Second-Price Proxy Auctions in Bidder-Seller Networks

A Game Theoretic and Experimental Analysis

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Contents

Chapter 1

Introduction

Auctions have a long tradition, dating back as far as 500 B.C. when women were auctioned for marriage in Babylonia. Auctions were also popular in the Roman Empire, ancient China, Japan, and Greece. In the last two centuries, they have been applied in many areas from philately and other collectibles and antiques to perishables like fish and tulips, treasury bills, construction contracts, and telecommunications licenses (Cassady, Jr., 1967; Milgrom, 1987; Lucking-Reiley, 2000b; Klemperer, 2004). With the success of the Internet, new applications for auctions arose.[1] Contrary to traditional auction formats, in auctions conducted via the Internet, it is not necessary that people meet in one place at one time.[2] This led to the development of many new auction formats employed, for example, to negotiate contracts in consumer-to-consumer or business-to-business relationships (see Anandalingam, Day, and Raghavan, 2005, for an overview).

The term auction is derived from the Latin word *augere*, which means *to augment* or *to increase*. It refers to the traditional type of auction where the price is determined by sequentially increasing a publicly announced bid until only one bidder is willing to pay that amount. An auction is defined as "a market institution with an explicit set of rules determining resource allocation and prices on the basis of bids from the market participants" (McAfee and McMillan, 1987, p. 701). Auctions are an appropriate selling mechanism if the seller is uncertain about the item's value to potential buyers and if the costs of running an auction are low (e.g., Ockenfels, Reiley, and Sadrieh,

[1] See Lucking-Reiley (2000a) for an overview of the years 1993–1999 with an ascertainment of the situation in 1998.
[2] Participation by phone was also possible before the Internet.

2006).[3]

Amongst private sellers, Internet auctions became popular as an instrument offering access to many potential buyers, mainly for small collectibles. They have the advantage of low transaction costs (of running the auction) and of reaching a large audience (Ockenfels et al., 2006). Nowadays, on the most popular auction sites, there are not only private sellers but also professional retailers offering almost anything one can imagine. The trade volume is huge, at least for some auction platforms. For example, on eBay Marketplace, the largest Internet auction platform in the USA, 55% of the gross merchandise volume of $ 60 billion comes from auctions (and the remaining 45% from a fixed-price listing format). The number of fixed price and auction listings on a single day can reach 140 million. At the end of 2008, 86.3 million active users[4] were counted (eBay Inc., 2008). eBay offers its auction platform in 38 markets.[5] Besides the USA, Germany and the United Kingdom are the largest eBay markets. In Germany, eBay.de has more than 14.5 million active users and on average more than 30 million items are offered concurrently.[6] In other countries, different companies are market leaders, for example Yahoo! Japan Auctions (auctions.yahoo.co.jp) in Japan[7] and ricardo.ch in Switzerland.[8] There are also auction platforms that have specialized in certain products, like my-hammer.de for craftsmen and services,[9] AuctionVine.com and WineBid.com for wines in the US, or Surplex.com for used machinery. Platform operators usually charge listing fees and/or transaction fees. In some cases, the platform operators are also sellers of the products (merchant sites like uBid.com). This may lead to a conflict of interest, especially with certain new selling procedures.

The auction formats applied on most auction sites (e.g., eBay and Yahoo!) are variants of an ascending second-price proxy auction (Lucking-Reiley, 2000a). That is, a bidder submits a bidding limit to an automatic proxy bidding agent, which bids on his behalf up to this limit.[10] As long as the auction is open, a bidder can submit

[3] Auctions are also used as buying mechanism, called a reverse auction, where sellers act as bidders and the lowest bid wins.

[4] An active user had bid, bought, or sold at least once in 2008.

[5] eBay.de, http://pages.ebay.de/aboutebay/thecompany/companyoverview.html, April 28, 2009.

[6] http://presse.ebay.de/news.exe?content=FD, April 28, 2009.

[7] www.japanauctioncenter.com/

[8] www.ricardo.ch/pages/about_company/de.php, April 28, 2009.

[9] There, however, the lowest offer does not win automatically (Stiftung Warentest, 2007).

[10] On some platforms, the bidder may choose if he wants to use a proxy bidder or the seller may choose if he offers a proxy bidding agent, e.g., at www.trademe.co.nz and azubo.de (bidders' choice) or uBid.com (sellers' choice).

new bids, i.e., he can increase his bidding limit. The bidder with the highest submitted bid (or bidding limit) is the current high bidder. His bidding limit is hidden to other bidders. The current standing bid equals the second-highest submitted bid (e.g., at WineBid.com) or the second-highest submitted bid plus one increment (e.g., in eBay auctions). When the auction ends, the current high bidder becomes the winner and the current standing bid is the price he has to pay.

There are two types of ending rules: soft close and hard close. eBay uses a hard close rule in which each auction has a fixed ending time. Amazon and Yahoo! auctions are often mentioned as examples of sites that implement flexible or soft close rules, under which an auctions' ending time is delayed whenever a new bid is submitted (e.g., Roth and Ockenfels, 2002; Houser and Wooders, 2005).[11]

Many variants of this auction format and other auction forms (e.g., sealed-bid auctions and Dutch auctions) are offered on the Internet. Sometimes sellers have additional options. An example is the "Buy-It-Now" option on eBay that allows a buyer to purchase an item at a posted price instead of starting the auction process.[12] Once a bid is submitted, this option disappears. On some platforms, sellers can offer a buyout price as an alternative fixed price during the whole auction process. Setting a secret reserve price is another popular option for sellers. Bidders are usually informed if the reserve price is met by the standing bid. If it is not met at the end, the item is either not sold or, on some platforms, the seller may decide nevertheless to sell to the high bidder.

In what follows, the term *Internet auctions* refers to consumer-to-consumer Internet auctions offered on platforms operated by independent companies.[13] However, some of the descriptions and also some of our results are not restricted to this environment.

Due to the success and the growing economic relevance, Internet auctions started to attract the interest of researchers (see Bajari and Hortaçsu, 2004, for an overview). Auctions offered on Internet platforms were typically analyzed as separate auctions of a single unit. But the typical presence of multiple, similar auctions should be

[11] The flexible or soft close rule is also called the going-going-gone rule. Yahoo! auctions offers both ending options. Yahoo! auctions closed in many countries (e.g., the US and Canada in 2007) and is mainly active in Japan and other Asian countries. Amazon auctions is not active anymore. uBid.com and compendo.de also have a soft close ending rule.

[12] See, e.g., Seifert (2006).

[13] Note that, for example, eBay has changed from a platform dominated by consumer-to-consumer auctions to one used by professional sellers and traditional stores. Besides offering items in auctions, sellers can use a pure fixed-price listing or offer the additional "Buy-It-Now" feature.

expected to have an effect on behavior and on outcomes. Recently, this aspect began to be included in theoretical and empirical research. In this thesis, such a model of multiple auctions is analyzed.

We assume these auctions are offered by independent sellers. That is, in contrast to other multi-unit auctions like the spectrum auctions, there is no single seller in whose interest the auctioneer can coordinate bids. In principle, the platform operator may act as an auctioneer who coordinates bids. The difference from the case of a single seller is that the auctioneer who acts on behalf of multiple sellers may be unable to act in the interest of *every* single seller. A decision may be in the interest of one seller but to the disadvantage of another seller. We call these auctions *independent*.

On an auction platform, a bidder can usually choose between many items offered for sale. Let us assume that he is interested in buying only one item. However, a bidder may evaluate many different items even though he wants to buy only one. Assume, for example, you want to buy a mountain bike. On an Internet auction platform you find a huge number of offers.[14] You may prefer a new bike to a used one, but if the seller of the used bike seems to be trustworthy (has a high rating in the reputation system) you may also consider buying the used bike. You may prefer a red bike to a yellow one, a full-suspension bike to a hard tail, a "Cannondale" to a "Specialized" bike, this year's model to last year's, and so on. But, eventually, you want to buy only one bike. In the end, the price you have to pay will determine which bike you prefer. This example illustrates the unit-demand preferences for heterogeneous items in our model.

Furthermore, different bidders may consider different items to be substitutes. Thus, a bidder interested solely in Cannondale and another one interested in Specialized bikes only will not compete. However, both of them may compete with bidders who are less restrictive, who in turn may be competitors in some auctions and not in others. This competition, or lack thereof, is included in unit-demand preferences for heterogeneous items because a bidder may regard an item as useless and assign it a value of zero. If one bidder has a low valuation for an item, this does not tell us anything about its ranking or absolute value for another bidder.

There are other reasons why the assumption of homogeneous items may be improper. A seller's reputation, shipping costs and terms seem to have an influence on

[14] For example, on April 22, 2009, 2,161 mountain bikes were listed on eBay.com in the category "Mountain Bikes."

prices (e.g., Melnik and Alm, 2002, 2005; Hossain and Morgan, 2006; Houser and Wooders, 2006; Resnick, Zeckhauser, Swanson, and Lockwood, 2006).[15] Moreover, items are presented differently, with more or less appealing layout and with or without pictures of the item. In contrast to some of the examples above, these factors probably make an item more or less attractive to *all* bidders.

The mountain bike example highlights another important aspect of Internet auctions. Some bidders may enter a search term like "Cannondale mountain bike" or "Specialized mountain bike,"[16] others may click through the system of categories to the subcategory "Mountain Bikes," while others may search for the term "mountain bike" in the category "Bicycles and Frames" and then refine for "Complete Bike" and "New."[17] Thus, a potential bidder's search strategy influences the set of auctions that he will know of. The necessity of knowing about an item to permit trade can be modeled via existent and non-existent links in a network of bidders and sellers. It motivates our analysis of incomplete bidder-seller networks in the last theoretical part of this thesis.

The mountain bike example shows, first, that bidders may consider items as substitutes while having different values for them, and second, that different bidders may consider different items as substitutes. Third, even if bidders have similar preferences, they may nevertheless have different search strategies and thus face different sets of auctions.

Internet auctions have another advantage over traditional auctions: bidders need not meet at a certain time and place. The auctions are typically open for bids over some time period (on eBay, for example, one to ten days). Bidders may check auctions that are interesting to them from time to time and can bid whenever they want.

Our basic model incorporates multiple independent auctions and assumes unit-demand preferences for heterogeneous items, as motivated above. The preference structure also allows different bidders to consider different items as substitutes. The auction format is a second-price proxy auction and the ending rule may be interpreted as a soft close rule. Bidders submit bids asynchronously. In an extension of the model, results concerning the incomplete bidder-seller network that most likely arises

[15]For a discussion and an overview of many empirical studies, see Bajari and Hortaçsu (2004).

[16]Note that these are proposed search terms if you enter "cann" or "spec" into the search field, respectively, which may increase the probability that potential bidders use them. The search terms gave 114 and 176 results on April 22, 2009, respectively, at eBay.com.

[17]This search gives 522 results.

on auction platforms are presented.

The structure of this thesis is as follows. In the next chapter, concepts used or referred to in the analysis are introduced and an overview of the related literature is given. Having provided this background, the main model is analyzed in Chapter 3. In Section 3.1, the model is presented. Section 3.2 provides the equilibrium analysis. A certain strategy is proposed, outcomes that result from following this strategy are analyzed, and, finally, incentives to follow this strategy are investigated. The results of Chapter 3 establish a relation to Vickrey outcomes. Hence, Chapter 4 is dedicated to the analysis of a certain aspect of Vickrey auctions: the impact of increasing a valuation on the outcome. The results are interesting in themselves, but also provide a basis for the investigation of the bidder-seller network model in Chapter 5. This chapter combines results of Chapters 3 and 4 to investigate auctions in incomplete bidder-seller networks. Results of an experimental study of the basic model of Chapter 3 are presented in Chapter 6. Chapter 7 concludes.

Chapter 2

Concepts and Related Literature

This chapter provides an overview of the methods and concepts used. We start with the concepts applied in the analysis of the basic model, and then introduce cooperative games, Vickrey auctions, and the assignment game since those will also play a role in the further analysis.

Then, related literature – in particular, results related to the assignment game, auction theory, and experiments – is presented and connections between our model and the literature are established.

2.1 Concepts and Definitions

In this section, we present the theoretical background, concepts, and definitions that are used in our model. The first part deals with the general, game-theoretic setting and the solution concept. After that, cooperative games, the Vickrey auction, and the assignment game are briefly introduced.

Game Theoretic Background The following formal description of a non-cooperative game in extensive form Γ closely follows the description of the extensive form representation of a game in Mas-Colell, Whinston, and Green (1995, p. 227).

Definition 2.1 (Game in extensive form) *A game in extensive form* $\Gamma = \{\mathcal{S}, u\}$ *is described by its extensive-form structure* $\mathcal{S} = \{\mathcal{I}, \mathcal{Y}, \mathcal{A}, prec(\cdot), a(\cdot), \mathcal{H}, H(\cdot), s(\cdot), \rho(\cdot)\}$ *and a collection of utility functions* $u = (u_i)_{i \in \mathcal{I}}$*. The elements are given by:*

- *A finite set of players* $\mathcal{I}$*, a finite set of nodes* $\mathcal{Y}$*, and a finite set of possible actions* $\mathcal{A}$*.*

- *A function prec*$(\cdot)$*:* $\mathcal{Y} \rightarrow \{\mathcal{Y} \cup \emptyset\}$ *that assigns a unique precedent node to every node of the game. The first node in the game tree is denoted by* y_0*, i.e.,* $prec(y_0) = \emptyset$*. The assignment of nodes to predecessors is such that the resulting structure is a tree.*[1]

- *A function* $a(\cdot)$*:* $\mathcal{Y}\backslash\{y_0\} \rightarrow \mathcal{A}$ *that assigns every node of the game to the action that leads to this node from its predecessor.*[2]

- *A collection of information sets* $\mathcal{H}$ *and the function* $H(\cdot)$*:* $\mathcal{Y} \rightarrow \mathcal{H}$ *that assigns every node* $y \in \mathcal{Y}$ *to an information set* $H(y) \in \mathcal{H}$*.*

- *A function* s*:* $\mathcal{H} \rightarrow \mathcal{I} \cup \{0\}$ *that assigns every information set to a player who moves at all decision nodes in the information set. 0 denotes the nature player. The collection of player i's information sets is denoted by* $\mathcal{H}_i = \{H \in \mathcal{H} : s(H) = i\}$ *for* $i \in \mathcal{I} \cup \{0\}$*.*

- *A function* ρ*:* $\mathcal{H}_0 \times \mathcal{A} \rightarrow [0, 1]$ *that assigns probabilities to actions at every information set of nature. Actions not available at an information set have probability zero; probabilities for available actions add up to one.*

- *A collection of utility functions* $u = (u_i)_{i\in\mathcal{I}}$ *with individual utility functions* $u_i(\cdot)$*:* $\mathcal{Z} \rightarrow \mathbb{R}$*, where* $\mathcal{Z}$ *denotes the set of terminal nodes in the game tree.*

We will, in a slight abuse of notation, denote player i's information sets $H(y) \in \mathcal{H}_i$ by H_i, and refer to information sets by $H(y)$, H, or H_i, depending on what is more suitable. When using the notation H_i, we will also write $y \in H_i$ if $H_i = H(y)$. The information sets $H(y) \in \mathcal{H}$ partition $\mathcal{Y}$. It is required that the same set of action choices be available at all nodes in the same information set (i.e., if $H(y) = H(y')$ then $a(y) = a(y')$). The set of available actions at an information set H is $A(H) := \{c \in \mathcal{A} : c \in a(y) \text{ for } y \in H\}$. Information sets represent knowledge of other players' past actions and of one's own type.

In the following, we describe important concepts in the context of the model that we are going to analyze. First, a common approach is to assume common knowledge in a game.

[1] A tree is a connected graph without cycles.

[2] The initial node does not have a predecessor and therefore the function a is not defined for node y_0.

Definition 2.2 (Common knowledge) *Some aspect of the game is common knowledge among players, if all players know it*[3] *and know that all other players know it, and know that all other players know that they know it, and so on.*

This intuitive definition of common knowledge is due to Lewis (2002, p. 52ff.). For early formal definitions of common knowledge see, for example, Schiffer (1972) and Aumann (1976). We assume common knowledge of the structure of the game and of rationality.

Besides the knowledge structure of a game, another related, important aspect is the information structure. It is common to distinguish between imperfect and incomplete information.

Definition 2.3 (Imperfect information) *A sequential game is a game of imperfect information if some player does not know all actions the players before him have chosen, when it is his turn. Considering information sets this is equivalent to at least one non-singleton information set existing.*

Definition 2.4 (Incomplete information) *A game is a game of incomplete information if at least some factor of the game, such as the set of players, the set of strategies of all players, the utility functions, or the structure of the game, is not common knowledge.*

If the full description of the game is common knowledge, the game is a game of complete information.[4]

If players do not have complete information in a game, further concepts are required to model the incomplete information. The most popular concept is that of a common prior. This assumption goes back to Harsanyi (1967, 1968a, 1968b).

Definition 2.5 (Common prior) *The assumption of a common prior indicates that the "state of the world" – an $|\mathcal{I}|$-tuple of types of the players – is drawn out of a common basic probability distribution.*

[3]That is, it is mutual knowledge (see, e.g., Aumann and Brandenburger, 1995).

[4]Harsanyi (1967) states that the basic ways in which incomplete information in a normal form game can occur are as incomplete information over the outcome function that assigns outcomes to strategy combinations, over utility functions (i.e., the evaluation of the outcomes by the players), or over strategy spaces. He argues that all these kinds of incomplete information can be expressed as incomplete information over utility functions.

In other words, the players' subjective probability distributions over states of the world can be derived as conditional probability distributions from some basic probability distribution – the common prior distribution. This assumption is often justified by the statement that all differences in information between players are included in their type, i.e., in their private information or attribute vector. Thus, the assumption asserts that a player's beliefs in different states of the world are posterior probabilities formed from a common prior probability distribution of all players, given his private information.[5]

Harsanyi (1967, 1968a, 1968b) shows that under the assumption of a common prior, a game with incomplete information can be transformed into a game with complete, but imperfect information with an initial move by nature. For the formulation of the model this means that we introduce a *nature player* that draws an $|\mathcal{I}|$-tuple of types or the attribute vectors of players (more generally, a "state of the world") using the common prior as probability distribution. After that, every player learns only his own type, but not those of the other players. Thus, every player uses a subjective posterior probability distribution that is derived from a common prior distribution over the unknown parameters, to analyze the game. Such a game can be analyzed with the usual equilibrium concepts for games with complete but imperfect information. Thus, the so-called *Harsanyi transformation* (Harsanyi, 1968b) provides a basis for the analysis of games with incomplete information.[6]

Another important concept for the analysis of extensive form games is that of perfect recall:

Definition 2.6 (Perfect recall) *A game Γ is a game with perfect recall if*

(1) $H(y) = H(\hat{y})$ implies that y is not a (direct or indirect) predecessor of $\hat{y}$, and $\hat{y}$ is not a predecessor of y, and

(2) if y, $\hat{y}$, and y' are decision nodes of i with $H(y) = H(\hat{y}) \neq H(y')$ and y' precedes y (directly or indirectly) and action $c \in a(y')$ is the action at y' on the path to y, then a predecessor $\hat{y}'$ of $\hat{y}$ must exist that is also element of $H(y')$ and the action that leads from $\hat{y}'$ to $\hat{y}$ must also be c.

[5] This assumption is not without controversy. For a discussion of the common prior assumption, see, for example, Morris (1995). For a recent, positive approach, see Heifetz (2006).

[6] See also Aumann and Heifetz (2002). Harsanyi (1968b) also shows how his theory can be extended to the "inconsistent case," the case without common prior.

The introduction of perfect recall as a restriction on the information partition goes back to Kuhn (1950, 1953).[7] Perfect recall is the assumption that a player remembers everything he knew at his earlier information sets of the game and he remembers all his previous action choices. As a consequence, all nodes that belong to some information set of player i are reached by the same sequence of choices made by player i, and i's information sets can be ordered by precedence. Perfect recall is a common assumption imposed on extensive-form games. In the following, when we talk about a game in extensive form, we always assume perfect recall.

So far, we considered concepts related to the extensive-form structure of a game. Now we consider concepts to describe the players' behavior in the game and their conjectures about other players' behavior. The first aspect is covered by the notion of a behavior strategy profile; the second aspect by the notion of a system of beliefs.

In the analysis of the model, we restrict ourselves to pure strategies. Therefore, we define a behavior strategy profile in pure strategies as follows.

Definition 2.7 (Behavior strategy profile $\sigma = (\sigma_i)_{i \in \mathcal{I}}$ in pure strategies) *A behavior strategy profile σ in pure strategies is a collection of behavior strategies $\sigma_i(\cdot)\colon \mathcal{H}_i \to A(H)_{H \in \mathcal{H}_i}$ that assign an available action $a \in A(H)$ to every information set $H \in \mathcal{H}_i$ of player i.*

A pure strategy of player i consists of action choices for all his information sets. For pure strategies, this coincides with the concept of behavior strategies. Σ_i denotes the set of pure strategies of player i and $\sigma \in \Sigma := \Sigma_1 \times \ldots \times \Sigma_n$.

Allowing a player to choose a probability distribution over pure strategies results in a mixed strategy. Mixed strategies differ from mixed behavior strategies in that they are probability distributions over strategies and mixed behavior strategies contain probability distributions over available actions at each information set of a player, i.e., they are locally randomized strategies. *Kuhn's theorem* (Kuhn, 1950, 1953) states that under the condition of perfect recall, each mixed behavior strategy can also be expressed as a mixture over pure strategies if strategies are uncorrelated between players.

To formalize the players' conjectures at non-singleton information sets caused by their lack of information, the concept of a system of beliefs is introduced.

[7]The definition given here follows Mas-Colell et al. (1995).

Definition 2.8 (System of beliefs) *A system of beliefs is a vector* $\mu = (\mu_H)_{H \in \mathcal{H}}$*, where* μ_H *for each* $H \in \mathcal{H}$ *is a function* $\mu_H \colon H \rightarrow [0,1]$ *such that* $\sum_{y \in H} \mu_H(y) = 1$*.*

The beliefs of a player at one of his information sets reflects his conjectures about the past play.

For the description of an equilibrium, the notion of an assessment, which goes back to Kreps and Wilson (1982), is necessary. It combines a strategy profile with a system of beliefs. A perfect Bayesian equilibrium is an assessment that fulfills certain conditions.

Definition 2.9 (Assessment) *An assessment is a pair* (σ, μ) *that consists of a behavior strategy profile* σ *and a system of beliefs* μ *at the information sets.*

Note that the set of assessments of a game in extensive form can be derived from the structure of the game $\mathcal{S}$, but does not depend on the utility function.

Consistency concepts define restrictions on beliefs, or the way beliefs are derived from previous play and beliefs in the game tree. We define the concept of Bayesian consistency that can be applied along the path induced by the strategy profile σ, as follows.

Definition 2.10 (Bayesian consistency) *An assessment* (σ, μ) *is Bayesian consistent if beliefs are derived from the strategies of other players by Bayes' rule wherever possible:*
At each information set $H(y)$ *the beliefs* $\mu_H(y)$ *for all* $y \in H(y)$ *are given by*

$$\mu_H(y) = \frac{\mathrm{Prob}^{\sigma}(y)}{\mathrm{Prob}^{\sigma}(H(y))}$$

whenever $\mathrm{Prob}^{\sigma}(H(y)) > 0$*, i.e., along the path induced by* σ*.* Prob^{σ} *denotes the prior probabilities induced by* σ*.*

Bayesian consistency is also referred to as weak consistency (e.g., Hendon, Jacobsen, and Sloth, 1996). An alternative consistency concept that eliminates contradictions between players' behavioral conjectures and beliefs is *updating consistency.*

Definition 2.11 (Updating consistency (Perea, 2002)) *An assessment* (σ, μ) *is updating consistent if for every player* i*, every two information sets* H_i *and* H_i' *of* i

with H_i' succeeding H_i, and every behavior strategy $\hat{\sigma}_i$ of i, beliefs fulfill the following condition for all $y \in H_i'$ whenever $\mathrm{Prob}^{(\hat{\sigma}_i,\sigma_{-i})}(H_i'|H_i, \mu_{H_i}) > 0$*:*

$$\mu_{H_i'}(y) = \frac{\mathrm{Prob}^{(\hat{\sigma}_i,\sigma_{-i})}(y|H_i, \mu_{H_i})}{\mathrm{Prob}^{(\hat{\sigma}_i,\sigma_{-i})}(H_i'|H_i, \mu_{H_i})}\ .$$

In this definition and in the following, σ_{-i} denotes the behavior strategies of all players except for i. In contrast to Bayesian consistency, updating consistency defines requirements on (σ, μ) at information sets H_i off the equilibrium path. It takes into account that from player i's point of view, the past behavior of the other players, given that H_i is reached, is described by his belief μ_{H_i}. His conjecture about their behavior on paths starting in H_i is given by σ_{-i}. Whenever reaching H_i' is compatible with these beliefs about the past and conjectures about future behavior of the other players evaluated at H_i and i's own strategy $\hat{\sigma}_i$, the beliefs of i at H_i' are required to be induced by Bayes' law (see Perea, 2001, p. 70 ff.).

The previous definitions of information and related properties of the extensive-form structure have been independent of the payoffs at the terminal nodes. We now come to the first concept that restricts the set of strategies by assuming some kind of rationality and, therefore, needs to take payoffs into account. In particular, we consider the concept of sequential rationality. In the following definition, $E[u_i(\sigma|H, \mu_H)]$ denotes the expected payoff of i if σ is played, given that H has been reached and beliefs at H are μ_H.

Definition 2.12 (Sequential rationality) *A strategy σ_i of player i is sequentially rational for beliefs μ if for all $H \in \mathcal{H}_i$*

$$\sigma_i \in \arg\max_{\tilde{\sigma}_i \in \Sigma_i} E[u_i(\tilde{\sigma}_i, \sigma_{-i})|H, \mu_H].$$

A strategy profile $\sigma = (\sigma_i)_{i\in\mathcal{I}}$ is sequentially rational if it consists of sequentially rational strategies σ_i for all $i \in \mathcal{I}$.

A property that helps to determine sequentially-rational strategies is the one-shot-deviation principle. The one-shot-deviation principle, a generalization of the one-stage-deviation principle, is due to Hendon et al. (1996). The validity of the one-stage-deviation principle for extensive-form games with observable actions goes back to Blackwell (1965). We define the one-shot-deviation principle and cite a proposition that reveals when the principle applies.

Definition 2.13 (One-shot-deviation principle) *In a finite extensive-form game* $\Gamma = (\mathcal{S}, u)$ *for a given* σ_{-i} *and* μ*, a strategy* $\sigma_i = (\sigma_{iH})_{H \in \mathcal{H}_i}$ *of player* i *is optimal if and only if player* i *cannot improve by choosing a strategy* $\hat{\sigma}_i$ *that differs from* σ_i *at* one *of* i*'s information sets.*

In the words of Perea (2002), (σ, μ) fulfills the one-shot-deviation principle if σ_i is globally optimal if and only if it is locally optimal. The following proposition gives a necessary and sufficient condition such that an assessment (σ, μ) in an extensive-form structure $\mathcal{S}$ satisfies the one-shot-deviation principle for every utility function u.

Proposition 2.1 (Perea (2002)) *An assessment* (σ, μ) *satisfies the one-shot-deviation principle if and only if* μ *is updating consistent.*

The next step is to define a suitable solution concept. The first proposed equilibrium concept that extends the idea of subgame perfect equilibrium to improper subgames (that start at non-singleton information sets), i.e., to games with imperfect information, is the sequential equilibrium of Kreps and Wilson (1982). Since this concept is hard to apply formally to our game, we solve for a perfect Bayesian equilibrium (PBE). The article of Fudenberg and Tirole (1991b) contains an early discussion of this concept (see also Fudenberg and Tirole, 1991a). The following definition of a PBE restricts beliefs off the equilibrium path by updating consistency. In the literature, several versions of PBE exist, which differ with respect to the restrictions on beliefs off the equilibrium path.[8]

Definition 2.14 (Perfect Bayesian equilibrium (PBE)) *An assessment* (σ^*, μ) *constitutes a perfect Bayesian equilibrium (PBE), if*

- σ^* *is sequentially rational for the given system of beliefs* μ*,*
- *beliefs* μ *are determined by Bayesian updating wherever possible, and*
- *off the equilibrium path* σ^* *is sequentially rational relative to some updating consistent beliefs.*

[8] The PBE without initial restrictions on beliefs off the equilibrium path is introduced in Mas-Colell et al. (1995) as *weak* PBE and is equivalent to the weak sequential equilibrium of Myerson (1991) (compare Mas-Colell et al., 1995, p. 283). Weak PBEs sometimes have undesirable properties, for example, the concept does not assure subgame perfection.

We also use the notion of an epsilon-equilibrium that goes back to Radner (1980).

Definition 2.15 (Epsilon-equilibrium (Radner, 1980)) *An* epsilon-equilibrium *is a combination of strategies such that the resulting payoffs of the players are within* ε *of the maximum possible expected payoff against the other players' strategies.*

Several approaches to judge the efficiency of the outcome of a game exist. When we talk about efficiency, we mean the simplest approach as defined below. Since in our model all players' payoffs are measured in monetary units (they are linear in money) and money is the means of exchange, the rather strong assumptions for such an efficiency measure are fulfilled.[9]

Definition 2.16 (Efficient outcome) *An outcome is efficient if it maximizes the sum of payoffs of all players.*

Cooperative Games The central assumption in models of cooperative game theory is that players can commit to binding agreements. A game in characteristic function form $\Gamma^c = (\mathcal{I}, c)$ is described by the set of players $\mathcal{I}$ and the characteristic function $c\colon \mathcal{P}(\mathcal{I}) \to \mathbb{R}$ with $c(S) = 0$ if $S = \emptyset$. The game Γ^c is also called a game in coalitional function form and c its coalitional function. The sets $S \subseteq \mathcal{I}$ are called coalitions if nonempty and $\mathcal{I}$ is called the grand coalition. We only consider coalitional games with transferable utility (so-called TU games). A coalition S can then divide its coalitional value $c(S)$ in any way among the players $i \in S$. A solution concept for cooperative games is a rule that assigns one or several *feasible* payoff vectors u, payoff vectors with the property $\sum_{i \in \mathcal{I}} u_i \leq c(\mathcal{I})$, to games $(\mathcal{I}, c)$.[10] One of the main solution concepts is the core, which was defined for the first time in Gillies (1959). Other popular solution concepts are the von Neumann/Morgenstern stable set, the Shapley value solution, and the nucleolus.[11]

In the following, we assume super-additivity[12] of c and thus we only consider so-called proper games (Holler and Illing, 2000; Owen, 1968).

[9] However, we show below that our efficient outcomes are also Pareto-optimal.

[10] For some solution concepts, the rule may also assign the empty set as the solution to some games. For example, the core or the von Neumann/Morgenstern stable set may be empty (Owen, 1968).

[11] For an introduction to the theory of cooperative games see for example Moulin (1991) and Peleg and Sudhölter (2007).

[12] Formally: for all S, $T \subset \mathcal{I}$ such that $S \cap T = \emptyset$, $c(S) + c(T) \leq c(S \cup T)$.

An *imputation* is an efficient and individually rational payoff vector. Hence, the imputation set is

$$\left\{ u \in \mathbb{R}^{\mathcal{I}} : \sum_{i \in \mathcal{I}} u_i = c(\mathcal{I}) \quad \text{and} \quad u_i \geq c(\{i\}) \quad \text{for each} \quad i \in \mathcal{I} \right\}.$$

If a payoff vector is an imputation, coalitions consisting of single players have no incentive to leave the grand coalition (non-improvability by individuals). Most solution concepts consider only imputations. The solution concept we use is the core.[13]

Definition 2.17 (Core) *The core* $\mathcal{C}(\Gamma^c)$ *of a game in characteristic function form* $\Gamma^c = (\mathcal{I}, c)$ *is a set of payoff vectors u:*

$$\mathcal{C}(\Gamma^c) = \left\{ u \in \mathbb{R}^{\mathcal{I}} : \sum_{i \in \mathcal{I}} u_i = c(\mathcal{I}) \quad \text{and} \quad \sum_{i \in S} u_i \geq c(S) \text{ for all } S \subset \mathcal{I} \right\}.$$

Payoff vectors in the core have the property that the value of the grand coalition is shared or goes to one player and no blocking sub-coalition S, i.e., no coalition S that may improve upon these payoffs, exists (non-improvability by coalitions). The core characterizes payoff combinations that are stable with respect to coalition formation. The core of a game may consist of a continuum of payoff vectors, a single payoff vector, or it may be empty.

In the characteristic function form associated with a non-cooperative game, the value of a coalition equals the maximum that the coalition can assure itself in the worst case, i.e., whatever players outside the coalition do (the maximin value) (see, e.g., Holler and Illing, 2000; Owen, 1968).

With respect to auctions, the core is considered an important concept (see Milgrom, 2004). If equilibrium payoffs of bidders and sellers in a non-cooperative auction game are in the core of a suitably defined cooperative game, the outcome is stable against renegotiation. Efficiency and stability against renegotiation are often aims of auction design.

Vickrey Auction Vickrey auctions are a subset of the non-cooperative games that are related to the famous Vickrey-Clarke-Groves (VCG) mechanisms (Vickrey, 1961;

[13]See, e.g., Slikker and van den Nouweland (2001); Milgrom (2004). The empty sum is assumed to equal zero.

Clarke, 1971; Groves, 1973). Other games where these mechanisms are applied are, for instance, models of public goods. A mechanism consists of a set of players, a message space for each player, and an outcome function that assigns a decision and transfers for each player to each message profile (Jackson, 2003). Combining a mechanism with utility functions we get a game. For an overview of VCG mechanisms and their specification as auction mechanisms see Milgrom (2004) or Krishna (2002).

The single-unit Vickrey auction, the multi-unit Vickrey auction, and the Vickrey package auction are all based on the VCG mechanism. The three specifications mentioned refer to auctions of single items, multi-unit auctions with bids for packages of items only related to the size of the package, and package auctions, where bidders' may bid for every package of items separately, depending on the items in the package. Vickrey auctions are one-shot, sealed-bid auctions that assign the items for sale efficiently with respect to the submitted bids. Bidders who do not win an item pay nothing and winning bidders' payoffs equal their respective marginal values to the grand coalition.

In the following definition, we assume that the seller does not have the option to act in the auction game. He accepts the payments and does not set limit prices strategically.

Definition 2.18 (Vickrey auction) *A Vickrey auction is a normal form game with incomplete information $G = (N \cup \{s\}, \Theta_1 \times \ldots \times \Theta_n, \mathcal{A}_1 \times \ldots \times \mathcal{A}_n, u)$, where the players are the seller s and the bidders $i \in N$ with $|N| = n$, each bidder i's action space $\mathcal{A}_i$ equals his type/valuation space Θ_i, and his utility function u_i is quasi-linear. On the basis of the bidders' actions (bids),*

- *items are assigned efficiently,*
- *Vickrey payments of bidders i are $p_i^V = c(N \backslash \{i\} \cup \{s\}) - c_{-i}(N \cup \{s\})$, and*
- *the seller receives all payments.*

The expression $c_{-i}(N \cup \{s\})$ denotes the value of coalition $(N \cup \{s\})$ without i's contribution to that coalition. Vickrey auctions are important because they are the only one-shot sealed-bid auctions that yield an efficient allocation of goods, where bidding truthfully is a dominant strategy, and losing bidders pay nothing.[14] Bidding

[14] This is because Vickrey-Clarke-Groves mechanisms are the only efficient and incentive compatible

truthfully means that each bidder i with type $\theta_i \in \Theta_i$ submits a bid $b_i \in \mathcal{A}_i$ with $b_i = \theta_i$.

We will refer to a variant of the Vickrey auction, where each bidder's bids (valuations) are such that he buys at most one item. Thus, bidders' payments may be related to individual items. We have multiple sellers, each offering one item. Then, it is possible to assign the payments to individual sellers and we use the appropriately extended definition, that each seller receives the payment of the bidder who buys his item.

Assignment Game Details on the historical background of the game are given in Section 2.2.1. For the assignment game, a characteristic function form and a more explicit market game interpretation are usually given.

The assignment game in characteristic function form $\Gamma^c_{ag} = (N \cup M, c)$ contains disjoint sets, N and M, of players (e.g., buyers and sellers) with $|N| = n$ and $|M| = m$. Each pair of players (i, j) with $i \in N$ and $j \in M$ can generate a value d_{ij}. The characteristic function c assigns the value d_{ij} to such pairs (i, j). Every other coalition's characteristic function value is either the maximum sum of pairs' values in the coalition, or zero if no pair (i, j) is part of the coalition. This characteristic function is super-additive.

We can interpret the assignment game as a market game by defining $d_{ij} := \max\{0, v_{ij} - v^S_j\}$ the gains from trade that the buyer-seller pair (i, j) may achieve in a market (N,M,V,v^S) with buyers' valuations for each item $V := (v_{ij})_{i\in N, j\in M}$ and sellers valuations for their item $v^S := (v^S_j)_{j\in M}$.[15] Utility functions u_i of bidder i and u^S_j of seller j are assumed to be quasi-linear. A price vector is denoted by $p = (p_1, \ldots, p_m)$. Thus, $d_{ij} = (v_{ij} - p_j) + (p_j - v^S_j)$ is the sum of payoffs $u_i := v_{ij} - p_j$ of the potential buyer i and $u^S_j := p_j - v^S_j$ of the potential seller j, if this sum is non-negative. The price is paid by the buyer and received by the seller and, therefore, has no influence on the conjoint value of a trading pair. If we assume that prices and values of players are continuous, there are always item prices such that a buyer is indifferent between

direct mechanisms (see, e.g., Milgrom (2004) and Jackson (2003); the result goes back to Groves (1973) and Green and Laffont (1977)).

[15] Our main focus is on the buyers or bidders. To distinguish them from the sellers, sellers' valuations and payoffs are denoted with a superscript S, whereas bidders' valuations v_{ij} and payoffs u_i have no superscript. The index i refers to buyers and j to sellers.

the items. This is an important property of this model.[16]

An assignment matches buyers and sellers. If a buyer i is assigned to a seller j, the seller j is also assigned to buyer i. An assignment is defined as a matrix of zeros and ones as follows.

Definition 2.19 (Assignment) *An assignment is a $n \times m$ matrix x with $x_{ij} \in \{0, 1\}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. If $x_{ij} = 1$, buyer i is assigned to seller j and vice versa; $x_{ij} = 0$ indicates that i and j are not matched.*

Not every assignment matrix represents a feasible assignment.

Definition 2.20 (Feasible assignment) *An assignment x is* feasible *if*

$$\sum_{j \in M} x_{ij} \leq 1 \quad \text{for all } i,$$
$$\sum_{i \in N} x_{ij} \leq 1 \quad \text{for all } j.$$

The set of all feasible assignments is denoted by X.

A feasible payoff-assignment combination is such that the sum of payoffs may be reached in the given assignment where side-payments between players are allowed.

The next definition considers optimal assignments.

Definition 2.21 (Optimal assignment) *A feasible assignment $x \in X$ is optimal if*

$$\sum_{i \in N} \sum_{j \in M} (v_{ij} - v_j^S) \cdot x_{ij} \geq \sum_{i \in N} \sum_{j \in M} (v_{ij} - v_j^S) \cdot x'_{ij} \quad \text{for all } x' \in X\,.^{17} \tag{2.1}$$

When we compare the set of efficient outcomes and the set of optimal assignments, we can conclude from

[16] This property transfers to our multiple-auctions game with discrete valuations and prices as defined in Section 3.1. In our formulation, with the restrictions on discrete valuations and prices given there, the following results for continuous valuations and prices are also valid.

[17] Alternatively, $\sum_{i \in N} \sum_{j \in M} d_{ij} \cdot x_{ij} \geq \sum_{i \in N} \sum_{j \in M} d_{ij} \cdot x'_{ij}$ for all $x' \in X$. Thus, optimal assignments maximize the sum of gains from trade achieved by bidder-seller pairs.

$$\max_{x\in X}\left(\sum_i u_i + \sum_j u_j^S\right) = \max_{x\in X}\left(\sum_i\sum_j (v_{ij}-p_j)\,x_{ij} - \sum_i\sum_j (p_j - v_j^S)\,x_{ij}\right)$$
$$= \max_{x\in X}\left(\sum_i\sum_j (v_{ij}-v_j^S)\,x_{ij}\right)$$

that the assignments in these sets are identical.[18] Optimal assignments combined with a price vector give an efficient outcome, and each efficient outcome consists of an optimal assignment (Shapley and Shubik, 1971).

An alternative approach to assess efficiency of outcomes is Pareto-efficiency or Pareto-optimality. In contrast to our efficiency measure, Pareto-efficiency does not require interpersonal comparability of utility.

Definition 2.22 (Pareto-optimal outcome) *An outcome (x, p) is Pareto-optimal if no other outcome $(\tilde{x}, \tilde{p})$ exists such that*

- $u_i(\tilde{x}, \tilde{p}) \geq u_i(x, p)$ *and* $u_j^S(\tilde{x}, \tilde{p}) \geq u_j^S(x, p)$ *for all* $i \in N$, $j \in M$ *and*
- $u_h(\tilde{x}, \tilde{p}) > u_h(x, p)$ *for at least one buyer* $h \in N$ *or* $u_k^S(\tilde{x}, \tilde{p}) > u_k^S(x, p)$ *for at least one seller* $k \in M$.

Allowing for side-payments, the set of efficient outcomes and the set of Pareto-optimal outcomes coincide. An outcome that is not efficient cannot be Pareto-optimal: otherwise, we may increase the sum of bidders' valuations minus sellers' valuations (the gains from trade) and rearrange payments such that every player gets the same or more than before.[19] Thus, any Pareto-optimal outcome is efficient. If an outcome is not Pareto-optimal, then there exists another outcome where every player has at least the same payoff and at least one player is better off. This is not possible by rearranging prices beacause a higher price makes a buyer worse off and a lower price makes a seller worse off. Thus, the gains from trade must be larger in a Pareto-improving outcome and this implies that the former outcome was not efficient. Hence, any efficient outcome is Pareto-optimal (in the current environment with transferable utility).

[18] The sets of efficient outcomes and optimal assignments also correspond to each other if the set of feasible assignments allows a bidder to buy more than one item even as he evaluates only one, that is, if $\sum_j x_{ij} \leq m$ for all i is feasible and $u_i = \max_j\{v_{ij}x_{ij}\}$.

[19] Under our later restrictions on the grid of valuations and the grid of bids (which consequently equals the grid of prices) such a rearrangement is possible even with discrete prices.

That is, with quasi-linear utility functions, efficient or Pareto-optimal outcomes are associated with optimal assignments and vice versa. It can be shown that side-payments are not necessary for this to hold (Roth and Sotomayor, 1990). In the following, we denote an optimal assignment by x^{eff}.

The assignment game is related to the theory of one-to-one matching,[20] where the main issue is stability of matchings (Roth and Sotomayor, 1990). A payoff-assignment combination is stable if it is individually rational and no blocking buyer-seller pair exists.[21]

Definition 2.23 (Stable payoff-assignment combination) *A feasible payoff-assignment combination* (u, x) *is* stable *if*

(1) $u_i \geq 0$ *and* $u_j^S \geq 0$ *for all* $i \in N$, $j \in M$, *and*

(2) $u_i + u_j^S \geq d_{ij}$ *for all* $(i, j) \in N \times M$.

The first part is the individual rationality condition. The second part of the definition states that no coalition of one buyer and one seller can improve upon the stable payoffs.

The characteristic function $c(\cdot)$ of the assignment game Γ_{ag}^c gives the value of an optimal assignment. It is defined as

$$c(S) = \max_{x \in X} \sum_{i \in S \cap N} \sum_{j \in S \cap M} d_{ij} \cdot x_{ij} \quad \text{for all} \quad S \subseteq (N \cup M).$$

The empty sum is assumed to equal zero. Thus, the characteristic function assigns the maximum value of possible trades between its members to a coalition. It follows that simple coalitions of only one player have a characteristic value of zero as no trade can be realized ($c(\{i\}) = 0$ for all $i \in N \cup M$). Note that coalitions consisting only of players from one side of the market also have a characteristic function value of zero, i.e., $c(S) = 0$ for all coalitions $S \subseteq N$ or $S \subseteq M$.

[20]Also called the marriage model. There, no payments between agents are possible.

[21]Roth and Sotomayor (1990, Definition 8.4, p. 205).

Therefore, the core of the assignment game Γ^c_{ag} can be described as

$$\mathcal{C}(\Gamma^c_{ag}) = \Bigg\{ (u, u^S) \in \mathbb{R}^{|N \cup M|} \, : \, \sum_{i \in N} u_i + \sum_{j \in M} u^S_j = \max_{x \in X} \sum_{i \in N} \sum_{j \in M} d_{ij} x_{ij} \, ,$$
$$\sum_{i \in S} u_i + \sum_{j \in T} u^S_j \geq \max_{x \in X} \sum_{i \in S} \sum_{j \in T} d_{ij} \, x_{ij} \text{ for all } S, T \neq \emptyset, S \subseteq N, T \subseteq M \, ,$$
$$\sum_{i \in S} u_i \geq 0 \text{ for } S \subseteq N \, , \quad \text{and} \quad \sum_{j \in T} u^S_j \geq 0 \text{ for } T \subseteq M \Bigg\} .$$

From this explicit description of the value of a coalition it becomes obvious that stable assignment-payoff combinations are those that fulfill the core conditions for coalitions consisting of a single buyer and a single seller or of a single buyer-seller pair.

2.2 Literature Review

In this section, we discuss models and results from different strands of literature that are related to our multiple-auctions game in Chapter 3.

As discussed in Milgrom (2004), the core of an auction game is useful because it contains all outcomes where no coalition of players has an incentive to renegotiate with the seller. The relevant associated game in coalitional form to our multiple-auctions game is the cooperative version of the assignment game, which is extensively analyzed in the literature. Considering auction theory, we distinguish between models that consider the case of multiple sellers and those where a single seller offers multiple items. A third category includes models that explicitly relate to Internet auctions. Furthermore, we describe experimental and empirical studies on auctions. Since eBay is by far the largest consumer-to-consumer auction platform in the US, most studies use eBay data and consider features of eBay auctions, like the fixed ending time. Our model, however, uses a soft close rule.

2.2.1 Assignment Game

Shapley and Shubik (1971) describe the assignment game as “a model for a two-sided market in which a product that comes in large, indivisible units (houses, cars, etc.) is exchanged for money, and in which each participant either supplies or demands exactly one unit. The units need not be alike, and the same unit may have different values to different participants." The definition of the assignment game as a cooperative market game goes back to Shapley (1955). Assignment problems had already occurred earlier in the literature. For example, Thorndike (1950) asks for a method for optimally assigning candidates to jobs. The underlying question of the optimal assignment problem is (in the current context) to determine which disjoint buyer-seller pairs generate the maximum overall gains from trade in the market. Disjoint pairs and unmatched potential buyers and sellers build a feasible assignment. Koopmans and Beckmann (1957) describe the mathematical background of the linear assignment problem and its solution, provide an early analysis as a market game, and present the relation to the zero-sum game of John von Neumann.[22] Roth and Sotomayor (1990) give an extensive and excellent overview of results concerning the assignment game. A formal description of the assignment game and the usual solution concepts applied to it is given in Section 2.1.

We begin our overview by considering the stability of assignment-payoff combinations. It turns out that in every stable payoff-assignment combination, a matched pair receives its coalitional value and every unmatched player receives a payoff of zero. Moreover, in the assignment game with continuous payoffs (and values of pairs), an optimal assignment is compatible with any stable payoff, and any stable outcome may only be associated with an assignment that is optimal (see, e.g., Roth and Sotomayor, 1990). Therefore, we also refer to stable payoffs instead of stable payoff-assignment combinations, keeping in mind that they are associated with the optimal assignment.

Shapley and Shubik (1971) show that the set of stable payoffs and the core of the cooperative assignment game Γ^c_{ag} coincide. Both are associated with an optimal assignment. As we already mentioned in the previous section, the stability conditions are a subset of the core conditions. Thus, any core payoff is stable. If an outcome is stable, then the payoffs are in the core: if the outcome is not in the core, then there exists a blocking coalition $T \subseteq N \cup M$ such that $\sum_{i \in T \cap N} u_i + \sum_{j \in T \cap M} u_j < c(T)$. Since x determines buyer-seller pairs as essential coalitions, we have $c(T) = \max_{x \in X} \sum_{i \in T \cap N} \sum_{j \in T \cap M} c(\{i,j\})\, x_{ij}$. Hence, a blocking coalition has at least one buyer-seller pair (i,j) with $u_i + u_j < c(\{i,j\}) = d_{ij}$ and the payoffs are not stable.[23]

[22] The Hungarian method of Kuhn (1955/2004) determines optimal assignments.

[23] For the proof of feasibility of the payoffs, see Shapley and Shubik (1971) or Roth and Sotomayor (1990).

Thus, the coincidence of core and stable payoffs is due to the bidder-seller pairs being the essential coalitions in this game.[24]

The core of the assignment game is non-empty.[25] Furthermore, the core with the partial order $\succsim_B$, which orders outcomes according to buyers preferences, forms a complete lattice (Shapley and Shubik, 1971). That is, if two payoff vectors $u = (u^B, u^S)$ and $\hat{u} = (\hat{u}^B, \hat{u}^S)$ are in the core, then the payoffs $(u^B,\hat{u}^S)$ and $(\hat{u}^B,u^S)$ are also in the core. The existence of buyer-optimal and seller-optimal payoffs as extreme points of the core is, thus, straightforward. We will mainly consider the buyer-optimal payoff vector, the payoffs that each buyer weakly prefers to any other payoff vector in the core.

Moreover, the core of the assignment game Γ^c_{ag} coincides with the set of payoffs in (appropriately defined) competitive (Walrasian) equilibria of the associated market game (Shapley and Shubik, 1971; Kaneko, 1982; Quinzii, 1984).[26]

Recent analyses of the assignment game consider the extreme points of its core (Hamers, Klijn, Tijs, and Villar, 2002; Núñez and Rafels, 2003b), singleton cores (Sotomayor, 2003; Wako, 2006), stable cores (Solymosi and Raghavan, 2001), buyer-seller exactness (Núñez and Rafels, 2003a), and the Shapley value in buyer-seller exact games (Hoffmann and Sudhölter, 2007).

2.2.2 Auctions

Our model is related to models of auctions with multiple independent sellers and to multi-unit auctions with one seller. There are also models that explicitly relate to Internet auctions. These auctions used to be analyzed as independent single-unit auctions. More recently, the important aspect that a potential bidder can usually choose between several auctions offered on an Internet auction platform (or even between auctions offered at different auctions platforms) started to be taken into account. We

[24] Side-payments are allowed, but it is shown that all payoffs in the core may be reached without side-payments between different buyer-seller pairs (i, j) and (h, k) in an optimal assignment (cp. e.g., Roth and Sotomayor, 1990).

[25] That is, it is a so-called balanced game (see, e.g., Shapley, 1967; Slikker and van den Nouweland, 2001). For TU games, a game with a non-empty core is a balanced game and vice versa.

[26] A price vector is competitive if there is at least one buyer interested in each item that has a price above the minimum price and if there is at most one buyer who strictly prefers it. Therefore, the market clears and each buyer receives an item in his demand set. A competitive price combined with a feasible assignment of items to buyers is a competitive equilibrium. In competitive equilibria, assignments are efficient.

review some of the results of these theoretical studies, starting with auctions offered by multiple sellers and central coordination of bids. Then we present the case of a single seller (but mostly, more general specifications of bidders' valuations), where coordination of bids is possible and is not a very critical assumption. Finally, we report studies that explicitly relate to Internet auctions.[27]

Multiple Sellers All models in this section assume that sellers offer one item each and that each bidder wants to buy at most one item. Demange (1982) and Leonard (1983) consider the non-cooperative, one-shot auctions game related to the assignment game that selects the bidder-optimal outcome. They were the first to investigate incentive compatibility for bidders to reveal their true valuations. Bidders submit one bid and an algorithm determines the outcome of the auction, depending on the bids. Bidding truthfully is a dominant strategy, because bidders' optimal payoffs in the core equal their payoffs in the efficient assignment with Vickrey payments; i.e., the auction is in fact a Vickrey auction.

Demange, Gale, and Sotomayor (1986) explicitly relate the assignment game to dynamic auction mechanisms.[28] In their "exact auction algorithm," bidders announce their demand and the auctioneer calculates overdemanded sets, chooses a minimal overdemanded set, and increases the prices in all auctions in this set by one price unit.[29] The bidders submit their demand again, and so on, until the auctioneer finds a matching of bidders to auctions such that each bidder receives an auction in his demand set. In their "approximate auction mechanism," bidders increase the standing bid by an increment in their preferred auction and the price equals the submitted bid. That is, they consider multiple English auctions. However, they do not consider the strategic behavior of bidders but the convergence of the described bidding behavior. They show that their exact auction algorithm converges to the competitive equilibrium with the minimum prices, which corresponds to the bidder-optimal outcome in the core. For the approximate auction algorithm, they find an

[27] For an introduction to auction theory see, e.g., Berninghaus, Ehrhart, and Güth (2006); Krishna (2002); Menezes and Monteiro (2005); Milgrom (2004).

[28] Auction algorithms similar to those used in these auctions to determine the outcomes are also used in Operations Research for example to determine shortest paths or to solve related problems (see, e.g., Bertsekas, 1991, 1992).

[29] An overdemanded set is a set of items (or auctions) such that the number of bidders who demand only items in this set is higher than the number of items. A minimal overdemanded set is an overdemanded set such that no subset of this set is also overdemanded.

upper bound on deviations from minimum competitive prices.

Generalizations with respect to valuations are due to Crawford and Knoer (1981) and Kelso and Crawford (1982). They do not consider incentives, but their adjustment process may also be interpreted as an auction algorithm. Crawford and Knoer (1981) analyze a model where the two sides of the market are represented by firms and the workers for whom they compete. Workers have firm specific job satisfaction and productivity, and the potential salaries are also specified for each firm-worker pair. They present a salary-adjustment process that converges to a core allocation. They generalize the assignment game by showing that the existence of the core and the convergence to a core allocation do not depend on the assumption of transferable utility. Furthermore, the process converges to the optimal outcome for the market side that proposes the salaries. Kelso and Crawford (1982) show that this convergence to the proposer-optimal strict core allocation of their adjustment process also holds for more general valuations – in particular, whenever their "workers are gross substitutes" condition (in the following, called *gross substitutes*) applies.[30] These models have in common that the adjustment processes are algorithms that determine an outcome under a behavioral assumption on the participants. These multi-stage processes consist at each stage of a certain proposal made by each seller to his most preferred worker given the current salary and a preliminary acceptance or refusal by a worker. The outcome of this algorithm is then analyzed with respect to its properties (competitive equilibrium, core).

Single Seller The literature on multi-unit auctions with indivisible items may be classified according to the following criteria: homogeneity and heterogeneity of items; unit demand, decreasing marginal valuations, or additive valuations in the case of homogeneous items; unit demand, gross substitutes or other substitutes conditions, additive package valuations, and general package preferences for heterogeneous items. Our model considers bidders with unit demand when items are heterogeneous. This is an extreme case of substitutes preferences. Thus, we concentrate on models that

[30]The gross substitutes condition (on bidders' valuations for packages) assures the existence of the core for these two-sided markets with multiple players on both market sides. Milgrom (2004) translates the definition of "bidder i's valuations satisfy the gross substitutes condition" of Kelso and Crawford (1982) to: $(p'_{-j} \geq p_{-j}$ and $p'_j = p_j) \rightarrow D_{ij}(p') \geq D_{ij}(p)$, where $D_{ij}(p)$ denotes i's demand for j at price vector p. See also Gul and Stacchetti (1999) for an alternative definition when utility functions are quasi-linear.

consider auctions for heterogeneous items and that allow for substitutes preferences.

In the multiple seller case with one item per seller, it is necessary that prices are determined per item. The market-clearing prices and assignment have to be found without central coordination. The single seller case gives rise to other kinds of package pricing, such as non-anonymous or non-additive pricing. The single seller or auctioneer can coordinate the bids. Thus, bidders' actions may be to announce demand sets and the auctioneer, as a central coordination device, clears the market and determines prices. In some cases, a competitive equilibrium with additive (unit) prices fails to exist. Bikhchandani and Ostroy (2002) show how to adjust pricing rules, packages of items and the definition of competitive equilibrium such that existence is restored. In the single seller case, the core is non-empty. At least the seller-optimal outcome, where the assignment is efficient and the single seller receives all gains from trade, is in the core.[31] In our case with multiple sellers and unit-demand preferences, the core differs from that when only one seller is present, but the Vickrey outcomes are the same.

Milgrom (2000) describes the simultaneous ascending auction that was applied to sell radio spectrum licenses in the USA in 1994. He shows that straightforward, myopic bidding – bidding on an auction in the demand set – leads to an outcome close to competitive equilibrium.

Gul and Stacchetti (2000) consider an ascending price auction with demand bids[32] for monotone gross substitutes valuations (as defined in Gul and Stacchetti, 1999) that generalizes the auction by Demange et al. (1986). Prices for all items in the excess demand set are increased by one unit. They show that truthful submission of demands at each stage is a PBE if the Vickrey outcome equals the efficient outcome at minimum competitive prices. This is true in our unit-demand preferences case. The existence of additive and anonymous prices[33]that correspond to a Vickrey outcome cannot be assured for all gross substitutes valuations. Thus, they also find that no incentive compatible, ascending auction exists that assigns such prices to items. A

[31] This is because all coalitions that do not contain the seller have a coalitional value of zero.

[32] With *demand bids* we refer to auctions where bidders submit their demand at given prices (like in an English auction) and the auctioneer adjusts prices in contrast to price bids where bidders submit a certain number for some or each package (like in a second-price auction or a first-price auction).

