Spectral multipliers, R-bounded homomorphisms, and analytic diffusion semigroups

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Preface

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Contents

1 Introduction

In this thesis, we investigate the smooth functional calculus for operators with spectrum in $\mathbb{R}_+$, more specifically spectral multiplier theorems.

Many differential operators of Laplacian type have such a functional calculus. The first cornerstone are the works of Mihlin and Hörmander around the year 1960, which give a sufficient condition for a function f to be a Fourier multiplier, expressed in terms of bounds on derivatives of f.

Since then, functional calculi modeled after Mihlin's and Hörmander's theorem have been established for a large variety of differential operators, and in the last thirty years, spectral multiplier theorems have been intensely studied in a large variety of contexts, including elliptic operators on domains and manifolds, Schrödinger operators, and Sublaplacians on Lie groups, mainly with methods from the theory of partial differential equations and harmonic analysis.

An important consequence of such functional calculi is that they immediately give information on bounds of operators associated with the differential operator, such as its resolvents, imaginary powers, the generated semigroup and wave operators. This in turn has been very successfully applied to solve associated problems such as wave and Schrödinger equations, and to deduce regularity of the solutions.

This thesis adds a unifying theme to the large class of examples and their applications by approaching them with a combination of methods from spectral theory and the geometry of Banach spaces.

The starting point is the construction of a functional calculus for Mihlin- and Hörmander type functions and certain 0-sectorial operators defined on some Banach space. The quality of the calculus (i.e. the order of the required smoothness) is then related to estimates of typical families of operators generated by the calculus, such as the above mentioned resolvents, imaginary powers, semigroup operators and wave operators. The dependency of the order of smoothness on the geometry of the underlying Banach space can be expressed in terms of the type and cotype of the space.

Furthermore, there are deep connections between the functional calculus and central results of harmonic analysis, such as the Paley-Littlewood theory and quadratic estimates.

One advantage of the Banach space approach is that it is also applicable to non-commutative L^p-spaces and diffusion operators defined in this context.

The following concept related to unconditionality will play a central role.

Let $(\varepsilon_n)_{n\geqslant 1}$ be a sequence of independent random variables on some probability space Ω such that $\text{Prob}(\{\varepsilon_n = 1\}) = \text{Prob}(\{\varepsilon_n = -1\}) = \frac{1}{2}$.

A set of bounded linear operators $\tau \subset B(X)$, where X is some Banach space, is called R-bounded if for some $C > 0$,

$$\int_\Omega \Big\| \sum_{n=1}^N \varepsilon_n(\omega) T_n x_n \Big\|_X^2 d\omega \leqslant C \int_\Omega \Big\| \sum_{n=1}^N \varepsilon_n(\omega) x_n \Big\|_X^2 d\omega \qquad (*)$$

for any finite families $T_1, \ldots, T_N \in \tau$ and $x_1, \ldots, x_N \in X$.

The notion of R-boundedness has been used implicitly in 1986 by Bourgain [13], and formalized for the first time in 1994 by Berkson and Gillespie [7].

If X is a Hilbert space, then

$$\int_\Omega \Big\| \sum_{n=1}^N \varepsilon_n(\omega) x_n \Big\|_X^2 d\omega = \sum_{n=1}^N \|x_n\|^2, \qquad (1.1)$$

due to the orthogonality of the ε_n's in $L^2(\Omega)$, so that $*$ is equivalent to the fact that τ is a bounded subset of $B(X)$. Furthermore, if X is an L^p-space, then R-boundedness can be expressed in terms of quadratic estimates.

For a general Banach space X, the left-hand side of 1.1 has proved to be an adequate unconditional substitute to the orthogonality in Hilbert spaces.

Since the beginning of the 2000s, with the works of Clément, de Pagter, Sukochev and Witvliet [22], and Weis [135], the concept of R-boundedness has become a highly powerful tool in vector-valued harmonic analysis, for maximal regularity for parabolic problems, for Schauder decompositions in Banach spaces, in spectral theory, and for functional calculi.

Let us now give a more detailed overview of our results chapter by chapter.

In chapter 3, we consider a certain class of abstract and optimal functional calculi, that is, homomorphisms

$$u : C(K) \to B(X)$$

where K is a compact set and $C(K)$ the algebra of all continuous functions $f : K \to \mathbb{C}$. Such homomorphisms appear naturally and play a major role in several fields of operator theory such as the classification of C^*-algebras, or spectral measures.

If X is a Hilbert space, then it is known that its tensor extension $\hat{u}$ with the commutant of the range E_u

$$\hat{u} : C(K) \otimes E_u \to B(X), \sum_{n=1}^N f_n \otimes T_n \mapsto \sum_{n=1}^N u(f_n) T_n$$

is again a bounded homomorphism, where $C(K) \otimes E_u$ is normed as a subspace of $C(K; B(X))$.

If X is a general Banach space, this theorem is not true anymore. The main result of this chapter gives an adequate Banach space analogue of the extension $\hat{u}$. A consequence of our theorem is that u is R-bounded in a certain matricial sense.

A first application concerns unconditional bases (e_n) in L^p spaces. Our result complements the work of Johnson and Jones [63] and Simard [119] and shows that under a suitable change of density, (e_n) becomes an unconditional basis in the Hilbert space L^2.

A second application concerns McIntosh's H^∞ functional calculus for sectorial operators A [96, 24]. If A has an H^∞ calculus to any positive angle which is uniformly bounded, in the sense that the norm of this calculus is independent of the angle, then also the operator valued H^∞ calculus of A is uniformly bounded. This is an analogue of a celebrated result of Kalton and Weis [73].

In chapter 4, we study a more common case of spectral multiplier theorems, and consider homomorphisms u defined on a Banach algebra E of functions defined on $\mathbb{R}_+$, above all the algebras defined by the Mihlin and Hörmander conditions

$$u : E \to B(X),\ f \mapsto u(f) = f(A).$$

Here, u is associated with an operator A whose spectrum is contained in $\mathbb{R}_+$. Of particular interest are the Laplace type operators A mentioned earlier. To show the boundedness of u for such A, one usually reconstructs the calculus from semigroup or wave operators. In the known proofs, simple norm bounds are not enough, and a variety of stronger assumptions are essential, such as Gaussian bounds, lattice positivity, contractivity on an L^p scale together with self-adjointness on L^2, and also information on wave propagation speed.

In contrast, we use an operator theoretic approach for Mihlin's and Hörmander's spectral multiplier theorems. We will show that to some extend, R-boundedness is an adequate unifying substitute for the above mentioned assumptions.

There are two indications for this:

Firstly, Mihlin functional calculus is strongly connected with bounds on the holomorphic calculus. This is already observed in the fundamental paper [24] on H^∞ calculus. Fortunately, by Kalton's and Weis' work [73], we already have a method at hand that allows us to characterize the H^∞ calculus in terms of R-boundedness.

Secondly, a basic ingredient in proving Hörmander's theorem is a decomposition of the function f into a series $f = \sum_{n \in \mathbb{Z}} f_n$, where f_n has its support in a dyadic interval $[2^{n-1}, 2^{n+1}]$. This extends the well-known Paley-Littlewood decomposition for the Laplace operator on Euclidean space. One usually proves first the boundedness of $f_n(A)$ and then reassembles the series and then recovers $f(A)$ as a sum of multipliers. This last step requires the unconditional strong convergence of the sum $\sum_n f_n(A)$, and this is where R-boundedness comes in.

The following are our central results.

Firstly, we are able to characterize the Hörmander functional calculus in terms of "averaged R-boundedness" of the classical operator families associated with A, such as resolvents, imaginary powers, semigroup and wave operators. This averaged R-boundedness is a weakened form of square function estimates.

The square function estimates themselves are characterized by the matricial R-boundedness of the functional calculus, which we had introduced in chapter 3.

Next, we obtain a Mihlin functional calculus under norm bounds and R-bounds of the classical operator families named above. Furthermore, we will see that information on the type and cotype of the underlying Banach space X allows to improve the quality of that functional calculus.

Finally, by the mentioned Paley-Littlewood decomposition, we obtain new characterizations of fractional domain spaces of A which reduce to the classical Triebel-Lizorkin spaces if A is Laplace, and also their real interpolation spaces, corresponding to Besov spaces in the classical case.

In chapter 5, we consider the functional calculus for strongly continuous groups $(U(t))_{t\in\mathbb{R}}$ on X. Boyadziev and deLaubenfels [15] have shown that if $U(t)$ has an exponential norm growth and X is a Hilbert space, then its generator has a bounded holomorphic calculus.

We examine groups of polynomial growth $\|U(t)\| \leqslant C(1+|t|)^\alpha$ having a bounded holomorphic calculus. By means of a transference principle, we show that the growth rate α can be equivalently expressed in terms of a functional calculus. The function space of this calculus differs from the classical ones, and thus its description demands particular attention. We show that it admits a certain atomic decomposition related to a Fourier series. This is a decisive difference to the spaces in chapter 4, which are closer related to bounds on smoothness.

If we choose $U(t)$ as the imaginary powers A^{it} of some sectorial operator A, that difference gives complementary information about the results of chapter 4 and it shows that the Mihlin calculus result mentioned above is optimal.

Further, we obtain norm bounds and R-bounds of semigroup operators and resolvents of A which are optimal within the class of polynomially bounded imaginary powers.

Finally, chapter 6 is devoted to diffusion semigroups T_t on non-commutative L^p-spaces. These are a generalization of the classical L^p-spaces, which are associated to a von Neumann algebra; as an example we mention the von Neumann Schatten classes S^p of compact operators on a Hilbert space with p-summable singular numbers. Here, the classes of operators with spectral multiplier theorems are not identified yet. However our methods give information on the optimal sector angle of sectorial diffusion semigroups.

In the case of classical L^p-spaces, the following is known: If a semigroup is defined on a scale of L^p-spaces, is contractive for all $1 \leqslant p < \infty$, and is self-adjoint on L^2, then the semigroup can be analytically extended to a sector in the complex plane, whose opening angle depends on

p. This was proved via complex interpolation by Stein [121]. Later, Liskevich and Perelmuter have improved Stein's result by enlarging the sector angle to

$$\omega_p = \frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}},$$

under the additional hypothesis that the operators T_t are (lattice) positive [92]. This angle is also optimal.

We extend Liskevich's and Perelmuter's result with methods of operator space theory to the case of non-commutative L^p-spaces. Surprisingly, the non-commutative theorem improves the commutative one, and one application of our result is that we can omit the positivity assumption in Liskevich's and Perelmuter's theorem.

For more detailed information, we refer to the first sections of chapters 3, 4, 5 and 6, which can be read independently to a large extend.

Chapter 2 contains some notational conventions, and the common background for chapters 3, 4 and 5.

2 Notations and Preliminaries

2.1 Notations

Let us record some notations.

- For the Kronecker symbol, we write $\delta_{n=k} \equiv \delta_{nk} = \begin{cases} 1 & n = k \\ 0 & n \neq k \end{cases}$.
- Further, for the characteristic function on I, we reserve the notation χ_I.
- We extensively use the notation $a \lesssim b$ for two non-negative expressions a and b to state that there is some constant $C > 0$ independent of a and b such that $a \leqslant Cb$. If we want to emphasise that C may depend on a third expression d, we write $a \lesssim_d b$.
- We also use $a \cong b$ in short for $a \lesssim b \lesssim a$.
- We denote $\mathbb{R}_+ = (0, \infty)$ the open half-line.
- The Fourier transform is denoted by $\mathcal{F}f$ or $\hat{f}$, and we use the convention $\mathcal{F}f(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$.
- The inverse Fourier transform is then denoted by $\mathcal{F}^{-1}$ or $\check{f}$.
- We will also use the so-called Mellin transform M mapping a function f defined on $\mathbb{R}_+$ to a function defined on $\mathbb{R}$, given by the formula $Mf(t) = \int_0^\infty t^{is} f(s) \frac{ds}{s}$.
- We will use the Gamma-function $\Gamma(z) = \int_0^\infty t^z e^{-t} \frac{dt}{t}$.
- The space of complex $n \times n$ matrices is denoted by M_n.
- If X is a Banach space, then Id_X denotes the identity mapping on X.
- The dual space of X is denoted by X^* in chapter 3, and by X' in chapters 4, 5 and 6.
- The notation $\langle x, y \rangle$ is used to denote an application of a distribution to some test function, and also to denote an application of a linear form to a Banach space element, depending on the context.
- If A is an (unbounded) operator on X, then its domain is denoted by $D(A)$, its range by $R(A)$ and its spectrum by $\sigma(A)$.
- If $\lambda \in \mathbb{C}\backslash\sigma(A)$, we denote the resolvent by $R(\lambda, A)$ or $(\lambda - A)^{-1}$.
- If $-A$ generates a semigroup, we denote that semigroup by $T(t)$ or T_t.

2.2 The holomorphic functional calculus

We now recall the basic notions on holomorphic functional calculus for sectorial and strip-type operators. For more information, we refer e.g. to [24, 73, 81, 86], and [52] for the strip-type case.

Definition 2.1 *For* $\omega \in (0, \pi)$, *let*

$$\Sigma_\omega = \{z \in \mathbb{C}\backslash\{0\} : \, |\arg z| < \omega\} \text{ and } \operatorname{Str}_\omega = \{z \in \mathbb{C} : \, |\operatorname{Im} z| < \omega\}.$$

We consider the following spaces of holomorphic functions.

$$\begin{aligned} H^\infty(\Sigma_\omega) &= \{f : \Sigma_\omega \to \mathbb{C} : \, f \text{ hol. and bounded}\} \\ H^\infty(\operatorname{Str}_\omega) &= \{f : \operatorname{Str}_\omega \to \mathbb{C} : \, f \text{ hol. and bounded}\} \\ H^\infty_0(\Sigma_\omega) &= \{f \in H^\infty(\Sigma_\omega) : \, \exists C, \varepsilon > 0 : \, |f(z)| \leqslant C|z(1+z)^{-2}|^\varepsilon\} \\ H^\infty_0(\operatorname{Str}_\omega) &= \{f \in H^\infty(\operatorname{Str}_\omega) : \, \exists C, \varepsilon > 0 : \, |f(z)| \leqslant C|e^z(1+e^z)^{-2}|^\varepsilon\} \\ \operatorname{Hol}(\Sigma_\omega) &= \{f : \Sigma_\omega \to \mathbb{C} : \, \exists n \in \mathbb{N} : \, f(z)(z(1+z)^{-2})^n \in H^\infty_0(\Sigma_\omega)\} \\ \operatorname{Hol}(\operatorname{Str}_\omega) &= \{f : \operatorname{Str}_\omega \to \mathbb{C} : \, \exists n \in \mathbb{N} : \, f(z)(e^z(1+e^z)^{-2})^n \in H^\infty_0(\operatorname{Str}_\omega)\} \end{aligned}$$

For $E \in \{H^\infty, H^\infty_0, \operatorname{Hol}\}$, we have $f \in E(\Sigma_\omega) \iff f \circ \exp \in E(\operatorname{Str}_\omega)$.

The spaces $H^\infty(\Sigma_\omega)$ and $H^\infty(\operatorname{Str}_\omega)$ are equipped with the norm

$$\|f\|_{\infty,\omega} = \sup_{z \in \Sigma_\omega} |f(z)| \text{ and } \|f\|_{\infty,\omega} = \sup_{z \in \operatorname{Str}_\omega} |f(z)|,$$

for which they are a Banach algebra.

By the identity theorem for holomorphic functions, for $\omega > \theta$, we can identify $H^\infty(\Sigma_\omega)$ with a subspace of $H^\infty(\Sigma_\theta)$ for and thus consider e.g. $\bigcup_{\omega > \omega_0} H^\infty(\Sigma_\omega)$. The same holds for the other five spaces above in place of $H^\infty(\Sigma_\omega)$.

Definition 2.2 *Let* $\omega \in (0, \pi)$. *Let* X *be a Banach space and let* A *be a densely defined operator on* X *such that its spectrum* $\sigma(A)$ *is contained in* $\overline{\Sigma_\omega}$. *Assume that for any* $\theta > \omega$, *there exists* C *such that*

$$\|\lambda(\lambda - A)^{-1}\| \leqslant C$$

for any $\lambda \in \mathbb{C}\backslash\{0\}$ *such that* $|\arg \lambda| \geqslant \theta$.

Then A *is called a sectorial operator. We denote* $\omega(A)$ *the infimum of all* ω *such that the above holds.*

Let B *be a densely defined operator on* X *such that* $\sigma(B) \subset \operatorname{Str}_\omega$ *for some* $\omega > 0$. *Assume that for any* $\theta > \omega$, *there exists* C *such that*

$$\|(\lambda - B)^{-1}\| \leqslant C$$

for any $\lambda \in \mathbb{C}\backslash \operatorname{Str}_\theta$.

Then B *is called strip-type operator. We denote* $\omega(B)$ *the infimum over all such* ω.

The definition of sectorial operators varies in the literature. Sometimes A is not supposed to be densely defined [24, 52] or is additionally supposed to be injective and to have dense range [81]. Also, in [52, p. 91], the definition of strip-type operators does not include the dense domain.

Assume that $(U(t))_{t\in\mathbb{R}}$ is a c_0-group and denote its generator by iB. Then B is a strip-type operator [52, exa 4.1.1]. In the sequel, we will only consider strip-type operators of this type.

For a sectorial operator A and $f \in H_0^\infty(\Sigma_\omega)$ with $\omega \in (\omega(A), \pi)$, one defines

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} f(\lambda)R(\lambda, A)d\lambda, \tag{2.1}$$

where $\omega(A) < \gamma < \omega$ and Γ_γ is the boundary $\partial\Sigma_\gamma$ oriented counterclockwise. Similarly, for a strip-type operator B and $f \in H_0^\infty(\mathrm{Str}_\omega)$ with $\omega > \omega(B)$, one defines

$$f(B) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} f(\lambda)R(\lambda, B)d\lambda, \tag{2.2}$$

where $\omega(B) < \gamma < \omega$ and Γ_γ is the boundary $\partial\,\mathrm{Str}_\gamma$ oriented counterclockwise.

The definitions do not depend on γ and the resulting mappings $H_0^\infty(\Sigma_\omega) \to B(X)$, $f \mapsto f(A)$ and $H_0^\infty(\mathrm{Str}_\omega) \to B(X)$, $f \mapsto f(B)$ are algebra homomorphisms.

We say that A has a (bounded) $H^\infty(\Sigma_\omega)$ calculus if this homomorphism is bounded, that is, there exists a constant $C > 0$ such that

$$\|f(A)\| \leqslant C\|f\|_{\infty,\omega} \quad (f \in H_0^\infty(\Sigma_\omega)). \tag{2.3}$$

Similarly, we say that B has a (bounded) $H^\infty(\mathrm{Str}_\omega)$ calculus if $H_0^\infty(\mathrm{Str}_\omega) \to B(X)$, $f \mapsto f(B)$ is bounded.

The boundedness of the $H^\infty(\Sigma_\omega)$ and $H^\infty(\mathrm{Str}_\omega)$ calculus depends in general on the angle ω. We say that A has an H^∞ calculus, if it has a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \in (0, \pi)$. We say that B has an H^∞ calculus, if it has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for some $\omega > 0$.

We now turn to sectorial operators with dense range. Note that such operators are automatically injective. This follows e.g. from [24, thm 3.8]. We cite the following proposition of holomorphic functional calculus, see e.g. [24], [81, sect 9,15B] or [52, sect 1,2].

Proposition 2.3 *Let A be a sectorial operator with dense range. There exists a unique mapping*

$$\bigcup_{\omega>\omega(A)} \mathrm{Hol}(\Sigma_\omega) \to \{\textit{closed and densely defined operators on } X\},\ f \mapsto f(A)$$

with the properties:

(1) *If $f \in H_0^\infty(\Sigma_\omega)$ for some $\omega > \omega(A)$, then $f(A)$ is given by 2.1.*

(2) *For $\lambda \in \mathbb{C}\backslash\overline{\Sigma_{\omega(A)}}$, $(\lambda - \cdot)^{-1}$ is mapped to the resolvent $R(\lambda, A)$, and the constant function 1 is mapped to the identity Id_X.*

(3) *Whenever* $f \in \mathrm{Hol}(\Sigma_\omega)$ *such that* $g(z) = f(z)\frac{z^n}{(1+z)^{2n}} \in H_0^\infty(\Sigma_\omega)$ *for some* $n \in \mathbb{N}$, *then* $f(A)x = g(A)(1+A)^{2n}A^{-n}x$ *for* x *in* $D(A^n) \cap R(A^n)$, *which moreover is a dense subset of* X.

An analogous statement holds for a strip-type operator B and Str_ω in place of A and Σ_ω, with a mapping

$$\bigcup_{\omega > \omega(B)} \mathrm{Hol}(\mathrm{Str}_\omega) \to \{\text{closed and densely defined operators on } X\},\ f \mapsto f(B),$$

where in particular $f(B)$ is given by 2.2 for $f \in H_0^\infty(\mathrm{Str}_\omega)$. We refer to [52, p. 91-96] for the details.

The mappings $f \mapsto f(A)$ and $f \mapsto f(B)$ are called extended holomorphic calculus (cf. [81, sect 15B]).

Remark 2.4

(1) *Note that the logarithm belongs to* $\mathrm{Hol}(\Sigma_\omega)$ *for any* $\omega \in (0, \pi)$.

Let B *be a strip-type operator and assume that there exists a sectorial operator* A *with dense range such that* $B = \log(A)$. *This is the case, e.g., if* B *has a bounded* $H^\infty(\mathrm{Str}_\omega)$ *calculus for some* $\omega < \pi$ *[52, prop 5.3.3]. Then by [52, thm 4.3.1],* $\omega(A) = \omega(B)$ *and by [52, thm 4.2.4], for any* $\omega \in (\omega(A), \pi)$, *one has*

$$f(B) = (f \circ \log)(A) \quad (f \in \mathrm{Hol}(\mathrm{Str}_\omega)). \tag{2.4}$$

(2) *If* A *is a sectorial operator with dense range and has a bounded* $H^\infty(\Sigma_\omega)$ *calculus for some* $\omega \in (\omega(A), \pi)$, *then 2.3 extends to all* $f \in H^\infty(\Sigma_\omega)$.

An important tool when dealing with the holomorphic functional calculus is the following proposition, which is often called convergence lemma. For a proof, we refer e.g. to [24, lem 2.1] in the sectorial case and [52, prop 5.1.7] in the strip case.

Proposition 2.5

(1) *Let* A *be a sectorial operator with dense range. Assume that* A *has a bounded* $H^\infty(\Sigma_\omega)$ *calculus for some* $\omega \in (\omega(A), \pi)$. *Let* $f \in H^\infty(\Sigma_\omega)$ *and* $(f_n)_n$ *be a sequence in* $H^\infty(\Sigma_\omega)$ *such that*

$$f_n(z) \to f(z) \text{ for all } z \in \Sigma_\omega \text{ and } \sup_n \|f_n\|_{\infty,\omega} < \infty.$$

Then $f_n(A)x \to f(A)x$ *for any* $x \in X$.

(2) *Let* B *be a strip-type operator. Assume that* B *has a bounded* $H^\infty(\mathrm{Str}_\omega)$ *calculus for some* $\omega > \omega(B)$. *Let* $f \in H^\infty(\mathrm{Str}_\omega)$ *and* $(f_n)_n$ *be a sequence in* $H^\infty(\mathrm{Str}_\omega)$ *such that*

$$f_n(z) \to f(z) \text{ for all } z \in \mathrm{Str}_\omega \text{ and } \sup_n \|f_n\|_{\infty,\omega} < \infty.$$

Then $f_n(B)x \to f(B)x$ *for any* $x \in X$.

2.3 R-boundedness and γ-boundedness

Let Ω_0 be a probability space and $(\varepsilon_k)_{k\geqslant 1}$ a sequence of independent Rademacher variables on Ω_0. That is, the ε_k's take values in $\{-1,1\}$ and $\mathrm{Prob}(\{\varepsilon_k = 1\}) = \mathrm{Prob}(\{\varepsilon_k = -1\}) = \frac{1}{2}$. For any Banach space X, we let $\mathrm{Rad}(X) \subset L^2(\Omega_0; X)$ be the closure of $\mathrm{Span}\{\varepsilon_k \otimes x \,:\, k \geqslant 1,\ x \in X\}$ in $L^2(\Omega_0; X)$. Thus for any finite family $x_1, \ldots, x_n$ in X, we have

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} = \left(\mathbb{E}\Big\|\sum_k \varepsilon_k(\cdot)x_k\Big\|_X^2\right)^{\frac{1}{2}} = \Big(\int_{\Omega_0}\Big\|\sum_k \varepsilon_k(\lambda)\,x_k\Big\|_X^2\,d\lambda\Big)^{\frac{1}{2}}.$$

In some cases, we shall also use Rademacher variables ε_k indexed by $k \in \mathbb{Z}$ or doubly indexed by $k \in \mathbb{Z} \times \mathbb{Z}$.

Parallely to this, we consider the case that Rademacher distribution is replaced by standard Gaussian distribution.

Namely, let $(\gamma_k)_{k\geqslant 1}$ be a sequence of independent standard Gaussian variables on Ω_0. Then we let $\mathrm{Gauss}(X) \subset L^2(\Omega_0; X)$ be the closure of $\mathrm{Span}\{\gamma_k \otimes x \,:\, k \geqslant 1,\ x \in X\}$ in $L^2(\Omega_0; X)$. For any finite family $x_1, \ldots, x_n$ in X, we have

$$\Big\|\sum_k \gamma_k \otimes x_k\Big\|_{\mathrm{Gauss}(X)} = \left(\mathbb{E}\Big\|\sum_k \gamma_k(\cdot)x_k\Big\|_X^2\right)^{\frac{1}{2}} = \Big(\int_{\Omega_0}\Big\|\sum_k \gamma_k(\lambda)\,x_k\Big\|_X^2\,d\lambda\Big)^{\frac{1}{2}}.$$

It will be convenient to let $\mathrm{Rad}_n(X)$ denote the subspace of $\mathrm{Rad}(X)$ of all finite sums $\sum_{k=1}^n \varepsilon_k \otimes x_k$, and $\mathrm{Gauss}_n(X)$ the subspace of $\mathrm{Gauss}(X)$ of all finite sums $\sum_{k=1}^n \gamma_k \otimes x_k$.

Now let $\tau \subset B(X)$. We say that τ is R-bounded if there is a constant $C \geqslant 0$ such that for any finite families $T_1, \ldots, T_n$ in τ, and $x_1, \ldots, x_n$ in X, we have

$$\Big\|\sum_k \varepsilon_k \otimes T_k x_k\Big\|_{\mathrm{Rad}(X)} \leqslant C\,\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}.$$

In this case, we let $R(\tau)$ denote the smallest possible C. It is called the R-bound of τ. By convention, we write $R(\tau) = \infty$ if τ is not R-bounded.

For $\sigma, \tau \subset B(X)$ one checks easily that

$$R(\sigma \circ \tau) \leqslant R(\sigma)R(\tau), \tag{2.5}$$

where $\sigma \circ \tau = \{S \circ T \,:\, S \in \sigma,\, T \in \tau\} \subset B(X)$.

Let $(\tau_n)_n$ be a sequence in $B(X)$ such that $\tau_n \subset \tau_{n+1}$ for any $n \in \mathbb{N}$. Then

$$R\left(\bigcup_n \tau_n\right) = \sup_n R(\tau_n). \tag{2.6}$$

Indeed, let $T_1, \ldots, T_L \in \bigcup_n \tau_n$. Then there exists n_0 such that $T_1, \ldots, T_L \in \tau_{n_0}$, so that $R(\{T_1, \ldots, T_L\}) \leqslant R(\tau_{n_0}) \leqslant \sup_n R(\tau_n)$. But clearly, $R(\bigcup_n \tau_n)$ equals the supremum of all such $R(\{T_1, \ldots, T_L\})$.

Similarly, replacing the Rademachers ε_k by Gaussian variables γ_k, the terms γ-bound(ed) and $\gamma(\tau)$ are defined.

There are some folklore results about R-boundedness and γ-boundedness which we shall recall for future reference.

Proposition 2.6

(1) *(Kahane's inequality) If one replaces in the definition of* $\mathrm{Rad}(X)$ *and* $\mathrm{Gauss}(X)$ *the space* $L^2(\Omega_0; X)$ *by* $L^p(\Omega_0; X)$ *for some* $1 \leqslant p < \infty$, *one gets the same spaces with an equivalent norm:*

$$\left(\int_{\Omega_0} \|\sum_k \varepsilon_k(\lambda)x_k\|_X^2 d\lambda\right)^{\frac{1}{2}} \cong_p \left(\int_{\Omega_0} \|\sum_k \varepsilon_k(\lambda)x_k\|_X^p d\lambda\right)^{\frac{1}{p}}$$

and

$$\left(\int_{\Omega_0} \|\sum_k \gamma_k(\lambda)x_k\|_X^2 d\lambda\right)^{\frac{1}{2}} \cong_p \left(\int_{\Omega_0} \|\sum_k \varepsilon_k(\lambda)x_k\|_X^p d\lambda\right)^{\frac{1}{p}}.$$

(2) *(Kahane's contraction principle) For any Banach space* X, *any* $x_1, \dots, x_n \in X$ *and any* $\alpha_1, \dots, \alpha_n \in \mathbb{C}$,

$$\Big\|\sum_k \gamma_k \otimes \alpha_k x_k\Big\|_{\mathrm{Gauss}(X)} \leqslant C \cdot \sup_k |\alpha_k| \Big\|\sum_k \gamma_k \otimes x_k\Big\|_{\mathrm{Gauss}(X)} \tag{2.7}$$

and

$$\Big\|\sum_k \varepsilon_k \otimes \alpha_k x_k\Big\|_{\mathrm{Rad}(X)} \leqslant C \cdot \sup_k |\alpha_k| \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}. \tag{2.8}$$

Here we can take $C = 1$ *in* 2.7, $C = 1$ *in* 2.8 *if the* α_k*'s are in addition real, and* $C = 2$ *in* 2.8 *for general* α_k*'s.*

(3) *For any* $n \in \mathbb{N}$ *and any scalar valued matrix* $a = [a_{ij}] \in M_n$, *we have*

$$\Big\|\sum_{i,j=1}^n a_{ij}\, \gamma_i \otimes x_j\Big\|_{\mathrm{Gauss}(X)} \leqslant \|a\|_{M_n} \Big\|\sum_{j=1}^n \gamma_j \otimes x_j\Big\|_{\mathrm{Gauss}(X)}, \tag{2.9}$$

where $\|a\|_{M_n} = \|a\|_{B(\ell^2_n)}$.

This is false when $\mathrm{Gauss}(X)$ *is replaced by* $\mathrm{Rad}(X)$ *(but see (7) below).*

(4) *Let* (Ω, μ) *be a measure space and let* $N : \Omega \to B(X)$ *be strongly measurable with values in some subset* $\tau \in B(X)$. *For* $h \in L^1(\Omega)$, *define* $N_h \in B(X)$ *by*

$$x \mapsto \int_\Omega h(\omega)N(\omega)x d\mu(\omega), \tag{2.10}$$

and set $\sigma = \{N_h : \|h\|_1 \leqslant 1\}$. *Then*

$$\gamma(\sigma) \leqslant \gamma(\tau) \text{ and } R(\sigma) \leqslant 2R(\tau).$$

(5) *Let (Ω, μ) be a measure space and let I be an index set. Further, for any $i \in I$, let $N_i : \Omega \to B(X)$ be strongly measurable such that*

$$\sup_{\omega \in \Omega} R(\{N_i(\omega) : i \in I\}) = C < \infty.$$

Let $h \in L^1(\Omega)$ and define $N_i \in B(X)$ by $N_i x = \int_\Omega h(\omega) N_i(\omega) x d\mu(\omega)$. Then

$$R(\{N_i : i \in I\}) \lesssim C\|h\|_1.$$

(6) *Let (Ω, μ) be a measure space and let $N : \Omega \to B(X)$ be strongly measurable. Assume that there exists a constant $C > 0$ such that*

$$\int_\Omega \left\|N(\omega)x\right\|_X d\mu(\omega) \leqslant C\|x\| \quad (x \in X).$$

For $h \in L^\infty(\Omega)$, define $N_h \in B(X)$ by 2.10, and set $\sigma = \{N_h : \|h\|_\infty \leqslant 1\}$. Then

$$\gamma(\tau) \lesssim C \text{ and } R(\tau) \lesssim 2C.$$

(7) *Assume that X is a Banach lattice with finite cotype (see 2.18 for the definition of cotype). Then for any finite family $x_1, \dots, x_n$ in X,*

$$\left\|\sum_k \varepsilon_k \otimes x_k\right\|_{\mathrm{Gauss}(X)} \cong \left\|\sum_k \gamma_k \otimes x_k\right\|_{\mathrm{Rad}(X)} \cong \left\|\left(\sum_k |x_k|^2\right)^{\frac{1}{2}}\right\|_X. \tag{2.11}$$

We remark once and for all that "$\cong$" in 2.11 does not depend on n, and that this is crucial for estimates of such norms.

For a general Banach space with finite cotype, the first equivalence is still true.

(8) *Let $x_1, \dots, x_n \in X$ and select some $x_{k_1}, \dots, x_{k_N}$ (with $N \leqslant n$ and $1 \leqslant k_i \neq k_j \leqslant n$ for $i \neq j$). Then*

$$\left\|\sum_{j=1}^N \varepsilon_{k_j} \otimes x_{k_j}\right\|_{\mathrm{Rad}(X)} \leqslant \left\|\sum_{k=1}^n \varepsilon_k \otimes x_k\right\|_{\mathrm{Rad}(X)} \tag{2.12}$$

and

$$\left\|\sum_{j=1}^N \gamma_{k_j} \otimes x_{k_j}\right\|_{\mathrm{Gauss}(X)} \leqslant \left\|\sum_{k=1}^n \gamma_k \otimes x_k\right\|_{\mathrm{Gauss}(X)}. \tag{2.13}$$

Proof. (1) See e.g. [30, 11.1] for $\mathrm{Rad}(X)$ and [131, thm 3.12] or [83, cor 3.4.1] for $\mathrm{Gauss}(X)$.

(2) For real $\alpha_1, \dots, \alpha_n$, a proof is in [30, 12.2] with constant $C = 1$. Then for complex $\alpha_1, \dots, \alpha_n$, 2.7 with constant $C = 1$ follows from the real case together with (3) applied to diagonal matrices, and 2.8 with constant $C = 2$ follows from the real case and the decomposition $\sum_k \alpha_k \varepsilon_k \otimes x_k = \sum_k \mathrm{Re}(\alpha_k) \varepsilon_k \otimes x_k + i \sum_k \mathrm{Im}(\alpha_k) \varepsilon_k \otimes x_k$.

(3) See [30, cor 12.17].

(4) In [81, cor 2.14], it is shown that $\sigma \subset \overline{\text{absco}(\tau)}^s$, where the latter is the strong closure of

$$\text{absco}(\tau) := \left\{ \sum_{k=1}^n \lambda_k T_k : \, n \in \mathbb{N},\, T_k \in \tau,\, \lambda_k \in \mathbb{C} \text{ with } \sum_{k=1}^n |\lambda_k| = 1 \right\}$$

in $B(X)$. On the other hand, $\gamma(\overline{\text{absco}(\tau)}^s) \leqslant \gamma(\tau)$ and $R(\overline{\text{absco}(\tau)}^s) \leqslant 2R(\tau)$. This can be deduced from (2), see e.g. [81, thm 2.13], where the $\text{Rad}(X)$ case is considered. The $\text{Gauss}(X)$ case works with the same proof.

(5) Choose $i_1, \ldots i_n \in I$ and $x_1, \ldots, x_n \in X$. Then by (1),

$$\begin{aligned} \Big\| \sum_k \varepsilon_k \otimes N_{i_k} x_k \Big\|_{\text{Rad}(X)} &= \Big\| \sum_k \varepsilon_k \otimes \int_\Omega h(\omega) N_{i_k}(\omega) x_k d\mu(\omega) \Big\|_{\text{Rad}(X)} \\ &\cong \int_{\Omega_0} \Big\| \sum_k \varepsilon_k(\lambda) \int_\Omega h(\omega) N_{i_k}(\omega) x_k d\mu(\omega) \Big\|_X d\lambda \\ &\leqslant \int_\Omega \int_{\Omega_0} |h(\omega)| \Big\| \sum_k \varepsilon_k(\lambda) N_{i_k}(\omega) x_k \Big\|_X d\lambda d\mu(\omega) \\ &\cong \int_\Omega |h(\omega)| \Big\| \sum_k \varepsilon_k \otimes N_{i_k}(\omega) x_k \Big\|_{\text{Rad}(X)} d\mu(\omega) \\ &\leqslant \int_\Omega |h(\omega)| \, C \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\text{Rad}(X)} d\mu(\omega) \\ &= C \|h\|_1 \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\text{Rad}(X)}. \end{aligned}$$

(6) See [81, cor 2.17] for the $\text{Rad}(X)$ case. Replacing 2.8 by 2.7 there, the same proof also works for the $\text{Gauss}(X)$ case.

(7) The inequality $\left\| \sum_k \gamma_k \otimes x_k \right\|_{\text{Gauss}(X)} \geqslant \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)}$ holds in any Banach space X, see [131, thm 3.2]. The converse inequality holds if X has finite cotype, see [131, thm 3.7] or [30, thm 12.27]. The second equivalence is proved in [30, thm 16.18].

(8) The random variables $Z = \sum_{j=1}^N \varepsilon_{k_j} \otimes x_{k_j}$ and $Y = \sum_{k=1}^N \varepsilon_k \otimes x_k - Z$ are independent and symmetric (i.e. have the same distribution as their negative). Thus, (see e.g. [131, prop 2.16])

$$\begin{aligned} \| \sum_{j=1}^N \varepsilon_{k_j} \otimes x_{k_j} \|_{\text{Rad}(X)} &= \left(\mathbb{E} \|Z\|_X^2 \right)^{\frac{1}{2}} = \frac{1}{2} \left(\mathbb{E} \|Z + Y + Z - Y\|^2 \right)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2} \left(\mathbb{E} \|Z + Y\|^2 \right)^{\frac{1}{2}} + \frac{1}{2} \left(\mathbb{E} \|Z - Y\|^2 \right)^{\frac{1}{2}} \\ &= \left(\mathbb{E} \|Z + Y\|^2 \right)^{\frac{1}{2}} \\ &= \| \sum_{k=1}^n \varepsilon_k \otimes x_k \|_{\text{Rad}(X)}. \end{aligned}$$

This shows 2.12. The same argument also applies to the Gaussian random variables, whence 2.13 follows. □

2.4 Gaussian function spaces

We recall the construction of Gaussian function spaces from [72], see also [71, sec 1.3]. Let H be a Hilbert space and let X be a Banach space. Then we let $\gamma_+(H,X)$ be the Banach space of all $u \in B(H,X)$ such that

$$\|u\|_{\gamma(H,X)} = \sup \Big\| \sum_k \gamma_k \otimes u(e_k) \Big\|_{\mathrm{Gauss}(X)} < \infty.$$

Here, the supremum is taken over all finite orthonormal systems (e_k) in H. Further we denote $\gamma(H,X)$ the closure of the finite dimensional operators in $\gamma_+(H,X)$. We refer to $\gamma(H,X)$ as a Gaussian function space. If X does not contain c_0 isomorphically, then $\gamma(H,X) = \gamma_+(H,X)$ [72, rem 4.2]. In the sequel, we will only make use of $\gamma(H,X)$ and not of $\gamma_+(H,X)$. If H is separable, then $\|u\|_{\gamma(H,X)} = \big\|\sum_k \gamma_k \otimes u(e_k)\big\|_{\mathrm{Gauss}(X)}$, where (e_k) is an orthonormal basis of H [72, rem 4.2].

Assume that (Ω,μ) is a σ-finite measure space and $H = L^2(\Omega,\mu)$. Then there is a particular subspace of $\gamma(H,X)$ which can be identified with a space of functions from Ω to X.

Namely, let $P_2(\Omega,X)$ denote the Bochner-measurable functions $f : \Omega \to X$ such that $x' \circ f \in L^2(\Omega)$ for all $x' \in X'$. To $f \in P_2(\Omega,X)$ we assign $u_f : H \to X''$ by

$$\langle u_f h, x' \rangle = \int_\Omega \langle f(t), x' \rangle h(t) d\mu(t). \tag{2.14}$$

An application of the uniform boundedness principle shows that, in fact, u_f belongs to $B(H,X)$ [72, sec 4], [43, sec 5.5]. Then we let

$$\gamma(\Omega,X) = \{ f \in P_2(\Omega,X) : \; u_f \in \gamma(H,X) \}$$

and set

$$\|f\|_{\gamma(\Omega,X)} = \|u_f\|_{\gamma(H,X)}.$$

The space $\{u_f : f \in \gamma(\Omega,X)\}$ is a proper subspace of $\gamma(H,X)$ in general. It is dense in $\gamma(H,X)$: Indeed, since the finite dimensional operators u are dense in $\gamma(H,X)$, we only have to check that such a u is of the form $u = u_f$ for an adequate $f \in \gamma(\Omega,X)$. Since u is finite dimensional, there exist $h_1,\ldots,h_N \in H$ and $x_1,\ldots,x_N \in X$ such that

$$u(h) = \sum_{n=1}^N \langle h, h_n \rangle x_n.$$

Then clearly $u = u_f$ for $f(\cdot) = \sum_n h_n(\cdot) x_n \in \gamma(\Omega,X)$.

In some cases, the spaces $\gamma(H,X)$ and $\gamma(\Omega,X)$ can be identified with more classical spaces.

If X is a Banach function space with finite cotype, e.g. an L^p space for some $p \in [1,\infty)$, then for any step function $f = \sum_{k=1}^n x_k \chi_{A_k} : \Omega \to X$, where $x_k \in X$ and the $A_k' s$ are measurable and disjoint with $\mu(A_k) \in (0,\infty)$, we have (cf. [72, rem 3.6, exa 4.6])

$$\begin{aligned} \Big\| \Big(\int_\Omega |f(t)(\cdot)|^2 d\mu(t) \Big)^{\frac{1}{2}} \Big\|_X &= \Big\| \Big(\sum_k \mu(A_k) |x_k|^2 \Big)^{\frac{1}{2}} \Big\|_X \\ &\cong \Big\| \sum_k \gamma_k \otimes \mu(A_k)^{\frac{1}{2}} x_k \Big\|_{\mathrm{Gauss}(X)} \\ &= \|f\|_{\gamma(\Omega,X)}. \end{aligned} \tag{2.15}$$

The second line follows from 2.11, and for the third line, note that $u_f(e_k) = \mu(A_k)^{\frac{1}{2}} x_k$ for $e_k = \mu(A_k)^{-\frac{1}{2}} \chi_{A_k}$. The e_k form an orthonormal system for $k = 1, \ldots, n$ in $L^2(\Omega)$ and we complete it by an orthonormal basis in $\{e_1, \ldots, e_n\}^\perp \subset L^2(\Omega)$. The expression in 2.15 is a (classical) square function (see e.g. [24, sec 6]), whence for an arbitrary space X, $\|u\|_{\gamma(H,X)}$ is called generalized square function [72, sec 4].

In particular, if X is a Hilbert space, then $\gamma(\Omega, X) = L^2(\Omega, X)$ with equal norms.

We record some properties which will be useful in subsequent chapters.

Lemma 2.7 *Let (Ω, μ) be a σ-finite measure space.*

(1) *Suppose that $f_n, f \in P_2(\Omega, X)$ and $f_n(t) \to f(t)$ for almost all $t \in \Omega$. Then $\|f\|_{\gamma(\Omega,X)} \leqslant \liminf_n \|f_n\|_{\gamma(\Omega,X)}$.*

(2) *Suppose that $(f_n)_n$ is a sequence in $L^\infty(\Omega)$ with $\sup_n \|f_n\|_\infty < \infty$ and $f_n(t) \to 0$ for almost all $t \in \Omega$.*

Then $\|f_n g\|_{\gamma(\Omega,X)} \to 0$ for all $g \in \gamma(\Omega, X)$.

(3) *For $f \in \gamma(\Omega, X)$ and $g \in \gamma(\Omega, X')$, we have*

$$\int_\Omega |\langle f(t), g(t) \rangle| d\mu(t) \leqslant \|f\|_{\gamma(\Omega,X)} \|g\|_{\gamma(\Omega,X')}.$$

(4) *If $t \mapsto N(t)$ is strongly continuous $\Omega \to B(X)$ and $\{N(t) : t \in \Omega\}$ is γ-bounded with constant C, then for any $g \in \gamma(\Omega, X)$, $h \in L^2(\Omega)$, $x \in X$, and $x' \in X'$, $N(t)g(t)$ and $h(t)N(t)x$ belong to $\gamma(\Omega, X)$, $h(t)N(t)'x'$ belongs to $\gamma(\Omega, X')$, and*

$$\begin{aligned} \|N(t)g(t)\|_{\gamma(\Omega,X)} &\leqslant C \|g\|_{\gamma(\Omega,X)}, \\ \|h(t)N(t)x\|_{\gamma(\Omega,X)} &\leqslant C \|h\|_2 \|x\|, \\ \|h(t)N(t)'x'\|_{\gamma(\Omega,X')} &\leqslant C \|h\|_2 \|x'\|. \end{aligned}$$

(5) *Let $T \in B(X)$ and $K \in B(H_1, H_2)$, where H_1, H_2 are Hilbert spaces. Then for $u \in \gamma(H_1, X)$, we have $TuK \in \gamma(H_2, X)$ and*

$$\|TuK\|_{\gamma(H_2,X)} \leqslant \|T\| \, \|u\|_{\gamma(H_1,X)} \|K\|.$$

In particular, to K we can assign the bounded operator $K^\otimes : \gamma(H_1, X) \to \gamma(H_2, X)$, $u \mapsto u \circ K'$.

(6) *Let* (Ω_1, μ_1) *and* (Ω_2, μ_2) *be* σ*-finite measure spaces. Let* $g : \Omega_1 \times \Omega_2 \to X$ *be weakly measurable and assume that for any* $x' \in X'$, *we have*

$$\int \left(\int |\langle g(t,s), x' \rangle| ds \right)^2 dt < \infty.$$

Then

$$\int g(\cdot, s) ds \in \gamma(\Omega_1, X) \text{ and } \| \int g(\cdot, s) ds \|_{\gamma(\Omega_1, X)} \leqslant \int \|g(\cdot, s)\|_{\gamma(\Omega_1, X)} ds$$

hold as soon as the right most expression is finite.

Proof. For a proof of (1)-(5), we refer to [72, lem 4.10, cor 5.5, prop 4.11, exa 5.7, prop 4.3].

(6) We have $\int g(\cdot, s) ds \in P_2(\Omega_1, X)$, since

$$\int \left| \int \langle g(t,s), x' \rangle ds \right|^2 dt \leqslant \int \left(\int |\langle g(t,s), x' \rangle| ds \right)^2 dt,$$

where the right expression is finite due to the assumptions.

Further, for any $x' \in X'$ and $h \in L^2(\Omega_1)$, we have

$$\int \int \langle g(t,s), x' \rangle h(t) ds dt = \int \int \langle g(t,s), x' \rangle h(t) dt ds.$$

Indeed, we can apply Fubini's theorem, since

$$\int \int |\langle g(t,s), x' \rangle h(t)| \, ds dt = \int \left(\int |\langle g(t,s), x' \rangle| ds \right) |h(t)| dt$$

is finite due to the same assumption as above.

Then the claim follows from the following calculation:

$$\begin{aligned}
\| \sum_k \gamma_k \otimes \int g(t,s) ds h_k(t) dt \|_{\mathrm{Gauss}(X)} &= \| \sum_k \gamma_k \otimes \int \int g(t,s) h_k(t) dt ds \|_{\mathrm{Gauss}(X)} \\
&= \left(\int \| \sum_k \gamma_k(\lambda) \int \int g(t,s) h_k(t) dt ds \|_X^2 d\lambda \right)^{\frac{1}{2}} \\
&\leqslant \int \left(\int \| \sum_k \gamma_k(\lambda) \int g(t,s) h_k(t) dt \|_X^2 d\lambda \right)^{\frac{1}{2}} ds \\
&\leqslant \int \|g(\cdot, s)\|_{\gamma(\Omega_1, X)} ds,
\end{aligned}$$

where $(\gamma_k)_k$ are independent standard Gaussians and $(h_k)_k$ form any orthonormal system in $L^2(\Omega_1)$. ☐

2.5 Some geometric properties of Banach spaces

2.5.1 Property (α)

Following [109], we say that X has property (α) if there is a constant $C \geqslant 1$ such that for any finite family (x_{ij}) in X and any finite family (t_{ij}) of complex numbers,

$$\Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes t_{ij} x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \leqslant C \sup_{i,j} |t_{ij}| \Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}. \tag{2.16}$$

Equivalently, X has property (α) if and only if we have a uniform equivalence

$$\Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \cong \Big\|\sum_{i,j} \varepsilon_{ij} \otimes x_{ij}\Big\|_{\mathrm{Rad}(X)},$$

where $(\varepsilon_{ij})_{i,j \geqslant 1}$ be a doubly indexed family of independent Rademacher variables.

Property (α) is inherited by closed subspaces and isomorphic spaces as one can see directly from the definition 2.16.

Let (Ω, μ) be a measure space and $1 \leqslant p < \infty$. Using Kahane's inequality from proposition 2.6, one can show that the spaces $L^p(\Omega)$ have property (α), and moreover, if X has property (α), then also $L^p(\Omega, X)$ has property (α) [81, rem 4.10].

2.5.2 Type and cotype

A Banach space X is said to have type $p \in [1, 2]$ if there exists $C > 0$ such that for any finite family $x_1, \ldots, x_n \in X$ we have

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \leqslant C \left(\sum_k \|x_k\|_X^p\right)^{\frac{1}{p}}. \tag{2.17}$$

Further, X is said to have cotype $q \in [2, \infty]$ if there exists $C > 0$ such that for any finite family $x_1, \ldots, x_n \in X$, we have

$$\left(\sum_k \|x_k\|_X^q\right)^{\frac{1}{q}} \leqslant C \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \tag{2.18}$$

(left hand side replaced by the sup norm for $q = \infty$).

Any Banach space X always has type 1 and cotype ∞, whence these are also called trivial type and cotype. The notions become more restrictive for larger p and smaller q.

A space X has type 2 and cotype 2 if and only if it is isomorphic to a Hilbert space [30, cor 12.20].

It is clear that type and cotype are inherited by subspaces and isomorphic spaces.

Let (Ω, μ) be a measure space and $1 \leqslant p < \infty$. Then $L^p(\Omega)$ has type $\min(p, 2)$ and cotype $\max(p, 2)$ [30, cor 11.7]. More generally, for a Banach space X, $L^p(\Omega, X)$ has type $\min(p, \operatorname{type} X)$ and cotype $\max(p, \operatorname{cotype} X)$ [30, thm 11.12]. This is false in general for $p = \infty$, and the space $L^\infty(\Omega)$ has trivial type 1 (and no better type), if it is infinite dimensional.

There is a relation between property (α) and finite cotype. Namely, if X has property (α), then it has finite cotype, and the converse holds if X is a Banach function space [109].

3 R-boundedness of $C(K)$-representations

3.1 Introduction

Throughout this chapter, we let K be a nonempty compact set and we let $C(K)$ be the algebra of all continuous functions $f\colon K \to \mathbb{C}$, equipped with the supremum norm. A representation of $C(K)$ on some Banach space X is a bounded unital homomorphism $u\colon C(K) \to B(X)$ into the algebra $B(X)$ of all bounded operators on X. Such representations appear naturally and play a major role in several fields of Operator Theory, including functional calculi, spectral theory and spectral measures, or classification of C^*-algebras. Several recent papers, in particular [73, 81, 33, 29], have emphasized the rich and fruitful interplays between the notion of R-boundedness, unconditionality and various functional calculi. The aim of this chapter is to establish new properties of $C(K)$-representations involving R-boundedness, and to give applications to H^∞ calculus (in the sense of [24, 73]) and to unconditionality in L^p-spaces.

We recall the definition of R-boundedness from chapter 2 and the spaces $\mathrm{Rad}_{(n)}(X)$ and $\mathrm{Gauss}_{(n)}(X)$. If $X = H$ is a Hilbert space, then $\mathrm{Rad}_n(H) = \ell^2_n(H)$ isometrically and all bounded subsets of $B(H)$ are automatically R-bounded. Conversely if X is not isomorphic to a Hilbert space, then $B(X)$ contains bounded subsets which are not R-bounded [3, Prop. 1.13].

In order to motivate the results in this chapter, we recall two well-known properties of $C(K)$-representations on Hilbert space H. First, any bounded homomorphism $u\colon C(K) \to B(H)$ is completely bounded, with $\|u\|_{cb} \leqslant \|u\|^2$. This means that for any integer $n \geqslant 1$, the tensor extension $\mathrm{Id}_{M_n} \otimes u\colon M_n(C(K)) \to M_n(B(H))$ satisfies $\| \mathrm{Id}_{M_n} \otimes u\| \leqslant \|u\|^2$, if $M_n(C(K))$ and $M_n(B(H))$ are equipped with their natural C^*-algebra norms. This implies that any bounded homomorphism $u\colon C(K) \to B(H)$ is similar to a $*$-representation, a result going back at least to [16]. We refer to [107, 112] and the references therein for some information on completely bounded maps and similarity properties.

Second, let $u\colon C(K) \to B(H)$ be a bounded homomorphism. Then for any $b_1, \cdots, b_n$ lying in the commutant of the range of u and for any $f_1, \ldots, f_n$ in $C(K)$, we have

$$\Big\| \sum_k u(f_k) b_k \Big\| \leqslant \|u\|^2 \sup_{t \in K} \Big\| \sum_k f_k(t) b_k \Big\|. \tag{3.1}$$

This property is essentially a rephrasing of the fact that $C(K)$ is a nuclear C^*-algebra. More precisely, nuclearity means that the above property holds true for $*$-representations (see e.g. [68, Chap. 11] or [107, Chap. 12]), and its extension to arbitrary bounded homomorphisms easily follows from the similarity property mentioned above (see [87] for more explanations and developments).

Now let X be a Banach space and let $u\colon C(K)\to B(X)$ be a bounded homomorphism. In Section 2, we will show the following analog of (3.1):

$$\Big\|\sum_k u(f_k)b_k\Big\| \leqslant \|u\|^2 R\Big(\Big\{\sum_k f_k(t)b_k : t\in K\Big\}\Big), \tag{3.2}$$

provided that the b_k's commute with the range of u.

Section 3 is devoted to sectorial operators A which have a uniformly bounded H^∞ calculus, in the sense that they satisfy an estimate

$$\|f(A)\| \leqslant C \sup_{t>0} |f(t)| \tag{3.3}$$

for any bounded analytic function on a sector Σ_θ surrounding $(0,\infty)$. Such operators turn out to have a natural $C(K)$-functional calculus. Applying (3.2) to the resulting representation $u\colon C(K)\to B(X)$, we show that (3.3) can be automatically extended to operator valued analytic functions f taking their values in the commutant of A. This is an analog of a remarkable result of Kalton-Weis [73, Thm. 4.4] saying that if an operator A has a bounded H^∞ calculus and f is an operator valued analytic function taking its values in an R-bounded subset of the commutant of A, then the operator $f(A)$ arising from 'generalized H^∞ calculus', is bounded.

In Section 4, we introduce matricially R-bounded maps $C(K)\to B(X)$, a natural analog of completely bounded maps in the Banach space setting. We show that if X has property (α), then any bounded homomorphism $C(K)\to B(X)$ is automatically matricially R-bounded. This extends both the Hilbert space result mentioned above, and a result of De Pagter-Ricker [29, Cor. 2.19] saying that any bounded homomorphism $C(K)\to B(X)$ maps the unit ball of $C(K)$ into an R-bounded set, provided that X has property (α).

In Section 5, we give an application of matricial R-boundedness to the case when $X=L^p$. A classical result of Johnson-Jones [63] asserts that any bounded operator $T\colon L^p\to L^p$ acts, after an appropriate change of density, as a bounded operator on L^2. We show versions of this theorem for bases (more generally, for FDD's). Indeed we show that any unconditional basis (resp. any R-basis) on L^p becomes an unconditional basis (resp. a Schauder basis) on L^2 after an appropriate change of density. These results rely on Simard's extensions of the Johnson-Jones Theorem established in [119].

We end this introduction with a few preliminaries and notation. For any Banach space Z, we let $C(K;Z)$ denote the space of all continuous functions $f\colon K\to Z$, equipped with the supremum norm

$$\|f\|_\infty = \sup\{\|f(t)\|_Z \,:\, t\in K\}.$$

We may regard $C(K)\otimes Z$ as a subspace of $C(K;Z)$, by identifying $\sum_k f_k\otimes z_k$ with the function $t\mapsto \sum_k f_k(t)z_k$, for any finite families $(f_k)_k$ in $C(K)$ and $(z_k)_k$ in Z. Moreover, $C(K)\otimes Z$ is dense in $C(K;Z)$. Note that for any integer $n\geqslant 1$, $C(K;M_n)$ coincides with the C^*-algebra $M_n(C(K))$ mentioned above.

We recall that any unital commutative C^*-algebra is a $C(K)$-space (see e.g. [68, Chap. 4]). Thus our results concerning $C(K)$-representations apply as well to all these algebras. For example we will apply them to ℓ^∞ in Section 5.

We let Id_X denote the identity mapping on a Banach space X, and we let χ_B denote the indicator function of a set B. If X is a dual Banach space, we let $w^*B(X) \subset B(X)$ be the subspace of all w^*-continuous operators on X.

3.2 The extension theorem

Let X be an arbitrary Banach space. For any compact set K and any bounded homomorphism $u\colon C(K) \to B(X)$, we let

$$E_u \;=\; \big\{b \in B(X) \,:\, bu(f) = u(f)b \ \text{ for any } \ f \in C(K)\big\}$$

denote the commutant of the range of u.

Our main purpose in this section is to prove (3.1). We start with the case when $C(K)$ is finite dimensional.

Proposition 3.1 *Let $N \geqslant 1$ and let $u\colon \ell_N^\infty \to B(X)$ be a bounded homomorphism. Let $(e_1, \ldots, e_N)$ be the canonical basis of ℓ_N^∞ and set $p_i = u(e_i)$, $i = 1, \ldots, N$. Then for any $b_1, \ldots, b_N \in E_u$, we have*

$$\Big\|\sum_{i=1}^{N} p_i b_i\Big\| \leqslant \|u\|^2 R(\{b_1, \ldots, b_N\}).$$

Proof. Since u is multiplicative, each p_i is a projection and $p_i p_j = 0$ when $i \neq j$. Hence for any choice of signs $(\alpha_1, \ldots, \alpha_N) \in \{-1, 1\}^N$, we have

$$\sum_{i=1}^{N} p_i b_i \;=\; \sum_{i,j=1}^{N} \alpha_i \alpha_j p_i p_j b_j .$$

Furthermore,

$$\Big\|\sum_i \alpha_i p_i\Big\| = \|u(\alpha_1, \ldots, \alpha_N)\| \leqslant \|u\|\, \|(\alpha_1, \ldots, \alpha_N)\|_{\ell_N^\infty} = \|u\|.$$

Therefore for any $x \in X$, we have the following chain of inequalities which prove the desired estimate:

$$\begin{aligned}
\Big\|\sum_i p_i b_i x\Big\|^2 &= \int_{\Omega_0} \Big\|\sum_i \varepsilon_i(\lambda) p_i \sum_j \varepsilon_j(\lambda) p_j b_j x\Big\|^2 \, d\lambda \\
&\leqslant \int_{\Omega_0} \Big\|\sum_i \varepsilon_i(\lambda) p_i\Big\|^2 \, \Big\|\sum_j \varepsilon_j(\lambda) p_j b_j x\Big\|^2 \, d\lambda \\
&\leqslant \|u\|^2 \int_{\Omega_0} \Big\|\sum_j \varepsilon_j(\lambda) b_j p_j x\Big\|^2 \, d\lambda \\
&\leqslant \|u\|^2\, R(\{b_1, \ldots, b_N\})^2 \int_{\Omega_0} \Big\|\sum_j \varepsilon_j(\lambda) p_j x\Big\|^2 \, d\lambda \\
&\leqslant \|u\|^4 R(\{b_1, \ldots, b_N\})^2 \|x\|^2.
\end{aligned}$$

□

The study of infinite dimensional $C(K)$-spaces requires the use of second duals and w^*-topologies. We recall a few well-known facts that will be used later on in this chapter. According to the Riesz representation Theorem, the dual space $C(K)^*$ can be naturally identified with the space $M(K)$ of Radon measures on K. Next, the second dual space $C(K)^{**}$ is a commutative C^*-algebra for the so-called Arens product. This product extends the product on $C(K)$ and is separately w^*-continuous, which means that for any $\xi \in C(K)^{**}$, the two linear maps

$$\nu \in C(K)^{**} \longmapsto \nu\xi \in C(K)^{**} \qquad \text{and} \qquad \nu \in C(K)^{**} \longmapsto \xi\nu \in C(K)^{**}$$

are w^*-continuous.

Let

$$\mathcal{B}^\infty(K) = \{f : K \to \mathbb{C} \mid f \text{ bounded, Borel measurable}\},$$

equipped with the sup norm. According to the duality pairing

$$\langle f, \mu \rangle = \int_K f(t)\, d\mu(t), \qquad \mu \in M(K),\ f \in \mathcal{B}^\infty(K),$$

one can regard $\mathcal{B}^\infty(K)$ as a closed subspace of $C(K)^{**}$. Moreover the restriction of the Arens product to $\mathcal{B}^\infty(K)$ coincides with the pointwise product. Thus we have natural C^*-algebra inclusions

$$C(K) \subset \mathcal{B}^\infty(K) \subset C(K)^{**}. \tag{3.4}$$

See e.g. [26, pp. 366-367] and [23, Sec. 9] for further details.

Let $\widehat{\otimes}$ denote the projective tensor product on Banach spaces. We recall that for any two Banach spaces Y_1, Y_2, we have a natural identification

$$(Y_1 \widehat{\otimes} Y_2)^* \simeq B(Y_2, Y_1^*),$$

see e.g. [31, VIII.2]. This implies that when X is a dual Banach space, $X = (X_*)^*$ say, then $B(X) = (X_* \widehat{\otimes} X)^*$ is a dual space. The next two lemmas are elementary.

Lemma 3.2 *Let $X = (X_*)^*$ be a dual space, let $S \in B(X)$, and let $R_S, L_S \colon B(X) \to B(X)$ be the right and left multiplication operators defined by $R_S(T) = TS$ and $L_S(T) = ST$. Then R_S is w^*-continuous whereas L_S is w^*-continuous if (and only if) S is w^*-continuous.*

Proof. The tensor product mapping $\mathrm{Id}_{X_*} \otimes S$ on $X_* \otimes X$ uniquely extends to a bounded map $r_S \colon X_* \widehat{\otimes} X \to X_* \widehat{\otimes} X$, and we have $R_S = r_S^*$. Thus R_S is w^*-continuous. Likewise, if S is w^*-continuous and if we let $S_* \colon X_* \to X_*$ be its pre-adjoint map, the tensor product mapping $S_* \otimes \mathrm{Id}_X$ on $X_* \otimes X$ extends to a bounded map $l_S \colon X_* \widehat{\otimes} X \to X_* \widehat{\otimes} X$, and $L_S = l_S^*$. Thus L_S is w^*-continuous. The 'only if' part (which we will not use) is left to the reader. □

Lemma 3.3 *Let $u \colon C(K) \to B(X)$ be a bounded map. Suppose that X is a dual space. Then there exists a (necessarily unique) w^*-continuous linear mapping $\widetilde{u} \colon C(K)^{**} \to B(X)$ whose restriction to $C(K)$ coincides with u. Moreover $\|\widetilde{u}\| = \|u\|$.*

If further u is a homomorphism, and u is valued in $w^*B(X)$, then $\widetilde{u}$ is a homomorphism as well.

Proof. Let $j\colon (X_*\widehat{\otimes}X) \hookrightarrow (X_*\widehat{\otimes}X)^{**}$ be the canonical injection and consider its adjoint $p = j^*\colon B(X)^{**} \to B(X)$. Then set

$$\widetilde{u} = p \circ u^{**}\colon C(K)^{**} \longrightarrow B(X).$$

By construction, $\widetilde{u}$ is w^*-continuous and extends u. The equality $\|\widetilde{u}\| = \|u\|$ is clear.

Assume now that u is a homomorphism and that u is valued in $w^*B(X)$. Let $\nu, \xi \in C(K)^{**}$ and let $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ be bounded nets in $C(K)$ w^*-converging to ν and ξ respectively. By both parts of Lemma 3.2, we have the following equalities, where limits are taken in the w^*-topology of either $C(K)^{**}$ or $B(X)$:

$$\widetilde{u}(\nu\xi) = \widetilde{u}(\lim_\alpha \lim_\beta f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha g_\beta) = \lim_\alpha \lim_\beta u(f_\alpha)u(g_\beta) = \lim_\alpha u(f_\alpha)\widetilde{u}(\xi) = \widetilde{u}(\nu)\widetilde{u}(\xi).$$

□

We refer e.g. to [57, Lem. 2.4] for the following fact.

Lemma 3.4 *Consider $\tau \subset B(X)$ and set $\tau^{**} = \{T^{**} : T \in \tau\} \subset B(X^{**})$. Then τ is R-bounded if and only if τ^{**} is R-bounded, and in this case,*

$$R(\tau) = R(\tau^{**}).$$

For any $F \in C(K; B(X))$, we set

$$R(F) = R\big(\{F(t) : t \in K\}\big).$$

Note that $R(F)$ may be infinite. If $F = \sum_k f_k \otimes b_k$ belongs to the algebraic tensor product $C(K) \otimes B(X)$, we set

$$\Big\|\sum_k f_k \otimes b_k\Big\|_R = R(F) = R\Big(\Big\{\sum_k f_k(t)b_k : t \in K\Big\}\Big).$$

Note that by (2.8), we have

$$\|f \otimes b\|_R \leqslant 2\|f\|_\infty \|b\|, \qquad f \in C(K),\ b \in B(X). \tag{3.5}$$

From this it is easy to check that $\|\ \|_R$ is finite and is a norm on $C(K) \otimes B(X)$.

Whenever $E \subset B(X)$ is a closed subspace, we let

$$C(K) \overset{R}{\otimes} E$$

denote the completion of $C(K) \otimes E$ for the norm $\|\ \|_R$.

Remark 3.5 *Since the R-bound of a set is greater than its uniform bound, we have $\|\ \|_\infty \leqslant \|\ \|_R$ on $C(K) \otimes B(X)$. Hence the canonical embedding of $C(K) \otimes B(X)$ into $C(K; B(X))$ uniquely extends to a contraction*

$$J\colon C(K) \overset{R}{\otimes} B(X) \longrightarrow C(K; B(X)).$$

Moreover J is 1-1 and for any $\varphi \in C(K) \overset{R}{\otimes} B(X)$, we have $R(J(\varphi)) = \|\varphi\|_R$. To see this, let $(F_n)_{n\geqslant 1}$ be a sequence in $C(K) \otimes B(X)$ such that $\|F_n - \varphi\|_R \to 0$ and let $F = J(\varphi)$. Then $\|F_n\|_R \to \|\varphi\|_R$ and $\|F_n - F\|_\infty \to 0$. According to the definition of the R-bound, the latter property implies that $\|F_n\|_R \to \|F\|_R$, which yields the result.

Theorem 3.6 *Let $u\colon C(K) \to B(X)$ be a bounded homomorphism.*

(1) For any finite families $(f_k)_k$ in $C(K)$ and $(b_k)_k$ in E_u, we have

$$\Big\|\sum_k u(f_k)b_k\Big\| \leqslant \|u\|^2 \Big\|\sum_k f_k \otimes b_k\Big\|_R .$$

(2) There is a (necessarily unique) bounded linear map

$$\widehat{u}\colon C(K) \overset{R}{\otimes} E_u \longrightarrow B(X)$$

such that $\widehat{u}(f \otimes b) = u(f)b$ for any $f \in C(K)$ and any $b \in E_u$. Moreover, $\|\widehat{u}\| \leqslant \|u\|^2$.

Proof. Part (2) clearly follows from part (1). To prove (1) we introduce

$$w\colon C(K) \longrightarrow B(X^{**}), \qquad w(f) = u(f)^{**}.$$

Then w is a bounded homomorphism and $\|w\| = \|u\|$. We let $\widetilde{w}\colon C(K)^{**} \to B(X^{**})$ be its w^*-continuous extension given by Lemma 3.3. Note that w is valued in $w^*B(X^{**})$, so $\widetilde{w}$ is a homomorphism. We claim that

$$\{b^{**} : b \in E_u\} \subset E_{\widetilde{w}}.$$

Indeed, let $b \in E_u$. Then for all $f \in C(K)$, we have

$$b^{**}\, w(f) = (bu(f))^{**} = (u(f)b)^{**} = w(f)b^{**}.$$

Next for any $\nu \in C(K)^{**}$, let $(f_\alpha)_\alpha$ be a bounded net in $C(K)$ which converges to ν in the w^*-topology. Then by Lemma 3.2, we have

$$b^{**}\widetilde{w}(\nu) = \lim_\alpha b^{**}w(f_\alpha) = \lim_\alpha w(f_\alpha)b^{**} = \widetilde{w}(\nu)b^{**},$$

and the claim follows.

Now fix some $f_1, \ldots, f_n \in C(K)$ and $b_1, \ldots, b_n \in E_u$. For any $m \in \mathbb{N}$, there is a finite family $(t_1, \ldots, t_N)$ of K and a measurable partition $(B_1, \ldots, B_N)$ of K such that

$$\Big\|f_k - \sum_{l=1}^N f_k(t_l)\chi_{B_l}\Big\|_\infty \leqslant \frac{1}{m}, \qquad k = 1, \ldots, n.$$

We set $f_k^{(m)} = \sum_{l=1}^N f_k(t_l)\chi_{B_l}$. Let $\psi\colon \ell_N^\infty \to \mathcal{B}^\infty(K)$ be defined by

$$\psi(\alpha_1, \ldots, \alpha_N) = \sum_{l=1}^N \alpha_l \chi_{B_l}.$$

Then ψ is a norm 1 homomorphism. According to (3.4), we can consider the bounded homomorphism

$$\widetilde{w} \circ \psi\colon \ell_N^\infty \longrightarrow B(X^{**}).$$

Applying Proposition 3.1 to that homomorphism, together with the above claim and Lemma 3.4, we obtain that

$$\begin{aligned}\Big\|\sum_k \widetilde{w}(f_k^{(m)})b_k{}^{**}\Big\| &= \Big\|\sum_{k,l} f_k(t_l)\,\widetilde{w}\circ\psi(e_l)b_k{}^{**}\Big\| \\ &\leqslant \|\widetilde{w}\circ\psi\|^2\,R\Big(\Big\{\sum_k f_k(t_l)b_k{}^{**}\ :\ 1\leqslant l\leqslant N\Big\}\Big) \\ &\leqslant \|u\|^2 R\Big(\Big\{\sum_k f_k(t)b_k{}^{**}\ :\ t\in K\Big\}\Big) \\ &\leqslant \|u\|^2\,\Big\|\sum_k f_k\otimes b_k\Big\|_R.\end{aligned}$$

Since $\|f_k^{(m)} - f_k\|_\infty \to 0$ for any k, we have

$$\Big\|\sum_k \widetilde{w}(f_k^{(m)})b_k{}^{**}\Big\| \longrightarrow \Big\|\sum_k w(f_k)b_k{}^{**}\Big\| = \Big\|\sum_k u(f_k)b_k\Big\|,$$

and the result follows at once. □

The following notion is implicit in several recent papers on functional calculi (see in particular [73, 29]).

Definition 3.7 *Let Z be a Banach space and let $v\colon Z\to B(X)$ be a bounded map. We set*

$$R(v)\ =\ R\big(\{v(z)\ :\ z\in Z,\ \|z\|\leqslant 1\}\big),$$

and we say that v is R-bounded if $R(v)<\infty$.

