Dependence Properties and Comparison Results for Lévy Processes

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Abstract

In this paper we investigate dependence properties and comparison results for multidimensional Lévy processes. In particular we address the questions, whether or not dependence properties and orderings of the copulas of the distributions of a Lévy process can be characterized by corresponding properties of the Lévy copula, a concept which has been introduced recently in Cont and Tankov (2004) and Kallsen and Tankov (2006). It turns out that *association*, *positive orthant* dependence and *positive supermodular* dependence of Lévy processes can be characterized in terms of the Lévy measure as well as in terms of the Lévy copula. As far as comparisons of Lévy processes are concerned we consider the supermodular and the concordance order and characterize them by orders of the Lévy measures and by orders of the Lévy copulas, respectively. An example is given that the Lévy copula does not determine dependence concepts like *multivariate total positivity of order 2* or *conditionally increasing in sequence*. Besides these general results we specialize our findings for subfamilies of Lévy processes. The last section contains some applications in finance and insurance like comparison statements for ruin times and probabilities and option prices which extends the current literature.

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1 Introduction

Recent considerations in finance and insurance have led to an increasing interest in multidimensional stochastic processes and to questions of dependence between the marginal processes. Whereas for random vectors stochastic comparisons and concepts of characterizing the dependence structure by means of copulas are well-established (see e.g. the books of Denuit et al. (2005), Joe (1997), Müller and Stoyan (2002) and Nelsen (2006)) there is still need of similar statements in case stochastic processes are involved. In the present paper we will address these questions for the class of multidimensional Lévy processes. A well-known property of this class of processes is that it can be characterized by the Lévy-Khintchine triplet (A, ν, γ), where A is a covariance matrix of a Brownian motion, γ is a drift parameter and ν is the Lévy-measure determining the frequency and size of jumps. The location parameter γ is not interesting when it comes to questions of dependence. Thus, the dependence structure of a multivariate Lévy process can be characterized completely by the Lévy measure and the covariance matrix of the Brownian motion. Since the continuous part and the jump part of a Lévy process are independent it suffices to consider the dependence structure of the continuous and the discontinuous part of Lévy processes separately. In our paper we will focus on the dependence structure of the jump part only since dependence properties of multivariate Brownian motion are well established.

As far as dependence properties are concerned some general results on *association* of Markov processes can be found in Liggett (2005). Moreover association properties of families of infinitely divisible distributions have already been investigated in Resnick (1988) and Samorodnitsky (1995). We generalize their findings to other notions of dependence like *positive orthant* dependence and *positive supermodular* dependence. It turns out that in the case of Lévy processes all three notions coincide and can be characterized by the property that the Lévy measure is concentrated on $\mathbb{R}^d_{++,--} = \{x \in \mathbb{R}^d \mid x_i \geq 0 \ \forall i \text{ or } x_i \leq 0 \ \forall i\}$, i.e. jumps in the components are jointly upwards or downwards.

In analogy to copulas for random vectors, Cont and Tankov (2004) have introduced the concepts of *Lévy copulas* which has been further refined in Kallsen and Tankov (2006). In the case of a multidimensional compound Poisson process the Lévy copula coincides (up to a constant) with the copula of the multivariate distribution of the jumps. Kallsen and Tankov (2006) generalize this to arbitrary Lévy processes. They suggest to use this concept in order to characterize the dependence among components of multidimensional Lévy processes. Indeed we show that it is possible to characterize association in terms of the Lévy copula as well, however, the Lévy copula fails to characterize other dependence properties like *multivariate total positivity of order 2* or *conditionally increasing in sequence*, even in the case of a compound Poisson process.

Another important issue of this paper is the comparison of Lévy processes with respect to the strength of dependence between the components. So far comparison results for Markov processes are mainly restricted to stochastic dominance relations (see e.g. Szekli (1995), chapter 2.4). We will address the *supermodular* and the *concordance order* here. For the supermodular order some results can already be found in Bergenthum and Rüschendorf (2007) who have investigated this question nicely in the general context of semimartingales.

Using an interpolation formula which can be obtained from Houdré (1998) and Houdré et al. (1998) we show that the supermodular ordering of the Lévy processes is equivalent to an adequately defined supermodular ordering of the Lévy measures. A similar result holds for the concordance order. Finally the concordance as well as the supermodular order can also be characterized in terms of the Lévy copulas. In the case of 2-dimensional processes where it is well-known that the supermodular order and the concordance order coincide, we obtain a particular simple characterization in terms of the Lévy copula: Namely these orderings hold if and only if the Lévy copulas under consideration can be ordered pointwise.
To outline our results we give a number of examples. For compound Poisson processes it turns out that life is easy and as expected these criteria reduce to criteria for the jump size distribution. In order to generate further examples, we define a family of Archimedean Lévy copulas which includes copulas with positive dependence. Last but not least we indicate applications of our results in finance and insurance. First we focus on ruin times and ruin probabilities of portfolios of risk processes extending results in Denuit et al. (2007) and Bregman and Klüppelberg (2005). Then we investigate consequences for exponential Lévy processes which typically arise as price processes of risky assets. Finally implications for option prices and for credit risk portfolios are indicated.
Our paper is organized as follows: In Section 2 we recall the dependence concepts and the dependence orderings we use in our paper and indicate in particular how they are used for Lévy processes. Section 3 investigates dependence properties of Lévy processes. We show that *association*, *positive orthant* dependence and *positive supermodular* dependence coincide and can be characterized by the property that the Lévy measure is concentrated on $\mathbb{R}^d_{++,--}$. In terms of Lévy copulas this is equivalent to the Lévy copula vanishing on $\mathbb{R}^d - \mathbb{R}^d_{++,--}$. The next section then addresses the question of comparisons. We consider the supermodular and the concordance order and characterize them by orders of the Lévy measures and by orders of the Lévy copulas respectively. An example in this section shows that the Lévy copula does not determine order relations like *multivariate total positivity of order 2* or *conditionally increasing in sequence* which are not induced by integral relations. Section 5 presents some examples. The extension of Archimedean copulas can be found here. Finally Section 6 contains some applications of our results in finance and insurance like comparison statements for ruin times and probabilities and option prices.

2 Dependence Concepts and Dependence Orders

In this section we summarize the basic definitions and properties of dependence concepts and dependence orders which we will use later. For an introduction and further properties of these concepts see e.g. Denuit et al. (2005), Joe (1997) or Müller and Stoyan (2002). One possibility to introduce a dependence concept is to consider the set of all random vectors X which are larger than $X^\perp$ with respect to some dependence order, where we denote by $X^\perp$ a random vector with the same marginals as X, but with independent components. Therefore we will first introduce some well known dependence orders, namely *upper orthant order*, *lower orthant order*, *concordance order* and *supermodular order*, which are defined below. In order to define the last concept, recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is called *supermodular*, if

$$f(x) + f(y) \leq f(x \vee y) + f(x \wedge y)$$

for all $x, y \in \mathbb{R}^d$ with $x \vee y$ and $x \wedge y$ denoting the componentwise maximum and minimum of x and y, respectively. Moreover, we denote by

$$F_X(t) = P(X_1 \leq t_1, \dots, X_d \leq t_d)$$

the distribution function of a random vector, and the survival function by

$$\bar{F}_X(t) = P(X_1 > t_1, \dots, X_d > t_d).$$

Definition 2.1. a) A random vector $X = (X_1, ..., X_d)$ is said to be smaller than the random vector $\tilde{X} = (\tilde{X}_1, ..., \tilde{X}_d)$ in the **supermodular order**, written $X \leq_{sm} \tilde{X}$, if $Ef(X) \leq Ef(\tilde{X})$ for all supermodular functions f such that the expectations exist.
b) A random vector $X = (X_1, ..., X_d)$ is said to be smaller than the random vector $\tilde{X} = (\tilde{X}_1, ..., \tilde{X}_d)$ in the **upper orthant order**, written $X \leq_{uo} \tilde{X}$, if $\bar{F}_X(t) \leq \bar{F}_{\tilde{X}}(t)$ for all $t \in \mathbb{R}^d$.
c) A random vector $X = (X_1, ..., X_d)$ is smaller than the random vector $\tilde{X} = (\tilde{X}_1, ..., \tilde{X}_d)$ in the **lower orthant order**, written $X \leq_{lo} \tilde{X}$, if $F_X(t) \leq F_{\tilde{X}}(t)$ for all $t \in \mathbb{R}^d$.
d) A random vector $X = (X_1, ..., X_d)$ is smaller than the random vector $\tilde{X} = (\tilde{X}_1, ..., \tilde{X}_d)$ in the **concordance order**, written $X \leq_c \tilde{X}$, if both $X \leq_{uo} \tilde{X}$ and $X \leq_{lo} \tilde{X}$ hold.

In the sequel we will frequently use the following properties, which are fulfilled by all these orders, see Theorem 3.3.19 and 3.9.14 in Müller and Stoyan (2002) for details.

(C) If X_1, X_2 and $\tilde{X}_1, \tilde{X}_2$ are independent, then $X_1 \preceq \tilde{X}_1$ and $X_2 \preceq \tilde{X}_2$ implies $X_1 + X_2 \preceq \tilde{X}_1 + \tilde{X}_2$;

(W) If $(X_n), (\tilde{X}_n)$ are sequences of random vectors, converging in distribution to X and $\tilde{X}$ respectively, then $X_n \preceq \tilde{X}_n$ for all n implies $X \preceq \tilde{X}$.