[33] That is, prices $p(X) = \sum_{j \in X} p_j$ for packages of items X, where p_j for all j are identical for all bidders.

preliminary assignment to a given auction, that exists in our multiple-auctions game, occurs in their game indirectly, since a bidder has to accept the price of the previous stage if the auction game ends.

The algorithmic primal-dual approach of de Vries, Schummer, and Vohra (2007) generalizes the model of Demange et al. (1986) to general valuations by introducing the concept of minimally undersupplied sets of bidders and by allowing for non-anonymous and non-additive prices.[34] They categorize auction algorithms as subgradient algorithms and primal-dual algorithms. The former require each participant to report one element of his demand correspondence at each iteration whereas the latter require each participant to report complete demand sets at each iteration. Our model indirectly implements a subgradient algorithm because we have price bids equivalent to reporting one element of the demand set at each iteration. Demange et al. (1986), Ausubel (2004), Gul and Stacchetti (2000), and de Vries et al. (2007) are related to primal-dual algorithms while Crawford and Knoer (1981), Kelso and Crawford (1982), Parkes (1999), Ausubel and Milgrom (2002) and the uniform-price auction implement subgradient algorithms.

Subgradient algorithms are either not exact or do not converge in a finite number of iterations (de Vries et al., 2007). Our model implements a subgradient algorithm that does not converge to the exact solution. It differs from other models in the asynchronous submission of bids and the random bidding order.

Ausubel and Milgrom (2002), Mishra and Parkes (2007), and de Vries et al. (2007) investigate whether and under which conditions ascending price auctions with appropriate adjustments of prices in each round lead to Vickrey outcomes[35] and find a "no gap" condition[36] on the joint preferences of bidders, or bidder-submodularity.[37]

[34]Similarly, Ausubel (2006) allows for non-linear prices for the prices for different quantities of a single type of commodity to restore incentive compatibility.

[35]Bikhchandani and Ostroy (2006) ask the same question for homogeneous, multi-unit demand with decreasing marginal valuations.

[36]The condition demands that Vickrey payments of each bidder equal the sum of the minimum competitive prices for all possible reports of valuations (where the set of possible reports is restricted).

[37]The coalitional function c is bidder-submodular (defined on the lattice given by the partially ordered set of coalitions, which are ordered by set inclusion), if $c(S \cup \{i\}) - c(S) \geq c(T \cup \{i\}) - c(T)$ for each bidder i and for all coalitions S and T, with $S \subset T$, that contain the seller. If all items are substitutes for all bidders, then c is bidder-submodular. These are exactly the valuations for which the Vickrey outcome of every subset of the players that contains the seller is in the core (Ausubel and Milgrom, 2002; Milgrom, 2004). Note that if bidder's valuations satisfy the gross substitutes condition, these *valuations* are also submodular, but (for more than two goods) the

Bidder-submodularity of the coalitional function guarantees that the Vickrey payoffs are in the core.[38] The valuations structure that we consider in our model is a special (and extreme) case of gross substitutes. With these valuations, the bidders are substitutes and the coalitional function is bidder-submodular.

If the submitted demand bids can be explained by truthful bidding according to some underlying valuations, then truthful bidding is an equilibrium for certain classes of auctions (primal-dual auctions in de Vries et al. (2007), other specific auction algorithms for more general valuations in Mishra and Parkes (2007)). Mishra and Parkes (2007) find an ex-post Nash equilibrium but do not consider sequential rationality off the equilibrium path. Ausubel and Milgrom (2002) characterize Nash equilibria of the proxy package auction, where the participant instructs a single proxy bidder for bidding in the whole game, if Vickrey payoffs are in the core. In our game, each new bid is a new instruction for a proxy bidder and each auction uses a different proxy bidder. Only Gul and Stacchetti (2000) explicitly solve for a PBE.

Internet Auctions Several models explicitly relate to Internet auctions. Theoretical and empirical studies on Internet auctions used to treat auctions offered on a platform as unrelated, single-unit auctions (e.g., Roth and Ockenfels, 2002; Ockenfels and Roth, 2006). Only recently has the multiple-auctions aspect of Internet auctions begun to be incorporated. Bajari and Hortaçsu (2004) and Ockenfels et al. (2006) give overviews of the literature on Internet auctions.

Often, the phenomena of late and multiple bids are the focus of this research. Sniping (last minute bidding) is an extreme version of late bidding which occurs in auctions with hard close rules (i.e., with fixed ending times). Sniping is modeled as a last round of bids submitted simultaneously, such that no bidder can react to the bids.

opposite is not true (Milgrom, 2004).

[38]In general, bidders' Vickrey payoffs determine an upper bound for every single bidder's payoff in the core. The cases where the Vickrey payoffs are in the core are exactly those, where a single bidder-optimal point in the core exists. This Pareto-optimal point for bidders consists of the Vickrey payoffs (Ausubel and Milgrom, 2002; Milgrom, 2004; Ausubel and Milgrom, 2006). With gross substitutes preferences or if bidders are substitutes, competitive equilibrium prices form a lattice and Vickrey prices are the lower bound on the sum of competitive prices a bidder pays (Gul and Stacchetti, 1999; Bikhchandani and Ostroy, 2002). Bidders are substitutes if $c(\mathcal{I}) - c(\mathcal{I}\backslash T) \geq \sum_{i \in T}(c(\mathcal{I}) - c(\mathcal{I}\backslash\{i\}))$ for all subsets of players $T \subset \mathcal{I}$ that do not contain the seller (e.g., Bikhchandani and Ostroy, 2002). The bidders-are-substitutes condition is necessary and sufficient for Vickrey payoffs to be in the core (de Vries et al., 2007).

Ockenfels and Roth (2006) explain sniping in a private-values model where last minute bids may get lost. Sniping avoids a bidding war with bidders who bid below their valuation but submit new bids when they are outbid. Bajari and Hortaçsu (2003) explain sniping in a common-values model with an attempt to withhold information on the item. Sniping can also be explained in a sequential auctions model (Wang, 2006b, 2006a). Wang (2006b) explains sniping as equilibrium behavior in an affiliated-private-values model with two sequential, second-price proxy (eBay) auctions[39] of the same kind of item. All symmetric equilibria in monotone and undominated strategies contain sniping in all but the last auction. The outcome is efficient, but with strict affiliation the expected prices increase, being lower than in a joint multi-unit Vickrey auction (expected prices are constant and equal to the multi-unit auction for independent values). With a soft close rule, sniping does not occur in equilibrium. Stryszowska (2006) concentrates on the independent-private-values setting and gets the same results. See Bajari and Hortaçsu (2004) for a discussion of the different hypotheses.

Late bidding is not surprising when considering overlapping auctions with hard close rules. This game has similarities to sequential auctions. The first auction ends and then bidders consider the next auctions. Thus, the number of bids in one auction increases when another auction has ended. This is an explanation for late bidding but not necessarily for sniping.

The submission of multiple bids from one bidder in the same auction (multiple bidding) is more interesting in the context of our model. Multiple bidding in single, second-price proxy auctions is not predicted by the standard independent-private-values approach. In a single auction model, multiple and late bidding can be explained by outside search for alternative offers (outside prices) (Vadovic, 2008). Bidders with low search costs bid late and search for outside prices whereas bidders with high search costs do not search unless they are outbid and bid early or submit multiple bids. Analyzing Internet auctions as multiple simultaneously offered auctions or as overlapping auctions often results in multiple bidding in equilibrium (Peters and Severinov, 2006; Stryszowska, 2006).

[39] Second-price proxy auctions are those where the bid is submitted to a proxy bidding agent that determines the current standing bid as the second highest submitted bid (probably plus one increment) and selects the bidder with the highest bid as the current high bidder. The submission of several bids is allowed.

Stryszowska (2006) considers several models of multiple auctions offering the same item in an independent-private-values environment, mostly with the hard close rule. Auctions progress in rounds in which bids are simultaneously submitted. She explains sniping in equilibrium when several auctions are available at the same time. Another approach compares simultaneous auctions with overlapping auctions under a hard close rule and with uncertain bid transmission in the last bidding stage. In equilibrium, both sniping and late bidding may occur in simultaneous auctions whereas bidders use multiple early bidding to coordinate in overlapping auctions, avoiding inefficiencies that occur under sniping.

Huang, Chen, Chen, and Chou (2007) propose a model of overlapping auctions. Auctions' time periods where bids are accepted overlap, but the auctions do not have to start or end at the same time. A new bidder arrives after each period and the winner leaves. In equilibrium, cross-bidding, switching between auctions that are open for bids at the same time, never occurs. Bidders only switch to another auction when the first auction closes, so in equilibrium auctions are effectively handled like sequential auctions and, therefore, late bidding occurs. Last minute bidding in the currently ending auctions occurs when new auctions arrive randomly, giving an explanation other than preventing revelation of information or coordination. The authors' empirical study supports this hypothesis.

Juda and Parkes (2006, 2009) address the sequential auction problem in online auctions that occurs due to the fixed ending time. Their empirical study suggests that the problem exists. They propose a bidding agent for the auction platform which uses an options system. Bidders submit valuations as well as starting and ending times of their bidding activity. A bidder's proxy agent wins options and decides which options to exercise at the ending time. Thus, a seller has to wait for his payment until a bidder exercises his option. The advantage of their proxy system is that it makes truthful bidding a dominant strategy, it allows for entry and departure of bidders in the market, and it simplifies bidding for bidders with certain demand structures. A disadvantage is that sellers have to agree to its application and that it may reduce the fascination of the auctions for bidders because they just tell the proxy agent their preferences.

The model most closely related to ours is that of Peters and Severinov (2006).

Our model is a generalization of theirs with respect to valuations.[40] A homogeneous good is offered in multiple, second-price proxy auctions and bidders, who have unit demand, bid sequentially in a given order until no new bids are submitted. In the perfect Bayesian equilibrium, the highest valuation of a losing bidder determines the price, which is unique for the whole auctions market.

Another closely related model is that of Bansal and Garg (2005). Their auctions are multiple English auctions with a fixed bid increment and arbitrary sequential (asynchronous) bidding. A bidder i's valuation for a single item j is given by v_{ij}. Bidders' valuations for packages of items P are the sum of the k_i maximum valuations for items in the package. The parameter $k_i \leq |P|$ may differ for each bidder i. These valuations are a special case of multi-unit substitutes preferences and a generalization of the unit-demand preferences in our model. They introduce a Local Greedy Bidding strategy (LGB) and give an upper bound for inefficiency of their LGB. Their concept of a marginal loser is equivalent to our price determining bidder. For the case $k_i = 1$, which is our unit-demand case, they discuss incentives to deviate from submitting the LGB to a proxy (in a one-shot auction), taking the increment into account. However, they do not consider sequential rationality.

2.2.3 Experimental and Empirical Studies

Online auctions are the subject of many laboratory experiments but they are also very suitable for field experiments where the experimenter acts as a seller and for empirical studies. Bajari and Hortaçsu (2004) and Ockenfels et al. (2006) review the experimental and empirical literature.

Many studies consider the timing of bids. Anwar, McMillan, and Zheng (2006) observe 40% of bids in the final 10% of auction time on eBay. Bajari and Hortaçsu (2003) observe late and very late bidding in an empirical study of common-value auctions.[41] 32% of all bids are submitted in the last 3% of time, which is, for example, in the last two hours of a three day auction and 25% of winning bids arrive within the last 0.2% of time (the last 8 minutes in a three day auction). They also find that auctions with low minimum bids attract more bidders. Roth and Ockenfels (2002)

[40] They also consider strategic choice of reserve prices by sellers and we do not.

[41] They empirically classify their eBay coin auctions as common-value auctions, as a negative correlation between the number of bidders and the height of bids in an auction suggests.

and Ockenfels and Roth (2006) report more late bidding in eBay (hard close) auctions than in Amazon (soft close) auctions (e.g., 20% vs. 7 % of last bids are submitted in the last hour). 8% of bids on eBay are last minute bids, often classified as sniping. 43% of the auctions received last minute bids. Ariely, Ockenfels, and Roth (2005) confirm with a laboratory experiment that the different ending rules cause such differences in behavior in single auctions. The hard close rule was represented by 100% and 80% probability of bid transmission in the second and last bidding stage, respectively. Furthermore, the outcomes in their soft close auctions are more efficient and prices are higher. Houser and Wooders (2005) conduct a field experiment with 15 pairs of auctions with different closing rules. They also find that the soft close rule leads to higher prices.

The impact of bidders' experience on their behavior is also investigated. In the study of Ockenfels and Roth (2006), experienced bidders bid later on eBay while on Amazon they bid earlier.[42] Borle, Boatwright, and Kandane (2006) allow for a non-linear relation between bidders' experience and bid submission time. They report that the more experienced bidders bid more often at the beginning and also at the end of the auction in an empirical study of eBay auctions.

Wang (2006a) compares sequential, one-shot, second-price, sealed-bid auctions and second-price, proxy auctions in the laboratory, where the submission of bids is possible whenever a bidder wants during a period of 60 seconds. He observes early and late bidding with sequential eBay auctions and declining prices under both auction formats (starting from prices above those predicted).

Hoppe (2008a) conducts an experiment related to the theory of Stryszowska (2006) and does not observe the predicted efficiency of outcomes, which he attributes to coordination failure by bidders. He conducts another experiment on overlapping, hard-close auctions of homogeneous items where bidders have unit demand, comparing high (five of six periods) and low (one of six periods) numbers of overlapping periods (Hoppe, 2008b). Bids are submitted simultaneously in each period. He observes that 30% of bidders are cross-bidders in the simultaneous-auctions treatment while about 60% are cross-bidders in both overlapping-auctions treatments. Efficiency is higher in both overlapping-auctions treatments compared to his simultaneous-auctions treatment. Sellers' revenues are higher than predicted by theory and also higher in the

[42] A bidder's experience is measured by a "feedback number" that is calculated from information gained from the reputation systems of the respective auction platforms.

overlapping-auctions treatments.

Anwar et al. (2006) test the predictions of Peters and Severinov (2006) –in particular, the existence of cross-bidding and bidding in the auction with the lowest standing bid when items are homogeneous.[43] They find that almost 20% of bidders are cross-bidders, excluding multi-unit bidders (who may have multi-unit demand). About 70% of bids are submitted in the auction with the lowest standing bid.[44] In groups of auctions where winning bidders are cross-bidders and non-cross-bidders, the winning cross-bidders paid on average 9% less.[45]

In the empirical study of Hayne, Smith, and Vijayasarathy (2003) covering more than 11,000 eBay auctions, 20% of bids are incremental bids and 61% of bidders submit multiple bids (in the same auction). In their field experiment, Haruvy and Popkowski Leszczyc (2008) find that 15.5% of bidders are cross-bidders in pairs of simultaneously offered auctions on eBay that end at the same time. About half of the bidders bid only once.

All the experiments mentioned consider unit demand for homogeneous items. Other experiments on single-seller, multi-unit auctions concentrate on demand reduction (Engelmann and Grimm, 2009), multiple round auction mechanisms for selling packages (Kagel, Lien, and Milgrom, 2009), or the appropriateness of simultaneous or sequential auctions or auctions that allow for package bids (Ledyard, Porter, and Rangel, 1997).

2.2.4 Buyer-Seller Networks

The economic literature on social and economic networks has vastly grown in recent years. Jackson (2007, 2008) gives an overview of theoretical results; Kosfeld (2004) surveys experimental results.

We consider a market where trading opportunities between sellers and bidders are

[43] They select a subsample of their data and consider groups of data points where a Central Processing Unit (CPU) is offered by the same seller, with the same description, the same starting price and delivery conditions, and that ends within a certain time slot (one day, one hour, or one minute).

[44] In particular, they find 20%, 20%, 14% cross-bidders and 76%, 72%, and 62% of submitted bids in the auction with the lowest standing bid in their minute, daily, and hourly samples, respectively. The percentage of bids is the average percentage of such bids over auctions.

[45] Tung, Gopal, and Whinston (2003) get contradicting results, but it is not clear on basis of which data or under which auction format.

restricted by an incomplete network of contacts between agents on the two sides of the market. In this environment, Kranton and Minehart (2000, 2001) consider market-wide, simultaneously ascending auctions of homogeneous items and buyers' and sellers' incentives to invest in connections. Corominas-Bosch (2004) investigates a model of multilateral bargaining in buyer-seller networks. Trading pairs leave the market and the remaining agents bargain for another round. She shows how the market can be decomposed in small components for which the equilibrium solution can be derived. The model predictions are experimentally tested by Charness, Corominas-Bosch, and Fréchette (2007). Both the auctions and the bargaining model use a prominent theorem called Hall's theorem or the marriage theorem (Hall, 1935).[46]

[46]Frobenius (1917) proved a related result on matchings in bipartite graphs.

Chapter 3

The Multiple-Auctions Game

The multiple-auctions game is a non-cooperative game in extensive form. Multiple, indivisible, heterogeneous items are offered by independent sellers. Bidders have unit-demand preferences, i.e., they may have positive valuations for many items, but want to own only one of them. Valuations for items are independent and private. Sellers set starting prices non-strategically and equal to their valuation. Total supply and demand in the auctions market does not change, so the entry of bidders or sellers is not considered.

This chapter starts with a description of the model (Section 3.1). In Section 3.2, a symmetric strategy for bidders is proposed (Section 3.2.1), all outcomes resulting from following this strategy are described, and one of them is chosen as a reference outcome (Section 3.2.2). The remainder of the analysis refers to this reference outcome, keeping the range of possible deviations from the reference outcome in mind. The solution concept applied is the perfect Bayesian epsilon-equilibrium (Section 3.2.2.5). The equilibrium strategy described is a best reply against many reasonable strategies, but not against all strategies – it is not a dominant strategy. Section 3.3 provides a discussion of the model's assumptions and the results. Furthermore, in Chapter 6, an experiment that tests the model and the analysis and, thus, gives a hint of the descriptive power of the analysis, is presented.

Our model adds to the existing literature in that it considers a combination of the following characteristics: independent auctions (no coordination of demand by the seller), price bids (in contrast to demand bids to a given price), asynchronous bidding with a random bidding order, second-price proxy auction format, unit-demand preferences, heterogeneous items, and the analysis of the bidders' incentives. All these single aspects have been addressed in the literature, however, not in this combination. The difference from the model of Peters and Severinov (2006) is mainly that we take heterogeneous items into account.

3.1 Introducing the Model

In this section, a formal description of the multiple-auctions game is given. Furthermore, an example of naive bidding is presented to illustrate some problems that a bidder faces.

3.1.1 Model Description

The multiple-auctions game Γ^a is a game in extensive form with imperfect and incomplete information. Applying the Harsanyi transformation results in a game with complete but imperfect information that can be analyzed with the familiar equilibrium concepts. We describe the model after the transformation. In the following, the model is explained in the context of ascending auctions.

Before describing the details formally, we give an overview of the course of the game. Nature moves first and determines all non-public parameters (bidding order, bidders' valuations, and sellers' valuations). Then, every seller non-strategically sets a starting price in his auction. Every auction is conducted as a second-price proxy auction. In each round, one bidder may act and can decide not to bid or to submit one bid in an auction of his choice. All bidders are always informed about the so-called current standing bid and the current high bidder in each auction. The current high bid is not made public. The game ends if all bidders have quit or hold a high bidder position, and no new bids are submitted. In each auction, the current high bidder becomes the winner and the current standing bid is the price he has to pay.

In what follows, we describe the model in detail and comment on the assumptions. First, the set of players and preferences (utility functions) are specified. Then, the bidding process is introduced and bidders' strategy sets are characterized. Finally, the role of the nature player is explained and the information structure is described.

Set of Players The finite set of players $\mathcal{I} = 0 \cup N \cup M$ consists of "nature," whom we denote player 0, the set of n bidders $N := \{B1, \ldots, Bn\}$, and the set of m sellers

$M := \{A1, \ldots, Am\}$. In what follows, we use the index i to refer to bidders and the index j to refer to sellers. Every auction is associated with exactly one seller. Thus, we will, in a slight abuse of notation, often refer to the auction of seller $j \in M$ simply as auction j.[1]

Preferences Bidders' and sellers' valuations are private knowledge and assumed to be independent. Every seller j offers one item for sale. His valuation for his item is given by v_j^S. The vector of sellers' valuations is $v^S \in \{0, \ldots, \bar{v}^S\}^m \subset \mathbb{N}^m$ with $\bar{v}^S \in \mathbb{N} < \infty$ denoting some sufficiently large number.

Every buyer wants to buy exactly one of the offered items. His marginal valuation for a second item is always equal to zero. From his point of view, the items may differ. A matrix of buyers' valuations gives each buyer's valuation for each item in the case that he buys only this item. The matrix of buyers' valuations is given by $V := (v_{ij})_{n \times m}$ with $v_{ij} \in \{0, \ldots, \bar{v}\} \subset \mathbb{N}$ for all $i \in N$, $j \in M$, and $\bar{v} \in \mathbb{N} < \infty$ denoting some sufficiently large number.

The assumption that the valuations of the bidders and sellers are integers may be understood as a normalization of the more general formulation that valuations lie on a grid with some fixed grid spacing (cp. Peters and Severinov, 2006). Thus, the grid spacing acts as a numeraire. This assumption is tantamount to the idea that bidders and sellers determine their valuations in a monetary unit, say in euros.[2]

An assignment matches bidders and sellers. The rules of the multiple-auctions game allow bidders to win at most one of the m auctions and sellers to offer only one item. Thus, we merely consider feasible assignments $x \in X \subset \{0, 1\}^{n \times m}$ as defined in Section 2.1, Definition 2.20. In analyzing the multiple-auctions game, we are not only interested in the resulting assignment but also in the final prices. Therefore, we define an outcome of the game as follows:

Definition 3.1 (Outcome) *An outcome (x, p) of the multiple-auctions game Γ^a consists of a feasible assignment $x \in X$ and a vector of prices $p = (p_1, \ldots, p_m) \in \mathbb{R}^m_{0,+}$.*

The collection of utility functions $u = \{u_1^S(\cdot), \ldots, u_m^S(\cdot), u_1(\cdot), \ldots, u_n(\cdot)\}$ consists of

[1]To simplify notation, we will often use the index i as $i = 1, \ldots, n$ instead of $i = B1, \ldots, Bn$.

[2]The assumption that these valuations are integers is quite common in the literature. See, for example, Shapley and Shubik (1971), Crawford and Knoer (1981), or Demange et al. (1986) for a discussion.

utility functions $u_j^S(\cdot)$, $u_i(\cdot)$: $X \times \mathbb{R}^m_{0,+} \rightarrow \mathbb{R}$ for sellers $j \in M$ and bidders $i \in N$.[3] The functions are defined over outcomes (x, p). Sellers' utility functions are specified as

$$u_j^S(x,p) = \begin{cases} p_j - v_j^S & \text{if } x_{ij} = 1 \text{ for some } i \\ 0 & \text{otherwise.} \end{cases}$$

Bidders' utility functions are given by

$$u_i(x,p) = \max_{j \in M}\{v_{ij} \cdot x_{ij}\} - \sum_{j=1}^{m} p_j \cdot x_{ij}.^4$$

Thus, we have an independent-private-values model with quasi-linear utility, no externalities,[5] unlimited bidder budgets, and free disposal.[6,7] The preferences that are reflected in our bidders' utility functions of bidders are sometimes called *unit-demand preferences* (see Gul and Stacchetti, 2000). Unit-demand preferences are an extreme kind of substitutes valuation. They are assumed in the popular assignment game and in auction models related to that game (e.g., Demange, 1982; Leonard, 1983; Demange and Gale, 1985; Demange et al., 1986).[8]

Bidding Process *Starting prices* are denoted by $b^0 = \{b_1^0, \ldots, b_m^0\}$ with $b^0 \in \mathbb{N}^m$. At the beginning, the *current standing bids* b_j^s are set equal to b_j^0 for all $j \in M$. As long as no bid or only one bid is submitted in auction j, $b_j^s = b_j^0$. If auction j has received at least two bids, the standing bid in j equals the second highest of these bids. The current standing bids are publicly announced. A new bid in an auction j has to exceed the current standing bid b_j^s at least by the *minimum bid increment*

[3]Since we only consider bidders' incentives and, thus, bidders' payoffs occur very often in the analysis, we indicate sellers' payoffs with superscript S, but do not use an analog superscript for bidders to simplify notation.

[4]With our restrictions on feasible assignments $x \in X$, $u_i(x,p) = \max_{j \in M}\{v_{ij} \cdot x_{ij}\} - \sum_{j=1}^m p_j \cdot x_{ij} = \sum_{j=1}^m (v_{ij} - p_j) \cdot x_{ij}$. We use the more general definition of unit-demand preferences $u_i(x,p) = \max_{j \in M}\{v_{ij} \cdot x_{ij}\} - \sum_{j=1}^m p_j \cdot x_{ij}$ because we use this utility function in Appendix A.2 where we consider assignments with $\sum_j x_{ij} \leq m$ for all i feasible.

[5]A bidder's utility payoff depends only on the price he pays and his valuation but not on the assignment of the other items to the other bidders.

[6]Also called monotonicity: a bidder's valuation for a package of items S is weakly higher than for T if $T \subset S$.

[7]These assumptions are common in related models (e.g., Ausubel and Milgrom, 2002; Ausubel, 2006).

[8]For a discussion of these utility functions, see, for example, Shapley and Shubik (1971).

ι. We assume that ι is smaller than one, the grid spacing of valuations. In addition we assume that ι equals $1/k$ for some $k \in \{2, 3, 4, \dots\}$ and that it thereby defines a refinement of the grid of valuations. All bids are restricted to this finer grid.[9] Besides the public standing bid, there is a hidden *current high bid* b_j^h in each auction where at least one bidder has submitted a bid. It equals the highest hitherto submitted bid in auction j. Thus, the high bids are proxy bids that are submitted to an automatic bidding agent who bids in the following on behalf of the bidder.[10] The vectors of current standing bids and current high bids are denoted by b^s and b^h, respectively.

The function B^h: $M \to N \cup M$ assigns every auction $j \in M$ to its *current high bidder* $i \in N$ or, by convention, assigns the seller to his own auction if no bidder has yet bid in this auction. In each auction, the bidder who submitted the current high bid is the current high bidder. At every decision node there is exactly one high bidder in every auction (a bidder $i \in N$ or the seller himself in his auction). If two bidders have submitted identical bids, the bidder who bid first is the current high bidder. In a slight abuse of notation, we sometimes use the notation B^h to refer to the vector of high bidders in auctions $j = 1, \dots, m$. Furthermore, in examples, B^h may refer to current high bidders or final winning bidders: whenever it is combined with a price vector, B^h refers to winning bidders.

At each stage or bidding round, one bidder i is selected to make a decision. Whenever it is bidder i's turn to bid and he holds no high-bidder position, he can decide not to bid or to submit a (proxy) bid in one auction of his choice. Bidder i is free to bid any amount b_{ij}, with $b_j^s + \iota \le b_{ij} \le \bar{b}$ on the grid determined by the increment size in auction j. That is, his bid has to be at least one increment ι higher than the current standing bid and it is bounded above by some sufficiently large number $\bar{b} < \infty$.[11] We

[9] The final prices are implicitly also restricted to this finer grid.

[10] In the current model, the usual system of Internet auctions is assumed. A bidder submits a bid for an auction to the bidding agent. The bidding agents outbid each other by minimum increments until only one bidding agent is left in the auction. Then the process stops until a bidder submits a new proxy bid. In the terminology of the model, the standing bid represents the currently highest bid submitted by a bidding agent on behalf of a bidder, and the current high bid is the highest currently submitted bid to a proxy bidding agent. In Internet auctions, the current standing bid is often determined as the second-highest submitted bid plus one increment (unlike our model), if the high bid is at least as high as this bid.

[11] Clearly in an auction the height of every bid is finite. The maximum bid $\bar{b}$ in every auction can be arbitrarily high but finite, so that the set of actions and the set of nodes are finite. For the same reason we restrict bidding to the grid given by the increment. We believe this assumption is justifiable because bids are usually measured in monetary units and, therefore, a smallest unit of measurement exists. The same argument holds for the starting prices and the valuations.

do not allow a bidder who is a current high bidder in an auction to bid again until he is outbid in this auction. That is, we assume that he does not bid because winning more than one auction is not in his interest. Due to this assumption, a bidder may be a high bidder in only one auction at any given bidding stage.[12]

If a new bid b_{ij} arrives, i becomes the new high bidder in j if $b_{ij} > b_j^h$. In this case, the current high bidder is adjusted to $B^h(j) = i$ and the current high bid is increased to the amount of his bid b_{ij}. If $b_{ij} \leq b_j^h$, the bid b_{ij} determines the new standing bid b_j^s. Whenever a new bid is submitted, either the standing bid or the high bidder changes as an observable component for all bidders. Bid withdrawal is not allowed.[13]

If a bidder decides not to bid and is not the current high bidder in any auction, then he will not be selected again to bid. His position in the bidding sequence is skipped. Thus, the decision not to bid is de facto an exit decision. We call the bidders that have not quit the game *active bidders* and denote the set of active bidders at a given bidding stage by B^a. If every bidder is either a current high bidder or has quit the game, all auctions end. If the standing bids in all auctions equal $\bar{b}$, the only available action for a bidder is to choose not to bid. Since bids may only increase, the finiteness of the game is thus assured. The standing bids b_j^s at the end of the auctions are also referred to as the prices p_j.

In what follows, we sometimes refer to high bids or standing bids at different bidding stages. To reduce the complexity of the notation, we use words like “current,” “previous,” “following,” “new,” etc., to clarify which high bid or standing bid we are referring to. However, where necessary, we employ an index for stages t.

Strategies In this analysis, we consider only bidders’ incentives. We assume that every seller sets his starting price b_j^0 equal to his valuation, i.e., $b_j^0 = v_j^S$ for all $j \in M$.[14]

A (behavior) bidding strategy σ_i for bidder i is a mapping from information sets

[12] The impact of relaxing this assumption is discussed in Appendix A.2.

[13] Compare, for example, the eBay rules: “A bid is a binding contract. All bids are active until the auction ends. If you win a listing, you’re obligated to complete the transaction. Except under special circumstances, bid retraction is not permitted. Furthermore, misuse of the bid retraction option to manipulate the bidding process is not permitted. This includes any manipulation of the bidding process to discover the maximum bid of the current high bidder or to uncover the reserve price.” (http://pages.ebay.com/help/policies/invalid-bid-retraction.html, May 9, 2009).

[14] The starting price is also called a limit price or an initial-bid price (Hall, 2002). Similar to a revealed starting price is the concept of a hidden reserve price, a lower limit on the price set by the seller but not announced to the bidders. In the literature, the term reserve price sometimes also refers to a revealed starting price.

to bids. It prescribes for each information set of bidder i what to bid and in which auction. A bid b_{ij} of bidder i in auction j has to be at least one increment above the current standing bid b_j^s and at most $\bar{b}$. Furthermore, the grid determined by the size of the increment restricts the set of possible bids. We choose this formulation to prevent cases where a bidder does not submit a new bid because of the size of the increment (when the standing bid is below his valuation but the standing bid plus an increment is above his valuation). In general, discrete bid increments may distort bidders' incentives (Chwe, 1989; David et al., 2007; Rogers, David, Jennings, and Schiff, 2007). In our formulation, a rational bidder is always interested in submitting a bid when the standing bid is below his valuation. A decision not to bid, if one does not hold a current high bidder position, is de facto an exit decision.

Formally, a strategy σ_i is described as follows: to each information set H_i it assigns a vector of bids $(b_{i1}, \ldots, b_{im}) \in \{0, \iota, \ldots, \bar{b}\}^m$. Each of these vectors contains at most one positive element $b_{ij} > 0$ with the following interpretation: if $b_{ij} > 0$ exists, i submits a bid of size b_{ij} in auction j; if $b_{ij} = 0$ for all j, i does not submit a bid. Thus, a vector of bids consisting solely of zeros corresponds to the decision not to bid. A bid $b_{ij} > 0$ has to respect the auction rules, i.e., $b_{ij} \geq b_j^s + \iota$. We consider only pure (behavior) strategies σ_i (see Definition 2.7). The strategy space Σ_i of bidder $i \in N$ consists of all strategies σ_i that fulfill these requirements.

Nature At the first decision node, nature selects bidder sequence o and valuations v^S and V out of a commonly known, joint, basic probability distribution $F(\cdot)$ that respects the assumption of independence of valuations and that assigns positive probability to each feasible combination. A bidder sequence or bidding order o describes the succession of bidders that are selected to bid in all possible rounds from round one to the maximum number of rounds,[15] i.e., the nature player determines the order of bidders in the whole game.[16] The bidders do not know the sequence. We say

[15] In case $n > m$, the maximum number of rounds is $2 \times \sum_{j=1}^{m}((\bar{b} - b_j^0)/\iota) + n - m$. Assuming that ι equals $1/k$ for some $k \in \{2, 3, \ldots\}$, $2\,(\bar{b} - b_j^0)/\iota$ is the maximum number of bids that may be submitted in an auction j. This is the case if bidding starts at b_j^0 and all bidders increase the standing bid by an increment. The auction ends no later than when two bidders have submitted the bid $\bar{b}$ in j. The $n - m$ additional rounds are due to the ending rule (the exit of the losing bidders).

[16] An alternative formulation closer to Definition 2.1 is that nature chooses the bidder who has the opportunity to bid after each round randomly. Our formulation is closer to the original Harsanyi model (Harsanyi, 1967), where nature plays only once, in the first stage. With our assumptions on

that a bidder is selected to bid or that it is a bidder's turn to bid when the sequence prescribes that he is the next bidder to act in the game (i.e., one of his information sets is reached).

This bidder sequence is meant to model bidders checking interesting auctions at randomly selected points in time. Some bidders do this more often than others. Thus, there is no fixed sequence of the n bidders that is repeated until the auctions end, but instead one of all possible sequences over the whole game occurs. The formulation is identical to nature randomly selecting one of the bidders after each round. The bidder sequence may select the same bidder i in subsequent bidding rounds.

Information (Nodes and Information Sets) We assume *perfect recall* (see Definition 2.6), which assures that every player remembers his own past bids in all auctions as well as all previous standing bids and high bidders.

A bidder i can, at each bidding stage, observe the vector of standing bids b^s (including the vector of starting prices b^0 in stage one), the identity of the bidders who submitted b^s_j for all j, the identities of the current high bidders B^h, his own current high bids b^h_j for all j with $B^h(j) = i$, and his own valuations v_{ij} for all j. However, the magnitude of high bids b^h_j for all j with $B^h(j) \neq i$ is unknown to i. He only knows that $b^h_j \geq b^s_j$. He usually observes the previous bidding order (and the stage in the game tree), but if an opponent does not bid when it is his turn, bidder i does not learn his identity because none of the observable variables change when the selected bidder does not place a bid. Thus, i also does not observe his opponents' exit decisions (by not bidding). So he cannot judge which or how many bidders are still active, and, thus, he does not know B^a or $|B^a|$. The information sets contain all possible constellations (decision nodes) that are consistent with this description of the bidder's private knowledge, perfect recall, and the common observation of b^s and B^h at each decision node.

For the sake of completeness, we state this formally. Bidder i's information at set $H^t_i \in \mathcal{H}_i$ is described by the history $h_i(t)$ as observed by i for bidding stages

information, both formulations are equivalent. With the alternative formulation, the restriction on a bidder's strategy space that he may not bid if he is high bidder, could be alternatively formulated by deleting the current high bidders from the set from which nature chooses the next bidder. An interpretation is that a high bidder has no incentive to visit the auction platform until he receives an e-mail that tells him that he has been outbid.

$t = 0, 1, \ldots, T$, with the final bidding stage denoted by T.[17] The history $h_i(t) := (h_i^0, h_i^1, \ldots, h_i^{t-1}; o_i^t; B_i^{a,t})$ is given by the information known in previous stages. Thus, $h_i^0 := (v_i, b^0)$ (with the assumption that $b^0 = v^S$) and $h_i^t = (b^{s,t}, b_i^{h,t}, B^{h,t}, B^{PD,t})$ for all $t = 1 \ldots, T$, where $b^{s,t}$, $b_i^{h,t}$, $B^{h,t}$, and $B^{PD,t}$ are the current standing bids, i's current high bids, the current high bidders, and the bidders who submitted the bids that determine the current standing bids at stage t, respectively. o_i^t is the information that i collected about the order of bidders during the previous bidding stages (and the current stage, if it is i's turn to bid), and $B_i^{a,t}$ (in a slight abuse of notation) is i's activity status at t, i.e., $B_i^{a,t} = 1$ if i is still allowed to bid and $B_i^{a,t} = 0$ otherwise. Note that $h_i(t)$ implicitly contains information about former high bids and the identities of bidders that have been outbid before stage t.[18]

Thus, the information set H_i^t contains all nodes at stage t with identical histories $h_i(t)$ from i's point of view. Note that all nodes in H_i^t share the same objectively observable information about current standing bids $b^{s,t}$ and current high bidders $B^{h,t}$, but not all nodes with the same standing bids and high bidders are in the same information set. The nodes in an information set differ with respect to bidders' valuations V_{-i}, bidding orders o, high bids b^h, and sets of active bidders B^a that i cannot distinguish.

The different elements of the model are summarized in the List of Symbols (see page 202). In the following, the indices g, h, and i refer solely to bidders and j, k, and l to sellers or auctions.

3.1.2 An Example of Naive Bidding

In a single second-price proxy auction with our restrictions on increments, bidding one's valuation is a weakly dominant strategy in an independent-private-values environment. eBay, for example, advises bidders to submit their true valuation to a bidding agent.[19] We present an example of naive bidding in the sense that the bidder

[17] T is not determined in advance. However, a final bidding stage exists because the game is finite.

[18] The combination of all bidders' information, $h(t) = (h^0, h^1, \ldots, h^{t-1}; o(t); B^{a,t})$, contains all available information, i.e., $h^0 = (V, b^0)$, $h_i^t = (b^{s,t}, b^{h,t}, B^{h,t})$, the complete previous order of bidders $o(t)$, and the current set of active bidders B^a.

[19] "Although we suggest that you bid the maximum amount that you're willing to pay for an item, you won't necessarily pay that amount." (http://pages.ebay.com/help/buy/bidding-overview.html, April 28, 2009)

ignores the presence of several auctions. Instead, he simply submits a bid equal to his valuation in the auction where his preferred item is sold and where his valuation is above the standing bid.

Figure 3.1 illustrates the result of naive bidding in an example with three sellers (auctions) and three bidders. All sellers' valuations and, thus, all starting prices equal zero: $v_S = b^0 = (0, 0, 0)$. The table contains the three bidders' valuations for the three items offered in auctions $A1$–$A3$ (i.e., the valuation matrix V), the resulting winners B^h, prices p, and bidders' payoffs u_i. For example, Bidder $B1$ has valuations 11, 6, and 3 for the items sold in auctions $A1$, $A2$, and $A3$, respectively. $A1$ is won by $B1$ at price $p_1 = 10$, resulting in the payoff $u_1 = 11 - 10 = 1$.

The four drawings below the table illustrate the bidding process. Bidder $B1$ bids first. He bids in auction $A1$, where his value of 11 is his highest of all the items. His bid is $b_{11} = v_{11} = \max_j\{v_{1j}\} = 11$. Then, it is $B2$'s turn. He bids $b_{21} = 10$ because item $A1$ has the highest value for him. Thus, the standing bid increases from 0 to 10, but $B1$ remains the current high bidder in $A1$. After that, $B3$ bids $b_{31} = 10$ and becomes current high bidder in $A3$ at standing bid $b_3^s = 0$. Then, $B2$ bids $b_{23} = 7$, his second highest valuation, because the standing bid $b_1^s = 10$ in $A1$ has reached $B2$'s highest valuation. Bidding above 10 in $A1$ is not rational for $B2$. His bid in $A2$ increases the standing bid b_3^s to 7. Since $B1$ and $B3$ are current high bidders, they do not bid, and the next bid is again submitted by $B2$. He bids $b_{22} = 2$, becomes the high bidder, and the game ends because no bidder is allowed to submit further bids. Thus, in the resulting outcome, $B1$ wins $A1$ at $p_1 = 10$ and has a payoff $u_1 = 1$, $B2$ wins $A2$ at $p_2 = 0$ and has a payoff $u_2 = 2$, and $B3$ wins $A3$ at $p_3 = 7$ and has a payoff $u_3 = 3$.

Note that the bidders are not content with this outcome. At prices $p = (10, 0, 7)$, $B1$ prefers to buy item $A2$. Thus, $B1$ and seller $A2$ could profitably deviate. If $B1$ buys $A2$ (instead of $A1$) at a price of 3, then both of them are better off. The same argument holds for $B3$ and $A2$. Furthermore, the final assignment in the example is not efficient. For example, if $B2$ and $B3$ exchange their items, the sum of payoffs increases by 2.[20]

[20] Note that naive bidding is also not an equilibrium if items are homogeneous. Consider an example with three bidders, $B1$, $B2$, and $B3$, with valuations $v_1 = 20$, $v_2 = 15$, and $v_3 = 10$ who participate in two auctions, $A1$ and $A2$ with starting prices of zero. $B1$ bids 20 in $A1$ and $B3$ bids 10 in $A2$. $B2$ only knows that in each auction a bidder has submitted a bid above the standing bids, which are still zero in both auctions. He bids 15 in $A1$ and, because he does not

	$A1$	$A2$	$A3$	u_i
$B1$	11	6	3	1
$B2$	10	2	7	2
$B3$	4	7	10	3
B^h	$B1$	$B2$	$B3$	
p	10	0	7	

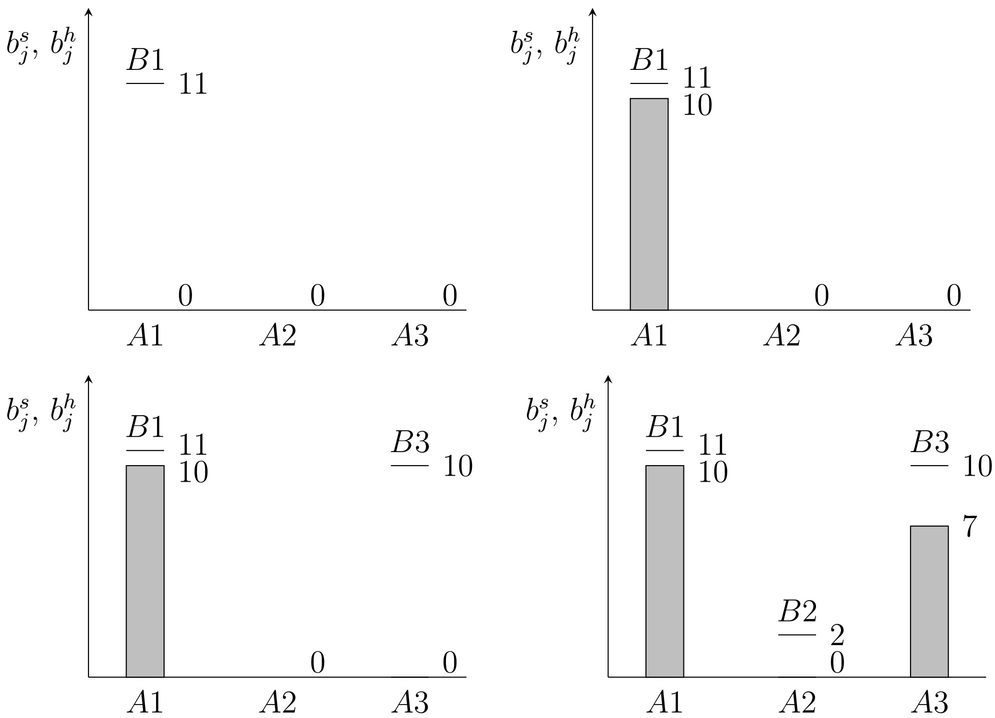


Figure 3.1: An example of naive bidding

In the following section, we characterize outcomes that do not have these undesirable properties and present a bidding strategy that is better suited for this environment.[21]

3.2 Analyzing the Multiple-Auctions Game

We introduce a specific behavior strategy (Section 3.2.1), describe and characterize the outcome (Section 3.2.2), and prove that our strategy configuration constitutes a perfect Bayesian epsilon-equilibrium (Section 3.2.2.5).

3.2.1 Strategy σ_i^*

In this section we define a particular behavior strategy that we denote by σ_i^* for $i \in N$. First, we introduce some elements used in the definition of the strategy.

From a bidder i's point of view, his current maximum potential payoff in an auction j is determined by the difference between his valuation for item j and the current standing bid b_j^s, which represents the lowest price at which he may win the auction. Thus, his current maximum potential payoff is defined as follows.

Definition 3.2 *The* current maximum potential payoff Δ_{ij} for bidder i in auction j *is*

$$\Delta_{ij} := v_{ij} - b_j^s \,.$$

A bidder has to outbid the current standing bid by at least ι to become the high bidder. If $b_j^h = b_j^s$ it may be possible to win j at price $p_j = b_j^s$.

To define our strategy σ_i^*, we need the current maximum potential payoff $\Delta_{i(1)}$ and the current second-highest potential payoff $\Delta_{i(2)}$ for bidder i at a given stage of the game as perceived by bidder i.

become high bidder, he bids 15 in $A2$. Then, the auctions end, $B1$ and $B2$ win $A1$ and $A2$, respectively, at prices $p_1 = 15$ and $p_2 = 10$. The minimum competitive price in this example is 10, which results in the PBE of Peters and Severinov (2006) with incremental bidding in the auction with the lowest standing bid. The outcome of naive bidding is efficient but it is not a competitive equilibrium since $B1$ prefers to buy item $A2$ if $p = (15, 10)$.

[21] Although suggesting entering the maximum willingness to pay to the proxy bidder, eBay also realizes the problem that arises in the example: "Don't bid on identical items in different listings if you just want one item. If you win both, you'll be obligated to buy both. If you're outbid on an item, wait until the auction has ended before placing a bid on an equivalent item. If the bidder who won the auction retracts the bid, your bid could become the winning bid." (http://pages.ebay.com/help/buy/bidding-overview.html#tips, April 28, 2009)

Definition 3.3 *The* current maximum potential payoff for bidder i *is given by*

$$\Delta_{i(1)} := \max_{j \in M} \Delta_{ij} .$$

Note that the auction that provides bidder i's current maximum potential payoff $\Delta_{i(1)}$ may be not unique.

Definition 3.4 *The* current second-highest potential payoff for bidder i *is*

$$\Delta_{i(2)} := \max_{j \in M} \{\Delta_{ij} : \Delta_{ij} \neq \Delta_{i(1)}\}.$$

The current maximum potential payoff $\Delta_{i(1)}$ for bidder i is the largest difference Δ_{ij} over all auctions j and $\Delta_{i(2)}$ is the second largest such difference with $\Delta_{i(2)} \neq \Delta_{i(1)}$. The maximum potential payoff $\Delta_{i(1)}$ leads to the definition of i's current demand set.

Definition 3.5 *Bidder i's* current demand set *is given by*

$$D_i := \{j \in M : \Delta_{ij} = \Delta_{i(1)} \quad \text{and} \quad \Delta_{ij} \geq 0\}.$$ [22]

If auctions with $B^h(j) = j$ (where no bid has been submitted yet) are contained in D_i, we denote the subset that consists of these auctions by D_i^0:

$$D_i^0 := \{j : j \in D_i \text{ and } B^h(j) = j\} .$$

Let us now define strategy σ_i^*.

Definition 3.6 (Strategy σ_i^*) *The* strategy σ_i^**: $\mathcal{H}_i \to A(H_i)_{H_i \in \mathcal{H}_i}$ for bidder $i \in N$ specifies that bidder i chooses the following action whenever he is selected to bid (i.e., whenever one of his information sets H_i is reached):*

(1) If $\Delta_{i(1)} \leq 0$, then he does not bid ($b_{ij} = 0$ for all j).

(2) If $\Delta_{i(1)} > 0$ and $|D_i| = 1$, then i makes the following bid in auction $j \in D_i$:

$$b_{ij} = v_{ij} - \max\{\Delta_{i(2)}, 0\} .$$

[22] Auctions j with $\Delta_{ij} = 0$ are included in i's demand set to guarantee that a current high bidder i, who has bid $b_{ij} = b_j^h$ and wins j at $p_j = b_j^s$, wins an auction in his demand set. On the other hand, for a bidder h who is not a current high bidder in j and $\Delta_{hj} = 0$, i.e., he is indifferent between winning j and not winning j, a definition with $\Delta_{ij} > 0$ would fit better with the intuition in the multiple-auctions game.

(3) *If $\Delta_{i(1)} > 0$ and $|D_i| > 1$, then bidder i bids in an auction $j \in D_i^0$ if $D_i^0 \neq \emptyset$ or in an auction $j \in D_i$, otherwise. In both cases, he chooses randomly (with uniform probability) one of the auctions in D_i^0 or D_i, respectively. In the selected auction j, he bids*

$$b_{ij} = b_j^s + \iota\,.$$

The strategy combination σ^* is given by the vector $(\sigma_i^*)_{i \in N}$. Remember that a bidder i who is a current high bidder cannot bid because we restrict his strategy space.

Part (1) of the definition simply states that bidder i does not submit a bid if all standing bids are so high that positive profit from winning an auction is impossible (i's maximum potential payoff Δ_{ij} in any auction j is negative). Thus, he does not bid and exits the game. This bidder will not be selected again to bid.

In cases (2) and (3) of Definition 3.6, positive profits from winning an auction seem to be achievable. In part (2), the current maximum potential payoff is associated with a unique auction j. Then, i determines his bid b_{ij} by the equation $v_{ij} - b_{ij} = \Delta_{i(2)}$, if $\Delta_{i(2)} > 0$. On the other hand, if $\Delta_{i(2)} < 0$, which means that j is the only profitable auction left, i determines his bid by the equation $v_{ij} - b_{ij} = 0$.[23] That is, bidder i selects his bid b_{ij} such that the minimum payoff he may achieve with this bid is equal to the maximum payoff he may achieve in any other auction at the current stage. The possible payoffs in other auctions may only become worse as bids increase while the minimum payoff he can achieve with his bid in auction j is constant as long as he is not outbid.[24] His payoff, if this bid is his last bid in the game, is the maximum payoff he can get from buying any item at the realized prices. Thus, bidding according to σ_i^*, a bidder will not regret his actions, as long as his bid does not influence the other bidders in an unfavorable way.

The same line of reasoning leads to bid $b_{ij} = b_j^s + \iota$ in part 3 of Definition 3.6. Here, we have several auctions j' with $\Delta_{ij'} = \Delta_{i(1)}$. Thus, the maximum payoff i may achieve in an other auction is equal to the maximum payoff in the selected auction j. In such cases, we will sometimes say that i is indifferent between bidding in the

[23] This situation is equivalent to bidding in a single second-price auction.

[24] Or as long as he does not revise his bid. Because current high bidders are never selected to bid, he does not have the chance to revise his bid as long as he is the high bidder. Note that a bidder who follows σ_i^* will always have the possibility to revise a bid when he is outbid and he does not need to revise his bid otherwise. The revision of bids in ascending auctions, if allowed, is usually only permitted if the bid is increased. Similarly, bid withdrawals are usually inadmissible.

auctions in D_i (or D_i^0). Solving $v_{ij} - b_{ij} = \Delta_{i(1)}$ results in $b_{ij} = v_{ij} - \Delta_{i(1)}$ and may be simplified to $b_{ij} = b_j^s$. A new bid in an auction has to be higher than the current standing bid, and, therefore, i cannot submit this bid $b_{ij} = b_j^s$. Not bidding results in quitting the game. To avoid this, i increases the current standing bid in j by one increment ι, which gives $b_{ij} = b_j^s + \iota$. Note that with this bid an exposure problem arises: i may win j at a price equal to $b_j^s + \iota$ even though he may have won a different auction k in D_i at a price of b_k^s. We refer to this problem as the increment problem and analyze its consequences in Section 3.2.2.

The preference for an auction in D_i^0 is because i's knows that with a bid $b_{ij} = b_j^s + \iota = b_j^0 + \iota$ he becomes the high bidder in j with an unchanged standing bid b_j^s.

If bidder i bids according to σ_i^* in an auction j and $b_j^h > b_{ij}$, he will not become a high bidder at this stage. But because he submitted a bid, he will have the chance to bid again at a later stage, if he so desires.

Consider part (2) with $\Delta_{i(2)} = v_{ik} - b_k^s > 0$. If bidder i bids more than described by σ_i^*, such as $b_{ij} = v_{ij} - \Delta_{i(2)} + \iota$, bidder i might win auction j at a price equal to this bid and receive a payoff of $u_{ij} = v_{ij} - v_{ij} + \Delta_{i(2)} - \iota = \Delta_{i(2)} - \iota$. If nothing has changed in the other auctions, bidder i would prefer to withdraw his bid and bid in auction k, where he still has the chance to win at a price of b_k^s and receive a payoff equal to $\Delta_{i(2)}$, which is clearly higher than the payoff $\Delta_{i(2)} - \iota$. By bidding according to σ_i^*, he avoids this exposure problem. This bidding strategy assures that he either wins the auction with the highest possible payoff or he is outbid and has the chance to bid again.

A current high bidder in an auction j could not improve his payoff by bidding in additional auctions if j is in his demand set, even if he was allowed to do so. But if j is not in his demand set, he may have an incentive to bid in another auction. We relax this assumption in Appendix A.2.

Throughout the following, we denote the outcome that results from the use of σ^* by (x^*, p^*).

An Example to Illustrate Strategy σ_i^* Figure 3.2 contains an example with three auctions and three bidders who follow strategy σ^*. This example builds upon the example on naive bidding of Figure 3.1.

Again, $B1$ submits the first bid in auction $A1$, where his value is highest. However, now the selection criterion is the largest difference between valuation and price, which,

	$A1$	$A2$	$A3$	u_i
$B1$	11	6	3	6
$B2$	10	2	7	5
$B3$	4	7	10	8
B^h	$B2$	$B1$	$B3$	
p	5	0	2	

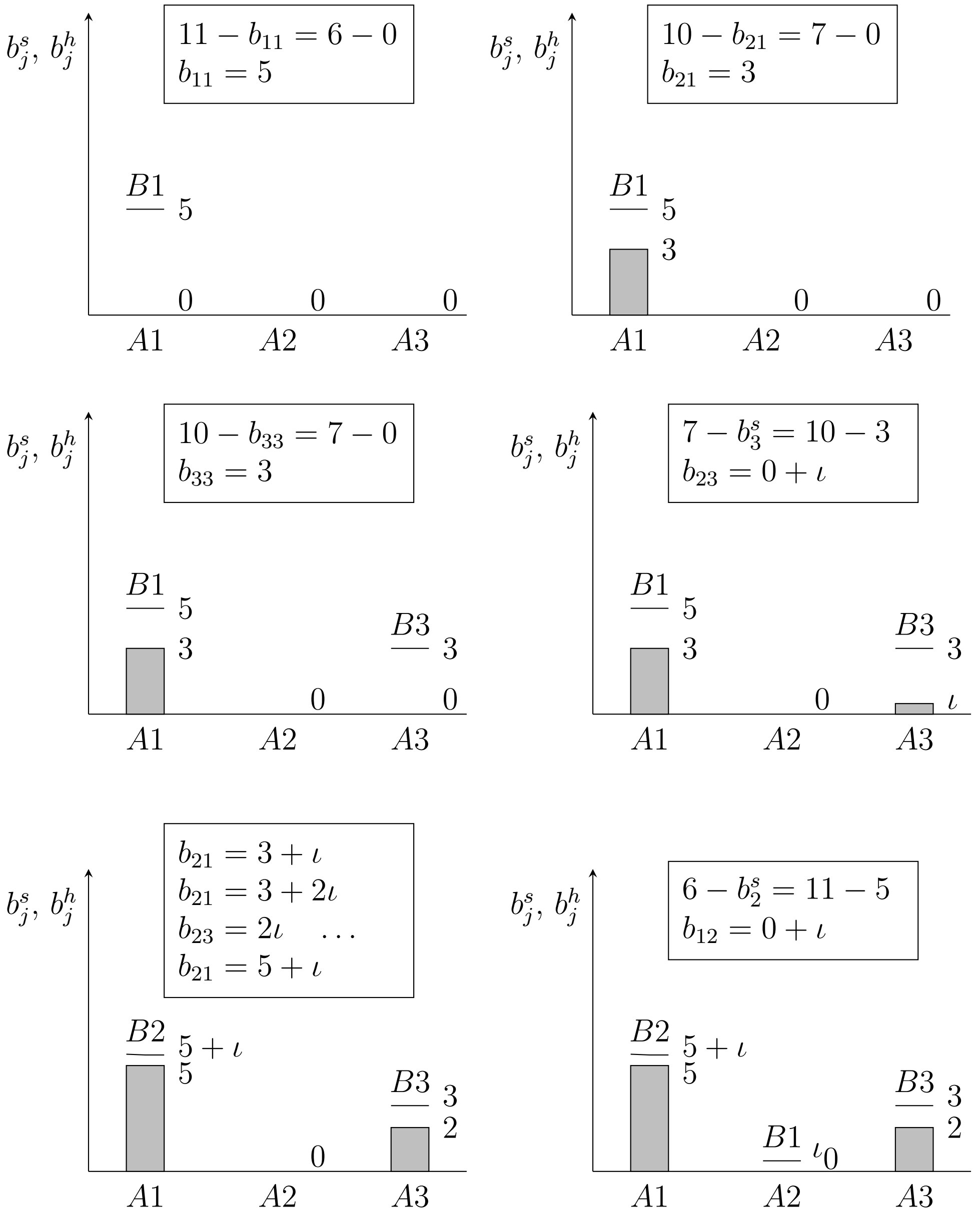


Figure 3.2: An example on bidding according to σ^*

in the beginning, is identical to the highest valuation because all starting prices equal zero. He calculates his bid according to $11 - b_{11} = 6 - 0$, where the right-hand side gives his maximum possible gain in any other auction, $A2$ in this case. Next, $B2$ bids in $A1 = \arg\max\{v_{2j} - b_j^s\}$ and submits the bid $b_{21} = 3$. He does not become high bidder, but the standing bid in $A1$ increases to $b_1^s = 3$. $B3$ submits $b_{33} = 3$ and becomes the high bidder in $A3$. Then, it is $B2$'s turn and he is indifferent between $A3$ with $v_{23} - b_3^s = 7 - 0 = 7$ and $A1$ with $v_{21} - b_1^s = 10 - 3 = 7$. That is, his demand set is not a singleton and part (3) of the definition of σ_i^* applies. In this situation, $B2$ randomly selects $A3$ and submits a bid that exceeds b_3^s by one increment: $b_{23} = b_3^s + \iota = \iota$. $B2$ does not become the high bidder and, with his next bid, increases the standing bid in $A1$ by ι: $b_{21} = b_1^s + \iota = 3 + \iota$. As long as he does not become the high bidder, he continues to bid in $A1$ and $A3$, increasing the standing bids one increment at a time. Finally, $B2$ becomes the high bidder in $A1$. It is $B1$'s turn and he, being indifferent between bidding in $A1$ and $A2$, submits the bid $b_{12} = \iota$. All bidders have a high-bidder position in some auction and are no longer allowed to bid.

