Coercivity and the Calderon Operator on an Unbounded Domain

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Introduction

The mathematical problem concerning scattering from a perfectly conducting obstacle in an unbounded two-layered medium has been approached by various authors using different methods. First and foremost, there were the results by P.-M. Cutzach and C. Hazard [**5**] in 1998: They were able to prove the existence and uniqueness of solutions to the time-harmonic Maxwell's equations in the two-layered lossless setup and the Silver-Müller radiation condition by reduction to a bounded scenario. This thesis presents several aspects regarding the weak formulation of electromagnetic scattering by reduction to an unbounded and dissipative half-space, such as a generalization to the Lax–Milgram Lemma, singular weighted Sobolev spaces, and coercivity of the sesquilinear form for the three-dimensional problem using a detailed analysis of the Calderon operator on an unbounded interface. An alternate approach was published by A. Kirsch [**17**] in 2007, cf. A. Kirsch and N. Grinberg [**18**] in 2008, without an artificial reduction of the domain using a new integral equation formulation.

The setup of a lossless half-space above a dissipative half-space with an infinite interface has been accepted as a suitable description for the medium in the discussion of the propagation of electromagnetic waves in air and earth [**26**] with a modern application to mine detection [**13**]. Besides this direct application, which will be the model problem in this text, the general approach presented here of using a Dirichlet-to-Neumann or Calderon operator to mask a layer is fundamental to rough surface scattering such as in [**10, 7, 24**] with its wide area of applications.

The general approach of using a Calderon operator in a non-local boundary condition for a weak formulation of unbounded electromagnetic scattering problems for bounded obstacles was introduced by A. Kirsch and P. Monk [**19**] in 1995. A reference for this general approach is the very helpful book of P. Monk [**26**] from 2003 which also discusses layered media following the line of P.-M. Cutzach and C. Hazard [**5**], where the unbounded problem is reduced to a bounded domain and important properties are deduced from the Green tensor. Green tensors were also discussed by M. Petry [**27**] in 1993 for dissipative multiple layers, and in two dimensions by J. Coyle [**3**] in 1998, cf. [**4**] in 2000, where he also considered the inverse problem for which there are many recent results, such as F. Delbary et al. [**11**] in 2008, and already mentioned [**13, 17, 18**]. An alternative approach to the layered medium using semigroups is presented in the book by M. Cessenat [**2**] from 1996, and in the case of dissipative media, C. H. Wilcox [**33**] obtained a coercive formulation in 1963.

Chapter 1 defines various concepts such as Sobolev spaces and trace theorems as they are needed throughout this text, as well as an important extension to the Lax-Milgram Lemma which presents new conditions for coercivity that are shown to be applicable to the regarded problems. Chapter 2 is devoted to the derivation of the mathematical models with an analysis of the conditions which allow the reduction to two-dimensional models. In Chapter 3, the two-dimensional problems are presented in the classical sense, and in weak formulations reduced to an unbounded half-space using a non-local boundary condition. They are proven to be uniquely solvable using the extended Lax-Milgram Lemma, followed by regularity results, and an additional weak formulation proves exponential decay of the solutions.

Based on the observations in the derivation of two-dimensional models and the mapping properties of Dirichlet-to-Neumann maps, Chapter 4 then introduces singular weighted Sobolev spaces with all properties necessary for the analysis of the non-local boundary condition for the full three-dimensional Maxwell's system. Chapter 5 presents classical definitions of the full three-dimensional scattering problems and combines the traditional setup with the results of the previous chapters to form the required framework. It introduces the Calderon map on the unbounded interface and proves its mapping properties using the weighted Sobolev spaces. The weak formulation is presented and proven to be coercive, and uniquely solvable, by detailed analysis of the Calderon operator and the new conditions that were introduced before. This text ends with regularity results that directly follow from the coercivity of the sesquilinear form.

Finally, I would like to express my deepest gratitude to my advisor Prof. Dr. Andreas Kirsch for guidance and helpful advice throughout my doctoral work, and to PD Dr. Frank Hettlich for sharing his thoughts and being the co-examiner of this thesis. I am much obliged to Prof. Dr. Simon Chandler-Wilde and his colleagues at the University of Reading for their support and hospitality during my two visits in the past years. Special thanks go to all current and former members of our workgroup, namely Dr. Tilo Arens, Monika Behrens, PD Dr. Natalia Grinberg, Andreas Helfrich-Schkarbanenko, Sven Heumann, Dr. Karsten Kremer, Dr. Armin Lechleiter, Dr. Wagner Muniz, Kai Sandfort, Susanne Schmitt, and Dr. Henning Schon, for their valuable help and fruitful discussions. Last but not least, it is a pleasure to thank Dr. Carsten Brockmann for his linguistic advice and, of course, my wife Sabine for her endless patience, support, and encouragement.

CHAPTER 1

Mathematical Foundations

1.1. Function Spaces

This part establishes notations and states results which will be needed later. See [**22, 32, 15**] for proofs and further detail. Here, we use the notation established by Schwartz [**29**], $\mathcal{E}(\mathbb{R}^n) := C^\infty(\mathbb{R}^n)$, to denote the space of infinitely continuously differentiable complex-valued functions over $\mathbb{R}^n$ and the subspace $\mathcal{D}(\mathbb{R}^n) := C_0^\infty(\mathbb{R}^n)$ of functions which additionally have compact support. Corresponding spaces of vector valued functions are denoted as $\mathcal{E}(\mathbb{R}^2, \mathbb{C}^2)$.

The topology and the concept of convergence on the Fréchet space $\mathcal{E}(\mathbb{R}^n)$, i.e., a complete, metrizable, locally convex topological vector space, is based on the countable family of semi-norms

$$|u|_{\mathcal{E},\alpha,\beta} := \sup_{|x|\leq\alpha} \left| \left(\frac{\partial}{\partial x}\right)^{\beta} u(x) \right| , \ \alpha \in \mathbb{N}, \ \beta \in \mathbb{N}_0^n$$

where convergence is defined as convergence with respect to all semi-norms, or uniform convergence on all compact sets in $\mathbb{R}^n$. The translation invariant metric

$$d_{\mathcal{E}(\mathbb{R}^n)}(x, y) := \sum_{\alpha,\beta} v_{\alpha,\beta} \frac{|x - y|_{\mathcal{E},\alpha,\beta}}{1 + |x - y|_{\mathcal{E},\alpha,\beta}}$$

for some fixed positive sequence $(v_{\alpha,\beta})_{\alpha\in\mathbb{N},\beta\in\mathbb{N}_0^n} \subset \mathbb{R}_{>0}$ with converging sum defines the same concept of convergence and the corresponding topology. Sequences in $\mathcal{D}(\mathbb{R}^n)$ are convergent if there is a compact set in which the supports of all elements are contained and the sequence converges in $\mathcal{E}(\mathbb{R}^n)$.

The Schwartz space of rapidly decreasing, $C^\infty(\mathbb{R}^n)$ functions is referred to by

$$\mathcal{S}(\mathbb{R}^n) := \{u \in \mathcal{E}(\mathbb{R}^n) : |u|_{\mathcal{S},\alpha,\beta} < \infty \text{ for all } \alpha \in \mathbb{N}_0, \beta \in \mathbb{N}_0^n\}$$
$$\text{where } |u|_{\mathcal{S},\alpha,\beta} := \sup_{x\in\mathbb{R}^n} \left|(1+|x|)^\alpha \left(\tfrac{\partial}{\partial x}\right)^\beta u(x)\right|.$$

This space with its semi-norms also is a Fréchet space and a corresponding translation invariant metric on $\mathcal{S}(\mathbb{R}^n)$ is given by

$$d_{\mathcal{S}(\mathbb{R}^n)}(x,y) := \sum_{\alpha,\beta} v_{\alpha,\beta} \frac{|x-y|_{\mathcal{S},\alpha,\beta}}{1+|x-y|_{\mathcal{S},\alpha,\beta}},$$

where $(v_{\alpha,\beta})_{\alpha\in\mathbb{N}_0,\beta\in\mathbb{N}_0^n} \subset \mathbb{R}_{>0}$ is a fixed positive sequence with converging sum, which again defines the topology and convergence in $\mathcal{S}(\mathbb{R}^n)$.

Above spaces are contained in each other by $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{E}(\mathbb{R}^n)$, and since they are metrizable, completion and continuity coincides with sequential completion and sequential continuity.

For each of these spaces we define their dual spaces of continuous linear functionals $\mathcal{D}^*(\mathbb{R}^n)$, $\mathcal{S}^*(\mathbb{R}^n)$, $\mathcal{E}^*(\mathbb{R}^n)$[1] equipped with the dual or weak topology, which are contained in each other as $\mathcal{E}^*(\mathbb{R}^n) \subset \mathcal{S}^*(\mathbb{R}^n) \subset \mathcal{D}^*(\mathbb{R}^n)$ as restrictions of the functionals to smaller function spaces. Of course, these spaces may also be defined on open subsets $D \subset \mathbb{R}^n$, such as $\mathcal{D}(D)$ or $\mathcal{D}^*(D)$, accordingly. Note, that $\mathcal{D}^*(\mathbb{R}^n) \subset \mathcal{D}^*(D)$ since functions in $\mathcal{D}(D)$ may always be extended by zero to functions in $\mathcal{D}(\mathbb{R}^n)$.

Convergence in these dual spaces is defined as point-wise convergence, and with respect to which they are complete, or, in other words, they are weakly sequentially complete.

For functions and functionals from corresponding dual spaces, e.g., $u \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$, the bilinear dual pairing is denoted by

[1]In this text the notation $\mathcal{E}(\mathbb{R})$ is preferred to $C^\infty(\mathbb{R})$ due to the notation $\mathcal{E}^*(\mathbb{R})$ for its dual. All dual spaces are understood to be the spaces of continuous functionals, there are no algebraic duals in this text.

$\langle u, \varphi \rangle = \langle \varphi, u \rangle := \varphi(u) \in \mathbb{C}$, which we identify with

$$\langle \varphi, u \rangle = \int\limits_{\mathbb{R}^n} \varphi(t)\, u(t)\, dt = (\varphi, \bar{u})_{L^2(\mathbb{R}^n)}$$

when $u \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ or $\varphi \in L^1(\mathbb{R}^n)$.

We define the Fourier transform by

$$\hat{u}(\tau) = \mathcal{F}u(\tau) := \frac{1}{(2\pi)^{n/2}} \int\limits_{\mathbb{R}^n} u(t)\, e^{-i(\tau \cdot t)}\, dt\,,\ \tau \in \mathbb{R}^n$$

for $n \in \mathbb{N}$. The transform and its inverse, are continuous linear operators $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{F} : L^1(\mathbb{R}^n) \to BC(\mathbb{R}^n)$. On the dual space $\mathcal{S}^*(\mathbb{R}^n)$, the Fourier transform is also continuous and continuously invertible, when defined using the dual pairing, i.e. if $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$, then $\langle \mathcal{F}\varphi, u \rangle := \langle \varphi, \mathcal{F}u \rangle$.

As the dual pairing is not sesquilinear, we have $\mathcal{F}u$ on the right hand side, which can be seen as an extension of the Fourier transform to generalized functions, as far as the definition coincides with the original definition for functions in $\mathcal{S}(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n)$.

Based on the properties of the Fourier transform, we may introduce the Bessel potential as a Fourier multiplier for functions in $\mathcal{S}(\mathbb{R}^n)$:

Definition 1.1.1. (Bessel Potential)

$$\mathcal{J}^s u := \mathcal{F}^{-1}(\psi^s \hat{u})\,,\ \text{where } \psi^s(\tau) = \left(1 + |\tau|^2\right)^{s/2}\,,\ s \in \mathbb{R},\ \tau \in \mathbb{R}^n$$

as a continuous and continuously invertible linear mapping

$$\mathcal{J}^s : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)\,.$$

On dual spaces $\mathcal{J}^s : \mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}^*(\mathbb{R}^n)$ the mapping is defined by the dual pairing: For $\varphi \in \mathcal{S}^*(\mathbb{R}^n)$ and $u \in \mathcal{S}(\mathbb{R}^n)$, let $\langle \mathcal{J}^s\varphi, u \rangle := \langle \varphi, \mathcal{J}^s u \rangle$.

Again, the latter definition may also be seen as an extension to $\mathcal{S}^*(\mathbb{R}^n)$, as the two definitions coincide for functions in $\mathcal{S}(\mathbb{R}^n)$. This leads to the

Definition 1.1.2. (Sobolev Spaces)

$$H^s(\mathbb{R}^n) := \{u \in \mathcal{S}^*(\mathbb{R}^n) : \mathcal{J}^s u \in L^2(\mathbb{R}^n)\},\ s \in \mathbb{R}.$$

Using the natural scalar product $(u, v)_{H^s(\mathbb{R}^n)} := (\mathcal{J}^s u, \mathcal{J}^s v)_{L^2(\mathbb{R}^n)}$ and the induced norm $||u||^2_{H^s(\mathbb{R}^n)} := (u, u)_{H^s(\mathbb{R}^n)}$ the spaces $H^s(\mathbb{R}^n)$ are Hilbert spaces and this extends to Sobolev spaces on subsets, most importantly based on $\mathbb{R}^n_\pm$ and Γ denoting the upper and lower half space $\mathbb{R}^n_\pm := \{x \in \mathbb{R}^n : x_n \gtrless 0\}$ and their associated boundary line $\Gamma = \{x \in \mathbb{R}^n : x_n = 0\}$:

Definition 1.1.3. Let $D \subset \mathbb{R}^n$ and $s \in \mathbb{R}$, then

$$H^s(D) := \{u \in \mathcal{D}^*(D) : u = U|_D\ ,\ \text{for some } U \in H^s(\mathbb{R}^n)\}.$$

The associated scalar product is defined with the help of a projection operator, see [**22**], leading to $||u||_{H^s(D)} = \inf\limits_{U|_D = u,\, U \in H^s(\mathbb{R}^n)} ||U||_{H^s(\mathbb{R}^n)}$.

Note that the Bessel potential is self-adjoint with respect to the above scalar products, so as an example for $\varphi \in \mathcal{S}^*(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ and test functions $u \in \mathcal{S}(\mathbb{R}^n)$ it holds that $(\mathcal{J}^s\varphi, u)_{L^2(\mathbb{R}^n)} = (\varphi, \mathcal{J}^s u)_{L^2(\mathbb{R}^n)}$. Since the Sobolev spaces were defined using $\mathcal{S}^*(\mathbb{R}^n)$, we have a continuous extension of the Fourier transform to these spaces.

1.2. Traces and Extensions

The existence of traces and estimates for them are very significant for the derivation of weak formulations, in fact a large part of this text is devoted to appropriate function spaces for the traces on the unbounded interface. The general theory of Sobolev spaces and their traces are thoroughly presented in the book of R. Adams and J. Fournier [**1**] but succinct in the discussion of Bessel potentials where this text follows the concepts of W. McLean [**22**] and W. Walter [**32**]. The concept of traces goes hand in hand with the notion of bounded extension of functions on boundaries or subsets to functions in sets with appropriate regularity. In the following are some of the results that will be needed later. The following theorem is a direct consequence of the Lemma 3.35 from [**22**]:

THEOREM 1.2.1. *(Trace Theorem) Let S be a strip $S = \{(x_1, x_2) \in \mathbb{R}^2 : \ -h < x_2 < 0\}$ and $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_2 = 0\}$ be a boundary line, then $||u||_{H^{1/2}(\Gamma)} \leq C_T \, ||u||_{H^1(S)}$ for some $C_T > 0$.*

A more general estimate is presented in the following Lemma which shows that the trace is mainly dependent on the gradient of the function in the set. First, we regard an unbounded rectangluar shaped set:

LEMMA 1.2.2. *Let $I \subseteq \mathbb{R}$ be a not necessarily bounded interval and $\Gamma_a = I \times \{a\}$ the upper boundary of $D_a = I \times (-\infty, a)$ for some $a \in \mathbb{R}$. If $u \in H^1(D_a)$, then for any $\varepsilon > 0$ there is an estimate*

$$||u||^2_{H^{1/2}(\Gamma_a)} \leq \varepsilon ||u||^2_{L^2(D_a)} + \left(\frac{C^2}{\varepsilon} + C \right) ||\nabla u||^2_{L^2(D_a)}$$

with some $C > 0$ independent of u and ε.

PROOF. For this result it suffices to prove the estimate for $u \in H^1(D_a) \cap C_0^\infty(D_a)$ since $C_0^\infty(D_a)$ is dense in $H^1(D_a)$. Using the estimate

$$\begin{aligned}
|u(x_1, x_2)|^2 &= |u(x_1, b)|^2 + \int_b^{x_2} \frac{\partial}{\partial x_2} |u(x_1, t)|^2 \, dt \\
&= |u(x_1, b)|^2 + 2 \operatorname{Re} \int_b^{x_2} \overline{u(x_1, t)} \, \frac{\partial}{\partial x_2} u(x_1, t) \, dt \\
&\leq |u(x_1, b)|^2 \\
&\quad + 2 \sqrt{\int_b^{x_2} |u(x_1, t)|^2 \, dt} \sqrt{\int_b^{x_2} |\frac{\partial}{\partial x_2} u(x_1, t)|^2 \, dt}
\end{aligned}$$

for fixed $x_1 \in I$ and since $u(x_1, b) \to 0$ for $b \to -\infty$ we obtain

$$|u(x_1, x_2)|^2 \leq 2 \sqrt{\int_{-\infty}^{x_2} |u(x_1, t)|^2 \, dt} \sqrt{\int_{-\infty}^{x_2} |\frac{\partial}{\partial x_2} u(x_1, t)|^2 \, dt} \, .$$

By the estimate

$$0 \leq \left(\sqrt{\delta \alpha} - \sqrt{\frac{\beta}{\delta}} \right)^2 = \delta \alpha - 2 \sqrt{\alpha \beta} + \frac{\beta}{\delta} \, , \quad \text{i.e. } 2 \sqrt{\alpha} \sqrt{\beta} \leq \delta \alpha + \frac{\beta}{\delta}$$

this yields

$$|u(x_1,x_2)|^2 \le \delta \int_{-\infty}^{x_2} |u(x_1,t)|^2 \, dt + \frac{1}{\delta} \int_{-\infty}^{x_2} |\frac{\partial}{\partial x_2} u(x_1,t)|^2 \, dt \, .$$

We integrate over the rectangle $R = I \times (a-h, a)$ of height $h > 0$ and gain

$$||u||^2_{L^2(R)} \le h\delta ||u||^2_{L^2(D_a)} + \frac{h}{\delta} ||\frac{\partial u}{\partial x_2}||^2_{L^2(D_a)} \le h\delta ||u||^2_{L^2(D_a)} + \frac{h}{\delta} ||\nabla u||^2_{L^2(D_a)} .$$

Since Γ_a is part of ∂R the Trace Theorem 1.2.1 finally yields

$$\begin{aligned} ||u||^2_{H^{1/2}(\Gamma_a)} &\le C ||u||^2_{H^1(D_a)} = C ||u||^2_{L^2(D_a)} + C ||\nabla u||^2_{L^2(D_a)} \\ &\le Ch\delta ||u||^2_{L^2(D_a)} + (C + C\frac{h}{\delta}) ||\nabla u||^2_{L^2(D_a)} \, . \end{aligned}$$

Choosing $h = 1$ and $\delta = \frac{\varepsilon}{C}$ results in the desired assertion. □

The next Lemma discusses a corresponding estimate for the H^1-norm on an arbitrary but bounded set.

LEMMA 1.2.3. *Let $D \subset \mathbb{R}^2$ be bounded and $\Gamma \subset \partial D$, then there exists $C > 0$ such that*

$$||u||^2_{H^1(D)} \le C \left(\varepsilon ||u||^2_{L^2(D)} + \frac{1}{\varepsilon} ||\nabla u||^2_{L^2(D)} + |\int_\Gamma u \, ds|^2 \right)$$

for all $u \in H^1(D)$ and $\varepsilon \in (0,1]$.

PROOF. Otherwise there exists a sequence $\varepsilon_j > 0$, $u_j \in H^1(D)$ with

$$||u_j||^2_{H^1(D)} > j \left(\varepsilon_j ||u_j||^2_{L^2(D)} + \frac{1}{\varepsilon_j} ||\nabla u_j||^2_{L^2(D)} + |\int_\Gamma u_j \, ds|^2 \right)$$

for all $j \in \mathbb{N}$. We set $\tilde{u}_j = \frac{1}{||u_j||_{H^1(D)}} u_j$. Then

$$1 > j \left(\varepsilon_j ||\tilde{u}_j||^2_{L^2(D)} + \frac{1}{\varepsilon_j} ||\nabla \tilde{u}_j||^2_{L^2(D)} + |\int_\Gamma \tilde{u}_j \, ds|^2 \right) \, , \text{ i.e.}$$

$$\begin{aligned} \frac{1}{j} &> \varepsilon_j ||\tilde{u}_j||^2_{L^2(D)} + \frac{1}{\varepsilon_j} ||\nabla \tilde{u}_j||^2_{L^2(D)} + |\int_\Gamma \tilde{u}_j \, ds|^2 \\ &\ge ||\nabla \tilde{u}_j||^2_{L^2(D)} + |\int_\Gamma \tilde{u}_j \, ds|^2 \, . \end{aligned}$$

Thus $\nabla \tilde{u}_j \to 0$ and $\int_\Gamma \tilde{u}_j \, ds \to 0$. There exists a convergent subsequence $\tilde{u}_{j_k} \to \tilde{u}$, and as we have seen $\nabla \tilde{u} \equiv 0$, so $\tilde{u}$ is constant and since $\int_\Gamma \tilde{u} \, ds = 0$ it is zero which is a contradiction to the definition of the sequence $(\tilde{u}_j)$. □

Only with the help of above estimates and the Trace Theorem we will be able to verify uniqueness of the weak problem by proving coercivity of the sesquilinear form, called a, of the weak formulation which will be introduced in Problems 3.3.1 and 3.3.2: The difficult part will be to find an estimate to $a(u, u)$ from below, because this term does not only contain terms related to the norm of u, but also terms of the trace of u which need to be estimated.

The detailed discussion of $a(u, u)$ will show that the trace component can be of opposite sign to the contribution of the L^2 norm of u in the lower half-space. Under some additional assumptions on the wave number it is possible to estimate the contribution of the trace by the L^2 norm of u, and therefore disregard the perturbation by the trace component and prove coercivity by the Lax Milgram Lemma 1.3.1.

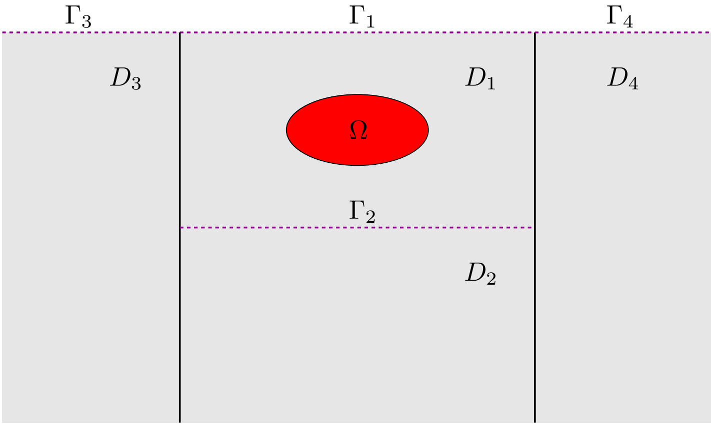
But by virtue of Lemma 1.3.3 we will prove coercivity of the sesquilinear form even without additional assumptions on the wave number. This is made possible by the following estimate of the trace primarily using the L^2 norm of the gradient of u in D by combining above results:

LEMMA 1.2.4. *Let $D = \mathbb{R}^2_- \backslash \bar{\Omega}$ for a bounded $\bar{\Omega} \subset \mathbb{R}^2_-$, $u \in H^1(D)$ and $\varepsilon \in (0, 1]$. Then there exist $C_1, C_2 > 0$ independent of ε and u such that*

$$||u||^2_{H^{1/2}(\Gamma)} \leq C_1 \left(\varepsilon ||u||^2_{L^2(D)} + \left(\frac{C_2^2}{\varepsilon} + C_2 \right) ||\nabla u||^2_{L^2(D)} \right)$$

for $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2 : \; x_2 = 0\}$.

PROOF. Choose $I = (a_1, a_2)$ and h such that $\bar{\Omega} \subset I \times (0, h)$. We split D into four rectangles $D_1 = D \cap (I \times (0, h))$, $D_2 = I \times (-\infty, h)$, $D_3 = (-\infty, a_1) \times (-\infty, 0)$, $D_4 = (a_2, \infty) \times (-\infty, 0)$ and with upper borders $\Gamma_1 = I \times \{0\}$, $\Gamma_2 = I \times \{h\}$, $\Gamma_3 = (-\infty, a_1) \times \{0\}$ and $\Gamma_4 = (a_2, \infty) \times \{0\}$.



We may apply Lemma 1.2.2 on set D_2, as well as D_3 and D_4 later on, yielding

$$||u||^2_{H^{1/2}(\Gamma_2)} \leq \varepsilon ||u||^2_{L^2(D_2)} + \left(\frac{C^2}{\varepsilon} + C\right) ||\nabla u||^2_{L^2(D_2)}.$$

Since $|\int_{\Gamma_2} u\, ds|^2 \leq ||u||^2_{L^2(\Gamma_2)} \leq ||u||^2_{H^{1/2}(\Gamma_2)}$, using Lemma 1.2.3 on set D_1 results in

$$||u||^2_{H^1(D_1)} \leq \tilde{C}\left(\varepsilon ||u||^2_{L^2(D_1 \cup D_2)} + \left(\frac{C^2}{\varepsilon} + C\right) ||\nabla u||^2_{L^2(D_1 \cup D_2)}\right)$$

and by the Trace Theorem 1.2.1

$$||u||^2_{H^{1/2}(\Gamma_1)} \leq C_T \tilde{C}\left(\varepsilon ||u||^2_{L^2(D_1 \cup D_2)} + \left(\frac{C^2}{\varepsilon} + C\right) ||\nabla u||^2_{L^2(D_1 \cup D_2)}\right).$$

Putting it all together we have

$$\begin{aligned} ||u||^2_{H^{1/2}(\Gamma)} &= ||u||^2_{H^{1/2}(\Gamma_1)} + ||u||^2_{H^{1/2}(\Gamma_3)} + ||u||^2_{H^{1/2}(\Gamma_4)} \\ &\leq C_1\left(\varepsilon ||u||^2_{L^2(D)} + \left(\frac{C_2^2}{\varepsilon} + C_2\right) ||\nabla u||^2_{L^2(D)}\right), \end{aligned}$$

where $C_1 = \max\{1, C_T\tilde{C}\}$ and C_2 the maximum value of the constants C occurring for D_2, D_3 and D_4. $\square$

1.3. Functional Analytic Results

LEMMA 1.3.1. *(Lax-Milgram) Let H be a Hilbert space, $a \in H^*$ and $b : H \times H \to \mathbb{C}$ a sesquilinear function, which is continuous, i.e., there exists a constant $C_1 \in \mathbb{R}$ such that for all $u, v \in H$*

$$|b(u,v)| \leq C_1 ||u||_H \cdot ||v||_H \,,$$

and coercive, i.e., there exist $\varphi \in \mathbb{R}$, $C_2 > 0$ such that for all $u \in H$

$$\mathrm{Re}\left(e^{-i\varphi} b(u,u)\right) \geq C_2 ||u||_H^2 \,.$$

Then the equation

$$b(u,v) = a(v)$$

for all $v \in H$ has a unique solution $u \in H$.

PROOF. This proof follows [**14**] but has an extension for the phase multiplication and restriction to the real part in the coercivity condition.

By the Riesz representation theorem there exists a linear bounded mapping $T : H \to H$ such that for all $u, v \in H$

$$e^{-i\varphi} b(u,v) = (Tu, v)_H \quad \text{and} \quad ||Tu||_H \leq C_1 ||u||_H \,.$$

By coercivity on the other hand, we have

$$\begin{aligned} C_2 ||u||_H^2 &\leq \mathrm{Re}\left(e^{-i\varphi} b(u,u)\right) = \mathrm{Re}\,(Tu, u)_H \\ &\leq |\,(Tu,u)_H\,| \leq ||u||_H \cdot ||Tu||_H \,. \end{aligned}$$

This leads to $C_2 ||u||_H \leq ||Tu||_H \leq C_1 ||u||_H$ for all $u \in H$, which implies that T is injective, has closed range and its inverse is bounded.

Let $z \in H$ such that $(Tu, z)_H = 0$ for all $u \in H$. Then $(Tz, z)_H = e^{-i\varphi} b(z,z) = 0$ and therefore $z = 0$, and this yields that T maps onto H. So T^{-1} is a bounded linear mapping on H. By Riesz there exists a unique $w \in H$ such that

$$e^{-i\varphi} a(v) = (w, v)_H$$

for all $v \in H$. Then the unique solution is $u = T^{-1}w$, as for all $v \in H$

$$e^{-i\varphi} b(T^{-1}w, v) = (w, v)_H = e^{-i\varphi} a(v)\,.$$

$\square$

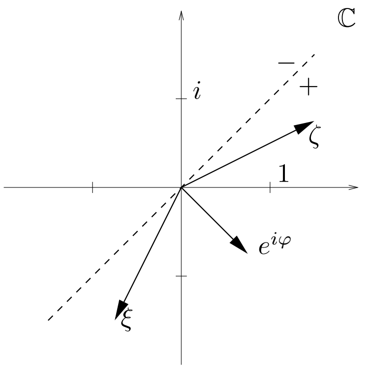
REMARK 1.3.2. If the square of the norm of a Hilbert space H is representable as the sum of two (or more) half norms such as $||u||_H^2 = |u|_A^2 + |u|_B^2$ for all $u \in H$, then all bounded sesquilinear forms with

$$b(u, u) = \xi |u|_A^2 + \zeta |u|_B^2,\ \ \xi, \zeta \in \mathbb{C}\backslash\{0\}$$

are coercive if ξ and ζ are on one side of a straight line through the origin in the complex plane.

For two half norms this is the case if and only if $\arg \xi \neq \arg(-\zeta)$, as then there exists an angle φ such that both $\mathrm{Re}(e^{-i\varphi}\xi) > 0$ and $\mathrm{Re}(e^{-i\varphi}\zeta) > 0$ and then

$$\mathrm{Re}\left(e^{-i\varphi}\, b(u, u)\right) \geq \min\{\mathrm{Re}(e^{-i\varphi}\xi), \mathrm{Re}(e^{-i\varphi}\zeta)\}\, ||u||_H^2\,.$$



The optimal value of φ with respect to the largest coercivity constant for the case $|\xi| = |\zeta|$ is given by the angle bisector between ξ and ζ:

$$\varphi = \frac{1}{2}\mathrm{Arg}\left(\frac{\xi}{\zeta}\right) + \mathrm{Arg}(\zeta)\,,$$

where $\mathrm{Arg} : \mathbb{C} \to (-\pi, \pi]$ denotes the principal value of the argument to determine the angle bisector of the enclosed (or smaller) angle between ξ and ζ. In general, any angle within the smaller angle of the arguments of ξ and ζ will do.

Of course the same concept holds if the bounded sesquilinear form has additional summands which are all on one side of a straight line through the origin. But it even applies to special cases, when an additional summand is opposite in the complex plane to the factor of the one semi-norm in the form, but is somehow bounded by the other semi-norm:

LEMMA 1.3.3. *Consider a Hilbert space H with a norm $||\cdot||_H$, the square of which is representable as the sum of the squares of two half norms $|\cdot|_A$ and $|\cdot|_B$, so $||u||_H^2 = |u|_A^2 + |u|_B^2$ for all $u \in H$. Also, let a be a bounded sesquilinear function such that*

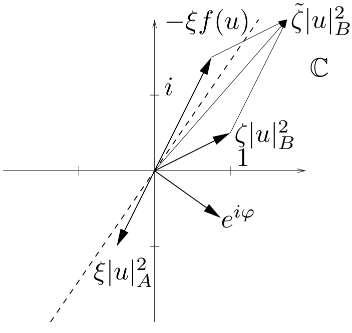
$$b(u,u) = \xi|u|_A^2 - \xi f(u) + \zeta|u|_B^2,\ \ \xi,\zeta \in \mathbb{C}\backslash\{0\},\ \arg\xi \neq \arg(\pm\zeta)$$

with $f : H \to \mathbb{R}$ for which there is a constant $D > 0$ such that $f(u) \leq D|u|_B^2$ for all $u \in H$. Then b is coercive, i.e., there exist $\varphi \in \mathbb{R}$ and $C > 0$, such that

$$\mathrm{Re}\left(e^{-i\varphi} b(u,u)\right) \geq C\left(|u|_A^2 + |u|_B^2\right) = C||u||_H^2 .$$

PROOF. Define $\tilde{\zeta} = \zeta - D\xi \neq 0$ and consider

$$\tilde{b}(u) := b(u,u) + \xi\left(f(u) - D|u|_B^2\right) = \xi|u|_A^2 + \tilde{\zeta}|u|_B^2 . \tag{1.1}$$



Since ξ and ζ are linearly independent, the same is true for ξ and $\tilde{\zeta}$. As in remark 1.3.2 one thus can find φ with $\mathrm{Re}(e^{-i\varphi}\xi) > 0$, $\mathrm{Re}(e^{-i\varphi}\tilde{\zeta}) > 0$ and a constant $C = \min\{\mathrm{Re}(e^{-i\varphi}\xi), \mathrm{Re}(e^{-i\varphi}\tilde{\zeta})\} > 0$ such that

$$\mathrm{Re}\left(e^{-i\varphi}\tilde{b}(u)\right) \geq C\left(|u|_A^2 + |u|_B^2\right)$$

Due to the choice of φ we have $\mathrm{Re}(e^{-i\varphi}\xi) > 0$, and therefore

$$\begin{aligned}\mathrm{Re}\left(e^{-i\varphi}b(u,u)\right) &= \mathrm{Re}\left(e^{-i\varphi}\tilde{b}(u)\right) + \mathrm{Re}\left(e^{-i\varphi}\xi\right)\left(D|u|_B^2 - f(u)\right)\\ &\geq \mathrm{Re}\left(e^{-i\varphi}\tilde{b}(u)\right) \geq C\left(|u|_A^2 + |u|_B^2\right).\end{aligned}$$

Note, that

$$\mathrm{Re}(e^{-\varphi}\zeta) = \mathrm{Re}(e^{-\varphi}(\tilde{\zeta} + D\xi)) = \mathrm{Re}(e^{-i\varphi}\tilde{\zeta}) + D\,\mathrm{Re}(e^{-i\varphi}\xi) > 0\,.$$

□

CONCLUSION 1.3.4. Let H be a Hilbert space with norm $||u||_H$, the square of which is representable as the sum of the squares of two half norms $|\cdot|_A$ and $|\cdot|_B$, so $||u||_H^2 = |u|_A^2 + |u|_B^2$ for all $u \in H$. Let $a \in H^*$ and $b: H \times H \to \mathbb{C}$ a continuous sesquilinear form which has a representation

$$b(u,u) = \xi|u|_A^2 + \zeta|u|_B^2 + g(u)$$

on its diagonal for some $\xi, \zeta \in \mathbb{C}\backslash\{0\}$, $\arg\xi \neq \arg(\pm\zeta)$ and a complex valued function $g: H \to \mathbb{C}$ which is bounded by

$$|g(u)| \leq D_1|u|_A^2 + D_2|u|_B^2$$

for all $u \in H$ with some $D_2 > 0$ and $0 \leq D_1 < |\xi|$, and its argument is limited to the half circle

$$\arg(g(u)) \in \{\arg(a_1\xi + a_2\zeta):\ a_1 \in \mathbb{R}^2\backslash\{0\}, a_2 \geq 0\}\,.$$

Then the equation $b(u,v) = a(v)$ for all $v \in H$ has a unique solution $u \in H$.

