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# A BAYESIAN APPROACH TO INCORPORATE MODEL AMBIGUITY IN A DYNAMIC RISK MEASURE

NICOLE BÄUERLE* AND ANDRÉ MUNDT‡

Abstract. In this paper we consider an explicit dynamic risk measure for discrete-time payment processes which have a Markovian structure. The risk measure is essentially a sum of conditional Average Value–at–Risks. Analogous to the static Average Value–at–Risk, this risk measures can be reformulated in terms of the value functions of a dynamic optimization problem, namely a so-called Markov decision problem. This observation gives a nice recursive computation formula. Afterwards, the definition of the dynamic risk measure is generalized to a setting with incomplete information about the risk distribution which can be seen as model ambiguity. We choose a parametric approach here. The dynamic risk measure is again defined as the sum of conditional Average Value–at–Risks or equivalently is the solution of a Bayesian decision problem. Finally, it is possible to discuss the effect of model ambiguity on the risk measure: Surprisingly, it may be the case that the risk decreases when additional "risk" due to parameter uncertainty shows up. All investigations are illustrated by a simple but useful coin tossing game proposed by Artzner and by the classical Cox–Ross-Rubinstein model.

## 1. Introduction and Motivation

Dealing with *risk* is particularly important for financial corporations such as banks and insurance companies, since they face uncertain events in various lines of their business. An important tool in risk management is the implementation of *risk measures*, in particular ones which go beyond the variance. Since the risk is modeled by random quantities, i. e. random variables or, in a dynamic framework, stochastic processes, the established procedure is first to estimate or approximate the distribution of such positions and then to quantify the risk due to the unknown outcome. The value of a risk measure can be interpreted as the present (and future in case of dynamic risk measures) monetary value which is necessary to keep the risk acceptable. However, in contrast to most approaches so far, it seems reasonable to incorporate the model risk which stems from calibrating the risk process into the risk assessment. We will present an approach here which deals with this topic and which could be extended in various directions.

In this paper we introduce an explicit dynamic risk measure, based on a proposal by Pflug and Ruszczyński (2005) for stochastic (risk) processes which have a Markovian structure. This class of risk processes is large enough to include the most popular financial models in discrete-time. The dynamic risk measure is a

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sum of conditional Average Value–at–Risk and it can be shown that it has a lot of desirable properties. Since the static Average Value–at–Risk is a solution of a (static) optimization problem we will show that our dynamic risk measure is a solution of a dynamic (Markovian) optimization problem. This dynamic problem turns out to be a Markov Decision Problem. The advantage of this representation is the recursive computation formula which we obtain. Moreover, we generalize the dynamic risk measure to situations with incomplete information about the risk distribution which can be seen as model risk. In order to do this we use a parametric model and take a Bayesian approach, i.e. we suppose that the random variables of the risk process depend on an unknown parameter $\theta \in \Theta$ and we have a prior distribution $\mu$ for this unknown parameter. Of course in a dynamic context, the information about the unknown parameter which is gained over time by observing the risk process should be incorporated into the risk measurement procedure. Thus, we can in principle use the same definition of the dynamic risk measure w.r.t. the corresponding "Bayesian" probability measure. It can be shown that this is equivalent to a Bayesian decision problem. Thus, we get again a recursive computation algorithm and this recursive representation enables us to discuss the influence of model ambiguity. Namely, we are interested in comparing the risk in a situation where the parameter is unknown with prior distribution $\mu$ to a situation where the parameter is known and equal to $\int \theta\mu(d\theta)$. It turns out that such a comparison in general is very hard. We only get some partial answers.

Let us illustrate the approach by a simple example due to Artzner which has also been cited in Pflug and Ruszczyński (2005): Imagine that it is possible to play one of the following two games. A coin is tossed three times and the player can win one Euro in two different ways. In the first game, this is the case if the last throw shows heads. The second variant awards the player with one Euro if at least two of the three throws show heads. Now the question arises which of the two games is less risky for the player? If we know that the coin is fair then the approach in Pflug and Ruszczyński (2005) returns a number which also incorporates the way information is gained in this game, leading to a smaller risk for game 2. Now assume that the probability for heads is not known but a prior distribution is available (e.g. an expert opinion). Here again it is crucial that information is gained (also about the unknown probability for heads) as the coin is tossed. This information should be included in the risk measurement procedure. It indeed turns out that if the prior distribution is $\mu = \mathcal{U}(0,1)$ then the risk (measured with our Bayesian risk measure) is smaller than in the case the probability for heads is known and equal to $\frac{1}{2}$.

One of the first papers to investigate the notion of dynamic risk measures was Wang (1999). By now there are quite a number of papers on dynamic risk measures in discrete time, we just mention e.g. Riedel (2004), Burgert (2005), Weber (2006), Delbaen (2006), Föllmer and Penner (2006), Frittelli and Scandolo (2006). These papers show properties or characterize dynamic risk measures. More explicit dynamic risk measures can be found in Boda and Filar (2006) and Roorda and Schumacher (2007) (both consider dynamic Average Value–at–Risk) or the papers by Pflug (2001, 2006) and Pflug and Ruszczyński (2005). In Pflug (2006) the role of information in risk measurement is highlighted. All these papers assume that there is no explicit model ambiguity. In Cont et al. (2006) the authors deal with model risk by investigating robustness properties and empirical risk measures.

The paper is organized as follows. Section 2 gives a short review of static risk measures and Section 3 highlights the role of information in measuring risk. In the next section a dynamic version of a risk measure for processes introduced by Pflug and Ruszczyński (2005) is defined. We will be able to give a reformulation in terms of Markov decision processes if the income processes considered possess a Markovian structure. This can be advantageous for computational purposes. In Section 5 we generalize the model to the case where the generating random variables of the income process depend on some unknown parameter. A Bayesian risk measure is introduced which is again a sum of condition Average Value–at–Risk. This time it can be solved by a Bayesian decision model. In Section 6 we compare the risk measures with and without model ambiguity. Some conditions are given under which the risk measures can be compared. It turns out that in the Cox–Ross-Rubinstein model these conditions are satisfied and uncertainty decreases the risk.

## 2. Static Risk Measures

In this section we recall some facts about static risk measures which we use in the sequel. (For a thorough treatment see e.g. Föllmer and Schied (2004)). Negative values of financial positions are interpreted as a loss. By $L^1(\Omega, \mathcal{F}, \mathbb{P})$ we denote the equivalence classes of integrable random variables.

**Definition 2.1.** A mapping $\rho : L^1(\Omega, \mathcal{F}, \mathbb{P}) \to \overline{\mathbb{R}}$ is called a *(static) risk measure* if it is *monotone* and *translation invariant.*

**Example 2.2.** A classical and famous risk measure is Value–at–Risk. For a risk $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and level $\gamma \in (0, 1)$ it is defined by

$$\mathrm{VaR}_\gamma(X) := q_\gamma^-(-X) = \inf\{x \in \mathbb{R} \mid \mathbb{P}(-X \leq x) \geq \gamma\}.$$

Consequently, $\mathrm{VaR}_\gamma(X)$ represents the smallest monetary value such that $-X$ does not exceed this value at least with probability $\gamma$. Hence, we are interested in calculating the Value–at–Risk for large values of $\gamma$, e. g. set $\gamma = 0.95$ or $\gamma = 0.99$.

**Example 2.3.** Another well–known risk measure called Average Value–at–Risk can be defined by using Value–at–Risk. For $\gamma \in (0, 1)$ it represents an average of the Value–at–Risk to all safety levels larger than $\gamma$, formally for $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$

$$\mathrm{AVaR}_\gamma(X) := \frac{1}{1-\gamma} \int_\gamma^1 \mathrm{VaR}_u(X) \mathrm{d}u, \quad \gamma \in (0, 1). \tag{2.1}$$

The definition, which is continuous and strictly increasing in the safety level $\gamma$, is also valid for $\gamma = 0$ and we obtain $\mathrm{AVaR}_0(X) = \mathbb{E}[-X]$. By Uryasev and Rockafellar (2002), Average Value–at–Risk can be represented by a simple convex optimization problem. Fix $\gamma \in (0, 1)$ and $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then we have

$$\mathrm{AVaR}_\gamma(X) = \inf_{b \in \mathbb{R}} \Big\{ b + \frac{1}{1-\gamma} \mathbb{E}\left[(-X - b)^+\right] \Big\}. \tag{2.2}$$

In fact, the infimum is attained in $b^* = \mathrm{VaR}_\alpha(X)$. Besides the properties mentioned in Definition 2.1, the Average-Value-at-Risk is also convex.

A law invariant risk measure $\rho$ (i.e. $\rho$ depends only on the distribution of the risk) such as Value–at–Risk or Average Value–at–Risk can be regarded as a mapping on $\mathcal{M}^1$, the set of all distributions on $\mathbb{R}$ with finite mean in the sense that $\rho(\mu) := \rho(X)$

whenever $\mathcal{L}(X) = \mu$ for $\mu \in \mathcal{M}^1$. Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $Y$ be another random variable on $\mathbb{R}$. Then the conditional distribution $\mathcal{L}(X \mid Y = y)$ exists for every $y \in \mathbb{R}$, is unique almost surely and we can define:

$$\rho(X \mid Y = y) := \rho(\mathcal{L}(X \mid Y = y)), \quad y \in \mathbb{R}. \tag{2.3}$$

Note the following result (for a proof see Corollary 1.1 in Mundt (2007)).