(ID) If $X \preceq \tilde{X}$, then $(X, \dots, X) \preceq (\tilde{X}, \dots, \tilde{X})$.

Now we will define the following three dependence concepts:

Definition 2.2. A random vector $X = (X_1, \dots, X_d)$ is said to be

a) **(positively) associated**, if for all increasing functions $f, g : \mathbb{R}^d \to \mathbb{R}$

$$\mathrm{Cov}(f(X), g(X)) \geq 0.$$

b) **positive orthant dependent (POD)** if $X^\perp \leq_c X$;

c) **positive supermodular dependent (PSMD)** if $X^\perp \leq_{sm} X$.

The notion of *association* is already well-established since the pioneering paper by Esary et al. (1967). *Positive orthant dependence* can be traced back to Lehmann (1966) whereas *positive supermodular dependence* has only been developed recently. All these dependence concepts have been proved fruitful for applications in finance and insurance.

Remark 2.3. Note that there are different ways of characterizing the concordance order. The following statements are equivalent (see Müller and Stoyan (2002), Theorem 3.3.15 and p. 112).

a) $X \le_c \tilde{X}$.

b) The inequality
$$Ef(X) \le Ef(\tilde{X}) \tag{2.1}$$
holds for all functions $f = \prod_{i=1}^d f_i$ where $f_i : \mathbb{R} \to [0,\infty),\ i = 1,\dots,d$ are all increasing or all decreasing.

c) The inequality (2.1) holds for all $f : \mathbb{R}^d \to \mathbb{R}$ such that f is *d-increasing* or $-f$ is *d-decreasing*. A function f is called d-increasing if for all $a, b \in \mathbb{R}^d$ with $a \le b$
$$V_f\big((a,b]\big) := \sum_{x \in \{a_1,b_1\}\times\ldots\times\{a_d,b_d\}} (-1)^{\sharp\{k: x_k = a_k\}} f(x) \ge 0$$
where $V_f(A)$ is called the *f-volume* of the set A. f is called d-decreasing if $f(-x)$ is d-increasing.

These dependence properties satisfy the following implication (for the first see Christofides and Vaggelatou (2004) and for the second e.g. Bäuerle (1997)):

$$association \quad \Rightarrow \quad PSMD \quad \Rightarrow \quad POD.$$

Association has the following properties (for a proof of these properties see Esary et al. (1967)):

Lemma 2.4. *a) If $X = (X_1,\dots,X_d)$ is associated, then $(f_1(X),\dots,f_k(X))$ is associated if the functions $f_1,\dots,f_k : \mathbb{R}^d \to \mathbb{R}$ are all increasing (or all decreasing).*

b) If $X_1,\dots,X_d$ are independent, then $X = (X_1,\dots,X_d)$ is associated.

c) If $X = (X_1,\dots,X_d)$ and $Y = (Y_1,\dots,Y_k)$ are associated and stochastically independent, then (X,Y) is associated.

d) If $\{X^{(n)} = (X_1^{(n)},\dots,X_d^{(n)}), n \in \mathbb{N}\}$ is a sequence of associated random vectors with $X^{(n)} \xrightarrow{d} X$, then X is again associated.

Now let $X = (X(t))_{t\ge 0}$ be a stochastic process with values in $\mathbb{R}^d$ and $X(0) = x_0 \in \mathbb{R}^d$. There are different possibilities for extending the dependence concepts from random vectors to stochastic processes (see also Szekli (1995)). A natural condition would be

Definition 2.5. The stochastic process $X = (X(t))_{t\geq 0}$ is said to be **associated (POD, PSMD)** if and only if $(X(t_1), \ldots, X(t_n))$ is associated (POD, PSMD) for all $0 \leq t_1 < t_2 < \ldots < t_n$ and all $n \in \mathbb{N}$.

Note that $(X(t_1), \ldots, X(t_n))$ is an nd-dimensional random vector. In case the process $X = (X(t))_{t\geq 0}$ has independent increments, the condition can be simplified:

Lemma 2.6. *Let $X = (X(t))_{t\geq 0}$ be a stochastic process with independent increments. X is associated (POD, PSMD) if and only if $X(t)$ is associated (POD, PSMD) for all $t \geq 0$.*

Proof. The only-if-part is trivial. Therefore suppose that $X(t)$ has one of the properties for all $t \geq 0$. Obviously $\Big(X(t_1), \ldots, X(t_n)\Big)$ can be written as

$$\Big(X(t_1), X(t_1) + \big(X(t_2) - X(t_1)\big), \ldots, X(t_1) + \big(X(t_2) - X(t_1)\big) + \ldots + \big(X(t_n) - X(t_{n-1})\big)\Big)$$

where $X(t_1), \big(X(t_2) - X(t_1)\big), \ldots, \big(X(t_n) - X(t_{n-1})\big)$ are independent due to the assumption of independent increments. Thus, it is sufficient to show that whenever a random vector X is associated (POD, PSMD) then $\hat{X} = \Big(0, \ldots, 0, X, \ldots, X\Big)$ with values in $\mathbb{R}^{nd}$ ($\hat{X}$ has kd zeros with an arbitrary $k \in \{0, 1, \ldots n\}$ and $n - k$ blocks of X) is associated (POD, PSMD) and that these properties are preserved under convolution, i.e. if X and Y are independent random vectors which are associated (POD, PSMD), then also $X + Y$. As far as association is concerned both statements follow from Lemma 2.4 a). For POD and PSMD this follows from the properties (C) and (ID) of the corresponding dependence orders. □

Remark 2.7. There are also other definitions in the literature for dependence of stochastic processes. In Ebrahimi (2002) for example the author defines that the 2-dimensional process $X = \big(X_1(t), X_2(t)\big)_{t\geq 0}$ is associated if and only if

$$\big(X_1(t_1), X_2(t_2)\big)$$

is associated for any time points $0 \leq t_1, t_2$. In case X has independent increments this is obviously equivalent to our definition. The same holds true for POD and PSMD if defined in an analogous way.

A similar definition with similar consequences can be stated for dependence orders.

Definition 2.8. Two stochastic processes $X = (X(t))_{t\geq 0}$ and $\tilde{X} = (\tilde{X}(t))_{t\geq 0}$ are said to be **comparable with respect to the order** $\preceq \in \{\leq_c, \leq_{uo}, \leq_{lo}, \leq_{sm}\}$ (written $X \preceq \tilde{X}$) if

$$(X(t_1), \ldots, X(t_n)) \preceq (\tilde{X}(t_1), \ldots, \tilde{X}(t_n))$$

for all $0 \leq t_1 < t_2 < \ldots < t_n$ and all $n \in \mathbb{N}$.

In case the processes have independent increments, the condition can be simplified.

Lemma 2.9. *Let $X = (X(t))_{t\geq 0}$ and $\tilde{X} = (\tilde{X}(t))_{t\geq 0}$ be stochastic processes with independent increments, and let $\preceq \in \{\leq_c, \leq_{uo}, \leq_{lo}, \leq_{sm}\}$. Then $X \preceq \tilde{X}$ if and only if $X(t) \preceq \tilde{X}(t)$ for all $t \geq 0$.*

Proof. The proof is very similar to the proof of Lemma 2.6. □

3 Dependence properties of Lévy processes

Let $X = (X(t))_{t\geq 0}$ be a d-dimensional Lévy-process, i.e. a stochastically continuous process with independent and stationary increments. From the Lévy-Itô-decomposition we know that the distribution of a Lévy process is uniquely determined by a characteristic triplet (A, ν, γ), where A is a covariance matrix of a Brownian motion, γ is a drift parameter and ν is the Lévy-measure determining the frequency and size of jumps. The characteristic function is then given by

$$E\exp(i\langle u, X_t\rangle) = \exp\Big(t\cdot\big(i\langle\gamma, u\rangle - \frac{1}{2}\langle Au, u\rangle + \int_{\mathbb{R}^d}(e^{i\langle u,x\rangle} - 1 - i\langle u, h(x)\rangle)\nu(dx)\big)\Big),$$

where the truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$ is given by $h(u) = (h_1(u), \ldots, h_d(u))$ with

$$h_i(u) = u_i \mathbb{1}_{\{|u_i|\leq 1\}}.$$

The dependence structure of a multivariate Lévy process can be characterized completely by the Lévy measure and the covariance matrix of the Brownian motion. Since the continuous part and the jump part of a Lévy process are independent (Sato (1999)[Theorem 19.2]) it suffices to consider the dependence structure of the continuous and the discontinuous part of Lévy processes separately. In the following we will focus on the dependence structure of the jump part of a Lévy process only since the Brownian part is easy to handle. Therefore from now on a Lévy process will be completely characterized by its Lévy measure ν. Recall that for any Borel set B the quantity $\nu(B)$ describes the expected number of jumps per time unit with jump size in B. In particular, in the case of a compound Poisson process with jump size distribution Q and intensity λ it holds $\nu(B) = \lambda \cdot Q(B)$.
The next result can be derived from Samorodnitsky (1995)[Theorem 3.1.] or from Proposition 6 in Resnick (1988). Here, we present a slightly different proof for this result using a Theorem of Liggett (2005):