Corollary 3.8 *Let $u\colon C(K)\to B(X)$ be a bounded homomorphism and let $v\colon Z\to B(X)$ be an R-bounded map. Assume further that $u(f)v(z) = v(z)u(f)$ for any $f\in C(K)$ and any $z\in Z$. Then there exists a (necessarily unique) bounded linear map*

$$u\cdot v\colon C(K;Z)\longrightarrow B(X)$$

such that $u\cdot v(f\otimes z) = u(f)v(z)$ for any $f\in C(K)$ and any $z\in Z$. Moreover we have

$$\|u\cdot v\|\leqslant \|u\|^2 R(v).$$

Proof. Consider any finite families $(f_k)_k$ in $C(K)$ and $(z_k)_k$ in Z and observe that

$$\Big\|\sum_k f_k\otimes v(z_k)\Big\|_R = R\Big(\Big\{v\Big(\sum_k f_k(t)z_k\Big)\ :\ t\in K\Big\}\Big)\leqslant\ R(v)\,\Big\|\sum_k f_k\otimes z_k\Big\|_\infty.$$

Then applying Theorem 3.6 and the assumption that v is valued in E_u, we obtain that

$$\Big\|\sum_k u(f_k)v(z_k)\Big\|\leqslant \|u\|^2 R(v)\,\Big\|\sum_k f_k\otimes z_k\Big\|_\infty,$$

which proves the result. □

Remark 3.9 *As a special case of Corollary 3.8, we obtain the following result due to De Pagter and Ricker ([29, Prop. 2.27]): Let K_1, K_2 be two compact sets, let*

$$u\colon C(K_1) \longrightarrow B(X) \qquad \text{and} \qquad v\colon C(K_2) \longrightarrow B(X)$$

be two bounded homomorphisms which commute, i.e. $u(f)v(g) = v(g)u(f)$ for all $f \in C(K_1)$ and $g \in C(K_2)$. Assume further that $R(v) < \infty$. Then there exists a bounded homomorphism

$$w\colon C(K_1 \times K_2) \longrightarrow B(X)$$

such that $w_{|C(K_1)} = u$ and $w_{|C(K_2)} = v$, where $C(K_j)$ is regarded as a subalgebra of $C(K_1 \times K_2)$ in the natural way.

3.3 Uniformly bounded H^∞ calculus

Recall the notions of sectorial operators and the H^∞ calculus from section 2.2. We will focus on sectorial operators A such that $\omega(A) = 0$.

Definition 3.10 *We say that a sectorial operator A with $\omega(A) = 0$ has a uniformly bounded H^∞ calculus, if there exists a constant $C > 0$ such that $\|f(A)\| \leqslant C\|f\|_{\infty,\theta}$ for all $\theta > 0$ and $f \in H_0^\infty(\Sigma_\theta)$.*

The space

$$C_\ell([0,\infty)) = \big\{f\colon [0,\infty) \to \mathbb{C} \,|\, f \text{ is continuous and } \lim_{\infty} f \text{ exists}\big\}.$$

is a unital commutative C^*-algebra when equipped with the natural norm

$$\|f\|_{\infty,0} = \sup\{|f(t)| \,:\, t \geqslant 0\}$$

and involution. For any $\theta > 0$, we can regard $H_0^\infty(\Sigma_\theta)$ as a subalgebra of $C_\ell([0,\infty))$, by identifying any $f \in H_0^\infty(\Sigma_\theta)$ with its restriction $f_{|[0,\infty)}$.

For any $\lambda \in \mathbb{C} \setminus [0,\infty)$, we let $R_\lambda \in C_\ell([0,\infty))$ be defined by $R_\lambda(t) = (\lambda - t)^{-1}$. Then we let $\mathcal{R}$ be the unital algebra generated by the R_λ's. Equivalently, $\mathcal{R}$ is the algebra of all rational functions of nonpositive degree, whose poles lie outside the half line $[0,\infty)$. We recall that for any $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$, the definition of $f(A)$ given by the Cauchy integral formula 2.1 coincides with the usual rational functional calculus.

The following lemma is closely related to [72, Cor. 6.9].

Lemma 3.11 *Let A be a sectorial operator on X with $\omega(A) = 0$. The following assertions are equivalent.*

(a) A has a uniformly bounded H^∞ calculus.

(b) There exists a (necessarily unique) bounded unital homomorphism

$$u\colon C_\ell([0,\infty)) \longrightarrow B(X)$$

such that $u(R_\lambda) = (\lambda - A)^{-1}$ for any $\lambda \in \mathbb{C} \setminus [0,\infty)$.

Proof. Assume (a). We claim that for any $\theta > 0$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$\|f(A)\| \leqslant C\|f\|_{\infty,0}.$$

Indeed, if $0 \neq f \in H_0^\infty(\Sigma_{\theta_0})$ for some $\theta_0 > 0$, then there exists some $t_0 > 0$ such that $f(t_0) \neq 0$. Now let $r < R$ such that $|f(z)| < |f(t_0)|$ for $|z| < r$ and $|z| > R$. Choose for every $n \in \mathbb{N}$ a $t_n \in \Sigma_{\theta_0/n}$ such that $|f(t_n)| = \|f\|_{\infty,\theta_0/n}$. Necessarily $|t_n| \in [r, R]$, and there exists a convergent subsequence t_{n_k}, whose limit t_∞ is real. Then

$$\|f\|_{\infty,0} \geqslant |f(t_\infty)| \geqslant \liminf_{\theta\to 0} \|f\|_{\infty,\theta} \geqslant C^{-1}\|f(A)\|.$$

This readily implies that the rational functional calculus $(\mathcal{R}, \| \ \|_{\infty,0}) \to B(X)$ is bounded. By Stone-Weierstrass, this extends continuously to $C_\ell([0,\infty))$, which yields (b). The uniqueness property is clear.

Assume (b). Then for any $\theta \in (0,\pi)$ and for any $f \in H_0^\infty(\Sigma_\theta) \cap \mathcal{R}$, we have

$$\|f(A)\| \leqslant \|u\|\|f\|_{\infty,\theta}.$$

By [86, Prop. 2.10] and its proof, this implies that A has a bounded $H^\infty(\Sigma_\theta)$ calculus, with a boundedness constant uniform in θ. □

Remark 3.12 *An operator A which admits a bounded $H^\infty(\Sigma_\theta)$ calculus for all $\theta > 0$ does not necessarily have a uniformly bounded H^∞ calculus. To get a simple example, consider*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \ell_2^2 \longrightarrow \ell_2^2.$$

Then $\sigma(A) = \{1\}$ and for any $\theta > 0$ and any $f \in H_0^\infty(\Sigma_\theta)$, we have

$$f(A) = \begin{pmatrix} f(1) & f'(1) \\ 0 & f(1) \end{pmatrix}.$$

Assume that $\theta < \frac{\pi}{2}$. Using Cauchy's Formula, it is easy to see that $|f'(1)| \leqslant (\sin(\theta))^{-1}\|f\|_{\infty,\theta}$ for any $f \in H_0^\infty(\Sigma_\theta)$. Thus A admits a bounded $H^\infty(\Sigma_\theta)$ calculus.

Now let h be a fixed function in $H_0^\infty(\Sigma_{\frac{\pi}{2}})$ such that $h(1) = 1$, set $g_s(\lambda) = \lambda^{is}$ for any $s > 0$, and let $f_s = hg_s$. Then $\|g_s\|_{\infty,0} = 1$, hence $\|f_s\|_{\infty,0} \leqslant \|h\|_{\infty,0}$ for any $s > 0$. Further $g_s'(\lambda) = is\lambda^{is-1}$ and $f_s' = h'g_s + hg_s'$. Hence $f_s'(1) = h'(1) + is$. Thus

$$\|f_s(A)\|\|f_s\|_{\infty,0}^{-1} \geqslant |f_s'(1)|\|h_s\|_{\infty,0}^{-1} \longrightarrow \infty$$

when $s \to \infty$. Hence A does not have a uniformly bounded H^∞ calculus.

The above result can also be deduced from Proposition 3.16 below. In fact we will show in that proposition and in Corollary 3.20 that operators with a uniformly bounded H^∞ calculus are 'rare'.

We now turn to the so-called generalized (or operator valued) H^∞ calculus. Throughout we let A be a sectorial operator. We let $E_A \subset B(X)$ denote the commutant of A, defined as the subalgebra of all bounded operators $T\colon X \to X$ such that $T(\lambda - A)^{-1} = (\lambda - A)^{-1}T$ for any λ belonging to the resolvent set of A. We let $H_0^\infty(\Sigma_\theta; B(X))$ be the algebra of all bounded analytic functions $F\colon \Sigma_\theta \to B(X)$ for which there exist $\varepsilon, C > 0$ such that $\|F(\lambda)\| \leqslant C\min(|\lambda|^\varepsilon, |\lambda|^{-\varepsilon})$

for any $\lambda \in \Sigma_\theta$. Also, we let $H_0^\infty(\Sigma_\theta; E_A)$ denote the space of all E_A-valued functions belonging to $H_0^\infty(\Sigma_\theta; B(X))$. The generalized H^∞ calculus of A is an extension of 2.1 to this class of functions. Namely for any $F \in H_0^\infty(\Sigma_\theta; E_A)$, we set

$$F(A) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} F(\lambda)(\lambda - A)^{-1}\, d\lambda\,, \tag{3.6}$$

where $\gamma \in (\omega(A), \pi)$. Again, this definition does not depend on γ and the mapping $F \mapsto F(A)$ is an algebra homomorphism. The following fundamental result is due to Kalton and Weis.

Theorem 3.13 *([73, Thm. 4.4],[81, Thm. 12.7].) Let $\omega_0 \geqslant \omega(A)$ and assume that A has a bounded $H^\infty(\Sigma_\theta)$ calculus for any $\theta > \omega_0$. Then for any $\theta > \omega_0$, there exists a constant $C_\theta > 0$ such that for any $F \in H_0^\infty(\Sigma_\theta; E_A)$,*

$$\|F(A)\| \leqslant C_\theta R\big(\{F(z) \,:\, z \in \Sigma_\theta\}\big). \tag{3.7}$$

Our aim is to prove a version of this result in the case when A has a uniformly bounded H^∞ calculus. We will obtain in Theorem 3.15 that in this case, the constant C_θ in (3.7) can be taken independent of θ.

The algebra $C_\ell([0,\infty))$ is a $C(K)$-space and we will apply the results of Section 2 to the bounded homomophism u appearing in Lemma 3.11. We recall Remark 3.5.

Lemma 3.14 *Let $J\colon C_\ell([0,\infty)) \overset{R}{\otimes} B(X) \to C_\ell([0,\infty); B(X))$ be the canonical embedding. Let $\theta \in (0,\pi)$, let $F \in H_0^\infty(\Sigma_\theta; B(X))$ and let $\gamma \in (0,\theta)$.*

(1) The integral

$$\varphi_F = \frac{1}{2\pi i} \int_{\Gamma_\gamma} R_\lambda \otimes F(\lambda)\, d\lambda \tag{3.8}$$

is absolutely convergent in $C_\ell([0,\infty)) \overset{R}{\otimes} B(X)$, and $J(\varphi_F)$ is equal to the restriction of F to $[0,\infty)$.

(2) The set $\{F(t) \,:\, t > 0\}$ is R-bounded.

Proof. Part (2) readily follows from part (1) and Remark 3.5. To prove (1), observe that for any $\lambda \in \partial\Sigma_\gamma$, we have

$$\|R_\lambda \otimes F(\lambda)\|_R \leqslant 2\|R_\lambda\|_{\infty,0}\|F(\lambda)\| \leqslant \frac{2}{\sin(\gamma)|\lambda|}\,\|F(\lambda)\|$$

by (3.5). Thus for appropriate constants $\varepsilon, C > 0$, we have

$$\|R_\lambda \otimes F(\lambda)\|_R \leqslant \frac{2C}{\sin(\gamma)}\, \min(|\lambda|^{\varepsilon-1}, |\lambda|^{-\varepsilon-1}).$$

This shows that the integral defining φ_F is absolutely convergent. Next, for any $t > 0$, we have

$$[J(\varphi_F)](t) = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \big(R_\lambda \otimes F(\lambda)\big)(t)\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \frac{F(\lambda)}{\lambda - t}\, d\lambda = F(t)$$

by Cauchy's Theorem. □

Theorem 3.15 *Let A be a sectorial operator with $\omega(A) = 0$ and assume that A has a uniformly bounded H^∞ calculus. Then there exists a constant $C > 0$ such that for any $\theta > 0$ and any $F \in H_0^\infty(\Sigma_\theta; E_A)$,*

$$\|F(A)\| \leqslant CR\big(\{F(t) \, : \, t > 0\}\big).$$

Proof. Let $u\colon C_\ell([0,\infty)) \to B(X)$ be the representation given by Lemma 3.11. It is plain that $E_u = E_A$. Then we let

$$\widehat{u}\colon C_\ell([0,\infty)) \overset{R}{\otimes} E_A \longrightarrow B(X)$$

be the associated bounded map provided by Theorem 3.6.

Let $F \in H_0^\infty(\Sigma_\theta; E_A)$ for some $\theta > 0$, and let $\varphi_F \in C_\ell([0,\infty)) \overset{R}{\otimes} E_A$ be defined by (3.8). We claim that

$$F(A) = \widehat{u}(\varphi_F).$$

Indeed for any $\lambda \in \partial\Sigma_\gamma$, we have $u(R_\lambda) = (\lambda - A)^{-1}$, hence $\widehat{u}\big(R_\lambda \otimes F(\lambda)\big) = (\lambda - A)^{-1}F(\lambda)$. Thus according to the definition of φ_F and the continuity of $\widehat{u}$, we have

$$\widehat{u}(\varphi_F) = \frac{1}{2\pi i}\int_{\Gamma_\gamma} \widehat{u}\big(R_\lambda \otimes F(\lambda)\big)\, d\lambda \; = \frac{1}{2\pi i}\int_{\Gamma_\gamma} (\lambda - A)^{-1}F(\lambda)\, d\lambda \; = F(A).$$

Consequently,

$$\|F(A)\| \leqslant \|\widehat{u}\|\|\varphi_F\|_R \leqslant \|u\|^2\|\varphi_F\|_R.$$

It follows from Lemma 3.14 and Remark 3.5 that $\|\varphi_F\|_R = R\big(\{F(t) \, : \, t > 0\}\big)$, and the result follows at once. □

In the rest of this section we will further investigate operators with a uniformly bounded H^∞ calculus. We start with the case when X is a Hilbert space.

Proposition 3.16 *Let H be a Hilbert space and let A be a sectorial operator on H, with $\omega(A) = 0$. Then A admits a uniformly bounded H^∞ calculus if and only if there exists an isomorphism $S\colon H \to H$ such that $S^{-1}AS$ is selfadjoint.*

Proof. Assume that A admits a uniformly bounded H^∞ calculus and let $u\colon C_\ell([0,\infty)) \to B(H)$ be the associated representation. According to [107, Thm. 9.1 and Thm. 9.7], there exists an isomorphism $S\colon H \to H$ such that the unital homomorphism $u_S\colon C_\ell([0,\infty)) \to B(H)$ defined by $u_S(f) = S^{-1}u(f)S$ satisfies $\|u_S\| \leqslant 1$. We let $B = S^{-1}AS$. For any $s \in \mathbb{R}^*$, we have $\|R_{is}\|_{\infty,0} = |s|$ and $u_S(R_{is}) = S^{-1}(is - A)^{-1}S = (is - B)^{-1}$. Hence

$$\|(is - B)^{-1}\| \leqslant |s|, \qquad s \in \mathbb{R}^*.$$

By the Hille-Yosida Theorem, this implies that iB and $-iB$ both generate contractive c_0-semigroups on H. Thus iB generates a unitary c_0-group. By Stone's Theorem, this implies that B is selfadjoint.

The converse implication is clear. □

In the non Hilbertian setting, we will first show that operators with a uniformly bounded H^∞ calculus satisfy a spectral mapping theorem with respect to continuous functions defined on the one-point compactification of $\sigma(A)$. Then we will discuss the connections with spectral measures and scalar-type operators. We mainly refer to [34, Chap. 5-7] for this topic.

For any compact set K and any closed subset $F \subset K$, we let

$$I_F = \{f \in C(K) \,:\, f_{|F} = 0\}.$$

We recall that the restriction map $f \mapsto f_{|F}$ induces a $*$-isomorphism $C(K)/I_F \to C(F)$.

Lemma 3.17 *Let $K \subset \mathbb{C}$ be a compact set and let $u\colon C(K) \to B(X)$ be a representation. Let $\kappa \in C(K)$ be the function defined by $\kappa(z) = z$ and put $T = u(\kappa)$.*

(1) Then $\sigma(T) \subset K$ and u vanishes on $I_{\sigma(T)}$.

Let $v\colon C(\sigma(T)) \simeq C(K)/\operatorname{Id}_{\sigma(T)} \longrightarrow B(X)$ be the representation induced by u.

(2) For any $f \in C(\sigma(T))$, we have $\sigma(v(f)) = f(\sigma(T))$.

(3) v is an isomorphism onto its range.

Proof. The inclusion $\sigma(T) \subset K$ is clear. Indeed, for any $\lambda \notin K$, $(\lambda - T)^{-1}$ is equal to $u((\lambda - \cdot\,)^{-1})$. We will now show that u vanishes on $I_{\sigma(T)}$.

Let $w\colon C(K) \to B(X^*)$ be defined by $w(f) = [u(f)]^*$, and let $\widetilde{w}\colon C(K)^{**} \to B(X^*)$ be its w^*-extension. Since w is valued in $w^*B(X^*) \simeq B(X)$, this is a representation (see Lemma 3.3). Let Δ_K be the set of all Borel subsets of K. It is easy to check that the mapping

$$P\colon \Delta_K \longrightarrow B(X^*), \qquad P(B) = \widetilde{w}(\chi_B),$$

is a spectral measure of class (Δ_K, X) in the sense of [34, p. 119]. According to [34, Prop. 5.8], the operator T^* is prespectral of class X (in the sense of [34, Def. 5.5]) and the above mapping P is its resolution of the identity. Applying [34, Lem. 5.6] and the equality $\sigma(T^*) = \sigma(T)$, we obtain that $\widetilde{w}(\chi_{\sigma(T)}) = P(\sigma(T)) = \operatorname{Id}_{X^*}$. Therefore for any $f \in \operatorname{Id}_{\sigma(T)}$, we have

$$u(f)^* = \widetilde{w}\big(f(1 - \chi_{\sigma(T)})\big) = \widetilde{w}(f)\widetilde{w}(1 - \chi_{\sigma(T)}) = 0.$$

Hence u vanishes on $I_{\sigma(T)}$.

The proofs of (2) and (3) now follow from [34, Prop. 5.9] and the above proof. □

In the sequel we consider a sectorial operator A with $\omega(A) = 0$. This assumption implies that $\sigma(A) \subset [0, \infty)$. By $C_\ell(\sigma(A))$, we denote either the space $C(\sigma(A))$ if A is bounded, or the space $\{f\colon \sigma(A) \to \mathbb{C} \,|\, f$ is continuous and $\lim_\infty f$ exists$\}$ if A is unbounded. In this case, $C_\ell(\sigma(A))$ coincides with the space of continuous functions on the one-point compactification of $\sigma(A)$. The following strengthens Lemma 3.11.

Proposition 3.18 *Let A be a sectorial operator on X with $\omega(A) = 0$. The following assertions are equivalent.*

(1) A has a uniformly bounded H^∞ calculus.

(2) There exists a (necessarily unique) bounded unital homomorphism

$$\Psi\colon C_\ell(\sigma(A)) \longrightarrow B(X)$$

such that $\Psi\big((\lambda - \cdot\,)^{-1}\big) = (\lambda - A)^{-1}$ *for any* $\lambda \in \mathbb{C} \setminus \sigma(A)$.

In this case, Ψ *is an isomorphism onto its range and for any* $f \in C_\ell(\sigma(A))$*, we have*

$$\sigma(\Psi(f)) = f(\sigma(A)) \cup f_\infty, \tag{3.9}$$

where $f_\infty = \varnothing$ *if* A *is bounded and* $f_\infty = \{\lim_\infty f\}$ *if* A *is unbounded.*

Proof. Assume (1) and let $u\colon C_\ell([0,\infty)) \to B(X)$ be given by Lemma 3.11. We introduce the particular function $\phi \in C_\ell([0,\infty))$ defined by $\phi(t) = (1+t)^{-1}$. Consider the $*$-isomorphism

$$\tau\colon C([0,1]) \longrightarrow C_\ell([0,\infty)), \qquad \tau(g) = g \circ \phi,$$

and set $T = (1+A)^{-1}$. If we let $\kappa(z) = z$ as in Lemma 3.17, we have $(u \circ \tau)(\kappa) = T$. Let $v\colon C(\sigma(T)) \to B(X)$ be the resulting factorisation of $u \circ \tau$. The spectral mapping theorem gives $\sigma(A) = \phi^{-1}(\sigma(T) \setminus \{0\})$ and $0 \in \sigma(T)$ if and only if A is unbounded. Thus the mapping

$$\tau_A\colon C(\sigma(T)) \longrightarrow C_\ell(\sigma(A))$$

defined by $\tau_A(g) = g \circ \phi$ also is a $*$-isomorphism. Put $\Psi = v \circ \tau_A^{-1}\colon C_\ell(\sigma(A)) \to B(X)$. This is a unital bounded hommormorphism. Note that $\phi^{-1}(z) = \frac{1-z}{z}$ for any $z \in (0,1]$. Then for any $\lambda \in \mathbb{C} \setminus \sigma(A)$,

$$\begin{aligned}
\Psi\big((\lambda - \cdot\,)^{-1}\big) &= v\big((\lambda - \cdot\,)^{-1} \circ \phi^{-1}\big) = v\Big(z \mapsto \Big(\lambda - \frac{1-z}{z}\Big)^{-1}\Big) \\
&= v\Big(z \mapsto \frac{z}{(\lambda+1)z - 1}\Big) \\
&= T\big((\lambda+1)T - 1\big)^{-1} = (\lambda - A)^{-1}.
\end{aligned}$$

Hence Ψ satisfies (2). Its uniqueness follows from Lemma 3.11. The fact that Ψ is an isomorphism onto its range, and the spectral property (3.9) follow from the above construction and Lemma 3.17. Finally the implication '(2) $\Rightarrow$ (1)' also follows from Lemma 3.11. □

Remark 3.19 *Let* A *be a sectorial operator with a uniformly bounded* H^∞ *calculus, and let* $T = (1+A)^{-1}$*. It follows from Lemma 3.17 and the proof of Proposition 3.18 that there exists a representation*

$$v\colon C(\sigma(T)) \longrightarrow B(X)$$

satisfying $v(\kappa) = T$ *(where* $\kappa(z) = z$*), such that* $\sigma(v(f)) = f(\sigma(T))$ *for any* $f \in C(\sigma(T))$ *and* v *is an isomorphism onto its range. Also, it follows from the proof of Lemma 3.17 that* T^* *is a scalar-type operator of class* X*, in the sense of [34, Def. 5.14].*

Next according to [34, Thm. 6.24], the operator T *(and hence* A*) is a scalar-type spectral operator if and only if for any* $x \in X$*, the mapping* $C(\sigma(T)) \to X$ *taking* f *to* $v(f)x$ *for any* $f \in C(\sigma(T))$ *is weakly compact.*

Corollary 3.20 *Let A be a sectorial operator on X, with $\omega(A) = 0$, and assume that X does not contain a copy of c_0. Then A admits a uniformly bounded H^∞ calculus if and only if it is a scalar-type spectral operator.*

Proof. The 'only if' part follows from the previous remark. Indeed if X does not contain a copy of c_0, then any bounded map $C(K) \to X$ is weakly compact [31, VI, Thm. 15]. (See also [115] and [29] for related approaches.) The 'if' part follows from [44, Prop. 2.7] and its proof. $\square$

Remark 3.21

(1) The hypothesis on X in Corollary 3.20 is necessary. Namely it follows from [32, Thm. 3.2] and its proof that if $c_0 \subset X$, then there is a sectorial operator A with a uniformly bounded H^∞ calculus on X which is not scalar-type spectral.

(2) Scalar-type spectral operators on Hilbert space coincide with operators similar to a normal one (see [34, Chap. 7]). Thus when $X = H$ is a Hilbert space, the above corollary reduces to Proposition 3.16.

3.4 Matricial R-boundedness

For any integer $n \geqslant 1$ and any vector space E, we will denote by $M_n(E)$ the space of all $n \times n$ matrices with entries in E. We will be mostly concerned with the cases $E = C(K)$ or $E = B(X)$. As mentioned in the introduction, we identify $M_n(C(K))$ with the space $C(K; M_n)$ in the usual way. We now introduce a specific norm on $M_n(B(X))$. Namely for any $[T_{ij}] \in M_n(B(X))$, we set

$$\big\|[T_{ij}]\big\|_R = \sup\Big\{\Big\|\sum_{i,j=1}^{n} \varepsilon_i \otimes T_{ij}(x_j)\Big\|_{\mathrm{Rad}(X)} \,:\, x_1,\ldots,x_n \in X,\ \Big\|\sum_{j=1}^{n} \varepsilon_j \otimes x_j\Big\|_{\mathrm{Rad}(X)} \leqslant 1\Big\}.$$

Clearly $\|\ \|_R$ is a norm on $M_n(B(X))$. Moreover if we consider any element of $M_n(B(X))$ as an operator on $\ell^2_n \otimes X$ in the natural way, and if we equip the latter tensor product with the norm of $\mathrm{Rad}_n(X)$, we obtain an isometric identification

$$\big(M_n(B(X)), \|\ \|_R\big) = B\big(\mathrm{Rad}_n(X)\big). \tag{3.10}$$

Definition 3.22 *Let $u\colon C(K) \to B(X)$ be a bounded linear mapping. We say that u is matricially R-bounded if there is a constant $C \geqslant 0$ such that for any $n \geqslant 1$, and for any $[f_{ij}] \in M_n(C(K))$, we have*

$$\big\|[u(f_{ij})]\big\|_R \leqslant C\|[f_{ij}]\|_{C(K;M_n)}. \tag{3.11}$$

Remark 3.23 *The above definition obviously extends to any bounded map $E \to B(X)$ defined on an operator space E, or more generally on any matricially normed space (see [39, 40]). The basic observations below apply to this general case as well.*

(1) In the case that $X = H$ is a Hilbert space, we have

$$\Big\|\sum_{j=1}^{n} \varepsilon_j \otimes x_j\Big\|_{\mathrm{Rad}(H)} = \Big(\sum_{j=1}^{n} \|x_j\|^2\Big)^{\frac{1}{2}}$$

for any $x_1, \ldots, x_n \in H$. Consequently, writing that a mapping $u\colon C(K) \to B(H)$ is matricially R-bounded is equivalent to writing that u is completely bounded (see e.g. [107]). See Section 5 for the case when X is an L^p-space.

(2) The notation $\| \ \|_R$ introduced above is consistent with the one considered so far in Section 2. Indeed let $b_1, \ldots, b_n$ in $B(X)$. Then the diagonal matrix $\mathrm{Diag}\{b_1, \ldots, b_n\} \in M_n(B(X))$ and the tensor element $\sum_{k=1}^n e_k \otimes b_k \in \ell^\infty_n \otimes B(X)$ satisfy

$$\big\|\mathrm{Diag}\{b_1, \ldots, b_n\}\big\|_R \ = \ R(\{b_1, \ldots, b_n\}) \ = \ \Big\|\sum_{k=1}^n e_k \otimes b_k\Big\|_R.$$

(3) If $u\colon C(K) \to B(X)$ is matricially R-bounded (with the estimate (3.11)), then u is R-bounded and $R(u) \leqslant C$. Indeed, consider $f_1, \ldots, f_n$ in the unit ball of $C(K)$. Then we have $\|\mathrm{Diag}\{f_1, \ldots, f_n\}\|_{C(K;M_n)} \leqslant 1$. Hence for any $x_1, \ldots, x_n$ in X,

$$\begin{aligned}\Big\|\sum_k \varepsilon_k \otimes u(f_k)x_k\Big\|_{\mathrm{Rad}(X)} \ &\leqslant \ \big\|\mathrm{Diag}\{u(f_1), \ldots, u(f_n)\}\big\|_R \Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \\ &\leqslant \ C\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)}.\end{aligned}$$

For any $n \geqslant 1$, introduce $\sigma_{n,X} : M_n \to B(\mathrm{Rad}_n(X))$ by letting

$$\sigma_{n,X}([a_{ij}]) = [a_{ij}\,\mathrm{Id}_X].$$

Recall the notion of property (α) from 2.16 in chapter 2, section 2.5. The following is a characterization of property (α) in terms of the R-boundedness of $\sigma_{n,X}$.

Lemma 3.24 *A Banach space X has property (α) if and only if*

$$\sup_{n\geqslant 1} R(\sigma_{n,X}) \ < \infty.$$

Proof. Assume that X has property (α). As mentioned in chapter 2, section 2.5, this implies that X has finite cotype. Hence X satisfies the equivalence property

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \cong \Big\|\sum_k \gamma_k \otimes x_k\Big\|_{\mathrm{Gauss}(X)}$$

from 2.11. Let $a(1), \ldots, a(N)$ be in M_n and let $z_1, \ldots, z_N$ be in $\mathrm{Rad}_n(X)$. Let x_{jk} in X such that $z_k = \sum_j \varepsilon_j \otimes x_{jk}$ for any k. We consider a doubly indexed family $(\varepsilon_{ik})_{i,k\geqslant 1}$ of independent Rademacher variables as well as a doubly indexed family $(\gamma_{ik})_{i,k\geqslant 1}$ of independent standard Gaussian variables. Then

$$\sum_k \varepsilon_k \otimes \sigma_{n,X}\big(a(k)\big) z_k \ = \ \sum_{k,i,j} \varepsilon_k \otimes \varepsilon_i \otimes a(k)_{ij} x_{jk}. \tag{3.12}$$

Hence using proposition 2.6, we have

$$\begin{aligned}
\Big\|\sum_k \varepsilon_k \otimes \sigma_{n,X}\big(a(k)\big) z_k\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} &\cong \Big\|\sum_{k,i,j} \varepsilon_{ik} \otimes a(k)_{ij} x_{jk}\Big\|_{\mathrm{Rad}(X)} \\
&\cong \Big\|\sum_{k,i,j} \gamma_{ik} \otimes a(k)_{ij} x_{jk}\Big\|_{\mathrm{Gauss}(X)} \\
&\lesssim \left\|\begin{pmatrix} a(1) & 0\dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots 0 & a(N)\end{pmatrix}\right\|_{M_{Nn}} \Big\|\sum_{k,j} \gamma_{jk} \otimes x_{jk}\Big\|_{\mathrm{Gauss}(X)} \\
&\lesssim \max_k \|a(k)\|_{M_n} \Big\|\sum_{k,j} \varepsilon_{jk} \otimes x_{jk}\Big\|_{\mathrm{Rad}(X)} \\
&\lesssim \max_k \|a(k)\|_{M_n} \Big\|\sum_{k,j} \varepsilon_k \otimes \varepsilon_j \otimes x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \\
&\lesssim \max_k \|a(k)\|_{M_n} \Big\|\sum_{k} \varepsilon_k \otimes z_k\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.
\end{aligned}$$

This shows that the $\sigma_{n,X}$'s are uniformly R-bounded.

Conversely, assume that for some constant $C \geqslant 1$, we have $R(\sigma_{n,X}) \leqslant C$ for all $n \geqslant 1$. Let $(t_{jk})_{j,k} \in \mathbb{C}^{n^2}$ with $|t_{jk}| \leqslant 1$ and for any $k = 1, \dots, n$, let $a(k) \in M_n$ be the diagonal matrix with entries $t_{1k}, \dots, t_{nk}$ on the diagonal. Then $\|a(k)\| \leqslant 1$ for any k. Hence applying (3.12), we obtain that for any $(x_{jk})_{j,k}$ in X^{n^2},

$$\begin{aligned}
\Big\|\sum_{j,k} \varepsilon_k \otimes \varepsilon_j \otimes t_{jk} x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} &\leqslant R\big(\{a(1), \dots, a(n)\}\big) \Big\|\sum_{j,k} \varepsilon_k \otimes \varepsilon_j \otimes x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \\
&\leqslant C \Big\|\sum_{j,k} \varepsilon_k \otimes \varepsilon_j \otimes x_{jk}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.
\end{aligned}$$

This means that X has property (α). □

Proposition 3.25 *Assume that X has property (α). Then any bounded homomorphism $u\colon C(K) \to B(X)$ is matricially R-bounded.*

Proof. Let $u\colon C(K) \to B(X)$ be a bounded homomorphism and let $w\colon C(K) \to B\big(\mathrm{Rad}_n(X)\big)$ be defined by

$$w(f) = \mathrm{Id}_{\mathrm{Rad}_n} \otimes u(f).$$

Clearly w is also a bounded homomorphism, with $\|w\| = \|u\|$. Recall the identification (3.10) and note that $w(f) = \mathrm{Diag}\{u(f), \dots, u(f)\}$ for any $f \in C(K)$. Then for any $a = [a_{ij}] \in M_n$, we have

$$w(f)\sigma_{n,X}(a) = [a_{ij} u(f)] = \sigma_{n,X}(a) w(f).$$

By Corollary 3.8 and Lemma 3.24, the resulting mapping $w \cdot \sigma_{n,X}$ satisfies

$$\big\|w \cdot \sigma_{n,X} : C(K; M_n) \longrightarrow B(\mathrm{Rad}_n(X))\big\| \leqslant C\|u\|^2$$

where C does not depend on n. Let E_{ij} denote the canonical matrix units of M_n, for $i,j = 1,\ldots,n$. Consider $[f_{ij}] \in C(K;M_n) \simeq M_n(C(K))$ and write this matrix as $\sum_{i,j} E_{ij} \otimes f_{ij}$. Then

$$w\cdot \sigma_{n,X}\big([f_{ij}]\big) = \sum_{i,j=1}^{n} w(f_{ij})\sigma_{n,X}(E_{ij}) \;=\; \sum_{i,j=1}^{n} u(f_{ij}) \otimes E_{ij} \;= [u(f_{ij})].$$

Hence $\big\|[u(f_{ij})]\big\|_R \leqslant C\|u\|^2\|[f_{ij}]\|_{C(K;M_n)}$, which proves that u is matricially R-bounded. □

In the case that $X = H$ is a Hibert space, it follows from Remark 3.23 (1) that the above proposition reduces to the fact that any bounded homomorphism $C(K) \to B(H)$ is completely bounded.

We also observe that applying the above proposition together with Remark 3.23 (3), we obtain the following corollary originally due to De Pagter and Ricker [29, Cor. 2.19]. Indeed, Proposition 3.25 should be regarded as a strengthening of their result.

Corollary 3.26 *Assume that X has property (α). Then any bounded homomorphism $u\colon C(K) \to B(X)$ is R-bounded.*

Remark 3.27 *The above corollary is nearly optimal. Indeed we claim that if X does not have property (α) and if K is any infinite compact set, then there exists a unital bounded homomorphism*

$$u\colon C(K) \longrightarrow B\big(\mathrm{Rad}(X)\big)$$

which is not R-bounded.

To prove this, let $(z_n)_{n\geqslant 1}$ be an infinite sequence of distinct points in K and let u be defined by

$$u(f)\Big(\sum_{k\geqslant 1} \varepsilon_k \otimes x_k\Big) \;=\; \sum_{k\geqslant 1} f(z_k)\,\varepsilon_k \otimes x_k.$$

According to (2.8), this is a bounded unital homomorphism satisfying $\|u\| \leqslant 2$. Assume now that u is R-bounded. Let $n \geqslant 1$ be an integer and consider families $(t_{ij})_{i,j}$ in $\mathbb{C}^{n^2}$ and $(x_{ij})_{i,j}$ in X^{n^2}. For any $i = 1,\ldots,n$, there exists $f_i \in C(K)$ such that $\|f_i\| = \sup_j |t_{ij}|$ and $f_i(z_j) = t_{ij}$ for any $j = 1,\ldots,n$. Then

$$\sum_i \varepsilon_i \otimes u(f_i)\Big(\sum_j \varepsilon_j \otimes x_{ij}\Big) \;=\; \sum_{i,j} t_{ij}\,\varepsilon_i \otimes \varepsilon_j \otimes x_{ij},$$

hence

$$\begin{aligned}\Big\|\sum_{i,j} t_{ij}\,\varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} &\leqslant R(u)\sup_i \|f_i\|\,\Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\ &\leqslant R(u)\sup_{i,j} |t_{ij}|\,\Big\|\sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.\end{aligned}$$

This shows (2.16).

3.5 Application to L^p-spaces and unconditional bases

Let X be a Banach lattice with finite cotype. By proposition 2.6, we have

$$\Big\|\sum_k \varepsilon_k \otimes x_k\Big\|_{\mathrm{Rad}(X)} \cong \Big\|\Big(\sum_k |x_k|^2\Big)^{\frac{1}{2}}\Big\|_X$$

for finite families $(x_k)_k$ of X. Thus a bounded linear mapping $u\colon C(K) \to B(X)$ is matricially R-bounded if there is a constant $C \geqslant 0$ such that for any $n \geqslant 1$, for any matrix $[f_{ij}] \in M_n(C(K))$ and for any $x_1, \ldots, x_n \in X$, we have

$$\Big\|\Big(\sum_i\Big|\sum_j u(f_{ij})x_j\Big|^2\Big)^{\frac{1}{2}}\Big\| \leqslant C\, \|[f_{ij}]\|_{C(K;M_n)} \Big\|\Big(\sum_j |x_j|^2\Big)^{\frac{1}{2}}\Big\|.$$

Mappings satisfying this property were introduced by Simard in [119] under the name of ℓ^2-cb maps. In this section we will apply a factorization property of ℓ^2-cb maps established in [119], in the case when X is merely an L^p-space.

Throughout this section, we let (Ω, μ) be a σ-finite measure space. By definition, a density on that space is a measurable function $g\colon \Omega \to (0, \infty)$ such that $\|g\|_1 = 1$. For any such function and any $1 \leqslant p < \infty$, we consider the linear mapping

$$\phi_{p,g}\colon L^p(\Omega,\mu) \longrightarrow L^p(\Omega, g d\mu), \qquad \phi_{p,g}(h) = g^{-1/p}h,$$

which is an isometric isomorphism. Note that $(\Omega, g d\mu)$ is a probability space. Passing from (Ω, μ) to $(\Omega, g d\mu)$ by means of the maps $\phi_{p,g}$ is usually called a change of density. A classical theorem of Johnson-Jones [63] asserts that for any bounded operator $T\colon L^p(\mu) \to L^p(\mu)$, there exists a density g on Ω such that $\phi_{p,g} \circ T \circ \phi_{p,g}^{-1}\colon L^p(g d\mu) \to L^p(g d\mu)$ extends to a bounded operator $L^2(g d\mu) \to L^2(g d\mu)$. The next statement is an analog of that result for $C(K)$-representations.

Proposition 3.28 *Let $1 \leqslant p < \infty$ and let $u\colon C(K) \to B(L^p(\mu))$ be a bounded homomorphism. Then there exists a density $g\colon \Omega \to (0,\infty)$ and a bounded homomorphism $w\colon C(K) \to B(L^2(g d\mu))$ such that*

$$\phi_{p,g} \circ u(f) \circ \phi_{p,g}^{-1} = w(f), \qquad f \in C(K),$$

where equality holds on $L^2(g d\mu) \cap L^p(g d\mu)$.

Proof. Since $X = L^p(\mu)$ has property (α), the mapping u is matricially R-bounded by Proposition 3.25. According to the above discussion, this means that u is ℓ^2-cb in the sense of [119, Def. 2]. The result therefore follows from [119, Thms. 3.4 and 3.6]. □

We will now focus on Schauder bases on separable L^p-spaces. We refer to [91, Chap. 1] for general information on this topic. We simply recall that a sequence $(e_k)_{k\geqslant 1}$ in a Banach space X is a basis if for every $x \in X$, there exists a unique scalar sequence $(a_k)_{k\geqslant 1}$ such that $\sum_k a_k e_k$ converges to x. A basis $(e_k)_{k\geqslant 1}$ is called unconditional if this convergence is unconditional for all $x \in X$. We record the following standard characterization.

Lemma 3.29 *A sequence* $(e_k)_{k\geqslant 1} \subset X$ *of non-zero vectors is an unconditional basis of* X *if and only if* $X = \overline{\text{Span}}\{e_k : k \geqslant 1\}$ *and there exists a constant* $C \geqslant 1$ *such that for any bounded scalar sequence* $(\lambda_k)_{k\geqslant 1}$ *and for any finite scalar sequence* $(a_k)_{k\geqslant 1}$,

$$\Big\|\sum_k \lambda_k a_k e_k\Big\| \leqslant C \sup_k |\lambda_k| \, \Big\|\sum_k a_k e_k\Big\|. \tag{3.13}$$

We will need the following elementary lemma.

Lemma 3.30 *Let* (Ω, ν) *be a* σ*-finite measure space, let* $1 \leqslant p < \infty$ *and let* $Q\colon L^p(\nu) \to L^p(\nu)$ *be a finite rank bounded operator such that* $Q_{|L^2(\nu)\cap L^p(\nu)}$ *extends to a bounded operator* $L^2(\nu) \to L^2(\nu)$. *Then* $Q(L^p(\nu)) \subset L^2(\nu)$.

Proof. Let $E = Q(L^p(\nu) \cap L^2(\nu))$. By assumption, E is a finite dimensional subspace of $L^p(\nu) \cap L^2(\nu)$. Since E is automatically closed under the L^p-norm and Q is continuous, we obtain that $Q(L^p(\nu)) = E$. □

Theorem 3.31 *Let* $1 \leqslant p < \infty$ *and assume that* $(e_k)_{k\geqslant 1}$ *is an unconditional basis of* $L^p(\Omega, \mu)$. *Then there exists a density* g *on* Ω *such that* $\phi_{p,g}(e_k) \in L^2(gd\mu)$ *for any* $k \geqslant 1$, *and the sequence* $(\phi_{p,g}(e_k))_{k\geqslant 1}$ *is an unconditional basis of* $L^2(gd\mu)$.

Proof. Property (3.13) implies that for any $\lambda = (\lambda_k)_{k\geqslant 1} \in \ell^\infty$, there exists a (necessarily unique) bounded operator $T_\lambda\colon L^p(\mu) \to L^p(\mu)$ such that $T_\lambda(e_k) = \lambda_k e_k$ for any $k \geqslant 1$. Moreover $\|T_\lambda\| \leqslant C\|\lambda\|_\infty$. We can therefore consider the mapping

$$u\colon \ell^\infty \longrightarrow B(L^p(\mu)), \qquad u(\lambda) = T_\lambda,$$

and u is a bounded homomorphism. By Proposition 3.28, there is a constant $C_1 > 0$, and a density g on Ω such that with $\phi = \phi_{p,g}$, the mapping

$$\phi T_\lambda \phi^{-1}\colon L^p(gd\mu) \longrightarrow L^p(gd\mu)$$

extends to a bounded operator

$$S_\lambda\colon L^2(gd\mu) \longrightarrow L^2(gd\mu)$$

for any $\lambda \in \ell^\infty$, with $\|S_\lambda\| \leqslant C_1\|\lambda\|_\infty$.

Assume first that $p \geqslant 2$, so that $L^p(gd\mu) \subset L^2(gd\mu)$. Let $\lambda = (\lambda_k)_{k\geqslant 1}$ in ℓ^∞ and let $(a_k)_{k\geqslant 1}$ be a finite scalar sequence. Then $S_\lambda(\phi(e_k)) = \phi T_\lambda \phi^{-1}(\phi(e_k)) = \lambda_k \phi(e_k)$ for any $k \geqslant 1$, hence

$$\Big\|\sum_k \lambda_k a_k \phi(e_k)\Big\|_{L^2(gd\mu)} = \Big\|S_\lambda\Big(\sum_k a_k \phi(e_k)\Big)\Big\|_{L^2(gd\mu)} \leqslant C_1\|\lambda\|_\infty \Big\|\sum_k a_k \phi(e_k)\Big\|_{L^2(gd\mu)}.$$

Moreover the linear span of the $\phi(e_k)$'s is dense in $L^p(gd\mu)$, hence in $L^2(gd\mu)$. By Lemma 3.29, this shows that $(\phi(e_k))_{k\geqslant 1}$ is an unconditional basis of $L^2(gd\mu)$.

Assume now that $1 \leqslant p < 2$. For any $n \geqslant 1$, let $f_n \in \ell^\infty$ be defined by $(f_n)_k = \delta_{n,k}$ for any $k \geqslant 1$, and let $Q_n\colon L^p(gd\mu) \to L^p(gd\mu)$ be the projection defined by

$$Q_n\Big(\sum_k a_k \phi(e_k)\Big) = a_n \phi(e_n).$$

Then $Q_n = \phi T_{f_n} \phi^{-1}$ hence Q_n extends to an L^2 operator. Therefore, $\phi(e_n)$ belongs to $L^2(gd\mu)$ by Lemma 3.30.

Let $p' = p/(p-1)$ be the conjugate number of p, let $(e_k^*)_{k\geqslant 1}$ be the biorthogonal system of $(e_k)_{k\geqslant 1}$, and let $\phi' = \phi^{*-1}$. (It is easy to check that $\phi' = \phi_{p',g}$, but we will not use this point.) The linear span of the e_k^*'s is w^*-dense in $L^{p'}(\mu)$. Equivalently, the linear span of the $\phi'(e_k^*)$'s is w^*-dense in $L^{p'}(gd\mu)$, hence it is dense in $L^2(gd\mu)$. Moreover for any $\lambda \in \ell^\infty$ and for any $k \geqslant 1$, we have $T_\lambda^*(e_k^*) = \lambda_k e_k^*$. Thus for any finite scalar sequence $(a_k)_{k\geqslant 1}$, we have

$$\sum_k \lambda_k a_k \phi'(e_k^*) \,=\, (\phi T_\lambda \phi^{-1})^* \Bigl(\sum_k a_k \phi'(e_k^*)\Bigr) \,=\, S_\lambda^* \Bigl(\sum_k a_k \phi'(e_k^*)\Bigr).$$

Hence

$$\Bigl\|\sum_k \lambda_k a_k \phi'(e_k^*)\Bigr\|_{L^2(gd\mu)} \,\leqslant\, C_1 \Bigl\|\sum_k a_k \phi'(e_k^*)\Bigr\|_{L^2(gd\mu)}.$$

According to Lemma 3.29, this shows that $(\phi'(e_k^*))_{k\geqslant 1}$ is an unconditional basis of $L^2(gd\mu)$. It is plain that $(\phi(e_k))_{k\geqslant 1} \subset L^2(gd\mu)$ is the biorthogonal system of $(\phi'(e_k^*))_{k\geqslant 1} \subset L^2(gd\mu)$. This shows that in turn, $(\phi(e_k))_{k\geqslant 1}$ is an unconditional basis of $L^2(gd\mu)$. □

We will now establish a variant of Theorem 3.31 for conditional bases. Recall that if $(e_k)_{n\geqslant 1}$ is a basis on some Banach space X, the projections $P_N \colon X \to X$ defined by

$$P_N\Bigl(\sum_k a_k e_k\Bigr) \,=\, \sum_{k=1}^N a_k e_k$$

are uniformly bounded. We will say that $(e_k)_{k\geqslant 1}$ is an R-basis if the set $\{P_N \,:\, N \geqslant 1\}$ is actually R-bounded. It follows from [22, Cor. 3.15] that any unconditional basis on L^p is an R-basis. See Remark 3.33 (2) for more on this.

Proposition 3.32 *Let $1 \leqslant p < \infty$ and let $(e_k)_{k\geqslant 1}$ be an R-basis of $L^p(\Omega, \mu)$. Then there exists a density g on Ω such that $\phi_{p,g}(e_k) \in L^2(gd\mu)$ for any $k \geqslant 1$, and the sequence $(\phi_{p,g}(e_k))_{k\geqslant 1}$ is a basis of $L^2(gd\mu)$.*

Proof. According to [89, Thm. 2.1], there exists a constant $C \geqslant 1$ and a density g on Ω such that with $\phi = \phi_{p,g}$, we have

$$\|\phi P_N \phi^{-1} h\|_2 \leqslant C\|h\|_2, \qquad N \geqslant 1,\ h \in L^2(gd\mu) \cap L^p(gd\mu).$$

Then the proof is similar to the one of Theorem 3.31, using [91, Prop. 1.a.3] instead of Lemma 3.29. We skip the details. □

Remark 3.33

(1) Theorem 3.31 and Proposition 3.32 can be easily extended to finite dimensional Schauder decompositions. We refer to [91, Sect. 1.g] for general information on this notion. Given a Schauder decomposition $(X_k)_{k\geqslant 1}$ of a Banach space X, let P_N be the associated projections, namely for any $N \geqslant 1$, $P_N \colon X \to X$ is the bounded projection onto $X_1 \oplus \cdots \oplus X_N$ vanishing on X_k for any $k \geqslant N+1$. We say that $(X_k)_{k\geqslant 1}$ is an R-Schauder decomposition if the set $\{P_N \,:\, N \geqslant 1\}$ is R-bounded. Then we obtain that for any $1 < p < \infty$ and for any finite dimensional R-Schauder (resp. unconditional)

decomposition $(X_k)_{k\geqslant 1}$ *of* $L^p(\mu)$*, there exists a density* g *on* Ω *such that* $\phi_{p,g}(X_k) \subset L^2(gd\mu)$ *for any* $k \geqslant 1$*, and* $(\phi_{p,g}(X_k))_{k\geqslant 1}$ *is a Schauder (resp. unconditional) decomposition of* $L^2(gd\mu)$*.*

(2) The concept of R*-Schauder decompositions can be tracked down to [7], and played a key role in [22] and then in various works on* L^p*-maximal regularity and* H^∞ *calculus, see in particular [70, 73]. Let* C_p *denote the Schatten spaces. For* $1 < p \neq 2 < \infty$*, an explicit example of a Schauder decomposition on* $L^2([0,1]; C_p)$ *which is not* R*-Schauder is given in [22, Sect. 5]. More generally, it follows from [70] that whenever a reflexive Banach space* X *has an unconditional basis and is not isomorphic to* ℓ^2*, then* X *admits a finite dimensional Schauder decomposition which is not* R*-Schauder. This applies in particular to* $X = L^p([0,1])$*, for any* $1 < p \neq 2 < \infty$*. However the question whether* $L^p([0,1])$ *admits a Schauder basis which is not* R*-Schauder, is apparently an open question.*

We finally mention that according to [73, Thm. 3.3], any unconditional decomposition on a Banach space X *with property* (Δ) *is an* R*-Schauder decomposition.*

4 The Mihlin and Hörmander functional calculus

4.1 Introduction

Consider the Laplace operator Δ on $\mathbb{R}^d$ and a bounded function $f : \mathbb{R}_+ \to \mathbb{C}$. By the spectral theory for self-adjoint operators, $f(-\Delta)$ is a bounded operator on $L^2(\mathbb{R}^d)$.

Let $q \in (1, \infty)$. A famous result of Mihlin is that $f(-\Delta)$ extends to a bounded operator on $L^q(\mathbb{R}^d)$ provided that for an integer m strictly larger than $\frac{d}{2}$, f is m-times continuously differentiable and satisfies

$$\sup_{t>0} |t^k \frac{d^k}{dt^k} f(t)| < \infty \quad (k = 0, \dots, m). \tag{4.1}$$

Later on, this condition was relaxed by Hörmander [58],[59, equ (7.9.8)]. Namely, $f(-\Delta)$ extends still to a bounded operator on $L^q(\mathbb{R}^d)$ provided that for an integer $m > \frac{d}{2}$,

$$\sup_{R>0} \int_{R/2}^{R} \left| t^k \frac{d^k}{dt^k} f(t) \right|^2 \frac{dt}{t} < \infty \quad (k = 0, \dots, m). \tag{4.2}$$

Condition 4.1 is known as the Mihlin condition, and 4.2 is known as the Hörmander condition. Clearly, 4.2 is weaker than 4.1.

In fact, there are several variants of 4.1 and 4.2, we refer to [60, sec 1] for an overview. The original ones in [97, 98] and [58, thm 2.5] consider functions $f : \mathbb{R}^d \to \mathbb{C}$ and the above versions then correspond to radial functions.

The Mihlin and Hörmander condition have a remarkable applicability to partial differential equations.

Many other authors have extended Mihlin's and Hörmander's theorem to other (differential) operators A instead of $-\Delta$ and spaces different from the Euclidean space $\mathbb{R}^d$, for example (Sub)laplacian operators on Lie groups of polynomial growth, Schrödinger operators, or operators satisfying Gaussian estimates [82, 95, 55, 21, 100, 1, 35, 99, 17, 37, 9, 104, 117, 106].

To obtain a Mihlin or Hörmander theorem for an operator A, one often uses estimates on

(I) the semigroup $T(z) = e^{-zA}$ generated by $-A$, and related operators like $A^{\frac{1}{2}}T(z)$,

(II) boundary values of the semigroup on the imaginary axis, often regularized by powers of resolvents, e.g. $(1 + A)^{-\alpha} e^{itA}$,

(III) or imaginary powers A^{it}.

(IV) Partially, also resolvents $R(\lambda, A) = (\lambda - A)^{-1}$, and related operators like $A^{\frac{1}{2}}R(\lambda, A)$ are useful.

Estimates on (I) have been decisively used to show a Mihlin theorem in [35], on (II) to show a Hörmander theorem in [100, 101] and estimates on (III) are considered in [118, 37].

Assign to 4.1 the following Banach algebra of functions.

$$\mathcal{M}^m = \{f : (0,\infty) \to \mathbb{C},\ f \text{ is } m-\text{times continuously differentiable}, \\ \|f\|_{\mathcal{M}^m} = \sup_{t>0,k=0,\dots,m} |t^k \frac{d^k}{dt^k} f(t)| < \infty\}. \tag{4.3}$$

Then the question whether Mihlin's theorem holds for an operator A on a space $X = L^q(\Omega)$ can be restated as a functional calculus problem. Namely, suppose that A is a self-adjoint operator on some space $L^2(\Omega)$, then Mihlin's theorem states that

$$\mathcal{M}^m \to B(X), f \mapsto f(A) \text{ is a bounded homomorphism} \tag{M}$$

for $m > \frac{d}{2}$, $X = L^q(\Omega)$ and any $q \in (1, \infty)$.

Similarly, to 4.2 we assign a Banach algebra. For our considerations, it will be convenient to allow an additional exponent $p \in (1, \infty)$.

$$\mathcal{H}^m_p = \left\{ f : (0,\infty) \to \mathbb{C},\ \|f\|_{\mathcal{H}^m_p} = \sup_{R>0,\, k=0,\dots,m} \left(\int_{R/2}^R \left| t^k \frac{d^k}{dt^k} f(t) \right|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty \right\}. \tag{4.4}$$

Then Hörmander's theorem states that

$$\mathcal{H}^m_p \to B(X), f \mapsto f(A) \text{ is a bounded homomorphism} \tag{H}$$

for $m > \frac{d}{2}$, $p = 2$ and $X = L^q(\Omega)$ for any $q \in (1, \infty)$.

In this chapter, we consider a sectorial operator A with dense range on some Banach space X. The central purpose is to study the connections of M and H with boundedness conditions for the operators in (I)-(IV). (We will see in section 4.2 how to reasonably define in this setting the operator $f(A)$.)

In fact, we modify the spaces 4.3 and 4.4 slightly by letting

$$\mathcal{M}^\alpha = \{f : (0,\infty) \to \mathbb{C},\ \|f\|_{\mathcal{M}^\alpha} = \|f \circ \exp\|_{\mathcal{B}^\alpha_{\infty,1}(\mathbb{R})} < \infty\} \tag{4.5}$$

where $\mathcal{B}^\alpha_{\infty,1}(\mathbb{R})$ is a Besov space, and

$$\mathcal{H}^\alpha_p = \{f : (0,\infty) \to \mathbb{C},\ \|f\|_{\mathcal{H}^\alpha_p} = \|f \circ \exp\|_{\mathcal{W}^\alpha_p} \equiv \sup_{n\in\mathbb{Z}} \|\varphi(\cdot - n)(f \circ \exp)\|_{W^\alpha_p(\mathbb{R})} < \infty\} \tag{4.6}$$

where $\varphi \in C_0^\infty(\mathbb{R})$ shall satisfy $\sum_{n\in\mathbb{Z}} \varphi(t-n) = 1$ for any $t \in \mathbb{R}$, and $W_p^\alpha(\mathbb{R})$ is a Sobolev space. The space $\mathcal{W}_p^\alpha$ is sometimes called a uniform Sobolev space [130, def 2.19].

The spaces $\mathcal{M}^\alpha$ and $\mathcal{H}_p^\alpha$ are defined also for non-integer $\alpha > 0$. We refer to remark 4.10 and proposition 4.11 in section 4.2 for the connection between 4.3 and 4.5 (4.4 and 4.6 resp).

Many of the theorems in the literature leading to M and H do not just use operator norm estimates for (I)-(IV), but also additional properties of the operators. Often, they are self-adjoint on L^2, and positive and contractive on the whole L^p-scale. Also kernel bounds, or estimates on L^1 play an important role. Since a variety of assumptions on A apparently lead to the same result, one might ask for a common denominator of the assumptions. Our intuition is that the notion of R-boundedness, which has turned out to be very fruitful in connection with functional calculus in the last several years, is the underlying reason.

Let us concretise the last sentence.

A first approach to M is the following theorem from [24, thm 4.10]. Recall the notion of a sectorial operator, the spaces $H^\infty(\Sigma_\theta)$ of bounded holomorphic functions on a sector Σ_θ, and the bounded $H^\infty(\Sigma_\theta)$ calculus from chapter 2.

Theorem 4.1 ([24]) *Let A be an injective sectorial operator on some Banach space X. Let $\alpha > 0$. Then the following conditions are equivalent.*

(1) *The sectorial angle $\omega(A)$ equals 0, A has a bounded $H^\infty(\Sigma_\theta)$ calculus for any $\theta > 0$, and there exists a constant $C > 0$ such that for any $\theta > 0$ and any $f \in H^\infty(\Sigma_\theta)$, $\|f(A)\| \leqslant C\theta^{-\alpha}\|f\|_{\infty,\theta}$.*

(2) *There exists a constant $C > 0$ such that for any $\theta > 0$ and $f \in H^\infty(\Sigma_\theta)$, $\|f(A)\| \leqslant C\|f\|_{\mathcal{M}^\alpha}$.*

Fortunately, to characterize the bounded $H^\infty(\Sigma_\theta)$ calculus, we have an already widely developed theory at hand, for which the above mentioned R-boundedness plays a key role [73, 81].

The boundedness of the calculus in H consists, loosely speaking, of two parts.

Firstly, observe that $\widetilde{W_p^\alpha} = \{f : (0,\infty) \to \mathbb{C},\ \|f\| = \|f \circ \exp\|_{W_p^\alpha} < \infty\}$ embeds continuously into $\mathcal{H}_p^\alpha$. Thus, if H holds, then necessarily the homomorphism

$$\widetilde{W_p^\alpha} \to B(X),\ f \mapsto f(A) \tag{W}$$

is bounded.

Secondly, if $\varphi \in C_c^\infty(\mathbb{R})$ is as in 4.6, it is easy to check that there exists a $C > 0$ such that for any choice of signs $a_n = \pm 1$, one has $\| \sum_{n\in\mathbb{Z}} a_n \varphi(\cdot - n)\|_{\mathcal{B}^\alpha_{\infty,1}(\mathbb{R})} \leqslant C$. Thus, if M holds, then $\left\|\sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \psi_n(A)x\right\|_{\mathrm{Rad}(X)} \leqslant C\|x\|$, where $\psi_n(t) = \varphi(e^{t-n})$. A duality reasoning shows then that M (or the stronger H) implies

$$\Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \psi_n(A)x\Big\|_{\mathrm{Rad}(X)} \cong \|x\|. \tag{PL}$$

If A is the Laplace operator on $X = L^p(\mathbb{R}^d)$, $1 < p < \infty$, then condition PL is known as the Paley-Littlewood theorem [123, VI.7.14].

Thus it seems natural to first establish W and PL in terms of (I)-(IV). It is easy to see that if X is a Hilbert space, then these two conditions will already imply H. However, to obtain a similar conclusion for more general spaces X, we will be lead to strengthen the mere boundedness in W to R-boundedness, a notion we already have seen in definition 3.7 of chapter 3.

The following are the central results.

We obtain a Mihlin calculus under the assumption of norm bounds on imaginary powers, or R-bounds on semigroup and wave operators (see proposition 4.65 and lemma 4.72). Here the bounds on the operators are motivated by the Mihlin norms of the corresponding functions. The order α of these Mihlin norms is however smaller than the order β of the obtained Mihlin calculus. The gap is essentially 1, and within the class of all Banach spaces X, we will show in chapter 5 that this is optimal when comparing the imaginary powers and the functional calculus.

A second result is that additional information on the type and cotype of the underlying Banach space X narrows the gap. We will show in proposition 4.79 that the just mentioned bounds on imaginary powers, the semigroup and wave operators imply a $\mathcal{H}^\beta_r$ calculus with differentiation order $\beta - \alpha \approx \max(\frac{1}{\text{type}\, X} - \frac{1}{\text{cotype}\, X}, \frac{1}{2})$ and exponent $\frac{1}{r} \approx \beta - \alpha$.

Replacing uniform norm and R-boundedness by averaged R-boundedness, a notion we will develop in this chapter, allows to characterize the Hörmander functional calculus. In theorem 4.73, we will show that R-bounded Hörmander functional calculus has an equivalent formulation in terms of any of the operator families in (I) - (IV) above.

This averaged R-boundedness is a weakened form of square function estimates. The square function estimates themselves are characterized by the matricial R-boundedness of the functional calculus, see proposition 4.50.

Let us give an overview of the organization of this chapter.

In section 4.2, we fix notations and settings. We introduce the needed function spaces, in particular $\mathcal{M}^\alpha$ and $\mathcal{H}^\alpha_p$, and discuss some of their properties.

In a second step, we define the operator A which will be the subject of our study. We also introduce an operator B corresponding to the logarithm of A. In the sequel we will study A and B parallely. The reason for the introduction of a second operator is that for some assertions, the formulation for B is simpler and more natural. In fact, we have already seen an example, in the use of the exponential map in the definitions of $\mathcal{M}^\alpha$ and $\mathcal{H}^\alpha_p$.

In the rest of section 4.2, we develop several functional calculi associated with A and B. Hereby, the holomorphic functional calculus from chapter 2 will serve as the fundament. The results obtained here should be seen as technical devices for the rest of the chapter.

Section 4.3 is concerned with the non-localized part W of the Hörmander calculus. We introduce the notion of operator families which are R-bounded in a certain averaged sense. This notion appeared implicitly in some recent work; we draw particular attention to [61, prop 4.1, rem 4.2] and [51, cor 3.19].

In our context, averaged R-boundedness turns out to be the adequate tool to handle the calculus $\widetilde{W^\alpha_p} \to B(X)$ from W. In the simplest case $p = 2$, we obtain equivalent characterizations of the R-boundedness of this calculus in terms of the operators from (I) - (IV) (see theorem 4.46).
Due to [51, cor 3.19] there is a relationship between estimates of the generalized square functions from chapter 2 section 2.4 and averaged R-boundedness, which we investigate in subsection 4.3.2.

Section 4.4 is then devoted to the study of PL. Apart from the passage from W to H, we also apply PL to obtain new characterizations of fractional domain spaces of A. These reduce to the classical Triebel-Lizorkin spaces if A is the Laplace operator. We also describe their real interpolation spaces, which correspond to Besov spaces in the classical case, in our abstract framework.

In subsection 4.5.1, we compare the Mihlin calculus with bounds on the distinguished operator families (I) - (IV).

In subsection 4.5.2, we then apply the results of the sections 4.3, 4.4 and 4.5.1 to the Hörmander calculus. In the main theorem 4.73, we set up conditions for the operators in (I) - (IV). These conditions contain (R)-bounds and also averaged R-bounds from section 4.3 and square function estimates. We put these conditions into a context with the Hörmander functional calculus and study their relations.

In section 4.6, we finally compare the results of our operator theoretic approach with Hörmander multiplier theorems from the literature.

We start in subsection 4.6.1 by an interpolation procedure of the Hörmander calculus, where the differentiation order m in H is lowered. This procedure applies to operators A which are defined on an L^p-scale and are self-adjoint on L^2, a situation which appears in many examples, and it actually yields the optimal differentiation order for the Hörmander calculus in the initial example $A = -\Delta$, for the spaces $X = L^q(\mathbb{R}^d)$, $1 < q < \infty$.

In subsection 4.6.2, we show that for the latter example, Hörmander's classical theorem extends to an R-bounded version (theorem 4.90).

Finally, in subsection 4.6.3, we consider operators A satisfying so-called (generalized) Gaussian estimates, which is a typical assumption for Hörmander calculus theorems in the literature, and compare our theory with Blunck's result [9].

4.2 Function spaces and functional calculus

This section has a twofold purpose. At first, we introduce function spaces over $\mathbb{R}$ and $\mathbb{R}_+$ and resume some simple properties that we will need in subsequent sections.

These include the classical Sobolev and Besov spaces, the Hörmander spaces $\mathcal{W}^\alpha_p$ and $\mathcal{H}^\alpha_p$, and the Mihlin spaces $\mathcal{B}^\alpha$ and $\mathcal{M}^\alpha$, which are of central interest for our functional calculus. We also calculate the Mihlin norms of some functions which are important for later considerations.

Secondly, we define 0-strip-type operators and 0-sectorial operators having their spectrum in $\mathbb{R}$ and $\mathbb{R}_+$, and construct functional calculi for them, i.e. we insert these operators into functions of the Sobolev, Besov, Mihlin and Hörmander spaces mentioned above.

These calculi are based on the holomorphic functional calculus (see chapter 2, section 2.2). We give some sufficient/necessary criteria for the boundedness of these calculi (see propositions 4.18, 4.22 and remark 4.23), which will be pursued and developed much deeper in subsequent sections.

We also introduce an unbounded version of the Besov and Sobolev calculus to include e.g. fractional powers of sectorial operators.

4.2.1 Function spaces

Let us start with some manipulations of functions which we will use frequently in this chapter.

Definition 4.2 *Let $f : \mathbb{R} \to \mathbb{C}$ be a function and $h \in \mathbb{R}$. We define $(\Delta_h f) : \mathbb{R} \to \mathbb{C}$ by $(\Delta_h f)(x) = f(x+h) - f(x)$. For $N \in \mathbb{N}$, we define the iterated difference recursively by*

$$(\Delta_h^N f)(x) = (\Delta_h(\Delta_h^{N-1} f))(x).$$

Definition 4.3 *Let $I \subset \mathbb{C}\backslash(-\infty, 0]$ and $J = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi,\ e^z \in I\}$. For a function $f : I \to \mathbb{C}$, we define $f_e = f \circ \exp : J \to \mathbb{C}, z \mapsto f(e^z)$.*

Notation 4.4 *For $t \in \mathbb{R}$ and $\alpha \geqslant 0$, we write*

$$\langle t \rangle = (1 + t^2)^{1/2}.$$

For most of our function spaces, partitions of unity play a key role. In some cases, the function itself is partitioned, in other cases, its Fourier transform is partitioned. It will be convenient to distinguish the two cases, and we reserve the letter φ for the first and the letter ϕ for the second case.

Definition 4.5

(1) *Let $\varphi \in C_c^\infty(\mathbb{R})$. Assume that $\operatorname{supp} \varphi \subset [-1, 1]$ and $\sum_{n=-\infty}^\infty \varphi(t - n) = 1$ for all $t \in \mathbb{R}$. For $n \in \mathbb{Z}$, we put $\varphi_n = \varphi(\cdot - n)$ and call $(\varphi_n)_n$ an equidistant partition of unity.*

(2) *Similarly, if $\operatorname{supp} \varphi \subset [\frac{1}{2}, 2]$ and $\sum_{n=-\infty}^\infty \varphi(2^{-n} t) = 1$ for all $t > 0$, we put $\varphi_n = \varphi(2^{-n} \cdot)$ and call $(\varphi_n)_n$ a dyadic partition of unity.*

(3) *Finally, we will make use of a third partition of unity. Let* $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R})$ *be such that* $\operatorname{supp}\phi_1 \subset [\frac{1}{2}, 2]$ *and* $\operatorname{supp}\phi_0 \subset [-1,1]$. *For* $n \geqslant 2$, *put* $\phi_n = \phi_1(2^{1-n}\cdot)$, *so that* $\operatorname{supp}\phi_n \subset [2^{n-2}, 2^n]$. *For* $n \leqslant -1$, *put* $\phi_n = \phi_{-n}(-\cdot)$. *We assume that* $\sum_{n=-\infty}^{\infty}\phi_n(t) = 1$ *for all* $t \in \mathbb{R}$. *Then we call* $(\phi_n)_n$ *a dyadic Fourier partition of unity, since we will use such a partition only for the Fourier image of a function.*

For the existence of such smooth partitions, we refer to the idea in [6, lem 6.1.7]. Whenever $(\varphi_n)_n$ *is a partition of unity of one of these three types, we put*

$$\widetilde{\varphi}_n = \sum_{k=-1}^{1} \varphi_{n+k}.$$

It will be very often useful to note that

$$\widetilde{\varphi}_m \varphi_n = \varphi_n \text{ for } m = n \text{ and } \widetilde{\varphi}_m \varphi_n = 0 \text{ for } |n-m| \geqslant 2. \tag{4.7}$$

We recall the following classical function spaces:

Definition 4.6 *Let* $m \in \mathbb{N}_0$ *and* $\alpha > 0$.

(1) $C_b^m = \{f : \mathbb{R} \to \mathbb{C} : f$ *m-times diff. and* $f, f', \dots, f^{(m)}$ *uniformly cont. and bounded*$\}$.

(2) $C_c^\infty = \{f : \mathbb{R} \to \mathbb{C} : f$ ∞*-times differentiable with compact support*$\}$.

(3) $W_p^\alpha = \{f \in L^p(\mathbb{R}) : \|f\|_{W_p^\alpha} = \|(\hat{f}(t)\langle t\rangle^\alpha)\check{}\|_p < \infty\}$, *for* $1 < p < \infty$. *We will always restrict to* $\alpha > \frac{1}{p}$, *so that* $W_p^\alpha \hookrightarrow C_b^0$ *[116, p. 222] and* W_p^α *is closed under pointwise multiplication.*

(4) $\mathcal{B}_{\infty,\infty}^\alpha$ *and* $\mathcal{B}_{\infty,1}^\alpha$, *the Besov spaces defined for example in [128, p. 45]: Let* $(\phi_n)_n$ *be a dyadic Fourier partition of unity. Then*

$$\mathcal{B}_{\infty,\infty}^\alpha = \{f \in C_b^0 : \|f\|_{\mathcal{B}_{\infty,\infty}^\alpha} = \sup_{n\in\mathbb{Z}} 2^{|n|\alpha}\|f * \check{\phi}_n\|_\infty < \infty\}$$

and

$$\mathcal{B}^\alpha = \mathcal{B}_{\infty,1}^\alpha = \{f \in C_b^0 : \|f\|_{\mathcal{B}_{\infty,1}^\alpha} = \sum_{n\in\mathbb{Z}} 2^{|n|\alpha}\|f * \check{\phi}_n\|_\infty < \infty\}. \tag{4.8}$$

The space W_p^α *is a Banach algebra with respect to pointwise multiplication if* $\alpha > \frac{1}{p}$, *and all the other spaces from definition 4.6 are Banach algebras for any* $\alpha > 0$ *(see [116, p. 222] for the Sobolev and Besov spaces).*

Further we also consider the local spaces

(5) $W_{p,\text{loc}}^\alpha = \{f : \mathbb{R} \to \mathbb{C} : f\varphi \in W_p^\alpha$ *for all* $\varphi \in C_c^\infty\}$ *for* $1 < p < \infty$ *and* $\alpha > \frac{1}{p}$.