**Lemma 2.4.** *Let $\rho = \mathrm{VaR}_\gamma$ or $\rho = \mathrm{AVaR}_\gamma$ for some $\gamma \in (0,1)$. Then the mapping $y \mapsto \rho(X \mid Y = y)$ defined in (2.3) is $(\mathcal{B}, \mathcal{B})$–measurable.*

## 3. Risk Measures and Information

Up to now most approaches in finding risk measurement procedures assume that the law governing the risk or the income process is known. In applications however the probability distribution is unknown and has to be estimated from data. This implies another risk coming from mis-specifying the model. Some authors (see e.g. Cont et al. (2006)) thus discuss *robustness* of the risk measure w.r.t. changes in the model. Obviously, if more data is available, then the range of possible models is smaller and hence more information helps to determine the risk more appropriately. Note however that the well-established coherent risk measures already involve some kind of model risk since the dual representation for a risk $X \in L^\infty$ is given by

$$\sup_{Q \in M} \mathbb{E}_Q[-X]$$

where $M$ is a certain class of probability measures. But often (see e.g the Average Value–at–Risk) this class $M$ is not sufficient.

When we turn to a dynamic risk measurement, the notion of *information* also has another dimension. This has e.g. been explained in Pflug (2006) and is illustrated with the coin tossing example of the introduction. Different information which is available for the decision maker in terms of different measurability conditions, leads to different risk assignments. In this paper now we are going to combine these two aspects. More precisely, we suppose that model ambiguity is given by an unknown parameter of the risk distribution. Further the risk is measured dynamically over a certain time horizon and thus the information about the unknown parameter which is obtained during the time range of the model is incorporated. This approach is explained in Section 5 for a Bayesian parametric model. However, it seems to be difficult to measure the "value of information" in terms of a more concentrated prior distribution (this information can be compared w.r.t. the Blackwell information order or simply the convex order).

## 4. A Risk Measure by Pflug and Ruszczyński and its MDP Version

We apply our risk measures only to special income processes in finite and discrete time which have a Markovian structure. This is not a severe restriction because many models in finance (e.g. stock price models, interest rate models) are Markovian or can be made Markovian with moderate effort.

4.1. **Income Process.** The income or risk processes we consider are defined in discrete time with time horizon $T \in \mathbb{N}$ and evolve as follows: Assume that we are given independent random variables $Y_1, \dots, Y_T$ on $(\Omega, \mathcal{F}, \mathbb{P})$ which generate a filtration on the probability space via

$$\mathcal{F}_0 := \{\emptyset, \Omega\}, \quad \mathcal{F}_t := \sigma(Y_1, \dots, Y_t), \quad t = 1, \dots, T.$$

We further assume that a Markov chain $(Z_t)_{t=0,1,\dots,T}$ is given through

$$Z_0 \equiv c \in \mathbb{R}, \quad Z_t := g_t(Z_{t-1}, Y_t), \quad t = 1, \dots, T,$$

where $g_t : \mathbb{R}^2 \to \mathbb{R}$ are $(\mathcal{B}^2, \mathcal{B})$–measurable functions. We restrict ourselves to income processes $I = (I_1, \dots, I_T) \in \times_{t=1}^T L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ for which $I_t$ only depends on $Z_{t-1}$ and on $Y_t$ for $t = 1, \dots, T$. In other words, we assume that there exist $(\mathcal{B}^2, \mathcal{B})$–measurable functions $h_t : \mathbb{R}^2 \to \mathbb{R}$, such that

$$I_t = h_t(Z_{t-1}, Y_t), \quad t = 1, \dots, T. \tag{4.1}$$

Denote the set of these income processes by $\mathcal{X}^{\mathrm{M}}$. Obviously $\mathcal{X}^{\mathrm{M}}$ includes all Markovian processes. Examples are given in Section 4.4.

4.2. **Definition and Properties of the Risk Measure.** Next we define a concrete dynamic risk measures. In general, a dynamic risk measure $\rho$ is a mapping $\rho : \Omega \times \{0, 1, \dots, T-1\} \times \mathcal{X}^{\mathrm{M}} \to \overline{\mathbb{R}}$ such that $(\rho_t(I))_{t=0,1,\dots,T-1}$ is an $(\mathcal{F}_t)_{t=0,1,\dots,T-1}$–adapted process and satisfies some reasonable properties. We are not going to discuss properties of dynamic risk measures in general, but will list a number of properties which are satisfied by the dynamic risk measure in Definition 4.1. For a general overview on dynamic risk measures and further properties see Riedel (2004), Burgert (2005) and Mundt (2007).

We are now ready to introduce a dynamic version of a risk measure proposed by Pflug and Ruszczyński (2005). There the authors considered more general income processes. The restriction to income processes $\mathcal{X}^{\mathrm{M}}$ seems to be natural.

Let $(c_t)_{t=1,\dots,T}$ be a sequence of discounting factors with $c_t > c_{t+1} > 0$, $t = 1, \dots, T$, and $(\gamma_t)_{t=1,\dots,T}$ be a sequence of safety levels with $\gamma_t \in (0, 1)$, $t = 1, \dots, T$. Furthermore set $\lambda_t := \frac{c_{t+1}}{c_t}$, $t = 1, \dots, T$, and define for every $t$ a static risk measure via

$$\rho^{(t)}(X) := \lambda_t \mathbb{E}[-X] + (1 - \lambda_t)\mathrm{AVaR}_{\gamma_t}(X), \quad X \in L^1(\Omega, \mathcal{F}, \mathbb{P}). \tag{4.2}$$

**Definition 4.1.** For $t = 0, 1, \dots, T-1$ set

$$\rho_t^{\mathrm{PR}}(I) := \mathbb{E}\Bigg[ \sum_{k=t+1}^{T} \frac{c_k}{c_t} \cdot \rho^{(k)}(I_k \mid \mathcal{F}_{k-1}) \,\Big|\, \mathcal{F}_t \Bigg], \quad I \in \mathcal{X}^{\mathrm{M}}.$$

**Remark 4.2.**
(a) This risk measure was defined in Pflug and Ruszczyński (2005) for $t = 0$ via a different optimization model than the one which we will consider below. In Pflug (2006), it is defined in a more general setting.
(b) Since $\frac{c_k}{c_t}$ is just the discount factor from time $k$ to $t$, we see that the risk measure at time $t$ is the conditional expectation under $\mathcal{F}_t$ of a discounted sum of convex mixtures of two conditional static risk measure applied to each component of the process $I \in \mathcal{X}^{\mathrm{M}}$.
(c) The risk measure may be seen as a dynamic version of the AVaR.

We next list a number of properties, the dynamic risk measure of Definition 4.1 has. Part of the properties are already shown in Pflug and Ruszczyński (2005), proofs for the others can be found in Mundt (2007).

**Proposition 4.3.** *The dynamic risk measure $\rho^{\mathrm{PR}}$ of Definition 4.1 has the following properties:*

(a) $\rho^{\text{PR}}$ *is* monotone, *i. e. for all* $I^{(1)}, I^{(2)} \in \mathcal{X}^M$ *with* $I_t^{(1)} \leq I_t^{(2)}$ *for all* $t = 1, \dots, T$ *it holds*

$$\rho_t^{\text{PR}}(I^{(1)}) \geq \rho_t^{\text{PR}}(I^{(2)}), \quad t = 0, 1, \dots, T-1.$$

(b) $\rho^{\text{PR}}$ *is* homogeneous, *i. e. for all* $t = 1, \dots, T$, $I \in \mathcal{X}^M$ *and* $\Lambda \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ *with* $\Lambda > 0$ *we have*

$$\rho_t^{\text{PR}}(\Lambda \cdot I) = \Lambda \cdot \rho_t^{\text{PR}}(I).$$

(c) $\rho^{\text{PR}}$ *is* translation invariant, *i.e. for all* $Y = (0, \dots, 0, Y_{t+1}, \dots, Y_T) \in \mathcal{X}^M$ *which are predictable and* $\sum_{k=t+1}^T c_k Y_k$ *is* $\mathcal{F}_t$*-measurable it holds that*

$$\rho_t^{\text{PR}}(I + Y) = \rho_t^{\text{PR}}(I) - \sum_{k=t+1}^{T} \frac{c_k}{c_t} Y_k.$$

(d) $\rho^{\text{PR}}$ *is* subadditive, *i. e. for all* $I^{(1)}, I^{(2)} \in \mathcal{X}^M$ *and* $t = 1, \dots, T$ *it holds*

$$\rho_t^{\text{PR}}(I^{(1)} + I^{(2)}) \leq \rho_t^{\text{PR}}(I^{(1)}) + \rho_t^{\text{PR}}(I^{(2)}), \quad t = 0, 1, \dots, T-1.$$

(e) $\rho^{\text{PR}}$ *is* independent of the past, *i. e. for every* $t = 0, 1, \dots, T-1$ *and* $I \in \mathcal{X}^M$, $\rho_t^{\text{PR}}(I)$ *does not depend on* $I_1, \dots, I_{t-1}$. *More precisely, for every* $I \in \mathcal{X}$ *it holds*

$$\rho_t^{\text{PR}}(I_1, \dots, I_T) = \rho_t^{\text{PR}}(0, \dots, 0, I_t, \dots, I_T), \quad t = 0, 1, \dots, T-1.$$