Proposition 3.1. *Let X be a d-dimensional Lévy process with Lévy measure ν. X is associated if and only if ν is concentrated on $\mathbb{R}^d_{++,--} = \{x \in \mathbb{R}^d \mid x_i \geq 0\ \forall i \text{ or } x_i \leq 0\ \forall i\}$, i.e. $\nu(\mathbb{R}^d - \mathbb{R}^d_{++,--}) = 0$.*

Proof. First note that every Lévy process is a Feller process (Applebaum (2004)[Theorem 3.1.9.]) and every Lévy process is stochastically monotone because of the independence of its increments. Thus by Liggett (2005)[Theorem 2.14] the following holds: X is associated if and only if

$$\mathcal{A}fg \geq g\mathcal{A}f + f\mathcal{A}g \tag{3.1}$$

for all increasing functions $f, g : \mathbb{R}^d \to \mathbb{R}$ in the domain of the generator $\mathcal{A}$ of X. By Sato (1999)[Theorem 31.5] the infinitesimal generator of a Lévy process is given by

$$\mathcal{A}f(x) = \int_{\mathbb{R}^d}\Big(f(x+y) - f(x) - \langle\nabla f(x), h(y)\rangle\Big)\nu(dy)$$

for $f \in C_0^2$, where we denote by ∇f the gradient of the function f. Using this representation, we obtain that inequality (3.1) is fulfilled if and only if

$$\begin{aligned}
&\mathcal{A}fg - f\mathcal{A}g - g\mathcal{A}f \geq 0 \\
\Leftrightarrow \quad & \int_{\mathbb{R}^d} \Big(f(x+y)g(x+y) - f(x)g(x) - f(x)g(x+y) + f(x)g(x) \\
& -g(x)f(x+y) + f(x)g(x) \Big) \nu(dy) \geq 0 \\
\Leftrightarrow \quad & \int_{\mathbb{R}^d} \big(g(x+y) - g(x)\big)\big(f(x+y) - f(x)\big) \nu(dy) \geq 0.
\end{aligned}$$

Let $i \neq j$. Choosing $f(y) = \mathbb{1}_{[y_i \geq 0]}$ and $g(y) = \mathbb{1}_{[y_j \geq 0]}$ and fixing x with $x_i < 0$ and $x_j > 0$, we get that

$$f(x+y) - f(x) = \begin{cases} 1 & , \quad x_i + y_i \geq 0 \quad (\Rightarrow y_i \geq -x_i > 0) \\ 0 & , \quad \text{otherwise} \end{cases}$$

and

$$g(x+y) - g(x) = \begin{cases} -1 & , \quad x_j + y_j < 0 \quad (\Rightarrow y_j < -x_j < 0) \\ 0 & , \quad \text{otherwise} \end{cases} .$$

Therefore

$$(f(x+y) - f(x))(g(x+y) - g(x)) = -\mathbb{1}_{[y_i \geq -x_i,\, y_j < -x_j]}.$$

As $x_i < 0$ and $x_j > 0$ were arbitrary, we must have $\nu(\{y : y_i > 0, y_j < 0\}) = 0$. Hence we see that ν has to be concentrated on the negative or positive orthant. Vice versa if ν is concentrated on $\mathbb{R}^d_{++,--}$ the integral is greater or equal zero since f, g are increasing. □

Remark 3.2. Samorodnitsky (1995) has shown that association of $X(t)$ for an arbitrary fixed $t > 0$ is not enough to deduce association of the process X.

Next we give a short review of Lévy copulas, a concept which has been introduced recently by Cont and Tankov (2004) and which can be used to characterize the dependence among components of multidimensional Lévy processes. For details concerning the following definitions see Kallsen and Tankov (2006). As usual $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$. Also for $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define $x_I := (x_{i_1}, \ldots, x_{i_k})$. Moreover, we define $\bar{\mathbb{R}} := (-\infty, \infty]$.

Definition 3.3. Let X be a $\mathbb{R}^d$-valued Lévy process with Lévy measure ν.

a) The **tail integral** of X is the function $U : (\mathbb{R}\backslash\{0\})^d \to \mathbb{R}$ defined by

$$U(x_1, \ldots, x_d) := \prod_{i=1}^d \text{sgn}(x_i) \nu\left(\prod_{j=1}^d \mathcal{I}(x_j) \right),$$

where $\mathcal{I}(x) = \begin{cases} (x, \infty) & , \quad x \geq 0 \\ (-\infty, x] & , \quad x < 0 \end{cases}$.

b) For $I \subset \{1, \dots, d\}$ non-empty, the **I-marginal tail integral** U^I of X is the tail integral of the process $X^I = (X^i)_{i \in I}$. To simplify notation, we denote one-dimensional margins by $U_i := U^{\{i\}}$.

Definition 3.4. For a measure defining function $F : \bar{\mathbb{R}}^d \to \bar{\mathbb{R}}$ we define for any non-empty $I \subset \{1, \dots, d\}$, the **$I$-margin F^I of F** by

$$F^I(u_I) = \lim_{a \to \infty} \sum_{u_{I^c} \in \{-a, \infty\}^{|I^c|}} F(u) \prod_{i \in I^c} \operatorname{sgn}(u_i).$$

Theorem 3.5. (and Definition). *Let X be a $\mathbb{R}^d$-valued Lévy process. Then there exists a measure defining function $F : \bar{\mathbb{R}}^d \to \bar{\mathbb{R}}$ with univariate marginals which are the identity functions on $\bar{\mathbb{R}}$ such that*

$$U^I(x_I) = F^I\big(U_{i_1}(x_{i_1}), \dots, U_{i_k}(x_{i_k})\big)$$

for all $I = \{i_1, \dots, i_k\} \subset \{1, \dots, d\}$ and all $x \in \big(\mathbb{R} - \{0\}\big)^{|I|}$. This function is unique on $\prod_{i=1}^d \overline{RanU_i}$. F is called **Lévy copula** *of X.*

In what follows we want to characterize association also in terms of the Lévy copula. The next lemma shows that the Lévy copula vanishing on $\mathbb{R}^d - \mathbb{R}^d_{++,--}$ is equivalent to the Lévy measure being concentrated on $\mathbb{R}^d_{++,--}$.

Proposition 3.6. *Let X be a $\mathbb{R}^d$-valued Lévy process with Lévy measure ν and Lévy copula F. The following statements are equivalent:*

a) $\nu(\mathbb{R}^d - \mathbb{R}^d_{++,--}) = 0$.

b) $F(u) = 0$ for $u \in (\mathbb{R}^d - \mathbb{R}^d_{++,--}) \cap \prod_{i=1}^d \overline{RanU_i}$.

c) For all $i, j \in \{1, \dots, d\}$ with $i \neq j$ we have $F^{\{i,j\}}(u_i, u_j) = 0$ for all $u_i \in \overline{RanU_i}, u_j \in \overline{RanU_j}$ with $\operatorname{sgn}(u_i) \cdot \operatorname{sgn}(u_j) < 0$.

Proof. The equivalence of a) and b) follows from the definition and Theorem 3.5. By Sklar's theorem for Lévy copulas (Kallsen and Tankov (2006)[Theorem 3.6]) and the definition of tail integrals we have

$$F^{\{i,j\}}(U_i(x_i), U_j(x_j)) = U^{\{i,j\}}(x_i, x_j) = \operatorname{sgn}(x_i)\operatorname{sgn}(x_j)\nu^{\{i,j\}}(\mathcal{I}(x_i) \times \mathcal{I}(x_j)). \tag{3.2}$$

Now assume that $F^{\{i,j\}}(u_i, u_j) = 0$ for all $i, j \in \{1, \dots, d\}$ when $\operatorname{sgn}(u_i) \cdot \operatorname{sgn}(u_j) < 0$. Then, using (3.2) and the fact that $x_i > 0$ if and only if $U_i(x_i) > 0$, it follows that $\nu^{\{i,j\}}(\mathcal{I}(x_i) \times \mathcal{I}(x_j)) = 0$ if $\operatorname{sgn}(x_i) \cdot \operatorname{sgn}(x_j) < 0$. Noticing that $\nu(\mathbb{R}^d - \mathbb{R}^d_{++,--}) > 0$ implies that $\nu^{\{i,j\}}(\mathcal{I}(x_i) \times \mathcal{I}(x_j)) > 0$ for some i, j, x_i, x_j, this yields $\nu(\mathbb{R}^d - \mathbb{R}^d_{++,--}) = 0$.
Conversely suppose that $\nu(\mathbb{R}^d - \mathbb{R}^d_{++,--}) = 0$, that is $\nu^{\{i,j\}}(\mathcal{I}(x_i) \times \mathcal{I}(x_j)) = 0$ if $\operatorname{sgn}(x_i) \cdot \operatorname{sgn}(x_j) < 0$. Thus by (3.2) we know that $F^{\{i,j\}}(u_i, u_j) = 0$ if $\operatorname{sgn}(u_i) \cdot \operatorname{sgn}(u_j) < 0$ and if $(u_i, u_j) \in \overline{\mathrm{Ran}U_i} \times \overline{\mathrm{Ran}U_j}$. □

Remark 3.7. From (3.2) it follows directly that whenever $(u_i, u_j) \in \overline{\text{Ran}U_i} \times \overline{\text{Ran}U_j}$ we obtain in general that $F^{\{i,j\}}(u_i, u_j) \geq 0$ if $\text{sgn}(u_i) \cdot \text{sgn}(u_j) > 0$ and $F^{\{i,j\}}(u_i, u_j) \leq 0$ if $\text{sgn}(u_i) \cdot \text{sgn}(u_j) < 0$.