PROOF. The function

$$\tilde{b}(u) = \xi\underbrace{\left(1 - \frac{D_1}{|\xi|}\right)}_{>0}|u|_A^2 + \left(\zeta - \xi\frac{D_2}{|\xi|}\right)|u|_B^2$$

is "coercive" by the proof of Lemma 1.3.3 Equation 1.1 for $f(u) = \frac{D_2}{|\xi|}|u|_B^2$, i.e., there exists φ such that

$$\mathrm{Re}\left(e^{-i\varphi}\tilde{b}(u)\right) \geq C||u||_H^2\,.$$

For such φ we have $\mathrm{Re}(e^{-i\varphi}\xi) \geq 0$ as well as $\mathrm{Re}(e^{-i\varphi}\zeta) \geq 0$, as noted above, and for $g(u) = a_1\xi + a_2\zeta$, $a_1 \neq 0$, $a_2 \geq 0$,

$$\begin{aligned} \mathrm{Re}\left(e^{-i\varphi}g(u)\right) &= \mathrm{Re}(e^{-i\varphi}a_1\xi) + \underbrace{a_2\,\mathrm{Re}(e^{-i\varphi}\zeta)}_{\geq 0} \geq -|a_1|\mathrm{Re}(e^{-i\varphi}\xi) \\ &\geq -\mathrm{Re}\left(e^{-i\varphi}\frac{\xi}{|\xi|}|g(u)|\right) \\ &\geq -\mathrm{Re}\left(e^{-i\varphi}D_1\frac{\xi}{|\xi|}|u|_A^2\right) - \mathrm{Re}\left(e^{-i\varphi}D_2\frac{\xi}{|\xi|}|u|_B^2\right). \end{aligned}$$

Accordingly,

$$\begin{aligned} \mathrm{Re}\left(e^{-i\varphi}b(u,u)\right) &= \mathrm{Re}\left(e^{-i\varphi}\left(\xi|u|_A^2 + \zeta|u|_B^2\right)\right) + \mathrm{Re}\left(e^{-i\varphi}g(u)\right) \\ &\geq \mathrm{Re}\left(e^{-i\varphi}\tilde{a}(u)\right) \geq C||u||_H^2\,, \end{aligned}$$

therefore the Lax-Milgram Lemma 1.3.1 is applicable. □

REMARK 1.3.5. It should be mentioned that above conclusion, besides additional summands, can be generalized to sesquilinear forms with a diagonal

$$b(u,u) = f_A(u) + f_B(u) + g(u)$$

where $f_A(u)$ and $f_B(u)$ are generalizations in modulus and phase of the squares of the half-norms $|\cdot|_A$ and $|\cdot|_B$: There should be constants n_A, n_B and N_A, N_B independent of $u \in H$ such that

$$n_A|u|_A^2 \leq |f_A(u)| \leq N_A|u|_A^2\,, \quad n_B|u|_B^2 \leq |f_B(u)| \leq N_B|u|_A^2\,,$$

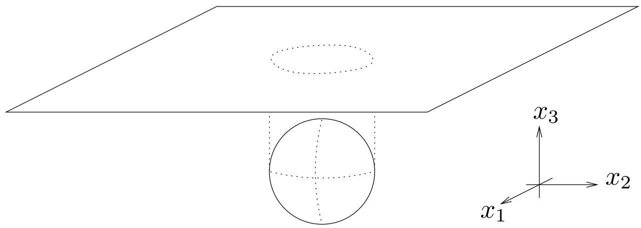
and the support of the arguments of f_A and f_B should be sufficiently small intervals, such that the conditions on the arguments in above conclusion are satisfied for any $\xi \in \overline{\arg(f_A(H))}$ and $\zeta \in \overline{\arg(f_B(H))}$ at the same time. If finally $|g(u)| \leq D_1|u|_A^2 + D_2|u|_B^2$ for some $D_2 > 0$ and $0 \leq D_1 < n_A$, then the weak problem is uniquely solvable as well.

CHAPTER 2

Modeling

2.1. General Geometry

The considered geometry is a two-layered background medium with a flat interface and a perfectly conducting scatterer buried in the lower half-space. The upper half-space is assumed to be homogeneous and lossless, while the lower half is dissipative. This geometry is a model for scattering problems at an air–sand interface.



2.2. Maxwell's Equations

The electromagnetic fields are modeled by time dependent vector functions in space, denoted by $\mathcal{E}$ and $\mathcal{H}$, which represent the electric and magnetic field intensities, and $\mathcal{D}$ and $\mathcal{B}$, the electric displacement and magnetic induction. These fields are put in relation to each other by Maxwell's equations:

$$\begin{aligned} \operatorname{curl}\mathcal{E} &= -\frac{\partial \mathcal{B}}{\partial t}\,, & \operatorname{curl}\mathcal{H} &= \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}\,, \\ \operatorname{div}\mathcal{D} &= \rho\,, & \operatorname{div}\mathcal{B} &= 0\,. \end{aligned}$$

Here, ρ denotes the scalar electric charge density and $\mathcal{J}$ the electric current density in space, which are in relation by $\operatorname{div}\mathcal{J} = -\frac{\partial \rho}{\partial t}$ if charge is conserved.

Following [**26**], in this text the fields $\mathcal{E}, \mathcal{H}, \mathcal{D}, \mathcal{B}, \mathcal{J}$ and ρ are assumed to be time harmonic with frequency ω, such as

$$\begin{aligned} \mathcal{E}(x,t) &= \operatorname{Re}\left(\exp(-i\omega t)\, E(x)\right) \\ \mathcal{H}(x,t) &= \operatorname{Re}\left(\exp(-i\omega t)\, H(x)\right) , \end{aligned}$$

which leads to the time harmonic Maxwell's equations:

$$\begin{aligned} \operatorname{curl} E &= i\omega B\,, & \operatorname{curl} H &= -i\omega D + J \\ \operatorname{div} D &= \varrho\,, & \operatorname{div} B &= 0\,. \end{aligned}$$

The next assumption is that we have linear isotropic materials, such that $D = \varepsilon E$ and $B = \mu H$, where the electric permittivity ε and the magnetic permeability μ are scalar valued functions of space, actually constant in two homogeneous half-spaces with the exception of a perfectly conducting scattering object in the lower layer.

Also, we assume that Ohm's law holds, and that there are no currents applied, as then $J = \sigma E$, where σ denotes the conductivity, which is also assumed to be isotropic and constant in each layer.

Then, the remaining first-order Maxwell system is

$$\operatorname{curl} E = i\omega\mu H\,, \qquad \operatorname{curl} H = -i\omega\varepsilon E + \sigma E\,. \tag{2.1}$$

Note that these two equations also include the divergence equations, since $\omega, \mu, \varepsilon > 0$ and $\operatorname{div} J = i\omega\varrho$ by charge conservation.

Elimination of E or H yields the two alternative second-order differential equations for the problem, which will both be discussed throughout the text, as they result in differing boundary and transmission conditions:

$$\operatorname{curl}\operatorname{curl} H = k^2\, H\,, \quad \text{or} \quad \operatorname{curl}\operatorname{curl} E = k^2\, E\,, \tag{2.2}$$

where $k^2 := \omega^2\varepsilon\mu + i\omega\sigma\mu$ is the so called wave number. Be aware that this is a slightly different definition as compared to [**26**], where the material properties are proper functions in space.

Here, they are constant, in both the upper layer and the lower layer by above assumptions, so we introduce the notation of two wave numbers k_+ and k_- for the wave numbers of the upper and lower layer, just as generally in this text the index + and − correspond to variables and fields of the upper or lower half-space.

Furthermore, we make the following approximations to the material properties of the two layers representing air and soil, where μ_0 and ε_0 represent the permeability and permittivity of free space:

$$\begin{aligned} \text{Air}: &\qquad \varepsilon_+ = \varepsilon_0,\ \sigma_+ = 0,\ \mu_+ = \mu_0 \\ \text{Soil}: &\qquad \varepsilon_- = \varepsilon_r \varepsilon_0,\ \sigma_- = \sigma_s > 0,\ \mu_- = \mu_0 \end{aligned}$$

For illustration, here are some approximate values for the physical constants and properties, mainly taken from [**30**]:

$$\varepsilon_0 \approx 8.854 \cdot 10^{-12} \frac{\mathrm{A\,s}}{\mathrm{V\,m}}, \quad \mu_0 = 4\pi \cdot 10^{-7} \frac{\mathrm{V\,s}}{\mathrm{A\,m}}$$

	ε_r	σ_s in $\frac{\mathrm{A}}{\mathrm{V\,m}}$
air	1	0
wet ground	10	10^{-3}
dry ground	5	10^{-5}
fresh water	81	10^{-3}
copper	1	$6 \cdot 10^{7}$

Therefore, in this model

$$k_+^2 = \omega^2 \varepsilon_0 \mu_0$$

is a positive real number, and

$$k_-^2 = \omega^2 \varepsilon_r \varepsilon_0 \mu_0 + i\omega\sigma_s\mu_0 = \left(\varepsilon_r + i\frac{\sigma_s}{\omega\varepsilon_0}\right) k_+^2 \tag{2.3}$$

is a complex number with an argument of $0 < \arg(k_-^2) < \frac{\pi}{2}$.

The transmission conditions for a surface with normal n are given by

$$[n \times E]_\pm = [n \times H]_\pm = 0\,,$$

which means that the tangential field components are continuous across the boundary. In terms of the E and H fields alone, this results in the transmission conditions

$$[n \times E]_\Gamma = [n \times \operatorname{curl} E]_\Gamma = 0\,,$$

as $H = -\frac{i}{\omega\mu_0}\operatorname{curl} E$ has no jump in the coefficients at the vicinity of the interface Γ, and

$$[n \times H]_\Gamma = [n \times k_\mp^2 \operatorname{curl} H]_\Gamma = 0$$

by $E = \frac{i}{\omega\varepsilon_\pm + i\sigma_\pm}\operatorname{curl} H$ and Equation (2.3). Since there is no electrical field in a perfectly conducting object, this finally leads to the boundary condition at a perfectly conducting scatterer Ω with normal n at its boundary $\partial\Omega$ of

$$n \times E|_{\partial\Omega} = 0$$

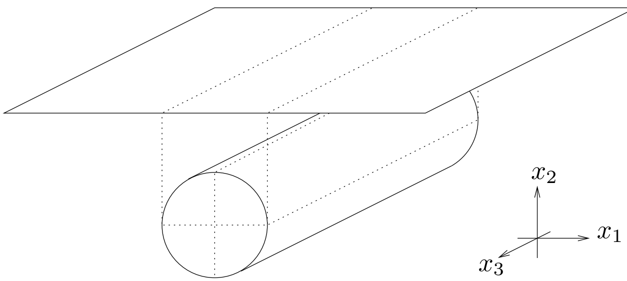
or

$$n \times \operatorname{curl} H|_{\partial\Omega} = 0$$

in terms of H.

2.3. Reduction to Two-Dimensional Models

In order to reduce the problem by one dimension, we assume that the geometry is invariant in one coordinate direction.



Please note the change in the spatial coordinate system in this section. In three dimensions it is most common to have the soil at $x_3 = 0$, in two dimensions it is expected to be $x_2 = 0$. In this, and only this section, the latter is assumed to determine the differential equation and boundary and transmission conditions for the two dimensional setup.

Let the geometry be invariant in the x_3 coordinate axis. We apply method of separation for the magnetic and electric field such as

$$E(x_1, x_2, x_3) = \begin{pmatrix} \tilde{E}_1(x_1, x_2)\, \Psi_1(x_3) \\ \tilde{E}_2(x_1, x_2)\, \Psi_2(x_3) \\ \tilde{E}_3(x_1, x_2)\, \Psi_3(x_3) \end{pmatrix} .$$

The divergence condition $\operatorname{div} E = 0$ leads to

$$\begin{aligned} \left(\frac{\partial \tilde{E}_1}{\partial x_1}(x_1, x_2) \right) \Psi_1(x_3) + \left(\frac{\partial \tilde{E}_2}{\partial x_2}(x_1, x_2) \right) \Psi_2(x_3) & \\ + \tilde{E}_3(x_1, x_2)\, \Psi_3'(x_3) &= 0 , \end{aligned}$$

this determines $\tilde{E}_3$ once $\tilde{E}_1$, $\tilde{E}_2$, and Ψ are known. On the other hand, the second order time harmonic Maxwell's Equation (2.2), using the divergence condition $\operatorname{div} E = 0$ and noting that $\Delta E + k^2 E = 0$, results in:

$$\begin{aligned} \left(\frac{\partial^2 \tilde{E}}{\partial x_1^2}(x_1, x_2) + \frac{\partial^2 \tilde{E}}{\partial x_2^2}(x_1, x_2) \right) \Psi(x_3) + \Psi''(x_3) \tilde{E}(x_1, x_2) & \\ + k^2 \tilde{E}(x_1, x_2) \Psi(x_3) &= 0 \end{aligned}$$

Since $\tilde{E}(x_1, x_2)$ and $\Psi(x_3)$ are functions of independent variables, by separation this results in two coupled differential equations:

$$\begin{cases} \frac{\partial^2 \tilde{E}}{\partial x_1^2}(x_1, x_2) + \frac{\partial^2 \tilde{E}}{\partial x_2^2}(x_1, x_2) + \kappa^2 \tilde{E}(x_1, x_2) = 0 \\ \Psi''(x_3) - k_\Psi^2 \Psi(x_3) = 0 \\ \kappa^2 - k_\Psi^2 = k^2 \end{cases}$$

The solutions to

$$\Psi''(x_3) = k_\Psi^2 \Psi(x_3) \tag{2.4}$$

are linear combinations of $e^{\pm k_\Psi x_3}$ in each component. Although k_Ψ may be any complex number, we will discuss later why only $k_\Psi = 0$ is considered. Until then, the wave number for the two dimensional model will be denoted as $\kappa^2 = k^2 + k_\Psi^2$.

So far, $\tilde{E}$ and Ψ are two vector valued functions, but it turns out that just two scalar functions are sufficient: To see this, we have to introduce an additional step of spatial decomposition. The representation of the electrical and magnetic fields, which we just separated as functions of (x_1, x_2) and x_3, are now represented as the sum of tangential and normal parts with respect to $p = (0, 0, 1)^\top$, the so called propagation direction,

$$E(x) = \underbrace{\begin{pmatrix} E_1(x) \\ E_2(x) \\ 0 \end{pmatrix}}_{:=E_\perp(x)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ E_3(x) \end{pmatrix}}_{:=E_{||}(x)},$$

$$H(x) = \underbrace{\begin{pmatrix} H_1(x) \\ H_2(x) \\ 0 \end{pmatrix}}_{:=H_\perp(x)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ H_3(x) \end{pmatrix}}_{:=H_{||}(x)}.$$

By Maxwell's equations (2.1) we then have

$$\begin{aligned} H(x) &= -\frac{i}{\omega\mu}\left(\operatorname{curl} E_\perp(x) + \operatorname{curl} E_{||}(x)\right) \\ &= -\frac{i}{\omega\mu}\left(\begin{pmatrix} -\frac{\partial E_2}{\partial x_3}(x) \\ \frac{\partial E_1}{\partial x_3}(x) \\ \frac{\partial E_2}{\partial x_1}(x) - \frac{\partial E_1}{\partial x_2}(x) \end{pmatrix} + \begin{pmatrix} \frac{\partial E_3}{\partial x_2}(x) \\ -\frac{\partial E_3}{\partial x_1}(x) \\ 0 \end{pmatrix}\right), \end{aligned}$$

$$\begin{aligned} E(x) &= \frac{1}{\sigma - i\omega\varepsilon}\left(\operatorname{curl} H_\perp(x) + \operatorname{curl} H_{||}(x)\right) \\ &= \frac{1}{\sigma - i\omega\varepsilon}\left(\begin{pmatrix} -\frac{\partial H_2}{\partial x_3}(x) \\ \frac{\partial H_1}{\partial x_3}(x) \\ \frac{\partial H_2}{\partial x_1}(x) - \frac{\partial H_1}{\partial x_2}(x) \end{pmatrix} + \begin{pmatrix} \frac{\partial H_3}{\partial x_2}(x) \\ -\frac{\partial H_3}{\partial x_1}(x) \\ 0 \end{pmatrix}\right), \end{aligned}$$

so the normal parts of the fields satisfy

$$\begin{aligned} H_\perp(x) &= -\frac{i}{\omega\mu}\left(\frac{\partial}{\partial x_3}(p \times E_\perp(x)) + \operatorname{curl} E_{||}(x)\right), \\ E_\perp(x) &= \frac{1}{\sigma - i\omega\varepsilon}\left(\frac{\partial}{\partial x_3}(p \times H_\perp(x)) + \operatorname{curl} H_{||}(x)\right). \end{aligned}$$

Application of both formulae into each other, using (2.4) and

$$p \times (p \times H_\perp(x)) = -H_\perp(x)$$

as well as

$$p \times (p \times E_\perp(x)) = -E_\perp(x)$$

is leading to

$$(1 + \frac{k_\Psi^2}{k^2})H_\perp(x) = -\frac{i}{\omega\mu}\operatorname{curl} E_{||}(x) + \frac{1}{k^2}\frac{\partial}{\partial x_3}(p\times \operatorname{curl} H_{||}(x)) \,, \quad (2.5)$$

$$(1 + \frac{k_\Psi^2}{k^2})E_\perp(x) = \frac{1}{\sigma - i\omega\varepsilon}\operatorname{curl} H_{||}(x) + \frac{1}{k^2}\frac{\partial}{\partial x_3}(p\times \operatorname{curl} E_{||}(x)) \quad (2.6)$$

where we see that under above assumptions the fields are completely determined by the field components $E_{||}$ and $H_{||}$ in the direction of propagation, if $k_\Psi^2 \neq -k^2$. Therefore we may split the total field into two components, the so called transverse electric (TE) mode defined by $H_{||}$ and the transverse magnetic (TM) mode defined by $E_{||}$.

The nomenclature of the modes is based on the observation that for example the electric field induced by $H_{||}$ has only components in the plane orthogonal (transverse) to the direction of propagation, see also [**16**]. The circumstances under which the two modes may be treated separately in this setup are discussed in the following:

2.3.1. Boundary and Transmission Conditions. To analyze the boundary and transmission condition let u represent the transverse magnetic and v the transverse electric potential:

$$E_{||}(x) = \begin{pmatrix} 0 \\ 0 \\ u(x_1,x_2)\psi_u(x_3) \end{pmatrix}, \quad H_{||}(x) = \begin{pmatrix} 0 \\ 0 \\ v(x_1,x_2)\psi_v(x_3) \end{pmatrix}.$$

Using (2.6) the total electric field intensity is then given by

$$E(x) = \begin{pmatrix} \frac{1}{\kappa^2}\left(i\omega\mu\frac{\partial v}{\partial x_2}(x_1,x_2)\psi_v(x_3) + \frac{\partial u}{\partial x_1}(x_1,x_2)\psi_u'(x_3)\right) \\ \frac{1}{\kappa^2}\left(-i\omega\mu\frac{\partial v}{\partial x_1}(x_1,x_2)\psi_v(x_3) + \frac{\partial u}{\partial x_2}(x_1,x_2)\psi_u'(x_3)\right) \\ u(x_1,x_2)\psi_u(x_3) \end{pmatrix}.$$

Consider a perfectly conducting object $\Omega \subset \mathbb{R}^3_-$ with normal vector $n = (n_1, n_2, 0)^\top$ and tangential $\tau = (-\tau_2, \tau_1, 0)^\top$ at its boundary $\partial\Omega$.

To compute the boundary conditions for u and v we regard

$$n \times E(x) = \begin{pmatrix} n_2 u(x_1,x_2)\psi_u(x_3) \\ -n_1 u(x_1,x_2)\psi_u(x_3) \\ \frac{1}{\kappa^2}\left(-\frac{\partial v}{\partial n}(x_1,x_2)\psi_v(x_3) + \frac{\partial u}{\partial \tau}(x_1,x_2)\psi_u'(x_3)\right) \end{pmatrix}$$

to satisfy $n \times E|_{\partial\Omega} = 0$: First of all, this leads to the boundary condition $u|_{\partial\Omega} = 0$, which implies $\frac{\partial u}{\partial \tau}|_{\partial\Omega} = 0$. Therefore, the second condition is $\frac{\partial v}{\partial n}|_{\partial\Omega} = 0$.

By equation (2.5) the total magnetic field intensity is

$$H(x) = \begin{pmatrix} \frac{1}{\kappa^2}\left((\sigma - i\omega\varepsilon)\frac{\partial u}{\partial x_2}(x_1,x_2)\psi_u(x_3) + \frac{\partial v}{\partial x_1}(x_1,x_2)\psi_v'(x_3)\right) \\ \frac{1}{\kappa^2}\left(-(\sigma - i\omega\varepsilon)\frac{\partial u}{\partial x_1}(x_1,x_2)\psi_u(x_3) + \frac{\partial v}{\partial x_2}(x_1,x_2)\psi_v'(x_3)\right) \\ v(x_1,x_2)\psi_v(x_3) \end{pmatrix}.$$

To discuss the transmission conditions for u and v at a surface $\Gamma = \{x \in \mathbb{R}^3 : \ x_2 = 0\}$ with normal vector $(0,1,0)^\top$, mind the change of spatial coordinates in this section, we compute

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times E(x) = \begin{pmatrix} u(x_1,x_2)\psi_u(x_3) \\ 0 \\ -\frac{1}{\kappa^2}\Big(i\omega\mu\frac{\partial v}{\partial x_2}(x_1,x_2)\psi_v(x_3) \\ \qquad + \frac{\partial u}{\partial x_1}(x_1,x_2)\psi_u'(x_3)\Big) \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times H(x) = \begin{pmatrix} v(x_1,x_2)\psi_v(x_3) \\ 0 \\ -\frac{1}{\kappa^2}\Big((\sigma - i\omega\varepsilon)\frac{\partial u}{\partial x_2}(x_1,x_2)\psi_u(x_3) \\ \qquad + \frac{\partial v}{\partial x_1}(x_1,x_2)\psi_v'(x_3)\Big) \end{pmatrix}.$$

It turns out that in contrast to the boundary conditions the transmission conditions for u and v are not separable if ψ_u' or ψ_v' do not vanish.

Thus, only due to the transmission condition in the regarded geometry, no separate discussion of the two modes will be possible if $k_\Psi \neq 0$. Therefore, we set $k_\Psi = 0$ and $\psi_u = \psi_v = 1$ to uncouple the two modes. Nevertheless, the general case of $k^2 \neq \kappa^2$ is included in the three dimensional setup, but of course with a bounded scatterer.

In both modes, the fields are then expressed by scalar potentials $\mathbb{R}^2 \to \mathbb{C}$, and Maxwell's equations reduce to a Helmholtz equation with corresponding transmission and boundary conditions dependent on the mode, which are presented in detail in the following.

2.3.2. The Transverse Magnetic Mode. Let $H_{||}(x) = 0$, $E_{||}(x) = (0, 0, u(x_1, x_2))^\top$ and $k_\Psi = 0$ as discussed, then the electric field given by Equation (2.6) is

$$E(x) = \begin{pmatrix} 0 \\ 0 \\ u(x_1, x_2) \end{pmatrix} .$$

The boundary condition for u at a perfectly conducting object Ω with outward normal $n = (n_1, n_2, 0)^\top$ is $n \times E|_{\partial\Omega} = 0$: As

$$\begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u(x_1, x_2) \end{pmatrix} = \begin{pmatrix} n_2 u(x_1, x_2) \\ -n_1 u(x_1, x_2) \\ 0 \end{pmatrix} ,$$

this results in $u|_{\partial\Omega} = 0$, as seen before. The corresponding H field is given by Equation (2.5) as

$$H(x) = -\frac{i}{\omega\mu} \begin{pmatrix} \frac{\partial u}{\partial x_2}(x_1, x_2) \\ -\frac{\partial u}{\partial x_1}(x_1, x_2) \\ 0 \end{pmatrix} .$$

Therefore, the transmission conditions for u at a surface $\Gamma = \{x \in \mathbb{R}^3 : \ x_2 = 0\}$ with a normal vector $(0, 1, 0)^\top$ are

$$\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times E(x) \right]_\Gamma = \left[\begin{pmatrix} u(x_1, x_2) \\ 0 \\ 0 \end{pmatrix} \right]_\Gamma = 0 ,$$

$$\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times H(x) \right]_\Gamma = \left[\frac{i}{\omega\mu} \begin{pmatrix} 0 \\ 0 \\ \frac{\partial u}{\partial x_2}(x_1, x_2) \end{pmatrix} \right]_\Gamma = 0 .$$

Since $\mu_+ = \mu_- = \mu_0$, they reduce to

$$[u]_\Gamma = \left[\frac{\partial u}{\partial x_2}\right]_\Gamma = 0\,.$$

Due to

$$\operatorname{curl}\operatorname{curl} E(x) = \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial^2 u}{\partial x_1^2}(x_1, x_2) - \frac{\partial^2 u}{\partial x_2^2}(x_1, x_2) \end{pmatrix},$$

Maxwell's equation yields the Helmholtz equation for $u : \mathbb{R}^2 \to \mathbb{C}$ in the TM Mode:

$$\Delta u + k^2 u = 0$$

2.3.3. The Transverse Electric Mode. Assume $E_{||}(x) = 0$, $H_{||}(x) = (0, 0, v(x_1, x_2))^\top$, and $k_\Psi = 0$. Then the magnetic field given by equation (2.5) is

$$H(x) = \begin{pmatrix} 0 \\ 0 \\ v(x_1, x_2) \end{pmatrix},$$

and the corresponding field E is given by equation (2.6) as

$$E(x) = \frac{1}{\sigma - i\omega\varepsilon} \begin{pmatrix} \frac{\partial v}{\partial x_2}(x_1, x_2) \\ -\frac{\partial v}{\partial x_1}(x_1, x_2) \\ 0 \end{pmatrix}.$$

The boundary condition for u at a perfectly conducting object Ω with outward normal $n = (n_1, n_2, 0)^\top$ is $n \times E|_{\partial\Omega} = 0$: As

$$\begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} \frac{\partial v}{\partial x_2}(x_1, x_2) \\ -\frac{\partial v}{\partial x_1}(x_1, x_2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial v}{\partial n}(x_1, x_2) \end{pmatrix}$$

this results in $\frac{\partial v}{\partial n}|_{\partial\Omega} = 0$.

The transmission conditions for u at a surface $\Gamma = \{x \in \mathbb{R}^3 : \; x_2 = 0\}$ with a normal vector $(0,1,0)^\top$ are

$$\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times H(x)\right]_\Gamma = \left[\begin{pmatrix} v(x_1,x_2) \\ 0 \\ 0 \end{pmatrix}\right]_\Gamma = 0\,,$$

$$\left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times E(x)\right]_\Gamma = \left[\frac{1}{\sigma_\pm - i\omega\varepsilon_\pm}\begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x_2}(x_1,x_2) \end{pmatrix}\right]_\Gamma = 0\,.$$

Since $-i\omega\varepsilon_0\frac{k_+^2}{k_-^2} = (\sigma_- - i\varepsilon_-\omega)^{-1}$, by (2.3) they reduce to

$$[v]_\Gamma = \left[k_\mp^2 \frac{\partial v}{\partial x_2}\right]_\Gamma = 0\,.$$

Due to

$$\operatorname{curl}\operatorname{curl} E(x) = \begin{pmatrix} 0 \\ 0 \\ -\frac{\partial^2 v}{\partial x_1^2}(x_1,x_2) - \frac{\partial^2 v}{\partial x_2^2}(x_1,x_2) \end{pmatrix},$$

Maxwell's equation yields the Helmholtz equation for $v : \; \mathbb{R}^2 \to \mathbb{C}$ in the TM Mode:

$$\Delta v + k^2 v = 0$$

2.4. Radiation Conditions

The traditional radiation condition to characterize the physically relevant solutions for Maxwell's equations is the Silver-Müller radiation condition:

DEFINITION 2.4.1. A scattered field (E,H) satisfies the *Silver-Müller radiation condition*, if the limit

$$\lim_{|x|\to\infty} (H \times x - |x|\, E) = 0$$

holds uniformly in all directions $x/|x|$.

This condition is generally used for homogeneous media, but it also applies to lossless two-layered media: P. M. Cutzach and C. Hazard

[**5**] prove uniqueness when the Silver-Müller radiation condition is required in each layer separately. If one layer is dissipative F. Delbary et al. [**11**] propose to demand the radiation condition for the lossless medium and an exponential decay condition in the dissipative medium.

Definition 2.4.2. [**11**] A scattered field (E, H) satisfies the *exponential decay condition* in a dissipative half space with wave number $k \in \mathbb{C}$ if

$$|E(x)| + |H(x)| \leq M \exp\left(-(\operatorname{Im} k)|x|\right)$$

for some constant $M > 0$.

For solutions to Maxwell's equations, the Silver-Müller radiation condition is equivalent to the Sommerfeld radiation condition in the Cartesian components [**6**], a standard radiation condition for Helmholtz problems:

Definition 2.4.3. A scattered field u satisfies the *Sommerfeld radiation condition* in $\mathbb{R}^d$, $d = 2, 3$ with respect to wave number k, if

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u}{\partial r} - iku\right) = 0\,, \;\; r = |x|$$

uniformly in all directions $x/|x|$.

Again, this condition was traditionally used for homogeneous media, but F. M. Odeh [**23**] was able to prove uniqueness of the Helmholtz problem in lossless half-spaces when the Sommerfeld radiation condition is required in each layer separately. In the case of one dissipative layer, it was demonstrated by J. Coyle [**3**] that uniqueness for the two-dimensional Helmholtz equation is established, when the following radiation condition for $\mathbb{R}^2$ is used:

Definition 2.4.4. (cf. [**3**], Theorem 3.3.1) A scattered field u satisfies the *two-layer radiation condition*, if u satisfies the Sommerfeld radiation condition in the lossless layer, and

$$\lim_{r\to\infty} \int\limits_{\Omega_r} \left(\bar{u}\frac{\partial u}{\partial \nu} - u\frac{\partial \bar{u}}{\partial \nu}\right)\, ds = 0$$

in the dissipative layer where Ω_r is the part of the boundary $\partial B_r(0)$ located in this layer.

In the lower half-space, this radiation condition seems considerably than the Silver-Müller radiation condition for the two-layered medium in the three-dimensional case and clearly the radiation condition of exponential decay proposed by F. Delbary et al. [**11**]. Even the condition of integrability, as it is used later, turns out to be stronger. Yet there is an even weaker condition for dissipative media proposed by Chandler-Wilde and Ross in [**9**] that should be sufficient and is well worth noting: It demands that the solution must not *grow stronger than exponentially* as $C\exp(\theta|x|)$, where $\theta < \operatorname{Im} k_-$. This condition is certainly met for bounded solutions.

Another more general radiation condition than the Sommerfeld radiation condition is the upward propagating radiation condition (UPRC) as proposed in [**10**]:

DEFINITION 2.4.5. The function $u:\ \mathbb{R}^2_+ \to \mathbb{C}$ satisfies the upward propagating radiation condition (UPRC) in $\mathbb{R}^2_+$ if, for some $h \geq 0$, $\varphi \in L^\infty(\Gamma_h)$ and $\Gamma_h := \{x \in \mathbb{R}^2 :\ x_2 = h\}$, there holds

$$u(x) = 2\int\limits_{\Gamma_h} \frac{\partial\Phi(x,y)}{\partial y_2}\,\varphi(y)\,ds(y)\,, \quad x \in \mathbb{R}^2\,,\ x_2 > h\,. \tag{2.7}$$

Here, $\Phi(x,y)$ denotes the free-space Green's function for the Helmholtz equation, and in two dimensions it is given by

$$\Phi(x,y) = \frac{i}{4}H_0^{(1)}(k_+|x-y|)\,,\ x,y \in \mathbb{R}^2\,,\ x \neq y\,,$$

or for $x_2 > y_2$ in a plane-wave spectral representation also by

$$\Phi(x,y) = \frac{i}{4\pi}\int\limits_{\mathbb{R}} \frac{\exp\left(i(x-y)\cdot\left(-\tau_1, \sqrt{k_+^2-\tau^2}\right)^\top\right)}{\sqrt{k_+^2-\tau^2}}\,d\tau\,, \tag{2.8}$$

where the imaginary part of the square root is chosen to be non-negative as generally throughout this text. This also ensures integrability for large $|\tau|$, and since the root singularity for τ around $\pm k_+$ is sufficiently weak, this representation is well defined.

In three dimensions the Green's function is

$$\Phi(x,y) = \frac{1}{4\pi}\frac{e^{ik_+|x-y|}}{|x-y|}\,,\ x,y\in\mathbb{R}^3\,,\ x\neq y\,,$$

or, for $x_3 > y_3$, it is also representable by

$$\Phi(x,y) = \frac{i}{8\pi^2}\int\limits_{\mathbb{R}^2} \frac{\exp\left(i(x-y)\cdot\left(-\tau_1,-\tau_2,\sqrt{k_+^2-|\tau|^2}\right)^\top\right)}{\sqrt{k_+^2-|\tau|^2}}\,d\tau\,. \tag{2.9}$$

You can find the details of derivation of above plane-wave spectral representations, or Weyl representations, in [**28**].

The definition of the UPRC for two dimensions is well defined due to the following estimates for the Hankel function. Also, these estimates will be very important for the derivation of the Dirichlet-to-Neumann operator and the Calderon operator.

