On The Equivalence Of Addition Formula Of Lattice Paths And The Kolmogorov Equation Of Birth And Death Processes

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Abstract

We establish the equivalence of the addition formula of weighted lattice paths and the Kolmogorov equation of birth and death processes and provide suitable illustrations.

1 Introduction

The enumeration of lattice paths consists of sequences of lattice points in the plane, beginning at the origin with steps restricted to (1, 1), (1, 0), (1, -1) and in which every point has a non-negative altitude. By using a simple decomposition, the enumerator for these paths with respect to steps of each type at each altitude is given by the continued fraction of Jacobi (called J-fraction) [2].

Approximations employing continued fractions (CFs) often provide a good representation for transcendental functions. CFs arise naturally in calculations of random walk probabilities, and are strongly connected with combinatorial configurations equivalent to lattice paths.

Birth and death processes (BDPs) are widely used to model a variety of applications including computer communications, production management, neutron propagation, chemical reactions, epidemics, population dynamics and so on. Parthasarathy and Lenin [4] have obtained closed form transient solutions of system size probabilities of BDPs by means of CFs.

Goulden and Jackson [2] have given an addition formula as follows: Let \( a(x) = \sum_{i \geq 0} a_i x^i / i! \) and \( \hat{a}(x) = \sum_{i \geq 0} a_i x^i \). If \( a(x) \) has the property that

\[
a(x + y) = \sum_{m \geq 0} p_m f_m(x)f_m(y),
\]

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where $p_m$ is independent of $x$ and $y$, and

$$f_m(x) = \frac{x^m}{m!} + q_{m+1} \frac{x^{m+1}}{(m+1)!} + O(x^{m+2})$$

then $a(x)$ has an addition formula with parameters \{$(p_m, q_{m+1})| m \geq 0$\}. The formal power series $\hat{f}(x)$ has an addition formula with parameters \{$(p_m, q_{m+1})| m \geq 0$\} if and only if

$$\frac{1}{x} \hat{f} \left( \frac{1}{x} \right) = \frac{1}{x - (q_1 - q_0)} - \frac{p_1 p_0^{-1}}{x - (q_2 - q_1)} - \frac{p_2 p_1^{-1}}{x - (q_3 - q_2)} - \cdots. \tag{1}$$

where we use the notation for CFs

$$\frac{a_1}{b_1} \frac{a_2}{b_2} \frac{a_3}{b_3} \cdots = \frac{a_1}{b_1 - \frac{a_2}{b_2 - \frac{a_3}{b_3 - \cdots}}}.$$

They have given a combinatorial proof of this theorem by considering the generating function for paths of non-negative length from altitude 0 to altitude 0, with respect to rises and levels.

It is well known that the transition function $P_{mn}(t) = P(X(t) = n| X(0) = m)$ of BDPs satisfies the Kolmogorov equation:

$$P_{mn}(t + u) = \sum_{k \in \mathbb{N}} P_{mk}(t) P_{kn}(u).$$

This equation states that in order to move from state $m$ to state $n$ in time $t + u$, $X(t)$ first moves to some state $k$ in time $t$ and then from $k$ to $n$ in the remaining time $u$.

In this paper, we present the equivalence of the addition formula of lattice paths and the Chapman-Kolmogorov equation of birth and death processes. We give examples to illustrate our result.

## 2 Birth and Death Processes

A BDP is a continuous time Markov chain \{X(t), t \geq 0\} defined on a probability space $(\Omega, \mathcal{F}, P)$ with stationary transition probability $P_{mn}(t)$ satisfying the forward Kolmogorov differential equations:

$$
P'_{m0}(t) = -a_0 P_{m0}(t) + a_1 P_{m1}(t)$$

$$P'_{mn}(t) = a_{2n-2} P_{m,n-1}(t) - (a_{2n-1} + a_{2n}) P_{mn}(t) + a_{2n-1} P_{m,n+1}(t), \tag{2}$$

for $n = 1, 2, 3, \ldots$. The parameters $a_{2n}$ ($n \geq 0$) and $a_{2n-1}$ ($n \geq 1$) are respectively called the birth and death rates (instead of the usual $\lambda_n$ and $\mu_n$). The reason for this notation will become apparent in the sequel.