The following Corollary is a direct consequence of Proposition 3.6 and Proposition 3.1.

Corollary 3.8. *A Lévy process is associated if and only if the Lévy copula satisfies $F(u) = 0$ for $u \in (\mathbb{R}^d - \mathbb{R}^d_{++,--}) \cap \prod_{i=1}^d \overline{RanU_i}$.*

Next we can characterize the POD property of Lévy processes. It turns out that association and POD coincide in this case.

Theorem 3.9. *Let X be a d-dimensional Lévy process. X is POD if and only if X is associated.*

Proof. Since association implies POD only the only if part requires an argument. Let X be POD and $I = \{i, j\}$ with $i, j \in \{1, \ldots, d\}$ and $i \neq j$. Thus X^I is POD. According to Lemma 3.8 we have to show that $F^{\{i,j\}}(u_i, u_j) \geq 0$ for all $i, j \in \{1, \ldots, d\}$, $i \neq j$ and $u_i, \in \overline{\text{Ran}U_i}, u_j \in \overline{\text{Ran}U_j}$ with $\text{sgn}(u_i) \cdot \text{sgn}(u_j) < 0$. W.l.o.g. suppose $I = \{1, 2\}$.
Before we proceed with the proof let us introduce some notations, analogously to Kallsen and Tankov (2006). Let $X(t) = (X_1(t), X_2(t))$ be a $\mathbb{R}^2$-valued random vector. We denote by $H_t^{(\alpha_1,\alpha_2)} : \mathbb{R}^2 \to [0, 1]$ the joint distribution function of $(-\alpha_1 X_1(t), -\alpha_2 X_2(t))$, by $C_t^{(\alpha_1,\alpha_2)} : [0, 1]^d \to [0, 1]$ an (ordinary) copula of $(-\alpha_1 X_1(t), -\alpha_2 X_2(t))$ and by $H_{t,i}^{(\alpha_i)} : \mathbb{R} \to [0, 1]$ the distribution function of $-\alpha_i X_i(t)$. Then by Sklar's Theorem we obtain

$$C_t^{(\alpha_1,\alpha_2)}\big(H_{t,1}^{(\alpha_1)}(z_1), H_{t,2}^{(\alpha_2)}(z_2)\big) = H_t^{(\alpha_1,\alpha_2)}(z_1, z_2).$$

Now let us return to the proof of our statement and fix $t > 0$. Due to the POD property we obtain for arbitrary $z_1, z_2 \in \mathbb{R}$:

$$P\big(X_1(t) \leq z_1, -X_2(t) \leq z_2\big) \leq P\big(X_1(t) \leq z_1\big) P\big(-X_2(t) \leq z_2\big).$$

By means of copulas we can write

$$C_t^{(-1,1)}\big(H_{t,1}^{(-1)}(z_1), H_{t,2}^{(1)}(z_2)\big) = H_t^{(-1,1)}(z_1, z_2) \leq H_{t,1}^{(-1)}(z_1) H_{t,2}^{(1)}(z_2).$$

Analogously we obtain

$$C_t^{(1,-1)}\big(H_{t,1}^{(1)}(z_1), H_{t,2}^{(-1)}(z_2)\big) = H_t^{(1,-1)}(z_1, z_2) \leq H_{t,1}^{(1)}(z_1) H_{t,2}^{(-1)}(z_2).$$

Note now that in general, when (X_1, X_2) is a POD random vector, then there exists a copula C for it such that $C(u_1, u_2) \geq u_1 u_2$ for all $u_1, u_2 \in [0, 1]$. Hence we can find copulas such that for all $u_1, u_2 \in [0, 1]$

$$C_t^{(-1,1)}\big(u_1, u_2\big) \leq u_1 u_2$$

$$C_t^{(1,-1)}\big(u_1, u_2\big) \leq u_1 u_2$$

In particular we obtain for $u_1 < 0, u_2 > 0$ and $t > 0$ small

$$0 \leq -C_t^{(-1,1)}\big(-tu_1, tu_2\big) - t^2 u_1 u_2$$

and for $u_1 > 0, u_2 < 0$ and $t > 0$ small

$$0 \leq -C_t^{(1,-1)}\big(tu_1, -tu_2\big) - t^2 u_1 u_2.$$

Applying Kallsen and Tankov (2006)[Theorem 5.1.] we obtain in both cases

$$0 \leq -\lim_{t\to 0} \frac{1}{t} C_t^{(\text{sgn }(u_1), \text{sgn }(u_2))}(t|u_1|, t|u_2|) - \lim_{t\to 0} \frac{1}{t} t^2 u_1 u_2 = F^{\{1,2\}}(u_1, u_2).$$

which yields the assertion in combination with Remark 3.7. □

Finally we investigate the case of positive supermodular dependence. However, since this is a property weaker than association and stronger than POD we obtain immediately

Corollary 3.10. *Let X be a d-dimensional Lévy process. The concepts of association, POD and PSMD of X coincide in this case. They can be characterized by the following two equivalent conditions*

a) the Lévy measure is concentrated on $\mathbb{R}^d_{++,--}$.

b) the Lévy copula vanishes on $\left(\mathbb{R}^d - \mathbb{R}^d_{++,--}\right) \cap \prod_{i=1}^d \overline{RanU_i}$.

There are other important concepts of dependence, which are stronger than association. An important one is *multivariate total positivity of order 2* (MTP_2) which holds for a random vector $X = (X_1, \ldots, X_d)$, if it has a density f with respect to a product measure fulfilling

$$f(x)f(y) \leq f(x \vee y)f(x \wedge y)$$

for all $x, y \in \mathbb{R}^d$. Another interesting concept is *conditionally increasing in sequence* (CIS), which holds if

$$P(X_i > t | X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$$

is an increasing function of $x_1, \ldots, x_{i-1}$ for all t. It is well known that MTP_2 implies CIS which in turn implies association, see e.g. Karlin and Rinott (1980) for the first statement and e.g. Müller and Stoyan (2002) Theorem 3.10.11 for the second and a general overview. These two concepts are preserved under monotone transformations of the marginals, and therefore they are properties of the copula of the random vector. In the next example we will show, however, that for a Lévy process these two concepts can not be characterized by the Lévy copula. As the example only uses compound Poisson processes, it even shows that these concepts can not be characterized by the copula of the jump size distributions in the case of a compound Poisson process. This demonstrates that there are important dependence properties of copulas, which in the case of a Lévy process can not be characterized by the Lévy copula.

Example 3.11. Assume that there is given a bivariate compound Poisson process with Lévy measure

$$\nu = \frac{1}{3}(\delta_{(1,0)} + \delta_{(2,1)} + \delta_{(3,3)}),$$

where as usual δ_x denotes the one-point measure in x. Then

$$\nu * \nu = \frac{1}{9}(\delta_{(2,0)} + 2\delta_{(3,1)} + \delta_{(4,2)} + 2\delta_{(4,3)} + 2\delta_{(5,4)} + \delta_{(6,6)}).$$

For t small the random vector $X(t)$ has mass $O(1)$ in the origin, $O(t)$ in the points of support of ν and mass $O(t^2)$ in the points of support of $\nu * \nu$, whereas the mass in all other points is of smaller magnitude. Therefore the conditional distribution of X_2 given $X_1 = 3$ is approximately δ_3, whereas the conditional distribution of X_2 given $X_1 = 4$ is approximately $(\delta_2 + \delta_3)/2$ and thus stochastically smaller. Hence $X(t)$ is not CIS and therefore also not MTP_2. On the other hand the Lévy copula of ν is the comonotone copula, and if one considers a Lévy process with the same Lévy copula, but with identical marginals (e.g. $\nu = (\delta_{(1,1)} + \delta_{(2,2)} + \delta_{(3,3)})/3$), then $X(t)$ obviously is comonotone and thus also MTP_2 and CIS. This shows that the property of being MTP_2 and CIS can not be characterized by the Lévy copula. It also depends on the marginals of the Lévy measure.

4 Comparison of Lévy Processes

In this section we deal with comparison of Lévy processes with respect to dependence orderings. In particular we consider the *supermodular order* $\leq_{sm}$ and the *concordance order*. For the supermodular order some results can be found in Bergenthum and Rüschendorf (2007) who have investigated this question for the more general case of semimartingales, which includes Lévy processes as a special case. However, our results are more explicit and we also give a characterization in terms of Lévy copulas.