LEMMA 2.4.6. *For $x \neq y$ the following estimates for the fundamental solution in two dimensions hold for some constants $C_1, C_2 > 0$ only dependent on the wave number k_+:*

$$\begin{aligned}\left|\frac{\partial\Phi(x,y)}{\partial y_2}\right| &\leq C_1|x_2-y_2|\left(|x-y|^{-2}+|x-y|^{-3/2}\right)\\ \left|\frac{\partial^2\Phi(x,y)}{\partial x_2\,\partial y_2}\right| &\leq C_2(|x-y|^{-2}+|x_2-y_2|^2\,|x-y|^{-5/2}\\ &\qquad + |x_1-y_1|^2\,|x-y|^{-7/2})\end{aligned}$$

And, as functions of y_1

$$\left|\frac{\partial\Phi(x,y)}{\partial y_2}\right| \leq C_3\,(1+|y_1|)^{-3/2}\,, \tag{2.10}$$

$$\left|\frac{\partial^2\Phi(x,y)}{\partial x_2\,\partial y_2}\right| \leq C_4\,(1+|y_1|)^{-3/2}\,, \tag{2.11}$$

where the constants are dependent of the remaining variables

$$\begin{aligned}C_3 &= 2^{3/4}\,C_1|x_2-y_2|\left(1+|x_2-y_2|^{-2}\right)\\ &\quad\cdot\left(1-\frac{1}{1+|x_2-y_2|^2}\right)^{-\frac{3}{4}}\left(\frac{1}{2+x_1^2}\right)^{-\frac{3}{4}}\,,\end{aligned}$$

$$\begin{aligned} C_4 &= 2^{3/4}\, C_2 \left(1 + |x_2 - y_2|^{-2} + |x_2 - y_2|^{-1/2}\right) \\ &\quad \cdot \left(1 - \frac{1}{1 + |x_2 - y_2|^2}\right)^{-\frac{7}{4}} \left(\frac{1}{2 + x_1^2}\right)^{-\frac{3}{4}} . \end{aligned}$$

PROOF. These estimates are based on the following asymptotic behavior of the Hankel functions for small and large positive arguments: For small $z > 0$

$$\begin{aligned} \frac{d}{dz}\, H_0^{(1)}(z) &= -H_1^{(1)}(z) = O\left(z^{-1}\right) , \\ \frac{d^2}{dz^2}\, H_0^{(1)}(z) &= H_0^{(1)}(z) - \frac{1}{z} H_1^{(1)}(z) = O\left(z^{-2}\right) \end{aligned}$$

and for large $z \to \infty$ we have

$$\left|\frac{d}{dz}\, H_0^{(1)}(z)\right| = O\left(z^{-1/2}\right) , \quad \left|\frac{d^2}{dz^2}\, H_0^{(1)}(z)\right| = O\left(z^{-1/2}\right) .$$

Therefore,

$$\begin{aligned} \frac{\partial \Phi(x,y)}{\partial y_2} &= \frac{ik_+}{4}\, \frac{d}{dz}\, H_0^{(1)}(k_+|x-y|)\, \frac{y_2 - x_2}{|x-y|} \\ &= \begin{cases} O\left(\frac{|x_2-y_2|}{|x-y|^2}\right) & \text{if } |x-y| \to 0\,, \\ O\left(\frac{|x_2-y_2|}{|x-y|^{3/2}}\right) & \text{if } |x-y| \to \infty \end{cases} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \Phi(x,y)}{\partial x_2\, \partial y_2} &= -\frac{ik_+^2}{4}\, \frac{d^2}{dz^2}\, H_0^{(1)}(k_+|x-y|)\, \frac{(x_2-y_2)^2}{|x-y|^2} \\ &\quad -\frac{ik_+}{4}\, \frac{d}{dz}\, H_0^{(1)}(k_+|x-y|)\, \frac{(x_1-y_1)^2}{|x-y|^3} \\ &= \begin{cases} O\left(\frac{1}{|x-y|^2}\right) & \text{if } |x-y| \to 0\,, \\ O\left(\frac{|x_2-y_2|^2}{|x-y|^{5/2}} + \frac{|x_1-y_1|^2}{|x-y|^{7/2}}\right) & \text{if } |x-y| \to \infty\,. \end{cases} \end{aligned}$$

For the reduction to a function in y_1 the following estimates were used: First, $|1 + y_1^2|^{-3/4} \le 2^{3/4}(1 + |y_1|)^{-3/2}$ holds since

$$\left(\frac{1 + y_1^2}{1 + 2|y_1| + y_1^2}\right)^{-3/4} = \left(1 + \frac{2|y_1|}{y_1^2 + 1}\right)^{3/4} \le 2^{3/4} .$$

Then note, that the inequality $|x_1 - y_1|^2 \geq \frac{1}{a}(1 + y_1^2)$ is fulfilled for $a > 0$, if $a(x_1-y_1)^2 - 1 - y_1^2 = (\sqrt{a-1}y_1 - \frac{a}{\sqrt{a-1}}x_1)^2 - \frac{a}{a-1}x_1^2 + a - 1 \geq 0$. This leads to the condition $a - 2 + \frac{1}{a} \geq x_1^2$, which is satisfied by $a = 2 + x_1^2$. Next,

$$\begin{aligned} |x_1 - y_1|^2 + |x_2 - y_2|^2 &\geq \min\{1, |x_2 - y_2|^2\}(1 + |x_1 - y_1|^2) \\ &\geq \frac{|x_2 - y_2|^2}{1 + |x_2 - y_2|^2}(1 + |x_1 - y_1|^2) \\ &= \left(1 - \frac{1}{1 + |x_2 - y_2|^2}\right)(1 + |x_1 - y_1|^2) \end{aligned}$$

and as $|x - y|^{-2} \leq |x_2 - y_2|^{-2}$ and $(a + b) \leq \max\{1, |a|\}(1 + |b|) \leq (1 + |a|)(1 + |b|)$ we finally gain

$$\begin{aligned} |x - y|^{-2} + |x - y|^{-\frac{3}{2}} &\leq (1 + |x_2 - y_2|^{-2})\left(1 - \frac{1}{1 + |x_2 - y_2|^2}\right)^{-\frac{3}{4}} \\ &\quad \cdot \left(\frac{1}{2 + x_1^2}\right)^{-\frac{3}{4}} 2^{3/4}(1 + |y_1|)^{-\frac{3}{2}} . \end{aligned}$$

The second estimate follows accordingly, additionally using

$$|x_2 - y_2|^2 |x - y|^{-5/2} \leq |x_2 - y_2|^2 \, |x_2 - y_2|^{-5/2} = |x_2 - y_2|^{-1/2} .$$

□

Straightforward computations yield the corresponding estimates for three dimensions:

LEMMA 2.4.7. *For $x \neq y$ the following estimates for the fundamental solution in three dimensions hold for some constants $C_1, C_2 > 0$ only dependent on the wave number k_+:*

$$\begin{aligned} \left|\frac{\partial \Phi(x,y)}{\partial y_3}\right| &\leq C_1 |x_3 - y_3| \, |x - y|^{-3} \\ \left|\frac{\partial^2 \Phi(x,y)}{\partial x_3 \, \partial y_3}\right| &\leq C_2(|x - y|^{-3} + |x_3 - y_3|^2 \, |x - y|^{-5}) \end{aligned}$$

Or, as functions of $\tilde{y} = (y_1, y_2)^\top$ *and constants* C_3, C_4 *depending on* $x \in \mathbb{R}^3$, $y_3 \in \mathbb{R}$, $x_3 \neq y_3$, *and wave number* k_+*:*

$$\begin{aligned} \left| \frac{\partial \Phi(x,y)}{\partial y_3} \right| &\leq C_3 \, (1 + |\tilde{y}|)^{-3} \\ \left| \frac{\partial^2 \Phi(x,y)}{\partial x_3 \, \partial y_3} \right| &\leq C_4 \, (1 + |\tilde{y}|)^{-3} \end{aligned}$$

The choice of h in (2.7) is arbitrary and we may very well choose $h = 0$, as the following Theorem will show:

THEOREM 2.4.8. *(cf.* [**10**], *Theorem 2.9) Given* $u : \mathbb{R}^2_+ \to \mathbb{C}$, *the following statements are equivalent:*

(1) $u \in C^2(\mathbb{R}^2_+)$, $u \in L^\infty(E_a)$ *for all* $E_a = \{x \in \mathbb{R}^2_+ : 0 < x_2 < a\}$, $a > 0$, $\Delta u + k_+^2 u = 0$ *in* $\mathbb{R}^2_+$ *and* u *satisfies the UPRC, Definition 2.4.5.*
(2) u *satisfies Equation (2.7) for* $h = 0$ *and some* $\varphi \in L^\infty(\Gamma)$*.*
(3) $u \in L^\infty(E_a)$ *for some* $a > 0$ *and* u *satisfies Equation (2.7) for each* $h > 0$ *with* $\varphi = u|_{\Gamma_h}$.

Please note that even if $h = 0$, Equation (2.7) is still only valid for $x_2 > h$. But since it is given as a double layer potential, it may be extended continuously from above to $x_2 = 0$ if φ is continuous.

2.5. Valid Incident Fields

For illustration, we present two examples of valid incident fields for the TE mode: The Green's function represents a point source field, then the field of a plane wave in the two-layered medium is computed.

2.5.1. The Green's Function for a Layered Medium. We have seen that the fundamental solution of the Helmholtz equation in free space can be expressed using its Fourier transform:

$$\begin{aligned} u(x) = \frac{i}{4} H_0^{(1)}(k|x|) &= \frac{i}{4\pi} \int_{-\infty}^{\infty} \frac{1}{\sqrt{k^2 - t^2}} e^{i|x_2|\sqrt{k^2 - t^2} + i x_1 t} \, dt \\ &= \frac{i}{2\pi} \int_0^{\infty} \frac{1}{\sqrt{k^2 - t^2}} e^{i|x_2|\sqrt{k^2 - t^2}} \cos(t|x_1|) \, dt \, . \end{aligned}$$

This motivates an Ansatz for the Green's function in the layered medium for $y_2 < 0$, as proposed by [**25**]:

$$\begin{aligned} G(x,y) &= \frac{i}{2\pi}\int_0^\infty \cos(t|x_1-y_1|) \\ &\quad \cdot\left[\frac{1}{\sqrt{k_+^2-t^2}}e^{i|x_2-y_2|\sqrt{k_-^2-t^2}}\right. \\ &\quad \left. + \alpha(t)e^{-i(x_2+y_2)\sqrt{k_-^2-t^2}}\right] dt \quad \text{for} \quad x_2 < 0 \end{aligned}$$

and

$$\begin{aligned} G(x,y) &= \frac{i}{2\pi}\int_0^\infty \cos(t|x_1-y_1|)\left[\frac{1}{\sqrt{k_+^2-t^2}}+\alpha(t)\right] \\ &\quad \cdot e^{ix_2\sqrt{k_+^2-t^2}-iy_2\sqrt{k_-^2-t^2}}\, dt \quad \text{for} \quad x_2 > 0 \end{aligned}$$

This approach is already continuous at $x_2 = 0$, continuity of the normal derivative is achieved for

$$\begin{aligned} \alpha(t) &= \frac{1}{\sqrt{k_-^2-t^2}}\frac{\sqrt{k_-^2-t^2}-\sqrt{k_+^2-t^2}}{\sqrt{k_-^2-t^2}+\sqrt{k_+^2-t^2}} \\ &= \frac{1}{\sqrt{k_-^2-t^2}}\frac{k_-^2-k_+^2}{\left(\sqrt{k_-^2-t^2}+\sqrt{k_+^2-t^2}\right)^2}. \end{aligned}$$

An important property of this Green's function in the lower layer was proven in [**3**]:

LEMMA 2.5.1. *(cf. [**3**], Lemma 3.4.10)*

Let both the source point $x = (x_1, x_2)^\top$ and the observation point $y = (y_1, y_2)^\top$ be located in the dissipative layer $\mathbb{R}^2_-$, such that $|y_2| = r\sin(\theta)$ and $y_1 = r\cos(\theta)$ for $\theta \in [0, \pi]$. Then,

$$G(x,y) = o\left(e^{-\mathrm{Im}(k_-)r}\right)$$

as $r \to \infty$ uniformly in θ.

Furthermore it was shown that classical scattering solutions to the two-layered medium problem satisfying the *two-layer radiation condition* with an analytic incoming field, as it will be considered here, can be bound by the Green's function and will therefore be integrable in the lower half-space.

For the derivation of the Green tensor for Maxwell's equations, please see [**11, 5, 26, 27**].

2.5.2. The Plane Wave in the Layered Medium. An incoming plane wave $v^i(x) = e^{ik_+(d\cdot x)}$ with direction $d = (d_1, d_2)$, $d_2 < 0$, $|d| = 1$ in $\mathbb{R}^2_+$ will yield a plane wave reflection $v^r(x) = re^{ik_+(d_1x_1 - d_2x_2)}$ in $\mathbb{R}^2_+$ and in $\mathbb{R}^2_-$ a transmitted plane wave $v^i(x) = te^{ik_-(e\cdot x)}$, $|e| = 1$, $e_2 < 0$ with a reflection factor $r \in \mathbb{C}$ and a transmission factor $t \in \mathbb{C}$. Demanding continuity for

$$u^i(x) = \begin{cases} v^i(x) + v^r(x)\,, & x \in \mathbb{R}^2_+ \\ v^t(x)\,, & x \in \mathbb{R}^2_- \end{cases}$$

at Γ illustrates the representation of the reflection and transmission, as well as the equations $1 + r = t$ and Snell's law of refraction $k_-e_1 = k_+d_1$. Therefore the direction of the transmitted plane wave is

$$e = \left(\frac{k_+}{k_-} d_1, -\sqrt{1 - \left(\frac{k_+}{k_-}\right)^2 d_1^2} \right)^\top .$$

Continuity of the normal derivative at Γ additionally yields $k_+d_2 - k_+d_2r = k_-e_2t$. Using the conditions on the directions this results in the linear system

$$\begin{array}{rcrcr} t & - & r & = & 1 \\ \sqrt{\left(\frac{k_-}{k_+}\right) - d_1^2}\, t & - & d_2\, r & = & -d_2 \end{array}$$

with the solutions for the transmission and reflection factors

$$t = \frac{2}{1 - \frac{1}{d_2}\sqrt{\left(\frac{k_-}{k_+}\right)^2 - d_1^2}} \text{ and } r = \frac{d_2 + \sqrt{\left(\frac{k_-}{k_+}\right)^2 - d_1^2}}{d_2 - \sqrt{\left(\frac{k_-}{k_+}\right)^2 - d_1^2}} .$$

CHAPTER 3

Two-Dimensional Problems

In this chapter we will discuss the two-dimensional reductions of the time-harmonic Maxwell's equations. The two-layered space is denoted as $\mathbb{R}^2 = \mathbb{R}^2_+ \cup \Gamma \cup \mathbb{R}^2_-$ with the interface $\Gamma = \mathbb{R} \times \{0\}$ and half-spaces $\mathbb{R}^2_\pm := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \gtrless 0\}$. In this setup we seek the solution to the Helmholtz equation $\Delta u + k^2_\pm u = 0$ with wave numbers $k^2 = k^2_+ > 0$ for $x_2 > 0$ and $k^2_- \in \mathbb{C}$, $\arg k^2_- \in (0, \frac{\pi}{2})$ for $x_2 < 0$. A bounded perfect electrically conducting obstacle $\Omega \subset \mathbb{R}^2_-$ is located in the lower half-space, and dependent on the mode, the transmission condition at the interface, and boundary condition on the obstacle are given.

We will split the total field u^t into the incoming field u^i, which itself is a solution to the Helmholtz equations in the two-layered space without the scatterer Ω, and the scattering field u. As the problem will be modeled by weak formulation, it is natural to discuss the fields in Sobolev spaces. Finally, the radiation condition on u will ensure physically relevant solutions to the problem; we will use the *two-layer radiation condition*, Definition 2.4.4, discussed in [**3**], and the *Upward Propagating Radiation Condition* (UPRC), Definition 2.4.5, which is discussed in [**10**].

First the classical problems in the transverse electric (TE) and transverse magnetic (TM) mode are defined on the whole plane, then corresponding weak formulations are presented on the reduced scenario of a half plane with the help of a non-local Dirichlet-to-Neumann boundary condition. By application of the Lax Milgram Lemma, and the extensions presented before, we will then be able to prove existence and uniqueness of the weak problems. Next, regularity of the weak solutions is shown and, finally, equivalence of the weak and classical formulations for both modes with the additional condition of integrability. Thanks to the results in [**3**] this additional condition is negligible, as already the classical problem was uniquely solvable.

3.1. Classical Problem Definitions

In all following problem definitions the total field u^t is the sum of the assumed to be known incoming field u^i and unknown scattering field u. The incoming field is assumed to be a solution the Helmholtz equation with transmission conditions, but does not necessarily need to fulfill a radiation condition. The discussion of the physical model in the previous chapter resulted in the boundary and transmission conditions summarized below:

	TM mode	TE mode
Transmission	$[u^t]_\Gamma = \left[\frac{\partial u^t}{\partial x_2}\right]_\Gamma = 0$	$[u^t]_\Gamma = \left[k_\mp^2 \frac{\partial u^t}{\partial x_2}\right]_\Gamma = 0$
Boundary	$u^t\|_{\partial\Omega} = 0$	$\left.\frac{\partial u^t}{\partial n}\right\|_{\partial\Omega} = 0$

Since u^i satisfies the transmission condition and $u^t = u^i + u$ the transmission and boundary conditions, we may now state the complete problem definition of the first Dirichlet boundary value problem (DBVP1) for the two dimensional TM mode setting:

PROBLEM 3.1.1. (DBVP1) Given a bounded scatterer $\Omega \subset \mathbb{R}^2_-$ with boundary $\partial\Omega \in C^2$ and an analytic incoming field u^i defined on some region G in the lower half plane around the scatterer, i.e. $\bar{\Omega} \subset G \subset \mathbb{R}^2_-$, the problem is to find the scattering field

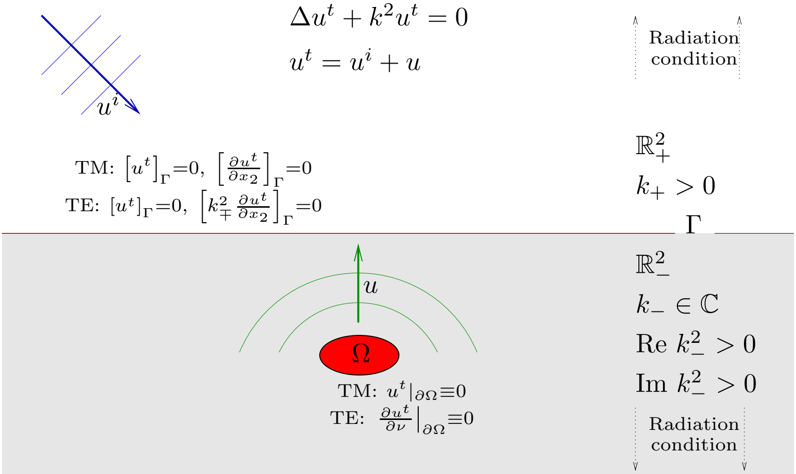
(i) $u \in C^2(\mathbb{R}^2 \backslash (\Gamma_0 \cup \bar{\Omega})) \cap BC(\mathbb{R}^2 \backslash \Omega)$ such that
(ii) $\Delta u + k_\pm^2 u = 0$ in $\mathbb{R}^2_\pm$,
(iii) $u|_{\partial\Omega} = -u^i|_{\partial\Omega}$,
(iv) $[u]_\Gamma = [\frac{\partial u}{\partial x_2}]_\Gamma = 0$ and
(v) u satisfies the *two-layer radiation condition* (Definition 2.4.4).

The corresponding definition for the TE mode is the first Neumann boundary value problem:

PROBLEM 3.1.2. (NBVP1) Given a bounded scatterer $\Omega \subset \mathbb{R}^2_-$ with boundary $\partial\Omega \in C^2$ and an analytic incoming field u^i defined on some region G in the lower half plane around the scatterer, i.e. $\bar{\Omega} \subset G \subset \mathbb{R}^2_-$, the problem is to find the scattering field

(i) $u \in C^2(\mathbb{R}^2 \backslash (\Gamma_0 \cup \bar{\Omega})) \cap BC(\mathbb{R}^2 \backslash \Omega)$ such that
(ii) $\Delta u + k^2_{\pm} u = 0$ in $\mathbb{R}^2_{\pm}$,
(iii) $\frac{\partial u}{\partial n}|_{\partial\Omega} = -\frac{\partial u^i}{\partial n}|_{\partial\Omega}$,
(iv) $[u]_\Gamma = [k^2_{\mp} \frac{\partial u}{\partial x_2}]_\Gamma = 0$ and
(v) u satisfies the *two-layer radiation condition* (Definition 2.4.4).

See the following diagram for illustration of Problems 3.1.1 (DBVP1) and 3.1.2 (NBVP1):



3.2. The Dirichlet-to-Neumann Map

Both full space problems will now be transformed to corresponding weak half-space problems using a Dirichlet-to-Neumann operator on the interface. One can derive the map in the form of a Fourier multiplier:

LEMMA 3.2.1. *The Dirichlet-to-Neumann map*

$$\begin{aligned} \Lambda : \; H^{1/2}(\mathbb{R}) &\rightarrow H^{-1/2}(\mathbb{R}) \\ \widehat{\Lambda\varphi}\,(\tau) &= i\sqrt{k^2 - \tau^2}\,\hat{\varphi}(\tau) \end{aligned}$$

is well defined and has an operator norm of $||\Lambda|| = \max\{1, |k|\}$. *Note that* $\sqrt{k^2 - \tau^2} := i\sqrt{\tau^2 - k^2}$ *for* $k < |\tau|$ *and* $k > 0$.

REMARK 3.2.2. Note, that the Fourier transform in above Lemma was defined for functions in Sobolev spaces in a distributional sense, and that this map is not only linear, but also translation invariant because it is in the form of a Fourier multiplier. This means that it will commute with a difference quotient, as needed later. Also note that if $\varphi \in \mathcal{S}(\mathbb{R})$, above definition corresponds to

$$\Lambda\varphi\,(t) = \frac{i}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \sqrt{k^2 - \tau^2}\,\hat{\varphi}(\tau)\,e^{i\tau t}\,d\tau\,.$$

PROOF. By the definition of fractional order Sobolev spaces using Bessel potentials $\psi^s(\tau) = \left(1 + |\tau|^2\right)^{s/2}$ and Parseval's identity,

$$\begin{aligned}
||\Lambda\varphi||^2_{H^{-1/2}(\mathbb{R})} &= \int\limits_{\mathbb{R}} \left|\psi^{-1/2}(\tau)\,\widehat{\Lambda\varphi}(\tau)\right|^2 d\tau \\
&= \int\limits_{\mathbb{R}} \frac{|k^2 - \tau^2|}{(1+\tau^2)^{1/2}} |\hat{\varphi}(\tau)|^2\,d\tau \\
&\leq \max\{1, |k|^2\} \int\limits_{\mathbb{R}} \frac{1+\tau^2}{(1+\tau^2)^{1/2}} |\hat{\varphi}(\tau)|^2\,d\tau \\
&= \max\{1, |k|^2\}\,||\varphi||^2_{H^{1/2}(\mathbb{R})}\,.
\end{aligned}$$

□

REMARK 3.2.3. In the case of $|k| > 1$, the operator norm $\max\{1, |k|\}$ is obtained for sequences of functions whose the support of the corresponding Fourier transforms collapse to $\pm k$, or, in the case of $|k| \leq 1$, the operator norm is obtained for sequences of functions whose support of Fourier transforms have lower bounds growing to infinity— in both cases the sequences are vanishing if they are convergent.

The following Lemma establishes its fundamental property of mapping Dirichlet data onto Neumann data on Γ.

LEMMA 3.2.4. *Let u be a solution to Problem 3.1.1 (DBVP1) with boundary data $u|_\Gamma \in H^{1/2}(\mathbb{R})$. Then the Dirichlet-to-Neumann operator Λ satisfies $\Lambda u|_\Gamma = \frac{\partial u}{\partial x_2}|_\Gamma$.*

PROOF. Since u is bounded and radiating, it will also satisfy the (UPRC), and the solution u will take the representation

$$u(x) = 2 \int_{\mathbb{R}} \frac{\partial \Phi(x_1, x_2, y_1, h)}{\partial y_2} \, u(y_1, h) \, dy_1$$

for all $x_2 > h > 0$ by Theorem 2.4.8 (3), which in this case extends to $h = 0$ by dominated convergence as u is bounded and the constant C_3 in (2.10) of Lemma 2.4.6 remains bounded for $h \to 0$. By exchanging differentiation and integration, which is again justified by dominated convergence since u is bounded and by the estimate (2.11) for Φ in Lemma 2.4.6, the partial derivative of u with respect to x_2 is given by

$$\frac{\partial u(x)}{\partial x_2} = 2 \int_{\mathbb{R}} \frac{\partial^2 \Phi(x_1, x_2, y_1, 0)}{\partial x_2 \, \partial y_2} \, u_0(y_1) \, dy_1 \, ,$$

where $u_0(y_1) := u(y_1, 0)$. Using (2.8) we can compute the partial derivatives for $x_2 > y_2$

$$\frac{\partial \Phi(x, y)}{\partial x_2} = - \frac{1}{4\pi} \int_{\mathbb{R}} e^{i(x_2 - y_2)\sqrt{k_+^2 - \tau^2} + i(x_1 - y_1)\tau} \, d\tau \, ,$$

$$\frac{\partial^2 \Phi(x, y)}{\partial x_2 \, \partial y_2} = \frac{i}{4\pi} \int_{\mathbb{R}} \sqrt{k_+^2 - \tau^2} \, e^{i(x_2 - y_2)\sqrt{k_+^2 - \tau^2} + i(x_1 - y_1)\tau} \, d\tau \, ,$$

both justified by dominated convergence, as the integrand is exponentially decreasing for large $|\tau|$ since $\sqrt{k_+^2 - \tau^2} := i\sqrt{\tau^2 - k_+^2}$ for $|\tau| > |k_+|$. Therefore, we gain the following representation valid for $x_2 > 0$:

$$\frac{\partial u(x)}{\partial x_2} = \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} u_0(y_1) \, \sqrt{k_+^2 - \tau^2} \, \cdot$$

$$e^{i x_2 \sqrt{k_+^2 - \tau^2} + i(x_1 - y_1)\tau} \, d\tau \, dy_1 \qquad (3.1)$$

To justify the change of the order of integration regard some positive increasing sequence (c_n), i.e., $c_n \to \infty$ and $c_{n+1} > c_n > 0$ for all $n \in \mathbb{N}$, and apply Fubini's Theorem for fixed n as the absolute integral over u is bound by $\int_{-c_n}^{c_n} |u_0(y_1)| dy_1 \le 2c_n ||u||_\infty$

$$\begin{aligned} &\lim_{n\to\infty} \int_{-c_n}^{c_n} \int_{\mathbb{R}} u_0(y_1) \sqrt{k_+^2 - \tau^2}\, e^{ix_2\sqrt{k_+^2-\tau^2}+i(x_1-y_1)\tau}\, d\tau\, dy_1 \\ = \quad &\lim_{n\to\infty} \int_{\mathbb{R}} \int_{-c_n}^{c_n} u_0(y_1) e^{-iy_1\tau}\, dy_1 \sqrt{k_+^2 - \tau^2}\, e^{ix_2\sqrt{k_+^2-\tau^2}+ix_1\tau}\, d\tau\,. \end{aligned}$$

The inner integral is a truncated Fourier transform and results in a sequence of L^2 functions converging to the Fourier transform in L^2 sense, so that the sequence (f_n)

$$f_n(\tau) := \int_{-c_n}^{c_n} u_0(y_1) e^{-iy_1\tau}\, dy_1 \sqrt{k_+^2 - \tau^2} e^{ix_2\sqrt{k_+^2-\tau^2}+ix_1\tau} \in L^1(\mathbb{R})$$

converges to $f(\tau) = \sqrt{2\pi}\,\hat{u}_0(\tau) \sqrt{k_+^2 - \tau^2} e^{ix_2\sqrt{k_+^2-\tau^2}+ix_1\tau} \in L^1(\mathbb{R})$. We may now choose (c_n) (or a sub-sequence thereof) such that

$$||f_n - f||_{L^1(\mathbb{R})} < 2^{-n}\,,$$

and define

$$g = |f| + \sum_{n=1}^{\infty} |f_n - f|$$

which converges to an $L^1(\mathbb{R})$ function as $||g||_{L^1(\mathbb{R})} \le ||f||_{L^1(\mathbb{R})} + 1$ by triangle inequality, which dominates the function sequence for all n. Therefore the theorem of dominated convergence applies and

$$\frac{\partial u(x)}{\partial x_2} = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}_0(\tau) \sqrt{k_+^2 - \tau^2} e^{ix_2\sqrt{k_+^2-\tau^2}+ix_1\tau}\, d\tau\,.$$

Since u_0 is in $H^{1/2}(\mathbb{R})$ and therefore $\hat{u}_0(\tau) \sqrt{k_+^2 - \tau^2}$ in $L^2(\mathbb{R})$. Finally, we may now let $x_2 \to 0$ for the desired result

$$\frac{\partial u(x_1, 0)}{\partial x_2} = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{u}_0(\tau) \sqrt{k_+^2 - \tau^2} e^{ix_1\tau}\, d\tau\,.$$

Note that this proof is closely related to the proof of Theorem 3.2 in [**7**] which applies convergence results on Fourier transforms of convolutions. □

So far we established the mapping properties of the Dirichlet-to-Neumann map, and its relationship to classical solutions: As expected, it maps Dirichlet trace data on Γ of a solution onto its Neumann trace. Since this map will be pivotal for reduction to the half-space, the next step is to derive properties of the mapping which will play an important role to prove solvability and uniqueness of the solutions to the weak formulation. The following property will be a key feature in discussion of the sesquilinear forms throughout this text, and will be illustrated below.

LEMMA 3.2.5. *For $u \in H^1(\mathbb{R}^2)$ the dual pairing*

$$I = \int_{\mathbb{R}} \bar{u}|_\Gamma \, \Lambda u|_\Gamma \, dx_1 \quad \textit{satisfies} \quad Re\, I \leq 0 \quad \textit{and} \quad Im\, I \geq 0\,.$$

PROOF.

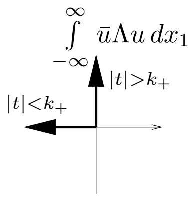
$$\begin{aligned} \int_{\mathbb{R}} \bar{u}|_\Gamma \, \Lambda u|_\Gamma \, dx_1 &= \int_{\mathbb{R}} \widehat{\bar{u}|_\Gamma} \, \widehat{\Lambda u|_\Gamma} \, dt = i \int_{\mathbb{R}} \sqrt{k_+^2 - t^2} |\widehat{u|_\Gamma}|^2 \, dt \\ &= i \int_{|t|<k_+} \sqrt{k_+^2 - t^2} |\widehat{u|_\Gamma}|^2 \, dt \\ &\quad - \int_{|t|>k_+} \sqrt{t^2 - k_+^2} |\widehat{u|_\Gamma}|^2 \, dt \end{aligned}$$

□

REMARK 3.2.6. This result can be seen as a phase property in the frequency domain: In the right hand side of the equation in the proof, we see that I can be expressed as the sum of two half norms of u with different complex factors:

$$\begin{aligned} I &= i\,||u||_L^2 + (-1)\,||u||_H^2\,, \text{ where} \\ ||u||_L^2 &= \int\limits_{|t|<k_+} \sqrt{k_+^2 - t^2}|\hat{u}|^2\,dt\,, \text{ and} \\ ||u||_H^2 &= \int\limits_{|t|>k_+} \sqrt{t^2 - k_+^2}|\hat{u}|^2\,dt\,. \end{aligned}$$

This fits exactly into the setup of Lemma 1.3.3, for which we will need to analyze the arguments or phases of the factors. To visualize this analysis, the arguments are illustrated as arrows in the complex plane, as you can see below:



The visualization of the two discrete phases of the factors -1 and i in the expression for the term depending on the frequency domain instead of the intermediate phase of the sum will be most useful for the treatment of the Calderon operator. Please note that the illustrations will show $\arg(k_-^2) \approx \frac{\pi}{4}$ for clarity and simplicity, but of course any permissible value for $\arg(k_-^2) \in (0, \frac{\pi}{2})$ will yield the same results.

3.3. Variational Approach

Since the lower half-space is dissipative, we may seek the solutions in the Sobolev space H^1 and formulate the weak problem definition:

PROBLEM 3.3.1. (NBVP2) For $D = \mathbb{R}_-^2 \backslash \bar{\Omega}$ find $u \in H^1(D)$ such that for all test functions $\varphi \in H^1(D)$ the equation $b_{\mathrm{TE}}(u, \varphi) = a_{TE}(\varphi)$

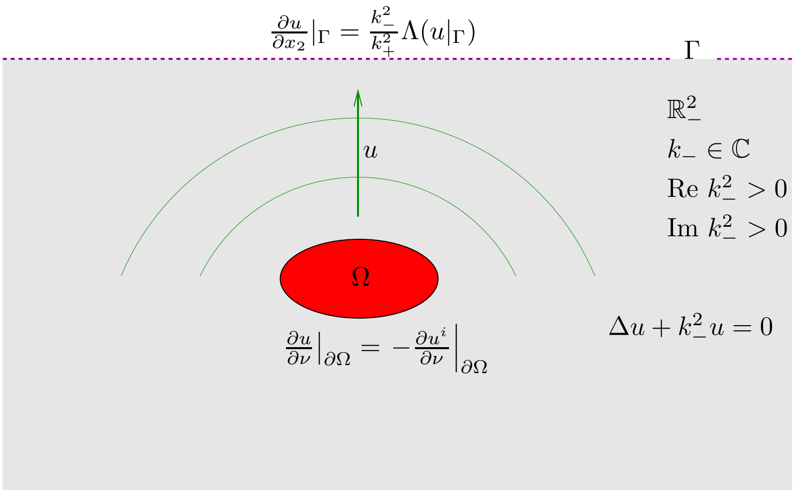
holds, where

$$b_{\mathrm{TE}}(u,\varphi) \;:=\; \int\limits_D \left(\nabla\bar{\varphi}\nabla u - k_-^2\bar{\varphi}u\right)\,dx - \frac{k_-^2}{k_+^2}\int\limits_\Gamma \bar{\varphi}\Lambda u\,ds\,,$$

$$a_{\mathrm{TE}}(\varphi) \;:=\; \int\limits_{\partial\Omega} \bar{\varphi}\frac{\partial u^i}{\partial\nu}\,ds\,,$$

and u^i represents an analytic incoming field at least defined around and on $\partial\Omega$. Note, that the second integral of b_{TE} exists in the sense of dual pairing.

See this diagram for an illustration of Problem 3.3.1 (NBVP2):



Since the variational formulation cannot yield Dirichlet boundary conditions natively, the idea is to express the solution as a sum $u^s = u+g$ using a function g which satisfies the boundary condition and to seek

u with a homogeneous boundary condition on $\partial\Omega$:

$$\begin{aligned} \Delta u + k_-^2 u &= -\Delta g - k_-^2 g \quad \text{in } D \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

PROBLEM 3.3.2. (DBVP2) For $D = \mathbb{R}^2_- \backslash \bar{\Omega}$ find $u \in H^1_0(D)$, where

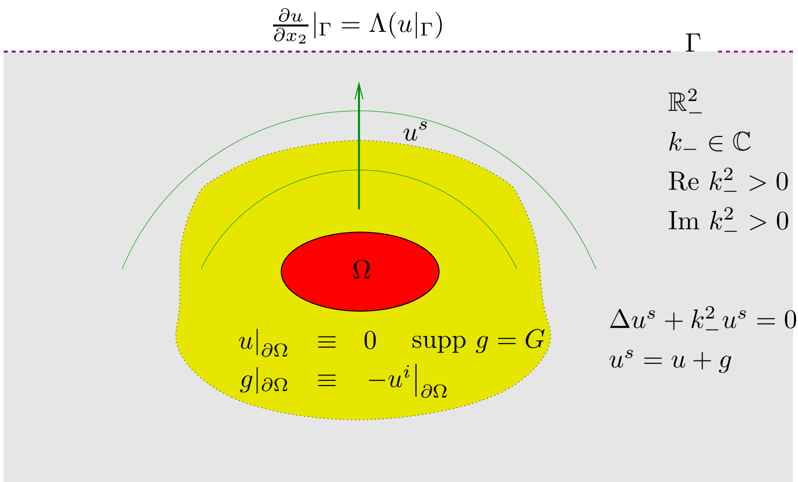
$$H^1_0(D) := \{u \in H^1(D) : u|_{\partial\Omega} = 0\}$$

such that for all test functions $\varphi \in H^1_0(D)$ the equation $b_{TM}(u, \varphi) = a_{TM}(\varphi)$ holds, where

$$\begin{aligned} b_{\mathrm{TM}}(u, \varphi) &:= \int_D \left(\nabla\bar{\varphi}\nabla u - k_-^2 \bar{\varphi} u\right) \, dx - \int_\Gamma \bar{\varphi}\Lambda u \, ds \\ a_{\mathrm{TM}}(\varphi) &:= -\int_G \left(\nabla\bar{\varphi}\nabla g - k_-^2 \bar{\varphi} g\right) \, ds \,, \end{aligned}$$

and $G \subset \mathbb{R}^2_-$ is the support around $\partial\Omega$ of a smooth rapidly decaying function g, for which $g|_{\partial\Omega} = -u^i|_{\partial\Omega}$, and u^i represents an analytic incoming field at least defined on $\bar{G}$. Note, that the second integral of b_{TM} exists in the sense of dual pairing.