4.3. **Formulation as a Markov decision problem (MDP).** It is well-known that the Average Value–at–Risk is the solution of an optimization problem (cf. Equation (2.2)). Analogously we show that $\rho^{\text{PR}}$ is the solution of a *dynamic* optimization problem, namely a so-called Markov decision problem which can be interpreted as a dynamic cash balance problem. The advantage of this representation is that we obtain a recursive computation algorithm (see (4.3)). For the general MDP theory compare e.g. Hernández-Lerma and Lasserre (1996), Puterman (1994) or Hinderer (1970). For abbreviation let

$$q_0 := 0, \quad q_t := \frac{c_t - c_{t+1}\gamma_t}{1 - \gamma_t} > c_t, \quad t = 1, \dots, T.$$

The data of the Markov decision problem is as follows:

- The *state space* is denoted by $S \subset \mathbb{R}^2$ and equipped with the $\sigma$–algebra $\mathcal{S} := \mathcal{B}_S^2$. Let $s := (w, z) \in S$ be an element of the state space, where $w, z$ represent realizations of a wealth process $(W_t)_{t=1,\dots,T}$ and the generating Markov chain $(Z_t)_{t=1,\dots,T}$, respectively, at time $t$.
- The *action space* is $\mathbb{R}$ equipped with $\mathcal{B}$. Then $a \in \mathbb{R}$ denotes the consumed amount at time $t$. Let $D := S \times \mathbb{R}$.
- The *disturbance* $Y_t$ has values in $E \subset \mathbb{R}$ equipped with $\mathcal{E} := \mathcal{B}_E$. The distribution of $Y_t$ is denoted by $Q_t$.
- The *transition function* $T_t : D \times E \to S$ at time $t = 1, \dots, T$ is given by

$$\begin{aligned} T_t(s, a, y) :&= (F_t(w, h_t(z, y), a), g_t(z, y)) \\ &= (w^+ + h_t(z, y) - a, g_t(z, y)). \end{aligned}$$

  for $(s, a, y) = (w, z, a, y) \in D \times E$. It gives the current state if the previous state is $s$, the action $a$ is chosen and if the disturbance attains the value $y$.

- The *one-step reward function* at time $t = 0, 1, \dots, T-1$ is a measurable mapping $r_t : D \to \mathbb{R}$ defined as
$$r_t(s,a) := -q_t w^- + c_{t+1} a, \quad (s,a) \in D.$$
- The *terminal reward* is a measurable mapping $V_T : S \to \mathbb{R}$ defined as
$$V_T(s) := c_{T+1} w^+ - q_T w^-, \quad s = (w,z) \in S.$$

Furthermore, let us denote by
$$F := \{f : S \to \mathbb{R} \mid f \text{ is } (\mathcal{S}, \mathcal{B})\text{–measurable}\}$$
the set of decision rules. Then $\pi \in F^t$ is a $t$-step Markov policy. For every $\pi = (f_0, \dots, f_{T-1}) \in F^T$ we recursively define the resulting Markov decision process $(X_t)_{t=1,\dots,T}$ via
$$X_0 \in S, \quad X_t := (W_t, Z_t) := T_t(X_{t-1}, f_{t-1}(X_{t-1}), Y_t), \quad t = 1, \dots, T.$$
and the value functions via
$$V_{t,\pi}(s) := \mathbb{E}^\pi_{t,s}\Big[\sum_{k=t}^{T-1} r_k(X_k, f_k(X_k)) \, + \, V_T(X_T)\Big], \quad s \in S,$$
for every $\pi = (f_t, \dots, f_{T-1}) \in F^{T-t}$ where $\mathbb{E}^\pi_{t,s}$ is the expectation w.r.t. the conditional probability $\mathbb{P}^\pi(\cdot \mid X_t = s)$ and
$$V_t(s) := \sup_{\pi \in F^{T-t}} V_{t,\pi}(s), \quad s \in S.$$
Note that
$$V_{t-1}(s) := \sup_{a \in \mathbb{R}} \big\{ r_{t-1}(s,a) + \int_E V_t(T_t(w,z,a,y)) Q_t(\mathrm{d}y) \big\}, \quad s \in S \tag{4.3}$$
holds and provides a recursive computation. We now obtain the following explicit formula for the value functions:

**Theorem 4.4.** *For $t = 0, 1, \dots, T$ and $(w,z) \in S$ the value function is given by*
$$V_t(w,z) = c_{t+1} w^+ - q_t w^- - \sum_{k=t+1}^{T} c_k \cdot \mathbb{E}[\rho^{(k)}(I_k \mid Z_{k-1}) \mid Z_t = z],$$
*and the optimal policy $\pi^* = (f^*_t, \dots, f^*_{T-1})$ is*
$$f^*_k(w,z) = w^+ - \mathrm{VaR}_{\gamma_k}(I_{k+1} \mid Z_k = z), \quad (w,z) \in S,$$
*for $k = t, \dots, T-1$.*

*Proof.* The proof is by backward induction on $t$. The case $t = T$ is trivial. We first consider the case $t = T-1$. By the value iteration (cf. e.g. Theorem 3.2.1 in Hernández-Lerma and Lasserre (1996)) we have for $(w,z) \in S$
$$\begin{aligned} &V_{T-1}(w,z) + q_{T-1} w^- \\ &= \sup_{a \in \mathbb{R}} \big\{ q_{T-1} w^- + r_{T-1}(w,z,a) + \int_E V_T(T_T(w,z,a,y)) Q_T(\mathrm{d}y) \big\} \\ &= \sup_{a \in \mathbb{R}} \big\{ c_T a + \mathbb{E}[c_{T+1}[w^+ + h_T(z,Y_T) - a]^+ - q_T [w^+ + h_T(z,Y_T) - a]^-] \big\} \\ &= c_T w^+ + c_{T+1} \mathbb{E}[h_T(z,Y_T)] - (c_T - c_{T+1}) \mathrm{AVaR}_{\gamma_T}(h_T(z,Y_T)) \\ &= c_T w^+ + c_{T+1} \mathbb{E}[I_T \mid Z_{T-1} = z] - (c_T - c_{T+1}) \mathrm{AVaR}_{\gamma_T}(I_T \mid Z_{T-1} = z), \end{aligned}$$

where we used Equation (2.2) in the last but one step. We note that the supremum is attained in

$$f^*_{T-1}(w,z) = -\mathrm{VaR}_{\gamma_T}(h_T(z,Y_T)+w^+) = w^+ - \mathrm{VaR}_{\gamma_T}(I_T \mid Z_{T-1}=z).$$

By Lemma 2.4, $V_{T-1}$ and $f_{T-1}$ are indeed measurable functions on $S$. Hence, the assertion is true for $t = T-1$.

Now assume that the assertion is true for $t \le T-1$. The value iteration yields for $(w,z) \in S$

$$\begin{aligned} &V_{t-1}(w,z) + q_{t-1}w^- \\ &= \sup_{a\in\mathbb{R}} \big\{ q_{t-1}w^- + r_{t-1}(w,z,a) + \mathbb{E}[V_t(X_t)|X_{t-1}=(w,z), a_{t-1}=a] \big\} \\ &= \sup_{a\in\mathbb{R}} \big\{ c_t a + \mathbb{E}\big[c_{t+1}[w^+ + h_t(z,Y_t)-a]^+ - q_t[w^+ + h_t(z,Y_t)-a]^-\big] \big\} \\ &\qquad + \sum_{k=t+1}^{T} c_k \mathbb{E}\big[\mathbb{E}[\rho^{(k)}(I_k|Z_{k-1}) \mid Z_t] \mid Z_{t-1}=z\big] \\ &= c_t w^+ + c_{t+1}\mathbb{E}[I_t|Z_{t-1}=z] - (c_t - c_{t+1})\mathrm{AVaR}_{\gamma_t}(I_t|Z_{t-1}=z) \\ &\qquad + \sum_{k=t+1}^{T} c_k \mathbb{E}\big[\rho^{(k)}(I_k|Z_{k-1}) \mid Z_{t-1}=z\big]. \end{aligned}$$

Hence, the statement is shown. □

As a direct consequence we have the following representation of the dynamic risk measure.

**Corollary 4.5.** *For every $t = 0,1,\dots,T-1$ it holds*

$$\rho^{PR}_t(I) = -\frac{1}{c_t}V_t(0,Z_t), \quad I \in \mathcal{X}^M.$$

This representation provides us with a way to calculate $\rho^{\mathrm{PR}}$ via (4.3).

