PAMM · Proc. Appl. Math. Mech. **7**, 2060057–2060058 (2007) / **DOI** 10.1002/pamm.200700799

On Tamir's algorithm for solving the nonlinear complementarity problem

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Some comments concerning Tamir's algorithm for solving the nonlinear complementarity problem are given.

1 Introduction

Given a vector $f = (f_1, ..., f_n)^{\mathrm{T}}$ of n real, nonlinear functions of a real vector $x = (x_1, ..., x_n)^{\mathrm{T}}$, the nonlinear complementarity problem $NCP(f)$ is to find a vector x such that

$$f(x) \geq 0, \quad x \geq 0, \quad x^{\mathrm{T}} f(x) = 0,$$

or to show that no such vector exists (see Facchinei and Pang [2] or Harker and Pang [4]). Here, the $\geq$-sign is meant componentwise.

In 1974, Tamir [7] published an algorithm for solving the $NCP(f)$ for the case that f is a so-called Z-function, where f is called a Z-function if for any $x \in \mathbb{R}^n$ the functions $\varphi_{ij}(t) := f_i(x + te_j)$, $i \neq j$, $i, j = 1, ..., n$ are antitone and e_j denotes the jth unit vector. Tamir's algorithm is a generalization of Chandrasekaran's algorithm which solves the linear complementarity problem for the case that the given matrix M is a so-called Z-matrix (see Chandrasekaran [1]).

2 Tamir's algorithm

Tamir's algorithm is given in Table 1, where $\mathbb{R}^k_+$ denotes the positive orthant of $\mathbb{R}^k$; i.e., $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x_j \geq 0,\, j = 1, ..., k\}$. We remark that the pseudocode in Table 1 is not the original pseudocode presented by Tamir. We have removed the modified Jacobi process. Instead, we use the lines 5-7.

begin
$k := 0\,;\ z := 0\,;\ J := \emptyset\,;$
if $f(z) \geq 0$ **then goto** 10

else repeat $k := k + 1;$
  choose $i_k \in \{1, ..., n\}$ with $f_{i_k}(z) < 0\,;$
  $J := J \cup \{i_k\}\,;$
  let $J = \{i_1, ..., i_k\}$ and $g^{(k)} : \mathbb{R}^k_+ \to \mathbb{R}^k$ be defined as

$$\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \mapsto \begin{pmatrix} f_{i_1}(\sum\limits_{j=1}^{k} t_j e_{i_j}) \\ \vdots \\ f_{i_k}(\sum\limits_{j=1}^{k} t_j e_{i_j}) \end{pmatrix};$$

5: let $M^{(k)} := \{t \in \mathbb{R}^k_+ : g^{(k)}(t) = 0,\ t_j \geq z_{i_j},\ j = 1, ..., k-1\};$
6: **if** $M^{(k)} \neq \emptyset$ **then**
7: **begin** $t^{(k)} := \inf M^{(k)};\ z := \sum\limits_{j=1}^{k} t_j^{(k)} e_{i_j}$ **end**
  else begin write('NCP(f) has no solution'); **goto** 20 **end**;
  until $f(z) \geq 0\,;$
10: **write**('The solution is ',z);
20: **end.**

Table 1 Tamir's algorithm

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n	$\tilde{s}$	running time
10	1.349931	0.001 s
50	1.372619	0.017 s
100	1.379208	0.114 s
150	1.390799	0.720 s
200	1.389587	1.507 s
250	1.388859	3.962 s
500	1.387397	20.478 s

$\varepsilon = 10^{-5}$

n	$\tilde{s}$	running time
10	1.349931	0.001 s
50	1.372619	0.028 s
100	1.393210	0.201 s
150	1.390799	0.831 s
200	1.389587	2.192 s
250	1.388859	4.577 s
500	1.393042	29.514 s

$\varepsilon = 10^{-11}$

Table 2

3 Numerical examples

We consider the ordinary free boundary problem:

$$\left.\begin{array}{c} \text{Find } s > 0 \text{ and } z(x) : [0,\infty) \to \mathbb{R} \text{ such that} \\ z''(x) = \sqrt{1 + z(x)^2}, \ \text{ for } x \in [0, s], \\ z(0) = 1, \quad z'(s) = 0, \\ z(x) = 0, \ \text{ for } x \in [s, \infty). \end{array}\right\} \tag{1}$$

One can show that (1) has a unique solution, say $\{\hat{s}, \hat{z}(x)\}$, and that $\hat{s} \leq \sqrt{2}$, see Schäfer [5] and Thompson [8]. Choosing $n \in \mathbb{N}$ and setting $l := \frac{1}{n+1}\sqrt{2}$, $x_i := i \cdot l$, $z_i :\approx \hat{z}(x_i)$, $i := 1, ..., n$, the $NCP(f)$ is arising with $f(z) = Mz + \Phi(z) + q$ where

$$M = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}, \ \Phi(z) = l^2 \begin{pmatrix} \sqrt{1+z_1^2} \\ \vdots \\ \vdots \\ \vdots \\ \sqrt{1+z_n^2} \end{pmatrix}, \ q = \begin{pmatrix} -1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}.$$

Obviously, f is a continuous Z-function. Furthermore, it is well-known that M is regular satisfying $M^{-1} \geq O$. Therefore, it is easy to see that $f(z)$, $z \geq 0$ is injective. As a result, applying Tamir's algorithm for solving $NCP(f)$, all sets $M^{(k)}$ are either empty or a singleton. In contrast to the original paper of Tamir [7], the method for calculating a zero of $g^{(k)}$ is not fixed in Table 1. So, it is left to the programmer which method for calculating a zero is chosen.

The results presented in Table 2 are based on the following implementation (see Hammer [3]): The input data are n and the tolerance $\varepsilon > 0$. As the method for calculating a zero of $g^{(k)}$ Newton's method was chosen, where

$$t_{start} := \begin{cases} 0 & \text{if } k = 1 \\ \begin{pmatrix} t^{(k-1)} \\ 0 \end{pmatrix} & \text{if } k > 1 \end{cases}$$

was taken as the starting point, respectively. If $z_i > 0$ and $z_{i+1} = 0$, then $\tilde{s} := \frac{1}{2}(x_i + x_{i+1})$ was taken as an approximation for $\hat{s}$. See Table 2 for some examples. Note, that the exact value of $\hat{s}$ satisfies $\hat{s} \in [1.393206, 1.397715]$; see Schäfer [6].

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