The winning bidders of auctions $A1$, $A2$, and $A3$ are $B2$, $B1$, and $B3$, respectively. The prices are $p = (5, 0, 2)$, weakly below the prices that result from naive bidding ($p = (10, 0, 7)$). Each bidder weakly prefers the auction he wins to any other auction at the final prices. Thus, no bidder-seller pair wants to jointly deviate. The assignment is efficient.

Note that these prices are not the unique outcome, due to the randomizing that occurs when bidders are indifferent. For instance, $B1$ may, at his last decision in the example above, decide to bid $b_{11} = 5 + \iota$ before submitting $b_{12} = \iota$. Then, the assignment is the same, but the price in $A1$ is an increment higher.

When we analyze outcomes in the following sections, we take these randomizations into account.

3.2.2 Characteristics of the Outcome $(x^*,\ p^*)$

In this section, we describe properties of the outcome $(x^*,\ p^*)$ that results if all bidders follow strategy σ^*. First, we analyze stability of the outcome, efficiency of the assignment, and the level of prices for increments $\iota \in \{1/2, 1/3, \ldots\}$. We show that the outcome may, but does not always, have these desirable properties and point out when, why, and to what extend the properties are violated. A further restriction on the increment avoids or constrains the deviations.[25]

In our model, some minimum increment is necessary so that standing bids increase. Otherwise, an indifferent bidder (part (3) of the definition of σ_i^*) would submit a bid equal to the standing bid and the activity rule implied by the rule for quitting the game would be useless. In a single, dynamic, second-price auction j (i.e., $m = 1$), the increment may lead to an outcome where two bidders who follow the strategy to bid up to their valuation (which is reflected in part (2) of σ_i^*), may, at the final prices, have strictly positive potential payoffs. For example, if $v_{ij} = 10$, $v_{hj} = 11$, $b_j^h = b_j^s = 9.6$, $B^h(j) = h$ and $\iota = 0.5$, bidder i would not submit an additional bid because $10.1 > 10$. However, at the resulting final price $p_j = 9.6$, both bidder i (0.4) and bidder h (1.4) have a potential or realized positive payoff, respectively. Thus, for bidder h it is advantageous that the standing bid is 9.6 instead of 9.5, because this prevents i from submitting an additional bid. On the other hand, if i submits the bid 10.1, he may win auction j at 9.6 if no other bid is submitted and, in this situation, he misses a possible gain by not bidding. Therefore, whether bidding 10.1 pays off for i in this situation depends on the other bidders.

Modeling the grid of feasible bids as a sub-grid of the grid of valuations and choosing the minimum increment equal to the grid size of the bidding grid, we avoid the problems that arise from the size of the discrete increment above. A standing bid of 9.6 is then impossible and at each standing bid, the next feasible bid (one minimum increment above the current standing bid) is either submitted by i if all his potential payoffs resulting from this bid are higher than or equal to zero, or he does not bid if all potential payoffs are lower than or equal to zero.[26]

Thus, distortions due to the increment that are of the kind illustrated by the example do not occur in our model. However, with multiple auctions there is another undesirable impact of a minimum bid increment. Consider an example with $m = 2$, $b^s = (7.5, 7.5)$, $b^h = (7.5, 7.5)$, $\iota = 0.5$, and $v_i = (10, 10)$. It is bidder i's turn to bid and he is not the high bidder in any auction. Suppose he is indifferent between bidding in the two auctions. Assume that if i bids in auction $A1$, no other bidder

[25] The restriction avoids deviations form efficiency and constrains deviations of prices.

[26] For a discussion of the influence of the discrete increment on the results of a single-unit English auction see, e.g., Rothkopf and Harstad (1994) and David et al. (2007). For problems that arise with a continuum of feasible bids when players' payoffs may change discontinuously and an analysis thereof, see Reny (1999).

will bid there, whereas if he bids in $A2$, another bidder will submit the final bid of 8 there. Thus, if i bids $b_{i1} = 8$ in $A1$, he gets the payoff $u_i = 2.5$, whereas a bid $b_{i2} = 8$ in $A2$ leads to $u_i = 2$. Such a price determining bid is possible under σ^*, even if we choose valuations and feasible bids as in our model.

Peters and Severinov (2006) have an additional specification in their equilibrium strategy that allows bidders with homogeneous unit-demand preferences ($v_{ij} = v_{ig}$ for all $i \in N$ and $j, g \in M$) to coordinate and to avoid this problem. We will describe later why their method does not apply when items are heterogeneous.

In the following, we analyze the outcome (x^*, p^*) and take the impact of the increment into account. The potential influence of the increment is manifold. By restricting ourselves to increments of the form $\iota = 1/k$ for $k \in \{2, 3, \ldots\}$, we avoid the case where a bidder does not submit a bid when the standing bid is below his valuation because the minimum allowed bid would exceed his valuation. But the problem that still occurs is caused by the need to increase some bid when a bidder is indifferent between several auctions. This may lead to mis-coordination that is resolved by additional bids or that is not resolved at all.

3.2.2.1 Stability of (x^*, p^*)

A key property of outcomes in the literature on two-sided matching is their stability (see Roth and Sotomayor, 1990). If an outcome is stable, other characteristics can be deduced. Thus, we start our analysis with an examination of the stability of outcomes that result from playing according to σ^*. For this purpose we repeat and check the requirements for stability given in Definition 2.23. An outcome (x, p) evaluated via u and u^S is stable if it is individually rational and no bidder-seller-pair wants to jointly deviate:

(1) $u_i(x, p) \geq 0$ and $u_j^S(x, p) \geq 0$ for all $i \in N$, $j \in M$ and

(2) $u_i(x, p) + u_j^S(x, p) \geq d_{ij} = \max\{v_{ij} - v_j^S, 0\}$ for all pairs $(i, j) \in N \times M$.

The first conditions are fulfilled for bidders and sellers. Following σ^*, no participant bids above his valuation and the price is less than or equal to the high bid. Thus, in the worst case for a winning bidder, the price equals his valuation and he gets a payoff of zero. A losing bidder – one who does not win any auction – also has a payoff of zero. A seller j who sets his reservation price b_j^0 equal to his valuation v_j^S

as assumed, realizes a price of at least b_j^0 if the item is sold, which gives him a payoff of at least zero, or he does not sell his good and has a payoff of zero.

To check if the second condition is fulfilled, we have to look at the sum of payoffs of every realized and every alternative bidder-seller pair at the resulting outcome (x^*, p^*). We distinguish five cases of pairs (i, j): i and j trade with each other, both i and j do not trade, i does not trade but j does, i trades but j does not and, finally, both i and j trade but not with each other.

Case 1. $x_{ij} = 1$**:** If $x_{ij} = 1$ we have $u_i = v_{ij} - p_j$ and $v_j^S = p_j - v_j^S$. Thus, $u_i + u_j^S = v_{ij} - v_j^S = d_{ij} \geq 0$ because a pair with $v_{ij} \leq v_j^S$ may not arise under σ^*.

Case 2. $x_{ik} = 0$ **for all** k**,** $x_{hj} = 0$ **for all** h**:** In this case $u_i = u_j^S = 0$. Since $u_i = 0 \geq v_{ik} - p_k$ for all k and $p_j = b_j^0 = v_j^S$ for all j who do not sell their items, we find that $u_i + u_j^S = 0 \geq v_{ij} - v_j^S$ and, thus, $u_i + u_j^S \geq d_{ij}$.

Case 3. $x_{ik} = 0$ **for all** k**,** $\exists\, h \neq i$ **such that** $x_{hj} = 1$**:** In this case, $u_i + u_j^S \geq 0$ and $u_i + u_j^S = 0 + p_j - v_j^S \geq v_{ij} - v_j^S$ since $v_{ij} \leq p_j$ if $x_{ik} = 0$ for all k, and, therefore $u_i + u_j^S \geq d_{ij}$.

Case 4. $\exists\, k \neq j$ **such that** $x_{ik} = 1$**,** $x_{hj} = 0$ **for all** h**:** Clearly, $u_i + u_j^S \geq 0$ and $u_i + u_j^S = v_{ik} - p_k + 0$. To ensure stability, the inequality $v_{ik} - p_k \geq v_{ij} - p_j = v_{ij} - b_j^0 = v_{ij} - v_j^S$ has to be valid. This is not assured, as we will discuss below.

Case 5. $\exists\, k \neq j$ **such that** $x_{ik} = 1$**,** $\exists\, h \neq i$ **such that** $x_{hj} = 1$**:** In this case, both agents trade, but not with each other. We have $u_i + u_j^S = v_{ik} - p_k + p_j - v_j^S \geq 0$. For $v_{ik} - p_k + p_j - v_j^S \geq v_{ij} - v_j^S$ to be valid, the condition $v_{ik} - p_k \geq v_{ij} - p_j$ has to be fulfilled. But we can show that this condition may be violated.

The last two cases allow for $0 \leq u_i + u_j^S < v_{ij} - v_j^S = d_{ij}$ for a bidder-seller pair (i, j), causing an instable outcome. As we have seen, this may only happen when bidder i trades with some seller k but prefers to trade with some other seller j. The reason for instability in this case is that bidder i may prefer to win auction j at price p_j to winning k at p_k, or that seller j may prefer to sell to i at a price higher than p_j. In our model, only the bidders can influence the prices and the assignment. Thus, we concentrate on the possible increase in payoff for bidder i when analyzing the possible blocking pair.

First, we show that the difference in the sum of payoffs that causes instability is of at most size ι, and thus, if such a profitable deviation exists, it is of exactly size ι since all feasible prices lie on a grid with grid space ι. How can such an instable outcome occur? Remember that when i bids in k, $v_{ik} - b_k^s \geq v_{ij} - b_j^s$ for all j and at the end of the game we postulate that an instability due to $v_{ik} - p_k < v_{ij} - p_j$ occurs. We have $b_{ik} = b_j^h \geq p_k \geq b_k^s$ and $p_j \geq b_j^s$. It may happen that bidder i, when deciding on his last bid in the game, is in the situation $v_{ik} - b_k^s = v_{ij} - b_j^s$. Let us first consider the fourth case from above. If, in the described situation of indifference between auctions k and j, both $B^h(k) = k$ and $B^h(j) = j$ (i.e., $b_k^s = b_k^0$ and $b_j^s = b_j^0$), then bidder i chooses one of them arbitrarily, in our case k, and bids $b_{ik} = b_k^s + \iota$. He becomes the current high bidder, another bidder submits a bid of the same size in k, and the auctions end. Then we have $v_{ik} - p_k = v_{ij} - b_j^0 - \iota = v_{ij} - p_j - \iota$, and i prefers to win j at price p_j to winning k. Thus, $u_i + u_j^S \geq \max\{v_{ij} - v_j^S - \iota, 0\}$, but we cannot assure that $u_i + u_j^S \geq d_{ij}$.

In the fifth case listed above, the argument is similar. Suppose $v_{ik} - b_k^s = v_{ij} - b_j^s$ and $B^h(j) \neq j$. Thus, bidder i is either indifferent between bidding in one of these auctions (if $B^h(k) \neq k$) or he prefers k (if $B^h(k) = k$). He decides to bid in k, bids $b_{ik} = b_k^s + \iota$, becomes the high bidder, and the standing bid does not change. After his turn one more bidder bids in k and the standing bid increases by one increment to the final price $p_k = b_{ik}$. The standing bid in j remains unchanged until the end of the game. In this situation, $v_{ik} - p_k = v_{ij} - p_j - \iota < v_{ij} - p_j$.

Since i's winning bid in both the forth and the fifth cases is an increase by no more that ι, the deviation may never exceed one increment ι. The situations discussed above are the only ways that such an instable situation can emerge: $v_{ij} - p_j < \Delta_{i(2)} - \iota$ can only hold if $b_{ij} > v_{ij} - \Delta_{i(2)} + \iota$ holds but a bidder i never submits such a bid b_{ij}, and subsequent bidders cannot increase the standing bid to a final price greater than $p_j = b_{ij}$ if i wins with this bid b_{ij}.

We summarize the result in the following Lemma 3.1:

Lemma 3.1 *An outcome (x^*, p^*) that results from playing σ^* may be instable.*

If (x^*, p^*) is instable,

- *the payoffs of at least one pair $(i, j) \in N \times M$ that is not matched under x^* can* *be combined, such that* $u_i(x^*, p^*) + u_j^S(x^*, p^*) = d_{ij} - \iota$ *and*

- *for all pairs* (i, j)*:* $u_i(x^*, p^*) + u_j^S(x^*, p^*) \geq d_{ij} - \iota$.

We conclude that stability cannot be assured if bidders follow σ^*, but we can restrict incentives to deviate to ι. The core of the assignment game equals the set of payoff vectors in stable payoff-assignment combinations. The cooperative game induced by the multiple-auctions game is the cooperative version of the assignment game. Thus, our analysis also shows that the outcome (x^*, p^*) is not always in the core.

We now shift the focus away from the stability of (x^*, p^*) and concentrate on the range of prices p^* that occur and the efficiency of x^*. Therefore, we recapitulate the main aspects of the occurrence of instable pairs in (x^*, p^*) and anticipate some of the results that also relate to the issue of stability:

- A stable outcome exists (see Shapley and Shubik (1971) or Roth and Sotomayor (1990, Theorem 8.5, p. 207)) and it may have been reached by following σ^*, i.e., the random selection of the bidder sequence and the random selection in case of indifference cause the deviation.

- If an instable outcome results, this depends on a decision made by a bidder who was indifferent between at least two auctions in his last move in the game.

- For an instable pair (i, j), we have $u_i + u_j^S = d_{ij} - \iota$. The deviation is never more (or less) than one increment.

- In i's finally preferred auction j, the standing bid does not change anymore after his final move.

- The price determining bidder $B^{PD}(k)$ (as defined below in Definition 3.7) in auction k (the auction won by bidder i) placed a bid after bidder i's final move.

3.2.2.2 Prices p^*

We consider the prices p^* that result from playing according to σ^* and characterize the range in which they may lie as well as the circumstances that lead to the respective prices. For this analysis, we first introduce so-called price determining bidders and define a reference outcome. Then, we analyze deviations from reference prices.

Definition 3.7 (Price determining bidder $B^{PD}(j)$ in auction j) *The price determining bidder $B^{PD}(j) \in \{N \cup j\}$ in auction j*

- *is bidder i with $b_{ij} = p_j$, if such a bid b_{ij} exists and is unique,*
- *if bidders i and h with bids $b_{ij} = b_{hj} = p_j$ exist and $x^*_{hj} = 1$, then $B^{PD}(j) = i$,*
- *if two bidders i and h with bids $b_{ij} = b_{hj} = p_j$ and $x^*_{ij} = x^*_{hj} = 0$ exist, and i submits his bid before h, then $B^{PD}(j) = i$, and*
- *if no bidder submits a bid in j, then $B^{PD}(j) = j$.*

A bidder i is called the price determining bidder[27] $B^{PD}(j) = i$ in auction j, if he submits the unique bid b_{ij} that equals the final price in auction j. If two bidders i and h both submit bids equal to p_j and h wins the auction, then $B^{PD}(j) = i$. If two bidders i and h both submit bids equal to p_j, i submitted his bid before h,[28] and another bidder wins the auction, then $B^{PD}(j) = i$. More than two bidders cannot submit the same bid in an auction because a new bid has to exceed the current standing bid. If no bidder submits a bid in j, then, in a slight abuse of language, the seller j is the price determining bidder. Thus, the price determining bidder in each auction is uniquely defined for each realized play of the multiple-auctions game.

We distinguish two kinds of price determining bidders: *internal price determining bidders* (those that are included in the set of winning bidders) and *external price determining bidders* (those that do not win any auction). Results concerning the price determining bidders are given in Proposition 3.1, from whose proof we derive a further proposition on the timing of submitted bids (Proposition 3.2).

Proposition 3.1 *If a bidder i follows σ^*_i and he is the price determining bidder in auction k, i.e., $B^{PD}(k) = i$, then*

- *$B^h(k) \neq i$, and*
- *either $x_{il} = 0$ for all $l \in M$ and $p_k = v_{ik}$ (external price determining bidder)*

[27] We will, in a slight misuse of notation, occasionally replace the long expression "price determining bidder" by B^{PD} in the text.

[28] Then, i was the high bidder in j with the bid b_{ij}, which was higher than the current standing bid, and h afterwards submitted the bid $b_{hj} = b_{ij}$, thereby increasing the standing bid.

- *or* $\exists\, j, j \neq k$ *such that* $x_{ij} = 1$ *and* $v_{ij} - p_j = v_{ik} - p_k$ *or* $v_{ij} - p_j = v_{ik} - p_k \pm \iota$ *(internal price determining bidder).*

Proof of Proposition 3.1: First, we show that $B^h(k) \neq i$ if $B^{PD}(k) = i$, i.e., i does not determine his own price. Assume that $B^h(k) = B^{PD}(k) = i$. Bidder i who follows σ_i^* does not bid in k if he is the current high bidder. Bidder i becomes $B^{PD}(k)$ by submitting a bid $b_{ik} = p_k > b_k^{s,t}$ at stage t and either $b_k^{h,t} \geq b_{ik}$ or $b_{ik} > b_j^{h,t}$, where $b_k^{s,t}$ and $b_k^{h,t}$ denote the current standing bid and the current high bid at stage t, respectively. If a bid b_{ik} with $b_k^{h,t} \geq b_{ik} > b_k^{s,t}$ makes him the price determining bidder, he must have bid in an auction in which he was the current high bidder, a contradiction. If his price determining bid is $b_{ik} > b_k^{h,t}$, he becomes the high bidder and the ultimate winner must submit his winning bid in a later round. If i is the winning bidder, he must bid in an auction where he is the current high bidder, again a contradiction.

Second, we show that if $B^{PD}(k) = i$ and i wins no auction, then the final price p_k in k equals his valuation for item k, i.e., his price determining bid is $b_{ik} = v_{ik}$. Since i does not win any auction, his payoff is zero. The auctions are over. Thus, i did not want to submit any more bids, because otherwise the game would not have ended. He never submits a bid $b_{ik} > v_{ik}$. Suppose his price determining bid in k is $b_{ik} < v_{ik}$. Then he would not stop bidding because at the end he is not the high bidder in any auction. Thus, his price determining bid has to be $b_{ik} = v_{ik} = p_k$.

Third, we assert that a price determining bidder i in k who wins an auction $j \neq k$ is approximately indifferent between j and k. At the stage t when i submits $b_{ik} = p_k$ either

(1) $v_{ik} - b_k^{s,t} > v_{il} - b_l^{s,t}$ for all $l \neq k$, or

(2) $\exists l \neq k$ such that $v_{ik} - b_k^{s,t} = v_{il} - b_l^{s,t}$.

In Case 1, the submitted bid is $b_{ik} = v_{ik} - \Delta_{i(2)}^t > b_k^{s,t}$. With this bid, i either

(a) becomes current high bidder or

(b) his bid determines the new standing bid b_k^{s,t^+},

where $t^+ > t$ indicates that b_k^{s,t^+} is the standing bid later in the auction than $b_k^{s,t}$. In each case the standing bid increases because new bids always increase the standing bid. Thus, $b_k^{s,t^+} > b_k^{s,t}$. If i becomes the current high bidder (Case (a)) with this bid b_{ik}, he must be outbid by another bidder, the winner of k, before he submits any other bid. Then, the bid b_{ik} also determines the current standing bid b_k^{s,t^+} in k, as in Case (b). Thus, both (a) and (b) lead to a situation where i may bid again and $v_{ik} - b_k^{s,t^+} \geq v_{il} - b_l^{s,t^+}$ because this was how i determined his bid $b_{ik} = b_k^{s,t^+}$ and standing bids in other auctions l may have increased (for $l \neq k$, $b_l^{s,t^+} \geq b_l^s$ is the standing bid at this bidding stage and $b_l^{s,t}$ can have changed or not due to other bidders' bids since the initial situation). Since we are considering i's price determining bid, we know that $b_k^{s,t^+} = p_k$. Bidder i submits his bid in j later and $b_k^{s,t^+} = p_k$ does not change anymore. Thus, when i bids in j we must have $v_{ij} - b_j^{s,t^+} \geq v_{il} - b_l^{s,t^+}$ for all l. Because standing bids can only increase, we have $v_{ik} - b_k^{s,t^+} = \Delta_{i(2)}^t \geq v_{il} - b_l^{s,t^+}$ for all l including j. Thus, $v_{ij} - b_j^{s,t^+} = v_{ik} - b_k^{s,t^+}$ and $b_j^{s,t^+} = b_j^{s,t}$, i.e., i is indifferent between k and j when he submits his winning bid in j, and the standing bid in j has not changed since the initial situation we considered. It follows that the winning bid is $b_{ij} = b_j^{s,t^+} + \iota$. Assume to the contrary that this bid was not his winning bid. Since i wins j, he would have to bid again in j. The standing bid in j would increase. Then, i would prefer k over j and bid in k. This is a contradiction to the assumption that the bid $b_{ik} = b_k^{s,t^+} = p_k$ is the price determining bid. Thus, in Case 1, when i submits his winning bid in j he is indifferent between j and k, and his winning bid in j is ι higher than the current standing bid in j. The worst case is that another bidder submits a bid equal to i's bid in j and, thus, $v_{ij} - p_j = v_{ik} - p_k$ or $v_{ij} - p_j = v_{ik} - p_k - \iota$. In each case, the winning bid of a bidder who follows σ_i^* is his last bid.

After submitting the price determining bid in k and before submitting the winning bid in j, i may submit other bids in auctions with a difference between valuation and standing bid equal to $\Delta_{i(2)}^t$, if such auctions exist. Bidder i may also be the price determining bidder in these auctions. Then, i is indifferent between them and the auction he wins at the final prices.

In Case 2, when $v_{ik} - b_k^{s,t} = v_{il} - b_l^{s,t}$ for at least one $l \neq k$, i bids one increment above $b_k^{s,t}$, i.e., $b_{ik} = b_k^{s,t} + \iota = v_{ik} - v_{il} + b_l^{s,t} + \iota = p_k$. With this bid, i either determines the new standing bid, or he becomes high bidder and is outbid by another bidder later. In both cases, there is some round $t^+ > t$ when $b_k^{s,t^+} = b_{ik} = p_k$ and i is not high bidder in k. Since i wins an auction by assumption, he submits another bid in at least one other auction, the auction j that he wins, after submitting his price determining bid. Similar to the argument before, we conclude from the fact that standing bids in all auctions other than j can only increase between i's decision to bid in k and his decision to bid in j, and the fact that i preferred k to j before, that either $v_{ij} - b_j^{s,t^+} = v_{ik} - b_k^{s,t^+} + \iota$ if the standing bid in j has not increased (i.e., $b_j^{s,t^+} = b_j^{s,t}$), or $v_{ij} - b_j^{s,t^+} = v_{ik} - b_k^{s,t^+}$ if the standing bid in j has increased by ι (i.e., $b_j^{s,t^+} = b_j^{s,t} + \iota$). (Of course, in the latter case, standing bids in all other auctions that were in i's demand set when he bid in k also increased by ι).) That means that j is now either better by ι than k for i, or i is indifferent. In either case, i submits the bid $b_{ij} = b_j^{s,t^+} + \iota$ in j. With this bid, he either becomes the high bidder or he does not. If he becomes the high bidder, he either wins j at $b_{ij} - \iota = p_j$ (if no other bidder submits a bid in j), or at $b_{ij} = p_j$ (if another bid is submitted after i's winning bid). If he does not become the high bidder with his bid (which can occur when $v_{ij} - b_j^{s,t^+} = v_{ik} - b_k^{s,t^+} + \iota$), he submits a new bid that is one increment higher. In this case, the final price may again equal his bid or be an increment lower.

Summarizing Case 2, the final price p_j may be such that when $v_{ij} - b_j^{s,t^+} = v_{ik} - b_k^{s,t^+}$ before i's final bid in j, we may have $v_{ij} - p_j = v_{ik} - p_k$ or $v_{ij} - p_j = v_{ik} - p_k - \iota$. When $v_{ij} - b_j^s = v_{ik} - b_k^{s,t^+} + \iota$ before i bids in j, we may have $v_{ij} - p_j = v_{ik} - p_k + \iota$ or $v_{ij} - p_j = v_{ik} - p_k$ (or $v_{ij} - p_j = v_{ik} - p_k - \iota$ if i increases the standing bid with two subsequent bids in k, both which rise the standing bid by ι). ■

From Proposition 3.1, it follows that $B^h(j) \neq B^{PD}(j)$ for all $j \in M$ if all bidders follow σ^*. The price determining bidder is approximately indifferent between his payoff and the payoff from winning the auction in which he determined the price. Thus, we get the following corollary.

Corollary 3.1 *A winning bidder i in auction j with high bid $b_{ij} = b_j^h > p_j + \iota$ is not the price determining bidder in any auction, i.e.,*

$$b_j^h > p_j + \iota \Rightarrow B^h(j) \neq B^{PD}(l) \text{ for all } l \in M.$$

From the argument in the proof we can also draw conclusions about some winning bids and timing. These are summarized in the following Proposition 3.2.

Proposition 3.2 *Whenever the price determining bidder $i = B^{PD}(j)$ in j wins another auction $l \neq j$, i.e., $x_{il} = 1$,*

- *he wins this auction by increasing the standing bid by an increment,*
- *his winning bid in l is submitted after his price determining bid in j, and*
- *the last bid submitted in j was submitted before the last bid in l.*

Proof of Proposition 3.2: The results follow from the argument in the proof of Proposition 3.1. ■

Clearly, if $p_j > b_j^0$ then a price determining bidder $i = B^{PD}(j) \notin M$ exists. We now define our reference outcome and analyze deviations from it.

Definition 3.8 (Reference outcome) *The* reference outcome $(\bar{x}^*, \bar{p}^*)$ *is the outcome which results from playing according to σ^* with the property that all internal and external price determining bidders are indifferent between their payoff at $(\bar{x}^*, \bar{p}^*)$ and the difference between their valuation and the price in the auction in which they determine the price.*

The reference outcome is one of usually several outcomes that may result from play according to σ^*. This outcome is not guaranteed by σ^* because of the random bidding order and the random selection of an auction according to part (3) of the definition of σ_i^*.

The following lemmas refine the statement of Proposition 3.1 and are derived from the argument in the proof of Proposition 3.1. They describe some properties of a deviation from the reference outcome. Lemma 3.2 describes what we call an upwards deviation from bidder i's reference outcome of indifference at final prices.

Lemma 3.2 *If all bidders follow σ^*, bidder i wins auction j, $B^{PD}(k) = i$ for some $k \neq j$, and $v_{ij} - p_j > v_{ik} - p_k$ then*

- $v_{ij} - p_j = v_{ik} - p_k + \iota$,
- *i prefers j by one increment over k (with $\Delta_{i(2)} = \Delta_{ik}$) when submitting the winning bid,*
- *i is indifferent between j and k when submitting the price determining bid in k.*

Lemma 3.3 describes a downwards deviation from internal price determining bidder i's reference result. This deviation is the one already considered in Section 3.2.2.1.[29]

[29] These deviations do not necessarily cause a deviation from the reference price because such upwards and downwards deviations may accumulate, as shown below.

Lemma 3.3 *If all bidders follow σ^*, bidder i wins auction j, $B^{PD}(k) = i$ for some $k \neq j$, and $v_{ij} - p_j < v_{ik} - p_k$ then*

- $v_{ij} - p_j + \iota = v_{ik} - p_k$,
- *i is indifferent between j and k when submitting the winning bid in j,*
- *$B^{PD}(j)$ submits his bid in j after bidder i.*

For an external price determining bidder, no such deviations occur.

Corollary 3.2 *If all bidders follow σ^*, bidder i wins no auction, and $B^{PD}(j) = i$ for some auction j, then $v_{ij} - p_j = 0 \geq v_{ik} - p_k$ for all $k \in M$ and, thus, $v_{ij} = p_j$.*

From this corollary, we conclude that externally determined prices (and prices p_j with $B^{PD}(j) = j$) are integers because all bidders' and sellers' valuations are integers. Note that all reference prices $\bar{p}_j^*$ are integers because they have to lie on the same grid as the valuations.

External price determining bidders with unit-demand preferences are concerned with the absolute level of the prices because they do not win an auction. One item at a time, they compare a payoff of zero with the payoff from buying each item. Internal price determining bidders, on the other hand, determine from their point of view the relative prices of items. Winning bidders are concerned with the relative advantage of the item they win over the other items at the given prices.

Lemmas 3.2, 3.3, and Corollary 3.2 consider individual bidders. Lemma 3.4 transfers the results to the resulting outcome (x^*, p^*).

Lemma 3.4 *If all bidders follow σ^*, then in the resulting outcome (x^*, p^*) the following inequalities hold:*

$$\left.\begin{array}{l} v_{ij} - p_j^* \geq v_{ik} - p_k^* - \iota \\ v_{ij} - p_j^* \geq 0 \end{array}\right\} \quad \text{for } i : \exists j \in M : x_{ij}^* = 1 \text{ and } k \in M,$$

$$0 \geq v_{ik} - p_k^* \quad \text{for } i : x_{ik}^* = 0 \text{ for all } k \in M,$$

$$p_k^* \geq v_k^S \quad \text{for all } k \in M.$$

Proof of Lemma 3.4: No bidder ever bids above v_{ij} in any auction j. Therefore, $v_{ij} - p_j^* \geq 0$ at the final prices for the winning bidder i in an auction j. From Lemmas

3.2 and 3.3, we already know that the inequalities in the first line hold for internal price determining bidders. For all submitted bids of some bidder i in j we have $p_j \leq b_{ij} \leq \Delta_{i(1)} + \iota$. Thus, the inequalities are valid for all winning bidders. Losing bidders i submit bids $b_{ij} \leq v_{ij}$ in auctions j and do not stop bidding until all standing bids are at least as high as their valuation. This proves the second row of inequalities. The third row considers the sellers. Prices equal starting prices if the item is not sold. If it is sold, at least one bidder has bid more than the starting price and he pays at least the starting price. $\square$

In the reference outcome $(\bar{x}^*, \bar{p}^*)$, where the increment problem does not occur, these equations would be

$$\left.\begin{array}{l} v_{ij} - \bar{p}_j^* \geq v_{ik} - \bar{p}_k^* \\ v_{ij} - \bar{p}_j^* \geq 0 \end{array}\right\} \quad \text{for } i: \exists j \in M: \bar{x}_{ij}^* = 1 \text{ and } k \in M,$$

$$0 \geq v_{ik} - \bar{p}_k^* \quad \text{for } i: \bar{x}_{ik}^* = 0 \ \text{ for all } k \in M,$$

$$\bar{p}_k^* \geq v_k^S \quad \text{for all } k \in M.$$

Before further analyzing the outcome (x^*, p^*), we look at a restriction on the number of deviations caused by internal price determining bidders.

Lemma 3.5 *The number of internal price determining bidders is less than or equal to* $\min\{n-1, m-1\}$.

Proof of Lemma 3.5: The last bid in the game is either a winning bid or the bid of an external price determining bidder. If it is a winning bid then either the auction is won at its starting price or the previous high bidder is outbid and wins no auction. In both cases, the price determining bidder is not internal. Thus, if any item is sold, then there is there is at least one item that is sold and does not have an internal price determining bidder. The maximum number of items sold is $\min\{n, m\}$. Combining these two arguments, we find that the maximum number of internal price determining bidders is at most $\min\{n-1, m-1\}$. $\square$

From this argument we can directly infer the following lemma:

Lemma 3.6 *The last bid in the multiple-auctions game* Γ^a *is submitted in an auction* j *with an external price determining bidder or with* $B^{PD}(j) = j$.

Individual bidders' deviations from the reference prices $\bar{p}^*$ as examined in Lemmas 3.2 and 3.3 may accumulate.

Proposition 3.3 *Price deviations from the reference price $\bar{p}^*$ may accumulate up to a maximum of* $(\min\{n, m\} - 1) \cdot \iota$.

Proof of Proposition 3.3: We know from Lemmas 3.2 and 3.3 and Corollary 3.2 that price deviations from the reference prices are always caused by internal price determining bidders. Combining this with the information on the number of internal price determining bidders in Lemma 3.5, we conclude that no more than $\min\{n-1, m-1\}$ deviations may occur. We now show how they may accumulate. The proof is given by an instructive example with increment $\iota = 0.5$. In Table 3.1, a valuation matrix is given. Bold valuations in italics indicate the efficient assignment.[30] Three outcomes are described by the winning bidders B^h and price vectors p^*, p', or p'' resulting from strategy profile σ^*. Additionally, the winning bidders' final high bids b^h and the price determining bidders B^{PD} are given. In the upper outcome, all items sell at a price of five and the assignment is efficient. This is the reference outcome $(\bar{x}^*, \bar{p}^*)$.[31] The outcome in the middle shows the maximum accumulation of deviations from the reference outcome. In auction $A5$, we have a price equal to seven, which is five plus four increments, i.e., $p_5 = 5 + (\min\{n, m\} - 1) \cdot \iota = 5 + 4 \cdot 0.5$. The lower outcome shows an accumulation of several downwards deviations of prices (going along with bidders' downwards deviations from indifference). At least in this example, the maximum downwards price deviation seems to be 1.5.[32]

An illustration of the three outcomes is given in Figure 3.3. In the graphs, an arrow pointing from a bidder to an auction indicates that the bidder wins that auction. Arrows originating in auctions indicate the respective price determining bidders. We call a path in the graph that starts at an auction or a bidder and follows the arrows

[30] In all examples, if not stated differently, we assume $v^S = \mathbf{0}$.

[31] For the connection between the efficient assignment x^{eff} and the reference outcome $\bar{x}^*$, see Proposition 3.4.

[32] In a simulation, the maximum accumulation of price deviations in this example occurs in 9 of 10,000 outcome calculations using σ^*. The outcome $(\bar{x}^*, \bar{p}^*)$ is observed 619 times. The lower outcome in Table 3.1 is observed 14 times. All other outcomes are combinations of $\bar{x}^*$ and prices $p_j \in \{4.0, 4.5, 5.0, 5.5, 6.0, 6.5\}$; the most frequently observed price vector is $p = (5.0, 5.5, 5.5, 5.5, 5.5)$ (630 times), and average prices are $p = (5.0, 5.3, 5.3, 5.2, 5.3)$. Price determining bidders in this example are unambiguous, as given in the table. Final high bids are variable. Note that this example is especially designed to facilitate the occurrence of maximum deviations.

Table 3.1: Example of the maximum price deviation from ($\bar{x}^*$, $\bar{p}^*$).

	$A1$	$A2$	$A3$	$A4$	$A5$
$B1$	***8***	8	0	0	0
$B2$	0	***10***	10	0	0
$B3$	0	0	***15***	15	0
$B4$	0	0	0	***16***	16
$B5$	0	0	0	0	***18***
$B6$	5	0	0	0	0
B^h	$B1$	$B2$	$B3$	$B4$	$B5$
b^h	5.5	5.0	5.0	5.5	18.0
$\bar{p}^*$	5.0	5.0	5.0	5.0	5.0
B^{PD}	$B6$	$B1$	$B2$	$B3$	$B4$
B^h	$B1$	$B2$	$B3$	$B4$	$B5$
b^h	5.5	6.0	6.5	7.0	18.0
p'	5.0	5.5	6.0	6.5	7.0
B^{PD}	$B6$	$B1$	$B2$	$B3$	$B4$
B^h	$B1$	$B2$	$B3$	$B4$	$B5$
b^h	5.5	5.0	4.5	4.0	18.0
p''	5.0	5.0	4.5	4.0	3.5
B^{PD}	$B6$	$B1$	$B2$	$B3$	$B4$

an *indifference path*: on such a path, a bidder is indifferent between the auction that an arrow points to and auctions from which arrows point to him. Note that an indifference path either ends at a bidder who does not win an auction and who, thus, is a potential external price determining bidder, or at an auction j with $B^{PD}(j) = j$ and, therefore, $p_j = b_j^0$.[33] The dashed arrows in the graph on the right in Figure 3.3 indicate that the price determining bidder slightly (by one increment) prefers the auction that he wins. On the other hand, arrows with double arrowheads from k to i indicate that $v_{ij} - p_j = v_{ik} - p_k - \iota$ for $x_{ij} = 1$, i.e., bidder i who wins j prefers to win k at price p_k by one increment. For a detailed description of how such an accumulation as exemplified in Table 3.1 can arise, see Appendix A.1. There, in a simpler example, each bid is stated.

We have shown by example that the maximum deviation of $(\min\{n, m\} - 1) \cdot \iota$ may occur. We have to show that the accumulated deviation cannot be higher. Since $(\bar{x}^*, \bar{p}^*)$ is an outcome, there are no losing bidders willing to bid at prices $\bar{p}^*$. Denote

[33] This kind of graph goes back to Demange (1982). See the proof of Proposition 3.9 for an alternative description.

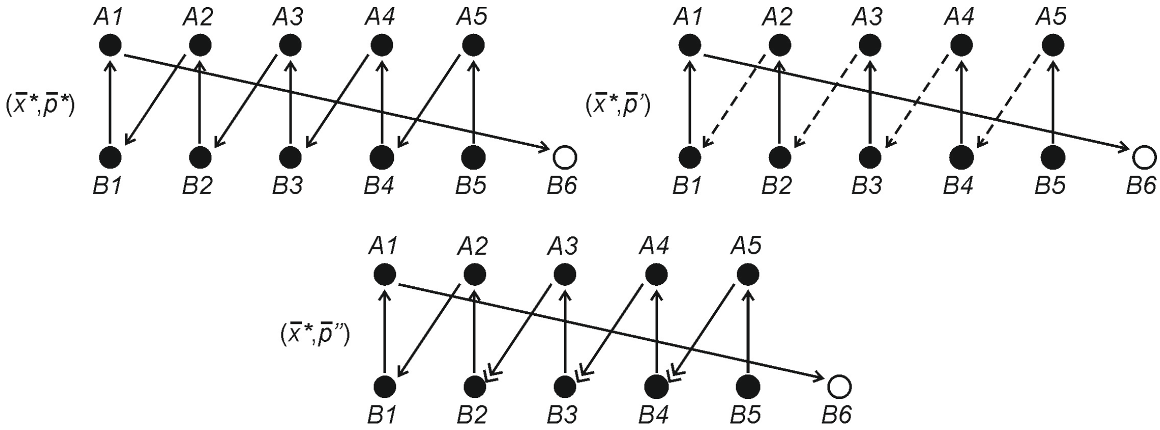


Figure 3.3: The assignment $\bar{x}^*$ with prices $\bar{p}^*$ (left), p' (right), p'' (below) in the example of Proposition 3.3.

the number of sold items at $(\bar{x}^*, \bar{p}^*)$ by w and the set of winning bidders by W. At standing bids $b^s \geq \bar{p}^*$ there are w or less bidders willing to bid (a subset of the winning bidders at $\bar{p}^*$). We have w or less different price determining bidders.

As soon as every remaining bidder holds a high bidder position, the game ends. At standing bids $b^s \geq \bar{p}^*$, w or more bidders hold a high bidder position.[34] For these high bidders i in j, either $v_{ij} - b^s_j = \Delta_{i(1)} \geq 0$ or $v_{ij} - b^s_j = \Delta_{i(1)} - \iota$. For bidding activity to occur, at least one of the winning bidders under $\bar{x}^*$ must not have a high bidder position. Thus, at least one of the losing bidders under $\bar{x}^*$ has to hold a high bidder position (or this auction k has $\bar{p}^*_k = b^0_k$). In the auctions k where this is the case, standing bids $b^s_k = \bar{p}^*_k$. These bidders will not bid anymore. Thus, all standing bids above $\bar{p}^*$ are determined by winning bidders $i \in W$ and, thus, by internal price determining bidders. A bidder cannot be his own B^{PD} (Proposition 3.1). Neighboring prices in the chain deviate by at most ι (Lemmas 3.2 and 3.3). Since there are no more than $w \leq \min\{n, m\} - 1$ internal price determining bidders (Lemma 3.5), the

[34]There may be more than w current high bidders. Consider the following example: $v^S = (0, 0, 0, 0, 3)$, $\iota = 0.5$, bidder Bi wins item Ai under $\bar{x}^*$, $i \in 1, \ldots, 4$, and the fifth item is not sold in the unique reference outcome with $\bar{p}^* = (2, 2, 2, 2, 3)$. Auction $A5$ has a starting price $b^0_5 = 3$. Bidder $B4$ has valuations $v_{44} = 4 = v_{45}$. In the reference outcome, Auction $A1$ has an external price determining bidder, Bidder $B5$. Now, if $B5$ holds the high bidder position in $A1$ at $b^s_1 = 2$ and $B1$ that of $A2$ and so on, and deviations occur and accumulate such that $b^s_4 = 3$, and it is $B4$'s turn to bid, then he may bid $b_{45} = 3.5$ and five auctions have a high bidder $B^h(j) \neq j$. Then, five items are sold due to the accumulation of deviations (and the assignment is inefficient). Our argument in the proof can be adapted to this case of more than w high bidders. We will not mention it separately anymore.

number of auctions with prices $p^*_j \geq \bar{p}^*$ is less than or equal to $\min\{n, m\} - 1$. The w winning bidders, of whom at most $w - 1$ are also price determining, cannot increase the prices above the described maximum deviation. This is because at every vector of standing bids $b^s \geq \bar{p}^*$, as soon as the external price determining bidders are outbid in auctions j with $b^s_j = \bar{p}^*_j$, all w high bidder positions are held by these winning bidders $i \in W$, who consequently do not submit new bids. Thus, "indifference circles" of bidders who sequentially increase the prices cannot exist.

Note that the last price on the path that is determined by an external bidder or the seller is always equal to the price given by $\bar{p}^*$. Similarly, all prices p^*_j have to be equal to or higher than all valuations of losing bidders, i.e., if $x^*_{ij} = 0$ for all j for a bidder i, then $v_{ij} \leq p^*_j$ for all p^*_j. For internal price determining bidders, the relation of prices is important for determining their demand, thus, they determine relative prices and may be less concerned about absolute prices in their bidding behavior. Thus, if the only bidders that demand items on an indifference path are internal price determining bidders, one winning bidder at the first node and one external price determining bidder at the last node, downwards price deviations may occur. One deviation is of size ι. If all internal price determining bidders cause such deviations, these accumulate to a maximum of $w - 1$ when w items are sold. Since $w \leq \min\{n, m\} - 1$, the result is proved.

Note that in the example above, the maximum accumulation of downwards deviations is $\min\{n, m\} - 2$, because the first internal B^{PD} cannot cause a deviation downwards. With this number of bidders, $B1$ will not be the chosen to bid for the first time at the end of the game. This is the precondition for such a deviation. But when $B1$ is chosen to bid for the first time, he will submit his only bid, $b_{11} = 5$. In more complex examples, the maximum deviation of a price could probably also be reached in downwards direction. ■

In order for the maximum deviation to occur, the valuations of the bidders have to be such that all but one determine the prices of the others and a special bidder order has to arise.

The deviations have to be seen as relative deviations: on the indifference path, neighboring prices may be an increment closer or farther away from the reference price than the price before. We have seen that these deviations may accumulate on indifference paths. On the other hand, they may also compensate for other deviations if they go in the opposite direction. Or they do not have to occur at all and exact reference prices may result.

A necessary condition for such price deviations is the existence of an indifference path on which several internal price determining bidders determine the prices of each other. The occurrence of such an indifference path depends on the existence of internal price determining bidders, i.e., on the valuation matrix, the order in which bids are submitted, and chance (because each bidder selects one of the auctions by chance when he is indifferent).

In more complex examples, the graph that displays the final outcome may contain several disconnected indifference paths, separated bidder-seller pairs, and single unconnected nodes (losing bidders or unsold items). On the other hand, if we not only display links between auctions and their price determining bidder but also include arrows from auctions to other indifferent bidders in the graph, such components may be connected. Then, each bidder has at most one outgoing directed edge, but may have several incoming edges. Similarly, an auction may have at most one incoming edge, but several outgoing edges. In the following, when using such graphs, it will be clear from the description if it is a reduced graph (only the respective price determining bidder has an incoming edge) or if it is a complete graph (all indifferent bidders who do not win the respective auction have an incoming edge). Note that bidders in separated components of a complete graph have no interest in items outside of their component (their indifference paths).

We find that a deviation from $\bar{p}_j^*$ to p_j^* is bounded by $\bar{p}_j^* + \iota \cdot (\min\{n, m\} - 1) \geq p_j^* \geq \bar{p}_j^* - \iota \cdot (\min\{n, m\} - 1)$. Demange et al. (1986, Theorem 4) find a similar maximum deviation of prices in their "approximate auction mechanism," which is related to the auction game considered here.[35] In their model, bidders increase the standing bid by an increment in their preferred auction and the price equals the submitted bid. They do not consider strategic behavior of bidders. In the approximate auction mechanism, for each resulting price p_j^*, the bounds are $p_j + \iota \cdot \min\{n, m\} \geq p_j^* \geq p_j - \iota \cdot \min\{n, m\}$. The similarity of the results in these two auction mechanisms occurs because, at the end of the multiple-auctions game, internal price determining bidders effectively increase standing bids by only an increment. The difference in the bounds arises because, in the multiple-auctions game, the winner and the price determining bidder

[35] Similar bounds have been derived, e.g., by Bertsekas (1992) and Bansal and Garg (2005). Due to the different auction formats, their bounds are larger than our bound of $(\min\{n, m\} - 1) \cdot \iota$.

have to submit bids such that a deviation occurs.

With a single seller, the maximum sum of accumulated deviations from $\bar{p}^*$ (i.e., the range of the sum of prices) may be of interest. The absolute value of the maximum sum of price deviations in the game is $\iota + 2\iota + \ldots + (\min\{m,n\}-1)\iota = \iota \cdot (\min\{m,n\} - 1)\min\{m,n\}/2$ because along the indifference path, the deviation may increase by at most ι from auction to auction.

A third kind of deviation (other than upwards and downwards deviations of prices) is a deviation from the reference assignment $\bar{x}^*$. We consider this deviation in the next section.

3.2.2.3 The Assignment x^* (Efficiency)

We now turn to the analysis of the assignment x^* that results from play of σ^*. We investigate whether x^* may deviate from the reference assignment $\bar{x}^*$ (see Definition 3.8). From the previous section, such a deviation may result if too many upwards price deviations accumulate. That is, accumulated price deviations induce a bidder to prefer (or to be indifferent to) an auction that would not be in his demand set otherwise. If this happens, he may win a different auction than he does under $\bar{x}^*$. We begin the analysis of the assignment x^* by characterizing the reference assignment $\bar{x}^*$.

Proposition 3.4 *If* $p^* = \bar{p}^*$ *then* $x^* = x^{\text{eff}}$.

That is, the assignment x^* that results from playing σ^* when the resulting prices are the reference prices $\bar{p}^*$, and thus all internal price determining bidders are indifferent between the auction they win and the auction in which they determine the price, is efficient. Remember that efficient assignments of the multiple-auctions game are optimal assignments under valuations $V$and v^S.

Proof of Proposition 3.4: We assume $n \geq m$. At first, we prove the result for $v^S = \mathbf{0}$, and then we show that it is also valid for $v^S \geq \mathbf{0}$. Bidders' payoffs have the following properties:

$$\begin{array}{llllll} \text{If} & \exists\, j \text{ such that } x^*_{ij} = 1, & \text{then} & u^B_i(x^*, \bar{p}^*) = v_{ij} - \bar{p}^*_j & \geq v_{ik} - \bar{p}^*_k & \forall\, k \in M. \\ \text{If} & x^*_{ij} = 0\ \forall\, j, & \text{then} & u^B_i(x^*, \bar{p}^*) = 0 & \geq v_{ik} - \bar{p}^*_k & \forall\, k \in M. \end{array} \tag{3.1}$$

Every auction is won by at most one bidder and every bidder i wins at most one auction because i may not submit a bid when $B^h(j) = i$ for some j. Therefore, the sum of all payoffs on the left hand side of Equations (3.1) is $\sum_{i \in N} u_i^B(x^*, \bar{p}^*) = \sum_{i \in N} \sum_{j \in M} v_{ij} x_{ij}^* - \sum_{j \in M} \bar{p}_j^*$.

Now consider some alternative assignment $\tilde{x}$ at prices $\bar{p}^*$. For $\tilde{x}$ we assume that every bidder wins at most one auction. For every auction k that is won by some bidder i, we can compare the difference $v_{ik} - \bar{p}_k^*$ for $\tilde{x}_{ik} = 1$ with i's payoff under x^*, which is $v_{ij} - \bar{p}_j^*$ for $x_{ij}^* = 1$ or zero if $x_{ij}^* = 0$ for all j. In each case, we can see from Equations (3.1) that for each auction's winner i under $\tilde{x}$, the payoff under $\tilde{x}$ is not higher than under x^*. A bidder who does not win an auction under $\tilde{x}$ has a payoff of zero, which is the minimum a bidder following σ_i^* can get. Hence it follows that $\sum_{i \in N} \sum_{j \in M} v_{ij} x_{ij}^* - \sum_{j \in M} \bar{p}_j^* \geq \sum_{i \in N} \sum_{j \in M} v_{ij} \tilde{x}_{ij} - \sum_{j \in M} \bar{p}_j^*$ and, thus,

$$\sum_{i \in N} \sum_{j \in M} v_{ij} x_{ij}^* \geq \sum_{i \in N} \sum_{j \in M} v_{ij} \tilde{x}_{ij} \tag{3.2}$$

for all alternative assignments $\tilde{x}$.[36] From this we conclude that x^* is an efficient assignment, i.e., $x^* = x^{\mathit{eff}}$.

Now we take $v^S \geq \mathbf{0}$ into account. We rearrange bidders and sellers such that the first η bidders and sellers trade and possibly some $n - \eta$ bidders and $m - \eta$ sellers do not trade in the outcome $(x^*, \bar{p}^*)$, i.e., $x_{ii}^* = 1$ for all $i = 1, \ldots, \eta$. Summing up the payoffs of all bidders and sellers we get

$$\underbrace{\sum_{i=1}^{\eta} (v_{ii} - \bar{p}_i)}_{\text{trading bidders under } x^*} + \underbrace{\sum_{i=1}^{\eta} \left(\bar{p}_i - v_i^S\right)}_{\text{trading sellers under } x^*} + \underbrace{\sum_{i=\eta+1}^{n} 0}_{\text{non-trading bidders under } x^*} + \underbrace{\sum_{i=\eta+1}^{m} \left(\bar{p}_i - v_i^S\right)}_{=0,\text{ non-trading sellers under } x^*} .$$

If no bidders or sellers who do not trade exist, this does not make a difference for this sum. Now we compare this outcome with a different outcome $(\tilde{x},\bar{p}^*)$, i.e., we keep prices but change the assignment. To keep notation simple, we assume that under $\tilde{x}$ μ pairs trade and rename bidders again such that $\tilde{x}_{kk} = 1$ for all $k = 1, \ldots, \mu$. Note, however, that these trading bidders and sellers may form different trading pairs

[36] If we allow a bidder to win more than one auction under $\tilde{x}$, resulting in a valuation of zero for the second item, the sum of valuations on the left hand side of Equation (3.2) would decrease or remain the same.

than under x^* and that other bidders and sellers may trade. Therefore, we denote valuations after renaming bidders by $\tilde{v}_{ij}$ for $i \in N$, $j \in M$ and those of sellers by $\tilde{v}^S_j$. Summing up payoffs under $\tilde{x}$ gives

$$\underbrace{\sum_{k=1}^{\mu} (\tilde{v}_{kk} - \bar{p}_k)}_{\text{trading bidders under } \tilde{x}} + \underbrace{\sum_{k=1}^{\mu} \left(\bar{p}_k - \tilde{v}^S_k\right)}_{\text{trading sellers under } \tilde{x}} + \underbrace{\sum_{k=\mu+1}^{n} 0}_{\text{non-trading bidders under } \tilde{x}} + \underbrace{\sum_{k=\mu+1}^{m} \left(\bar{p}_k - \tilde{v}^S_k\right)}_{=0,\text{ non-trading sellers under } \tilde{x}} .$$

Subtracting the payoffs under $\tilde{x}$ from those under x^* results in

$$\begin{aligned}
&\left(\sum_{i=1}^{\eta} (v_{ii} - \bar{p}_i) + \sum_{i=1}^{m} \left(\bar{p}_i - v^S_i\right)\right) - \left(\sum_{k=1}^{\mu} (\tilde{v}_{kk} - \bar{p}_k) + \sum_{k=1}^{m} \left(\bar{p}_k - \tilde{v}^S_k\right)\right) \\
&= \sum_{i=1}^{\eta} (v_{ii} - \bar{p}_i) - \sum_{k=1}^{\mu} (\tilde{v}_{kk} - \bar{p}_k) = \sum_{j=1}^{m}\sum_{i=1}^{n} (v_{ij} - \bar{p}_j) \cdot x^*_{ij} - \sum_{j=1}^{m}\sum_{i=1}^{n} (v_{ij} - \bar{p}_j) \cdot \tilde{x}_{ij}.
\end{aligned}$$

Note that $\sum_{i=1}^{m}(\bar{p}_i - v^S_i) = \sum_{k=1}^{m}(\bar{p}_k - \tilde{v}^S_k)$ as renaming the sellers results in a permutation $\tilde{v}^S$ of v^S. Thus, asking for efficiency when bidders follow σ^* in the case of $v^S \geq \mathbf{0}$ can be reduced to a comparison of bidders' payoffs. This has been considered above. ■

Upwards deviations in prices may result in an inefficient assignment x^*.

Proposition 3.5 *Deviations from $\bar{p}^*$ may result in an inefficient assignment. In this inefficient assignment, the sum of payoffs is at least*[37]

$$\sum_{i \in N}\sum_{j \in M} \left(\left(v_{ij} - v^S_j\right) \cdot x^{\mathit{eff}}_{ij}\right) - (min\{n,m\} - 1) \cdot \iota.$$

Proof of Proposition 3.5: The first statement is proved by an example with increment $\iota = 0.5$. In Table 3.2, bidders' valuations are presented twice. The respective valuation for the item won by a bidder is bold and in italics. Beneath each (of the identical) valuation matrices, two exemplary outcomes are described by the winning bidders B^h, the final high bids b^h, the price vector p^*, and the price determining bidders B^{PD}. The resulting outcome x^* on the left side of the example equals the

[37] Bertsekas (1992) and Bansal and Garg (2005) find a bound of $\min\{n, m\} \cdot \iota$ for auction algorithms similar to ours except that they are both with English (pay your bid) auctions.

efficient assignment x^{eff}, whereas on the right hand side we have a deviation from x^{eff}. Figures 3.4 and 3.5 illustrate the examples on the left and the right side of Table

Table 3.2: An example where accumulated deviations may lead to an inefficient outcome.

	$A1$	$A2$	$A3$	$A4$	$A5$		$A1$	$A2$	$A3$	$A4$	$A5$
$B1$	***8***	8	0	0	0	$B1$	8	***8***	0	0	0
$B2$	0	***10***	10	0	0	$B2$	0	10	***10***	0	0
$B3$	0	0	***15***	15	0	$B3$	0	0	15	***15***	0
$B4$	0	0	0	***16***	16	$B4$	0	0	0	16	***16***
$B5$	17	0	0	0	***18***	$B5$	***17***	0	0	0	18
$B6$	5	0	0	0	0	$B6$	5	0	0	0	0
B^h	$B1$	$B2$	$B3$	$B4$	$B5$	B^h	$B5$	$B1$	$B2$	$B3$	$B4$
b^h	5.5	5.5	5.0	5.0	6.0	b^h	5.5	5.0	5.5	6.0	6.5
$\bar{p}^*$	5.0	5.0	5.0	5.0	5.0	p''	5.0	5.0	5.5	6.0	6.5
B^{PD}	$B6$	$B1$	$B2$	$B3$	$B4$	B^{PD}	$B6$	$B2$	$B3$	$B4$	$B5$
B^h	$B1$	$B2$	$B3$	$B4$	$B5$	B^h	$B5$	$B1$	$B2$	$B3$	$B4$
b^h	5.5	5.0	5.0	5.0	4.5	b^h	5.5	5.5	5.5	6.0	6.0
p'	5.0	5.0	4.5	5.0	4.5	p'''	5.0	5.5	5.5	5.5	6.0
B^{PD}	$B6$	$B1$	$B2$	$B3$	$B4$	B^{PD}	$B6$	$B2$	$B3$	$B4$	$B5$

3.2, respectively. Remember, arrows with double arrowheads indicate that the price determining bidder prefers the auction in which he determines the price over the auction that he wins by one increment. As before, dashed arrows indicate the opposite preference (for the auction he wins). In Figure 3.5, the dotted arrow to bidder $B1$ means that although he is not the price determining bidder, he has these approximate indifference characteristics. The deviation from $(x^{eff}, \bar{p}^*)$ may increase, remain constant, or decrease by an increment between adjacent auctions on the indifference path (auctions separated by a single bidder). These deviations are those described in Lemmas 3.2 and 3.3. In the outcome on the right side of Figure 3.4, $B2$ is worse off winning $A2$ than $A3$, the auction in which he determines the price. $B3$ prefers winning $A3$ to winning $A4$, and $B4$ would improve his payoff by buying the item in $A5$. In this example, $p' = (5, 5, 4.5, 5, 4.5)$. The deviation caused by $B3$ compensates for the downwards deviation caused by $B2$. $B4$ causes another downwards deviation. This example shows how combinations of deviations as described in Lemmas 3.2 and 3.3 may occur on an indifference path such that the resulting prices deviate from $\bar{p}^*$.