Let us take Laplace transforms

$$\hat{P}_{mn}(s) = \int_0^\infty e^{-st} P_{mn}(t) dt, \quad n = 0, 1, 2, \ldots,$$
for $\text{Re}(s) > 0$ of the system of equations given by (2). We assume that $m = 0$, and in that case $\hat{P}_{00}(s)$ simplifies to the expression

$$\hat{P}_{00}(s) = \frac{1}{s + a_0 - a_1 \hat{P}_{11}(s)},$$

and

$$\frac{\hat{P}_{0n}(s)}{\hat{P}_{0,n-1}(s)} = \frac{a_{2n-2}}{s + a_{2n-1} + a_{2n}} \cdot \frac{\hat{P}_{0n+1}(s)}{\hat{P}_{0n}(s)} \cdot \frac{a_{2n+1}a_{2n+2}}{s + a_{2n+3} + a_{2n+4}} \cdots (3)$$

Using (3),

$$\hat{P}_{00}(s) = \frac{1}{s + a_0 - a_1 - a_2 - a_3 - \cdots} \cdots \cdots (4)$$

We shall now give a fundamental correspondence between the addition formula and the Kolmogorov equation of a birth-death process. For this, we first refer to the following theorem of Parthasarathy and Sudhesh [5].

**THEOREM 1.** If $X(0) = m$, then the power series representation for the state probabilities $P_{mn}(t), \ n = 0, 1, 2, \ldots$, corresponding to a state-dependent BDP with birth and death rates respectively $a_{2n}$ and $a_{2n+1}, \ (0 \leq n < \infty)$ are given by

$$P_{mn}(t) = \begin{cases} \frac{M_m}{M_n} \sum_{r=0}^{\infty} (-1)^r \frac{t^{r+n-m}}{(r+n-m)!} \sum_{k=0}^{\min(r,n)} (-1)^k \phi_k(2n+1)A(r-k, 2m), & n = 0, 1, 2, \ldots, m \\ \frac{M_m}{L_{n-1}} \sum_{r=0}^{\infty} (-1)^r \frac{t^{r+n-m}}{(r+n-m)!} \sum_{k=0}^{\min(r,m)} (-1)^k \phi_k(2m+1)A(r-k, 2n), & n = m, m+1, \ldots. \end{cases} (5)$$

where $L_{n-1} = 1, L_{n-1} = a_0a_2a_4\cdots a_{2n-2}, M_0 = 1, M_n = a_1a_3a_5\cdots a_{2n-1}, \phi_0(n) = 1, \phi_r(n) = 0$ if $r \geq \lceil \frac{n+1}{2} \rceil$, for $r \geq 1, n = 1, 2, 3, \ldots,$

$$\phi_r(n+1) = \sum_{i_1=0}^{n-2r} a_{i_1} \sum_{i_2=i_1+2}^{n-2r+2} a_{i_2} \sum_{i_3=i_2+2}^{n-2r+4} a_{i_3} \cdots \sum_{i_{r-1}=i_{r-2}+2}^{n-2} a_{i_r}, (6)$$

and

$$A(r, n) = \sum_{i_1=0}^{n} a_{i_1} \sum_{i_2=i_1+2}^{i_1+1} a_{i_2} \sum_{i_3=i_2+2}^{i_2+1} a_{i_3} \cdots \sum_{i_r=i_{r-1}+2}^{i_{r-1}+1} a_{i_r}, \forall n \in \mathbb{N}, A(0, n) = 1. \quad (7)$$

From (5), we observe that the transition function $P_{mn}(t)$ satisfies

$$\pi_m P_{mn}(t) = \pi_n P_{mn}(t), \quad m, n \in \mathcal{N}, \quad t \geq 0, \quad (8)$$

where

$$\pi_m \equiv 1, \pi_n \equiv \frac{a_0a_2\cdots a_{2n-2}}{a_1a_3\cdots a_{2n-1}}, \quad n = 1, 2, 3, \ldots,$$
are the potential coefficients and (8) shows that \( P_{mn}(t) \) is symmetric. Therefore \( \{X(t), t \geq 0\} \) is reversible [1].

THEOREM 2. The transition probability \( P_{00}(t) \) has an addition formula with parameters \( \{\prod_{i=0}^{2m-1} a_i, -\sum_{i=0}^{2m} a_i\} ; m \geq 0\} \).

