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Verification Methods for the Horizontal Linear Complementarity Problem

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1 Introduction

Given $N, M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$ the horizontal linear complementarity problem (HLCP) is to find two vectors $w,\, z \in \mathbb{R}^n$ satisfying

$$Nw - Mz = q, \quad w \geq o,\, z \geq o, \quad w^{\mathrm{T}} z = 0$$

or to show that no such vectors exist. Here, the $\geq$-sign is meant componentwise. If N is the identity, then the HLCP is a classical LCP. For an application of the HLCP we refer to [3].

Sometimes by reordering of the columns of N and of M the new N is invertible and the HLCP can be reduced to the classical LCP. See [5]. However, nothing guarantees the existence of such a reordering.

Example 1: Let

$$N = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

Then no reordering of any columns of N and of M results in an invertible matrix. □

Moreover, if the reduction is possible, the matrix inversion might be ill-conditioned. So we attack the HLCP directly. See also [3], [4], [7].

Lemma 1: *Let $N, M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. Furthermore, let*

$$f(x) := (N + M)x - (N - M)|x| + q, \tag{1}$$

where $|x| = (|x_i|) \in \mathbb{R}^n$. If $f(x^*) = o$, then

$$w := |x^*| - x^*, \quad z := |x^*| + x^* \tag{2}$$

is a solution of HLCP defined by N, M, q.

Proof: From (2) we have

$$w \geq o, \quad z \geq o. \tag{3}$$

On the other hand, it is

$$\left.\begin{array}{lll} x_i^* \geq 0 & \Rightarrow & w_i = 0 \\ x_i^* < 0 & \Rightarrow & z_i = 0 \end{array}\right\} \quad i = 1, ..., n. \tag{4}$$

Finally, we have $o = f(x^*) = (N + M)x^* - (N - M)|x^*| + q$. This is equivalent to

$$N(|x^*| - x^*) - M(|x^*| + x^*) = q. \tag{5}$$

Therefore, due to (2)-(5), w and z solve the HLCP defined by N, M and q. □

If N is the identity; i.e., $N = I$, and if $I + M$ is regular, then

$$f(x^*) = o \quad \Leftrightarrow \quad x^* = (I + M)^{-1}\Big((I - M) \cdot |x^*| - q\Big). \tag{6}$$

The fixed point iteration based on (6) was developed in [6] and was called the modulus algorithm. See also [2].

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for $j := 1$ **to** n **do begin**
if $\underline{x}_j \geq 0$ **then** $G(\hat{x},[x]).{}_j := 2 \cdot M._j$
else if $\overline{x}_j \leq 0$ **then** $G(\hat{x},[x]).{}_j := 2 \cdot N._j$
else if $\hat{x}_j \geq 0$ **then** $G(\hat{x},[x]).{}_j := N._j + M._j - [\frac{\hat{x}_j + \underline{x}_j}{\hat{x}_j - \underline{x}_j}, 1] \cdot (N._j - M._j)$
else $G(\hat{x},[x]).{}_j := N._j + M._j - [-1, \frac{\overline{x}_j + \hat{x}_j}{\overline{x}_j - \hat{x}_j}] \cdot (N._j - M._j)$
end.

Table 1 Algorithm for calculating $G(\hat{x},[x])$

2 Verification methods

Let $[x] = ([\underline{x}_i, \overline{x}_i])$ be an n-dimensional interval vector; i.e., $[x]$ is a vector where the components are intervals. Furthermore, let $f : [x] \to \mathbb{R}^n$ be a continuous function. If there exists an interval matrix $G(\hat{x},[x])$ such that for fixed $\hat{x} \in [x]$ one can conclude that

$$\forall\, y \in [y]\ \exists\, G(\hat{x},y) \in G(\hat{x},[x]) :\ f(y) - f(\hat{x}) = G(\hat{x},y) \cdot (y - x), \tag{7}$$

we define the so-called Krawczyk-Operator $K(\hat{x},R,[x]) := \hat{x} - R \cdot f(\hat{x}) + (I - R \cdot G(\hat{x},[x])) \cdot ([x] - \hat{x})$, where $R \in \mathbb{R}^{n \times n}$ has to be a regular matrix. If

$$K(\hat{x},R,[x]) \subseteq [x], \tag{8}$$

then there exists $\xi \in K(\hat{x},R,[x])$ with $f(\xi) = o$. See [1]. Concerning the function f from (1) a corresponding interval matrix $G(\hat{x},[x])$ satisfying (7) is given in Table 1. The proof of this fact will be published elsewhere.

Example 2: Let $N = 1$, $M = \frac{1}{2}$ and $q = -\frac{1}{10}$. Then, $f(x) = (N + M) \cdot x - (N - M) \cdot |x| + q = \frac{3}{2}x - \frac{1}{2}|x| - \frac{1}{10}$. Since $f(\frac{1}{20}) = -\frac{1}{20}$ we have that $\tilde{x} := \frac{1}{20}$ is an ε-approximation for $\varepsilon = 0.1$ and for $f(x) \overset{!}{=} 0$. Defining $\hat{x} := \tilde{x} = \frac{1}{20}$ and $[x] := [\tilde{x} - \varepsilon, \tilde{x} + \varepsilon] = [-\frac{1}{20}, \frac{3}{20}]$, Table 1 gives $G(\hat{x},[x]) = 1 + \frac{1}{2} - [\frac{\frac{1}{20} - \frac{1}{20}}{\frac{1}{20} + \frac{1}{20}}, 1](1 - \frac{1}{2}) = [1, \frac{3}{2}]$. Choosing $R := \frac{4}{5}$ we get

$$K(\hat{x},R,[x]) = \frac{1}{20} - \frac{4}{5}(-\frac{1}{20}) + (1 - \frac{4}{5} \cdot [1, \frac{3}{2}]) \cdot ([-\frac{1}{20}, \frac{3}{20}] - \frac{1}{20}) = [\frac{7}{100}, \frac{11}{100}] \subseteq [x].$$

So, by [1], within $[\frac{7}{100}, \frac{11}{100}]$ there is a zero of f. Using Lemma 1 we get that $w^* = 0$ and some $z^* \in [0.14, 0.22]$ are a solution of the HLCP. For the sake of completeness, we mention that the unique solution is $w^* = 0$, $z^* = \frac{2}{10}$. □

Bigger examples, where the verification test (8) is done by a machine, will be published elsewhere.

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