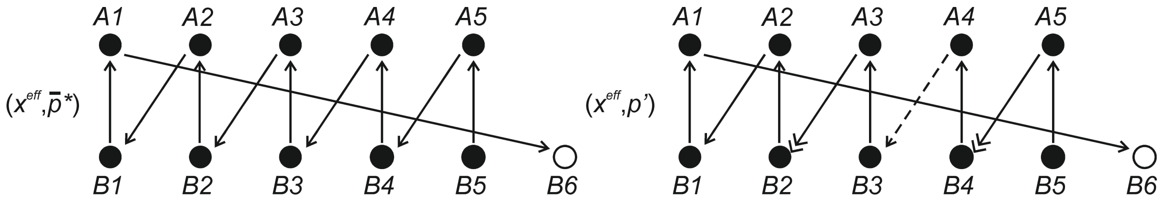


Figure 3.4: Efficient assignment x^{eff} at prices $\bar{p}^*$ and deviating prices p' in the example in Table 3.2.

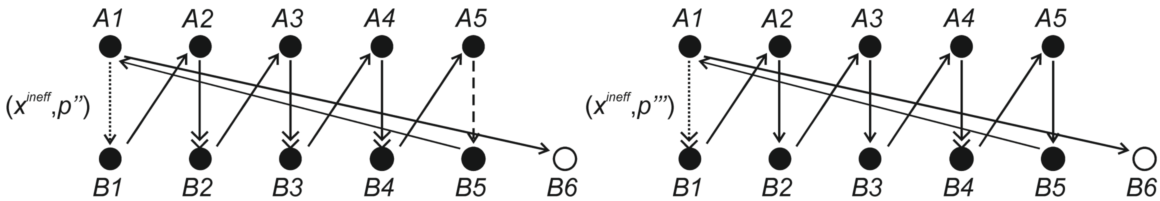


Figure 3.5: An inefficient assignment x^{ineff} at prices p'' and p''' that may arise in the example in Table 3.2.

Note that the last price on the path that is determined by an external bidder is always equal to the price given by $\bar{p}^*$.

The right-hand side of Table 3.2 and Figure 3.5 exemplify another kind of deviation: an accumulation of upwards deviations that results in an inefficient assignment. The valuation matrix is the same as that considered in Figure 3.4. The assignment is inefficient. This is a result of the accumulated deviations that are reflected in $p = (5, 5, 5.5, 6, 6.5)$. The dotted arrow from $A1$ to $B1$ in the figure indicates $B1$'s indifference. The arrow is not solid because $B1$ is not the price determining bidder in $A1$. We see that if $B5$ is the last bidder to submit a bid, then he prefers bidding in $A1$ at a standing bid of 5 to bidding in $A5$ at a standing bid of 6.5. The accumulation of upwards price deviations leads to a decision of bidder $B5$ that determines his final payoff because all winning bidders are tied to some auction and $B6$ is not bidding anymore. The cascade of upwards deviations may for some bidder (in our case $B5$) result in the perceived attractiveness of an auction that he does not win in any efficient allocation because less upwards deviations have accumulated there.

We now turn to the second statement in Proposition 3.5. We allow for deviations from $\bar{p}^*$ so that we can investigate inefficient assignments. As we know from Proposition 3.4, the reference price vector $\bar{p}^*$ guarantees an efficient assignment $x^* = x^{eff}$. In Table 3.2 an example of an inefficient assignment x^* is given. A precondition for an inefficient assignment is an internal price determining bidder with $v_{ij} - p^*_j < v_{ik} - p^*_k$ when $x^*_{ij} = 1$. We now construct an outcome (x^*, p^*) with the maximum number of such price deviations.

We assume $n > m$[38] and that all items are sold. Thus, we can rearrange bidders and items, such that $B1$ wins $A2$, $B2$ wins $A3$, and so on for the last $m-1$ auctions, and Bm wins $A1$. We assume that $B2$ determines the price in $A2$, $B3$ that in $A3$, and so on, but the price in $A1$ is determined by an external price determining bidder $B(m+1)$. The existence of at least one non-internally determined price (by an external price determining bidder or by the seller's starting price) is assured by Lemma 3.5. We want to construct the maximum number of deviations. Because deviations only occur for internal price determining bidders, we assume the maximum number of $m-1$ internal price determining bidders. From Lemma 3.6 we know that either $B(m+1)$ or Bm submits the final bid in the game. If Bm submits the last bid in the game, then we know that he does not prefer a different auction in the final outcome. If $B(m+1)$ submits the last bid in the game in $A1$, the induced increase in the standing bid in $A1$ leads to the preference of Bm for Am over $A1$ in the final outcome. We know that Bm's price determining bid in Am was submitted before his last bid in $A1$. Since Bm's price determining bid in Am causes bidder $B(m-1)$ to prefer $A(m-1)$ by an increment over Am in the final outcome, we know that $B(m-1)$ submitted his bid in Am before Bm (see Lemma 3.3) and his bid in $A(m-1)$ before that in Am, because he wins Am. Arguing like that, we find that $B1$ submitted his bid in $A2$ before $B2$ and his bid in $A1$ before his final bid in $A2$. Since after this final bid of $B1$ in $A2$ the standing bid in $A1$ increased, caused by Bm's last bid in the game, bidder $B1$ cannot prefer $A1$ over $A2$ in the final outcome. From this argument, we find that all winning bidders aside from $B1$ prefer their alternative auction by an increment over the auction they win. With this, we get the maximum amount of deviations that may

[38] The inverse case, $n \geq m$, can be investigated analogously. The same conclusions are valid for the case where not all items are sold due to high starting prices.

cause an inefficiency

$$
\begin{aligned}
v_{12} - p_2^* &= v_{11} - p_1^* \geq v_{1k} - p_k^* \quad \text{for all } k \in M \\
v_{23} - p_3^* + \iota &= v_{22} - p_2^* \geq v_{2k} - p_k^* \quad \text{for all } k \in M \\
&\vdots \\
v_{m-1,m} - p_m^* + \iota &= v_{m-1,m-1} - p_{m-1}^* \geq v_{m-1,k} - p_k^* \quad \text{for all } k \in M \\
v_{m1} - p_1^* + \iota &= v_{mm} - p_m^* \geq v_{mk} - p_k^* \quad \text{for all } k \in M \\
0 &= v_{m+1,1} - p_1^* \geq v_{m+1,k} - p_k^* \quad \text{for all } k \in M \\
0 &\geq v_{m+2,k} - p_k^* \quad \text{for all } k \in M \\
&\vdots \\
0 &\geq v_{nk} - p_k^* \quad \text{for all } k \in M\,.
\end{aligned}
$$

Summing up the left- and right-hand sides of the first m equations, respectively, we get

$$\sum_{j=1}^{m-1}(v_{j,j+1} - p_{j+1}^*) + v_{m1} - p_1^* + (m-1)\iota = \sum_{j=1}^{m}(v_{jj} - p_j).$$

If we now take the inequalities into account in the same way as we did above, we find that $\sum_{j=1}^{m}(v_{jj} - p_j)$ is the value of an efficient assignment. Thus, we constructed a case with the maximum sum of deviations and can restrict the range of the inefficiency to $(m-1)\iota$ in the case $n > m$. With $n \leq m$, the result of an equivalent analysis is an inefficiency of size $(n-1)\iota$. ■

Thus, upwards deviations in prices from $\bar{p}^*$ may cause inefficient assignments. This may only happen under certain constellations of bidders' valuations: accumulated deviations, which may occur on a path of internal price determining bidders, increase the price in j, causing the winner of j under x^{eff} to submit a winning bid in an auction $k \neq j$. At all prices above the reference prices, the number of active bidders is never higher than the number of items sold under the efficient assignment. Thus, all active bidders hold a high bidder position. For components of the graph (separated indifference paths), we can state the following. At prices below the reference prices, the accumulated price deviations cannot lead to an inefficient assignment (in that component): there would be at least one unmatched active bidder, who, with his bid, initiates bidding activity that resolves the inefficient assignments. Furthermore, deviations may lead to a connection of separated indifference paths.

3.2.2.4 Restricting the Size of the Increment

From Propositions 3.3 and 3.4, we know the maximum sum of upwards deviations from $\bar{p}^*$ and we know that, without any deviation in prices, the assignment is efficient. Together with Proposition 3.5, this means that a restriction of the increment ι can assure an efficient assignment x^*. The following proposition establishes this supposition.

Proposition 3.6 *An assignment x^* that results from playing σ^* with $\iota < 1/\min\{n-1, m-1\}$ is efficient, i.e., $x^* = x^{\mathit{eff}}$.*

Proof of Proposition 3.6: We refer to the argument and inequalities in the proofs of Propositions 3.4 and 3.5. From these proofs we know that we can reduce the question to a comparison of bidder payoffs. From Lemma 3.4 we know that for every i either

$$\begin{array}{llcrcll} \exists\, j \text{ such that } x^*_{ij} = 1 & \text{and} \quad u_i(x^*, p^*) & = & v_{ij} - p^*_j + \iota & \geq & v_{ik} - p^*_k & \forall\, k \in M \text{ or} \\ x^*_{ij} = 0 \;\forall\, j & \text{and} \quad u_i(x^*, p^*) & = & 0 & \geq & v_{ik} - p^*_k & \forall\, k \in M. \end{array} \tag{3.3}$$

Following Proposition 3.3, the number of equalities in the first row is at most $\min\{n-1, m-1\}$. Summing up both sides of the inequalities (as explained and done before, for example, in the proof of Proposition 3.5) we find for a comparison of (x^*, p^*) with any alternative feasible assignment x' at the same prices p^* that in the worst case (when the maximum of $\min\{n-1, m-1\}$ winning bidders prefer the auction in which they are the price determining bidder over the auction they win at the final prices p^*)

$$\sum_{i \in N} \sum_{j \in M} v_{ij} x^*_{ij} + \min\{n-1, m-1\} \cdot \iota - \sum_{j \in M} p^*_j \geq \sum_{i \in N} \sum_{j \in M} v_{ij} x'_{ij} - \sum_{j \in M} p^*_j. \tag{3.4}$$

Note that in Equation (3.4) equality is not possible, because the valuations and, therefore, the sums of valuations are integers. The sum of deviations, $\min\{n-1, m-$

$1\} \cdot \iota$, is smaller than one by assumption ($\iota < 1/\min\{n-1, m-1\}$). Thus,

$$\begin{aligned}\sum_{i\in N}\sum_{j\in M} v_{ij}x^*_{ij} + 1 &> \sum_{i\in N}\sum_{j\in M} v_{ij}x^*_{ij} + \min\{n-1, m-1\} \cdot \iota \\ &> \sum_{i\in N}\sum_{j\in M} v_{ij}x'_{ij}.\end{aligned}$$

From this we get

$$1 > \sum_{i\in N}\sum_{j\in M} v_{ij}x'_{ij} - \sum_{i\in N}\sum_{j\in M} v_{ij}x^*_{ij}.$$

Since valuations are integers we can conclude that

$$\sum_{i\in N}\sum_{j\in M} v_{ij}x^*_{ij} \geq \sum_{i\in N}\sum_{j\in M} v_{ij}x'_{ij}. \tag{3.5}$$

Since an efficient assignment x^{eff} exists, it is either $x^* = x^{eff}$ or it exists $x' = x^{eff}$. From Equation 3.5 we see that if $x' = x^{eff}$, then $\sum_{i\in N}\sum_{j\in M} v_{ij}x^*_{ij} = \sum_{i\in N}\sum_{j\in M} v_{ij}x'_{ij}$ and both x^* and x' are efficient. We considered the case of the maximum number of ι-deviations from indifference. Clearly, this argument is valid for all other outcomes (x^*, p^*) with fewer deviations. Thus, we have shown the result.

For a discussion of the case of several efficient assignments, see Appendix A.3. From there we know that in all efficient assignments, all bidders and sellers have the same payoff. ■

Thus, the precondition for a resulting inefficient assignment (i.e., the accumulation of price deviations from $\bar{p}^*$ such that a bidder finally bids in an auction that he is not assigned to under an efficient assignment) cannot be fulfilled with this restriction on the increment ι. From Proposition 3.3 and the knowledge that all deviations from $\bar{p}^*$ are accumulations by $\pm\iota$ from the price in a neighboring auction on an indifference path, we get the following corollary.

Corollary 3.3 *With $\iota < 1/\min\{n-1, m-1\}$, all prices p^*_j are in the range $\bar{p}^*_j + 1 > p^*_j > \bar{p}^*_j - 1$. For $\iota \to 0$, $p^*_j \to \bar{p}^*_j$.*

Thus, by restricting ι, we assure an efficient outcome and prices that are close to the reference outcome. However, the potential instability of an outcome (x^*, p^*) as described in Section 3.2.2.1 is not prevented. In the following, we concentrate on the outcome $(x^{eff}, \bar{p}^*)$ and neglect deviations thereof. Furthermore, we neglect the potential that the bidder winning auction j prefers another auction k by one increment ι. With ι small enough, every outcome from playing according to σ^* is approximately equal to the reference outcome, independent of the realization of the bidding order and the bidders' randomized choice rules (part (3) of σ_i^*).

Note that the price deviations may be reduced by additional selection rules in the strategy σ_i^*. For example, Peters and Severinov (2006) present a rule that assures a unique market price when bidders have homogeneous valuations, i.e., for every i, $v_{ij} = v_{ik}$ for all $j, k \in M$. Their rule does not help in our context. Other rules are conceivable. For example, an indifferent bidder may bid preferably in the auction where his valuation is highest (because higher valuations have a better chance of being part of the efficient assignment), or in the auction with the higher auction number (which would then have to be made public) to avoid inefficiencies caused by deviations on a closed indifference path. These rules avoid some of the deviations but not all of them. For example, inefficiency may occur if we introduce the rule to bid in the auction with the lower auction number, as the upper outcome on the right in the example in Table 3.2 shows. There, $B5$ prefers $A1$ over $A5$ by an increment when he submits his final bid. It seems that in our environment with heterogeneous unit-demand preferences, it is not possible to collect or infer the information needed to differentiate between auctions to assure that the outcome does not deviate from $(x^{eff}, \bar{p}^*)$. Thus, we allow for the deviations but restrict the increment to at least avoid inefficient assignments.

3.2.2.5 Analyzing the Reference Outcome ($\bar{x}^*$, $\bar{p}^*$)

From the former analysis, we know that $\bar{x}^* = x^{eff}$ and, with $\iota < \min\{n, m\} - 1$, all prices p^* are in the range $\bar{p}^* + 1 > p^* > \bar{p}^* - 1$. Furthermore, $(\bar{x}^*, \bar{p}^*)$ is stable. To see this, note that the deviations, which cause unstable outcomes (x,p), do not occur under $(\bar{x}^*, \bar{p}^*)$.

The following definitions of quasi-competitive prices and competitive equilibrium are based on those given in Roth and Sotomayor (1990, p. 209).

Definition 3.9 (Quasi-competitive prices) *A price vector p is called quasi-competitive if a feasible assignment $x \in X$ exists such that*

(1) $x_{ij} = 1 \Rightarrow j \in \arg\max_k\{v_{ik} - p_k\}$

(2) $x_{ij} = 0$ *for all* $j \Rightarrow \max_k\{v_{ik} - p_k\} \leq 0$.

Then, x is *compatible* with p. Thus, at a quasi-competitive price vector every buyer can be assigned to an item in his demand set and a bidder that does not win an auction is either indifferent between winning and not winning at p or strictly prefers not to buy.

Definition 3.10 (Competitive equilibrium) *A feasible outcome* (x, p) *is a competitive equilibrium if*

(1) p *is quasi-competitive,*

(2) x *is compatible with* p*, and*

(3) if $x_{ij} = 0$ *for all* i*, then* $p_j = b_j^0$.

In the literature, a competitive equilibrium is also called Walrasian equilibrium. We call quasi-competitive prices p that are part of a competitive equilibrium (x, p) *competitive prices.*

Proposition 3.7 *The outcome* $(\bar{x}^*, \bar{p}^*)$ *is a competitive equilibrium.*

Proof of Proposition 3.7: Let us first show that the prices $\bar{p}^*$ are quasi-competitive. Following σ_i^*, no bidder wins more than one auction. Thus, $\bar{x}^*$ is feasible. For every bidder i, $v_{ij} - \bar{p}_j^* \geq v_{ik} - \bar{p}_k^*$ for $\bar{x}_{ij}^* = 1$. Thus, j is in i's demand set and Condition (1) of Definition 3.9 is fulfilled for all winning bidders. A losing bidder h does not stop bidding until $v_{hj} - b_j^s \leq 0$. Therefore, $v_{hj} - \bar{p}_j^* \leq 0$ for losing bidders h (Condition (2) of Definition 3.9).

Since in $\bar{x}^*$ every bidder is assigned to an auction in his demand set, $\bar{x}^*$ is compatible with the resulting prices.

An unsold item j has the price $p_j = b_j^0$. This can easily be seen from the rules of the auction since any such auction j has not received any bids. As soon as any bidder submits a bid in an auction the item will be sold because withdrawing bids is not allowed. Thus, no unsold item has a price above (or below) the starting price and Condition (3) is also fulfilled. ■

Gul and Stacchetti (1999, Lemma 6) show that this result holds more generally. They prove for more general substitutes valuations ("gross substitutes") that if p is any competitive price vector and x is any efficient assignment, then (x, p) is a competitive equilibrium.

The remainder of this section reproduces important results that go back to Demange (1982), Leonard (1983), and Demange et al. (1986) for the multiple-auctions game. First, a result of Demange et al. (1986, p. 868, Theorem 1) for their auction mechanism is also valid in the multiple-auctions game.

Proposition 3.8 *Prices $\bar{p}^*$ are minimum competitive prices.*

Proof of Proposition 3.8[39]: Assume that q is a quasi-competitive price vector and $\bar{p}^* \nleq q$. That is, there exists at least one auction j where $\bar{p}_j^* > q_j$. We show that if $\bar{p}^*$ (which is reached when following σ^*) is a vector of quasi-competitive prices (which is the case according to Proposition 3.7), then q may not be competitive. To make clear to which stage in the auction we refer, we denote by $b^{s,t}$ the vector of standing bids at stage t. In the beginning, we have $b_j^0 = v_j^S$. Thus $b^0 \leq q$. Then there exists an auction j and a stage t^+ where we observe for the first time a standing bid above q, i.e., $b_j^{s,t} \leq q_j < b_j^{s,t^+}$ and $b_k^{s,t} = b_k^{s,t^+} \leq q_k$ for all $k \neq j$. When this happens, there exists a high bidder $B^h(j) := i$ and a bidder who determines the standing bid b_j^{s,t^+} that we denote by h (remember, $h \neq i$). Bidder i has bid at least b_j^{s,t^+} and when he chose this bid his second best auction, denoted by l, was at most as good as auction j. Thus, if nothing has changed in that second best auction l, he may be indifferent between auction j at price b^{s,t^+} and auction l, or prefer j: $v_{ij} - b_j^{s,t^+} \geq v_{il} - b_l^{s,t^+}$. Additionally we have $v_{ij} - b_j^{s,t^+} < v_{ij} - q_j$ and $v_{ik} - b_k^{s,t^+} \geq v_{ik} - q_k$ for all $k \neq j$. Thus, $v_{ij} - q_j > v_{ij} - b_j^{s,t^+} \geq v_{ik} - b_k^{s,t^+} \geq v_{ik} - q_k$. Bidder h has bid b_j^{s,t^+} in auction j in or before round t^+. He chose his bid in auction j such that he was indifferent between $v_{hj} - b_{hj}$ $(= v_{hj} - b_j^{s,t^+})$ and his second best auction at the current standing bid. So we know that $v_{hj} - b_j^{s,t^+} \geq v_{hk} - b_k^{s,t^+}$ for all $k \neq j$. Relating this to q, we find $v_{hj} - q_j > v_{hj} - b_j^{s,t^+} \geq v_{hk} - b_k^{s,t^+} = v_{hk} - q_k$. It follows that we have two bidders who both prefer winning auction j to winning any other auction when the price in auction j is higher than q_j and prices in all other auctions k are lower than or equal to q_k. Thus, at q there also exists at least two bidders who strictly prefer winning auction j to winning any other auction and, therefore, q cannot be a quasi-competitive price vector. ■

[39]The proof follows the idea of Demange et al. (1986) as stated in Roth and Sotomayor (1990).

Corollary 3.4 $(x^{\textit{eff}}, \bar{p}^*)$ *is the bidder-optimal competitive equilibrium.*

The competitive equilibrium with the lowest price is the bidder-optimal equilibrium. That is, at $(x^{\textit{eff}}, \bar{p}^*)$ all bidders get their maximum payoff compared to all other competitive equilibria. For more on the existence of a bidder-optimal competitive equilibrium, see Shapley and Shubik (1971) or Roth and Sotomayor (1990).

Another important result that may now be reproduced for the multiple-auctions game is that the minimum competitive prices are Vickrey prices.

Prices Resulting from σ^* are Vickrey Prices We claim that prices $\bar{p}^*$ are Vickrey prices, i.e., every bidder has to pay a price that equals the loss of the other bidders caused by his entry into the game. In other words, prices are such that i's utility is equal to the increase in the value of the grand coalition caused by his entry.

Remember, the value of a bidder-seller coalition is[40]

$$c(S,T) = \max_{x \in X} \sum_{i \in S} \sum_{j \in T} (v_{ij} - v_j^s) \cdot x_{ij} \quad \text{for } \emptyset \neq S \subseteq N \text{ and } \emptyset \neq T \subseteq M$$

(see Section 2.1, p. 22). Let $c_{-i}(N,M)$ denote the value of the grand coalition without i's contribution to $c(N,M)$. That is,

$$c_{-i}(N,M) = \begin{cases} c(N,M) - v_{ij} & \text{for } x_{ij}^{\textit{eff}} = 1 \\ c(N \backslash \{i\}, M) & \text{if } x_{ij}^{\textit{eff}} = 0 \text{ for all } j \in M. \end{cases}$$

With this, we define bidder i's Vickrey payment p_i^V as follows (see Definition 2.18).

Definition 3.11 (Vickrey payment p_i^V) *Bidder i's Vickrey payment is*

$$p_i^V = c(N \backslash \{i\}, M) - c_{-i}(N,M).$$

That is, for winning bidders i the price $p_i^V = c(N \backslash \{i\}, M) - (c(N,M) - v_{ij})$ for $x_{ij}^{\textit{eff}} = 1$ and for losing bidders i $p_i^V = 0$. Because i's payoff is given by $u_i(p^V, x^{\textit{eff}}) = v_{ij} - p_i^V$ for $x_{ij}^{\textit{eff}} = 1$ or $u_i(p^V, x^{\textit{eff}}) = 0$, the Vickrey payoff is represented by the following formula.

[40] In the following, we write $c(S,T)$ instead of $c(S \cup T)$ where $S \subseteq N$ and $T \subseteq M$.

Definition 3.12 (Vickrey payoff) *Bidder i's Vickrey payoff is*

$$u_i(p^V, x^{\text{eff}}) = c(N, M) - c(N\backslash\{i\}, M).$$

Since each bidder wins at most one auction, we can in this environment relate the Vickrey prices uniquely to auctions. Therefore, we also define Vickrey prices p_j^V.

Definition 3.13 (Vickrey prices p_j^V) *Prices p^V are Vickrey prices if*

- $p_j^V = c(N\backslash\{i\}, M) - (c(N, M) - v_{ij})$ *for all* $j \in M$ *and for the* $i \in N$ *with* $x_{ij}^{\text{eff}} = 1$, *and*
- $p_j^V = v_j^S$, *otherwise.*

In this definition, we artificially introduce Vickrey prices p_j^V for items j that are not sold. Note that we similarly defined $\bar{p}_j^* = v_j^S$ if j is not sold.

We use Vickrey payments and Vickrey prices p^V interchangeably when it is clear from context which we mean.

Proposition 3.9 *The prices $\bar{p}^*$ are Vickrey prices: $\bar{p}_j^* = p_j^V$ for j.*

The proof follows that of Demange (1982). We draw upon the presentation in Roth and Sotomayor (1990). Another early proof of incentive compatibility is due to Leonard (1983).

Proof of Proposition 3.9: Suppose x^{eff} is an optimal assignment for (N, M, V,v^S). Construct a graph with nodes $N \cup M$ and a directed edge from $i \in N$ to $j \in M$ if $x_{ij}^{\text{eff}} = 1$. If bidder i is indifferent between the auction j that he wins and a different auction k at $\bar{p}^*$, then add a directed edge from k to i.[41] WE consider paths given by directed edges in this graph.

Let j be an auction with price $\bar{p}_j^*$ larger than b_j^0 on such a path. Then, there exists a further directed path starting at j and ending at a bidder who buys nothing or at an auction l with price $\bar{p}_l^*$ equal to its reservation price b_l^0. To see this, consider first the bidders on the path. By definition of the path, it ends if a losing bidder is reached (who, being indifferent between winning an auction and a payoff of zero,

[41] For example, if bidder i determines the price in auction k, then he is indifferent in this way. There may also be other indifferent (and, depending on the bidding sequence, potentially price determining) bidders.

is a potential external price determining bidder) and it does not end if the bidder wins an auction (he is indifferent between his payoff and another auction and, thus, is a potential internal price determining bidder). Thus, the path cannot end at an internal price determining bidder. Next, consider an auction l with $\bar{p}^*_l > b^0_l$. In such an auction, a price determining bidder exists, and, thus, the path may not end. On the other hand, if l is sold at $\bar{p}^*_l = b^0_l$ then a bidder who is indifferent may exist but the path may also end at auction l. An unsold auction l has $\bar{p}^*_l = b^0_l$ and no incoming link. The node may be isolated or an indifferent bidder exists. Thus, either a losing bidder or an auction l with $\bar{p}^*_l = b^0_l$ is at the end of a path starting at j.

On the other hand, such a path cannot start at an auction j with price larger than b^0_j. Otherwise, j would have a price determining bidder but no winner. However, for every auction with price larger than b^0_j we have $B^h(j) \neq j$ and thus, a winning bidder.

All bidders that are not reached by any path from j are not interested in buying an item on the path starting at j at price $\bar{p}^*$. Otherwise, they would have an outgoing (as buyers) or incoming (as potential price determining bidders) edge with an auction on the path.

Assume w.l.o.g. that bidder i_1 is the winner of auction j_1 with $p_{j_1} > b^0_{j_1}$. Thus, there exists at least one directed path from j_1 that ends at a bidder i_k or an auction j_k, $k \geq 1$, as described above. Suppose that the path consists of the following nodes in the given order:

- $j_1, i_2, j_2, i_3, \ldots, j_{k-1}, i_k$, or
- $j_1, i_2, j_2, i_3, \ldots, j_{k-1}, i_k, j_k$ and i_k wins j_k at price $\bar{p}^*_{j_k} = b^0_{j_k}$

Now consider the assignment x' under $(N\backslash\{i_1\}, M, V, v^S)$ with $x'_{i_2 j_1} = 1$, $x'_{i_3 j_2} = 1, \ldots, x'_{i_k j_{k-1}} = 1$ and leave j_k unassigned if it is on the path (see Figure 3.6 for an illustration).[42] In addition, all other matchings in x' are equal to the original assignments under x^{eff} for all $i \in N\backslash\{i_1, \ldots, i_k\}$.

We continue by showing that $(x', \bar{p}^*)$[43] is a stable outcome of the game without i_1, thus it is an optimal assignment (by Corollary 8.8 of Roth and Sotomayor (1990), p. 207) and the related sum of payoffs is equal to the value of the characteristic function. By the definition of stability (see Definition 2.23), the outcome $(x', \bar{p}^*)$ is

[42] For example, this reassignment x' might be to assign every good to the bidder who determines its price under x^{eff}.

[43] Note that price vector $\bar{p}^*$ is unchanged.

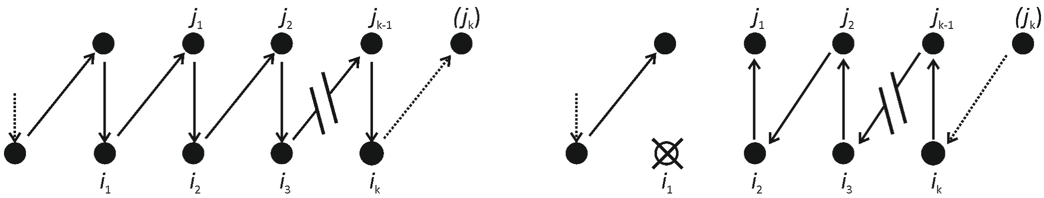


Figure 3.6: Illustration of indifference paths in the proof of Proposition 3.9.

stable if all payoffs are larger than or equal to zero and $u_i' + u_j^{S,\prime} \geq d_{ij}$ for all i, j, with $u_i' := u_i(x', \bar{p}^*)$ and $u_j^{S,\prime} := u_j^S(x', \bar{p}^*)$. Note that under $(x', \bar{p}^*)$ $u_i' = u_i^*$ for all $i \neq i_1$ because those bidders that are reassigned under x' are indifferent between their assigned item under $(x^{eff}, \bar{p}^*)$ and $(x', \bar{p}^*)$. With this, the individual rationality condition is clearly satisfied because under $(x^{eff}, \bar{p}^*)$ all bidders $i \neq i_1$ and all sellers have payoffs larger than or equal to zero and their payoff does not change. Secondly, assume that i wins k under x' and j under x^{eff}. Then $u_i' + u_l^{S,\prime} = v_{ik} - \bar{p}_k^* + \bar{p}_l^* - v_l^S = v_{ij} - \bar{p}_j^* + \bar{p}_l^* - v_l^S \geq v_{il} - \bar{p}_l^* + \bar{p}_l^* - v_l^S = d_{il}$ because $v_{ik} - \bar{p}_k^* = v_{ij} - \bar{p}_j^* \geq v_{il} - \bar{p}_l^*$ for all l, including $l = j$ and $l = k$ (for k we have $u_i' + u_k^{S,\prime} = d_{ik}$). Therefore, $(x', \bar{p}^*)$ is stable and thus x' is an optimal assignment. We find that

$$c(N\backslash\{i_1\}, M) = \sum_{i \in N\backslash\{i_1\}, j \in M} d_{ij} x'_{ij}.$$

Since $u_i' + u_j^{S,\prime} = d_{ij}$ for $x'_{ij} = 1$, $u_i' = u_i^*$ for all $i \neq i_1$, and $u_j^{S,\prime} = u_j^{S,*}$ for all j, it follows that

$$\sum_{i \in N\backslash\{i_1\}, j \in M} d_{ij} x'_{ij} = \sum_{i \in N\backslash\{i_1\}} u_i' + \sum_{j \in M} u_j^{S,\prime} = \sum_{i \in N} u_i^* - u_{i_1}^* + \sum_{j \in M} u_j^{S,*} = c(N, M) - u_{i_1}^*$$

and thus we have

$$u_{i_1}^* = c(N, M) - c(N\backslash\{i_1\}, M).$$

Together with $u_{i_1}^* = v_{i_1 j_1} - p_{j_1}$ this results in

$$\begin{aligned} \bar{p}_{j_1}^* &= v_{i_1 j_1} - u_{i_1} = v_{i_1 j_1} - c(N, M) + c(N\backslash\{i_1\}, M) \\ &= c(N\backslash\{i_1\}, M) - (c(N, M) - v_{i_1 j_1}) = p_{j_1}^V. \end{aligned}$$

■

Let us mention some properties of the cooperative game associated with the multiple-auctions game. The core of the multiple-auctions game equals that of the assignment game. All payoff vectors on the grid determined by the increment size are achievable without side payments because the only necessary transfers in the core are the prices. Since the bidding rules of the multiple-auctions game restrict the set of prices, the remaining core payoff vectors are only achievable via side-payments. However, all corners (extreme points of the simplex) of the core do not require side-payments. This holds in particular for the bidder-optimal (and the seller-optimal) outcome in the core.

3.2.3 Equilibrium Analysis

We show that the strategy combination σ^* is a perfect Bayesian epsilon-equilibrium (ε-PBE) of the multiple-auctions game Γ^a.

If not stated otherwise, all outcomes considered in this section are reference outcomes (see Definition 3.8). This implies that we assume $\iota < 1/\min\{n-1, m-1\}$ as discussed in Section 3.2.2.4. Deviations from the reference outcome, which are due to the random factor in bids and bidding order, are analyzed in detail in Section 3.2.2. All prices may deviate from the reference prices by at most $\pm\,\iota \cdot \min\{n-1, m-1\}$, but the assignment is efficient. As a consequence, at outcome (x^*, p^*) bidders may exist that prefer to win a different auction by ι. This deviation might be prevented if they changed their randomly selected auction in case of indifference (part (3) of σ_i^*). On the other hand, a bid in an auction outside of demand set D_i may prevent a deviation caused by another bidder, or it may cause downwards price deviations. We do not analyze such strategic options, which may have a positive probability to minimally increase a bidder's payoff. We account for the deviations by solving for an epsilon-equilibrium (see Definition 2.15). The deviations analyzed in Section 3.2.2.4 determine the size of epsilon.

In order for an assessment (σ^*, μ) to constitute a PBE of the multiple-auctions game Γ^a, it is sufficient to require beliefs to be updating consistent and to apply the one-shot-deviation principle to σ^*.[44] That is, for all information sets H_i of i we have to show that σ_i^* is at least as profitable against σ_{-i}^* as any other strategy of i that

[44]For definitions of these concepts, see Section 2.1.

deviates from σ_i^* only at the information set H_i. Since players are symmetric, we consider a representative bidder i.

Prior to the equilibrium analysis, we explain some details about our assumptions on beliefs at information sets. This is merely for the sake of completeness.

3.2.3.1 Beliefs

Beliefs μ_{H_i} of bidder i at his information sets $H_i \in \mathcal{H}_i$ are probability distributions over the nodes in H_i. We do not specify these distributions explicitly, but merely describe conditions that these probability distributions have to fulfill.

Let us characterize nodes y in an information set $H(y)$. All nodes y in $H(y)$ are described by the same values for v^S, b^s, B^h, and B^{PD}. That means that nodes in an information set differ only with respect to the remaining characteristics V_{-i}, o, b^h, and B^a. The relevant beliefs are therefore probability distributions over nodes that differ with respect to the latter. Bidder i updates his beliefs (by Bayesian updating) at H_i according to the relevant information gained during the course of the game.

In game Γ^a, the updating of beliefs and the consistency requirement for beliefs on the equilibrium path result in the following restrictions on the beliefs of bidder i at information set H_i^t.

- Beliefs about current high bids:
  - $\text{Prob}\{b_j^{h,t} \geq b_j^{s,t}\} = 1$ for all auctions j with $B^{h,t}(j) \neq i$.
  - $\text{Prob}\{b_j^{h,t} = b_j^{s,t}\} = 1$ for all auctions j with $B^{h,t}(j) = j$.
- Beliefs about valuations:
  - $\text{Prob}\{v_{hj} \geq b_j^{s,t}\} = 1$ for all h and j with $B^{h,t}(j) = h$, $h \neq i$.
  - $\text{Prob}\{v_{hj} \geq \tilde{b}_{hj}\} = 1$ for all h and j with $B^{h,t}(j) \neq h$, $h \neq i$ and $\tilde{b}_{hj} \in \mathbb{N}$, where $\tilde{b}_{hj}$ denotes the highest bid of bidder h that i observed in auction j up to the current stage t. Thus, $0 \leq \tilde{b}_{hj} \leq b_j^s$.
- Beliefs about active bidders:
  $\text{Prob}\{B_h^{a,t} = 1\} > 0$ for $h \neq i \in N$ if $\text{Prob}\{v_{hj} \geq b_j^{s,t}\} > 0$ for at least one auction j. That is, i assigns positive probability to bidder h still being active if he assigns positive probability to h having a valuation above the standing bid in an auction.

- Beliefs about bidding orders: The beliefs assign positive probability to all bidding orders that are consistent with o_i^t.

Note that all these restrictions on beliefs are consistent with bidder i's observations during the game. In order for beliefs to be consistent with σ^*_{-i}, bidder i also has to include information about an opponent's differences between his valuations, which this opponent uses to determine his bids.[45] Since we will argue that our PBE is a PBE for many realizations of beliefs and that the exact specification is not important, we do not analyze beliefs in any more detail. In particular, we do not specify concrete probability distributions over nodes in H_i. *All probability distributions that are compatible with the described restrictions and, more restrictively, with σ^* and updating consistency, are feasible systems of beliefs μ for our equilibrium analysis.*

Beliefs off the equilibrium path cannot be inferred from opponents' equilibrium behavior. The restrictions on beliefs explicitly stated above are, however, compatible with behavior according to σ^*_{-i} of appropriately modified opponents. In other words, the objectively observable components $v_j^S = b_j^0$, b^s, B^h, and B^{PD} at some information set off the equilibrium path are compatible with bidders who play according to σ^*_{-i} with different valuations than those that are really drawn by nature. The modification assigns the opponents these different valuations. It is possible, however, that the bidding behavior of these modified bidders at earlier stages (before H_i) can not be reconciled with σ^*_{-i}. At those information sets H_i, we assume an adjustment of beliefs such that they fulfill the restrictions above but do not have to include the information about the differences in opponents' valuations. We demand updating consistency of bidder i's beliefs at all his information sets that follow such an information set H_i with adjusted beliefs.

3.2.3.2 A Perfect Bayesian Epsilon-Equilibrium (ε-PBE)

In this section, we prove that σ^* combined with the appropriate beliefs constitutes a ε-PBE. To explain why the details of beliefs do not play a role, we introduce the concept of ex-post equilibrium before moving on to the equilibrium analysis.

[45] For example, if there are two items and bidder $h \neq i$ bids $b_{h1} = 4$ in auction $A1$ when the standing bids are $b^s = (1,1)$, then i's beliefs about h's valuations when h is outbid and i learns his bid should reflect the fact that h determined this bid either from $v_{h1} - 4 = v_{h2} - 1$ or from $v_{h1} - 4 = 0$ when $v_{h1} \geq 4$.

With strategy σ^*, the bidding order determined by nature has no influence on the reference outcome. If an information set off the prospective equilibrium path is reached, the outcome (x, p) that results from further play of σ^* depends on the node in $y_{H_i} \in H_i$ that is reached. We consider the strategies σ_h^* of players $h \neq i$, which depend mainly on valuations v_h. Thus, we write $u_i(x, p) = u_i(\sigma_i, \sigma_{-i}, v^S, V, y_{H_i})$ and include valuations explicitly, but have the other factors specified above implicitly included in y_{H_i}. If $\sigma_i^*(v_i)$ is part of a PBE, then

$$\sigma_i^*(v_i) \in \max_{\sigma_i} \mathrm{E}_{\mu_{H_i}} \left(u_i(\sigma_i, \sigma_{-i}^*, v^S, v_i, V_{-i}, y_{H_i}) \right) \tag{3.6}$$

for all v_i and at all information sets H_i with beliefs μ_{H_i}. That is, for given beliefs μ_{H_i} over nodes $y_{H_i} \in H_i$, σ_i^* maximizes i's expected payoff if, from H_i on, all opponents play according to σ_{-i}^*. We do not calculate the expected payoffs, but rather show that σ_i^* is a best reply against each realization of v^S and V_{-i} for all types v_i, starting at an arbitrary node $y_{H_i} \in H_i$. Concretely,

$$\sigma_i^*(v_i) \in \max_{\sigma_i} u_i(\sigma_i, \sigma_{-i}^*, v^S, v_i, V_{-i}, y_{H_i}) \tag{3.7}$$

for all v_i and V_{-i} at all nodes y_{H_i} in information sets H_i. Clearly, (3.6) follows from (3.7) for i's decisions at all nodes $y_{H_i} \in H_i$. Then, σ_i^* is a best reply against σ_{-i}^* for all realizations of v^S, V and o. An equilibrium with this property is also called an *ex-post equilibrium* (see, e.g., Crémer and McLean, 1985). In the words of Holzman and Monderer (2004, p. 88), a strategy builds a symmetric ex-post equilibrium if it has the property that "if an agent assumes that the other agents use this strategy, it is optimal for him to use it as well, regardless of the other agents' valuations and of the number of them who actually participate in the auction."[46] The set of ex-post Nash equilibria of a game is a subset of the Bayes-Nash equilibria and a superset of the ex-post dominant strategy equilibria, but in general not a superset of the dominant strategy equilibria (Crémer and McLean, 1985). Kalai (2004) notes that the ex-post Nash equilibria of a simultaneous move game are a superset of its extensively robust equilibria. We extend the definition of ex-post Nash equilibrium to the case

[46] Note that strategy σ_i^* does not depend on the number of bidders and sellers. We made no assumption about the observability of these numbers at the beginning of the game. But it is clear that a bidder does not know the number of active bidders and he does not know how many bidders have valuations below v^S.

of an extensive form game with imperfect information in the manner of Peters and Severinov (2006). Since an ex-post equilibrium does not depend on beliefs, we do not characterize them in more detail but simply assume that they have the consistency properties described above for feasible systems of beliefs.

With strategy σ^* and the system of beliefs μ we can now specify the ε-PBE of the multiple-auctions game Γ^a.

Theorem 3.1 (ε-PBE of Γ^a) *The symmetric bidding strategies σ_i^* for all bidders $i \in N$ and the system of beliefs μ constitute a perfect Bayesian epsilon-equilibrium of the multiple-auctions game Γ^a.*

We begin the proof of Theorem 3.1 with some general considerations. Then, we show that a unilateral deviation by bidder i from σ^* cannot lead to an improvement for i. The next step considers all information sets off the equilibrium path, that is, information sets that are not reached if all players play according to σ^*. Taking the situation at the off-equilibrium information set as starting point, a unilateral deviation by bidder i from σ^* also does not pay. Finally, we use the one-shot-deviation principle to argue that then no combination of deviations by bidder i can be profitable.

Lemma 3.7 *If bidder i bids in auction j, this bid influences bids of bidders $h \neq i$ and $B^h(j) \neq h$ who follow strategies σ_h^* as follows:*

- *The probability that bidder h bids in auction j decreases or is unchanged. The bid of bidder h in case he bids in auction j is unchanged.*
- *The probability that bidder h bids in auction $k \neq j$ increases or is unchanged. The bid of bidder h in case he bids in an auction $k \neq j$ increases or is unchanged.*

Proof of Lemma 3.7: The probability of a bid in auction j by bidder h depends on the probability that $j \in \arg\max_{k \in M}\{v_{hk} - b_k^s\}$. Clearly, this probability is negatively influenced by an increase in b_j^s. When bidder i bids in auction j, b_j^s increases or does not change. Thus, the probability that bidder h bids in auction j decreases or is not influenced by i's bid. On the other hand, this negative influence on the probability that $j \in \arg\max_{k \in M}\{v_{hk} - b_k^s\}$ has the reverse impact on auctions $k \neq j$. The probability that bidder h bids in $k \neq j$ increases or remains the same.

According to strategy σ_h^*, the magnitude of bid b_{hk} of bidder h in auction k is $v_{hk} - \Delta_{h(2)}$ or $b_k^s + \iota$. The bid $b_{hk} = b_k^s + \iota$ is clearly not influenced by an increase in the standing bid b_j^s. In the case $k = j$, the bid b_{hj} is not influenced by a change in b_j^s because choosing to bid in j implies $\Delta_{h(1)} = \Delta_{hj} \neq \Delta_{h(2)}$ if h follows σ_h^*. In the case $k \neq j$, it may be that $\Delta_{h(2)} = \Delta_{hj}$. If the standing bid b_j^s increased when i bid in j, the bid b_{hk} would then increase too. It may also be the case that the increase in b_j^s causes bidder h's bid to change from $b_{hk} = b_k^s + \iota$ to $b_{hk} = v_{hk} - \Delta_{h(2)}$ if $\Delta_{hj} = \Delta_{h(1)}$ becomes $\Delta_{hj} = \Delta_{h(2)}$ (due to the relative impairment of auction j for h). The argument in this case is the same as before. $\square$

From Lemma 3.7 we can conclude that a bid by bidder i in auction j fosters competition in auctions $k \neq j$ and reduces competition in auction j or has no effect. Bidder i does not know if his bid will have an effect or not. By bidding in j, i makes auction j less attractive and the other auctions more attractive.

Unilateral Deviation on the Equilibrium Path We state and prove the following lemma that considers only the potential equilibrium path.

Lemma 3.8 *Strategy σ_i^* is a best reply of i against σ_{-i}^* for all $i \in N$.*

Proof of Lemma 3.8: The proof follows the idea of Gul and Stacchetti (2000).

Bidding such that $v_{ij} - p_j < 0$ for $x_{ij} = 1$ or quitting the auctions game when a price is below i's valuation can never be optimal because σ_i^* assures $u_i \geq 0$. Therefore, we only consider positive payoffs for i in what follows.

Consider unilateral deviations by i that result either in another stable outcome or in an unstable outcome. Since the outcome $(\bar{x}^*, \bar{p}^*)$ is the bidder-optimal core outcome, there is no other core outcome where any bidder has a higher payoff and, thus, also no stable outcome where any bidder has a higher payoff.[47] Therefore, a unilateral deviation that results in a stable outcome cannot be better for i.

Next, suppose i's unilateral deviation leads to unstable payoffs $(u(x', p'), u^S(x', p'))$ associated with an unstable outcome (x', p'). The instability may only be due to $0 \leq v_{ik} - p'_k < \max_{l \neq k \in M}\{v_{il} - p'_l\}$ for $x'_{ik} = 1$ because σ_h^* avoids instabilities for bidders $h \neq i$ in the reference outcome. Let $j \in M$ be the auction that i wins in the outcome $(\bar{x}^*, \bar{p}^*)$, i.e., $\bar{x}_{ij}^* = 1$. Furthermore, replace v_i by $\tilde{v}_i$ with $\tilde{v}_{il} = v_{il}$ for $l \neq k$ and $\tilde{v}_{ik} = \lceil \max_{l \in M}\{v_{il} - p'_l\} + p'_k \rceil > v_{ik}$. That is, $\tilde{v}_{ik}$ is an integer

[47] Remember that we analyze reference outcomes and that, therefore, the set of stable outcomes equals the set of core outcomes.

equal to $\max_{l\in M}\{v_{il} - p_l\} + p'_k$ or the next highest integer. With valuation $\tilde{v}_i$, the outcome (x', p') is stable and prices are competitive prices, but possibly not minimum competitive prices. Therefore,

$$p'_k \geq \tilde{p}^*_k, \tag{3.8}$$

where $\tilde{p}^*_k$ denotes the minimum competitive price in the economy where i's valuation is replaced by $\tilde{v}_i$. Let $\tilde{p}^V_i$ and $\tilde{p}^V_k$ denote i's Vickrey payment and the Vickrey price for item k if i bids $\tilde{v}_i$, respectively. They are identical because $\tilde{x}^*_{ik} = 1$. Since the Vickrey price equals the minimum competitive price, we have

$$\tilde{p}^*_k = \tilde{p}^V_k = \tilde{p}^V_i. \tag{3.9}$$

On the other hand, if i bids according to σ^*_i and v_i in the multiple-auctions game, he wins j in the outcome $(\bar{x}^*, \bar{p}^*)$ and thus also in the Vickrey outcome. Since $\bar{p}^*_j$ is the minimum competitive price, we have

$$\bar{p}^*_j = p^V_j = p^V_i. \tag{3.10}$$

Now, consider bidding $\tilde{v}_i$ instead of v_i as a deviation from truthful bidding in a Vickrey auction. Since bidding truthfully is a weakly dominant strategy, we know that

$$v_{ik} - \tilde{p}^V_i \leq v_{ij} - p^V_i. \tag{3.11}$$

From (3.8), (3.9), (3.11), and (3.10) it follows that

$$v_{ik} - p'_k \leq v_{ik} - \tilde{p}^*_k = v_{ik} - \tilde{p}^V_i \leq v_{ij} - p^V_i = v_{ij} - \bar{p}^*_j .$$

Thus, $u_i(x', p') = v_{ik} - p'_k \leq v_{ij} - \bar{p}^*_j = u_i(\bar{x}^*, \bar{p}^*)$.

Hence it follows that no unilateral deviation from σ^*_i is profitable against σ^*_{-i}.

□

We call the paths in the game tree that result from the play of σ^* *equilibrium paths.*

Unilateral Deviation at an Information Set H_i Off the Equilibrium Path

We consider an arbitrary information set H_i and investigate if bidding according to σ^*_i at H_i is a best reply to σ^*_{-i}. That is, we assume that all bidders $h \neq i$ follow σ^*_h

from H_i on. As alternatives for bidder i, we consider only behavior strategies that deviate from σ_i^* solely at H_i.

An information set H_i at a stage t of the game contains nodes which are characterized by the same standing bids b^s, high bidders B^h, price determining bidders B^{PD}, and history $h(i)^t$.

At an information set H_i where i has a choice between actions, he is still active (i.e., $i \in B^a$) and he does not hold any high bidder position. Not bidding when σ_i^* prescribes i to bid results in quitting the game with a payoff of zero, but an increase in i's payoff at no risk (of decreasing it) is possible. Thus, bidding in this case is optimal. On the other hand, bidding when σ_i^* prescribes i not to bid (i.e., if $\Delta_{i(1)} < 0$) means risking a loss without having a chance of a positive payoff. Then, i's best reply is to quit the game (by not bidding).

Before we discuss where and how much to bid in deviations from σ_i^*, we look at differences between information sets on and off the equilibrium path. If some H_i off the equilibrium path is reached, at least one bidder has bid differently than prescribed by σ^*. This may result in a constellation of price determining bidders, standing bids, high bids, high bidders, or a set of active bidders not consistent with an equilibrium path. The consequences of each of these elements on further play by bidders $h \neq i$ are as follows. The identity of a price determining bidder has no impact. The standing bid may affect the selection of an auction and the height of future bids. Since bid withdrawals are not allowed, the current standing bids at H_i have to be considered as the given lowest possible prices. For the same reason, the current high bids b^h at H_i cannot decrease. If an auction is not in the demand set of its high bidder or if the standing bid increases such that the item leaves the demand set, we call the preliminary assignment of the high bidder to the auction a (preliminary) mis-assignment.

The strategy σ_h^* avoids further mis-assignments of any bidder h. None of i's actions can induce a bidder $h \neq i$ to become mis-assigned. Therefore, no further mis-assignment may occur.

A preliminary mis-assignment of h to j at H_i may lead to the omission of bids of some of the mis-assigned bidder h compared to the situation without his mis-assignment. Moreover, it may cause more and higher bids in auctions $l \neq j$ by other bidders (Lemma 3.7). Either it becomes a final mis-assignment if h is not outbid or h is outbid and the mis-assignment is dissolved.

Let us consider i's impact on dissolving the mis-assignment of h in j. We omit the case in which σ_i^* prescribes i to outbid h in j because this is considered later. Instead, we concentrate on the possibility that i can prevent a bidder g from outbidding h. Note that it can never be profitable for i to induce a bidder g to outbid a mis-assigned bidder h because this may lead to additional bids by h. Now, if i wants to prevent a bid of g that would outbid h in j, he has to decrease the probability that g bids in j or decrease g's bid in j. From Lemma 3.7, the probability that g bids in j decreases with a bid in j and increases with a bid in $k \neq j$. Thus, i cannot decrease this probability with a bid in k. A bid in j that does not outbid h but increases the standing bid prevents bids that would be below h's high bid anyway (and thus have no impact on the relevant bid of g). Not bidding is not a profitable option for i either because this is an exit decision. Thus, i has no possibility of reducing or preventing any bid that would correct a mis-assignment.

Before, we classified some possible deviations from σ_i^* as unprofitable. Now, we consider the remaining conceivable deviations by taking a different point of view. We know that a deviation on the equilibrium path does not pay. For the rest of the analysis, we describe the game tree following H_i as part of an equilibrium path of a manipulated game $\tilde{\Gamma}^a$, where $\tilde{V}_{-i}$ replaces V_{-i} and $\tilde{N}$ replaces N.

First, all bidders that are not active anymore at H_i are deleted from N if they are not price determining bidder in any auction. Deleting them is possible because they do not play a role on any path beginning at H_i. If a bidder h with $h \notin B^a$ is a price determining bidder in one or several auctions j, we replace him by a virtual bidder $\tilde{h}_j$ for each such auction j with $v_{\tilde{h}_j j} = b_j^s$ and $v_{\tilde{h}_j k} = 0$ for all items $k \neq j$. Like the original bidder h, these bidders are inactive, but considering game $\tilde{\Gamma}^a$, bidding up to b_j^s and quitting when selected to bid next time, is prescribed for $\tilde{h}_j$ by strategy $\sigma_{\tilde{h}_j}^*$. Those current price determining bidders h at H_i who are still active are supplemented by virtual bidders $\tilde{h}_j$ with valuations as described before.[48] Thus, the current standing bid b_j^s at H_i in each auction is characterized as the result of a bid according to σ^*.[49]

[48]Of course, these standing bids do not necessarily contradict σ^*. To avoid further differentiation, we introduce more virtual bidders $\tilde{h}$ than necessary.

[49]Note that this kind of manipulation is incompatible with the assumption that bidders can observe the identity of the high bidders and price determining bidders. We abstract from this problem by either assuming that they cannot observe the identities (which is not consistent with the common practice in most auctions) or by arguing that the identity does not play a role in the strategy σ^* and that a deviation to a strategy that takes bidders' identities into account does not pay off.

Similarly, since we do not manipulate the valuations of the active bidders, their bids on the paths starting at H_i are not changed in $\tilde{\Gamma}^a$ compared to Γ^a. Thus, we have put H_i on the equilibrium path by manipulating the set of bidders and the valuation matrix. The newly introduced bidders are inactive in the remainder of the game.

At information sets off the equilibrium path, mis-assigned high bidders, that is, bidders that are current high bidder in auctions that are not in their demand set, and deviating high bids may occur. A deviation in a high bid of h is one that cannot have been calculated according to σ_h^*. If the high bid is too low, bidder h will either be outbid and can bid again or he will win with this bid. In both cases, the deviation is not relevant to the resulting prices and winners. Thus, we forgo a further examination of this case.[50] The magnitude of the bid is irrelevant if the auction is in the demand set of the winner at the final price. Thus, even if the bid is too high and bidder h wins the auction, there is not necessarily a mis-assignment at the end. However, if h is still the high bidder when the standing bid in j increases[51] by so much that auction j is no longer in his demand set, he becomes mis-assigned. Although the classification as a mis-assignment is caused by bids submitted after H_i, this is similar to mis-assigned bidders at H_i. Thus, we deal with both deviations in an analogous manner as follows. For integrating a mis-assigned bidder h at j into the considerations, we again introduce a virtual bidder $\tilde{g}$ with $v_{\tilde{g}j} = b_j^h$ (b_j^h denoting the high bid at H_i) and $v_{\tilde{g}k} = 0$ for $k \neq j$. If h is not outbid during the following bidding process, he is deleted from the set of bidders.[52] However, if h is outbid during the following bidding process, h is not deleted. The mis-assignment may prevent the original bidder h from bidding. However, if he is outbid, he is still an active bidder (because he had a high bidder position) and he has the opportunity submit new bids. These bids are consistent with those that the original bidder h would have submitted if he was not mis-assigned. However, we assume that the bidding order does not give him the right to bid until he is outbid in j.

With this manipulation, we introduce the bids that were too high into the game $\tilde{\Gamma}^a$ such that the future bidding behavior of h is guaranteed to be described by σ_h^*. However, there is one decisive difference from the manipulations considered for the

[50] In fact, a strategy that equals σ_i^* except for the height of the bids may also be an equilibrium strategy. Our argument is also valid for bids between $b_j^s + \iota$ and b_{ij} as prescribed by σ_i^*.

[51] Of course, the standing bids in the other auctions may also increase.

[52] His price determining bids and his winning bid are assured by the virtual bidders $\tilde{h}$ and $\tilde{g}$.

case of standing bids and price determining bidders: if a bidder is mis-assigned, the kind of manipulation of the matrix depends on whether h is outbid. That is, at information set H_i it is not yet determined which manipulation for h will be conducted. Therefore, which manipulated valuation matrix is relevant depends on the bidding behavior of the other bidders, including i.

As we argued above, i cannot prevent other bidders from outbidding a mis-assigned bidder h and it is never in i's interest to induce others to outbid h. Thus, it remains to consider the case that i's own bid is decisive for dissolving a mis-assignment at H_i. We consider the following simplified situation: all bidders with valuations lower than or equal to the current standing bids have quit the game and all remaining active bidders are current high bidders in one or several auctions. Thus, now that it is i's turn to bid, no bidder besides i is willing or able to submit a bid. If i's decision prescribed by σ_i^* at H_i is decisive for dissolving the mis-assignment of h to j, the relevant analysis compares outcomes where h is mis-assigned with outcomes that result if h is outbid during the course of the auctions game. The outcomes are unambiguous because in the game following H_i all bidders bid according to σ_{-i}^*. Thus, our argument for the selected simplest information set is still valid in more complicated situations, for example, if some bidders that will quit have not quit yet or are still willing to submit bids.

Let us compare the outcomes resulting from i's decision at the described information set H_i. Note that the comparison of these outcomes may be seen as a decision between bidding against sets of bidders with different valuation matrices: one matrix where the mis-assigned bidder h does not bid anymore (describing the situation where he is mis-assigned) and another matrix where h bids according to his original valuations.