4.4. **Examples.**

**Example 4.6.** Consider the example by Artzner with general but known success probability $\theta \in (0,1)$. We have $T = 3$ and $Y_1, Y_2, Y_3$ are independent with $\mathbb{P}(Y_t = 1) = \theta = 1 - \mathbb{P}(Y_t = 0)$, $t = 1,2,3$. Furthermore,

$$Z_0 \equiv 0, \quad Z_t = Z_{t-1} + Y_t, \quad t = 1,2,3.$$

We investigate the two resulting income processes $I^{(1)}_{\mathrm{A}} = (0,0,Y_3)$ and $I^{(2)}_{\mathrm{A}} = (0,0,1_{\{Z_2+Y_3\ge 2\}})$. Direct calculations (see also Section 6.2) show that for $\theta \in (0,1)$

$$\rho^{\mathrm{PR}}_1(I^{(1)}_{\mathrm{A}})(\omega) \begin{cases} < \rho^{\mathrm{PR}}_1(I^{(2)}_{\mathrm{A}})(\omega) &, \quad \omega \in \{Y_1 = 0\}, \\ > \rho^{\mathrm{PR}}_1(I^{(2)}_{\mathrm{A}})(\omega) &, \quad \omega \in \{Y_1 = 1\}, \end{cases}$$

If $\theta$ is chosen appropriately, all orderings are possible for the case $t = 0$. But in the most important situation $\theta = \frac{1}{2}$ or equivalently, when $I^{(1)}_{\mathrm{A}}$ and $I^{(2)}_{\mathrm{A}}$ are identically distributed, we have

$$\rho^{\mathrm{PR}}_0(I^{(1)}_{\mathrm{A}}) > \rho^{\mathrm{PR}}_0(I^{(2)}_{\mathrm{A}}).$$

In particular, we see that the dynamic risk measure $\rho^{\mathrm{PR}}$ is not law invariant. The interpretation of this last inequality is as follows: The risk at time $t = 0$ is the expectation of the conditional static risk measure $\rho^{(3)}$ of the final value $I^{(i)}_{\mathrm{A},3}$, $i = 1,2,$

given the information $\mathcal{F}_2$. When calculating the risk of $I_{\mathrm{A}}^{(1)}$, the final payment $I_{\mathrm{A},3}^{(1)} = Y_3$ is stochastically independent from this information (namely the random variable $Z_2$). Thus, the information is not used in this case. On the other hand, calculating the risk of $I_{\mathrm{A}}^{(2)}$ uses this additional information that is generated over time in order to diminish the risk of the process at time $t = 0$.

**Example 4.7.** In the first example, $I_t$ depends only on $Z_t$. Now, we consider the standard Cox-Ross-Rubinstein (CRR)-model to generate an income process, where $I_t$ depends on $Z_{t-1}$ and $Y_t$. Define the distribution of $Y_t$ by

$$\mathbb{P}(Y_t = u) = \theta = 1 - \mathbb{P}(Y_t = d),$$

where $0 < d < 1 < u$ and $\theta \in (0,1)$. Let the price process of an asset be given by $Z_0 \equiv 1$ and $Z_t := Z_{t-1} \cdot Y_t$ for $t = 0, 1, \ldots, T$. If a policy holder has one unit of this asset in her portfolio, her income from $t-1$ to $t$ is given by

$$I_t = Z_t - Z_{t-1} = (Y_t - 1) \cdot Z_{t-1}, \quad t = 1, \ldots, T.$$

The random variable $I_t$ can attain negative values of course. The income in period $(t-1, t]$ can not be formulated as a function of $Z_t$, so we have to include $Z_{t-1}$ in defining $I_t$. This is why we assume (4.1). We easily compute for $t \in \{1, \ldots, T\}$

$$\begin{aligned} \mathbb{E}[Y_t] &= d + \theta(u - d), \\ \mathrm{AVaR}_\gamma(Y_t) &= -d - 1_{[0,\theta]}(\gamma) \frac{\theta - \gamma}{1 - \gamma}(u - d), \qquad \gamma \in [0,1]. \end{aligned}$$

Then, the dynamic risk measure becomes by a straightforward calculation

$$\rho_t^{\mathrm{PR}}(I) = Z_t \cdot \sum_{k=t+1}^{T} \frac{c_k}{c_t} \cdot \mathbb{E}[Y_1]^{k-t-1} \cdot \big(1 + \rho^{(k)}(Y_k)\big),$$

concluding the example.

## 5. Extension to model ambiguity

In this section, we generalize the definition of the dynamic risk measure from Section 4 to models with unknown risk distribution. Indeed, in reality it is seldom the case that a risk distribution is given, instead we have data. There are now different possible approaches to deal with this situation: in a non-parametric model one can consider an empirical risk measure or measure the risk of the empirical distribution. In a parametric model one can for example estimate the parameter and then measure the risk (for these approaches see Cont et al. (2006)). But in a dynamic context this estimate has to be updated. In what follows we consider a parametric model and take a Bayesian approach.

5.1. **Model Setup.** We assume that all generating random variables $Y_t$, $t = 1, \ldots, T$, depend on a parameter $\vartheta \in \Theta \subset \mathbb{R}$ which might be unknown and therefore is modeled as a random variable on the given probability space with unknown distribution $\mathcal{L}(\vartheta)$. If $\vartheta$ is known, for example with value $\theta \in \Theta$, its distribution reduces to $\mathcal{L}(\vartheta) = \delta_\theta$, where $\delta_\theta$ denotes the distribution concentrated in $\theta$. We make the usual assumption:

**Assumption 1:** Under $\vartheta$, the random variables $Y_1, \dots, Y_T$ are (conditionally) independent.

Sometimes, the parameter $\vartheta$ can be interpreted as an unknown probability. In this case, we choose $\Theta = (0, 1)$. In the Artzner–example and the Cox–Ross-Rubinstein model, this is the probability for heads or an upward development of the asset respectively.

Let $\mathcal{P}(\Theta)$ be the set of all probability measures on $\Theta$ so that we have $\mathcal{L}(\vartheta) \in \mathcal{P}(\Theta)$, and equip $\mathcal{P}(\Theta)$ with the standard $\sigma$-algebra $\mathcal{M}_\Theta$. We assume that a *prior distribution* $\mu \in \mathcal{P}(\Theta)$ for the unknown parameter is given. In a dynamic context it does not seem to be reasonable to estimate $\theta$ at the beginning and fix it. Instead, the information about the unknown parameter which is gained over time by observing $(Y_t)$, should of course be incorporated into the risk measurement procedure. Indeed we define the dynamic risk measure as before where the conditional distribution is now taken w.r.t. a new probability measure. In order to explain this, suppose $(\Omega, \mathcal{F})$ is the underlying measurable space and $\mathbb{P}^\theta$ is the probability measure if $\vartheta$ is known and equal to $\theta$. Let now $\mathbb{P} := \mu_0 \otimes P^\theta$ be the unique probability measure on $\Theta \times \Omega$ defined by $\mu_0$ and the transition probability $(\theta, B) \mapsto \mathbb{P}^\theta(B)$. Moreover, define

$$\bar{\mathbb{P}}(B) := \mathbb{P}(\Theta \times B) = \int \mu_0(d\theta)\mathbb{P}^\theta(B)$$

on $(\Omega, \mathcal{F})$. In order to simplify the notation we assume that

**Assumption 2:** For every $t = 1, \dots, T$ the law $Q_t^\theta(\,\cdot\,)$ either has a counting density or a Lebesgue–density w.r.t. a $\sigma$-finite measure $\nu$ which we will denote by $q_t^\theta(\,\cdot\,)$.

From our assumption it follows that $\bar{\mathbb{P}}$ has a density $\bar{p}$ which satisfies

$$\begin{aligned} \bar{p}(y) \quad = \quad & \bar{p}(y_1, \dots, y_T) = \int \mu_0(d\theta) q_1^\theta(y_1) \int \mu_1(d\theta|y_1) q_2^\theta(y_2) \dots \\ & \dots \int \mu_{T-1}(d\theta|y_1, \dots, y_{T-1}) q_T^\theta(y_T) \end{aligned} \tag{5.1}$$

see e.g. Rieder (1975). By $\bar{\rho}^{(k)}$ we denote the the same risk measure as in (4.2) with $\mathbb{P}$ replaced by $\bar{\mathbb{P}}$.

**Definition 5.1.** For every $I \in \mathcal{X}^{\mathrm{M}}$ we define a Bayesian dynamic risk measure by

$$\rho_t^{\mathrm{B},\mu_0}(I) \quad = \quad \bar{\mathbb{E}}\Big[ \sum_{k=t+1}^{T} \frac{c_k}{c_t} \cdot \bar{\rho}^{(k)}(I_k \mid \mathcal{F}_{k-1}) \,\Big|\, \mathcal{F}_t \Big], \quad t = 0, 1, \dots, T-1.$$

**Remark 5.2.** Obviously every $\rho_t^{\mathrm{B},\mu_0}(I)$ is $\mathcal{F}_t-$measurable. Consequently, the dynamic risk measure $(\rho_t^{\mathrm{B},\mu_0}(I))_{t=0,1,\dots,T-1}$ is an $(\mathcal{F}_t)_{t=0,1,\dots,T-1}-$adapted process. Moreover, it is not difficult to see from the definition that $\rho^{\mathrm{B},\mu_0}$ satisfies the properties of Proposition 4.3 w.r.t. the new probability measure $\bar{\mathbb{P}}$ and is thus a reasonable dynamic risk measure.