(6) $\mathcal{B}^\alpha{}_{\text{loc}} = \{f : \mathbb{R} \to \mathbb{C} : f\varphi \in \mathcal{B}^\alpha$ *for all* $\varphi \in C_c^\infty\}$ *for* $\alpha > 0$.

These spaces are closed under pointwise multiplication. Indeed, if $\phi \in C_c^\infty$ *is given, choose* $\psi \in C_c^\infty$ *such that* $\psi\phi = \phi$. *For* $f, g \in W_{p,\text{loc}}^\alpha$ $(\mathcal{B}^\alpha{}_{\text{loc}})$, *we have* $(fg)\phi = (f\phi)(g\psi) \in W_{p,\text{loc}}^\alpha$ $(\mathcal{B}^\alpha{}_{\text{loc}})$.

Aside from these classical spaces we introduce the following Mihlin class and Hörmander class.

Definition 4.7

(1) *Let* $\alpha > 0$. *We define the Mihlin class*

$$\mathcal{M}^\alpha = \{f : \mathbb{R}_+ \to \mathbb{C} : \ f_e \in \mathcal{B}^\alpha\},$$

equipped with the norm $\|f\|_{\mathcal{M}^\alpha} = \|f_e\|_{\mathcal{B}^\alpha}$.

(2) *Let* $(\varphi_n)_n$ *be an equidistant partition of unity,* $p \in (1, \infty)$ *and* $\alpha > \frac{1}{p}$.

We define the Hörmander class

$$\mathcal{W}_p^\alpha = \{f \in L^p_{loc}(\mathbb{R}) : \ \|f\|_{\mathcal{W}_p^\alpha} = \sup_{n \in \mathbb{Z}} \|\varphi_n f\|_{W_p^\alpha} < \infty\}$$

and equip it with the norm $\|f\|_{\mathcal{W}_p^\alpha}$. *Further, we set*

$$\mathcal{H}_p^\alpha = \{f \in L^p_{loc}(\mathbb{R}_+) : \ \|f\|_{\mathcal{H}_p^\alpha} = \|f \circ \exp\|_{\mathcal{W}_p^\alpha} < \infty\}$$

with the norm $\|f\|_{\mathcal{H}_p^\alpha}$.

The parameter α *will sometimes be referred to as the differentiation order of the Mihlin (Hörmander) class.*

We have the following elementary properties of Mihlin and Hörmander spaces.

Proposition 4.8

(1) *The spaces* $\mathcal{M}^\alpha, \mathcal{W}_p^\alpha, \mathcal{H}_p^\alpha$ *are Banach algebras.*

(2) *Different partitions of unity* $(\varphi_n)_n$ *give the same spaces* $\mathcal{W}_p^\alpha$ *and* $\mathcal{H}_p^\alpha$ *with equivalent norms.*

(3) *For any* $a \in \mathbb{R}$, *the mapping* $f \mapsto f(\cdot - a)$ *is an isomorphism* $\mathcal{B}^\alpha \to \mathcal{B}^\alpha$ *and* $\mathcal{W}_p^\alpha \to \mathcal{W}_p^\alpha$. *Similarly, for any* $b > 0$, *the mapping* $f \mapsto f(b(\cdot))$ *is an isomorphism* $\mathcal{M}^\alpha \to \mathcal{M}^\alpha$ *and* $\mathcal{H}_p^\alpha \to \mathcal{H}_p^\alpha$.

The isomorphism constants are independent of a *and* b.

(4) *For any* $a > 0$, *the mapping* $f \mapsto f(a\cdot)$ *is an isomorphism* $\mathcal{B}^\alpha \to \mathcal{B}^\alpha$ *and* $\mathcal{W}_p^\alpha \to \mathcal{W}_p^\alpha$, *and also the mapping* $f \mapsto f((\cdot)^a)$ *is an isomorphism* $\mathcal{M}^\alpha \to \mathcal{M}^\alpha$ *and* $\mathcal{H}_p^\alpha \to \mathcal{H}_p^\alpha$.

The space $\mathcal{M}^\alpha$ coincides with the space $\Lambda^\alpha_{\infty,1}(\mathbb{R}_+)$ in [24, p. 73].

Proof. (1) The Mihlin class $\mathcal{M}^\alpha$ is a Banach algebra, since $\mathcal{B}^\alpha$ is a Banach algebra.

Let us show the completeness of $\mathcal{W}_p^\alpha$. Consider a sequence $(f_n)_{n \geqslant 1}$ in $\mathcal{W}_p^\alpha$ such that

$$\sum_{n=1}^\infty \|f_n\|_{\mathcal{W}_p^\alpha} < \infty. \tag{4.9}$$

We have to show that $\sum_n f_n$ converges in $\mathcal{W}_p^\alpha$. From 4.9, we immediately deduce that the sum $\sum_n \|f_n \varphi_k\|_{W_p^\alpha}$ is finite for any $k \in \mathbb{Z}$, so that by the completeness of W_p^α, there exists a $g_k \in W_p^\alpha$ such that the identity $\sum_n f_n \varphi_k = g_k$ holds in W_p^α. Also $\sum_n \sup_k \|f_n \varphi_k\|_{W_p^\alpha} < \infty$, so that the convergence of $\sum_n f_n \varphi_k$ in W_p^α is uniformly in $k \in \mathbb{Z}$.

Set $f(t) = \sum_{k\in\mathbb{Z}} g_k(t)$. This sum converges at least pointwise, since the support condition on $(\varphi_n)_n$ implies that for fixed $t \in \mathbb{R}$, at most 3 summands do not vanish. Note that we have

$$f\varphi_l = \sum_k (\sum_n f_n \varphi_k)\varphi_l = \sum_k (\sum_n f_n \varphi_k \varphi_l) = \sum_n f_n \varphi_l = g_l.$$

Then

$$\|f - \sum_{n=1}^N f_n\|_{\mathcal{W}_p^\alpha} = \sup_{l\in\mathbb{Z}} \|(f - \sum_{n=1}^N f_n)\varphi_l\|_{W_p^\alpha} = \sup_{l\in\mathbb{Z}} \|g_l - \sum_{n=1}^N f_n\varphi_l\|_{W_p^\alpha} \to 0$$

by the mentioned uniform convergence. We have shown the completeness of $\mathcal{W}_p^\alpha$.

Let us show its multiplicativity. Pick $f, g \in \mathcal{W}_p^\alpha$. As W_p^α is a Banach algebra,

$$\|\varphi_n fg\|_{W_p^\alpha} = \|(\widetilde{\varphi}_n f)(\varphi_n g)\|_{W_p^\alpha} \lesssim \|\widetilde{\varphi}_n f\|_{W_p^\alpha} \|\varphi_n g\|_{W_p^\alpha} \leqslant \sum_{k=-1}^{1} \|\varphi_{n+k} f\|_{W_p^\alpha} \|\varphi_n g\|_{W_p^\alpha}.$$

Consequently,

$$\begin{aligned} \|fg\|_{\mathcal{W}_p^\alpha} &= \sup_n \|\varphi_n fg\|_{W_p^\alpha} \lesssim \sup_n \sum_{k=-1}^{1} \|\varphi_{n+k} f\|_{W_p^\alpha} \|\varphi_n g\|_{W_p^\alpha} \\ &\leqslant 3 \sup_n \|\varphi_n f\|_{W_p^\alpha} \sup_n \|\varphi_n g\|_{W_p^\alpha} \cong \|f\|_{\mathcal{W}_p^\alpha} \|g\|_{\mathcal{W}_p^\alpha}. \end{aligned}$$

Thus, $\mathcal{W}_p^\alpha$ is a Banach algebra.

We immediately deduce that also $\mathcal{H}_p^\alpha$ is a Banach algebra.

(2) Consider two equidistant partitions of unity $(\varphi_n)_n$ and $(\psi_n)_n$. Then 4.7 still holds for $\widetilde{\psi}_n$ and φ_n, i.e. $\widetilde{\psi}_n \varphi_n = \varphi_n$ for any $n \in \mathbb{Z}$. Thus,

$$\sup_n \|\varphi_n f\|_{W_p^\alpha} = \sup_n \|\widetilde{\psi}_n \varphi_n f\|_{W_p^\alpha} \leqslant \sup_n \|\widetilde{\psi}_n f\|_{W_p^\alpha} \|\varphi_n\|_{W_p^\alpha} \lesssim \sup_n \|\widetilde{\psi}_n f\|_{W_p^\alpha} \leqslant 3 \sup_n \|\psi_n f\|_{W_p^\alpha}.$$

Interchanging the roles of $(\varphi_n)_n$ and $(\psi_n)_n$ shows the claim.

(3) and (4) The statements for $\mathcal{M}^\alpha$ and $\mathcal{H}_p^\alpha$ follow directly from the statements for $\mathcal{B}^\alpha$ and $\mathcal{W}_p^\alpha$. The latter in turn follow easily from the definition of $\mathcal{B}^\alpha$ and $\mathcal{W}_p^\alpha$. □

We will often compare the spaces from definitions 4.6 and 4.7. The following (continuous) embeddings will be useful.

Proposition 4.9

(1) *Let $m, n \in \mathbb{N}_0$ and $\alpha, \beta > 0$ such that $m > \beta > \alpha > n$. Then*

$$C_b^m \hookrightarrow \mathcal{B}_{\infty,\infty}^\beta \hookrightarrow \mathcal{B}^\alpha \hookrightarrow \mathcal{B}_{\infty,\infty}^\alpha \hookrightarrow C_b^n.$$

(2) *Let $\beta > \alpha > 0$ and $E \in \{W, \mathcal{B}, \mathcal{B}_{\infty,\infty}, \mathcal{W}\}$. Then*

$$E^\beta \hookrightarrow E^\alpha.$$

(3) *Let* $p \in (1, \infty)$ *and* $\alpha > \frac{1}{p}$. *Then*

$$W_p^\alpha \hookrightarrow \mathcal{W}_p^\alpha.$$

(4) *Let* $\alpha, \beta, \gamma, \varepsilon > 0$ *and* $1 < p < q < \infty$ *such that* $\alpha \geqslant \beta + \frac{1}{p} - \frac{1}{q}$ *and* $\beta > \gamma + \frac{1}{q}$. *Then*

$$\mathcal{B}^{\alpha+\varepsilon} \hookrightarrow \mathcal{W}_q^\alpha \hookrightarrow \mathcal{W}_p^\alpha \hookrightarrow \mathcal{W}_q^\beta \hookrightarrow \mathcal{B}^\beta$$

Proof. (1) The first embedding follows from [128, 2.3.5 (1), 2.2.2. (6)]. The second and third embedding are shown in [128, 2.3.2 prop 2], and the last one follows from [128, 2.5.12 thm].

(2) This follows from [128, 2.3.2 prop 2].

(3) Let $f \in W_p^\alpha$ and $(\varphi_n)_{n\in\mathbb{Z}}$ be an equidistant partition of unity. As W_p^α is a Banach algebra and $\|\varphi_n\|_{W_p^\alpha} = \|\varphi_0(\cdot - n)\|_{W_p^\alpha} = \|\varphi_0\|_{W_p^\alpha}$ does not depend on $n \in \mathbb{Z}$, we have

$$\|f\|_{\mathcal{W}_p^\alpha} = \sup_{n\in\mathbb{Z}} \|\varphi_n f\|_{W_p^\alpha} \lesssim \sup_{n\in\mathbb{Z}} \|\varphi_n\|_{W_p^\alpha} \|f\|_{W_p^\alpha} \lesssim \|f\|_{W_p^\alpha}.$$

(4) For a function g with compact support, one has $\|g\|_{W_q^\alpha} \leqslant C\|g\|_{\mathcal{B}^{\alpha+\varepsilon}_{\infty,\infty}}$, where C depends on the width of the support. This follows from the local representation of the $\mathcal{B}^{\alpha+\varepsilon}_{\infty,\infty}$ norm [128, thm 2.5.12]. Further, for $f \in \mathcal{B}^{\alpha+\varepsilon}_{\infty,\infty}$, by [129, p. 124], one has $\|f\|_{\mathcal{B}^{\alpha+\varepsilon}_{\infty,\infty}} \cong \sup_n \|\varphi_n f\|_{\mathcal{B}^{\alpha+\varepsilon}_{\infty,\infty}}$. Then

$$\|f\|_{\mathcal{W}_q^\alpha} = \sup_n \|\varphi_n f\|_{W_q^\alpha} \lesssim \sup_n \|\varphi_n f\|_{\mathcal{B}^{\alpha+\varepsilon}_{\infty,\infty}} \cong \|f\|_{\mathcal{B}^{\alpha+\varepsilon}_{\infty,\infty}} \lesssim \|f\|_{\mathcal{B}^{\alpha+\varepsilon}},$$

and thus, $\mathcal{B}^{\alpha+\varepsilon} \hookrightarrow \mathcal{W}_q^\alpha$.

For $f \in \mathcal{W}_q^\alpha$, we have with $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$,

$$\|f\|_{\mathcal{W}_p^\alpha} \cong \sup_n \|f\varphi_n\widetilde{\varphi}_n\|_{W_p^\alpha} \lesssim \sup_n \|f\varphi_n\|_{W_q^\alpha} \|\widetilde{\varphi}_n\|_{W_r^\alpha} \lesssim \|f\|_{\mathcal{W}_q^\alpha},$$

which yields $\mathcal{W}_q^\alpha \hookrightarrow \mathcal{W}_p^\alpha$.

The embedding $\mathcal{W}_p^\alpha \hookrightarrow \mathcal{W}_q^\beta$ follows from the Sobolev embedding theorem $W_p^\alpha \hookrightarrow W_q^\beta$ [6, thm 6.5.1].

Finally, for $\gamma' > \gamma$ such that $\beta \geqslant \gamma' + \frac{1}{q}$, by (1) and the fact that $W_q^\beta \hookrightarrow \mathcal{B}^{\gamma'}_{\infty,\infty}$ [6, chap 6],

$$\|f\|_{\mathcal{B}^\gamma} \lesssim \|f\|_{\mathcal{B}^{\gamma'}_{\infty,\infty}} \cong \sup_n \|\varphi_n f\|_{\mathcal{B}^{\gamma'}_{\infty,\infty}} \lesssim \sup_n \|\varphi_n f\|_{W_q^\beta} = \|f\|_{\mathcal{W}_q^\beta}.$$

Hence, $\mathcal{W}_q^\beta \hookrightarrow \mathcal{B}^\gamma$. □

Remark 4.10 *The names "Mihlin and Hörmander class" are justified by the following facts.*

A classical form of the Mihlin condition for a β*-times differentiable function* $f : \mathbb{R}_+ \to \mathbb{C}$ *is (cf. [35, (1)])*

$$\sup_{t>0,k=0,\ldots,\beta} |t|^k |f^{(k)}(t)| < \infty. \tag{4.10}$$

If f *satisfies 4.10, then* $f \in \mathcal{M}^\alpha$ *for* $\alpha < \beta$ *[24, p. 73]. Conversely, if* $f \in \mathcal{M}^\alpha$*, then* f *satisfies 4.10 for* $\alpha \geqslant \beta$*. The proof of this can be found in [46, thm 3.1], where also the case* $\beta \notin \mathbb{N}$ *is considered.*

Let $1 < p < \infty$. The classical Hörmander condition (see [59, equ (7.9.8)] for $p = 2$) with a parameter $\alpha_1 \in \mathbb{N}$ reads as follows:

$$\sum_{k=0}^{\alpha_1} \sup_{R>0} \int_{R/2}^{2R} |R^k f^{(k)}(t)|^p dt/R < \infty. \tag{4.11}$$

Furthermore, consider the following condition for some $1 < p < \infty$ and $\alpha > \frac{1}{p}$:

$$\sup_{t>0} \|\psi f(t\cdot)\|_{W_p^\alpha} < \infty, \tag{4.12}$$

where ψ is a fixed function in $C_c^\infty(\mathbb{R}_+)\backslash\{0\}$.

This condition appears in [21, (1)],[55], [95, (1.1)], [100, 99, 9] with $p = 2$, and [37], [104, (7.66) and (7.68)] with different values of p.

With a reasoning as in proposition 4.8, one can check that 4.12 does not depend on the particular choice of ψ (see also [37, p. 445]).

By the following proposition, the norm $\|\cdot\|_{\mathcal{H}_p^\alpha}$ expresses condition 4.12 and generalizes the classical Hörmander condition 4.11.

Proposition 4.11 *Let $f \in L^1_{loc}(\mathbb{R}_+)$. Consider the conditions*

(1) *f satisfies 4.11,*

(2) *f satisfies 4.12,*

(3) $\|f\|_{\mathcal{H}_p^\alpha} < \infty$.

Then $(1) \Rightarrow (2)$ *if $\alpha_1 \geqslant \alpha$ and* $(2) \Rightarrow (1)$ *if $\alpha \geqslant \alpha_1$. Further,* $(2) \Leftrightarrow (3)$.

Proof. The proof of the statement concerning (1) and (2) is simple and is omitted.

(2) $\Leftrightarrow$ (3)

Let $(\varphi_n)_n$ be an equidistant partition of unity and $\varphi = \varphi_0$. Then

$$\begin{aligned} \|f\|_{\mathcal{H}_p^\alpha} &= \sup_n \|\varphi_n f_e\|_{W_p^\alpha} \cong \sup_{s\in\mathbb{R}} \|\varphi(\cdot - s) f_e\|_{W_p^\alpha} \\ &= \sup_{t>0} \|\varphi f_e(\cdot + \log t)\|_{W_p^\alpha} = \sup_{t>0} \|\varphi f(te^{(\cdot)})\|_{W_p^\alpha}. \end{aligned}$$

So we have to show that

$$\sup_{t>0} \|\varphi f(te^{(\cdot)})\|_{W_p^\alpha} \cong \sup_{t>0} \|\psi f(t\cdot)\|_{W_p^\alpha}. \tag{4.13}$$

where $\psi \in C_c^\infty(\mathbb{R}_+)\backslash\{0\}$. We distinguish the cases $\alpha \in \mathbb{N}_0$ and $\alpha \notin \mathbb{N}_0$.

1st case: $\alpha = m \in \mathbb{N}_0$.
We start with "$\lesssim$" in 4.13.
Fixing some $t > 0$, we have

$$\|\varphi f(te^{(\cdot)})\|_{W_p^m} \cong \sum_{k=0}^{m} \| \left[\varphi f(te^{(\cdot)})\right]^{(k)} \|_p.$$

Fixing some $k \in \{0, \ldots, m\}$, we have in turn

$$\| \left[\varphi f(te^{(\cdot)})\right]^{(k)} \|_p \lesssim \sum_{j=0}^{k} \|\varphi^{(k-j)} f(te^{(\cdot)})^{(j)}\|_p.$$

Fix some $j \in \{0, \ldots, k\}$, and write $\widetilde{\varphi} = \varphi^{(k-j)}$. Then

$$\|\widetilde{\varphi} f(te^{(\cdot)})^{(j)}\|_p \lesssim \sum_{l=0}^{j} \|\widetilde{\varphi} f^{(l)}(te^{(\cdot)}) t^l e^{l(\cdot)}\|_p.$$

Fix some $l \in \{0, \ldots, j\}$ and write $K = \operatorname{supp} \varphi \circ \log \subset \mathbb{R}_+$. Then

$$\begin{aligned}\|\widetilde{\varphi} f^{(l)}(te^{(\cdot)}) t^l e^{l(\cdot)}\|_p^p &= \int_{\mathbb{R}} |\widetilde{\varphi}(x)|^p \, |f^{(l)}(te^x)(te^x)^l|^p dx \\ &= \int_0^\infty |\widetilde{\varphi}(\log s)|^p \, |f^{(l)}(ts)|^p \, |ts|^{pl} \frac{ds}{s} \\ &\cong \int_K |\widetilde{\varphi}(\log s)|^p \, |f^{(l)}(ts)|^p \, |t|^{pl} ds \\ &\cong \int_K |t^l f^{(l)}(ts)|^p ds.\end{aligned}$$

Putting things together yields

$$\|\varphi f(te^{(\cdot)})\|_{W_p^m}^p \lesssim \sum_{l=0}^{m} \int_K |t^l f^{(l)}(ts)|^p ds \leqslant \|\psi f(t\cdot)\|_{W_p^m}^p, \tag{4.14}$$

if $\psi = 1$ on K. Thus, "$\lesssim$" in 4.13 follows for such a ψ, and hence for any $\psi \in C_c^\infty(\mathbb{R}_+)\backslash\{0\}$.

For the estimate "$\gtrsim$" in 4.13, we first proceed similarly: For fixed $t > 0$, we have

$$\|\psi f(t\cdot)\|_{W_p^m} \lesssim \sum_{k=0}^{m} \|(\psi f(t\cdot))^{(k)}\|_p \lesssim \sum_{k=0}^{m} \sum_{l=0}^{k} \|\psi^{(k-l)} f^{(l)}(t\cdot) t^l\|_p.$$

For fixed k and l, put $\widetilde{\psi} = \psi^{(k-l)}$ and $K = \operatorname{supp} \psi \circ \exp \subset \mathbb{R}$. Then

$$\begin{aligned}\|\psi^{(k-l)} f^{(l)}(t\cdot) t^l\|_p^p &= \int_0^\infty |\widetilde{\psi}(s) f^{(l)}(ts) t^l|^p ds = \int_{\mathbb{R}} |\widetilde{\psi}(e^x) f^{(l)}(te^x) t^l|^p e^x dx \\ &\cong \int_K |\widetilde{\psi}(e^x) f^{(l)}(te^x)(te^x)^l|^p dx \lesssim \int_K |f^{(l)}(te^x)(te^x)^l|^p dx.\end{aligned}$$

We want to replace $g_l(x) = f^{(l)}(te^x)(te^x)^l$ by $h_l(x) = f(te^{(\cdot)})^{(l)}(x)$. We develop the l-th derivative:

$$h_l(x) = \sum_{j=0}^{l} \beta_j g_j(x). \tag{4.15}$$

with coefficients $\beta_j \in \mathbb{N}$ and $\beta_l = 1$. Thus, $g_l = h_l - \sum_{j=0}^{l-1} \beta_j g_j$ and

$$\int_K |g_l(x)|^p dx \lesssim \int_K |h_l(x)|^p dx + \sum_{j=0}^{l-1} \int_K |g_j(x)|^p dx.$$

Now proceed inductively and estimate with the same argument the last sum by

$$\sum_{j=0}^{l-1} \int_K |g_j(x)|^p dx \lesssim \sum_{j=0}^{l-1} \int_K |h_j(x)|^p dx + \sum_{j=0}^{l-2} \int_K |g_j(x)|^p dx \lesssim \ldots \lesssim \sum_{j=0}^{l-1} \int_K |h_j(x)|^p dx.$$

This gives $\|\psi^{(k-l)} f^{(l)}(t\cdot)t^l\|_p \lesssim \sum_{j=0}^{l} \|f(te^{(\cdot)})^{(j)}\|_{L^p(K)}$, and finally

$$\|\psi f(t\cdot)\|_{W_p^m} \lesssim \|\widetilde{\varphi} f(te^{(\cdot)})\|_{W_p^m} \tag{4.16}$$

for any $\widetilde{\varphi} \in C_c^\infty(\mathbb{R})$ such that $\widetilde{\varphi} \equiv 1$ on K. There exists an $N \in \mathbb{N}$ such that $\widetilde{\varphi} = \sum_{k=-N}^{N} \varphi_k$ has this property. But

$$\sup_{t>0} \|\varphi_k f(te^{(\cdot)})\|_{W_p^m} = \sup_{t>0} \|\varphi f(te^k e^{(\cdot)})\|_{W_p^m} = \sup_{t>0} \|\varphi f(te^{(\cdot)})\|_{W_p^m}.$$

Therefore,

$$\sup_{t>0} \|\psi f(t\cdot)\|_{W_p^m} \lesssim \sup_{t>0} \sum_{k=-N}^{N} \|\varphi_k f(te^{(\cdot)})\|_{W_p^m} \lesssim \sup_{t>0} \|\varphi f(te^{(\cdot)})\|_{W_p^m},$$

which shows 4.13 in the 1st case.

2nd case $\alpha = m + \theta$ *with* $m \in \mathbb{N}_0$ *and* $\theta \in (0, 1)$.
Fix $t > 0$. For any function $f : \mathbb{R} \to \mathbb{C}$, denote $\overline{f} : \mathbb{R}_+ \to \mathbb{C}$ the function given by

$$\overline{f}(t\cdot) = f(te^{(\cdot)}).$$

In 4.14, we have shown that

$$\|\varphi \overline{f}\|_{W_p^m} \lesssim \|f\|_{W_p^m(K)}, \tag{4.17}$$
$$\|\varphi \overline{f}\|_{W_p^{m+1}} \lesssim \|f\|_{W_p^{m+1}(K)}, \tag{4.18}$$

where $K = \text{supp}(\varphi \circ \log) \subset \mathbb{R}_+$ and $W^m(K)$, $W^{m+1}(K)$ are Sobolev spaces with domain K.

By the complex interpolation result $[W_p^m(K), W_p^{m+1}(K)]_\theta = W_p^\alpha(K)$, we deduce from 4.17 and 4.18 that

$$\|\varphi \overline{f}\|_{W_p^\alpha} \lesssim \|f\|_{[W_p^m(K), W_p^{m+1}(K)]_\theta} \lesssim \|\psi f\|_{W_p^\alpha}$$

for any ψ with $\psi = 1$ on K. This gives "$\lesssim$" in 4.13.

For "$\gtrsim$" in 4.13, interchange the roles of f and $\overline{f}$, use 4.16 instead of 4.14 and argue similarly. □

At the end of this subsection, we estimate the Mihlin norms of some typical functions.

Proposition 4.12

(1) *Let $\alpha > 0$ and $\beta \geqslant 0$. Then for any $\varepsilon > 0$, there exists a $C > 0$ such that for any $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,*

$$\|t \mapsto t^\beta \exp(-e^{i\theta}t)\|_{\mathcal{M}^\alpha} \leqslant C(\cos\theta)^{-(\alpha+\beta+\varepsilon)}. \tag{4.19}$$

(2) *Let $\alpha, \varepsilon > 0$. Then there exists $C > 0$ such that for any $a \in \mathbb{R}$,*

$$\left\|t \mapsto \frac{1}{(1+t)^{\alpha+\varepsilon}} e^{ita}\right\|_{\mathcal{M}^\alpha} \leqslant C\langle a\rangle^{\alpha+\varepsilon}.$$

(3) *Let $\alpha, \varepsilon > 0$ and $\psi \in C^\infty(\mathbb{R}_+)$ such that for some $0 < t_0 < t_1 < \infty$, $\psi(t) = \begin{cases} 0 & \text{for } t < t_0 \\ 1 & \text{for } t > t_1 \end{cases}$.*

Then

$$\left\|t \mapsto \frac{t^{\alpha+\varepsilon} - (1+t)^{\alpha+\varepsilon}}{t^{\alpha+\varepsilon}(1+t)^{\alpha+\varepsilon}} \psi(t) e^{it}\right\|_{\mathcal{M}^\alpha} < \infty.$$

(4) *For any $\alpha > 0$,*

$$\|t \mapsto t^{is}\|_{\mathcal{M}^\alpha} \cong C\langle s\rangle^\alpha.$$

Proof. (1) For $m \in \mathbb{N}_0$, we denote by $\|f\|_{\widetilde{\mathcal{M}^m}}$ the expression of the "classical Mihlin condition" in 4.10, i.e.

$$\|f\|_{\widetilde{\mathcal{M}^m}} = \max(\sup_{t>0} |f(t)|, \sup_{t>0} |t^m f^{(m)}(t)|).$$

At first, assume that $\alpha + \varepsilon \in \mathbb{N}$ and set $m = \alpha + \varepsilon$. We show 4.19 with $\widetilde{\mathcal{M}^m}$ in place of $\mathcal{M}^\alpha$ and with $\varepsilon = 0$, i.e. $\|t \mapsto t^\beta \exp(-e^{i\theta}t)\|_{\widetilde{\mathcal{M}^m}} \lesssim (\cos\theta)^{-(m+\beta)}$.

On the one hand, we have

$$\begin{aligned} \sup_{t>0} |t^\beta \exp(-e^{i\theta}t)| &= \sup_{t>0} t^\beta \exp(-\cos\theta \cdot t) \\ &= \sup_{t>0} \left[(\cos\theta \cdot t)^\beta \exp(-\cos\theta \cdot t)\right] (\cos\theta)^{-\beta} \\ &\lesssim (\cos\theta)^{-\beta}. \end{aligned} \tag{4.20}$$

On the other hand,

$$\begin{aligned} \sup_{t>0} t^m \left|\frac{d^m}{dt^m} \left(t^\beta \exp(-e^{i\theta}t)\right)\right| &\lesssim \sup_{t>0} \sum_{k=0}^m t^m |t^{\beta-k}(-e^{i\theta})^{m-k} \exp(-e^{i\theta}t)| \\ &\lesssim \sum_{k=0}^m (\cos\theta)^{-(m+\beta-k)} \\ &\lesssim (\cos\theta)^{-(m+\beta)}. \end{aligned}$$

The second inequality follows from 4.20 with $m + \beta - k$ in place of β.

By $\|f\|_{\mathcal{M}^\alpha} \lesssim \|f\|_{\widetilde{\mathcal{M}^{\alpha+\varepsilon}}}$ from remark 4.10, we deduce that 4.19 holds for $\alpha + \varepsilon \in \mathbb{N}$.

To extend this to arbitrary $\alpha > 0$, we use complex interpolation.

Consider the function

$$f(z;t) = (\cos\theta)^{z+\beta} t^\beta \exp(-e^{i\theta}t) \quad (z \in \mathbb{C},\ t > 0).$$

Clearly, f is analytic in z. Since $|(\cos\theta)^{m+i\tau+\beta}| = (\cos\theta)^{m+\beta}$, we have shown above that

$$\sup_{|\theta|<\frac{\pi}{2}} \sup_{\tau\in\mathbb{R}} \|f(m+i\tau;\cdot)\|_{\mathcal{M}^{m-\varepsilon}} < \infty \quad (m \in \mathbb{N}). \tag{4.21}$$

We want to include the case $m = 0$. Extend definition 4.7 of the Mihlin class to negative exponents, and set $\mathcal{M}^{-\varepsilon} = \{f \circ \exp : \ f \in \mathcal{B}^{-\varepsilon}_{\infty,1}\}$. The space C_b embeds continuously into the Besov space with negative exponent $\mathcal{B}^{-\varepsilon}_{\infty,1}$ (combine e.g. [128, 2.3.2] and [116, 2.2.2 thm (iv)]). We have $f(i\tau; e^{(\cdot)}) \in C_b$ and

$$\|f(i\tau; e^{(\cdot)})\|_{C_b} = \sup_{t>0} |(\cos\theta)^\beta t^\beta \exp(-e^{i\theta}t)| \lesssim 1,$$

so 4.21 also holds for $m = 0$.

According to [127, 2.4.1 thm], we have the complex interpolation identity

$$[\mathcal{M}^{m-\varepsilon}, \mathcal{M}^{m+1-\varepsilon}]_\delta = \mathcal{M}^{m+\delta-\varepsilon}.$$

Thus it follows from 4.21 that

$$\sup_{|\theta|<\frac{\pi}{2}} \|f(m+\delta;\cdot)\|_{\mathcal{M}^{m+\delta-\varepsilon}} < \infty \quad (m \in \mathbb{N}_0, \delta \in [0,1]),$$

or

$$\sup_{|\theta|<\frac{\pi}{2}} \|t \mapsto (\cos\theta)^{m+\delta+\beta} t^\beta \exp(-e^{i\theta}t)\|_{\mathcal{M}^{m+\delta-\varepsilon}} < \infty.$$

Choosing δ such that $m + \delta = \alpha + \varepsilon$, 4.19 follows.

(2) We argue similarly as in (1). More precisely, let $a \in \mathbb{R}$ and

$$f(z;t) = \frac{1}{(1+t)^z} \frac{1}{\langle a \rangle^z} e^{ita}.$$

By an elementary calculation, for any $m \in \mathbb{N}_0$ and any $\tau \in \mathbb{R}$, $\sup_{a\in\mathbb{R}} \|f(m+i\tau;\cdot)\|_{\widetilde{\mathcal{M}^m}} \lesssim \langle\tau\rangle^m$, whence by the same argument as in (1),

$$\sup_{a\in\mathbb{R}} \|f(m+i\tau;\cdot)\|_{\mathcal{M}^{m-\varepsilon}} \lesssim \langle\tau\rangle^m \quad (m \in \mathbb{N}_0).$$

By Stein's complex interpolation [122, thm 1], we deduce that

$$\sup_{a\in\mathbb{R}} \|f(\alpha+\varepsilon;\cdot)\|_{\mathcal{M}^\alpha} \lesssim 1 \quad (\alpha > 0),$$

or equivalently, $\|t \mapsto \frac{1}{(1+t)^{\alpha+\varepsilon}} e^{ita}\|_{\mathcal{M}^\alpha} \lesssim \langle a \rangle^\alpha$.

(3) We let $f(z;t) = \dfrac{t^z - (1+t)^z}{t^z(1+t)^z} \psi(t) e^{it}$. Then by an elementary calculation, $\|f(m+i\tau;\cdot)\|_{\widetilde{\mathcal{M}^{m+1}}} \lesssim \langle\tau\rangle^{m+1}$. Now argue as in (1) and (2).

(4) We have $\|t \mapsto t^{is}\|_{\mathcal{M}^\alpha} = \|t \mapsto e^{ist}\|_{\mathcal{B}^\alpha}$. Take a dyadic Fourier partition of unity $(\phi_n)_{n\in\mathbb{Z}}$. Then $e^{is(\cdot)} * \check{\phi}_n(t) = \phi_n(s) e^{ist}$. Thus, by the dyadic supports of ϕ_n,

$$\|t \mapsto e^{ist}\|_{\mathcal{B}^\alpha} = \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \|e^{is(\cdot)} * \check{\phi}_n\|_\infty \cong \langle s \rangle^\alpha.$$

□

4.2.2 0-sectorial and 0-strip-type operators

Recall the notions of sectorial operators A, strip-type operators B and holomorphic functional calculus from chapter 2. We will now focus on the case $\omega(A) = \omega(B) = 0$ and define the operators which are subject for our functional calculi.

Definition 4.13 *Let A be a sectorial operator with $\omega(A) = 0$. Assume in addition that A has dense range. Then we call A a 0-sectorial operator.*

Let B be a strip-type operator with $\omega(B) = 0$. Then we call B a 0-strip-type operator.

In the next proposition, we show the correspondence between 0-sectorial operators and 0-strip-type operators.

Proposition 4.14

(1) *Let A be a 0-sectorial operator. Then $B = \log(A)$ is a 0-strip-type operator.*

(2) *Conversely, if B is a 0-strip-type operator which has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for some $\omega \in (0, \pi)$, then there exists a 0-sectorial operator A such that $B = \log(A)$.*

(3) *Let A be a 0-sectorial operator and $B = \log(A)$. Let $\omega \in (0, \pi)$. Then A has a bounded $H^\infty(\Sigma_\omega)$ calculus iff B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus. In that case, iB generates the group $U(t) = A^{it}$.*

Proof. (1) We have $\log(z) \cdot \frac{z}{(1+z)^2} \in H_0^\infty(\Sigma_\omega)$ for $\omega \in (\omega(A), \pi)$, so that by proposition 2.3, $D(A) \cap R(A) \subset D(\log(A))$ and $\log(A)$ is densely defined. Then by [52, exa 4.1.1], $B = \log(A)$ is a strip-type operator.

(2) This follows from [52, prop 5.3.3].

(3) The first statement follows again from [52, prop 5.3.3]. Then iB generates the c_0-group A^{it} by [52, cor 3.5.7]. □

4.2.3 Functional calculus on the line and half-line

Let E be a Sobolev space or Besov space as in definition 4.6. We want to define an E functional calculus for A and B from subsection 4.2.2 by tracing it back to the holomorphic functional calculus from chapter 2. The following observation on density of holomorphic functions will be useful.

Lemma 4.15

(1) *Let $E \in \{W_p^\alpha, \mathcal{B}^\alpha\}$, where $\alpha > 0$ and $p \in (1, \infty)$ such that $\alpha > \frac{1}{p}$. Then $\bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap E$ is dense in E. More precisely, if $\psi \in C_c^\infty$ such that $\psi(t) = 1$ for $|t| \leqslant 1$ and $\psi_n = \psi(2^{-n}(\cdot))$, then*

$$f * \check{\psi}_n \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap E \text{ and } f * \check{\psi}_n \to f \text{ in } E. \tag{4.22}$$

Thus if f happens to belong to several spaces E as above, then it can be simultaneously approximated by a holomorphic sequence in any of these spaces.

(2) *Let $f \in \mathcal{B}^\alpha$ and $(\phi_n)_n$ a dyadic Fourier partition of unity. Then $f = \sum_{n\in\mathbb{Z}} f * \check{\phi}_n$ converges in $\mathcal{B}^\alpha$.*

Proof. (1) Let $f \in E$ and ψ, ψ_n as in the lemma. As $f * \check{\psi}_n = (\hat{f}\psi_n)\check{}$ is the Fourier transform of a distribution with compact support, the Paley-Wiener theorem yields that $f * \check{\psi}_n$ is an entire function, given by

$$f * \check{\psi}_n(t + is) = \int_{\mathbb{R}} f(r - t)[e^{-s(\cdot)}\psi_n]\check{}(r)dr.$$

If $E = \mathcal{B}^\alpha$, we have in particular $f \in L^\infty(\mathbb{R})$, and if $E = W^\alpha_p$, then $f \in L^p(\mathbb{R})$. Putting $q = \infty$ or p, we get for $|s| \leqslant \omega$

$$\|f * \check{\psi}(\cdot + is)\|_{L^\infty(\mathbb{R})} \leqslant \|f\|_{L^q(\mathbb{R})}\|[e^{-s(\cdot)}\psi_n]\check{}\|_{L^{q'}(\mathbb{R})} \leqslant C(f, \omega, n).$$

Therefore, $f * \check{\psi}_n \in H^\infty(\text{Str}_\omega)$ for any $\omega > 0$.

For the convergence $f * \check{\psi}_n \to f$, we first consider the case $E = \mathcal{B}^\alpha$. Note that for $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, $\psi_n(t) = 1$ for $t \in [2^{-n}, 2^n]$ and $\operatorname{supp}\phi_k \subset [-2^{|k|+2}, 2^{|k|+2}]$. Thus, $\check{\psi}_n * \check{\phi}_k = \check{\phi}_k$ for $n \geqslant |k| + 2$ and

$$\begin{aligned}\|f - f * \check{\psi}_n\|_{\mathcal{B}^\alpha} &= \sum_{k\in\mathbb{Z}} 2^{|k|\alpha}\|(f - f * \check{\psi}_n) * \check{\phi}_k\|_\infty \\ &= \sum_{|k|>n-2} 2^{|k|\alpha}\|f * \check{\phi}_k - f * \check{\psi}_n * \check{\phi}_k\|_\infty \\ &\leqslant \sum_{|k|>n-2} 2^{|k|\alpha}\|f * \check{\phi}_k\|_\infty(1 + \|\check{\psi}_n\|_1). \end{aligned} \tag{4.23}$$

Now the convergence follows, since $\|\check{\psi}_n\|_1 = \|\check{\psi}\|_1$ is constant. The case $E = \mathcal{B}^\alpha_{p,p}$ is shown in precisely the same manner.

For the case $E = W^\alpha_p$, note first that

$$\|f * \check{\psi}_n\|_{W^\alpha_p} = \|[\langle t\rangle^\alpha \hat{f}(t)\psi(t)]\check{}\|_{L^p} \leqslant \|[\langle t\rangle^\alpha \hat{f}(t)]\check{}\|_{L^p}\|\check{\psi}_n\|_{L^1} = \|f\|_{W^\alpha_p}\|\check{\psi}\|_1,$$

so $f * \check{\psi}_n$ is a bounded sequence in W^α_p. Therefore, we can assume that f is in the dense subset $\mathcal{B}^\beta_{p,p}$ of W^α_p [128, 2.3.1 def 2], where $\beta > \alpha$. By the same calculation as above, $\|f - f * \check{\psi}_n\|_{\mathcal{B}^\beta_{p,p}} \to 0$. Thus, $\|f - f * \check{\psi}_n\|_{W^\alpha_p} \lesssim \|f - f * \check{\psi}_n\|_{\mathcal{B}^\beta_{p,p}} \to 0$.

(2) Note that $\sup_{n\in\mathbb{Z}} \|\check{\phi}_n\|_{L^1(\mathbb{R})} < \infty$ for any $n \in \mathbb{Z}$. Thus,

$$\sum_{k\in\mathbb{Z}} 2^{|k|\alpha}\|f * \check{\phi}_n * \check{\phi}_k\|_\infty = \sum_{k=n-1}^{n+1} 2^{|k|\alpha}\|f * \check{\phi}_n * \check{\phi}_k\|_\infty \lesssim \|f\|_\infty,$$

so that $f * \check{\phi}_n \in \mathcal{B}^\alpha$. Now using the disjointness of the supports of ϕ_n and ϕ_m for $|n-m| \geqslant 2$, we get

$$\begin{aligned} \|f - \sum_{n=-N}^{M} f * \check{\phi}_n\|_{\mathcal{B}^\alpha} &= \sum_{k\in\mathbb{Z}} 2^{|k|\alpha} \|(f - \sum_{n=-N}^{M} f * \check{\phi}_n) * \check{\phi}_k\|_\infty \\ &= \sum_{\substack{k \leqslant -N-2,\\ k \geqslant M+2}} 2^{|k|\alpha} \|f * \check{\phi}_k\|_\infty + \sum_{\substack{|k-(-M)|\leqslant 1,\\ |k-N|\leqslant 1}} 2^{k\alpha} \|(f - \sum_{|n-k|\leqslant 1} f * \check{\phi}_n) * \check{\phi}_k\|_\infty. \end{aligned}$$

As $f \in \mathcal{B}^\alpha$, the first sum goes to 0 as $N, M \to \infty$. The second sum can be estimated by

$$4\|\check{\phi}_1\|_\infty \sum_{\substack{|k-(-M)|\leqslant 1,\\ |k-N|\leqslant 1}} 2^{k\alpha} \|f * \check{\phi}_k\|_\infty,$$

which converges to 0 as $N, M \to \infty$ by the same reason. □

Remark 4.16 *In the case $E = \mathcal{B}^\alpha$, we have even $H^\infty(\mathrm{Str}_\omega) \hookrightarrow \mathcal{B}^\alpha$ for any $\omega > 0$.*

Indeed, let $f \in H^\infty(\mathrm{Str}_\omega)$. Then for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$, $f^{(n)}(t) = \frac{n!}{2\pi i} \int_{B(t,\omega/2)} \frac{f(z)}{(z-t)^n} dz$, and consequently, $f \in C_b^m(\mathbb{R})$. So the remark follows from proposition 4.9 by the embedding $C_b^m(\mathbb{R}) \hookrightarrow \mathcal{B}^\alpha$ for $m > \alpha$.

Similarly, we have $H^\infty(\Sigma_\omega) \hookrightarrow \mathcal{M}^\alpha$ for any $\omega \in (0, \pi)$ and $\alpha > 0$.

By lemma 4.15, it is clear how the E functional calculus should be defined.

Definition 4.17 *Let B be a 0-strip-type operator and $E \in \{\mathcal{B}^\alpha, W_p^\alpha\}$, where $\alpha > 0$ and $1 < p < \infty$ such that $\alpha > \frac{1}{p}$.*

We say that B has a (bounded) E calculus if there exists a constant $C > 0$ such that

$$\|f(B)\| \leqslant C\|f\|_E \quad (f \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap E). \tag{4.24}$$

In this case, by the just proved density of $\bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap E$ in E, the algebra homomorphism $u : \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap E \to B(X)$ given by $u(f) = f(B)$ can be continuously extended in a unique way to a bounded algebra homomorphism

$$u : E \to B(X),\ f \mapsto u(f).$$

We write again $f(B) = u(f)$ for any $f \in E$.

Assume that $E_1, E_2 \in \{\mathcal{B}^\alpha, W_p^\alpha\}$ and that B has an E_1 calculus and an E_2 calculus. Then for $f \in E_1 \cap E_2$, $f(B)$ is defined twice by the above. However, the second part of lemma 4.15 shows that these definitions coincide.

If A is a 0-sectorial operator, we say that A has a (bounded) $\mathcal{M}^\alpha$ calculus if $B = \log(A)$ has a bounded $\mathcal{B}^\alpha$ calculus, and put

$$f(A) = (f \circ \exp)(B), \quad (f \in \mathcal{M}^\alpha),$$

in accordance with 2.4.

Let us now have a closer look on the functional calculi in the above definition.

4.2.4 The $\mathcal{M}^\alpha/\mathcal{B}^\alpha$ calculus

We have the following characterization of the $\mathcal{M}^\alpha$ calculus, which is essentially from [24, thm 4.10].

Proposition 4.18 *Let A be a 0-sectorial operator and $\alpha > 0$. The following are equivalent.*

(1) *There exists $C > 0$ such that for all $\omega \in (0, \frac{\pi}{2})$ and $f \in H_0^\infty(\Sigma_\omega)$*

$$\|f(A)\| \leqslant C\omega^{-\alpha}\|f\|_{\infty,\omega}.$$

(2) *The above estimate $\|f(A)\| \leqslant C\omega^{-\alpha}\|f\|_{\infty,\omega}$ holds for all $f \in \bigcup_{\omega>0} H^\infty(\Sigma_\omega)$.*

(3) *A has a bounded $\mathcal{M}^\alpha$ calculus.*

Similarly, if B is a 0-strip-type operator, then B has a bounded $\mathcal{B}^\alpha$ calculus if and only if $\|f(B)\| \leqslant C\omega^{-\alpha}\|f\|_{\infty,\omega}$ for any $\omega > 0$ and $f \in H^\infty(\mathrm{Str}_\omega)$, if and only if $\|f(B)\| \leqslant C\omega^{-\alpha}\|f\|_{\infty,\omega}$ for any $\omega > 0$ and $f \in H_0^\infty(\mathrm{Str}_\omega)$.

Proof. The equivalence of (2) and (3) is proved in [24, thm 4.10], and (2) $\Rightarrow$ (1) is clear, so that (1) $\Rightarrow$ (2) remains. Let $\tau(z) = \frac{z}{(1+z)^2}$ and $f \in H^\infty(\Sigma_\omega)$. Then $\tau^{1/n}(z) \to 1$ for any $z \in \Sigma_\omega$ and $\sup_n \|\tau^{1/n}\|_{\infty,\omega} < \infty$. Thus, by proposition 2.5, $f(A)x = \lim_n(\tau^{1/n}f)(A)x$ for all $x \in X$. Since $\tau^{1/n}f \in H_0^\infty(\Sigma_\omega)$, by (1),

$$\begin{aligned}\|f(A)\| &\leqslant \limsup_n \|(\tau^{1/n}f)(A)\| \lesssim \omega^{-\alpha}\|\tau^{1/n}f\|_{\infty,\omega} \\ &\leqslant \omega^{-\alpha}\|\tau\|_{\infty,\omega}^{1/n}\|f\|_{\infty,\omega} \leqslant \omega^{-\alpha}\max(\|\tau\|_{\infty,\pi/2}, 1)\|f\|_{\infty,\omega}.\end{aligned}$$

The strip-type case follows then from proposition 4.14 and 2.4. □

As for the H^∞ calculus, there is a convergence lemma for the Mihlin calculus.

Note that the space $\mathcal{B}^\alpha$ (and thus also $\mathcal{M}^\alpha$) is not stable with respect to pointwise and norm-bounded convergence, i.e. for any $\alpha > 0$, there is $(f_n)_n \subset \mathcal{B}^\alpha$ such that $\|f_n\|_{\mathcal{B}^\alpha} \leqslant 1$ and $f_n(t) \to f(t)$ for all $t \in \mathbb{R}$, but $f \notin \mathcal{B}^\alpha$. Therefore, in the next proposition, we replace the class $\mathcal{B}^\alpha$ by $\mathcal{B}^\beta_{\infty,\infty}$, in which the described defect does not occur.

Proposition 4.19 (Convergence Lemma) *Let B be a 0-strip-type operator with bounded $\mathcal{B}^\alpha$ calculus for some $\alpha > 0$. Then the following convergence property holds. Let $\beta > \alpha$ and $(f_n)_n$ be a sequence in $\mathcal{B}^\beta_{\infty,\infty} \subset \mathcal{B}^\alpha$ such that*

(a) $\sup_n \|f_n\|_{\mathcal{B}^\beta_{\infty,\infty}} < \infty$,

(b) *$f_n(t) \to f(t)$ pointwise on $\mathbb{R}$ for some function f.*

Then

(1) $f \in \mathcal{B}^\beta_{\infty,\infty}$,

(2) *$f_n(B)x \to f(B)x$ for all $x \in X$.*

The corresponding statement holds for a 0-sectorial operator A in place of B and $\mathcal{M}^\alpha$ in place of $\mathcal{B}^\alpha$.

Proof. We give the proof for the strip-type case. By [128, thm 2.5.12], the norm on $\mathcal{B}^\beta_{\infty,\infty}$ is equivalent to the following norm:

$$\|f\|_{L^\infty(\mathbb{R})} + \sup_{x,t\in\mathbb{R},t\neq 0} |t|^{-\beta}|\Delta_t^N f(x)|$$

for a fixed $N \in \mathbb{N}$ such that $N > \beta$. Here, Δ_t^N is the iterated difference from definition 4.2. More precisely, by [128, rem 3], if $f \in L^\infty(\mathbb{R})$ such that the above expression is finite, then $f \in \mathcal{B}^\beta_{\infty,\infty}$. We have

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} \lim_n |f_n(x)| \leqslant \sup_{x\in\mathbb{R}} \sup_n |f_n(x)| = \sup_n \|f_n\|_\infty < \infty$$

and

$$\sup_{x,t\in\mathbb{R},t\neq 0} |t|^{-\beta}|\Delta_t^N f(x)| = \sup_{x,t} |t|^{-\beta} \lim_n |\Delta_t^N f_n(x)| \leqslant \sup_{x,t,n} |t|^{-\beta}|\Delta_t^N f_n(x)| \leqslant \sup_n \|f_n\|_{\mathcal{B}^\beta_{\infty,\infty}} < \infty.$$

Therefore, assertion (1) of the proposition follows.

Let $(\phi_n)_n$ be a dyadic Fourier partition of unity. Note that by the boundedness of the $\mathcal{B}^\alpha$ calculus and lemma 4.15, $f_n(B) = \sum_k f_n * \check{\phi}_k(B)$. We first show the stated convergence for each summand and claim that for any $x \in X$ and fixed $k \in \mathbb{Z}$,

$$f_n * \check{\phi}_k(B)x \to f * \check{\phi}_k(B)x. \tag{4.25}$$

Indeed, this follows from proposition 2.5. Fix some strip height $\omega > 0$. Firstly,

$$\|f_n * \check{\phi}_k\|_{\infty,\omega} \leqslant \|f_n\|_{L^\infty(\mathbb{R})} \sup_{|\theta|<\omega} \|\check{\phi}_k(i\theta - s)\|_{L^1(\mathbb{R})} \leqslant C.$$

Secondly, for any $z \in \text{Str}_\omega$,

$$f_n * \check{\phi}_k(z) = \int_{\mathbb{R}} f_n(s)\check{\phi}_k(z-s)ds \to \int_{\mathbb{R}} f(s)\check{\phi}_k(z-s)ds = f * \check{\phi}_k(z)$$

by dominated convergence, and 4.25 follows.

For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$, put $x_{n,k} = f_n * \check{\phi}_k(B)x$, where $x \in X$ is fixed. For any $N \in \mathbb{N}$,

$$\begin{aligned} \|f_n(B)x - f(B)x\| &\leqslant \sum_{k\in\mathbb{Z}} \|f_n * \check{\phi}_k(B)x - f * \check{\phi}_k(B)x\| \\ &\leqslant \sum_{|k|\leqslant N} \|x_{n,k} - f * \check{\phi}_k(B)x\| + \sum_{|k|>N} \left(\|x_{n,k}\| + \|f * \check{\phi}_k(B)x\|\right). \end{aligned}$$

By 4.25, we only have to show that

$$\lim_N \sup_n \sum_{|k|>N} \|x_{n,k}\| = 0. \tag{4.26}$$

Fix some $\gamma \in (\alpha, \beta)$.

$$\begin{aligned}\|x_{n,k}\| &\lesssim \|f_n * \check{\phi}_k\|_{\mathcal{B}^\alpha}\|x\| \lesssim \|f_n * \check{\phi}_k\|_{\mathcal{B}^\gamma_{\infty,\infty}}\|x\| \\ &= \sup_{l\in\mathbb{Z}} 2^{|l|\gamma}\|f_n * \check{\phi}_k * \check{\phi}_l\|_\infty \|x\| = \sup_{|l-k|\leqslant 1} 2^{|l|\gamma}\|f_n * \check{\phi}_k * \check{\phi}_l\|_\infty\|x\| \\ &\lesssim 2^{|k|\gamma}\|f_n * \check{\phi}_k\|_\infty\|x\|.\end{aligned}$$

By assumption of the proposition, $\sup_n \|f_n\|_{\mathcal{B}^\beta_{\infty,\infty}} = \sup_{n,k} 2^{|k|\beta}\|f_n * \check{\phi}_k\|_\infty < \infty$. Therefore, $\|x_{n,k}\| \lesssim 2^{|k|\gamma}2^{-|k|\beta}\|x\|$, and 4.26 follows. □

We spell out a particular case of the Convergence Lemma 4.19 which will be frequently used in the sequel.

Corollary 4.20 *Let $(\varphi_n)_n$ be an equidistant partition of unity and B a 0-strip-type operator. Assume that B has a $\mathcal{B}^\alpha$ calculus for some $\alpha > 0$. Then*

$$x = \sum_{n\in\mathbb{Z}} \varphi_n(B)x \quad (\textit{convergence in } X).$$

Proof. For $k \in \mathbb{N}$, set $f_k = \sum_{-k}^{k} \varphi_n$. Then the sequence $(f_k)_{k\geqslant 1}$ satisfies the assumptions of the preceding proposition with limit $f \equiv 1$. Indeed, the poinwise convergence is clear. For the uniform boundedness $\sup_{k\geqslant 1}\|f_k\|_{\mathcal{B}^\alpha} < \infty$, see e.g. lemma 4.52 in section 4.4.

Then the corollary follows from the Convergence Lemma 4.19 by the fact that $1(B) = \mathrm{Id}_X$ (proposition 2.3 (2)). □

Another application of the Convergence Lemma is the following.

Corollary 4.21 *Let $\beta > \alpha > 0$ and B be a 0-strip-type operator with bounded $\mathcal{B}^\alpha$ calculus. Further let A be a 0-sectorial operator with bounded $\mathcal{M}^\alpha$ calculus.*

(1) *Let $f \in \mathcal{B}^\beta_{\infty,\infty}$. Then for any $x \in X$, the mapping $t \mapsto f(t+(\cdot))(B)x \equiv f(t+B)x$ is continuous. Similarly, if $g : (0,\infty) \to \mathbb{C}$ is such that $g_e \in \mathcal{B}^\beta_{\infty,\infty}$, then for any $x \in X$, $t \mapsto g(tA)x$ is continuous.*

(2) *Let $f \in \mathcal{B}^{\beta+1}_{\infty,\infty}$.*

Then for any $x \in X$, the mapping $t \mapsto f(t+B)x$ is differentiable and

$$\frac{d}{dt}[f(t+B)x] = (f')(t+B)x.$$

Similarly, if $g : (0,\infty) \to \mathbb{C}$ is such that $g_e \in \mathcal{B}^{\beta+1}_{\infty,\infty}$, then for any $x \in X$ and $t > 0$

$$\frac{d}{dt}[g(tA)x] = [(\cdot)g'(t(\cdot))](A)x.$$

Proof. Note first that the sectorial claims follow easily from the strip-type claims by setting $f = g_e$ and $B = \log(A)$.

(1) Since $\beta > 0$, f is a continuous function and thus, for any $t \in \mathbb{R}$, $f(t+h) \to f(t)$ as $h \to 0$. Also $\sup_{h \neq 0} \|f(\cdot + h)\|_{\mathcal{B}^\beta_{\infty,\infty}} = \|f\|_{\mathcal{B}^\beta_{\infty,\infty}} < \infty$, so that we can appeal to proposition 4.19 with $f_n = f(\cdot + h_n)$ and h_n a null sequence.

(2) Let us check the claimed differentiability of $f(t+B)x$, and fix some $x \in X$ and $t_0 \in \mathbb{R}$. As $\mathcal{B}^{\beta+1}_{\infty,\infty} \hookrightarrow C^1_b$ by proposition 4.9, f is continuously differentiable. Hence for any $t \in \mathbb{R}$,

$$\lim_{h \to 0} \frac{1}{h}(f(t_0 + t + h) - f(t_0 + t)) = f'(t_0 + t).$$

Further, $\frac{1}{h}(f(t_0 + \cdot + h) - f(t_0 + \cdot))$ is uniformly bounded in $\mathcal{B}^\beta_{\infty,\infty}$ [116, sec 2.3 prop]. Then the claim follows at once from proposition 4.19. □

4.2.5 The W^α_p calculus

The W^α_p calculus for B can best be characterized in terms of the c_0-group generated by iB, see 4.27 below. Here, we have to restrict to the case $p \leqslant 2$. For these p, condition 4.27 below is sufficient for the W^α_p calculus, and if $p = 2$, it is also necessary (see remark 4.23).

Proposition 4.22 *Let X be a Banach space with dual X', and let $p \in (1, 2]$. Let $\alpha > \frac{1}{p}$, so that W^α_p is a Banach algebra. We assume that B is a 0-strip-type operator with c_0-group $U(t) = \exp(itB)$ such that for some $C > 0$ and all $x \in X$, $x' \in X'$*

$$\|\langle t\rangle^{-\alpha}\langle U(t)x, x'\rangle\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |\langle t\rangle^{-\alpha}\langle U(t)x, x'\rangle|^p dt\right)^{1/p} \leqslant C\|x\| \, \|x'\|. \tag{4.27}$$

Then B has a bounded W^α_p calculus. Moreover, $f(B)$ is given by

$$\langle f(B)x, x'\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)\langle U(t)x, x'\rangle dt \quad (f \in W^\alpha_p). \tag{4.28}$$

Proof. For $f \in W^\alpha_p, x \in X$, and $x' \in X'$, set

$$\langle \Phi(f)x, x'\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)\langle U(t)x, x'\rangle dt.$$

Denote $p' = \frac{p}{p-1}$ the conjugated exponent. We have

$$\begin{aligned} |\langle \Phi(f)x, x'\rangle| &\lesssim \int_{\mathbb{R}} |\hat{f}(t)\langle U(t)x, x'\rangle| dt = \int_{\mathbb{R}} |\langle t\rangle^\alpha \hat{f}(t)\langle t\rangle^{-\alpha}\langle U(t)x, x'\rangle| dt \\ &\leqslant \|\langle t\rangle^\alpha \hat{f}(t)\|_{p'}\|\langle t\rangle^{-\alpha}\langle U(t)x, x'\rangle\|_p \overset{4.27}{\lesssim} \|\langle t\rangle^\alpha \hat{f}(t)\|_{p'}\|x\| \, \|x'\| \\ &\leqslant \|f\|_{W^\alpha_p}\|x\| \, \|x'\|, \end{aligned} \tag{4.29}$$

so that Φ defines a bounded operator $W_p^\alpha \to B(X, X'')$. Let

$$K = \bigcap_{\omega>0} \{f \in H^\infty(\mathrm{Str}_\omega) : \exists C > 0 : |f(z)| \leqslant C(1 + |\operatorname{Re} z|)^{-2} \text{ and } \hat{f} \text{ has comp. supp.}\}$$

We have that K is a dense subset of W_p^α.

Indeed, by the Cauchy integral formula, $K \subset W_p^\alpha$. We now approximate a given $f \in W_p^\alpha$ by elements of K. Since $C_c^\infty(\mathbb{R})$ is dense in W_p^α, we can assume $f \in C_c^\infty(\mathbb{R})$. Let ψ and ψ_n be as in the density lemma 4.15 and put $f_n = f * \check{\psi}_n$. Then $\hat{f}_n = \hat{f}\psi_n$ has compact support. Further, the estimate $|f_n(z)| \leqslant C(1+|\operatorname{Re} z|)^{-2}$ for z in a given strip Str_ω follows from the Paley-Wiener theorem and the fact that $\hat{f}_n = \hat{f}\psi_n \in C_c^\infty(\mathbb{R})$. Thus, $f * \check{\psi}_n \in K$, and K is dense in W_p^α.

Assume for a moment that

$$\Phi(f) = f(B) \quad (f \in K). \tag{4.30}$$

Then by 4.29, there exists $C > 0$ such that for any $f \in K$, $\|f(B)\| \lesssim \|f\|_{W_p^\alpha}$. By the density of K in W_p^α, B has a W_p^α calculus. Then for any $f \in W_p^\alpha$ and $(f_n)_n$ a sequence in K such that $f = \lim_n f_n$,

$$f(B) = \lim_n f_n(B) = \lim_n \Phi(f_n) = \Phi(f),$$

where limits are in $B(X, X'')$. Thus, $f(B) = \Phi(f)$ for arbitrary f, and 4.28 follows.

We show 4.30. Let $f \in K$. We argue as in [53, lem 2.2]. Choose some $\omega > 0$. According to the representation formula

$$R(\lambda, B)x = -\operatorname{sgn}(\operatorname{Im}\lambda)i \int_0^\infty e^{i\operatorname{sgn}(\operatorname{Im}\lambda)\lambda t} U(-\operatorname{sgn}(\operatorname{Im}\lambda)t)x\,dt, \tag{4.31}$$

we have

$$\begin{aligned}
\langle f(B)x, x'\rangle &= \frac{1}{2\pi i}\int_{\mathbb{R}} f(s-i\omega)\langle R(s-i\omega, B)x, x'\rangle ds - \frac{1}{2\pi i}\int_{\mathbb{R}} f(s+i\omega)\langle R(s+i\omega, B)x, x'\rangle ds \\
&= \frac{1}{2\pi i}\left[\int_{\mathbb{R}} f(s-i\omega)\cdot i\int_0^\infty e^{-i(s-i\omega)t}\langle U(t)x, x'\rangle dt ds\right. \\
&\left. + \int_{\mathbb{R}} f(s+i\omega)\cdot i \int_{-\infty}^0 e^{-i(s+i\omega)t}\langle U(t)x, x'\rangle dt ds\right] \\
&\overset{(*)}{=} \frac{1}{2\pi}\left[\int_0^\infty \left(\int_{\mathbb{R}} f(s-i\omega)e^{-i(s-i\omega)t} ds\right)\langle U(t)x, x'\rangle dt\right. \\
&\left. + \int_{-\infty}^0 \left(\int_{\mathbb{R}} f(s+i\omega)e^{-i(s+i\omega)t} ds\right)\langle U(t)x, x'\rangle dt\right] \\
&\overset{(**)}{=} \frac{1}{2\pi}\left[\int_0^\infty \int_{\mathbb{R}} f(s)e^{-ist} ds \langle U(t)x, x'\rangle dt + \int_{-\infty}^0 \int_{\mathbb{R}} f(s)e^{-ist} ds \langle U(t)x, x'\rangle dt\right] \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)\langle U(t)x, x'\rangle dt \\
&= \langle \Phi(f)x, x'\rangle.
\end{aligned} \tag{4.32}$$

As $f \in K$, we could apply Fubini's theorem in $(*)$ and shift the contour of the integral in $(**)$. Hence 4.30 follows. □

Remark 4.23

(1) *For the case* $p = 2$, *there is a converse of proposition 4.22.*

Namely, assume that B *is a* 0*-strip-type operator having a bounded* W_2^α *calculus. Then 4.27 holds necessarily.*
Indeed, the calculation 4.32 in the proof of proposition 4.22 still holds, and thus

$$\langle f(B)x, x'\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)\langle U(t)x, x'\rangle dt \quad (f \in K).$$

Therefore, by the density of K *in* W_2^α,

$$\begin{aligned}\|\langle t\rangle^{-\alpha}\langle U(t)x, x'\rangle\|_2 &= 2\pi \sup\{|\langle f(B)x, x'\rangle| : \ f \in K, \ \|\langle t\rangle^\alpha \hat{f}(t)\|_2 \leqslant 1\} \\ &\lesssim \|x\| \, \|x'\|.\end{aligned}$$

(2) *Assume that the assumptions of proposition 4.22 hold, and in addition, for some* $x \in X$,

$$\|\langle t\rangle^{-\alpha} U(t)x\|_{L^p(\mathbb{R})} < \infty.$$

Then for $f \in W_p^\alpha$, $\int_{\mathbb{R}} |\hat{f}(t)| \, \|U(t)x\| dt \leqslant \|\hat{f}(t)\langle t\rangle^\alpha\|_{p'} \|\langle t\rangle^{-\alpha}U(t)x\|_p < \infty$, *so that the integral* $\int_{\mathbb{R}} \hat{f}(t)U(t)x dt$ *exists in* X *and*

$$f(B)x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)U(t)x dt \quad (f \in W_p^\alpha).$$

4.2.6 The extended calculus and the $\mathcal{W}_p^\alpha$ and $\mathcal{H}_p^\alpha$ calculus

Let $\alpha > 0$ and $p \in (1, \infty)$ such that $\frac{1}{p} < \alpha$. Throughout this section, we let B be a 0-strip-type operator which shall have either a $\mathcal{B}^\alpha$ calculus, or both a W_p^α calculus and a $\mathcal{B}^\beta$ calculus for some (large) $\beta > 0$.

As for the H^∞ calculus in proposition 2.3, there exists an "extended version" of the $\mathcal{B}^\alpha$ calculus (W_p^α calculus), which is defined for $f \in {\mathcal{B}^\alpha}_{\mathrm{loc}}$ ($f \in W^\alpha_{p,\mathrm{loc}}$).

Let $(\varphi_n)_n$ be an equidistant partition of unity and

$$D = D_B = \{x \in X : \exists N \in \mathbb{N} : \ \varphi_n(B)x = 0 \quad (|n| \geqslant N)\}. \tag{4.33}$$

Then D_B is a dense subset of X. Indeed, for any $x \in X$ let $x_N = \sum_{k=-N}^N \varphi_k(B)x$. Then for $|n| \geqslant N+1$, $\varphi_n(B)x_N = \sum_{k=-N}^N (\varphi_n\varphi_k)(B)x = 0$, so that x_N belongs to D. On the other hand, by corollary 4.20, x_N converges to x for $N \to \infty$. Clearly, D is independent of the choice of $(\varphi_n)_n$. If A is a 0-sectorial operator such that $B = \log(A)$, we shall also use the notation $D_A = D_B$ in subsequent sections. We call D_B the calculus core of B.

Definition 4.24 *Assume that* B *has a* $\mathcal{B}^\alpha$ *calculus or a* W_p^α *calculus and that* f *belongs to the corresponding local space* ${\mathcal{B}^\alpha}_{\mathrm{loc}}$ *or* $W^\alpha_{p,\mathrm{loc}}$. *We define the operator* $\Phi(f)$ *to be the closure of*

$$\begin{cases} D_B \subset X & \longrightarrow X \\ x & \longmapsto \sum_{n\in\mathbb{Z}} (f\varphi_n)(B)x \end{cases}. \tag{4.34}$$

Since for $x \in D_B$ *and large* $|n|$, $(\varphi_n f)(B)x = (\widetilde{\varphi}_n\varphi_n f)(B)x = (\widetilde{\varphi}_n f)(B)\varphi_n(B)x = 0$, *the above sum is finite. The fact that the operator in 4.34 is indeed closable will be proved in a moment.*

Proposition 4.25

(a) *The operator* $\Phi(f)$ *is closed and densely defined with domain*

$$D(\Phi(f)) = \{x \in X : \sum_{k=-n}^{n} (f\varphi_k)(B)x \text{ converges in } X \text{ as } n \to \infty\},$$

and it is independent of the choice of the partition of unity $(\varphi_n)_n$.

The sets D_B *and* $\{g(B)x : g \in C_c^\infty(\mathbb{R}), x \in X\}$ *are both cores for* $\Phi(f)$.

(b) *If further* $f \in \mathcal{B}^\alpha$ *or* $f \in W_p^\alpha$, *then* $\Phi(f)$ *coincides with the* $\mathcal{B}^\alpha$ *calculus or* W_p^α *calculus of* B. *If* $f \in \text{Hol}(\text{Str}_\omega)$ *for some* $\omega > 0$, *then* $\Phi(f)$ *coincides with the (unbounded) holomorphic calculus of* B.

(c) *Let* g *be a further function in* ${\mathcal{B}^\alpha}_{\text{loc}}$ *or* $W^\alpha_{p,\text{loc}}$. *Then* $\Phi(f)\Phi(g) \subset \Phi(fg)$, *where* $\Phi(f)\Phi(g)$ *is equipped with the natural domain* $\{x \in D(\Phi(g)) : \Phi(g)x \in D(\Phi(f))\}$. *If* $\Phi(g)$ *is a bounded operator, then even* $\Phi(f)\Phi(g) = \Phi(fg)$.

Proof. It is clear from 4.34 that D_B is a core. We divide the proof into the following parts.

(1) $\Phi(f)|_D$ is closable, and $D(\Phi(f)) = D' := \{x : \sum_{-n}^{n}(f\varphi_k)(B)x \text{ converges}\}$.

(2) The definition of $\Phi(f)$ is independent of the choice of $(\varphi_n)_n$.

(3) If $f \in \mathcal{B}^\alpha$ or $f \in W_p^\alpha$, then $\Phi(f) = f(B)$.

(4) If $f \in \text{Hol}(\text{Str}_\omega)$, then $\Phi(f) = f(B)$.

(5) The multiplicativity statement (c)

(6) $\{g(B)x : g \in C_c^\infty(\mathbb{R}), x \in X\}$ is a core for $\Phi(f)$.

(1) Let $(x_n)_n \subset D$ a sequence converging to 0 such that $\Phi(f)x_n$ converges to y for some $y \in X$. We have to show that $y = 0$. We claim that for any $x \in D$ and $l \in \mathbb{Z}$,

$$(\varphi_l f)(B)x = \varphi_l(B)\Phi(f)x. \tag{4.35}$$

Indeed, by corollary 4.20 and the multiplicativity of the $\mathcal{B}^\alpha$ calculus or W_p^α calculus,

$$\begin{aligned}(\varphi_l f)(B)x &= \sum_{k\in\mathbb{Z}} \varphi_k(B)(\varphi_l f)(B)x \\ &= \sum_{k\in\mathbb{Z}} (\varphi_k\varphi_l f)(B)x = \sum_{k\in\mathbb{Z}} \varphi_l(B)(\varphi_k f)(B)x \\ &= \varphi_l(B)\Phi(f)x.\end{aligned}$$

Thus, $0 = \lim_n(\varphi_l f)(B)x_n = \varphi_l(B)\lim_n \Phi(f)x_n = \varphi_l(B)y$. Again by corollary 4.20, $y = \sum_l \varphi_l(B)y = \sum_l 0 = 0$, so that $\Phi(f)|_D$ is closable.

Note that 4.35 extends to $x \in D(\Phi(f))$, since D is a core for $\Phi(f)$. Then for $x \in D(\Phi(f))$,

$$\sum_{-n}^{n}(f\varphi_k)(B)x = \sum_{-n}^{n}\varphi_k(B)\Phi(f)x \to \Phi(f)x.$$

Thus, $D(\Phi(f)) \subset D'$.

On the other hand, if $x \in D'$, then $x_n = \sum_{-n}^n \varphi_k(B)x$ is a sequence in $D \subset D(\Phi(f))$ which converges to x in the graph norm of $\Phi(f)$. Indeed, by corollary 4.20, $x_n \to x$. Further, as $x \in D'$, $\Phi(f)x_n = \sum_{-n}^n (f\varphi_k)(B)x$ is a Cauchy sequence. Thus, $x \in D(\Phi(f))$.