This figure illustrates Problem 3.3.2 (DBVP2):



3.4. Existence, Uniqueness, and Equivalence

Using the variational formulation we may now prove existence and uniqueness of solutions of the variational problems. Since the variational and classical problems turn out to be equivalent this also proves the unique solvability of classical problems. First, we discuss the TE mode:

LEMMA 3.4.1. *The sesquilinear form b_{TE} is bounded and coercive, therefore the Problem 3.3.1 (NBVP2) is uniquely solvable for for all* $\frac{\partial u^i}{\partial \nu}|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$.

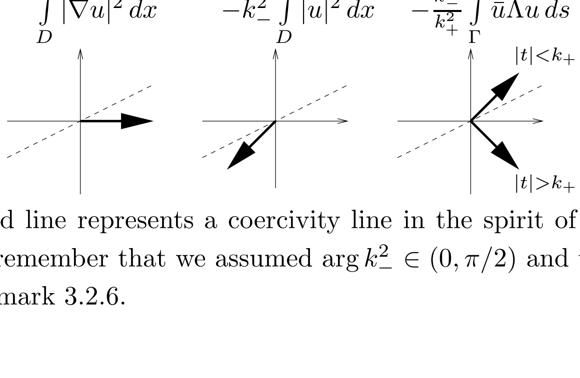
PROOF. Let $u, v \in H^1(D)$, then the sesquilinear form is bounded by

$$\begin{aligned} b_{\mathrm{TE}}(u,v) \quad &\leq \quad ||\nabla u||_{L^2(D)}||\nabla v||_{L^2(D)} \\ &\quad +|k_-|^2\,||u||_{L^2(D)}||v||_{L^2(D)} \\ &\quad +\frac{|k_-^2|}{k_+^2}\,\max\{1,k_+\}\,||u||_{H^{1/2}(\Gamma)}||v||_{H^{1/2}(\Gamma)} \\ &\leq \quad C\,||u||_{H^1(D)}||v||_{H^1(D)} \end{aligned}$$

for $C = 1+|k_-|^2+C_T^2\,|k_-^2|\,\min\{1,k_+^{-1}\}$. Conclusion 1.3.4 then yields coercivity and uniqueness: The expression

$$b_{\mathrm{TE}}(u,u) = \int\limits_D \left(|\nabla u|^2 - k_-^2|u|^2\right)\,dx - \frac{k_-^2}{k_+^2}\int\limits_\Gamma \bar{u}\Lambda u\,ds$$

has three terms, and the phases are illustrated in the following diagram:



The dashed line represents a coercivity line in the spirit of Lemma 1.3.3 and remember that we assumed $\arg k_-^2 \in (0,\pi/2)$ and the findings of Remark 3.2.6.

Since $k_-^2 \notin \mathbb{R}$ and

$$\left| \frac{k_-^2}{k_+^2} \int_\Gamma \bar{u} \Lambda u \, ds \right| \leq \left| \frac{k_-^2}{k_+^2} \right| \max\{1, k_+\} \, ||u||^2_{H^{1/2}(\Gamma)} ,$$

which is bounded by $\varepsilon ||u||_{L^2(D)}$ and $C_\varepsilon \, ||\nabla u||^2_{L^2(D)}$ for some $C_\varepsilon > 0$ just as needed using Lemma 1.2.4, the conclusion is applicable and therefore b_{TE} is coercive. □

REMARK 3.4.2. As mentioned earlier, we may prove coercivity directly if k_+ is large: When $\max\{1, k_+\} > C_T^2$ is satisfied

$$||u||^2_{L^2(D)} - \left| \frac{1}{k_+^2} \int_\Gamma \bar{u} \Lambda u \, ds \right| > \left(1 - \frac{C_T^2}{\max\{1, k_+\}} \right) ||u||^2_{L^2(D)}$$

holds, and the Lax-Milgram Lemma 1.3.1 may be applied directly.

The equivalence of Problems 3.1.2 (NBVP1) and 3.3.1 (NBVP2) is discussed in the following two Lemmata:

LEMMA 3.4.3. *The restriction $u|_D$ of a solution u to Problem 3.1.2 (NBVP1) with $u|_D \in H^1(D)$ is a solution to Problem 3.3.1 (NBVP2).*

PROOF. Let u be a solution to (NBVP1) and $u|_D \in H^1(D)$, then $u|_\Gamma \in H^{1/2}(\Gamma)$ by trace theorems, so Λu is well defined. By Green's theorem, Lemma 3.2.4, and test functions $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned} 0 &= -\int_D \left(\Delta u \bar{\varphi} + k_-^2 u \bar{\varphi} \right) dx \\ &= \int_D \left(\nabla u \nabla \bar{\varphi} - k_-^2 u \bar{\varphi} \right) dx - \frac{k_-^2}{k_+^2} \int_\Gamma \bar{\varphi} \Lambda u \, ds - \int_{\partial\Omega} \bar{\varphi} \frac{\partial u^i}{\partial \nu} \, ds , \end{aligned}$$

thus u satisfies NBVP2, as the test function space is dense in $H^1(D)$. □

The main part in proving the converse result will be to prove regularity of the weak solution using the coercivity of the sesquilinear form as in [**22**].

LEMMA 3.4.4. *The extension v of a solution u to the Problem 3.3.1 (NBVP2) defined by*

$$v(x) := \begin{cases} \frac{2k_-^2}{k_+^2} \int\limits_{\mathbb{R}} \frac{\partial \Phi(x_1,x_2,y_1,h)}{\partial y_2} u(y_1,0)\, dy_1 & \textit{for} \quad x_2 > 0 \\ u(x) & \textit{for} \quad x_2 \in \bar{D} \end{cases}$$

is a solution to Problem 3.1.2 (NBVP1).

PROOF. We denote the difference quotient into the j-th coordinate for functions in $H^1(D)$ as

$$\left(\Delta_j^h u\right)(x) := \frac{1}{h}\left(u(x+he_j) - u(x)\right) \in H^1(D)\,.$$

Since $(\Delta_j^h u, v)_{L^2(\mathbb{R}^n)} = -(u, \Delta_j^{-h} v)_{L^2(\mathbb{R}^n)}$ for $u, v \in L^2(\mathbb{R}^n)$, and Λ commutes with the difference quotient Δ_1^h

$$\begin{aligned} \int\limits_\Gamma \bar{\varphi} \Lambda \Delta_1^h u\, ds &= \frac{i}{h\sqrt{2\pi}} \int\limits_{\mathbb{R}} \bar{\hat{\varphi}}(\tau) \sqrt{k_+^2 - \tau^2} (e^{ih\tau} - 1) \hat{u}(\tau)\, d\tau \\ &= \frac{i}{h\sqrt{2\pi}} \int\limits_{\mathbb{R}} \overline{\hat{\varphi}(\tau)\,(e^{-ih\tau} - 1)} \sqrt{k_+^2 - \tau^2} \hat{u}(\tau)\, d\tau \\ &= -\int\limits_\Gamma \overline{\Delta_1^{-h}\varphi} \Lambda u\, ds\,, \end{aligned}$$

as noted earlier for $u, \varphi \in H^1(D)$ in Remark 3.2.2, thus it holds that

$$b_{\mathrm{TE}}(\Delta_1^h u, \varphi) = -b_{\mathrm{TE}}(u, \Delta_1^{-h}\varphi) = -a_{\mathrm{TE}}(\Delta_1^{-h}\varphi) \qquad (3.2)$$

for solutions u of (NBVP2).

Let u be a solution and $f \in H^2(D)$ denote a smooth rapidly decaying function with compact support G around $\bar{\Omega}$, $f|_{\partial\Omega} = u^i|_{\partial\Omega}$, $\frac{\partial f}{\partial \nu}|_{\partial\Omega} = \frac{\partial u^i}{\partial \nu}|_{\partial\Omega}$, and satisfying the condition $||f||_{H^2(D)} \leq C||u^i||_{H^{3/2}(\partial\Omega)}$, then we are able to prove regularity of $\tilde{u} := u - f$, because it satisfies the variational equation with the modified but bounded right hand side

$\tilde{a}_{\mathrm{TE}}$ for all $\varphi \in H^1(D)$:

$$\begin{aligned} b_{\mathrm{TE}}(u-f,\varphi) &= -\int\limits_{\partial\Omega} \bar{\varphi}\frac{\partial u^i}{\partial \nu}\,ds + \int\limits_{G} (-\nabla\bar{\varphi}\nabla f + k_-^2\bar{\varphi} f)\,dx \\ &= \int\limits_{G} \bar{\varphi}(\Delta f + k_-^2 f)\,dx =: \tilde{a}_{\mathrm{TE}}(\varphi)\,. \end{aligned}$$

Therefore, Equation (3.2) holds for $\tilde{u}$,

$$b_{\mathrm{TE}}(\Delta_1^h\tilde{u},\varphi) = -b_{\mathrm{TE}}(\tilde{u},\Delta_1^{-h}\varphi) = -\tilde{a}_{\mathrm{TE}}(\Delta_1^{-h}\varphi)\,,$$

and by coercivity of b_{TE} and estimation of the difference quotient exploiting the compact support of f,

$$\begin{aligned} C||\Delta_1^h\tilde{u}||^2_{H^1(D)} &\leq |b_{\mathrm{TE}}(\Delta_1^h\tilde{u},\Delta_1^h\tilde{u})| \\ &\leq C_2||\Delta_1^{-h}\tilde{u}||_{L^2(D)}||f||_{H^2(G)} \leq C_2||\tilde{u}||_{H^1(D)}||f||_{H^2(D)} \end{aligned}$$

where $C_2 = \max\{1,|k_-^2|\}$ for all $h > 0$, and thus $\frac{\partial}{\partial x_1}\tilde{u}$ and $\frac{\partial}{\partial x_1}u$ are in $H^1(D)$ by Theorem 3 in Chapter 5 in [**12**].

Since $\tilde{u}$ satisfies $b_{\mathrm{TE}}(\tilde{u},\varphi) = \tilde{a}_{\mathrm{TE}}(\varphi)$ for all $\varphi \in H^1(D)$, we may now express the second weak derivative as a functional of φ:

$$\begin{aligned} -\int\limits_D \frac{\partial\bar{\varphi}}{\partial x_2}\frac{\partial\tilde{u}}{\partial x_2}\,dx &= \int\limits_D (\frac{\partial\bar{\varphi}}{\partial x_1}\frac{\partial\tilde{u}}{\partial x_1} - k_-^2\bar{\varphi}\tilde{u})\,dx \\ &\qquad - \frac{k_-^2}{k_+^2}\int\limits_\Gamma \bar{\varphi}\Lambda\tilde{u}\,ds - \tilde{a}_{TE}(\varphi) \\ &= -\int\limits_D (\bar{\varphi}\frac{\partial^2\tilde{u}}{\partial x_1^2} + k_-^2\bar{\varphi}\tilde{u})\,dx \\ &\qquad - \frac{k_-^2}{k_+^2}\int\limits_\Gamma \bar{\varphi}\Lambda\tilde{u}\,ds - \tilde{a}_{TE}(\varphi) \end{aligned}$$

where the last equation is due to an integration by parts with respect to the integral of x_1 over $\mathbb{R}$, where both φ and $\tilde{u}$ and derivatives vanish towards $\pm\infty$ due to their integrability.

Thus $\frac{\partial^2 \tilde{u}}{\partial x_2^2}$ and $\frac{\partial^2 u}{\partial x_2^2}$ are in $L^2(D)$ by definition with bound

$$\begin{aligned} ||\frac{\partial^2 \tilde{u}}{\partial x_2^2}||_{L^2(D)} &\leq ||\frac{\partial^2 \tilde{u}}{\partial x_1^2}||_{L^2(D)} + ||f||_{H^2(D)} \\ &+|k_-|^2 \left(||\tilde{u}||_{L^2(D)} + \frac{C_T}{k_+^2} \max\{1, k_+\} ||\tilde{u}||_{H^1(D)} \right) . \end{aligned}$$

So $u \in H^2(D) \subset C(D)$, and by smoothness of u^i and $\partial\Omega$ the same argumentation extends to $u \in C^2(D)$, solving (NBVP1) in $\mathbb{R}^2_-$.

As a double layer potential in the upper half space with continuous density $u|_\Gamma$, also the extension v is twice continuously differentiable and solving the (NBVP1) in $\mathbb{R}^2_+$, satisfying the transmission condition as well. □

This concludes the discussion of the TE mode. Since the variational problem is uniquely solvable and both the variational and classical problems are equivalent, we have shown the unique solvability of the classical problem. Next, we discuss the TM mode:

LEMMA 3.4.5. *The sesquilinear form b_{TM} is bounded and coercive, therefore the Problem 3.3.2 (DBVP2) is uniquely solvable for all $u^i|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$.*

PROOF. For $u, v \in H^1(D)$ the sesquilinear form is bounded by

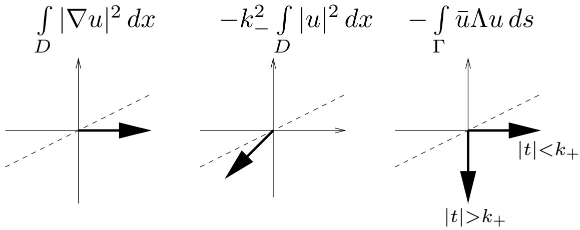
$$\begin{aligned} b_{\mathrm{TM}}(u, v) &\leq ||\nabla u||_{L^2(D)} ||\nabla v||_{L^2(D)} \\ &+|k_-|^2 \, ||u||_{L^2(D)} ||v||_{L^2(D)} \\ &+ \max\{1, k_+\} \, ||u||_{H^{1/2}(D)} ||v||_{H^{1/2}(D)} \\ &\leq (1 + |k_-^2| + C_T^2 \max\{1, k_+\}) \, ||u||_{H^1(D)} ||v||_{H^1(D)} . \end{aligned}$$

Conclusion 1.3.4 then yields coercivity and uniqueness:

The expression

$$b_{\mathrm{TM}}(u, u) = \int\limits_D \left(|\nabla u|^2 - k_-^2 |u|^2 \right) \, dx - \int\limits_\Gamma \bar{u} \Lambda u \, ds \, ,$$

has three terms, and the phases are illustrated in the following diagram:



Since $k_-^2 \notin \mathbb{R}$ and

$$\left|\int_\Gamma \bar{u}\Lambda u\,ds\right| \leq \max\{1, k_+^2\}\,||u||^2_{H^{1/2}(\Gamma)}$$

which is bound by $\varepsilon||u||_{L^2(D)}$ and $C_\varepsilon||\nabla u||^2_{L^2(D)}$ for some $C_\varepsilon > 0$ as needed, using Lemma 1.2.4, the conclusion is applicable and therefore b_{TM} is coercive. Note that in this case, one can apply Remark 1.3.2 directly, since the phase of the third term is inside the coercivity half plane of the two semi-norms in the first two terms. □

The equivalence of Problems 3.1.1 (DBVP1) and 3.3.2 (DBVP2) is discussed in the following two Lemmata: The first lemma proves that an integrable classical solution will satisfy the weak formulation as well. The second lemma proves regularity of weak solutions, and that they therefore will be classical solutions as well.

Lemma 3.4.6. *Let u be a solution of Problem 3.1.1 (DBVP1) with $u|_D \in H^1(D)$. Then $u|_D - g$ is a solution to Problem 3.3.2 (DBVP2).*

Proof. Let u be a solution to (DBVP1) and g be smooth, rapidly decreasing, with a support G around $\bar{\Omega}$ such that $u|_D - g \in H^1_0(D)$, then $u|_\Gamma \in H^{1/2}(\Gamma)$ by the trace theorem, so Λu is well defined.

By Green's theorem, Lemma 3.2.4, and test functions $\varphi \in C^\infty(\overline{D})$, $\varphi|_{\partial\Omega} = 0$ we have

$$
\begin{aligned}
0 &= -\int_D \left((\Delta u + k_-^2 u)\bar{\varphi}\right) dx \\
&= \int_D \left(\nabla(u-g)\nabla\bar{\varphi} - k_-^2(u-g)\bar{\varphi}\right) dx - \int_\Gamma \bar{\varphi}\Lambda u \, ds \\
&\quad + \int_G \left(\nabla\bar{\varphi}\nabla g - k_-^2 \bar{\varphi} g\right) dx \,,
\end{aligned}
$$

thus u satisfies (DBVP2) as the test function space is dense in $H_0^1(D)$.

□

LEMMA 3.4.7. *The extension $\tilde{u}$ of a solution u to Problem 3.3.2 (DBVP2) defined by*

$$
\tilde{u}(x) := \begin{cases} 2\int_{\mathbb{R}} \frac{\partial \Phi(x_1,x_2,y_1,h)}{\partial y_2} u(y_1,0)\, dy_1 & \text{for} \quad x_2 > 0 \\ u(x) & \text{for} \quad x_2 \in \bar{D} \end{cases}
$$

is a solution to Problem 3.1.1 (DBVP1).

PROOF. For $u, \varphi \in H_0^1(D)$, it holds that

$$
b_{\mathrm{TM}}(\Delta_1^h u, \varphi) = -b_{\mathrm{TM}}(u, \Delta_1^{-h}\varphi) = -a_{\mathrm{TM}}(\Delta_1^{-h}\varphi)
$$

for solutions u of (DBVP2).

By coercivity of b_{TM} and estimation of the difference quotient

$$
\begin{aligned}
C||\Delta_1^h u||_{H^1(D)}^2 &\leq |b_{\mathrm{TM}}(\Delta_1^h u, \Delta_1^h u)| \\
&\leq C_2||\Delta_1^{-h} u||_{L^2(D)}||g||_{H^2(G)} \\
&\leq C_2||u||_{H^1(D)}||g||_{H^2(D)}
\end{aligned}
$$

for all $h > 0$ and thus $\frac{\partial}{\partial x_1} u \in H_0^1(D)$.

Since u satisfies $b_{\mathrm{TM}}(u,\varphi) = a_{\mathrm{TM}}(\varphi)$ for all $\varphi \in H_0^1(D)$, we may now express the second weak derivative as a functional of φ:

$$\begin{aligned} -\int_D \frac{\partial\bar{\varphi}}{\partial x_2}\frac{\partial u}{\partial x_2}\,dx &= \int_D \Big(\frac{\partial\bar{\varphi}}{\partial x_1}\frac{\partial u}{\partial x_1} - k_-^2\bar{\varphi}u\Big)\,dx \\ &\quad - \int_\Gamma \bar{\varphi}\Lambda u\,ds - a_{\mathrm{TM}}(\varphi)\,, \\ &= -\int_D \Big(\bar{\varphi}\frac{\partial^2 u}{\partial x_1^2} + k_-^2\bar{\varphi}u\Big)\,dx \\ &\quad - \int_\Gamma \bar{\varphi}\Lambda u\,ds - a_{\mathrm{TM}}(\varphi)\,, \end{aligned}$$

thus by definition $\frac{\partial^2 u}{\partial x_2^2} \in L^2(D)$ with bound

$$\begin{aligned} \Big|\Big|\frac{\partial^2 u}{\partial x_2^2}\Big|\Big|_{L^2(D)} &\leq \Big|\Big|\frac{\partial^2 u}{\partial x_1^2}\Big|\Big|_{L^2(D)} + |k_-|^2||u||_{L^2(D)} \\ &\quad + C_T \max\{1, k_+\}||u||_{H^1(D)} + ||g||_{H^2(D)}\,. \end{aligned}$$

So $u \in H^2(D) \subset C(D)$, and by smoothness of u^i and $\partial\Omega$ the same argumentation extends to $u \in C^2(D)$, solving (DBVP1) in $\mathbb{R}^2_-$.

As a double layer potential with continuous density, also the extension $\tilde{u}$ is twice continuously differentiable and solving the (DBVP1) in $\mathbb{R}^2_+$, satisfying the transmission condition as well. □

We have proven existence and uniqueness of solutions to the weak problem formulations representing the TE and TM mode, and the equivalence of the weak formulations to the classical formulations in the whole space for solutions integrable in the lower half-space.

In [**3**] it was shown, that the classical problem as above is uniquely solvable and the solution in the lower half-space was found to be integrable by Lemma 2.5.1. So the radiation condition of H^1 integrability is no restriction to the statements. In fact, for $x_2 \to -\infty$ we can prove exponential decay of the solution using a coercive sesquilinear form, as it will be shown in the following section. It should be noted that the integrability condition is indeed stronger than the radiation condition of the *two-layer radiation condition* in the lower half space, since

$$\begin{aligned}\lim_{r\to\infty}\left|\int\limits_{\Omega_r}\left(\bar{u}\frac{\partial u}{\partial\nu}-u\frac{\partial\bar{u}}{\partial\nu}\right)ds\right| &= 2\lim_{r\to\infty}\left|\int\limits_{\Omega_r}\operatorname{Im}\left(\bar{u}\frac{\partial u}{\partial\nu}\right)ds\right| \\ &\leq 2\lim_{r\to\infty}\int\limits_{\Omega_r}\left(||u||^2+||\nabla u||^2\right)ds=0\end{aligned}$$

when $\Omega_r=\partial B_r(0)\cap D$ is the half-circle with sufficiently large radius r in the lower half-space and $u\in H^1(D)\cap C(D)$.

3.5. Exponential Decay

We expect the solution to be exponentially decreasing in the $-x_2$ coordinate direction. This motivates the question whether it is possible to include this assumption into the weak formulation without losing coercivity.

Consider a smooth weight function $\varrho:\mathbb{R}^2\to\mathbb{R}_{>0}$ only dependent on the second variable with $\varrho(\cdot,0)=1$, such as $\varrho(x)=e^{-ax_2}$. In the TM case, we will show that a ϱ weighted solution exists by proving coercivity of the resulting weighted sesquilinear form $\tilde{b}$.

Since ϱ is only dependent on the second variable $\frac{\partial}{\partial x_1}\varrho(x)=0$ holds, and we compute the derivatives of the weighted functions $u(x)=\varrho(x)v(x)$ to be

$$\nabla u(x)=v(x)\nabla\varrho(x)+\varrho(x)\nabla v(x)$$

and for $\varphi(x)=\frac{\phi(x)}{\varrho(x)}$ to be

$$\nabla\varphi(x)=-\phi(x)\frac{\nabla\varrho(x)}{\varrho^2(x)}+\frac{\nabla\phi(x)}{\varrho(x)}\,.$$

Now the ϱ weighted sesquilinear form in v and ϕ may be defined as

$$\begin{aligned}\tilde{b}_{\mathrm{TM}}(v,\phi) &:= b_{\mathrm{TM}}(u,\varphi)\\ &= \int\limits_D \Big[(-\bar{\phi}\frac{\nabla\varrho}{\varrho^2}+\frac{\nabla\bar{\phi}}{\varrho})(v\nabla\varrho+\varrho\nabla v)-k_-^2\bar{\phi}v\Big]dx\\ &\quad -\int\limits_\Gamma \bar{\phi}\,\Lambda v\,ds\\ &= \int\limits_D \Big[\nabla\bar{\phi}\nabla v+\frac{\nabla\varrho}{\varrho}(v\nabla\bar{\phi}-\bar{\phi}\nabla v)-(\frac{|\nabla\varrho|^2}{\varrho^2}+k_-^2)\bar{\phi}v\Big]dx\\ &\quad -\int\limits_\Gamma \bar{\phi}\,\Lambda v\,ds\,.\end{aligned}$$

As we are expecting an exponential decay of the solution in the $-x_2$ coordinate direction, from now on we will assume a weight function of $\varrho(x)=e^{-ax_2}$, $a>0$ and see that then $\frac{\nabla\varrho}{\varrho}=-\binom{0}{a}$.

THEOREM 3.5.1. *Let $a<\mathrm{Im}(k_-)$, then the weighted sesquilinear form $\tilde{b}_{\mathrm{TM}}$ is coercive, that is, for*

$$\begin{aligned}\tilde{b}_{\mathrm{TM}}(v,v) &= \int\limits_D \left[|\nabla v|^2-2i\begin{pmatrix}0\\ a\end{pmatrix}\mathrm{Im}(v\nabla\bar{v})-(a^2+k_-^2)|v|^2\right]dx\\ &\quad -\int\limits_\Gamma \bar{v}\,\Lambda v\,ds\end{aligned}$$

there exist constants C and α such that

$$\mathrm{Re}\left(e^{i\alpha}\tilde{b}_{\mathrm{TM}}(v,v)\right)\geq C||v||^2_{H^1(D)}$$

for all $v\in H^1(D)$.

PROOF. Let $k_-=|k_-|e^{i\kappa}$ and $||\cdot||$ denote the L^2-Norm on D. Then $\kappa\in(0,\frac{\pi}{4})$ by Equation 2.3. We will now show that $\mathrm{Re}(e^{i\alpha}\tilde{b}_{\mathrm{TM}}(v,v))>C(||\nabla v||^2+||v||^2)$ holds for $\alpha=\frac{\pi}{2}-\kappa$ and $a<\mathrm{Im}(k_-)$. Note that for such α we have $\cos(\alpha)=\sin(\kappa)>0$ as well as $\sin(\alpha)=\cos(\kappa)>0$ and $\mathrm{Re}\left((a^2+|k_-|^2e^{i2\kappa})\cdot e^{i(\frac{\pi}{2}-\kappa)}\right)=$

$(\sin\kappa)(a^2-|k_-|^2)$. Thus

$$\begin{aligned}\mathrm{Re}\left(e^{i\alpha}\tilde{b}_{\mathrm{TM}}(v,v)\right) &= (\sin\kappa)||\nabla v||^2-(\sin\kappa)(a^2-|k_-|^2)||v||^2\\ &\quad +2a(\cos\kappa)\int\limits_D\begin{pmatrix}0\\1\end{pmatrix}\cdot\mathrm{Im}(v\nabla\bar{v})\,dx+r\end{aligned}$$

where $r=-\mathrm{Re}(e^{i\alpha}\int\limits_\Gamma\bar{v}\,\Lambda v\,ds)=-\mathrm{Re}(e^{i\alpha}\int\limits_\Gamma\bar{u}\,\Lambda u\,ds)>0$ for chosen α by Lemma 3.2.4. By Cauchy-Schwartz inequality, we have

$$|\int\limits_D\begin{pmatrix}0\\1\end{pmatrix}\cdot\mathrm{Im}(v\nabla\bar{v})\,dx|\le|\int\limits_D\begin{pmatrix}0\\1\end{pmatrix}\cdot(v\nabla\bar{v})\,dx|\le||v||\cdot||\nabla v||$$

and an estimation into which we will introduce a parameter $t\in\mathbb{R}$ and complete a square from the mixed term:

$$\begin{aligned}\mathrm{Re}\left(e^{i\alpha}\tilde{b}_{\mathrm{TM}}(v,v)\right) &\ge (\sin\kappa)||\nabla v||^2-(\sin\kappa)(a^2-|k_-|^2)||v||^2\\ &\quad +2a(\cos\kappa)||v||\cdot||\nabla v||+r\\ &= (\sin\kappa)||\nabla v||^2-(\sin\kappa)(a^2-|k_-|^2)||v||^2\\ &\quad +2\left(ta(\cos\kappa)||v||\right)\cdot\left(\frac{1}{t}||\nabla v||\right)+r\\ &= \left[ta(\cos\kappa)||v||-\frac{1}{t}||\nabla v||\right]^2\\ &\quad +\left(\sin\kappa-\frac{1}{t^2}\right)||\nabla v||^2\\ &\quad -\left(t^2a^2\cos^2\kappa+(\sin\kappa)(a^2-|k_-|^2)\right)||v||^2+r\end{aligned}$$

Now assuming $\frac{1}{t^2}=\sin\kappa-\varepsilon$, we have $\sin\kappa-\frac{1}{t^2}=\varepsilon>0$ and it remains to show the inequality

$$\begin{aligned}&a^2\cos^2\kappa+\frac{1}{t^2}(\sin\kappa)(a^2-|k_-|^2)\\ =\;&a^2-(\sin^2\kappa)|k_-|^2-\varepsilon(\sin\kappa)(a^2-|k_-|^2)<0\,,\end{aligned}$$

that is,

$$\begin{aligned}a^2 &< \sin^2\kappa|k_-|^2+\varepsilon(\sin\kappa)(a^2-|k_-|^2)\\ &= (\mathrm{Im}(k_-))^2+\varepsilon(\sin\kappa)(a^2-|k_-|^2)\,,\end{aligned}$$

which is fulfilled for

$$a < \mathrm{Im}(k_-) \quad \text{and} \quad \varepsilon = \frac{1}{2}\left(\frac{a^2 - (\mathrm{Im}(k_-))^2}{\sin\kappa(a^2 - |k_-|^2)}\right).$$

□

CHAPTER 4

Distributions and Weighted Sobolev Spaces

4.1. Motivation

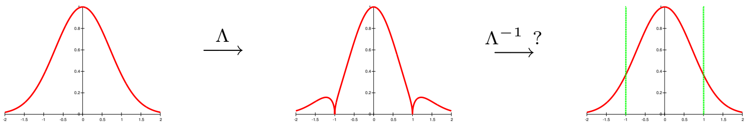
As we have seen in Lemma 3.2.4, the Dirichlet-to-Neumann Map Λ yields the normal derivative of the boundary data on Γ, that is $\Lambda\varphi := \frac{\partial u(x_1,x_2)}{\partial x_2}|_{x_2=0}$ of the solution u to the problem in the upper half-space when the Dirichlet data φ on Γ is given. The mapping properties on Sobolev spaces were discussed in Lemma 3.2.1— these results were sufficient for two-dimensional problems but are lacking for the spatial model:

The derivation of two-dimensional models showed that we had to establish an additional condition to ensure separability of the TE and TM mode. For discussion what will be needed in the general case, let us assume for now that this condition is not satisfied, and thus we have a coupled system of the two modes. This means that we lose the freedom to express the problem in terms of either the electric or magnetic field intensity to ensure that we only have to deal with a Dirichlet-to-Neumann operator as we have done in the TE and TM mode. Even in this simplified coupled system we will already need both the Dirichlet-to-Neumann as well as its inverse, the Neumann-to-Dirichlet operator. So it turns out that in the general case of three dimensional Maxwell's equations it will be necessary to properly characterize the Neumann-to-Dirichlet operator and its mapping properties:

In order to define the inverse mapping, we need to identify the image set $\Lambda\left(H^{1/2}(\mathbb{R})\right)$, which turns out to be a proper subspace of

$H^{-1/2}(\mathbb{R})$, as e.g. the Fourier transform of the $H^{-1/2}(\mathbb{R})$ function $\psi(t) = \exp(-t^2/2)$ has no roots at $\pm k$ and is therefore not in the image of the Dirichlet-to-Neumann mapping.

EXAMPLE 4.1.1. Example plots of $|\mathcal{F}u(\tau)|$ and $|\mathcal{F}\Lambda u(\tau)|$:



This formal definition of the Neumann-to-Dirichlet mapping will help to determine the image of Λ:

DEFINITION 4.1.2. (Formal Neumann-to-Dirichlet)

$$\Lambda^{-1}\psi\,(t) := \frac{1}{i\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\hat{\psi}(\tau)}{\sqrt{k^2-\tau^2}}\, e^{i\tau t}\, d\tau$$

As expected, we see that the condition $\psi \in H^{-1/2}(\mathbb{R})$, or $\mathcal{J}^{-1/2}\psi \in L^2(\mathbb{R})$, is simply too weak due to the singularities at $\pm k$. This is the motivation to define a modified Bessel Potential similar to Definition 1.1.1:

DEFINITION 4.1.3. (Modified Bessel k,α-Potential)

Let $s, \alpha < 2$, $k > 0$, and $n = 1$ or $n = 2$, then for $u \in \mathcal{S}(\mathbb{R}^n)$ we define

$$\begin{aligned} \mathcal{J}^s_{k,\alpha} u &:= \mathcal{F}^{-1}(\psi^s_{k,\alpha}\hat{u})\,, \\ \text{where } \psi^s_{k,\alpha}(\tau) &:= \underbrace{\left(1+|\tau|^2\right)^{\frac{s}{2}}}_{=\psi^s(\tau)} \left|\frac{1+|\tau|^2}{k^2-|\tau|^2}\right|^{\frac{\alpha}{2}}, \ \tau \in \mathbb{R}^n\,, \end{aligned}$$

and $\hat{u}$ and $\mathcal{F}^{-1}$ denote the n-dimensional Fourier transform and inverse.

Note that $\psi^s_{k,\alpha}$ has the same growth as ψ^s for large $|\tau|$, and that this definition is only valid for functions in $\mathcal{S}(\mathbb{R})$ in general under the additional condition of $\alpha < 2$ to ensure integrability of the potential at the singularity. Yet, this is just a formal definition and it is desirable to gain the same mapping properties for $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^*(\mathbb{R})$ as for the original Bessel Potential, but this is not necessarily the case:

LEMMA 4.1.4. *For $\alpha \in (-\infty, 2)$, $\alpha \neq 0$, the modified Bessel k, α-Potential is a continuous linear operator*

$$\mathcal{J}^s_{k,\alpha} : \mathcal{S}(\mathbb{R}) \to \mathcal{E}(\mathbb{R}) \text{ and } \mathcal{J}^s_{k,\alpha}\left(\mathcal{S}(\mathbb{R})\right) \not\subseteq \mathcal{S}(\mathbb{R})\,.$$

PROOF. Let $u \in \mathcal{S}(\mathbb{R})$, then we see for $\beta \in \mathbb{N}_0$ and $q \in \mathbb{N}$, $q > (1-\alpha)/2$

$$\begin{aligned}\left(\frac{d}{dt}\right)^{\beta}\left(\mathcal{J}^s_{k,\alpha}u\right)(t) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \underbrace{(i\tau)^{\beta}(1+\tau^2)^{\frac{s+\alpha}{2}+q}\hat{u}(\tau)}_{\in\mathcal{S}(\mathbb{R})\subset BC(\mathbb{R})} \\ &\quad\cdot \underbrace{\frac{1}{|k^2-\tau^2|^{\alpha/2}(1+\tau^2)^q}}_{\in L^1(\mathbb{R})} \cdot e^{i\tau t}d\tau\,,\end{aligned}$$

so $\left(\frac{d}{dt}\right)^{\beta}\mathcal{J}^s_{k,\alpha}u \in C(\mathbb{R})$ and therefore $\mathcal{J}^s_{k,\alpha}u \in \mathcal{E}(\mathbb{R}) = C^{\infty}(\mathbb{R})$.

For continuity, consider a sequence $(u_n) \subset \mathcal{S}(\mathbb{R})$ with $u_n \overset{\mathcal{S}(\mathbb{R})}{\longrightarrow} 0$, so (u_n) and all sequences of derivatives converge to 0 uniformly. Since then $\hat{u}_n \overset{\mathcal{S}(\mathbb{R})}{\longrightarrow} 0$, it follows that $\mathcal{J}^s_{k,\alpha}u \overset{\mathcal{E}(\mathbb{R})}{\longrightarrow} 0$ as for all $\beta \in \mathbb{N}_0$ the integral is bounded by the L^1-Norm of the second factor and the converging L^{∞}-Norm of the first factor. Since $\mathcal{S}(\mathbb{R})$ is metrizable, this proves continuity of $\mathcal{J}^s_{k,\alpha}$ as a mapping $\mathcal{S}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$.

