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A Fixed Point Theorem in Infinite-Dimensional Spaces

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A generalization of the theorem of Miranda is given.

1 Introduction

In 1940, Miranda published the following theorem ([4]).

Theorem 1 *Let* $\Omega = \{x \in R^n : |x_i| \le L,\ i = 1, ..., n\}$ *and* $f : \Omega \to R^n$ *be continuous satisfying*

$$f_i(x_1, x_2, ..., x_{i-1}, -L, x_{i+1}, ..., x_n) \ge 0,\ f_i(x_1, x_2, ..., x_{i-1}, +L, x_{i+1}, ..., x_n) \le 0 \text{ for all } i \in \{1, ..., n\}. \quad (1)$$

Then, $f(x) = o$ *has a solution in* Ω*.*

For $n = 1$ Theorem 1 reduces to the well-known intermediate-value theorem illustrated in Figure 1. Miranda proved his

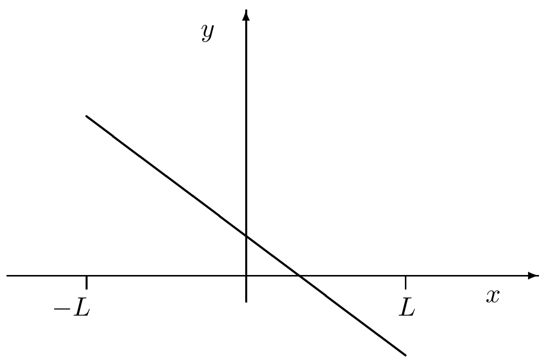

Fig. 1

theorem using the Brouwer fixed point theorem. Using the Brouwer degree of a mapping, Vrahatis gave another short proof of Theorem 1 (see [7]). Following this proof it is easy to see that Theorem 1 is also true, if L is dependent of i; i.e., Ω can also be a rectangle and need not to be a cube. Even some L_i can be zero. Very often, the theorem of Miranda is stated as in the following corollary (see also [2], [3], [5]), which is not the theorem of Miranda in its original form, but a consequence of it.

Corollary 1 *Let* $\hat{x} \in R^n$, $L \in R^n$, $L \ge o$, Ω *be the rectangle* $\Omega := \{x \in R^n : |x_i - \hat{x}_i| \le L_i,\ i = 1, ..., n\}$ *and* $f : \Omega \to R^n$ *be a continuous function on* Ω*. Also let*

$$F_i^+ := \{x \in \Omega : x_i = \hat{x}_i + L_i\}, \quad F_i^- := \{x \in \Omega : x_i = \hat{x}_i - L_i\}, \quad i = 1, ..., n$$

be the pairs of parallel opposite faces of the rectangle Ω*. If for all* $i = 1, ..., n$

$$f_i(x) \cdot f_i(y) \le 0 \quad \text{for all } x \in F_i^+ \text{ and for all } y \in F_i^+,$$

then there exists some $x^* \in \Omega$ *satisfying* $f(x^*) = o$*.*

In principle, Corollary 1 says that Theorem 1 is also true if the $\le$-sign and the $\ge$-sign are exchanged with each other in (1). Corollary 1 also says that Theorem 1 is not restricted to a rectangle with 0 as its center. For $n = 1$ Corollary 1 is illustrated in Figure 2.

2 The infinite-dimensional case

It is well-known that extending the techniques from the finite-dimensional to the infinite-dimensional spaces presents a serious problem (see [1]). The basic reason for this is that the unit sphere (more generally any closed bounded set) in the finite-dimensional case is compact while the unit sphere in the infinite-dimensional case is not compact. Therefore, we only get a