First, note that i does not profit from dissolving h's mis-assignment if h stays mis-assigned under σ_i^*. Dissolving the mis-assignment leads to bidding activity by h, which, compared to the situation without new bids by h, results in weakly higher prices.

Assume that σ_i^* prescribes i to outbid h in j. Then, j is in i's demand set. Thus, not outbidding h in j means bidding in an auction not in i's demand set (or, if several auctions are in i's demand set, i may bid in one of these auctions, but bidding there is not considered a deviation from σ_i^*). Bidder i, when submitting a bid in j, does not know if his bid is high enough to become the high bidder, i.e., he does not know if $b^*_{ij} > b^h_j$ or $b^*_{ij} \leq b^h_j$. Our assumption that i is decisive for dissolving h's mis-assignment implies that $b^*_{ij} > b^h_j$. From b^*_{ij} being prescribed by σ^*_i, we conclude that $v_{ij} - b^s_j > v_{ij} - b^*_{ij} \geq v_{ik} - b^s_k$ for all $k \neq j$. Combining this with $b^*_{ij} > b^h_j$ and $b^s_j \leq b^h_j$ we get

$$v_{ij} - b^s_j \geq v_{ij} - b^h_j > v_{ij} - b^*_{ij} \geq v_{ik} - b^s_k. \tag{3.12}$$

Therefore, whenever h is still mis-assigned to j in the final outcome, i does not win his most preferred auction at the final prices. Hence, to investigate a possible profitable deviation from σ^*_i by not dissolving a mis-assignment, we have to compare an unstable (and inefficient) outcome, where h wins j and i prefers to win j to his assignment, with an outcome where h bids weakly more but i wins his preferred auction.

Let us denote the relevant variables in the two situations as follows. If i deviates in H_i, we denote the auction in which he bids by k and the final price vector in the resulting outcome by p^1. If i follows σ^*_i at H_i, the resulting prices are given by p^2. The symbols b^h and b^s always refer to the situation at H_i. Remember, the mis-assigned bidder is denoted by h. He is mis-assigned to auction j. Bidder i is supposed to bid b^*_{ij} in j according to σ^*_i.

From this, we conclude for the auction k that i may win by deviating, using

- $b^*_{ij} > b^h_j \geq b^s_j$,
- $v_{ij} - b^*_{ij} = \max_{l \neq j}\{v_{il} - b^s_l\}$, and
- $p^1_k = b^h_k \geq b^s_k$,

that

$$v_{ij} - b^s_j \geq v_{ij} - b^h_j > v_{ij} - b^*_{ij} \geq v_{ik} - b^s_k \geq v_{ik} - p^1_k. \tag{3.13}$$

Note that bidder i, who deviates from σ^*_i may either win k with his bid or he will bid in j later if he is outbid in k or immediately at the stage after H_i if he does not become high bidder in k.[53] Thus, we do not have to consider i winning an auction different from k. We also know that $p^2_j \geq b^h_j$ since h is outbid by i in this case. In both cases, since we concentrate on information sets H_i with the characteristics described above, the bidding order in the game following H_i is determined by the order in which

[53] In this case, bidder i will be selected to bid again because all other active bidders are current high bidders in H_i and, therefore, cannot bid in additional auctions.

high bidders are outbid. That is, whenever a bidder is outbid, he is the next bidder who may submit a bid because he is the only active bidder who holds no current high bidder position.

If i wins k and the following play of the other bidders results in dissolving h's mis-assignment to j, then the deviation cannot pay for i. The bids of h, which i's deviation seeks to avoid, will nevertheless become relevant. Therefore, if i's deviation from σ_i^* is profitable, then h wins j and none of the bidders who bid after i bid more than b_j^h in j. That means that if i bids at H_i in k and wins k with this bid, then the bidding dynamics that follow (in a deterministic bidding order) lead to a maximum bid equal to b_j^h in j, resulting in $p_j^1 \leq b_j^h$.

Assume i follows σ_i^* and wins j with b_{ij}^*. Then, from Equation (3.13), $v_{ij} - b_{ij}^* \geq v_{ik} - p_k^1$ so the deviation does not pay. Hence, in order for the deviation to be profitable, i has to be outbid in j and, therefore, $p_j^2 \geq b_{ij}^*$. Then, dissolving h's mis-assignment led to a bid higher than b_{ij}^*. That is, if h gets rid of his mis-assignment in j, he bids in some auction l such that this bid induces a chain of bids in the auctions that includes the bid above b_{ij}^* in j. A chain of bids in auctions or a chain of bidders in auctions from l to j is a visualization of the following: h bids in l, then the former high bidder of l bids in l_1, the former high bidder of l_1 bids in l_2 and so on, until a bidder bids in j.[54] The auction k, which i wins if deviating, cannot be part of this chain starting at l because outbidding the high bidder in k does not lead to a bid in j higher than b_j^h. On the other hand, the auction l, in which h bids if outbid in j, may also not be part of the chain of bidders and auctions that starts at k: whenever the high bidder in l is outbid, the reactions of the others lead to the higher bid in j. Therefore, these two chains of auctions and bids are separable.

Hence, if h is outbid in j by i and afterwards i is also outbid in j, the auction k still has the same high bidder and high bid as in H_i. Bidder i may still bid in k. If a deviation in H_i leads to winning k with a bid b_{ik}, i could now win k with the same bid b_{ik}. Therefore, we conclude that a deviation from σ_i^* aiming at avoiding the dissolution of a mis-assignment is never profitable for i. Note that we did not exclude the possibility of further dissolved mis-assignments in the considered chains of auctions and bidders.

[54]This neglects unsuccessful bids (considering the capture of a high bidder position). Such a bid is followed by another bid of the former bidder or the end of the game if this bidder quits.

One-Shot-Deviation Principle Since beliefs μ_i are required to be updating consistent, the one-shot-deviation principle applies. Thus, since σ_i^* is a best reply against σ_{-i}^* when considering only unilateral deviations at single information sets, the assessment (σ^*, μ) forms a PBE of Γ^a.

Epsilon-Equilibrium Taking neglected deviations from prices $\bar{p}^*$ due to the random factor in the strategy σ^* and the random bidder sequence into account, an improvement by ι may be possible for a bidder. If he could anticipate accumulations of deviations, he might be able to use such accumulations to his advantage. Note, however, that even these manipulations cannot increase his payoff by more than $\min\{n-1, m-1\} \cdot \iota < 1$. We call the maximum deviations ε-deviations and the equilibrium an ε-equilibrium.

Note that any bid between $b_j^s + \iota$ and b_{ij}, as described by σ_i^*, would also be a best reply. Strategy σ_i^* is the strategy in this class of bids with the lowest number of submitted bids.

Since the strategy σ^* does not depend on beliefs about V_{-i}, v^S, and o, but is optimal against all realizations thereof, the ε-PBE is also an ex-post equilibrium, neglecting epsilon-deviations.[55] Note, however, that for a PBE, optimal *expected* payoffs are decisive (cp. Equation 3.6). Thus, it is possible that beliefs μ exist such that (σ^*, μ) is a PBE. That is, the restriction to ε-PBE may be dispensable. Since handling beliefs in more detail seems difficult and unlikely to lead to promising new insights, we concentrate on the more practical approach of allowing for epsilon-deviations.

3.3 Discussion and Further Issues

We relate our results to the literature and discuss briefly some further issues.

Relating the Results to Other Models Our model fills a gap in the literature. Albeit many related models and analyses exist, our independent second-price proxy auction model with heterogeneous items provides new insights.

[55] We prefer to refer to our equilibrium as ε-PBE because the deviations would be hard to motivate if we assumed that valuations V_{-i} are known by i. Deviations may then be avoided. However, as mentioned before, it is not clear if this is always in the interest of a bidder. Analyzing the game assuming full information is also not desirable because this assumption is hard to justify.

The multiple-auctions game is a generalization of the model of Peters and Severinov (2006) with respect to bidders' preferences.[56] They consider only homogeneous items, which means that a bidder's valuations for all items are identical. With heterogeneous items, the efficient outcome does not only depend on the sets of winning and losing bidders, but on who wins which auction. Thus, the independent bidding process has to deal with a more complex coordination or assignment problem.

In our model, we exclude the kind of impact of the increment typically considered for single unit auctions (see Chwe, 1989; David et al., 2007; Rogers et al., 2007; Rothkopf and Harstad, 1994) because we restrict bids to a certain grid of values. However, in the multi-unit case, minimum bid increments evoke other problems. We provide an upper bound on deviations and give insights into why they occur.[57] Peters and Severinov (2006) show how these deviations can be avoided in the case of homogeneous items. Bidders can coordinate their beliefs by observing the bidding dynamics (changes in current high bidders and standing bids), such that the unique price vector in every outcome in their PBE is our reference price. Their method does not extend to the heterogeneous case. It is an open question whether a similar method exists for our environment. Several attempts we made have not been successful. The occurrence of deviations can be reduced, but they do not disappear completely and not for all valuation matrices or bidding sequences. The decisive differences to Peters and Severinov (2006) are that in the homogeneous items case a losing bidder's valuation determines all prices (even though a winning bidder may submit the price determining bid) and, therefore, all deviations lead to prices above the minimum competitive price. These deviations are not in the interest of any winning bidder. With heterogeneous items, deviations may include prices below minimum competitive prices. Note that such a lower price may even be to the advantage of a bidder in an unstable pairing. It follows that it is not clear whether a bidder would have an incentive to avoid these deviations, even if we found a way to coordinate exactly on our reference outcome. In our analysis, we assume that increments are small and that bidders accept the deviations that occur due to the increment. Thus, we solve for an ε-equilibrium.

Most related papers concentrate on Nash equilibria or do not consider strategic bidding. Gul and Stacchetti (2000), de Vries et al. (2007), and Mishra and Parkes (2007) describe PBE in auctions with simultaneous submission of complete demand

[56] And, e.g., the random bidding order.

[57] See Bansal and Garg (2005), Bertsekas (1992), and Demange et al. (1986), for related analyses.

sets and central bid coordination. The solution concept that we apply is the ε-PBE.

Our equilibrium reference outcome is an efficient assignment at minimum competitive prices. By following the equilibrium strategy, bidders coordinate themselves on the bidder-optimal solution. If all bidders follow the equilibrium strategy or a similar straightforward bidding strategy (bidding one increment above the standing bid), the coordination problem of assigning the items to bidders efficiently is solved without a central coordination advice (which acts similarly to a Walrasian auctioneer).

The application of the second-price rule accounts for the close relation to models with demand bids. Second-price proxy bidding can be interpreted as indicating one element of the demand set.[58] Therefore, the multiple-auctions game is related to the category of auctions where bidders simultaneously announce one element of their demand set at each bidding stage (e.g., Ausubel and Milgrom, 2002; Crawford and Knoer, 1981; Kelso and Crawford, 1982; Parkes, 1999).[59] However, bidders bid asynchronously and we allow for a random bidding order. Our model has these features in common with the work of Bansal and Garg (2005), who address similar questions on the increment problem in independent English auctions. A minimum increment is needed for the process to proceed. This increment causes deviations because the winning bidders have to solve a coordination problem. Thus, the predicted prices are not unique but lie within a bounded range. Another difference from models with demand bids and coordination by a central seller is that high bidders lose their status only when they are explicitly outbid. If high bidders are not allowed to bid, as in our model, strategy σ_i^* is nevertheless a best reply off the equilibrium path and against all realizations of bidders' valuations. However, if high bidders are allowed to bid, a bidder may hold several high bidder positions off the equilibrium path. Then, even the extended version of σ_i^* given in Appendix A.2 is not a best reply for all valuations (see the example in the appendix).

Our model reproduces multiple bidding (by a bidder in one auction), cross-bidding, and incremental bidding[60] as part of the equilibrium strategy. Multiple and incre-

[58]Price bids may be seen as indicating the maximum price at which the item is still in the demand set of the bidder in the worst case (when the other standing bids do not change).

[59]Milgrom and Strulovici (2009) show that in an auction with substitutes valuations and where bidding is submitting the demand at current prices, it is enough to submit one of the demanded packages per bidder at each simultaneous bidding stage to reach an approximate competitive equilibrium. This is closely related to our result because the price bids submitted here mark the maximum price at which the item is still in the demand set.

[60]Remember that multiple bidding refers to multiple bids by a bidder in the same auction, cross-

mental bidding have often been considered as irrational or caused by a misunderstanding of the auction format. Empirical studies on eBay find almost 20% cross-bidders (Anwar et al., 2006),[61] 20% incremental bids, and 61% bidders with multiple bids (Hayne et al., 2003). In a field experiment with pairs of simultaneously offered items, 15.5% of participants are cross-bidders (Haruvy and Popkowski Leszczyc, 2008). Hoppe (2008a) observes 30% (in simultaneous auctions) to 60% (in overlapping auctions) cross-bidders in his experiment. In the theory of Peters and Severinov (2006) for homogeneous items, as soon as incoming bids have leveled out the differences in standing bids among auctions, only incremental bidding occurs until the end. In contrast, in our model, both incremental bidding and larger bidding steps may appear in each bidding phase.

Contrary to many models that relate to Internet auctions and especially to eBay auctions, we do not consider hard close auctions. The ending rule resembles a soft close rule because auctions only end if no new bids are submitted. An example of an auction site that comes close to our model is Bidshares (www.bidshares.com), where many auctions are scheduled to end at the same time, but bidding activity within the last 30 minutes extends the ending time.

Other Equilibria Consider the following strategy $\hat{\sigma}_i$: bid $b_{ij} = \bar{b}$ in an auction j where $B^h(j) = j$ and $j \in \arg\max_{j:B^h(j)=j}\{v_{ij} - b_j^0\}$, and do not bid if $B^h(j) \neq j$ for all $j \in M$. In other words, bid in the auction where your potential profit is highest of those where no one has bid yet, and quit if no such auction exists. If you bid, submit the maximum bid allowed. All bidders choosing this strategy, constitutes a Nash equilibrium. All items sell at starting prices. Losing bidders have no chance to improve because all high bids are maximal when it is their turn to bid. All winning bidders receive the item they appreciate most among those available for them.

However, the equilibrium depends on the credibility of the winners' risky high bids. A losing bidder's deviation, for example to σ^*, can be expensive for the winner. The deviating bidder, however, does not lose anything by doing so. The resulting assignment is in general not efficient. Furthermore, the equilibrium relies on the

bidding is switching between auctions, and incremental bidding is bidding just one increment above the standing bid.

[61] They exclude multi-unit bidders who hold several high bidder positions because they probably have multi-unit demand.

existence of a maximum bid $\bar{b}$.

Off the equilibrium path, deviating from $\hat{\sigma}_i$ may be profitable. Suppose a bidder h has submitted a very low bid in auction j, for example $b_{hj} = b_j^0 + \iota$, instead of $b_{hj} = \bar{b}$. It is bidder i's turn, and in all auctions a bid has been submitted. The strategy prescribes that he quit. However, with a valuation $v_{hj} > b_j^0 + \iota$, he may improve his payoff by bidding in j.

One may assume that a bidder i would never lose by deviating from $\hat{\sigma}_i$ to σ_i^*. However this is not true. Assume a bidder i is selected to bid and auction j is the only auction where no bid has yet been submitted. According to $\hat{\sigma}_i$, he bids $b_{ij} = \bar{b}$ in j and gains $u_i = v_{ij} - b_j^0$. Strategy σ_i^* may tell him, however, to bid in $k \neq j$. With this bid, he does not become the high bidder because the high bid is $b_k^h = \bar{b}$. Then, another bidder h is selected and, bidding according to $\hat{\sigma}_h$, bids $b_{hj} = \bar{b}$. When it is i's turn to bid the next time, all high bids are equal to $\bar{b}$ and he cannot win any auction. His payoff is zero, and, thus, less than he receives by following $\hat{\sigma}_i$. Thus, this example shows that σ^* is not a dominant strategy.

Strategic Sellers Haruvy et al. (2008) promote the analysis of competing auctions. One type of competition between auctions is that between independent auctions offered by competing sellers. One decision a seller faces is how to optimally set his starting price or his (unrevealed) reserve price. This may decrease efficiency if b_j^0 is set above v_j^S. A strategic seller j in the game would never set b_j^0 below v_j^S because he only gains additional sales that result in a negative payoff. Thus, $b_j^0 \geq v_j^S$.

In the assignment game, it is in general not possible to align incentives for bidders and sellers to reveal their valuations (see Roth and Sotomayor, 1990). A mechanism may only assure incentive compatibility for *one* side of the market. We considered the bidders side. A mechanism that implements the seller-optimal outcome would be incentive compatible for the sellers' side. Usually, the bidder-optimal and the seller-optimal outcome differ.

For large markets, however, strategic sellers may set the starting price equal to their valuation in equilibrium. For example, Peters and Severinov (2006) find that in their model with homogeneous unit-demand preferences, sellers have an incentive to reveal their reservation value truthfully. However, this result only holds for a sufficiently large number of traders. It is not clear if or how the result of Peters and Severinov (2006) extends to our model.

More General Substitutes Valuations Auctions suited for more general preferences have been analyzed and applied (e.g., by the Federal Communications Commission in spectrum auctions). An example is the simultaneous ascending auction, which is an auction format with open (price) bids, proceeding in rounds (Milgrom, 2000). In this model, there is only one seller and straightforward bidding converges to a competitive equilibrium if all items are substitutes for all bidders. However, a competitive equilibrium with independent prices for each item, which is required in the case of multiple sellers, may fail to exist. Bikhchandani and Ostroy (2006) show that for more general substitutes valuations so-called pricing equilibria exist where prices refer to packages or bidders and are, thus, nonlinear or even non-anonymous.

For more general demand structures in multi-unit auctions, using other strategic options may pay: for example, signaling via bids (see Klemperer, 2002) or profitable demand reduction (e.g., Ausubel and Cramton, 2002).

Our analysis is also valid for the case of simple additive valuations for packages (the borderline case between substitutes valuations and complementary valuations of bidders, also called neutral package valuations; the value of a package equals the sum of values of items in the package). Then, every bidder's demand for an item is independent of the prices in other auctions and they treat each auction like a single unit auction. For unit-demand preferences and additive valuations, prices in competitive equilibrium are anonymous and can be assigned to items.

Note, however, that our analysis does not extend to a bidder population with a mixture of unit-demand preferences and additive valuations for packages. This is illustrated by the example below. In this mixed population, it makes a difference whether a bidder with additive valuations for the m items participates or if m bidders participate who demand only one item each. The strategic situation with the single bidder with additive valuations is different. He may increase his payoff the in competitive equilibrium by reducing his demand when playing against bidders with unit-demand preferences.

Consider the example in Table 3.3 with two bidders, $B1$ with unit demand and $B2$ with additive valuations, and two items $A1$ and $A2$.

The efficient assignment allocates both items to Bidder $B2$. $B2$'s Vickrey price is $p^V_{B2} = 4$ and the minimum competitive prices are $p^{CE}_{A1} = 2$ and $p^{CE}_{A2} = 4$, resulting in Vickrey payoffs $u^V_{B1} = 0$ and $u^V_{B2} = 5$ and competitive payoffs $u^{CE}_{B1} = 0$ and $u^{CE}_{B2} = 3$.

The outcome in the middle illustrates what happens if $B2$ is split up into two bid-

Table 3.3: The impact of splitting a bidder's demand between two bidders (i.e., shill bidding) and of demand reduction in a mixed population with a unit-demand bidder $B1$ and a bidder $B2$ with additive valuations.

	$A1$	$A2$	$(A1, A2)$	p^V	p^{CE}	u^V	u^{CE}
$B1$	2	4	4	0	0	0	0
$B2$	3	6	***9***	4	2+4	5	3
$B1$	2	4	4	0	0	0	0
$B2a$	***3***	0	3	2	2	1	1
$B2b$	0	***6***	6	4	4	2	2
$B1$	***2***	4	4	0	0	2	2
$B2$	0	***6***	6	2	2	4	4

ders $B2a$ and $B2b$ with unit-demand preferences. Adding these two bidders' payoffs gives a lower Vickrey payoff (1+2<5) than $B2$ has in the upper outcome. However, the payoffs in the minimum competitive equilibrium equal those that $B2$ receives by bidding truthfully (1+2=3).

However, as stated above, $B2$ has an incentive to reduce demand in an ascending auction. A deviation of $B2$ by bidding $(0, 6, 6)$ in a Vickrey auction or by bidding up to these valuations in an ascending auction, results in the assignment of $A1$ to $B1$ and $A2$ to $B2$ with prices and payoffs $p^V_{B1} = 0$, $p^V_{B2} = 2$, $p^{CE}_{A1} = 0$, $p^{CE}_{A2} = 2$, $u^V_{B1} = 2$, $u^V_{B2} = 4$, $u^{CE}_{B1} = 2$, and $u^{CE}_{B2} = 4$. Thus, not surprisingly, $B2$'s Vickrey payoff decreases. However, his demand reduction increases his payoff in competitive equilibrium from 3 to 4.

In a population of bidders with unit-demand preferences, bidders have an incentive to bid straightforwardly in an ascending auction. The same is true for bidders in a population with additive valuations. But if we mix these two populations, this is no longer true for the bidders with the additive valuations, as the example shows.

Outside Options In our model, sellers' valuations may be considered as outside options. If each bidder is also assigned an outside option with value v_i^0, a result of Demange and Gale (1985) states that the bidders' optimal payoff vector in the core weakly increases in v^0 and weakly decreases in v^S. The opposite statement applies to sellers' payoffs in the bidder-optimal outcome.

Effect of Additional Bidders and Sellers The effect of new entrants in the assignment game is summarized in Roth and Sotomayor (1990). We apply those results to our model.

If a bidder is added to the game, every bidders' payoff in the Vickrey outcome weakly decreases and every seller's payoff weakly increases. Adding a seller has the reverse effect.

If an added bidder i wins an auction, then the indifference path leading to this auction is reversed. The seller j whose item is won by i profits the most by his entry. The sellers' gains are decreasing in their distance from i on the reversed path. On the other hand, the bidder who used to win j is harmed most by i's entry, and bidders along the reversed indifference path are harmed less the farther they are away (Mo, 1988).

Asking for the impact of additional bidders is related to the phenomenon considered next paragraph: profitable shill bidding.

Shill Bidding If a bidder pretends to be more than one bidder or a seller pretends to be a bidder (called *shill bidding*), this may increase prices. On eBay, for example, shill bidding by sellers is prohibited,[62] but it is hard to control.

Consider the following matrix of valuations of five bidders for items offered in four auctions ($v^S = (0, 0, 0, 0)$):

	$A1$	$A2$	$A3$	$A4$
$B1$	10	0	0	0
$B2$	10	20	0	0
$B3$	0	20	30	0
$B4$	0	0	30	40
$B5$	0	0	0	39

Consider first the minimum competitive equilibrium (or Vickrey outcome) if $B5$ does not participate. Bidders $B1$–$B4$ win $A1$–$A4$, respectively, at prices $p = (0, 0, 0, 0)$. Thus, the sum of the sellers' revenues is zero and all gains from trade (which amount to 100) go to the bidders.

[62]See http://pages.ebay.com/help/policies/seller-shill-bidding.html.

This changes drastically if one of the sellers creates a false identity and enters as bidder $B5$. Bidders $B1$–$B4$ win the same auction. However, the items are sold at prices $p = (9, 19, 29, 39)$, giving the sellers a total revenue of 96. Thus, almost all gains from trade go to the sellers.

The example shows how the entry of a (losing) bidder may shift all gains from trade from the buyers' side to the sellers' side. One may assume that seller $A4$ has introduced this shill bidder. However, all sellers profit from $B5$'s existence. Thus, they may also have introduced $B5$ as a shill bidder and have bid in the auction of a different seller to decrease the chance of detection of the manipulation. The additional bidder raises the price level in the whole marketplace, not only in one auction. Only a bidder that does not win any auction and does not determine any price does not influence the results (like in a second-price auction).

We gave an extreme example that illustrates the value of additional (artificially introduced or real) bidders for the auction platform. One may think that increasing bids has a similar effect. However, the situation is not quite as simple as it seems, as the following chapter shows.

Chapter 4

Monotonicity of Vickrey Payoffs and Prices

Vickrey auctions have nice theoretical properties, but also many drawbacks.[1] Despite its weaknesses, the Vickrey auction plays an important role in the design and analysis of auctions.

In the previous chapter, we identified a PBE of the multiple-auctions game Γ^a with the following properties. The equilibrium (reference) outcome is efficient and prices are minimum competitive prices. Moreover, they equal Vickrey prices.[2] That is, every bidder pays an amount equal to the loss of social surplus due to his participation. Note that the Vickrey outcome is the bidder-optimal outcome in the core if the coalitional function is bidder-submodular (e.g., Milgrom, 2004). A population of bidders with unit-demand preferences fulfills the bidder-submodularity condition. Thus, the payoffs in our equilibrium outcome equal the bidder-optimal payoffs in the core of the associated cooperative game.

In this chapter, we consider the effect of an increase or decrease in a single valuation v_{ij} on the reference outcome $(\bar{x}^*, \bar{p}^*)$ in the PBE of the multiple-auctions game. We use the results of Propositions 3.4 and 3.9 that $(\bar{x}^*, \bar{p}^*) = (x^{\mathit{eff}}, p^V)$. Hence, we directly analyze the outcome of a Vickrey auction (x^{eff}, p^V) where all bidders' bids equal their valuations, which is a weakly dominant strategy. Note that increasing a bidder's

[1] For a discussion of advantages and disadvantages of the Vickrey auction see, e.g., Ausubel and Milgrom (2006), Lucking-Reiley (2000b), Milgrom (2004), Rothkopf (2007), and Rothkopf, Teisberg, and Kahn (1990).

[2] In the following, if we refer to Vickrey prices, payments, or outcomes, we mean equilibrium prices, payments, and outcomes in the equilibrium of the Vickrey auction where all bidders truthfully submit their valuation.

valuation for an item is equal to assuming that he increases his bid in a Vickrey auction but different from assuming that he submits a higher bid at a final stage of the multiple-auctions game. That is, the situation that we consider is an increase in a valuation before the multiple-auction game starts, but not an increase that occurs during bidding. For example, an increase during the bidding process may result in an outcome where a bidder determines a price with an earlier bid that he would not submit under his new valuations.

Bidders' and sellers' optimal payoffs are monotone with respect to the entry of bidders: each seller's payoff weakly weakly increases and each bidder's payoff weakly decreases (e.g., Roth and Sotomayor, 1990). Moreover, monotonicity of the (single) seller's Vickrey revenue in bidders holds for all valuation structures that are bidder-submodular (e.g., Milgrom, 2004). It has been assumed that in this case the single seller's Vickrey revenue is also monotone increasing in bidders' valuations (Day and Milgrom, 2008). In our environment, this would imply that the sum of sellers' revenues in the Vickrey outcome is weakly monotone increasing in reported valuations of bidders. However, we show that this is not true.

In the single seller case, a property of the Vickrey auction is non-monotonicity of seller's revenue, mainly in connection with complementary valuations for packages (e.g., Milgrom, 2004; Beck, 2009) or budget restrictions (e.g., Day and Milgrom, 2008). We add another critical point: non-monotonicity with respect to increasing single valuations even if items are substitutes. However, considering individual Vickrey payments, we find that they are either weakly monotone increasing or decreasing.

We use a comparative statics approach to compare Vickrey outcomes for different valuation matrices, assuming that bidders report valuations truthfully. This may be considered an examination of the effect of a unilateral deviation in a Vickrey auction on prices. The deviation considered is the submission of a higher bid in a single auction. The results are relevant for the analysis of the bidder-seller networks in Chapter 5, but they are also of independent interest because they reveal characteristics of the Vickrey auction.

4.1 Impact of Increasing Valuations on Vickrey Payoffs and Prices

In the following, we do not distinguish between valuations and bids. For instance, in the following example, the increased valuation may also be interpreted as an increased bid.

To illustrate the subject of this section, we give an example of the reaction of prices to an increased valuation. In the example in Figure 4.1, $\tilde{v}_{41} = 6 > 1 = v_{41}$,

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$
$B1$	5	3	0		5	3	0
$B2$	0	10	10		0	10	10
$B3$	0	0	15		0	0	15
$B4$	***1***	7	0	$\rightsquigarrow$	***6***	7	0
B^h	$B1$	$B2$	$B3$		$B4$	$B2$	$B3$
p^V	1	7	7		5	6	6

Figure 4.1: Example of the reaction of prices to an increasing valuation $v_{ij} = v_{41}$.

$\tilde{p}_1^V = 5 > 1 = p_1^V$, and $\tilde{p}_l^V = 6 < 7 = p_l^V$ for $l \in \{A2, A3\}$. This illustrates that prices may increase or decrease. In the following we analyze the impact of increasing valuations in detail.

For this, we also make use the concept of a price determining bidder but in a different context than in the multiple-auctions game. There, $B^{PD}(j)$ is the bidder who submitted the bid that equals the price in auction j. In this section, we use the term in a slightly different way and introduce a different notation: $B^I(j)$ is a correspondence that gives all bidders (except for the winning bidder of auction j) that are indifferent between winning j and the payoff that they receive in the outcome (x^{eff}, p^V). Thus, these are all the possible price determining bidders in different bidder sequences in the multiple-auctions game (except for the case $B^{PD}(j) = j$ and $i \in B^I(j)$). Notice that the price determining bidder $B^I(j)$ may be not unique and it may be that $B^I(j) = \emptyset$.[3]

In this section, we mainly refer to Vickrey prices p_j^V instead of Vickrey payments p_i^V (see Section 3.2.2.5 for definitions). Remember that we defined Vickrey prices p_j^V

[3]In the examples, we indicate $B^I(j) = \emptyset$ by "–".

for unsold items j as $p_j^V = v_j^S$. These artificial prices serve mainly as a benchmark if j is sold after the considered change in a valuation.

Throughout this section, we write x instead of x^{eff} to simplify notation since all considered assignments are efficient. For the same reason, we write for example $N\backslash i$ instead of $N\backslash\{i\}$. If the efficient assignment x refers to a market different from (N, M, V, v^S), for example to $(N\backslash i, M, V, v^S)$, we denote this by $x(N\backslash i, M)$. The coalitional values for (N, M, V, v^S) and $(N\backslash i, M, V, v^S)$ are denoted by $c(N, M)$ and $c(N\backslash i, M)$, respectively.

The terms $\tilde{v}_{ij}$ "is part of," "contributes to," "is contained in," or "is a summand of" $\tilde{c}(N, M)$ refers to $\tilde{c}(N, M) = \sum_{h \in N\backslash i, k \in M\backslash j} (v_{hk} - v_k^S)\, x_{hk} + (v_{ij} - v_j^S)$ (equivalently for other coalitional values).

Before we consider further implications of an increased valuation, we present Lemma 4.1, which shows the important role of the price determining bidder.[4]

Lemma 4.1 *Suppose $x_{ij} = 1$ in the market (N, M, V, v^S) and $g \in B^I(j)$. Then*

(1) g wins j in the market $(N\backslash i, M, V, v^S)$ if $g = B^I(j)$ or g is one of several possible winners j in $(N\backslash i, M, V, v^S)$ if $B^I(j)$ is not uniquely defined,

(2) the summands of $c(N\backslash i, M)$ are determined by replacing the summands of $c(N, M)$ by the respective bidders' valuations along an indifference path in (x, p^V) starting at auction j, and retaining summands that are off the considered indifference path.

Lemma 4.1 includes the following statements:

- The bidder g with $x'_{gj} = 1$ in the efficient assignment $x'(N\backslash i, M)$ is indifferent between winning j at p_j^V and his payoff u_g^V.
- Every bidder g who is indifferent between winning an auction k at p_k^V and his payoff u_g^V, such that k has j as a predecessor on an indifference path, may be assigned to k under an efficient assignment $x'(N\backslash i, M)$.

Proof of Lemma 4.1: Lemma 4.1 follows from the argument used in the proof of Proposition 3.9. □

[4] Furthermore, Lemma 4.1 is used in the proofs of Propositions 4.2 and 4.3.

To employ Lemma 4.1 in this chapter, remember that the Vickrey outcome is a competitive equilibrium if bidders have unit-demand preferences and that the results on the indifference paths of the previous chapter also apply to the Vickrey outcome.

The following three propositions, 4.1, 4.2, and 4.3, consider all possible points of departure of an increase in a single valuation. Note that we do not distinguish between a valuation and a bid. The distinction may be relevant because the assignment and the prices are calculated on the basis of the bids, and the payoff is based on the valuation, the assignment, and the price. We keep this in mind and distinguish between valuations and bids whenever this is relevant in the proofs.

The notation v_{ij} refers to i's valuation before the change, and $\tilde{v}_{ij}$ to the same valuation after the increase. In the following, all parameters and function values that relate to the situation after the change from v_{ij} to $\tilde{v}_{ij}$ are labeled with a tilde.[5]

We start our analysis with an increasing valuation in the auction that bidder i wins.

Proposition 4.1 *If a valuation v_{ij} with $x_{ij} = 1$ increases ceteris paribus to $\tilde{v}_{ij} > v_{ij}$, this results in*

(1) an unaltered price p^V_j and assignment x: $\tilde{p}^V_j = p^V_j$ and $\tilde{x} = x$,

(2) lower or unchanged prices in auctions $l \in M \backslash j$: $\tilde{p}^V_l \leq p^V_l$.

Proof of Proposition 4.1: First, we prove (1). It can be easily seen that the efficient assignment x is not influenced by the change in v_{ij}: a reassignment would be a contradiction to efficiency of x. Bidder i remains the winner of auction j. Since the winner of an auction does not influence the price he has to pay in a Vickrey outcome, p^V_j is unaltered.

Next, we prove (2) and consider prices in auctions $k \in M \backslash j$. Since the efficient assignment is unaltered, $p^V_l = c(N \backslash h, M) - c_{-h}(N, M)$ and $\tilde{p}^V_l = \tilde{c}(N \backslash h, M) - \tilde{c}_{-h}(N, M)$ for $x_{hl} = \tilde{x}_{hl} = 1$ and $h \in N \backslash i$.[6] Let $\delta := \tilde{v}_{ij} - v_{ij} > 0$. Since $\tilde{x} = x$,

[5]The coalitional values $c(N, M)$ and $\tilde{c}(N, M)$ refer to markets (N, M, V, v^S) and $(N, M, \tilde{V}, v^S)$, respectively.

[6]Remember that the expression $c_{-h}(N, M)$ represents the coalitional value of the coalition (N, M) reduced by the payoff of player h. In our game, this means $c_{-h}(N, M) = c(N, M) - v_{hl}$.

$\tilde{c}_{-h}(N, M) = c_{-h}(N, M) + \delta$ and we get

$$\begin{aligned} p_l^V - \tilde{p}_l^V &= c(N\backslash h, M) - c_{-h}(N, M) - \tilde{c}(N\backslash h, M) + \tilde{c}_{-h}(N, M) \\ &= \delta - (\tilde{c}(N\backslash h, M) - c(N\backslash h, M)). \end{aligned}$$

We know

$$\delta \geq \tilde{c}(N\backslash h, M) - c(N\backslash h, M),$$

because otherwise $c(N\backslash h, M)$ is not maximal (the increase beyond δ may be reached under the original valuations too). With this, we conclude that $p_l^V - \tilde{p}_l^V \geq 0$ and, thus, $p_l^V \geq \tilde{p}_l^V$, which completes the proof. ■

We can derive some more information from the last inequalities in the proof. First, note that $\tilde{c}(N\backslash h, M) - c(N\backslash h, M) \geq 0$ because the maximum sum of valuations of all bidders except for h cannot decrease if the valuation of one bidder $i \neq h$ increases. Thus, $p_l^V - \tilde{p}_l^V \leq \delta = \tilde{v}_{ij} - v_{ij}$. The difference $p_l^V - \tilde{p}_l^V$ can take any integer value between δ and 0 because $\tilde{c}(N\backslash h, M) - c(N\backslash h, M)$ takes an integer value between 0 and δ.[7]

The difference $\tilde{c}(N\backslash h, M) - c(N\backslash h, M)$ equals δ if $\tilde{v}_{ij}$ is contained in $\tilde{c}(N\backslash h, M)$ and v_{ij} in $c(N\backslash h, M)$. This happens if $x_{ij}(N\backslash h, M) = \tilde{x}_{ij}(N\backslash h, M) = 1$,[8] i.e., i wins j regardless of whether h participates. An example is given in Figure 4.2.[9] There, $i = j = 2$, $h = l = 1$ or $h = l = 3$, and $\delta = 2$. The change in $B2$'s valuation v_{22} does not impact the prices in $A1$ and $A3$.

We have $\tilde{c}(N\backslash h, M) - c(N\backslash h, M) = 0$ if neither v_{ij} nor $\tilde{v}_{ij}$ is contained in the respective sums of valuations. That is, if h does not participate, then i wins a different auction than j. This is, for example, the case if i is the price determining bidder in l with $x_{hl} = 1$. An example is given in Figure 4.3 (left example). There, $i = j = 2$, $h = l = 3$, and $\delta = 2$. Then, the decrease in the price for $B3$ in $A3$, due to the increase in the valuation of $B2$, is maximal and equal to 2.

Price decreases between 0 and δ may occur if $\tilde{v}_{ij}$ is part of $\tilde{c}(N\backslash h, M)$ but v_{ij}

[7] The restriction to integers is due to all v_{ij} being integers. If $v_{ij} \in \mathbb{R}$, then $p_l^V - \tilde{p}_l^V$ is in the interval $[0, \delta] \subset \mathbb{R}$.

[8] Remember that we write $x_{ij}(N\backslash h, M)$ to make it clear which market the efficient assignment relates to when it is not the full market (N, M, V, v^S).

[9] Unless we say otherwise, $v^S = \mathbf{0}$ in all examples in the current chapter.

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$
$B1$	5	8	0		5	8	0
$B2$	0	***10***	6	$\rightsquigarrow$	0	***12***	6
$B3$	11	0	15		11	0	15
$B4$	1	0	0		1	0	0
$B5$	0	0	3		0	0	3
B^h	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$
p^V	1	4	3		1	4	3
$g \in B^I$	$B4$	$B1$	$B5$		$B4$	$B1$	$B5$

Figure 4.2: Example of Proposition 4.1 (prices p^V_{A1} and p^V_{A3} are unchanged).

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$
$B1$	5	8	0		5	8	0	5	8	0		5	8	0
$B2$	0	***10***	10	$\rightsquigarrow$	0	***12***	10	0	***10***	10	$\rightsquigarrow$	0	***12***	10
$B3$	11	0	15		11	0	15	11	0	15		11	0	15
$B4$	1	0	0		1	0	0	1	0	0		1	0	0
$B5$	0	0	0		0	0	0	0	0	3		0	0	3
B^h	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$
p^V	1	4	4		1	4	2	1	4	4		1	4	3
B^I	$B4$	$B1$	$B2$		$B4$	$B1$	$B2$	$B4$	$B1$	$B2$		$B4$	$B1$	$B5$

Figure 4.3: Examples of Proposition 4.1 (maximum and intermediate changes in p^V_{A3}).[11]

is not contained in $c(N\backslash h, M)$ (i.e., if $\tilde{x}_{ij}(N\backslash h, M) = 1$ and, thus, $\tilde{c}(N\backslash h, M) = \sum_{g\neq h, g\neq i}\sum_{k\neq j}(v_{gk} - v^S_k)\tilde{x}_{gk} + \tilde{v}_{ij} - v^S_j$, but $x_{ij}(N\backslash h, M) = 0$ and, thus, $c(N\backslash h, M) > \sum_{g\neq h, g\neq i}\sum_{k\neq j}(v_{gk} - v^S_k)x_{gk} + v_{ij} - v^S_j$). The example on the right in Figure 4.3 illustrates this for $i = j = 2$, $h = l = 3$, and $\delta = 2$. With $B2$'s valuation $v_{22} = 10$, we have $c(N\backslash h, M) = v_{41} + v_{12} + v_{23} = 1 + 8 + 10 = 19$. But if the valuation changes to $\tilde{v}_{22} = 12$, then $\tilde{c}(N\backslash h, M) = v_{11} + v_{22} + v_{53} = 5 + 12 + 3 = 20$, which leads to a decrease in the price of $A3$ to $\tilde{p}^V_3 = 3 = p_3 - 1$.[10]

The intuition behind the possible decrease in prices p_l is that the increase in i's

[10] Note that it is impossible for v_{ij} to be contained in $c(N\backslash h, M)$ but $\tilde{v}_{ij}$ not to be containd in $\tilde{c}(N\backslash h, M)$, so this combination is irrelevant.

[11] We write B^I to save space. Price determining bidders $g \in B^I(j)$ are unique for all j in these examples anyway.

valuation makes it less likely that he prefers to switch to a different item if an opponent quits. He either does not want to switch before the increase, as in the example in Figure 4.2, or he switches to a different auction before and after the increase, as in Figure 4.3 on the left, or he switches before the increase but does not wish to do so after the increase, as in Figure 4.3 on the right. Whether a bidder prefers to switch to a different auction can be deduced from his position as price determining bidder.

From another point of view, the increase in i's valuation leads to a reduction in his demand for other items.

The following proposition considers the increase in a valuation v_{ij} if i wins $k \neq j$.

Proposition 4.2 *If a valuation v_{ij} such that $x_{ij} = 0$ and $x_{ik} = 1$ for $k \neq j$ increases ceteris paribus to $\tilde{v}_{ij} > v_{ij}$, we get Vickrey prices $\tilde{p}^V$ with*

(1) $\tilde{p}^V_j \geq p^V_j$,

(2) $\tilde{p}^V_k = p^V_k$ if $\tilde{x} = x$,

(3) $\tilde{p}^V_k \leq p^V_k$ if $\tilde{x} \neq x$,

(4) $\tilde{p}^V_l \geq p^V_l$ for $l \in M \backslash \{j, k\}$ if $\tilde{x} = x$, and

(5) three disjoint categories L_1, L_2, $L_3 \subset M \backslash \{j, k\}$ of auctions with prices

- *$\tilde{p}^V_l \leq p^V_l$ and $\tilde{p}^V_l < p^V_l$ for a high enough value $\tilde{v}_{ij}$ if $l \in L_1$ and $\tilde{x} \neq x$,*
- *$\tilde{p}^V_l = p^V_l$ for all $\tilde{v}_{ij} > v_{ij}$ if $l \in L_2$ and $\tilde{x} \neq x$, and*
- *$\tilde{p}^V_l \geq p^V_l$ and $\tilde{p}^V_l > p^V_l$ for a high enough value $\tilde{v}_{ij}$ if $l \in L_3$ and $\tilde{x} \neq x$.*

Figure 4.4 contains a numerical example with $i = B1$, $j = A3$, and $k = A1$. In the example, valuation v_{13} increases stepwise. When it reaches 18, a change in the efficient assignment occurs. Therefore, the figure provides examples of cases (1)–(5). The last two price vectors in the figure indicate that $A2$ is in category L_1.

Proof of Proposition 4.2: If $\tilde{x}_{ij} = 0$ then $\tilde{x} = x$, because whenever the increase in v_{ij} to $\tilde{v}_{ij}$ leads to a different assignment, $\tilde{x}_{ij}$ makes up one of the differences. If, on the other hand, $\tilde{x}_{ij} = 1$, then $\tilde{x}$ and x differ in two or more entries. Since $x_{ik} = 1$, the change from $x_{ij} = 0$ to $\tilde{x}_{ij} = 1$ induces $\tilde{x}_{ik} = 0$. Notice the following impact of $\tilde{v}_{ij}$ on coalitional values if $x_{ij} = 0$.

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$		$A1$	$A2$	$A3$	
$B1$	5	5	***0***	$\rightsquigarrow$	5	5	***4***	$\rightsquigarrow$	5	5	***6***	$\rightsquigarrow$
$B2$	0	10	10		0	10	10		0	10	10	
$B3$	0	0	15		0	0	15		0	0	15	
$B4$	2	0	0		2	0	0		2	0	0	
B^h	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$		$B1$	$B2$	$B3$	
p^V	2	2	2		2	2	2		2	2	3	
$g \in B^I$	$B4$	$B1$	$B2$		$B4$	$B1$	$B2$		$B4$	$B1$	$B1$	

$B1$	5	5	***18***	$\rightsquigarrow$	5	5	***19***	$\rightsquigarrow$	5	5	***20***
$B2$	0	10	10		0	10	10		0	10	10
$B3$	0	0	15		0	0	15		0	0	15
$B4$	2	0	0		2	0	0		2	0	0
B^h	$B1$	$B2$	$B3$		$B4$	$B2$	$B1$		$B4$	$B2$	$B1$
	$B4$	$B2$	$B1$								
p^V	2	2	15		1	1	15		0	0	15
$g \in B^I$	$B4$	$B1$	$B1$		$B1$	$B1$	$B3$		$B1$	$B1$	$B3$
	$B1$	$B1$	$B3$								

Figure 4.4: A numerical example of the impact of a stepwise increase in v_{13}.

- For both $\tilde{x}_{ij} = 0$ and $\tilde{x}_{ij} = 1$

$$\tilde{c}(N \backslash i, M) = c(N \backslash i, M). \tag{4.1}$$

- If $\tilde{x}_{ij} = 0$, then $\tilde{x} = x$, and

$$\tilde{c}(N, M) = c(N, M), \tag{4.2}$$
$$\tilde{c}_{-g}(N, M) = c_{-g}(N, M) \text{ for all } g \in N, \tag{4.3}$$
$$\tilde{c}(N \backslash g, M) \geq c(N \backslash g, M) \text{ for all } g \in N \backslash i. \tag{4.4}$$

- If $\tilde{x}_{ij} = 1$, then $\tilde{x} \neq x$ and

$$\tilde{c}(N, M) \geq c(N, M), \tag{4.5}$$
$$\tilde{c}_{-g}(N, M) \gtreqless c_{-g}(N, M) \text{ for all } g \in N, \tag{4.6}$$
$$\tilde{c}(N \backslash g, M) \geq c(N \backslash g, M) \text{ for all } g \in N \backslash i. \tag{4.7}$$

Because a change in a coalitional value may only occur if $\tilde{v}_{ij}$ is part of this coalitional value, (4.1) follows. This is obviously impossible for $\tilde{c}(N \backslash i, M)$. Equations (4.2) and (4.3) follow from $\tilde{x} = x$ for the same reason.

Coalitional values are weakly monotone increasing in valuations because the maximum sum of valuations (minus sellers valuations), which equals the coalitional value, may not decrease if valuations increase. Therefore, (4.4), (4.5), and (4.7) are valid. The strict inequality $\tilde{c}(N \backslash g, M) > c(N \backslash g, M)$ in (4.4) and (4.7) applies if $\tilde{v}_{ij}$ is part of the sum that is represented by $\tilde{c}(N \backslash g, M)$. This is the case if $i \in B^I(l)$ for $x_{gl} = 1$ or if i is in the set $i \in B^I(q)$ for some other auction q on the respective part of the indifference path, as considered in Lemma 4.1. Note that v_{ij} may be contained in $c(N \backslash g, M)$ or not. We have $\tilde{c}(N \backslash g, M) = c(N \backslash g, M)$ if $\tilde{v}_{ij}$ and v_{ij} are irrelevant for these coalitional values. Equality holds in (4.5) at a single value $\tilde{v}_{ij} = \hat{v}_{ij}$, as explained below. For higher values $\tilde{v}_{ij} > \hat{v}_{ij}$, we have $\tilde{c}(N, M) > c(N, M)$ because $\tilde{v}_{ij}$ is part of $\tilde{c}(N, M)$ if $\tilde{x}_{ij} = 1$.

Note, however, that $\tilde{c}_{-g}(N, M)$ is not a coalitional value. Because $\tilde{c}_{-g}(N, M) = \tilde{c}(N, M) - v_{gl}$ for $\tilde{x}_{gl} = 1$ (or $\tilde{c}_{-g}(N, M) = \tilde{c}(N, M)$), a change in the assignment may imply that g wins an item l' with value $v_{gl'} \gtreqless v_{gl}$. Thus, $\tilde{c}_{-g}(N, M)$ may be higher, lower or equal to $c_{-g}(N, M)$, as given in (4.6).

Let us divide the increase from v_{ij} to $\tilde{v}_{ij}$ into incremental steps of size one. In a slight abuse of notation, we denote all valuations during this increase by $\tilde{v}_{ij}$. Since $x_{ij} = 0$ at v_{ij} and $\tilde{x}_{ij} = 1$ at the final value of $\tilde{v}_{ij}$, a valuation $\hat{v}_{ij} \in (v_{ij}, \tilde{v}_{ij}]$ exists that is the lowest value such that $\tilde{x}_{ij} = \hat{x}_{ij} = 1$.

A stepwise increase in $\tilde{v}_{ij}$ may change the assignment only once because $\tilde{v}_{ij} = \hat{v}_{ij}$ is a summand of $\tilde{c}(N, M)$ and the optimal assignment also has to assign i to j for values $\tilde{v}_{ij} > \hat{v}_{ij}$ to achieve the maximum sum of payoffs. Thus, once $\tilde{v}_{ij}$ is part of $\tilde{c}(N, M)$, the optimal assignment does not change anymore.

We claim that at $\hat{v}_{ij}$ both the assignment x and the new assignment, denoted by $\hat{x}$, are optimal. Assume to the contrary that $\hat{c}(N, M) > c(N, M)$. Since we only consider integers, this implies that $\hat{c}(N, M) \geq c(N, M) + 1$. An increase $v_{ij} + 1 = \hat{v}_{ij}$ leads to a new unique optimal assignment. However, this increase of one implies that $\hat{c}(N, M) \leq c(N, M) + 1$ because the maximum sum cannot increase by more than v_{ij} increases. From $\hat{c}(N, M) \geq c(N, M) + 1 \geq \hat{c}(N, M)$ it follows that $c(N, M) + 1 = \hat{c}(N, M)$. Now, consider the valuations whose sum gives $\hat{c}(N, M)$. We know that $\hat{v}_{ij}$ is one of them. In this case, decreasing $\hat{v}_{ij}$ by one and adding the same valuations as before results in an amount equal to $c(N, M)$. Thus, the optimal assignment associated with $\hat{c}(N, M)$ is also an optimal assignment with valuation $v_{ij} = \hat{v}_{ij} - 1$. This is a contradiction to the definition of $\hat{v}_{ij}$.

Thus, at valuation $\hat{v}_{ij}$ both assignment x and $\tilde{x}$ are optimal. Therefore, at $\tilde{v}_{ij} = \hat{v}_{ij}$, i may win either k or j. Other reassignments may also occur. With this, we further differentiate the impact on diverse coalitional values. Remember that $\tilde{c}(N\backslash i, M) = c(N\backslash i, M)$ is valid independent of $\tilde{v}_{ij}$ and any possible change in assignment $\tilde{x}$. We distinguish three cases for values $\tilde{v}_{ij}$: $\tilde{v}_{ij} < \hat{v}_{ij}$, $\tilde{v}_{ij} = \hat{v}_{ij}$, and $\tilde{v}_{ij} > \hat{v}_{ij}$. Cases 1 and 3 are as before.

- Case 1: $\tilde{v}_{ij} < \hat{v}_{ij}$ ($\tilde{x} = x$)

$$\begin{aligned}
\tilde{c}(N, M) &= c(N, M), \\
\tilde{c}_{-g}(N, M) &= c_{-g}(N, M) \text{ for all } g \in N, \\
\tilde{c}(N\backslash g, M) &\geq c(N\backslash g, M) \text{ for all } g \in N\backslash i.
\end{aligned}$$

- Case 2: $\tilde{v}_{ij} = \hat{v}_{ij}$ ($\tilde{x} = x$, $\tilde{x} \neq x$)

$$\begin{aligned}\tilde{c}(N,M) &= \hat{c}(N,M) = c(N,M),\\ \tilde{c}_{-g}(N,M) &= \hat{c}_{-g}(N,M) = c_{-g}(N,M) \text{ for all } g \in N,\\ \tilde{c}(N\backslash g,M) &= \hat{c}(N\backslash g,M) \geq c(N\backslash g,M) \text{ for all } g \in N\backslash i.\end{aligned}$$

- Case 3: $\tilde{v}_{ij} > \hat{v}_{ij}$ ($\tilde{x} \neq x$)

$$\begin{aligned}\tilde{c}(N,M) &> \hat{c}(N,M),\\ \tilde{c}_{-g}(N,M) &\gtreqless \hat{c}_{-g}(N,M) \text{ for all } g \in N,\\ \tilde{c}(N\backslash g,M) &\geq \hat{c}(N\backslash g,M) \text{ for all } g \in N\backslash i.\end{aligned}$$

For item's Vickrey prices this has the following implications. In case 1 since no reassignment occurs, prices are described by

$$\tilde{p}_j^V = \tilde{c}(N\backslash h,M) - \tilde{c}(N,M) + v_{hj} \geq p_j^V \tag{4.8a}$$
$$\tilde{p}_j^V = v_j^S \qquad = p_j^V \tag{4.8b}$$
$$\tilde{p}_k^V = \tilde{c}(N\backslash i,M) - \tilde{c}(N,M) + v_{ik} \ = p_k^V \tag{4.8c}$$
$$\tilde{p}_l^V = \tilde{c}(N\backslash g,M) - \tilde{c}(N,M) + v_{gl} \geq p_l^V, \quad g \in N\backslash\{i,h\}, l \in M\backslash\{j,k\} \tag{4.8d}$$
$$\tilde{p}_l^V = v_l^S \qquad = p_l^V, \quad l \in M\backslash\{j,k\} \tag{4.8e}$$

If a bidder exists that wins j under x, we denote him by h. The comparison to the price p associated with v_{ij} is given. The results follow directly from the impact of $\tilde{v}_{ij}$ on the coalitional values as characterized above and the definition $p_l^V = v_j^S$ if item l is unsold.

Note that the inequalities and equalities hold for all starting valuations $v_{ij} < \hat{v}_{ij}$. Therefore, in (4.8a) and (4.8d), the inequality indicates that $\tilde{p}_j^V$ and $\tilde{p}_l^V$ are weakly monotone increasing in $\tilde{v}_{ij}$ on the range v_{ij} to $\hat{v}_{ij}$.

In case 2, $\tilde{x} = x$ and $\tilde{x} = \hat{x}$ are both efficient. If several efficient assignments exist, Vickrey prices for all efficient assignments are identical (Lemma A.1 in Appendix A.3).[12] However, prices may be associated with different winners or some items may

[12]Here, it is important to note that we refer to Vickrey prices instead of Vickrey payments; Vickrey *payments* may differ.

be sold or unsold for higher values $\tilde{v}_{ij}$. We provide calculations of prices for both assignments (left: $\tilde{x} = x$, right: $\tilde{x} \neq x$).[13] Case 2 is relevant for the transition between assignments. The equations and inequalities from case 1 are also valid here, because it is just a special case of $\tilde{x} = x$.

$$\hat{p}_j^V = \tilde{c}(N\backslash h, M) - \tilde{c}(N, M) + v_{hj} = \tilde{c}(N\backslash i, M) - \tilde{c}(N, M) + \tilde{v}_{ij} \tag{4.9a}$$
$$\hat{p}_j^V = v_j^S = \tilde{c}(N\backslash i, M) - \tilde{c}(N, M) + \tilde{v}_{ij} \tag{4.9b}$$
$$\hat{p}_k^V = \tilde{c}(N\backslash i, M) - \tilde{c}(N, M) + v_{ik} = \tilde{c}(N\backslash g, M) - \tilde{c}(N, M) + v_{gj}, \quad g \in N\backslash i \tag{4.9c}$$
$$\hat{p}_k^V = \tilde{c}(N\backslash i, M) - \tilde{c}(N, M) + v_{ik} = v_k^S \tag{4.9d}$$
$$\hat{p}_l^V = \tilde{c}(N\backslash g, M) - \tilde{c}(N, M) + v_{gl} = \tilde{c}(N\backslash f, M) - \tilde{c}(N, M) + v_{fl}, \quad g \in N\backslash \{i, h\},\ f \in N\backslash i,\ l \in M\backslash \{j, k\} \tag{4.9e}$$
$$\hat{p}_l^V = \tilde{c}(N\backslash g, M) - \tilde{c}(N, M) + v_{gl} = v_l^S, \quad g \in N\backslash \{i, h\},\ l \in M\backslash \{j, k\} \tag{4.9f}$$
$$\hat{p}_l^V = v_l^S = \tilde{c}(N\backslash g, M) - \tilde{c}(N, M) + v_{gl}, \quad g \in N\backslash i,\ l \in M\backslash \{j, k\} \tag{4.9g}$$
$$\hat{p}_l^V = v_l^S, \quad l \in M\backslash \{j, k\} \tag{4.9h}$$

In case 3, the unique assignment is $\tilde{x} \neq x$, i.e., the items are assigned to the bidders on the right in equalities (4.9a)–(4.9c), (4.9e), and (4.9g), and are unsold in (4.9d), (4.9f), and (4.9h). Remember that we consider a stepwise increase in $\tilde{v}_{ij}$ starting at a valuation $v_{ij} \leq \hat{v}_{ij}$. For the analysis of the overall impact on prices, we therefore have to take the earlier changes (from cases 1 and 2) into account. In the following equations, however, we only consider the relation to prices at $\hat{v}_{ij}$.

$$\tilde{p}_j^V = \tilde{c}(N\backslash i, M) - \tilde{c}(N, M) + \tilde{v}_{ij} = \hat{p}_j^V \tag{4.10a}$$
$$\tilde{p}_k^V = \tilde{c}(N\backslash g, M) - \tilde{c}(N, M) + v_{gj} \leq \hat{p}_k^V, \quad g \in N\backslash i \tag{4.10b}$$
$$\tilde{p}_k^V = v_k^S = \hat{p}_k^V \tag{4.10c}$$
$$\tilde{p}_l^V = \tilde{c}(N\backslash g, M) - \tilde{c}(N, M) + v_{gl} \leq \hat{p}_l^V, \quad g \in N\backslash i,\ l \in M\backslash \{j, k\} \tag{4.10d}$$
$$\tilde{p}_l^V = v_l^S = \hat{p}_l^V, \quad l \in M\backslash \{j, k\} \tag{4.10e}$$

Equations (4.10c) and (4.10e) need no further explanation: the item is not sold and the price is determined by definition. Equations (4.10a), (4.10b), and (4.10d) are

[13] We assume that no more than two efficient assignments exist. If there are more than two, we may deal with this in an analogous manner.