We will now give an appropriate definition of supermodular order for general (possibly infinite) Lévy measures, which will yield the result that supermodular ordering of the Lévy measures implies supermodular ordering of the corresponding Lévy processes. As we have to take care of the possible singularity of the measures at zero, we need some technical conditions on the functions to ensure that the occurring integrals are finite. We will denote

$$\mathcal{B}_0 := \left\{ f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ is measurable and bounded and } \limsup_{x \to 0} \frac{|f(x)|}{\|x\|^2} < \infty \right\}.$$

As any Lévy measure ν fulfills $\int (\|x\|^2 \wedge 1)\nu(dx) < \infty$, this implies that for any Lévy measure ν and any $f \in \mathcal{B}_0$ the integral $\int f d\nu$ is finite. Thus the integrals in the following definition are all well-defined.

Definition 4.1. Lévy measures $\nu, \tilde{\nu}$ are said to be comparable with respect to the **supermodular order** (written $\nu \leq_{sm} \tilde{\nu}$), if for all supermodular $f \in \mathcal{B}_0$:

$$\int f d\nu \leq \int f d\tilde{\nu}.$$

There are other classes of supermodular functions, which generate the same order. We denote

$$\mathcal{B}_{00} := \left\{ f : \mathbb{R}^d \to \mathbb{R} \mid f \text{ is measurable and bounded and } \exists \varepsilon > 0 \text{ s.t. } f(x) = 0, \text{ if } \|x\| < \epsilon \right\}$$

the set of all bounded, measurable functions f with the property that f vanishes around zero.

Lemma 4.2. *For Lévy measures $\nu, \tilde{\nu}$ the following conditions are equivalent:*

a) $\nu \leq_{sm} \tilde{\nu}$;

b) ν and $\tilde{\nu}$ have the same marginal tail integrals and $\int f d\nu \leq \int f d\tilde{\nu}$ for all increasing supermodular $f \in \mathcal{B}_0$;

c) $\int f d\nu \leq \int f d\tilde{\nu}$ for all infinitely differentiable supermodular $f \in \mathcal{B}_0$;

d) $\int f d\nu \leq \int f d\tilde{\nu}$ for all supermodular $f \in \mathcal{B}_{00}$.

Proof. The proof of the equivalence of a) and b) follows the same lines as the proof of Theorem 3.3 and Theorem 3.4 in Müller and Scarsini (2000). The proof of the equivalence of b) and c) is easily adapted from a similar result in Denuit and Müller (2002). As trivially a) implies d), we can finish the proof by showing that d) implies a). Therefore assume that $f \in \mathcal{B}_0$ is a supermodular function. Let us define

$$f_n(x) := f(round(nx)/n),$$

where as usual $round(x)_i = n_i \in \mathbb{Z}$ for $n_i - 1/2 < x_i \leq n_i + 1/2$. Then f_n is a bounded supermodular function vanishing around zero, and thus $f_n \in \mathcal{B}_{00}$. Therefore $\int f_n d\nu \leq \int f_n d\tilde{\nu}$. As f_n converges to f pointwise, by dominated convergence we get $\int f d\nu \leq \int f d\tilde{\nu}$. □

Remark 4.3. Notice that in part b) we require ν and $\tilde{\nu}$ to have the same marginal tail integrals and not the same marginals. Indeed it may happen that the marginals differ concerning their point masses in zero. As an example consider for $\tilde{\nu}$ a Lévy measure concentrated on the diagonal $\{x : x_1 = \ldots = x_d\}$ and let ν be the Lévy measure with the same marginal tail integrals but with mass concentrated on the axis (so that the corresponding Lévy process has independent components). Then indeed $\nu \leq_{sm} \tilde{\nu}$ (for a proof see Remark 4.6), but the marginals of ν have point masses in zero, whereas the marginals of $\tilde{\nu}$ don't.

Next we show that for finite Lévy measures the condition $f \in \mathcal{B}_0$ can be removed, which in particular implies that for probability measures our new definition coincides with the classical one.

Lemma 4.4. *For Lévy measures $\nu, \tilde{\nu}$ with finite total mass the following conditions are equivalent:*

a) $\nu \leq_{sm} \tilde{\nu}$;

b) ν *and* $\tilde{\nu}$ *have the same marginal tail integrals and* $\int f d\nu \leq \int f d\tilde{\nu}$ *for all bounded increasing supermodular* $f : \mathbb{R}^d \to \mathbb{R}$*;*

Proof. We only prove that a) implies the ordering of the Lévy measures for all bounded continuous supermodular functions. The rest of the proof is as in Lemma 4.2. Thus let us assume that $\nu \leq_{sm} \tilde{\nu}$ holds and that f is an arbitrary bounded continuous supermodular function. As ν and $\tilde{\nu}$ must have the same total mass, $\int f d\nu \leq \int f d\tilde{\nu}$ holds if and only if $\int (f - f(0))d\nu \leq \int (f - f(0))d\tilde{\nu}$. Thus we can assume without loss of generality that $f(0) = 0$. As in the proof of Lemma 4.2 let us define

$$f_n(x) := f(round(nx)/n).$$

Then f_n is a bounded supermodular function vanishing around zero, and thus $f_n \in \mathcal{B}_0$. Therefore $\int f_n d\nu \leq \int f_n d\tilde{\nu}$. As f_n converges to f pointwise, by dominated convergence we get $\int f d\nu \leq \int f d\tilde{\nu}$. □

Now we can state the main result about supermodular comparison of Lévy processes. The implication from a) to b) can already be found in Bergenthum and Rüschendorf (2007), where a different proof is given.

Theorem 4.5. *For Lévy processes* X, $\tilde{X}$ *with Lévy measures* $\nu, \tilde{\nu}$*, the following conditions are equivalent:*

a) $\nu \leq_{sm} \tilde{\nu}$*;*

b) $X \leq_{sm} \tilde{X}$*.*

Proof. For the proof we will use the following interpolation formula, which can be derived from Houdré (1998) and Houdré et al. (1998): let f be a bounded twice continuously differentiable function (written $f \in \mathcal{C}_b^2$) and denote by $\mathcal{A}$ and $\tilde{\mathcal{A}}$ the generators of the Lévy processes X and $\tilde{X}$ respectively. Then

$$\begin{aligned} &Ef(\tilde{X}(t)) - Ef(X(t)) \\ &\quad = \int_0^1 E[(\tilde{\mathcal{A}} - \mathcal{A})f(X^{(\alpha)}(t))]d\alpha \\ &\quad = \int_0^1 \int \int_{\mathbb{R}^d} (f(x+u) - f(x) - \langle h(u), \nabla f(x)\rangle)(t\tilde{\nu} - t\nu)(du)P_{X^{(\alpha)}(t)}(dx)d\alpha \end{aligned} \tag{4.1}$$

where $(X^{(\alpha)})$ is a Lévy process with Lévy measure $\alpha\tilde{\nu} + (1-\alpha)\nu$.
Now assume that a) holds. To show b), it is sufficient to show $Ef(X(t)) \leq Ef(\tilde{X}(t))$ for supermodular functions $f \in \mathcal{C}_b^2$ (see Denuit and Müller (2002)). But if $f \in \mathcal{C}_b^2$ then

$$u \mapsto g_x(u) := f(x+u) - f(x) - \langle h(u), \nabla f(x)\rangle$$

is a supermodular function in $\mathcal{B}_0$. To see supermodularity, notice that we have chosen h such that $u \mapsto -\langle h(u), \nabla f(x)\rangle$ is a sum of functions depending only on one variable and

therefore supermodular, and $g_x \in \mathcal{B}_0$ follows easily by developing f in a Taylor series around x. Thus $\nu \le_{sm} \tilde{\nu}$ implies that the inner integral in (4.1) is non-negative, and therefore the whole expression in (4.1) must be non-negative. This shows that a) implies b).
Next assume that b) holds and that $f \in \mathcal{C}_b^2 \cap \mathcal{B}_0$. From (4.1) we get

$$\begin{aligned} 0 &\le \lim_{t\to 0} \frac{Ef(\tilde{X}(t)) - Ef(X(t))}{t} \\ &= \lim_{t\to 0} \int_0^1 \int_{\mathbb{R}^d} E\Big(f(X^{(\alpha)}(t)+u) - f(X^{(\alpha)}(t)) - \langle h(u), \nabla f(X^{(\alpha)}(t))\rangle \Big)(\tilde{\nu}-\nu)(du)d\alpha \\ &= \int f(u)(\tilde{\nu}-\nu)(du). \end{aligned}$$

The last equality here follows from the fact that $X_t^{(\alpha)} \to 0$ a.s. for $t \to 0$ and from the fact that for $f \in \mathcal{C}_b^2 \cap \mathcal{B}_0$ we have that f and ∇f are continuous with $f(0) = 0$ and $\nabla f(0) = 0$. Thus we have $\int f d\nu \le \int f d\tilde{\nu}$ for all $f \in \mathcal{C}_b^2 \cap \mathcal{B}_0$ and hence especially for all infinite differentiable $f \in \mathcal{B}_0$. It follows now from Lemma 4.2 that b) implies a). □

Remark 4.6. The PSMD result from Theorem 3.10 is an easy corollary from Theorem 4.5. Notice that any univariate function is supermodular and therefore supermodular ordering can only hold for Lévy measures with the same marginal tail integrals. Thus for supermodular functions f we have $\int f d\tilde{\nu} - \int f d\nu = \int \overline{f} d\tilde{\nu} - \int \overline{f} d\nu$, where

$$\overline{f}(x_1,\ldots,x_d) = f(x) - f(x_1,0,\ldots,0) - f(0,x_2,0,\ldots,0) - \ldots - f(0,\ldots,0,x_d) + (d-1)f(0).$$

is a function vanishing on all axis. But a supermodular function vanishing on all axis must be non-negative on $\mathbb{R}^d_{++,--}$. Next notice that if X is a Lévy process with Lévy measure ν, then the Lévy process with the same marginals and independent components has a Lévy measure $\nu^\perp$ which is concentrated on the axis. Therefore for ν concentrated on $\mathbb{R}^d_{++,--}$ and f supermodular we get

$$\int f d\nu - \int f d\nu^\perp = \int \overline{f} d\nu - \int \overline{f} d\nu^\perp = \int \overline{f} d\nu \ge 0$$

and thus that the Lévy process X is PSMD if ν is concentrated on $\mathbb{R}^d_{++,--}$.