5.2. **Representation as a Bayesian Decision Problem.** Analogously to the first part of the paper we will now show that the Bayesian dynamic risk measure is the solution of a Bayesian MDP. The advantage of the MDP formulation is its recursive computations which allows in Section 6 to establish some comparison results between models with and without model ambiguity. Hence, we proceed as follows:

For every fixed $\theta \in \Theta$, we can formulate an optimization problem as in Section 4 by replacing the reference probability $\mathbb{P}$ with $\mathbb{P}^\theta$. We obtain a family of value functions $(V_t^\theta)_{t\in\{0,1,\dots,T\}}$ for every $\theta \in \Theta$. In what follows we define

$$V_t^\mu(s) := \sup_\pi \int_\Theta V_{t,\pi}^\theta(s)\mu(\mathrm{d}\theta), \quad s \in S \tag{5.2}$$

where $t \in \{0, 1, \dots, T-1\}$, $\mu \in \mathcal{P}(\Theta)$ and the supremum is taken over all policies which may depend on the total available history so far. It is well-known that problems like this can be solved by a Bayesian decision model (see e.g. Rieder (1975)). We will first recall this technique and then show how to derive the dynamic risk measure by this approach. The Bayesian decision model is defined as follows:

- The *state space* is denoted by $S \subset \mathbb{R}^2 \times \mathcal{P}(\Theta)$ and equipped with the corresponding $\sigma$–algebra $\mathcal{S} := (\mathcal{B}^2 \otimes \mathcal{M}_\Theta)_S$. For convenience, we sometimes use the notation $S = S' \times M = S_1 \times S_2 \times M$. Let $(s, \mu) := (w, z, \mu) \in S$ be an element of the state space, where $w, z, \mu$ represent realizations of the wealth process $W_t$, the generating Markov chain $Z_t$ and the conditional distribution of $\vartheta$ given the history up to time $t$, respectively.
- The *action space* is $\mathbb{R}$ equipped with $\mathcal{B}$. Let $D := S \times \mathbb{R}$.
- The distribution $Q_t$ of the *disturbance* $Y_t$ now depends on $\mu_t$. More precisely, let $Q_t : M \times \mathcal{E} \to [0, 1]$ at time $t = 1, \dots, T$ be the probability kernel between $(M, \mathcal{M}_\Theta)$ and $(E, \mathcal{E})$ defined by

  $$Q_t(\mu; B) := \int_\Theta Q_t^\theta(B)\mu(\mathrm{d}\theta), \quad B \in \mathcal{E},$$

  where $Q_t^\theta(B) = \mathbb{P}^\theta(Y_t \in B)$.
- The *transition function* $T_t : D \times E \to S$ at time $t = 1, \dots, T$ is given by

  $$T_t(s, \mu, a, y) := (T_t'(s, a, y), \Phi_t(\mu, y))$$

  with

  $$T_t'(s, a, y) = (w^+ + h_t(z, y) - a, g_t(z, y))$$

  for $(s, \mu, a, y) = (w, z, \mu, a, y) \in D \times E$ and $\Phi_t : \mathcal{P}(\Theta) \times E \to \mathcal{P}(\Theta)$ is the so-called *Bayes operator*, which updates the estimated distribution $\mu \in \mathcal{P}(\Theta)$ by using the new observation $y \in E$. It will be defined and further investigated below.
- The *one-step reward function* is as before and does not depend on $\mu$, i.e.

  $$r_t(s, \mu, a) = r_t(s, a) = -q_t w^- + c_{t+1} a.$$

- The *terminal reward function* is is as before and does not depend on $\mu$, i.e.

  $$V_T(s, \mu) = V_T(s) = c_{T+1} w^+ - q_T w^-.$$

As mentioned above, let us further describe the Bayes operator. For $t = 1, \dots, T$ the Bayes operator $\Phi_t : \mathcal{P}(\Theta) \times E \to \mathcal{P}(\Theta)$ is defined via

$$\Phi_t(\mu, y)(B) := \frac{\int_B q_t^\theta(y)\mu(\mathrm{d}\theta)}{\int_\Theta q_t^\theta(y)\mu(\mathrm{d}\theta)}, \quad B \in \mathcal{B}_\Theta,\ (\mu, y) \in \mathcal{P}(\Theta) \times E.$$

The set of decision rules $F_B$, the Markov decision process $(X_t)_{t=0,1,\dots,T}$ and the value functions $(V_t)_{t=0,1,\dots,T}$ are defined analogously to Section 4 but with the extended state space and the extended transition function. The third component of $(X_t)_{t=0,1,\dots,T}$ which is independent of the chosen policy $\pi \in F_B^T$ can be defined recursively via

$$\mu_0 \in \mathcal{P}(\Theta), \quad \mu_t = \Phi_t(\mu_{t-1}, Y_t), \quad t = 1, \dots, T. \tag{5.3}$$

A chosen initial distribution $\mu_0 \in \mathcal{P}(\Theta)$ is the *prior distribution*, while the $\mu_t$, $t = 1, \dots, T$, are called *posterior distributions*. They can be interpreted as the distribution of $\vartheta$ at time $t$ given the history $(Y_1, \dots, Y_t)$ if $\vartheta$ is drawn according to $\mu_0$ at time 0. Under some mild integrability assumptions it holds for all $t = 0, 1, \dots T$ that

$$V_t(s, \mu) = V_t^{\mu}(s), \quad (s, \mu) \in S,$$

(see Rieder (1975)), i.e. the value functions solve problem (5.2). Note that this implies a recursion for $V_t$ analogously to (4.3). The following result is crucial.

**Theorem 5.3.** *Let $t = 0, 1, \dots, T$ and $w \in S_1$. Then*

$$V_t(w, Z_t, \mu_t) = c_{t+1} w^+ - q_t w^- - \bar{\mathbb{E}}\Big[ \sum_{k=t+1}^{T} c_k \cdot \bar{\rho}^{(k)}(I_k \mid \mathcal{F}_{k-1}) \Big| \mathcal{F}_t \Big]$$

*and the optimal policy $\pi^* = (f_t^*, \dots, f_{T-1}^*)$ is*

$$f_k^*(w, Z_k, \mu_k) = w^+ - \mathrm{VaR}_{\gamma_{k+1}}\big(I_{k+1} \,\big|\, \mathcal{F}_k\big)$$

*for $k = t, \dots, T-1$ and $w \in S_1$.*

*Proof.* The proof is analogous to the proof of Theorem 4.4 and done by induction: $t = T$ is trivial. Let us look at $t = T-1$ (note that $\mu_{T-1} = \mu_{T-1}(\cdot \,|y_1, \dots, y_{T-1})$):

$$\begin{aligned}
&V_{T-1}(w, z, \mu_{T-1}) + q_{T-1} w^- \\
&= \sup_{a \in \mathbb{R}} \Big\{ q_{T-1} w^- + r_{T-1}(w, z, a) \\
&\qquad + \int_E \int_\Theta V_T(T_T(w, z, a, y)) q_T^\theta(y) \nu(dy) \mu_{T-1}(d\theta | y_1, \dots, y_{T-1}) \Big\} \\
&= \sup_{a \in \mathbb{R}} \Big\{ c_T a + \int_E \int_\Theta \big( c_{T+1}(w^+ + h_T(z, y) - a)^+ - q_T(w^+ + h_T(z, y) - a)^- \big) \\
&\qquad\qquad q_T^\theta(y) \nu(dy) \mu_{T-1}(d\theta | y_1, \dots, y_{T-1}) \Big\}.
\end{aligned}$$

Now from (5.1) it follows that

$$\bar{p}(y_T | y_1, \dots, y_{T-1}) = \int \mu_{T-1}(d\theta | y_1, \dots, y_{T-1}) q_T^\theta(y_T).$$

Hence

$$\begin{aligned}
&V_{T-1}(w, z, \mu_{T-1}) + q_{T-1} w^- = \sup_{a \in \mathbb{R}} \big\{ c_T a + \\
&+ \int_E \big( c_{T+1}(w^+ + h_T(z, y) - a)^+ - q_T(w^+ + h_T(z, y) - a)^- \big) \bar{p}(y | y_1, \dots, y_{T-1}) \nu(dy) \big\}
\end{aligned}$$

and we obtain

$$\begin{aligned}
V_{T-1}(w, z, \mu_{T-1}) &= c_T w^+ - q_{T-1} w^- \\
&- c_T \bar{\mathbb{E}}\Big[ \bar{\rho}^{(T)}(I_T \mid Y_1 = y_1, \dots, Y_{T-1} = y_{T-1}\big) \mid Y_1 = y_1, \dots, Y_{T-1} = y_{T-1} \Big].
\end{aligned}$$

The remaining part of the proof is basically the same as for Theorem 4.4, only with some more cumbersome notational effort. □

From Definition 5.4 and Theorem 5.3 we obtain the following representation.

**Corollary 5.4.** *Consider the Bayesian decision model for an income process $I \in \mathcal{X}^M$. Furthermore, let an initial distribution $\mu_0 \in \mathcal{P}(\Theta)$ be given from which we obtain the sequence of posterior distributions $(\mu_t)_{t=1,\dots,T}$ via* (5.3). *It holds that*

$$\rho_t^{\mathrm{B},\mu_0}(I) := -\frac{1}{c_t} V_t(0, Z_t, \mu_t), \quad t = 0, 1, \dots, T-1. \tag{5.4}$$

The representation of the risk measure is now of the same structure as the one in the case where $\vartheta$ is known, compare Corollary 4.5.