(2) Let $(\psi_n)_n$ be another partition of unity. For $x \in D$, we have

$$\sum_n (\psi_n f)(B)x = \sum_{n,k} \varphi_k(B)(\psi_n f)(B)x = \sum_{n,k} (\varphi_k \psi_n f)(B)x = \sum_k (\varphi_k f)(B)x.$$

Since D is a core for $\Phi(f)$, the claim follows.

(3) If $f \in \mathcal{B}^\alpha$ or $f \in W_p^\alpha$, then for $x \in D$,

$$\Phi(f)x = \sum_n (\varphi_n f)(B)x = \sum_n \varphi_n(B) f(B)x = f(B)x.$$

Since $f(B) \in B(X)$, also $\Phi(f) \in B(X)$ by the closed graph theorem and $\Phi(f) = f(B)$.

(4) Let $f \in \operatorname{Hol}(\operatorname{Str}_\omega)$ for some $\omega > 0$. We show firstly that $f(B)$ and $\Phi(f)$ coincide on D and secondly, that D is a core for $f(B)$. Let $\tau(z) = e^z/(1+e^z)^2$ and $n \in \mathbb{N}$ be sufficiently large so that $f\tau^n \in H_0^\infty(\operatorname{Str}_\omega)$. $\tau^n(B)$ maps $D \to D$ bijectively. Indeed, if $x \in D$, then for k sufficiently large, $\varphi_k(B)\tau^n(B)x = \tau^n(B)\varphi_k(B)x = 0$, so $\tau^n(B)(D) \subset D$. On the other hand, for a given $x \in D$, there is $\psi = \sum_{-N}^N \varphi_n \in C_c^\infty(\mathbb{R})$ such that $\psi(B)x = x$. Then with $y = (\psi\tau^{-n})(B)x \in D$, $\tau^n(B)y = \psi(B)x = x$, so that $\tau^n(B)(D) = D$. For $x \in D$, we have

$$\Phi(f)\tau^n(B)x = \sum_k (f\varphi_k)(B)\tau^n(B)x = \sum_k (f\tau^n\varphi_k)(B)x = \Phi(f\tau^n)x.$$

Further,

$$\Phi(f\tau^n)(B)x \overset{(3)}{=} (f\tau^n)(B)x = f(B)\tau^n(B)x,$$

so we have shown that $\Phi(f)|_D = f(B)|_D$.

It is clear from the definition of $\Phi(f)$ that D is a core for $\Phi(f)$. Thus it remains to approximate a given $x \in D(f(B))$ by $(x_N)_N \subset D$ such that $\|x - x_N\| \to 0$ and $\|f(B)x - f(B)x_N\| \to 0$. By [81, thm 15.8], $R(\tau^n(B))$ is a core for $f(B)$, so that we can assume $x = \tau^n(B)y \in R(\tau^n(B))$. Put $x_N = \sum_{k=-N}^N \varphi_k(B)\tau^n(B)y$. By corollary 4.20, $\|x - x_N\| \to 0$ and also

$$f(B)x_N = \sum_{k=-N}^N f(B)\varphi_k(B)\tau^n(B)y = \sum_{k=-N}^N (f\tau^n)(B)\varphi_k(B)y \to (f\tau^n)(B)y = f(B)x.$$

(5) Let $x \in D(\Phi(f)\Phi(g))$. Let us show that $x \in D(\Phi(fg))$. For any $K > M \in \mathbb{N}$,

$$\sum_{m=-M}^M (fg\varphi_m)(B)x = \sum_{m=-M}^M \sum_{n=-K}^K (fg\varphi_m\varphi_n)(B)x = \sum_{m=-M}^M (f\varphi_m)(B) \sum_{n=-K}^K (g\varphi_n)(B)x.$$

The assumption $x \in D(\Phi(g))$ implies that the inner sum converges for $K \to \infty$. Thus,

$$\sum_{m=-M}^{M} (fg\varphi_m)(B)x = \sum_{m=-M}^{M} (f\varphi_m)(B)\Phi(g)x,$$

which converges for $M \to \infty$ by the assumption $\Phi(g)x \in D(\Phi(f))$. This shows that $x \in D(\Phi(fg))$ and also $\Phi(fg)x = \Phi(f)\Phi(g)x$.

Assume now in addition that $\Phi(g)$ is a bounded operator. Let $x \in D(\Phi(fg))$. We have to show that $\Phi(g)x \in D(\Phi(f))$.

$$\begin{aligned}\sum_{m=-M}^{M} (f\varphi_m)(B)\Phi(g)x &= \sum_{m=-M}^{M} (f\varphi_m)(B) \sum_{n=-\infty}^{\infty} (g\varphi_n)(B)x \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-M}^{M} (f\varphi_m)(B)(g\varphi_n)(B)x \\ &= \sum_{n=-M-1}^{M+1} \sum_{m=-M}^{M} (f\varphi_m g\varphi_n)(B)x \\ &= \sum_{m=-M}^{M} (f\varphi_m g)(B)x.\end{aligned}$$

The last sum converges for $M \to \infty$ by the assumption $x \in D(\Phi(fg))$, and so the first sum also converges, which means by (1) that $\Phi(g)x \in D(\Phi(f))$. Further, the calculation also shows $\Phi(f)\Phi(g)x = \Phi(fg)x$.

(6) We have $D_B \subset \{g(B)x \, g \in C_c^\infty(\mathbb{R}),\, x \in X\} \subset D(\Phi(f))$. The first inclusion is trivial and the second follows from (5). Then (6) follows immediately. □

Henceforth we write $f(B)$ instead of $\Phi(f)$, since possible ambiguity with the other calculi has been ruled out.

Note that the Hörmander class $\mathcal{W}_p^\alpha$ is contained in $W_{p,\mathrm{loc}}^\alpha$. Thus the $W_{p,\mathrm{loc}}^\alpha$ calculus in proposition 4.25 enables us to define the $\mathcal{W}_p^\alpha$ calculus, whose boundedness is a main object of investigation in this chapter.

Definition 4.26 *Let $p \in (1, \infty)$, $\alpha > \frac{1}{p}$, let B be a 0-strip type operator and let A be a 0-sectorial operator.*

(1) *We say that B has a (bounded) $\mathcal{W}_p^\alpha$ calculus if there exists a constant $C > 0$ such that*

$$\|f(B)\| \leqslant C\|f\|_{\mathcal{W}_p^\alpha} \quad (f \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap \mathcal{W}_p^\alpha). \tag{4.36}$$

(2) *We say that A has a (bounded) $\mathcal{H}_p^\alpha$ calculus if there exists a constant $C > 0$ such that*

$$\|f(A)\| \leqslant C\|f\|_{\mathcal{H}_p^\alpha} \quad (f \in \bigcap_{\omega>0} H^\infty(\Sigma_\omega) \cap \mathcal{H}_p^\alpha). \tag{4.37}$$

Remark 4.27 *Let $p \in (1, \infty)$ and $\alpha > \frac{1}{p}$.*

(1) *Let B be a 0-strip-type operator having a $\mathcal{W}_p^\alpha$ calculus in the sense of definition 4.26. Then B has a W_p^α calculus and a $\mathcal{B}^\beta$ calculus for any $\beta > \alpha$. Thus, we can apply proposition 4.25 and consider the unbounded $W_{p,\mathrm{loc}}^\alpha$ calculus of B, and in particular $f(B)$ is defined for $f \in \mathcal{W}_p^\alpha \subset W_{p,\mathrm{loc}}^\alpha$.*

Then condition 4.36 extends automatically to all $f \in \mathcal{W}_p^\alpha$.

(2) *Let A be a 0-sectorial operator having a $\mathcal{H}_p^\alpha$ calculus in the sense of definition 4.26. Then $B = \log(A)$ has a bounded $\mathcal{W}_p^\alpha$ calculus. For $f : \mathbb{R}_+ \to \mathbb{C}$ such that $f \circ \exp \in W_{p,\mathrm{loc}}^\alpha$, we put $f(A) = (f \circ \exp)(B)$, thus extending 2.4.*

Then by (1), 4.37 extends to all $f \in \mathcal{H}_p^\alpha$.

Proof of (1). According to proposition 4.9, B has a W_p^α calculus and a $\mathcal{B}^\beta$ calculus for any $\beta > \alpha$.

We show the claimed estimate first for $f \in \mathcal{W}_p^\alpha$ with compact support. In this case, $f \in W_p^\alpha$, and thus, according to lemma 4.15, f can be approximated by a sequence $(f_k)_k$ in $\bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega)$. This approximation holds first in W_p^α, and by the embedding $W_p^\alpha \hookrightarrow \mathcal{W}_p^\alpha$, also in $\mathcal{W}_p^\alpha$. Then

$$\|f(B)\| = \lim_k \|f_k(B)\| \lesssim \liminf_k \|f_k\|_{\mathcal{W}_p^\alpha} = \|f\|_{\mathcal{W}_p^\alpha},$$

where the first identity follows from the bounded W_p^α calculus, and the second estimate follows from 4.36. For a general $f \in \mathcal{W}_p^\alpha$, by definition 4.24, $f(B)x = (\sum_n f\varphi_n)(B)x$. If x belongs to the calculus core D_B, the latter sum can be taken finite, and thus, $\sum_n f\varphi_n$ has compact support. It follows $\|f(B)x\| \lesssim \|\sum_n f\varphi_n\|_{\mathcal{W}_p^\alpha}\|x\| \lesssim \|f\|_{\mathcal{W}_p^\alpha}\|x\|$. □

4.2.7 Duality

Throughout this subsection, let A be a 0-sectorial operator and B a 0-strip-type operator on some Banach space X which are linked by the equation $B = \log(A)$. In the subsequent sections, we will need the functional calculus of the dual operators of A and B. Since A' and B' need not to be densely defined, one has to restrict them to a subspace $X^\#$ of X'. This subspace is still large enough to norm X.

Definition 4.28 *[81, def 15.3] Let $X^\# = \overline{D(A')} \cap \overline{R(A')}$, which is a closed subspace of X'. Define the moon dual operator $A^\#$ on $X^\#$ by $A^\# x' = A'x'$, $D(A^\#) = \{x' \in D(A') \cap \overline{R(A')} : A'x' \in \overline{D(A')}\}$.*

Proposition 4.29

(1) *$X^\#$ norms X, i.e. $\|x\| \cong \sup\{|\langle x, x'\rangle| : x' \in X^\#, \|x'\| \leqslant 1\}$.*

(2) *$A^\#$ is a 0-sectorial operator.*

(3) *$f(A^\#) \subset f(A)'|_{X^\#}$ for $f \in \bigcup_{\omega>0} \mathrm{Hol}(\Sigma_\omega)$, and equality holds if $f(A) \in B(X)$.*

In particular, by proposition 4.14, we can define $B^\# = \log(A^\#)$, which is a 0-strip-type operator. We have $f(B^\#) \subset f(B)'|_{X^\#}$ for $f \in \bigcup_{\omega>0} \mathrm{Hol}(\mathrm{Str}_\omega)$.

Proof. (1) is shown in [43, Satz 6.2.3], see also [81, prop 15.4].
(2) follows from [43, Satz 6.2.4], see also [81, prop 15.2].
(3) Let $f \in \bigcup_{\omega>0} \mathrm{Hol}(\Sigma_\omega)$. By [52, prop 2.6.5], $f(A^\#)$ is the restriction of $f(A')$ to $\{x' \in X^\# : f(A)'x' \in X^\#\}$, where the definition of $f(A')$ is covered by [52, sect 2.3.4], since $\overline{R(A)} = X$ and hence A' is injective. Then by [52, prop 2.6.3], $f(A^\#) \subset f(A')|_{X^\#} = f(A)'|_{X^\#}$. If $f(A) \in B(X)$, then $f(A)'|_{X^\#} \in B(X^\#, X')$ and equality must hold, since $f(A^\#)$ is closed and densely defined. □

The functional calculi for $B^\#$ are covered by the next proposition.

Proposition 4.30 *Let $p \in (1, \infty), \alpha > \frac{1}{p}$ and $\beta > 0$.*

(1) *Assume that B has a bounded $\mathcal{B}^\beta$ calculus. Then $B^\#$ has a bounded $\mathcal{B}^\beta$ calculus and $f(B^\#) = f(B)'|_{X^\#}$ for $f \in \mathcal{B}^\beta$.*

(2) *Assume that $\alpha > \frac{1}{p}$ and that B has a bounded W^α_p calculus. Then $B^\#$ has a W^α_p calculus such that $f(B^\#) = f(B)'|_{X^\#}$ for all $f \in W^\alpha_p$.*

(3) *Assume that B has a $\mathcal{B}^\beta$ calculus, or a $\mathcal{B}^\beta$ calculus and a W^α_p calculus. Then by (1) and (2), also $B^\#$ has such a calculus, and for the (unbounded) $\mathcal{B}^\beta_{\mathrm{loc}}$ ($W^\alpha_{p,\mathrm{loc}}$) calculi, we have*

$$f(B^\#) \subset f(B')|_{X^\#}.$$

Proof. (1) For $\omega > 0$ and $f \in H^\infty(\mathrm{Str}_\omega)$, we have by the preceding proposition $f(B^\#) = f(B)'|_{X^\#}$. Thus, $\|f(B^\#)\| \leqslant \|f(B)\| \lesssim \|f\|_{\mathcal{B}^\beta}$ and $B^\#$ has a $\mathcal{B}^\beta$ calculus. Further, for $f \in \mathcal{B}^\beta$ and $f_n \in H^\infty(\mathrm{Str}_\omega)$ an approximating sequence,

$$f(B^\#) = \lim_n f_n(B^\#) = \lim_n (f_n(B)'|_{X^\#}) = (\lim_n f_n(B)')|_{X^\#} = f(B)'|_{X^\#}.$$

(2) Copy of the proof of (1).

(3) Let $f \in \mathcal{B}^\beta_{\mathrm{loc}}$ ($f \in W^\alpha_{p,\mathrm{loc}}$). Put

$$D^\# = D_{B^\#} = \{x' \in X^\# : \exists N \in \mathbb{N} : \varphi_n(B^\#)x' = 0 \quad (|n| \geqslant N)\}.$$

Note that $D^\# \subset D(f(B)')$, since for $x' \in D^\#$ and any $x \in D$

$$|\langle x', f(B)x\rangle| = |\langle x', \sum_n (f\varphi_n)(B)x\rangle| = |\langle \sum_n (f\varphi_n)(B)'x', x\rangle| \leqslant \| \sum_n (f\varphi_n)(B)'x'\| \, \|x\|.$$

This also shows that for $x' \in D^\#$,

$$f(B^\#)x' = \sum_n (f\varphi_n)(B^\#)x' = \sum_n (f\varphi_n)(B)'x' = f(B)'x'.$$

As $D^\#$ is by definition a core for $f(B^\#)$, the claim follows. □

4.3 Averaged and matricial R-boundedness and the W_2^α calculus

Recall the notions of the Rademacher and Gaussian spaces $\mathrm{Rad}(X)$ and $\mathrm{Gauss}(X)$, R-boundedness as well as the generalized square functions $\gamma(\Omega, X)$ from chapter 2.

In subsection 4.3.1, we will develop these notions more deeply. We consider R-bounded families of operators which are averaged by integrating them against elements of some function space E. This averaged R-boundedness, or $R[E]$-boundedness, is implicitly contained already in some previous work [81, sec 2], [61], see proposition 4.32 below.

In subsection 4.3.2, we show how the averaged R-boundedness compares to typical generalized square function estimates which appear e.g. in characterizations of the bounded H^∞ calculus as in [24, sec 6], [72, thms 2.1, 2.2, 6.2, 7.2]. Moreover, we reveal how these notions are related to the matricial R-boundedness which we met in section 3.4 of chapter 3, and extend a result of Le Merdy [88, prop 3.3], and Haak and Kunstmann [51, cor 3.19].

Averaged R-boundedness will play a key role in the characterization of the functional calculus. We focus in subsection 4.3.3 on the W_2^α calculus for a 0-strip-type operator B. To be able to pass from the W_2^α calculus to the $\mathcal{W}_2^\alpha$ calculus, we introduce two new boundedness notions of a functional calculus related to R-boundedness. Subsequently, we characterize the W_2^α calculus in terms of operator families associated with B, using thereby the results of subsections 4.3.1 and 4.3.2.

The main results are theorem 4.46 and proposition 4.50.

4.3.1 Averaged R-boundedness

Let (Ω, μ) be a σ-finite measure space. Throughout the section, we consider spaces E which are subspaces of the space $\mathcal{L}$ of equivalence classes of measurable functions on (Ω, μ). Here, equivalence classes refer to identity modulo μ-null sets. We require that the dual E' of E can be realized as the completion of

$$E_0' = \{f \in \mathcal{L} : \exists C > 0 : |\langle f, g \rangle| = |\int_\Omega f(t)g(t)d\mu(t)| \leqslant C\|g\|_E\} \tag{4.38}$$

with respect to the norm $\|f\| = \sup_{\|g\|_E \leqslant 1} |\langle f, g\rangle|$. This is clearly the case in the following examples:

$$\begin{aligned} &E = L^p(\Omega, w d\mu) \text{ for } 1 \leqslant p \leqslant \infty \text{ and a weight } w, \\ &E = W_p^\alpha = W_p^\alpha(\mathbb{R}) \text{ for } 1 < p < \infty \text{ and } \alpha > \frac{1}{p}, \\ &E = \{f : \mathbb{R}_+ \to \mathbb{C} : f_e \in W_p^\alpha\}. \end{aligned} \tag{4.39}$$

The definition of averaged R-boundedness now reads as follows. For applications in subsection 4.3.3, it is reasonable also to include unbounded operators.

Definition 4.31 *Let (Ω, μ) be a σ-finite measure space. Let E be a function space on (Ω, μ) as in 4.39. Let $(N(t) : t \in \Omega)$ be a family of closed operators on a Banach space X such that*

(1) *There exists a dense subspace $D_N \subset X$ which is contained in the domain of $N(t)$ for any $t \in \Omega$.*

(2) *For any $x \in D_N$, the mapping $\Omega \to X$, $t \mapsto N(t)x$ is measurable.*

(3) *For any $x \in D_N$, $x' \in X'$ and $f \in E$, $t \mapsto f(t)\langle N(t)x, x'\rangle$ belongs to $L^1(\Omega)$.*

Then $(N(t) : t \in \Omega)$ is called R-bounded on the E-average or $R[E]$-bounded, if for any $f \in E$, there exists $N_f \in B(X)$ such that

$$\langle N_f x, x'\rangle = \int_\Omega f(t)\langle N(t)x, x'\rangle d\mu(t) \quad (x \in D_N,\, x' \in X') \tag{4.40}$$

and further

$$R[E](N(t) : t \in \Omega) = R(\{N_f : \|f\|_E \leqslant 1\}) < \infty.$$

In the following proposition, we illustrate the $R[E]$-boundedness for several spaces E.

Proposition 4.32 *Let (Ω, μ) be a σ-finite measure space and let $(N(t) : t \in \Omega)$ be a family of closed operators on X satisfying (1) and (2) of definition 4.31.*

($E = L^1$) *If $N(t)$ is a bounded operator for all $t \in \Omega$, and $(N(t) : t \in \Omega)$ is R-bounded, then it is also $R[L^1(\Omega)]$-bounded, and*

$$R[L^1(\Omega)](N(t) : t \in \Omega) \leqslant 2R(\{N(t) : t \in \Omega\}).$$

Conversely, assume in addition that Ω is a metric space, μ is a σ-finite strictly positive Borel measure and $t \mapsto N(t)$ is strongly continuous. If $(N(t) : t \in \Omega)$ is $R[L^1(\Omega)]$-bounded, then it is also R-bounded.

($E = L^\infty$) *Assume that there exists $C > 0$ such that*

$$\int_\Omega \|N(t)x\| d\mu(t) \leqslant C\|x\| \quad (x \in D_N).$$

Then $(N(t) : t \in \Omega)$ is $R[L^\infty(\Omega)]$-bounded with constant at most $2C$.

($E = L^2$) *Assume that X has property (α). If $N(t)x$ belongs to $\gamma(\Omega, X)$ for all $x \in D_N$ and if there exists $C > 0$ such that*

$$\|N(t)x\|_{\gamma(\Omega,X)} \leqslant C\|x\| \quad (x \in D_N),$$

then $(N(t) : t \in \Omega)$ is $R[L^2(\Omega)]$-bounded and there exists a constant $C_0 = C_0(X)$ such that

$$R[L^2(\Omega)](N(t) : t \in \Omega) \leqslant C_0 C.$$

($E = L^{r'}$) *Assume that X has type $p \in [1, 2]$ and cotype $q \in [2, \infty]$. Let $1 \leqslant r, r' < \infty$ with $\frac{1}{r} = 1 - \frac{1}{r'} > \frac{1}{p} - \frac{1}{q}$.*

Assume that $N(t) \in B(X)$ for all $t \in \Omega$, that $t \mapsto N(t)$ is strongly measurable, and that

$$\|N(t)\|_{B(X)} \in B(L^r(\Omega)).$$

Then $(N(t) : t \in \Omega)$ is $R[L^{r'}(\Omega)]$-bounded and there exists a constant $C_0 = C_0(r, p, q, X)$ such that

$$R[L^{r'}(\Omega)](N(t) : t \in \Omega) \leqslant C_0 C.$$

(general E) If E is a space as in 4.39 and $R[E](N(t) : t \in \Omega) = C < \infty$, then

$$\|\langle N(\cdot)x, x'\rangle\|_{E'} \leqslant C\|x\| \, \|x'\| \quad (x \in D_N, \, x' \in X'). \tag{4.41}$$

In particular, if $1 \leqslant p, p' \leqslant \infty$ are conjugated exponents and

$$R[L^{p'}(\Omega)](N(t) : t \in \Omega) = C < \infty,$$

then

$$\left(\int_\Omega |\langle N(t)x, x'\rangle|^p d\mu(t) \right)^{1/p} \leqslant C\|x\| \, \|x'\| \quad (x \in D_N, \, x' \in X').$$

If X is a Hilbert space, then also the converse holds: Condition 4.41 implies that $(N(t) : t \in \Omega)$ is $R[E]$-bounded.

Proof. ($E = L^1$) Assume that $(N(t) : t \in \Omega)$ is R-bounded. Then it follows from proposition 2.6 (4) that $R[L^1(\Omega)](N(t) : t \in \Omega) \leqslant 2R(\{N(t) : t \in \Omega\})$.

Let us show the converse under the mentioned additional hypotheses. Suppose that $R(\{N(t) : t \in \Omega\}) = \infty$. We will deduce that also $R[L^1(\Omega)](N(t) : t \in \Omega) = \infty$. Choose for a given $N \in \mathbb{N}$ some $x_1, \ldots, x_n \in X \backslash \{0\}$ and $t_1, \ldots, x_n \in \Omega$ such that

$$\Big\| \sum_k \varepsilon_k \otimes N(t_k)x_k \Big\|_{\mathrm{Rad}(X)} > N \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)}.$$

It suffices to show that

$$\Big\| \sum_k \varepsilon_k \otimes \int_\Omega f_k(t) N(t) x_k d\mu(t) \Big\|_{\mathrm{Rad}(X)} > N \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)} \tag{4.42}$$

for appropriate $f_1, \ldots, f_n$. It is easy to see that by the strong continuity of N, 4.42 holds with $f_k = \frac{1}{\mu(B(t_k, \varepsilon))} \chi_{B(t_k,\varepsilon)}$ for ε small enough. Here the fact that μ is strictly positive and σ-finite guarantees that $\mu(B(t_k, \varepsilon)) \in (0, \infty)$ for small ε.

($E = L^\infty$) By the proof of proposition 2.6 (6),

$$\Big\| \sum_{k=1}^n \varepsilon_k \otimes N_{f_k} x_k \Big\|_{\mathrm{Rad}(X)} \leqslant 2C \Big\| \sum_{k=1}^n \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)}$$

for any finite family $N_{f_1}, \ldots, N_{f_n}$ from 4.40 such that $\|f_k\|_1 \leqslant 1$, and any finite family $x_1, \ldots, x_n \in D_N$. Since D_N is a dense subspace of X, the set of $\sum_{k=1}^n \varepsilon_k \otimes x_k$ with such $x_k' s$ is dense in $\mathrm{Rad}_n(X)$. This clearly implies that $\{N_f : \|f\|_1 \leqslant 1\}$ is R-bounded.

($E = L^2$) For $x \in D_N$, set $\varphi(x) = N(\cdot)x \in \gamma(\Omega, X)$. By assumption, φ extends to a bounded operator $X \to \gamma(\Omega, X)$. Then the assertion follows at once from [51, cor 3.19].

($E = L^{r'}$) This is a result of Hytönen and Veraar, see [61, prop 4.1, rem 4.2].

(general E) We have

$$\begin{aligned} &R[E](N(t):\, t \in \Omega) \\ &\geqslant \sup\{\|N_f\|_{B(X)}:\, \|f\|_E \leqslant 1\} \\ &= \sup\left\{ \left| \int_\Omega f(t)\langle N(t)x, x'\rangle d\mu(t) \right| :\, \|f\|_E \leqslant 1,\, x \in D_N, \|x\| \leqslant 1,\, x' \in X', \|x'\| \leqslant 1 \right\} \\ &= \sup\{\|\langle N(\cdot)x, x'\rangle\|_{E'}:\, x \in D_N, \|x\| \leqslant 1,\, x' \in X', \|x'\| \leqslant 1\}. \end{aligned} \tag{4.43}$$

If X is a Hilbert space, then bounded subsets of $B(X)$ are R-bounded, and thus, "$\geqslant$" in 4.43 is in fact "$=$". □

An $R[E]$-bounded family yields a new averaged R-bounded family under a linear transformation in the function space variable. This simple observation, resumed in the following proposition, will be extremely useful.

Proposition 4.33 *For $i = 1, 2$, let (Ω_i, μ_i) be a σ-finite measure space and E_i a function space on Ω_i as in 4.39, and $K \in B(E_1', E_2')$ such that its adjoint K' maps E_2 to E_1.*

Let further $(N(t) : t \in \Omega_1)$ be an $R[E_1]$-bounded family of closed operators with common dense subset D_N. Assume that there exists a family $(M(t) : t \in \Omega_2)$ of closed operators with the same common dense subset $D_M = D_N$ such that $t \mapsto M(t)x$ is measurable for all $x \in D_N$ and

$$\langle M(\cdot)x, x'\rangle = K(\langle N(\cdot)x, x'\rangle) \quad (x \in D_N, x' \in X').$$

Then $(M(t) :\, t \in \Omega_2)$ is $R[E_2]$-bounded and

$$R[E_2](M(t) :\, t \in \Omega_2) \leqslant \|K\| R[E_1](N(t) :\, t \in \Omega_1).$$

Proof. Let $x \in D_N$ and $x' \in X'$. By 4.41 in proposition 4.32, we have $\langle N(\cdot)x, x'\rangle \in E_1'$, and thus, $\langle M(\cdot)x, x'\rangle \in E_2'$. For any $f \in E_2$,

$$\int_{\Omega_2} \langle M(t)x, x'\rangle f(t) d\mu_2(t) = \int_{\Omega_1} \langle N(t)x, x'\rangle (K'f)(t) d\mu_1(t) = \langle N_{K'f}x, x'\rangle.$$

By assumption, the operator $N_{K'f}$ belongs to $B(X)$, and therefore also M_f belongs to $B(X)$. Furthermore,

$$\begin{aligned} R[E_2](M(t) :\, t \in \Omega_2) &= R(\{M_f :\, \|f\|_{E_2} \leqslant 1\}) \\ &= R(\{N_{K'f} :\, \|f\|_{E_2} \leqslant 1\}) \\ &\leqslant \|K'\| R(\{N_{K'f} :\, \|K'f\|_{E_1} \leqslant 1\}) \\ &\leqslant \|K\| R(\{N_g :\, \|g\|_{E_1} \leqslant 1\}) \\ &= \|K\| R[E_1](N(t) :\, t \in \Omega_1). \end{aligned}$$

□

In the following lemma, we collect some further simple manipulations of $R[E]$-boundedness.

Lemma 4.34 *Let (Ω, μ) be a σ-finite measure space, let E be as in 4.39 and let $(N(t) :\, t \in \Omega)$ satisfy (1) and (2) of definition 4.31.*

(1) *Let* $f \in L^\infty(\Omega)$ *and* $(N(t) : t \in \Omega)$ *be* $R[L^p(\Omega)]$*-bounded for some* $1 \leqslant p \leqslant \infty$. *Then*

$$R[L^p(\Omega)](f(t)N(t) : t \in \Omega) \leqslant \|f\|_\infty R[L^p(\Omega)](N(t) : t \in \Omega).$$

In particular, $R[L^p(\Omega_1)](N(t) : t \in \Omega_1) \leqslant R[L^p(\Omega)](N(t) : t \in \Omega)$ *for any measurable subset* $\Omega_1 \subset \Omega$.

(2) *For* $n \in \mathbb{N}$, *let* $\varphi_n : \Omega \to \mathbb{R}_+$ *with* $\sum_{n=1}^\infty \varphi_n(t) = 1$ *for all* $t \in \Omega$. *Then*

$$R[E](N(t) : t \in \Omega) \leqslant \sum_{n=1}^\infty R[E](\varphi_n(t)N(t) : t \in \Omega).$$

(3) *Let* $w : \Omega \to (0, \infty)$ *be measurable. Then for* $1 \leqslant p \leqslant \infty$ *and* p' *the conjugate exponent,*

$$R[L^p(\Omega, w(t)d\mu(t))](N(t) : t \in \Omega) = R[L^p(\Omega, d\mu)](w(t)^{\frac{1}{p'}} N(t) : t \in \Omega).$$

(4) *For* $n \in \mathbb{N}$, *let* $\phi_n : \Omega \to [0, 1]$ *with* $\phi_n(t) \to 1$ *monotonically as* $n \to \infty$ *for all* $t \in \Omega$. *Then for* $1 \leqslant p \leqslant \infty$,

$$R[L^p(\Omega)](N(t) : t \in \Omega) = \sup_n R[L^p(\Omega)](\phi_n(t)N(t) : t \in \Omega).$$

Proof. Parts (1)-(3) can easily be checked using definition 4.31. For part (4), we show first that if one of the sides of the equation is finite, then $N_f = \lim_n(\phi_n N)_f$ in the norm topology. If the left hand side is finite, then by proposition 4.32 (5), $\|\langle N(t)x, x'\rangle\|_{p'} \lesssim \|x\| \, \|x'\|$ for $x \in D_N$ and $x' \in X'$. If the right hand side is finite, this is also the case: By dominated convergence,

$$\begin{aligned}\|\langle N(t)x, x'\rangle\|_{p'} &= \sup_{\|f\|_p \leqslant 1} \int_\Omega |f(t)\langle N(t)x, x'\rangle| dt = \sup_{\|f\|_p \leqslant 1, n \in \mathbb{N}} \int_\Omega |f(t)\phi_n(t)\langle N(t)x, x'\rangle| dt \\ &\leqslant \sup_n R[L^p(\Omega)](\phi_n(t)N(t) : t \in \Omega)\|x\| \, \|x'\|.\end{aligned}$$

Then for $f \in L^p(\Omega)$,

$$\begin{aligned}\|N_f - (\phi_n N)_f\| &= \sup_{\substack{x \in D_N, \\ \|x\|, \|x'\| \leqslant 1}} \int_\Omega |f(t)(1 - \phi_n(t))| |\langle N(t)x, x'\rangle| d\mu(t) \\ &\leqslant \sup_{x, x'} \|f(t)(1 - \phi_n(t))\|_p \|\langle N(t)x, x'\rangle\|_{p'} \\ &\lesssim \|f(t)(1 - \phi_n(t))\|_p \to 0\end{aligned}$$

as $n \to \infty$, by dominated convergence. Now the claim follows from 2.6, since $\phi_n(t)$ is monotone, and therefore, the sets $\{(\phi_n N)_f : \|f\|_p \leqslant 1\}$ are directed by inclusion. □

4.3.2 Matricial R-boundedness and square functions

Let $E = L^2(\Omega)$ be a Hilbert space. Consider a family $(N(t) : t \in \Omega)$ of closed operators on X satisfying (1) and (2) of definition 4.31 such that for some $C > 0$ and some dense subset $D \subset X$

$$\|\langle N(\cdot)x, x'\rangle\|_2 = \left(\int_\Omega |\langle N(t)x, x'\rangle|^2 d\mu(t)\right)^{\frac{1}{2}} \leqslant C\|x\| \, \|x'\| \quad (x \in D, \, x' \in X'). \tag{4.44}$$

To $(N(t) : t \in \Omega)$, we associate the operator

$$u : L^2(\Omega) \longrightarrow B(X),\ f \mapsto \int_\Omega f(t)N(t)d\mu(t) = N_f, \tag{4.45}$$

where the integral has to be understood in the strong sense. Clearly, condition 4.44 implies that u is bounded. If X is a Banach space with property (α), then by proposition 4.32 ($E = L^2$), the following chain of implications holds.

$$\|N(t)x\|_{\gamma(\Omega,X)} \lesssim \|x\| \quad (x \in D) \Longrightarrow (N(t) : t \in \Omega) \text{ is } R[L^2(\Omega)]\text{-bounded} \Longrightarrow u \text{ is bounded.} \tag{4.46}$$

Further, if X is a Hilbert space, then by proposition 4.32 (general E), the last two conditions in 4.46 are equivalent.

We now have a closer look at 4.46.

Recall definition 3.7 of R-bounded maps. Namely, we say that u is R-bounded if

$$R(\{u(f) : \|f\|_{L^2(\Omega)} \leqslant 1\}) < \infty.$$

Further recall definition 3.22 of matricially R-bounded maps. There, a mapping $u : C(K) \to B(X)$ was considered, and both $Z = C(K)$ and $Z = B(X)$ were equipped with certain matricial norms, i.e. for any $n \in \mathbb{N}$, $M_n(Z) = \{[z_{ij}]_{i,j=1,\ldots,n} : z_{ij} \in Z\}$ is normed in a specific way.

Now we replace $C(K)$ by the Hilbert space $H = L^2(\Omega)$. We equip H with the so-called row norm (cf. [113])

$$\left\|[z_{ij}]\right\|_{M_n(H)} = \left\|\left[\sum_k \langle z_{ik}, z_{jk}\rangle\right]_{ij}\right\|_{M_n}^{\frac{1}{2}}. \tag{4.47}$$

This norm is induced by first embedding $H \hookrightarrow B(H)$, $h \mapsto e \otimes \overline{h}$, where e is an arbitrary element of H of unit length, and then considering the natural C^*-algebra norm of $M_n(B(H)) \cong B(\ell_n^2(H))$. If H is finite dimensional, say $H = \ell_N^2$, then the embedding corresponds to the identification of ℓ_N^2 with a row of M_N, which explains the name row norm. Following the usual notation, we also write $M_n(H) = M_n(H_r)$ for the space normed by 4.47.

Definition 4.35 *Recall that $B(X)$ is equipped with the matricial norm from 3.10, i.e. $M_n(B(X)) = B(\mathrm{Rad}_n(X))$ isometrically. Consistently with 3.11, we call $u : L^2(\Omega)_r \to B(X)$ matricially R-bounded, if there exists $C > 0$ such that for any $n \in \mathbb{N}$ and $[z_{ij}] \in M_n(L^2(\Omega))$, we have*

$$\|[u(z_{ij})]\|_{B(\mathrm{Rad}_n(X))} \leqslant C\|[z_{ij}]\|_{M_n(L^2(\Omega)_r)}. \tag{4.48}$$

The following theorem is a strengthening of [88, prop 3.3] and [51, cor 3.19].

Theorem 4.36 *Consider the above family $(N(t) : t \in \Omega)$ of operators on a Banach space X satisfying 4.44, and the associated mapping $u : L^2(\Omega) \to B(X)$ from 4.45.*

(1) *$u : L^2(\Omega) \to B(X)$ is R-bounded if and only if $(N(t) : t \in \Omega)$ is $R[L^2(\Omega)]$-bounded.*

(2) *Assume that X has property (α). Then*

$$u : L^2(\Omega)_r \to B(X) \text{ is matricially } R\text{-bounded} \tag{4.49}$$

if and only if there exists $C > 0$ such that

$$\|N(t)x\|_{\gamma(\Omega,X)} \leqslant C\|x\| \quad (x \in D). \tag{4.50}$$

Proof. (1) The map u is R-bounded iff $\{u(f) : \|f\|_2 \leqslant 1\}$ is R-bounded. By 4.45, the latter set equals $\{N_f : \|f\|_2 \leqslant 1\}$, which by definition is R-bounded iff $(N(t) : t \in \Omega)$ is $R[L^2(\Omega)]$-bounded.

(2) Let $(e_k)_{k\geqslant 1}$ be an orthonormal basis of $L^2(\Omega)$. Set

$$N_k = \int_\Omega e_k(t)N(t)d\mu(t) \in B(X).$$

For $f \in L^2(\Omega)$, set $f^{(k)} = \langle f, e_k\rangle$. Then $u(f) = \sum_{k=1}^\infty f^{(k)} N_k$. Let $n \in \mathbb{N}$ and $[f_{ij}] \in M_n(L^2(\Omega))$. Then

$$\left\|[u(f_{ij})]_{ij}\right\|_{B(\mathrm{Rad}_n(X))} = \left\|\left[\sum_{k=1}^\infty f_{ij}^{(k)} N_k\right]_{ij}\right\|_{B(\mathrm{Rad}_n(X))} = \lim_m \left\|\left[\sum_{k=1}^m f_{ij}^{(k)} N_k\right]_{ij}\right\|_{B(\mathrm{Rad}_n(X))}$$

and

$$\left\|[f_{ij}]_{ij}\right\|_{M_n(L^2(\Omega)_r)} = \lim_m \left\|\left[\sum_{k=1}^m f_{ij}^{(k)} e_k\right]_{ij}\right\|_{M_n(L^2(\Omega)_r)} = \lim_m \left\|\left[\left(f_{ij}^{(k)}\right)_k\right]_{ij}\right\|_{M_n(\ell^2_{m,r})},$$

where $M_n(\ell^2_{m,r})$ is the space $M_n(\ell^2_m)$ again equipped with the row norm 4.47. Consequently, condition 4.49 reads

$$\sup_{n,m\in\mathbb{N}} \left\|u_{n,m} : M_n(\ell^2_{m,r}) \to B(\mathrm{Rad}_n(X)),\ \left[\left(f_{ij}^{(k)}\right)_k\right]_{ij} \mapsto \left[\sum_{k=1}^m f_{ij}^{(k)} N_k\right]_{ij}\right\| < \infty. \tag{4.51}$$

On the other hand,

$$\|N(t)x\|_{\gamma(\Omega,X)} = \sup_n \left\|\sum_{k=1}^n \gamma_k \otimes N_k x\right\|_{\mathrm{Gauss}_n(X)} \cong \sup_n \left\|\sum_{k=1}^n \varepsilon_k \otimes N_k x\right\|_{\mathrm{Rad}_n(X)},$$

where the last equivalence follows from the fact that X has property (α) (and thus finite cotype), see proposition 2.6 (7). Hence we see that 4.50 is equivalent to

$$\sup_{n\in\mathbb{N}} \left\|\sum_{k=1}^n \varepsilon_k \otimes N_k x\right\|_{\mathrm{Rad}_n(X)} \leqslant C\|x\| \quad (x \in X). \tag{4.52}$$

We thus have to show 4.51 $\iff$ 4.52.

4.52 $\Longrightarrow$ 4.51:
Define the operators

$$V : X \to \mathrm{Rad}_m(X),\ x \mapsto \sum_{k=1}^m \varepsilon_k \otimes N_k x, \quad W : \mathrm{Rad}_m(X) \to X,\ \sum_{k=1}^m \varepsilon_k \otimes x_k \mapsto x_1.$$

By assumption 4.52, the operator V is bounded with a constant C independent of m. Recall the mapping $\sigma_{m,X} : M_m \to B(\mathrm{Rad}_m(X))$, $\sigma_{m,X}([a_{ij}]) = [a_{ij}\,\mathrm{Id}_X]$ from chapter 3, section 3.4. Denote

$$i_m : \ell^2_m \hookrightarrow M_m,\ (f^{(1)}, \ldots, f^{(m)}) \mapsto \begin{bmatrix} f^{(1)} & \ldots & f^{(m)} \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix}$$

and $\pi_m = \sigma_{m,X} \circ i_m : \ell^2_m \to B(\mathrm{Rad}_m(X))$. By definition of $M_n(\ell^2_{m,r})$, $i_m : \ell^2_{m,r} \hookrightarrow M_m$ is a completely isometric embedding, i.e. for any $n \in \mathbb{N}$,

$$i_m \otimes \mathrm{Id}_{M_n} : M_n(\ell^2_{m,r}) \hookrightarrow M_n(M_m),\ [f_{ij}]_{ij} \mapsto [i_m(f_{ij})]_{ij}$$

is isometric. For $f = [f_{ij}]_{ij} \in M_n(\ell^2_{m,r})$, we have

$$\begin{aligned} u_{n,m}(f) &= \begin{bmatrix} W & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & W \end{bmatrix} \begin{bmatrix} \pi_m(f_{11}) & \ldots & \pi_m(f_{1n}) \\ \vdots & \ddots & \vdots \\ \pi_m(f_{n1}) & \ldots & \pi_m(f_{nn}) \end{bmatrix} \begin{bmatrix} V & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & V \end{bmatrix} \\ &=: W_n \Pi V_n. \end{aligned}$$

We estimate $\|V_n : \mathrm{Rad}_n(X) \to \mathrm{Rad}_n(\mathrm{Rad}_m(X))\|$, $\|\Pi : \mathrm{Rad}_n(\mathrm{Rad}_m(X)) \to \mathrm{Rad}_n(\mathrm{Rad}_m(X))\|$ and $\|W_n : \mathrm{Rad}_n(\mathrm{Rad}_m(X)) \to \mathrm{Rad}_n(X)\|$.

For $\sum_k \varepsilon_k \otimes x_k \in \mathrm{Rad}_n(X)$, by assumption 4.52

$$\begin{aligned} \Big\| V_n\big(\sum_k \varepsilon_k \otimes x_k\big) \Big\|_{\mathrm{Rad}_n(\mathrm{Rad}_m(X))} &= \Big\| \sum_{l,k} \varepsilon_l \otimes \varepsilon_k \otimes N_l x_k \Big\|_{\mathrm{Rad}_n(\mathrm{Rad}_m(X))} \\ &= \left(\int_{\Omega_0} \Big\| \sum_l \varepsilon_l \otimes N_l \Big(\sum_k \varepsilon_k(\lambda) x_k \Big) \Big\|^2_{\mathrm{Rad}_m(X)} d\lambda \right)^{\frac{1}{2}} \\ &\leqslant \left(\int_{\Omega_0} C^2 \Big\| \sum_k \varepsilon_k(\lambda) x_k \Big\|^2_X d\lambda \right)^{\frac{1}{2}} \\ &= C \Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}_n(X)}. \end{aligned}$$

Thus, $\|V_n\| \leqslant C$.

By lemma 3.24, the fact that X has property (α) implies that $\sigma_{m,X}$ is R-bounded with a constant C_0 independent of m. Therefore,

$$\|\Pi\|_{B(\mathrm{Rad}_n(\mathrm{Rad}_m(X)))} \leqslant C_0 \left\| \begin{bmatrix} i_m(f_{11}) & \dots & i_m(f_{1n}) \\ \vdots & \ddots & \vdots \\ i_m(f_{n1}) & \dots & i_m(f_{nn}) \end{bmatrix} \right\|_{M_n(M_m)}$$
$$= C_0 \left\| [f_{ij}]_{ij} \right\|_{M_n(\ell^2_{m,r})}.$$

Finally, for $\sum_{k,l} \varepsilon_k \otimes \varepsilon_l \otimes x_{kl} \in \mathrm{Rad}_n(\mathrm{Rad}_m(X))$,

$$\left\| W_n\Big(\sum_{k,l} \varepsilon_k \otimes \varepsilon_l \otimes x_{kl}\Big) \right\|_{\mathrm{Rad}_n(X)} = \left\| \sum_l \varepsilon_l \otimes x_{1l} \right\|_{\mathrm{Rad}_n(X)}$$
$$\leqslant \left\| \sum_{k,l} \varepsilon_k \otimes \varepsilon_l \otimes x_{kl} \right\|_{\mathrm{Rad}_n(\mathrm{Rad}_m(X))}.$$

This shows that $\|W_n\| \leqslant 1$.

Altogether, we have $\|u_{n,m}(f)\| \leqslant \|W_n\| \, \|\Pi\| \, \|V_n\| \leqslant C_0 C \|f\|$. This shows 4.51.

4.51 $\Longrightarrow$ 4.52:
Choose $n = m \in \mathbb{N}$ and $f = [f_{ij}]_{ij} \in M_n(\ell^2_{n,r})$ with $f_{ij} = \delta_{j1} e_i$, where $(e_i)_i$ is the standard basis of ℓ^2_n. By definition of the row norm 4.47,

$$\|f\|_{M_n(\ell^2_{n,r})} = \left\| \left[\sum_k \langle f_{ik}, f_{jk} \rangle \right]_{ij} \right\|_{M_n}^{\frac{1}{2}}$$
$$= \left\| \left[\sum_k \delta_{k1} \langle e_i, e_j \rangle \right]_{ij} \right\|_{M_n}^{\frac{1}{2}}$$
$$= \left\| [\delta_{ij}]_{ij} \right\|_{M_n}^{\frac{1}{2}}$$
$$= 1.$$

Then by assumption 4.51, there is $C < \infty$ such that

$$C \geqslant \|u_{n,n}\| \geqslant \|u_{n,n}(f)\|_{B(\mathrm{Rad}_n(X))} = \left\| \begin{bmatrix} N_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ N_n & 0 & \dots & 0 \end{bmatrix} \right\|_{B(\mathrm{Rad}_n(X))}$$
$$= \sup \left\{ \left\| \sum_k \varepsilon_k \otimes N_k x_1 \right\|_{\mathrm{Rad}_n(X)} : \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\mathrm{Rad}_n(X)} \leqslant 1 \right\}$$
$$= \sup \left\{ \left\| \sum_k \varepsilon_k \otimes N_k x \right\|_{\mathrm{Rad}_n(X)} : \|x\|_X \leqslant 1 \right\}.$$

Letting $n \to \infty$ shows that 4.52 holds. □

Remark 4.37 *If X is a Hilbert space in the above theorem, then the conditions in (1) are also equivalent to the weak estimate*

$$\|\langle N(\cdot)x, x'\rangle\|_{L^2(\Omega)} \leqslant C\|x\|\,\|x'\|,$$

whereas the conditions in (2) are equivalent to the strong estimate

$$\|N(t)x\|_{L^2(\Omega,X)} \leqslant C\|x\|.$$

Clearly, (2) implies (1), but the converse does not hold, as the following simple example shows.

Let $\Omega = \mathbb{N}$ and $X = \ell^2(\mathbb{N})$. Denote $(e_k)_k$ the canonical orthonormal basis of X. For any $n \in \Omega$, let $N(n) \in B(X)$ be defined by

$$N(n)((x_k)_k) = x_1 e_n.$$

Then for $x, y \in X$,

$$\|\langle N(n)x, y\rangle\|_{\ell^2(\mathbb{N})} = |x_1|\,\|y\| \leqslant \|x\|\,\|y\|,$$

but

$$\|N(n)x\|_{\ell^2(\mathbb{N},X)} = \left(\sum_{n\geqslant 1} \|N(n)x\|_X^2\right)^{\frac{1}{2}} = \left(\sum_{n=1}^{\infty} |x_1|^2\right)^{\frac{1}{2}} = \infty$$

as soon as $x_1 \neq 0$.

4.3.3 Application to operator families associated with 0-sectorial and 0-strip-type operators

Let A be a 0-sectorial operator with bounded imaginary powers and B be the 0-strip-type operator $B = \log(A)$. We consider the following distinguished families of operators associated with A and B.

- imaginary powers $\{A^{it} = e^{itB} : t \in \mathbb{R}\}$,
- resolvents $\{R(\lambda, A) : \lambda \in \mathbb{C}\backslash[0,\infty)\}$ and $\{R(\lambda, B) : \lambda \in \mathbb{C}\backslash\mathbb{R}\}$,
- semigroup operators $\{e^{-zA} : \operatorname{Re} z > 0\}$, and
- their boundary values $\{e^{isA} : s \in \mathbb{R}\}$.

Pick the first operator family and fix some $\alpha > \frac{1}{2}$. We have seen in proposition 4.22 and remark 4.23 that B has a W_2^α calculus if and only if there exists $C > 0$ such that

$$\|t \mapsto \langle\langle t\rangle^{-\alpha} e^{itB}x, x'\rangle\|_{L^2(\mathbb{R})} \leqslant C\|x\|\,\|x'\| \quad (x \in X,\ x' \in X'). \tag{4.53}$$

However, as we will see in sections 4.4 and 4.5, if we want to pass from the W_2^α calculus to the $\mathcal{W}_2^\alpha$ calculus, mere boundedness of the calculus is not sufficient, but R-boundedness is needed.

This motivates the following definition.

Definition 4.38 *Let $E \in \{\mathcal{W}_p^\alpha, \mathcal{B}^\alpha, W_p^\alpha\}$ (resp. $E \in \{\mathcal{H}_p^\alpha, \mathcal{M}^\alpha\}$). We say that B (resp. A) has an R-bounded E calculus if B (resp. A) has an E calculus, which is an R-bounded mapping in the sense of definition 3.7, i.e.*

$$R(\{f(B) : \|f\|_E \leqslant 1\}) < \infty$$

(resp. $R(\{f(A) : \|f\|_E \leqslant 1\}) < \infty$)).
Assume in addition that E is a Hilbert space. Then we say that B (resp. A) has a matricially R-bounded E calculus if B (resp. A) has an E calculus, which is a matricially R-bounded mapping in the sense of 4.48, i.e. there is $C > 0$ such that for any $n \in \mathbb{N}$ and $[f_{ij}] \in M_n(E)$,

$$\|[f_{ij}(B)]\|_{B(\mathrm{Rad}_n(X))} \leqslant C\|[f_{ij}]\|_{M_n(E_r)}$$

(resp. $\|[f_{ij}(A)]\|_{B(\mathrm{Rad}_n(X))} \leqslant C\|[f_{ij}]\|_{M_n(E_r)}$).

Let us see how both the R-bounded W_2^α calculus and the matricially R-bounded W_2^α calculus of B can be equivalently expressed in terms of e^{itB} by strengthenings of the weak $L^2(\mathbb{R})$ estimate 4.53.

Consider the family $(N(t) = \langle t\rangle^{-\alpha}e^{itB} : t \in \mathbb{R})$. Then the associated mapping $u : L^2(\mathbb{R}) \to B(X)$ in the sense of 4.45 is

$$u(f) = \int_{\mathbb{R}} f(t)\langle t\rangle^{-\alpha}e^{itB}dt.$$

By proposition 4.22, $u(f) = 2\pi g(B)$ with $g = [f\langle\cdot\rangle^{-\alpha}]\check{} \in W_2^\alpha$. Using the isometry $L^2(\mathbb{R}) \to W_2^\alpha$, $f \mapsto g = [f\langle\cdot\rangle^{-\alpha}]\check{}$, we immediately deduce from theorem 4.36 the following corollary.

Corollary 4.39 *Let $\alpha > \frac{1}{2}$ and B be a 0-strip type operator on a Banach space X generating the c_0-group e^{itB}.*

Then B has an R-bounded W_2^α calculus if and only if

$$(\langle t\rangle^{-\alpha}e^{itB} : t \in \mathbb{R}) \text{ is } R[L^2(\mathbb{R})]\text{-bounded.} \tag{4.54}$$

Assume in addition that X has property (α).

Then B has a matricially R-bounded W_2^α calculus if and only if there exists $C > 0$ such that

$$\|\langle t\rangle^{-\alpha}e^{itB}x\|_{\gamma(\mathbb{R},X)} \leqslant C\|x\| \quad (x \in X). \tag{4.55}$$

The rest of this section is devoted to extend corollary 4.39 by finding conditions similar to 4.54 and 4.55, but containing the other operator families mentioned at the beginning of this subsection instead of e^{itB}.

Let us start with 4.54. The idea is to transfer that condition by means of suitable mappings. To this end, we recall proposition 4.33, which transforms $R[E_1]$-bounded families to $R[E_2]$-bounded families by means of an operator $K \in B(E_1', E_2')$. The counterpart when dealing with the functional calculus is proposition 4.40 below.

The families $(N(t) : t \in \Omega)$ that are subject to averaged R-boundedness in the sequel are of the form $N(t) = g_t(B)$, where B has a $W^\alpha_{2,\mathrm{loc}}$ calculus and $g_t \in W^\alpha_{2,\mathrm{loc}}$. When dealing with the operators e^{itA}, it may occur that $N(t)$ is unbounded. However, any such family does have a dense subspace D_N contained in the domain of $N(t)$ for any $t \in \Omega$ as required in the defintion 4.31 of averaged R-boundedness. We remark once and for all that in view of proposition 4.25, we may and do always take $D_N = D_B$, the calculus core from 4.33.

Proposition 4.40 *Let B be a 0-strip-type operator having a bounded $\mathcal{B}^\beta$ calculus and a W_2^α calculus for some $\alpha > \frac{1}{2}, \beta > 0$. For $i = 1, 2$, let (Ω_i, μ_i) be a σ-finite measure space and E_i a space as in 4.39. Let $K \in B(E_1', E_2')$ and $g : \Omega_1 \times \mathbb{R} \to \mathbb{C}$ a measurable function such that*

(1) *$g(t, \cdot) \in W^\alpha_{2,\mathrm{loc}}$ for almost all $t \in \Omega_1$.*

(2) *$g(\cdot, s) \in E_1'$ for almost all $s \in \mathbb{R}$.*

(3) *$(Kg) : \Omega_2 \times \mathbb{R} \to \mathbb{C}$ is measurable.*

(4) *$(Kg)(t, \cdot) \in W^\alpha_{2,\mathrm{loc}}$ for almost all $t \in \Omega_2$.*

(5) *There exists a weakly dense subset D of E_2 with*

 (a) *$\int_{\Omega_1} \|g(t,\cdot)\varphi\|_{W_2^\alpha} |K' f(t)| d\mu_1(t) < \infty$ for all $f \in D$, $\varphi \in C_c^\infty(\mathbb{R})$.*

 (b) *$\int_{\Omega_2} \|Kg(t,\cdot)\varphi\|_{W_2^\alpha} |f(t)| d\mu_2(t) < \infty$ for all $f \in D$, $\varphi \in C_c^\infty(\mathbb{R})$.*

(6) *For any $\varphi \in C_c^\infty(\mathbb{R})$, we have $\mathcal{F}K[g(t,s)\varphi(s)] = K\mathcal{F}[g(t,s)\varphi(s)]$, where $\mathcal{F}$ is the Fourier transform applied in the variable s, and K is applied in the variable t.*

Then the following holds.

(i) *If for all $x \in D_B$ and $x' \in X'$, $\langle g(\cdot, B)x, x'\rangle \in E_1'$, then $\langle (Kg)(\cdot, B)x, x'\rangle \in E_2'$ and*

$$\langle (Kg)(\cdot, B)x, x'\rangle = K(\langle g(\cdot, B)x, x'\rangle).$$

(ii) *If $(g(t, B) : t \in \Omega_1)$ is $R[E_1]$-bounded and $t \mapsto Kg(t,B)x$ is measurable for all $x \in D_B$, then $((Kg)(t, B) : t \in \Omega_2)$ is $R[E_2]$-bounded.*

Remark 4.41 *We give two situations where the above proposition applies.*

(1) *Consider $\Omega_1 = \Omega_2 = \mathbb{R}$ with the usual Lebesgue measure, $E_1 = L^2(\mathbb{R}, \langle t\rangle^{2\gamma} dt)$, $E_2 = W_2^\gamma$ for some $\gamma \geqslant 0$ and $K : L^2(\mathbb{R}, \langle t\rangle^{-2\gamma} dt) \to (W_2^\gamma)'$ the Fourier transform or its inverse. We choose*

$$D = \operatorname{span}\{t^n e^{-t^2/2} : n \in \mathbb{N}_0\},$$

so that $D = K(D)$ is dense in $L^2(\mathbb{R}, \langle t\rangle^{2\gamma} dt)$ and W_2^γ. In view of the estimate $|t|^n e^{-t^2/2} \lesssim_{n,N} e^{-N|t|}$ for any $n, N \in \mathbb{N}$, it is easy to see that the assumptions of proposition 4.40 are satisfied, if for some $l > \alpha$, g and Kg are l-times continuously differentiable in the second variable and for $k = 0, \ldots, l$, any $C \subset \mathbb{R}$ compact and some $N \in \mathbb{N}$

$$\int_{\mathbb{R}} \sup_{s \in C} |\partial_s^k g(t,s)| e^{-N|t|} dt < \infty \text{ and } \int_{\mathbb{R}} \sup_{s \in C} |\partial_s^k Kg(t,s)| e^{-N|t|} dt < \infty.$$

(2) *Consider $\Omega_1 = \mathbb{R}_+$ with $\mu_1 = \frac{ds}{s}$, and $\Omega_2 = \mathbb{R}$ with usual Lebesgue measure μ_2. We take the Mellin transform isometry*

$$K : L^2(\mathbb{R}_+, \frac{ds}{s}) \to L^2(\mathbb{R}, dt),\ f \mapsto \int_0^\infty s^{it} f(s) \frac{ds}{s} = \mathcal{F}(f_e)(t).$$

Consider the 0-sectorial operator $A = e^B$. Suppose that the functions $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{C}$ and its Mellin transform in the first variable $Kg : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$ both satisfy the following conditions:

For some $l > \alpha$, g and Kg are l-times continuously differentiable in μ, and for $k = 0, \ldots, l$, for any compact $C \subset (0, \infty)$ and some $N \in \mathbb{N}$, we have

$$\int_0^\infty \sup_{\mu \in C} |\partial_\mu^k g(s,\mu)| \min(s^N, s^{-N}) ds < \infty, \tag{4.56}$$

$$\int_{-\infty}^\infty \sup_{\mu \in C} |\partial_\mu^k Kg(t,\mu)| \, e^{-N|t|} dt < \infty. \tag{4.57}$$

Then with the same D as in the first part of the remark, proposition 4.40 implies that

$$K(\langle g(\cdot, A)x, x'\rangle) = \langle Kg(\cdot, A)x, x'\rangle,$$

provided that one side of the equation belongs to $L^2(\mathbb{R})$.

Proof of proposition 4.40. Let $x \in D_B$ and $x' \in X'$. Write $U(t) = e^{itB}$. There exists $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(B)x = x$. We put $\tilde{g}(t,s) = g(t,s)\varphi(s)$.

For the first statement, it suffices to show that for any $f \in D$

$$\int_{\Omega_2} K[\langle g(\cdot, B)x, x'\rangle](t) f(t) d\mu_2(t) = \int_{\Omega_2} \langle (Kg)(t,B)x, x'\rangle f(t) d\mu_2(t).$$

We have

$$\begin{aligned}
\int_{\Omega_2} K[\langle g(\cdot, B)x, x'\rangle](t) f(t) d\mu_2(t) &= \int_{\Omega_1} \langle g(t,B)x, x'\rangle K' f(t) d\mu_1(t) \\
&= \frac{1}{2\pi} \int_{\Omega_1} \int_{\mathbb{R}} \mathcal{F}\tilde{g}(t,s) \langle U(s)x, x'\rangle ds K' f(t) d\mu_1(t) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\Omega_1} \mathcal{F}\tilde{g}(t,s) \langle U(s)x, x'\rangle K' f(t) d\mu_1(t) ds \qquad (4.58) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\Omega_2} K\mathcal{F}\tilde{g}(t,s) \langle U(s)x, x'\rangle f(t) d\mu_2(t) ds \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\Omega_2} \mathcal{F}K\tilde{g}(t,s) \langle U(s)x, x'\rangle f(t) d\mu_2(t) ds \\
&= \frac{1}{2\pi} \int_{\Omega_2} \int_{\mathbb{R}} \mathcal{F}K\tilde{g}(t,s) \langle U(s)x, x'\rangle f(t) ds d\mu_2(t) \qquad (4.59) \\
&= \int_{\Omega_2} \langle Kg(t,B)x, x'\rangle f(t) d\mu_2(t).
\end{aligned}$$

For the change of order of integration in 4.58, we use assumption (5)(a), and in 4.59, we use assumption (5)(b). Namely, since B has a W_2^α calculus, $\langle s\rangle^{-\alpha}\langle U(s)x, x'\rangle$ belongs to $L^2(\mathbb{R})$, and

$$\begin{aligned}
&\int_{\Omega_1} \int_{\mathbb{R}} |\mathcal{F}\tilde{g}(t,s) \langle U(s)x, x'\rangle K' f(t)| ds d\mu_1(t) \\
&\leqslant \int_{\Omega_1} \left(\int_{\mathbb{R}} |\mathcal{F}\tilde{g}(t,s) \langle s\rangle^\alpha|^2 ds \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |\langle s\rangle^{-\alpha} \langle U(s)x, x'\rangle|^2 ds \right)^{\frac{1}{2}} |K' f(t)| d\mu_1(t) \\
&\lesssim \int_{\Omega_1} \|g(t,\cdot)\varphi\|_{W_2^\alpha} |K' f(t)| d\mu_1(t) < \infty.
\end{aligned}$$

Furthermore,

$$\int_{\Omega_2}\int_{\mathbb{R}} |\mathcal{F}K\tilde{g}(t,s)\langle U(s)x,x'\rangle f(t)|ds d\mu_2(t) \lesssim \int_{\Omega_2} \|K\tilde{g}(t,\cdot)\|_{W_2^\alpha}|f(t)|d\mu_2(t) < \infty.$$

Thus, the first statement is shown. Then the second statement follows at once from proposition 4.33. □

We now turn to the characterization of the R-bounded W_2^α calculus of $B = \log(A)$ in terms of the operators

$$\{e^{isA} : s \in \mathbb{R}\},$$

which are sometimes called wave operators. Let us give some background information.

The wave operators are unbounded in general. It is for example a classical result of Hörmander [58], that if $A = -\Delta$ is the Laplace operator on $L^p(\mathbb{R}^d)$, then e^{isA} is bounded if and only if $s = 0$ or $p = 2$.

One method to obtain a uniformly bounded operator family is to regularize by multiplication with powers of resolvents of A. More precisely, for the case of the Laplace operator on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, we have for $\alpha > d|\frac{1}{p} - \frac{1}{2}|$

$$\sup_{s\in\mathbb{R}} \|(1+s)^{-\alpha}(1+A)^{-\alpha}e^{isA}\| < \infty \tag{4.60}$$

(see [14, prop 1.1] and also [56, 120]).

Similar results for different A can be found in [104, thm 7.20] for $A = \sqrt{-\Delta}$ on $\mathbb{R}^d$, in [101] and [99, (3.1)] for $A = \sqrt{-\Delta}$ on Heisenberg and related groups, and in [17] for A such that e^{-tA} satisfies Gaussian estimates.

Condition 4.60 can be used to obtain solutions of the Cauchy problem of the associated Schrödinger equation

$$i\partial_t u + Au = 0, \quad u(t=0) = u_0$$

(see [56, cor 4.3 and exa 4.4]), or to derive a Hörmander functional calculus [99, thm 3, Subordination principle and thm 2], and is linked to the growth rate of the analytic semigroup e^{-zA} when z approaches the imaginary axis (see [14, thm 2.2 and 2.3] and lemma 4.72 in section 4.5).

We consider a related expression in proposition 4.42 (2) below. We will discuss in section 4.5, remark 4.76, the link between 4.60 and the condition (2) in proposition 4.42.

The reformulation of condition 4.54 in terms of e^{isA} now reads as follows.

Proposition 4.42 *Let A be a 0-sectorial operator having a $\mathcal{M}^\beta$ calculus for some $\beta > 0$. Let $\alpha > \frac{1}{2}$ and m some integer, $m > \alpha - \frac{1}{2}$. Then the following are equivalent:*

(1) *$(\langle t\rangle^{-\alpha}A^{it} : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded.*

(2) *$(|s|^{-\alpha}A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m : s \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded.*

We will need the following three lemmas for the proof.

Lemma 4.43 *Let* $m \in \mathbb{N}$ *and* $\operatorname{Re} z \in (-m, 0)$. *Then*

$$\int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s} = \Gamma(z) f_m(z),$$

with the entire function

$$f_m(z) = \sum_{k=1}^m \binom{m}{k} (-1)^{m-k} k^{-z}. \tag{4.61}$$

Note that $\Gamma(z) f_m(z)$ *is a holomorphic function for* $\operatorname{Re} z \in (-m, 0)$.

Proof. We proceed by induction over m. In the case $m = 1$, we obtain by integration by parts

$$\int_0^\infty s^z (e^{-s} - 1) \frac{ds}{s} = \left[\frac{1}{z} s^z (e^{-s} - 1) \right]_0^\infty + \int_0^\infty \frac{1}{z} s^z e^{-s} ds = 0 + \frac{1}{z} \Gamma(z + 1) = \Gamma(z) = \Gamma(z) f_1(z).$$

Next we claim that for $\operatorname{Re} z > -m$,

$$\int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} = \Gamma(z) \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k + 1)^{-z}.$$

Note that the left hand side is well-defined and holomorphic for $\operatorname{Re} z > -m$ and the right hand side is meromorphic on $\mathbb{C}$. By the identity theorem for meromorphic functions, it suffices to show the claim for e.g. $\operatorname{Re} z > 0$. For these z in turn, we can develop

$$\int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \int_0^\infty s^z e^{-ks} e^{-s} \frac{ds}{s},$$

which gives the claim.

Assume now that the lemma holds for some m. Let first $\operatorname{Re} z \in (-m, 0)$. In the following calculation, we use both the claim and the induction hypothesis in the second equality, and the convention $\binom{m}{m+1} = 0$ in the third.

$$\begin{aligned} \int_0^\infty s^z (e^{-s} - 1)^{m+1} \frac{ds}{s} &= \int_0^\infty s^z (e^{-s} - 1)^m e^{-s} \frac{ds}{s} - \int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s} \\ &= \Gamma(z) \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k + 1)^{-z} - \Gamma(z) f_m(z) \\ &= \Gamma(z) \sum_{k=1}^{m+1} \binom{m}{k-1} (-1)^{m+1-k} k^{-z} + \Gamma(z) \sum_{k=1}^{m+1} \binom{m}{k} (-1)^{m+1-k} k^{-z} \\ &= \Gamma(z) \sum_{k=1}^{m+1} \left[\binom{m}{k-1} + \binom{m}{k} \right] (-1)^{m+1-k} k^{-z} \\ &= \Gamma(z) f_{m+1}(z). \end{aligned}$$

Thus, the lemma holds for $m + 1$ and $\operatorname{Re} z \in (-m, 0)$. For $\operatorname{Re} z \in (-(m + 1), -m]$, we appeal again to the identity theorem. □

Lemma 4.44 *Let* $\operatorname{Re} z \in (-m, 0)$ *and* $\operatorname{Re} \lambda \geqslant 0$. *Then*

$$\int_0^\infty s^z (e^{-\lambda s} - 1)^m \frac{ds}{s} = \lambda^{-z} \int_0^\infty s^z (e^{-s} - 1)^m \frac{ds}{s}.$$

Proof. This is an easy consequence of the Cauchy integral theorem. □

Lemma 4.45 *Let* $\beta \in \mathbb{R}$ *and* $f(t) = f_m(\beta + it)$ *with* f_m *as in* 4.61. *Then there exist* $C, \varepsilon, \delta > 0$ *such that for any interval* $I \subset \mathbb{R}$ *with* $|I| \geqslant C$ *there is a subinterval* $J \subset I$ *with* $|J| \geqslant \delta$ *so that* $|f(t)| \geqslant \varepsilon$ *for* $t \in J$. *Consequently, for* $N > C/\delta$,

$$\sum_{k=-N}^{N} |f(t + k\delta)| \gtrsim 1.$$

Proof. Suppose for a moment that

$$\exists C, \varepsilon > 0 \, \forall I \text{ interval with } |I| \geqslant C \, \exists t \in I : \; |f(t)| \geqslant \varepsilon. \tag{4.62}$$

It is easy to see that $\sup_{t \in \mathbb{R}} |f'(t)| < \infty$, so that for such a t and $|s - t| \leqslant \delta = \delta(\|f'\|_\infty, \varepsilon)$, $|f(s)| \geqslant \varepsilon/2$. Thus the lemma follows from 4.62 with $J = B(t, \delta/2)$.

It remains to show 4.62. Suppose that this is false. Then

$$\forall C, \varepsilon > 0 \, \exists I \text{ interval with } |I| \geqslant C : \; \forall t \in I : \; |f(t)| < \varepsilon. \tag{4.63}$$

Since f'' is bounded and $\|f'\|_{L^\infty(I)} \leqslant \sqrt{8 \|f\|_{L^\infty(I)} \|f''\|_{L^\infty(I)}}$, we deduce that 4.63 holds for f' in place of f, and successively also for $f^{(n)}$ for any n. But there is some $n \in \mathbb{N}$ such that $\inf_{t \in \mathbb{R}} |f^{(n)}(t)| > 0$. Indeed,

$$f^{(n)}(t) = \sum_{k=1}^{m} \alpha_k (-i \log k)^n e^{-it \log k},$$

with $\alpha_k = \binom{m}{k} (-1)^{m-k} k^{-\beta} \neq 0$, whence

$$|f^{(n)}(t)| \geqslant |\alpha_m| \, |\log m|^n - \sum_{k=1}^{m-1} |\alpha_k| \, |\log k|^n > 0$$

for n large enough. This contradicts 4.63, so that the lemma is proved. □

Proof of proposition 4.42. First of all, note that by lemma 4.34 (3), condition (2) of the proposition is equivalent to

$$((sA)^{\frac{1}{2} - \alpha} (e^{\pm isA} - 1)^m : \; s > 0) \text{ is } R[L^2(\mathbb{R}_+, \frac{ds}{s})]\text{-bounded}. \tag{4.64}$$

Let $\mu > 0$ be fixed. Combining lemmas 4.43 and 4.44 with $\lambda = \pm i\mu$, we get

$$\int_0^\infty s^z (e^{\mp i\mu s} - 1)^m \frac{ds}{s} = (e^{\pm i \frac{\pi}{2}} \mu)^{-z} \Gamma(z) f_m(z).$$

Put now $z = \frac{1}{2} - \alpha + it$ for $t \in \mathbb{R}$, so that $\operatorname{Re} z \in (-m, 0)$ by the assumptions of the proposition. Then

$$\int_0^\infty s^{it+\frac{1}{2}-\alpha}(e^{\mp i\mu s} - 1)^m \frac{ds}{s} = e^{\mp i\frac{\pi}{2}(\frac{1}{2}-\alpha+it)}\mu^{-it}\mu^{-(\frac{1}{2}-\alpha)}\Gamma(z)f_m(z),$$

so that

$$M\left[((\cdot)\mu)^{\frac{1}{2}-\alpha}(e^{\mp i\mu(\cdot)} - 1)^m\right](t) = h_\mp(t)\mu^{-it}. \tag{4.65}$$

Here, $M : L^2(\mathbb{R}_+, \frac{ds}{s}) \to L^2(\mathbb{R}, dt),\ f \mapsto \int_0^\infty f(s)s^{it}\frac{ds}{s}$, is the Mellin transform isometry and we have written

$$h_\mp(t) = e^{\mp i\frac{\pi}{2}(\frac{1}{2}-\alpha)}e^{\pm\frac{\pi}{2}t}\Gamma(\frac{1}{2} - \alpha + it)f_m(\frac{1}{2} - \alpha + it) \tag{4.66}$$

in short.

Our next goal is to insert $\mu = A$ in 4.65, and thus we check that proposition 4.40 in the form of remark 4.41 (2) applies whenever condition (1) or (2) of the proposition holds.

