \quad x \in R.$$

If $I = [\underline{x}, \overline{x}]$, then it is not clear how (1) can be performed by a computer, because $\sqrt{2}$ cannot be represented exactly as a decimal number. It can only be enclosed in some interval, say $[1.4142, 1.4143]$.

In fact, we have to consider the following problem: Let $f : D \subseteq R^n \to R^n$ be continuous and additionally we assume that f contains real parameters $a^{(1)}, ..., a^{(m)}$, which are not exactly known and which cannot be represented as a machine number, respectively. We assume that we only know that these parameters are contained in some given intervals; i.e., $f = f(x; a^{(1)}, ..., a^{(m)})$, $x \in D$, $a^{(j)} \in [a^{(j)}]$, $j = 1, ..., m$. Then, we consider the following problem:

Prove that for any $a^{(j)} \in [a^{(j)}]$, $j = 1, ..., m$,
there exists some $x^* \in [x]$ with $f(x^*; a^{(1)}, ..., a^{(m)}) = o$.

Here, $[x]$ denotes an interval vector which is a vector with an interval in each component (see [1]).

EXAMPLE 3. Let

$$y = f(x; a) = x - a, \quad x \in R,$$

where a is a parameter which is enclosed in $[a] = [1.4142, 1.4143]$. If $I = [\underline{x}, \overline{x}] = [1, 2]$, then (1) is modified to

$$f(\underline{x}, [a]) = [-0.4143, -0.4142], \quad f(\overline{x}, [a]) = [0.5857, 0.5858].$$

Therefore, for any fixed $a \in [a]$ we have

$$f(\underline{x}; a) < 0, \quad f(\overline{x}; a) > 0,$$

and again the intermediate-value theorem guarantees that for any $a \in [a]$ there exists an $x^* \in I$ satisfying $f(x^*; a) = 0$.

For $n > 1$ we need some generalizations of the intermediate-value theorem. The existence tests considered here are based on Brouwer's fixed point theorem [6] and the theorem of Miranda [7], respectively. Although those theorems are equivalent, the resulting existence tests are not; i.e., if an existence test based on Miranda's theorem can be applied successfully, then this does not imply that also an existence test based on Brouwer's fixed point theorem can be applied successfully.

We consider three existence tests: the Moore test [8], the Frommer, Lang, and Schnurr test [3], and the Moore and Kioustelidis test [9]. The first test is based on Brouwer's fixed point theorem, whereas the other ones are based on Miranda's theorem.

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2 Comparison

If the success of test A implies the success of test B, we write $\boxed{\text{A}} \Rightarrow \boxed{\text{B}}$. We do not compare the running times, here. Under some assumptions one can conclude

$$\boxed{Moore} \Longrightarrow \boxed{\begin{matrix} Frommer \\ Lang \\ Schnurr \end{matrix}} \Longrightarrow \boxed{\begin{matrix} Moore \\ Kioustelidis \end{matrix}}$$

Fig. 1

For example, if f does not depend on parameters $a^{(j)}$ and the existence tests are using the derivative of f, then Fig. 1 is true. The first part was shown in [3] and the second one in [2]. However, if the Jacobian matrix used within the existence tests is replaced by a so-called slope matrix (see [4], [5]), then the second implication arrow in Fig. 1 is wrong, in general. See Example 3.1 in [10].

The second implication arrow in Fig. 1 is also wrong, in general, if some parameter $a^{(j)}$ appears more than once within the expression of f. See Example 3.2 in [10].

For the case that f contains some parameters $a^{(j)}$ and a slope matrix is used, one can also find assumptions such that the second implication arrow in Fig. 1 is valid. For details we refer to [10]. One interesting application where those assumptions are fulfilled can be found in [11].

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