**Remark 5.5.** Note that $\rho^{\mathrm{B},\mu_0}$ can easily be extended to so-called *Hidden Markov Models*. There it is assumed that $\Theta = \{\theta_1, \dots, \theta_m\}$ is finite, however, the unknown parameter $\vartheta \in \Theta$ is not fixed over time but varies according to a Markov chain $(\xi_t)$ with state space $\Theta$ and transition probabilities $(p_{ij})$. If $\xi_t = \theta_j$, then the true parameter at time $t$ is $\theta_j$, however, cannot be observed directly. Again under $(\xi_1, \dots, \xi_T)$ the random variables $Y_1, \dots, Y_T$ are independent. In this case we define

$$\rho_t^{\mathrm{HMM},\mu_0}(I) := -\frac{1}{c_t} V_t(0, Z_t, \mu_t), \quad t = 0, 1, \dots, T-1,$$

where $V_t(0, Z_t, \mu_t)$ is given as before only with a different Bayes operator which is now

$$\Phi(\mu, y)(\theta_k) = \frac{\sum_{j=1}^m \mu(\theta_j) p_{jk} q_t^{\theta_j}(y)}{\sum_{j=1}^m \mu(\theta_j) q_t^{\theta_j}(y)}.$$

## 6. Influence of model ambiguity

In this section we investigate the influence of model ambiguity in some special cases by comparing the risk measure $\rho_t^{\mathrm{B},\mu_0}(I)$ with $\rho_t^{\mathrm{B},\delta_{\bar\mu}}(I) = \rho_t^{\mathrm{PR},\bar\mu}(I)$ where $\bar\mu = \int \theta \mu_0(d\theta)$. It turns out that model ambiguity may lower the risk.

6.1. **Comparison Results.** In what follows it turns out that general statements about comparison results are quite difficult and not intuitive. First we give a simple criterion which leads to the desired comparison, however this criterion is quite restrictive. We use the following abbreviations: For any distribution $\mu$ on $\mathbb{R}$, we denote, if they exist, by $m_\mu$ and $\sigma_\mu^2$ its first moment and its variance respectively.

**Proposition 6.1.** *Let $t = 0, 1, \dots, T-1$ and $s \in S'$. If $\theta \mapsto V_{t,\pi}^\theta(s)$ is convex on $\Theta = (0,1)$ for every $\pi \in F^{T-t}$, then*

$$V_t(s, \mu) \geq V_t(s, \delta_{m_\mu}), \quad \mu \in \mathcal{P}(0,1).$$

*Proof.* Let $\mu \in \mathcal{P}(0,1)$. Since $\theta \to V_{t,\pi}^\theta(s)$ is convex, Jensen's inequality yields

$$\int_\Theta V_{t,\pi}^\theta(s)\, \mu(\mathrm{d}\theta) \geq V_{t,\pi}^{m_\mu}(s) = V_{t,\pi}(s, \delta_{m_\mu}), \quad \pi \in F^{T-t}. \tag{6.1}$$

Consequently,

$$V_t(s, \mu) = \sup_\pi \int_\Theta V_{t,\pi}^\theta(s)\, \mu(\mathrm{d}\theta) \overset{(6.1)}{\geq} \sup_{\pi \in F^{T-t}} V_{t,\pi}(s, \delta_{m_\mu}) = V_t(s, \delta_{m_\mu}).$$

□

In the following section of examples, we will see that our models do in general not fulfill the strong assumption of Proposition 6.1. We also see, that it is usually not enough to assume convexity of $\theta \mapsto V^{\theta}_{t,\pi^*}(s)$ if $\pi^*$ is chosen such that

$$\sup_{\pi \in F^{T-t}} V_{t,\pi}(s, \delta_{m_\mu}) = V_{t,\pi^*}(s, \delta_{m_\mu}),$$

because the optimal policies for $\theta$ differ in general. On the other hand we obtain:

**Proposition 6.2.** *Let $t = 0, 1, \dots, T$, $s \in S'$ and assume that the inequality $V_t(s, \delta_{m_\mu}) \leq V_t(s, \mu)$ holds for all $\mu \in \mathcal{P}(0,1)$. Then $\theta \mapsto V^{\theta}_t(s)$ is convex on $\Theta$.*

*Proof.* It is well known that $V_t(s, \cdot)$ is convex on $\mathcal{P}(0,1)$. Now take $\theta_1, \theta_2 \in \Theta$, $\lambda \in [0,1]$ and note that $m_{\lambda\delta_{\theta_1}+(1-\lambda)\delta_{\theta_2}} = \lambda\theta_1 + (1-\lambda)\theta_2$. The assumption and convexity of $V_t(s, \cdot)$ then yield the assertion:

$$\begin{aligned} V_t^{\lambda\theta_1+(1-\lambda)\theta_2}(s) &= V_t(s, \delta_{\lambda\theta_1+(1-\lambda)\theta_2}) \leq V_t(s, \lambda\delta_{\theta_1} + (1-\lambda)\delta_{\theta_2}) \\ &\leq \lambda V_t(s, \delta_{\theta_1}) + (1-\lambda)V_t(s, \delta_{\theta_2}) = \lambda V_t^{\theta_1}(s) + (1-\lambda)V_t^{\theta_2}(s). \end{aligned}$$

□

We now state the main comparison result of this section. Here we consider the binomial model, i.e. we interpret $\vartheta$ as a probability, i.e. choose $\Theta = (0,1)$ and assume that

$$\mathbb{P}(Y_t = u \,|\, \vartheta = \theta) = \theta = 1 - \mathbb{P}(Y_t = d \,|\, \vartheta = \theta), \quad \theta \in \Theta,\ t = 1, \dots, T,$$

for some $0 \leq d < 1 \leq u$. In this setting, the Beta distribution is a conjugate prior distribution for the binomial distribution. If we have no information about the parameter $\vartheta$ at time 0, we usually start with $\mu_0 = \mathcal{U}(0,1)$, which is a special Beta distribution, namely $\mathcal{U}(0,1) = \text{Beta}(1,1)$. In order to compute the risk measure we have to calculate the Bayes operator and the transition kernel. It is well known that the Bayes operator preserves the class of Beta distributions and that we have for $\alpha, \beta > 0$ and $\mu := \text{Beta}(\alpha, \beta)$

$$\Phi(\mu, y) = \text{Beta}(\alpha + 1_{\{u\}}(y), \beta + 1_{\{d\}}(y)), \quad y \in \{u, d\}. \tag{6.2}$$

Note that $\Phi$ does not depend on the time index $t$, thus we skip it here. Consequently, with such an initial distribution, we only need to calculate the transition kernel for this class of distributions. It is also well known and easily calculated that

$$Q(\mu; u) = m_\mu = \frac{\alpha}{\alpha+\beta}, \quad Q(\mu; d) = 1 - m_\mu = \frac{\beta}{\alpha+\beta}, \tag{6.3}$$

where $m_\mu$ denotes the mean of the distribution $\mu$.

Next we have to recall the following definition: A function $g : B \to \mathbb{R}$, $B \subset \mathbb{R}^2$, is called *supermodular*, if

$$g(x \wedge y) + g(x \vee y) \geq g(x) + g(y)$$

for all $x, y, x \vee y, x \wedge y \in B$, where $\wedge$ denotes the componentwise minimum and $\vee$ the componentwise maximum of the vectors.

The following theorem gives a manageable criterion for the desired comparison result. It will be applied in the next subsection to the CRR model.

**Theorem 6.3.** *Consider the Bayesian decision model from above. If for all* $t = 0, 1, \dots, T$ *and* $s \in S'$, $a \in \mathbb{R}$ *the function*

$$(y, \theta) \mapsto V_t^{\theta}(T_t'(s, a, y)), \quad (y, \theta) \in \{u, d\} \times (0, 1),$$

*is supermodular, then it holds for all* $t \in \{0, 1, \dots, T\}$

$$V_t(s, \mu) \geq V_t(s, \delta_{m_\mu}), \quad (s, \mu) \in S.$$

*Proof.* The proof of the theorem is by backward induction on $t$. For $t = T$ we even have equality

$$V_T(s, \mu) = V_T(s, \delta_{m_\mu}), \quad (s, \mu) \in S.$$

Now assume that the assertion holds for fixed $t = 1, \dots, T$ and let $(s, \mu) \in S$, where $\mu \in \mathcal{P}(0, 1)$ with $\mu \notin \{\delta_\theta \mid \theta \in \Theta\}$. Note that then $\Phi(\mu, y) \in \mathcal{P}(0, 1)$ for $y \in \{u, d\}$. Furthermore, by definition of the transition kernel we have in the same way as in (6.3)

$$Q_t(\mu; u) = m_\mu = 1 - Q_t(\mu; d). \tag{6.4}$$

Moreover, note that $r$ is independent of $\mu$. Consequently, the induction hypothesis (I.H.) yields with $\mu_u := \Phi_t(\mu, u)$, $\mu_d := \Phi_t(\mu, d)$ and with $m_u := m_{\Phi_t(\mu,u)}$, $m_d := m_{\Phi_t(\mu,d)}$

$$\begin{aligned}
&V_{t-1}(s, \mu) \\
&= \sup_{a \in \mathbb{R}} \big\{ r_{t-1}(s, a) + \int V_t(T_t(s, \mu, a, y)) Q_t(\mu; \mathrm{d}y) \big\} \\
&\overset{(6.4)}{=} \sup_{a \in \mathbb{R}} \big\{ r_{t-1}(s, a) + m_\mu \cdot V_t(T_t'(s, a, u), \mu_u) + (1 - m_\mu) \cdot V_t(T_t'(s, a, d), \mu_d) \big\} \\
&\overset{\text{I.H.}}{\geq} \sup_{a \in \mathbb{R}} \big\{ r_{t-1}(s, a) + m_\mu \cdot V_t(T_t'(s, a, u), \delta_{m_u}) + (1 - m_\mu) \cdot V_t(T_t'(s, a, d), \delta_{m_d}) \big\}.
\end{aligned}$$