Put $g(s,\mu) = (s\mu)^{\frac{1}{2}-\alpha}(e^{\mp i\mu s} - 1)^m$ and $Mg(t,\mu) = h_\mp(t)\mu^{-it}$. Clearly, g and Mg are arbitrarily many times differentiable in μ. Let us check that g and Mg satisfy 4.56 and 4.57 respectively.

For any $C \subset (0,\infty)$ compact and any $a,b,c,d \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that

$$\int_0^\infty \sup_{\mu\in C} |\underbrace{\mu^a s^b (e^{\mp i\mu s} - 1)^c \exp(\mp i\mu s)^d}_{(*)}| \min(s^N, s^{-N})ds < \infty.$$

Note that for any $k \in \mathbb{N}_0$, $\partial_\mu^k g(s,\mu)$ is a linear combination of expressions of the form $(*)$, so that g satisfies the assumption 4.56.

On the other hand, the Γ-function admits the following estimate [90, p. 15]:

$$|\Gamma(\frac{1}{2} - \alpha + it)| \cong e^{-\frac{\pi}{2}|t|}|t|^{-\alpha} \quad (|t| \geqslant 1). \tag{4.67}$$

Since $\sup_{t\in\mathbb{R}} |f_m(\frac{1}{2} - \alpha + it)| < \infty$ and $\Gamma(z)f_m(z)$ is locally bounded, we get

$$|f_m(\frac{1}{2} - \alpha + it)|e^{\frac{\pi}{2}(\pm t - |t|)}\langle t\rangle^{-\alpha} \lesssim |h_\mp(t)| \lesssim \langle t\rangle^{-\alpha} \quad (t \in \mathbb{R}). \tag{4.68}$$

Hence $|\partial_\mu^k Mg(t,\mu)| \lesssim \langle t\rangle^{-\alpha+k}\mu^{-k}$ and Mg satisfies 4.57. Thus, whenever condition (1) or (2) holds, proposition 4.40 applies to g and Mg, and consequently, 4.65 yields

$$\begin{aligned} &((sA)^{\frac{1}{2}-\alpha}(e^{\mp isA} - 1)^m : s > 0) \text{ is } R[L^2(\mathbb{R}_+, \frac{ds}{s})]\text{-bounded} \\ &\iff (h_\mp(t)A^{-it} : t \in \mathbb{R}) \text{ is } R[L^2(\mathbb{R}, dt)]\text{-bounded.} \end{aligned} \tag{4.69}$$

Suppose now that (1) holds, i.e. $(\langle t\rangle^{-\alpha}A^{it} : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded. By the upper estimate in 4.68 and lemma 4.34 (1), the right hand side of 4.69 is $R[L^2(\mathbb{R})]$-bounded, and thus, 4.64, i.e. (2), holds.

Conversely, suppose that (2) holds. By 4.64 and 4.69, $(h_{\mp}(t)A^{-it} : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded. Thus by the lower estimate in 4.68 and 4.34 (1),

$$(\langle t\rangle^{-\alpha} f_m(\frac{1}{2} - \alpha + it)A^{-it} : t \in \mathbb{R}) \text{ is } R[L^2(\mathbb{R})]\text{-bounded.} \tag{4.70}$$

To get rid of f_m in this expression, we apply lemma 4.45. Write $f(t) = f_m(\frac{1}{2} - \alpha + it)$. According to lemma 4.45, for suitable $N \in \mathbb{N}$ and $\delta > 0$, $\sum_{k=-N}^{N} |f(t + k\delta)| \gtrsim 1$. In view of lemma 4.34 (1), it now suffices to show that $(\sum_{k=-N}^{N} f(t + k\delta)\langle t\rangle^{-\alpha} A^{-it} : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded. Write

$$\sum_{k=-N}^{N} f(t + k\delta)\langle t\rangle^{-\alpha} A^{-it} = \sum_{k=-N}^{N} \left[\frac{\langle t + k\delta\rangle^{\alpha}}{\langle t\rangle^{\alpha}} A^{-ik\delta} \right] \left[f(t + k\delta)\langle t + k\delta\rangle^{-\alpha} A^{i(t+k\delta)} \right].$$

By 4.70, the term in the second brackets is $R[L^2(\mathbb{R})]$-bounded. The term in the first brackets is a bounded function times a bounded operator, due to the assumption that A has a bounded $\mathcal{M}^\beta$ calculus. Thus, the right hand side is $R[L^2(\mathbb{R})]$-bounded, and condition (1) follows. □

In the following theorem, we summarize further averaged R-boundedness conditions for the remaining operator families.

Theorem 4.46 *Let B be a 0-strip-type operator on a Banach space X generating the c_0-group $U(t) = e^{itB}$. Assume that B has a bounded $\mathcal{B}^\beta$ calculus for some (large) $\beta > 0$. Let $A = e^B$ be the corresponding 0-sectorial operator and $T(z)$ the analytic semigroup generated by A. Let $\alpha > \frac{1}{2}$. Consider the following conditions.*

Sobolev Calculus

(1) *B has an R-bounded W_2^α calculus.*

Imaginary powers

(2) *$(\langle t\rangle^{-\alpha} U(t) : t \in \mathbb{R}) = (\langle t\rangle^{-\alpha} A^{it} : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded.*

Resolvents

(3) *$(R(t + ia, B) : t \in \mathbb{R})$ is uniformly $R[W_2^\alpha]$-bounded for all $a \in \mathbb{R}\backslash\{0\}$.*

(4) *$(|a|^{\alpha-\frac{1}{2}} e^{-|a|} R(t + ia, B) : t \in \mathbb{R},\ a \neq 0)$ is $R[L^2(\mathbb{R} \times \mathbb{R}\backslash\{0\})]$-bounded.*

(5) *For some $b > 0$, $(|a|^{\alpha-\frac{1}{2}} R(t + ia, B) : t \in \mathbb{R},\ a \in [-b, b]\backslash\{0\})$ is $R[L^2(\mathbb{R} \times [-b, b]\backslash\{0\})]$-bounded.*

(6) *There exists $C > 0$ such that for all $a \neq 0$: $R[L^2(\mathbb{R})](R(t + ia, B) : t \in \mathbb{R}) \leqslant C|a|^{-\alpha}$.*

(7) *There exists $C > 0$ such that for all $\theta \in (-\pi, \pi)\backslash\{0\}$: $R[L^2(\mathbb{R}_+)](A^{1/2} R(e^{i\theta} t, A) : t > 0) \leqslant C|\theta|^{-\alpha}$.*

(8) *For some $\theta_0 \in (0, \pi]$, $(|\theta|^{\alpha-\frac{1}{2}} A^{\frac{1}{2}} R(e^{i\theta} t, A) : 0 < |\theta| \leqslant \theta_0, t > 0)$ is $R[L^2((0, \infty) \times [-\theta_0, \theta_0]\backslash\{0\}, dtd\theta)]$-bounded.*

Analytic Semigroup

(9) *There exists* $C > 0$ *such that for all* $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$: $R[L^2(\mathbb{R}_+)](A^{1/2}T(e^{i\theta}t) : t > 0) \leqslant C(\frac{\pi}{2} - |\theta|)^{-\alpha}$.

(10) $(\langle \frac{x}{y} \rangle^{\alpha} |x|^{-\frac{1}{2}} A^{1/2} T(x+iy) : x > 0, y \in \mathbb{R})$ *is* $R[L^2(\mathbb{R}_+ \times \mathbb{R})]$*-bounded.*

Wave Operators

(11) *For some* $m > \alpha - \frac{1}{2}$, $(|s|^{-\alpha} A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m : s \in \mathbb{R})$ *is* $R[L^2(\mathbb{R})]$*-bounded.*

Then the following conditions are equivalent:

$$(1), (2), (3), (4), (5), (8), (10), (11).$$

These conditions imply the remaining ones (6), (7) and (9). If X *has property* (α) *then, conversely, these three conditions imply that* B *has an* R*-bounded* $W_2^{\alpha+\varepsilon}$ *calculus for any* $\varepsilon > 0$.

Remark 4.47 *If one omits the conditions (5) and (11), then the theorem could also be stated under the weaker assumption that* B *has an* H^∞ *calculus instead of a* $\mathcal{B}^\beta$ *calculus. In that case, we cannot appeal to proposition 4.40 in the proof, which justifies that the Fourier, Mellin and Laplace transformations behave well with respect to the functional calculus, and we have to replace that proposition in each use by an alternative argument.*

Proof of theorem 4.46.
(1) $\Leftrightarrow$ (2): This has been proved already in corollary 4.39.

(2) $\Leftrightarrow$ (3): For $a \neq 0$, let $g_a(t,s) = -i\,\mathrm{sgn}(a)\chi_{(-\infty,0)}(at)e^{at}e^{ist}$. Let

$$K : L^2(\mathbb{R}, \langle t \rangle^{-2\alpha} dt) \to (W_p^\alpha)'$$

be the Fourier transform isometry. Then $g_a(t,B) = -i\,\mathrm{sgn}(a)\chi_{(-\infty,0)}(at)e^{at}U(t)$ and $(Kg_a)(t,B) = R(t+ia,B)$. By lemma 4.34 and proposition 4.40,

$$\begin{aligned} R[L^2(\mathbb{R})](\langle t \rangle^{-\alpha} U(t) : t \in \mathbb{R}) &= R[L^2(\mathbb{R}, \langle t \rangle^{-2\alpha} dt)](U(t)) = \sup_{a>0} R[L^2(\langle t \rangle^{-2\alpha})](e^{-a|t|}U(t)) \\ &= \sup_{a\neq 0} R[L^2(\langle t \rangle^{-2\alpha})](g_a(t,B)) = \sup_{a \neq 0} R[W_p^\alpha]((Kg_a)(t,B)) \\ &= \sup_{a\neq 0} R[W_p^\alpha](R(t+ia,B) : t \in \mathbb{R}). \end{aligned}$$

(2) $\Leftrightarrow$ (4): Consider the Laplace transform isometry

$$K : L^2(\mathbb{R}_+, v(s)ds) \to L^2(\mathbb{R}_+ \times \mathbb{R}, w(a)dadt), f \mapsto \int_0^\infty e^{-(a+it)s} f(s) ds$$

with $w(a) = a^{2\alpha-1}e^{-2a}$ and $v(s) = \int_0^\infty w(a) e^{-2as} da = (2s+2)^{-2\alpha}\Gamma(2\alpha) \cong \langle s \rangle^{-2\alpha}$. Then $\langle U(\pm s)x, x' \rangle$ and $-i\langle R(ia - t, \pm B)x, x'\rangle$ are mapped to each other by K and K^{-1} whenever one of them is in L^2, and by proposition 4.33

$$\begin{aligned} &R[L^2(\mathbb{R})](\langle s \rangle^{-\alpha} U(s) : s \in \mathbb{R}) = R[L^2(\mathbb{R}, \langle s \rangle^{-2\alpha})](U(s)) \\ &\cong R[L^2(\mathbb{R}_+, v(s))](U(s) : s > 0) + R[L^2(\mathbb{R}_+, v(s))](U(-s) : s > 0) \\ &= R[L^2(\mathbb{R}_+ \times \mathbb{R}, w(a)dadt)](-iR(ia - t, B)) + R[L^2(\mathbb{R}_+ \times \mathbb{R}, w(a)dadt)](-iR(ia-t, -B)) \\ &\cong R[L^2(\mathbb{R} \times \mathbb{R}\backslash\{0\})](|a|^{\alpha - \frac{1}{2}} e^{-|a|} R(t+ia, B) : t \in \mathbb{R}, a \neq 0). \end{aligned}$$

(4) $\Leftrightarrow$ (5): Let $b > 0$ be fixed. Since $1 \lesssim e^{-|a|}$ for $a \in [-b, b]\backslash\{0\}$, "$\Rightarrow$" follows. Conversely, assume that (5) holds. We only need to show that $\{|a|^{\alpha-\frac{1}{2}}e^{-|a|}R(t+ia, B) : t \in \mathbb{R}, |a| > b\}$ is $R[L^2(\mathbb{R}\times\mathbb{R}\backslash[-b, b])]$-bounded. Since B has a $\mathcal{B}^\beta$ calculus, by proposition 4.9 (4), it has a $\mathcal{W}_2^\gamma$ calculus for $\gamma > \beta + \frac{1}{2}$. By "(1) $\Rightarrow$ (4)" for γ in place of α, the result follows from $|a|^{\alpha-\frac{1}{2}} \lesssim_b |a|^{\gamma-\frac{1}{2}}$ for $\gamma > \alpha$.

(2) $\Leftrightarrow$ (8): Consider

$$K : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R} \times (-\pi, \pi), dsd\theta),\ f(s) \mapsto (\pi - |\theta|)^{\alpha-\frac{1}{2}} \frac{1}{\cosh(\pi s)} e^{\theta s}\langle s\rangle^\alpha f(s). \tag{4.71}$$

K is an isomorphic embedding. Indeed,

$$\|Kf\|_2^2 = \int_{\mathbb{R}}\int_{-\pi}^{\pi} \left((\pi - |\theta|)^{\alpha-\frac{1}{2}}e^{\theta s}\right)^2 d\theta \frac{1}{\cosh^2(\pi s)}\langle s\rangle^{2\alpha}|f(s)|^2 ds$$

and

$$\begin{aligned}\int_{-\pi}^{\pi} (\pi - |\theta|)^{2\alpha-1}e^{2\theta s}d\theta &\cong \int_0^\pi \theta^{2\alpha-1}e^{2(\pi-\theta)|s|}d\theta \\ &\cong \cosh^2(\pi s)\int_0^\pi \theta^{2\alpha-1}e^{2\theta|s|}d\theta.\end{aligned}$$

The last integral is bounded from below uniformly in $s \in \mathbb{R}$, and for $|s| \geqslant 1$,

$$\int_0^\pi \theta^{2\alpha-1}e^{2\theta|s|}d\theta = (2|s|)^{-2\alpha}\int_0^{2|s|\pi} \theta^{2\alpha-1}e^\theta d\theta \cong |s|^{-2\alpha}.$$

This clearly implies that $\|Kf\|_2 \cong \|f\|_2$. Applying proposition 4.33, we get

$$R[L^2(\mathbb{R}, ds)]\left(\langle s\rangle^{-\alpha}U(s)\right) \cong R[L^2(\mathbb{R}\times(-\pi,\pi), dsd\theta)]\left((\pi - |\theta|)^{\alpha-\frac{1}{2}}\frac{1}{\cosh(\pi s)}e^{\theta s}U(s)\right).$$

In [81, p. 228 and thm 15.18], the following formula is derived for $x \in A(D(A^2))$ and $|\theta| < \pi$:

$$\frac{\pi}{\cosh(\pi s)}e^{\theta s}U(s)x = \int_0^\infty t^{is}\left[e^{i\frac{\theta}{2}}t^{\frac{1}{2}}A^{\frac{1}{2}}(e^{i\theta}t + A)^{-1}x\right]\frac{dt}{t}. \tag{4.72}$$

(One could also use proposition 4.40 to deduce 4.72 for $x \in D_{\log A}$ from the identity of the corresponding functions [81, (15.5)].) Note that $A(D(A^2))$ is a dense subset of X. As the Mellin-transform $f(s) \mapsto \int_0^\infty t^{is}f(s)\frac{ds}{s}$ is an isometry $L^2(\mathbb{R}_+, \frac{ds}{s}) \to L^2(\mathbb{R}, t)$, we get

$$\begin{aligned}R[L^2(\mathbb{R})](\langle s\rangle^{-\alpha}U(s)) &\cong R[L^2(\mathbb{R}_+\times(-\pi,\pi), \frac{dt}{t}d\theta)]\left((\pi - |\theta|)^{\alpha-\frac{1}{2}}t^{\frac{1}{2}}A^{\frac{1}{2}}(e^{i\theta}t + A)^{-1}\right) \\ &\cong R[L^2(\mathbb{R}_+\times(0,2\pi), dtd\theta)](|\theta|^{\alpha-\frac{1}{2}}A^{\frac{1}{2}}R(e^{i\theta}t, A)),\end{aligned}$$

so that (2) $\Leftrightarrow$ (8) for $\theta_0 = \pi$.

For a general $\theta_0 \in (0, \pi]$, consider K from 4.71 with restricted image, i.e.

$$K : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}\times(-\pi, -(\pi - \theta_0)] \cup [\pi - \theta_0, \pi), dsd\theta).$$

Then argue as in the case $\theta_0 = \pi$.

(8) $\Leftrightarrow$ (10): The proof of (2) $\Leftrightarrow$ (8) above shows that condition (8) is independent of $\theta_0 \in (0, \pi]$. Put $\theta_0 = \pi$. The equivalence follows again from proposition 4.40, using the fact that for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mu > 0$,

$$(e^{i\theta}\mu + it)^{-1} = K[\exp(-se^{i\theta}\mu)\chi_{(0,\infty)}(s)](t),$$

where $K : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, dt)$ is the Fourier transform.

(7) $\Leftrightarrow$ (9): We use the same argument as right above.

(2) $\Rightarrow$ (7): We use a similar K_θ as in the proof of (2) $\Leftrightarrow$ (8), fixing $\theta \in (-\pi, \pi)$:

$$K_\theta : L^2(\mathbb{R}, ds) \to L^2(\mathbb{R}, ds),\ f(s) \mapsto (\pi - |\theta|)^\alpha \frac{1}{\cosh(\pi s)} e^{\theta s} \langle s \rangle^\alpha f(s).$$

We have $\sup_{|\theta|<\pi} \|K_\theta\| = \sup_{|\theta|<\pi,\, s \in \mathbb{R}} \langle s \rangle^\alpha (\pi - |\theta|)^\alpha \frac{e^{\theta s}}{\cosh(\pi s)} \lesssim \sup_{\theta, s} \langle s(\pi - |\theta|)\rangle^\alpha e^{-|s|(\pi-|\theta|)} < \infty$. Thus, by 4.72,

$$\begin{aligned}\sup_{0<|\theta|\leqslant\pi} |\theta|^\alpha R[L^2(\mathbb{R}_+, dt)](A^{\frac{1}{2}} R(te^{i\theta}, A)) &= \sup_{|\theta|<\pi} (\pi - |\theta|)^\alpha R[L^2(\mathbb{R}_+, \frac{dt}{t})](t^{\frac{1}{2}} A^{\frac{1}{2}} (e^{i\theta} t + A)^{-1}) \\ &= \sup_{|\theta|<\pi} (\pi - |\theta|)^\alpha R[L^2(\mathbb{R})](\frac{\pi}{\cosh(\pi s)} e^{\theta s} U(s)) \qquad (4.73) \\ &\lesssim R[L^2(\mathbb{R})](\langle s \rangle^{-\alpha} U(s)).\end{aligned}$$

(7), $\alpha \Rightarrow$ (2), $\alpha + \varepsilon$: First we consider $(\langle s \rangle^{-(\alpha+\varepsilon)} U(s) :\ s \geqslant 1)$.

$$R[L^2(\mathbb{R})](\langle s \rangle^{-(\alpha+\varepsilon)} U(s) :\ s \geqslant 1) \leqslant \sum_{n=0}^\infty 2^{-n\varepsilon} R[L^2](\langle s \rangle^{-\alpha} U(s) :\ s \in [2^n, 2^{n+1}]). \qquad (4.74)$$

For $s \in [2^n, 2^{n+1}]$, we have

$$\langle s \rangle^{-\alpha} \lesssim 2^{-n\alpha} \lesssim 2^{-n\alpha} e^{-2^{-n}s} \lesssim (\pi - \theta_n)^\alpha \frac{e^{\theta_n s}}{\cosh(\pi s)},$$

where $\theta_n = \pi - 2^{-n}$. Therefore

$$\begin{aligned}R[L^2](\langle s \rangle^{-\alpha} U(s) :\ s \in [2^n, 2^{n+1}]) &\lesssim (\pi - \theta_n)^\alpha R[L^2(\mathbb{R})](\frac{\pi}{\cosh(\pi s)} e^{\theta_n s} U(s)) \\ &\overset{4.73}{\lesssim} \sup_{0<|\theta|\leqslant\pi} |\theta|^\alpha R[L^2(\mathbb{R}_+)](A^{\frac{1}{2}} R(te^{i\theta}, A)) < \infty.\end{aligned}$$

Thus, the sum in 4.74 is finite.

The part $(\langle s \rangle^{-(\alpha+\varepsilon)} U(s) :\ s \leqslant -1)$ is treated similarly, whereas $R[L^2](\langle s \rangle^{-\alpha} U(s) :\ |s| < 1) \cong R[L^2](U(s) :\ |s| < 1)$. It remains to show that the last expression is finite. We have assumed that X has property (α). Then the fact that B has an H^∞ calculus implies that $\{U(s) : |s| < 1\}$ is R-bounded [72, cor 6.6]. For $f \in L^2([-1,1])$, we have $\|f\|_1 \leqslant C\|f\|_2$, and consequently,

$$\{\int_{-1}^{1} f(s)U(s)ds : \|f\|_2 \leqslant 1\} \subset C\{\int_{-1}^{1} f(s)U(s)ds : \|f\|_1 \leqslant 1\}.$$

In other words, $(U(s) : |s| < 1)$ is $R[L^2]$-bounded.

(2) $\Rightarrow$ (6): Let $R_a = |a|^\alpha R[L^2](R(t+ia,B) : t \in \mathbb{R})$. We have to show $\sup_{a\neq 0} R_a < \infty$. Applying proposition 4.40 with K the Fourier transform and its inverse, we get

$$R_a = \begin{cases} R[L^2](a^\alpha e^{at}U(t) : t < 0), & a > 0, \\ R[L^2](|a|^\alpha e^{at}U(t) : t > 0), & a < 0. \end{cases}$$

For $t < 0$, $\sup_{a>0} a^\alpha e^{at} = \sup_{a>0} \langle t\rangle^{-\alpha}\langle t\rangle^\alpha a^\alpha e^{-|at|} \lesssim \langle t\rangle^{-\alpha}$. Thus, $\sup_{a>0} R[L^2](a^\alpha e^{at}U(t) : t < 0) \lesssim R[L^2](\langle t\rangle^{-\alpha}U(t) : t < 0) < \infty$. The part $a < 0$ is estimated similarly.

(6), $\alpha \Rightarrow$ (2), $\alpha+\varepsilon$: Let R_a be as before. Split $\langle t\rangle^{-(\alpha+\varepsilon)}U(t)$ as in "(7) $\Rightarrow$ (2)" into the parts $t \geqslant 1, t \leqslant -1, |t| < 1$, and further $t \geqslant 1$ into $t \in [2^n, 2^{n+1}], n \in \mathbb{N}_0$. Then $\langle t\rangle^{-\alpha} \lesssim 2^{-n\alpha} \lesssim 2^{-n\alpha}e^{-2^{-n}t}$, and

$$\begin{aligned} R[L^2](\langle t\rangle^{-(\alpha+\varepsilon)}U(t) : t \geqslant 1) &\leqslant \sum_{n=0}^\infty 2^{-n\varepsilon} R[L^2](2^{-n\alpha}e^{-2^{-n}t}U(t) : t \in [2^n, 2^{n+1}]) \\ &\leqslant \sum_{n=0}^\infty 2^{-n\varepsilon} \sup_{a<0} R_a < \infty. \end{aligned}$$

The estimates for $t \leqslant -1$ and $|t| < 1$ can be handled similarly, c.f. "(7) $\Rightarrow$ (2)".

(2) $\Leftrightarrow$ (11): see proposition 4.42. □

Let us turn to the characterization of the matricially R-bounded W_2^α calculus of B in terms of generalized square functions. We have the following analogue of the transformation property for averaged R-boundedness from proposition 4.33.

Proposition 4.48 *For $i = 1, 2$, let (Ω_i, μ_i) be σ-finite measure spaces and $K \in B(L^2(\Omega_1), L^2(\Omega_2))$. Assume that $f \in \gamma(\Omega_1, X)$, and that there exists a Bochner-measurable $g : \Omega_2 \to X$ such that*

$$\langle g(\cdot), x'\rangle = K(\langle f(\cdot), x'\rangle) \quad (x' \in X').$$

Then $g \in \gamma(\Omega_2, X)$ and

$$\|g\|_{\gamma(\Omega_2,X)} \leqslant \|K\| \, \|f\|_{\gamma(\Omega_1,x)}.$$

Proof. By assumption, $\langle g(\cdot), x'\rangle = K(\langle f(\cdot), x'\rangle) \in L^2(\Omega_2)$ for any $x' \in X'$, and we can consider the associated operator $u_g : H \to X$ as in 2.14. We have $u_g = u_f \circ K'$, where $K' : B(L^2(\Omega_2), L^2(\Omega_1))$ is the Banach space adjoint of K. Thus, by lemma 2.7, $\|u_g\|_{\gamma(L^2(\Omega_2),X)} \leqslant \|K'\| \, \|u_f\|_{\gamma(L^2(\Omega_1),X)}$, and the claim follows. □

Furthermore, in theorem 4.46 we have made repeated use of lemma 4.34. The square function version of this lemma reads as follows.

Lemma 4.49 *Let (Ω, μ) be a σ-finite measure space and $g : \Omega \to X$ measurable.*

(1) *Let $f \in L^\infty(\Omega)$ and $g \in \gamma(\Omega, X)$. Then $f \cdot g \in \gamma(\Omega, X)$ and*

$$\|f \cdot g\|_{\gamma(\Omega,X)} \leqslant \|f\|_\infty \|g\|_{\gamma(\Omega,X)}.$$

In particular, if $\Omega_1 \subset \Omega_2$, with $f = \chi_{\Omega_1} \in L^\infty(\Omega_2)$, we get $\|g\|_{\gamma(\Omega_1,X)} \leqslant \|g\|_{\gamma(\Omega_2,X)}$.

(2) *For $n \in \mathbb{N}$, let $\varphi_n : \Omega \to \mathbb{R}_+$ measurable with $\sum_{n=1}^\infty \varphi_n(t) = 1$ for all $t \in \Omega$. Then*

$$\|g\|_{\gamma(\Omega,X)} \leqslant \sum_{n=1}^\infty \|\varphi_n g\|_{\gamma(\Omega,X)}.$$

(3) *Let $w : \Omega \to (0, \infty)$ measurable. Then*

$$\|g\|_{\gamma(\Omega,w(t)d\mu(t),X)} = \|w^{\frac{1}{2}} \cdot g\|_{\gamma(\Omega,d\mu)}.$$

(4) *For $n \in \mathbb{N}$, let $\phi_n : \Omega \to [0, 1]$ with $\phi_n(t) \to 1$ monotonically for all $t \in \Omega$. Then*

$$\|g\|_{\gamma(\Omega,X)} = \sup_n \|\phi_n \cdot g\|_{\gamma(\Omega,X)}.$$

Proof. (1) This follows from lemma 2.7 (5) with $T = \mathrm{Id}_X$ and $K : L^2(\Omega) \to L^2(\Omega)$, $h \mapsto fh$.

(2) For $n \in \mathbb{N}$, put $\phi_n = \sum_{k=1}^n \varphi_k$. Then $\phi_n : \Omega \to [0, 1]$ and $\phi_n(t) \to 1$ monotonically for all $t \in \Omega$. Thus by (4), $\|g\|_\gamma = \sup_n \|\phi_n g\|_\gamma = \sup_n \| \sum_{k=1}^n \varphi_k g\|_\gamma \leqslant \sum_{k=1}^\infty \|\varphi_k g\|_\gamma$.

(3) This follows again from lemma 2.7 (5) with the isometry $K : L^2(\Omega, w(t)d\mu(t)) \to L^2(\Omega, d\mu)$, $h \mapsto w^{\frac{1}{2}} h$.

(4) Assume first that $g \in \gamma(\Omega, X)$. Then by (1), we have $\phi_n g \in \gamma(\Omega, X)$ and $\sup_n \|\phi_n \cdot g\|_\gamma \leqslant \sup_n \|\phi_n\|_\infty \|g\|_\gamma = \|g\|_\gamma$.

Conversely, assume that for any $n \in \mathbb{N}$, $\phi_n \cdot g \in \gamma(\Omega, X)$ and that $\sup_n \|\phi_n \cdot g\|_\gamma < \infty$.

Let us show first that $g \in P_2(\Omega, X)$, i.e. for any $x' \in X'$, $\langle g(\cdot), x'\rangle \in L^2(\Omega)$. By assumption, we have $|\langle g(t), x'\rangle| = \lim_n \phi_n(t) |\langle g(t), x'\rangle|$ for any $t \in \Omega$, and this convergence is monotone. Then by Beppo Levi's theorem,

$$\|\langle g(\cdot), x'\rangle\|_{L^2(\Omega)} = \lim_n \|\langle \phi_n(\cdot) g(\cdot), x'\rangle\|_{L^2(\Omega)} \leqslant \limsup_n \|\phi_n \cdot g\|_{\gamma(\Omega,X)} \|x'\|.$$

Let us show the last inequality, i.e. $\|\langle f(\cdot), x'\rangle\|_{L^2(\Omega)} \leqslant \|f\|_{\gamma(\Omega,X)}$ for any $f \in \gamma(\Omega, X)$ and $\|x'\| \leqslant 1$. One has $\|f\|_{\gamma(\Omega,X)} = \| \sum_k \gamma_k \otimes u_f(e_k)\|_{\mathrm{Gauss}(X)}$. Choosing the orthonormal basis $(e_k)_k$ of $L^2(\Omega)$ such that $e_1 = \langle f(\cdot), x'\rangle / \|\langle f(\cdot), x'\rangle\|_2$, we deduce from 2.13

$$\|f\|_{\gamma(\Omega,X)} \geqslant \|u_f(e_1)\|_X \geqslant |\langle u_f(e_1), x'\rangle| = \|\langle f(\cdot), x'\rangle\|_{L^2(\Omega)}.$$

Thus we have shown that $g \in P_2(\Omega, X)$. Then by lemma 2.7 (1)

$$\|g\|_{\gamma(\Omega,X)} \leqslant \liminf_n \|\phi_n \cdot g\|_{\gamma(\Omega,X)} \leqslant \sup_n \|\phi_n \cdot g\|_{\gamma(\Omega,X)}.$$

□

Replacing proposition 4.33 and lemma 4.34 by proposition 4.48 and lemma 4.49, we get with the same proof as for theorem 4.46:

Proposition 4.50 *Let X be a Banach space with property (α). Let B be a 0-strip-type operator on X generating the c_0-group $U(t) = e^{itB}$. Assume that B has a bounded $\mathcal{B}^\beta$ calculus for some (large) $\beta > 0$. Let $A = e^B$ be the corresponding 0-sectorial operator and $T(z)$ the analytic semigroup generated by A. Let $\alpha > \frac{1}{2}$. Consider the following conditions.*

Sobolev calculus

(1) *B has a matricially R-bounded W_2^α calculus.*

Imaginary powers

(2) $\|\langle t\rangle^{-\alpha} U(t)x\|_{\gamma(\mathbb{R},dt,X)} = \|\langle t\rangle^{-\alpha} A^{it}x\|_{\gamma(\mathbb{R},dt,X)} \lesssim \|x\| \quad (x \in X).$

Resolvents

(3) $\| \, |a|^{\alpha-\frac{1}{2}} e^{-|a|} R(t+ia,B)x\|_{\gamma(\mathbb{R}\times\mathbb{R}\backslash\{0\},dtda,X)} \lesssim \|x\| \quad (x \in X).$

(4) *For some* $b > 0$, $\| |a|^{\alpha-\frac{1}{2}} R(t+ia,B)x\|_{\gamma(\mathbb{R}\times[-b,b]\backslash\{0\},dtda,X)} \lesssim \|x\| \quad (x \in X).$

(5) *For* $a \neq 0$, $|a|^\alpha \|R(t+ia,B)x\|_{\gamma(\mathbb{R},X)} \lesssim \|x\| \quad (x \in X).$

(6) *For* $\theta \in (-\pi,\pi)\backslash\{0\}$, $|\theta|^\alpha \|A^{1/2}R(e^{i\theta}t,A)x\|_{\gamma(\mathbb{R}_+,dt,X)} \lesssim \|x\| \quad (x \in X).$

(7) *For some* $\theta_0 \in (0,\pi]$, $\| \, |\theta|^{\alpha-\frac{1}{2}} A^{\frac{1}{2}} R(e^{i\theta}t,A)x\|_{\gamma(\mathbb{R}_+\times[-\theta_0,\theta_0],dtd\theta,X)} \lesssim \|x\| \quad (x \in X).$

Analytic semigroup

(8) *For* $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$, $(\frac{\pi}{2} - |\theta|)^\alpha \|A^{1/2}T(e^{i\theta}t)x\|_{\gamma(\mathbb{R}_+,dt,X)} \lesssim \|x\| \quad (x \in X).$

(9) $\|\langle \frac{a}{b}\rangle^\alpha |a|^{-\frac{1}{2}} A^{1/2} T(a+ib)x\|_{\gamma(\mathbb{R}_+\times\mathbb{R},dadb,X)} \lesssim \|x\| \quad (x \in X).$

Wave operator

(10) *For some* $m > \alpha - \frac{1}{2}$, $\| \, |s|^{-\alpha} A^{-\alpha+\frac{1}{2}} (e^{isA} - 1)^m x\|_{\gamma(\mathbb{R},ds,X)} \lesssim \|x\| \quad (x \in D_B).$

Then the conditions $(1),(2),(3),(4),(7),(9),(10)$ *are equivalent. Further, these conditions imply the remaining ones* $(5),(6),(8)$, *which conversely imply that B has a matricially R-bounded $W_2^{\alpha+\varepsilon}$ calculus for any* $\varepsilon > 0$.

Remark 4.51

(1) *Similarly to theorem 4.46, omitting conditions (4) and (10), we could have stated the proposition under the weaker assumption that B has a bounded H^∞ calculus instead of a $\mathcal{B}^\beta$ calculus.*

(2) *Besides the R-boundedness characterizations and the stronger matricial R-boundedness characterizations of the W_2^α calculus, one could also formulate a weaker "bounded" version of proposition 4.50, replacing in (2) - (10) $\gamma(\Omega, X)$ by a "weak L^2 estimate" as e.g. in 4.53 and in (1) matricial R-boundedness by boundedness.*

4.4 Paley-Littlewood decomposition and interpolation spaces

Throughout the section, we let B be a 0-strip-type operator on some Banach space X. We will always assume that B has a $\mathcal{B}^\alpha$ calculus for some $\alpha > 0$. Parallely, we also consider a 0-sectorial operator A having a bounded $\mathcal{M}^\alpha$ calculus.

We will show in theorem 4.53 that the $\mathcal{B}^\alpha$ calculus implies that an abstract Paley-Littlewood decomposition holds for B.

Such decompositions involve partitions of unity, and we fix the following notation.

(1) An equidistant partition of unity (on $\mathbb{R}$) in the sense of definition 4.5 is denoted by $(\varphi_n^{equi})_{n\in\mathbb{Z}}$.

(2) A dyadic partition of unity (on $\mathbb{R}_+$) in the sense of definition 4.5 is denoted by $(\dot{\varphi}_n^{dyad})_{n\in\mathbb{Z}}$.

(3) Let $(\dot{\varphi}_n^{dyad})_{n\in\mathbb{Z}}$ be a dyadic partition of unity. We define the inhomogeneous dyadic partition of unity $(\varphi_n^{dyad})_{n\in\mathbb{N}_0}$ by

$$\varphi_n^{dyad} = \begin{cases} \dot{\varphi}_n^{dyad} & n \geqslant 1 \\ \sum_{k=-\infty}^0 \dot{\varphi}_k^{dyad} & n = 0 \end{cases}. \tag{4.75}$$

Definition 4.5 included that these partitions are C^∞ functions. However, in this section, we only need that they belong to $\mathcal{B}^\beta_{\infty,\infty}$, where $\beta > \alpha$ and α is the index of the $\mathcal{B}^\alpha$ calculus of B.

The classical Paley-Littlewood theorem states that if $A = -\Delta$ is the Laplace operator on $L^p(\mathbb{R}^d)$ for some $1 < p < \infty$, then for any partitions $(\dot{\varphi}_n^{dyad})_{n\in\mathbb{Z}}$ and $(\varphi_n^{dyad})_{n\in\mathbb{N}_0}$ as above, one has the equivalences [123, VI.7.14]

$$\|x\|_{L^p(\mathbb{R}^d)} \cong \left\| \left(\sum_{n\in\mathbb{Z}} |\dot{\varphi}_n^{dyad}(A)x|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \cong \left\| \left(\sum_{n\in\mathbb{N}_0} |\varphi_n^{dyad}(A)x|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)}. \tag{4.76}$$

This decomposition has many applications in harmonic analysis and 4.76 has been generalized to a large variety of operators A. We will show how our results are related to [45, 136, 103].

According to 2.11, for an operator A on L^p with $p < \infty$, we have the equivalence

$$\left\| \left(\sum_{n\in F} |\varphi_n^{dyad}(A)x|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \cong \left\| \sum_{n\in F} \varepsilon_n \otimes \varphi_n^{dyad}(A)x \right\|_{\mathrm{Rad}(L^p)}, \tag{4.77}$$

which is uniform in finite index sets F. If e.g. $\sum_{n=0}^\infty \varphi_n^{dyad}(A)x$ converges unconditionally in L^p, then the right hand side is uniformly bounded in F, so that by monotone convergence, $\left(\sum_{n\in\mathbb{N}_0} |\varphi_n^{dyad}(A)x|^2\right)^{\frac{1}{2}}$ belongs to L^p and 4.77 holds for $\mathbb{N}_0$ in place of F. Of course, a similar statement of 4.77 is true for $\dot{\varphi}_n^{dyad}(A)x$.

We will deal with expressions as the right hand side of 4.77 instead of the left hand side. In subsection 4.4.1, after establishing 4.76 for operators on X having a bounded $\mathcal{B}^\alpha$ calculus, we give an extension, where the norm of $\|g(B)x\|$ instead of $\|x\|$ is decomposed. Here, $g(B)$ arises from the $\mathcal{B}^{\alpha}{}_{\text{loc}}$ calculus of B. We will see that a norm equivalence similar to 4.76 holds, where the summands on the right hand side are weighted by the local Besov norms of g.

In particular, if $g(t) = 2^{t\theta}$ for some $\theta \in \mathbb{R}$, $g(B)$ equals the fractional power of a sectorial operator. Consequently, in subsection 4.4.2, we obtain characterizations of fractional domain spaces, and in 4.4.3, we discuss their real interpolation spaces. Finally, in subsection 4.4.4, we show how the Paley-Littlewood decomposition 4.76 (see 4.81 for its more general form) can be applied to the functional calculus. We will see that the spectral localization allows to pass from the Sobolev spaces W^β_p on $\mathbb{R}$ or $\mathbb{R}_+$ to the "localized" counterparts $\mathcal{W}^\beta_p$ and $\mathcal{H}^\beta_p$.

4.4.1 Spectral decompositions

The basic tool for all our considerations is the following lemma.

Lemma 4.52 *Let $\beta > 0$ and $(g_n)_n$ be a bounded sequence in $\mathcal{B}^\beta_{\infty,\infty}$ such that the supports satisfy the following overlapping condition:*

There exist $a > 1$ and $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}$: $\#\{n : \operatorname{supp} g_n \cap [x-a, x+a] \neq \varnothing\} \leqslant N$.

Then $\sum_{n=1}^\infty g_n \in \mathcal{B}^\beta_{\infty,\infty}$ and

$$\|\sum_n g_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \sup_n \|g_n\|_{\mathcal{B}^\beta_{\infty,\infty}}. \tag{4.78}$$

Proof. According to [128, thm 2.5.12], $\mathcal{B}^\beta_{\infty,\infty}$ has the equivalent norm

$$\|g\|_{\mathcal{B}^\beta_{\infty,\infty}} \cong \|g\|_{L^\infty(\mathbb{R})} + \sup_{|h|\in(0,\delta)} |h|^{-\beta} \|\Delta^M_h g\|_\infty. \tag{4.79}$$

Here, $\delta > 0$ and $M > \alpha$ are fixed and Δ^M_h is the iterated difference from definition 4.2. Note that $\triangle^M_h g(x)$ depends only on $g(x), g(x+1\cdot h), \ldots, g(x+Mh)$. Thus, by the overlapping assumption of the lemma, for $|h| \leqslant \frac{a}{M}$, there exist $g_{j_1}, \ldots, g_{j_N}$ such that

$$|\triangle^M_h(\sum_n g_n)(x)| = |\triangle^M_h(\sum_{j=1}^N g_{n_j})(x)| = |\sum_{j=1}^N \triangle^M_h g_{n_j}(x)| \leqslant N \sup_n \|\triangle^M_h g_n\|_\infty.$$

Hence

$$\sup_{|h|\in(0,\delta)} |h|^{-\beta} \|\Delta^M_h \sum_n g_n\|_\infty \leqslant N \sup_{|h|\in(0,\delta)} \sup_n |h|^{-\beta} \|\Delta^M_h g_n\|_\infty.$$

Similarly, $\sup_{x\in\mathbb{R}} |\sum_n g_n(x)| \leqslant N \sup_n \|g_n\|_\infty$, so that by 4.79

$$\|\sum_n g_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \sup_n \|g_n\|_\infty + \sup_n \sup_{|h|\in(0,\delta)} |h|^{-\beta} \|\Delta^M_h g_n\|_\infty \cong \sup_n \|g_n\|_{\mathcal{B}^\beta_{\infty,\infty}}.$$

□

Assume now that B has a $\mathcal{B}^\beta_{\infty,\infty}$ calculus and g_n is a sequence as above. Then for any choice of signs $a_n = \pm 1$, we have by the above lemma

$$\| \sum_n a_n g_n(B)x\|_X \lesssim \sup_n |a_n| \, \|g_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \|x\| \lesssim \|x\|. \tag{4.80}$$

This observation leads to the following theorem of Paley-Littlewood type.

Theorem 4.53 (Paley-Littlewood decomposition) *Let B be a 0-strip-type operator on some Banach space X having a $\mathcal{B}^\alpha$ calculus for some $\alpha > 0$. Let further $(\varphi_n^{equi})_n$ be an equidistant partition of unity.*

(1) *The norm on X has an equivalent description:*

$$\|x\| \cong \| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \varphi_n^{equi}(B)x\|_{\mathrm{Rad}(X)} \cong \sup\left\{ \| \sum_{n\in\mathbb{Z}} a_n \varphi_n^{equi}(B)x\| : \, |a_n| \leqslant 1 \right\}. \tag{4.81}$$

The claim includes that for any $|a_n| \leqslant 1$ and $x \in X$, $\sum_{n\in\mathbb{Z}} a_n \varphi_n^{equi}(B)x$ converges in X.

(2) *For any $f \in \mathcal{B}^\alpha$ and $x \in X$,*

$$f(B)x = \sum_{n\in\mathbb{Z}} (f\varphi_n^{equi})(B)x \tag{4.82}$$

converges unconditionally in X. Moreover, if N denotes the norm of the $\mathcal{B}^\alpha$ calculus,

$$R(\{(f\varphi_n^{equi})(B) : \, n \in \mathbb{Z}\}) \leqslant C_{\alpha,\beta} N^2 \|f\|_{\mathcal{B}^\alpha}. \tag{4.83}$$

Proof. We write in short $\varphi_n = \varphi_n^{equi}$.

(1) Choose some $\beta > \alpha$. Then B has a $\mathcal{B}^\beta_{\infty,\infty}$ calculus. Let $(a_n)_n$ be a sequence such that $|a_n| \leqslant 1$. Then $g_n = a_n \varphi_n \in C_c^\infty \subset \mathcal{B}^\beta_{\infty,\infty}$ satisfies the assumptions of lemma 4.52, so that $\varphi_{(a_n)} := \sum_{n\in\mathbb{Z}} a_n \varphi_n \in \mathcal{B}^\beta_{\infty,\infty}$ and

$$\|\varphi_{(a_n)}\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \sup_n \|a_n \varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \leqslant \sup_n \|\varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}} = \|\varphi_0\|_{\mathcal{B}^\beta_{\infty,\infty}}.$$

By the same argument, also the partial sums $\sum_{n=-N}^M a_n \varphi_n$ are bounded in $\mathcal{B}^\beta_{\infty,\infty}$, so that by the Convergence Lemma 4.19, $\sum_{n=-N}^M a_n \varphi_n(B)x$ converges to $\varphi_{(a_n)}(B)x$ as $N, M \to \infty$. Thus,

$$\| \sum_{n\in\mathbb{Z}} a_n \varphi_n(B)x\| = \|\varphi_{(a_n)}(B)x\| \lesssim \|\varphi_{(a_n)}\|_{\mathcal{B}^\beta_{\infty,\infty}} \|x\| \lesssim \|\varphi_0\|_{\mathcal{B}^\beta_{\infty,\infty}} \|x\|.$$

Since $|\varepsilon_n(\omega)| = 1$ for any $n \in \mathbb{Z}$ and $\omega \in \Omega_0$, the estimate

$$\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \varphi_n(B)x\|_{\mathrm{Rad}(X)} \leqslant \sup\left\{ \| \sum_{n\in\mathbb{Z}} a_n \varphi_n(B)x\| : \, |a_n| \leqslant 1 \right\}$$

is clear. It only remains to show

$$\|x\| \lesssim \| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \varphi_n(B)x\|_{\mathrm{Rad}(X)}. \tag{4.84}$$

By corollary 4.20, $x = \lim_N \sum_{n=-N}^N \varphi_n(B)x$. Let $x' \in X^\# \subset X'$ from subsection 4.2.7. Let $(\varepsilon_n)_{n\in\mathbb{Z}}$ be a sequence of independent Rademacher variables. Then for any $c \in \mathbb{C}$, $\mathbb{E}(\varepsilon_n\varepsilon_k c) = \delta_{n=k}c$, and

$$\begin{aligned}|\langle x, x'\rangle| &= |\sum_n \langle \varphi_n(B)x, x'\rangle| = |\mathbb{E}\sum_{n,k}\varepsilon_n\varepsilon_k\langle\varphi_n(B)x, \widetilde{\varphi}_k(B)'x'\rangle| \\ &= |\mathbb{E}\langle\sum_n \varepsilon_n\varphi_n(B)x, \sum_k \varepsilon_k\widetilde{\varphi}_k(B^\#)x'\rangle| \\ &\leqslant \|\sum_n \varepsilon_n \otimes \varphi_n(B)x\|_{\mathrm{Rad}(X)}\|\sum_k \varepsilon_k \otimes \widetilde{\varphi}_k(B^\#)x'\|_{\mathrm{Rad}(X')}.\end{aligned}$$

As the moon dual operator $B^\#$ has by proposition 4.30 also a $\mathcal{B}^\alpha$ calculus, we can repeat the above argument and get

$$\|\sum_k \varepsilon_k \otimes \widetilde{\varphi}_k(B^\#)x'\|_{\mathrm{Rad}(X')} \lesssim \|x'\|.$$

Since $X^\# \subset X'$ norms X by proposition 4.29, we finally deduce 4.84.

(2) By the convergence property of the $\mathcal{B}^\beta_{\infty,\infty}$ calculus from corollary 4.20, $\sum_n (f\varphi_n)(B)x = \sum_n \varphi_n(B)[f(B)x]$ converges to $f(B)x$. Further, this convergence is unconditional, since for any finitely supported sequence $(a_n)_n$ with $|a_n| \leqslant 1$,

$$\|\sum_n a_n(f\varphi_n)(B)x\|_X \lesssim \|\sum_n a_n\varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}}\|f(B)x\|_X \lesssim \sup_n \|\varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}}\|f(B)x\|_X$$

according to lemma 4.52.

According to [48, lem 3.12], for any X-valued array $(x_{kl})_{k,l\in\mathbb{Z}}$, one has

$$\Big\|\sum_k \varepsilon_k \otimes x_{kk}\Big\|_{\mathrm{Rad}(X)} \lesssim \Big\|\sum_{k,l}\varepsilon_k\otimes\varepsilon_l\otimes x_{kl}\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))}.$$

Pick some finite family $x_1, \ldots, x_N \in X$ and set $x_{kl} = \varphi_k(B)x_l$. Then

$$\begin{aligned}\Big\|\sum_k \varepsilon_k \otimes (\varphi_k f)(B)x_k\Big\|_{\mathrm{Rad}(X)} &\lesssim \Big\|\sum_{k,l}\varepsilon_k\otimes\varepsilon_l\otimes(\varphi_k f)(B)x_l\Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \\ &\cong \int_{\Omega_0}\Big\|\sum_k \varepsilon_k \otimes (\varphi_k f)(B)[\sum_l \varepsilon_l(\lambda)x_l]\Big\|_{\mathrm{Rad}(X)} d\lambda \\ &\overset{(*)}{\lesssim} N\int_{\Omega_0}\Big\|f(B)[\sum_l \varepsilon_l(\lambda)x_l]\Big\|_X d\lambda \\ &\leqslant N^2\|f\|_{\mathcal{B}^\alpha}\int_{\Omega_0}\Big\|\sum_l \varepsilon_l(\lambda)x_l\Big\|_X d\lambda \\ &\cong N^2\|f\|_{\mathcal{B}^\alpha}\Big\|\sum_l \varepsilon_l\otimes x_l\Big\|_{\mathrm{Rad}(X)}.\end{aligned}$$

In $(*)$, we have used (1) of the theorem with $f(B)\sum_l \varepsilon_l(\lambda)x_l$ instead of x. This shows that $\{(f\varphi_n)(B) : n \in \mathbb{Z}\}$ is R-bounded. □

Remark 4.54

(1) *The identity 4.82 is an analogue to the spectral decomposition of selfadjoint operators on Hilbert spaces. The spectral projections, which correspond to intervals* $[a, b)$, *are here replaced by a smoothed spectral expansion* $\varphi_n^{equi}(B)$ *with the properties*

$$\sum_n \varphi_n^{equi}(B) = \mathrm{Id}_X, \quad \varphi_n^{equi}(B)\varphi_m^{equi}(B) = 0 \quad (|n-m| \geqslant 2)$$
$$\text{and } \varphi_n^{equi}(B)f(B) = f(B)\varphi_n^{equi}(B).$$

(2) *Compare the* R*-boundedness 4.83 of the theorem to corollary 3.26. There we had seen that if* X *has property* (α), *then any bounded homomorphism* $u : C(K) \to B(X)$ *is* R*-bounded, i.e. satisfies*

$$R(\{u(f) : \|f\|_\infty \leqslant 1\}) < \infty.$$

If $C(K)$ *is the space of bounded and uniformly continuous functions, and* u *is given by the functional calculus of* B, *then this clearly implies 4.83, i.e.*

$$R(\{u(f\varphi_n^{equi}) : n \in \mathbb{Z}\}) \lesssim \|f\|_\infty.$$

Conversely, we do not know if every bounded $\mathcal{B}^\beta_{\infty,\infty}$ *calculus or* $\mathcal{B}^\alpha$ *calculus over a Banach space* X *with property* (α) *is* R*-bounded.*

The norm equivalence for $\|\mathrm{Id}_X(x)\|$ in part (1) of theorem 4.53 can be extended to more general operators $g(B)$ instead of Id_X, if $g \in \mathcal{B}^\beta_{\mathrm{loc}}$ doesn't vary too much on intervals $[n-1, n+1]$, $n \in \mathbb{Z}$. This is the content of the next proposition.

Proposition 4.55 *Let* B *be a* 0*-strip-type operator having a* $\mathcal{B}^\alpha$ *calculus and let* $\beta > \alpha$. *Further let* $g \in \mathcal{B}^\beta_{\mathrm{loc}}$ *(see definition 4.6) such that* g *is invertible and* g^{-1} *also belongs to* $\mathcal{B}^\beta_{\mathrm{loc}}$. *Assume that*

$$\sup_{n\in\mathbb{Z}} \|\widetilde{\varphi_n^{equi}} g\|_{\mathcal{B}^\beta_{\infty,\infty}} \cdot \|\varphi_n^{equi} g^{-1}\|_{\mathcal{B}^\beta_{\infty,\infty}} < \infty. \tag{4.85}$$

Let $(c_n)_{n\in\mathbb{Z}}$ *be a sequence in* $\mathbb{C}\backslash\{0\}$ *satisfying*

$$|c_n| \cong \|\widetilde{\varphi_n^{equi}} g\|_{\mathcal{B}^\beta_{\infty,\infty}}.$$

Then for any $x \in D(g(B))$, $\sum_{n\in\mathbb{Z}} c_n \varphi_n^{equi}(B)x$ *converges unconditionally in* X *and*

$$\|g(B)x\| \cong \Big\|\sum_n \varepsilon_n \otimes c_n \varphi_n^{equi}(B)x\Big\|_{\mathrm{Rad}(X)} \cong \sup\left\{\Big\|\sum_n a_n c_n \varphi_n^{equi}(B)x\Big\|_X : |a_n| \leqslant 1\right\}. \tag{4.86}$$

Proof. Write in short $\varphi_n = \varphi_n^{equi}$. Let us show the unconditional convergence of $\sum_n c_n\varphi_n(B)x$ for $x \in D(g(B))$. We have by proposition 4.25 (c) $\varphi_n(B)x = (g^{-1}\varphi_n)(B)g(B)x$. Thus for any choice of scalars $|a_n| \leqslant 1$,

$$\sum_{n=-N}^{N} a_n c_n \varphi_n(B)x = \left[\sum_{n=-N}^{N} a_n c_n (g^{-1}\varphi_n)\right](B)g(B)x.$$

The term in brackets is a sequence of functions indexed by N which clearly converges pointwise for $N \to \infty$. It is also uniformly bounded in $\mathcal{B}^\beta_{\infty,\infty}$, because

$$\|\sum_{-N}^{N} a_n c_n g^{-1}\varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \sup_n |c_n| \, \|g^{-1}\varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \sup_n \|g\widetilde{\varphi}_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \, \|g^{-1}\varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}} < \infty \tag{4.87}$$

by assumption 4.85. Thus, the Convergence Lemma 4.19 yields the unconditional convergence. Estimate 4.87 also shows that

$$\|\sum_n a_n c_n \varphi_n(B)x\| = \|\sum_n a_n c_n (g^{-1}\varphi_n)(B) g(B)x\| \lesssim \sup_n |c_n| \, \|g^{-1}\varphi_n\|_{\mathcal{B}^\beta_{\infty,\infty}} \|g(B)x\|$$

for any choice of scalars $|a_n| \leqslant 1$, so that one inequality in 4.86 is shown.

For the reverse inequality, we argue as in the proof of theorem 4.53: For any $x \in D(g(B))$ and any $x' \in X^\# \subset X'$,

$$\begin{aligned}\langle g(B)x, x'\rangle &= \sum_n \langle c_n\varphi_n(B)x, c_n^{-1}\widetilde{\varphi}_n(B^\#)x'\rangle \\ &\leqslant \Big\|\sum_n \varepsilon_n \otimes c_n\varphi_n(B)x\Big\|_{\mathrm{Rad}(X)} \Big\|\sum_k \varepsilon_k \otimes c_k^{-1}(g\widetilde{\varphi}_k)(B^\#)x'\Big\|_{\mathrm{Rad}(X^\#)}.\end{aligned}$$

Since $B^\#$ has again a $\mathcal{B}^\alpha$ calculus, for any scalars $|a_k| \leqslant 1$, we have, similarly to 4.87,

$$\Big\|\sum_k a_k c_k^{-1}(g\widetilde{\varphi}_k)(B^\#)x'\Big\|_{X^\#} \lesssim \Big\|\sum_k a_k c_k^{-1} g\widetilde{\varphi}_k\Big\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \sup_k |c_k|^{-1}\|g\widetilde{\varphi}_k\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim 1$$

for $\|x'\| \leqslant 1$, where we use the assumption 4.85 in the last step. This shows $\Big\|\sum_k \varepsilon_k \otimes c_k^{-1}(g\widetilde{\varphi}_k)(B^\#)x'\Big\|_{\mathrm{Rad}(X^\#)} \lesssim \|x'\|_{X^\#}$, and thus, 4.86 follows. □

Remark 4.56 *A variant of proposition 4.55 is the following fact: Let B have a $\mathcal{B}^\alpha$ calculus and $(\varphi_n)_n$ be an equidistant partition of unity. Let $x \in X$ and $(d_n)_{n\in\mathbb{Z}}$ be a scalar sequence such that $\sum_n d_n\varphi_n(B)x$ converges unconditionally. Let $(c_n)_{n\in\mathbb{Z}}$ be a further scalar sequence with*

$$|c_n| \lesssim |d_n| \quad (n \in \mathbb{Z}). \tag{4.88}$$

Then also $\sum_n c_n\varphi_n(B)x$ converges unconditionally and

$$\Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n \otimes c_n\varphi_n(B)x\Big\|_{\mathrm{Rad}(X)} \lesssim \Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n \otimes d_n\varphi_n(B)x\Big\|_{\mathrm{Rad}(X)}. \tag{4.89}$$

Indeed, let $|a_n| \leqslant 1$, and choose a finite subset $F \subset \mathbb{Z}$. Writing $\sum' \,/\, \sum''$ for summation over even/odd indices, we have $\|\sum_{n\in F} a_n c_n \varphi_n(B)x\| \leqslant \|\sum'_{n\in F} a_n c_n\varphi_n(B)x\| + \|\sum''_{n\in F} a_n c_n\varphi_n(B)x\|$. Then for k, n both even numbers, $\widetilde{\varphi}_k\varphi_n = \delta_{n=k}\varphi_n$, and therefore

$$\begin{aligned}\|\sum_{n\in F}{}' a_n c_n\varphi_n(B)x\| &= \|\sum_{k\in F}{}'\sum_{n\in F}{}' a_n \frac{c_k}{d_k}\widetilde{\varphi}_k(B) d_n\varphi_n(B)x\| \\ &= \|\left(\sum_{k\in\mathbb{Z}}{}' \frac{c_k}{d_k}\widetilde{\varphi}_k(B)\right)\sum_{n\in F}{}' a_n d_n\varphi_n(B)x\| \\ &\leqslant \|\sum_{k\in\mathbb{Z}}{}' \frac{c_k}{d_k}\widetilde{\varphi}_k(B)\| \, \|\sum_{n\in F}{}' a_n d_n\varphi_n(B)x\|,\end{aligned}$$

where $\sum'_{k\in\mathbb{Z}} \frac{c_k}{d_k}\widetilde{\varphi}_k(B)$ *is a bounded operator due to the assumption 4.88 and lemma 4.52. A similar estimate holds for* $\sum''_F a_n c_n \varphi_n(B)x$. *This shows that* $\sum_n c_n\varphi_n(B)x$ *converges unconditionally, and it also shows 4.89.*

4.4.2 Fractional powers of 0-sectorial operators

The Paley-Littlewood decomposition in the form of proposition 4.55 can be used to characterize the domains of fractional powers of a 0-sectorial operator A, and moreover to interpolate between them.

We give a short overview on fractional powers. Let A be a 0-sectorial operator and $\theta \in \mathbb{C}$. Since $\lambda \mapsto \lambda^\theta \in \mathrm{Hol}(\Sigma_\omega)$ for $\omega > 0$, A^θ is defined by the extended holomorphic calculus from proposition 2.3. If $B = \log(A)$ has a $\mathcal{B}^\alpha$ calculus, then $A^\theta = e^{\theta(\cdot)}(B)$ is also given by the $\mathcal{B}^\alpha_{\mathrm{loc}}$ calculus from proposition 4.25. To A^θ, one associates the following two scales of extrapolation spaces ([81, def 15.21, lem 15.22], see also [69, sec 2] and [41, II.5])

$$\dot{X}_\theta = (D(A^\theta), \|A^\theta \cdot\|_X)^{\tilde{}} \qquad (\theta \in \mathbb{C})$$
$$\text{and } X_\theta = (D(A^\theta), \|A^\theta \cdot\|_X + \|\cdot\|_X)^{\tilde{}} \quad (\mathrm{Re}\,\theta \geqslant 0).$$

Here, $\tilde{}$ denotes completion with respect to the indicated norm. Clearly, $X = \dot{X}_0 = X_0$. If $A = -\Delta$ is the Laplace operator on $L^p(\mathbb{R}^d)$, then $\dot{X}_\theta$ is the Riesz or homogeneous potential space, whereas X_θ is the Bessel or inhomogeneous potential space.

For two different values of θ, the completions can be realized in a common space. More precisely, if $m \in \mathbb{N}$, $m \geqslant \max(|\theta_0|, |\theta_1|)$, then $\dot{X}_{\theta_j}$ and X_{θ_j} can be viewed as subspaces of

$$(X, \|(A(1+A)^{-2})^m \cdot\|_X)^{\tilde{}} \tag{4.90}$$

for $j = 0, 1$. Then for $\theta > 0$, one has $X_\theta = \dot{X}_\theta \cap X$ with equivalent norms [81, prop 15.25 and 15.26]. Thus, $\{\dot{X}_{\theta_0}, \dot{X}_{\theta_1}\}$ and $\{X_{\theta_0}, X_{\theta_1}\}$ form an interpolation couple. We will pursue this in the next subsection.

It is known that if A has bounded imaginary powers, so that $\|A^\theta \cdot\| \cong \|A^{\mathrm{Re}\,\theta} \cdot\|$, we have for the complex interpolation method

$$[\dot{X}_{\theta_0}, \dot{X}_{\theta_1}]_r = \dot{X}_{(1-r)\theta_0 + r\theta_1}$$

for any $\theta_0, \theta_1 \in \mathbb{R}$ and $r \in (0, 1)$ [69, prop 2.2]. The connection of complex interpolation of the spaces $\dot{X}_\theta$ and the H^∞ functional calculus has been studied e.g. in [69].

Let $(\varphi_n^{equi})_{n\in\mathbb{Z}}$ be an equidistant partition of unity and $(\dot{\varphi}_n^{dyad})_{n\in\mathbb{Z}}$ the dyadic partition of unity given by $\dot{\varphi}_n^{dyad}(2^t) = \varphi_n^{equi}(t)$. Then for $A = 2^B$, $\dot{\varphi}_n^{dyad}(A) = \varphi_n^{equi}(B)$, and proposition 4.55 can be transferred from B to A. We obtain as a special case:

Proposition 4.57 *Let* A *be a* 0*-sectorial operator having a bounded* $\mathcal{M}^\alpha$ *calculus for some* $\alpha > 0$. *Let further* $(\dot{\varphi}_n^{dyad})_{n\in\mathbb{Z}}$ *be a dyadic partition of unity and* $(\varphi_n^{dyad})_{n\in\mathbb{N}_0}$ *be the corresponding inhomogeneous dyadic partition from 4.75. Then for* $\theta \in \mathbb{R}$,

$$\|A^\theta x\| \cong \Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes 2^{n\theta} \dot{\varphi}_n^{dyad}(A)x \Big\|_{\mathrm{Rad}(X)} \quad (x \in D(A^\theta)) \tag{4.91}$$

and for $\theta > 0$,

$$\|A^\theta x\| + \|x\| \cong \Big\| \sum_{n \in \mathbb{N}_0} \varepsilon_n \otimes 2^{n\theta} \varphi_n^{dyad}(A)x \Big\|_{\mathrm{Rad}(X)} \quad (x \in D(A^\theta)). \tag{4.92}$$

Proof. Let $\beta > \alpha$ and $g \in \mathcal{B}^\beta_{\infty,\infty}$ be given by $g(t) = 2^{t\theta}$. Recall the embedding $C^m_b \hookrightarrow \mathcal{B}^\beta_{\infty,\infty} \hookrightarrow C^0_b$ for $m \in \mathbb{N}$, $m > \beta > 0$ from proposition 4.9. We have for all $n \in \mathbb{Z}$

$$2^{n\theta} \leqslant \|g\widetilde{\varphi_n^{equi}}\|_{C^0_b} \lesssim \|g\widetilde{\varphi_n^{equi}}\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \|g\widetilde{\varphi_n^{equi}}\|_{C^m_b} \lesssim 2^{n\theta}$$

and $\|g^{-1}\varphi_n^{equi}\|_{\mathcal{B}^\beta_{\infty,\infty}} \lesssim \|g^{-1}\varphi_n^{equi}\|_{C^m_b} \lesssim 2^{-n\theta}$. (Here, equivalence constants may depend on θ.) Consequently, $\sup_{n\in\mathbb{Z}} \|g\widetilde{\varphi_n^{equi}}\|_{\mathcal{B}^\beta_{\infty,\infty}} \|g^{-1}\varphi_n^{equi}\|_{\mathcal{B}^\beta_{\infty,\infty}} < \infty$. Put $B = \log(A)/\log(2)$. Then B has a $\mathcal{B}^\alpha$ calculus, and $g(B) = A^\theta$ and $\varphi_n^{equi}(B) = \dot{\varphi}_n^{dyad}(A)$. By proposition 4.55, we have with $c_n = 2^{n\theta}$,

$$\|A^\theta x\| = \|g(B)x\| \cong \Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n \otimes 2^{n\theta}\varphi_n^{equi}(B)x\Big\|_{\mathrm{Rad}(X)} = \Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n \otimes 2^{n\theta}\dot{\varphi}_n^{dyad}(A)x\Big\|_{\mathrm{Rad}(X)}$$

for $x \in D(A^\theta)$, so that 4.91 follows.

By [81, lem 15.22] (set $A = A^\theta$ and $\alpha = 1$ there), the left hand side of 4.92 satisfies

$$\|A^\theta x\| + \|x\| \cong \|(1 + A^\theta)x\| \quad (x \in D(A^\theta)), \tag{4.93}$$

whereas by proposition 2.6 (8), the right hand side of 4.92 is equivalent to

$$\Big\|\sum_{n\geqslant 1} \varepsilon_n \otimes 2^{n\theta}\varphi_n^{dyad}(A)x\Big\|_{\mathrm{Rad}(X)} + \|\varphi_0^{dyad}(A)x\|. \tag{4.94}$$

"$\lesssim$" in 4.92: We use the equivalent expressions from 4.93 and 4.94. We set $g(t) = 1 + 2^{t\theta}$. Then one checks similarly to the first part that $\|g\widetilde{\varphi_n^{equi}}\|_{\mathcal{B}^\beta_{\infty,\infty}} \cong \max(1, 2^{n\theta})$ for $n \in \mathbb{Z}$ and that $\sup_{n\in\mathbb{Z}} \|g\widetilde{\varphi_n^{equi}}\|_{\mathcal{B}^\beta_{\infty,\infty}} \|g^{-1}\varphi_n^{equi}\|_{\mathcal{B}^\beta_{\infty,\infty}} < \infty$. Thus, by proposition 4.55,

$$\begin{aligned}\|(1 + A^\theta)x\| = \|g(B)x\| &\cong \Big\|\sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \max(1, 2^{n\theta})\dot{\varphi}_n^{dyad}(A)x\Big\|_{\mathrm{Rad}(X)} \\ &\leqslant \Big\|\sum_{n\leqslant -1} \dots\Big\|_{\mathrm{Rad}(X)} + \Big\|\sum_{n\geqslant 1}\dots\Big\|_{\mathrm{Rad}(X)} + \|\dot{\varphi}_0^{dyad}(A)x\|. \end{aligned} \tag{4.95}$$

We estimate the three summands. By 4.75, we have $\dot{\varphi}_n^{dyad}\varphi_0^{dyad} = \dot{\varphi}_n^{dyad}$ for any $n \leqslant -1$, so that

$$\Big\|\sum_{n\leqslant -1} \varepsilon_n \otimes \dot{\varphi}_n^{dyad}(A)x\Big\| = \Big\|\sum_{n\leqslant -1} \varepsilon_n \otimes \dot{\varphi}_n^{dyad}(A)\varphi_0^{dyad}(A)x\Big\| \lesssim \|\varphi_0^{dyad}(A)x\|,$$

where we use 4.81 from the Paley-Littlewood decomposition in the last step. Since $\dot{\varphi}_n^{dyad} = \varphi_n^{dyad}$ for $n \geqslant 1$, also the second summand is controlled by 4.94. Finally, we have $\dot{\varphi}_0^{dyad} = \dot{\varphi}_0^{dyad}[\varphi_0^{dyad} + \dot{\varphi}_1^{dyad}]$, so that

$$\|\dot{\varphi}_0^{dyad}(A)x\| \leqslant \|\varphi_0^{dyad}(A)x\| + \|\dot{\varphi}_0^{dyad}(A)\dot{\varphi}_1^{dyad}(A)x\| \lesssim \|\varphi_0^{dyad}(A)x\| + \|\dot{\varphi}_1^{dyad}(A)x\|,$$

and the last term is controlled by the second summand of 4.95. This shows "$\lesssim$" in 4.92.

"$\gtrsim$" in 4.92: We use again the expression in 4.94. By proposition 2.6 (8) and the first part of the corollary,

$$\Big\|\sum_{n\geqslant 1}\varepsilon_n\otimes 2^{n\theta}\varphi_n^{dyad}(A)x\Big\|_{\mathrm{Rad}(X)}\leqslant\Big\|\sum_{n\in\mathbb{Z}}\varepsilon_n\otimes 2^{n\theta}\dot\varphi_n^{dyad}(A)x\Big\|_{\mathrm{Rad}(X)}\lesssim\|A^\theta x\|.$$

Finally, $\|\varphi_0^{dyad}(A)x\|\lesssim\|x\|$ because φ_0^{dyad} belongs to $\mathcal{M}^\alpha$. □

Remark 4.58 *The operator $\dot\varphi_n^{dyad}(A):X\to X$ can be continuously extended to $\dot X_\theta\to\dot X_\theta$. Indeed, by proposition 4.25, for $x\in D(A^\theta)$,*

$$\|\dot\varphi_n^{dyad}(A)x\|_{\dot X_\theta}=\|A^\theta\dot\varphi_n^{dyad}(A)x\|_X=\|\dot\varphi_n^{dyad}(A)A^\theta x\|_X\leqslant\|\dot\varphi_n^{dyad}(A)\|_{X\to X}\|A^\theta x\|_X.$$

One even has $\dot\varphi_n^{dyad}(A)(\dot X_\theta)\subset X$. Then by the density of $D(A^\theta)$ in $\dot X_\theta$, 4.91 can be extended to all of $\dot X_\theta$ in the following way: There exists $C>0$ such that

$$\frac{1}{C}\|x\|_{\dot X_\theta}\leqslant\sup_{F\subset\mathbb{Z}\text{ finite}}\Big\|\sum_{n\in F}\varepsilon_n\otimes 2^{n\theta}\dot\varphi_n^{dyad}(A)x\Big\|_{\mathrm{Rad}(X)}\leqslant C\|x\|_{\dot X_\theta}\quad(x\in\dot X_\theta).$$

Remark 4.59 *Let us compare proposition 4.57 to some results in the literature.*

(1) *Consider the situation in [45]. There, $A=\mathcal{L}$ is the sub-Laplacian on a connected Lie group G of polynomial growth, considered as an operator on $L^p(G)$ for some $1<p<\infty$. By [1, thm], A has a $\mathcal{M}^\alpha$ calculus for suitable α (see also illustration 4.87 (2) below). Thus by the equivalence in 4.77 and the comments after, one can deduce the Paley-Littlewood theorem in [45, thm 4.4] from 4.91 with $\theta=0$.*

(2) *Consider a Schrödinger operator $A=-\Delta+V$ on $L^p(\mathbb{R}^d)$, where V is a potential function subject to further conditions. In [136] and [103], abstract Triebel-Lizorkin spaces $\dot F_p^{\alpha,q}$ and $F_p^{\alpha,q}$ associated with A are considered. For $q=2$, their norms are defined by the right hand sides of 4.91 and 4.92 (with $\alpha=\theta$ and using 4.77). Thus, proposition 4.57 shows that these spaces coincide with fractional domain spaces of A, once a $\mathcal{M}^\alpha$ calculus is established. For the latter, see e.g. remark 4.94 (2) in section 4.6.*

(3) *In 4.91, instead of a partition $\dot\varphi_n^{dyad}$ summing up to 1, one can also use a partition $(\varphi_n)_{n\in\mathbb{Z}}$ such that $\sum_{n\in\mathbb{Z}}\varphi_n=\Phi$ satisfies $\inf_{t>0}|\Phi(t)|>0$ and Φ_e belongs to $\mathcal{B}^\beta_{\infty,\infty}$. Indeed, the estimate "$\gtrsim$" in 4.91 is proved in the same way, and to show "$\lesssim$", one can use a dual dyadic partition $(\psi_n)_{n\in\mathbb{Z}}$ such that $\psi_{n,e}\in\mathcal{B}^\beta_{\infty,\infty}$ and $\sum_{n\in\mathbb{Z}}\psi_n=\Phi^{-1}$. Note that $\inf_t|\Phi(t)|>0$ implies that Φ_e^{-1} also belongs to $\mathcal{B}^\beta_{\infty,\infty}$.*

A partition of this type is considered e.g. in [136].