	$A1$	$A2$		$A1$	$A2$
$B1$	2	**2**	$\rightsquigarrow$	2	**5**
$B2$	0	2		0	2
B^h	$B1$	$B2$		–	$B1$
p^V	1	1		1	2
$g \in B^I$	–	$B1$		–	$B2$
v^S	1	0		1	0

Figure 4.5: Example of (P4): $i = B1$ wins auction $k = A1$ before the change from $v_{12} = 2$ to $\tilde{v}_{12} = 5$ and he wins auction $j = A2$ after the change. Seller $A1$'s item is not sold after the increase.

derived from Proposition 4.1.

Consider the impact of the stepwise increase in $\tilde{v}_{ij}$ starting at value v_{ij}. Possible paths connecting the equations in case 1 to those in case 3 are the following:

$$
\begin{array}{ll}
j & \left\{ \begin{array}{llllll} (4.8a) & \longrightarrow & (4.9a) & \longrightarrow & (4.10a) & (P1) \\ (4.8b) & \longrightarrow & (4.9b) & \longrightarrow & (4.10a) & (P2) \end{array} \right. \\[2ex]
k & \left\{ \begin{array}{llllll} \multirow{2}{*}{(4.8c)} & \nearrow & (4.9c) & \longrightarrow & (4.10b) & (P3) \\ & \searrow & (4.9d) & \longrightarrow & (4.10c) & (P4) \end{array} \right. \\[2ex]
l & \left\{ \begin{array}{llllll} (4.8d) & \nearrow & (4.9e) & \longrightarrow & (4.10d) & (P5) \\ & \searrow & (4.9f) & \longrightarrow & (4.10e) & (P6) \\ (4.8e) & \nearrow & (4.9g) & \longrightarrow & (4.10d) & (P7) \\ & \searrow & (4.9h) & \longrightarrow & 4.10e & (P8) \end{array} \right.
\end{array}
$$

On paths (P1) and (P2), price $\tilde{p}_j^V$ increases weakly until $\hat{p}_j^V$ and is then constant for further increases in $\tilde{p}_j^V$. With this, part (1) of Proposition 4.2 is proved. Price $\tilde{p}_k^V$ is constant until $\tilde{v}_{ij} = \hat{v}_{ij}$ and decreases weakly afterwards, as can be concluded from the equations on paths (P3) and (P4). This confirms Proposition 4.2 (2) and (3). Figure 4.5 illustrates (4.10c).

Let us consider the remaining paths (P5), (P6), (P7), and (P8). The inequalities along these may be combined as follows. We denote the values of $\tilde{p}_l^V$ in case 1 by $\tilde{p}_l^{V,1}$

and in case 3 by $\tilde{p}_l^{V,3}$,[14] and get

$$\begin{aligned}
&(\text{P5}) \quad p_l^V \leq \tilde{p}_l^{V,1} = \tilde{c}(N\backslash g, M) - \tilde{c}(N,M) + v_{gl} \\
&\qquad\qquad \leq \hat{p}_l^V = \tilde{c}(N\backslash f, M) - \tilde{c}(N,M) + v_{fl} \geq \tilde{p}_l^{V,3}, \quad g \in N\backslash\{i,h\},\ f \in N\backslash i\,, \\
&(\text{P6}) \quad p_l^V \leq \tilde{p}_l^{V,1} = \tilde{c}(N\backslash g, M) - \tilde{c}(N,M) + v_{gl} \leq \hat{p}_l^V = v_l^S = \tilde{p}_l^{V,3}, \quad g \in N\backslash\{i,h\}\,, \\
&(\text{P7}) \quad p_l^V = \tilde{p}_l^{V,1} = v_l^S = \hat{p}_l^V = \tilde{c}(N\backslash g, M) - \tilde{c}(N,M) + v_{gl} \geq \tilde{p}_l^{V,3}, \quad g \in N\backslash i\,, \\
&(\text{P8}) \quad p_l^V = \tilde{p}_l^{V,1} = v_l^S = \hat{p}_l^V = \tilde{p}_l^{V,3}.
\end{aligned}$$

Part (4) of Proposition 4.1 follows from the comparison between p_l^V and $\tilde{p}_l^{V,1}$ in (P5)–(P8). It remains to prove part (5) of the proposition.

Firstly, consider path (P8). The price in auction l is not influenced at all by the increase in $\tilde{v}_{ij}$. Thus, these auctions are of category L_2. We denote this by $l_{(P8)} \in L_2$.

On (P7), price $\tilde{p}_l^{V,1}$ in assignment x equals v_l^S. Thus, the inequality ($\geq \tilde{p}_l^{V,3}$) may only be fulfilled with equality, $v_j^S = \tilde{p}_l^{V,3}$, because a decrease below v_j^S is impossible.[15] Therefore, price $\tilde{p}_l^V$ is constant and $l_{(P7)} \in L_2$.

Similarly, along path (P6), price $\tilde{p}_l^{V,3}$ in the new assignment equals v_l^S. Thus, even though item l is sold when $\tilde{v}_{ij} \leq \hat{v}_{ij}$, price $\tilde{p}_l^{V,1}$ calculated by g's payment formula may not be lower than $\tilde{p}_l^{V,3}$. It follows that price $\tilde{p}_l^V$ is constant and $l_{(P6)} \in L_2$.

However, along path (P5), we find $p_l^V \leq \tilde{p}_l^{V,1} \leq \hat{p}_l^V \geq \tilde{p}_l^{V,3}$. To analyze this case in detail, we distinguish the subcases $f \neq g$ (for values $\tilde{v}_{ij}$ above $\hat{v}_{ij}$ bidder g is replaced by a different bidder f as the winner of l) and $f = g$ (g remains the winner of l).

Let us first consider the subcase $f \neq g$, i.e., the assignment of l changes at $\tilde{v}_{ij} = \hat{v}_{ij}$. In this case, prices are calculated by $\tilde{p}_l^V = \tilde{p}_l^{V,1} = \tilde{c}(N\backslash g, M) - \tilde{c}(N,M) + v_{gl}$ for $\tilde{v}_{ij} \leq \hat{v}_{ij}$ and by $\tilde{p}_l^V = \tilde{p}_l^{V,3} = \tilde{c}(N\backslash f, M) - \tilde{c}(N,M) + v_{fl}$ for $\tilde{v}_{ij} \geq \hat{v}_{ij}$. If $\tilde{v}_{ij}$ is not part of $\tilde{c}(N\backslash g, M)$ for all $\tilde{v}_{ij} \leq \hat{v}_{ij}$, then the price $\tilde{p}_l^V = \tilde{p}_l^{V,1}$ is constant until $\hat{v}_{ij}$ is reached. If, however, $\tilde{v}_{ij}$ contributes to $\tilde{c}(N\backslash g, M)$ for some value of $\tilde{v}_{ij}$, then $\tilde{p}_l^V = \tilde{p}_l^{V,1}$ is weakly increasing.[16] For bidder f's payment, the reverse holds: if $\tilde{v}_{ij}$ is part of $\tilde{c}(N\backslash f, M)$ for $\tilde{v}_{ij} = \hat{v}_{ij}$, then $\tilde{p}_l^V = \tilde{p}_l^{V,3}$ is constant for all $\tilde{v}_{ij} \geq \hat{v}_{ij}$. If, on the other hand, $\tilde{v}_{ij}$ is not part of $\tilde{c}(N\backslash f, M)$, price $\tilde{p}_l^V = \tilde{p}_l^{V,3}$ decreases from $\tilde{v}_{ij} = \hat{v}_{ij}$ on,

[14] Following this logic, one might introduce $\tilde{p}_l^{V,2} = \hat{p}_l^V$.

[15] Suppose g is assigned to l in the optimal assignment $\tilde{x}$. Then $\tilde{c}(N,M) = \tilde{c}(N\backslash g, M\backslash l) + v_{gl} - v_l^S$. With $\tilde{c}(N\backslash g, M\backslash l) \leq \tilde{c}(N\backslash g, M)$ we get $\tilde{c}(N,M) - v_{gl} = \tilde{c}(N\backslash g, M\backslash l) - v_l^S \leq \tilde{c}(N\backslash g, M) - v_l^S$. From this, $v_l^S \leq \tilde{c}(N\backslash g, M) - \tilde{c}(N,M) + v_{gl} = \tilde{p}_l^V$ follows for Vickrey prices p_l^V.

[16] It increases strictly once $\tilde{v}_{ij}$ enters $\tilde{c}(N\backslash g, M)$ and is constant for lower values of $\tilde{v}_{ij}$ if $\tilde{v}_{ij} = v_{ij}$ is not already part of $\tilde{c}(N\backslash g, M)$.

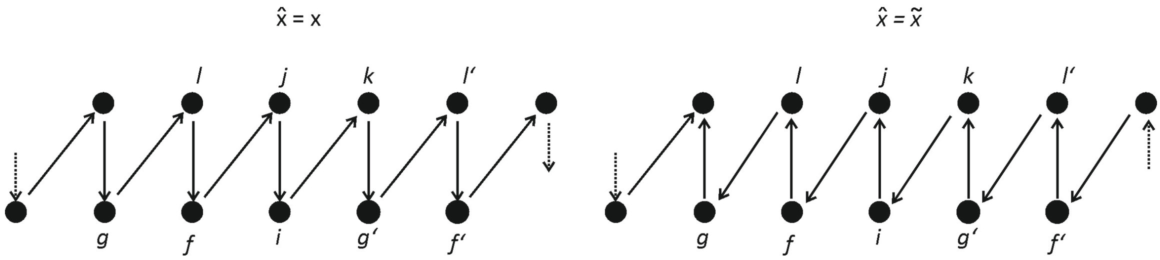


Figure 4.6: A reassignment of i, g, f, g', and f' at the transition between two efficient assignments at $\tilde{v}_{ij} = \hat{v}_{ij}$.

until eventually $\tilde{v}_{ij}$ enters $\tilde{c}(N\backslash f, M)$.

In the current subcase $f \neq g$, the transition between prices $\tilde{p}_l^{V,1}$ below and $\tilde{p}_l^{V,3}$ above $\hat{p}_l^V$ not only depends on coalitional values but also on v_{gl} and v_{fl}. However, remember that Vickrey prices for all efficient assignments are identical. Therefore, to analyze the impact of the increase in $\tilde{v}_{ij}$ we can again concentrate on the development of the differences $\tilde{c}(N\backslash g, M) - \tilde{c}(N, M)$ and $\tilde{c}(N\backslash f, M) - \tilde{c}(N, M)$. It is clear that $\tilde{c}(N, M)$ is constant for $\tilde{v}_{ij} \leq \hat{v}_{ij}$ and increasing for $\tilde{v}_{ij} \geq \hat{v}_{ij}$. Thus, we explore the relationship between $\tilde{c}(N\backslash g, M)$ and $\tilde{c}(N\backslash f, M)$.

If $g \neq f$ a reassignment of l takes place at value $\tilde{v}_{ij} = \hat{v}_{ij}$. Since both assignments are efficient and bidders have the same Vickrey payments in all efficient assignments, we conclude that the reassignment occurs along an indifference path. Figure 4.6 depicts such an indifference path both for $\hat{x} = x$ (left hand side) and for $\hat{x} = \tilde{x}$ (right hand side). In (P5), l is reassigned from g to f at $\tilde{v}_{ij} = \hat{v}_{ij}$. Thus, in the figure, $f \in B^I(l)$, i.e., f is the first bidder on the indifference path starting at l. In the following argument, we distinguish between situations where g and f are predecessors or successors of i and j on the indifference path. Therefore, Figure 4.6 contains bidders g and f and item l who are predecessors of i and j as well as g', f', and l' who are successors.

From Lemma 4.1(2) we conclude that $\tilde{c}(N\backslash g, M)$ and $\tilde{c}(N\backslash f, M)$ are both derived from $\tilde{c}(N, M)$ by replacing the summands on the indifference path starting at g and f, respectively. Figure 4.7 illustrates how to construct the new indifference path when a bidder is absent. In the upper left corner, bidder g does not participate. The path formerly starting at g is reversed and stops at l.[17] In the lower left corner, the

[17]This describes the situation at prices $\hat{p}$. However, note that if g is absent, prices of items on this

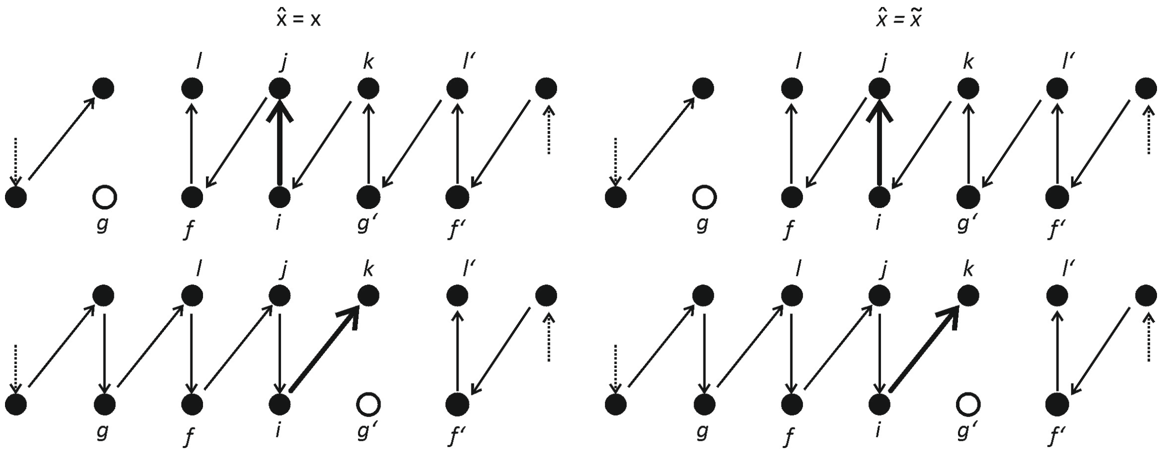


Figure 4.7: The assignments associated with $\tilde{c}(N\backslash g, M)$ (upper row) and $\tilde{c}(N\backslash g', M)$ (lower row) at the two efficient assignments $\hat{x} = x$ (left) and $\hat{x} = \tilde{x}$ (right).[18]

indifference path constructed from x if g' leaves is illustrated. The graphs on the right correspond to the adaptation of $\tilde{x}$ when g or g' are absent.

In the following, we need to distinguish between $\tilde{v}_{ij}$ contributing to $\tilde{c}(N\backslash g, M)$, $\tilde{c}(N\backslash f, M)$, $\tilde{c}(N\backslash g', M)$, and $\tilde{c}(N\backslash f', M)$. In Figure 4.7 a bold arrow indicates if i wins j in coalitions $(N\backslash g, M)$ and $(N\backslash g', M)$ or if i wins k. Consider the graph in the upper left corner. If g is absent, i wins j and therefore $\tilde{v}_{ij} = \hat{v}_{ij}$ is part of $\tilde{c}(N\backslash g, M)$. Obviously a rearrangement when f is absent would also assign i to j. The same applies to the graph in the upper right corner. Thus, if g and f are predecessors of i and j in the graph that describes $\hat{x} = x$, then $\tilde{v}_{ij} = \hat{v}_{ij}$ is part of both $\tilde{c}(N\backslash g, M)$ and $\tilde{c}(N\backslash f, M)$. Similarly, considering successors g' and f' of i and j, we conclude from the graphs in the lower row that $\tilde{v}_{ij} = \hat{v}_{ij}$ is neither part of $\tilde{c}(N\backslash g', M)$ nor $\tilde{c}(N\backslash f', M)$.

Therefore, it is impossible for $\tilde{v}_{ij}$ to be part of $\tilde{c}(N\backslash g, M)$ ($\tilde{p}_l^{V,1}$ increases) but not of $\tilde{c}(N\backslash f, M)$ ($\tilde{p}_l^{V,3}$ decreases). If $\tilde{v}_{ij}$ is neither part of $\tilde{c}(N\backslash g, M)$ nor $\tilde{c}(N\backslash f, M)$ at $\tilde{v}_{ij} = \hat{v}_{ij}$, then $\tilde{p}_l^V = \tilde{p}_l^{V,1}$ is constant and $\tilde{p}_l^V = \tilde{p}_l^{V,3}$ decreases until eventually $\tilde{v}_{ij}$ contributes to $\tilde{c}(N\backslash f, M)$. Then, $l_{(P5)} \in L_1$. On the other hand, if $\tilde{v}_{ij}$ is both part of

reversed indifference path may be decreased until some other bidder outside the indifference path becomes indifferent or an auction reaches a price equal to the seller's valuation. In the current analysis, we are only interested in coalitional values and not in market prices without g.

[18] The respective assignments for f and f' are derived equivalently.

Table 4.1: Evolution of prices along paths (P1)–(P8) for increasing valuations $\tilde{v}_{ij}$.

	$v_{ij} \leq \tilde{v}_{ij} \leq \hat{v}_{ij}$	$\tilde{v}_{ij} \geq \hat{v}_{ij}$	Proof of part
(P1)	$(\longrightarrow) \nearrow$	$\longrightarrow$	(1)
(P2)	$(\longrightarrow) \nearrow$	$\longrightarrow$	(1)
(P3)	$\longrightarrow$	$\searrow \longrightarrow$	(2), (3)
(P4)	$\longrightarrow$	$\longrightarrow$	(2), (3)
(P5)	$\longrightarrow$	$\searrow \longrightarrow$	(4), (5) L_1
	$(\longrightarrow) \nearrow$	$\longrightarrow$	(4), (5) L_3
	$\longrightarrow$	$\longrightarrow$	(4), (5) L_2
(P6)	$\longrightarrow$	$\longrightarrow$	(4), (5) L_2
(P7)	$\longrightarrow$	$\longrightarrow$	(4), (5) L_2
(P8)	$\longrightarrow$	$\longrightarrow$	(4), (5) L_2

$\tilde{c}(N\backslash g, M)$ and of $\tilde{c}(N\backslash f, M)$ at $\tilde{v}_{ij} = \hat{v}_{ij}$, then $\tilde{p}_l^V = \tilde{p}_l^{V,1}$ increases and $\tilde{p}_l^V = \tilde{p}_l^{V,3}$ is constant, i.e., $l_{(P5)} \in L_3$. A special case occurs if $\tilde{v}_{ij}$ enters $\tilde{c}(N\backslash g, M)$ at $\tilde{v}_{ij} = \hat{v}_{ij}$. Then, price $\tilde{p}_l^V$ is constant, i.e., $l_{(P5)} \in L_2$ in this case.

If $f = g$, the assignment of l does not change. It follows that g is not on the indifference path of i and j at $\tilde{v}_{ij} = \hat{v}_{ij}$. Bidder g may be part of a different indifference path but not that which contains i and j. According to part (2) of Lemma 4.1, the coalitional value $\tilde{c}(N\backslash g, M)$ is thus calculated by replacing summands on g's path and retaining those off this path. Therefore, summands that are associated with the indifference path of i and j are retained. Hence it follows that $\tilde{v}_{ij} \leq \hat{v}_{ij}$ is not part of $\tilde{c}(N\backslash g, M)$ but $\tilde{v}_{ij} \geq \hat{v}_{ij}$ contributes to $\tilde{c}(N\backslash g, M)$. Thus, prices $\tilde{p}_l^{V,1}$ are constant because $\tilde{v}_{ij}$ is not part both of $\tilde{c}(N, M)$ and $\tilde{c}(N\backslash g, M)$. At $\tilde{v}_{ij} = \hat{v}_{ij}$, $\tilde{v}_{ij}$ enters both $\tilde{c}(N, M)$ and $\tilde{c}(N\backslash g, M)$, and, therefore, prices $\tilde{p}_l^{V,3}$ are also constant. This gives auctions l in category L_2, i.e., $l_{(P5)} \in L_2$ if $g = f$.

With this, part (5) of the proposition is proved. ■

Table 4.1 gives an overview of the results related to paths (P1)–(P8) in the proof.

The comprehensive example in Figures 4.8 and 4.9 shows the different classes of Proposition 4.2 for $k = A1$, $j = A3$, $L_1 = \{A2\}$, $L_2 = \{A5, A6\}$, and $L_3 = \{A4\}$.

Figure 4.9 illustrates selected coalitional value functions for increasing values of $\tilde{v}_{ij}$. Furthermore, the evolution of prices p_2^V, p_4^V, and p_5^V is given, where $\hat{v}_{13} = 16$ and, for example, $p_4^V = \tilde{c}(N\backslash B4, M) - \tilde{c}(N, M) + v_{44}$ for $\tilde{v}_{13} \leq 16$ and $p_4^V = \tilde{c}(N\backslash B3, M) -$

	A1	A2	A3	A4	A5	A6
B1	5	5	***0***	0	0	0
B2	0	10	10	0	0	0
B3	0	0	15	5	0	0
B4	0	0	0	3	0	0
B5	2	0	0	0	0	0
B6	1	0	0	0	0	0
B7	0	0	0	0	3	1
B8	0	0	0	0	2	1
B^h	B1	B2	B3	B4	B7	B8
p^V	2	2	2	0	1	0
$g \in B^I$	B5	B1	B2	B5	B8	B5

$\rightsquigarrow$

	A1	A2	A3	A4	A5	A6
B1	5	5	***6***	0	0	0
B2	0	10	10	0	0	0
B3	0	0	15	5	0	0
B4	0	0	0	3	0	0
B5	2	0	0	0	0	0
B6	1	0	0	0	0	0
B7	0	0	0	0	3	1
B8	0	0	0	0	2	1
B^h	B1	B2	B3	B4	B7	B8
p^V	2	2	3	0	1	0
$g \in B^I$	B5	B1	B2	B5	B8	B5

$\rightsquigarrow$

	A1	A2	A3	A4	A5	A6
B1	5	5	***14***	0	0	0
B2	0	10	10	0	0	0
B3	0	0	15	5	0	0
B4	0	0	0	3	0	0
B5	2	0	0	0	0	0
B6	1	0	0	0	0	0
B7	0	0	0	0	3	1
B8	0	0	0	0	2	1
B^h	B1	B2	B3	B4	B7	B8
p^V	2	2	11	1	1	0
$g \in B^I$	B5	B1	B2	B3	B8	B5

$\rightsquigarrow$

	A1	A2	A3	A4	A5	A6
B1	5	5	***16***	0	0	0
B2	0	10	10	0	0	0
B3	0	0	15	5	0	0
B4	0	0	0	3	0	0
B5	2	0	0	0	0	0
B6	1	0	0	0	0	0
B7	0	0	0	0	3	1
B8	0	0	0	0	2	1
B^h	B1	B2	B3	B4	B7	B8
	B5	B2	B1	B3	B7	B8
p^V	2	2	13	3	1	0
$g \in B^I$	B5	B1	B2	B3	B8	B5
	B1	B1	B3	B4	B8	B5

$\rightsquigarrow$

	A1	A2	A3	A4	A5	A6
B1	5	5	***17***	0	0	0
B2	0	10	10	0	0	0
B3	0	0	15	5	0	0
B4	0	0	0	3	0	0
B5	2	0	0	0	0	0
B6	1	0	0	0	0	0
B7	0	0	0	0	3	1
B8	0	0	0	0	2	1
B^h	B5	B2	B1	B3	B7	B8
p^V	1	1	13	3	1	0
$g \in B^I$	B6	B1	B2	B4	B8	B6

$\rightsquigarrow$

	A1	A2	A3	A4	A5	A6
B1	5	5	***20***	0	0	0
B2	0	10	10	0	0	0
B3	0	0	15	5	0	0
B4	0	0	0	3	0	0
B5	2	0	0	0	0	0
B6	1	0	0	0	0	0
B7	0	0	0	0	3	1
B8	0	0	0	0	2	1
B^h	B5	B2	B1	B3	B7	B8
p^V	1	0	13	3	1	0
$g \in B^I$	B6	B6	B2	B4	B8	B6

Figure 4.8: A numerical example of the different kinds of influences of a stepwise increase in $v_{ij} = v_{13}$ on prices: p_1^V and p_2^V increase, p_3^V and p_4^V decrease, and p_5^V and p_6^V are constant.

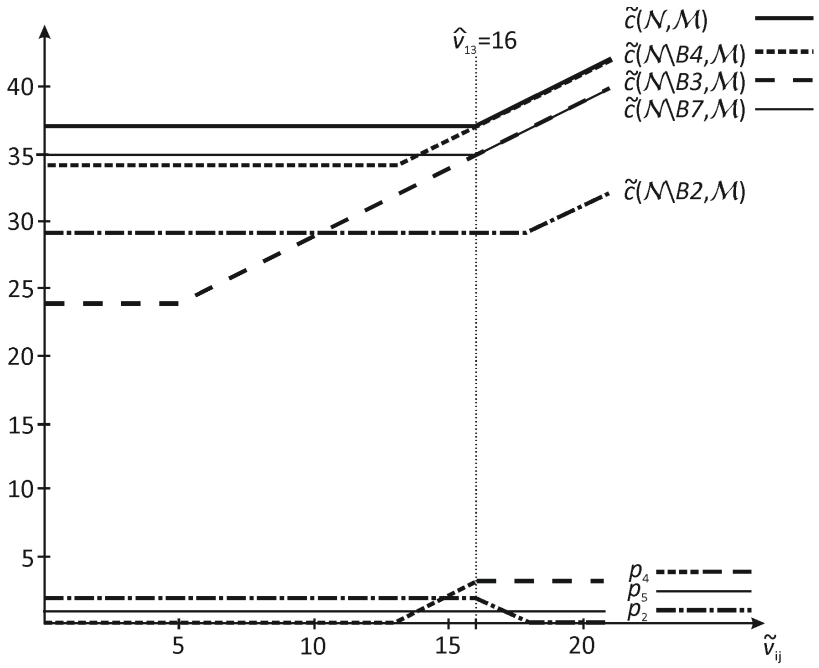


Figure 4.9: Illustration of some selected prices of the example in Figure 4.8.

$\tilde{c}(N, M) + v_{34}$ for $\tilde{v}_{13} \geq 16$.

Remember that the outcome (x, p^V) is the bidder-optimal outcome in the core, which is the worst outcome for all sellers, and that the set of outcomes related to core payoffs equals the set of competitive equilibria.

We illustrate the set of competitive equilibria of the example in Figure 4.10 with prices p_1 and p_2 as axes in Figure 4.11. The lines define the range of prices at which the bidders prefer the efficient assignment. For example, in the first graph, $B3$ prefers not to buy an item whenever both prices are higher than one. This is the area above/to the right of his lines. $B2$ prefers to buy item $A2$ for all price combinations below his line, and $B1$ prefers $A1$ for all prices to the left of his line. The gray area is the set of competitive equilibria. The set shrinks as $B1$'s line moves upwards until it is only a single point. This is when $\hat{v}_{12} = 4$ is reached. Then the efficient assignment changes and $B1$'s line is reflected. Then it moves upwards again and the set increases. The outcome (x, p^V) is given by the lower left corner of the set of competitive equilibria. This price vector is at first not influenced by $\tilde{v}_{12}$, then it moves upwards, and finally to the left.

The cores of the example of Figure 4.10, neglecting the seller's payoffs, are illus-

	$A1$	$A2$		$A1$	$A2$		$A1$	$A2$		$A1$	$A2$	
$B1$	2	***0***	$\rightsquigarrow$	2	***1***	$\rightsquigarrow$	2	***2***	$\rightsquigarrow$	2	***3***	$\rightsquigarrow$
$B2$	0	3		0	3		0	3		0	3	
$B3$	1	1		1	1		1	1		1	1	
B^h	$B1$	$B2$		$B1$	$B2$		$B1$	$B2$		$B1$	$B2$	
p^V	1	1		1	1		1	1		1	2	
$g \in B^I$	$B3$	$B3$		$B3$	$B3$		$B3$	$B3$		$B3$	$B3$	

$B1$	2	***4***	$\rightsquigarrow$	2	***5***
$B2$	0	3		0	3
$B3$	1	1		1	1
B^h	$B1$ $B3$	$B2$ $B1$		$B3$	$B1$
p^V	1	3		0	3
$g \in B^I$	$B3$ $B1$	$B1$ $B2$		$B1$	$B2$

Figure 4.10: Example of the impact of a stepwise increase in $\tilde{v}_{ij}$ on prices.

trated in Figure 4.12. The axes give payoffs u_1, u_2, and u_3 of bidders $B1$, $B2$, and $B3$. Note that for the calculation of bidder $B1$'s payoff we use the respective valuation $\tilde{v}_{12}$, i.e., we compare the core for different levels of $B1$'s valuation for $A2$. The core is the gray area and corresponds to the payoffs determined by the competitive prices given in Figure 4.11 in the efficient assignment. The bidders' part of the core shrinks as $\tilde{v}_{12}$ increases until it is only a single point at $\tilde{v}_{12} = 4$. Since sellers' payoffs in this example equal the prices in Figure 4.11, the core at $\tilde{v}_{12} = 4$ is a singleton. Then the core expands but now on the plane spanned by u_1 and u_3. The bidder-optimal outcome is the upper right corner in the first three graphs and the equivalent corner in the other plane in the last two graphs. The core for $\tilde{v}_{12} = 0$, which is not depicted, equals that for $\tilde{v}_{12} = 1$. The first increase in the figure from $\tilde{v}_{12} = 1$ to $\tilde{v}_{12} = 2$ decreases the core but not the bidder-optimal outcome in the core. With a further increase, the bidder-optimal outcome decreases for $B2$ until $\tilde{v}_{12} = 4$. Then $B1$'s and $B3$'s payoffs increase until $\tilde{v}_{12} = 5$. From then on only $B1$'s payoff increases, which is caused by the increase in his valuation.

The next proposition considers the remaining case of an increase in v_{ij} when i is a

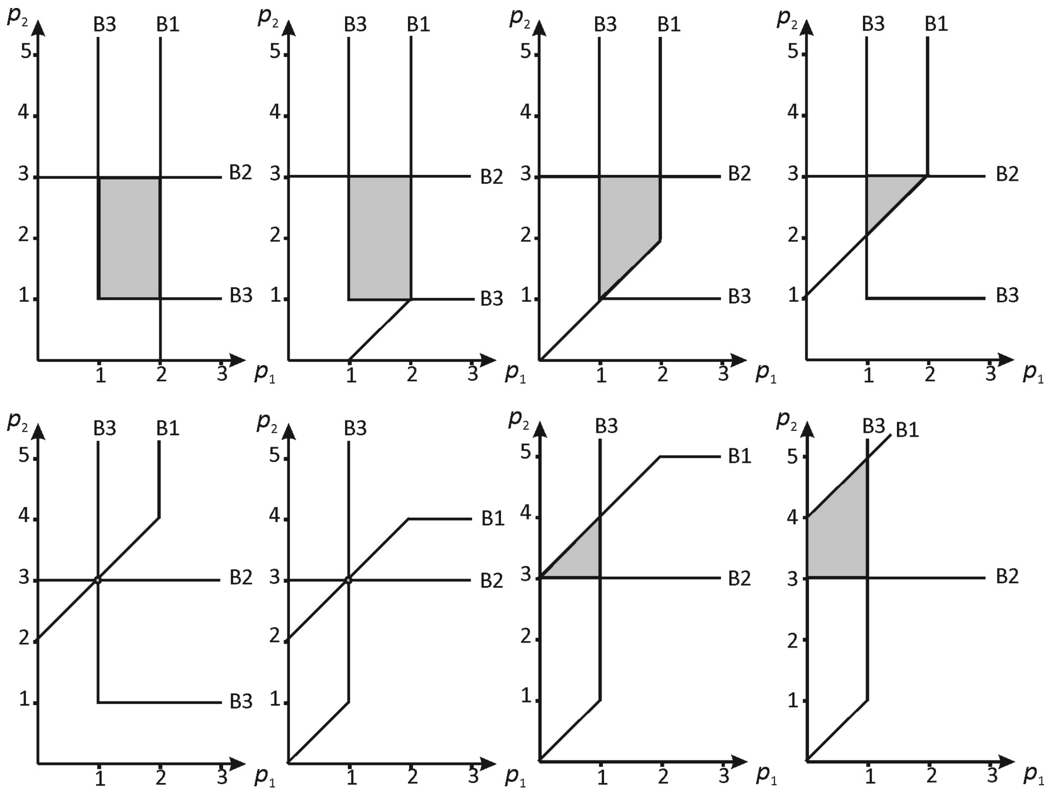


Figure 4.11: Evolution of the set of competitive prices in the example in Figure 4.10 for values $\tilde{v}_{12} = 0, 1, 2, 3, 4, 4, 5, 6$ (axes are prices).

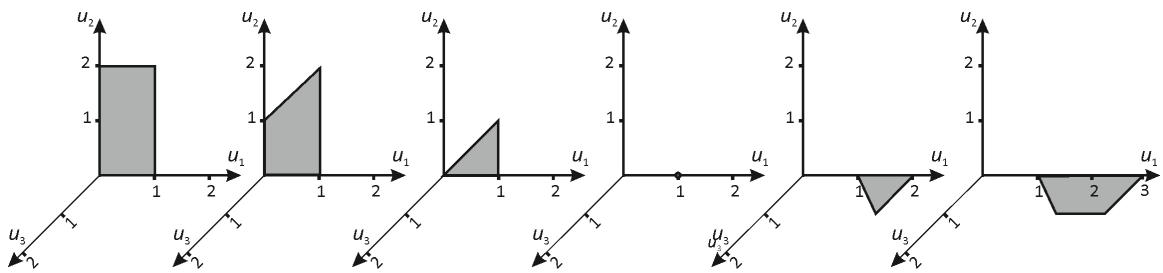


Figure 4.12: Evolution of bidders' core payoffs in the example in Figure 4.10 for values $\tilde{v}_{12} = 1, \ldots, 6$ (axes are payoffs).

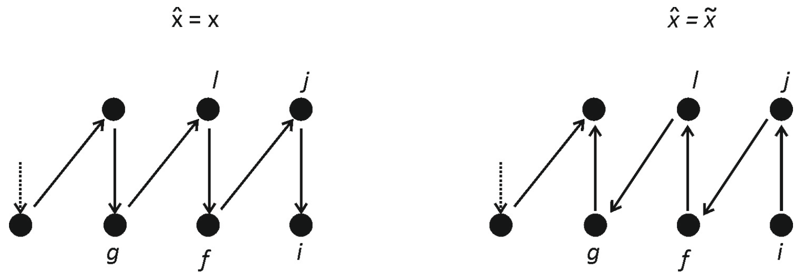


Figure 4.13: A reassignment of i, g, and f at the transition between two efficient assignments at $\tilde{v}_{ij} = \hat{v}_{ij}$.

losing bidder.

Proposition 4.3 *If the valuation v_{ij} of a bidder i with $x_{il} = 0$ for all l increases ceteris paribus to $\tilde{v}_{ij} > v_{ij}$, we get prices $\tilde{p}$ with*

(1) $\tilde{p}_j^V \geq p_j^V$,

(2) $\tilde{p}_l^V \geq p_l^V$ *for* $l \in M \backslash j$ *if* $\tilde{x} = x$, *and*

(3) three disjoint categories L_1, L_2, $L_3 \subset M \backslash \{j\}$ *of auctions with prices*

- $\tilde{p}_l^V \leq p_l^V$ *and* $\tilde{p}_l^V < p_l^V$ *for a high enough value* $\tilde{v}_{ij}$ *if* $l \in L_1$ *and* $\tilde{x} \neq x$,
- $\tilde{p}_l^V = p_l^V$ *for all* $\tilde{v}_{ij} > v_{ij}$ *if* $l \in L_2$ *and* $\tilde{x} \neq x$, *and*
- $\tilde{p}_l^V \geq p_l^V$ *and* $\tilde{p}_l^V > p_l^V$ *for a high enough value* $\tilde{v}_{ij}$ *if* $l \in L_3$ *and* $\tilde{x} \neq x$.

Proof of Proposition 4.3: As in Proposition 4.2, the assignments x and $\tilde{x}$ only differ if i wins j with $\tilde{v}_{ij}$. Otherwise, x is unchanged by the increase in v_{ij}.

We prove the current proposition by referring to the proof of Proposition 4.2. The only difference is that in the current proposition bidder i does not win an auction if his valuation is v_{ij} whereas i wins an auction k in Proposition 4.2. Therefore, equations (4.8a), (4.8b), (4.8d), and (4.8e) of case 1 also apply here. Equivalently, equations (4.9a), (4.9b), (4.9e), (4.9g), and (4.9h) of case 2, as well as (4.10a), (4.10d), and (4.10e) of case 3 remain valid. Thus, paths (P1), (P2), (P5), (P7), and (P8) have to be considered. The indifference path at $\tilde{v}_{ij} = \hat{v}_{ij}$ used to prove (P5) is given in Figure 4.6. For $\hat{x} = x$, i is indifferent between winning j and not winning j, but i does not win any auction. In the graph on the right, i wins j at $\hat{x} = \tilde{x}$. Thus, all bidders g and f in the proof of Proposition 4.2 are predecessors of i and j on the indifference path

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$		$A1$	$A2$	$A3$	
$B1$	5	5	0		5	5	0		5	5	0	
$B2$	0	10	10		0	10	10		0	10	10	
$B3$	0	0	15		0	0	15		0	0	15	
$B4$	2	***1***	1	$\rightsquigarrow$	2	***3***	1	$\rightsquigarrow$	2	***10***	1	$\rightsquigarrow$
B^h	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$		$B1$	$B2$	$B3$	
									$B1$	$B4$	$B3$	
p^V	2	2	2		2	3	3		2	10	10	
$g \in B^I$	$B4$	$B1$	$B2$		$B4$	$B4$	$B2$		$B4$	$B4$	$B2$	
									$B4$	$B2$	$B2$	

$B1$	5	5	0		5	5	0		5	5	0	
$B2$	0	10	10		0	10	10		0	10	10	
$B3$	0	0	15		0	0	15		0	0	15	
$B4$	2	***11***	1	$\rightsquigarrow$	2	***12***	1	$\rightsquigarrow$	2	***20***	1	
B^h	$B1$	$B4$	$B3$		$B1$	$B4$	$B3$		$B1$	$B4$	$B3$	
p^V	1	10	10		0	10	10		0	10	10	
$g \in B^I$	$B4$	$B2$	$B2$		$B4$	$B2$	$B2$		$B2$	$B2$	$B2$	

Figure 4.14: Example of Proposition 4.3: $i = B4$ wins auction $j = A2$ for values $\tilde{v}_{42} \geq \hat{v}_{42} = 10$ and prices finally decrease (p_1^V) or increase (p_2^V and p_3^V).

associated with $\hat{x} = x$ and successors in the graph associated with $\hat{x} = \tilde{x}$. However, the argument is the same as before for all relevant paths (P1), (P2), (P5), (P7), and (P8) and the result follows immediately. ■

Note that the results in Propositions 4.2 and 4.3 allow for arbitrary combinations of price increases, decreases and unchanged prices in the population, depending on the respective valuation matrix. Compare, for example, the prices for the first and the last valuation matrix in Figure 4.14. An increase in v_{42} has lead to a decrease in p_1^V but to an increase in p_2^V and p_3^V.

In contrast, in Figure 4.15 all prices increase. In this example, Bidder $i = B4$ displaces $B1$ as the winner of $A1$ when $\tilde{v}_{41} < v_{11}$, i.e., $B4$'s valuation for item $A1$ is lower than that of the former winner. Note that for all values $\tilde{v}_{41} \geq \hat{v}_{41} = 2$ there are no changes in prices. This is because $B4$ is the last bidder on the only indifference path in this example (which includes all bidders and auctions) before the reassignment, and

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$		$A1$	$A2$	$A3$		$A1$	$A2$	$A3$
$B1$	5	8	0		5	8	0		5	8	0		5	8	0
$B2$	0	10	20		0	10	20		0	10	20		0	10	20
$B3$	0	0	15		0	0	15		0	0	15		0	0	15
$B4$	***1***	0	0	$\rightsquigarrow$	***2***	0	0	$\rightsquigarrow$	***3***	0	0	$\rightsquigarrow$	***10***	0	0
B^h	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$								
					$B4$	$B1$	$B2$		$B4$	$B1$	$B2$		$B4$	$B1$	$B2$
p^V	1	4	14		2	5	15		2	5	15		2	5	15
B^I	$B4$	$B1$	$B2$		$B4$	$B1$	$B2$								
					$B1$	$B2$	$B3$		$B1$	$B2$	$B3$		$B1$	$B2$	$B3$

Figure 4.15: Example of Proposition 4.3: $i = B4$ wins auction $j = A1$ for values $\tilde{v}_{41} \geq \hat{v}_{41} = 2$ and all prices increase.

the first bidder on the indifference path afterwards. Thus, for values $\tilde{v}_{41}$ above $\hat{v}_{41} = 2$, $\tilde{v}_{41}$ is part of all coalitional values $\tilde{c}(N\backslash g, M)$ for $g \in \{B1, B2, B3\}$.

Keep in mind that in this environment Vickrey prices are the lowest competitive prices. The intuition behind all of these price decreasing results is that increasing valuation v_{ij} may reduce i's demand in other auctions if i wins j. Thus, the price in these auctions may decrease and other bidders may therefore also reduce their demands in other auctions.[19] If i does not win auction j, prices may only increase in $\tilde{v}_{ij}$.

The following results are based on Propositions 4.1–4.3. There, we considered three starting points for the increase in v_{ij}. Now we differentiate the results by the direction of the impact on individual items' Vickrey prices. Corollaries 4.1, 4.2, 4.3, and 4.5 contain monotonicity results.

Corollary 4.1 *The price p_j^V is weakly monotone increasing in v_{ij} if $x_{ij} = 0$. If $x_{ij} = 1$, p_j^V is constant (x being the optimal assignment before the increase).*

The first statement of Corollary 4.1 follows from Proposition 4.2(1) and Proposition 4.3(1) by considering arbitrary valuations v_{ij}. The second part follows from the property of a Vickrey auction that a bidder cannot influence the price he pays and

[19] In the dynamic variant (the multiple-auctions game) i may bid less in other auctions, which reduces the prices in the other auctions whenever he was the unique price determining bidder or indirectly influenced the price by being part of an indifference path.

from the fact that a further increase in v_{ij} does not change the assignment, so the Vickrey price p_j^V is associated with i's Vickrey payment (or from Proposition 4.1(1)).

Corollary 4.2 follows directly from Proposition 4.1(2).

Corollary 4.2 *Prices p_l^V for $l \in M\backslash j$ are weakly monotone decreasing in v_{ij} if $x_{ij} = 1$ (x being the optimal assignment before the increase in v_{ij}).*

As can be seen from Propositions 4.2(5) and 4.3(3) category L_1, condition $x_{ij} = 1$ is not a necessary condition but is a sufficient condition for the price to be weakly monotone increasing.

On the other hand, we also identify a condition that characterizes increasing prices.

Corollary 4.3 *Prices p_l^V for $l \in M$ are weakly monotone increasing in v_{ij} if $\tilde{x}_{ij} = 0$ ($\tilde{x}$ being the optimal assignment after the increase in v_{ij}).*

From the condition $\tilde{x}_{ij} = 0$ it follows that $x_{ij} = 0$ and, as we know from the proofs of Propositions 4.2 and 4.3, $\tilde{v}_{ij} \leq \hat{v}_{ij}$ and $\tilde{x} = x$. With this, Corollary 4.3 follows from Proposition 4.2 (1), (2), (4), and Proposition 4.3 (1) and (2), because these propositions cover all cases for $x_{ij} = 0$.

From the possibility of decreasing prices and Corollary 4.3, we get Corollary 4.4.

Corollary 4.4 *An increase in v_{ij} may lead to a decrease in prices p_l^V for $l \in M\backslash j$ only if $\tilde{x}_{ij} = 1$.*

In addition we find limits on the price increases or decreases resulting from an increase in v_{ij} to $\tilde{v}_{ij}$.

Proposition 4.4 *An increase of v_{ij} to $\tilde{v}_{ij}$ may lead to a maximum increase in the sum of prices of $\min\{n, m\}(\tilde{v}_{ij} - v_{ij})$ and a maximum decrease in the sum of prices of $(\min\{n, m\} - 1)(\tilde{v}_{ij} - v_{ij})$.*

Proof of Proposition 4.4: A price increase is due to an increase in the difference $\tilde{p}_l^V = \tilde{c}(N\backslash g, M) - \tilde{c}_{-g}(N, M)$ for $\tilde{x}_{gl} = 1$. Since both coalitional values weakly increase in v_{ij}, to get an increase in $\tilde{p}_l^V$, $\tilde{c}(N\backslash g, M)$ has to increase while $\tilde{c}_{-g}(N, M)$ is constant. This is possible if $\tilde{x}_{ij} = 0$. Thus, as all changes in the assignment occur at $\tilde{v}_{ij} = \hat{v}_{ij}$, we do not have to consider a change in the winning bidder in auction l (see Table 4.1 for the range of $\tilde{v}_{ij}$ in which prices $\tilde{p}_l^V$ may increase). But $\tilde{c}(N\backslash g, M)$ cannot increase by more than $\tilde{v}_{ij}$ increases. Furthermore, if $n > m$ then an increase

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$
$B1$	3	3	0		3	3	0	***3***	3	0	$\rightsquigarrow$	***5***	3	0
$B2$	0	5	5		0	5	5	0	5	5		0	5	5
$B3$	0	0	10		0	0	10	0	0	10		0	0	10
$B4$	***1***	0	0	$\rightsquigarrow$	***3***	0	0	2	0	0		2	0	0
B^h	$B1$	$B2$	$B3$		$B4$	$B2$	$B3$	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$
p^V	1	1	1		3	3	3	2	2	2		2	0	0
B^I	$B4$	$B1$	$B2$		$B1$	$B1$	$B2$	$B4$	$B1$	$B2$		$B4$	–	–

Figure 4.16: Example of the maximum influence of an increase in $\tilde{v}_{ij}$ on the sum of prices (see Proposition 4.4).

can only occur in the m auctions and if $m > n$ then prices can only increase in the n auctions in which items are sold. Thus, the increase is less than $\min\{n, m\}(\tilde{v}_{ij} - v_{ij})$. Figure 4.16 shows on the left-hand side an example where the maximum increase of $\min\{n, m\}(\tilde{v}_{ij} - v_{ij}) = 3 \cdot (3 - 1) = 6$ occurs.

The argument for the maximum decrease is similar. In this case, $\tilde{c}(N\backslash g, M)$ has to stay constant if $\tilde{c}_{-g}(N, M)$ increases. Thus, $\tilde{x}_{ij} = 1$. The coalitional value $\tilde{c}_{-g}(N, M)$ increases by weakly less than v_{ij} increases. The number of affected prices is restricted to $(\min\{n, m\} - 1)$ because the price in j does not decrease (Corollary 4.1). An example of the maximum decrease in the sum of prices is given in Figure 4.16 on the right-hand side: $(\min\{n, m\} - 1)(\tilde{v}_{ij} - v_{ij}) = 2 \cdot (5 - 3) = 4$. ■

The maximum increase occurs if i does not win j at v_{ij} and is the last bidder on all indifference paths that together contain all bidders. The maximum decrease occurs if i wins j at v_{ij} and is the last winning bidder on all indifference paths that together contain all bidders, the losing bidders valuations are zero for items $l \neq j$, and the seller's reservation values v_j^S are also zero. The price in j can never decrease.

According to Proposition 4.4, even if there is a single seller who offers all items, his payoff (the sum of prices) may decrease in v_{ij}. It may even decrease by more than the increase in v_{ij}.

The next corollary summarizes the monotonicity results of Propositions 4.1–4.3. It follows from the impact of the stepwise increase in v_{ij} on each price $\tilde{p}_l^V$ for all values of $\tilde{v}_{ij}$.

Corollary 4.5 (Individual price monotonicity) *Every price p_l^V, $l \in M$, is either*

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$		$A1$	$A2$	$A3$
$B1$	5	8	0		5	8	0		5	8	0
$B2$	0	10	***15***	$\rightsquigarrow$	0	10	***19***	$\rightsquigarrow$	0	10	***21***
$B3$	2	0	15		0	0	15		0	0	15
$B4$	1	0	0		1	0	0		1	0	0
B^h	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$		$B3$	$B1$	$B2$
p^V	1	4	***9***		1	4	***13***		***1***	3	14
B^I	$B4$	$B1$	$B2$		$B4$	$B1$	$B2$		$B4$	$B2$	$B3$

Figure 4.17: The Vickrey payment of $B3$ is not monotone in $\tilde{v}_{23}$.

weakly monotone increasing or weakly monotone decreasing in v_{ij}.

Note, however, that Vickrey payments p_g^V are not necessarily monotone. That is, the smooth transition at $\tilde{v}_{ij} = \hat{v}_{ij}$, which we used for the argument above, does not mean that the price a bidder g pays either increases or decreases monotonically in $\tilde{v}_{ij}$. If $\tilde{v}_{ij}$ is increased stepwise, a reassignment of g may occur at $\tilde{v}_{ij} = \hat{v}_{ij}$. Thus, $\tilde{c}_{-g}(N, M)$ may change because it depends on g's assignment. If g wins k for values $\tilde{v}_{ij} \leq \hat{v}_{ij}$ and he wins l for $\tilde{v}_{ij} \geq \hat{v}_{ij}$, then the associated values of $\tilde{c}_{-g}(N, M)$ differ by $v_{gk} - v_{gl}$.

An example of such a discontinuity in Vickrey payments is given in Figure 4.17. $B3$'s Vickrey payment increases from 9 to 13 if v_{23} increases from 15 to 19. But if v_{23} continues to increase from 19 to 21, $B3$ wins $A1$ instead of $A3$ and his Vickrey payment decreases from 13 to 1.

We can, however, transfer the individual price monotonicity result, which implies monotonicity in the sellers' payoffs, to bidders' payoffs.

Proposition 4.5 (Individual payoff monotonicity) *(1) Every bidder g's Vickrey payoff u_g^V for $g \in N\backslash i$ is either weakly monotone increasing or weakly monotone decreasing in v_{ij}.*

(2) Every seller k's Vickrey payoff u_k^V for $k \in M$ is either weakly monotone increasing or weakly monotone decreasing in v_{ij}.

Proof of Proposition 4.5: Part (2) follows directly from Corollary 4.5 because a seller's payoff is determined by the price he receives, or it is equal to zero and may then only increase in v_{ij}.

Concerning part (1), remember that $u_g^V = c(N,M) - c(N\backslash g, M)$. If the stepwise increased v_{ij} is part of $c(N,M)$ or $c(N\backslash g, M)$, it remains part of the coalitional values for all higher values of v_{ij}. Thus, if it enters $c(N,M)$ before it enters $c(N\backslash g, M)$, g's payoff weakly increases in v_{ij} and it weakly decreases otherwise. Since the payoff only depends on these two coalitional values and they do not depend on the underlying assignment, there is no other case to be considered and (1) is proved. ■

Notice that bidder i's payoff $u_i^V = c(N,M) - c(N\backslash i, M)$ is weakly monotone increasing because $\tilde{c}(N,M) \geq c(N,M)$ and $\tilde{c}(N\backslash i, M) = c(N\backslash i, M)$. On the other hand, if bidder i falsely reported a $\tilde{v}_{ij}$ above his true valuation v_{ij}, then i's payoff weakly decreases because submitting v_i truthfully is a weakly dominant strategy.

An important implication of Propositions 4.2 and 4.3 is recorded in the following Corollary 4.6.

Corollary 4.6 *In the Vickrey auction with multiple heterogeneous units and unit demand, profitable collusion may be possible by increasing bids.*

The corollary follows directly from the possibility of decreasing prices (see, for example, Propositions 4.2 and 4.3). Then, a bidder i may, by increasing his bid, decrease the price another bidder h has to pay without increasing his own payment (see the example in Figure 4.18 described below). Note that without side payments only bidder h may strictly profit from the deviation. However, if side payments are allowed, h can transfer some of his gains to i.

Consider the example in Figure 4.18, where the valuation matrix on the left gives the true valuations of the bidders and the valuation matrix on the right shows a deviation of $B1$ by bidding $b_1 = (15, 10, 10)$ instead of $b_1 = (10, 10, 10)$. The result is that $B2$ and $B3$ pay less. Via side payments, $B1$ may also profit from his deviation if he colludes with $B2$ or $B3$.

In the single-unit Vickrey auction a profitable joint deviation as described in Corollary 4.6 is clearly not possible. There, every increase in a bid weakly increases the price.

The contribution of Corollary 4.6 is that profitable collusion is possible by *increasing* a bid when items are substitutes (with side payments if one requires strict improvements for all deviating bidders, without side payments if one allows for deviating bidders that do not strictly improve their payoffs). In the Vickrey auction, it is

	$A1$	$A2$	$A3$		$A1$	$A2$	$A3$
$B1$	10	10	10	$\rightsquigarrow$	***15***	10	10
$B2$	0	15	0		0	15	0
$B3$	0	0	15		0	0	15
$B4$	5	0	0		5	0	0
B^h	$B1$	$B2$	$B3$		$B1$	$B2$	$B3$
p^V	5	5	5		5	0	0
$g \in B^I$	$B4$	$B1$	$B1$		$B4$	$B1$	$B1$

Figure 4.18: Example where profitable collusion by increasing a bid is possible.

of course possible for groups of bidders to deviate profitable by *decreasing* bids. For example, in the single-unit second-price auction, the bidder with the second highest valuation may collude with the winner: he decreases his bid and thereby decreases the price.

A crucial property of such collusion is that not all colluding bidders' payoffs strictly increase. At least one bidder will have the same payoff as before. Therefore, only if side payments are allowed may all bidders strictly profit from colluding.

Remember that equilibrium payoffs in this Vickrey auction with substitutes are in the core. That is, no coalition of players may profitably deviate. However, profitable collusion by bidders is possible (Corollary 4.6). The reason for this difference is that any profitable deviation in the current game requires that coalitions contain the seller because the coalitional value of any group of bidders is zero without a seller. Therefore, it is important to remember that the core considers stability of the outcome of the auction and excludes profitable renegotiations after the auction (see, e.g., Milgrom (2004) and Day and Milgrom (2008)) but it does not exclude profitable deviations by groups of bidders.

Note that our results also hold for a unilateral *decrease* in a valuation v_{ij}. This may be deduced from the proof of Proposition 4.1 and the use of a stepwise increase in $\tilde{v}_{ij}$ in the proofs of Propositions 4.2 and 4.3. This implies that decreasing a bid in the Vickrey auction with substitutes may increase prices and the sum of prices (Proposition 4.4): if $B1$ in Figure 4.18 replaces his bid $b_1 = (10, 10, 10)$ by $\tilde{b}_1 = (0, 10, 10)$, Vickrey prices change from $p^V = (5, 5, 5)$ to $\tilde{p}^V = (0, 10, 10)$. The sum of prices increases from 15 to 20.

4.2 Summary and Relation to the Multiple-Auctions Game

Summary and Outlook It is well known that if all bidders have substitutes valuations adding a bidder can only lead to an increase in prices (see e.g., Shapley and Shubik (1971) for the case of unit-demand preferences and Milgrom (2004) for general substitutes). The results of this chapter add the following insights.

- An increase in a bid may cause a decrease in Vickrey prices (e.g., Corollary 4.2).
- Adding a bid (interpreted as increasing a former zero-bid) may cause a decrease in Vickrey prices (e.g., Figures 4.4, 4.10, and 4.8).
- An increase in a bid may cause a decrease in the sum of Vickrey prices that may even be larger than the increase in the bid (e.g., Figure 4.18). This implies that the sum of the sellers' revenues may decrease. Thus, a single seller's revenue is not monotonic in valuations.
- An increase in the bid of a bidder who did not win any item may also cause a decrease in Vickrey prices. Therefore, adding a bidder really means adding a new bidder that did not participate before and cannot be loosely interpreted as increasing the bids of losing bidders (Proposition 4.3(3)).
- If the increased bid causes a price decrease, then bidder i wins item j (Corollary 4.4).
- If i wins j before the bid increases, prices cannot increase (Corollary 4.2).
- Vickrey prices and payments are individually monotone increasing or decreasing in v_{ij} (Corollary 4.5 and Proposition 4.5).

A special case of such an increase is a change from $v_{ij} = 0$ to $v_{ij} > 0$. This special case is the basis of the analysis in Chapter 5, where we examine incomplete bidder-seller networks. Trade is restricted to bidder-seller pairs that are linked by an edge in the graph that represents the network. As in the example in the introductory chapter, we assume that bidders have to know about auctions to be able to participate. Finding an item that is considered a substitute means adding a link to the network. Thus, the results of the current chapter provide the basis for the analysis of bidder-seller networks in the following chapter.

Relation to the Multiple-Auctions Game Note that there is a difference between "bidding higher" in the Vickrey auction and "bidding higher" in the multiple-auctions game. In the multiple-auctions game, the prices weakly increase if a bidder submits a higher bid and the other bidders submit the same bids as before. We also know that a higher bid does not decrease the others' bids (Lemma 3.7, Section 3.2.3.2). A higher bid $\tilde{v}_{ij}$ in the Vickrey auction is equivalent to bidding according to σ_i^*, using valuation $\tilde{v}_{ij}$ (instead of v_{ij}) as an input in the multiple-auctions game. Then, it may be that i bids less in auctions $l \in M \backslash j$ and, if he is the price determining bidder in l, the price there may decrease. The results of the current section carry over in this way to the multiple-auctions game in equilibrium σ^*.

