

A Fixed Point Theorem in Infinite-Dimensional Spaces

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A generalization of the theorem of Miranda is given.

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1 Introduction

In 1940, Miranda published the following theorem ([4]).

Theorem 1 Let $\Omega = \{x \in R^n : |x_i| \leq L, i = 1, \dots, n\}$ and $f : \Omega \rightarrow R^n$ be continuous satisfying

$$f_i(x_1, x_2, \dots, x_{i-1}, -L, x_{i+1}, \dots, x_n) \geq 0, f_i(x_1, x_2, \dots, x_{i-1}, +L, x_{i+1}, \dots, x_n) \leq 0 \text{ for all } i \in \{1, \dots, n\}. \quad (1)$$

Then, $f(x) = o$ has a solution in Ω .

For $n = 1$ Theorem 1 reduces to the well-known intermediate-value theorem illustrated in Figure 1. Miranda proved his

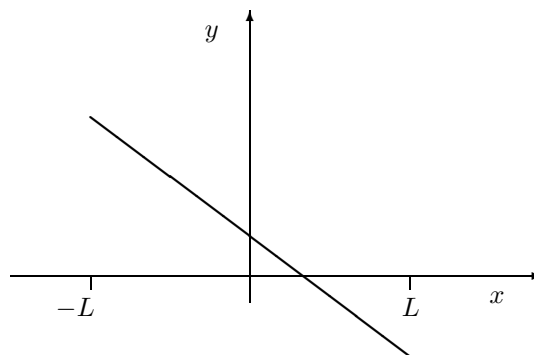


Fig. 1

theorem using the Brouwer fixed point theorem. Using the Brouwer degree of a mapping, Vrahatis gave another short proof of Theorem 1 (see [7]). Following this proof it is easy to see that Theorem 1 is also true, if L is dependent of i ; i.e., Ω can also be a rectangle and need not to be a cube. Even some L_i can be zero. Very often, the theorem of Miranda is stated as in the following corollary (see also [2], [3], [5]), which is not the theorem of Miranda in its original form, but a consequence of it.

Corollary 1 Let $\hat{x} \in R^n, L \in R^n, L \geq o, \Omega$ be the rectangle $\Omega := \{x \in R^n : |x_i - \hat{x}_i| \leq L_i, i = 1, \dots, n\}$ and $f : \Omega \rightarrow R^n$ be a continuous function on Ω . Also let

$$F_i^+ := \{x \in \Omega : x_i = \hat{x}_i + L_i\}, \quad F_i^- := \{x \in \Omega : x_i = \hat{x}_i - L_i\}, \quad i = 1, \dots, n$$

be the pairs of parallel opposite faces of the rectangle Ω . If for all $i = 1, \dots, n$

$$f_i(x) \cdot f_i(y) \leq 0 \quad \text{for all } x \in F_i^+ \text{ and for all } y \in F_i^+,$$

then there exists some $x^* \in \Omega$ satisfying $f(x^*) = o$.

In principle, Corollary 1 says that Theorem 1 is also true if the \leq -sign and the \geq -sign are exchanged with each other in (1). Corollary 1 also says that Theorem 1 is not restricted to a rectangle with 0 as its center. For $n = 1$ Corollary 1 is illustrated in Figure 2.

2 The infinite-dimensional case

It is well-known that extending the techniques from the finite-dimensional to the infinite-dimensional spaces presents a serious problem (see [1]). The basic reason for this is that the unit sphere (more generally any closed bounded set) in the finite-dimensional case is compact while the unit sphere in the infinite-dimensional case is not compact. Therefore, we only get a

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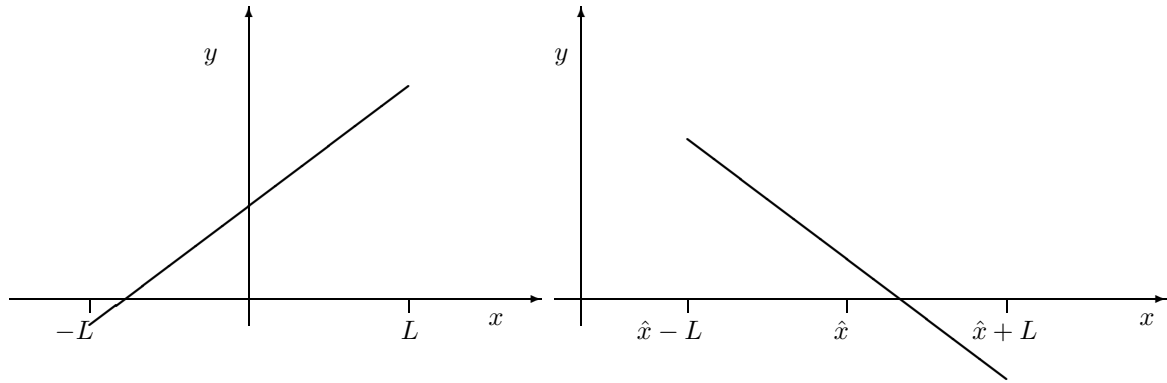


Fig. 2

fixed point version of Theorem 1, and only compact mappings can be considered, where a mapping $g : V \rightarrow V$ is called compact, if it is continuous and if the closure $\overline{f(A)}$ of $f(A)$ is compact whenever A is bounded in V .