Now, if $u(t) = e^{-t^2/2} \in \mathcal{S}(\mathbb{R})$, then $\widehat{\mathcal{J}^s_{k,\alpha}u}\,(\tau) = \frac{e^{-\tau^2/2}}{|k^2-\tau^2|^{\alpha/2}}$ is certainly not in $\mathcal{S}(\mathbb{R})$ for $\alpha \neq 0$, and therefore $\mathcal{J}^s_{k,\alpha}u \notin \mathcal{S}(\mathbb{R})$, since the Fourier transforms of all functions in $\mathcal{S}(\mathbb{R})$ are in $\mathcal{S}(\mathbb{R})$. □

This lemma clarifies why the traditional test function space $\mathcal{S}(\mathbb{R})$ is not sufficient. Still, it is possible to define the operator on generalized function spaces, although this will prove to be of no use for the given model:

DEFINITION 4.1.5. For $\alpha \in (-\infty, 2)\backslash\{0\}$ the Bessel k, α-Potential is defined on distributions $\mathcal{J}^s_{k,\alpha} : \mathcal{E}^*(\mathbb{R}) \to \mathcal{S}^*(\mathbb{R})$, where

$$\langle \mathcal{J}^s_{k,\alpha}\varphi, u\rangle_{\mathcal{S}(\mathbb{R})} := \langle \varphi, \underbrace{\mathcal{J}^s_{k,\alpha}u}_{\in\mathcal{E}(\mathbb{R})}\rangle_{\mathcal{E}(\mathbb{R})}$$

for $u \in \mathcal{S}(\mathbb{R})$ and $\varphi \in \mathcal{E}^*(\mathbb{R})$.

REMARK 4.1.6. This definition for generalized functions in $\mathcal{E}^*(\mathbb{R})$ can be seen as an extension of the definition for functions in $\mathcal{S}(\mathbb{R})$ in the sense that they coincide for functions in both sets in the sense of distributions: Let $u \in \mathcal{S}(\mathbb{R}) \cap \mathcal{E}^*(\mathbb{R})$ and $v \in \mathcal{S}(\mathbb{R})$ any test function, then by definition 4.1.3

$$\begin{aligned} \langle \mathcal{J}^s_{k,\alpha}\varphi, v\rangle_{\mathcal{S}(\mathbb{R})} &= (\mathcal{J}^s_{k,\alpha}\varphi, \bar{v})_{L^2(\mathbb{R})} \\ &= (\mathcal{F}\,\mathcal{J}^s_{k,\alpha}\varphi, \mathcal{F}\bar{v})_{L^2(\mathbb{R})} = (\psi^s_{k,\alpha}\,\mathcal{F}\varphi, \mathcal{F}\bar{v})_{L^2(\mathbb{R})} \end{aligned}$$

by Parseval's theorem. As $\psi^s_{k,\alpha}$ is real,

$$\begin{aligned} (\psi^s_{k,\alpha}\,\mathcal{F}\varphi, \mathcal{F}\bar{v})_{L^2(\mathbb{R})} &= (\mathcal{F}\varphi, \psi^s_{k,\alpha}\mathcal{F}\bar{v})_{L^2(\mathbb{R})} \\ &= (\mathcal{F}\varphi, \mathcal{F}\mathcal{J}^s_{k,\alpha}\bar{v})_{L^2(\mathbb{R})} = (\varphi, \mathcal{J}^s_{k,\alpha}\bar{v})_{L^2(\mathbb{R})}\,, \end{aligned}$$

again by Parseval's theorem. Now since $\psi^s_{k,\alpha}$ is real and even,

$$\overline{\mathcal{J}^s_{k,\alpha}\bar{v}} = \overline{\mathcal{F}^{-1}\psi^s_{k,\alpha}\mathcal{F}\bar{v}} = \mathcal{F}\psi^s_{k,\alpha}\mathcal{F}^{-1}v = \mathcal{F}^{-1}\psi^s_{k,\alpha}\mathcal{F}v = \mathcal{J}^s_{k,\alpha}v\,.$$

This finally leads to

$$(\varphi, \mathcal{J}^s_{k,\alpha}\bar{v})_{L^2(\mathbb{R})} = \langle \varphi, \overline{\mathcal{J}^s_{k,\alpha}\bar{v}}\rangle_{\mathcal{E}(\mathbb{R})} = \langle \varphi, \mathcal{J}^s_{k,\alpha}v\rangle_{\mathcal{E}(\mathbb{R})}\,,$$

which coincides with definition 4.1.5.

So based on the modified Bessel Potential and traditional test function spaces, at most, one might try to somehow define a space based on the subset of generalized functions of $\mathcal{E}^*(\mathbb{R})$, the potentials of which are in $L^2(\mathbb{R})$. But this is not advisable, as $\mathcal{E}^*(\mathbb{R})$ can be characterized as the temperate distributions with essentially compact support ([**22**], Theorem 3.8), a somewhat less desirable property for the treatment of electromagnetic fields on unbounded domains.

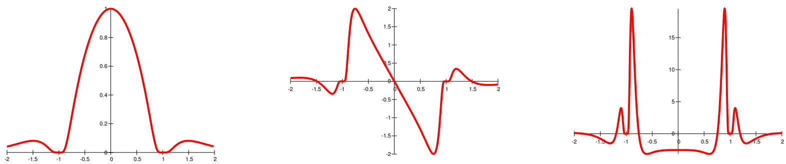
4.2. A Modified Set of Test Functions

A reduced set of test functions will help extending the Bessel Potential to more suitable function spaces:

DEFINITION 4.2.1. For $k > 0$ and $\varrho_k(\tau) := |k^2 - \tau^2|^{-1/2}$,

$$\mathcal{S}_k(\mathbb{R}) := \{u \in \mathcal{S}(\mathbb{R}) : \ \varrho_k \hat{u} \in \mathcal{S}(\mathbb{R})\} .$$

EXAMPLE 4.2.2. Elements of $\mathcal{S}_1(\mathbb{R})$: $\hat{u}(\tau) = e^{-\frac{\tau^2}{\arctan|\tau^2-1|}}$ and its first and second derivative:



The topology and convergence in this space is based on the convergence of the Fourier transforms in $\mathcal{S}(\mathbb{R})$:

DEFINITION 4.2.3. The topology of $\mathcal{S}_k(\mathbb{R})$ is defined by a countable family of semi-norms

$$|u|_{\mathcal{S}_k,\alpha,\beta} := |\hat{u}|_{\mathcal{S},\alpha,\beta} + |\varrho_k \hat{u}|_{\mathcal{S},\alpha,\beta} .$$

This means a sequence (u_n) of functions $u_n \in \mathcal{S}_k(\mathbb{R})$ converges to 0 in $\mathcal{S}_k(\mathbb{R})$ if both sequences $(\hat{u}_n)$ and $(\varrho_k \hat{u}_n)$ converge to 0 in $\mathcal{S}(\mathbb{R})$. This also defines $u_n \to u$ as $(u_n - u) \to 0$ in $\mathcal{S}_k(\mathbb{R})$.

LEMMA 4.2.4. *$\mathcal{S}_k(\mathbb{R})$ is a Fréchet space.*

PROOF. Since the topology of $\mathcal{S}_k(\mathbb{R})$ is given by a countable family of semi-norms, it is locally convex and metrizable, [**15**] 2 §4, [**21**] §18 2.2, and sequential completeness implies topological completeness, [**31**] Proposition 8.2.

Consider a Cauchy sequence $(u_n) \subset \mathcal{S}_k(\mathbb{R})$. Then the sequences $(\hat{u}_n)$ and $(\varrho_k \, \hat{u}_n)$ are Cauchy sequences in $\mathcal{S}(\mathbb{R})$ and have limits therein. As the multiplication with ϱ_k is continuous, the limit of the latter sequence is $\varrho_k \, \hat{u}$ and by continuity of the Fourier transform the limit $u \in \mathcal{S}_k(\mathbb{R})$. Thus $\mathcal{S}_k(\mathbb{R})$ is complete and therefore a Fréchet space. $\square$

The following technical lemmata will be needed to show that Fourier transforms of functions in this space are rapidly decreasing at $\pm k$.

LEMMA 4.2.5. *If $u, v \in C^\infty(-\varepsilon, \varepsilon)$ where $v(t) = |t|^{-1/2}u(t)$, then u, v and all derivatives of u and v vanish at 0.*

PROOF. Let $f(t) := u^2(t)$ and $g(t) := v^2(t)$. Then also $f, g \in C^\infty(-\varepsilon, \varepsilon)$ and $f(t) = |t|g(t)$, or $f(t) = t\,g(t)$ for $t \geq 0$ and $f(t) = -t\,g(t)$ for $t \leq 0$. Of course, $f(0) = 0$ and

$$f'(t) = \begin{cases} g(t) + t\,g'(t) & \text{for } t > 0, \\ -g(t) - t\,g'(t) & \text{for } t < 0\,. \end{cases}$$

Thus by continuity of the derivative, $f'(0) = g(0) = -g(0)$, thus $f'(0) = 0$ and $g(0) = 0$. Now let $n \in \mathbb{N}$, then, by induction,

$$f^{(n)}(t) = \begin{cases} n\,g^{(n-1)}(t) + t\,g^{(n)}(t) & \text{for } t > 0, \\ -n\,g^{(n-1)}(t) - t\,g^{(n)}(t) & \text{for } t < 0 \end{cases}$$

and therefore, by continuity of the derivatives,

$$f^{(n)}(0) = n\,g^{(n-1)}(0) = -n\,g^{(n-1)}(0)$$

and $g^{(n-1)}(0) = 0$, so f and g and all their derivatives vanish at 0. This carries over to u and v by induction:

For $n = 0$, we have $u^2(0) = f(0) = 0$. Now let $n \in \mathbb{N}$ and $u^{(k)}(0) = 0$ for $k = 0, \ldots, n-1$. By iterated product rule on $f(t) = u(t) \cdot u(t)$, we have

$$0 = f^{(2n)}(0) = \sum_{k=0}^{2n} \binom{2n}{k} u^{(k)}(0) \cdot u^{(2n-k)}(0) = \binom{2n}{n} \left(u^{(n)}(0)\right)^2 ,$$

as all other summands contain $u^{(k)}(0)$ for $k < n$. Therefore $u^{(n)}(0) = 0$ and the same holds for v. $\square$

LEMMA 4.2.6. *If $u \in C^\infty(-\varepsilon, \varepsilon)$ and u and all its derivatives vanish at 0, then for all $l \in \mathbb{R}$ the functions*

$$v_l(t) := \begin{cases} |t|^l u(t) & \textit{for } t \neq 0 \\ 0 & \textit{for } t = 0 \end{cases}$$

are in $C^\infty(-\varepsilon, \varepsilon)$ with $v_l^{(n)}(0) = 0$ for $n \in \mathbb{N}$.

PROOF. As v_l are compositions of smooth functions, it holds that $v_l \in C^\infty((-\varepsilon,\varepsilon)\backslash\{0\})$. For the case $l \geq 0$, the functions v_l are clearly continuous at 0. If $l < 0$, choose $n \in \mathbb{N}$ such that $-l < n$, then by Taylor theorem for u at $t_0 = 0$ and $u^{(k)}(0) = 0$:

$$\begin{aligned} u(t) &= \frac{1}{n!}\int_0^t (s-t)^n\, u^{(n+1)}(s)\, ds \\ &= \frac{t^n}{n!}\int_0^t \left(\frac{s}{t}-1\right)^n u^{(n+1)}(s)\, ds \\ &= \frac{t^{n+1}}{n!}\int_0^1 (\tau-1)^n\, u^{(n+1)}(\tau t)\, d\tau\,, \text{ by } s=\tau t. \end{aligned}$$

Therefore, v_l is continuous at $t = 0$:

$$\lim_{t\to 0}\left||t|^l u(t)\right| = \lim_{t\to 0}\left|\underbrace{\frac{|t|^{l+n+1}}{n!}}_{\to 0}\int_0^1 (\tau-1)^n\, u^{(n+1)}(\tau t)\, d\tau\right| = 0\,.$$

Now, for all $n \in \mathbb{N}$ the derivatives $v_l^{(n)}$ of v_l for $t \neq 0$ are finite sums of functions of the same type as v_l. Each of the summands is continuous and vanishes at 0 and therefore $v_l^{(n)}$ is continuous and vanishes at $t = 0$ as well. □

PROPOSITION 4.2.7.

$$\begin{aligned} \text{(i)}\quad \mathcal{S}_k(\mathbb{R}) &= \{u \in \mathcal{S}(\mathbb{R}) : \hat{u}^{(n)}(\pm k) = 0 \textit{ for all } n \in \mathbb{N}_0\} \\ \text{(ii)}\quad \mathcal{S}_k(\mathbb{R}) &= \{u \in \mathcal{S}(\mathbb{R}) : \varrho_{k,\alpha}\hat{u} \in \mathcal{S}(\mathbb{R}) \textit{ for all } \alpha \in \mathbb{R}\}, \\ &\quad \textit{where } \varrho_{k,\alpha}(\tau) := |k^2-\tau^2|^{\alpha/2} \end{aligned}$$

PROOF. *(ii) ⇒ Definition:* Obviously the condition in (ii) implies the condition of Definition 4.2.1 for $\alpha = -1$ as $\varrho_{k,-1} = \varrho_k$.

Definition ⇒ (i): On the other hand, the condition presented in (i) follows from Definition 4.2.1, where we have $\hat{u} \in \mathcal{S}(\mathbb{R}) \subset \mathcal{E}(\mathbb{R})$ and $\varrho_k\hat{u} \in \mathcal{S}(\mathbb{R}) \subset \mathcal{E}(\mathbb{R})$. As $|k^2-\tau^2|^{-1/2} = |k+\tau|^{-1/2}\cdot|k-\tau|^{-1/2}$, we may apply Lemma 4.2.5 on neighborhoods around $\pm k$ to prove that the Fourier transform and all of its derivatives vanish at $\pm k$, as stated in (i).

(i) $\Rightarrow$ (ii): Now, the condition in (i) yields $\varrho_{k,\alpha}\hat{u} \in \mathcal{E}(\mathbb{R})$ by Lemma 4.2.6 on neighborhoods around $\pm k$ for any $\alpha \in \mathbb{R}$. For any $\alpha \in \mathbb{R}$, the factor $|k^2 - \tau^2|^{\alpha/2}$ and all its derivatives are at most slowly increasing and therefore the product $\varrho_{k,\alpha}\hat{u}$ still is rapidly decreasing and thus $\varrho_{k,\alpha}\hat{u} \in \mathcal{S}(\mathbb{R})$. □

The following lemma will help in the discussion of continuity of the Bessel k, α-Potential.

LEMMA 4.2.8. *Let $p \in \mathbb{N}_0$. If $(u_n) \subset C^{p+1}(-\varepsilon, \varepsilon)$ with*

$$0 = u_n(0) = u_n'(0) = \cdots = u_n^{(p)}(0)$$

for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} ||u_n^{(p+1)}||_\infty = 0$ then $\lim\limits_{n\to\infty} ||v_n||_\infty = 0$ where $v_n(t) = \frac{u_n(t)}{|t|^{p+1}}$.

PROOF. By Taylor's theorem

$$|t|^{p+1}\, v_n(t) = u_n(t) = \underbrace{\sum_{k=0}^{p} \frac{t^k}{k!} u_n^{(k)}(0)}_{=0} + \frac{t^{p+1}}{(p+1)!}\, u_n^{(p+1)}(\tau)$$

for some $\tau \in (-\varepsilon, \varepsilon)$ and therefore $||v_n||_\infty \leq \frac{1}{(p+1)!}||u_n^{(p+1)}||_\infty$. □

THEOREM 4.2.9. *For $\alpha, s \in \mathbb{R}$, the Bessel k, α-Potential*

$$\begin{aligned} \mathcal{J}_{k,\alpha}^s : \mathcal{S}_k(\mathbb{R}) &\rightarrow \mathcal{S}_k(\mathbb{R}) \\ u &\mapsto \mathcal{F}^{-1}(\psi_{k,\alpha}^s \hat{u})\,, \end{aligned}$$

where $\psi_{k,\alpha}^s(\tau) = (1+\tau^2)^{\frac{s}{2}} \left|\frac{1+\tau^2}{k^2-\tau^2}\right|^{\frac{\alpha}{2}}$, is well defined and continuous. The continuous inverse of $\mathcal{J}_{k,\alpha}^s$ is $\mathcal{J}_{k,-\alpha}^{-s}$.

PROOF. Let $u \in \mathcal{S}_k(\mathbb{R})$, then $\varrho_{k,\alpha}\hat{u} \in \mathcal{S}(\mathbb{R})$ by proposition 4.2.7 and therefore

$$\psi_{k,\alpha}^s\, \hat{u} = \psi^{s+\alpha} \underbrace{\varrho_{k,\alpha}\hat{u}}_{\in \mathcal{S}(\mathbb{R})} \in \mathcal{S}(\mathbb{R})\,,$$

where $\psi^{s+\alpha}(\tau) = (1+\tau^2)^{(s+\alpha)/2}$ denotes the traditional Bessel potential from definition 1.1.1, so $\mathcal{F}^{-1}\psi^s_{k,\alpha}\hat{u} \in \mathcal{S}(\mathbb{R})$ is well defined, and $\mathcal{J}^s_{k,\alpha}$ a linear operator $\mathcal{S}_k(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$. As additionally

$$\varrho_k\, \psi^s_{k,\alpha}\, \hat{u} = \psi^{s+\alpha} \underbrace{\varrho_{k,\alpha+\frac{1}{2}}\hat{u}}_{\in\mathcal{S}(\mathbb{R})} \in \mathcal{S}(\mathbb{R})\,,$$

we have $\mathcal{J}^s_{k,\alpha}u = \mathcal{F}^{-1}\psi^s_{k,\alpha}\hat{u} \in \mathcal{S}_k(\mathbb{R})$.

Since $\mathcal{J}^s_{k,\alpha}u$ is well defined for $k,\alpha \in \mathbb{R}$ and $u \in \mathcal{S}_k(\mathbb{R})$, so is

$$\mathcal{J}^{-s}_{k,-\alpha} \underbrace{\left(\mathcal{J}^s_{k,\alpha}u\right)}_{\in\mathcal{S}_k(\mathbb{R})} = u$$

and therefore $\left(\mathcal{J}^s_{k,\alpha}\right)^{-1} = \mathcal{J}^{-s}_{k,-\alpha}$.

Consider a sequence $(u_n) \subset \mathcal{S}_k(\mathbb{R})$ with $u_n \overset{\mathcal{S}_k(\mathbb{R})}{\longrightarrow} 0$. To prove continuity of $\mathcal{J}^s_{k,\alpha}$, we have to verify that this implies $\mathcal{J}^s_{k,\alpha}u_n \overset{\mathcal{S}_k(\mathbb{R})}{\longrightarrow} 0$. Since $\mathcal{J}^s u_n \overset{\mathcal{S}(\mathbb{R})}{\longrightarrow} 0$, it is sufficient to discuss uniform convergence of all derivatives of $(\widehat{\mathcal{J}^s_{k,\alpha}u_n})$ and $(\varrho_k\widehat{\mathcal{J}^s_{k,\alpha}u_n})$ around $\pm k$, or equivalently, uniform convergence of all derivatives of $(\varrho_{k,-\alpha}\hat{u}_n)$ and $(\varrho_{k,-\alpha-1,}\hat{u}_n)$ in fixed small neighborhoods of $\pm k$ if $-\alpha-1$ is negative at all. The derivatives are finite sums of functions which are bounded by functions as discussed in Lemma 4.2.8 for $p > \frac{\alpha+1}{2}$ and higher, and because of this they will all converge uniformly to zero. □

REMARK 4.2.10. The Bessel k,α Potential is self-adjoint with respect to the L^2 scalar product, i.e., for any $u,v \in \mathcal{S}_k(\mathbb{R})$ we have

$$(\mathcal{J}^s_{k,\alpha}u, v)_{L^2} = (u, \mathcal{J}^s_{k,\alpha}v)_{L^2}\,.$$

This is an effect of the definition as a multiplication with a real valued multiplier in Fourier space.

DEFINITION 4.2.11. $\mathcal{S}^*_k(\mathbb{R})$ denotes the dual space of continuous linear functionals on $\mathcal{S}_k(\mathbb{R})$.

LEMMA 4.2.12. *$\mathcal{S}^*_k(\mathbb{R})$ is weakly sequentially complete, i.e., a sequence $(\varphi_n) \subset \mathcal{S}^*_k(\mathbb{R})$ of linear functionals such that $(\varphi_n(u))$ converges in $\mathbb{C}$ for any $u \in \mathcal{S}_k(\mathbb{R})$ is convergent in $\mathcal{S}^*_k(\mathbb{R})$.*

Proof. Since $\mathcal{S}_k(\mathbb{R})$ is a Fréchet space, it is barreled by Baire's Theorem, and thus such sequences of continuous functionals are convergent to a continuous linear form on $\mathcal{S}_k(\mathbb{R})$, [**15**] 3 §6 Proposition 5 and Corollary. See also the proofs for $\mathcal{D}^*(\mathbb{R})$ and $\mathcal{S}^*(\mathbb{R})$ in [**32**] §4, II and §11, VII. □

Definition 4.2.13. The Bessel k,α-Potential has a natural continuous extension $\mathcal{J}^s_{k,\alpha} : \mathcal{S}^*_k(\mathbb{R}) \to \mathcal{S}^*_k(\mathbb{R})$, where

$$\langle \mathcal{J}^s_{k,\alpha}\varphi, u\rangle := \langle \varphi, \underbrace{\mathcal{J}^s_{k,\alpha}u\rangle}_{\in \mathcal{S}_k(\mathbb{R})} \quad \text{for } \varphi \in \mathcal{S}^*_k(\mathbb{R}),\ u \in \mathcal{S}_k(\mathbb{R}).$$

Remark 4.2.14. Again, this definition really is an extension because $\mathcal{J}^s_{k,\alpha}$ is self-adjoint and its Fourier multiplier is real: For $\varphi \in \mathcal{S}^*_k(\mathbb{R}) \cap \mathcal{S}_k(\mathbb{R})$ and $u \in \mathcal{S}_k(\mathbb{R})$ we identify the dual pairing with the $L^2(\mathbb{R})$ scalar product,

$$\langle \mathcal{J}^s_{k,\alpha}\varphi, u\rangle = (\mathcal{J}^s_{k,\alpha}\varphi, \bar{u})_{L^2(\mathbb{R})} = (\varphi, \mathcal{J}^s_{k,\alpha}\bar{u})_{L^2(\mathbb{R})} = \langle \varphi, \mathcal{J}^s_{k,\alpha}u\rangle$$

and this coincides with the above definition.

4.3. Weighted Sobolev Spaces

Now weighted Sobolev spaces may be defined using the modified Bessel k,α-Potential:

Definition 4.3.1. For $s \in \mathbb{R}$, $\alpha \in \mathbb{R}$ and $k > 0$, let

$$H^s_{k,\alpha}(\mathbb{R}) := \{\varphi \in \mathcal{S}^*_k(\mathbb{R}) :\ \mathcal{J}^s_{k,\alpha}\,\varphi \in L^2(\mathbb{R})\}\,.$$

Remark 4.3.2. We equip this space with the inner product

$$(\varphi, \psi)_{H^s_{k,\alpha}(\mathbb{R})} := (\mathcal{J}^s_{k,\alpha}\varphi, \mathcal{J}^s_{k,\alpha}\psi)_{L^2(\mathbb{R})}$$

and the induced norm

$$||\varphi||^2_{H^s_{k,\alpha}(\mathbb{R})} := (\varphi, \varphi)_{H^s_{k,\alpha}(\mathbb{R})}\,.$$

The necessary properties of the inner product directly follow from the inner product in $L^2(\mathbb{R})$. Since $\mathcal{J}^s_{k,0} = \mathcal{J}^s$, the spaces $H^s_{k,0}(\mathbb{R})$ and $H^s(\mathbb{R})$ coincide in the sense which will be discussed in Lemma 4.3.4. Note that an increasing α denotes additional regularity.

THEOREM 4.3.3. *$H^s_{k,\alpha}(\mathbb{R})$ is a Hilbert space.*

PROOF. It remains to prove completeness: Consider a Cauchy sequence $(\varphi_n) \subset H^s_{k,\alpha}(\mathbb{R})$. Since $L^2(\mathbb{R})$ is complete, the $L^2(\mathbb{R})$ Cauchy sequence $(\mathcal{J}^s_{k,\alpha}\varphi_n)$ converges to a function $y \in L^2(\mathbb{R})$. Let $u \in \mathcal{S}_k(\mathbb{R})$ be an arbitrary test function, then the sequence $\Big((\mathcal{J}^s_{k,\alpha}\varphi_n, u)_{L^2(\mathbb{R})}\Big)$, which is the same complex valued sequence as $\Big((\varphi_n, \mathcal{J}^s_{k,\alpha}u)_{L^2(\mathbb{R})}\Big)$, will converge to $(y, u)_{L^2}$. Since u was arbitrary and $\mathcal{J}^s_{k,\alpha}u \in \mathcal{S}_k(\mathbb{R})$, the sequence (φ_n) will converge to $\varphi \in \mathcal{S}^*_k(\mathbb{R})$ by Lemma 4.2.12, for which for all $u \in \mathcal{S}_k(\mathbb{R})$

$$(\varphi, \mathcal{J}^s_{k,\alpha}u)_{L^2(\mathbb{R})} = (\mathcal{J}^s_{k,\alpha}\varphi, u)_{L^2(\mathbb{R})} = (y, u)_{L^2(\mathbb{R})}\,.$$

Therefore, $y = \mathcal{J}^s_{k,\alpha}\varphi$ with respect to the test function space, and thus $\varphi \in H^s_{k,\alpha}(\mathbb{R})$, as $y = \mathcal{J}^s_{k.\alpha}\varphi \in L^2(\mathbb{R})$. □

LEMMA 4.3.4. *If $\alpha > 0$, $s \in \mathbb{R}$,*

$$H^s_{k,\alpha}(\mathbb{R}) \subset H^s(\mathbb{R})$$

and $||u||_{H^s(\mathbb{R})} \leq \max\{1, k^\alpha\}\, ||u||_{H^s_{k,\alpha}(\mathbb{R})}$ for all $u \in H^s_{k,\alpha}(\mathbb{R})$.

PROOF. This relationship is not as obvious as it seems, because $\mathcal{S}^*_k(\mathbb{R}) \supset \mathcal{S}^*(\mathbb{R})$. Thus a distribution $u \in \mathcal{S}^*_k(\mathbb{R})$ itself may not have a meaning in $\mathcal{S}^*(\mathbb{R})$. But as $\mathcal{J}^s_{k,\alpha}u \in L^2(\mathbb{R})$ for $u \in H^s_{k,\alpha}(\mathbb{R})$, we may define

$$v := \mathcal{J}^0_{k,-\alpha}\underbrace{\mathcal{J}^s_{k,\alpha}u}_{\in L^2(\mathbb{R})} = \mathcal{F}^{-1}\psi^0_{k,-\alpha}\widehat{\mathcal{J}^s_{k,\alpha}u} \in L^2(\mathbb{R})\,,$$

since the Fourier multiplier $\psi^0_{k,-\alpha}(\tau) = \left|\frac{k^2-\tau^2}{1+\tau^2}\right|^{\alpha/2}$ is an $L^\infty(\mathbb{R})$ function for $\alpha \geq 0$ with $||\psi^0_{k,-\alpha}||_{L^\infty(\mathbb{R})} = \max\{1, k^\alpha\}$.

This defines a continuous linear functional $\vartheta : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ by

$$\vartheta(f) := (\underbrace{\mathcal{J}^{-s}f}_{\in\mathcal{S}(\mathbb{R})}, \bar{v})_{L^2(\mathbb{R})}\,, \quad f \in \mathcal{S}(\mathbb{R})\,,$$

which coincides with u as functionals on $\mathcal{S}_k(\mathbb{R})$: For $f \in \mathcal{S}_k(\mathbb{R})$,

$$\vartheta(f) = (\mathcal{J}^{-s}f, \bar{v})_{L^2(\mathbb{R})} = \langle \mathcal{J}^{-s}f, \mathcal{J}^s u\rangle_{\mathcal{S}_k(\mathbb{R})} = \langle f, u\rangle_{\mathcal{S}_k(\mathbb{R})} = u(f)\,.$$

This extends u to a functional $u \in \mathcal{S}^*(\mathbb{R})$, and since $\mathcal{J}^s u \in L^2(\mathbb{R})$, we have $u \in H^s(\mathbb{R})$. This diagram illustrates the use of the different potentials in the case of $\alpha \geq 0$:

$$\begin{array}{ccc} H^s_{k,\alpha}(\mathbb{R}) & \overset{\mathcal{J}^s}{\rightarrow} & L^2(\mathbb{R}) \\ & \mathcal{J}^s_{k,\alpha}\searrow & \downarrow \mathcal{J}^0_{k,\alpha} \\ & & L^2(\mathbb{R}) \end{array}$$

Finally,

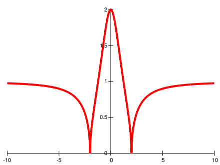
$$\begin{aligned} ||u||_{H^s(\mathbb{R})} &= ||\mathcal{J}^s u||_{L^2(\mathbb{R})} = ||\psi^s \hat{u}||_{L^2(\mathbb{R})} = ||\psi^0_{k,-\alpha}\psi^s_{k,\alpha}\hat{u}||_{L^2(\mathbb{R})} \\ &\leq ||\psi^0_{k,-\alpha}||_{L^\infty(\mathbb{R})}\, ||\mathcal{J}^s_{k,\alpha} u||_{L^2(\mathbb{R})} \\ &\overset{\alpha\geq 0}{=} \max\{1, k^\alpha\}\, ||u||_{H^s_{k,\alpha}(\mathbb{R})}\,. \end{aligned}$$

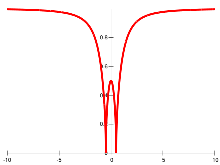
□

For an impression of the weight functions $\psi^s_{k,\alpha}$ see the following plots. The parameter s specifies the growth to infinity in case of $s > 0$ and α sets the order of the poles for $\alpha > 0$ or roots for $\alpha < 0$ at $\pm k$. The value of k sets the position of the roots or poles, and it does also modify the graph as a whole, but these changes are negligible.

EXAMPLE 4.3.5. Illustrative plots of $\psi^0_{2,-1}$ and $\psi^0_{1/2,-1}$:

$$\psi^0_{2,-1}(\tau) = \left|\frac{2^2-\tau^2}{1+\tau^2}\right|^{1/2} \qquad \psi^0_{1/2,-1}(\tau) = \left|\frac{(1/2)^2-\tau^2}{1+\tau^2}\right|^{1/2}$$





LEMMA 4.3.6. *If $\alpha > \beta$, then*

$$H^s_{k,\alpha}(\mathbb{R}) \subset H^s_{k,\beta}(\mathbb{R})$$

and $||u||_{H^s_{k,\beta}(\mathbb{R})} \leq \max\{1, k^{\alpha-\beta}\}\, ||u||_{H^s_{k,\alpha}(\mathbb{R})}$ for all $u \in H^s_{k,\alpha}(\mathbb{R})$. This includes $H^s(\mathbb{R}) \subset H^s_{k,\beta}(\mathbb{R})$ if $\beta < 0$ and

$$||u||_{H^s_{k,\beta}(\mathbb{R})} \leq \max\{1, k^{-\beta}\}\, ||u||_{H^s(\mathbb{R})}$$

for all $u \in H^s(\mathbb{R})$.

PROOF. Since $H^s(\mathbb{R}) = H^s_{k,0}(\mathbb{R})$, it remains to prove the second norm inequality: For $u \in H^s_{k,\alpha}(\mathbb{R})$,

$$\begin{aligned} & ||u||_{H^s_{k,\beta}(\mathbb{R})} = ||\mathcal{J}^s_{k,\beta} u||_{L^2(\mathbb{R})} \\ = \; & ||\psi^s_{k,\beta}\hat{u}||_{L^2(\mathbb{R})} = ||\psi^0_{k,\beta-\alpha}\psi^s_{k,\alpha}\hat{u}||_{L^2(\mathbb{R})} \\ \leq \; & ||\psi^0_{k,\beta-\alpha}||_{L^\infty(\mathbb{R})}\, ||\mathcal{J}^s_{k,\alpha} u||_{L^2(\mathbb{R})} \overset{\beta-\alpha\leq 0}{=} \max\{1, k^{\alpha-\beta}\}\, ||u||_{H^s_{k,\alpha}(\mathbb{R})} \,. \end{aligned}$$

□

Finally, these spaces characterize the mapping properties of the Dirichlet-to-Neumann map:

THEOREM 4.3.7. *The Dirichlet-to-Neumann map is a bounded linear operator*

$$\Lambda : H^s_{k,\alpha}(\mathbb{R}) \to H^{s-1}_{k,\alpha+1}(\mathbb{R}) \,, \quad s, \alpha \in \mathbb{R}$$

and $||\Lambda\varphi||_{H^{s-1}_{k,\alpha+1}(\mathbb{R})} = ||\varphi||_{H^s_{k,\alpha}(\mathbb{R})}$ *for all* $\varphi \in H^s_{k,\alpha}(\mathbb{R})$.

PROOF. Let $\varphi \in H^s_\alpha(\mathbb{R})$, then

$$|\widehat{\Lambda\varphi}(\tau)| = |\sqrt{\tau^2 - k^2}\hat{\varphi}(\tau)| = \underbrace{|\tau^2 - k^2|^{1/2}}_{=\varrho_{k,1}(\tau)} |\hat{\varphi}(\tau)| \,,$$

and since $\varrho_{k,1} = \psi^1_{k,-1}$, we have

$$\begin{aligned} ||\Lambda\varphi||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R})} &= ||\mathcal{J}^{s-1}_{k,\alpha+1}\Lambda\varphi||^2_{L^2(\mathbb{R})} = ||\psi^{s-1}_{k,\alpha+1}\, \psi^1_{k,-1}\, \hat{\varphi}||^2_{L^2(\mathbb{R})} \\ &= ||\psi^s_{k,\alpha}\hat{\varphi}||^2_{L^2(\mathbb{R})} = ||\mathcal{J}^s_{k,\alpha}\varphi||^2_{L^2(\mathbb{R})} = ||\varphi||^2_{H^s_{k,\alpha}(\mathbb{R})} \,. \end{aligned}$$

□

And the mapping is onto, thus we may now give the formal definition of the Neumann-to-Dirichlet map a meaning:

THEOREM 4.3.8. *The Neumann-to-Dirichlet operator is a bounded linear operator*

$$\Lambda^{-1} : H^{s-1}_{k,\alpha+1}(\mathbb{R}) \to H^s_{k,\alpha}(\mathbb{R}) \,, \quad s, \alpha \in \mathbb{R} \,,$$

and $||\Lambda^{-1}\varphi||_{H^s_{k,\alpha}(\mathbb{R})} = ||\varphi||_{H^{s-1}_{k,\alpha+1}(\mathbb{R})}$ *for all* $\varphi \in H^{s-1}_{k,\alpha+1}(\mathbb{R})$.

PROOF. Let $\varphi \in H^{s-1}_{k,\alpha+1}(\mathbb{R})$, then

$$|\widehat{\Lambda^{-1}\varphi}(\tau)| = |\frac{1}{\sqrt{\tau^2 - k^2}}\hat{\varphi}(\tau)| = \underbrace{|\tau^2 - k^2|^{-1/2}}_{=\varrho_{k,-1}(\tau)} |\hat{\varphi}(\tau)|$$

and since $\varrho_{k,-1} = \psi^{-1}_{k,1}$, we have

$$\begin{aligned} ||\Lambda^{-1}\varphi||^2_{H^s_{k,\alpha}(\mathbb{R})} &= ||\mathcal{J}^s_{k,\alpha}\Lambda^{-1}\varphi||^2_{L^2(\mathbb{R})} = ||\psi^s_{k,\alpha}\,\psi^{-1}_{k,1}\,\hat{\varphi}||^2_{L^2(\mathbb{R})} \\ &= ||\psi^{s-1}_{k,\alpha+1}\hat{\varphi}||^2_{L^2(\mathbb{R})} = ||\mathcal{J}^{s-1}_{k,\alpha+1}\varphi||^2_{L^2(\mathbb{R})} \\ &= ||\varphi||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R})}\,. \end{aligned}$$

□

4.4. Extension to Two Dimensions

So far we have established suitable function spaces for discussion of a two-dimensional coupled TE, TM mode system, that is, the functions on the interface were functions of one variable. In the full three-dimensional Maxwell setting, the traces on the interface will be functions of two variables, so we will have to extend the concept above spaces to $\mathbb{R}^2$. We will see that this does not exhibit additional challenges: The general approach for extension will be to apply polar coordinates, and the same method could be used to derive spaces for higher dimensions by spherical coordinates. The following results for functions defined on $\mathbb{R}^2$ follow the same order as in the previous section and only the differences in the proofs compared to their earlier counterparts are presented in detail.