PROOF. From (5),

\[
P_{0m}(t) = L_{m-1}f_m(t), \quad P_{m0}(t) = M_{m}f_m(t),
\]

where

\[
f_m(t) = \frac{t^m}{m!} - A(1, 2m) \frac{t^{m+1}}{(m+1)!} + A(2, 2m) \frac{t^{m+2}}{(m+2)!} - A(3, 2m) \frac{t^{m+3}}{(m+3)!} + \cdots \tag{9}
\]

Using the Kolmogorov equation,

\[
P_{00}(t + u) = \sum_{m=0}^{\infty} P_{0m}(t)P_{m0}(u) = \sum_{m=0}^{\infty} p_m f_m(t)f_m(u),
\]

where

\[
p_m = \prod_{i=0}^{2m-1} a_i.
\]

From (9),

\[
q_{m+1} = -A(1, 2m) = -\sum_{i=0}^{2m} a_i.
\]

Therefore, for \( m = 0, 1, 2, \ldots \),

\[
q_{m+1} - q_m = -(a_{2m-1} + a_{2m}), \quad a_{-1} = 0 \quad \text{and} \quad p_{m+1}p_{m}^{-1} = a_{2m}a_{2m+1}.
\]

Hence, \( P_{00}(t) = f_0(t) \) has an addition formula with parameters

\[
\left\{ \left( \prod_{i=0}^{2m-1} a_i, -\sum_{i=0}^{2m} a_i \right) ; m \geq 0 \right\}
\]

and the corresponding \( J \)-fraction coincides with (4).

The following two examples present the addition formula representation of the transition probability \( P_{00}(t) \).

EXAMPLE 1 (Linear rates). Consider a BDP with birth and death rates

\[
a_{2n} = a_{2n+1} = n + 1, \quad n = 0, 1, 2, \ldots
\]

so that

\[
\pi_n = 1, \quad n = 0, 1, 2, \ldots
\]

The corresponding \( J \)-fraction for \( \hat{P}_{00}(s) \) is

\[
\hat{P}_{00}(s) = \frac{1}{s+1} - \frac{1^2}{s+3} - \frac{2^2}{s+5} - \cdots = \int_0^\infty \frac{e^{-t}}{s + t} \, dt = \sum_{r=0}^{\infty} (-1)^r \frac{r!}{s^{r+1}},
\]
This leads to the transition probability,
\[ P_{00}(t) = \frac{1}{1 + t}. \]

From Theorem 1,
\[ P_{0m}(t) = \frac{t^m}{(1 + t)^{m+1}}, \quad m \geq 0. \]

Using (8),
\[ P_{0m}(t) = \frac{\pi_0}{\pi_m} P_{0m}(t) = \frac{t^m}{(1 + t)^{m+1}}. \]

Therefore,
\[ P_{00}(t + u) = \frac{1}{1 + t + u} = \sum_{m=0}^{\infty} P_{0m}(t)P_{m0}(u) = \sum_{m=0}^{\infty} \frac{m!}{t} f_m(t) f_m(u), \]
where
\[ f_m(t) = \frac{1}{m!} \frac{t^m}{(1 + t)^{m+1}} = \frac{t^m}{m!} \left(1 - \frac{1}{1 + t}\right)^2 \frac{t^{m+1}}{(m+1)!} + O(t^{m+2}). \]

Hence \( P_{00}(t) \) has an addition formula with parameters \( \{(m!)^2, -(m + 1)^2 \}\). \( m \geq 0 \).

**EXAMPLE 2** (Infinite server queue) Consider a BDP with birth and death rates
\[ a_{2n} = 1, \quad n = 0, 1, \ldots, \] and \[ a_{2n-1} = n, \quad n = 1, 2, \ldots, \]
so that
\[ \pi_n = \frac{1}{n!}, \quad n = 0, 1, 2, \ldots. \]

This rates corresponds to the \( M/M/\infty \) (self service) model. It is well known that,
\[ P_{0m}(t) = \frac{(1 - e^{-t})^m \exp(e^{-t} - 1)}{m!}, \quad m \geq 0. \]

Using (8),
\[ P_{m0}(t) = (1 - e^{-t})^m \exp(e^{-t} - 1), \quad m \geq 0. \]

Therefore,
\[ P_{00}(t + u) = \exp[e^{-(t+u)} - 1] = \sum_{m=0}^{\infty} P_{0m}(t)P_{m0}(u) = \sum_{m=0}^{\infty} m! f_m(t) f_m(u), \]
where
\[ f_m(t) = \frac{(1 - e^{-t})^m \exp(e^{-t} - 1)}{m!} = \frac{t^m}{m!} - \frac{(m + 1)(m + 2)}{2} \frac{t^{m+1}}{(m+1)!} + O(t^{m+2}). \]

Hence \( P_{00}(t) \) has an addition formula with parameters \( \{(m!,-(m+1)(m+2)/2)\}\). \( m \geq 0 \).

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