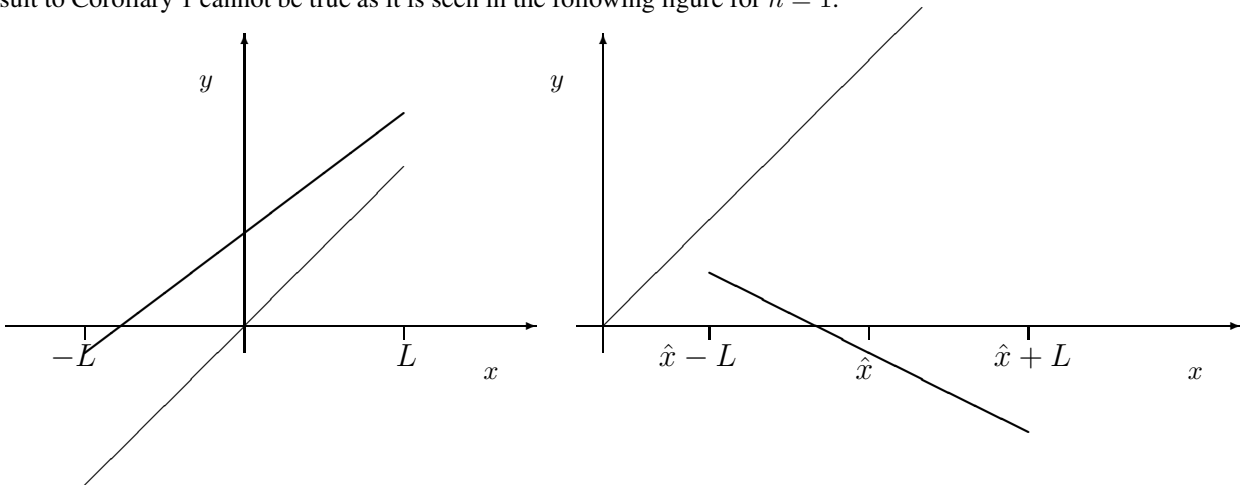
We consider the infinite-dimensional Hilbert space of all square summable sequences of real numbers, denoted by l^2 and equipped with the natural order $x, y \in l^2 : x \leq y \Leftrightarrow x_n \leq y_n \forall n \in N$.

Theorem 2 Let $L = \{l_n\}_{n=1}^\infty \in l^2, l_n > 0 \forall n \in N$. Let $\Omega = \{x \in l^2 : |x_n| < l_n \forall n \in N\}$ and suppose that the mapping $g : \overline{\Omega} \rightarrow l^2$ is compact on the closure $\overline{\Omega}$ of Ω , and

$$g_n(x_1, x_2, \dots, x_{n-1}, -l_n, x_{n+1}, \dots) \geq 0, \quad g_n(x_1, x_2, \dots, x_{n-1}, +l_n, x_{n+1}, \dots) \leq 0 \text{ for all } n \in N.$$

Then, $g(x) = x$ has a solution in Ω .

The proof is done by the Leray-Schauder degree of a mapping ([6]) and it is also true in R^n of course. However, an analogous result to Corollary 1 cannot be true as it is seen in the following figure for $n = 1$.



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References

- [1] J. Cronin, Fixed Points and Topological Degree in Nonlinear Analysis, Am. Math. Soc., 1964.
- [2] J. B. Kioustelidis, Algorithmic error estimation for approximate solutions of nonlinear systems of equations, Computing, **19**, 313–320 (1978).
- [3] J. Mayer, A generalized theorem of Miranda and the theorem of Newton-Kantorovich, Num. Funct. Anal. Optim., **23**, 333–357 (2002).
- [4] C. Miranda, Un osservazione su un theorema di Brower, Belletino Unione Math. Ital. Ser. II, 5–7 (1940).
- [5] R. E. Moore and J. B. Kioustelidis, A simple test for accuracy of approximate solutions to nonlinear (or linear) systems, SIAM J. Numer. Anal., **17**, 521–529 (1980).
- [6] U. Schäfer, On Miranda’s theorem in infinite-dimensional spaces, Preprint No. 11, Fakultät für Mathematik, Universität Karlsruhe, 2005.
- [7] M. N. Vrahatis, A short proof and a generalization of Miranda’s existence theorem, Proc. Am. Math. Soc., **107**, 701–703 (1989).