DEFINITION 4.4.1. For $k > 0$ and $\varrho_k(\tau) := |k^2 - |\tau|^2|^{-1/2}$,

$$\mathcal{S}_k(\mathbb{R}^2) := \{u \in \mathcal{S}(\mathbb{R}^2) : \ \varrho_k\hat{u} \in \mathcal{S}(\mathbb{R}^2)\}$$

where $\hat{u}$ denotes the two-dimensional Fourier transform of u. The topology of $\mathcal{S}_k(\mathbb{R}^2)$ is defined by a countable family of semi-norms

$$|u|_{\mathcal{S}_k,\alpha,\beta} := |\hat{u}|_{\mathcal{S},\alpha,\beta} + |\varrho_k\hat{u}|_{\mathcal{S},\alpha,\beta}\,.$$

This means a sequence (u_n) of functions $u_n \in \mathcal{S}_k(\mathbb{R}^2)$ converges to 0 in $\mathcal{S}_k(\mathbb{R}^2)$ if both sequences $(\hat{u}_n)$ and $(\varrho_k \hat{u}_n)$ converge to 0 in $\mathcal{S}(\mathbb{R}^2)$. This also defines $u_n \to u$ as $(u_n - u) \to 0$ in $\mathcal{S}_k(\mathbb{R}^2)$.

LEMMA 4.4.2. *$\mathcal{S}_k(\mathbb{R}^2)$ is a Fréchet space.*

PROOF. This is true due to the same reasoning as in the proof for Lemma 4.2.4. □

PROPOSITION 4.4.3.

$$\begin{aligned} \text{(i)}\quad \mathcal{S}_k(\mathbb{R}^2) &= \{u \in \mathcal{S}(\mathbb{R}^2):\ |\hat{u}^{(n)}(\tau)| = 0 \textit{ for all } |\tau| = k,\, n \in \mathbb{N}_0\} \\ \text{(ii)}\quad \mathcal{S}_k(\mathbb{R}^2) &= \{u \in \mathcal{S}(\mathbb{R}^2):\ \varrho_{k,\alpha}\hat{u} \in \mathcal{S}(\mathbb{R}^2) \textit{ for all } \alpha \in \mathbb{R}\}, \\ & \qquad \textit{where } \varrho_{k,\alpha}(\tau) := |k^2 - |\tau|^2|^{\alpha/2} \end{aligned}$$

PROOF. *(ii) ⇒ Definition:* As before in the proof of Proposition 4.2.7, the condition in (ii) includes the condition in the Definition 4.4.1 for $\alpha = -1$, as $\varrho_{k,-1} = \varrho_k$.

Definition ⇒ (i): Now, let $u \in \mathcal{S}(\mathbb{R}^2)$ and $v \in \mathcal{S}(\mathbb{R}^2)$ with $v = \varrho_k u$. Transformation to polar coordinates yields

$$v(r,\varphi) = |k^2 - r^2|^{-1/2}\, u(r,\varphi)\,.$$

Lemma 4.2.5 applied on shifted restrictions

$$u_\varphi(r+k) := u(r+k,\varphi) \quad \text{and} \quad v_\varphi(r+k) := v(r+k,\varphi)$$

for $r \in (-\varepsilon, \varepsilon)$ and fixed φ proves

$$u(k,\varphi) = v(k,\varphi) = 0 \;\text{ and }\; \left(\frac{\partial}{\partial r}\right)^n u(k,\varphi) = \left(\frac{\partial}{\partial r}\right)^n v(k,\varphi) = 0$$

for all $n \in \mathbb{N}$. Now since this is true for all φ and $n \in \mathbb{N}_0$, the functions $\left(\frac{\partial}{\partial r}\right)^n u(k,\varphi)$ and $\left(\frac{\partial}{\partial r}\right)^n v(k,\varphi)$ are constant in φ, and all $\left(\frac{\partial}{\partial r}\right)^n \left(\frac{\partial}{\partial \varphi}\right)^m u(k,\varphi) \equiv 0$ for $m, n \in \mathbb{N}_0$, thus the condition in (i) is fulfilled.

(i) ⇒ (ii): Assume the conditions in (i) are satisfied for u, that is $u \in \mathcal{S}(\mathbb{R}^2)$ with $|u^{(n)}(\tau)| = 0$ for all $|\tau| = k$ and $n \in \mathbb{N}_0$. As a composition of smooth functions $v := \varrho_{k,\alpha} u$, the function v is

$$v \in C^\infty(\mathbb{R}^2 \backslash \{\tau \in \mathbb{R}^2 : |\tau| = k\})$$

for all $\alpha \in \mathbb{R}$. In polar coordinates, we see by Lemma 4.2.6 that for fixed φ the function $v_\varphi(r) := v(r,\varphi)$ is in $C^\infty(\mathbb{R}_{>0})$ and $v_\varphi(k) = 0$ for all φ. Therefore, $v(r,\varphi)$ and all partial radial derivatives of v are zero and constant for $r = k$, thus all derivatives exist and are continuous also at $|\tau| = k$ and therefore $v \in C^\infty(\mathbb{R}^2)$. Again, as $\varrho_{k,\alpha}$ is at most slowly increasing, $v(\tau)$ is rapidly decreasing for $|\tau| \to \infty$ as $u(\tau)$ and therefore $v \in \mathcal{S}(\mathbb{R}^2)$. □

THEOREM 4.4.4. *Let $\alpha, s \in \mathbb{R}$, $k > 0$. Then the Bessel k,α-Potential for scalar functions on $\mathbb{R}^2$*

$$\begin{aligned} \mathcal{J}^s_{k,\alpha} : \mathcal{S}_k(\mathbb{R}^2) &\rightarrow \mathcal{S}_k(\mathbb{R}^2) \\ u &\mapsto \mathcal{F}^{-1}(\psi^s_{k,\alpha}\hat{u}) \,, \end{aligned}$$

where $\psi^s_{k,\alpha}(\tau) := (1+|\tau|^2)^{\frac{s}{2}} \left| \frac{1+|\tau|^2}{k^2-|\tau|^2} \right|^{\frac{\alpha}{2}}$, is well defined and continuous. Its continuous inverse is $\mathcal{J}^{-s}_{k,-\alpha}$.

PROOF. Analogous to the proof of Theorem 4.2.9 but for $\mathcal{S}_k(\mathbb{R}^2)$ and use of Lemma 4.2.8 with polar coordinates as in the proof of Proposition 4.4.3. □

DEFINITION 4.4.5. The Bessel k,α-Potential for scalar functions on $\mathbb{R}^2$ has a natural continuous extension $\mathcal{J}^s_{k,\alpha} : \mathcal{S}^*_k(\mathbb{R}^2) \to \mathcal{S}^*_k(\mathbb{R}^2)$.

By this we may define the corresponding weighted Sobolev spaces:

THEOREM 4.4.6. *The spaces*

$$H^s_{k,\alpha}(\mathbb{R}^2) := \{u \in \mathcal{S}^*_k(\mathbb{R}^2) : \mathcal{J}^s_{k,\alpha} u \in L^2(\mathbb{R})\}$$

for $s \in \mathbb{R}$ and $k > 0$ equipped with the inner product

$$(u,v)_{H^s_{k,\alpha}(\mathbb{R}^2)} := (\mathcal{J}^s_{k,\alpha}u, \mathcal{J}^s_{k,\alpha}v)_{L^2(\mathbb{R})}$$

and the induced norms

$$||u||^2_{H^s_{k,\alpha}(\mathbb{R}^2)} := (u,u)_{H^s_{k,\alpha}(\mathbb{R}^2)}$$

are Hilbert spaces.

CHAPTER 5

Full Three-Dimensional Maxwell System

We will now discuss the scattering problem in the two-layered space in $\mathbb{R}^3 = \mathbb{R}^3_+ \cup \Gamma \cup \mathbb{R}^3_-$ with the interface $\Gamma = \mathbb{R}^2 \times \{0\}$ and half-spaces $\mathbb{R}^3_\pm := \{(x_1, x_2, x_3) \in \mathbb{R}^2 : x_3 \gtrless 0\}$, we seek the solution to the time harmonic Maxwell's equations for the electric or magnetic field intensities with wave numbers $k^2 = k_+^2 > 0$ for $x_2 > 0$ and $k_-^2 \in \mathbb{C}$, $\arg k_-^2 \in (0, \frac{\pi}{2})$ for $x_2 < 0$, the transmission condition at the interface, and boundary condition on a bounded perfect electrically conducting obstacle $\Omega \subset \mathbb{R}^2_-$ within the lower half-space, and a radiation condition. As before we split the total field into an incoming field, which itself is a solution to the Helmholtz equations in the two-layered space without the scatterer Ω, and the scattering field.

First the classical problems for the electric and magnetic field intensities are defined on the whole space, then integral equalities and functions and trace spaces suitable for the corresponding weak formulation are presented. This leads to the definition of the Calderon operator and its mapping properties, with which it will be possible to reduce the scenario to the lower half-space using weak formulations. By virtue of the Lax-Milgram Lemma we will be able to prove unique solvability of the weak problems using specific singular weighted Sobolev spaces. For these spaces we have equivalence of the weak and classical formulations under the additional condition of integrability which is no real restriction as the original problem was shown to be uniquely solvable by Delbary et al. [**11**].

5.1. Classical Problem Definitions

As outlined in Chapter 2, we may express the problem in terms of both the electric and the magnetic field intensity, with differing boundary and transmission conditions. First, we formulate the definition of the classical boundary value problem for the electric field intensity:

PROBLEM 5.1.1. (EBVP1) Given a bounded scatterer $\Omega \subset \mathbb{R}^3_-$ with boundary $\partial\Omega \in C^2$, an analytic incoming field E^i (at least defined on some region G in the lower half-space enclosing the scatterer, i.e., $\bar{\Omega} \subset G \subset \mathbb{R}^3_-$) the problem is to find the scattering field

(i) $E \in C^2(\mathbb{R}^3 \backslash (\Gamma \cup \bar{\Omega}), \mathbb{C}^3) \cap BC(\mathbb{R}^3_\pm \backslash \Omega, \mathbb{C}^3)$ such that
(ii) $\operatorname{curl}\operatorname{curl} E - k_\pm^2 E = 0$ in $\mathbb{R}^3_\pm$,
(iii) $n \times E|_{\partial\Omega} = -n \times E^i|_{\partial\Omega}$,
(iv) $[n \times E]_\Gamma = [n \times \operatorname{curl} E]_\Gamma = 0$, and
(v) components E_1, E_2 of $E = (E_1, E_2, E_3)^\top$ satisfy the UPRC, E_3 the Sommerfeld radiation condition 2.4.3 in $\mathbb{R}^3_+$, and the exponential decay condition 2.4.2 in $\mathbb{R}^3_-$.

Secondly, we add the definition for the magnetic field intensity:

PROBLEM 5.1.2. (HBVP1) Given a bounded scatterer $\Omega \subset \mathbb{R}^3_-$ with boundary $\partial\Omega \in C^2$, an analytic incoming field H^i (at least defined on some region G in the lower half-space around the scatterer, i.e., $\bar{\Omega} \subset G \subset \mathbb{R}^3_-$) the problem is to find the scattering field

(i) $H \in C^2(\mathbb{R}^3 \backslash (\Gamma \cup \bar{\Omega}), \mathbb{C}^3) \cap BC(\mathbb{R}^3_\pm \backslash \Omega, \mathbb{C}^3)$ such that
(ii) $\operatorname{curl}\operatorname{curl} H - k_\pm^2 H = 0$ in $\mathbb{R}^3_\pm$,
(iii) $n \times \operatorname{curl} H|_{\partial\Omega} = -n \times \operatorname{curl} H^i|_{\partial\Omega}$,
(iv) $[n \times H]_\Gamma = [k_\mp^2 n \times \operatorname{curl} H]_\Gamma = 0$, and
(v) components H_1, H_2 of $H = (H_1, H_2, H_3)^\top$ satisfy the UPRC, H_3 the Sommerfeld radiation condition 2.4.3 in $\mathbb{R}^3_+$, and the exponential decay condition 2.4.2 in $\mathbb{R}^3_-$.

REMARK 5.1.3. The commonly used Silver-Müller radiation condition is equivalent to the Sommerfeld radiation condition of the Cartesian components as proven in [**6**]. In general, solutions satisfying the Sommerfeld radiation condition will also satisfy the UPRC, but the converse is not necessarily the case. Therefore, above radiation conditions for the upper layer are more general than the traditional Silver-Müller radiation condition. The radiation condition of exponential decay for the lower layer

$$|E(x)| + |\operatorname{curl} E(x)| \leq M \exp\left(-\operatorname{Im} k_- \, |x|\right)$$

for some $M > 0$ and correspondingly for H, follows [**11**] to ensure existence and uniqueness of the classical problems.

As in the two dimensional case, we will be able to transform these problems to weak formulations using Green's formula, which is outlined below.

5.2. Integral Equalities and Function Spaces

In this section, integral equalities and function spaces most suitable for Maxwell's equations are established. The following notation for vectors, which will also be used for functions, clarifies the switch between $\mathbb{R}^2$ representing Γ and $\mathbb{R}^3$ when it is needed:

$$\text{If } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$$\text{then } \tilde{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \underset{\sim}{y} = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix}$$

Please note, that in earlier chapters we already used a similar notation of a tilde above a symbol in completely different contexts and meanings. The reader should not be confused by this minor inconvenience, and in this chapter this notation will consistently only have above meaning.

To derive suitable spaces for the variational formulation we first need to extend the following well known Theorems to unbounded interfaces:

THEOREM 5.2.1. *Let $\Omega \subset \mathbb{R}^3$ a bounded domain with Lipschitz continuous boundary $\partial\Omega$ with unit outward normal n. Then the Divergence Theorem holds for functions $f \in C^1(\overline{\Omega}, \mathbb{C}^3)$:*

$$\int\limits_{\Omega} \operatorname{div} f \, dx = \int\limits_{\partial\Omega} f \cdot n \, dA$$

Choosing $f = \psi u$ and the product rule for one, and using the identities $\operatorname{div}(u \times \varphi) = \varphi \cdot \operatorname{curl} u - u \cdot \operatorname{curl} \varphi$, as well as $(u \times \varphi) \cdot n = (n \times u) \cdot \varphi$, and the substitution $f = u \times \varphi$ for the other, we get, cf. [**14**]:

THEOREM 5.2.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$ with unit outward normal n. Then, for functions $u, v, \varphi \in C^1(\overline{\Omega}, \mathbb{C}^3)$ and $\psi \in C^1(\bar{\Omega})$, the following equalities hold by the Divergence Theorem:*

$$\int\limits_{\Omega} (\psi \operatorname{div} u + u \cdot \operatorname{grad} \psi) \, dx = \int\limits_{\partial\Omega} (n \cdot u) \, \psi \, dA \,, \tag{5.1}$$

$$\int\limits_{\Omega} (\varphi \cdot \operatorname{curl} v - v \cdot \operatorname{curl} \varphi) \, dx = \int\limits_{\partial\Omega} (n \times v) \cdot \varphi \, dA \,. \tag{5.2}$$

This motivates the definition of function spaces suitable for Maxwell's Equations:

DEFINITION 5.2.3. For $\Omega \subset \mathbb{R}^3$ not necessarily bounded, we define

$$\begin{aligned} H(\operatorname{div}, \Omega) &:= \left\{ v \in L^2(\Omega, \mathbb{C}^3) \,:\, \operatorname{div} v \in L^2(\Omega) \right\} , \\ H(\operatorname{curl}, \Omega) &:= \left\{ v \in L^2(\Omega, \mathbb{C}^3) \,:\, \operatorname{curl} v \in L^2(\Omega, \mathbb{C}^3) \right\} . \end{aligned}$$

Indeed as shown in [**26**], Equations (5.1) and (5.2) hold for functions $u \in H(\operatorname{div}, \Omega)$, $v, \varphi \in H(\operatorname{curl}, \Omega)$, and $\psi \in H^1(\Omega)$, when Ω is bounded with a Lipschitz boundary. To discuss the validity of above integral identities on the unbounded domains $\mathbb{R}^3_\pm$ with boundary Γ we will make use of the following density argument.

LEMMA 5.2.4. *$C^\infty(\overline{\mathbb{R}^3_\pm}, \mathbb{C}^3) \cap H(\operatorname{div}, \mathbb{R}^3_\pm)$ is dense in $H(\operatorname{div}, \mathbb{R}^3_\pm)$, and $C^\infty(\overline{\mathbb{R}^3_\pm}, \mathbb{C}^3) \cap H(\operatorname{curl}, \mathbb{R}^3_\pm)$ is dense in $H(\operatorname{curl}, \mathbb{R}^3_\pm)$.*

PROOF. Both assertions for unbounded domains are justified by the same technique of diagonal sequences and corresponding theorems on bounded domains, for example from [**26**]. Here, only the proof for $H(\mathrm{curl}, \mathbb{R}^3_\pm)$ is presented for brevity. Let $\Omega_R = B_R(0) \cap \mathbb{R}^3_\pm$ and u_R the restriction of u to Ω_R and continued by 0 outside Ω_R. Then $||u_R - u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)} \to 0$ for $R \to \infty$ and there exists a sequence (R_k) such that $||u_{R_k} - u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)} < \frac{1}{2k}$. Since Ω_{R_k} is bounded for all k, there exists a function $\varphi_k \in C^\infty(\overline{\mathbb{R}^3_\pm}, \mathbb{C}^3)$ such that $||\varphi_k - u_{R_k}||_{H(\mathrm{curl},\mathbb{R}^3_\pm)} < \frac{1}{2k}$ by standard density results on bounded domains and $\varphi_k \in H(\mathrm{curl}, \mathbb{R}^3_\pm)$. This defines a sequence (φ_k) in $C^\infty(\overline{\mathbb{R}^3_\pm}, \mathbb{C}^3)$ which is convergent to u with respect to the $H(\mathrm{curl}, \mathbb{R}^3_\pm)$ norm:

$$\begin{aligned} ||\varphi_k - u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)} &\leq ||\varphi_k - u_{R_k}||_{H(\mathrm{curl},\mathbb{R}^3_\pm)} + ||u_{R_k} - u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)} \\ &\leq \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \quad \stackrel{k\to\infty}{\longrightarrow} \quad 0\,. \end{aligned}$$

□

Next, we need to discuss the definition of the four different traces showing up on the right hand sides of the identities:

DEFINITION 5.2.5. Let $n = (0, 0, \pm 1)^\top$ denote the unit outward normal to Γ, depending on the context. Then the trace operator onto Γ is denoted by

$$\gamma_0 u := u|_\Gamma\,,$$

the surface traces are

$$\begin{aligned} \pi_\Gamma u &:= n \times (u \times n)\,|_\Gamma\,, \\ \gamma_t u &:= n \times u|_\Gamma\,, \end{aligned}$$

and the normal component trace is expressed as

$$\gamma_n u := n \cdot u|_\Gamma\,.$$

Of course, by adaption of Trace Theorem 1.2.1 to $\mathbb{R}^3$, the trace γ_0 is a bounded operator $H^1(\mathbb{R}^3) \to H^{1/2}(\mathbb{R}^2 \times \{0\})$.

Due to the special form of Γ the traces will often be discussed as functions on $\mathbb{R}^2$ instead of $\mathbb{R}^2 \times \{0\}$, making use of the notations introduced before. Note that they may implicitly reflect above traces as, such as $\widetilde{\pi_\Gamma u} = \tilde{u}|_\Gamma$ as well as $\widetilde{\gamma_0 u} = \tilde{u}|_\Gamma$ and $\widetilde{\gamma_t u} = \widetilde{n \times u}|_\Gamma$.

To state the mapping properties of the other traces, we need to define the surface divergence, denoted by Div, and the surface curl, denoted by Curl, following [**26, 6**]:

DEFINITION 5.2.6. Let S denote a C^2 surface in $\mathbb{R}^3$ parametrized by $\Psi(v)$ and let g denote the determinant of the first fundamental matrix defined by $g_{ij} := \frac{\partial\Psi}{\partial v_i} \cdot \frac{\partial\Psi}{\partial v_j}$. Then, for a smooth tangential field u decomposed into $u = a_1 \frac{\partial\Psi}{\partial v_1} + a_2 \frac{\partial\Psi}{\partial v_2}$, the surface divergence and curl are defined as

$$\begin{aligned} \operatorname{Div} u &= \frac{1}{\sqrt{g}} \left(\frac{\partial(\sqrt{g}a_1)}{\partial v_1} + \frac{\partial(\sqrt{g}a_2)}{\partial v_2} \right) , \\ \operatorname{Curl} u &= \frac{1}{\sqrt{g}} \left(\frac{\partial(\sqrt{g}a_2)}{\partial v_1} - \frac{\partial(\sqrt{g}a_1)}{\partial v_2} \right) . \end{aligned}$$

As in the regarded setup $\Psi(v) = \underset{\sim}{v}$ and therefore $g = 1$, the following definition on Γ will suffice, defined for vector fields in $\mathbb{R}^3$, or on $\mathbb{R}^2$.

DEFINITION 5.2.7. For $u \in \mathcal{D}(\mathbb{R}^2, \mathbb{C}^2)$ or $u \in \mathcal{D}(\mathbb{R}^3, \mathbb{C}^3)$, the surface curl and divergence on Γ are defined as

$$\operatorname{Div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} , \ \operatorname{Curl} u = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} ,$$

and are extended continuously by density arguments as operators on Sobolev spaces $H^s(\mathbb{R}^2, \mathbb{C}^2)$ and $H^s(\mathbb{R}^3, \mathbb{C}^3)$, $s \in \mathbb{R}$.

The following spaces will aid to define the ranges of the traces:

DEFINITION 5.2.8. For $s \in \mathbb{R}$ let

$$\begin{aligned} H^s(\mathrm{Div};\Gamma) &:= \left\{ v \in H^s(\Gamma, \mathbb{C}^3) \ : \ n \cdot v = 0, \ \operatorname{Div} v \in H^s(\Gamma) \right\} , \\ H^s(\mathrm{Curl};\Gamma) &:= \left\{ v \in H^s(\Gamma, \mathbb{C}^3) \ : \ n \cdot v = 0, \ \operatorname{Curl} v \in H^s(\Gamma) \right\} . \end{aligned}$$

Now we can state the first integral equality on the unbounded sets $\mathbb{R}^3_\pm$ and the mapping properties of the γ_n trace operator:

LEMMA 5.2.9. *Let* $u \in H(\mathrm{div}, \mathbb{R}^3_\pm)$, $\varphi \in H^1(\mathbb{R}^3{}_\pm)$ *and* $n = (0, 0, \mp 1)^\top$, *then*

$$\int_{\mathbb{R}^3_\pm} (u \cdot \mathrm{grad}\, \varphi + \varphi \, \mathrm{div}\, u) \; dx = \int_\Gamma (n \cdot u) \, \varphi \, dA \,, \tag{5.3}$$

with a bounded dual pairing of the traces on the right hand side, and the trace

$$\begin{aligned} \gamma_n \; : \; H(\mathrm{div}, \mathbb{R}^3_\pm) \quad &\rightarrow \quad H^{-1/2}(\Gamma)\,, \\ u \quad &\mapsto \quad n \cdot u|_\Gamma \end{aligned}$$

is bounded.

PROOF. By the above density argument, and as for the corresponding density in $H^1(\mathbb{R}^3_\pm)$, it is sufficient to prove the assertion under the additional assumption of $u, \varphi \in C^\infty(\mathbb{R}^3_\pm, \mathbb{C}^3)$. Also note that the left hand side of the equation is bound by the product of the $H(\mathrm{div}, \mathbb{R}^3_\pm)$ and $H^1(\mathbb{R}^3_\pm)$ norms of u and φ.

Let $\Omega_R = B_R(0) \cap \mathbb{R}^3_\pm$ and $\Gamma_{0,R} = B_R(0) \cap \Gamma_0$, then

$$\begin{aligned} &\int_{\Omega_R} (u \cdot \mathrm{grad}\, \varphi + \varphi \, \mathrm{div}\, u) \; dx \\ &\qquad = \int_{\Gamma_{0,R}} (n \cdot u) \, \varphi \, dA + \int_{\partial\Omega_R \backslash \Gamma_{0,R}} (n \cdot u) \, \varphi \, dA \,. \end{aligned}$$

Therefore, for the equality it remains to show for u (and φ, too), that

$$\int_{\partial\Omega_R \backslash \Gamma_{0,R}} |u|^2 \, dA \overset{R \to \infty}{\longrightarrow} 0 \,.$$

This is a consequence of $u \in H(\mathrm{div}, \mathbb{R}^3_\pm) \subset L^2(\mathbb{R}^3_\pm)$ and the assumed continuity of u: If the limit were not true, there would be a radius $\hat{R}$ from which on all above integrals for $R > \hat{R}$ were greater than some $\varepsilon > 0$. Then the integral of $|u|^2$ over $\mathbb{R}^3_\pm \backslash B_{\hat{R}}(0)$ would diverge, which contradicts $u \in L^2(\mathbb{R}^3_\pm)$.

To prove the mapping properties of the trace, let $\Phi \in H^{1/2}(\Gamma)$ and assume that $\varphi = E\Phi \in H^1(\mathbb{R}^3_\pm)$ is the image of a bounded extension operator E such that $\gamma_0 \varphi = \Phi$ and $||\varphi||_{H^1(\mathbb{R}^3_\pm)} \le C_E \, ||\Phi||_{H^{1/2}(\Gamma)}$.

Then, by above findings,

$$\begin{aligned}\left|\int_\Gamma \gamma_n u\,\Phi\,dA\right| &= \left|\int_\Gamma (n\cdot u)\,\varphi\,dA\right| \\ &\le ||u||_{H(\mathrm{div},\mathbb{R}^3_\pm)}\,||\varphi||_{H^1(\mathbb{R}^3_\pm)} \\ &\le C_E\,||u||_{H(\mathrm{div},\mathbb{R}^3_\pm)}\,||\Phi||_{H^{1/2}(\Gamma)}\end{aligned}$$

and therefore $\gamma_n : H(\mathrm{div},\mathbb{R}^3_\pm) \to H^{-1/2}(\Gamma)$ is bounded by C_E as a mapping onto the dual space. □

LEMMA 5.2.10. *Let $u,\varphi \in H(\mathrm{curl},\mathbb{R}^3_\pm)$ and $n = (0,0,\mp 1)^\top$, then*

$$\int_{\mathbb{R}^3_\pm} (\varphi\cdot\mathrm{curl}\,u - u\cdot\mathrm{curl}\,\varphi)\;dx = \int_\Gamma (n\times u)\cdot\varphi\,dA\,, \tag{5.4}$$

with a bounded dual pairing on the image spaces of the traces on the right hand side, and the traces are bounded as mappings

$$\begin{aligned}\gamma_t\;:\;H(\mathrm{curl},\mathbb{R}^3_\pm) &\to H^{-1/2}(\mathrm{Div};\Gamma)\,, \\ u &\mapsto n\times u|_\Gamma\,, \\ \pi_\Gamma\;:\;H(\mathrm{curl},\mathbb{R}^3_\pm) &\to H^{-1/2}(\mathrm{Curl};\Gamma)\,, \\ u &\mapsto n\times(u\times n)|_\Gamma\,.\end{aligned}$$

PROOF. By the above density argument it is sufficient to prove the assertion assuming $u,\varphi\in C^\infty(\mathbb{R}^3_\pm,\mathbb{C}^3)\cap H(\mathrm{curl},\mathbb{R}^3_\pm)$. Also note that the left hand side of the equation is bound by the product of the $H(\mathrm{curl},\mathbb{R}^3_\pm)$ norms of u and φ.

As above, let $\Omega_R = B_R(0)\cap\mathbb{R}^3_\pm$ and $\Gamma_{0,R} = B_R(0)\cap\Gamma_0$, then

$$\begin{aligned}&\int_{\Omega_R} (\varphi\cdot\mathrm{curl}\,u - u\cdot\mathrm{curl}\,\varphi)\;dx \\ &\qquad = \int_{\Gamma_{0,R}} (n\times u)\cdot\varphi\,dA + \int_{\partial\Omega_R\setminus\Gamma_{0,R}} (n\times u)\cdot\varphi\,dA\,.\end{aligned}$$

Therefore, for the equality it remains to show for u (and also φ), that

$$\int\limits_{\partial\Omega_R\setminus\Gamma_{0,R}} |u|^2\, dA \overset{R\to\infty}{\longrightarrow} 0\,.$$

This is a consequence of $u \in H(\mathrm{curl}, \mathbb{R}^3_\pm) \subset L^2(\mathbb{R}^3_\pm)$ and the assumed continuity of u: If the limit were not true, there would be a radius $\hat{R}$ from which on all above integrals for $R > \hat{R}$ were greater than some $\varepsilon > 0$. Then the integral of $|u|^2$ over $\mathbb{R}^3_\pm \setminus B_{\hat{R}}(0)$ would diverge, which contradicts $u \in L^2(\mathbb{R}^3_\pm)$.

To prove the mapping properties of γ_t, we first assume that $\varphi = E\,\Phi \in H^1(\mathbb{R}^3, \mathbb{C}^3)$ is an extension of $\Phi \in H^{1/2}(\mathbb{R}^3, \mathbb{C}^3)$, such that $\varphi|_\Gamma = \Phi$ and $||\varphi||_{H^1(\mathbb{R}^3,\mathbb{C}^3)} \leq C_E\, ||\Phi||_{H^{1/2}(\mathbb{R}^3,\mathbb{C}^3)}$. Then

$$\begin{aligned}
\left|\int\limits_\Gamma \gamma_t u \cdot \Phi\, dA\right| &= \left|\int\limits_\Gamma (n \times u)\cdot \varphi\, dA\right| \\
&\leq ||u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)}\, ||\varphi||_{H^1(\mathbb{R}^3_\pm,\mathbb{C}^3)} \\
&\leq C_E\, ||u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)}\, ||\Phi||_{H^{1/2}(\Gamma,\mathbb{C}^3)}\,,
\end{aligned}$$

which means that $\gamma_t : H(\mathrm{curl}, \mathbb{R}^3_\pm) \to H^{-1/2}(\Gamma, \mathbb{C}^3)$ is bounded. Exactly the same argument holds for $\gamma_T : H(\mathrm{curl}, \mathbb{R}^3_\pm) \to H^{-1/2}(\Gamma, \mathbb{C}^3)$.

Furthermore, we now assume that $\varphi = \nabla\Psi$, for some $\Psi \in H^2(\mathbb{R}^3_\pm)$. Of course $\mathrm{curl}\, u \in H(\mathrm{div}, \mathbb{R}^3_\pm)$, as $\mathrm{div}\,\mathrm{curl}\, u = 0$ and by (5.3) we have on the other hand

$$\begin{aligned}
\int\limits_{\mathbb{R}^3_\pm} \mathrm{grad}\,\Psi \cdot \mathrm{curl}\, u\, dx &= \int\limits_\Gamma (n \cdot \mathrm{curl}\, u)\, \Psi\, dA \\
&= \int\limits_\Gamma (\mathrm{Curl}\, u)\ \Psi\, dA = -\int\limits_\Gamma (\mathrm{Div}\,(n\times u))\ \Psi\, dA.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\int\limits_\Gamma (\mathrm{Div}\,\gamma_t u)\ \Psi\, dA\right| &= \left|\int\limits_\Gamma (\mathrm{Div}\,(n\times u))\ \Psi\, dA\right| \\
&\leq ||u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)}\, ||\Psi||_{H^1(\mathbb{R}^3_\pm,\mathbb{C}^3)} \\
&\leq C_E\, ||u||_{H(\mathrm{curl},\mathbb{R}^3_\pm)}\, ||\Psi||_{H^{1/2}(\Gamma,\mathbb{C}^3)}\,,
\end{aligned}$$

so $\gamma_t : H(\mathrm{curl}, \mathbb{R}^3_\pm) \to H^{-1/2}(\mathrm{Div}; \Gamma)$ is bounded, using the fact that the third component of γ_t vanishes. Note that since $\mathrm{curl}\,\varphi = 0$, we have $\varphi \in H(\mathrm{curl}, \mathbb{R}^3_\pm)$ and thus by above identity

$$\begin{aligned}\int_{\mathbb{R}^3_\pm} \mathrm{grad}\,\Psi \cdot \mathrm{curl}\,u\,dx &= \int_{\mathbb{R}^3_\pm} \varphi \cdot \mathrm{curl}\,u\,dx = \int_\Gamma (n \times u) \cdot \varphi\,dA \\ &= \int_\Gamma (n \times u) \cdot \mathrm{grad}\,\Psi\,dA\,,\end{aligned}$$

so $-\langle \mathrm{Div}\,\gamma_t u, \Psi\rangle = \langle \gamma_t u, \mathrm{grad}\,\Psi\rangle$ as a dual product. By symmetry of the left hand side of (5.4), a similar argumentation holds for γ_T, with the simplification that $n \cdot \mathrm{curl}\,\varphi = \mathrm{Curl}\,\pi_\Gamma \varphi$ on Γ. Therefore, π_Γ is as well bounded as a mapping $\pi_\Gamma : H(\mathrm{curl}, \mathbb{R}^3_\pm) \to H^{-1/2}(\mathrm{Curl}; \Gamma)$. □

Although we just found that the $H^{-1/2}(\Gamma)$ norm of the surface projection trace $\pi_\Gamma u$ and its surface curl $\mathrm{Curl}\,\pi_\Gamma u$ of a vector field $u \in H(\mathrm{curl}, \mathbb{R}^3_\pm)$ are bounded by the $H(\mathrm{curl}, \mathbb{R}^3_\pm)$ norm of u, we will need the following more specific result for the surface curl alone:

LEMMA 5.2.11. *For $u \in H(\mathrm{curl}, \mathbb{R}^3_\pm)$ the surface curl of u may be estimated by just the $L^2(\mathbb{R}^3_\pm)$ norm of* $\mathrm{curl}\,u$*:*

$$||\mathrm{Curl}\,\pi_\Gamma u||_{H^{-1/2}(\Gamma)} \le C_E\,||\mathrm{curl}\,u||_{L^2(\mathbb{R}^3_\pm)}$$

PROOF. Since $\mathrm{div}\,\mathrm{curl}\,u = 0$, $\mathrm{curl}\,u \in H(\mathrm{div}, \mathbb{R}^3_\pm)$ and therefore

$$\begin{aligned}||\mathrm{Curl}\,\pi_\Gamma u||_{H^{-1/2}(\Gamma)} &= ||\gamma_n\,\mathrm{curl}\,u||_{H^{-1/2}(\Gamma)} \\ &\le C_E\,||\mathrm{curl}\,u||_{H(\mathrm{div},\mathbb{R}^3_\pm)} = C_E\,||\mathrm{curl}\,u||_{L^2(\mathbb{R}^3_\pm)}\,,\end{aligned}$$

using the operator norm C_E of the extension operator E from above proofs. □

REMARK 5.2.12. Note, that this result is surprisingly stronger than corresponding estimate in the two-dimensional case! Recall that in the two-dimensional case we only had

$$||u||^2_{H^{1/2}(\Gamma)} \le \varepsilon\,C_1||u||^2_{L^2(D)} + \left(\frac{C_1}{\varepsilon} + C_1C_2\right)||\nabla u||^2_{L^2(D)}$$

for fixed C_1, C_2 but arbitrary ε in Lemma 1.2.4. In both cases the idea is to estimate the traces mainly by the derivative terms in the lower half-space, but in two dimension we can only make the dependence of the trace with respect to the L^2 norm of u arbitrary small, with the cost of bearing an arbitrary large $\frac{1}{\varepsilon}$. So here, we have a far more fortunate estimate, and it is due to less regularity of the chosen function space $H(\mathrm{curl}, \mathbb{R}^3_\pm)$ compared to $H^1(\mathbb{R}^3_\pm, \mathbb{C}^3)$.