Let us now turn to the concordance order. A natural definition to compare Lévy measures in that way is

Definition 4.7. Lévy measures $\nu, \tilde{\nu}$ are said to be comparable with respect to the **concordance order** (written $\nu \le_c \tilde{\nu}$), if for all $f \in \mathcal{B}_0$ which are d-increasing or d-decreasing:

$$\int f d\nu \le \int f d\tilde{\nu}.$$

The preceding results for the supermodular order (in particular Lemma 4.4 and Theorem 4.5) can be shown for the concordance order in completely the same way. The analogy to Lemma 4.2 follows from Denuit and Müller (2002) Theorem 3.2. The translation of Lemma 4.4 is straightforward. It is important to note that the concordance order also implies that the marginal tail integrals of the Lévy measures have to be equal. Since Theorem 4.5 is the main result for the supermodular order in this section we will formulate it for the concordance order also. The proof follows the same lines.

Theorem 4.8. *For Lévy processes $X, \tilde{X}$ with Lévy measures $\nu, \tilde{\nu}$, the following conditions are equivalent:*

a) $\nu \le_c \tilde{\nu}$;

b) $X \le_c \tilde{X}$.

For $d = 2$ it is well-known that the orders $\le_{sm}$ and $\le_c$ coincide. This property carries over to the comparison of Lévy measures. Moreover in this case the characterization of $\le_{sm}$ and $\le_c$ simplifies. This is due to the fact that in this case the order can be generated by products of functions of the form

$$x \mapsto \mathbb{1}_{[x \ge t]} \text{ if } t > 0 \quad \text{and} \quad x \mapsto -\mathbb{1}_{[x \le t]} \text{ if } t < 0$$

$$x \mapsto -\mathbb{1}_{[x \ge t]} \text{ if } t > 0 \quad \text{and} \quad x \mapsto \mathbb{1}_{[x \le t]} \text{ if } t < 0.$$

We obtain here

Lemma 4.9. *Let $d = 2$. For Lévy measures $\nu, \tilde{\nu}$ with tail integrals $U, \tilde{U}$ the following conditions are equivalent:*

a) $\nu \le_c \tilde{\nu}$;

b) $\nu \le_{sm} \tilde{\nu}$;

c) ν *and* $\tilde{\nu}$ *have the same marginal tail integrals and* $U(x) \le \tilde{U}(x)$ *for all* $x \in (\mathbb{R}\backslash\{0\})^2$.

Next we will express the two considered orders in terms of the Lévy copulas. We will show that the supermodular and the concordance ordering of Lévy processes holds if and only if they have the same marginals and their Lévy copulas fulfill an appropriate condition. To define that condition we need the following classes of functions.

Definition 4.10. a) Let $\mathcal{S}^{SM}$ be the class of bounded measurable functions $f : (\mathbb{R}\backslash\{0\})^d \to \mathbb{R}$ with the property that the function $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\tilde{f}(u_1, \dots, u_d) = \begin{cases} f(\frac{1}{u_1}, \dots, \frac{1}{u_d}), & \text{if } u \in (\mathbb{R}\backslash\{0\})^d, \\ 0, & \text{else}, \end{cases} \tag{4.2}$$

is supermodular with $\tilde{f} \in \mathcal{B}_{00}$.

b) Let $\mathcal{S}_d^I$ be the class of bounded measurable functions $f : (\mathbb{R}\backslash\{0\})^d \to \mathbb{R}$ with the property that the function $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ defined by (4.2) is d-increasing or $-\tilde{f}$ is d-decreasing with $\tilde{f} \in \mathcal{B}_{00}$.

Functions in $\mathcal{S}^{SM}$ are supermodular, if they are restricted to an orthant, but in general they are not supermodular on the whole domain. A typical example of a function $f \in \mathcal{S}^{SM}$ in case $d = 2$ is

$$f(x_1, x_2) = \mathbb{1}_{\{(0,a)\times(0,b)\}}(x_1, x_2)$$

for $a, b > 0$.
In what follows note that since a Lévy copula is d-increasing and continuous there exists a unique measure μ_F on $\bar{\mathbb{R}}^d - \{(\infty, \dots, \infty)\}$ such that $\mu_F\big([a_1, b_1) \times \ldots \times [a_d, b_d)\big)$ is equal to the F-volume of $[a_1, b_1) \times \ldots \times [a_d, b_d)$.
Definition 4.10 in particular implies that for $f \in \mathcal{S}^{SM}$ there is some K such that $f(x) = 0$ if $\min_{i=1,\dots,d} |x_i| > K$. Together with the boundedness this has the consequence that $\int f d\mu_F$ is finite for any Lévy copula measure μ_F and all $f \in \mathcal{S}^{SM}$. Using this class $\mathcal{S}^{SM}$ we can show now that supermodular ordering of Lévy processes can be characterized in terms of Lévy copulas.

Theorem 4.11. *For Lévy processes $X, \tilde{X}$ with Lévy measures $\nu, \tilde{\nu}$ and Lévy copula measures $\mu_F, \mu_{\tilde{F}}$ the following conditions are equivalent:*

a) $X \leq_{sm} \tilde{X}$;

b) ν and $\tilde{\nu}$ have the same marginal tail integrals and $\int f d\mu_F \leq \int f d\mu_{\tilde{F}}$ for all $f \in \mathcal{S}^{SM}$.

Proof. Assume that a) holds. As $X \leq_{sm} \tilde{X}$ implies that X and $\tilde{X}$ have the same marginals, it is clear that ν and $\tilde{\nu}$ have the same marginal tail integrals $U_1, \dots, U_d$. Moreover, it follows from Theorem 4.5 that $\nu \leq_{sm} \tilde{\nu}$ and hence $\int f d\nu \leq \int f d\tilde{\nu}$ for all supermodular $f \in \mathcal{B}_{00}$. ν $(\tilde{\nu})$ is the image of μ_F $(\mu_{\tilde{F}})$ under the mapping

$$T : (u_1, \dots, u_d) \mapsto (U_1^{-1}(u_1), \dots, U_d^{-1}(u_d))$$

where

$$U_i^{-1}(u) = \begin{cases} \inf\{x > 0 \mid u \geq U_i(x)\}, & u \geq 0 \\ \inf\{x < 0 \mid u \geq U_i(x)\} \wedge 0, & u < 0 \end{cases}$$

is the canonical inverse. Thus

$$\begin{aligned} \int f d\nu = \int f d\mu_F^T = \int f \circ T d\mu_F \\ = \int f(U_1^{-1}(u_1), \dots, U_d^{-1}(u_d))\mu_F(d(u_1, \dots, u_d)) \\ = \int \hat{f}(\frac{1}{u_1}, \dots, \frac{1}{u_d})\mu_F(d(u_1, \dots, u_d)), \end{aligned}$$

and

$$\int f d\tilde{\nu} = \int \hat{f}(\frac{1}{u_1}, \dots, \frac{1}{u_d})\mu_{\tilde{F}}(d(u_1, \dots, u_d)),$$

where $\hat{f}(x) = f(h_1(x_1), \ldots, h_d(x_d))$ with

$$h_i(x) = \begin{cases} U_i^{-1}(\frac{1}{x}), & x \neq 0, \\ 0, & x = 0, \end{cases} \quad i = 1, \ldots, d.$$

As the functions $h_i, i = 1, \ldots, d$ are increasing and continuous in the origin, $\hat{f}$ is supermodular and in $\mathcal{B}_{00}$, if and only if this holds for f. Thus

$$\int f d\nu \leq \int f d\tilde{\nu}$$

is equivalent to

$$\int g d\mu_F \leq \int g d\mu_{\tilde{F}}$$

for all $g \in \mathcal{S}^{SM}$. Hence a) implies b), and the implication from b) to a) follows the same lines. □

There is an important practical case where we can circumvent the use of class $\mathcal{S}^{SM}$. We obtain

Corollary 4.12. *Let X and $\tilde{X}$ be associated Lévy processes with corresponding Lévy copula measures μ_F and $\mu_{\tilde{F}}$. Then the following conditions are equivalent:*

a) $X \leq_{sm} \tilde{X}$.

b) the Lévy measures have the same marginal tail integrals and $\int_{\mathbb{R}^d_{++,--}} f d\mu_F \leq \int_{\mathbb{R}^d_{++,--}} f d\mu_{\tilde{F}}$ *for all bounded and supermodular* $f : \mathbb{R}^d_{++,--} \to \mathbb{R}$ *with* $f(x) = 0$ *for all* $x \in \mathbb{R}^d_{++,--}$ *with* $\min_{i=1,\ldots,d} |x_i| > K$.