The next goal is a continuous variant of the above result. We state the following preparatory lemma, whose H^∞ calculus variant is a well-known result of McIntosh (see [96], [81, lem 9.13], [52, thm 5.2.6]).

Lemma 4.60 *Let A be a* 0*-sectorial operator having a bounded $\mathcal{M}^\alpha$ calculus and $D \subset X$ be its calculus core from 4.33. Let further $g : (0, \infty) \to \mathbb{C}$ be a function with compact support (not containing* 0*) such that $g \circ \exp \in \mathcal{B}^\beta_{\infty,\infty}$ for some $\beta > \alpha$. Assume that $\int_0^\infty g(t) \frac{dt}{t} = 1$. Then for any $x \in D$,*

$$x = \int_0^\infty g(tA)x \frac{dt}{t}. \tag{4.96}$$

Proof. Let $x \in D$. Then there exists $\rho \in C_c^\infty(\mathbb{R}_+)$ such that $\rho(A)x = x$. As g has by assumption compact support, there exist $b > a > 0$ such that

$$g(tA)\rho(A) = 0 \quad (t \in [a, b]^c). \tag{4.97}$$

By corollary 4.21, $t \mapsto g(tA)x$ is continuous, and 4.97 implies that $\int_0^\infty g(tA)x \frac{dt}{t} = \int_a^b g(tA)x \frac{dt}{t}$. Also,

$$\left(\int_a^b g(t\cdot) \frac{dt}{t} \right)(A)x = \left(\int_a^b g(t\cdot) \frac{dt}{t} \rho \right)(A)x = \left(\int_0^\infty g(t\cdot) \frac{dt}{t} \rho \right)(A)x = \rho(A)x = x.$$

In the third equality we have used the assumption $\int_0^\infty g(t) \frac{dt}{t} = 1$, which extends by substitution to

$$\int_0^\infty g(ts) \frac{dt}{t} = 1 \quad (s > 0).$$

It remains to show

$$\left(\int_a^b g(t\cdot) \frac{dt}{t} \right)(A)x = \int_a^b g(tA)x \frac{dt}{t}. \tag{4.98}$$

If g belongs to $H^\infty(\Sigma_\omega)$ for some ω, then this is shown in [81, lem 9.12]. For a general g, we use the approximation from lemma 4.15. More precisely, set $f = g_e$ and let $f_n = f * \check{\psi}_n$, where $(\psi_n)_{n \in \mathbb{N}}$ is the sequence in $C_c^\infty(\mathbb{R})$ from that lemma. Then $g_n \equiv f_n \circ \log \to g$ in $\mathcal{M}^\alpha$. We claim that also $\int_a^b g_n(t\cdot) \frac{dt}{t} \to \int_a^b g(t\cdot) \frac{dt}{t}$ in $\mathcal{M}^\alpha$. Indeed, a straightforward application of Fubini's theorem shows that $\left(\int_a^b g_n(t\cdot) \frac{dt}{t} \right)_e = \left(\int_a^b g(t\cdot) \frac{dt}{t} \right)_e * \check{\psi}_n$, and we appeal again to lemma 4.15. Therefore,

$$\left(\int_a^b g(t\cdot) \frac{dt}{t} \right)(A)x = \lim_n \left(\int_a^b g_n(t\cdot) \frac{dt}{t} \right)(A)x = \lim_n \int_a^b g_n(tA)x \frac{dt}{t} = \int_a^b g(tA)x \frac{dt}{t}.$$

Hence 4.98 holds and the lemma is shown. □

Proposition 4.61 *Let A be a* 0*-sectorial operator having a bounded $\mathcal{M}^\alpha$ calculus on some space X with finite cotype. Let $\psi : (0, \infty) \to \mathbb{C}$ be a non-zero function with compact support (not containing* 0*) such that $\psi \circ \exp \in \mathcal{B}^{\beta+1}_{\infty,\infty}$ for some $\beta > \alpha$.*

Then for any $\theta \in \mathbb{R}$, we have

$$\|A^\theta x\| \cong \|t^{-\theta} \psi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}, X)} \quad (x \in D(A^\theta)) \tag{4.99}$$

and for any $\theta > 0$,

$$\|A^\theta x\| + \|x\| \cong \|(t^{-\theta} + 1)\psi(tA)x\|_{\gamma(\mathbb{R}_+, \frac{dt}{t}, X)} \quad (x \in D(A^\theta)). \tag{4.100}$$

Proof. We first reduce 4.99 to the case $\theta = 0$. Set temporarily $\psi_\theta(t) = t^{-\theta}\psi(t)$. Clearly ψ satisfies the hypotheses of the proposition if and only if ψ_θ does. By proposition 4.25 (c), for $x \in D(A^\theta)$, $t^{-\theta}\psi(tA)x = \psi_\theta(tA)A^\theta x$. Thus, if 4.99 holds for $\theta = 0$, also

$$\|t^{-\theta}\psi(tA)x\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)} = \|\psi_\theta(tA)A^\theta x\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)} \cong \|A^\theta x\| \quad (x \in D(A^\theta)).$$

Assume now $\theta = 0$. By corollary 4.21 and the fundamental theorem of calculus, we have for $n \in \mathbb{Z}$ and $t \in [2^n, 2^{n+1})$

$$\psi(tA)x = \psi(2^nA)x + \int_1^2 \chi_{[2^n,t]}(2^ns)2^nsA\psi'(2^nsA)x\frac{ds}{s}.$$

Writing $\chi_n = \chi_{[2^n,2^{n+1})}$ and $\psi(tA)x = \sum_{n\in\mathbb{Z}}\chi_n(t)\psi(tA)x$, this yields by lemma 2.7 (6) (that the assumption there is satisfied follows easily from 4.102 below):

$$\|\psi(tA)x\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)} \leqslant \Big\|\sum_{n\in\mathbb{Z}}\chi_n(t)\psi(2^nA)x\Big\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)} + \int_1^2 \Big\|\sum_{n\in\mathbb{Z}}\chi_n(t)2^nsA\psi'(2^nsA)x\Big\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)}\frac{ds}{s}. \tag{4.101}$$

Since $\|\chi_n\|_{L^2(\mathbb{R}_+,\frac{dt}{t})}$ does not depend on n, by [72, exa 4.6 a)] (see also the calculation in 2.15), we have

$$\Big\|\sum_{n\in\mathbb{Z}}\chi_n(t)\psi(2^nA)x\Big\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)} \cong \Big\|\sum_{n\in\mathbb{Z}}\gamma_n\otimes\psi(2^nA)x\Big\|_{\mathrm{Gauss}(X)} \cong \Big\|\sum_{n\in\mathbb{Z}}\varepsilon_n\otimes\psi(2^nA)x\Big\|_{\mathrm{Rad}(X)},$$

where the last equivalence follows from the fact that X has finite cotype (see 2.11). Since ψ has compact support, the latter expression in turn can be estimated by $\|x\|$ according to 4.80.

Replacing ψ by $\psi_1 = s(\cdot)\psi'(s(\cdot))$, by the same arguments, we also have

$$\Big\|\sum_{n\in\mathbb{Z}}\chi_n(t)2^nsA\psi'(2^nsA)x\Big\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)} \leqslant C\|x\|. \tag{4.102}$$

Note that $\|\psi_1\circ\exp\|_{\mathcal{B}^\beta_{\infty,\infty}}$ is independent of s, and thus also the above constant C is. We have shown that $\|\psi(tA)x\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)} \leqslant c_1\|x\|$.

For the reverse inequality, we assume first that x belongs to the calculus core D. By lemma 4.60,

$$cx = \int_0^\infty |\psi|^2(tA)x\frac{dt}{t}$$

with $c = \displaystyle\int_0^\infty |\psi(t)|^2\frac{dt}{t} > 0$. Thus, by lemma 2.7, for any $x' \in X^\# \subset X'$,

$$|\langle x,x'\rangle| = |c^{-1}\int_0^\infty \langle\psi(tA)x,\overline{\psi}(tA^\#)x'\rangle\frac{dt}{t}| \lesssim \|\psi(tA)x\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)}\|\psi(tA^\#)x'\|_{\gamma(\mathbb{R}_+,\frac{dt}{t},X)}.$$

Applying the first part to $A^\#$, we deduce $\|\psi(tA^\#)x'\| \lesssim \|x'\|$. Since $X^\#$ norms X (see proposition 4.29), this shows $\|x\| \leqslant c_2\|\psi(tA)x\|_\gamma$ for $x \in D$.

For a general $x \in X$, let $(x_n)_n \subset D$ with $x_n \to x$. Then

$$\|\psi(tA)x\| \geqslant \|\psi(tA)x_n\| - \|\psi(tA)(x - x_n)\| \geqslant c_2^{-1}\|x_n\| - c_1\|x - x_n\|,$$

and letting $n \to \infty$ shows 4.99 for all $x \in X$.

Finally, 4.100 is a simple consequence of 4.99. Just note that the right hand side of 4.100 satisfies

$$\|(1+t^{-\theta})\psi(tA)x\| \cong \|\psi(tA)x\| + \|t^{-\theta}\psi(tA)x\|.$$

Indeed, "$\lesssim$" is the triangle inequality and "$\gtrsim$" follows from the two inequalities

$$\|\psi(tA)x\|_\gamma \leqslant \|(1+t^{-\theta})^{-1}\|_\infty \|(1+t^{-\theta})\psi(tA)x\|_\gamma$$

and

$$\|t^{-\theta}\psi(tA)x\|_\gamma \leqslant \|t^{-\theta}(1+t^{-\theta})^{-1}\|_\infty \|(1+t^{-\theta})\psi(tA)x\|_\gamma.$$

□

4.4.3 Real interpolation of fractional domain spaces

We now turn to the description of real interpolation spaces in the scales $\dot{X}_\theta$ and X_θ. For $A = -\Delta$ on $L^p(\mathbb{R}^d)$, these interpolation spaces correspond to homogeneous and inhomogeneous Besov spaces.

Abstract Besov spaces for sectorial operators have been studied by Komatsu, see e.g. [76] and also the monograph [94]. For the connection of Besov spaces with semigroup theory, we refer to [127] and [93], and with the H^∞ calculus to [52, chapt 6].

For Schrödinger operators $A = -\Delta + V$ on $L^p(\mathbb{R}^d)$, where V is a real potential function subject to further conditions, the abstract spaces have been studied e.g. in [136] in the homogeneous case and in [103] in the inhomogeneous case. Furthermore, in [45, sec 5] such Besov spaces are investigated for sub-Laplacians on Lie groups with polynomial growth, in connection with the Paley-Littlewood decomposition.

Proposition 4.63 below shows that these spaces are real interpolation spaces for any operator in our abstract framework with a bounded Mihlin calculus.

In the case that iA generates a c_0-group with polynomial growth, i.e. $\|e^{itA}\|_{B(X)} \lesssim \langle t\rangle^\alpha$, formally similar results can be found in [2, theorem 3.6.2].

Let us briefly recall some facts on real interpolation. If (Y, Z) is an interpolation couple (i.e. both are continuously embedded into a third Banach space and thus, the sum $Y + Z$ is meaningful), then one considers for $x \in Y + Z$ and $t > 0$ the K-functional

$$K(t, x) = \inf(\|y\|_Y + t\|z\|_Z),$$

where the infimum runs over all decompositions $x = y + z$ with $y \in Y$ and $z \in Z$. Then for $s \in (0, 1)$ and $q \in [1, \infty]$, the real interpolation space $(Y, Z)_{s,q}$ consists of all elements $x \in Y + Z$ such that

$$\|x\|_{s,q} = \left(\sum_{n\in\mathbb{Z}} 2^{nsq} K(2^{-n}, x)^q\right)^{\frac{1}{q}} \tag{4.103}$$

is finite [6, lem 3.1.3] (standard modification for $q = \infty$). The space $(Y, Z)_{s,q}$ is a Banach space equipped with the norm $\|x\|_{s,q}$.

The calculus core D_A is a dense subset of any of the spaces $\dot{X}_\theta$ and X_θ. Indeed, if $x \in D(A^\theta)$, then it is straightforward to check that $x_n = \sum_{-n}^n \dot{\varphi}_n^{dyad}(A)x \in D_A$ approximates x in $\dot{X}_\theta$ and in X_θ. Since $D(A^\theta)$ itself is dense in these spaces, also D_A is. Moreover, by the same reason, D_A is also dense in the space from 4.90, and thus in $\dot{X}_{\theta_0} + \dot{X}_{\theta_1}$ for any θ_0 and θ_1. This enables us in the sequel to work with D_A when dealing with $K(t, x)$ and determining interpolation norms.

Let us state a preliminary lemma.

Lemma 4.62 *Let A be a 0-sectorial operator having a $\mathcal{M}^\alpha$ calculus for some $\alpha > 0$. Let $\theta_0, \theta_1 \in \mathbb{R}$ with $\theta_0 < \theta_1$ and consider homogeneous and inhomogeneous partitions of unity $(\dot{\varphi}_n^{dyad})_{n\in\mathbb{Z}}$ and $(\varphi_n^{dyad})_{n\in\mathbb{N}_0}$.*

(1) *The K-functional corresponding to the interpolation couple $(\dot{X}_{\theta_0}, \dot{X}_{\theta_1})$ has the equivalent expression*

$$K(t, x) \cong \Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{\theta_0 n}, 2^{\theta_1 n} t) \dot{\varphi}_n^{dyad}(A)x \Big\|_{\mathrm{Rad}(X)} \quad (t > 0, x \in D_A).$$

(2) *The K-functional corresponding to the interpolation couple $(X_{\theta_0}, X_{\theta_1})$ has the equivalent expression*

$$K(t, x) \cong \Big\| \sum_{n\in\mathbb{N}_0} \varepsilon_n \otimes \min(2^{\theta_0 n}, 2^{\theta_1 n} t) \varphi_n^{dyad}(A)x \Big\|_{\mathrm{Rad}(X)} \quad (t > 0, x \in D_A).$$

Proof. (1) Fix some $t > 0$ and $x \in D_A$. Write in short $\|\cdot\|_j = \|\cdot\|_{\dot{X}_{\theta_j}}$, $j = 0, 1$ and $\varphi_n = \dot{\varphi}_n^{dyad}$. By definition, $K(t, x) = \inf_{x=y+z}(\|y\|_0 + t\|z\|_1)$. We set

$$n_t = \min\{n \in \mathbb{Z} : \; 2^{n\theta_0} \leqslant 2^{n\theta_1} t\}$$

and choose $y = \sum_{n \geqslant n_t} \varphi_n(A)x$ and $z = \sum_{n<n_t} \varphi_n(A)x$. Then by the description of the norms

$\| \cdot \|_j$ from proposition 4.57,

$$
\begin{aligned}
\|y\|_0 &\cong \Big\| \sum_{k\in\mathbb{Z}} \varepsilon_k \otimes 2^{k\theta_0} \varphi_k(A) y \Big\|_{\mathrm{Rad}(X)} \\
&= \Big\| \sum_{k\in\mathbb{Z}} \sum_{n \geqslant n_t} \varepsilon_k \otimes 2^{k\theta_0} \varphi_k(A) \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)} \\
&= \Big\| \varepsilon_{n_t} \otimes 2^{n_t\theta_0} [\varphi_{n_t}(A) + \varphi_{n_t+1}(A)] \varphi_{n_t}(A) x + \varepsilon_{n_t-1} \otimes 2^{(n_t-1)\theta_0} \varphi_{n_t-1}(A) \varphi_{n_t}(A) x \\
&+ \sum_{n \geqslant n_t+1} \varepsilon_n \otimes 2^{n\theta_0} \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)} \\
&\lesssim \| 2^{n_t\theta_0} \varphi_{n_t}(A) x \| + \| 2^{(n_t-1)\theta_0} \varphi_{n_t}(A) x \| + \Big\| \sum_{n \geqslant n_t+1} \varepsilon_n \otimes 2^{n\theta_0} \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)} \\
&\lesssim \Big\| \sum_{n \geqslant n_t} \varepsilon_n \otimes 2^{n\theta_0} \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)} \\
&= \Big\| \sum_{n \geqslant n_t} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)} \\
&\leqslant \Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)}.
\end{aligned}
$$

In the same way, we obtain $t\|z\|_1 \lesssim \left\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) x \right\|_{\mathrm{Rad}(X)}$. This shows

$$
K(t,x) \lesssim \Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)}.
$$

For the reverse inequality, let $x = y + z$ be an arbitrary decomposition. Since $x \in D_A$ and D_A is dense in $\dot{X}_{\theta_0} + \dot{X}_{\theta_1}$, we can assume that $y, z \in D_A$. We have by remark 4.56

$$
\Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) y \Big\|_{\mathrm{Rad}(X)} \lesssim \Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes 2^{n\theta_0} \varphi_n(A) y \Big\|_{\mathrm{Rad}(X)} \cong \|y\|_0,
$$

and similarly, $\left\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) z \right\|_{\mathrm{Rad}(X)} \lesssim t\|z\|_1$. This shows that

$$
\Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)} \lesssim \|y\|_0 + t\|z\|_1,
$$

and taking the infimum over y and z shows

$$
\Big\| \sum_{n\in\mathbb{Z}} \varepsilon_n \otimes \min(2^{n\theta_0}, 2^{n\theta_1} t) \varphi_n(A) x \Big\|_{\mathrm{Rad}(X)} \lesssim K(t,x).
$$

(2) The inhomogeneous statement can be proved as the homogeneous statement, replacing $(\dot{\varphi}_n^{dyad})_n$ by $(\varphi_n^{dyad})_n$, and the description of $\| \cdot \|_{\theta_j}$ from 4.91 by 4.92. Note that for $t \geqslant 1$, $n_t = 0$ so that $K(t,x) \cong \|x\|_{X_{\theta_0}}$ in that case. □

Proposition 4.63 *Let A have a $\mathcal{M}^\alpha$ calculus. Let further $\theta_0, \theta_1 \in \mathbb{R}$ with $\theta_0 < \theta_1$, $s \in (0,1)$ and $q \in [1,\infty]$. We set $\theta = (1-s)\theta_0 + s\theta_1$.*

(1) *For the real interpolation space* $(\dot{X}_{\theta_0}, \dot{X}_{\theta_1})_{s,q}$, *we have*

$$\|x\|_{s,q} \cong \left(\sum_{n\in\mathbb{Z}} 2^{n\theta q} \|\dot{\varphi}_n^{dyad}(A)x\|_X^q \right)^{\frac{1}{q}} \qquad (x \in D_A)$$

with the standard modification for $q = \infty$.

(2) *Assume in addition that* $\theta_0, \theta_1 \geqslant 0$. *For the real interpolation space* $(X_{\theta_0}, X_{\theta_1})_{s,q}$, *we have*

$$\|x\|_{s,q} \cong \left(\sum_{n\in\mathbb{N}_0} 2^{n\theta q} \|\varphi_n^{dyad}(A)x\|_X^q \right)^{\frac{1}{q}} \qquad (x \in D_A)$$

with the standard modification for $q = \infty$.

Proof. (1) To simplify the notation, we assume $\theta_1 - \theta_0 = 1$. Write $\varphi_n = \dot{\varphi}_n^{dyad}$. We consider first $q < \infty$ and let $x \in D_A$. We have $2^{n\theta_0} = \min(2^{n\theta_0}, 2^{n\theta_1}2^{-n})$, and consequently by lemma 4.62 for $t = 2^{-n}$,

$$2^{n\theta_0}\|\varphi_n(A)x\| \leqslant \left\| \sum_k \varepsilon_k \otimes \min(2^{k\theta_0}, 2^{k\theta_1}2^{-n})\varphi_k(A)x \right\|_{\mathrm{Rad}(X)} \lesssim K(2^{-n}, x).$$

Therefore, by 4.103, with $2^{n\theta} = 2^{n\theta_0}2^{ns}$,

$$\left(\sum_{n\in\mathbb{Z}} 2^{n\theta q} \|\varphi_n(A)x\|_X^q \right)^{\frac{1}{q}} \lesssim \left(\sum_{n\in\mathbb{Z}} 2^{nsq} K(2^{-n}, x)^q \right)^{\frac{1}{q}} \cong \|x\|_{s,q}.$$

Conversely,

$$\begin{aligned}
\|x\|_{s,q} &\cong \left(\sum_{n\in\mathbb{Z}} 2^{nsq} K(2^{-n}, x)^q \right)^{\frac{1}{q}} \\
&\cong \left(\sum_{n\in\mathbb{Z}} 2^{nsq} \left\| \sum_{k\in\mathbb{Z}} \varepsilon_k \otimes \min(2^{k\theta_0}, 2^{k\theta_1}2^{-n})\varphi_k(A)x \right\|_{\mathrm{Rad}(X)}^q \right)^{1/q} \\
&\lesssim \left(\sum_{n\in\mathbb{Z}} 2^{nsq} \left(\sum_{k\in\mathbb{Z}} 2^{k\theta_0} \min(1, 2^{k-n}) \|\varphi_k(A)x\| \right)^q \right)^{1/q} \\
&= \left(\sum_{n\in\mathbb{Z}} 2^{n\theta q} \left(\sum_{k\in\mathbb{Z}} 2^{k\theta_0} \min(1, 2^{k}) \|\varphi_{k+n}(A)x\| \right)^q \right)^{1/q} \\
&\leqslant \sum_{k\in\mathbb{Z}} 2^{k\theta_0} \min(1, 2^k) \left(\sum_{n\in\mathbb{Z}} 2^{n\theta q} \|\varphi_{k+n}(A)x\|^q \right)^{1/q} \\
&= \underbrace{\left(\sum_{k\in\mathbb{Z}} \min(1, 2^k) 2^{-ks} \right)}_{<\infty} \left(\sum_{n\in\mathbb{Z}} 2^{n\theta q} \|\varphi_n(A)x\|^q \right)^{1/q}.
\end{aligned}$$

The case $q = \infty$ can be treated in a similar way.

(2) As mentioned at the end of the proof of lemma 4.62, we have for $t \geqslant 1$ that $K(t,x) \cong \|x\|_{\theta_0} \cong K(2^0, x)$. Thus, by 4.103,

$$\|x\|_{s,q} \cong \left(\sum_{n=0}^{\infty} 2^{nsq} K(2^{-n}, x)^q \right)^{\frac{1}{q}} + \left(\sum_{n=-\infty}^{-1} 2^{nsq} \right)^{\frac{1}{q}} \|x\|_{\theta_0} \cong \left(\sum_{n=0}^{\infty} 2^{nsq} K(2^{-n}, x)^q \right)^{\frac{1}{q}}$$

(standard modification for $q = \infty$).

Then replacing part (1) of lemma 4.62 by part (2), we get with the same proof as in the homogeneous case that $\left(\sum_{n=0}^{\infty} 2^{n\theta q} \|\varphi_n^{dyad}(A)x\|^q\right)^{\frac{1}{q}} \lesssim \|x\|_{s,q}$. Also the proof for the reverse inequality can be formally repeated, replacing $(2^{nsq})_{n\in\mathbb{Z}}$ by $(2^{nsq}\delta_{n\geqslant 0})_{n\in\mathbb{Z}}$ and $(\|\dot{\varphi}_k^{dyad}(A)x\|)_{k\in\mathbb{Z}}$ by $(\|\varphi_k^{dyad}(A)x\|\delta_{k\geqslant 0})_{k\in\mathbb{Z}}$. □

4.4.4 The localization principle of the functional calculus

At the end of this section, we apply the Paley-Littlewood decomposition from theorem 4.53 to the functional calculus.

In section 4.3, we had studied the R-bounded W_2^β calculus. The inconvenience of the Sobolev class W_2^β is that it requires a certain decay at infinity, e.g. $1 \notin W_2^\beta$.

However, if the Paley-Littlewood decomposition from theorem 4.53 is at hand, e.g. if B has a $\mathcal{B}^\alpha$ calculus for some bad, i.e. large α, then the R-bounded W_p^β calculus automatically improves to the Hörmander class $\mathcal{W}_2^\beta$ of functions belonging locally to the Sobolev space W_2^β.

More precisely, we have the following corollary. We include the general index $p \in (1,\infty)$ instead of $p = 2$ for applications in section 4.5. Recall definition 4.26.

Corollary 4.64 (Localization principle) *Let $p \in (1,\infty)$, $\beta > \frac{1}{p}$ and $\alpha > 0$.*

(1) *Assume that B has a $\mathcal{B}^\alpha$ calculus and an R-bounded W_p^β calculus. Then B has a bounded $\mathcal{W}_p^\beta$ calculus. If in addition X has property (α), then B has an R-bounded $\mathcal{W}_p^\beta$ calculus.*

(2) *Assume that A has a $\mathcal{M}^\alpha$ calculus and that*

$$R(\{f(2^n A) :\ f \in C_c^\infty(\mathbb{R}_+),\, \operatorname{supp} f \subset [\frac{1}{2}, 2],\, \|f\|_{W_p^\beta} \leqslant 1,\, n \in \mathbb{Z}\}) < \infty.$$

Then A has a $\mathcal{H}_p^\beta$ calculus. If in addition X has property (α), then A has an R-bounded $\mathcal{H}_p^\beta$ calculus.

Proof. (1) Let $(\varphi_n)_n$ be an equidistant partition of unity. For any $f \in \mathcal{W}_p^\beta$ with $\|f\|_{\mathcal{W}_p^\beta} \leqslant 1$,

$$\|\varphi_n f\|_{W_p^\beta} \leqslant \sup_{k\in\mathbb{Z}} \|\varphi_k f\|_{W_p^\beta} = \|f\|_{\mathcal{W}_p^\beta}.$$

Thus, the assumption implies that

$$R(\{(\varphi_n f)(B) :\ \|f\|_{\mathcal{W}_p^\beta} \leqslant 1\}) = C < \infty. \tag{4.104}$$

Assume first that X has property (α). Let $f_k \in \mathcal{W}^\beta_p$ such that $\|f_k\|_{\mathcal{W}^\beta_p} \leqslant 1\,(k = 1, \ldots, K)$. Then for x_k belonging to the calculus core D of B,

$$\begin{aligned}\|\sum_k \varepsilon_k \otimes f_k(B)x_k\|_{\mathrm{Rad}(X)} &\cong \|\sum_n\sum_k \varepsilon_n \otimes \varepsilon_k \otimes (\varphi_n f_k)(B)x_k\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\ &= \|\sum_n\sum_k \varepsilon_n \otimes \varepsilon_k \otimes (\varphi_n f_k)(B)\widetilde{\varphi}_n(B)x_k\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\ &\lesssim C\|\sum_n\sum_k \varepsilon_n \otimes \varepsilon_k \otimes \widetilde{\varphi}_n(B)x_k\|_{\mathrm{Rad}(\mathrm{Rad}(X))}\\ &\cong C\|\sum_k \varepsilon_k \otimes x_k\|_{\mathrm{Rad}(X)}.\end{aligned}$$

Here, we have used the Paley-Littlewood decomposition from theorem 4.53 in the first and the last step, and 4.104 and property (α) in the third step. If X does not have property (α), then pick only one element $f_1 \in \mathcal{W}^\beta_p$ and repeat the above calculation. Then the sum in k reduces to one summand and property (α) is not needed any more in step 3. It follows that $\|f_1(B)x_1\| \leqslant C\|x_1\|$. By the density of D in X, part (1) follows.

(2) Let $(\varphi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity. Again assume first that X has property (α). Let $f_1, \ldots, f_K \in C^\infty(\mathbb{R}_+)$ with $\sup_k \|f_k\|_{\mathcal{H}^\beta_p} \leqslant 1$. We put $f_{k,n} = f_k(2^n(\cdot))\varphi_0$. Then $\operatorname{supp} f_{k,n} \subset [\frac12, 2]$, and $f_{k,n}(2^{-n}A) = (f_k\varphi_n)(A)$. Thus, by the assumption,

$$R(\{f_k\varphi_n(A) :\ k \in \{1, \ldots, K\},\, n \in \mathbb{Z}\}) \lesssim \sup_{k,n} \|f_{k,n}\|_{W^\beta_p} \cong \sup_k \|f_k\|_{\mathcal{H}^\beta_p} \tag{4.105}$$

where we have used proposition 4.11 in the last step. Then for $x_1, \ldots, x_K$ belonging to the calculus core D of A

$$\begin{aligned}\|\sum_k \varepsilon_k \otimes f_k(A)x_k\| &\cong \|\sum_n\sum_k \varepsilon_k \otimes \varepsilon_n \otimes (f_k\varphi_n)(A)\widetilde{\varphi}_n(A)x_k\|\\ &\lesssim \|\sum_n\sum_k \varepsilon_k \otimes \varepsilon_n \otimes \widetilde{\varphi}_n(A)x_k\|\\ &\cong \|\sum_k \varepsilon_k \otimes x_k\|.\end{aligned}$$

In the first and third step, we used the Paley-Littlewood decomposition, and in the second step, we used 4.105 and property (α). By the density of D in X, we deduce that

$$\{f(A) :\ f \in C^\infty(\mathbb{R}_+),\ \|f\|_{\mathcal{H}^\beta_p} \leqslant 1\} \tag{4.106}$$

is R-bounded. In particular, since $C^\infty(\mathbb{R}_+) \supset \bigcap_{\omega>0} H^\infty(\Sigma_\omega)$, A has a bounded $\mathcal{H}^\beta_p$ calculus in the sense of definition 4.26, and by taking the closure of 4.106, this calculus is R-bounded.

If X does not have property (α), then, as in (1), the same calculation with $K = 1$ shows that $\|f(A)\| \lesssim \|f\|_{\mathcal{H}^\beta_p}$ for $f \in C^\infty(\mathbb{R}_+)$, so that A has a bounded $\mathcal{H}^\beta_p$ calculus. □

4.5 Boundedness criteria for the Mihlin and Hörmander calculus

As in the preceding sections, we consider a 0-sectorial operator A on some Banach space X.

We shall develop necessary and sufficient conditions for the bounded and R-bounded Mihlin and Hörmander calculus of A. The aim is to compare the calculus with the boundedness of distinguished operator families we have already investigated in section 4.3. Here, the word boundedness refers to uniform (norm) and R-boundedness as well as to the averaged and matricial R-boundedness from section 4.3.

An overview will be given in the main result of the section, see theorem 4.73.

4.5.1 Mihlin calculus

We give in this subsection sufficient conditions for the $\mathcal{M}^\gamma$ calculus in terms of uniform and R-bounds of some selected operator families associated with A.

Namely, motivated by the Mihlin norms given in proposition 4.12 (1) and (4), we consider the following conditions for some $\alpha > 0$:

$$\text{There exists some } C > 0 \text{ such that } \|A^{it}\| \leqslant C(1+|t|)^\alpha \qquad (N_{\mathrm{BIP}})_\alpha$$

and

$$\text{There exists some } C > 0 \text{ such that } R\left(\{T(e^{i\theta}2^k t) : \, k \in \mathbb{Z}\}\right) \leqslant C(\frac{\pi}{2} - |\theta|)^{-\alpha} \qquad (N_T)_\alpha$$

for all $t > 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Proposition 4.65 *Let A be a 0-sectorial operator on a Banach space X having an $H^\infty(\Sigma_\sigma)$ calculus for some $\sigma \in (0, \pi)$. Let further $\alpha > 0$. Then*

$$(N_{\mathrm{BIP}})_\alpha \text{ or } (N_T)_\alpha \text{ imply that } A \text{ has a } \mathcal{M}^\gamma \text{ calculus for } \gamma > \alpha + 1. \qquad (4.107)$$

We postpone the proof and first give a remark, an example, and two preparatory lemmas.

Remark 4.66

(1) *The connection of the Mihlin calculus with the growth rate of the analytic semigroup when approaching the imaginary axis as in condition $(N_T)_\alpha$ has been studied by Duong in [35]. There, the calculus of a Laplacian operator on a Nilpotent Lie group is investigated. A Mihlin calculus is obtained from a kernel estimate of the analytic semigroup.*

(2) *Note that by proposition 4.12 one can only expect the reverse of 4.107 to hold for $\alpha > \gamma$. Thus proposition 4.65 contains a loss of 1 in the differentiation order. We will see in remark 4.75 that for $(N_{\mathrm{BIP}})_\alpha$, this loss is optimal in general, and in proposition 4.89 (2) and theorem 4.90 (3) that*

$$(N_T)_\alpha \Longrightarrow A \text{ has a } \mathcal{M}^\gamma \text{ calculus}$$

is false in general for $\gamma < \alpha + \frac{1}{2}$.

(3) *We will apply the above proposition to use the localization principle in corollary 4.64 to pass from the W_r^β calculus to the $\mathcal{W}_r^\beta$ calculus. Note that a bad (i.e. large) differentiation order γ is sufficient in corollary 4.64. In this respect, the restriction $\gamma > \alpha + 1$ is insignificant and in particular auto-improves in many cases.*

(4) *The bounded H^∞ calculus in the assumption of proposition 4.65 cannot be omitted. We give two examples of 0-sectorial operators without $H^\infty(\Sigma_\sigma)$ calculus for any $\sigma \in (0, \pi)$, the first one satisfying $(N_{\text{BIP}})_{\alpha=0}$ and the second one satisfying $(N_T)_{\alpha=1}$. By remark 4.16, $H^\infty(\Sigma_\sigma) \hookrightarrow \mathcal{M}^\gamma$ for any σ and γ, so that these operators cannot have a $\mathcal{M}^\gamma$ calculus.*

Firstly, consider the 0-sectorial operator A on $L^p(\mathbb{R})$ such that A^{it} is the shift group, i.e. $A^{it}g = g(\cdot + t)$. By [24, lem 5.3], A does not have an $H^\infty(\Sigma_\sigma)$ calculus to any positive σ, unless $p = 2$. However, A^{it} is even uniformly bounded, so that $(N_{\text{BIP}})_0$ holds.

An operator satisfying $(N_T)_{\alpha=1}$ without H^∞ calculus is given in the example below.

Example 4.67 *We give an example of a bounded 0-sectorial operator A on a Hilbert space without bounded H^∞ calculus such that its semigroup satisfies*

$$\|T(e^{i\theta}t)\| \lesssim (\frac{\pi}{2} - |\theta|)^{-1} \quad (t > 0,\, |\theta| < \frac{\pi}{2}).$$

In [86, thm 4.1], the following situation is considered, based on an idea of Baillon and Clément. Let X be an infinite dimensional space admitting a Schauder basis $(e_n)_{n\geqslant 1}$. Let V denote the span of the e_n's. For a sequence $a = (a_n)_{n\geqslant 1}$, the operator $T_a : V \to V$ is defined by letting $T_a(\sum_n \alpha_n e_n) = \sum_n a_n \alpha_n e_n$ for any finite family $(\alpha_n)_{n\geqslant 1} \subset \mathbb{C}$. Let $a^{(N)} = (a_n^{(N)})_{n\geqslant 1}$ be the sequence defined by $a_n^{(N)} = \delta_{n\leqslant N}$. It is well-known that for any Schauder basis (even conditional), $T_{a^{(N)}}$ extends to a bounded projection on X and $\sup_N \|T_{a^{(N)}}\| < \infty$ [91, Chap. 1]. This readily implies that for any sequence $a = (a_n)_{n\geqslant 1}$ of bounded variation, T_a extends to a bounded operator, and

$$\|T_a\| \lesssim \|a\|_{BV} := \|a\|_{\ell^\infty} + \sum_{n=1}^{\infty} |a_n - a_{n+1}|. \tag{4.108}$$

In [86], it is shown that for $a_n = 2^{-n}$, the bounded linear extension $A : X \to X$ of T_a is a 0-sectorial (injective) operator, and that for $f \in H^\infty(\Sigma_\sigma)$, V is a subset of $D(f(A))$. Further, for $x \in V$, one has

$$f(A)x = T_{f(a)}x, \tag{4.109}$$

where $f(a)_n = f(a_n)$. Finally, it is shown in [86] that if the Schauder basis is conditional, then A does not have a bounded H^∞ calculus.

Now assume that X is a separable Hilbert space, so that the R-boundedness in condition $(N_T)_\alpha$ reduces to norm boundedness. Clearly, X admits a Schauder basis, and as mentioned in [86] even a conditional one. We take a conditional basis and the above operator A without bounded H^∞ calculus. By 4.108 and 4.109, $(N_T)_{\alpha=1}$ will follow from

$$\|(\exp(-te^{i\theta}2^{-n}))_n\|_{BV} \lesssim (\frac{\pi}{2} - |\theta|)^{-1} \quad (t > 0,\, |\theta| < \frac{\pi}{2}). \tag{4.110}$$

It is easy to check that

$$|\exp(-te^{i\theta}2^{-n}) - \exp(-te^{i\theta}2^{-(n+1)})| \lesssim 2^{-(n+1)}t\exp(-t\cos(\theta)2^{-(n+1)}).$$

Thus,

$$\begin{aligned}\|(\exp(-te^{i\theta}2^{-n}))_n\|_{BV} &\lesssim 1+\sum_{n=1}^{\infty} 2^{-(n+1)} t\exp(-t\cos(\theta)2^{-(n+1)})\\ &\lesssim 1+\int_0^1 st\exp(-t\cos(\theta)s)\frac{ds}{s}\\ &\lesssim 1+(\cos\theta)^{-1}\int_0^\infty \exp(-s)ds\\ &\cong (\frac{\pi}{2}-|\theta|)^{-1}.\end{aligned}$$

This shows 4.110, and thus A satisfies $(N_T)_{\alpha=1}$ without having an H^∞ calculus.

We state a lemma as an intermediate step of the proof of proposition 4.65.

Lemma 4.68 *Let A be a 0-sectorial operator on some Banach space X having an $H^\infty(\Sigma_\sigma)$ calculus for some $\sigma \in (0, \frac{\pi}{2})$. Assume that A satisfies*

$$\sup_{\lambda\in\partial\Sigma_\theta\setminus\{0\}} R\left(\{\lambda^{1/2}(2^kA)^{1/2}R(\lambda,2^kA)\,:\,k\in\mathbb{Z}\}\right)\lesssim\theta^{-\gamma} \tag{4.111}$$

for some $\gamma > 0$. Then A has an $\mathcal{M}^\beta$ calculus for any $\beta > \gamma$.

Proof. By the characterization of the $\mathcal{M}^\beta$ calculus from proposition 4.18, it suffices to show that

$$\|f(A)\|\lesssim\theta^{-\beta}\|f\|_{\infty,\theta}\text{ for any } f\in\bigcup_{\theta>0}H_0^\infty(\Sigma_\theta). \tag{4.112}$$

To show 4.112, we use the Kalton-Weis characterization of the bounded $H^\infty(\Sigma_\theta)$ calculus in terms of R-bounded operator families ([73], see also [81, thm 12.7]). More precisely, we follow that characterization in the form of the proof of [81, thm 12.7] and keep track of the dependence of appearing constants on the angle θ. It is shown there that for $f \in H_0^\infty(\Sigma_{2\theta})$, $x \in X$ and $x' \in X'$,

$$\begin{aligned}|\langle f(A)x,x'\rangle| &= |\frac{1}{2\pi i}\int_{\partial\Sigma_\theta}\langle\lambda^{-\frac12}f(\lambda)A^{\frac12}R(\lambda,A)x,x'\rangle d\lambda|\\ &\leqslant\frac{1}{2\pi}\sum_{j=\pm1}\int_0^\infty|\langle f(te^{ij\theta})(tA)^{\frac12}R(e^{ij\theta},tA)x,x'\rangle|\frac{dt}{t}\\ &=(*).\end{aligned}$$

We put

$$\phi_{j\theta}(\lambda)=\frac{\lambda^{\frac14}(1+\lambda)^{\frac12}}{e^{ij\theta}-\lambda}\text{ and }\psi(\lambda)=\left(\frac{\lambda}{(1+\lambda)^2}\right)^{\frac18},$$

so that $(tA)^{\frac{1}{2}}R(e^{ij\theta},tA) = \phi_{j\theta}(tA)\psi(tA)\psi(tA)$. By [81, lem 12.6], the integral $(*)$ can be controlled by Rad-norms. More precisely, we have

$$(*) \lesssim \sup_{j=\pm 1} \sup_{t>0} \sup_N \| \sum_{k=-N}^{N} \varepsilon_k \otimes f(2^k t e^{ij\theta}) \phi_{j\theta}(2^k tA)\psi(2^k tA)x\|_{\mathrm{Rad}(X)} \tag{4.113}$$

$$\cdot \| \sum_{k=-N}^{N} \varepsilon_k \otimes \psi(2^k tA)' x' \|_{\mathrm{Rad}(X')}$$

$$\lesssim \|f\|_{\infty,\theta} \sup_{j,t} R\left(\{\phi_{j\theta}(2^k tA) :\ k \in \mathbb{Z}\}\right) \sup_{N,t} \| \sum_{k=-N}^{N} \varepsilon_k \otimes \psi(2^k tA)x\|_{\mathrm{Rad}(X)}$$

$$\cdot \sup_{N,t} \| \sum_{k=-N}^{N} \varepsilon_k \otimes \psi(2^k tA)'x'\|_{\mathrm{Rad}(X')}.$$

By [81, thm 12.2], the fact that A has a bounded H^∞ calculus implies that $\sup_{N,t} \| \sum_{k=-N}^{N} \varepsilon_k \otimes \psi(2^k tA)x\|_{\mathrm{Rad}(X)} \lesssim \|x\|$ and $\sup_{N,t} \| \sum_{k=-N}^{N} \varepsilon_k \otimes \psi(2^k tA)'x'\|_{\mathrm{Rad}(X')} \lesssim \|x'\|$. Note that there is no dependence on θ in these two inequalities. It remains to show that

$$\sup_{j=\pm 1, t>0} R(\{\phi_{j\theta}(2^k tA) :\ k \in \mathbb{Z}\}) \lesssim \theta^{-\beta}. \tag{4.114}$$

We have

$$\phi_{j\theta}(2^k tA) = \frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{\theta}{2}}} \phi_{j\theta}(\lambda)\lambda^{\frac{1}{2}}(2^k tA)^{\frac{1}{2}} R(\lambda, 2^k tA)\frac{d\lambda}{\lambda}$$
$$= \frac{1}{2\pi i} \int_{\partial \Sigma_{\frac{\theta}{2}}} \phi_{j\theta}(t\lambda)\lambda^{\frac{1}{2}}(2^k A)^{\frac{1}{2}} R(\lambda, 2^k A)\frac{d\lambda}{\lambda}.$$

By proposition 2.6 (5),

$$\sup_{j=\pm 1, t>0} R(\{\phi_{j\theta}(2^k tA) :\ k \in \mathbb{Z}\}) \lesssim \sup_{j=\pm 1, t>0} \|\phi_{j\theta}(t\lambda)\|_{L^1(\partial\Sigma_{\frac{\theta}{2}}, |\frac{d\lambda}{\lambda}|)}$$
$$\times \sup_{\lambda \in \partial \Sigma_{\theta/2}\setminus\{0\}} R\left(\{\lambda^{\frac{1}{2}}(2^k A)^{\frac{1}{2}} R(\lambda, 2^k A) :\ k \in \mathbb{Z}\}\right).$$

By assumption, it suffices to show that for any $\varepsilon > 0$

$$\sup_{t>0} \|\phi_{j\theta}(t\lambda)\|_{L^1(\partial\Sigma_{\frac{\theta}{2}}, |\frac{d\lambda}{\lambda}|)} \leqslant C_\varepsilon \theta^{-\varepsilon}.$$

$$\int_{\partial \Sigma_{\frac{\theta}{2}}} |\phi_{j\theta}(t\lambda)| \left|\frac{d\lambda}{\lambda}\right| = \int_{\partial \Sigma_{\frac{\theta}{2}}} |\phi_{j\theta}(\lambda)| \left|\frac{d\lambda}{\lambda}\right| = \sum_{l=\pm 1} \int_0^\infty \left| \frac{s^{\frac{1}{4}}(1 + e^{il\frac{\theta}{2}}s)^{\frac{1}{2}}}{e^{ij\theta} - e^{il\frac{\theta}{2}}s} \right| \frac{ds}{s}.$$

The denominator is estimated from below by

$$|e^{ij\theta} - e^{il\frac{\theta}{2}}s| = |e^{i\theta(j-\frac{l}{2})} - s| \gtrsim |\cos(\theta(j-\frac{l}{2})) - s| + |\sin(\theta(j-\frac{l}{2}))|$$
$$\gtrsim |1-s| - |\cos(\theta(j-\frac{l}{2})) - 1| + \theta$$
$$\gtrsim |1-s| - \theta^2 + \theta \gtrsim |1-s| + \theta$$

for the crucial case of small θ. Thus

$$\int_{\partial\Sigma_{\frac{\theta}{2}}} |\phi_{j\theta}(\lambda)| \left|\frac{d\lambda}{\lambda}\right| \lesssim \int_0^\infty \frac{s^{\frac{1}{4}}(1+s)^{\frac{1}{2}}}{\theta+|1-s|}\frac{ds}{s}.$$

We split the integral into the parts $\int_0^\infty = \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{1-\theta} + \int_{1-\theta}^{1+\theta} + \int_{1+\theta}^{2} + \int_2^\infty$.

$$\int_0^{\frac{1}{2}} \frac{s^{\frac{1}{4}}(1+s)^{\frac{1}{2}}}{\theta+|1-s|}\frac{ds}{s} \leqslant \int_0^{\frac{1}{2}} \frac{s^{\frac{1}{4}}(1+s)^{\frac{1}{2}}}{|1-s|}\frac{ds}{s} < \infty$$

is independent of θ. The same estimate applies to $\int_2^\infty$.

$$\int_{\frac{1}{2}}^{1-\theta} \frac{s^{\frac{1}{4}}(1+s)^{\frac{1}{2}}}{\theta+|1-s|}\frac{ds}{s} \lesssim \int_{\frac{1}{2}}^{1-\theta} \frac{1}{\theta+|1-s|}ds \leqslant \int_{\frac{1}{2}}^{1-\theta} \frac{1}{1-s}ds \lesssim |\log\theta|.$$

Similarly,

$$\int_{1+\theta}^{2} \frac{s^{\frac{1}{4}}(1+s)^{\frac{1}{2}}}{\theta+|1-s|}\frac{ds}{s} \lesssim \int_{1+\theta}^{2} \frac{1}{s-1}ds \lesssim |\log\theta|.$$

Finally,

$$\int_{1-\theta}^{1+\theta} \frac{s^{\frac{1}{4}}(1+s)^{\frac{1}{2}}}{\theta+|1-s|}\frac{ds}{s} \lesssim \int_{1-\theta}^{1+\theta} \frac{1}{\theta}ds \lesssim 1.$$

Since $1+|\log\theta| \leqslant C_\varepsilon \theta^{-\varepsilon}$, the lemma is shown. □

Remark 4.69 *Let us have a look at 4.113. As done in [81, thm 12.7], this estimate works also for $F \in H_0^\infty(\Sigma_\sigma; E_A)$, (cf. chapter 3 section 3.3 for the operator valued space $H_0^\infty(\Sigma_\sigma; E_A)$), where $\|f\|_{\infty,\theta}$ has to be replaced by $R(\{F(z) : z \in \Sigma_\theta\})$. Hence, lemma 4.68 also shows that there exists $C > 0$ such that*

$$\|F(A)\| \leqslant C\theta^{-\beta} R(\{F(z) : z \in \Sigma_\theta\}) \quad (\theta \in (0,\pi),\ F \in H_0^\infty(\Sigma_\sigma; E_A)). \tag{4.115}$$

If X has in addition property (α), then in [81, thm 12.8], it is deduced from 4.115 for a fixed $\theta \in (0,\pi)$ (and with $\widetilde{A} = \mathrm{Id}_{\mathrm{Rad}} \otimes A$ in place of A) that A has an R-bounded $H^\infty(\Sigma_\theta)$ calculus.

By that same method, one can show that under the assumptions of lemma 4.68, A has an R-bounded $\mathcal{M}^\beta$ calculus provided that X has property (α). Since, in the setting of proposition 4.65, we will show a stronger R-bounded Hörmander calculus later with a different approach, we only sketch the proof of this claim.

So assume that A satisfies the assumptions of lemma 4.68. Then also the operator $\widetilde{A} = \mathrm{Id}_{\mathrm{Rad}} \otimes A$ on $\mathrm{Rad}(X)$ does (see [81, thm 12.8] for details).

Take a $\beta' > \beta$ and a bounded sequence $(f_k)_{k\geqslant 1} \subset \mathcal{M}^{\beta'} \subset \mathcal{M}^\beta$. We show that $\{f_k(A) : k \geqslant 1\}$ is R-bounded, which is clearly sufficient.

Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic Fourier decomposition of unity and define the holomorphic function $f_{k,n}$ by $(f_{k,n})_e = (f_k)_e * \check{\phi}_n$. By lemma 4.15 (2), $f_k = \sum_{n\in\mathbb{Z}} f_{k,n}$ in $\mathcal{M}^{\beta'}$ and by 4.115 for $\widetilde{A}$, one deduces as in [81, thm 12.8] that

$$R(\{f_k(A) : k \geqslant 1\}) \leqslant \sum_{n\in\mathbb{Z}} R(\{f_{k,n}(A) : k \geqslant 1\}) \lesssim \sum_{n\in\mathbb{Z}} \theta_n^{-\beta} \sup_k \|f_{k,n}\|_{\infty,\theta_n},$$

where, in the last step, one combines 4.115 for $\widetilde{A}$ in place of A with the proof of [81, thm 12.8]. By a Paley-Wiener argument (cf. [24, proof of thm 4.10]), one has $\|f_{k,n}\|_{\infty,\theta_n} \leqslant e^{2^{|n|+1}\theta_n}\|f_{k,n}\|_{\infty,0}$. Choosing $\theta_n = 2^{-|n|}$, we get with $\varepsilon = \beta' - \beta > 0$

$$\begin{aligned} R(\{f_k(A):\ k \geqslant 1\}) &\lesssim \sum_{n\in\mathbb{Z}} 2^{|n|\beta} e^2 \sup_k \|f_{k,n}\|_{\infty,0} \\ &\lesssim \sum_{n\in\mathbb{Z}} 2^{-|n|\varepsilon} \sup_k 2^{|n|\beta'} \|f_{k,n}\|_{\infty,0} \\ &\leqslant \sum_{n\in\mathbb{Z}} 2^{-|n|\varepsilon} \sup_{k,n'} 2^{|n'|\beta'} \|f_{k,n'}\|_{\infty,0} \\ &= \sum_{n\in\mathbb{Z}} 2^{-|n|\varepsilon} \sup_k \|f_{k,e}\|_{\mathcal{B}^{\beta'}_{\infty,\infty}} \\ &\lesssim \sup_k \|f_k\|_{\mathcal{M}^{\beta'}}. \end{aligned}$$

Thus, A has an R-bounded $\mathcal{M}^{\beta'}$ calculus.

We need yet another preparatory lemma for the proof of proposition 4.65.

Lemma 4.70 *Let A be a 0-sectorial operator on some Banach space X having an H^∞ calculus. Let $\omega \in (0,\pi)$ and assume that $\{\lambda R(\lambda, A):\ -\lambda \in \Sigma_\theta\}$ is R-bounded for some $\theta > \pi - \omega$.*

Let $f \in H^\infty(\Sigma_\omega)$ such that f is holomorphic in a neighborhood of 0 and that there exists $\varepsilon > 0$ such that $|f(\lambda)| \lesssim |\lambda|^{-\varepsilon}$ for $\lambda \in \Sigma_\omega,\ |\lambda| \geqslant 1$.

Then $\{f(tA):\ t > 0\}$ is R-bounded.

Proof. Replacing f by $f(t_0\cdot)$ if necessary, we can suppose that $f(\lambda)$ is holomorphic for $|\lambda| < 2$. Let $\Gamma = \{se^{i(\pi-\theta)}:\ s > 1\} \cup \{e^{i\varphi}:\ \varphi \in [\pi-\theta, \pi+\theta]\} \cup \{se^{i(\pi+\theta)}:\ s > 1\}$. The operator $f(tA)$ can be expressed by a Cauchy integral formula (see e.g. [81, exa 9.8]):

$$\begin{aligned} f(tA) &= \frac{1}{2\pi i}\int_\Gamma f(\lambda)R(\lambda, tA)d\lambda \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{1}{\lambda} f(\lambda)\frac{\lambda}{t}R(\frac{\lambda}{t}, A)d\lambda. \end{aligned}$$

Since $f(\lambda)$ decreases at ∞ by assumption, $\frac{1}{\lambda}f(\lambda)$ is integrable on Γ. Then $\{f(tA):\ t > 0\}$ is R-bounded by the R-sectoriality of A and proposition 2.6 (5). □

Proof of proposition 4.65. We show that A satisfies 4.111 with $\gamma = \alpha + 1$, so that the proposition will follow from lemma 4.68. Assume first that $(N_{\text{BIP}})_\alpha$ holds, i.e.

$$\|A^{it}\| \lesssim \langle t\rangle^\alpha.$$

Recall formula 4.72, i.e. for $x \in A(D(A^2))$ and $|\omega| < \pi$

$$\frac{\pi}{\cosh(\pi s)} e^{\omega s} A^{is} x = \int_0^\infty t^{is}\left[e^{i\frac{\omega}{2}} t^{\frac{1}{2}} A^{\frac{1}{2}} (e^{i\omega}t + A)^{-1} x\right]\frac{dt}{t}.$$

Substituting $t = e^u$, the right hand side becomes a Fourier inverse integral. Applying the Fourier transform, we get with $\psi_\omega(\lambda) = \lambda^{\frac{1}{2}}(e^{i\omega} + \lambda)^{-1}$,

$$\psi_\omega(e^{-u}A)x = c\mathcal{F}(\frac{e^{\omega(\cdot)}}{\cosh(\pi(\cdot))}A^{i(\cdot)}x)(u),$$

where $|c|$ does not depend on ω. By assumption $(N_{\mathrm{BIP}})_\alpha$, we have

$$\int_{\mathbb{R}} \|\frac{e^{\omega s}}{\cosh(\pi s)}A^{is}x\|ds \lesssim \int_{\mathbb{R}} e^{(|\omega|-\pi)|s|}(1+|s|^\alpha)ds = 2\int_0^\infty e^{(|\omega|-\pi)s}(s+s^{\alpha+1})\frac{ds}{s}$$
$$= 2((\pi-|\omega|)^{-1}\Gamma(1) + (\pi-|\omega|)^{-(\alpha+1)}\Gamma(\alpha+1)) \lesssim (\pi-|\omega|)^{-(\alpha+1)}.$$

As $A(D(A^2)) \supset D(A^3) \cap R(A^3)$ is dense in X [81, prop 9.4], proposition 2.6 implies that $\{\psi_\omega(e^{-u}A) : u \in \mathbb{R}\}$ is R-bounded with constant $\lesssim (\pi - |\omega|)^{-(\alpha+1)}$. Thus, with $\omega = \pm(\pi - \theta)$,

$$\begin{aligned}\sup_{\lambda\in\partial\Sigma_\theta\backslash\{0\}} R(\{\lambda^{\frac{1}{2}}(2^kA)^{\frac{1}{2}}R(\lambda,2^kA) : k \in \mathbb{Z}\}) &= R(\{(2^{-k}\lambda)^{\frac{1}{2}}A^{\frac{1}{2}}R(2^{-k}\lambda,A) : k\in\mathbb{Z}\})\\ &\leqslant R(\{\lambda^{\frac{1}{2}}A^{\frac{1}{2}}R(\lambda,A) : \lambda\in\partial\Sigma_\theta\backslash\{0\}\})\\ &= R(\{-\lambda^{\frac{1}{2}}A^{\frac{1}{2}}(te^{ij(\pi-\omega)}+A)^{-1} : t>0,\, j=\pm1\})\\ &= R(\{\psi_{j\omega}(tA) : t>0,\, j=\pm1\})\\ &\lesssim \theta^{-(\alpha+1)}.\end{aligned}$$

This is precisely 4.111 for $\gamma = \alpha + 1$.

Assume now that $(N_T)_\alpha$ holds, i.e. for $t > 0$ and $|\theta| < \frac{\pi}{2}$,

$$R\left(\{T(e^{i\theta}2^kt) : k\in\mathbb{Z}\}\right) \leqslant C(\frac{\pi}{2} - |\theta|)^{-\alpha}.$$

We show first that for $\omega \in (0, \frac{\pi}{2})$,

$$R(\{(2^ktA)^{\frac{1}{2}}T(e^{\pm i(\frac{\pi}{2}-\omega)}2^kt) : k\in\mathbb{Z}\}) \lesssim \omega^{-(\alpha+\frac{1}{2})}. \tag{4.116}$$

Decompose

$$e^{\pm i(\frac{\pi}{2}-\omega)}t = s + e^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})}r,$$

where $s, r > 0$ are uniquely determined by t and ω. Then

$$(2^ktA)^{\frac{1}{2}}T(e^{\pm i(\frac{\pi}{2}-\omega)}2^kt) = \left(\frac{t}{s}\right)^{\frac{1}{2}} \cdot (2^ksA)^{\frac{1}{2}}T(2^ks)\cdot T(e^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})}2^kr),$$

and consequently,

$$\begin{aligned}R(\{(2^ktA)^{\frac{1}{2}}T(e^{\pm i(\frac{\pi}{2}-\omega)}2^kt) : k\in\mathbb{Z}\}) &\leqslant \sup_t (t/s)^{\frac{1}{2}} \times R(\{(2^ksA)^{\frac{1}{2}}T(2^ks) : k\in\mathbb{Z}\})\\ &\quad\times R(\{T(e^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})}2^kr) : k\in\mathbb{Z}\}).\end{aligned} \tag{4.117}$$

We will show that the right hand side of 4.117 can be estimated by $\lesssim \omega^{-\frac{1}{2}} \times 1 \times \omega^{-\alpha}$.

The estimate for the first factor follows from the law of sines

$$t/s = \sin(\frac{\pi}{2} + \omega/2)/\sin(\omega/2) \cong \omega^{-1}.$$

For the second estimate, note that by [81, exa 2.16], $(N_T)_\alpha$ implies that $\{T(z) : z \in \Sigma_\delta\}$ is R-bounded for any $\delta < \frac{\pi}{2}$ and consequently, by [81, thm 2.20, $(iii) \Rightarrow (i)$], $\{\lambda R(\lambda, A) : -\lambda \in \Sigma_\theta\}$ is R-bounded for any $\theta \in (\frac{\pi}{2}, \pi)$. By lemma 4.70 with $f(\lambda) = \lambda^{\frac{1}{2}} e^{-\lambda}$ and $\omega \in (0, \frac{\pi}{2})$, $\{(tA)^{\frac{1}{2}} T(t) : t > 0\}$ and thus, $\{(2^k tA)^{\frac{1}{2}} T(2^k t) : k \in \mathbb{Z}\}$ is R-bounded.

The estimate for the third factor in 4.117 follows from the assumption $(N_T)_\alpha$. Thus, 4.116 follows.

Now we write the expression in 4.111 as an integral of the expression in 4.116. Let $\theta \in (0, \frac{\pi}{2})$, $\lambda = te^{i\theta}$ and set $\varphi = \frac{\pi}{2} - \frac{\theta}{2}$, so that $\mathrm{Re}(e^{i\varphi}\lambda) < 0$. Then

$$\begin{aligned} \lambda^{\frac{1}{2}} (2^k A)^{\frac{1}{2}} (\lambda - 2^k A)^{-1} &= \lambda^{\frac{1}{2}} (2^k A)^{\frac{1}{2}} e^{i\varphi} (e^{i\varphi}\lambda - e^{i\varphi} 2^k A)^{-1} \\ &= -e^{i\varphi} \int_0^\infty s^{-\frac{1}{2}} \lambda^{\frac{1}{2}} (2^k sA)^{\frac{1}{2}} \exp(e^{i\varphi}\lambda s) T(2^k e^{i\varphi} s) ds. \end{aligned} \tag{4.118}$$

In view of proposition 2.6 (5), we want to estimate $\sup_{\arg\lambda=\theta} \int_0^\infty s^{-\frac{1}{2}} |\lambda^{\frac{1}{2}} \exp(e^{i\varphi}\lambda s)| ds$.

$$\begin{aligned} \int_0^\infty s^{-\frac{1}{2}} |\lambda^{\frac{1}{2}} \exp(e^{i\varphi}\lambda s)| ds &= \int_0^\infty s^{-\frac{1}{2}} |\exp(e^{i\varphi} e^{i\theta} s)| ds \\ &= \int_0^\infty s^{-\frac{1}{2}} \exp(\cos(\frac{\pi}{2} + \frac{\theta}{2}) s) ds \\ &= \int_0^\infty s^{-\frac{1}{2}} \exp(-s) ds \, |\cos(\frac{\pi}{2} + \frac{\theta}{2})|^{-\frac{1}{2}} \\ &\lesssim \theta^{-\frac{1}{2}}. \end{aligned} \tag{4.119}$$

Combining 4.116, 4.118 and 4.119, we get by the submultiplicativity of R-bounds (see 2.5)

$$\sup_{\arg\lambda=\theta} R(\lambda^{\frac{1}{2}} (2^k A)^{\frac{1}{2}} R(\lambda, 2^k A) : k \in \mathbb{Z}) \lesssim \theta^{-(\alpha+1)}.$$

Replacing θ by $-\theta$ and setting $\varphi = -\frac{\pi}{2} + \frac{\theta}{2}$, we finally get with a similar estimate 4.111 with $\gamma = \alpha + 1$. □

Let us give an application of proposition 4.65. Suppose we are given a 0-strip-type operator B on a space X with property (α), having an R-bounded W^α_p calculus. If we tempt to apply the localization principle from corollary 4.64 to deduce an R-bounded $\mathcal{W}^\alpha_p$ calculus, we have to take care of the crucial assumption there: We assumed in the localization principle that B has a $\mathcal{B}^\gamma$ calculus for some (large) $\gamma > 0$. Since $\mathcal{B}^\gamma \not\hookrightarrow W^\alpha_p$, this might not be clear.

If however B has a bounded $H^\infty(\mathrm{Str}_\sigma)$ calculus for some $\sigma > 0$, then B does have a $\mathcal{B}^\gamma$ calculus, as the following proposition shows. Its main application will be the situation described above.

Proposition 4.71 *Let $p \in (1, \infty)$, $\alpha > \frac{1}{p}$, and let B be a 0-strip-type operator having an R-bounded W^α_p calculus on a space with property (α). Assume moreover that B has a bounded $H^\infty(\mathrm{Str}_\sigma)$ calculus for some $\sigma > 0$.*

Then $A = e^B$ satisfies $(N_T)_\beta$ from proposition 4.65 for some $\beta > 0$, and consequently, by that proposition, A has a $\mathcal{M}^\gamma$ calculus, i.e. B has a $\mathcal{B}^\gamma$ calculus, for some $\gamma > 0$.

Proof. Replacing B by a multiple if necessary, we may and do assume that $\sigma < \frac{\pi}{4}$.

Let $A = e^B$ be the associated 0-sectorial operator. Since X has property (α), the fact that A has a bounded H^∞ calculus extends by [81, thm 12.8] to

$$R\left(\{g(A) : \|g\|_{H^\infty(\Sigma_\theta)} \leqslant 1\}\right) < \infty \tag{4.120}$$

for a $\theta < \frac{\pi}{4}$. By assumption, also

$$R\left(\{h(A) : \|h \circ \exp\|_{W_p^\alpha} \leqslant 1\}\right) < \infty. \tag{4.121}$$

For $\operatorname{Re} z > 0$, let $f_z(\lambda) = \exp(-z\lambda)$. Then $(N_T)_\beta$ reads as $R(\{f_{2^k z}(A) : k \in \mathbb{Z}\}) \lesssim (z/\operatorname{Re} z)^\beta$. By 4.120 and 4.121, it suffices to decompose $f_z = g_z + h_z$ such that $\|g_z\|_{H^\infty(\Sigma_\theta)} \lesssim (|z|/\operatorname{Re} z)^\beta$ and $\|h_z \circ \exp\|_{W_p^\alpha} \lesssim (|z|/\operatorname{Re} z)^\beta$. By a simple scaling argument we may assume that $|z| = 1$. We choose the decomposition

$$f_z(\lambda) = f_z(\lambda)e^{-\lambda} + f_z(\lambda)(1 - e^{-\lambda}).$$

Then $\|f_z(\lambda)e^{-\lambda}\|_{H^\infty(\Sigma_\theta)} = \|\exp(-(z+1)\lambda)\|_{H^\infty(\Sigma_\theta)} \lesssim 1$, since $\theta + |\arg(z+1)| \leqslant \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$. Further, taking e.g. β an integer larger than α, it is a simple matter to check that $\|h_z \circ \exp\|_{W_p^\alpha} \lesssim \|h_z \circ \exp\|_{W_p^\beta} \lesssim (\operatorname{Re} z)^{-\beta}$. □

At the end of this subsection, let us compare condition $(N_T)_\alpha$ for the analytic semigroup with related ones for its boundary values on the imaginary axis. We consider the following two variants for an operator A and $\alpha > 0$.

A has a $\mathcal{M}^\gamma$ calculus for some (large) $\gamma > 0$ and $\{(1 + 2^k|s|A)^{-\alpha}e^{i2^k A} : k \in \mathbb{Z}\}$ is R-bounded by a constant independent of $s \in \mathbb{R}$. $\quad (N_W)_\alpha$

A has a $\mathcal{M}^\gamma$ calculus for some (large) $\gamma > 0$ and $\{\langle s\rangle^{-\alpha}(1 + 2^k A)^{-\alpha}e^{is2^k A} : k \in \mathbb{Z}\}$ is R-bounded by a const. independent of $s \in \mathbb{R}$. $\quad (\widetilde{N_W})_\alpha$

The reason for the $\mathcal{M}^\gamma$ calculus in these conditions is to ensure that the occuring operators are well-defined by proposition 4.25.

Further, we consider analogous stronger conditions, where R-bounds over dyadic arguments 2^k are replaced by R-bounds over all positive numbers t :

A has a $\mathcal{M}^\gamma$ calculus for some (large) $\gamma > 0$ and $\{(1 + |s|A)^{-\alpha}e^{isA} : s \in \mathbb{R}\}$ is R-bounded. $\quad (R_W)_\alpha$

A has a $\mathcal{M}^\gamma$ calculus for some (large) $\gamma > 0$ and $\{\langle s\rangle^{-\alpha}(1 + A)^{-\alpha}e^{isA} : s \in \mathbb{R}\}$ is R-bounded. $\quad (\widetilde{R_W})_\alpha$

$\{(\frac{\pi}{2} - |\theta|)^\alpha T(e^{i\theta}t) : t > 0, |\theta| < \frac{\pi}{2}\}$ is R-bounded. $\quad (R_T)_\alpha$

Clearly, the $(R_{\dots})_\alpha$ conditions imply their weak counterparts $(N_{\dots})_\alpha$.

The next lemma shows how these conditions compare further.

Lemma 4.72 *Let A be a 0-sectorial operator on a space X with property (α) having an H^∞ calculus. Then for $\beta > \alpha > 0$,*

$$(R_W)_\alpha \Longrightarrow (R_T)_\alpha \Longrightarrow (\widetilde{R_W})_\beta$$

and

$$(N_W)_\alpha \Longrightarrow (N_T)_\alpha \Longrightarrow (\widetilde{N_W})_\beta.$$

Proof. $(R_W)_\alpha \Longrightarrow (R_T)_\alpha$: Note first that for any $\omega \in (0,\pi)$, we have

$$\{f(A) : \|f\|_{\infty,\omega} \leqslant 1\} \text{ is } R\text{-bounded.} \tag{4.122}$$

Indeed, by assumption, A has a $\mathcal{M}^\gamma$ calculus for some $\gamma > 0$, so in particular a bounded $H^\infty(\Sigma_\omega)$ calculus for any $\omega \in (0,\pi)$. Since X has property (α), A has also an R-bounded $H^\infty(\Sigma_\omega)$ calculus for any $\omega \in (0,\pi)$ [81, thm 12.8] (see also remark 4.69), so that 4.122 follows. In particular,

$$\Big\{(\frac{\pi}{2} - |\theta|)^\alpha T(e^{i\theta}t) : t > 0,\ |\theta| \leqslant \frac{\pi}{4}\Big\} \text{ is } R\text{-bounded.}$$

Thus it remains to show that

$$\Big\{(\frac{\pi}{2} - |\theta|)^\alpha T(e^{i\theta}t) : t > 0, |\theta| \in (\frac{\pi}{4},\frac{\pi}{2})\Big\} \text{ is } R\text{-bounded.} \tag{4.123}$$

We write $e^{i\theta}t = r + is$ with real r and s. Then

$$\left(\frac{r}{|s|}\right)^\alpha T(r+is) = \left[\left(\frac{r}{|s|}\right)^\alpha (1+rA)^{-\alpha}(1+|s|A)^\alpha\right] \circ \left[(1+|s|A)^{-\alpha}e^{-isA}\right] \circ \left[(1+rA)^\alpha T(r)\right].$$

We show that all three brackets form R-bounded sets for $r + is$ varying in $\{z \in \mathbb{C}\backslash\{0\} : |\arg z| \in (\frac{\pi}{4},\frac{\pi}{2})\}$. Note that for $|\theta| \in (\frac{\pi}{4},\frac{\pi}{2})$, we have $\frac{\pi}{2} - |\theta| \cong \frac{r}{|s|}$, so that this will imply 4.123 by Kahane's contraction principle (see proposition 2.6).

We show in a moment that

$$\left(\frac{r}{|s|}\right)^\alpha (1+r(\cdot))^{-\alpha}(1+|s|(\cdot))^\alpha \text{ is uniformly bounded in } H^\infty(\Sigma_\sigma) \tag{4.124}$$

for $\sigma = \frac{\pi}{2}$, say. Then the fact that the first bracket is R-bounded follows from 4.122. For $\lambda \in \Sigma_{\frac{\pi}{2}}$,

$$\begin{aligned}\left(\frac{r}{|s|}\right)^\alpha \left|(1+r\lambda)^{-\alpha}(1+|s|\lambda)^\alpha\right| &= \left|\frac{\frac{1}{|s|}+\lambda}{\frac{1}{r}+\lambda}\right|^\alpha \\ &\cong \left(\frac{\frac{1}{|s|}+|\lambda|}{\frac{1}{r}+|\lambda|}\right)^\alpha \\ &\lesssim 1,\end{aligned}$$

since $|s| > r$ by the restriction $|\theta| \in (\frac{\pi}{4},\frac{\pi}{2})$. Thus, 4.124 follows.

The assumption $(R_W)_\alpha$ implies that the second bracket is R-bounded with s varying in $\mathbb{R}$.

Finally, 4.122 with $f(\lambda) = (1+\lambda)^\alpha e^{-\lambda}$ implies that the third bracket is R-bounded with r varying in $(0,\infty)$.

Now $(R_T)_\alpha$ follows.

$(N_W)_\alpha \implies (N_T)_\alpha$: The proof is similar to $(R_W)_\alpha \implies (R_T)_\alpha$. After restricting to the case $|\theta| \in (\frac{\pi}{4}, \frac{\pi}{2})$, decompose

$$\left(\frac{r}{|s|}\right)^\alpha T(2^k(r+is)) = \left[\left(\frac{r}{|s|}\right)^\alpha (1+r2^kA)^{-\alpha}(1+|s|2^kA)^\alpha\right] \circ \left[(1+|s|2^kA)^{-\alpha}e^{-is2^kA}\right] \circ \left[(1+r2^kA)^\alpha T(2^kr)\right].$$

All three brackets form R-bounded sets with k varying in $\mathbb{Z}$, and this uniformly in $r+is \in \{z \in \mathbb{C}\backslash\{0\} : |\arg z| \in (\frac{\pi}{4}, \frac{\pi}{2})\}$ by the same arguments as before.