5.3. The Calderon Operator

The Calderon operator maps given boundary data such as the tangential E field λ on Γ onto the tangential trace of the H field on Γ, when the E, H fields are a radiating solution to the time harmonic Maxwell's equations in the upper half-space with given boundary data.

It is strongly related to the Dirichlet-to-Neumann operator, as it not only takes a similar role in the weak formulation of the problem, but in fact, the reduction of the Calderon operator to the simplified TE or TM setting yields just the Dirichlet-to-Neumann operator, or the Neumann-to-Dirichlet operator, respectively.

Due to the symmetry in Maxwell's equations, there are two Calderon operators: The Calderon operator which maps E to H is denoted as $G_{EH}\lambda = \widetilde{n \times H}|_\Gamma = \widetilde{\gamma_t H}$ where $\lambda = \widetilde{n \times E}|_\Gamma$ and n is orthogonal to the surface, that is $n = (0,0,1)^\top$, thus $(\lambda_1, \lambda_2, \lambda_3)^\top = (-E_2, E_1, 0)$. The corresponding operator G_{HE} maps $\lambda = \widetilde{n \times H}|_\Gamma$ onto $\widetilde{n \times E}|_\Gamma$.

THEOREM 5.3.1. *Let E and $H = -\frac{i}{\omega\mu}\mathrm{curl}\, E$ be a (classical) solution to the full space Maxwell's Problem 5.1.1 (EBVP1), then the E to H Calderon operator is given by the Fourier multiplier*

$$\widehat{G_{EH}\lambda}(\tau) = \frac{1}{\omega\mu\sqrt{k_+^2 - |\tau|^2}} \begin{pmatrix} \tau_1\tau_2 & \tau_2^2 - k_+^2 \\ k_+^2 - \tau_1^2 & -\tau_1\tau_2 \end{pmatrix} \cdot \hat{\lambda}(\tau)$$

with $\tau \in \mathbb{R}^2$ satisfies $G_{EH}\lambda = \widetilde{n \times H}|_\Gamma$, for $\lambda = \widetilde{n \times E}|_\Gamma$ if the Fourier transforms exist.

REMARK 5.3.2. Later specific conditions for existence will be given. Most importantly, there is a condition on λ to ensure integrability, but we also need integrability of the E and H fields and their curls in the proof. If $\lambda \in \mathcal{S}_{k_+}(\mathbb{R}^2, \mathbb{C}^2)$ then above definition corresponds to

$$G_{EH}\lambda\,(x) = \frac{1}{2\pi\omega\mu}\int\limits_{\mathbb{R}^2} \frac{\exp\left(i\,(x\cdot\tau)\right)}{\sqrt{k_+^2 - |\tau|^2}} \begin{pmatrix} \tau_1\tau_2 & \tau_2^2 - k_+^2 \\ k_+^2 - \tau_1^2 & -\tau_1\tau_2 \end{pmatrix} \cdot \hat{\lambda}(\tau)\,d\tau\,.$$

PROOF. We split the transmitted divergence-free field

$$E = E^{(1)} + E^{(2)} = (E_1, E_2, v)^\top$$

into two divergence-free components such that

$$E^{(1)} = \begin{pmatrix} E_1 \\ 0 \\ v_1 \end{pmatrix},\ \operatorname{div} E^{(1)} = 0, \quad E^{(2)} = \begin{pmatrix} 0 \\ E_2 \\ v_2 \end{pmatrix},\ \operatorname{div} E^{(2)} = 0\,.$$

This means that $\frac{\partial v_j}{\partial x_3} = -\frac{\partial E_j}{\partial x_j}$ for $j = 1, 2$ and $v = v_1 + v_2$. We will now compute the corresponding $H^{(j)} = -\frac{i}{\omega\mu}\operatorname{curl} E^{(j)}$:

$$\begin{aligned} H^{(1)} &= -\frac{i}{\omega\mu}\operatorname{curl} E^{(1)} \\ &= -\frac{i}{\omega\mu}\operatorname{curl}\left(e_1 E_1\right) - \frac{i}{\omega\mu}\operatorname{curl}\left(e_3 v_1\right) \\ &= -\frac{i}{\omega\mu}\left(e_2\frac{\partial E_1}{\partial x_3} - e_3\frac{\partial E_1}{\partial x_2}\right) - \frac{i}{\omega\mu}\left(e_1\frac{\partial v_1}{\partial x_2} - e_2\frac{\partial v_1}{\partial x_1}\right) \\ &= \frac{i}{\omega\mu}\left[-e_1\frac{\partial v_1}{\partial x_2} + e_2\left(\frac{\partial v_1}{\partial x_1} - \frac{\partial E_1}{\partial x_3}\right) + e_3\frac{\partial E_1}{\partial x_2}\right] \end{aligned}$$

The second field results in:

$$\begin{aligned} H^{(2)} &= -\frac{i}{\omega\mu}\operatorname{curl} E^{(2)} \\ &= -\frac{i}{\omega\mu}\operatorname{curl}\left(e_2 E_2\right) - \frac{i}{\omega\mu}\operatorname{curl}\left(e_3 v_2\right) \\ &= -\frac{i}{\omega\mu}\left(-e_1\frac{\partial E_2}{\partial x_3} + e_3\frac{\partial E_2}{\partial x_1}\right) - \frac{i}{\omega\mu}\left(e_1\frac{\partial v_2}{\partial x_2} - e_2\frac{\partial v_2}{\partial x_1}\right) \\ &= \frac{i}{\omega\mu}\left[e_1\left(\frac{\partial E_2}{\partial x_3} - \frac{\partial v_2}{\partial x_2}\right) + e_2\frac{\partial v_2}{\partial x_1} - e_3\frac{\partial E_2}{\partial x_1}\right] \end{aligned}$$

As the components E_1 and E_2 satisfy the conditions for Theorem 2.4.8 and are solutions to the Helmholtz equation themselves, we can represent the solution as double layer potentials for $x_3 > 0$.

Using the Fourier representation (2.9) of the fundamental solution in three dimensions and its estimates in Lemma 2.4.7, we can apply the same arguments as in Lemma 3.2.4.

In analogy to Equation 3.1 this yields the representation

$$E_j(x) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \exp\left(i\left[x_3\sqrt{k^2 - |\tau|^2} + \tilde{x}\cdot\tau\right]\right)\hat{E}_j(\tau)\,d\tau\,,\ j = 1, 2, 3$$

for $x_3 > 0$, where $\hat{E}_j$ denotes the two-dimensional Fourier transform of E_j at Γ. Additionally, since E is divergence free, the Fourier transforms of the three components of E satisfy

$$\tau_1\hat{E}_1(\tau) + \tau_2\hat{E}_2(\tau) + \sqrt{k^2 - |\tau|^2}\hat{E}_3(\tau) = 0\,.$$

Therefore, it is a reasonable Ansatz to assume the following representations for the fields $E^{(j)}$, j=1, 2 in the upper layer:

$$\begin{aligned} E_U^{(j)}(x) \;:=\; & \frac{e_j}{2\pi} \int\limits_{\mathbb{R}^2} \exp\left(i\left[x_3\sqrt{k^2 - |\tau|^2} + \tilde{x}\cdot\tau\right]\right)\hat{E}_j(\tau)\,d\tau \\ & - \frac{e_3}{2\pi} \int\limits_{\mathbb{R}^2} \frac{\tau_j}{\sqrt{k^2 - |\tau|^2}} \\ & \qquad \cdot \exp\left(i\left[x_3\sqrt{k^2 - |\tau|^2} + \tilde{x}\cdot\tau\right]\right)\hat{E}_j(\tau)\,d\tau \end{aligned}$$

where we identify E_j and v_j by $E_U^{(j)} = e_jE_j + e_3v_j$.

Since we defined v_j in the upper layer using the divergence condition, there might still be some function $u(x_1, x_2)$, that is, constant in the x_3 coordinate direction, which itself satisfies the three-dimensional Helmholtz equation such that

$$E_3 = v_1 + v_2 + u\,.$$

Note that this is different to the two dimensional setup, where the transmission conditions are relatively stronger in comparison to this case. It turns out that we cannot have a unique solution without a radiation condition preventing such contributions that are constant in the x_3 coordinate direction, for example plane waves propagating parallel to the interface Γ. By the Sommerfeld radiation condition in E_3, we ensure that there is no constant contribution in any direction to infinity, so $u \equiv 0$ and $v_1 + v_2 = E_3$, as expected.

By above findings and time harmonic Maxwell's equations

$$H^{(1)} = -\frac{i}{\omega\mu}\operatorname{curl} E^{(1)}$$

and differentiation rules for the Fourier transform the field $H^{(1)}$ in the upper layer is given by

$$\begin{aligned} H_U^{(1)}(x) &= \frac{i}{\omega\mu}\Bigg[e_1 \frac{i}{2\pi}\int_{\mathbb{R}^2} \frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} \\ &\qquad \cdot \exp\left(i\left[x_3\sqrt{k^2-\tau^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_1(\tau)\,d\tau \\ &\quad - e_2\frac{i}{2\pi}\int_{\mathbb{R}^2}\left(\frac{\tau_1^2}{\sqrt{k^2-|\tau|^2}}+\sqrt{k^2-\tau^2}\right) \\ &\qquad \cdot \exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_1(\tau)\,d\tau \\ &\quad + e_3\frac{i}{2\pi}\int_{\mathbb{R}^2}\tau_2\exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_1(\tau)\,d\tau\Bigg] \end{aligned}$$

and therefore

$$\begin{aligned} H_U^{(1)}(x) &= \frac{1}{2\pi\omega\mu}\int_{\mathbb{R}^2}\begin{pmatrix} -\frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} \\ \frac{k^2-\tau_2^2}{\sqrt{k^2-|\tau|^2}} \\ \tau_2 \end{pmatrix} \\ &\qquad \cdot \exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_1(\tau)\,d\tau\,. \end{aligned}$$

For the second field $H^{(2)}$ we get

$$H_U^{(2)}(x) = \frac{i}{\omega\mu}\Bigg[e_1 \frac{i}{2\pi}\int\limits_{\mathbb{R}^2}\left(\frac{\tau_2^2}{\sqrt{k^2-|\tau|^2}}+\sqrt{k^2-|\tau|^2}\right) \cdot \exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_2(\tau)\,d\tau$$
$$-e_2\frac{i}{2\pi}\int\limits_{\mathbb{R}^2}\frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} \cdot \exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_2(\tau)\,d\tau$$
$$-e_3\frac{i}{2\pi}\int\limits_{\mathbb{R}^2}\tau_1\exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_2(\tau)\,d\tau\Bigg]$$

and thus

$$H_U^{(2)}(x) = \frac{1}{2\pi\omega\mu}\int\limits_{\mathbb{R}^2}\begin{pmatrix} -\frac{k^2-\tau_1^2}{\sqrt{k^2-|\tau|^2}} \\ \frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} \\ \tau_1 \end{pmatrix} \cdot \exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\hat{E}_2(\tau)\,d\tau\,.$$

Altogether, this yields

$$H_U(x) = \frac{1}{2\pi\omega\mu}\int\limits_{\mathbb{R}^2}\begin{pmatrix} -\frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} & -\frac{k^2-\tau_1^2}{\sqrt{k^2-|\tau|^2}} \\ \frac{k^2-\tau_2^2}{\sqrt{k^2-|\tau|^2}} & \frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} \\ \tau_2 & \tau_1 \end{pmatrix} \cdot \hat{E}(\tau) \cdot \exp\left(i\left[x_3\sqrt{k^2-|\tau|^2}+\tilde{x}\cdot\tau\right]\right)\,d\tau\,.$$

Therefore, we have for $x_3 = 0$, and using $\tilde{z} = G_{EH}\lambda$

$$
\begin{aligned}
z(x) &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times H_U(\lambda) \\
&= \frac{1}{2\pi\omega\mu} \int\limits_{\mathbb{R}^2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \frac{k^2-\tau_1^2}{\sqrt{k^2-|\tau|^2}} & -\frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} \\ -\frac{\tau_1\tau_2}{\sqrt{k^2-|\tau|^2}} & \frac{k^2-\tau_2^2}{\sqrt{k^2-|\tau|^2}} \\ 0 & 0 \end{pmatrix} \cdot \hat{\lambda}(\tau) \\
&\qquad \cdot \exp\left(i\,\tilde{x}\cdot\tau\right)\, d\tau \\
&= \frac{1}{2\pi\omega\mu} \int\limits_{\mathbb{R}^2} \frac{\exp\left(i\,\tilde{x}\cdot\tau\right)}{\sqrt{k^2-|\tau|^2}} \begin{pmatrix} \tau_1\tau_2 & \tau_2^2 - k^2 \\ k^2 - \tau_1^2 & -\tau_1\tau_2 \\ 0 & 0 \end{pmatrix} \cdot \hat{\lambda}(\tau)\, d\tau\,.
\end{aligned}
$$

□

5.4. Weighted Spaces for the Calderon Operator

In contrast to the two-dimensional models where the weak problems for the unbounded domains are discussed using traditional Sobolev spaces, we will see that it is impossible to achieve coercivity of the sesquilinear form for the corresponding three-dimensional problem without using weighted spaces. The following decomposition of the Fourier multiplier matrix in the Calderon operator is a key point in finding the appropriate weights:

REMARK 5.4.1. For $|\tau| \neq 0$, the matrix in the Fourier multiplier of the Calderon operator has the following revealing decomposition:

$$\begin{pmatrix} \tau_1\tau_2 & -k^2+\tau_2^2 \\ k^2-\tau_1^2 & -\tau_1\tau_2 \end{pmatrix} = \frac{1}{|\tau|^2}\begin{pmatrix} \tau_2 & \tau_1 \\ -\tau_1 & \tau_2 \end{pmatrix}\begin{pmatrix} |\tau|^2-k^2 & 0 \\ 0 & k^2 \end{pmatrix}\begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & -\tau_1 \end{pmatrix}$$

By this decomposition and the division by $\sqrt{k^2-|\tau|^2}$, we see both the integrating as well the differentiating part of the Calderon operator: The matrix on the right hand side represents a linear map performing the surface divergence and curl in Fourier space, the matrix on the left hand side is the inverse to the map performing surface curl and divergence, so in other words the curl components map onto divergence components and vice versa. The diagonal matrix divided by $\sqrt{k^2-|\tau|^2}$ finally features the Fourier multipliers of the Dirichlet-to-Neumann, and the Neumann-to-Dirichlet map. The detailed discussion of the mapping properties of both operators in the previous chapter, will now lead us to appropriate function spaces describing the mapping properties of the Calderon operator on the unbounded domain Γ.

In Definition 5.2.8 the domain and ranges of the functions in the set were artificially truncated from $\mathbb{R}^3$ to the two dimensional set Γ and by the additional condition of orthogonality to n. It seems natural to treat the traces as functions $\mathbb{R}^2 \to \mathbb{C}^2$. To facilitate the discussion of the mapping properties and enable us to use the previously introduced weighted Sobolev spaces directly, we will now introduce equivalent and corresponding weighted function spaces of functions with domains and ranges in $\mathbb{R}^2$:

DEFINITION 5.4.2. Let $s \in \mathbb{R}$, $k > 0$, and $\alpha \in \mathbb{R}$, then

$$\begin{aligned}
H^s(\mathrm{Div};\mathbb{R}^2) &:= \{u \in H^s(\mathbb{R}^2,\mathbb{C}^2) : \ \mathrm{Div}\, u \in H^s(\mathbb{R}^2)\} \\
H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) &:= \{u \in H^s(\mathbb{R}^2,\mathbb{C}^2) : \ \mathrm{Div}\, u \in H^s_{k,\alpha}(\mathbb{R}^2), \\
&\qquad \mathrm{Curl}\, u \in H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)\} \\
H^s(\mathrm{Curl};\mathbb{R}^2) &:= \{u \in H^s(\mathbb{R}^2,\mathbb{C}^2) : \ \mathrm{Curl}\, u \in H^s(\mathbb{R}^2)\} \\
H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2) &:= \{u \in H^s(\mathbb{R}^2,\mathbb{C}^2) : \ \mathrm{Curl}\, u \in H^s_{k,\alpha}(\mathbb{R}^2), \\
&\qquad \mathrm{Div}\, u \in H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)\} .
\end{aligned}$$

REMARK 5.4.3. These spaces are normed spaces by

$$\begin{aligned}
||u||^2_{H^s(\mathrm{Div};\mathbb{R}^2)} &= ||u||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)} + ||\mathrm{Div}\, u||^2_{H^s(\mathbb{R}^2)} , \\
||u||^2_{H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)} &= ||u||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)} + ||\mathrm{Div}\, u||^2_{H^s_{k,\alpha}(\mathbb{R}^2)} \\
&\quad + ||\mathrm{Curl}\, u||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)} , \\
||u||^2_{H^s(\mathrm{Curl};\mathbb{R}^2)} &= ||u||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)} + ||\mathrm{Curl}\, u||^2_{H^s(\mathbb{R}^2)} , \\
||u||^2_{H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2)} &= ||u||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)} + ||\mathrm{Curl}\, u||^2_{H^s_{k,\alpha}(\mathbb{R}^2)} \\
&\quad + ||\mathrm{Div}\, u||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)} .
\end{aligned}$$

Note that rotation of one space results in the other, in the sense that if $s \in \mathbb{R}$ and n represents the normal vector on Γ,

$$\begin{aligned}
H^s(\mathrm{Div};\mathbb{R}^2) &= \left\{ \widetilde{n \times \underset{\sim}{v}} \ : \ v \in H^s(\mathrm{Curl};\mathbb{R}^2) \right\} , \\
H^s(\mathrm{Curl};\mathbb{R}^2) &= \left\{ \widetilde{n \times \underset{\sim}{v}} \ : \ v \in H^s(\mathrm{Div};\mathbb{R}^2) \right\} , \\
H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) &= \left\{ \widetilde{n \times \underset{\sim}{v}} \ : \ v \in H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2) \right\} , \\
H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2) &= \left\{ \widetilde{n \times \underset{\sim}{v}} \ : \ v \in H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) \right\} .
\end{aligned}$$

Choosing some $\varphi \in \mathcal{S}_k(\mathbb{R}^2)$ illustrates that the weighted spaces are not empty: For any u in $H^s(\mathrm{Div};\mathbb{R}^2)$ or $H^s(\mathrm{Curl};\mathbb{R}^2)$ the convolution $\varphi * u$ is not only in same space by regularity of φ, but also in the weighted versions $H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$ or $H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2)$ for any α, since φ handles the additional weights by Proposition 4.4.3.

The following lemma establishes the norms of above spaces in the frequency domain:

LEMMA 5.4.4. *In the frequency domain, above unweighted norms have the representations*

$$||u||^2_{H^s(\mathrm{Div};\mathbb{R}^2)} = \int\limits_{\mathbb{R}^2} \left(1+|\tau|^2\right)^s \left(|\hat{u}(\tau)|^2 + |\tau\cdot\hat{u}(\tau)|^2\right) d\tau\,,$$

$$||u||^2_{H^s(\mathrm{Curl};\mathbb{R}^2)} = \int\limits_{\mathbb{R}^2} \left(1+|\tau|^2\right)^s \left(|\hat{u}(\tau)|^2 + |\tau_1\hat{u}_2(\tau)-\tau_2\hat{u}_1(\tau)|^2\right) d\tau$$

and the corresponding weighted norms are

$$\begin{aligned}
||u||^2_{H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)} = & \int\limits_{\mathbb{R}^2} \left(1+|\tau|^2\right)^s |\hat{u}(\tau)|^2 d\tau \\
& + \int\limits_{\mathbb{R}^2} \frac{\left(1+|\tau|^2\right)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha}} |\tau\cdot\hat{u}(\tau)|^2 d\tau \\
& + \int\limits_{\mathbb{R}^2} \frac{\left(1+|\tau|^2\right)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha+1}} |\tau_1\hat{u}_2(\tau)-\tau_2\hat{u}_1(\tau)|^2 d\tau\,,
\end{aligned}$$

$$\begin{aligned}
||u||^2_{H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2)} = & \int\limits_{\mathbb{R}^2} \left(1+|\tau|^2\right)^s |\hat{u}(\tau)|^2 d\tau \\
& + \int\limits_{\mathbb{R}^2} \frac{\left(1+|\tau|^2\right)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha}} |\tau_1\hat{u}_2(\tau)-\tau_2\hat{u}_1(\tau)|^2 d\tau \\
& + \int\limits_{\mathbb{R}^2} \frac{\left(1+|\tau|^2\right)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha+1}} |\tau\cdot\hat{u}(\tau)|^2 d\tau
\end{aligned}$$

where $\hat{u}$ is the two-dimensional Fourier transform of u.

PROOF. Since $\widehat{\mathrm{Div}\,u} = \tau\cdot\hat{u}$ and $\widehat{\mathrm{Curl}\,u} = \tau_1\hat{u}_2 - \tau_2\hat{u}_1$ by Definition 5.2.7 and the differentiation rules of the Fourier transform, the assertion just combines the definition of traditional and weighted

Sobolev spaces using the traditional and weighted Bessel potentials in the frequency domain of the norms of u, $\operatorname{Div} u$ and $\operatorname{Curl} u$. $\square$

As we expected, it turns out that the space $H^s_k(\mathrm{Div};\mathbb{R}^2)$ is very well suited for the Calderon operator at the unbounded interface Γ that was introduced before:

THEOREM 5.4.5. *The Calderon operator G_{EH} at Γ, given by*

$$\widehat{G_{EH}\lambda}\,(\tau) = \frac{1}{\omega\mu\,\sqrt{k_+^2 - |\tau|^2}} \begin{pmatrix} \tau_1\tau_2 & -k_+^2+\tau_2^2 \\ k_+^2-\tau_1^2 & -\tau_1\tau_2 \end{pmatrix} \cdot \hat{\lambda}(\tau)\,,$$

is a bounded mapping $G_{EH} : H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) \to H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$ for $s \in \mathbb{R}$ and $\alpha \geq 0$.

PROOF. This proof only validates the mapping properties, and for brevity let $k = k_+$: Recall that for $\lambda \in H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$,

$$\begin{aligned} ||\lambda||^2_{H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)} &= ||\lambda||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)} + ||\operatorname{Div}\lambda||^2_{H^s_{k,\alpha}(\mathbb{R}^2)} + ||\operatorname{Curl}\lambda||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)} \\ &= \int_{\mathbb{R}^2} (1+|\tau|^2)^s |\hat{\lambda}(\tau)|^2\, d\tau \\ &\quad + \int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha}} |\tau_1\hat{\lambda}_1(\tau) + \tau_2\hat{\lambda}_2(\tau)|^2\, d\tau \\ &\quad + \int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha+1}} |\tau_2\hat{\lambda}_1(\tau) - \tau_1\hat{\lambda}_2(\tau)|^2\, d\tau\,. \end{aligned}$$

Then the image $\mu(t) = G_{EH}\lambda(t)$ is bounded by these norms. First, the norm of $||\operatorname{Curl} G_{EH}\lambda||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)}$ is

$$||\operatorname{Curl} G_{EH}\lambda||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha+1}} \left|\tau_2\hat{\mu}_1(\tau) - \tau_1\hat{\mu}_2(\tau)\right|^2 d\tau$$

where

$$\hat{\mu}(\tau) = \frac{1}{2\pi\omega\mu}\,\frac{1}{\sqrt{k^2-|\tau|^2}} \begin{pmatrix} \tau_1\tau_2 & -k^2+\tau_2^2 \\ k^2-\tau_1^2 & -\tau_1\tau_2 \end{pmatrix} \cdot \hat{\lambda}(\tau)\,.$$

Since

$$\begin{pmatrix} \tau_2 \\ -\tau_1 \end{pmatrix}^\top \cdot \begin{pmatrix} \tau_1\tau_2 & -k^2+\tau_2^2 \\ k^2-\tau_1^2 & -\tau_1\tau_2 \end{pmatrix} = (\tau_1^2+\tau_2^2-k^2)\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$$

we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha+1}} \left|\tau_2\hat{\mu}_1(\tau)-\tau_1\hat{\mu}_2(\tau)\right|^2 d\tau \\ = \quad & \frac{1}{(2\pi\omega\mu)^2}\int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha+2}} \left|k^2-|\tau|^2\right|^2 \left|\tau_1\hat{\lambda}_1(\tau)+\tau_2\hat{\lambda}_2(\tau)\right|^2 d\tau\,, \end{aligned}$$

and therefore

$$\begin{aligned} ||\mathrm{Curl}\, G_{EH}\lambda||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)} \quad &= \quad \frac{1}{(2\pi\omega\mu)^2}\int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha}} \\ &\qquad \cdot \left|\tau_1\hat{\lambda}_1(\tau)+\tau_2\hat{\lambda}_2(\tau)\right|^2 d\tau \\ &= \quad \frac{1}{(2\pi\omega\mu)^2}\,||\mathrm{Div}\,\lambda||^2_{H^s_{k,\alpha}(\mathbb{R}^2)}\,. \end{aligned}$$

Next, the norm of $||\mathrm{Div}\, G_{EH}\lambda||^2_{H^s_{k,\alpha}(\mathbb{R}^2)}$:

$$||\mathrm{Div}\, G_{EH}\lambda||^2_{H^s_{k,\alpha}(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha}} \left|\tau_1\hat{\mu}_1(\tau)+\tau_2\hat{\mu}_2(\tau)\right|^2 d\tau$$

Since

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}^\top \cdot \begin{pmatrix} \tau_1\tau_2 & -k^2+\tau_2^2 \\ k^2-\tau_1^2 & -\tau_1\tau_2 \end{pmatrix} = k^2\begin{pmatrix} \tau_2 \\ -\tau_1 \end{pmatrix}$$

we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s+\alpha}}{|k^2-|\tau|^2|^{\alpha}} \left|\tau_1\hat{\mu}_1(\tau)+\tau_2\hat{\mu}_2(\tau)\right|^2 d\tau \\ = \quad & \frac{1}{(2\pi\omega\mu)^2}\int_{\mathbb{R}^2} \frac{(1+|\tau|^2)^{s}}{|k^2-|\tau|^2|^{\alpha+1}} k^4 \left|\tau_2\hat{\lambda}_1(\tau)-\tau_1\hat{\lambda}_2(\tau)\right|^2 d\tau\,. \end{aligned}$$

This results in

$$||\mathrm{Div}\, G_e\lambda||^2_{H^s_{k,\alpha}(\mathbb{R}^2)} = \frac{k^4}{(2\pi\omega\mu)^2}||\mathrm{Curl}\,\lambda||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)}\,.$$

The two remaining norms are both dissected into two parts:

$$\int_{\mathbb{R}^2}(1+|\tau|^2)^s|\hat{\mu}_p(\tau)|^2 d\tau \;=\; \int_K (1+|\tau|^2)^s|\hat{\mu}_p(\tau)|^2 d\tau + \int_{\mathbb{R}^2\setminus K}(1+|\tau|^2)^s|\hat{\mu}_p(\tau)|^2 d\tau$$

where $K = \{\tau \in \mathbb{R}^2 : |\tau| < \sqrt{2}k\}$ and $p = 1, 2$.

Outside K, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2\setminus K}(1+|\tau|^2)^s|\hat{\mu}_1(\tau)|^2 d\tau \\
&= \frac{1}{(2\pi\omega\mu)^2}\int_{\mathbb{R}^2\setminus K}\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|\tau_1\tau_2\hat{\lambda}_1(\tau)+(-k^2+\tau_2^2)\hat{\lambda}_2(\tau)\right|^2 d\tau \\
&= \frac{1}{(2\pi\omega\mu)^2}\int_{\mathbb{R}^2\setminus K}\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|\tau_2(\tau_1\hat{\lambda}_1(\tau)+\tau_2\hat{\lambda}_2(\tau))-k^2\hat{\lambda}_2(\tau)\right|^2 d\tau\,.
\end{aligned}$$

Since on the one hand

$$\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|\tau_2(\tau_1\hat{\lambda}_1(\tau)+\tau_2\hat{\lambda}_2(\tau))\right|^2 \le \frac{(1+|\tau|^2)^{s+1}}{|k^2-|\tau|^2|}\left|\tau_1\hat{\lambda}_1(\tau)+\tau_2\hat{\lambda}_2(\tau)\right|^2$$

and on the other hand for $|\tau|^2 > 2k^2$

$$\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|k^2\hat{\lambda}_2(\tau)\right|^2 \le (1+|\tau|^2)^s\left|k\hat{\lambda}_2(\tau)\right|^2$$

the summands can be estimated by $||\mathrm{Div}\lambda||^2_{H^s_{k.\alpha}(\mathbb{R}^2)}$ and $||\lambda||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)}$ using Minkowski's inequality

$$\begin{aligned}\int\limits_{\mathbb{R}^2\setminus K}(1+|\tau|^2)^s|\hat{\mu}_1(\tau)|^2d\tau &\leq \frac{\Big(C_1||\mathrm{Div}\lambda||_{H^s_{k.\alpha}(\mathbb{R}^2)}+k||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\Big)^2}{(2\pi\omega\mu)^2}\\ &= \frac{C_1^2}{(2\pi\omega\mu)^2}||\mathrm{Div}\lambda||^2_{H^s_{k.\alpha}(\mathbb{R}^2)}\\ &\quad+\frac{2C_1k}{(2\pi\omega\mu)^2}||\mathrm{Div}\lambda||_{H^s_{k,\alpha}(\mathbb{R}^2)}||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\\ &\quad+\frac{k^2}{(2\pi\omega\mu)^2}||\lambda||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\,,\end{aligned}$$

where $C_1^2=\max\limits_{|\tau|\geq\sqrt{2}k}\left|\frac{|\tau|^2}{k^2-|\tau|^2}\right|\cdot\left|\frac{k^2-|\tau|^2}{1+|\tau|^2}\right|^\alpha$ and likewise

$$\begin{aligned}&\int\limits_{\mathbb{R}^2\setminus K}(1+|\tau|^2)^s|\hat{\mu}_2(\tau)|^2d\tau\\ &= \frac{1}{(2\pi\omega\mu)^2}\int\limits_{\mathbb{R}^2\setminus K}\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|(k^2-\tau_1^2)\hat{\lambda}_1(\tau)-\tau_1\tau_2\hat{\lambda}_2(\tau)\right|^2d\tau\\ &= \frac{1}{(2\pi\omega\mu)^2}\int\limits_{\mathbb{R}^2\setminus K}\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|k^2\hat{\lambda}_1(\tau)-\tau_1(\tau_1\hat{\lambda}_1(\tau)+\tau_2\hat{\lambda}_2(\tau))\right|^2d\tau\,,\end{aligned}$$

thus

$$\begin{aligned}\int\limits_{\mathbb{R}^2\setminus K}(1+|\tau|^2)^s|\hat{\mu}_2(\tau)|^2d\tau &\leq \frac{\Big(C_1||\mathrm{Div}\lambda||_{H^s_{k.\alpha}(\mathbb{R}^2)}+k||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\Big)^2}{(2\pi\omega\mu)^2}\\ &= \frac{C_1^2}{(2\pi\omega\mu)^2}||\mathrm{Div}\lambda||^2_{H^s_{k.\alpha}(\mathbb{R}^2)}\\ &\quad+\frac{2C_1k}{(2\pi\omega\mu)^2}||\mathrm{Div}\lambda||_{H^s_{k,\alpha}(\mathbb{R}^2)}||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\\ &\quad+\frac{k^2}{(2\pi\omega\mu)^2}||\lambda||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\,.\end{aligned}$$

Within K, it is

$$\begin{aligned}
&\int\limits_K (1+|\tau|^2)^s|\hat{\mu}_1(\tau)|^2 d\tau \\
=\ & \frac{1}{(2\pi\omega\mu)^2}\int\limits_K \frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|\tau_1\tau_2\hat{\lambda}_1(\tau)+(-k^2+\tau_2^2)\hat{\lambda}_2(\tau)\right|^2 d\tau \\
=\ & \frac{1}{(2\pi\omega\mu)^2}\int\limits_K \frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|\tau_1\left(\tau_2\hat{\lambda}_1(\tau)-\tau_1\hat{\lambda}_2(\tau)\right)\right. \\
&\qquad\qquad\qquad\qquad \left.-(k^2-|\tau|^2)\hat{\lambda}_2(\tau)\right|^2 d\tau
\end{aligned}$$

and similar to before we have on the one hand

$$\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|\tau_1(\tau_2\hat{\lambda}_1(\tau)-\tau_1\hat{\lambda}_2(\tau))\right|^2 \leq \frac{(1+|\tau|^2)^{s+1}}{|k^2-|\tau|^2|}\left|\tau_2\hat{\lambda}_1(\tau)-\tau_1\hat{\lambda}_2(\tau)\right|^2$$

and on the other hand

$$\frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|(k^2-|\tau|^2)\hat{\lambda}_2(\tau)\right|^2 = (1+|\tau|^2)^s\left|\hat{\lambda}_2(\tau)\right|^2\left|k^2-|\tau|^2\right| .$$

Thus the summands can be estimated using $||\mathrm{Curl}\lambda||^2_{H^{s-1}_{k.\alpha+1}(\mathbb{R}^2)}$ and $||\lambda||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)}$

$$\begin{aligned}
\int\limits_K (1+|\tau|^2)^s|\hat{\mu}_1(\tau)|^2 d\tau \quad &\leq \quad \left(C_2||\mathrm{Curl}\lambda||_{H^{s-1}_{k.\alpha+1}(\mathbb{R}^2)}+||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\right)^2 \\
&= \quad C_2^2||\mathrm{Curl}\,\lambda||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)} \\
&\quad +2C_2||\mathrm{Curl}\,\lambda||_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)}||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)} \\
&\quad +||\lambda||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)}
\end{aligned}$$

where $C_2^2 = \max\limits_{|\tau|\leq\sqrt{2}k}|\tau|^2\cdot\left|k^2-|\tau|^2\right|^\alpha$. Finally,

$$\begin{aligned}
&\int\limits_K (1+|\tau|^2)^s|\hat{\mu}_2(\tau)|^2 d\tau \\
=\ & \int\limits_K \frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|(k^2-\tau_1^2)\hat{\lambda}_1(\tau)-\tau_1\tau_2\hat{\lambda}_2(\tau)\right|^2 d\tau \\
=\ & \int\limits_K \frac{(1+|\tau|^2)^s}{|k^2-|\tau|^2|}\left|(k^2-|\tau|^2)\hat{\lambda}_1(\tau)-\tau_2\left(\tau_2\hat{\lambda}_1(\tau)-\tau_1\hat{\lambda}_2(\tau)\right)\right|^2 d\tau
\end{aligned}$$

so likewise

$$\begin{aligned}\int_K (1+|\tau|^2)^s|\hat{\mu}_2(\tau)|^2 d\tau &\leq \left(C_2||\mathrm{Curl}\lambda||_{H^{s-1}_{k.\alpha+1}(\mathbb{R}^2)} + ||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\right)^2 \\ &= C_2^2||\mathrm{Curl}\,\lambda||^2_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)} \\ &\quad +2C_2||\mathrm{Curl}\,\lambda||_{H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)}||\lambda||_{H^s(\mathbb{R}^2,\mathbb{C}^2)} \\ &\quad +||\lambda||^2_{H^s(\mathbb{R}^2,\mathbb{C}^2)}\,.\end{aligned}$$

□

By the same argumentation, an identical result holds for the second Calderon operator:

THEOREM 5.4.6. *The Calderon operator G_{HE} at Γ, given by*

$$\widehat{G_{HE}\lambda}\,(\tau) = \frac{-1}{\omega\varepsilon_+\sqrt{k_+^2-|\tau|^2}}\begin{pmatrix} \tau_1\tau_2 & -k_+^2+\tau_2^2 \\ k_+^2-\tau_1^2 & -\tau_1\tau_2 \end{pmatrix}\cdot\hat{\lambda}(\tau)\,,$$

is a bounded mapping $G_{HE} : H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) \to H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$ for $s\in\mathbb{R}$ and $\alpha\geq 0$.