Proof. Since X is associated we have

$$\int_{\mathbb{R}^d} f d\nu = \int_{\mathbb{R}^d_{++}} f d\nu + \int_{\mathbb{R}^d_{--}} f d\nu$$

for all $f \in \mathcal{B}_{00}$ and the same for $\tilde{\nu}$. Now we have $f : \mathbb{R}^d_{++,--} \to \mathbb{R}$ is supermodular if and only if f is supermodular as a function on $\mathbb{R}^d_{++}$ and $\mathbb{R}^d_{--}$ separately. Thus we obtain that $X \leq_{sm} \tilde{X}$ in this case is equivalent to

$$\int_{\mathbb{R}^d_{++}} f d\nu \leq \int_{\mathbb{R}^d_{++}} f d\tilde{\nu} \quad \text{and} \quad \int_{\mathbb{R}^d_{--}} g d\nu \leq \int_{\mathbb{R}^d_{--}} g d\tilde{\nu}$$

for all $f : \mathbb{R}^d_{++} \to \mathbb{R}$, $g : \mathbb{R}^d_{--} \to \mathbb{R}$ supermodular and zero around $x = 0$. But on $\mathbb{R}^d_{++}$ and $\mathbb{R}^d_{--}$ separately the transformation used in the proof of Theorem 4.11 leads to a supermodular function again. Using the same arguments yields the statement. □

Obviously we obtain for the concordance order a similar result. The proof is omitted since it follows the same lines.

Theorem 4.13. *For Lévy processes X, $\tilde{X}$ with Lévy measures ν, $\tilde{\nu}$ and Lévy copula measures μ_F, $\mu_{\tilde{F}}$ the following conditions are equivalent:*

a) $X \leq_c \tilde{X}$.

b) ν *and* $\tilde{\nu}$ *have the same marginal tail integrals and* $\int f d\mu_F \leq \int f d\mu_{\tilde{F}}$ *for all* $f \in \mathcal{S}_d^I$.

In the case $d = 2$ we obtain from Lemma 4.9 and Sklar's Theorem for Lévy copulas the following simple characterization:

Corollary 4.14. *Let $d = 2$. For Lévy processes X, $\tilde{X}$ with Lévy measures ν, $\tilde{\nu}$ and Lévy copulas F, $\tilde{F}$ the following conditions are equivalent:*

a) $X \leq_c \tilde{X}$.

b) $X \leq_{sm} \tilde{X}$.

c) ν *and* $\tilde{\nu}$ *have the same marginal tail integrals and* $F \leq \tilde{F}$.

5 Examples

In this section we analyze the dependence properties of some popular Lévy processes.

Compound Poisson process
Let X be a d-dimensional compound Poisson process with intensity λ and jump size distribution Q. The Lévy measure on $\mathbb{R}^d$ is given by $\nu = \lambda Q$. Thus due to Theorem 3.9 X is associated (POD, PSMD) if and only if the common jumps have the same direction i.e. the jumps size distribution is concentrated on $\mathbb{R}_{++,--}$.
As far as the comparison of compound Poisson processes is concerned, suppose that we have two d-dimensional compound Poisson processes X and $\tilde{X}$ with intensities λ and $\tilde{\lambda}$ respectively and jump size distributions Q and $\tilde{Q}$ respectively and that $\tilde{\lambda} \leq \lambda$. Then we can w.l.o.g. assume that the intensity of the tilde process is λ and it has jump size distribution $\tilde{\tilde{Q}} = \frac{\tilde{\lambda}}{\lambda+\tilde{\lambda}}\tilde{Q} + \frac{\lambda}{\lambda+\tilde{\lambda}}\delta_0$ since this leads to the same process in probability. Thus, by Theorem 4.5 and Lemma 4.4 X and $\tilde{X}$ can be compared in the supermodular order if and only if the ordinary distributions Q and $\tilde{\tilde{Q}}$ can be compared in the supermodular order.

Archimedean Lévy copulas
Kallsen and Tankov (2006) have introduced a family of Archimedean Lévy copula, however, it can be checked easily that the class of Lévy processes generated by their family of Archimedean Lévy copulas in case $d > 2$ does not include positive dependent Lévy processes in terms of association. Thus, we present another definition here. The idea is as follows. We first define an Archimedean copula for spectrally positive Lévy measures on $(0, \infty)^d$, and then we extend this to the whole Euclidean space by defining such an Archimedean copula for each orthant and combining them in an appropriate way. On $(0, \infty)^d$ we use the following

definition, which is a straightforward extension of the definition of an Archimedean copula for probability measures: Let $\phi : (0, \infty) \to (0, \infty)$ be a strictly decreasing function with alternating signs of derivatives up to order d and with $\lim_{t \to 0} \phi(t) = \infty$ and $\lim_{t \to \infty} \phi(t) = 0$ and define

$$F_\phi(u_1, \ldots, u_d) = \phi\left(\sum_{i=1}^d \phi^{-1}(u_i)\right), \quad u_1, \ldots, u_d > 0. \tag{5.1}$$

Then F_ϕ satisfies the properties of a Lévy copula on $(0, \infty)^d$.

Remark 5.1. Notice that we use the more familiar *additive generator* approach in contrast to the *multiplicative generator* approach used in Kallsen and Tankov (2006), but they differ only by a logarithmic transformation, see Nelsen (2006) for the case of Archimedean copulas of probability measures.

Now let $I = \{-1, 1\}^d$ and notice that for each $\mathbf{i} = (i_1, \ldots, i_d) \in I$ we have an orthant

$$O_\mathbf{i} = \{x \in \mathbb{R}^d : \operatorname{sgn}(x_j) = i_j,\ j = 1, \ldots, d\}$$

Given a set of functions $F_{\phi_\mathbf{i}}, \mathbf{i} \in I$, and a weight function $\eta : I \to [0, 1]$ having the property that for each $k \in \{1, \ldots, d\}$

$$\sum_{\{\mathbf{i}: i_k = -1\}} \eta(\mathbf{i}) = \sum_{\{\mathbf{i}: i_k = 1\}} \eta(\mathbf{i}) = 1, \tag{5.2}$$

we can then define an Archimedean Lévy copula on $\mathbb{R}^d$ by

$$F(u_1, \ldots, u_d) = \begin{cases} \sum_{\mathbf{i} \in I} \left(\eta(\mathbf{i}) F_{\phi_\mathbf{i}}(|u_1|, \ldots, |u_d|) \mathbb{1}_{\{u \in O_\mathbf{i}\}} \prod_{j=1}^d \operatorname{sgn}(u_j) \right) & , \text{if } |u_j| > 0,\ j = 1, \ldots, d, \\ 0 & , \text{else.} \end{cases} \tag{5.3}$$

In contrast to the definition of Kallsen and Tankov (2006) our proposal allows for positive dependence. It follows immediately from Lemma 3.6 and Theorem 3.10:

Lemma 5.2. *An Archimedean Lévy copula as defined in* (5.3) *is associated if and only if*

$$\eta(1, 1, \ldots, 1) = \eta(-1, -1, \ldots, -1) = 1.$$

Example 5.3. In case we choose $\varphi(u) = u^{-\frac{1}{\theta}}$ in (5.1) and $\eta(1, 1) = \eta(-1, -1) = 1$, then we obtain the Clayton-Lévy copula

$$F(u) = F_\theta(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta})^{-\frac{1}{\theta}} \mathbb{1}_{[u \in \mathbb{R}^2_{++}]} + ((-u_1)^{-\theta} + (-u_2)^{-\theta})^{-\frac{1}{\theta}} \mathbb{1}_{[u \in \mathbb{R}^2_{--}]}$$

By Theorem 4.14 we know that $X_\theta \leq_c X_{\theta'} \Leftrightarrow F_\theta \leq F_{\theta'} \Leftrightarrow \theta \leq \theta'$.