Note that we have $m_d < m_\mu < m_u$ and it can be shown that if $\sigma_\mu^2 > 0$ then:

$$\frac{m_\mu}{1 - m_\mu} (m_{\Phi(\mu,u)} - m_\mu) = m_\mu - m_{\Phi(\mu,d)}.$$

This yields

$$\frac{m_\mu}{1 - m_\mu} = \frac{m_\mu - m_d}{m_u - m_\mu}. \tag{6.5}$$

Furthermore, by assumption and since it can be shown that $\theta \mapsto V_t(\hat{s}, \delta_\theta)$ is convex on $\Theta$ for every $\hat{s} \in S'$ we obtain

$$\frac{V_t(\hat{s}, \delta_{m_\mu}) - V_t(\hat{s}, \delta_{m_d})}{m_\mu - m_d} \leq \frac{V_t(\hat{s}, \delta_{m_u}) - V_t(\hat{s}, \delta_{m_\mu})}{m_u - m_\mu}. \tag{6.6}$$

Again, we get with $\hat{s} = T_t'(s, a, u)$, $a \in \mathbb{R}$, by the supermodularity assumption

$$\begin{aligned}
&V_t(T_t'(s, a, d), \delta_{m_\mu}) - V_t(T_t'(s, a, d), \delta_{m_d}) \\
&\leq V_t(T_t'(s, a, u), \delta_{m_\mu}) - V_t(T_t'(s, a, u), \delta_{m_d}) \\
&\overset{(6.6)}{\leq} \frac{m_\mu - m_d}{m_u - m_\mu} \big( V_t(T_t'(s, a, u), \delta_{m_u}) - V_t(T_t'(s, a, u), \delta_{m_\mu}) \big) \\
&\overset{(6.5)}{=} \frac{m_\mu}{1 - m_\mu} \big( V_t(T_t'(s, a, u), \delta_{m_u}) - V_t(T_t'(s, a, u), \delta_{m_\mu}) \big),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}&m_\mu \cdot V_t(T_t'(s,a,u),\delta_{m_u}) + (1-m_\mu)\cdot V_t(T_t'(s,a,d),\delta_{m_d})\\ &\geq m_\mu \cdot V_t(T_t'(s,a,u),\delta_{m_\mu}) + (1-m_\mu)\cdot V_t(T_t'(s,a,d),\delta_{m_\mu}).\end{aligned}$$

Combining this with the preceding inequality for $V_{t-1}$, we get

$$\begin{aligned}&V_{t-1}(s,\mu)\\ &\geq \sup_{a\in\mathbb{R}} \big\{ r_{t-1}(s,a) + m_\mu \cdot V_t(T_t'(s,a,u),\delta_{m_u}) + (1-m_\mu)\cdot V_t(T_t'(s,a,d),\delta_{m_d})\big\}\\ &\geq \sup_{a\in\mathbb{R}} \big\{ r_{t-1}(s,a) + m_\mu \cdot V_t(T_t'(s,a,u),\delta_{m_\mu}) + (1-m_\mu)\cdot V_t(T_t'(s,a,d),\delta_{m_\mu})\big\}\\ &= V_{t-1}(s,\delta_{m_\mu}),\end{aligned}$$

thus completing the proof of the theorem. □

The result can be used to compare the two dynamic risk measures with and without model ambiguity. The proof follows directly from Theorem 6.3.

**Corollary 6.4.** *Let the assumption of Theorem 6.3 be fulfilled and suppose* $(\mu_t)$ *is the sequence of posterior distributions defined via* (5.3). *Then*

$$\rho_t^{\mathrm{PR},m_{\mu_t}}(I) \geq \rho_t^{\mathrm{B},\,\mu_0}(I), \quad I \in \mathcal{X}^M.$$

Thus, we see that if the supermodularity assumption is fulfilled the dynamic risk measure $\rho^{\mathrm{PR}}$ is more conservative than $\rho^{\mathrm{B}}$ since it assigns a higher risk to every process $I \in \mathcal{X}^{\mathrm{M}}$. This result is somehow counterintuitive since additional "risk" due to model ambiguity leads to a lower value of the risk measure.

6.2. **Examples.** In this section we treat the coin–tossing game proposed by Artzner and the standard Cox–Ross–Rubinstein model. These will complement the results from the previous section and the remarks that were left open. Furthermore we will see how to calculate the dynamic risk measures in concrete situations.

**The Artzner–game**

Consider the Artzner–example. We have $E = \{0,1\}$ and $S' = \mathbb{R}\times\mathbb{N}_0$, whereas because of $T = 3$ also $S' = \mathbb{R}\times\{0,1,2,3\}$ can be used. But this makes no difference for our investigations. Take the process $I_1 = I_2 = 0$, $I_3 = 1_{\{Z_2+Y_3\geq 2\}}$. It can be shown that $\theta \mapsto V_t^\theta(s)$, $s \in S'$, is convex for this model, however we will find $(s,\mu) \in S'\times\mathcal{P}(0,1)$ such that

$$V_1(s,\mu) < V_1(s,\delta_{m\mu})$$

which means that the comparison of Theorem 6.3 is not valid in this situation. We can assume without loss of generality that $w = 0$ and have to consider the two cases $z = 0$ and $z = 1$, which are just the possible realizations of $Z_1 = Y_1$. First, we choose $s = (0,1)$ and consequently $\mu = \mathrm{Beta}(2,1)$, such that $m_\mu = \frac{2}{3}$. Furthermore let $\gamma_3 > \frac{2}{3}$, e. g. $\gamma_3 = 0.95$. In this case, we have $\mathrm{AVaR}_{\gamma_3}^{\theta}(Y_3) = 0$ for all $\theta \leq 0.95$. It follows that

$$\begin{aligned}V_1\big(0,1,\delta_{\frac{2}{3}}\big) &= -c_3\cdot\mathbb{E}^{\frac{2}{3}}\big[\rho_{\frac{2}{3}}^{(3)}(1_{\{Z_2+Y_3\geq 2\}}\,|\,Z_2 = 1+Y_2)\big]\\ &= -c_3\cdot\Big(\frac{2}{3}\rho_{\frac{2}{3}}^{(3)}(1_{\{Y_3\geq 0\}}) + \frac{1}{3}\rho_{\frac{2}{3}}^{(3)}(1_{\{Y_3\geq 1\}})\Big)\\ &= -c_3\cdot\Big(\frac{2}{3}\cdot(-1) - \frac{1}{3}\cdot\frac{2}{3}\cdot\lambda_3\Big) = \frac{2}{9}c_4 + \frac{2}{3}c_3,\end{aligned}$$

while we obtain

$$\begin{aligned} V_1\big(0,1,\mathrm{Beta}(2,1)\big) &= -c_3 \cdot \mathbb{E}_{1,2,1}\Big[\rho^{(3)}_{1,2,1}\big(1_{\{Z_2+Y_3\geq 2\}} \,\big|\, Y_2, Z_1 = 1\big)\Big] \\ &= -c_3 \cdot \mathbb{E}\Big[\rho^{(3)}\Big(1_{\{Y_2+Y_3\geq 1\}} \,\big|\, Y_2, \vartheta = \frac{2+1_{\{1\}}(Y_2)}{4}\Big) \,\big|\, \vartheta = \frac{2}{3}\Big] \\ &= -c_3 \cdot \Big(\frac{2}{3}\cdot \rho^{(3)}_{\frac{3}{4}}(1_{\{Y_3\geq 0\}}) + \frac{1}{3}\cdot \rho^{(3)}_{\frac{1}{2}}(1_{\{Y_3\geq 1\}})\Big) \\ &= -c_3 \cdot \Big(\frac{2}{3}\cdot(-1) - \frac{1}{3}\cdot\frac{1}{2}\lambda_3\Big) = \frac{1}{6}c_4 + \frac{2}{3}c_3 < V_1\big(0,1,\delta_{\frac{2}{3}}\big). \end{aligned}$$

Similar calculations show that for $s = (0,0)$ and thus $\mu = \mathrm{Beta}(1,2)$ the reverse inequality holds:

$$V_1\big(0,0,\delta_{\frac{1}{3}}\big) = \frac{1}{9}c_4 < \frac{1}{6}c_4 = V_1\big(0,0,\mathrm{Beta}(1,2)\big).$$

Indeed, no general comparison result can be stated in the Artzner–example. But since the problem is very simple, all quantities can easily be computed. So let us also look at the important case $t = 0$, $s = (0,0)$ and $\mu = \mathrm{Beta}(1,1) = \mathcal{U}(0,1)$, therefore $m_\mu = \frac{1}{2}$,

$$V_0\big(0,0,\delta_{\frac{1}{2}}\big) = \frac{1}{4}c_3 + \frac{1}{4}c_4$$

and

$$V_0\big(0,0,\mathcal{U}(0,1)\big) = \frac{1}{3}c_3 + \frac{1}{6}c_4 > V_0\big(0,0,\delta_{\frac{1}{2}}\big),$$

since $c_4 < c_3$. We conclude that the risk measures have the order:

$$\rho_0^{\mathrm{PR},\frac{1}{2}}(I) > \rho_0^{\mathrm{B},\mathcal{U}(0,1)}(I).$$

In Table 1 we give some numerical values of the risk measures, where the variables $c_k = 0.95^{k-1}$, $k = 1,2,3$, $c_4 = 0.93^2$ and $q_k = 1.2 \cdot c_k$, $k = 1,2,3$, are chosen analogously to Pflug and Ruszczyński (2001). In that work, the values for $\rho_0^{\mathrm{PR}}(I^{(1)})$ and $\rho_0^{\mathrm{PR}}(I^{(2)})$ can already be found.