$(R_T)_\alpha \implies (\widetilde{R_W})_\beta$: By proposition 4.65, $(N_T)_\alpha$, and thus also $(R_T)_\alpha$ implies that A has a bounded $\mathcal{M}^\gamma$ calculus for $\gamma > \alpha + 1$.

Let D denote the calculus core of A. Then for $x \in D$, we have the Taylor expansion for the function $u \mapsto e^{-(u+is)A}x$, where $s \in \mathbb{R}$ is fixed:

$$e^{-isA}x = \sum_{j=0}^{m-1} c_j A^j T(1+is)x + c_m \int_0^1 u^{m-1}A^m T(u+is)x du,$$

where $m \in \mathbb{N}$ is chosen such that $m-1 \leqslant \beta \leqslant m$, and $c_j \in \mathbb{R}$ are constants arising from the Taylor expansion.

To show $(\widetilde{R_W})_\beta$, we split $\langle s\rangle^{-\beta}(1+A)^{-\beta}e^{-isA}$ into two parts according to the above formula. $A^j(1+A)^{-\beta}$ are bounded operators for $j \leqslant \beta$. Moreover, $\{\langle s\rangle^{-\beta}T(1+is) : s \in \mathbb{R}\}$ is R-bounded by assumption $(R_T)_\alpha$, since $\langle s\rangle^{-1} \cong \frac{\pi}{2} - |\arg(1+is)|$ and $\beta > \alpha$. Thus,

$$R(\{\sum_{j=0}^{m-1} c_j \langle s\rangle^{-\beta} A^j(1+A)^{-\beta}T(1+is) : s \in \mathbb{R}\}) < \infty. \tag{4.125}$$

Next we claim that

$$R(\{\langle s\rangle^{-\beta}(1+A)^{-\beta}A^m T(u+is) : s \in \mathbb{R}\}) \lesssim u^{-m+\beta-\alpha}. \tag{4.126}$$

Split

$$\langle s\rangle^{-\beta}(1+A)^{-\beta}A^m T(u+is) = \left[\langle s\rangle^{-\beta}T(u/2+is)\right] \circ \left[A^{m-\beta}T(u/2)\right] \circ \left[A^\beta(1+A)^{-\beta}\right].$$

Write the first bracket as

$$(\langle s/u\rangle^\alpha \langle s\rangle^{-\beta})(\langle s/u\rangle^{-\alpha}T(u/2+is)).$$

Note that $|\langle s/u\rangle^\alpha\langle s\rangle^{-\beta}| \lesssim u^{-\alpha}$ for $u \in (0,1)$. Further, by $(R_T)_\alpha$, $\{\langle s/u\rangle^{-\alpha}T(u/2+is) : s \in \mathbb{R}\}$ is R-bounded. Thus, $R(\{\langle s\rangle^{-\beta}T(u/2+is) : s \in \mathbb{R}\}) \lesssim u^{-\alpha}$.

The second bracket is bounded by $u^{-m+\beta}$, since $m \geqslant \beta$. Hence 4.126 follows.

Now by proposition 2.6 (5),

$$R(\{\langle s\rangle^{-\beta}\int_0^1 u^{m-1}(1+A)^{-\beta}A^m T(u+is)du : s \in \mathbb{R}\}) \leqslant \int_0^1 u^{m-1}u^{-m+\beta-\alpha}du < \infty. \tag{4.127}$$

Combining 4.125 and 4.127 shows $(\widetilde{R_W})_\beta$.

$(N_T)_\alpha \implies (\widetilde{N_W})_\beta$: The proof is similar to $(R_T)_\alpha \implies (\widetilde{R_W})_\beta$. Use the Taylor expansion for $u \mapsto e^{-(u+is)2^k A}$:

$$\langle s\rangle^{-\beta}(1+2^kA)^{-\beta}e^{-is2^kA}x = \sum_{j=0}^{m-1} c_j(2^kA)^j(1+2^kA)^{-\beta}\langle s\rangle^{-\beta}T((1+is)2^k)x$$
$$+ c_m\int_0^1 u^{m-1}(2^kA)^m(1+2^kA)^{-\beta}\langle s\rangle^{-\beta}T((u+is)2^k)x du.$$

Unlike before, we need R-boundedness instead of mere boundedness for $\{(2^kA)^j(1+2^kA)^{-\beta} : k \in \mathbb{Z}\}$. This follows from property (α) and [81, thm 12.8]. The argument for the R-boundedness of the integral part is similar to the preceding case. Once again we need R-boundedness instead of boundedness, for $\{(2^kA)^{m-\beta}T(\frac{u}{2}2^k) : k \in \mathbb{Z}\}$ and $\{(2^kA)^\beta(1+2^kA)^{-\beta} : k \in \mathbb{Z}\}$, and refer again to [81, thm 12.8]. □

4.5.2 Hörmander calculus

The following theorem summarizes the connections between the R-bounded Hörmander calculus and boundedness conditions on operator families. Here we complement the norm boundedness, dyadic R-boundedness and R-boundedness conditions that we investigated in the preceding subsection by averaged R-boundedness and square function conditions from section 4.3. Its proof is postponed to the end of the section.

Theorem 4.73 *Let A be a 0-sectorial operator with H^∞ calculus on a Banach space X with property (α). For $r \in (1,2]$ and $\alpha > \frac{1}{r}$, we consider the condition*

A has an R-bounded $\mathcal{H}^\alpha_r(\mathbb{R}_+)$ calculus. $(C_r)_\alpha$

Furthermore, we consider the following four groups of conditions, where each group contains statements about boundedness of the same type. Generally speaking, these boundedness notions become more restrictive when passing step by step from (I) to (IV), and each single condition becomes more restrictive if α gets smaller.

(I) Norm-boundedness and dyadic R-boundedness conditions: for $\alpha \geqslant 0$,

$(N_{\mathrm{BIP}})_\alpha$ *There exists $C > 0$ such that for all $t \in \mathbb{R}$, $\|A^{it}\| \leqslant C(1+|t|)^\alpha$.*

$(N_T)_\alpha$ *The set $\{(\frac{\pi}{2} - |\theta|)^\alpha T(e^{i\theta}2^kt) : k \in \mathbb{Z}\}$ is R-bounded with bound independent of $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $t > 0$.*

$(N_W)_\alpha$ *A has a $\mathcal{M}^\gamma$ calculus for some (large) $\gamma > 0$ and $\{(1+2^k|s|A)^{-\alpha}e^{i2^ksA} : k \in \mathbb{Z}\}$ is R-bounded with bound independent of $s \in \mathbb{R}$.*

(II) R-boundedness conditions: for $\alpha \geqslant 0$,

$(R_{\mathrm{BIP}})_\alpha$ *The set* $\{\langle t\rangle^{-\alpha}A^{it} : t \in \mathbb{R}\}$ *is R-bounded.*

$(R_T)_\alpha$ *The set* $\{(\frac{\pi}{2} - |\theta|)^\alpha T(e^{i\theta}t) : t > 0, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})\}$ *is R-bounded.*

$(R_W)_\alpha$ *A has a* $\mathcal{M}^\gamma$ *calculus for some* $\gamma > 0$ *and the set* $\{(1 + |s|A)^{-\alpha}e^{isA} : s \in \mathbb{R}\}$ *is R-bounded.*

(III) Averaged R-boundedness conditions: for $\alpha > \frac{1}{2}$,

$(R(L2)_{\mathrm{BIP}})_\alpha$ *The family* $(\langle t\rangle^{-\alpha}A^{it} : t \in \mathbb{R})$ *is* $R[L^2(\mathbb{R})]$*-bounded.*

$(R(L2)_R)_\alpha$ *The family* $(A^{1/2}R(e^{i\theta}t, A) : t > 0)$ *is* $R[L^2(\mathbb{R}_+, dt)]$*-bounded for* $\theta \in (-\pi, \pi)\backslash\{0\}$, *and its bound grows at most like* $|\theta|^{-\alpha}$ *for* $\theta \to 0$.

$(R(L2)_T)_\alpha$ *The family* $(A^{1/2}T(e^{i\theta}t) : t > 0)$ *is* $R[L^2(\mathbb{R}_+, dt)]$*-bounded for* $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and its bound grows at most like* $(\frac{\pi}{2} - |\theta|)^{-\alpha}$ *for* $|\theta| \to \frac{\pi}{2}$.

$(R(L2)_W)_\alpha$ *A has a* $\mathcal{M}^\gamma$ *calculus for some (large)* $\gamma > 0$ *and* $(s^{-\alpha}A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m : s \in \mathbb{R})$ *is* $R[L^2(\mathbb{R})]$*-bounded.*

(IV) Square function conditions: for $\alpha > \frac{1}{2}$,

$(S_{\mathrm{BIP}})_\alpha$ *The function* $t \mapsto \langle t\rangle^{-\alpha}A^{it}x$ *belongs to* $\gamma(\mathbb{R}, X)$ *with norm* $\lesssim \|x\|$ *for* $x \in X$.

$(S_R)_\alpha$ *The function* $t \mapsto A^{1/2}R(e^{i\theta}t, A)x$ *belongs to* $\gamma(\mathbb{R}_+, dt, X)$ *with norm* $\lesssim |\theta|^{-\alpha}\|x\|$ *for* $x \in X$ *and* $\theta \in (-\pi, \pi)$.

$(S_T)_\alpha$ *The function* $t \mapsto A^{1/2}T(e^{i\theta}t)x$ *belongs to* $\gamma(\mathbb{R}_+, dt, X)$ *with norm* $\lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha}\|x\|$ *for* $x \in X$ *and* $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

$(S_W)_\alpha$ *A has a* $\mathcal{M}^\gamma$ *calculus for some* $\gamma > 0$ *and the function* $s \mapsto s^{-\alpha}A^{-\alpha+\frac{1}{2}}(e^{isA} - 1)^m x$ *belongs to* $\gamma(\mathbb{R}, X)$ *with norm* $\lesssim \|x\|$ *for* $x \in D$, *where* m *is a fixed integer strictly larger than* $\alpha - \frac{1}{2}$.

Then the following hold.

(a) How to get the Hörmander calculus
Let $r \in (1, 2]$ *such that* $\dfrac{1}{r} > \dfrac{1}{\operatorname{type} X} - \dfrac{1}{\operatorname{cotype} X}$ *and* $\beta > \alpha + \frac{1}{r}$. *Then any of the conditions in (I) implies* $(C_r)_\beta$.

(b) From R-bounded sets to square functions
Consider $\alpha_{(II)}, \alpha_{(IV)} \geqslant 0$ *with* $\alpha_{(IV)} > \alpha_{(II)} + \frac{1}{2}$. *Then any of the conditions in (II) with* $\alpha = \alpha_{(II)}$ *implies any of the conditions in (IV) with* $\alpha = \alpha_{(IV)}$.

(c) Square functions ⇒ Hörmander calculus ⇒ R-bounded sets
Let $\alpha_{(II)} \geqslant 0$ *and* $\alpha_{(III)}, \alpha_{(IV)} > \frac{1}{2}$, *and for* j *=II,III,IV consider the conditions in the group (j) with parameter* $\alpha = \alpha_{(j)}$.

If $\alpha_{(IV)} < \alpha_{(III)} < \alpha_{(II)}$, *then any of the conditions in (IV) implies any of the conditions in (III), which in turn imply any of the conditions in (II).*

(d) *Equivalent conditions for the Hörmander calculus*
Condition $(C_2)_\alpha$ is essentially equivalent to any of the conditions in (III).

More precisely, for $\alpha > \frac{1}{2}$ and any $\varepsilon > 0$,

$$(C_2)_\alpha \iff (R(L2)_{\mathrm{BIP}})_\alpha \iff (R(L2)_W)_\alpha \implies (R(L2)_T)_\alpha,\, (R(L2)_R)_\alpha \implies (C_2)_{\alpha+\varepsilon}.$$

(e) *Square function equivalences*
The conditions in (IV) are essentially equivalent.

More precisely, for $\alpha > \frac{1}{2}$ and any $\varepsilon > 0$, $(S_{\mathrm{BIP}})_\alpha \iff (S_W)_\alpha \implies (S_T)_\alpha, (S_R)_\alpha \implies (S_{\mathrm{BIP}})_{\alpha+\varepsilon}$.

The strip-type version of the theorem reads as follows.

Theorem 4.74 *Let B be 0-strip-type operator with H^∞ calculus on some Banach space with property (α). Denote $U(t)$ the c_0-group generated by iB and $R(\lambda, B)$ the resolvents of B. For $r \in (1, 2]$ and $\alpha > \frac{1}{r}$, consider the condition*

$$B \text{ has an } R\text{-bounded } \mathcal{W}_r^\alpha(\mathbb{R}) \text{ calculus.} \qquad (C_r)_\alpha$$

Furthermore, for $\alpha \geqslant 0$, we consider the conditions

(I)$_\alpha$ *There exists $C > 0$ such that for all $t \in \mathbb{R}$, $\|U(t)\| \leqslant C(1 + |t|)^\alpha$.*

(II)$_\alpha$ *The set $\{\langle t\rangle^{-\alpha} U(t) : t \in \mathbb{R}\}$ is R-bounded.*

(IIIa)$_\alpha$ *The family $(\langle t\rangle^{-\alpha} U(t) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded.*

(IIIb)$_\alpha$ *The family $(R(t + ic, B) : t \in \mathbb{R})$ is $R[L^2(\mathbb{R})]$-bounded for any $c \neq 0$ and its bound grows at most like $|c|^{-\alpha}$ for $c \to 0$.*

(IVa)$_\alpha$ *The function $t \mapsto \langle t\rangle^{-\alpha} U(t)x$ belongs to $\gamma(\mathbb{R}, X)$ with norm $\lesssim \|x\|$ for $x \in X$.*

(IVb)$_\alpha$ *The function $t \mapsto R(t + ic, B)x$ belongs to $\gamma(\mathbb{R}, X)$ with norm $\lesssim |c|^{-\alpha} \|x\|$ for $x \in X$ and $c \neq 0$.*

Then the following hold.

(a) *Let $r \in (1, 2]$ such that $\frac{1}{r} > \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$ and $\beta > \alpha + \frac{1}{r}$. Then (I)$_\alpha$ implies $(C_r)_\beta$.*

(b) *Consider $\alpha, \beta \geqslant 0$ with $\beta > \alpha + \frac{1}{2}$. Then (II)$_\alpha$ implies (IVa)$_\beta$ and (IVb)$_\beta$.*

(c) *The following hold for $\alpha > \frac{1}{2}$ and $\varepsilon > 0$.*

$$\begin{aligned}
&(IVa)_\alpha \implies (IIIa)_\alpha \implies (II)_\alpha \implies (I)_\alpha \\
&(IVb)_\alpha \implies (IIIb)_\alpha \implies (II)_{\alpha+\varepsilon} \\
&(C_2)_\alpha \iff (IIIa)_\alpha \implies (IIIb)_\alpha \implies (C_2)_{\alpha+\varepsilon} \\
&(IVa)_\alpha \implies (IVb)_\alpha \implies (IVa)_{\alpha+\varepsilon}
\end{aligned}$$

Before proceeding to the proofs, let us compare the theorems with some simple examples.

Remark 4.75 *In theorems 4.73 and 4.74, comparing different conditions yields a loss in the differentiation parameter α. We give a list of examples showing that this loss is close to be optimal.*

(1) *Jordan block. Gap between $(R_{\mathrm{BIP}})_\alpha$, $(R_T)_\alpha$, $(R_W)_\alpha$ and $(C_2)_{\alpha+\frac{1}{2}}$.*
Let $\alpha = m \in \mathbb{N}$ be given and consider the Jordan block

$$B = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \in \mathbb{C}^{(m+1)\times(m+1)}$$

as an operator on the Hilbert space $X = \ell^2_{m+1}$. Then

$$f(B) = \begin{pmatrix} f(0) & \frac{f'(0)}{1!} & \dots & \frac{f^{(m)}(0)}{m!} \\ 0 & \ddots & \ddots & \end{pmatrix},$$

at least for $f \in \bigcup_{\omega>0} \mathrm{Hol}(\mathrm{Str}_\omega)$. Thus, for $x = (x_0, \dots, x_m)$ and $y = (y_0, \dots, y_m) \in \ell^2_{m+1}$,

$$|\langle e^{itB}x, y\rangle| = |\sum_{k=0}^{m}\sum_{l=0}^{k} \frac{(it)^{l-k}}{(l-k)!} x_l y_k| \lesssim \langle t\rangle^m \|x\|\,\|y\|,$$

and taking $x = (0, \dots, 0, 1)$, $y = (1, 0, \dots, 0)$ shows that the exponent m is optimal in this estimate. Thus, $t \mapsto \langle t\rangle^{-\beta}\langle e^{itB}x, y\rangle$ belongs to $L^2(\mathbb{R})$ for all $x, y \in X$ if and only if $\beta > m + \frac{1}{2}$, so that $A = e^B$ cannot have a $\mathcal{H}^\beta_2$ calculus for $\beta \leqslant m + \frac{1}{2}$.

On the other hand, $\|e^{itB}\| \cong \langle t\rangle^m$, so that $(N_{\mathrm{BIP}})_m$ holds, and since X is a Hilbert space, $(N_{\mathrm{BIP}})_m$ is equivalent to $(R_{\mathrm{BIP}})_m$. Also $(R_W)_m$ and thus, by lemma 4.72, $(R_T)_m$ hold, because

$$\|(1+|t|A)^{-m}e^{itA}\| \cong |\frac{d^m}{ds^m}\left[(1+|t|e^s)^{-m}\exp(ite^s)\right]_{s=0}| \leqslant C.$$

Therefore, in this example

$$(R_{\mathrm{BIP}})_m,\ (R_T)_m,\ (R_W)_m, \text{ hold, but } (C_2)_{m+\frac{1}{2}} \text{ does not hold.}$$

This shows that in theorems 4.73 and 4.74, (a) is sharp for $\frac{1}{r} = \frac{1}{2}$, and (b) is sharp.

(2) *Multiplication operator on W^α_2. Gap between $(C_2)_\alpha$ and $(S_{\mathrm{BIP}})_{\alpha+\frac{1}{2}}$.*
Let $X = W^\alpha_2$ for some $\alpha > \frac{1}{2}$. Consider the c_0-group $U(t)g = e^{it(\cdot)}g$ on X. Note that

$$\|e^{it(\cdot)}g\|_X = \|\hat{g}(\cdot - t)\langle\cdot\rangle^\alpha\|_2 = \|\hat{g}(\cdot)\langle(\cdot)+t\rangle^\alpha\|_2 \cong \langle t\rangle^\alpha\|g\|_X.$$

In particular, $\|U(t)\| \cong \langle t\rangle^\alpha$. $U(t)$ is strongly continuous, since $\|U(t)g - g\|_X = \|\hat{g}(\cdot)(\langle(\cdot + t)\rangle^\alpha - \langle\cdot\rangle^\alpha)\|_2 \to 0$ for $t \to 0$. Denote the generator by iB. Then B is of 0-strip-type. It is easy to check that $f(B)g = fg$ for any $g \in X$ and $f \in \bigcup_{\omega>0} H^\infty(\mathrm{Str}_\omega)$. Since W^α_2 is a Banach algebra for $\alpha > \frac{1}{2}$, we have

$$\|f(B)g\| = \|fg\|_X \lesssim \|f\|_{W^\alpha_2}\|g\|_X$$

and consequently, B has a W_2^α calculus. Furthermore, since $\mathcal{B}^\alpha \cdot W_2^\alpha \hookrightarrow W_2^\alpha$ [128, p. 141], B has also a $\mathcal{B}^\alpha$ calculus. Then by corollary 4.64, the W_2^α calculus extends to a (R-bounded) $\mathcal{W}_2^\alpha$ calculus, so that for $A = e^B$, condition $(C_2)_\alpha$ holds.

On the other hand, since X is a Hilbert space, the square function condition $(S_{\text{BIP}})_\alpha$ can be expressed as

$$\|\langle t\rangle^{-\beta} e^{itB} g\|_{\gamma(\mathbb{R},X)} = \|\langle t\rangle^{-\beta} e^{itB} g\|_{L^2(\mathbb{R},X)} \cong \left(\int_{\mathbb{R}} \langle t\rangle^{-2\beta+2\alpha} dt \right)^{\frac{1}{2}} \|g\|,$$

which is finite if and only if $\beta > \alpha + \frac{1}{2}$. In summary, we have

$$(C_2)_\alpha \text{ holds, but } (S_{\text{BIP}})_{\alpha+\frac{1}{2}} \text{ does not hold.}$$

(3) *Multiplication operator on E_p^α, $p \in [2,\infty)$. Gap between $(N_{\text{BIP}})_\alpha$ and $(C_r)_{\alpha+\frac{1}{r}}$, $r \in (1,2]$.*
In section 5.6 of chapter 5, we will consider a multiplication operator B on a different space $X = E_p^\alpha$ of functions defined on $\mathbb{R}$. Here, $\alpha \geqslant 0$ and $p \in [2,\infty)$ are some parameters. We will see in theorem 5.26 that in this case B satisfies $\|U(t)\| \leqslant \langle t\rangle^\alpha$, and that (a) of theorem 4.74 is false for $\beta < \alpha + \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$. Since $\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X} = 1 - \frac{1}{p}$ can be chosen arbitrary close to 1, this shows the optimality of (a).

(4) *Gauss and Poisson semigroup on $L^p(\mathbb{R}^d)$. Gap between $(N_{\text{BIP}})_\alpha$ and $(N_T)_\alpha$, $(R_{\text{BIP}})_\alpha$ and $(R_T)_\alpha$.*
Consider some A having an H^∞ calculus on a space with property (α). Then theorem 4.73 yields that for $\alpha \geqslant 0$ and $\beta > \beta' > \alpha + \frac{1}{2}$, we have $(R_{\text{BIP}})_\alpha \overset{(b)}{\Longrightarrow} (S_{\text{BIP}})_{\beta'} \overset{(c)}{\Longrightarrow} (R_T)_\beta$. Conversely, the theorem also shows that $(R_T)_\alpha \Longrightarrow (R_{\text{BIP}})_\beta$. If X satisfies in addition $\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X} < \frac{1}{2}$, e.g. if X is an L^p space for some $p \in (1,\infty)$, then for the same α and β as before, $(N_{\text{BIP}})_\alpha \overset{(a)}{\Longrightarrow} (R(L2)_{\text{BIP}})_{\beta'} \overset{(c)}{\Longrightarrow} (R_T)_\beta \Longrightarrow (N_T)_\beta$. Similarly, also $(N_T)_\alpha \Longrightarrow (N_{\text{BIP}})_\beta$.

Denote $\alpha(N_T) = \alpha_A(N_T)$ the infimum over all α such that $(N_{\text{BIP}})_\alpha$ holds for the operator A. Similarly, we define $\alpha(R_{\text{BIP}})$, $\alpha(N_T)$ and $\alpha(R_T)$. Then summarizing the above, we get

$$|\alpha(R_T) - \alpha(R_{\text{BIP}})| \leqslant \frac{1}{2} \text{ and } |\alpha(N_T) - \alpha(N_{\text{BIP}})| \leqslant \frac{1}{2}. \tag{4.128}$$

An indication that the gaps of $\frac{1}{2}$ in 4.128 are not artificial is given by the following example.

Let $X = L^p(\mathbb{R}^d)$ for some $d \in \mathbb{N}$ and $p \in (1,\infty)$ and consider the 0-sectorial operators $G = -\Delta$ generating the Gaussian semigroup, and $P = (-\Delta)^{\frac{1}{2}}$ generating the Poisson semigroup. By the identity $G^{it} = P^{2it}$, we have

$$\alpha_G(N_{\text{BIP}}) = \alpha_P(N_{\text{BIP}}) \text{ and } \alpha_G(R_{\text{BIP}}) = \alpha_P(R_{\text{BIP}}).$$

However, we will see in section 4.6 (proposition 4.89 and theorem 4.90) that

$$\alpha_G(N_T) = \alpha_G(R_T) = d\left|\frac{1}{p} - \frac{1}{2}\right|$$

and, in contrast,

$$\alpha_P(N_T) = (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|.$$

Thus, $\alpha_G(N_T) - \alpha_P(N_T) = \left|\frac{1}{p} - \frac{1}{2}\right| \to \frac{1}{2}$ for $p \to 1$ or $p \to \infty$.

Remark 4.76 *Let $\alpha - \frac{1}{2} \notin \mathbb{N}_0$. Under the assumption that A has an R-bounded $\mathcal{M}^{\alpha+1-\varepsilon}$ calculus for some $\varepsilon > 0$, (note that since $\mathcal{M}^{\alpha+1-\varepsilon} \subset \mathcal{H}_r^{\alpha+\frac{1}{r}+\varepsilon}$ for r as in (a) of theorem 4.73, $(N_W)_\alpha$ and $(R_W)_\alpha$ imply an R-bounded $\mathcal{M}^{\alpha+1-\varepsilon}$ calculus), the conditions $(S_W)_\alpha$, $(R(L2)_W)_\alpha$, $(R_W)_\alpha$ and $(N_W)_\alpha$ can be restated in a uniform way as follows: Let $m \in \mathbb{N}_0$ such that $\alpha - \frac{1}{2} \in (m, m+1)$ and*

$$w_\alpha(s) = s^{-\alpha}(e^{is} - \sum_{j=0}^{m-1} \frac{(is)^j}{j!}).$$

Let

$(S_W)'_\alpha$ $\|A^{\frac{1}{2}} w_\alpha(sA)x\|_{\gamma(\mathbb{R},X)} \lesssim \|x\|, \quad (x \in D_A)$.

$(R(L2)_W)'_\alpha$ $R[L^2(\mathbb{R})](A^{\frac{1}{2}} w_\alpha(sA) : s \in \mathbb{R}) < \infty$.

$(R_W)'_\alpha$ $R(\{w_\alpha(sA) : s \in \mathbb{R}\}) < \infty$.

$(N_W)'_\alpha$ $\sup_{s\in\mathbb{R}} R(\{w_\alpha(2^k sA) : k \in \mathbb{Z}\}) < \infty$.

Then each of the above four conditions is equivalent to the corresponding non-primed condition.

Proof. $(S_W)'_\alpha \iff (S_W)_\alpha$: We show that $(S_W)'_\alpha \iff (S_{\mathrm{BIP}})_\alpha$. The proof is similar to that of proposition 4.42. We determine the Mellin transform of $s^{\frac{1}{2}} w_\alpha(s)$: By a contour shift of the integral $s \rightsquigarrow is$,

$$\int_0^\infty s^{it} s^{\frac{1}{2}} w_\alpha(s) \frac{ds}{s} = \int_0^\infty (is)^{it} (is)^{\frac{1}{2}} w_\alpha(is) \frac{ds}{s}$$
$$= i^{-\alpha+\frac{1}{2}+it} \int_0^\infty s^{-\alpha+\frac{1}{2}+it} (e^{-s} - \sum_{j=0}^{m-1} \frac{(-s)^j}{j!}) \frac{ds}{s}.$$

Applying partial integration, one sees that this expression equals $i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha+\frac{1}{2}+it)$. Thus,

$$M(s^{\frac{1}{2}} w_\alpha(s))(t) = i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha+\frac{1}{2}+it),$$

and by proposition 4.33,

$$M(\langle (sA)^{\frac{1}{2}} w_\alpha(sA)x, x'\rangle)(t) = i^{-\alpha+\frac{1}{2}+it}\Gamma(-\alpha+\frac{1}{2}+it)\langle A^{-it}x, x'\rangle.$$

Since $\alpha - \frac{1}{2} \notin \mathbb{N}_0$,

$$|i^{-\alpha+\frac{1}{2}+it} \cdot \Gamma(-\alpha+\frac{1}{2}+it)| \cong e^{-\frac{\pi}{2}t} \cdot e^{-\frac{\pi}{2}|t|} \langle t\rangle^{-\alpha}$$

for $t \in \mathbb{R}$. Thus, with lemma 4.49, for $x \in D_A$,

$$(S_{\mathrm{BIP}})_\alpha \iff \|(\pm sA)^{\frac{1}{2}} w_\alpha(\pm sA)x\|_{\gamma(\mathbb{R}_+,\frac{ds}{s},X)} \lesssim \|x\| \iff \|A^{\frac{1}{2}} w_\alpha(sA)\|_{\gamma(\mathbb{R},ds,X)} \lesssim \|x\|.$$

$(R(L2)_W)'_\alpha \iff (R(L2)_W)_\alpha$: This is shown in the same manner as $(S_W)'_\alpha \iff (S_W)_\alpha$, replacing lemma 4.49 by lemma 4.34.

$(R_W)'_\alpha \iff (R_W)_\alpha$: Put $v_\alpha^\pm(t) = (1+t)^{-\alpha} e^{\pm it}$, so that

$$(R_W)_\alpha \iff R(\{v_\alpha^+(tA), v_\alpha^-(tA) : t > 0\}) < \infty,$$

whereas

$$(R_W)'_\alpha \iff R(\{w_\alpha(tA), w^\alpha(-tA) : t > 0\}) < \infty.$$

The claimed equivalence follows if $R(\{w_\alpha(tA) - v_\alpha^+(tA), w_\alpha(-tA) - v_\alpha^-(tA) : t > 0\}) < \infty$, or by the R-bounded $\mathcal{M}^{\alpha+1-\varepsilon}$ calculus, if $w_\alpha(\pm(\cdot)) - v_\alpha^\pm \in \mathcal{M}^{\alpha+1-\varepsilon}$. Let $\psi \in C^\infty(\mathbb{R}_+)$ be as in proposition 4.12 (3). We decompose

$$\begin{aligned} w_\alpha(t) - v_\alpha^+(t) &= \frac{t^\alpha - (1+t)^\alpha}{t^\alpha(1+t)^\alpha} e^{it} + t^{-\alpha} \sum_{j=0}^{m-1} \frac{(it)^j}{j!}. \\ &= \left[\frac{t^\alpha - (1+t)^\alpha}{t^\alpha(1+t)^\alpha} e^{it} \psi(t) \right] \\ &+ \left[\frac{t^\alpha - (1+t)^\alpha}{t^\alpha(1+t)^\alpha} e^{it}(1-\psi(t)) + t^{-\alpha} \sum_{j=0}^{m-1} \frac{(it)^j}{j!} \right]. \end{aligned}$$

By the just mentioned proposition 4.12 (3), the expression in the first brackets belongs to $\mathcal{M}^{\alpha+1-\varepsilon}$, and it is easy to check that also the second bracket, and thus, $w_\alpha - v_\alpha^+$ belong to that space. In the same way one checks that also $w_\alpha(-(\cdot)) - v_\alpha^- \in \mathcal{M}^{\alpha+1-\varepsilon}$.

$(N_W)'_\alpha \iff (N_W)_\alpha$ follows from the same argument. □

We now prepare the proof of theorem 4.73. First we have a look at (a), i.e. the implication

$$(N_{\text{BIP}})_\alpha,\ (N_T)_\alpha,\ (N_W)_\alpha \implies A \text{ has an } R\text{-bounded } \mathcal{H}_r^\beta \text{ calculus}$$

for suitable β and r.

As a preparatory lemma, we show a representation formula of the functional calculus in terms of the operators e^{itA}.

Lemma 4.77 *Let A be a 0-sectorial operator with a $\mathcal{M}^\gamma$ calculus for some $\gamma > 0$ such that*

$$t \mapsto \langle t \rangle^{-\beta} \| e^{itA}(1+A)^{-\alpha} \| \text{ is dominated by a function in } L^r(\mathbb{R}) \tag{4.129}$$

for some $\alpha, \beta > 0$ and $r \in (1,2]$. Then for x belonging to the calculus core D of A,

$$t \mapsto e^{itA}(1+A)^{-\alpha}x \text{ is differentiable (in particular measurable)} \tag{4.130}$$

and for $f \in C_c^\infty(0,\infty)$,

$$f(A)(1+A)^{-\alpha}x = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(t) e^{itA}(1+A)^{-\alpha} x\, dt. \tag{4.131}$$

Proof. Fix an integer $M > \gamma$. Let K be the set of all functions $f \in H^\infty(\Sigma_\omega)$ for some $\omega \in (0,\pi)$ satisfying the following properties: $|f(\lambda)| \lesssim |\lambda|^{M+1}$ for $|\lambda| \leqslant 1$, $|f(\lambda)| \lesssim |\lambda|^{-1}$ for $|\lambda| \geqslant 1$ and $f \in W_r^\beta$, where we extend $f(t)$ for negative t by putting $f(t) = 0$.

We will show the following:

(1) For $f \in K$, 4.131 holds.

(2) For $f \in C_c^\infty(0,\infty)$, there exists a sequence $(g_n)_n \subset K$ converging to f in W_r^β and in $\mathcal{M}^\gamma$.

(3) For $f \in C_c^\infty(0,\infty)$, 4.131 holds.

(1) We write $y = (1+A)^{-\alpha}x$. Let $f \in K$. In particular, $f \in H_0^\infty$ so that for some $\omega \in (0, \frac{\pi}{2})$, according to 2.1,

$$f(A)y = \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} f(\lambda)(\lambda - A)^{-1} y d\lambda.$$

We claim that $(\lambda - A)^{-1}$ can be expressed in terms of e^{itA}, more precisely, for $s > 0$,

$$(se^{-i\omega} - A)^{-1}y = i\int_0^\infty \exp[(-ise^{-i\omega} + iA)t]ydt. \tag{4.132}$$

Since x and thus also y belongs to D, there exists $N \in \mathbb{N}$ such that $\varphi(A)y = y$ for any $\varphi \in C_c^\infty(0,\infty)$ satisfying $\varphi(t) = 1$ for $t \in [2^{-N}, 2^N]$. It is also easy to see that for any $t_0 > 0$ there exists $\varphi_{t_0} \in C_c^\infty(0,\infty)$ such that $\varphi_{t_0}(tA)y = y$ and $\varphi_{t_0}'(tA)y = 0$ for any t in some neighborhood of t_0.

Fix some t_0 and let t be in that neighborhood. Then by corollary 4.21, $t \mapsto e^{itA}y = e^{itA}\varphi_{t_0}(tA)y$ is differentiable. Indeed, $g(\lambda) = \exp(i\lambda)\varphi_{t_0}(\lambda)$ belongs to $C_c^\infty(0,\infty)$, so that $g_e \in \mathcal{B}^{\gamma+1}_{\infty,\infty}$. Further, $g'(\lambda) = ig(\lambda) + \exp(i\lambda)\varphi_{t_0}'(\lambda)$, and consequently, by that corollary,

$$\frac{d}{dt}[e^{itA}y] = \frac{d}{dt}[g(tA)y] = iAg(tA)y = iAe^{itA}y. \tag{4.133}$$

This shows the claim 4.130. The integral on the right hand side of 4.132 is absolutely convergent. Indeed, replacing M if necessary by a larger number, we can take some $\delta > 0$ such that $M > \alpha + \delta > \gamma$. By the bounded $\mathcal{M}^\gamma$ calculus and proposition 4.12 (2), $\|e^{itA}(1+A)^{-(\alpha+\delta)}\| \lesssim \langle t\rangle^M$. Thus, $\|e^{itA}y\| \lesssim \langle t\rangle^M \|(1+A)^\delta x\|$, and consequently,

$$\|\exp[(-ise^{-i\omega} + iA)t]y\| \lesssim \langle t\rangle^M e^{-st\sin(\omega)}. \tag{4.134}$$

Therefore,

$$\int_0^\infty \|\exp[(-ise^{-i\omega} + iA)t]y\|dt \lesssim \int_0^\infty \langle t\rangle^M e^{-st\sin(\omega)}dt \lesssim_\omega s^{-(M+1)}. \tag{4.135}$$

Now 4.132 follows from the fundamental theorem of calculus and 4.133, noting that by the Convergence Lemma 4.19, $\lim_{t\to 0}(se^{-i\omega} - A)^{-1}\exp[(-ise^{-i\omega} + iA)t]y = (se^{-i\omega} - A)^{-1}y$, and by 4.134, $\lim_{t\to\infty}(se^{-i\omega} - A)^{-1}\exp[(-ise^{-i\omega} + iA)t]y = 0$. In a similar way, one also shows

$$(se^{i\omega} - A)^{-1}y = -i\int_{-\infty}^0 \exp[(-ise^{i\omega} + iA)t]ydt.$$

Plugging this into the Cauchy integral formula, we get

$$\begin{aligned} f(A)y &= -\frac{e^{i\omega}}{2\pi i}\int_0^\infty f(se^{i\omega})(se^{i\omega}-A)^{-1}yds + \frac{e^{-i\omega}}{2\pi i}\int_0^\infty f(se^{-i\omega})(se^{-i\omega}-A)^{-1}yds \\ &= \sum_{j=\pm 1}(-1)^{j+1}\frac{e^{ij\omega}}{2\pi}\int_0^\infty\int_0^{j\infty} f(se^{ij\omega})\exp[(-ise^{ij\omega}+iA)t]ydtds \\ &\overset{(*)}{=} \sum_{j=\pm 1}(-1)^{j+1}\frac{e^{ij\omega}}{2\pi}\int_0^{j\infty}\int_0^\infty f(se^{ij\omega})\exp[(-ise^{ij\omega}+iA)t]ydsdt \\ &\overset{(**)}{=} \frac{1}{2\pi}\int_{-\infty}^0\int_0^\infty f(s)\exp[-ist+itA]ydsdt + \frac{1}{2\pi}\int_0^\infty\int_0^\infty f(s)\exp[-ist+itA]ydsdt \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \hat{f}(t)e^{itA}ydt. \end{aligned}$$

In $(*)$, we applied Fubini's theorem: Note that by 4.135

$$\int_0^\infty\int_0^{\mp\infty}|f(se^{\pm i\omega})|\cdot\|\exp[(-ise^{\pm i\omega}+iA)t]y\|dtds \lesssim \int_0^\infty |f(se^{\pm i\omega})|s^{-(M+1)}ds,$$

which is finite due to the assumption $f\in K$. In $(**)$, we made a contour shift $se^{\pm i\omega}\rightsquigarrow se^{0\omega}$ which is allowed again due to the assumption $f\in K$.

Up to now we have shown that for $f\in K$ and $x\in D$,

$$f(A)(1+A)^{-\alpha}x = \frac{1}{2\pi}\int_{\mathbb{R}}\hat{f}(t)e^{itA}(1+A)^{-\alpha}xdt.$$

(2) Let $f\in C_c^\infty(0,\infty)$. Recall the holomorphic approximation $f_n = f*\check{\psi}_n$ as in lemma 4.15, i.e. $\psi_n = \psi(2^{-n}(\cdot))$, where $\psi\in C_c^\infty$ with $\psi(t)=1$ for $|t|\leqslant 1$. We have seen in this lemma, that $f_n\to f$ in W_r^β. Let us check that also $f_n|_{(0,\infty)}\to f$ in $\mathcal{M}^\gamma$. It clearly suffices to show that

$$\sup_{t>0}|t^k(f_n-f)^{(k)}(t)|\to 0 \quad (k\in\mathbb{N}_0). \tag{4.136}$$

For $k=0$, this follows from lemma 4.15. For $k\geqslant 1$,

$$t^k(f_n-f)^{(k)}(t) = t^k(f*\check{\psi}_n - f)^{(k)}(t) = (-1)^k[(\xi^k\hat{f}(\xi)(\psi_n(\xi)-1))^{(k)}]\check{}\,(t).$$

Since $f\in C_c^\infty$, $\xi^k\hat{f}(\xi)$ belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$. Now it clearly suffices to show that for any $\tilde{f}\in\mathcal{S}(\mathbb{R})$, $[\tilde{f}(\xi)\psi_n^{(k)}(\xi)]\check{}\,(t)$ converges to 0 uniformly in $t\in\mathbb{R}$. By the definition of ψ_n, we have $|\tilde{f}(\xi)\psi_n^{(k)}(\xi)| = 2^{-nk}|\tilde{f}(\xi)\psi^{(k)}(2^{-n}\xi)|$, which clearly converges to 0 in $L^1(\mathbb{R})$. Hence, 4.136 follows.

In order to get an approximation in K, we consider a modification $g_n = f_n\cdot h_n$. Put $h_n(\lambda) = \exp(-\frac{1}{n\lambda}-\frac{\lambda^2}{n^2})$. One checks that there exist $c_1,c_2>0$ such that for $\lambda\in\Sigma_{\frac{\pi}{8}}$,

$$|h_n(\lambda)|\leqslant \exp(-\frac{c_1}{n\operatorname{Re}\lambda}-\frac{c_2(\operatorname{Re}\lambda)^2}{n^2}). \tag{4.137}$$

In particular, $h_n \in H^\infty(\Sigma_{\frac{\pi}{8}})$. Continuing h_n by 0 for negative arguments, we have $h_n \in C^\infty(\mathbb{R})$. We check that $g_n = f_n \cdot h_n \in K$: Clearly, g_n is a holomorphic function on $\Sigma_{\frac{\pi}{8}}$. By the Paley-Wiener theorem, $|f_n(\lambda)| \lesssim e^{C|\operatorname{Im}\lambda|}$, so that by 4.137, the product with h_n is bounded on $\Sigma_{\frac{\pi}{8}}$ and satisfies the claimed decay properties at 0 and ∞. We have $g_n \in W^\beta_r$, since $f_n \in W^\beta_r$ and h_n and all its derivatives are bounded on $\mathbb{R}$.

Furthermore, $h_n - 1$ and all its derivatives converge locally uniformly to 0 on $\mathbb{R}_+$. Thus, $f_n \cdot h_n - f = f_n \cdot (h_n - 1) + f_n - f$ converges to 0 in W^β_r and in $\mathcal{M}^\gamma$ (as functions on $\mathbb{R}$ and $\mathbb{R}_+$ resp).

(3) Let $f \in C^\infty_c(0,\infty)$ and g_n the approximating sequence as in (2). Then $g_n(A)(1+A)^{-\alpha} \to f(A)(1+A)^{-\alpha}$ by the $\mathcal{M}^\gamma$ calculus, and $\int_\mathbb{R} \hat{g_n}(t)e^{itA}(1+A)^{-\alpha}xdt \to \int_\mathbb{R} \hat{f}(t)e^{itA}(1+A)^{-\alpha}xdt$, since

$$\int_\mathbb{R} |\hat{g_n}(t) - \hat{f}(t)|\, \|e^{itA}(1+A)^{-\alpha}x\| dt \lesssim \|(\hat{g_n}(t) - \hat{f}(t))\langle t\rangle^\beta\|_{r'} \lesssim \|g_n - f\|_{W^\beta_r} \to 0.$$

By (1), the claim follows. □

A key role in the proof of theorem 4.73 (a) is the following result due to Hytönen and Veraar, which produces R-bounded sets by integrating operator families against functions in $L^{r'}(\mathbb{R})$.

Theorem 4.78 *[61, prop 4.1, rem 4.2] Let X be a Banach space and (Ω,μ) a σ-finite measure space. Let $r \in [1,\infty)$ satisfy $\dfrac{1}{r} > \dfrac{1}{\text{type}\,X} - \dfrac{1}{\text{cotype}\,X}$. Further let $T \in L^r(\Omega, B(X))$. Denote r' the conjugated exponent to r.*

Then $(T(\omega) : \omega \in \Omega)$ is $R[L^{r'}(\Omega)]$-bounded, i.e. the set

$$\{T_f : \|f\|_{L^{r'}(\Omega)} \leqslant 1\}$$

is R-bounded, where $T_f x = \int_\Omega f(\omega)T(\omega)xd\omega$.

As remarked in [61, rem 4.2], the condition $T \in L^r(\Omega, B(X))$ can be relaxed to: $T : \Omega \to B(X)$ is strongly measurable and $\omega \mapsto \|T(\omega)\|_{B(X)}$ is dominated by a function in $L^r(\Omega)$. We will make use of this relaxation in the following proposition which essentially proves (a) of theorem 4.73.

Proposition 4.79 *Let A be a 0-sectorial operator on a Banach space with property (α) having an H^∞ calculus. Recall the conditions*

$$\|A^{it}\| \leqslant C(1+|t|)^\alpha, \qquad (N_{\text{BIP}})_\alpha$$

A has a $\mathcal{M}^\gamma$ calculus for some (large) $\gamma > 0$ and $\qquad (\widetilde{N_W})_\alpha$

$$\sup_{s\in\mathbb{R}} R(\{\langle s\rangle^{-\alpha}(1+2^kA)^{-\alpha}e^{is2^kA} : k \in \mathbb{Z}\}) < \infty,$$

A has an R-bounded $\mathcal{H}^\beta_r$ calculus. $\qquad (C_r)_\beta$

Let $r \in (1,2]$, $\frac{1}{r} > \frac{1}{\text{type}\,X} - \frac{1}{\text{cotype}\,X}$ and $\beta > \alpha + \frac{1}{r}$. Then

$$(N_{\text{BIP}})_\alpha \text{ or } (\widetilde{N_W})_\alpha \text{ imply } (C_r)_\beta.$$

Proof. By proposition 4.65, the assumptions imply that A has a $\mathcal{M}^\gamma$ calculus for some $\gamma > 0$.

Consider first the case that A satisfies $(N_{\text{BIP}})_\alpha$. Write $B = \log(A)$ and $U(t) = A^{it}$. By the localization principle 4.64, it suffices to check that

$$R(\{f(B) : \|f\|_{W^\beta_r} \leqslant 1\}) < \infty.$$

Clearly, $t \mapsto \langle t\rangle^{-\beta}\|U(t)\|$ is dominated by a function in $L^r(\mathbb{R})$. Indeed,

$$\langle t\rangle^{-\beta}\|U(t)\| = \langle t\rangle^{-(\beta-\alpha)}(\langle t\rangle^{-\alpha}\|U(t)\|),$$

and the first factor is in $L^r(\mathbb{R})$ by the choice of β, and the second factor is bounded by the assumption $(N_{\text{BIP}})_\alpha$. In particular, by proposition 4.22, B has a W^β_r calculus, and for $f \in W^\beta_r$, $\int_\mathbb{R} \hat{f}(t)U(t)dt$ exists even as a strong integral according to remark 4.23 (2), so that

$$f(B)x = \frac{1}{2\pi}\int_\mathbb{R} \langle t\rangle^\beta \hat{f}(t)\langle t\rangle^{-\beta}U(t)x dt \quad (x \in X).$$

Since $r \leqslant 2$, we have by the Hausdorff-Young inequality $\|\hat{f}(t)\langle t\rangle^\beta\|_{L^{r'}(\mathbb{R})} \lesssim \|f\|_{W^\beta_r}$. Further, by the assumption $\frac{1}{r} > \frac{1}{\text{type}\, X} - \frac{1}{\text{cotype}\, X}$, we can apply theorem 4.78 and consequently,

$$R(\{f(B) : \|f\|_{W^\beta_r} \leqslant 1\}) \lesssim R(\{f(B) : \|\hat{f}(t)\langle t\rangle^\beta\|_{L^{r'}(\mathbb{R})} \leqslant 1\}) < \infty.$$

Consider now the case that A satisfies $(\widetilde{N_W})_\alpha$. We want to apply the second part of the localization principle 4.64. Let

$$S_k(t) = \langle t\rangle^{-\alpha}(1 + 2^k A)^{-\alpha}e^{it2^k A}$$

and $S(t)$ be the operator on $\text{Rad}(X)$ defined by

$$S(t)\left(\sum_{k\in\mathbb{Z}} \varepsilon_k \otimes x_k\right) = \sum_{k\in\mathbb{Z}} \varepsilon_k \otimes S_k(t)x_k.$$

Strictly speaking, the right hand side might not be convergent in $\text{Rad}(X)$, and this minor problem can be solved by considering the finite dimensional subspaces

$$\text{span}\,\{\varepsilon_k \otimes x_k : k \in \{-N, \dots, N\}\} \subset \text{Rad}(X),$$

taking care of the (in)dependence of estimates on N in the following, and passing to the limit $N \to \infty$ only after 4.139.

Then $(\widetilde{N_W})_\alpha$ means that $\sup_{t\in\mathbb{R}} \|S(t)\|_{B(\text{Rad}(X))} < \infty$. The space $\text{Rad}(X)$ inherits from X the property (α) and its type and cotype. Indeed, by definition, $\text{Rad}(X)$ is a closed subspace of some $L^2(\Omega_0, X)$, and thus, as mentioned in chapter 2, section 2.5, $\text{Rad}(X)$ has property (α) and the same type and cotype as X. Put

$$\delta := \beta - \alpha > \frac{1}{r}.$$

Then

$$t \mapsto \langle t\rangle^{-\delta}\|S(t)\| \text{ is dominated by a function in } L^r(\mathbb{R}). \tag{4.138}$$

For $f \in L^{r'}(\mathbb{R})$ and $k \in \mathbb{Z}$, put

$$S_k^f : X \to X,\, x \mapsto \int_{\mathbb{R}} f(t)\langle t\rangle^{-\delta} S_k(t)x dt$$

and

$$S^f : \operatorname{Rad}(X) \to \operatorname{Rad}(X),\, y \mapsto \int_{\mathbb{R}} f(t)\langle t\rangle^{-\delta} S(t)y dt.$$

Strictly speaking, we define S_k^f first for $x \in D$, so that the measurability of $S_k^f(t)x$ is ensured by 4.130 from lemma 4.77, and then extend continuously to X by 4.138. A similar comment holds for S^f. By theorem 4.78 and 4.138, $\{S^f : \|f\|_{L^{r'}(\mathbb{R})} \leqslant 1\}$ is R-bounded in $B(\operatorname{Rad}(X))$. This implies that also

$$\{S_k^f : \|f\|_{L^{r'}(\mathbb{R})} \leqslant 1,\, k \in \mathbb{Z}\} \text{ is } R\text{-bounded in } B(X). \tag{4.139}$$

Indeed, let $x_i \in X$, $k_i \in \mathbb{Z}$ and $f_i \in L^{r'}(\mathbb{R})$ with $\|f_i\|_{r'} \leqslant 1$. Put $y_i = \varepsilon_{k_i} \otimes x_i \in \operatorname{Rad}(X)$. Then

$$\begin{aligned}\|\sum_i \varepsilon_i \otimes S_{k_i}^{f_i} x_i\|_{\operatorname{Rad}(X)} &= \|\sum_i \varepsilon_i \otimes \varepsilon_{k_i} \otimes S_{k_i}^{f_i} x_i\|_{\operatorname{Rad}(\operatorname{Rad}(X))} \\ &= \|\sum_i \varepsilon_i \otimes S^{f_i} y_i\|_{\operatorname{Rad}(\operatorname{Rad}(X))} \\ &\lesssim \|\sum_i \varepsilon_i \otimes y_i\|_{\operatorname{Rad}(\operatorname{Rad}(X))} \\ &= \|\sum_i \varepsilon_i \otimes x_i\|_{\operatorname{Rad}(X)}.\end{aligned}$$

Let $g \in C_c^\infty(0,\infty)$. By lemma 4.77, for $x \in D$,

$$\begin{aligned} g(2^k A)(1+2^k A)^{-\alpha} x &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(t) e^{it2^k A}(1+2^k A)^{-\alpha} x dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}(t)\langle t\rangle^{\beta}\langle t\rangle^{-\delta} S_k(t) x dt \\ &= S_k^f x,\end{aligned}$$

with $f(t) = \hat{g}(t)\langle t\rangle^\beta$. Thus by 4.139,

$$R(\{g(2^k A)(1+2^k A)^{-\alpha} : g \in C_c^\infty(0,\infty),\, \|g\|_{W_r^\beta} \leqslant 1,\, k \in \mathbb{Z}\}) < \infty.$$

Put $\psi(\lambda) = (1+\lambda)^\alpha \varphi(\lambda)$, where $\varphi \in C_c^\infty(0,\infty)$ equals 1 on $[\frac{1}{2}, 2]$.

We claim that

$$\{\psi(2^k A) : k \in \mathbb{Z}\} \text{ is } R\text{-bounded.} \tag{4.140}$$

Indeed, by assumption, A has a $\mathcal{M}^\gamma$ calculus for some $\gamma > 0$, and thus, by proposition 4.12, in particular $(N_{\mathrm{BIP}})_\gamma$ holds. The first part of the proof with γ instead of α in turn implies that A has an R-bounded $\mathcal{H}_r^{\gamma'}$ calculus. By proposition 4.8, $\sup_{k\in\mathbb{Z}} \|\psi(2^k \cdot)\|_{\mathcal{H}_r^{\gamma'}} = \|\psi\|_{\mathcal{H}_r^{\gamma'}}$, which is clearly finite, so that 4.140 follows. Hence

$$R(\{g(2^k A) : g \in C_c^\infty(\mathbb{R}_+),\, \operatorname{supp} g \subset [\frac{1}{2}, 2],\, \|g\|_{W_r^\beta} \leqslant 1,\, k \in \mathbb{Z}\}) < \infty.$$

The claim now follows from corollary 4.64. □

The next proposition essentially proves (b) of theorem 4.73.

Proposition 4.80 *Let A be a 0-sectorial operator having an H^∞ calculus on a Banach space with property (α). Recall the conditions*

$$\text{The set } \{\langle t\rangle^{-\alpha}A^{it} : t \in \mathbb{R}\} \text{ is } R\text{-bounded.} \qquad (R_{\text{BIP}})_\alpha$$

$$\text{The set } \{(\frac{\pi}{2}-|\theta|)^\alpha T(e^{i\theta}t) : t>0,\, \theta \in (-\frac{\pi}{2},\frac{\pi}{2})\} \text{ is } R\text{-bounded.} \qquad (R_T)_\alpha$$

$$\text{The function } t \mapsto \langle t\rangle^{-\alpha}A^{it}x \text{ belongs to } \gamma(\mathbb{R},X) \text{ with norm } \lesssim \|x\| \text{ for } x \in X. \qquad (S_{\text{BIP}})_\alpha$$

$$\text{The function } t \mapsto A^{1/2}T(e^{i\theta}t)x \text{ belongs to } \gamma(\mathbb{R}_+,dt,X) \qquad (S_T)_\alpha$$

$$\text{with norm } \lesssim (\frac{\pi}{2}-|\theta|)^{-\alpha}\|x\| \text{ for } x\in X \text{ and } \theta \in (-\frac{\pi}{2},\frac{\pi}{2}).$$

Then

$$(R_{\text{BIP}})_\alpha \Longrightarrow (S_{\text{BIP}})_\beta \quad (\beta > \alpha + \frac{1}{2})$$

and

$$(R_T)_\alpha \Longrightarrow (S_T)_\beta \quad (\beta \geqslant \alpha + \frac{1}{2}).$$

Proof. Assume that $(R_{\text{BIP}})_\alpha$ holds. Then by lemma 2.7,

$$\|\langle t\rangle^{-\beta}A^{it}x\|_{\gamma(\mathbb{R},X)} \leqslant \|\langle t\rangle^{-(\beta-\alpha)}\|_{L^2(\mathbb{R})}R(\{\langle t\rangle^{-\alpha}A^{it} : t\in\mathbb{R}\})\|x\| \lesssim \|x\|.$$

Thus, $(S_{\text{BIP}})_\beta$ follows.

Assume that $(R_T)_\alpha$ holds. By [72, thm 7.2, prop 7.7], the fact that A has an H^∞ calculus implies that for $x \in D(A)\cap R(A)$, $\|A^{\frac12}T(t)x\|_{\gamma(\mathbb{R}_+,X)} \lesssim \|x\|$, so that by lemma 2.7 with the isomorphic mapping $L^2(\mathbb{R}_+,dt) \to L^2(\mathbb{R}_+,dt), t\mapsto ts$

$$\|A^{\frac12}T(ts)x\|_{\gamma(\mathbb{R}_+,X)} \lesssim s^{-\frac12}\|x\|$$

for $s>0$. Decompose

$$e^{\pm i(\frac{\pi}{2}-\omega)} = re^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})} + s$$

where $s,r>0$ are uniquely determined. By the law of sines, $s \cong \omega$ for $\omega \to 0+$. Then

$$A^{\frac12}T(te^{\pm i(\frac{\pi}{2}-\omega)}) = T(tre^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})})\circ A^{\frac12}T(ts).$$

Therefore, by assumption $(R_T)_\alpha$,

$$\begin{aligned}\|A^{\frac12}T(te^{\pm i(\frac{\pi}{2}-\omega)})x\|_{\gamma(\mathbb{R}_+,X)} &\leqslant R(\{T(tre^{\pm i(\frac{\pi}{2}-\frac{\omega}{2})}) : t>0\})\|A^{\frac12}T(ts)x\|_{\gamma(\mathbb{R}_+,X)}\\ &\lesssim \omega^{-\alpha}\|A^{\frac12}T(ts)x\|_{\gamma(\mathbb{R}_+,X)}\\ &\lesssim \omega^{-\alpha}\omega^{-\frac12}\|A^{\frac12}T(t)x\|_{\gamma(\mathbb{R}_+,X)}\\ &\lesssim \omega^{-\alpha-\frac12}\|x\|.\end{aligned}$$

Now $(S_T)_{\alpha+\frac12}$ follows. □

We are finally in the position to prove theorems 4.73 and 4.74.

Proof of theorem 4.73. (d) This is proved in theorem 4.46, except the equivalence $(R(L2)_{\mathrm{BIP}})_\alpha \Leftrightarrow (C_2)_\alpha$. We do already know from theorem 4.46 that $(R(L2)_{\mathrm{BIP}})_\alpha$ is equivalent to the R-bounded W_2^α calculus of $B = \log(A)$. By proposition 4.71, B has a $\mathcal{B}^\gamma$ calculus for some (large) $\gamma > 0$. Consequently, by the localization principle from corollary 4.64, this is equivalent to the R-bounded $\mathcal{W}_2^\alpha$ calculus of B, i.e. to $(C_2)_\alpha$.

(e) This is entirely covered by proposition 4.50.

(a) For $(N_{\mathrm{BIP}})_\alpha$ this follows directly from proposition 4.79, for $(N_T)_\alpha$ and $(N_W)_\alpha$, we appeal in addition to the implications $(N_W)_\alpha \Longrightarrow (N_T)_\alpha \Longrightarrow (\widetilde{N_W})_{\alpha+\varepsilon}$ from lemma 4.72.

(b) For $(R_{\mathrm{BIP}})_\alpha$ and $(R_T)_\alpha$, the statement is directly covered by proposition 4.80 and (e). By

$$(R_W)_\alpha \Longrightarrow (R_T)_\alpha \Longrightarrow (\widetilde{R_W})_{\alpha+\varepsilon} \tag{4.141}$$

from lemma 4.72, we also have $(R_W)_\alpha \Rightarrow (R_T)_\alpha \Rightarrow (S_T)_{\alpha+\frac{1}{2}}$.

(c) We let $\alpha_{\text{(IV)}} < \alpha_{\text{(III)}} < \alpha_{\text{(II)}} = \alpha_{\text{(I)}}$.

Let us show that any of the conditions in (IV) implies any of the conditions in (III) respectively.

By (d) and (e), it suffices to know this for one condition in (IV) and (III). Now e.g. $(S_{\mathrm{BIP}})_{\alpha_{\text{(IV)}}} \Longrightarrow (R(L2)_{\mathrm{BIP}})_{\alpha_{\text{(III)}}}$ follows from theorem 4.36.

Let us show that any of the conditions in (III) implies any of the conditions in (II).

Again by (d), we may take $(C_2)_{\alpha_{\text{(III)}}}$ instead of any of the conditions under (III). To show $(R_W)_{\alpha_{\text{(II)}}}$, it suffices to estimate $\|\lambda \mapsto (1+|t|\lambda)^{-\alpha} e^{it\lambda}\|_{\mathcal{H}_2^{\alpha'}}$ for $\alpha = \alpha_{\text{(II)}} > \alpha' = \alpha_{\text{(III)}}$. By propositions 4.9 and 4.12 (4), we have

$$\|\lambda \mapsto (1+|t|\lambda)^{-\alpha} e^{it\lambda}\|_{\mathcal{H}_2^{\alpha'}} \lesssim \|\lambda \mapsto (1+|t|\lambda)^{-\alpha} e^{it\lambda}\|_{\mathcal{M}^{\alpha''}} = \|\lambda \mapsto (1+\lambda)^{-\alpha} e^{\pm i\lambda}\|_{\mathcal{M}^{\alpha''}} < \infty$$

for any auxiliary $\alpha'' \in (\alpha', \alpha)$. This clearly implies that $\{(1+|t|A)^{-\alpha} e^{itA} : t \in \mathbb{R}\}$ is R-bounded, i.e. $(R_W)_\alpha$ holds.

In the same way, with (4) instead of (2) in proposition 4.12, one shows that also $(R_{\mathrm{BIP}})_\alpha$ holds. Condition $(R_T)_\alpha$ follows then from 4.141. □

Proof of theorem 4.74. Considering the 0-sectorial operator $A = e^B$, (a) and (b) follow from the sectorial theorem 4.73. Recall the first stated chain of implications in (c):

$$(\text{IVa})_\alpha \Longrightarrow (\text{IIIa})_\alpha \Longrightarrow (\text{II})_\alpha \Longrightarrow (\text{I})_\alpha.$$

This also follows from the sectorial theorem (the last implication is trivial), except that in $(\text{IIIa})_\alpha \Longrightarrow (\text{II})_\alpha$ there is in fact no loss of ε in the parameter α : From the sectorial theorem, we know that $(\text{IIIa})_\alpha$ implies that B has an R-bounded $\mathcal{W}_2^\alpha$ calculus. It now suffices to estimate

$\|\langle t\rangle^{-\alpha} e^{it(\cdot)}\|_{\mathcal{W}_2^\alpha}$, without the detour of the Mihlin norm as in the proof of the sectorial theorem. We have for an equidistant partition of unity $(\varphi_n)_n$ as in the definition of $\mathcal{W}_2^\alpha$,

$$\begin{aligned}\|e^{it(\cdot)}\|_{\mathcal{W}_2^\alpha} &= \sup_{n\in\mathbb{Z}} \|e^{it(\cdot)}\varphi_n\|_{W_2^\alpha} \\ &\cong \sup_n \|\hat{\varphi}(\cdot - t)e^{-in(\cdot)}e^{int}\langle\cdot\rangle^\alpha\|_2 \\ &= \|\hat{\varphi}(\cdot)\langle\cdot + t\rangle^\alpha\|_2 \\ &\leqslant \|\hat{\varphi}(\cdot)\langle\cdot\rangle^\alpha\|_2\,\|\langle\cdot + t\rangle^\alpha/\langle\cdot\rangle^\alpha\|_\infty \\ &\lesssim \langle t\rangle^\alpha,\end{aligned}$$

so that $(\mathrm{II})_\alpha$ follows.

The second line in (c) was

$$(\mathrm{IVb})_\alpha \Longrightarrow (\mathrm{IIIb})_\alpha \Longrightarrow (\mathrm{II})_{\alpha+\varepsilon}.$$

The first implication follows from proposition 4.50, and the second implication from the sectorial theorem.

The third line in (c) was

$$(C_2)_\alpha \Longleftrightarrow (\mathrm{IIIa})_\alpha \Longrightarrow (\mathrm{IIIb})_\alpha \Longrightarrow (C_2)_{\alpha+\varepsilon}.$$

The first equivalence is covered by the sectorial theorem, and the other implications follow again from proposition 4.50, together with the localization principle in the last step.

Finally, the fourth line in (c) is covered again by proposition 4.50. □

4.6 Examples for the Hörmander functional calculus

In this section, we investigate how our theory developed in sections 4.4 and 4.5 compares to spectral multiplier theorems in the literature. We show that many Hörmander multiplier results for particular operators are covered by theorem 4.73, and thus the latter provides a unified view of these results. Moreover, we obtain R-bounded multiplier results instead of simple boundedness, and in some cases (see remark 4.96 below), the differentiation order of the Hörmander calculus can also be lowered compared to known results.

4.6.1 Interpolation

In many interesting examples for the Hörmander functional calculus, the operator A is defined on an interpolation scale like $L^p(\Omega)$ for all p belonging to some interval (p_0, p_1) containing 2. Suppose that for all these p, A has a $\mathcal{H}_q^\alpha$ calculus on $L^p(\Omega)$ for $\alpha = \alpha(p)$ and $q = q(p)$. Suppose moreover that A is self-adjoint and positive on $L^2(\Omega)$, so that A has the classical calculus for self-adjoint operators, allowing bounded Borel functions, which is larger than the $\mathcal{H}_q^\alpha$ calculus. Then by complex interpolation, the classical calculus improves the $\mathcal{H}_q^\alpha$ calculus on $L^p(\Omega)$. In this subsection, we work out this interpolation procedure in an abstract framework.

On the other hand, in some examples, the operator A satisfies the assumptions of 0-sectorialness in definition 4.13, except the injectivity and the dense range. We will call such an operator pseudo-0-sectorial. This is not a substantial obstruction for the functional calculus. Indeed, if the underlying space X is reflexive, then there exists a decomposition method such that A becomes injective on a subspace of X (see [81, prop 15.2], [24, thm 4.1]). This decomposition behaves well with respect to the complex interpolation procedure mentioned above, see lemma 4.86.

We start with an interpolation result for the Hörmander function spaces.

Recall that if (X, Y) is a complex interpolation couple, then we let $F(X, Y)$ be the space of all functions f defined on the strip $S = \{0 \leqslant \operatorname{Re} z \leqslant 1\}$ and with values in $X + Y$ such that f is continuous on S, analytic in the interior of S, $f|_{\operatorname{Re} z=0}$ has values in X, and $f|_{\operatorname{Re} z=1}$ has values in Y. Then $F(X, Y)$ is normed by $\|f\|_{F(X,Y)} = \sup_{\tau\in\mathbb{R}}(\|f(i\tau)\|_X, \|f(1 + i\tau)\|_Y)$.

For $\theta \in (0, 1)$, the complex interpolation space $(X, Y)_\theta$ equals $\{z \in X+Y : \exists f \in F(X, Y),\ f(\theta) = z\}$ and is normed by $\|z\|_{(X,Y)_\theta} = \inf\{\|f\|_{F(X,Y)} :\ f(\theta) = z\}$ (cf. [6, sect 4.1]).

Lemma 4.81 *Let $p_0, p_1 \in (1, \infty)$ and $\alpha_0 > \frac{1}{p_0}, \alpha_1 > \frac{1}{p_1}$. Then the spaces $\mathcal{W}^{\alpha_0}_{p_0}$ and $\mathcal{W}^{\alpha_1}_{p_1}$ form a complex interpolation couple by the embedding $\mathcal{W}^{\alpha_j}_{p_j} \subset L^\infty(\mathbb{R})$. For $\theta \in (0, 1)$, set $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\alpha_\theta = (1 - \theta)\alpha_0 + \theta\alpha_1$. Then*

$$(\mathcal{W}^{\alpha_0}_{p_0}, \mathcal{W}^{\alpha_1}_{p_1})_\theta \supset \mathcal{W}^{\alpha_\theta}_{p_\theta,0} := \{f \in \mathcal{W}^{\alpha_\theta}_{p_\theta} :\ \|f\varphi_n\|_{W^{\alpha_\theta}_{p_\theta}} \to 0 \text{ for } |n| \to \infty\}$$

and for $f \in \mathcal{W}^{\alpha_\theta}_{p_\theta,0}$, $\|f\|_{\mathcal{W}^{\alpha_\theta}_{p_\theta}} \cong \|f\|_{(\mathcal{W}^{\alpha_0}_{p_0},\mathcal{W}^{\alpha_1}_{p_1})_\theta}$. Here, of course, $(\varphi_n)_n$ is an equidistant partition of unity.

Proof. Let $f \in \mathcal{W}^{\alpha_\theta}_{p_\theta,0}$. We show that $f \in \mathcal{W}(\theta) := (\mathcal{W}^{\alpha_0}_{p_0}, \mathcal{W}^{\alpha_1}_{p_1})_\theta$ and

$$\|f\|_{\mathcal{W}(\theta)} \lesssim \|f\|_{\mathcal{W}^{\alpha_\theta}_{p_\theta}}. \tag{4.142}$$

Since the elements of compact support are dense in $\mathcal{W}^{\alpha_\theta}_{p_\theta,0}$, we can assume that $f\varphi_n = 0$ for $|n| > N$ for some $N \in \mathbb{N}$. As $f\varphi_n \in W^{\alpha_\theta}_{p_\theta} \subset W^{\alpha_0}_{p_0} + W^{\alpha_1}_{p_1}$, it is clear that $f \in \mathcal{W}^{\alpha_0}_{p_0} + \mathcal{W}^{\alpha_1}_{p_1}$. For $n \in \mathbb{Z}$ fixed, $f\varphi_n$ belongs to $W^{\alpha_\theta}_{p_\theta}$, which equals $(W^{\alpha_0}_{p_0}, W^{\alpha_1}_{p_1})_\theta$ according to [127, section 2.4.2]. Thus for any $\varepsilon > 0$, there is $\tilde{g}_n \in F(W^{\alpha_0}_{p_0}, W^{\alpha_1}_{p_1})$ with $\tilde{g}_n(\theta) = f\varphi_n$ and

$$\|\tilde{g}_n\|_{F(W^{\alpha_0}_{p_0},W^{\alpha_1}_{p_1})} \leqslant \|f\varphi_n\|_{W^{\alpha_\theta}_{p_\theta}} + \varepsilon.$$

Put now $g_n(\cdot) = \tilde{g}_n(\cdot) \cdot \psi_n$, where $\psi_n \in C^\infty_c$ with $\operatorname{supp} \psi_n \subset [n - 2, n + 2]$ and $\psi_n = 1$ on $[n - 1, n + 1]$, so that still $g_n(\theta) = f\varphi_n$, and moreover $g_n(z)$ and φ_j have disjoint supports for any z belonging to the strip S and any $|n - j| > 2$. Assemble $g(z) = \sum_{n=-N}^{N} g_n(z)$. As a finite sum, g clearly belongs to $F(\mathcal{W}^{\alpha_0}_{p_0}, \mathcal{W}^{\alpha_1}_{p_1})$. Then

$$\begin{aligned}
\|g\|_{F(\mathcal{W}^{\alpha_0}_{p_0},\mathcal{W}^{\alpha_1}_{p_1})} &= \|\sum_n g_n\|_{F(\mathcal{W}^{\alpha_0}_{p_0},\mathcal{W}^{\alpha_1}_{p_1})} = \sup_j \|\sum_n g_n\varphi_j\|_{F(W^{\alpha_0}_{p_0},W^{\alpha_1}_{p_1})} \\
&= \sup_j \sum_{|n-j|\leqslant 2} \|g_n\varphi_j\|_{F(W^{\alpha_0}_{p_0},W^{\alpha_1}_{p_1})} \lesssim \sup_n \|g_n\|_{F(W^{\alpha_0}_{p_0},W^{\alpha_1}_{p_1})} \\
&\lesssim \sup_n \|f\varphi_n\|_{W^{\alpha_\theta}_{p_\theta}} + \varepsilon = \|f\|_{\mathcal{W}^{\alpha_\theta}_{p_\theta}} + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ yields $\|f\|_{\mathcal{W}(\theta)} \lesssim \|f\|_{\mathcal{W}^{\alpha_\theta}_{p_\theta}}$.

On the other hand,

$$\begin{aligned}
\|f\|_{\mathcal{W}^{\alpha_\theta}_{p_\theta}} &= \sup_{n\in\mathbb{Z}} \|f\varphi_n\|_{W^{\alpha_\theta}_{p_\theta}} \\
&= \sup_n \|f\varphi_n\|_{\mathcal{W}(\theta)} \\
&= \sup_n \inf\{\|g_n\|_{F(W^{\alpha_0}_{p_0}, W^{\alpha_1}_{p_1})} : g_n \in F(W^{\alpha_0}_{p_0}, W^{\alpha_1}_{p_1}),\, g_n(\theta) = f\varphi_n\} \\
&= \sup_n \inf\{\|g\varphi_n\|_{F(W^{\alpha_0}_{p_0}, W^{\alpha_1}_{p_1})} : g \in F(\mathcal{W}^{\alpha_0}_{p_0}, \mathcal{W}^{\alpha_1}_{p_1}),\, g(\theta) = f\} \\
&\leqslant \inf\{\sup_n \|g\varphi_n\|_{F(W^{\alpha_0}_{p_0}, W^{\alpha_1}_{p_1})} : g \in F(\mathcal{W}^{\alpha_0}_{p_0}, \mathcal{W}^{\alpha_1}_{p_1}),\, g(\theta) = f\} \\
&= \|f\|_{\mathcal{W}(\theta)}.
\end{aligned}$$

□

The restriction to functions vanishing at ∞ in the above interpolation lemma disappears when dealing with the functional calculus, as the next lemma shows.