Consider the numerical example in Figure 4.18. Assume that all three items are offered by the same seller in independent Internet auctions on a platform. Bidder $B1$ asks the seller for some additional information about the item offered in $A1$ such as the exact name of the color, the product code, or information about certain interfaces of an electronic device and this information changes $B1$'s valuation for the item. If $B1$'s valuation increases to $\tilde{v}_{11} = 15$, then this may be a disadvantage for the seller because the resulting sum of prices in equilibrium is 5 instead of 15. On the other hand, if the seller's answer is dissatisfying to $B1$ and his valuation decreases to $\tilde{v}_{11} = 0$, prices change from $p = (5, 5, 5)$ to $\tilde{p} = (0, 10, 10)$ and the seller's payoff increases from 15 to 20 due to the negative answer.

If an increase in a valuation occurs before the bidding starts, the results are as just described. However, if the increase occurs during the bidding process, it may be too late to lower prices in the auctions where the price should decrease in equilibrium if the bidder has already submitted his (higher) price determining bids. Such an increase during the bidding process may be due to uncertain private valuations and the pseudo-endowment effect (Ehrhart, Ott, and Abele, 2008), uncertain private valuations and information acquisition (e.g., Rasmusen, 2006; Compte and Jehiel, 2007), or simply new information that the seller in an online auction adds to the description. Such an increase in valuations is captured by the current analysis only if it occurs early enough in the bidding process.

Chapter 5

Bidder-Seller Networks

The previous analysis dealt with complete bidder-seller networks: each bidder was able to bid in every auction. In this chapter, incomplete networks of bidders and sellers are examined.

Consider, for example, an online auction market. There are many reasons why the sets of auctions that bidders know about or consider relevant may differ. Usually, a seller assigns his item to one or several categories and he gives his item a long description with pictures as well as a brief description to be displayed on overview pages. A potential bidder, who is looking for an object, may click through categories or enter a search term. On many platforms, he can choose whether to search only the brief descriptions or the long descriptions as well. He may restrict his search to new articles or include used articles.[1] Since sellers sometimes have typos in their descriptions, a bidder may try to find such items by inserting different search terms (see, e.g., Sinclair, 2007).[2]

This description of offering and searching for Internet auctions illustrates why not all bidders find the same auctions. This motivates our investigation of incomplete bidder-seller networks. Other models that analyze bipartite networks of buyers and sellers are those of Kranton and Minehart (2000, 2001) and Corominas-Bosch (2004). Kranton and Minehart (2000, 2001) investigate an English button auction whereas Corominas-Bosch (2004) analyze bilateral bargaining.[3]

We define the network-restricted multiple-auctions game $\Gamma^a(G)$ and solve for an

[1] Remember the example with mountain bikes given in the introduction (Chapter 1).

[2] Auctions with typos in the description are usually found by fewer bidders. Thus, a bidder may hope to have an advantage by finding these items.

[3] The questions considered in the current chapter are close to those posed and answered for the homogeneous unit-demand case in Kranton and Minehart (2001).

ε-PBE. Using a comparative statics approach, we compare equilibrium outcomes in different networks. Specifically, we analyze the impact of additional links in the network on bidders' payoffs and prices.

5.1 The Network-Restricted Multiple-Auctions Game

A bidder-seller network is modeled as a bipartite graph. A bipartite graph is a graph whose vertices or nodes can be divided into two disjoint sets such that every edge in the graph connects nodes from different sets. In the following, we use the words network and graph interchangeably. In the bidder-seller network, the disjoint sets of vertices are the set of bidders N and the set of sellers M. We describe the bidder-seller network by a $n \times m$-matrix G of zeros and ones, $G := (g_{ij})_{i \in N, j \in M}$ with $g_{ij} \in \{0, 1\}$. The entry $g_{ij} = 1$ indicates an edge (also called a link or a connection) between bidder i and auction j; $g_{ij} = 0$ indicates that i and j have no connection. A bidder-seller network is incomplete if $g_{ij} = 0$ for at least one pair (i, j) of vertices. If $g_{ij} = 1$ for all (i, j), the network is complete.

A bidder only knows about auctions to which he is linked. Thus, i cannot bid in j if $g_{ij} = 0$. For the analysis we use the entry-wise product of V and G, $V^G := V \circ G = (v_{ij} g_{ij})_{i \in N, j \in M}$.[4] We call V^G the network-restricted valuation matrix. Since the entries of G are only zeros and ones, $v_{ij} g_{ij} = v_{ij}$ if $g_{ij} = 1$ (i.e., if i and j are connected) and $v_{ij} g_{ij} = 0$ otherwise.

According to σ_i^*, a bidder i with $v_{ij} = 0$ never bids in j in the multiple-auctions game. Thus, i neither wins j nor is he the price determining bidder in j. If $g_{ij} = 0$, he *cannot* bid in j. Even if i has a virtual valuation $v_{ij} > 0$, this is not relevant for him: he does not know that item j is offered in the market. For the equilibrium analysis (on the equilibrium path), it does not make a difference whether i does not bid because $v_{ij} = 0$ or because $g_{ij} = 0$. Note, however, that in general there is a difference between a valuation $v_{ij} = 0$ and a non-existent link $g_{ij} = 0$: if $g_{ij} = 0$, i does not have the option to bid in j. Thus, this condition is stronger because it restricts the strategy space and the set of feasible assignments. We formally define a network-restricted feasible assignment as follows.

[4] This operation is also called the Hadamard product.

Definition 5.1 (Network-restricted feasible assignment) *An assignment x is* feasible under network G *if*

$$\sum_{j \in M} x_{ij} \leq 1 \quad \forall i,$$
$$\sum_{i \in N} x_{ij} \leq 1 \quad \forall j,$$
$$x_{ij} = 1 \quad \Rightarrow \quad g_{ij} = 1.$$

We denote the set of network-restricted feasible assignments by X^G.

A *network-restricted efficient assignment* is an efficient assignment x in the market (N, M, V, v^S, G) that accounts for the (incomplete) network G (i.e., $x \in X^G$). Thus, the set of efficient assignments in (N, M, V, v^S, G) is a subset of the set of efficient assignments in (N, M, V^G, v^S).[5]

Next, consider the network-restricted Vickrey auction. Since i cannot buy j if $g_{ij} = 0$, it follows that $x_{ij} = 0$ in the efficient assignment x, even if $x_{ij} = 1$ in the complete network. Assume that i bids zero in j. Then this bid is not contained in any coalitional value (in the calculation of Vickrey payoffs) and i does not win j.[6] Thus, a zero bid results in the same outcome as a non-existent edge.

The network-restricted multiple-auctions game, denoted by $\Gamma^a(G)$, equals Γ^a with G added to the description. In our interpretation, $\Gamma^a(G)$ equals Γ^a with V replaced by V^G. We neglect the difference between $v_{ij} = 0$ and $g_{ij} = 0$ whenever it does not matter for our analysis.

To clarify the relation to the model of Kranton and Minehart (2001), we describe their basic model. Buyers in their market have homogeneous valuations. Several sellers exist who may sell one unit of the good. Buyers want to buy only one unit of the good. The market is replaced by an incomplete buyer-seller network so that each buyer can only buy from a subset of sellers and sellers may only sell to a subset of potential buyers. Kranton and Minehart implement an ascending auction mechanism where the price simultaneously and continuously increases in all auctions with at

[5] In (N, M, V^G, v^S) additional efficient assignments may exist in which i with $v_{ij}^G = 0$ is assigned to j if $v_{hj}^G = 0$ for all h.

[6] We neglect the borderline cases in which a bidder with a bid of zero can either win the item at a price of zero or not win it. In these cases, we assume that the network-restricted feasible efficient assignment, which then also exists, is chosen.

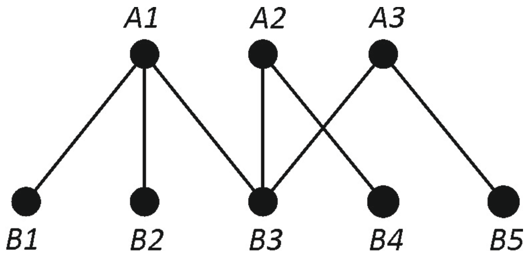


Figure 5.1: The incomplete network in the example taken from Kranton and Minehart (2001).

least two active bidders. Buyers indicate if they are still interested in buying at the given price from any seller they are connected to, which is very similar to an English button auction. It is a network-restricted English button auction. Since each bidder indicates all items in which he is interested at the given prices, this is equal to announcing demand.

Let us compare this model with the network-restricted multiple-auctions game $\Gamma^a(G)$. In $\Gamma^a(G)$, we consider heterogeneous instead of homogeneous unit-demand preferences, we analyze an (epsilon-) PBE or ex-post (epsilon-) equilibrium, the auctions are asynchronous second-price proxy auctions instead of simultaneous English button auctions, and sellers' reservation values and starting prices greater than zero are allowed.

We show by means of an example from Kranton and Minehart (2001) how the two models are related. Suppose three items $A1$, $A2$, and $A3$ are offered. Five bidders $B1, \ldots, B5$ participate in the market. Their valuations are $v_1 > v_2 > v_3 > v_4 > v_5$, where valuation v_i of bidder Bi is assumed to be identical for all three items (homogeneous unit-demand preferences). The incomplete network is illustrated in Figure 5.1. The efficient assignment is $x_{11} = x_{42} = x_{33} = 1$. Note that the assignments $x_{11} = x_{22} = x_{33} = 1$ and $x_{11} = x_{22} = x_{43} = 1$, which would yield higher surpluses, are not feasible in the given network. Equilibrium prices are $p_1 = v_2$ and $p_2 = p_3 = v_5$. In a complete network, a uniform market price $p = v_4$ would result.

We transfer the example into the context of our model and use the homogeneous valuations stipulated in the example. The original valuation matrix V is given in Figure 5.2 on the left, the network G is in the middle, and the network-restricted valuation matrix $V^G = V \circ G$ is on the right. The equilibrium outcomes (which are the same as the Vickrey outcomes) of the multiple-auctions game for valuation matrices

V	$A1$	$A2$	$A3$
$B1$	v_1	v_1	v_1
$B2$	v_2	v_2	v_2
$B3$	v_3	v_3	v_3
$B4$	v_4	v_4	v_4
$B5$	v_5	v_5	v_5
B^h	$B1$	$B2$	$B3$
p^V	v_4	v_4	v_4

G	$A1$	$A2$	$A3$
$B1$	1	0	0
$B2$	1	0	0
$B3$	1	1	1
$B4$	0	1	0
$B5$	0	0	1

V^G	$A1$	$A2$	$A3$
$B1$	v_1	0	0
$B2$	v_2	0	0
$B3$	v_3	v_3	v_3
$B4$	0	v_4	0
$B5$	0	0	v_5
B^h	$B1$	$B4$	$B3$
p^V	v_2	v_5	v_5

Figure 5.2: Adaptation of a valuation matrix V to the network-restricted valuation matrix V^G for the analysis of an incomplete bidder-seller network G.

V and V^G are given in the lower two rows (B^h and p^V). Obviously, the outcome of our network-restricted auctions game corresponds to the equilibrium outcome in Kranton and Minehart's example.

5.1.1 Equilibrium Analysis

To transfer the results of Chapters 3 and 4 to the game $\Gamma^a(G)$, we first state some technical notes. The network is an additional element in the multiple-auctions game. After determining bidders' valuations, nature selects an $n \times m$-matrix of zeros and ones. This gives the bidder-seller network G. At the beginning of the game, the network G is unknown to the bidders. Each bidder has private information about his links to auctions. During the bidding process, a bidder learns about other bidders' links by observing the high bidders. He updates his beliefs according to his observations under the assumption that all bidders follow σ^*. If a bidder observes a bid off the equilibrium path that contradicts his beliefs, he may either assume that the other bidder's valuation is higher or that the network is different from what he believed. We assume that he forms his beliefs over V^G such that he does not distinguish between $v_{ij} = 0$ and $g_{ij} = 0$.

We denote strategy σ^* by $\sigma^*(G)$ when we apply it in the network-restricted multiple-auctions game $\Gamma^a(G)$. Under $\sigma^*_i(G)$, bidder i with $g_{ij} = 0$ for some j orientates his bids at his network-restricted valuations v^G_i instead of v_i. Thus, $\sigma^*_i(G)$ is equal to σ^*_i if V is replaced by V^G.

Proposition 5.1 *Given a network G, in the reference outcome $(\bar{x}^{*,G}, \bar{p}^{*,G})$ that results from playing according to $\sigma^*(G)$ in a network-restricted multiple-auctions game*

$\Gamma^a(G)$, for every realization of valuations V and v^S

(1) the assignment $\bar{x}^{*,G}$ is an optimal assignment in X^G and

(2) $\bar{p}^{*,G} = p^V$ for Vickrey prices p^V in the market (N, M, V^G, v^S).

Proof of Proposition 5.1: From Section 3.2.2 we know that the reference outcome $(\bar{x}^*, \bar{p}^*)$ is a possible outcome if all bidders follow σ^* with valuations V.[7] Thus, $(\bar{x}^{*,G}, \bar{p}^{*,G})$ is also an outcome of the game $\Gamma^a(G)$ when bidders play according to σ^* using V^G to determine bids, i.e., when bidders play according to $\sigma^*(G)$. According to σ_i^*, a bidder i with valuation $v_{ij} = 0$ submits no bid in j in the game Γ^a. Thus, the resulting assignment $\bar{x}^{*,G}$ is feasible under the network G, $\bar{x}^{*,G} \in X^G$. From the definition of $\sigma_i^*(G)$, the resulting assignment $\bar{x}^{*,G}$ and prices $\bar{p}^{*,G}$ are exactly those of a game Γ^a with $V = V^G$. Hence, from Proposition 3.4, $\bar{x}^{*,G}$ is an optimal or efficient assignment for (V^G, v^S). Thus, all improvements to efficiency that are possible under V are inhibited by the incomplete network. Property (2) follows directly from Proposition 3.9. The graph G determines, for every feasible realization of valuations V, a feasible matrix V^G. Thus, the results are valid for all V and v^S. ■

Proposition 5.2 *Given a network G, $(\sigma^*(G), \mu)$ is an ε-PBE of $\Gamma^a(G)$ for $\iota < 1/\min\{n-1, m-1\}$.*

Proof of Proposition 5.2: According to Theorem 3.1, (σ^*, μ) is an ε-PBE of Γ^a for all V and $\iota < 1/\min\{n-1, m-1\}$. Thus, it is an ε-PBE for $V = V^G$. Bidder i is not allowed to bid in j if $g_{ij} = 0$, even if he has a potential valuation $v_{ij} > 0$. Thus, the transfer from a valuation matrix V to V^G only restricts i's options but does not change his incentives. It follows that $(\sigma^*(G), \mu)$ is an ε-PBE of $\Gamma^a(G)$.

For beliefs μ, we assume updating consistency to apply the one-shot-deviation principle as in Section 3.2.3.2. ■

From the argument in Section 3.2.3.2, we conclude that $\sigma^*(G)$ is also an ex-post ε-equilibrium.

[7] And for $\iota < 1/\{\min\{n-1, m-1\}$ all outcomes are close to the reference outcome. For details, see Section 3.2.2.5.

5.1.2 Network Analysis

In this section, we analyze the impact of adding links to a given network on the equilibrium (reference) outcome. We transfer and extend results of Kranton and Minehart (2001) to our multiple-auctions model and the case of unit-demand preferences for heterogeneous items.

Firstly, consider the following result of Kranton and Minehart (2001):

> "For a buyer, increasing its access to supply by adding a link to another seller weakly decreases the price it expects to pay, and vice versa for a seller." (Kranton and Minehart, 2001, p. 491)

We now investigate whether a similar statement is valid in the general unit-demand preferences case of the network-restricted multiple-auctions game. We analyze the bidders' side (Proposition 5.3) and then the sellers' side (Proposition 5.4). With heterogeneous items, the price alone does not decide bidders' payoffs. Thus, we consider bidders' payoffs instead of prices in the propositions. To see why a bidder's price (Vickrey payment) is not the relevant factor, consider the example in Figure 5.3. There, the link $\tilde{g}_{21} = 1$ is added to the network. As a consequence, the price that $B1$ pays increases from 1 (in $A1$) to 3 (in $A2$) but his payoff increases from $5 - 1 = 4$ to $8 - 3 = 5$. Thus, we concentrate on bidders' payoffs.

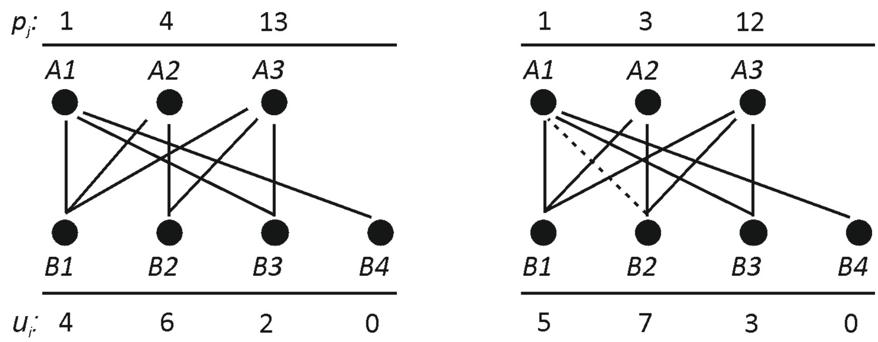
Denote the network that results from adding a link from i to j to network G by $\widetilde{G}$. We have $V^G = V \circ G$ and $V^{\widetilde{G}} = V \circ \widetilde{G}$. Thus, adding the link $\tilde{g}_{ij} = 1$ results in a change in the network-restricted valuation matrix: $v_{ij}^G = 0$ is replaced by $v_{ij}^{\widetilde{G}} \geq 0$. Analyzing the impact of an additional link is therefore equivalent to analyzing the effect of increasing a zero valuation. Hence, the relation to the analysis in Chapter 4 is obvious.

Adding a link to a network has the following consequences for bidders' equilibrium payoffs $u(\bar{x}^*, \bar{p}^*)$.

Proposition 5.3 *Suppose two bidder-seller networks $G \neq \widetilde{G}$ differ only in the entry $g_{ij} = 0$ and $\tilde{g}_{ij} = 1$ (i.e., a link between bidder i and auction j is added). Then, in the PBE outcomes $(\bar{x}^{*,G}, \bar{p}^{*,G})$ of $\Gamma^a(G)$ and $(\bar{x}^{*,\widetilde{G}}, \bar{p}^{*,\widetilde{G}})$ of $\Gamma^a(\widetilde{G})$,*

(1) $u_i(\bar{x}^{*,\widetilde{G}}, \bar{p}^{*,\widetilde{G}}) \geq u_i(\bar{x}^{*,G}, \bar{p}^{*,G})$, and

(2) $u_g(\bar{x}^{*,\widetilde{G}}, \bar{p}^{*,\widetilde{G}})$ could be lower, higher, or the same as $u_g(\bar{x}^{*,G}, \bar{p}^{*,G})$ for bidders $g \in N \backslash i$.



	V^G A1	A2	A3	$V^{\widetilde{G}}$ A1	A2	A3
$B1$	5	8	0	5	8	0
$B2$	0	10	19	***8***	10	19
$B3$	2	0	15	2	0	15
$B4$	1	0	0	1	0	0
B^h	$B1$	$B2$	$B3$	$B2$	$B1$	$B3$
p^V	1	4	13	1	3	12

Figure 5.3: Vickrey outcomes for one valuation matrix V and two networks G and $\widetilde{G}$ that differ only in that $g_{21} = 0$ while $\tilde{g}_{21} = 1$ (a link $\tilde{g}_{21}$ is added to the network G).

The analysis of adding a link is comparable to the analysis in Chapter 4. There, we considered the change from v_{ij} to $\tilde{v}_{ij} > v_{ij}$ both as a real increase and as a deviation by i from bidding truthfully. Concerning the network game, only the first point is relevant. Thus, a bidder i who obtains a new link to j, which is modeled as weakly increasing his valuation from $v_{ij}^G = 0$ to $v_{ij}^{\widetilde{G}} \geq 0$, calculates his payoff using $v_{ij}^{\widetilde{G}}$.[8]

Proof of Proposition 5.3: First we prove (1) that an additional link for bidder i weakly increases his payoff. From Proposition 5.1, the outcome in the PBE of $\Gamma^a(\widetilde{G})$ is the Vickrey outcome for bids $V^{\widetilde{G}}$. Bidding $v_i^{\widetilde{G}}$ is a weakly dominant strategy in the Vickrey auction. Thus, for a bidder with valuations $v_i^{\widetilde{G}}$, bidding v_i^G instead of $v_i^{\widetilde{G}}$ results in a weakly lower payoff. From this, i's payoff under G is weakly lower than under $\widetilde{G}$, i.e., $u_i(\bar{x}^{*,G}, \bar{p}^{*,G}) \leq u_i(\bar{x}^{*,\widetilde{G}}, \bar{p}^{*,\widetilde{G}})$, which completes the proof of (1).[9]

[8]Stated differently, in the Vickrey analysis of a deviation from truthful bidding, a bidder who bids $\tilde{v}_{ij}$ instead of his true valuation 0 nevertheless has to calculate his payoff as $0 - \tilde{p}_j$ if he wins j. However, in the current analysis, the increase in the valuation represents a new possible profitable trade for i and, winning j, his payoff equals $\tilde{v}_{ij} - \tilde{p}_j$.

[9]Alternatively, we can derive (1) from $u_i(\bar{x}^{*,G}, \bar{p}^{*,G}) = c^G(N, M) - c^G(N\backslash i, M)$, $c^{\widetilde{G}}(N, M) \geq c^G(N, M)$, and $c^{\widetilde{G}}(N\backslash i, M) = c^G(N\backslash i, M)$.

	V^G	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$V^{\widetilde{G}}$	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$
$B1$		5	5	***0***	0	0	0		5	5	***20***	0	0	0
$B2$		0	10	10	0	0	0		0	10	10	0	0	0
$B3$		0	0	15	5	0	0		0	0	15	5	0	0
$B4$		0	0	0	3	0	0		0	0	0	3	0	0
$B5$		2	0	0	0	0	0		2	0	0	0	0	0
$B6$		1	0	0	0	0	0		1	0	0	0	0	0
$B7$		0	0	0	0	3	1		0	0	0	0	3	1
$B8$		0	0	0	0	2	1		0	0	0	0	2	1
B^h		$B1$	$B2$	$B3$	$B4$	$B7$	$B8$		$B5$	$B2$	$B1$	$B3$	$B7$	$B8$
p^V		2	2	2	0	1	0		1	0	13	3	1	0

Figure 5.4: Example of Proposition 5.3.

Part (2) follows directly from Proposition 4.5(1) for $v_{ij} = 0$ and $\tilde{v}_{ij} = v_{ij}^{\widetilde{G}}$. ■

Figures 5.3 and 5.4[10] provide examples of result (1) in Proposition 5.3: the payoff of $i = B2$ increases from 6 to 7 and that of $i = B1$ from 3 to 7, respectively.

Moreover, Figure 5.4 illustrates result (2) of Proposition 5.3: the link between $B1$ and $A3$ is added and $B2$ and $B5$'s payoffs increase from 8 to 10 and from 0 to 1, respectively. $B4$'s payoff decreases from 3 to 0 and the payoffs of $B6$, $B7$, and $B8$ are unchanged.[11]

Note in particular that (2) is valid for a winning bidder $h \neq i$ of auction j in G (j is where the link is added): in Figure 5.4 the payoff of $h = B3$ decreases from 13 to 2, but in Figure 5.3 the payoff of $h = B1$ increases from 4 to 5.

A detailed analysis of this bidder h may be inferred from Corollary 4.1 and Proposition 4.5 (Chapter 4):[12] the price p_j^V in auction j weakly increases when a link is added to j. From individual payoff monotonicity, either p_j^V increases weakly and bidder h's payoff decreases,[13] or p_j^V remains constant and h's payoff weakly increases.[14] For values above the critical value $\hat{v}_{ij}$ (as determined in Chapter 4), no further change

[10] This example replicates the first and last valuation matrices considered in Figure 4.8.

[11] Note however, that $B5$'s payoff increases from 0 to $2 - 1 = 1$ and his Vickrey payment also increases from 0 to 1.

[12] Weakly monotone increasing price p_j in v_{ij} if $x_{ij} = 0$ (Corollary 4.1) and individual payoff monotonicity (Proposition 4.5).

[13] A decrease occurs for values $v_{ij}^{\widetilde{G}} \leq \hat{v}_{ij}$. For the meaning of $\hat{v}_{ij}$, see Chapter 4: $\hat{v}_{ij}$ indicates the minimum valuation such that i wins j and the assignment is altered.

[14] An increase in h's payoff may occur for values $v_{ij}^{\widetilde{G}} \geq \hat{v}_{ij}$.

in price p_j^V compared to the situation with $v_{ij}^{\widetilde{G}} = \hat{v}_{ij}$ occurs because i has no impact on p_j^V if the assignment is unaltered (because we consider a Vickrey auction).

Proposition 5.3(1) and (2) imply the respective results in Corollary 5.1. Part (3) of the corollary is due to the argument for bidder h above (in the corollary, i takes the former position of h).

Corollary 5.1 *Consider a bidder $i \in N$ in the network G.*

(1) Additional links to bidder i weakly increase his payoff.

(2) Additional links to other bidders in the network may increase or decrease i's payoff.

(3) If bidder i wins auction j before and after links to j are added, these additional links weakly decrease i's payoff.

The following proposition considers the sellers' side. It states that additional links to j weakly increase seller j's payoff, but additional links to other auctions in the network may increase or decrease j's payoff.

Proposition 5.4 *Suppose two bidder-seller networks $G \neq \widetilde{G}$ differ only in the entry $g_{ij} = 0$ and $\tilde{g}_{ij} = 1$ (i.e., a link between bidder i and auction j is added). Then, in the PBE outcomes $(\bar{x}^{*,G}, \bar{p}^{*,G})$ of $\Gamma^a(G)$ and $(\bar{x}^{*,\widetilde{G}}, \bar{p}^{*,\widetilde{G}})$ of $\Gamma^a(\widetilde{G})$,*

(1) $u_j^S(\bar{x}^{*,\widetilde{G}}, \bar{p}^{*,\widetilde{G}}) \geq u_j^S(\bar{x}^{*,G}, \bar{p}^{*,G})$ and $\bar{p}_j^{*,\widetilde{G}} \geq \bar{p}_j^{*,G}$, and

(2) $u_k^S(\bar{x}^{*,\widetilde{G}}, \bar{p}^{*,\widetilde{G}})$ could be lower than, equal to, or higher than $u_k^S(\bar{x}^{*,G}, \bar{p}^{*,G})$ for sellers $k \in M \backslash j$.

Proof of Proposition 5.4: Remember that $\bar{p}_j^{*,G}$ and $\bar{p}_j^{*,\widetilde{G}}$ equal the Vickrey payments of the winner of j. We use this in the following and denote the respective prices by p_j^V and $\tilde{p}_j^V$. The weak increase in the price p_j^V to $\tilde{p}_j^V$ follows directly from the result for case $x_{ij} = 0$ in Corollary 4.1. Thus, if j sells his item, we have $u_j^S(\bar{x}^{*,\widetilde{G}}, \tilde{p}^V) = \tilde{p}_j^V - v_j^S \geq p_j^V - v_j^S = u_j^S(\bar{x}^{*,G}, p^V)$. If j does not sell his item, his payoff is zero, and therefore may only increase. It is impossible for j to sell his item before the link is added but not afterwards: whenever $\bar{x}^{*,G} \neq \bar{x}^{*,\widetilde{G}}$ this implies that $\bar{x}_{ij}^{*,\widetilde{G}} = 1$.

Part (2) follows from Proposition 4.5(2) for $v_{ij} = 0$ and $\tilde{v}_{ij} = v_{ij}^{\tilde{G}}$. ∎

These results imply that an adaptation of the above-quoted statement of Kranton and Minehart (2001) applies to our environment: a buyer i's payoff weakly increases if his access to supply is increased by adding a link to another seller (Proposition 5.3(1)) and a seller's payoff weakly increases by adding a link to a bidder (Proposition 5.4(1)). However, i's payment may both increase and decrease (even if i is a winning bidder, i.e., a "buyer" in the quotation, before the link $\tilde{g}_{ij} = 1$ is added; see Figures 5.3 and 5.4).

Our results also refer to the indirect impact of a link on other agents (besides i and j if link $g_{ij} = 1$ is added) in the network. Before we further illustrate these results with examples, let us consider the following quote:

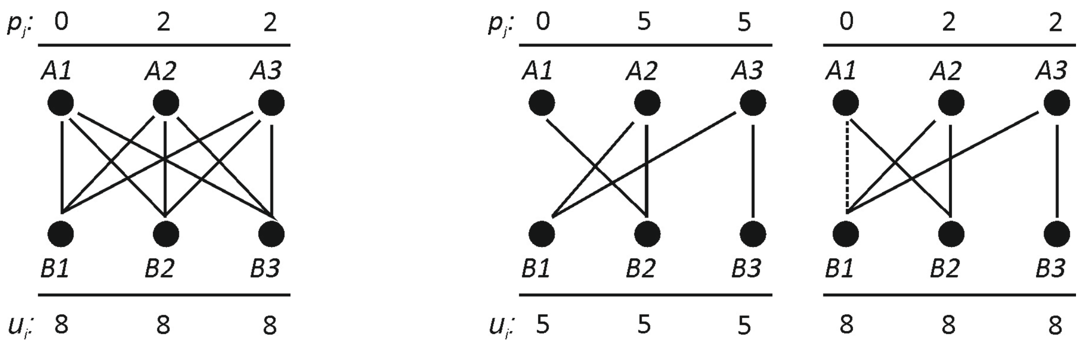
> "The payoffs of other agents also change in natural ways, e.g., the payoffs of other buyers linked to the seller with the additional link weakly decrease."
> (Kranton and Minehart, 2001, p. 491)

This claim is not true for the heterogeneous valuations case. For example, we already showed that the result in Proposition 5.3(2) is also valid for a bidder h who wins j before $\tilde{g}_{ij} = 1$ is added. The existence of link $g_{hj} = 1$ is a precondition for winning j and, thus, h is one of the "buyers linked to the seller with the additional link." In Figures 5.3 and 5.5, *every* bidder's payoff, independent of being linked to j or not, weakly increases.

On the other hand, Figure 5.6 shows that adding a link may weakly reduce all bidders' payoffs.[15] Figure 5.4 exemplifies a mixture of increasing and decreasing payoffs. Furthermore, these examples illustrate that all *prices* may strictly increase (Figure 5.6) and all prices may weakly decrease (Figure 5.5) if a link $\tilde{g}_{ij} = 1$ is added (weakly because the price p_j^V may only increase or be unaltered). A mixture of increasing and decreasing prices may also occur (Figures 5.3 and 5.4). For example, in Figure 5.3 $B1$ pays 1 and 3 before and after the link $\tilde{g}_{21} = 1$ is added, respectively, whereas $B3$ pays 13 and 12.

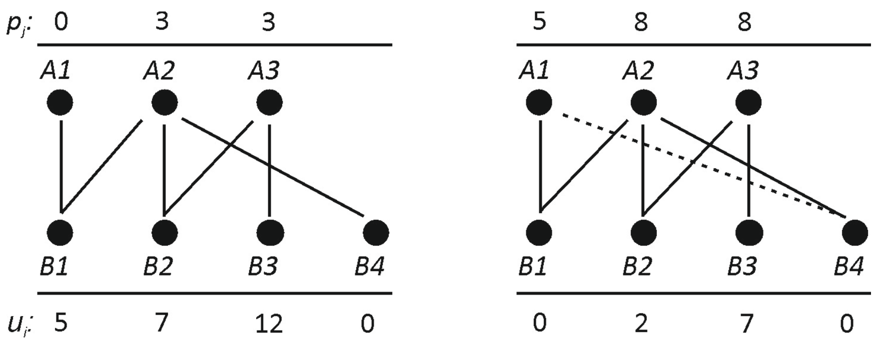
The quotation refers to bidders $h \neq i$ that are connected with j before $\tilde{g}_{ij} = 1$ is added. From the three examples, we may infer that in our environment all mixtures

[15]Specifically, in Figure 5.6 two efficient assignments exist. If $v_{41} = 6$, then $B4$'s payoff increases by one when the link $\tilde{g}_{41} = 1$ is added.



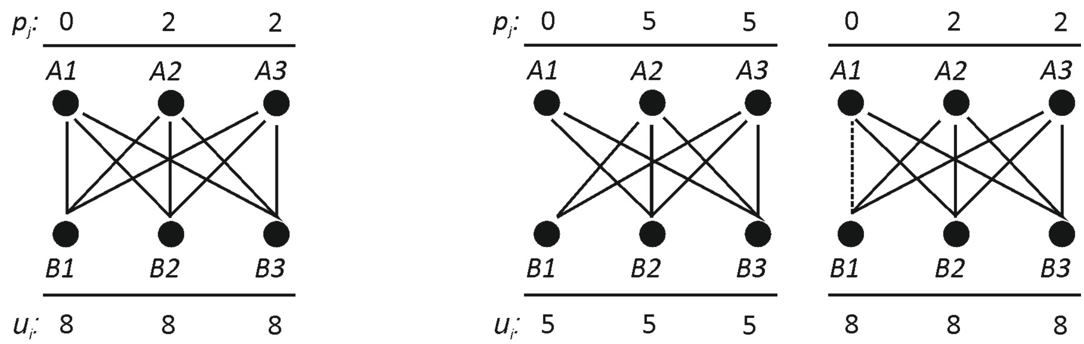
V	$A1$	$A2$	$A3$	V^G: $A1$	$A2$	$A3$	$V^{\tilde{G}}$: $A1$	$A2$	$A3$
$B1$	8	10	10	***0***	10	10	***8***	10	10
$B2$	5	10	5	5	10	0	5	10	0
$B3$	5	5	10	0	0	10	0	0	10
B^h	$B1$	$B2$	$B3$	$B2$	$B1$	$B3$	$B1$	$B2$	$B3$
p^V	0	2	2	0	5	5	0	2	2

Figure 5.5: An example where adding link $\tilde{g}_{11} = 1$ weakly increases all bidders' payoffs.



	V^G: $A1$	$A2$	$A3$	$V^{\tilde{G}}$: $A1$	$A2$	$A3$
$B1$	5	8	0	5	8	0
$B2$	0	10	10	0	10	10
$B3$	0	0	15	0	0	15
$B4$	***0***	4	0	**5**	4	0
B^h	$B1$	$B2$	$B3$	$B4$	$B2$	$B3$
p^V	0	3	3	5	8	8

Figure 5.6: An example where adding link $\tilde{g}_{41} = 1$ weakly decreases all bidders' payoffs.



$\widehat{V}$	$A1$	$A2$	$A3$
$B1$	8	10	10
$B2$	5	10	0
$B3$	0	0	10
B^h	$B1$	$B2$	$B3$
p^V	0	2	2

$\widehat{V}^G$	$A1$	$A2$	$A3$
	0	10	10
	5	10	0
	0	0	10
	$B2$	$B1$	$B3$
	0	5	5

$\widehat{V}^{\widetilde{G}}$	$A1$	$A2$	$A3$
	8	10	10
	5	10	0
	0	0	10
	$B1$	$B2$	$B3$
	0	2	2

Figure 5.7: Adaptation of the example in Figure 5.5.
Left: complete valuation matrix. Right: incomplete networks.

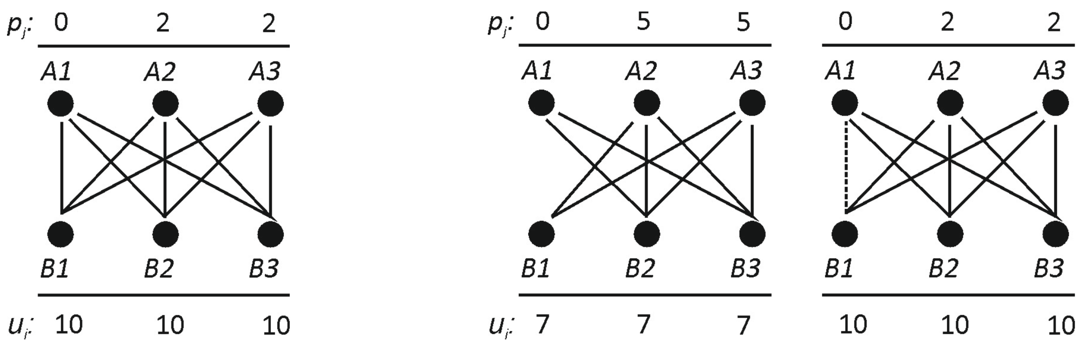
of price impacts may occur even when restricting the analysis to such agents h. The valuation matrices V^G in the examples can be interpreted as valuation matrices $\widehat{V}$ associated with a complete network. Therefore, the networks in the associated graphical illustrations may also be adapted to networks where g_{ij} is the *only* non-existent link. We exemplify this in Figure 5.7 for the example in Figure 5.5.

The results derived from Figures 5.3–5.6 may seem artificial because they are based on links that connect bidders with items that they do not value. However, we may complement the example by adding a constant to all valuations in the matrix (see Figure 5.8). We still get qualitatively the same outcomes. Thus, results that hold for matrix $\widehat{V}$, that is based on links associated with zero valuations, matrices $\widehat{\widehat{V}}$ with strictly positive valuations can be constructed such that the result still holds.

From Corollaries 4.3 and 4.4, respectively, we derive a sufficient condition for increasing prices in the network and a necessary condition for decreasing prices.

Corollary 5.2 *Suppose that a link between bidder i and seller j is added to the network.*

(1) If bidder i does not win auction j after the link is added, prices p_k for all $k \in M$ weakly increase.



$\hat{\hat{V}}$	$A1$	$A2$	$A3$
$B1$	10	12	12
$B2$	7	12	2
$B3$	2	2	12
B^h	$B1$	$B2$	$B3$
p^V	0	2	2

$\hat{\hat{V}}^G$	$A1$	$A2$	$A3$
	0	12	12
	7	12	2
	2	2	12
	$B2$	$B1$	$B3$
	0	5	5

$\hat{\hat{V}}^{\tilde{G}}$	$A1$	$A2$	$A3$
	10	12	12
	7	12	2
	2	2	12
	$B1$	$B2$	$B3$
	0	2	2

Figure 5.8: Further adaptation of the examples in Figures 5.5 and 5.8 (all valuations are 2 units higher).
Left: complete valuation matrix. Right: incomplete networks.

(2) Prices p_k for $k \neq j \in M$ may decrease if i wins j after the link is added.

Let us now consider adding several links sequentially. In Figure 5.5, the underlying valuation matrix V is presented. The overall efficient assignment associated with V equals the efficient assignment under $V^{\widetilde{G}}$. Thus, adding the three missing links to $V^{\widehat{G}}$ does not change the assignment. In this example, prices do not change anymore when further links are added to $V^{\widehat{G}}$.

Note, however, that individual prices are not monotone in the added links. Consider Figure 5.9. There, the price in $A3$ increases from 0 to 2 if we add link $\tilde{g}_{13}$. However, adding another link $\tilde{g}_{11}$ leads to a decrease in p_3^V. If we change the order in which we add links $\tilde{g}_{13}$ and $\tilde{g}_{11}$, prices behave monotonically. Replace matrix $V^{\widetilde{G}}$ by $V^{\widetilde{G}\prime}$ and prices change from $p^V = (0, 2, 0)$ to $(1, 1, 0)$ and finally to $(1, 1, 1)$. Thus, all prices either increase or decrease when links are added in this order.

The same non-monotonicity (or path dependence) in adding links applies to bidders' payoffs: in Figure 5.9, $B3$'s payoff decreases from 3 to 1 and then it increases to 2.

Kranton and Minehart (2001) examine a network formation game. In the first stage, bidders simultaneously decide about forming links at a cost $c^\ell > 0$ per link. In

V^G	$A1$	$A2$	$A3$	$V^{\tilde{G}}$	$A1$	$A2$	$A3$	V	$A1$	$A2$	$A3$	$V^{\tilde{G}\prime}$	$A1$	$A2$	$A3$
$B1$	0	2	0		0	2	$\boldsymbol{2}$		$\boldsymbol{2}$	2	2		$\boldsymbol{2}$	2	0
$B2$	0	3	0		0	3	0		0	3	0		0	3	0
$B3$	0	0	3		0	0	3		0	0	3		0	0	3
B^h	–	$B2$	$B3$		–	$B2$	$B3$		$B1$	$B2$	$B3$		$B1$	$B2$	$B3$
p^V	0	2	0		0	2	2		1	1	1		1	1	0

Figure 5.9: Example where prices (p_3^V) and payoffs (u_3) do not change monotonically if links are added sequentially.

stage two, bidders learn their (homogeneous) valuations and play a network button auction. They find that for any c^ℓ, each efficient network is an outcome in a PBE of the game (Kranton and Minehart, 2001, Proposition 2). Furthermore, they state that this result also holds for "any competitive process that yields an efficient allocation of goods and in which buyers' revenues are the marginal surplus from exchange" (Kranton and Minehart, 2001, p. 494). Our model fulfills these conditions. Thus, efficient networks are equilibrium networks of the network formation game combined with the network-restricted multiple-auctions game $\Gamma^a(G)$.

However, in the heterogeneous items environment, it is not straightforward to characterize ex-ante efficient networks. For homogeneous items, Kranton and Minehart find that least-link allocatively complete (LAC) networks are ex-ante efficient for low costs c^ℓ. LAC networks have the property that the overall efficient assignment is feasible for each realization of valuations. They characterize LAC networks using Hall's Theorem (Hall, 1935). Furthermore, using order statistics for prices, they prove that only LAC networks are equilibrium outcomes if, in the first stage, bidders simultaneously build links to sellers (assuming that the button auction takes place in the second stage of the game and learning about private valuations takes place between the two stages). However, in our environment the only network that assures the overall efficient assignment for all realizations of valuations is the complete network. Thus, this network formation game will have to be analyzed in more detail.

We abstain from doing this but simply state some more results without proof. The sparsest overall efficient network is defined as the network with the least number of links that allows for the efficiency level of the complete network. In a game with given valuation matrix V, it is the network that connects solely those bidders who win under the optimal assignment with their seller. In other words, only links $g_{ij} = 1$

with v_{ij} being part of $c(N, M)$ exist and $g_{ij} = 0$ otherwise. This can be easily seen because this network allows for all bidder-seller pairs that are part of the efficient assignment and no superfluous links exist. The sparsest overall efficient network with Vickrey prices equal to those of the complete network is the sparsest overall efficient network with additional links between a bidder in $B^I(j)$ and j for each auction j with $B^I(j) \neq j$. Then, the indifference paths (or, if several piecewise exchangeable indifference paths exist, choose one of them) are also part of the network and prices are determined like Vickrey prices in the complete network (cp. Lemma 4.1 in Chapter 4).

Adding a bidder to the network weakly decreases all other bidders' payoffs and weakly increases all sellers' payoffs, whereas adding a seller has the reverse effect. This result is independent of the number or distribution of the additional bidder's or seller's links. It follows directly from results on the assignment game, where adding bidders and sellers has these implications (cp. Section 3.3). This means that even though a bidder does not necessarily prefer less *links* between other bidders and the seller he is linked to or less links in the network in general, he does prefer to have less *bidders* in the network.

5.2 Discussion

We provide a framework for the analysis of bidder-seller networks. The equilibrium results of Section 3.2 extend to the case of incomplete networks. Furthermore, results of a comparative statics analysis investigating the effect of adding links (or analogously deleting links) are given. An informal analysis provides insights into strategic network formation in our environment.

One of the surprising results in this chapter is that a bidder may profit from an increase in the number of connections to sellers that he is linked to. Even though he competes with the additional bidders, his payoff may increase. Thus, a bidder in an Internet auction may be better off if items that he is interested in are found by other bidders (if those bidders know about more than these particular auctions). Furthermore, a seller is not necessarily hurt by bidders who are interested in his item searching for other items. With homogeneous items, these kinds of counterintuitive price effects do not occur.

However, we replicate the result of the homogeneous item case that bidders and sellers profit from having more connections. That is, executing costless search is always worthwhile for bidders and improving accessibility and presentation of one's own auction is always worthwhile for sellers.

Chapter 6

Experimental Test of Model Predictions

We report a laboratory experiment on bidding behavior and outcomes in a multiple-auctions game. We investigate whether participants bid according to strategy σ^* and if the predicted outcome (x^*, p^*) occurs (see Chapter 3).

6.1 Organization of the Experiment

The experiment was run at the University of Karlsruhe, Germany, with students from various disciplines randomly selected from the database of our experimental laboratory. Three sessions with 20 subjects per session were conducted. The subjects were partitioned into four groups of five bidders each. Every subject participated in three periods. A period consisted of one multiple-auctions game, played within the respective group. After every period, groups were recomposed. Thus, in total we have 36 observations of differently composed groups' results. Independence of observations is only given for the three session results. Since we do not compare treatments but analyze individual behavior and equilibrium results descriptively, we consider this to be unimportant. We assume that subjects consider the three periods to be independent because they are assured that they play in a different group in each period.

The experiments are computerized. Each subject is seated at a computer terminal that is separated from the other subjects' terminals. The subjects receive written instructions, which are also read out loud by an experimental assistant. Before the experiment starts, each subject has to answer several questions at her computer terminal concerning the instructions. After the subjects have answered all questions correctly, they receive their private information – their valuations for the three goods to be auctioned off in the first period. Then the experiment starts. Communication is not permitted. Subjects cannot identify the participants they play against. In each period, groups are composed of different subjects. At the end of an experimental session, the subjects are paid in cash according to their profits in all three periods. The experimental sessions last between 2 hours and 15 minutes and 2 hours and 50 minutes.

6.2 Experimental Design

For the experiment we choose $n = 5$ and $m = 3$ so that the multiple-auctions game consists of five bidders and three sellers. Only the bidders' roles are assigned to participants. The sellers are represented by three auctions with given starting prices. We conduct private value auctions where each bidder has private information about his three different valuations for the three hypothetical goods. In each period, every bidder receives new valuations. A total of five different valuation matrices are used inn the experiment. Our three sessions differ by the order and selection of the valuation matrices.

The valuations of the five bidders are drawn before the experiment from a uniform distribution over the even numbers between 40 and 140. We determine several 5×3 valuation matrices of bidders by drawing 15 valuations independently, and then select five appropriate valuation matrices.[1] To have some variation in the valuation matrices used (different characteristics of the expected results), three of the five selected matrices are used in each session. The five selected valuation matrices $V1$–$V5$ are shown in Table 6.1. To make the induced preferences (unit-demand preferences) easier to understand, valuations are described as prices at which the participant can sell the item after the auction. But he is only able to sell one item, the one with the highest price of those he owns after the auction.

[1]Selection criteria applied are the following: valuation matrices differ with respect to the number and composition of internal and external price determining bidders, winning bidders' names vary, the variance of expected payoffs of participants is not too high and the number of participants that do not win any auction is low.

Table 6.1: The five valuation matrices $V1$–$V5$ used in the experiment

V1	$A1$	$A2$	$A3$
$B1$	70	96	128
$B2$	134	136	64
$B3$	42	48	48
$B4$	76	116	124
$B5$	72	42	52

V2	$A1$	$A2$	$A3$
$B1$	104	128	130
$B2$	46	120	68
$B3$	114	52	72
$B4$	118	126	106
$B5$	58	76	64

V3	$A1$	$A2$	$A3$
$B1$	80	112	124
$B2$	98	60	56
$B3$	76	126	96
$B4$	84	46	102
$B5$	60	44	70

V4	$A1$	$A2$	$A3$
$B1$	66	112	110
$B2$	128	108	114
$B3$	118	70	90
$B4$	58	136	98
$B5$	134	52	130

V5	$A1$	$A2$	$A3$
$B1$	130	74	46
$B2$	90	72	48
$B3$	92	64	66
$B4$	96	102	94
$B5$	112	72	88

The order in which the five different matrices of Table 6.1 are used in the three periods in Sessions 1–3 is given in Table 6.2. Altogether, the game is played in nine periods, once with valuation matrix $V2$ and twice with valuation matrices $V1$, $V3$, $V4$, and $V5$.

Table 6.2: Order of valuation matrices in the different sessions

Period	1	2	3
Session 1	$V1$	$V2$	$V3$
Session 2	$V5$	$V4$	$V3$
Session 3	$V4$	$V1$	$V5$

After each period, the groups change such that no participant plays two rounds in the same group. With 20 participants per session, it is not possible to rematch groups such that no two participants ever meet again. But it is assured that no player ever plays twice in the same group. The matching protocol can be found in Table B.2 in Appendix B.1. Valuation matrices and group changes are arranged in such a way that almost each participant wins an auction in at least one round in equilibrium.[2]

[2] Only three of the 60 participants do not win in equilibrium – see Table B.4: participants 20, 4, and 18 in sessions 1, 2, and 3, respectively.

Next we describe the auction format, which is held as close as possible to the theoretical model. For more details, see the original instructions (in German) in Appendix B.3. Differences between the design and the model in Chapter 3 are discussed below.

The five bidders in a group are denoted by A, B, C, D, and E. Subjects usually have a different bidder name in each period so that they are reminded that each game is independent from the previous game. The starting price in all auctions is $b_j^0 = 40$ ExCU and there is a constant increment $\iota = 1$ ExCU. Bidders are allowed to submit bids in alphabetical order (A–E). When it is a bidder's turn to bid, he can choose between all auctions in which he is not the current high bidder (outbidding oneself is not allowed) or decide not to bid. In the chosen auction he can enter a bid of his choice that exceeds the current standing bid by at least one increment. Current standing bids and final prices are determined by the second-price rule. Bid withdrawal is not allowed.

Participants are always informed about their private valuations, the current standing bids, the current high bidder in all three auctions, and their high bids when they are current high bidders. In addition, they can press a button to see the bidding history in each auction, i.e., the previous high bidders and any previous high bids that have been outbid. Current high bids of other bidders are not revealed.

The auctions end when all bidders state that they do not want to submit a new bid. Thus, it is assured that all bidders accept the final outcome. When the auctions end, winners are paid their maximum valuation out of all goods they won minus the price plus a lump-sum payment that all subjects receive. After rematching, the next period begins. Different lump-sum payments per period in the three sessions (16.7, 8, and 8 ExCU[3]) and exchange ratios (1 ExCU equals €0.20, €0.30, and €0.25, respectively) are chosen to level out the different predicted payoffs in ExCU. The average expected payoffs without lump-sum payments in ExCU in the three sessions are 52.0, 43.4, and 58.6, respectively (see Table B.4). With lump-sum payments and exchange ratios this gives expected average payments of €20.40, €20.22, and €20.65, respectively. The average, minimum, and maximum realized payments of all subjects are €18.90, €0, and €39, respectively.

[3]In the first session, we planned to conduct 5 periods. Since participants in the slowest group were much slower in submitting their bids than expected, we had to restrict ourselves to three periods. To get a reasonable payment we increased the lump-sum payment from 5 to 16.7. In the following, we will deal with this session as if this lump-sum payment was planned beforehand.

The differences between the design and the model in Chapter 3 are the following. In the experiment, the valuations are even integers and the increment equals one. Compared to the theoretical model, this is equivalent to integer valuations and an increment of 0.5. We use this scaling to avoid decimals. In the model, deciding not to bid when selected is an exit decision. This activity rule assures finiteness of the game. We forgo this assumption in the experiment to avoid pressuring the subjects to bid. In decentralized Internet auctions, a bidder may submit a bid as long as the auction is offered and he is not forced to bid whenever he checks the current status of an auction (however, in centralized auctions activity rules are common). Another difference from the theoretical model is the fixed bidding sequence. Being randomly selected to bid in the model represents occasional checking of auction sites. For the experiment, a random bidding order is not adequate. For example, a bidder who does not want to bid may be selected several times without a change in the standing bids. Thus, this inadequateness of the random bidding order is also due to the different consequence of a decision not to bid. For the same reason, a slight change in the ending rule compared to the model is necessary. In the experiment, a bidder who is the current high bidder is allowed to bid in auctions where he is not the high bidder. This allows us to analyze the adequacy of our model assumption (see Hypothesis 1).

6.3 Experimental Hypotheses and Results

We specify three hypotheses on bidding behavior as well as three hypotheses on outcomes and present a descriptive analysis of the results with respect to these hypotheses.

6.3.1 Bidding Behavior

In the analysis of bidding behavior, the last decision not to bid (i.e., the second subsequent round in which all bidders decide not to bid) is not counted.

First, we ask if our model assumption that current high bidders do not submit additional bids is justifiable.

Hypothesis 1 *A bidder does not submit a bid when it is his turn to bid if he holds a current high bidder position in an auction.*

Out of 1,770 situations where a bidder is the current high bidder in at least one auction he does not bid in 1,668. That is, in 94.2% of the relevant cases, our hypothesis is confirmed and bidders behave in line with our model assumption. Therefore, we believe our model assumption is justified.

In the following, we turn to the situation where the selected bidder is not a current high bidder in any auction. The next hypothesis considers whether and where such a bidder submits a bid.

Hypothesis 2 *A bidder i whose turn it is to bid and who is not the current high bidder in any auction*

(a) bids in an auction that is in his demand set D_i if $|D_i| \geq 1$,

(b) does not bid if $D_i = \emptyset$.

In the experiment, 1,946 relevant situations, that is, situations in which the bidder is not a high bidder, occur. Of the decisions in these situations, 71.4% are in line with the hypothesis. 50.2% of all decisions correspond to Hypothesis 2(a), and 21.2% correspond to the prediction in 2(b). Looking closer at the result for 2(a), it consists of 46.2% where $|D_i| = 1$ and 4.0% where $|D_i| > 1$.[4]

The remaining cases (28.6%) involve the following decisions. In 14.3% of these cases a bidder i does not bid although $\Delta_{i(1)} > 0$. Notice that a bidder who does not bid in an early round but plans to bid in later rounds may do so without consequences. This difference between the experimental design and the theoretical model may explain part of these decisions. Furthermore, the rules in the experiment require a bidder who decides not to bid anymore to affirm this decision in each bidding round. Another 3.8% are bids in the auction j where i's valuation v_{ij} is maximal but j is not in i's demand set. 9.8% of bids are submitted in auctions j where v_{ij} is neither maximal nor is j contained in D_i, and 0.7% are bids when a positive payoff is impossible ($D_i = \emptyset$). Table 6.3 contains these results.

Nevertheless, we classify the observation that 71.4% of decisions are consistent with our predictions as strong support of Hypothesis 2.

Next, we consider the subset of situations of Hypothesis 2(a), where the bidder who is selected to bid has a non-empty demand set and submits a bid. From above,

[4]Analyzing Hypothesis 2 for each subcase separately, we find that 2(a) holds in 64% (976 of 1520) and 2(b) holds in 96.9% (413 of 426) of the relevant cases.

Table 6.3: Whether and where participants who are not current high bidders bid.

Observation	%	(of 1946)
In line with Hypothesis 2	71.4	(1389)
H2(a) ($\lvert D_i \rvert = 1$)	46.2	(898)
H2(a) ($\lvert D_i \rvert > 1$)	4.0	(78)
H2(b) ($D_i = \emptyset$)	21.2	(413)
Not in line with Hypothesis 2	28.6.(557)	
No bid when $\Delta_{i(1)} > 0$	14.3	(278)
Bid in $j = \arg\max v_{ij}$, $j \notin D_i$	3.8	(74)
Bid in $j \neq \arg\max v_{ij}$, $j \notin D_i$	9.8	(191)
Bid unless $D_i = \emptyset$	0.7	(14)

this applies to 50.2% of those situations in which the bidder is not a high bidder, or to 26% of all observed situations.

Hypothesis 3 *A bidder i whose turn it is to bid and who is not current the high bidder in any auction and who bids in an auction $j \in D_i$, $|D_i| \geq 1$, submits a bid that*

(a) is less than or equal to $b^*_{ij} := v_{ij} - \Delta_{i(2)}$ if $|D_i| = 1$,

(b) equals $b^*_{ij} := b^s_j + \iota$ if $|D_i| > 1$.

As we mention in the analysis of the theoretical model, besides the bid b^*_{ij} prescribed by strategy σ^*_i, bids between $b^s_j + \iota$ and b^*_{ij} are also possible in equilibrium. Strategy σ^*_i is the strategy that distinguishes itself mainly by minimizing the number of bids submitted. Thus, we consider all those bids below and equal to b^*_{ij} to be in line with our model predictions. We nevertheless display the analysis for bids equal to b^*_{ij} separately.

Table 6.4 contains the results. In the case of Hypothesis 3(a), when the demand set is a singleton, 64.9% of the bids are consistent with the prediction. A bid that is below b^*_{ij} deviates on average by 17.2 ExCU from b^*_{ij}. Higher than predicted bids are 35.1%. They exceed b^*_{ij} on average by 8.2 ExCU. For case 3(b), bidding below $b^*_{ij} = b^s_j + \iota$ is impossible. The predicted bid is submitted in only 11.5% of the 78 relevant cases. However, the upwards deviation in the other 88.5% of the relevant cases is on average only 3.6 ExCU. In these cases, the participants seem to prefer submitting a slightly higher bid to increasing the bid increment by increment. In doing this, they risk preferring a different item in the final outcome, so instability may occur.

Table 6.4: Magnitude of submitted bids (% (abs.); avg. dev. in ExCU).

Hypothesis 3	$< b^*_{ij}$	$= b^*_{ij}$	$> b^*_{ij}$	avg. dev. −	avg. dev. +
(a) $\lvert D_i\rvert = 1$ (898)	54.9 (493)	10.0 (90)	35.1 (315)	17.2	8.2
(b) $\lvert D_i\rvert > 1$ (78)	–	11.5 (9)	88.5 (69)	–	3.6
All (976)	50.5 (493)	10.1 (99)	39.3 (384)	17.2	7.4

In total, 60.6% of the relevant decisions are consistent with Hypothesis 3. This is again a result in favor of our theory.

Taking into account that a decision not to bid is also in line with the prediction, we find that participants in 413 (Hypothesis 2(b)) plus 592 (Hypothesis 3) of 1946 situations (where the selected bidder holds no high bidder position) act in line with the predictions, i.e., in 51.6% of the cases. Combining this with decisions not to bid when a bidder holds a high bidder position, we have 2,673 of 3,716, or 71.9% predicted actions.