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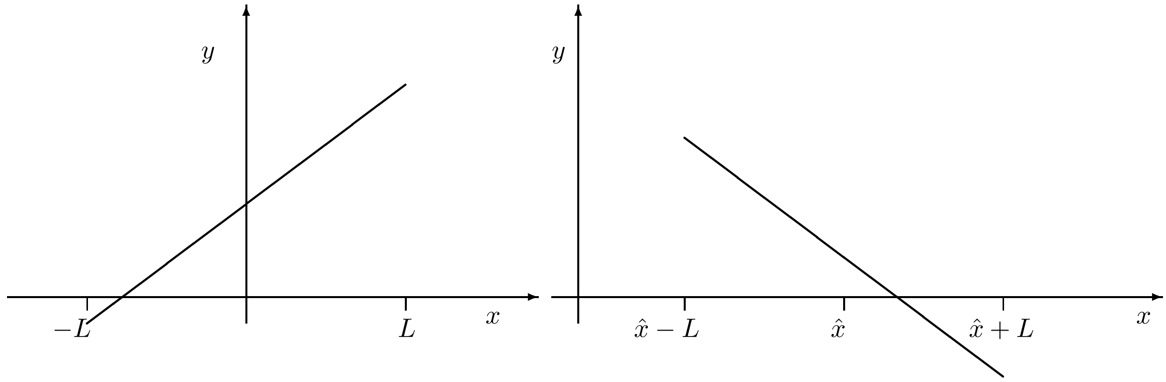


Fig. 2

fixed point version of Theorem 1, and only compact mappings can be considered, where a mapping $g : V \to V$ is called compact, if it is continuous and if the closure $\overline{f(A)}$ of $f(A)$ is compact whenever A is bounded in V.

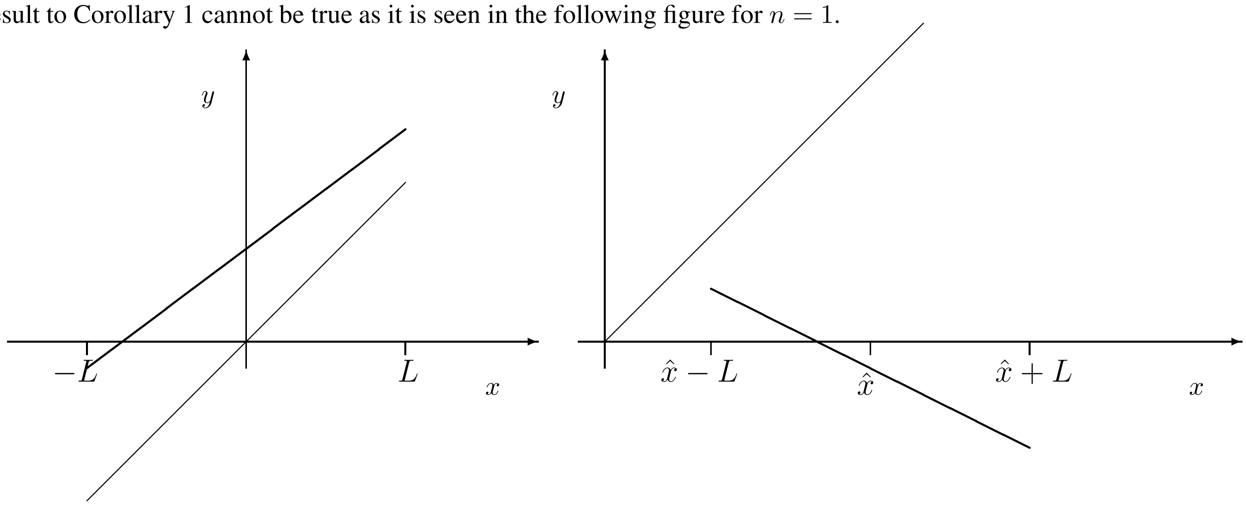
We consider the infinite-dimensional Hilbert space of all square summable sequences of real numbers, denoted by l^2 and equipped with the natural order $x, y \in l^2 : x \le y :\Leftrightarrow x_n \le y_n \, \forall n \in N$.

Theorem 2 *Let* $L = \{l_n\}_{n=1}^{\infty} \in l^2$, $l_n > 0 \, \forall n \in N$. *Let* $\Omega = \{x \in l^2 : |x_n| < l_n \, \forall n \in N\}$ *and suppose that the mapping* $g : \overline{\Omega} \to l^2$ *is compact on the closure* $\overline{\Omega}$ *of* Ω*, and*

$$g_n(x_1, x_2, ..., x_{n-1}, -l_n, x_{n+1}, ...) \ge 0, \quad g_n(x_1, x_2, ..., x_{n-1}, +l_n, x_{n+1}, ...) \le 0 \text{ for all } n \in N.$$

Then, $g(x) = x$ *has a solution in* Ω.

The proof is done by the Leray-Schauder degree of a mapping ([6]) and it is also true in R^n of course. However, an analogous result to Corollary 1 cannot be true as it is seen in the following figure for $n = 1$.



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