**The Cox–Ross–Rubinstein model**

Now, let us treat the CRR model. First, we want to show that the assumption of Proposition 6.1 is too strong. It is easily seen that $\theta \mapsto V^\theta_{T-1,\pi}(s)$ is linear, therefore convex, for every $s \in S'$ and $\pi = f_T \in F$. But for $t = T-2$, it can be shown that there exists $\pi \in F^2$ such that $\theta \mapsto V^\theta_{T-2,\pi}(s)$ is concave (cf. Mundt (2007)) and consequently Proposition 6.1 cannot be applied to obtain comparison results in the CRR model. But as we will see now, the assumption of Theorem 6.3 is fulfilled.

Note that the state space is $S' = \mathbb{R}\times\mathbb{R}_+$, therefore $Z_t > 0$ for all $t = 0,1,\ldots,T$. Let us first calculate the value functions $V^\theta_t(s)$ for $s \in S'$ and $\theta \in \Theta$ at time $t \in \{1,\ldots,T\}$. As in Example 4.7 we obtain

$$\begin{aligned} &V^\theta_t(w,z) - c_{t+1}w^+ + q_t w^- \\ &= z\sum_{k=t+1}^{T} c_k \mathbb{E}^\theta[Y_1]^{k-(t+1)}\Big(\lambda_k\theta(u-d) + d - 1 + (1-\lambda_k)1_{[0,\theta]}(\gamma_k)\frac{\theta-\gamma_k}{1-\gamma_k}(u-d)\Big). \end{aligned}$$

Now, let $(s,a) = (w,z,a) \in S' \times A$. Recall that

$$T'_t(s,a,y) = (w^+ + z(y-1) - a, zy), \quad y \in \{u,d\}.$$

| | | | |
|---|---|---|---|
| $\rho_0^{\mathrm{PR}}(I^{(1)})$ | -0.4325 | | |
| $\rho_0^{\mathrm{B},\mathcal{U}(0,1)}(I^{(1)})$ | -0.4325 | | |
| $\rho_0^{\mathrm{PR}}(I^{(2)})$ | -0.4419 | | |
| $\rho_0^{\mathrm{B},\mathcal{U}(0,1)}(I^{(2)})$ | -0.445 | | |
| | $Y_1 = 1$ | | $Y_1 = 0$ |
| $\rho_1^{\mathrm{PR}}(I^{(1)})$ | -0.5766 | | -0.2883 |
| $\rho_1^{\mathrm{B},\mathcal{U}(0,1)}(I^{(1)})$ | -0.5766 | | -0.2883 |
| $\rho_1^{\mathrm{PR}}(I^{(2)})$ | -0.7939 | | -0.0961 |
| $\rho_1^{\mathrm{B},\mathcal{U}(0,1)}(I^{(2)})$ | -0.7458 | | -0.1442 |
| | $Y_1+Y_2=2$ | $Y_1+Y_2=1$ | $Y_1+Y_2=0$ |
| $\rho_2^{\mathrm{PR}}(I^{(1)})$ | -0.6828 | -0.4552 | -0.2276 |
| $\rho_2^{\mathrm{B},\mathcal{U}(0,1)}(I^{(1)})$ | -0.6828 | -0.4552 | -0.2276 |
| $\rho_2^{\mathrm{PR}}(I^{(2)})$ | -0.95 | -0.4552 | 0 |
| $\rho_2^{\mathrm{B},\mathcal{U}(0,1)}(I^{(2)})$ | -0.95 | -0.4552 | 0 |

TABLE 1. The dynamic risk measures in the Artzner–game

This yields

$$\begin{aligned} & V_t^\theta(T_t'(s,a,u)) - V_t^\theta(T_t'(s,a,d)) \\ & = c_{t+1}(w^+ + z(u-1) - a)^+ - q_t(w^+ + z(u-1) - a)^- \\ & \quad - c_{t+1}(w^+ + z(d-1) - a)^+ + q_t(w^+ + z(d-1) - a)^- \\ & \quad + z(u-d) \cdot \sum_{k=t+1}^{T} c_k \cdot \mathbb{E}^\theta[Y_1]^{k-(t+1)}(\lambda_k \theta(u-d) + d - 1) \\ & \quad + z(u-d) \cdot \sum_{k=t+1}^{T} c_k \cdot \mathbb{E}^\theta[Y_1]^{k-(t+1)}(1-\lambda_k) 1_{[0,\theta]}(\gamma_k) \frac{\theta - \gamma_k}{1-\gamma_k}(u-d). \end{aligned}$$

We have to show that this term is non–decreasing in $\theta$. To this extend, we only have to consider the two sums. Since $\mathbb{E}^\theta[Y_1] = \theta(u-d) + d$, the last one is the sum of non–negative products of non–decreasing functions, therefore clearly non–decreasing. The first sum is just the value function $V_t^\theta(0, z(u-d))$, if $\theta \le \min\{\gamma_{t+1}, \dots, \gamma_T\}$. So we only have to show that $V_t^\theta(s)$ is non–decreasing in $\theta$ on this interval for every $s \in S'$. This can be seen as follows. The case $t = T-1$ is obvious, so assume $t \le T-2$. Furthermore, without loss of generality, we can take $(w,z) = (0,1)$. Let us write the value function as

$$\begin{aligned} V_t^\theta(w,z) = & \, c_{t+1}(\lambda_{t+1}\theta(u-d) + d - 1) + c_{t+2}(\theta(u-d) + d)(\lambda_{t+2}\theta(u-d) + d - 1) \\ & + \sum_{k=t+3}^{T} c_k \cdot \mathbb{E}^\theta[Y_1]^{k-(t+1)}(\lambda_k \theta(u-d) + d) - \sum_{k=t+3}^{T} c_k \cdot \mathbb{E}^\theta[Y_1]^{k-(t+1)}. \end{aligned}$$

One easily obtains

$$\frac{\partial V_t^\theta(w,z)}{\partial \theta} \ge c_{T+1}(u-d)\mathbb{E}^\theta[Y_1]^{T-t-1}(T-t) \ge 0.$$

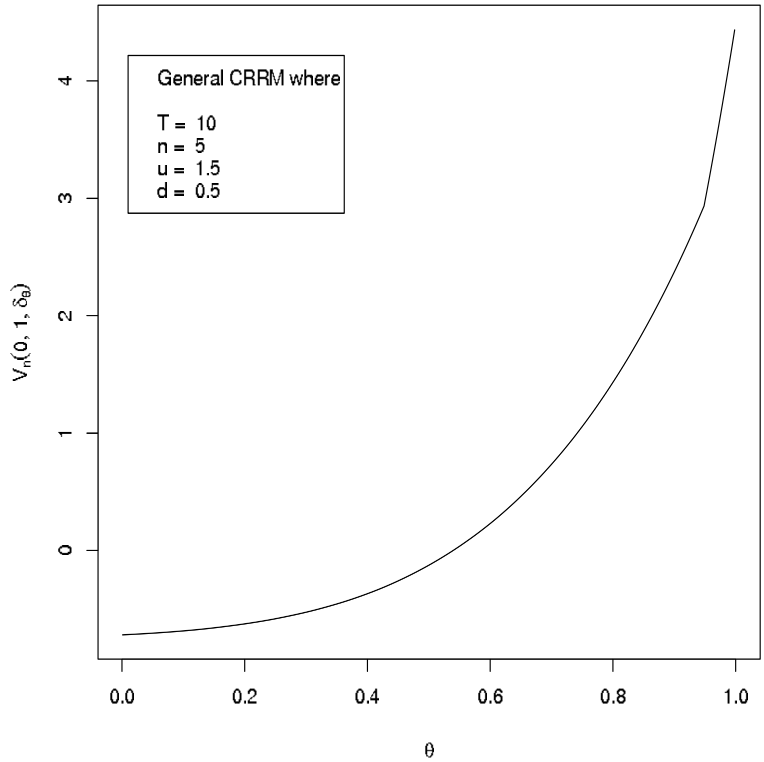


FIGURE 1. The value function in the CRR model

To visualize the behavior of the value function, we provide a simple graph in Figure 1. We set $c_k = 0.95^{k-1}$, $k = 1, \dots, T$, $c_{T+1} = 0.93^{T-1}$ and $\gamma_k = 0.95$, $k = 1, \dots, T$. Notice the point of non–differentiability at $\theta = 0.95 = \gamma_T$.

**Corollary 6.5.** *The supermodularity assumption is satisfied in the CRR model and thus the comparison result of Corollary 6.4 holds, i.e.*

$$\rho_t^{\mathrm{PR}, m_{\mu_t}}(I) \geq \rho_t^{\mathrm{B}, \mu_0}(I), \quad I \in \mathcal{X}^M.$$

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(N. Bäuerle) Institute for Stochastics, University of Karlsruhe (TH), D-76128 Karlsruhe, Germany
*E-mail address*: baeuerle@stoch.uni-karlsruhe.de
*URL*: http://www.mathematik.uni-karlsruhe.de/stoch/∼baeuerle/

(A. Mundt) Institute for Stochastics, University of Karlsruhe (TH), D-76128 Karlsruhe, Germany
*E-mail address*: mundt@stoch.uni-karlsruhe.de
*URL*: http://www.mathematik.uni-karlsruhe.de/stoch/∼mundt/