The last two theorems present the same mapping properties of the Calderon operators as in [**26, 19**] where they were used at bounded interfaces, but they will not be sufficient for the discussion at the infinite boundary plane. The following Lemma will be most useful for the discussion of coercivity of the weak formulation, as it defines the Calderon operators for even less regular spaces than before:

LEMMA 5.4.7. *For $s\geq -\frac{1}{2}$,*

$$G_{HE}, G_{EH} : H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) \quad\to\quad \left(H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2)\right)^*$$

are bounded with respect to the $L^2(\mathbb{R}^2)$ scalar product if and only if $\alpha\geq -\frac{1}{2}$.

PROOF. Let $u, v \in H^s_{k,\alpha}(\mathrm{Curl};\mathbb{R}^2)$, then $\widetilde{n \times \underset{\sim}{u}} \in H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$ and

$$\begin{aligned}
&\langle v, G_{HE}\,\widetilde{n \times \underset{\sim}{u}}\rangle_{\mathbb{R}^2} \\
=\;& C \int_{\mathbb{R}^2} \frac{\exp\left(i(t\cdot\tau)\right)}{\sqrt{k_+^2 - |\tau|^2}} (\hat{v}_1, \hat{v}_2) \begin{pmatrix} \tau_1\tau_2 & -k_+^2 + \tau_2^2 \\ k_+^2 - \tau_1^2 & -\tau_1\tau_2 \end{pmatrix} \begin{pmatrix} \hat{u}_2 \\ -\hat{u}_1 \end{pmatrix} d\tau \\
=\;& C \int_{|\tau|\le 2k_+} \frac{\exp\left(i(t\cdot\tau)\right)}{\sqrt{k_+^2 - |\tau|^2}} \left((\tau\cdot\hat{u})(\tau\cdot\hat{v}) + (k_+^2 - |\tau|^2)(\hat{u}\cdot\hat{v})\right) d\tau \\
&+ C \int_{|\tau|> 2k_+} \frac{\exp\left(i(t\cdot\tau)\right)}{\sqrt{k_+^2 - |\tau|^2}} \left((\tau_2\hat{u}_1 - \tau_1\hat{u}_2)(\tau_2\hat{v}_1 - \tau_1\hat{v}_2) + k_+^2(\hat{u}\cdot\hat{v})\right) d\tau\,.
\end{aligned}$$

This integration will only be successful for $|\tau| \le 2k_+$ for all u and v if and only if the surface divergences $\tau\cdot\hat{u}$ and $\tau\cdot\hat{v}$ are sufficiently regular: So if $\mathrm{Div}\,u, \mathrm{Div}\,v \in H^{s-1}_{k,\alpha+1}(\mathbb{R}^2)$, then the equivalent condition is $2\frac{\alpha+1}{2} \ge \frac{1}{2}$ or $\alpha \ge -\frac{1}{2}$. For $|\tau| > 2k_+$, the condition for the surface curls $\mathrm{Curl}\,u, \mathrm{Curl}\,v \in H^s_{k,\alpha}(\mathbb{R}^2)$ and $u, v \in H^s(\mathbb{R}^2)$ is $2\frac{s}{2} > -\frac{1}{2}$, so $s \ge -\frac{1}{2}$ is needed and suffices for the remainder of the integral. □

5.5. Variational Approach

Now that suitable integral equalities, the Calderon operator, and its mapping properties are established, we may lay out the weak problems after the next definition of the solution space of $H(\mathrm{curl};\mathbb{R}^3_-)$ functions with traces in $H^{-1/2}_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$ appropriate for the Calderon operator:

DEFINITION 5.5.1. For $k > 0$, $\alpha \in \mathbb{R}$, let

$$H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};\mathbb{R}^3_-) := \left\{ u \in H(\mathrm{curl}, \mathbb{R}^3_-) :\ \widetilde{n \times u}|_\Gamma \in H^{-\frac{1}{2}}_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) \right\}.$$

REMARK 5.5.2. The space $H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};\mathbb{R}^3_-)$ is normed by

$$\begin{aligned}
||u||^2_{H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};\mathbb{R}^3_-)} =\;& ||u||^2_{L^2(\mathbb{R}^3_-,\mathbb{C}^3)} + ||\mathrm{curl}\,u||^2_{L^2(\mathbb{R}^3_-,\mathbb{C}^3)} \\
&+ ||\widetilde{n \times u}|_\Gamma||^2_{H^{-1/2}(\mathbb{R}^2,\mathbb{C}^2)} \\
&+ ||\mathrm{Curl}\,\tilde{u}|_\Gamma||^2_{H^{-1/2}_{k,\alpha}(\mathbb{R}^2)} + ||\mathrm{Div}\,\tilde{u}|_\Gamma||^2_{H^{-3/2}_{k,\alpha+1}(\mathbb{R}^2)}
\end{aligned}$$

and is non-empty for $\alpha \geq 0$, as trace functions in $H^{-1/2}_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$ are in $H^{-1/2}(\mathrm{Div};\mathbb{R}^2)$ as well, and therefore have extensions to functions in $H(\mathrm{curl};\mathbb{R}^3_-)$. Thus all spaces $H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};\mathbb{R}^3_-)$ are non-empty, as $H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2) \subseteq H^s_{k,\beta}(\mathrm{Div};\mathbb{R}^2)$ for all $\beta \leq \alpha$. Since $H^s(\mathrm{Div};\mathbb{R}^2) \subseteq H^s_{k,\alpha}(\mathrm{Div};\mathbb{R}^2)$ for $\alpha \leq 0$ and since $H^s(\mathrm{Div};\mathbb{R}^2)$ is the trace space of $H(\mathrm{curl};\mathbb{R}^3_-)$ in the regarded context, we even have

$$H(\mathrm{curl};\mathbb{R}^3_-) = H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};\mathbb{R}^3_-) \text{ for } \alpha \leq -1$$

as sets with differing norms.

At first, it seems natural to assume $\alpha = 0$ or $\alpha = -1$ to seek a solution in a space closest to $H(\mathrm{curl};\mathbb{R}^3_-)$, as then either the surface curl or the surface divergence resembles the usual trace space. In order to discuss which α are appropriate, the problem definitions are posed for general $\alpha \geq -1/2$ for which the sesquilinear form is well defined:

Problem 5.5.3. (EBVP2)
For $D = \mathbb{R}^3_- \backslash \bar{\Omega}$ and $\alpha \geq -1/2$, find $u \in H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};D)$ such that for all test functions $\varphi \in H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};D)$ the equation

$$b_E(u,\varphi) = a_E(\varphi)$$

holds, where

$$\begin{aligned} b_E(u,\varphi) \quad = \quad & (\mathrm{curl}\, u, \mathrm{curl}\, \varphi)_{L^2(D)} - k_-^2 (u,\varphi)_{L^2(D)} \\ + \quad & i\omega\mu \langle G_{EH} \widetilde{\gamma_t u}, \widetilde{\pi_\Gamma \bar{\varphi}} \rangle_{\mathbb{R}^2} \end{aligned}$$

using the traces $\gamma_t u = n \times u|_\Gamma$, $\pi_\Gamma \bar{\varphi} = n \times (\bar{\varphi} \times n)|_\Gamma$, and

$$a_E(\varphi) = -\langle n \times E^i, (n \times \bar{\varphi}) \times n \rangle_{\partial\Omega}$$

for an analytic incoming field E^i defined at least around Ω.

The weak problem for the magnetic field intensity for $\alpha \in \mathbb{R}$ is defined as:

Problem 5.5.4. (HBVP2)
For $D = \mathbb{R}^3_- \backslash \bar{\Omega}$ and $\alpha \geq -1/2$, find $u \in H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};D)$ such that for all test functions $\varphi \in H_{k,\alpha}(\mathrm{curl};\mathrm{Curl};D)$ the equation

$$b_H(u,\varphi) = a_H(\varphi)$$

holds, where

$$\begin{aligned} b_H(u,\varphi) &= (\operatorname{curl} u, \operatorname{curl}\varphi)_{L^2(D)} - k_-^2(u,\varphi)_{L^2(D)} \\ &+ (\sigma_- - i\omega\varepsilon_-)\langle G_{HE}\widetilde{\gamma_t u}, \widetilde{\pi_\Gamma \bar\varphi}\rangle_{\mathbb{R}^2} \end{aligned}$$

and

$$a_H(\varphi) = (\operatorname{curl} g, \operatorname{curl}\varphi)_{L^2(D)} - k_-^2(g,\varphi)_{L^2(D)}\,,$$

where g is a smooth rapidly decaying vector field such that its support is located in a small domain around Ω and $g|_{\partial\Omega} = -H^i|_{\partial\Omega}$, an analytic incoming field, defined at least around Ω.

5.6. Existence, Uniqueness, and Equivalence

By the discussion of the Calderon operator we found out that the weak problem definitions are only well posed for $\alpha \geq -1/2$:

THEOREM 5.6.1.

$$\sup_{u,v\in H_{k,\alpha}(\text{curl};\text{Curl},\mathbb{R}^3_-)} \frac{|b_{E/H}(u,v)|}{||u||\,||v||} < \infty$$

if and only if $\alpha \geq -\frac{1}{2}$.

PROOF. As $u, v \in H(\text{curl}, \mathbb{R}^3_-)$, we only need to consider the last part of the sesquilinear forms. Lemma 5.4.7 then proves the desired assertion. □

In the following, we will concentrate on the weak problem for the H field for two reasons: First, only b_H exposes suitable phase conditions for application of the extended Lax Milgram Lemma, and second, a solution to one of the weak problems will yield a solution to the other problem.

PROPOSITION 5.6.2. *In the following, two representations of* $b_H(u,u)$ *will be used, where* $\hat u_T = \hat{\hat u} = (\hat u_1, \hat u_2)^\top$ *denotes the two dimensional*

Fourier transform of the first two components of u *at* Γ.

$$b_H(u,u) \overset{(i)}{=} ||\operatorname{curl} u||^2_{L^2(D)} - k_-^2 ||u||^2_{L^2(D)} - i\frac{k_-^2}{k_+^2} \int\limits_{\mathbb{R}^2} \frac{|\tau \cdot \hat{u}_T|^2}{\sqrt{k_+^2 - |\tau|^2}}\, d\tau - i\frac{k_-^2}{k_+^2} \int\limits_{\mathbb{R}^2} \sqrt{k_+^2 - |\tau|^2}\, |\hat{u}_T|^2 \, d\tau$$

$$b_H(u,u) \overset{(ii)}{=} ||\operatorname{curl} u||^2_{L^2(D)} - k_-^2 ||u||^2_{L^2(D)} + i\frac{k_-^2}{k_+^2} \int\limits_{\mathbb{R}^2} \frac{|\tau_2 \hat{u}_1 - \tau_1 \hat{u}_2|^2}{\sqrt{k_+^2 - |\tau|^2}}\, d\tau - ik_-^2 \int\limits_{\mathbb{R}^2} \frac{|\hat{u}_T|^2}{\sqrt{k_+^2 - |\tau|^2}}\, d\tau \,.$$

Proof. The first two summands are obvious since

$$\begin{aligned} b_H(u,u) &= (\operatorname{curl} u, \operatorname{curl} u)_{L^2(D)} - k_-^2 (u,u)_{L^2(D)} \\ &+ (\sigma_- - i\omega\varepsilon_-)\langle G_{HE}\widetilde{\gamma_t u}, \widetilde{\pi_\Gamma \bar{u}}\rangle_{\mathbb{R}^2} \,, \end{aligned}$$

so it remains to analyze the term $\frac{\sigma_- - i\omega\varepsilon_-}{-\omega\varepsilon_+} = i\frac{\omega^2\varepsilon_- + i\omega\sigma_-}{\omega^2\varepsilon_+} = i\frac{k_-^2}{k_+^2}$ using Equation (2.3) and the term

$$\frac{1}{\sqrt{k_+^2 - |\tau|^2}} (\overline{\hat{u}_1}, \overline{\hat{u}_2}) \begin{pmatrix} \tau_1\tau_2 & -k_+^2 + \tau_2^2 \\ k_+^2 - \tau_1^2 & -\tau_1\tau_2 \end{pmatrix} \begin{pmatrix} -\hat{u}_2 \\ \hat{u}_1 \end{pmatrix} .$$

First of all,

$$\begin{aligned} (\overline{\hat{u}_1}, \overline{\hat{u}_2}) \begin{pmatrix} \tau_1\tau_2 & -k_+^2 + \tau_2^2 \\ k_+^2 - \tau_1^2 & -\tau_1\tau_2 \end{pmatrix} \begin{pmatrix} \hat{u}_2 \\ -\hat{u}_1 \end{pmatrix} &= -\tau_1\tau_2 \left(\overline{\hat{u}_1}\hat{u}_2 + \hat{u}_1\overline{\hat{u}_2}\right) \\ &\quad - \left(k_+^2 - \tau_2^2\right)|\hat{u}_1|^2 \\ &\quad - \left(k_+^2 - \tau_1^2\right)|\hat{u}_2|^2 \,. \end{aligned}$$

From this we get the first representation by

$$
\begin{aligned}
& -\tau_1\tau_2\left(\overline{\hat{u}_1}\hat{u}_2+\hat{u}_1\overline{\hat{u}_2}\right)-\left(k_+^2-\tau_2^2\right)|\hat{u}_1|^2-\left(k_+^2-\tau_1^2\right)|\hat{u}_2|^2 \\
= & -\left(k_+^2-|\tau|^2\right)|\hat{u}_T|^2-\tau_1^2|\hat{u}_1|^2-\tau_1\tau_2\left(\overline{\hat{u}_1}\hat{u}_2+\hat{u}_1\overline{\hat{u}_2}\right)-\tau_2^2|\hat{u}_2|^2 \\
= & -\left(k_+^2-|\tau|^2\right)|\hat{u}_T|^2-|\tau\cdot\hat{u}_T|^2
\end{aligned}
$$

and the second representation by

$$
\begin{aligned}
& -\tau_1\tau_2\left(\overline{\hat{u}_1}\hat{u}_2+\hat{u}_1\overline{\hat{u}_2}\right)-\left(k_+^2-\tau_2^2\right)|\hat{u}_1|^2-\left(k_+^2-\tau_1^2\right)|\hat{u}_2|^2 \\
= & -k_+^2|\hat{u}_T|^2+\left(\tau_2^2|\hat{u}_1|^2+\tau_1\tau_2\left(\overline{\hat{u}_1}\hat{u}_2+\hat{u}_1\overline{\hat{u}_2}\right)+\tau_1^2|\hat{u}_2|^2\right) \\
= & -k_+^2|\hat{u}_T|^2+|\tau_2\hat{u}_1-\tau_1\hat{u}_2|^2\,.
\end{aligned}
$$

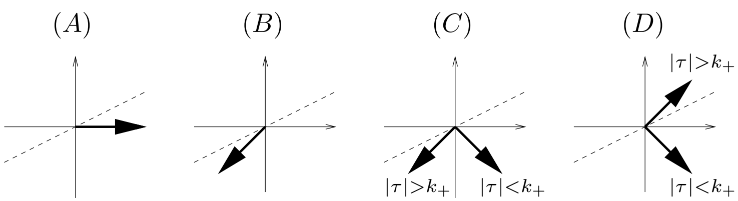
□

REMARK 5.6.3. We will need to analyze the structure of the sesquilinear forms by plots of the phases of the different terms of b_H in the complex plane. As before, we assume $\arg k_-^2 \in (0, \frac{\pi}{2})$. In the plots we set $\arg k_-^2 \approx \frac{\pi}{4}$ for illustration purposes. As before, the arguments of square roots are either 0 for $|\tau| \le k_+$ or $\frac{\pi}{2}$ for $|\tau| > k_+$.

Representation (i) of $b_H(u,u) =$

$$
\underbrace{||\operatorname{curl} u||^2_{L^2(\mathbb{R}^3_-)}}_{(A)}\ \underbrace{-k_-^2\,||u||^2_{L^2(\mathbb{R}^3_-)}}_{(B)}
$$

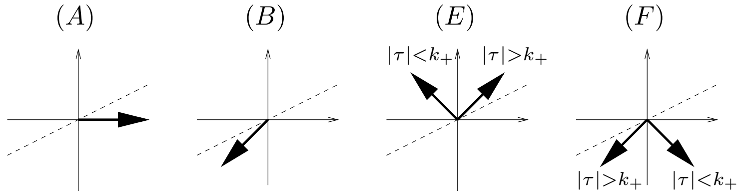
$$
\underbrace{-i\frac{k_-^2}{k_+^2}\int\limits_{\mathbb{R}^2}\frac{|\tau\cdot\hat{u}_T|^2}{\sqrt{k_+^2-|\tau|^2}}\,d\tau}_{(C)}\ \underbrace{-i\frac{k_-^2}{k_+^2}\int\limits_{\mathbb{R}^2}\sqrt{k_+^2-|\tau|^2}\,|\hat{u}_T|^2\,d\tau}_{(D)}
$$



Representation (ii) of $b_H(u,u) =$

$$\underbrace{||\operatorname{curl} u||^2_{L^2(\mathbb{R}^3_-)}}_{(A)} \underbrace{-k_-^2 ||u||^2_{L^2(\mathbb{R}^3_-)}}_{(B)}$$

$$\underbrace{+i\frac{k_-^2}{k_+^2}\int\limits_{\mathbb{R}^2} \frac{|\tau_2\hat{u}_1 - \tau_1\hat{u}_2|^2}{\sqrt{k_+^2 - |\tau|^2}}\,d\tau}_{(E)} \underbrace{-ik_-^2 \int\limits_{\mathbb{R}^2} \frac{|\hat{u}_T|^2}{\sqrt{k_+^2 - |\tau|^2}}\,d\tau}_{(F)}$$



The dashed lines in the plots approximate the coercivity plane in the spirit of Lemma 1.3.3 for the proof of the next theorem. It turns out that even though the problem is well defined for $\alpha \geq -\frac{1}{2}$, coercivity is impossible for $\alpha > -\frac{1}{2}$, leaving only one choice of α:

LEMMA 5.6.4. *For all constants $c > 0$ and $\alpha > -\frac{1}{2}$ there exists a function $u \in H_{k,\alpha}(\operatorname{curl}; \operatorname{Curl}; D)$ such that*

$$|b_H(u,u)| < c||u||^2_{H_{k,\alpha}(\operatorname{curl};\operatorname{Curl};D)}\,.$$

PROOF. First, we need some continuous extension operator E from $\mathcal{S}_k(\mathbb{R}^2, \mathbb{C}^2)$ to $H_{k,\alpha}(\operatorname{curl}; \operatorname{Curl}; D)$, such as $E\hat{\varphi}(x) = \hat{\varphi}(\tilde{x})e^{-x_3^2}$ for $\varphi \in \mathcal{S}_k(\mathbb{R}^2, \mathbb{C}^2)$. Please note, that the extension operator will be used together with the Fourier transform. For arbitrary but from now on fixed $\varepsilon > 0$ let

$$\varphi_0 \in \mathcal{S}_k(\mathbb{R}^2, \mathbb{C}^2) \quad \text{with} \quad \operatorname{supp} \varphi_0 \in B_{k+\varepsilon}(0) \backslash B_{k+\varepsilon/2}(0)$$

and normalized by $||E\hat{\varphi}_0||_{H_{k,\alpha}(\operatorname{curl};\operatorname{Curl};D)} = 1$. Starting from φ_0 we will define a sequence of functions which extended Fourier transforms will eventually yield a function with the desired properties by induction:

Assume that $\varphi_n \in \mathcal{S}_k(\mathbb{R}^2, \mathbb{C}^2)$ and

$$\operatorname{supp} \varphi_n \in B_{k+\varepsilon/2^n}(0) \backslash B_{k+\varepsilon/2^{n+1}}(0)\,,$$

then we define

$$\psi_{n+1}(re^{i\beta}) := \varphi_n \left(\frac{r+k}{2}\, e^{i\beta} \right)\,,$$

which inherits $\psi_{n+1} \in \mathcal{S}_k(\mathbb{R}^2, \mathbb{C}^2)$ and moves the support to

$$\operatorname{supp} \psi_{n+1} \in B_{k+\varepsilon/2^{n+1}}(0) \backslash B_{k+\varepsilon/2^{n+2}}(0)\,.$$

We therefore have an extension $E\hat{\psi}_{n+1} \in H_{k,\alpha}(\text{curl}; \text{Curl}; D)$, and we may normalize

$$\varphi_{n+1} := \frac{\psi_{n+1}}{||E\hat{\psi}_{n+1}||_{H_{k,\alpha}(\text{curl};\text{Curl};D)}}\,.$$

Note that such a sequence of $(E\hat{\varphi}_n)$ is not a Cauchy sequence in $H_{k,\alpha}(\text{curl}; \text{Curl}; D)$, since by construction

$$||\text{Div}\hat{\varphi}_{n+1} - \text{Div}\hat{\varphi}_n||_{H^{-3/2}_{k,\alpha+1}(\mathbb{R}^2)}$$

will not vanish. Thus it does not contradict the completeness of the singular weighted Sobolev spaces.

Regarding the individual parts of the $H_{k,\alpha}(\text{curl}; \text{Curl}; D)$ norm, we see that for $\alpha > -1$ the $||\text{Div}\, \hat{\varphi}_n||_{H^{-3/2}_{k,\alpha+1}(\mathbb{R}^2)}$ will grow for increasing n compared to the other parts due to the moving support and the highest order singular weight function. By representation (i) of $a_H(E\hat{\varphi}_n, E\hat{\varphi}_n)$, it is bound from above by several norms, from which the norm with the most singular weight is $||\text{Div}\, \hat{\varphi}_n||^2_{H^{-1/2}_{k,1/2}}$. Therefore, $a_H(E\hat{\varphi}_n, E\hat{\varphi}_n)$ converges to 0 if $\alpha > -\frac{1}{2}$ as it has a less singular weight, and thus for n sufficiently large there exists a function $u = E\hat{\varphi}_n$ such that the assertion is fulfilled for any previously chosen $c > 0$. □

THEOREM 5.6.5. *There are constants C and φ dependent on k_- and k_+ such that*

$$\operatorname{Re}\left(e^{i\varphi} b_H(u, u)\right) \geq C||u||^2_{H_{k_+,\alpha}(\text{curl};\text{Curl};D)}$$

for all $u \in H_{k_+,\alpha}(\text{curl};\text{Curl};D)$ *if and only if* $\alpha = -\frac{1}{2}$.

Proof. The norms are treated in Fourier space within three sections: The first section will treat large frequencies, the second section frequencies just above the wave number k_+, and the last section will cover frequencies below the wave number k_+.

First, we consider the segment

$$\Sigma_1 := \{\tau \in \mathbb{R}^2 : \ |\tau| > \sqrt{k_+^2 + \frac{k_+^4 C_T^2}{4|k_-|^4}}\},$$

where C_T is a trace constant, such that

$$C_T ||\tilde{u}|_\Gamma||^2_{H^{-1/2}(\mathbb{R}^2)} \leq ||u||^2_{H(\text{curl};\mathbb{R}^3_-)}$$

for all $u \in H(\text{curl};\mathbb{R}^3_-)$.

Using the representation (ii) of $b_H(u,u)$, we see that all terms are on one side of a coercivity plane for this segment, except for the term (E) for $|\tau| > k_+$. Since the term

$$\frac{1+|\tau|^2}{k_+^2 - |\tau|^2} = 1 + \frac{1+k_+^2}{k_+^2 - |\tau|^2}$$

is monotonically decreasing for $|\tau| > k_+$ which is the case for $\tau \in \Sigma_1$ and noting that

$$\left| k_+^2 - \left(k_+^2 + \frac{k_+^4 C_T^2}{4|k_-|^4} \right) \right|^{1/2} = \frac{k_+^2 C_T}{2|k_-^2|}$$

we may estimate

$$\frac{\left|1+|\tau|^2\right|^{1/2}}{\left|k_+^2 - |\tau|^2\right|^{1/2}} < 2\frac{|k_-^2|}{k_+^2 C_T} \sqrt{1 + k_+^2 + \frac{k_+^4 C_T^2}{4|k_-|^4}}$$

if $\tau \in \Sigma_1$ and therefore by Lemma 5.2.11

$$\left| \frac{k_-^2}{k_+^2} \int_{\Sigma_1} \frac{|\tau_2 \hat{u}_1 - \tau_1 \hat{u}_2|^2}{\sqrt{k_+^2 - |\tau|^2}} \, d\tau \right|$$

$$\begin{aligned}
&\leq \quad 2\frac{|k_-|^4}{k_+^4 C_T}\sqrt{1+k_+^2+\frac{k_+^4 C_T^2}{4|k_-|^4}}||\operatorname{Curl} u||^2_{H^{-1/2}(\mathbb{R}^2)} \\
&\leq \quad \underbrace{2\frac{|k_-|^4 C_E}{k_+^4 \quad C_T}\sqrt{1+k_+^2+\frac{k_+^4 C_T^2}{4|k_-|^4}}}_{=:C_1}||\operatorname{curl} u||^2_{L^2(D)} .
\end{aligned}$$

As discussed in Lemma 1.3.3, we now choose φ such that

$$\operatorname{Re}\left(e^{i\varphi} i \frac{k_-^2}{k_+^2}\int\limits_{\Sigma_1}\frac{|\tau_2\hat{u}_1-\tau_1\hat{u}_2|^2}{\sqrt{k_+^2-|\tau|^2}}\,d\tau\right)\leq \frac{1}{4}||\operatorname{curl} u||^2_{L^2(\mathbb{R}^3_-)} ,$$

then by above considerations there exists a constant C_2 such that

$$\begin{aligned}
&\operatorname{Re}\left[e^{i\varphi}\left(\frac{k_-^2}{k_+^2}\int\limits_{\Sigma_1}\frac{|\tau_2\hat{u}_1-\tau_1\hat{u}_2|^2}{\sqrt{k_+^2-|\tau|^2}}\,d\tau\right.\right. \\
&\qquad\left.\left.+\frac{1}{2}||\operatorname{curl} u||^2_{L^2(\mathbb{R}^3_-)}+\frac{k_-^2}{2}||u||^2_{L^2(\mathbb{R}^3_-)}\right)\right] \\
&\qquad\qquad \geq C_2\left(||\operatorname{curl} u||^2_{L^2(\mathbb{R}^3_-)}+||u||^2_{L^2(\mathbb{R}^3_-)}\right) .
\end{aligned}$$

The remaining norms are bounded due to the usual traces because $|\tau|>k_++\varepsilon$.

For the second segment consider

$$\Sigma_2 := \{\tau\in\mathbb{R}^2 :\ k_+<|\tau|<\sqrt{k_+^2+\frac{k_+^4 C_T^2}{4|k_-|^4}}\}$$

and representation (i) of $b_H(u,u)$. Here, the $|\tau|>k_+$ part of the (D) term is on the wrong side of the coercivity plane, but by monotonicity of the square root term and the specific choice of Σ_2, we see that

$$\begin{aligned}
\left|\frac{k_-^2}{k_+^2}\int\limits_{\Sigma_2}\sqrt{k_+^2-|\tau|^2}\,|\hat{u}_T|^2\,d\tau\right| \quad &\leq \quad \frac{C_T}{2}||\tilde{u}||_{H^{-1/2}(\mathbb{R}^2)} \\
&\leq \quad \frac{1}{2}\left(||u||^2_{L^2(\mathbb{R}^3_-)}+||\operatorname{curl} u||^2_{L^2(\mathbb{R}^3_-)}\right)
\end{aligned}$$

it is sufficiently small in Σ_2, thus

$$\operatorname{Re}\left[e^{i\varphi}\left(-i\frac{k_-^2}{k_+^2}\int\limits_{\Sigma_2}\sqrt{k_+^2-|\tau|^2}\,|\hat{u}_T|^2\,d\tau\right.\right.$$
$$\left.\left.+\frac{1}{2}||\operatorname{curl}u||^2_{L^2(\mathbb{R}^3_-)}+\frac{k_-^2}{2}||u||^2_{L^2(\mathbb{R}^3_-)}\right)\right]\geq 0\,.$$

In this segment the estimation of the α trace norms are most important; it turns out that

$$\operatorname{Re}\left(e^{i\varphi}(-i)\frac{k_-^2}{k_+^2}\int\limits_{\Sigma_2}\frac{|\tau\cdot\hat{u}_T|^2}{\sqrt{k_+^2-|\tau|^2}}\,d\tau\right)$$
$$\geq C_4\underbrace{\int\limits_{\Sigma_2}\frac{\left(1+|\tau|^2\right)^{\alpha-1/2}}{\left|k_+^2-|\tau|^2\right|^{\alpha+1}}|\tau\cdot\hat{u}_T|^2\,d\tau}_{||\operatorname{Div}u||^2_{H^{-3/2}_{k,\alpha+1}(\operatorname{Div};\mathbb{R}^2)|_{\Sigma_2}}}\,,$$

for some constant C_4 if and only if $\alpha\leq-\frac{1}{2}$: This is the necessary condition under which the denominator of the Bessel potential factor of the $||\operatorname{Div}u||^2_{H^{-3/2}_{k,\alpha+1}(\operatorname{Div};\mathbb{R}^2)}$ norm has at most the same singularity as the left-hand side. The estimate of the surface curl norm of u follows from Lemma 4.3.6 as $\alpha<0$.

For the third segment consider

$$\Sigma_3:=\{\tau\in\mathbb{R}^2:\ |\tau|<k_+\}$$

and representation (i) of $b_H(u,u)$. Here, all summands multiplied by $e^{i\varphi}$ have a positive real part, thus estimates for $||u||$ and $||\operatorname{curl}u||$ are given, and as before

$$\operatorname{Re}\left(e^{i\varphi}(-i)\frac{k_-^2}{k_+^2}\int\limits_{\Sigma_3}\frac{|\tau\cdot\hat{u}_T|^2}{\sqrt{k_+^2-|\tau|^2}}\,d\tau\right)$$
$$\geq C_4\underbrace{\int\limits_{\Sigma_3}\frac{\left(1+|\tau|^2\right)^{\alpha-1/2}}{\left|k_+^2-|\tau|^2\right|^{\alpha+1}}|\tau\cdot\hat{u}_T|^2\,d\tau}_{||\operatorname{Div}u||^2_{H^{-3/2}_{k,\alpha+1}(\operatorname{Div};\mathbb{R}^2)|_{\Sigma_3}}}$$

if and only if $\alpha \le -\frac{1}{2}$, and the estimate for the surface curl norm of u is again easily fulfilled by Lemma 4.3.6 as $\alpha < 0$. □

CONCLUSION 5.6.6. The weak problem for the magnetic field intensity (HBVP2) is uniquely solvable for $\alpha = -1/2$ by the Lax-Milgram Lemma. This also yields a solution for (EBVP2) for the same α, and the solutions may be extended to solutions in the full space $\mathbb{R}^3$.

LEMMA 5.6.7. *A solution H to Problem 5.1.2 (HBVP1) satisfying the integrability condition $H|_D \in H_{k,-1/2}(\mathrm{curl};\mathrm{Curl};D)$ is a solution to Problem 5.5.4 (HBVP2).*

PROOF. If H is a solution to HBVP1, then a function $u := H+g$, where g is chosen in HBVP2, will satisfy

$$\mathrm{curl}\,\mathrm{curl}\,u + k_-^2\,u = \mathrm{curl}\,\mathrm{curl}\,g + k_-^2\,g$$

in $\mathbb{R}^3_-$, and $n \times \mathrm{curl}\,u|_{\partial\Omega} = 0$. Let

$$\varphi \in H_{k,-1/2}(\mathrm{curl};\mathrm{Curl};D) \cap C^\infty(D,\mathbb{C}^3)$$

be an arbitrary test function. Then, by multiplication of above equation by φ, integration over D and the integral identity (5.4), we gain

$$\begin{aligned} & (\mathrm{curl}\,u, \mathrm{curl}\,\varphi)_{L^2(D)} - k_-^2\,(u,\varphi)_{L^2(D)} - \langle n \times \mathrm{curl}\,u, \varphi\rangle_\Gamma \\ = \; & (\mathrm{curl}\,g, \mathrm{curl}\,\varphi)_{L^2(D)} - k_-^2\,(g,\varphi)_{L^2(D)}\,. \end{aligned}$$

Since the term $n \times \mathrm{curl}\,u$ may be expressed by the Calderon operator, using Theorem 5.3.1 and density proven in Lemma 5.2.4, the vector field H satisfies $b_H(H,\varphi) = a_H(\varphi)$ for any test function $\varphi \in H_{k,-1/2}(\mathrm{curl};\mathrm{Curl};D)$. □

LEMMA 5.6.8. *The extension U of a solution u to Problem 5.5.4 (HBVP2) defined by*

$$U(x) = \begin{cases} v(x) & \textit{for} \quad x \in \mathbb{R}^3_+\,, \\ u(x) & \textit{for} \quad x \in \mathbb{R}^3_- \cup \Gamma\,, \end{cases}$$

where v is defined as a divergence free vector field using a double layer potential

$$\begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} = \frac{2k_-^2}{k_+^2} \int\limits_\Gamma \frac{\partial \Phi(x,y)}{\partial y_3} \begin{pmatrix} u_1(y) \\ u_2(y) \end{pmatrix} dy \,,$$

$$v_3(x) = \int\limits_{x_3}^{\infty} \left(-\frac{\partial v_1(x_1, x_2, y_3)}{\partial x_1} - \frac{\partial v_2(x_1, x_2, y_3)}{\partial x_2} \right) dy_3$$

is a solution to Problem 5.1.2 (HBVP1).

Proof. Since the Calderon operator is defined as a Fourier multiplier, it is as translation invariant and linear like the Dirichlet-to-Neumann operator Λ. Therefore,

$$b_H(\Delta_j^h u, \varphi) = -b_H(u, \Delta_j^{-h}) = -a_H(\Delta_j^{-h}\varphi)\,, \qquad j = 1, 2$$

if u is a solution to HBVP2. A corresponding equation holds for $u-g$, and thus we may bound the difference quotients by the coercivity of b_H

$$C||\Delta_j^h u||_{H^1(D)}^2 \le ||u||_{H^1(D)}||g||_{H^1(D)}$$

for all $h > 0$, so $\frac{\partial}{\partial x_j} u \in H^1(D)$. Using the sesquilinear form a_H to find a bound for the derivative of u with respect to x_3, the third component is also in $H^1(D)$ and therefore $u \in H^2(D) \subset C(D)$. By smoothness of H^i and $\partial\Omega$, we may repeat the argumentation resulting in $u \in C^2(\bar{D})$. The double layer potential Ansatz for $\mathbb{R}_+^3$ with such continuous and bounded potential therefore fulfills the radiation conditions and Maxwell's equations and the jump conditions. □

Please note that the uniqueness result of Delbary et al. [**11**] is applicable using $u \in H(\mathrm{curl}, D) \cap C(D)$ instead of the exponential decay condition in the proof. Therefore, the weak and classical problems are equivalent.

Index

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