6 Applications

6.1 Ruin times

Suppose that the purely discontinuous Lévy process $X = (X_1(t), \ldots, X_d(t))_{t\geq 0}$ represents the evolution of d risk reserve processes of different business lines. Of particular interest are the ruin times (or more general first hitting times) of these processes. Denote by

$$\tau_j := \inf\{t \geq 0 \mid X_j(t) \leq 0\}$$

the ruin time of risk reserve $j, j = 1, \ldots, d$. If the risk reserve processes are positive dependent, then the ruin time points are positive dependent. This statement can be formalized as follows:

Theorem 6.1. *Let $X = (X_1(t), \ldots, X_d(t))_{t\geq 0}$ be a $\mathbb{R}^d$-valued Lévy process. If X is associated (or POD or PSMD) then the ruin time points $\tau = (\tau_1, \ldots, \tau_d)$ are associated (or POD or PSMD).*

Proof. Let us consider a discretisation of our model. For $\triangle > 0$, the discrete ruin time $\tau_j^{\triangle}$ of the j-th risk process is then defined as

$$\tau_j^{\triangle} = \min\{m\triangle \geq 0 \mid X_j(m\triangle) < 0\} \text{ for } j = 1, \ldots, d, m \in \mathbb{N}$$

and

$$\tau_j^{[n],\triangle} = \min\{\tau_j^{\triangle}, n\}$$

the ruin time truncated at n. Similarly to Denuit et al. (2007) $\tau_j^{[n],\triangle}$ can be written as

$$\tau_j^{[n],\triangle} = \sum_{l=1}^{n} \prod_{m=1}^{l} \mathbb{1}_{\{X_j(m\triangle)\geq 0\}} + 1.$$

Note that we know from Lemma 2.6 that the vector $\Big(X_j(m\triangle) - X_j((m-1)\triangle),\ j = 1, \ldots, d, m = 1, \ldots, M\Big)$ is associated. Now, $\tau_j^{[n],\triangle}$ is a conjunction of non-decreasing functions of this vector, and consequently $\Big(\tau_1^{[n],\triangle}, \ldots, \tau_d^{[n],\triangle}\Big)$ is associated (see Lemma 2.4 a)). We now change over to the continuous setting and let $\triangle$ tend to 0. Note that we are using the (unique) càdlàg version of a Lévy process as customary. As the partition gets finer, $\tau_j^{[n],\triangle}$ converges a.s. to $\tau_j^{[n]}$, which implies the convergence in distribution. According to Lemma 2.4 d) we conclude that $\tau_j^{[n]}$ is associated. Letting $n \to \infty$ and using the same arguments again, we obtain that the ruin time vector $\tau = (\tau_1, \ldots, \tau_d)$ is associated. $\square$

It is also possible to derive a comparison result. It follows from Proposition 2.6 in Denuit et al. (2007):

Theorem 6.2. *Let $X = (X_1(t), \dots, X_d(t))_{t\geq 0}$ and $\tilde{X} = (\tilde{X}_1(t), \dots, \tilde{X}_d(t))_{t\geq 0}$ be two $\mathbb{R}^d$-valued Lévy processes. If $X \leq_{sm} \tilde{X}$ then the ruin time points are ordered:*

$$\tau = (\tau_1, \dots, \tau_d) \leq_{sm} \tilde{\tau} = (\tilde{\tau}_1, \dots, \tilde{\tau}_d).$$

Another popular risk model is to consider a portfolio of risk processes whose sum describes the risk of an insurance company. Models of this type have for example been investigated in Juri (2002) and Bregman and Klüppelberg (2005). Suppose now that the portfolio is described by an $\mathbb{R}^d$-valued Lévy process $X = (X_1(t), \dots, X_d(t))_{t\geq 0}$. By $X_t^+ := \sum_{i=1}^d X_i(t)$ we denote the one-dimensional risk process for the insurance company. Note that this is again a Lévy process. By ψ_X we denote its probability of ruin, i.e.

$$\psi_X(u) = P\left(\inf_{t\geq 0} X_t^+ < 0 \mid X_0^+ = u\right).$$

A question of interest is how dependence between the components influences the probability of ruin. We obtain:

Theorem 6.3. *Suppose we have two portfolios of risk processes $X = (X_1(t), \dots, X_d(t))_{t\geq 0}$ and $\tilde{X} = (\tilde{X}_1(t), \dots, \tilde{X}_d(t))_{t\geq 0}$ which are both $\mathbb{R}^d$-valued Lévy processes. If $X \leq_{sm} \tilde{X}$ then*

$$\int_u^\infty \psi_X(s)ds \leq \int_u^\infty \psi_{\tilde{X}}(s)ds.$$

This statement can be shown in the same way as Theorem 1 in Bäuerle and Rolski (1998). Moreover Proposition 5.1. in Asmussen et al. (1995) then implies that the corresponding adjustment coefficients R and $\tilde{R}$, whenever they exist, are ordered by $R \geq \tilde{R}$. Now let us consider an explicit example: Suppose $d = 2$, and the components of the risk processes are linked by the Clayton Lévy copula in Example 5.3 with parameter θ. From Theorem 6.3, Theorem 4.14 and Example 5.3 it follows that the corresponding adjustment coefficient is decreasing in θ. This is reasonable since we obtain for $\theta \to \infty$ the complete dependence case and for $\theta \to 0$ the independence copula. This generalizes results given in Theorem 3.12 in Bregman and Klüppelberg (2005).

6.2 Option pricing and credit risk

Suppose we have a financial market with d risky assets whose vector of price processes is denoted by $\big(S_1(t), \dots, S_d(t)\big)_{t\geq 0}$. Prices of risky assets are often modelled as exponential Lévy processes. Here we suppose that $\big(X_1(t)), \dots, X_d(t)\big)_{t\geq 0}$ is a d-dimensional Lévy process with purely discontinuous paths and that the price processes satisfy the following stochastic differential equation

$$\begin{aligned} dS_i(t) &= S_i(t-)\big[\mu_i(t)dt + \sigma_i(t-)dX_i(t)\big] \\ S_i(0) &= 1 \end{aligned} \tag{6.1}$$

where $\sigma_i(t) > 0, \mu_i(t)$ are bounded deterministic càdlàg functions. Further we assume for all $i = 1, \dots, d$

(A1) $E[exp(-hX_i(1))] < \infty$ for all $h \in (-h_1, h_1)$, for some $h_1 > 0$.

(A2) $\sigma_i(t)\big(X_i(t) - X_i(t-)\big) \geq -1$ for all $t \geq 0$.

The first assumption guarantees that $X_i(t)$ has finite moments of all orders and the second assumption implies that the jumps of $X_i(t)$ are bounded from below and that $S_i(t) \geq 0$ for all t. A solution of the stochastic differential equation (6.1) is given by (see e.g. Protter (1990))

$$S_i(t) \;=\; \exp\left[\int_0^t \mu_i(s) - \frac{1}{2}\sigma_i^2(s)ds\right] \times \prod_{0<s\leq t}\Big(1 + \sigma_i(s)(X_i(s) - X_i(s-))\Big)$$

Lemma 6.4. *If the Lévy process X is associated (or POD or PSMD), then the price processes are associated (and thus also POD and PSMD).*

Proof. That $S(t)$ is associated for any $t \geq 0$ has been shown in Bäuerle (2002). Since the price processes have independent increments, the statement follows from Lemma 2.6. □

This lemma can be applied when option prices on more than one stock are computed. Suppose for example that we have two stocks where the stock price processes under the risk neutral probability measure Q are two exponential Lévy processes as before and a deterministic bond with price process $B = \big(B(t)\big)_{t\geq 0}$. We look at a contingent claim with pay-off $H = h(S_1(T), S_2(T))$. Its price is given by $\pi(H) = B_T^{-1}E_Q[h(S_1(T), S_2(T))]$. The following lemma now follows easily form our results

Lemma 6.5. *If h is a supermodular function and S_1 and S_2 are associated, then*

$$\pi(H) \geq \pi(H)^{\perp}$$

where $\pi(H)^{\perp}$ is the price of the same option under the assumption of independent price processes.

Typical functions h which are supermodular are

$$h(x,y) = \big(\min(x,y) - K\big)^+, \quad h(x,y) = \big(-\max(x,y) - K\big)^+, \quad h(x,y) = \big(x + y - K\big)^+.$$

Obviously the preceding Lemma also holds when the pay-off $h(S_1, S_2)$ of the option is path-dependent and h has enough structure. Examples are here

$$\begin{aligned}
h(s_1, s_2) &= \Big(\min\big(\inf_{t\in[0,T]} s_1(t), \inf_{t\in[0,T]} s_2(t)\big) - K\Big)^+ \\
h(s_1, s_2) &= \Big(-\max(\sup_{t\in[0,T]} s_1(t), \sup_{t\in[0,T]} s_1(t)) - K\Big)^+ \\
h(s_1, s_2) &= \big(s_1(T) - K\big)^+ \mathbb{1}_{[\inf_{t\in[0,T]} s_2(t)\geq b]}.
\end{aligned}$$

As far as the comparison of the stock prices are concerned, we obtain the following result (for the current model and under the assumptions made so far).

Lemma 6.6. *Let X and $\tilde{X}$ be two Lévy processes with $X \leq_{sm} \tilde{X}$. Then the corresponding price processes satisfy $S \leq_{sm} \tilde{S}$.*

Proof. Define $\log S := \big(\log S_1(t), \ldots, \log S_d(t)\big)_{t\geq 0}$. Note that the supermodular order is preserved under increasing transformations of the margins. Thus, as in the proof of Lemma 2.6 and Lemma 2.9 respectively we obtain that $X \leq_{sm} \tilde{X}$ implies

$$\log\Big(1+\sigma_i(t)\big(X_i(t)-X_i(t-h)\big)\Big) \leq_{sm} \log\Big(1+\sigma_i(t)\big(\tilde{X}_i(t)-\tilde{X}_i(t-h)\big)\Big),\ \forall t,h>0$$

and thus

$$S(t) \leq_{sm} \tilde{S}(t)\ \forall t>0 \quad \Rightarrow \quad S \leq_{sm} \tilde{S}.$$

□

As a consequence, a comparison of option prices can be done along the lines of Lemma 6.5. Another application of the preceding considerations are structural models of credit risk. Here a default of one counterparty is triggered whenever its stock price falls below a certain threshold.

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