Recapitulating, our results strongly support all three hypotheses on bidding behavior. Table 6.5 summarizes the results for the relevant subsets of decisions.

Table 6.5: Summary of observed bidding behavior in line with Hypotheses 1–3.

Relevant subset	prediction concerns	%
i is high bidder	decision not to bid	94.2
i is not high bidder	selection of auction	71.4
i is not high bidder and submits a bid in $j \in D_i$	height of bid	60.5
i is not high bidder	action chosen	51.6
All decisions	action chosen	71.9

6.3.2 Outcomes

The predicted outcomes for valuation matrices $V1$–$V5$ are given in Table 6.6. A graphical illustration of the theoretical results is given in Figure B.1 in Appendix B.2.

Table 6.6: Expected results with valuation matrices $V1$–$V5$.[5]

	V1			**V2**			**V3**		
Auction	$A1$	$A2$	$A3$	$A1$	$A2$	$A3$	$A1$	$A2$	$A3$
Winner	$B2$	$B4$	$B1$	$B3$	$B4$	$B1$	$B2$	$B3$	$B1$
Price p^*	72	73–75	80–84	111–113	120	99–101	84	89–91	102
Payoff	62	41–43	44–48	1–3	6	29–31	14	35–37	22
B^{PD}	$B5$	$B2$	$B4$	$B4$	$B2$	$B4$	$B4$	$B1$	$B4$

	V4			**V5**		
Auction	$A1$	$A2$	$A3$	$A1$	$A2$	$A3$
Winner	$B2$	$B4$	$B5$	$B1$	$B4$	$B5$
Price p^*	118	112	110	92	72	66
Payoff	10	24	20	38	30	22
B^{PD}	$B3$	$B1$	$B1$	$B3$	$B2$	$B3$

B^{PD}: Price determining bidder

Note that the increment $\iota = 1$ used in the experiment does not assure efficient outcomes for all possible valuation matrices with three auctions and five bidders and even valuations.[6] However, all expected results are efficient (see Table 6.6), i.e., possible deviations in bids following σ^* do not lead to inefficient outcomes. Prices are exactly predictable only for valuation matrices $V4$ and $V5$ because these are the only cases where all prices are determined by external price determining bidders. In valuation matrices $V1$ to $V3$, some prices are predicted to lie in a range, where the largest price range of 80–84 in $A3$ in valuation matrix $V1$ is due to possible accumulations of deviations after a deviation in $A2$ (see Proposition 3.3, Section 3.2.2.2).[7]

Our three hypotheses regarding outcomes are the following.

[5]The minimum competitive prices or Vickrey prices for valuations $V1$–$V5$ are (72,74,82), (112,120,100), (84,90,102), (118,112,110), and (92,72,66), respectively.

[6]See Section 3.2.2.4. The constraint that assures efficiency is $\iota < 1/\min\{n-1, m-1\}$. In the experiment, since we have only even valuations, the equivalent restriction is $\iota < 2/\min\{n-1, m-1\}$, which is not fulfilled: $1 > 2/3$.

[7]In the proof of Proposition 3.3, we mention that the maximum downward deviation calculated there may sometimes not apply. In the experiment, a bidder's decision not to bid does not change his activity status so the argument from the lemma does not apply to the experiment. Even though the fixed bidding order prevents a bidder from being selected late in the game for the first time, he may nevertheless submit his first bid late and, thus, cause the considered downward deviation. Therefore, we take the whole price range of Proposition 3.3 into account. However, it does not make a difference for our results.

Hypothesis 4 *The observed assignments are efficient.*

31 (86%) of 36 games end with an efficient assignment. From Table 6.7, the number of efficient outcomes is high in all sessions and for all valuation matrices. The average degree of efficiency (the sum of the valuations of winning bidders divided by the maximum sum of valuations) is 98%. We consider these results to be consistent with Hypothesis 4.

Table 6.7: Results concerning efficiency ordered by session and by valuation matrix.

Session	No. of games	Games with an efficient outcome
1	12	10 (83%)
2	12	10 (83%)
3	12	11 (92%)
Valuations	No. of games	Games with an efficient outcome
$V1$	8	6 (75%)
$V2$	4	4 (100%)
$V3$	8	7 (88%)
$V4$	8	7 (88%)
$V5$	8	7 (88%)
Sum	36	31 (86%)

Analyzing efficient group results with respect to participants, we find that 62% (37), 35% (21), 3% (2) or 0% of the 60 participants experience efficient group results in the three periods three times, twice, once, or never, respectively. In two auction games, a bidder wins two auctions (at prices above zero).

To analyze the outcome, we next consider the prices in the games with efficient outcomes. First, we investigate if the observed prices are competitive prices. Note that the outcomes of the five inefficient games cannot include competitive price vectors. Thus, at most 31 competitive equilibrium outcomes may occur. We state the following hypothesis.

Hypothesis 5 *The resulting outcome is a competitive equilibrium.*

We compare each bidder's outcome with his maximum potential outcome at final prices. In order for the outcome to be a competitive equilibrium, all winning bidders

Table 6.8: Competitive equilibria (CE), competitive equilibria at minimum competitive prices (mCE), and the relaxed requirements rCE ($u^B \in \{u^B_{max} - 3, u^B_{max}\}$) and its subset rmCE ($p^* \pm 1$).

Session	No. of games	CE	mCE	rCE	thereof rmCE
1	12	0	0	4	1
2	12	7	3	8	3
3	12	4	0	8	3
Valuations					
$V1$	8	1	0	2	0
$V2$	4	0	0	1	1
$V3$	8	1	0	3	0
$V4$	8	5	2	7	4
$V5$	8	4	1	7	2
Sum	36	11 (31%)	3 (8%)	20 (56%)	7 (19%)

have to win an auction in their demand set at the final prices and the losing bidders must have a maximum potential payoff below zero.

In 11 games (31% of all games), the expected competitive equilibrium is observed. Hence, 35% of the 31 efficient outcomes have competitive prices. If we relax the requirements and allow each bidder's payoff to differ by up to three ExCU from his maximum potential payoff, i.e., losing bidders quit too early and winning bidders either bid too high in the auction they win or too low in the auction in which they determine the price, we can classify 20 outcomes as approximate competitive equilibria (56% of all games or 65% of the 31 efficient outcomes). We believe this result is mainly in line with Hypothesis 5.

Table 6.8 contains the results with respect to competitive equilibria. We attribute the higher numbers of competitive equilibria in Session 3 and especially Session 2 to the use of valuation matrices $V4$ and $V5$ (cp. Table 6.2). That is, we assume a valuation matrix effect rather than a session effect. Remember that matrices $V4$ and $V5$ have the property that all prices are determined by external price determining bidders.

Notice that 11 group results are in the core of the respective associated cooperative games (Kaneko, 1982; Quinzii, 1984; Shapley and Shubik, 1971).

Since the previous results reveal that the model predicts the experimental results rather well, we investigate how many groups exactly replicate the predicted outcome.

Hypothesis 6 *The observed outcome is a competitive equilibrium at minimum prices, i.e., an efficient assignment at prices p^*.*

In addition to requiring the outcome to be a competitive equilibrium, we now ask for minimum competitive prices. From the theoretical analysis in Section 3.2.2.2, we know that due to the increment the predicted prices in the multiple-auctions game may deviate a bit from minimum competitive prices $\bar{p}^*$. In the analysis of Hypothesis 6, we allow for these deviations that may occur with our given bidding order and no activity rule. Thus, we refer to the price as p^* instead of $\bar{p}^* = p^V$. Table 6.6 contains the predicted price ranges for $V1$–$V5$.

The results are given in Table 6.8. We find that three outcomes (8% of all outcomes or 27% of the 11 competitive outcomes) are minimum competitive equilibrium outcomes. Allowing for a deviation from the maximum potential payoff of one increment, we have five groups (14% or 45%) with minimum competitive prices and an efficient assignment. Combining this relaxation with that on competitive prices (realized payoff 3 ExCU less than maximum potential payoff) seven groups (19% of all groups) are close to a competitive equilibrium at Vickrey prices.

Thus, for the most detailed prediction in Hypothesis 6, the results seem less encouraging. Note, however, that one deviating bidder in a group is enough to prevent an efficient assignment, competitive prices, and of course, minimum competitive prices. This puts our three observations of the most restrictive prediction of minimum competitive prices (associated with the bidder-optimal outcome in the core) into perspective.

The average price of 93.64 ExCU is slightly above the predicted average price of 92.22 ExCU.[8] Thus, on average the bidder-optimal prices (of efficient outcomes) are met rather well. This result is similar if we restrict to the 31 groups with efficient assignments: the average observed price is 92.83 ExCU.

[8] Note that in the multiple-auctions game with all valuations drawn independently from a uniform distribution on integers between zero and 100, a reserve price of zero, $\iota = 0.5$, $n = 5$ and $m = 3$, the average equilibrium auction price is around 58 (by simulation with 5000 instances). In a comparable auction with homogeneous valuations, the average predicted price is $\mathrm{E}(W_{(4:5)}) = 33.3$, the expected value of the forth order statistic $W_{(4:5)}$ of valuations. Thus, our predicted average price for our valuation matrices (92.22) for a uniform distribution on even integers between 40 and 140 is close to the expected value for randomly drawn valuations (58+40=98).

6.4 Conclusion

Our descriptive analysis of the experimental results reveals that strategy σ_i^* and related equilibrium strategies (lower bids) describe the observed bidding behavior rather well.

Our result that 71.4% of bidding in an auction occurs in the demand set is similar to the observation that 70% of bids are submitted in the auction with the lowest standing bid by Anwar et al. (2006) (for homogeneous items).

The efficiency in the experiment is very high. This suggests that the design of the multiple-auctions game has advantages over auctions with fixed ending times (Hoppe, 2008a). In our environment, the high degree of efficiency may even be more surprising because items are heterogeneous, compared to the homogeneous items in Hoppe (2008a). As expected, sequential bidding and the open ending rule help to coordinate the bidders and to avoid inefficiency.

We conclude that the experimental results are broadly in line with the theory of the multiple-auctions game presented in Chapter 3.

Chapter 7

Summary and Conclusion

A model that has several features that are typical for Internet auctions (second-price proxy auctions, independent sellers, multiple auctions, heterogeneity of items) is analyzed. By means of an analysis of all possible outcomes of the auction game, the range of outcomes is determined. A restriction on the increment allows us to derive a perfect Bayesian epsilon-equilibrium. The equilibrium outcome is associated with the minimum competitive equilibrium and the Vickrey outcome. Moreover, it corresponds to the bidder-optimal payoffs in the core.

Furthermore, new results concerning monotonicity of Vickrey auctions are presented. A single seller's revenue in a Vickrey auction is non-monotone in bids. However, we find a certain kind of monotonicity with respect to single prices and payoffs. Those either increase or decrease if one single valuation increases. If several valuations increase, this monotonicity does not hold. These results are also relevant for the application of so-called core-selecting auctions. This term usually refers to auctions that select the bidder-optimal outcome in the core as auction outcome. These auctions have advantages over Vickrey auctions because they are less vulnerable to shill bidding (Day and Milgrom, 2008). However, our findings show that they are not monotone in bids, even in the usually less problematic case of substitutes valuations. This has to be taken into account when these auctions are applied.

In addition to the multiplicity of auctions, the incomplete network character of the market is taken into account. The analysis of the bidder-seller network game is based on the equilibrium analysis and the investigation of Vickrey outcomes in the previous chapters. A reinterpretation of valuation matrices allows us to analyze incomplete bidder-seller networks within the framework of the multiple-auctions game for heterogeneous items. In contrast to the case of homogeneous items, adding a link may result in increasing payoffs for bidders linked to the seller who has the new link.

An experimental investigation indicates the practical relevance of the equilibrium strategy in Chapter 3. Observed outcomes are highly efficient. The phenomena of multiple-bidding, cross-bidding, and incremental bidding frequently reported in empirical studies are both predicted by the theoretical analysis and observed in the experiment.

Thus, we conclude that the results both provide new theoretical insights and are relevant for the implementation and application of auctions in the considered environment.

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Appendix A

Addenda to Chapters 3 and 4

A.1 Example of Proposition 3.3

The tables in Figure A.1 exemplify the bidding that leads to the maximum accumulated price deviation. The example is similar to that in the proof of Proposition 3.3 (Section 3.2.2.2) but a bit simpler. The increment ι equals 0.5. In each table, there is one block of b^h and b^s per submitted bid, but the first three bids can be found in the first entry. The current high bidders B^h are listed whenever a change occurs. From the third bid on, only one bidder does not hold a high bidder position. Thus, it is clear who submits the new bids.

From the tables we see that the upwards deviations in prices are the result of necessary additional bids that resolve the the mis-assignments. Downwards price deviations are due to early submission of winning bids.

	$A1$	$A2$	$A3$
$B1$	2	2	0
$B2$	0	3	3
$B3$	0	0	10
$B4$	1	0	0
B^h	$B4$	$B2$	$B3$
b^h	1	0.5	10
b^s	0	0	0
b^h	1	0.5	10
b^s	0.5	0.5	0
B^h	$B4$	$B1$	$B3$
b^h	1	1	10
b^s	0.5	0.5	0
b^h	1	1	10
b^s	0.5	0.5	0.5
b^h	1	1	10
b^s	0.5	1	0.5
b^h	1	1	10
b^s	0.5	1	1
B^h	$B4$	$B2$	$B3$
b^h	1	1.5	10
b^s	0.5	1	1
b^h	1	1.5	10
b^s	1	1	1
B^h	$B1$	$B2$	$B3$
b^h	1.5	1.5	10
b^s	1	1	1

	$A1$	$A2$	$A3$
$B1$	2	2	0
$B2$	0	3	3
$B3$	0	0	10
$B4$	1	0	0
B^h	$B4$	$B1$	$B3$
b^h	1	0.5	10
b^s	0	0	0
b^h	1	0.5	10
b^s	0	0.5	0
b^h	1	0.5	10
b^s	0	0.5	0.5
B^h	$B4$	$B2$	$B3$
b^h	1	1	10
b^s	0	0.5	0.5
b^h	1	1	10
b^s	0.5	0.5	0.5
b^h	1	1	10
b^s	0.5	1	0.5
b^h	1	1	10
b^s	1	1	0.5
B^h	$B4$	$B1$	$B3$
b^h	1	1.5	10
b^s	1	1	0.5
b^h	1	1.5	10
b^s	1	1	1
b^h	1	1.5	10
b^s	1	1	1.5
b^h	1	1.5	10
b^s	1	1.5	1.5
b^h	1	1.5	10
b^s	1	1.5	2
B^h	$B4$	$B2$	$B3$
b^h	1	2	10
b^s	1	1.5	2
B^h	$B1$	$B2$	$B3$
b^h	1.5	2	10
b^s	1	1.5	2

	$A1$	$A2$	$A3$
$B1$	2	2	0
$B2$	0	3	3
$B3$	0	0	10
$B4$	1	0	0
B^h	$B4$	$B1$	$B3$
b^h	1	0.5	10
b^s	0	0	0
b^h	1	0.5	10
b^s	0	0.5	0.5
B^h	$B4$	$B2$	$B3$
b^h	1	1	10
b^s	0	0.5	0.5
b^h	1	1	10
b^s	0.5	0.5	0.5
b^h	1	1	10
b^s	1	0.5	0.5
b^h	1	1	10
b^s	1	1	0.5
B^h	$B1$	$B2$	$B3$
b^h	1.5	1	10
b^s	1	1	0.5

Figure A.1: Evolution of the reference outcome $(\bar{x}^*, \bar{p}^*)$ to the maximum upwards price deviation and to an accumulated downwards price deviation.

A.2 Allowing Current High Bidders to Bid

In the main model, the current high bidders are not allowed to submit bids. Clearly, it is not in their interest to do so. Thus, the assumption does not seem to be crucial. In this section, we briefly consider the case when such bidding is allowed. We state the adapted strategy $\tilde{\sigma}^*$ and provide an example where the new bidding options cause a problem in the equilibrium analysis.

Most of the analysis of the main model also holds for this extension. In particular, the best reply property on the equilibrium path is not affected. In fact, the example that we provide illustrates the only case in which the results from the main model do not hold.

As a consequence of allowing high bidders to bid, bidders can be assigned to multiple auctions. Now, an assignment x is feasible if

$$\sum_{j \in M} x_{ij} \leq 1 \quad \forall\, i$$
$$\sum_{i \in N} x_{ij} \leq 1 \quad \forall\, j.$$

Accordingly, we have to add a bidding rule to σ_i^* by an action for the case when multiple high bidder positions are held.

Definition A.1 (Strategy $\tilde{\sigma}_i^*$) *The* strategy $\tilde{\sigma}_i^*$*:* $\mathcal{H}_i \to A(H_i)_{H_i \in \mathcal{H}_i}$ *for bidder* $i \in N$ *specifies that bidder* i *chooses the following action whenever he is selected to bid (i.e., whenever one of his information sets* H_i *is reached):*

(1) If $\Delta_{i(1)} \leq 0$*, then he does not bid (*$b_{ij} = 0$ *for all* j*).*

(2) If $\Delta_{i(1)} > 0$ *and* i *is the current high bidder in some auctions* $J_i := \{j : B^h(j) = i\}$ *then*

a) i *does not bid if* $|J_i \cap D_i| > 0$*.*

b) If $|J_i \cap D_i| = 0$ *and* $|D_i| = 1$*,* i *bids in* $j' \in D_i$ *if* $v_{ij'} - b^s_{j'} > \max_{j \in J_i}\{v_{ij}\}$[1] *according to*

$$b_{ij'} = v_{ij'} - \max\{\max_{j \in J_i}\{v_{ij}\}, \max_{j'' \in M \setminus (J_i \cup j')}\{v_{ij''} - b^s_{j''}\}\}$$

[1] This case is relevant only off the equilibrium path.

or does not bid if such a j' does not exist.

c) *If $|J_i \cap D_i| = 0$ and $|D_i| > 1$, i chooses randomly (with uniform probability) one of the auctions $j' \in D_i$ with $v_{ij'} > \max_{j \in J_i}\{v_{ij}\}$*[2] *and bids according to*[3]

$$b_{ij'} = b^s_{j'} + \iota$$

or does not bid if such a j' does not exist.

(3) *If i is not the current high bidder in any auction j then i bids as follows:*

a) *If $\Delta_{i(1)} > 0$ and $|D_i| = 1$, then i bids in auction $j \in D_i$. He determines his bid by*

$$b_{ij} = v_{ij} - \max\{\Delta_{i(2)}, 0\}\,.$$

b) *If $\Delta_{i(1)} > 0$ and $|D_i| > 1$, then bidder i chooses randomly (with uniform probability) one of the auctions in D_i. In the selected auction j he bids:*

$$b_{ij} = b^s_j + \iota\,.$$

On the equilibrium path, the strategy $\tilde{\sigma}^*_i$ prescribes the same behavior as σ^*_i. Thus, the results when all bidders follow $\tilde{\sigma}^*$ are identical to those in Section 3.2.3. Furthermore, off the equilibrium path we can make arguments similar to those in Section 3.2.3. However, we have to make adjustments when a bidder is mis-assigned. A mis-assigned bidder h deviates from his prescribed bidding behavior if he was not mis-assigned by bidding lower amounts and less often. In Section 3.2.3, such a bidder does not bid while being mis-assigned. Nevertheless, the idea of the proof is transferable to the current model. A deviation from $\tilde{\sigma}^*_i$ that aims to dissolve a mis-assigned bidder or to prevent a dissolution either directly or indirectly via other bidders does not increase bidder i's payoff. However, $\tilde{\sigma}^*$ is not an ex-post epsilon-equilibrium. We show this by an example in which deviating from $\tilde{\sigma}^*$ is profitable for a given valuation matrix.

If i is himself mis-assigned at H_i, $\tilde{\sigma}^*_i$ prescribes him to bid whenever his (possibly negative) current payoff can be increased. Thus, he does not miss a chance to increase

[2]This case is relevant only off the equilibrium path.

[3]In this case, $\max\{\max_{j \in J_i}\{v_{ij}\}, \max_{j'' \in M \setminus (J_i \cup j')}\{v_{ij''} - b^s_{j''}\}\} = \max_{j'' \in M \setminus (J_i \cup j')}\{v_{ij''} - b^s_{j''}\} = v_{ij'} - b^s_{j'}$.

his payoff. He has one more factor to consider, namely if his actions can influence the dissolution of his mis-assignment. This might only be done by bidding more than $\tilde{\sigma}_i^*$ prescribes at the risk of a further mis-assignment. But such a further mis-assignment may be profitable for i: the resulting higher standing bid may induce another bidder to dissolve i's mis-assignment in an auction where the price is high. We illustrate this case with the following example.

The outcome if all bidders follow $\tilde{\sigma}^*$ is described in the following Tables A.1 and A.2. In Table A.1, on the left we find the three bidders' valuation matrix, in the middle the situation at information set H_{B1} is described, and on the right an outcome is given.

Table A.1: Valuation matrix, situation at H_{B1}, and outcome.

V	$A1$	$A2$
$B1$	50	10
$B2$	105	15
$B3$	0	5

H_{B1}	$A1$	$A2$
b^h	100	5
b^s	100	5
B^h	$B1$	$B3$

	$A1$	$A2$
b^h	100	10
p	100	5
B^h	$B1$	$B2$

The bidding dynamics starting at H_{B1} are as follows. Bidder $B1$ does not bid because $v_{11} = 50 > 10 = v_{12}$. Bidder $B3$ does not submit anymore bids because $b_j^s \geq v_{3j}$ for $j = 1, 2$. He is the high bidder in his preferred auction $A2$. Bidder $B2$ submits the bid $b_{22} = 10$ and becomes the high bidder in auction $A2$ at $b_2^s = 5$. Then, the game ends with bidders' payoffs $u_1 = -50$, $u_2 = 10$, and $u_3 = 0$ and prices $p = (100, 5)$.

Now consider a deviation of $B1$ from $\tilde{\sigma}_{B1}^*$ at H_{B1} with the result given in Table A.2. First, $B1$ bids $b_{12} = 12$ in $A2$ and becomes the high bidder in $A2$ at $b_2^s = 5$.

Table A.2: The result of a deviation of $B1$ at H_{B1}.

	$A1$	$A2$	$A1$	$A2$	$A1$	$A2$	$A1$	$A2$
b^h	100	5	100	12	100	12	100+ι	12
b^s	100	5	100	5	100	10	100	10
B^h	$B1$	$B3$	$B1$	$B1$	$B1$	$B1$	$B2$	$B1$

Bidder $B3$ quits. Bidder $B2$ bids 10 in $A2$, does not become the high bidder and the standing bid b_2^s increases to 10. Having both $A1$ and $A2$ in his demand set, $B2$ bids one increment above b_1^s in $A1$. With this bid $b_{21} = 100 + \iota$, $B2$ becomes the high bidder in $A1$ and the game ends with payoffs $u_1 = 0$, $u_2 = 5$, and $u_3 = 0$ at prices $p = (100, 10)$. Thus, due to his deviation, $B1$'s payoff increases by 50.

Note that $B1$'s deviation might also have led to a decrease in his payoff. For example, if $B2$'s valuations were $v_2 = (0, 11)$, he would have bid 11 in $A2$ and quit. $B1$'s deviation results in a payoff $u_1 = 50 - 100 - 11 = -61 < -50$. Since bidder $B1$ does not know $B2$'s valuations, he does not know how $B2$ will react to the increased standing bid in $A2$. If he knew the valuations, he would not behave according to $\tilde{\sigma}^*$.

In this example, a profitable deviation from $\tilde{\sigma}^*$ exists for certain realizations of valuations. Thus, $\tilde{\sigma}^*$ does not constitute an ex-post equilibrium in the model where high bidders are allowed to bid in other auctions. Note, however, having a mis-assigned bidder i at the information set H_i is the only case in which the strategy $\tilde{\sigma}_i^*$ is not a best reply. The assumption in the main model that a high bidder is not allowed to submit further bids prevents the occurrence of a bidder with several high bidder positions, and, thus, solves the considered problem.

A.3 Multiplicity of Efficient Assignments

Suppose two efficient assignments x and y exist. Bidders' payoffs $u_i(x, \bar{p}^{*,x})$ and $u_i(y, \bar{p}^{*,y})$ in the associated outcomes equal Vickrey payoffs $u^V(x, p^{V,x}) = c(N, M) - c(N\backslash\{i\}, M) = u^V(y, p^{V,y})$. Vickrey payoffs are calculated using coalitional values that depend only on the maximum sum of valuations but not on the underlying assignment. Thus,

$$u_i(x, \bar{p}^{*,x}) = u_i(y, \bar{p}^{*,y}) \quad \text{for all } i \in N.$$

In the following, $p^x := \bar{p}^{*,x} = p^{V,x}$ and $p^y := \bar{p}^{*,y} = p^{V,y}$. Both assignments are optimal and thus both are associated with efficient outcomes. Therefore,

$$\sum_{i\in N} u_i(x,p^x)+\sum_{j\in M} u_j^S(x,p^x) = \sum_{i\in N} u_i(y,p^y)+\sum_{j\in M} u_j^S(y,p^y) = \sum_{i\in N} u_i(x,p^x)+\sum_{j\in M} u_j^S(y,p^y)$$

and

$$\sum_{j\in M} u_j^S(x,p^x) = \sum_{j\in M} u_j^S(y,p^y).$$

Both payoff-assignment combinations are stable. We have $u_i(x,p^x) + u_j^S(x,p^x) \geq d_{ij}$

	$A1$	$A2$
$B1$	1	2
$B2$	2	3
B^h	$B1$	$B2$
or	$B2$	$B1$
p^V	0	1

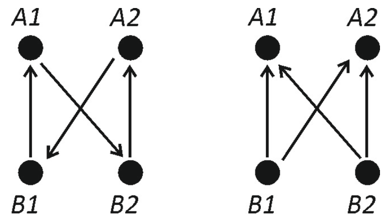


Figure A.2: An example where two efficient assignments exist.

for all i and j and $u_i(x, p^x) + u_j^S(x, p^x) = d_{ij}$ for pairs (i, j) with $x_{ij} = 1$, and the same is valid for $(y, p^{V,y})$. Hence, for all i and j with $x_{ij} = 1$, we know

$$u_i(x, p^x) + u_j^S(x, p^x) = d_{ij} \leq u_i(y, p^y) + u_j^S(y, p^y) = u_i(x, p^x) + u_j^S(y, p^y)$$

and therefore $u_j^S(x, p^x) \leq u_j^S(y, p^y)$. The same argument applies for h and j with $y_{hj} = 1$, so we find that $u_j^S(x, p^x) \geq u_j^S(y, p^y)$ and therefore

$$u_j^S(x, p^x) = u_j^S(y, p^y) \quad \text{for all } j \in M.$$

Because either $u_j^S(x, p^x) = v_j^S - p_j^x = v_j^S - p_j^y = u_j^S(y, p^y)$ or $u_j^S(x, p^x) = v_j^S - p_j^x = 0 = u_j^S(y, p^y)$ or $u_j^S(x, p^x) = 0 = v_j^S - p_j^y = u_j^S(y, p^y)$ and because $p_j = v_j^S$ if an item is unsold, we get

$$p_j^x = p_j^y \quad \text{for all } j \in M.$$

We record this in the following lemma.

Lemma A.1 *Suppose $(x, p^{V,x})$ and $(y, p^{V,y})$ are two efficient outcomes at Vickrey prices. Then $p_j^{V,x} = p_j^{V,y}$ for all auctions $j \in M$.*

Corollary A.1 follows directly from the equality in prices.

Corollary A.1 *A bidder who wins j under (x, p^V) and k under (y, p^V) is indifferent between winning j and k at prices p^V.*

Thus, the winner i of j under y is a potential price determining bidder in j under x.

An example in which several efficient assignments exist is given in Figure A.2. The graph illustrates that assignments $(x_{11} = 1, x_{22} = 1)$ and $(x_{12} = 1, x_{21} = 1)$ are both efficient. In both cases, the sum of valuations equals 4.

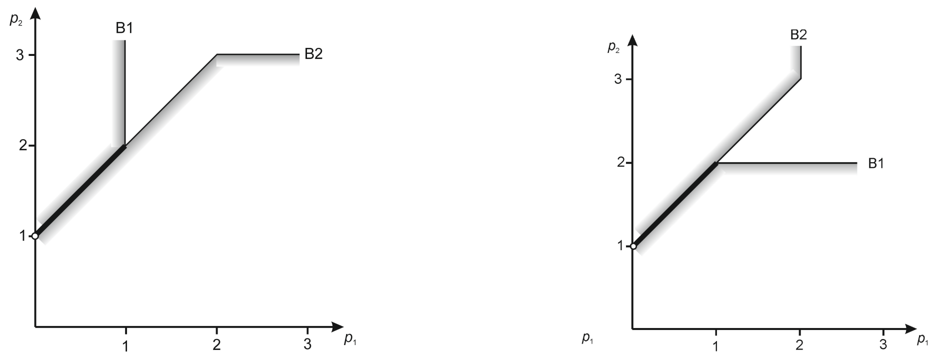


Figure A.3: The core of the example in Figure A.2 illustrated for both efficient assignments.

Sotomayor (2003) shows that infinitely many payoffs exist in the core if there is only one efficient assignment. This implies that when the core is a singleton, more than one efficient assignment exists. Wako (2006) complements this analysis with another proof using complementary slackness of the dual linear programs. From the example in Figure A.2, we can see that the core does not have to be a singleton when several optimal assignments exist. The prices in the core are given by the convey hull conv$\{(0,1),(1,2)\}$, that is, by the bold line in both drawings in Figure A.3.

Appendix B

Supplements to the Experiment

B.1 Organization of the Experiment

Table B.1 gives the schedule of the sessions.

Table B.1: Timetable of the sessions

Session	Date	Time
1	Tuesday, June 26, 2007	13.30
2	Thursday, June 28, 2007	10.00
3	Thursday, June 28, 2007	16.30

The participants were seated in two rooms, each room with 10 terminals. Participants 1–10 were in one room and participants 11–20 in the other room.

Table B.2 shows the composition of the groups in each period. This is internal information not known to the participants. They knew only their bidder name in each round but neither their group identifier nor their internal player number. The information given was identical in all three sessions.

In Table B.3, the last entry in the first row, $4E(1) - 1A(2) - 5D(3)$, means that in Session 3, the participant with the internal number 1 had the bidder name E in the first period and played the game with valuation matrix $V4$, in the second period he was Bidder A in $V1$, and in Period 3 he was Bidder D in $V5$. The participant with internal number 1 played the game in groups with internal numbers 1, 2, and 3.

Table B.2: Group matching: Internal player numbers (1–20) assigned to bidder names (A–E)

		A	B	C	D	E
Period 1	Group 1	5	11	20	6	1
	Group 2	2	7	10	13	19
	Group 3	9	14	18	3	16
	Group 4	17	15	4	12	8
Period 2	Group 1	4	20	19	5	10
	Group 2	1	6	9	12	18
	Group 3	8	13	17	2	15
	Group 4	16	14	3	11	7
Period 3	Group 1	3	19	18	4	9
	Group 2	10	5	8	11	17
	Group 3	7	12	16	1	14
	Group 4	15	13	2	20	6

Table B.3: Valuation matrices, bidder names, and internal group numbers in periods 1–3 of the three sessions (example 2E(1): valuation matrix $V2$, bidder E, group 1)

Part.	Session 1	Session 2	Session 3
1	$1E(1) - 2A(2) - 3D(3)$	$5E(1) - 4A(2) - 3D(3)$	$4E(1) - 1A(2) - 5D(3)$
2	$1A(2) - 2D(3) - 3C(4)$	$5A(2) - 4D(3) - 3C(4)$	$4A(2) - 1D(3) - 5C(4)$
3	$1D(3) - 2C(4) - 3A(1)$	$5D(3) - 4C(4) - 3A(1)$	$4D(3) - 1C(4) - 5A(1)$
4	$1C(4) - 2A(1) - 3D(1)$	$5C(4) - 4A(1) - 3D(1)$	$4C(4) - 1A(1) - 5D(1)$
5	$1A(1) - 2D(1) - 3B(2)$	$5A(1) - 4D(1) - 3B(2)$	$4A(1) - 1D(1) - 5B(2)$
6	$1D(1) - 2B(2) - 3E(4)$	$5D(1) - 4B(2) - 3E(4)$	$4D(1) - 1B(2) - 5E(4)$
7	$1B(2) - 2E(4) - 3A(3)$	$5B(2) - 4E(4) - 3A(3)$	$4B(2) - 1E(4) - 5A(3)$
8	$1E(4) - 2A(3) - 3C(2)$	$5E(4) - 4A(3) - 3C(2)$	$4E(4) - 1A(3) - 5C(2)$
9	$1A(3) - 2C(2) - 3E(1)$	$5A(3) - 4C(2) - 3E(1)$	$4A(3) - 1C(2) - 5E(1)$
10	$1C(2) - 2E(1) - 3A(2)$	$5C(2) - 4E(1) - 3A(2)$	$4C(2) - 1E(1) - 5A(2)$
11	$1B(1) - 2D(4) - 3D(2)$	$5B(1) - 4D(4) - 3D(2)$	$4B(1) - 1D(4) - 5D(2)$
12	$1D(4) - 2D(2) - 3B(3)$	$5D(4) - 4D(2) - 3B(3)$	$4D(4) - 1D(2) - 5B(3)$
13	$1D(2) - 2B(3) - 3B(4)$	$5D(2) - 4B(3) - 3B(4)$	$4D(2) - 1B(3) - 5B(4)$
14	$1B(3) - 2B(4) - 3E(3)$	$5B(3) - 4B(4) - 3E(3)$	$4B(3) - 1B(4) - 5E(3)$
15	$1B(4) - 2E(3) - 3A(4)$	$5B(4) - 4E(3) - 3A(4)$	$4B(4) - 1E(3) - 5A(4)$
16	$1E(3) - 2A(4) - 3C(3)$	$5E(3) - 4A(4) - 3C(3)$	$4E(3) - 1A(4) - 5C(3)$
17	$1A(4) - 2C(3) - 3E(2)$	$5A(4) - 4C(3) - 3E(2)$	$4A(4) - 1C(3) - 5E(2)$
18	$1C(3) - 2E(2) - 3C(1)$	$5C(3) - 4E(2) - 3C(1)$	$4C(3) - 1E(2) - 5C(1)$
19	$1E(2) - 2C(1) - 3B(1)$	$5E(2) - 4C(1) - 3B(1)$	$4E(2) - 1C(1) - 5B(1)$
20	$1C(1) - 2B(1) - 3D(4)$	$5C(1) - 4B(1) - 3D(4)$	$4C(1) - 1B(1) - 5D(4)$

B.2 Details on Predictions and Results of the Experiment

Table B.4: Expected payoffs of participants over Periods 1–3 without lump-sum payments.

Part.	Session 1		Session 2		Session 3	
1	0 + 30 + 0 =	**30**	22 + 0 + 0 =	**22**	20 + 45 + 30 =	**95**
2	45 + 6 + 37 =	**88**	38 + 24 + 37 =	**99**	0 + 42 + 0 =	**42**
3	42 + 2 + 22 =	**66**	30 + 0 + 22 =	**52**	24 + 0 + 38 =	**62**
4	0 + 30 + 0 =	**30**	0 + 0 + 0 =	**0**	0 + 45 + 30 =	**75**
5	45 + 6 + 14 =	**65**	38 + 24 + 14 =	**76**	0 + 42 + 0 =	**42**
6	42 + 0 + 0 =	**42**	30 + 10 + 0 =	**40**	24 + 62 + 22 =	**108**
7	62 + 0 + 22 =	**84**	0 + 20 + 22 =	**42**	10 + 0 + 38 =	**48**
8	0 + 30 + 37 =	**67**	22 + 0 + 37 =	**59**	20 + 45 + 0 =	**65**
9	45 + 2 + 0 =	**47**	38 + 0 + 0 =	**38**	0 + 0 + 22 =	**22**
10	0 + 0 + 22 =	**22**	0 + 20 + 22 =	**42**	0 + 0 + 38 =	**38**
11	62 + 6 + 0 =	**68**	0 + 24 + 0 =	**24**	10 + 42 + 30 =	**82**
12	42 + 6 + 14 =	**62**	30 + 24 + 14 =	**68**	24 + 42 + 0 =	**66**
13	42 + 0 + 14 =	**56**	30 + 10 + 14 =	**54**	24 + 62 + 0 =	**86**
14	62 + 0 + 0 =	**62**	0 + 10 + 0 =	**10**	10 + 62 + 22 =	**94**
15	62 + 0 + 22 =	**84**	0 + 20 + 22 =	**42**	10 + 0 + 38 =	**48**
16	0 + 30 + 37 =	**67**	22 + 0 + 37 =	**59**	20 + 45 + 0 =	**65**
17	45 + 2 + 0 =	**47**	38 + 0 + 0 =	**38**	0 + 0 + 22 =	**22**
18	0 + 0 + 37 =	**37**	0 + 20 + 37 =	**57**	0 + 0 + 0 =	**0**
19	0 + 2 + 14 =	**16**	22 + 0 + 14 =	**36**	20 + 0 + 0 =	**20**
20	0 + 0 + 0 =	**0**	0 + 10 + 0 =	**10**	0 + 62 + 30 =	**92**
Avg.		**52.0**		**43.4**		**58.6**

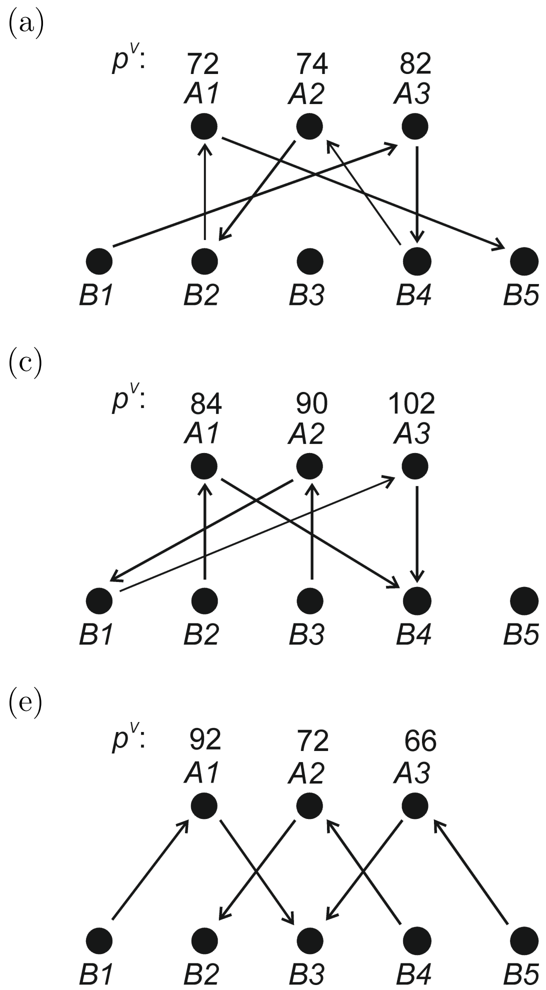


Figure B.1: Illustration of the expected results for valuations (a) $V1$ – (e) $V5$

B.3 Experimental Instructions

This section contains the original instructions of the experiment in German. In the three sessions the instructions differ with respect to the calculation of payoffs. We use lump-sum payments of 16.7, 8, and 8 as well as exchange ratios of 20, 30, and 25 Cents per ExCU, respectively. This corrects for the differences in predicted payoffs due to the use of the different valuation matrices. The following instructions belong to Session 1.[1] Deviations in sessions 2 and 3 are given in squared brackets, where the first entry corresponds to Session 2 and the second to Session 3.

Anleitung

Im Folgenden nehmen Sie an einem wirtschaftswissenschaftlichen Experiment teil. Hierbei treffen Sie Ihre Entscheidungen isoliert von den anderen Teilnehmern an Ihrem Computerterminal. In diesem Experiment können Sie bares Geld verdienen. Wie viel Sie verdienen, hängt von Ihren Entscheidungen und den Entscheidungen der anderen Teilnehmer ab. Die monetären Recheneinheiten im Experiment sind so genannte Geldeinheiten (GE).

Das Experiment besteht aus mehreren Perioden. In jeder Periode bilden Sie mit vier weiteren Teilnehmern eine **Fünfergruppe**. Nach jeder Periode werden die Gruppen neu gebildet, so dass Sie in jeder Periode mit anderen Teilnehmern eine Gruppe bilden. Interaktion findet nur innerhalb der Gruppe statt. Es werden **3 Perioden** gespielt. Eine Periode läuft wie folgt ab.

Ausgangssituation

In jeder Periode können Sie an **3 Auktionen** teilnehmen in denen je 1 Gut angeboten wird. Die Auktionen finden **gleichzeitig** statt. Je nachdem, wie Sie und die anderen Gruppenmitglieder bieten, können Sie bis zu 3 Auktionen gewinnen.
Wenn die Auktionen beendet sind kauft der Spielleiter Ihnen automatisch genau eines der Güter zu einem vorgegebenen Weiterverkaufspreis ab, wenn Sie in den Auktionen mindestens eines ersteigert haben. **Güter, die Sie behalten, haben keinerlei Wert für Sie.** Damit endet die Periode.
Im Folgenden werden die einzelnen Teile einer Periode genauer beschrieben, zunächst der Ablauf und die Regeln der Auktionen, danach der automatische Verkauf an den

[1] Footnote 3 gives the reason for the lump-sum payment of 16.7 ExCU in Session 1.

Spielleiter und die Berechnung Ihrer Auszahlung. Die Regeln sind für alle 3 Perioden dieselben. Von Periode zu Periode ändern sich nur die Zusammensetzung der Gruppe und die vorgegebenen Weiterverkaufspreise an den Spielleiter.

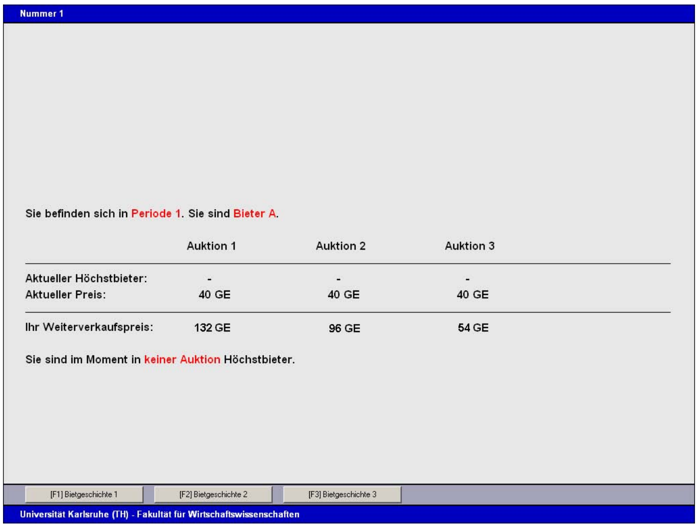
Auktionen

Ablauf

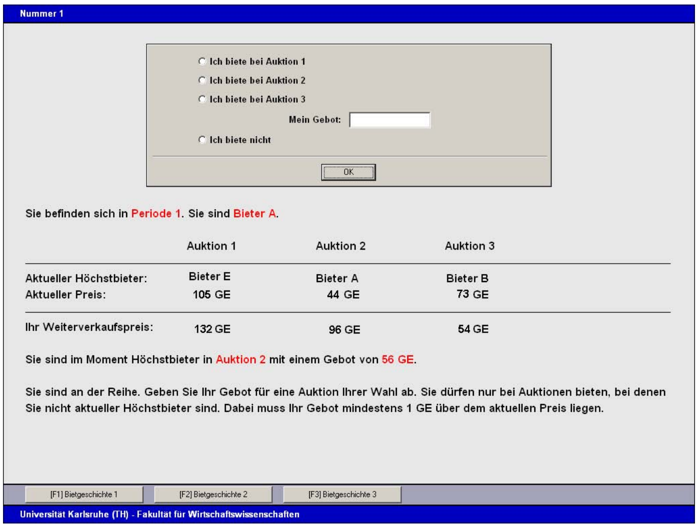
Die 5 Mitglieder Ihrer Gruppe werden mit Bieter A, Bieter B,..., Bieter E bezeichnet. Ihre eigene Bezeichnung wir Ihnen zu Beginn einer Periode am Bildschirm mitgeteilt und kann sich von Periode zu Periode unterscheiden.

Die Auktionen, bei denen Sie die Güter erwerben können, besitzen die folgenden Regeln. Es werden 3 Auktionen angeboten, bei denen die Mitglieder Ihrer Gruppe bieten können. In jeder Auktion wird ein Gut angeboten, in Auktion 1 Gut 1, in Auktion 2 Gut 2 und in Auktion 3 Gut 3. Die Bieter bieten immer nacheinander gemäß Ihrer Bezeichnung A bis E.

Während des Experiments sehen Sie einen Bildschirm, der wie folgt aussieht (die Zahlen sind nur ein Beispiel):



Wenn Sie an der Reihe sind, sehen Sie zusätzlich ein Eingabefenster:



Sie können wählen, in welcher Auktion Sie bieten möchten und wie hoch Sie in dieser Auktion bieten möchten. Wenn Sie an der Reihe sind können Sie auf **maximal eine** Auktion bieten. Wenn Sie bieten, muss Ihr Gebot eine **ganze Zahl** sein und **mindestens eine Geldeinheit (GE) über dem aktuellen Preis** in der jeweiligen Auktion liegen. In einer Auktion, in der Sie selbst aktueller Höchstbieter sind, dürfen Sie nicht bieten. Wenn Sie Ihre Entscheidung getroffen haben, ist der nächste Bieter Ihrer Gruppe an der Reihe. Wenn alle anderen Bieter in Ihrer Gruppe an der Reihe waren, erscheint bei Ihnen wieder das Eingabefenster.
Wenn Sie in einer Auktion bieten, können zwei Fälle eintreten:
1.) Ihr Gebot ist das **höchste bisher abgegebene Gebot in dieser Auktion**. Dann sind Sie der **aktuelle Höchstbieter** in dieser Auktion. Der angezeigte **aktuelle Preis** entspricht dem **zweithöchsten bisher eingegangenen Gebot in dieser Auktion**. Das heißt, wenn Sie aktueller Höchstbieter werden, wird der aktuelle Preis automatisch auf den Wert des zweithöchsten bisher eingegangen Gebots festgelegt. Ihr tatsächliches Gebot wird den anderen Bietern nicht angezeigt, solange Sie nicht überboten werden.
2.) Ihr Gebot liegt über dem bisherigen aktuellen Preis, aber unter dem tatsächlichen Gebot des aktuellen Höchstbieters. Dann steigt der aktuelle Preis auf den Wert Ihres Gebots, da dieses dann das zweithöchste bisher eingegangene Gebot ist. In diesem Fall behält der bisherige Höchstbieter seine Position bei.
Sind höchstes und zweithöchstes Gebot gleich, so ist der Bieter, der es zuerst abge-

geben hat, aktueller Höchstbieter und der aktuelle Preis entspricht dem Wert dieses höchsten und gleichzeitig zweithöchsten Gebots. Der aktuelle Preis wird somit nie durch den aktuellen Höchstbieter selbst, sondern immer durch das Gebot eines anderen Bieters bestimmt.

Wenn die Auktionen enden, gewinnen die Bieter, die zu diesem Zeitpunkt aktuelle Höchstbieter in den einzelnen Auktionen sind, die Auktionen zum zugehörigen aktuellen Preis. Das heißt, am Ende wird der aktuelle Höchstbieter zum Gewinner der Auktion und der aktuelle Preis zum Preis, den der Gewinner zahlen muss.

Zu Beginn der Auktion ist der aktuelle Preis gleich dem Startpreis von 40 GE. Wenn in einer Auktion keine Gebote eingehen, bleibt der Preis auf dem Startpreis und das Gut wird nicht verkauft. Wenn ein erstes Gebot in einer Auktion eingeht, bleibt der aktuelle Preis auf dem Startpreis stehen, der in dieser Situation die Rolle des zweithöchsten eingegangenen Gebots einnimmt.

Informationen

Ihnen und allen anderen Bietern in Ihrer Gruppe werden immer die drei Auktionen mit aktuellem Höchstbieter und aktuellem Preis angezeigt. Zu jeder Auktion wird auch Ihr Weiterverkaufspreis an den Spielleiter angezeigt. Zudem können Sie jederzeit über die mit **„Bietgeschichte“** bezeichneten Buttons die **überbotenen Gebote** in einer Auktion mit den zugehörigen Bietern abrufen. Wenn Sie die Bietgeschichte einer Auktion aufrufen, sehen Sie eine Tabelle. In dieser stehen die überbotenen Bieter von unten nach oben geordnet nach der **Reihenfolge des Gebotseingangs** sowie ihr jeweiliges **tatsächliches Gebot**, das mittlerweile ja überboten wurde. Ganz unten steht der Startpreis, darüber das erste abgegebene Gebot, dann das nächste usw. Darüber werden der aktuelle Höchstbieter und der aktuelle Preis angezeigt. Das tatsächliche Gebot des aktuellen Höchstbieters wird dabei nicht gezeigt. Dies erfahren Sie erst, wenn dieser überboten wurde.

In der Bietgeschichte werden die Informationen zusammengefasst, die Sie auch durch Beobachtung der Preisentwicklung sammeln können.

Ende der Auktionen

Die Auktionen **enden alle gleichzeitig**, wenn in Ihrer Gruppe keine **Bietaktivität** mehr beobachtet wird. Wenn alle fünf Bieter direkt nacheinander „Nicht bieten“ gewählt haben, bekommt der erste, der nicht geboten hatte und jetzt wieder an der

Reihe ist, einen Hinweis auf mangelnde Bietaktivität. Entscheidet er sich dann wieder, nicht zu bieten, bekommt auch der folgende Bieter den Hinweis usw. Wenn alle fünf Bieter auf den Hinweis hin nacheinander nicht bieten, enden alle Auktionen. Wenn auf den Hinweis hin einer der Bieter wieder ein Gebot abgibt, gehen alle Auktionen weiter und das Programm registriert die erneute Bietaktivität. Der Hinweis erscheint später nach den gleichen Regeln, wenn wieder alle fünf Bieter nacheinander nicht geboten haben. **Die Auktionen enden also nur, wenn direkt nach einer Runde ohne Gebote eine weitere Runde mit dem Hinweis auf mangelnde Bietaktivität ohne Gebote verläuft.** Dies bedeutet, dass die Auktionen nur dann enden, wenn sich alle Bieter entscheiden, nicht mehr bieten zu wollen und somit die **aktuelle Situation als Endergebnis der Periode akzeptieren**.

Für Sie bedeutet das, dass die Auktionen nur dann enden können, wenn Sie auf den Hinweis hin entscheiden, nicht zu bieten. Wenn Sie zu dem Zeitpunkt aktueller Höchstbieter in einer Auktion sind und Sie bieten nicht, laufen Sie nicht Gefahr, dass Sie noch überboten werden, ohne darauf reagieren zu können. Entweder alle Auktionen enden oder jeder kann wieder bieten.

Die Bezahlung der ersteigerten Güter erfolgt nach dem Weiterverkauf automatisch.

Weiterverkauf

Ihre **Weiterverkaufspreise** an den Spielleiter werden Ihnen während der Auktion immer am Bildschirm angezeigt. Die Weiterverkaufspreise, die Ihnen angezeigt werden, gelten nur für Sie. **Die anderen Bieter haben andere Weiterverkaufspreise.** Die Weiterverkaufspreise wurden **für jedes Gut, für jeden Bieter und für jede Periode unabhängig voneinander aus einer Gleichverteilung über den geraden Zahlen aus dem Intervall [40,140] gezogen**. Wenn Sie in den drei Auktionen ein Gut oder mehrere Güter erworben haben, kauft Ihnen der Spielleiter automatisch **genau eines** ab. Wenn Sie mehrere Güter ersteigert haben, kauft Ihnen der Spielleiter automatisch das mit dem **höheren Weiterverkaufspreis** ab. Dieser Verkauf ist Ihre einzige Möglichkeit, einen Nutzen aus den ersteigerten Gütern zu ziehen.

Der Spielleiter kauft Ihnen nur genau ein Gut ab, die anderen Güter haben daher keinen Wert für Sie.

Beispiel: Nehmen Sie an, Ihre Weiterverkaufspreise wären wie folgt gegeben:

	Auktion 1	Auktion 2	Auktion 3
Weiterverkaufspreis	60 GE	40 GE	50 GE

Hätten Sie Auktion 1 gewonnen, würde er Ihnen Gut 1 zum Weiterverkaufspreis 60 GE abkaufen. Hätten Sie Auktion 2 und Auktion 3 gewonnen, würde er Ihnen nur Gut 3 für 50 GE abkaufen. Dieser Verkauf läuft **automatisch** ab. Haben Sie kein Gut gewonnen, so kann Ihnen der Spielleiter auch keines abkaufen.

Berechnung Ihrer Auszahlung

Am Ende der Periode werden der Weiterverkauf und die Bezahlung der Auktionen automatisch durchgeführt. Daraus ergibt sich Ihr Periodengewinn wie folgt: 1.) Falls Sie mindestens eine Auktion gewonnen haben:

Periodengewinn = (Ihr Weiterverkaufspreis des Gutes, das der Spielleiter Ihnen abkauft) - (Preise, die Sie in Ihren gewonnenen Auktionen zahlen müssen)

2.) Falls Sie keine Auktion gewonnen haben: **Periodengewinn = Null**
Zusätzlich erhält jeder Teilnehmer in jeder Periode eine sichere Zahlung von 16.7 [8, 8] GE. Somit beträgt Ihre

Periodenauszahlung = 16.7 [8, 8] GE + Periodengewinn.

Ihre Auszahlung am Ende des Experiments ist die Summe Ihrer Periodenauszahlungen.

Beginn einer neuen Periode

Zu Beginn einer Periode werden die Gruppen neu gebildet. Aus programmtechnischen Gründen müssen Sie sich dazu umsetzen. Bitte bleiben Sie am Ende einer Periode zunächst ruhig an Ihrem Platz sitzen, bis der Experimentleiter in Ihrem Raum Ihnen Ihren neuen Platz zuweist. Bitte verhalten Sie sich dabei ruhig und sprechen Sie nicht mit anderen Teilnehmern. Lassen Sie den letzten Bildschirm der Periode geöffnet, d.h. drücken Sie **nicht den OK-Button**, wenn der Bildschirm mit den Hinweisen zum Umsetzen erscheint. Das Umsetzen erfolgt nur innerhalb Ihres Raums nach einfachen Regeln.
Nehmen Sie beim Platzwechsel bitte alle Zettel und den Stift mit.
In den Zettel mit der Tabelle für die Auszahlungen tragen Sie bitte nach jeder Periode

Ihre Auszahlung ein und nehmen ihn an den neuen Platz mit. Dieser Zettel dient nur der Kontrolle. Ihre Gesamtauszahlung wird unabhängig davon ermittelt.
Nach Periode 3 lassen Sie bitte alles außer der „Erklärung des Vertragnehmers“ und dem Zettel, den Sie am Eingang gezogen hatten, am Platz.

Auszahlung

Am Ende des Experiments erfolgt die Auszahlung in bar. Ihre Auszahlung wird in EUR umgerechnet wobei 1 GE einen Gegenwert von 20 [30, 25] Cent hat. Die Auszahlung erfolgt individuell und anonym.

Sollten Sie Fragen haben, melden Sie sich bitte. Der Experimentleiter kommt dann zu Ihnen an den Platz und beantwortet Ihre Frage dort. Bevor die erste Periode beginnt, werden Ihnen auf dem Bildschirm einige Fragen zu den Regeln gestellt. Damit möchten wir sichergehen, dass alle Teilnehmer die Anleitung verstanden haben.

List of Symbols

N	Set of bidders, $\lvert N\rvert = n$
M	Set of sellers, $\lvert M\rvert = m$
$\mathcal{I} = N \cup M$	Set of players
g, h, i	Indices referring to bidders
j, k, l	Indices referring to sellers
v_{ij}	Bidder i's valuation for good j (private information)
$V = (v_{ij})_{i\in N, j\in M}$	Matrix of bidders' valuations
v_j^S	Seller j's valuation for his item
$v^S = (v_j^S)_{j\in M}$	Vector of sellers' valuations
d_{ij}	Maximum gains from trade that a pair (i,j) can generate: $d_{ij} = \max\{0, v_{ij} - v_j^S\}$
$x = (x_{ij})_{i\in N, j\in M}$	Assignment matrix, $x_{ij} \in \{0, 1\}$
$p = (p_1, \ldots, p_m)$	Price vector
(x, p)	Outcome
(x^*, p^*)	Outcome resulting from play of σ^*
$u_i(x, p)$	Bidder i's payoff at the outcome (x,p)
$u_j^S(x, p)$	Seller j's payoff at the outcome (x,p)
$(\bar{x}^*, \bar{p}^*)$	Reference outcome
x^{eff}	Efficient assignment (optimal assignment)
p_i^V	Bidder i's Vickrey payment
p_j^V	Vickrey price in auction j: $p_j^V = p_i^V$ if $x_{ij}^{\mathit{eff}} = 1$

b_j^0	Starting price (and reserve price) in auction j: $\quad b_j^0 = v_j^S$ for all $j \in M$
b_{ij}	Bidder i's bid in auction j
b_j^s	Current standing bid in auction j (observable second highest submitted bid in auction j up to the current stage)
b_j^h	Current high bid in auction j (non-observable highest submitted bid in j up to the current stage)
$B^h(j)$	Current high bidder in auction j
ι	Minimum bid increment
B^a	Set of remaining active bidders
$B^{PD}(j)$	Price determining bidder in auction j
$B^I(j)$	Bidders i with $j \in D_i$ and $B^h(j) \neq i$ at $(\bar{x}^*, \bar{p}^*)$
$c(\cdot)$	Coalitional function, $c(S)$ is the value of coalition S