Lemma 4.82 *Let B be a 0-strip-type operator having a $\mathcal{B}^\beta$ calculus for some $\beta > 0$. Let $p \in (1, \infty)$ and $\alpha > \frac{1}{p}$.*

If there exists a constant $C > 0$ such that

$$\|f(B)\| \leqslant C\|f\|_{\mathcal{W}^\alpha_p} \quad (f \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap \mathcal{W}^\alpha_{p,0}), \tag{4.143}$$

then B has a bounded $\mathcal{W}^\alpha_p$ calculus.

If in addition

$$\{f(B) : f \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega) \cap \mathcal{W}^\alpha_{p,0},\ \|f\|_{\mathcal{W}^\alpha_p} \leqslant 1\} \text{ is } R\text{-bounded}, \tag{4.144}$$

then B has an R-bounded $\mathcal{W}^\alpha_p$ calculus.

Proof. Let $(\varphi_k)_k$ be an equidistant partition of unity and set $\psi_n = \sum_{k=-n}^n \varphi_k$ for any $n \in \mathbb{N}$. Then $\psi_n(B)x \to x$ by the Convergence Lemma 4.19 for any $x \in X$.

Let $f \in \bigcap_{\omega>0} H^\infty(\mathrm{Str}_\omega)$. We have $f\psi_n \in \mathcal{W}^\alpha_{p,0}$ and $\|f\psi_n\|_{\mathcal{W}^\alpha_p} \lesssim \|f\|_{\mathcal{W}^\alpha_p}\|\psi_n\|_{\mathcal{W}^\alpha_p} \lesssim \|f\|_{\mathcal{W}^\alpha_p}$. Thus,

$$\|f(B)x\| = \lim_n \|f(B)\psi_n(B)x\| = \lim_n \|(f\psi_n)(B)x\| \lesssim \|f\|_{\mathcal{W}^\alpha_p}\|x\|,$$

and the first claim follows directly from definition 4.26.

The second claim is proved by the same approximation procedure. □

The interpolation of 0-sectorial operators and their Hörmander functional calculi now reads as follows.

Proposition 4.83

(1) *Let X and Y be Banach spaces which form a complex interpolation couple. Assume that A_X and A_Y are 0-sectorial operators on X and Y respectively such that their resolvents are compatible, in the sense that*

$$R(\lambda, A_X)x = R(\lambda, A_Y)x \quad (x \in X \cap Y,\, \lambda \notin [0,\infty)). \tag{4.145}$$

Then for any $\theta \in (0,1)$, there is a 0-sectorial operator A_θ on $(X,Y)_\theta$ which is compatible in the sense of 4.145.

(2) *Assume in addition that A_X has a $\mathcal{H}^{\alpha_0}_{p_0}$ calculus and A_Y has a $\mathcal{H}^{\alpha_1}_{p_1}$ calculus for some $p_0, p_1 \in (1,\infty)$ and $\alpha_0 > \frac{1}{p_0}$ and $\alpha_1 > \frac{1}{p_1}$.*

Then for any $\theta \in (0,1)$, A_θ has a $\mathcal{H}^{\alpha_\theta}_{p_\theta}$ calculus, where $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\alpha_\theta = (1-\theta)\alpha_0 + \theta\alpha_1$ are the usual interpolated parameters. Further, the calculi of A_X, A_Y and A_θ are consistent, i.e. $f(A_X)x = f(A_Y)x = f(A_\theta)x$ for $f \in \mathcal{H}^{\alpha_0}_{p_0} \cap \mathcal{H}^{\alpha_1}_{p_1}$ and $x \in X \cap Y$.

(3) *If the $\mathcal{H}^{\alpha_0}_{p_0}$ calculus of A_X and the $\mathcal{H}^{\alpha_1}_{p_1}$ calculus of A_Y is in addition R-bounded and X and Y have type > 1, then the $\mathcal{H}^{\alpha_\theta}_{p_\theta}$ calculus of A_θ is also R-bounded.*

Proof. (1): We write $X_\theta = (X,Y)_\theta$. For $\lambda \in \mathbb{C}\backslash[0,\infty)$, denote $R(\lambda) \in B(X_\theta)$ the interpolated operator of $R(\lambda, A_X)$ and $R(\lambda, A_Y)$. Then $R(\lambda) - R(\mu) = -(\lambda - \mu)R(\lambda)R(\mu)$ for any $\lambda, \mu \in \mathbb{C}\backslash[0,\infty)$, so that $\lambda \mapsto R(\lambda)$ is a pseudo resolvent. By [24, thm 3.8] and the fact that A_X is densely defined, we have $-nR(-n)x = -nR(-n, A_X)x \to x$ in X for any $x \in X \cap Y$. Similarly, $-nR(-n)x \to x$ in Y for any $x \in X \cap Y$. Since $X \cap Y$ is dense in X_θ and $-nR(-n)$ is uniformly bounded in $B(X_\theta)$, $-nR(-n)x \to x$ in X_θ for any $x \in X_\theta$. Then [108, p.37 cor 9.5] yields that $R(\lambda)$ is the resolvent of a closed and densely defined operator A_θ, and it is clear that A_θ inherits the resolvent growth condition for 0-sectorial operators from A_X and A_Y. It remains to show that A_θ has dense range. By [24, thm 3.8], this is the case if and only if

$$\lim_{n\to\infty} \frac{1}{n} R(-\frac{1}{n}, A_\theta)x = 0 \text{ for any } x \in X_\theta. \tag{4.146}$$

Note that again by [24, thm 3.8], the corresponding statement holds for A_X and A_Y on X and Y. This yields that 4.146 holds for $x \in X \cap Y$, and thus for any $x \in X_\theta$ by the uniform boundedness of $\frac{1}{n}R(-\frac{1}{n}, A_\theta)$ in $B(X_\theta)$.

(2): Write in short $E^0 = \mathcal{H}^{\alpha_0}_{p_0}$ and similarly $E^1 = \mathcal{H}^{\alpha_1}_{p_1}$. Consider the bounded bilinear mappings

$$X \times E^0 \to X,\, (x,f) \mapsto f(A_X)x \quad \text{and} \quad Y \times E^1 \to Y,\, (x,f) \mapsto f(A_Y)x. \tag{4.147}$$

We claim that they are compatible, i.e. if $(x,f) \in (X \cap Y) \times (E^0 \cap E^1)$, then $f(A_X)x = f(A_Y)x$. This is clear by assumption if $f(t) = (\lambda - t)^{-1}$ for some $\lambda \in \mathbb{C}\backslash[0,\infty)$. By the Cauchy integral formula, the same holds true for $f \in \bigcup_{\omega>0} H^\infty_0(\Sigma_\omega)$, and then also for $f \in \bigcup_{\omega>0} H^\infty(\Sigma_\omega)$ by the Convergence Lemma of the H^∞ calculus, proposition 2.5. By the density lemma 4.15, this extends to $f \circ \exp \in W^{\alpha_0}_{p_0} \cap W^{\alpha_1}_{p_1}$ and finally to $E^0 \cap E^1$ by definition 4.24.

By multilinear complex interpolation [6, thm 4.4.1], for $\theta \in (0,1)$, the mappings in 4.147 interpolate to a bounded bilinear mapping

$$T : X_\theta \times (E^0, E^1)_\theta \to X_\theta,\, (x,f) \mapsto T(x,f).$$

By the same reasoning as before, $T(x,f) = f(A_\theta)x$ for $x \in X \cap Y$, first for all $f \in \bigcup_{\omega>0} H_0^\infty(\Sigma_\omega)$ and then for $f \in \bigcup_{\omega>0} H^\infty(\Sigma_\omega)$. Thus we have

$$\|f(A_\theta)\|_{B(X_\theta)} \lesssim \|f\|_{(E^0,E^1)_\theta} \quad (f \in \bigcup_{\omega>0} H^\infty(\Sigma_\omega)). \tag{4.148}$$

By the embedding $\mathcal{W}^{\alpha_\theta}_{p_\theta,0} \subset (\mathcal{W}^{\alpha_0}_{p_0}, \mathcal{W}^{\alpha_1}_{p_1})_\theta$ from lemma 4.81, 4.148 implies that $\log(A_\theta)$ satisfies 4.143.

Note that A_X and A_Y have a $\mathcal{M}^{\alpha_0+\varepsilon}$ and $\mathcal{M}^{\alpha_1+\varepsilon}$ calculus according to proposition 4.9 (4). Set $E^0 = \mathcal{M}^{\alpha_0+\varepsilon}$ and $E^1 = \mathcal{M}^{\alpha_1+\varepsilon}$, so that $(E^0,E^1)_\theta = \mathcal{M}^{\alpha_\theta+\varepsilon}$. Then the same reasoning as above yields that A_θ has a $\mathcal{M}^{\alpha_\theta+\varepsilon}$ calculus. Consequently, we can apply the first part of lemma 4.82 to $\log(A_\theta)$ and deduce that A_θ has a $\mathcal{H}^{\alpha_\theta}_{p_\theta}$ calculus.

(3): Assume now that the calculi of A_X and A_Y are R-bounded. Write again $E^0 = \mathcal{H}^{\alpha_0}_{p_0}$ and $E^1 = \mathcal{H}^{\alpha_1}_{p_1}$. Then it is easy to see that

$$c_0(E^0) \times \mathrm{Rad}(X) \to \mathrm{Rad}(X),\ ((f_k)_k, \sum_k \varepsilon_k \otimes x_k) \mapsto \sum_k \varepsilon_k \otimes f_k(A_X)x_k$$

and

$$c_0(E^1) \times \mathrm{Rad}(Y) \to \mathrm{Rad}(Y),\ ((f_k)_k, \sum_k \varepsilon_k \otimes x_k) \mapsto \sum_k \varepsilon_k \otimes f_k(A_Y)x_k$$

define bounded bilinear mappings. Note that $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$ form a complex interpolation couple as subspaces of $L^2(\Omega_0, X+Y)$. Since X and Y have type > 1, $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$ are complemented subspaces of $L^2(\Omega_0,X)$ and $L^2(\Omega_0,Y)$, and thus

$$(\mathrm{Rad}(X), \mathrm{Rad}(Y))_\theta = \mathrm{Rad}((X,Y)_\theta)$$

([111], [30, chapt 13], see also [69, prop 3.7]). Also by [127, 1.18 rem 3], $(c_0(E^0), c_0(E^1))_\theta = c_0((E^0,E^1)_\theta)$. Then again by multilinear complex interpolation [6, thm 4.4.1], the above mappings extend to

$$c_0(\mathcal{H}^{\alpha_\theta}_{p_\theta,0}) \times \mathrm{Rad}((X,Y)_\theta) \to \mathrm{Rad}((X,Y)_\theta),\ ((f_k)_k, \sum_k \varepsilon_k \otimes x_k) \mapsto \sum_k \varepsilon_k \otimes f_k(A_\theta)x_k.$$

This in combination with lemma 4.82 implies that A_θ has an R-bounded $\mathcal{H}^{\alpha_\theta}_{p_\theta}$ calculus. □

For an operator A which is self-adjoint on X and has a Hörmander calculus on Y, we obtain the following "self-improvement" of the calculus as mentioned at the beginning of the subsection.

Corollary 4.84 *Let X be a Hilbert space, Y a space (of type > 1) and (X,Y) be a complex interpolation couple. Let A_X and A_Y be 0-sectorial operators on X and Y such that their resolvents are compatible in the sense of 4.145. Assume that A_X is self-adjoint positive and that A_Y has a (R-bounded) $\mathcal{H}^\alpha_p$ calculus for some $p \in (1,\infty)$ and $\alpha > \frac{1}{p}$.*

Then for $\theta \in (0,1)$, the interpolated operator A_θ on $(X,Y)_\theta$ has a (R-bounded) $\mathcal{H}^{\alpha_\theta}_{p_\theta}$ calculus for any $\alpha_\theta > \theta\alpha$ and $p_\theta > p/\theta$.

Proof. Since A_X is self-adjoint positive, it has a $\mathcal{H}^\delta_q$ calculus for any $q \in (1,\infty)$ and $\delta > \frac{1}{q}$ (see illustration 4.87 (2) below). For given $\alpha_\theta > \theta\alpha$ and $p_\theta > p/\theta$ choose $1/q$ and δ sufficiently small and apply proposition 4.83. □

In some examples, the operator A is not injective, e.g. the Neumann Laplace operator on $L^p(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^d$. If X is a reflexive space, then there is the following decomposition $X = X_0 \oplus X_1$ together with $A = A_0 \oplus 0$, such that A_0 is 0-sectorial on X_0. For the proof, we refer to [81, prop 15.2] and [24, thm 4.1].

Lemma 4.85 *Let X be a reflexive Banach space and A a pseudo-0-sectorial operator on X, i.e. A is closed and densely defined such that $\sigma(A) \subset [0,\infty)$, and the resolvent growth condition holds: for any $\omega \in (0,\pi)$ there exists $C_\omega > 0$ such that $\|\lambda R(\lambda, A)\| \leqslant C_\omega$ for $\lambda \in \mathbb{C}\backslash\overline{\Sigma_\omega}$.*

Then there exists a decomposition $X = X_0 \oplus X_1$, where $X_0 = \overline{R(A)}$ and $X_1 = \mathrm{Ker}(A)$ are closed subspaces. The projections on X_0 and X_1 are given by

$$P_0 : X \to X_0,\, x \mapsto \lim_{\lambda\to 0-} -AR(\lambda, A)x, \quad P_1 : X \to X_1,\, x \mapsto \lim_{\lambda\to 0-} \lambda R(\lambda, A)x. \tag{4.149}$$

Furthermore, for $x = x_0 \oplus x_1 \in D(A)$ with $x_j \in X_j$, $j = 0, 1$, A decomposes as $Ax = A_0x_0 \oplus 0$, where A_0 is a 0-sectorial operator (in particular injective and with dense range) on X_0, with domain $D(A_0) = \{x \in X_0 \cap D(A) : \; Ax \in X_0\}$.

This decomposition method and the complex interpolation method of proposition 4.83 (1) can be combined, as the next lemma shows.

Lemma 4.86 *Let X and Y be reflexive Banach spaces and A_X, A_Y operators on X and Y as in lemma 4.85. Assume that (X,Y) is a complex interpolation couple and that the resolvents of A_X and A_Y are compatible in the sense of 4.145. Then for $\theta \in (0,1)$, there exists A_θ on the interpolation space $X_\theta = (X,Y)_\theta$ as in lemma 4.85 with compatible resolvents to A_X and A_Y. Further, for the decomposition $X_\theta = (X_\theta)_0 \oplus (X_\theta)_1$, we have*

$$(X_j, Y_j)_\theta = (X_\theta)_j \quad (j = 0, 1) \tag{4.150}$$

and the resolvents of $(A_X)_0, (A_\theta)_0$ and $(A_Y)_0$ are compatible.

Proof. The existence of A_θ can be proved as in proposition 4.83. We now show 4.150. Denote P_j^X the projection on X_j for A_X as in 4.149, and similarly P_j^Y, P_j^θ for A_Y and A_θ. Since the resolvents of A_X, A_Y and A_θ are compatible, also P_j^X, P_j^Y and P_j^θ are, so that we can extend them to $P_j : X + Y \to X + Y$.

We have $x \in (X_j, Y_j)_\theta$ if and only if

$$x = g(\theta) \text{ for some } g \in F(X_j, Y_j). \tag{4.151}$$

This is the case if and only if

$$x = f(\theta) \text{ for some } f \in F(X,Y) \text{ and } x = P_j^\theta x. \tag{4.152}$$

Indeed, if $x = g(\theta)$ for some $g \in F(X_j, Y_j)$, then put $f = g \in F(X_j, Y_j) \subset F(X,Y)$. Since $P_j \circ g = g$ on the boundary of S, we have by the three lines lemma $P_j \circ g = g$ on S. Therefore,

$P_j^\theta x = P_j(g(\theta)) = g(\theta) = x$. Conversely, if 4.152 holds, then $g = P_j \circ f \in F(X_j, Y_j)$ and $x = g(\theta)$.

Finally, 4.151 means precisely $x \in (X_\theta)_j$, so that 4.150 is shown.

For the last statement, take $x \in X \cap Y$. Then

$$R(\lambda, (A_X)_0)x = R(\lambda, A_X)P_0 x = R(\lambda, A_Y)P_0 x = R(\lambda, (A_Y)_0)x = R(\lambda, (A_\theta)_0)x.$$

□

Illustration 4.87 (1) *Let $-A$ be a generator of an analytic semigroup $T(z)$ on a reflexive space X which is bounded on each sector $z \in \Sigma_\omega$, $\omega < \frac{\pi}{2}$. Then A satisfies the conditions of lemma 4.86. The semigroup decomposes on $X = X_0 \oplus X_1$ as*

$$T(z) = \begin{pmatrix} e^{-zA_0} & 0 \\ 0 & \mathrm{Id}_{X_1} \end{pmatrix}$$

where e^{-zA_0} is the semigroup generated by A_0. Further, it is easy to see that for any $\alpha > 0$, A satisfies $(N_T)_\alpha$ or $(R_T)_\alpha$ from theorem 4.73 iff A_0 does.

(2) *If A is a self-adjoint positive operator on a Hilbert space X, also A_0 is self-adjoint positive on the Hilbert space X_0, and for a bounded Borel function $f : [0, \infty) \to \mathbb{C}$, we have the decomposition*

$$f(A) = \begin{pmatrix} f(A_0) & 0 \\ 0 & f(0)\,\mathrm{Id}_{\mathrm{Ker}(A)} \end{pmatrix}.$$

Assume that Y is a reflexive space forming an interpolation couple with X. Then $f(A)$ extends to a bounded operator on Y for any $f \in \mathcal{H}_p^\alpha$ iff there is a pseudo-0-sectorial operator A_Y on Y such that its injective part $A_{Y,0}$ has a $\mathcal{H}_p^\alpha$ calculus and $f(A_{Y,0})x = f(A_0)x$ for $x \in X_0 \cap Y_0$ and $f : [0, \infty) \to \mathbb{C}$ such that $f|_{(0,\infty)} \in \mathcal{H}_p^\alpha$. In particular, our abstract definition 4.26 of the Hörmander functional calculus matches the usual one given in the vast literature for spectral multipliers on $L^p(\Omega)$ of self-adjoint operators. Further, lemma 4.86 allows to apply corollary 4.84 for non-injective A.

4.6.2 The Laplace operator on $L^p(\mathbb{R}^d)$

Let us start our observations with an overview of the guiding example of 0-sectorial operators, the Laplace operator $A = -\Delta$ on $L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$ and $d \in \mathbb{N}$.

It is well-known that Δ equipped with domain $D(\Delta) = W_p^2(\mathbb{R}^d)$ generates a semigroup which is analytic on $\Sigma_{\frac{\pi}{2}}$ and bounded on each subsector Σ_ω, $\omega \in (0, \frac{\pi}{2})$. Since $-\Delta$ is injective and has dense domain, it is thus a 0-sectorial operator.

We also want to consider fractional powers $(-\Delta)^\beta$, $\beta > 0$, and record the following fact.

Lemma 4.88 *Let $E = \mathcal{M}^\alpha$ for some $\alpha > 0$ or $E = \mathcal{H}_p^\alpha$ for some $p \in (1, \infty)$ and $\alpha > \frac{1}{p}$. Let further A be a 0-sectorial operator having an E calculus and $\beta > 0$. Then A^β is a 0-sectorial operator. Further, A^β also has an E calculus and*

$$f(A^\beta) = f((\cdot)^\beta)(A) \text{ for } f \in E. \tag{4.153}$$

Proof. By [81, thm 15.16], A^β is 0-sectorial. By the composition rule [52, thm 4.2.4], 4.153 holds for any $f \in \bigcup_{\omega>0} H^\infty(\Sigma_\omega)$ and thus, by proposition 4.8, A^β has an E calculus such that 4.153 holds for any $f \in E$. $\square$

We recall some known facts on the functional calculus of $-\Delta$.

Proposition 4.89 *Let Δ be the Laplace operator on $L^p(\mathbb{R}^d)$ for some $p \in (1,\infty)$ and $d \in \mathbb{N}$.*

(1) *$-\Delta$ has a $\mathcal{H}^\alpha_q$ calculus for $\alpha > d\left|\frac{1}{p} - \frac{1}{2}\right|$ and $\frac{1}{q} < \left|\frac{1}{p} - \frac{1}{2}\right|$.*

(2) *$-\Delta$ has no $\mathcal{H}^\alpha_q$ or $\mathcal{M}^\alpha$ calculus for any $\alpha < d\left|\frac{1}{p} - \frac{1}{2}\right|$ and any $q \in (1,\infty)$.*

(3) $\|\exp(te^{i\theta}\Delta)\| \cong (\frac{\pi}{2} - |\theta|)^{-d\left|\frac{1}{p}-\frac{1}{2}\right|} \quad (|\theta| < \frac{\pi}{2})$.

(4) *For any $\alpha > (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$, $\|\exp(-te^{i\theta}(-\Delta)^{\frac{1}{2}})\| \lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha} \quad (|\theta| < \frac{\pi}{2})$.*

Proof. (1) Clearly, $-\Delta$ is self-adjoint on $L^2(\mathbb{R}^d)$, so that by the interpolation corollary 4.84, it suffices to establish a $\mathcal{H}^\alpha_2$ calculus for any $\alpha > \frac{d}{2}$. (For $1 < p < 2$, interpolate then between $X = L^2(\mathbb{R}^d)$ and $Y = L^q(\mathbb{R}^d)$ with $q \to 1$, and for $2 < p < \infty$, take $Y = L^q(\mathbb{R}^d)$ with $q \to \infty$). This in turn is Hörmander's classical result [58, thm 2.5] in combination with proposition 4.11.

(2) By the embedding proposition 4.9 (4), it suffices to consider the case of the $\mathcal{M}^\alpha$ calculus. If $-\Delta$ had a $\mathcal{M}^\alpha$ calculus for some $\alpha < d\left|\frac{1}{p} - \frac{1}{2}\right|$, then we would have

$$\|\exp(-te^{i\theta}(-\Delta))\| \lesssim \|\exp(-te^{i\theta}(\cdot))\|_{\mathcal{M}^\alpha} \lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha'}$$

for some $\alpha' < d\left|\frac{1}{p} - \frac{1}{2}\right|$ according to proposition 4.12. This contradicts (3).

(3) This is shown in [4, (2.2) and lem 2.2].

(4) The claim is shown in [104, (7.54) and (7.55)] for $\alpha = \frac{d-1}{2}$ if $d \geqslant 2$ and $\alpha > 0$ if $d = 1$. Then considering the analytic family of operators $(\frac{\pi}{2} - |\theta|)^{\alpha(z)} \exp(-te^{i\theta}(-\Delta)^{\frac{1}{2}})$ on $F(L^q(\mathbb{R}^d), L^2(\mathbb{R}^d))$, where q is near 1 or ∞ and $\alpha(z) = (1-\frac{q}{2})^{-1} - (\frac{1}{q} - \frac{1}{2})^{-1}z$, (4) follows by complex interpolation [122]. $\square$

Note that by lemma 4.88, $-\Delta$ and $(-\Delta)^{\frac{1}{2}}$ have the same Hörmander functional calculus. On the other hand, proposition 4.89 shows that the growth rate of the semigroups generated by $-\Delta$ and $(-\Delta)^{\frac{1}{2}}$ differs, and thus, the growth rate of the semigroup cannot characterize the optimal exponent of the Hörmander calculus.

The next theorem shows that the results of proposition 4.89 remain true when boundedness is replaced by R-boundedness. Note that once the R-boundedness of the Poisson semigroup in (3) is shown, one obtains the sharp parameter α for the Hörmander calculus by our general theory of sections 4.4 and 4.5. In contrast, the R-bound of the Gaussian semigroup in (2) has essentially the same growth rate as the (R-bounded) Hörmander calculus in (1) and thus we could only deduce an R-bounded calculus with parameter $\alpha > (d+1)|\frac{1}{p} - \frac{1}{2}|$ from (2).

Theorem 4.90 *Let* Δ *be the Laplace operator on* $L^p(\mathbb{R}^d)$ *for some* $p \in (1, \infty)$ *and* $d \in \mathbb{N}$.

(1) $-\Delta$ *has an* R*-bounded* $\mathcal{H}^\alpha_q$ *calculus for* $\alpha > d\left|\frac{1}{p} - \frac{1}{2}\right|$ *and* $\frac{1}{q} < \left|\frac{1}{p} - \frac{1}{2}\right|$.

(2) $R(\{(\frac{\pi}{2} - |\theta|)^\alpha \exp(te^{i\theta}\Delta) : t > 0, |\theta| < \frac{\pi}{2}\}) < \infty$ *for* $\alpha > d\left|\frac{1}{p} - \frac{1}{2}\right|$.

(3) $R(\{\exp(-te^{i\theta}(-\Delta)^{\frac{1}{2}}) : t > 0\}) \lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha}$ *for* $\alpha > (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$.

In particular, $(N_T)_\alpha$ *from theorem 4.73 holds for* $A = (-\Delta)^{\frac{1}{2}}$ *and* $\alpha > (d-1)\left|\frac{1}{p} - \frac{1}{2}\right|$.

Proof. (1) Since $L^p(\mathbb{R}^d)$ has type > 1, again by corollary 4.84, it suffices to show that $-\Delta$ has an R-bounded $\mathcal{H}^\alpha_2$ calculus for any $\alpha > \frac{d}{2}$ (compare the proof of proposition 4.89 (1)). By lemma 4.88, we can replace $-\Delta$ by $(-\Delta)^{\frac{1}{2}}$. By theorem 4.73, implication $(N_T)_\beta \Longrightarrow (C_2)_{\beta+\frac{1}{2}+\varepsilon}$ (note that for the space $X = L^p(\mathbb{R}^d)$, $\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X} < \frac{1}{2}$ for any $p \in (1, \infty)$), $(-\Delta)^{\frac{1}{2}}$ does have an R-bounded $\mathcal{H}^\alpha_2$ calculus provided (3) of the theorem is shown.

(2) This follows from (1) and the fact that

$$\|(\frac{\pi}{2} - |\theta|)^\alpha \exp(-te^{i\theta}(\cdot))\|_{\mathcal{H}^{\alpha-2\varepsilon}_q} \lesssim \|(\frac{\pi}{2} - |\theta|)^\alpha \exp(-te^{i\theta}(\cdot))\|_{\mathcal{M}^{\alpha-\varepsilon}} \lesssim 1$$

according to proposition 4.12.

(3) Suppose that we have shown

$$R(\{\exp(-te^{i\theta}(-\Delta)^{\frac{1}{2}}) : t > 0\}) \lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha_d}, \tag{4.154}$$

where $\alpha_d > 0$ if $d = 1$ and $\alpha_d = \frac{d-1}{2}$ if $d \geqslant 2$. Then by an interpolation argument as in the proof of (4) of the preceding proposition, (3) follows. It is well known (see e.g. [104, (7.53)]) that $\exp(-e^{i\theta}t(-\Delta)^{\frac{1}{2}})f = p_{t,\theta} * f$ with

$$p_{t,\theta}(x) = c_d \frac{e^{i\theta}t}{((e^{i\theta}t)^2 + |x|^2)^{(d+1)/2}}. \tag{4.155}$$

We have

$$|p_{t,\theta}(x)| \cong \frac{t}{|e^{2i\theta}t^2 + x^2|^{(d+1)/2}} = t^{-d}\phi_\theta(\frac{x}{t}),$$

where $\phi_\theta(x) = |e^{2i\theta} + x^2|^{-\frac{d+1}{2}}$ is a non-negative radial function which is radially decreasing for the interesting case $|\theta| \in (\frac{\pi}{4}, \frac{\pi}{2})$. Then by a maximal inequality [123, p.57 (16)], for any $t > 0$, $x \in \mathbb{R}^d$ and $f \in L^p(\mathbb{R}^d)$,

$$|p_{t,\theta}| * |f|(x) \leqslant t^{-d}\phi_\theta(\frac{\cdot}{t}) * |f|(x) \leqslant \|\phi_\theta\|_{L^1(\mathbb{R}^d)} Mf(x),$$

where M is the maximal operator $Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy$, which satisfies $\|Mf\|_p \leqslant C_p\|f\|_p$ for $p \in (1, \infty)$. Thus, for $t_1, \ldots, t_n > 0$ and $f_1, \ldots, f_n \in L^p(\mathbb{R}^d)$,

$$\left(\sum_{k=1}^n (|p_{t_k,\theta}| * |f_k|(x))^2\right)^{\frac{1}{2}} \leqslant \|\phi_\theta\|_1 \left(\sum_{k=1}^n (Mf_k(x))^2\right)^{\frac{1}{2}}.$$

By proposition 2.6, the $\text{Rad}(L^p(\mathbb{R}^d))$ norm has the equivalent description $\| \sum_k \varepsilon_k \otimes f_k \|_{\text{Rad}(L^p(\mathbb{R}^d))} \cong \| \left(\sum_k |f_k|^2\right)^{\frac12} \|_{L^p(\mathbb{R}^d)}$. Then

$$\begin{aligned}
\left\| \sum_{k=1}^n \varepsilon_k \otimes \exp(-e^{i\theta} t_k (-\Delta)^{\frac12}) f_k \right\|_{\text{Rad}(L^p(\mathbb{R}^d))} &\cong \left\| \left(\sum_k |\exp(-e^{i\theta} t_k (-\Delta)^{\frac12}) f_k|^2 \right)^{\frac12} \right\|_{L^p(\mathbb{R}^d)} \\
= \left\| \left(\sum_k |p_{t_k,\theta} * f_k|^2 \right)^{\frac12} \right\|_{L^p(\mathbb{R}^d)} &\leqslant \left\| \left(\sum_k (|p_{t_k,\theta}| * |f_k|)^2 \right)^{\frac12} \right\|_{L^p(\mathbb{R}^d)} \\
\leqslant \|\phi_\theta\|_1 \left\| \left(\sum_k (M f_k)^2 \right)^{\frac12} \right\|_{L^p(\mathbb{R}^d)} &\cong \|\phi_\theta\|_1 \left\| \sum_k \varepsilon_k \otimes M f_k \right\|_{\text{Rad}(L^p(\mathbb{R}^d))} \\
\leqslant \|\phi_\theta\|_1 C_p \left\| \sum_k \varepsilon_k \otimes f_k \right\|_{\text{Rad}(L^p(\mathbb{R}^d))} &.
\end{aligned}$$

Thus, 4.154 follows from $\|\phi_\theta\|_1 \lesssim (\frac{\pi}{2} - |\theta|)^{-\alpha_d}$, which is shown in [47, p. 348]. □

Remark 4.91 *More generally, let A be a self-adjoint operator on $L^2(\mathbb{R}^d)$ such that the semigroup $\exp(-e^{i\theta} tA)$ has a kernel $p_{t,\theta}$ satisfying the complex Poisson bound*

$$|p_{t,\theta}(x)| \lesssim \frac{t}{|e^{2i\theta} t^2 + x^2|^{(d+1)/2}} \quad (x \in \mathbb{R}^d,\, t > 0,\, \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})).$$

It is shown in [38, thm 3.4] that A has an H^∞ calculus on $L^p(\mathbb{R}^d)$ for any $1 < p < \infty$. Then with the same proof as above, A satisfies $(N_T)_\alpha$ on $L^p(\mathbb{R}^d)$ for $\alpha > (d-1)\left|\frac1p - \frac12\right|$ and A has an R-bounded $\mathcal{H}^\alpha_q$ calculus on $L^p(\mathbb{R}^d)$ for $\alpha > d\left|\frac1p - \frac12\right|$ and $\frac1q < \left|\frac1p - \frac12\right| \quad (1 < p < \infty)$.

4.6.3 Generalized Gaussian estimates

Most of the spectral multiplier theorems for operators with spectrum in $[0, \infty)$ in the literature consider operators A which act on a scale of $L^p(\Omega)$-spaces, $p \in (p_0, p_1)$, and assume that A is self-adjoint positive on $L^2(\Omega)$. Further, the proofs often rely on Gaussian estimates GE or generalized Gaussian estimates GGE for the analytic semigroup generated by $-A$ (see e.g. [37, sec 8.2]).

We will give a setting which covers many of these examples and show that our theory gives a (R-bounded) Hörmander calculus for such an operator A. The price of the R-boundedness of the $\mathcal{H}^\alpha_p$ calculus is that the derivation order α in our abstract framework will be worse (i.e. higher) in most of the cases compared to the results for the concrete operators in the literature [95, 21, 100, 37]. In one special case however, the derivation order obtained by our methods is better than what was known [9] (see remark 4.96).

Definition 4.92 *Let Ω be a topological space which is equipped with a distance ρ and a Borel measure μ. Let $d \geqslant 1$ be an integer. Ω is called a homogeneous space of dimension d if there exists $C > 0$ such that for any $x \in \Omega$, $r > 0$ and $\lambda \geqslant 1$:*

$$\mu(B(x, \lambda r)) \leqslant C \lambda^d \mu(B(x, r)).$$

Typical cases of homogeneous spaces are open subsets of $\mathbb{R}^d$ with Lipschitz boundary and Lie groups with polynomial volume growth, in particular stratified nilpotent Lie groups (see e.g. [42]).

In the rest of section 4.6 we will assume the following:

Assumption 4.93 *A is a self-adjoint positive (unbounded) operator on $L^2(\Omega)$, where Ω is an open subspace of a homogeneous space Ω of a certain dimension d. Further, there exists some $p_0 \in [1,2)$ such that the semigroup generated by $-A$ satisfies the so called generalized Gaussian estimate (see e.g. [9, (GGE)]):*

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_0 \to p_0'} \leqslant C\mu(B(x,r_t))^{\frac{1}{p_0'} - \frac{1}{p_0}} \exp(-c(\rho(x,y)/r_t)^{\frac{m}{m-1}}) \quad (x,y \in \Omega,\, t>0). \tag{GGE}$$

Here, p_0' is the conjugated exponent to p_0, $C, c > 0$, $m \geqslant 2$ and $r_t = t^{\frac{1}{m}}$, χ_B denotes the characteristic function of B, $B(x,r)$ is the ball $\{y \in \Omega : \rho(y,x) < r\}$ and $\|\chi_{B_1} T \chi_{B_2}\|_{p_0 \to p_0'} = \sup_{\|f\|_{p_0} \leqslant 1} \|\chi_{B_1} \cdot T(\chi_{B_2} f)\|_{p_0'}$.

Remark 4.94

(1) *If $p_0 = 1$, then it is proved in [12] that GGE is equivalent to the usual Gaussian estimate, i.e. e^{-tA} is given by a kernel $k_t(x,y)$ satisfying the pointwise estimate (cf. e.g. [37, ass 2.2])*

$$|k_t(x,y)| \lesssim \mu(B(x,t^{\frac{1}{m}}))^{-1} \exp\left(-c\left(\rho(x,y)/t^{\frac{1}{m}}\right)^{\frac{m}{m-1}}\right) \quad (x,y \in \Omega,\, t>0). \tag{GE}$$

This is satisfied in particular by sub-Laplacian operators on Lie groups of polynomial growth [132] as considered e.g. in [95, 21, 1, 100, 35], or by more general elliptic and sub-elliptic operators [28, 104], and Schrödinger operators [105]. Assumption 4.93 is also satisfied by all the operators in [37, sec 2].

(2) *Examples of operators satisfying a generalized Gaussian estimate for $p_0 > 1$ are higher order operators with bounded coefficients and Dirichlet boundary conditions on domains of $\mathbb{R}^d$, Schrödinger operators with singular potentials on $\mathbb{R}^d$ and elliptic operators on Riemannian manifolds as listed in [9, sec 2] and the references therein.*

The theorem on the R-boundedness of the semigroup and the Hörmander calculus for operators satisfying assumption 4.93 now reads as follows.

Theorem 4.95 *Let assumption 4.93 hold for some homogeneous space Ω of dimension d and a self-adjoint operator A on $L^2(\Omega)$. Then for any $p \in (p_0, p_0')$, the operator A satisfies $(R_T)_\alpha$ from theorem 4.73 on $L^p(\Omega)$ for $\alpha = d\left|\frac{1}{p_0} - \frac{1}{2}\right|$.*

Consequently, A has an R-bounded $\mathcal{H}_2^\alpha$ calculus on $L^p(\Omega)$ for any $\alpha > d\left|\frac{1}{p_0} - \frac{1}{2}\right| + \frac{1}{2}$.

Proof. By [11, prop 2.1], the assumption GGE implies that

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_0 \to 2} \leqslant C_1\mu(B(x,r_t))^{\frac{1}{2} - \frac{1}{p_0}} \exp(-c_1(\rho(x,y)/r_t)^{\frac{m}{m-1}}) \quad (x,y \in \Omega,\, t>0)$$

for some $C_1, c_1 > 0$. By [10, thm 2.1], this can be extended from real t to complex $z = te^{i\theta}$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$:

$$\|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\|_{p_0 \to 2} \leqslant C_2\mu(B(x,r_z))^{\frac{1}{2} - \frac{1}{p_0}} (\cos\theta)^{-d(\frac{1}{p_0} - \frac{1}{2})} \exp(-c_2(\rho(x,y)/r_z)^{\frac{m}{m-1}}),$$

for $r_z = (\cos\theta)^{-\frac{m-1}{m}} t^{\frac{1}{m}}$, and some $C_2, c_2 > 0$. By [11, prop 2.1 (i) (1) ⇒ (3) with $R = e^{-zA}$, $\gamma = \alpha = \frac{1}{p_0} - \frac{1}{2}$, $\beta = 0$, $r = r_z$, $u = p_0$ and $v = 2$], this gives for any $x \in \Omega$, $\operatorname{Re} z > 0$ and $k \in \mathbb{N}_0$

$$\|\chi_{B(x,r_z)} e^{-zA} \chi_{A(x,r_z,k)}\|_{p_0 \to 2} \leqslant C_3 \mu(B(x,r_z))^{\frac{1}{2}-\frac{1}{p_0}} (\cos\theta)^{-d(\frac{1}{p_0}-\frac{1}{2})} \exp(-c_3 k^{\frac{m}{m-1}}),$$

where $A(x, r_z, k)$ denotes the annular set $B(x,(k+1)r_z) \backslash B(x, kr_z)$. By [79, thm 2.2 with $q_0 = p_0$, $q_1 = s = 2$, $\rho(z) = r_z$ and $S(z) = (\cos\theta)^{d(\frac{1}{p_0}-\frac{1}{2})} e^{-zA}$], we deduce that

$$\{(\cos\theta)^{d(\frac{1}{p_0}-\frac{1}{2})} e^{-zA} : \operatorname{Re} z > 0\}$$

is R_2-bounded in the sense of [79, (1)] on $L^p(\Omega)$ for any $p \in (p_0, 2)$, and by duality for any $p \in (p_0, p_0')$. Note that by proposition 2.6, R_2-boundedness is equivalent to R-boundedness, so that $(R_T)_{d|\frac{1}{p_0}-\frac{1}{2}|}$ holds.

For the Hörmander calculus consequence, it only remains to check that A has an H^∞ calculus on $L^p(\Omega)$, which is shown in [10, cor 2.3]. □

Remark 4.96 (1) *Theorem 4.95 improves on [9, thm 1.1] in that it includes the R-boundedness of the Hörmander calculus. However, [9] obtains also a weak-type result for $p = p_0$.*

(2) *If p_0 is strictly larger than 1, then theorem 4.95 improves the order of derivation α of the calculus from $\frac{d}{2} + \frac{1}{2} + \varepsilon$ in [9] to $d\left|\frac{1}{p_0} - \frac{1}{2}\right| + \frac{1}{2} + \varepsilon$.*

(3) *Again complex interpolation of the Hörmander calculus with the self-adjoint calculus on the Hilbert-space is possible to improve the calculus. Consequently, A has an R-bounded $\mathcal{H}^\alpha_q$ calculus on $L^p(\Omega)$ for $p \in (p_0, p_0')$, for*

$$\alpha > \left((d(\frac{1}{p_0} - \frac{1}{2}) + \frac{1}{2}\right) \frac{\frac{1}{p} - \frac{1}{2}}{\frac{1}{p_0} - \frac{1}{2}} \text{ and } \frac{1}{q} < \frac{1}{2} \frac{\frac{1}{p} - \frac{1}{2}}{\frac{1}{p_0} - \frac{1}{2}}.$$

(4) *The theorem also holds for the weaker assumption that Ω is an open subset of a homogeneous space $\tilde{\Omega}$. In that case, the ball $B(x, r_t)$ on the right hand side in GGE is the one in $\tilde{\Omega}$. The idea of the modification of the proof is from Duong and McIntosh [36, p. 245], see also [9]: One replaces an operator $T : L^p(\Omega) \to L^q(\Omega)$ by $S : L^p(\tilde{\Omega}) \to L^q(\tilde{\Omega})$, where*

$$Sf(x) = \begin{cases} T(\chi_\Omega f)(x) : & x \in \Omega \\ 0 : & x \in \tilde{\Omega} \backslash \Omega \end{cases}.$$

Then $\|S\| = \|T\|$, and also the R-bound of a family $\{T_t : t \in \tau\} \subset B(L^p(\Omega), L^q(\Omega))$ is the same as the R-bound of the corresponding continuated family $\{S_t : t \in \tau\} \subset B(L^p(\tilde{\Omega}), L^q(\tilde{\Omega}))$. This variant can be applied to elliptic operators on irregular domains $\Omega \subset \mathbb{R}^d$ as discussed in [9, sec 2].

(5) *In [37], for many examples, a Hörmander functional calculus is proved for the better order $\alpha > \frac{d}{2}$ instead of $\alpha > \frac{d+1}{2}$ as in theorem 4.95. Note however that the assumption 4.93 cannot imply in general a $\mathcal{H}^\alpha_2$ calculus for $\alpha > \frac{d}{2}$. Indeed, in [126], it is shown that certain Riesz-means of the harmonic oscillator $A = -\Delta + x^2$, which satisfies assumption 4.93 on $\mathbb{R}^1$, do not converge for all $p \in (1, \infty)$. One can deduce that $-\Delta + x^2$ does not have a $\mathcal{H}^\alpha_2$ calculus for $\alpha < \frac{1}{2} + \frac{1}{6} = \frac{d}{2} + \frac{1}{6}$ (see also [37, p. 473] and [9, p. 450]).*

5 Functional calculus for c_0-groups of polynomial growth

5.1 Introduction

In this chapter, we consider a c_0-group $(U(t))_{t\in\mathbb{R}}$ on some Banach space X. We are interested in the functional calculus for B, where iB is the generator of $(U(t))_{t\in\mathbb{R}}$.

If X is a Hilbert space, then Boyadzhiev and deLaubenfels [15] have shown that

$$U(t) \text{ has an exponential growth } \|U(t)\|_{B(X)} \leqslant Ce^{\theta|t|} \quad (t \in \mathbb{R}) \tag{5.1}$$

if and only if

$$B \text{ admits an } H^\infty(\mathrm{Str}_\omega) \text{ calculus for any } \omega > \theta. \tag{5.2}$$

If X is a general Banach space, this is false. A group generator need not to have a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus at all, even for large $\omega > 0$. A counterexample is the shift group $U(t)f(s) = f(s+t)$ on $X = L^p(\mathbb{R})$. In this case, $\|U(t)\|_{B(X)} = 1$, so that one could even choose $\theta = 0$ in 5.1, but the generator has a bounded H^∞ calculus if and only if $p = 2$ [24].

Nevertheless, Kalton and Weis have found an adequate replacement of 5.1 to transfer Boyadzhiev and deLaubenfels' result to the Banach space case. They have shown in [72, thm 6.8] that if $\{e^{-\theta|t|}U(t) : t \in \mathbb{R}\}$ is γ-bounded, then still 5.2 holds, and if X has Pisier's property (α), then also the converse holds. Note that γ-boundedness is the same as (norm) boundedness for operators on Hilbert spaces, so that [72, thm 6.8] reduces to Boyadzhiev and deLaubenfels's result in that case.

Later on, Haase ([54], see also [52]) used a different approach to generalize Boyadzhiev and deLaubenfels' result, especially to the case that X is a UMD space. He shows a transference principle for groups satisfying 5.1, which limits the norm of $f(B)$ on X to a Fourier multiplier norm on a vector valued space $L^p(\mathbb{R}, X)$ [54, thm 3.2]. This method also works for operators without bounded H^∞ calculus. The price is that the mentioned Fourier multiplier norm is harder to control (see [54, lem 3.5, thm 3.6]).

We are concerned with groups of polynomial growth instead of exponential, that is, we assume that there exists $\alpha \geqslant 0$ such that

$$\|U(t)\|_{B(X)} \leqslant C(1 + |t|)^\alpha \quad (t \in \mathbb{R}). \tag{5.3}$$

Further, we do assume that B has an H^∞ calculus. An example is the group of imaginary powers $U(t) = (-\Delta)^{it}$ of the Laplace operator on $L^p(\mathbb{R}^d)$, where $\alpha = d\left|\frac{1}{p} - \frac{1}{2}\right|$.

We show in theorem 5.10 below that 5.3 together with the boundedness of the H^∞ calculus of B is equivalent to a certain E^α_∞ functional calculus of B, if X has property (α) or $\{U(t) : t \in I\}$ is γ-bounded on an interval I of finite length.

The proof combines both Kalton and Weis' theory and Haase's method, in that it uses a transference principle on the Gaussian function space $\gamma(L^2(\mathbb{R}), X)$.

The class E^α_∞ consists of continuous and bounded functions on the real line such that

$$\sum_{n\in\mathbb{Z}} (1 + |n|)^\alpha \|f * \check{\phi}_n\|_{L^\infty(\mathbb{R})} < \infty \tag{5.4}$$

(see definition 5.1 in section 5.2). Here $(\phi_n)_{n\in\mathbb{Z}}$ is a smooth partition of unity on the real line such that $\operatorname{supp}\phi_n \subset [n-1, n+1]$. This resembles the definition of the Besov space $\mathcal{B}^\alpha$ (see 4.8), the difference is that the partition of unity for $\mathcal{B}^\alpha$ has its support in dyadic intervals $\pm[2^{|n|-1}, 2^{|n|+1}]$.

In section 5.2, we investigate the spaces E^α_∞ (also variants E^α_p, $1 \leqslant p < \infty$, where the $L^\infty(\mathbb{R})$ norm in 5.4 is replaced by the $L^p(\mathbb{R})$ norm). We show that, in the case $p \in [2, \infty]$, they admit a sort of atomic decomposition

$$f(t) = \sum_{m,n} a_{mn} e^{int} \psi(t - \pi m)$$

where ψ is the Fourier transform of a test function, in particular rapidly decreasing.

Section 5.3 contains the construction of the E^α_∞ calculus and the equivalence to the polynomial growth 5.3 is shown.

In section 5.4, we consider an operator valued extension of the E^α_∞ calculus. As for the operator valued H^∞ calculus [73], if X has property (α), this yields R-bounded families $\{f(B) : f \in \tau\}$ for suitable $\tau \subset E^\alpha_\infty$.

Subsequently, in section 5.5 we apply the (operator valued) E^α_∞ calculus to some typical functions corresponding to semigroup operators and resolvents of the 0-sectorial operator $A = e^B$. Further we deduce (R-)bounds for these operators which are optimal in the class of groups satisfying 5.3.

In section 5.6, we compare the results from chapters 5 and 4, where we had dealt with the Mihlin functional calculus for 0-sectorial operators A, which is the same as the Besov $\mathcal{B}^\alpha$ calculus for the group generator $B = \log(A)$.

By the embedding $E^\alpha_\infty \hookrightarrow \mathcal{B}^\alpha$ (see proposition 5.5), for the same order α, the Besov functional calculus is a stronger condition than the E^α_∞ functional calculus. Moreover, the Besov norm is closer related to differentiability properties than the E^α_∞ norm, so that it is easier to handle.

In theorem 4.74, it was shown that the growth condition 5.3 implies a $\mathcal{B}^\beta$ functional calculus. The differentiation order β in this Besov calculus is strictly larger than the α from 5.3, more precisely,

$$\beta > \alpha + \max(\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}, \frac{1}{2}). \tag{5.5}$$

With the E_∞^α calculus from theorem 5.10, we can show that 5.5 is optimal for $\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X} \geqslant \frac{1}{2}$. More precisely, for $2 \leqslant p < \infty$ and $X = E_p^\alpha$, the right hand side of 5.5 equals $\alpha + 1 - \frac{1}{p}$, and if β is smaller than this quantity, the pointwise multiplication $\mathcal{B}^\beta \cdot E_p^\alpha$ does not map into E_p^α. Stated differently, the multiplication operator $f(t) \mapsto tf(t)$ on E_p^α is an example of a generator for a group with 5.3 where condition 5.5 is sharp.

5.2 The spaces E_p^α

In this section, we introduce the function spaces E_∞^α for the functional calculus of c_0-groups of polynomial growth. They are defined by a summability condition for a decomposition in the Fourier image, see 5.6, in a similar way to the Besov spaces $\mathcal{B}^\alpha = \mathcal{B}^\alpha_{\infty,1}$. Recall that

$$\mathcal{B}^\alpha = \{f : \mathbb{R} \to \mathbb{C},\ f \text{ uniformly continuous and bounded, } \|f\|_{\mathcal{B}^\alpha} = \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \|f * \check{\phi}_n\|_\infty < \infty\},$$

where $(\phi_n)_{n\in\mathbb{Z}}$ is a Fourier partition of unity (definition 4.5), i.e. $\phi_n \in C_c^\infty(\mathbb{R})$ and $\operatorname{supp} \phi_n \subset [2^{n-2}, 2^n]$ for $n \geqslant 1$. The fundamental difference is that the Fourier decomposition for E_∞^α is not dyadic as in the Besov space case, but equidistant.

We consider the range $\alpha \geqslant 0$, since E_∞^α is a multiplication algebra for these α (proposition 5.2), which is a natural requirement for a functional calculus. This is in contrast to our Besov spaces in chapter 4, where one has to restrict to $\alpha > 0$ (see [75] for the negative statement for $\alpha = 0$). For later sections, we also include the spaces E_p^α for finite p (which correspond to $\mathcal{B}^\alpha_{p,1}$).

After some elementary properties, we show that there is a sort of atomic decomposition for functions $f \in E_p^\alpha$ in the case $2 \leqslant p \leqslant \infty$. Namely, f can be written as an (infinite) linear combination of one rapidly decreasing function shifted in space and phase 5.7. This resembles the atomic decomposition of Besov spaces, but reduces for $p = \infty$ to the Fourier series if f is periodic. The decomposition enables us subsequently to show optimal embedding results between E_∞^α and $\mathcal{B}^\alpha$.

We recall the notation $\langle t \rangle = (1 + t^2)^{\frac{1}{2}}$ of chapter 4. We have the following elementary inequalities: $\langle k + l \rangle \lesssim \langle k \rangle + \langle l \rangle \lesssim \langle k \rangle \cdot \langle l \rangle$.

Furthermore, we consider an equidistant Fourier partition of unity $(\phi_n)_n$.

That is, we let $\phi \in C_c^\infty(\mathbb{R})$, assume that $\operatorname{supp} \phi \subset [-1, 1]$ and $\sum_{n=-\infty}^\infty \phi(t - n) = 1$ for all $t \in \mathbb{R}$. For $n \in \mathbb{Z}$ we put $\phi_n = \phi(\cdot - n)$ and $\widetilde{\phi}_n = \sum_{k=-1}^1 \phi_{n+k}$. Note that $\widetilde{\phi}_m \phi_n = \phi_n$ for $m = n$ and $\widetilde{\phi}_m \phi_n = 0$ for $|n - m| \geqslant 2$.

Definition 5.1 *Let $\alpha \geqslant 0$ be a parameter, $1 \leqslant p \leqslant \infty$ and $(\phi_n)_n$ an equidistant Fourier partition of unity. We define*

$$E_p^\alpha = \{f \in L^p(\mathbb{R}) : \|f\|_{E_p^\alpha} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_p < \infty\} \tag{5.6}$$

and equip it with the norm $\|f\|_{E_p^\alpha}$. Further, we set

$$E_0^\alpha = \{f \in E_\infty^\alpha : \lim_{|t|\to\infty} f(t) = 0\}$$

with the norm $\|f\|_{E_0^\alpha} = \|f\|_{E_\infty^\alpha}$.

Let us record some elementary properties of the spaces E_p^α and E_0^α.

Proposition 5.2 *Let $\alpha \geqslant 0$ and $1 \leqslant p \leqslant \infty$.*

(1) *E_p^α and E_0^α are Banach spaces.*

(2) *For $f \in E_\infty^\alpha$ and $g \in E_p^\alpha$, we have $fg \in E_p^\alpha$ and $\|fg\|_{E_p^\alpha} \lesssim \|f\|_{E_\infty^\alpha}\|g\|_{E_p^\alpha}$. Similarly, for $f \in E_\infty^\alpha$ and $g \in E_0^\alpha$, we have $fg \in E_0^\alpha$.*

(3) *For any $\omega > 0$, $H^\infty(\mathrm{Str}_\omega) \cap E_p^\alpha$ is dense in E_p^α. More precisely, for $f \in E_p^\alpha$, $\sum_{n=-N}^M f * \check{\phi}_n$ converges to f in E_p^α $(N, M \to \infty)$.*

(4) *E_p^α is independent of the choice of the equidistant Fourier partition of unity and different choices give equivalent norms.*

(5) *Translations $f \mapsto f(\cdot - t)$ and dilations $f \mapsto f(a\cdot)$ $(t \in \mathbb{R}, a > 0)$ are isomorphisms $E_p^\alpha \to E_p^\alpha$.*

(6) *Let $J_\alpha(f) = (\langle\cdot\rangle^{-\alpha}\hat{f})\check{}$. Then J_α is an isomorphism $E_p^0 \to E_p^\alpha$ and $\mathcal{B}^0 \to \mathcal{B}^\alpha$.*

Proof. (1): For E_p^α, this can be shown as for Besov spaces. Then E_0^α is a Banach space because $\|\cdot\|_\infty \leqslant \|\cdot\|_{E_\infty^\alpha}$, and thus, convergence to 0 at infinity is preserved by limits in E_0^α.

(2): Let $f \in E_\infty^\alpha$ and $g \in E_p^\alpha$. For $k, l \in \mathbb{Z}$, we put $f_k = f * \check{\phi}_k$ and $g_l = g * \check{\phi}_l$. By (3) below, it suffices to consider the case that there exist $K, L \in \mathbb{N}$ such that $f = \sum_{-K}^K f_k$, $g = \sum_{-L}^L g_l$. For $n \in \mathbb{Z}$, we have

$$\|(f_k g_l) * \check{\phi}_n\|_p \leqslant \|\check{\phi}_n\|_1 \|f_k g_l\|_p \leqslant \|\check{\phi}_1\|_1 \|f_k\|_\infty \|g_l\|_p.$$

Note that

$$\mathrm{supp}(f_k g_l)\check{}\ = \mathrm{supp}(\check{f}_k * \check{g}_l) \subset \mathrm{supp}(\phi_k) + \mathrm{supp}(\phi_l) \subset [k + l - 2, k + l + 2].$$

Thus if $|n - (k + l)| > 2$, we have $(f_k g_l) * \check{\phi}_n = 0$. If $|n - (k + l)| \leqslant 2$, then $\langle n\rangle^\alpha \lesssim \langle k\rangle^\alpha + \langle l\rangle^\alpha \lesssim \langle k\rangle^\alpha \langle l\rangle^\alpha$. Therefore,

$$\begin{aligned}\|fg\|_{E_p^\alpha} &= \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(fg) * \check{\phi}_n\|_p \leqslant \sum_{|k|\leqslant K} \sum_{|l|\leqslant L} \sum_{|n-(k+l)|\leqslant 2} \langle n\rangle^\alpha \|f_k g_l * \check{\phi}_n\|_p \\ &\lesssim \sum_{|k|\leqslant K} \sum_{|l|\leqslant L} \langle k\rangle^\alpha \langle l\rangle^\alpha \|f_k\|_\infty \|g_l\|_p = \|f\|_{E_\infty^\alpha} \|g\|_{E_p^\alpha}.\end{aligned}$$

If $g \in E_0^\alpha$, then $fg \in E_\infty^\alpha$ by the above, and clearly $\lim_{|t|\to\infty} f(t)g(t) = 0$. Thus, $fg \in E_0^\alpha$.

(3): The density of H^∞ in E_p^α can be shown as in lemma 4.15, with the same observations that $\phi_n\phi_m = 0$ for $|n-m| \geqslant 2$ and that $\|\check{\phi}_n\|_1 < \infty$ is independent of $n \in \mathbb{Z}$.

(4): Let $(\phi_n)_n$ and $(\psi_n)_n$ be two equidistant Fourier partitions of unity. Then $f*\check{\phi}_n = f*\check{\phi}_n*\widetilde{\psi_n}\check{}$ and consequently,

$$\sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha\|f*\check{\phi}_n\|_p \leqslant \sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha\|\check{\phi}_n\|_1\|f*\widetilde{\psi_n}\check{}\|_p \leqslant \sum_{k=-1}^{1}\sum_{n+k\in\mathbb{Z}}\langle n+k\rangle^\alpha\|f*\check{\psi}_{n+k}\|_p$$
$$\lesssim \sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha\|f*\check{\psi}_n\|_p.$$

Exchanging the roles of $(\phi_n)_n$ and $(\psi_n)_n$ gives the result.

(5): Consider the case $E_p^\alpha \to E_p^\alpha$. For translations, the isomorphism follows directly from $f(\cdot - t)*\check{\phi}_n = (f*\check{\phi}_n)(\cdot - t)$. For dilations, note that $f(a\cdot)*\check{\phi}_n = (f*\check{\psi}_n)(a\cdot)$ with $\psi_n = \phi_n(a\cdot)$. There exists $N = N(a)$ such that $\psi_n = \psi_n\sum_{k=-N}^N \phi_{\lfloor n/a\rfloor + k}$. Since $\|\check{\psi}_n\|_1 = \|\check{\psi}_0\|_1$ and $\langle n\rangle \cong \langle\lfloor n/a\rfloor + k\rangle$ for $|k| \leqslant N$,

$$\|f(a\cdot)\|_{E_p^\alpha} = \sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha\|f(a\cdot)*\check{\phi}_n\|_p \lesssim \sum_{n,k}\langle n\rangle^\alpha\|f*\check{\phi}_{\lfloor n/a\rfloor+k}\|_p \lesssim \sum_n\langle n\rangle^\alpha\|f*\check{\phi}_n\|_p = \|f\|_{E_p^\alpha}.$$

Exchanging a and $1/a$ gives the estimate in the other way.

(6): If $\langle n\rangle^\alpha\|J_\alpha f*\check{\phi}_n\|_p \cong \|f*\check{\phi}_n\|_p$ for any $f\in E_p^0$, then clearly, $J_\alpha : E_p^0 \to E_p^\alpha$ is an isomorphism. We have

$$\|J_\alpha f*\check{\phi}_n\|_p = \|f*\check{\phi}_n*(\langle\cdot\rangle^{-\alpha}\widetilde{\phi}_n)\check{}\|_p$$
$$\leqslant \|f*\check{\phi}_n\|_p\|(\langle\cdot\rangle^{-\alpha}\widetilde{\phi}_n)\check{}\|_1$$

and

$$\|(\langle\cdot\rangle^{-\alpha}\widetilde{\phi}_n)\check{}\|_1 \leqslant \|\langle\cdot\rangle^{-\alpha}\widetilde{\phi}_n\|_{W_p^1} \lesssim \langle n\rangle^{-\alpha}.$$

Since this is also valid for negative α, we get

$$\|J_\alpha f*\check{\phi}_n\|_p \lesssim \langle n\rangle^{-\alpha}\|f*\check{\phi}_n\|_p = \langle n\rangle^{-\alpha}\|J_{-\alpha}J_\alpha f*\check{\phi}_n\|_p \lesssim \|J_\alpha f*\check{\phi}_n\|_p.$$

The case $J_\alpha : \mathcal{B}^0 \to \mathcal{B}^\alpha$ is proved in the same manner, see [6, lem 6.2.7 and thm 6.2.7]. □

We now turn to the mentioned atomic decomposition of E_p^α $(2 \leqslant p \leqslant \infty)$. This decomposition is not independent of the partition $(\phi_n)_n$, so we fix one. We further assume that $\operatorname{supp}\phi_0 \subset (-1,1)$ and choose some $\overline{\phi_0} \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp}\overline{\phi_0} \subset [-1,1]$ and $\overline{\phi_0}(t) = 1$ for all $t \in \operatorname{supp}\phi_0$. Put $\overline{\phi_n} = \overline{\phi_0}(\cdot - n)$.

The norm of E_p^α will be decribed on the atomic side by

$$\ell_\alpha^1(\ell^p) = \ell^1(\mathbb{Z},\langle n\rangle^\alpha,\ell^p(\mathbb{Z})) = \{(a_{nm}) \in \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}} : \|a_{nm}\|_{\ell_\alpha^1(\ell^p)} = \sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha\|(a_{nm})_m\|_{\ell^p(\mathbb{Z})} < \infty\},$$

equipped with the norm $\|a_{nm}\|_{\ell_\alpha^1(\ell^p)}$.

Proposition 5.3 *Let $\alpha \geqslant 0$ and $2 \leqslant p \leqslant \infty$.*

(1) *The linear mappings*

$$T : E_p^\alpha \to \ell_\alpha^1(\ell^p),\ f \mapsto (Tf)_{nm} = \frac{1}{2}\int_{\mathbb{R}} f(t)e^{-int}\overline{\phi_0}\hat{\ }(t-\pi m)dt$$

and

$$S : \ell_\alpha^1(\ell^p) \to E_p^\alpha,\ (a_{nm}) \mapsto \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_{nm}e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m) \tag{5.7}$$

are well-defined, continuous and $ST = \mathrm{Id}_{E_p^\alpha}$. Here, the sum over m in the definition of S converges locally uniformly, and the sum over n converges in E_p^α. In particular, every $f \in E_p^\alpha$ has a decomposition

$$f(t) = \sum_{nm} a_{nm}e^{int}\check{\phi}_0(t-\pi m)$$

such that

$$\|f\|_{E_p^\alpha} \cong \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(a_{nm})_m\|_p.$$

(2) *In the particular case that $p = \infty$ and $f \in E_\infty^\alpha$ is π-periodic, $(Tf)_{nm} = (-1)^{nm}a_n$ with $a_n = \frac{1}{2}\int_{\mathbb{R}} f(t)e^{-int}\overline{\phi_0}\check{\ }(t)dt$, and the above decomposition of f reduces to the Fourier series:*

$$f(t) = STf(t) = \sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{Z}}(-1)^{nm}\check{\phi}_0(t-\pi m)\right)a_n e^{int} = \sum_{n\in 2\mathbb{Z}} a_n e^{int}.$$

Conversely, any π-periodic function $f(t) = \sum_{n\in 2\mathbb{Z}} a_n e^{int}$ belongs to E_∞^α if and only if we have $\sum_{n\in 2\mathbb{Z}}\langle n\rangle^\alpha |a_n| < \infty$, and in this case, $\|f\|_{E_\infty^\alpha} \cong \sum_{n\in 2\mathbb{Z}}\langle n\rangle^\alpha |a_n|$.

Proof. (1): Let $f \in E_p^\alpha$ and $a_{nm} = (Tf)_{nm}$. For any $n \in \mathbb{Z}$, $\|(a_{nm})_m\|_p \lesssim \|f * \widetilde{\phi_n}\check{\ }\|_p$. Indeed, if $p = \infty$, then

$$\begin{aligned}|a_{nm}| &= \frac{1}{2}\Big|\int_{\mathbb{R}} f(t)[\overline{\phi_n}e^{i\pi m(\cdot)}]\hat{\ }(t)dt\Big| \\ &\leqslant |\langle \hat{f}, \overline{\phi_n}e^{i\pi m(\cdot)}\rangle| \\ &= |\langle \hat{f}\widetilde{\phi}_n, \overline{\phi_n}e^{i\pi m(\cdot)}\rangle| \\ &= |\langle f * \widetilde{\phi}_n\check{\ }, [\overline{\phi_n}e^{i\pi m(\cdot)}]\hat{\ }\rangle| \\ &\leqslant \|f * \widetilde{\phi}_n\check{\ }\|_\infty \|[\overline{\phi_n}e^{i\pi m(\cdot)}]\hat{\ }\|_1,\end{aligned}$$

and the last factor is a constant. Thus, $\|a_{nm}\|_\infty \lesssim \|f * \widetilde{\phi}_n\check{\ }\|_\infty$. If $p = 2$, then by the Plancherel identity for both Fourier series and Fourier transforms,

$$\|(a_{nm})_m\|_2 = \|\hat{f}\overline{\phi_n}\|_{L^2(\mathbb{R})} = \|f * \overline{\phi_n}\check{\ }\|_{L^2(\mathbb{R})} \lesssim \|f * \check{\phi}_n\|_{L^2(\mathbb{R})}.$$

Finally, the case $2 < p < \infty$ follows by complex interpolation. Thus,

$$\|Tf\|_{\ell_\alpha^1(\ell^p)} = \sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha\|(a_{nm})_m\|_p \lesssim \sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha \|f * \widetilde{\phi}_n\check{\ }\|_p \leqslant \sum_{k=-1}^{1}\sum_{n\in\mathbb{Z}}\langle n+k\rangle^\alpha\|f * \check{\phi}_n\|_p \lesssim \|f\|_{E_p^\alpha},$$

and T is a bounded operator.

We turn to the operator S and consider $(a_{nm}) \in \ell_\alpha^1(\ell^p)$.

- Step 1: For any $n \in \mathbb{Z}$, $\sum_{m=-M}^{M} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m)$ converges locally uniformly as $M \to \infty$ and $\| \sum_{m\in\mathbb{Z}} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m)\|_p \lesssim \|(a_{nm})_m\|_p$.
- Step 2: $\sum_{n=-N}^{N} \sum_{m\in\mathbb{Z}} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m)$ converges in E_p^α as $N \to \infty$.
- Step 3: If $a_{nm} = (Tf)_{nm}$ for some $f \in E_p^\alpha$, then $\sum_{m\in\mathbb{Z}} a_{nm} e^{in(\cdot)} \check{\phi}_0(\cdot - \pi m) = f * \check{\phi}_n$ for any $n \in \mathbb{Z}$.
- Step 4: Conclusion.

Step 1: Since $\phi_0 \in C_0^\infty(\mathbb{R})$, $\check{\phi}_0$ is rapidly decreasing, and thus in particular, for any $R > 0$,

$$\sum_{m\in\mathbb{Z}} \sup_{|t|\leqslant R} |\check{\phi}_0(t - \pi m)| \lesssim \sum_{m\in\mathbb{Z}} \sup_{|t|\leqslant R} \langle t - \pi m\rangle^{-2} < \infty.$$

Hence for any $F \subset \mathbb{Z}$ finite,

$$\sup_{|t|\leqslant R} |\sum_{m\in F} a_{nm} e^{int} \check{\phi}_0(t - \pi m)| \leqslant \|(a_{nm})_m\|_\infty \sum_{m\in F} \sup_{|t|\leqslant R} |\check{\phi}_0(t - \pi m)|.$$

This shows the locally uniform convergence (note that $\|(a_{nm})_m\|_\infty \leqslant \|(a_{nm})_m\|_p$). For the claimed estimate, we proceed again by complex interpolation. If $p = \infty$, for any $t \in \mathbb{R}$ fixed,

$$\begin{aligned} |\sum_{m\in\mathbb{Z}} a_{nm} e^{int} \check{\phi}_0(t - \pi m)| &\leqslant \|(a_{nm})_m\|_\infty \sum_{m\in\mathbb{Z}} |\check{\phi}_0(t - \pi m)| \\ &\leqslant \|(a_{nm})_m\|_\infty \sum_{m\in\mathbb{Z}} \sup_{|t|\leqslant\pi} |\check{\phi}_0(t - \pi m)| \\ &\lesssim \|(a_{nm})_m\|_\infty. \end{aligned}$$

For $p = 1$,

$$\begin{aligned} \int_{\mathbb{R}} |\sum_{m\in\mathbb{Z}} a_{nm} e^{int} \check{\phi}_0(t - \pi m)| dt &\leqslant \sum_{m\in\mathbb{Z}} |a_{nm}| \int_{\mathbb{R}} |\check{\phi}_0(t - \pi m)| dt \\ &\lesssim \|(a_{nm})_m\|_1, \end{aligned}$$

since $\int_{\mathbb{R}} |\check{\phi}_0(t - \pi m)| dt$ does not depend on m and is finite.

Step 2: Let $n, n_0 \in \mathbb{Z}$. Since the series in step 1 converges locally uniformly and is bounded in $L^\infty(\mathbb{R})$, by dominated convergence

$$[\sum_{m\in\mathbb{Z}} a_{nm} \check{\phi}_0(\cdot - \pi m) e^{in(\cdot)}] * \check{\phi}_{n_0} = \sum_{m\in\mathbb{Z}} a_{nm} [\check{\phi}_0(\cdot - \pi m) e^{in(\cdot)} * \check{\phi}_{n_0}].$$

The support condition on $(\phi_n)_n$ yields that the sum vanishes for $|n - n_0| \geqslant 2$. For $|n - n_0| \leqslant 1$, by step 1,

$$\begin{aligned} \|[\sum_{m\in\mathbb{Z}} a_{nm} \check{\phi}_0(\cdot - \pi m) e^{in(\cdot)}] * \check{\phi}_{n_0}\|_p &\leqslant \|\sum_{m\in\mathbb{Z}} a_{nm} \check{\phi}_0(\cdot - \pi m) e^{in(\cdot)}\|_p \|\check{\phi}_{n_0}\|_1 \\ &\lesssim \|(a_{nm})_m\|_\infty. \end{aligned}$$

Thus for any $F \subset \mathbb{Z}$ finite and $F^* = \{n+k : n \in F, k = -1, 0, 1\}$,

$$\begin{aligned} \| \sum_{n\in F}\sum_{m\in\mathbb{Z}} a_{nm}\check{\phi}_0(\cdot - \pi m)e^{in(\cdot)}\|_{E_p^\alpha} &= \sum_{n_0\in\mathbb{Z}} \langle n_0\rangle^\alpha \| \sum_{n\in F}\sum_{m\in\mathbb{Z}} a_{nm}\check{\phi}_0(\cdot - \pi m)e^{in(\cdot)} * \check{\phi}_{n_0}\|_p \\ &\lesssim \sum_{k=-1}^{1}\sum_{n\in F} \langle n+k\rangle^\alpha \|(a_{nm})_m\|_p \\ &\lesssim \sum_{n\in F^*} \langle n\rangle^\alpha \|(a_{nm})_m\|_p. \end{aligned}$$

This shows that the stated convergence of the double series defining $S(a_{nm})$ and that S is a bounded operator.

Step 3: Let $\psi \in C_c^\infty(\mathbb{R})$.

$$\begin{aligned} \langle f * \check{\phi}_n, \psi\rangle &= \langle \hat{f}\phi_n, \check{\psi}\rangle \\ &= \langle \hat{f}\overline{\phi_n}, \phi_n\check{\psi}\rangle \\ &= \langle \hat{f}\overline{\phi_n}, \sum_{m\in\mathbb{Z}} \frac{1}{2}\langle e^{-i\pi m(\cdot)}, \phi_n\check{\psi}\rangle e^{i\pi m(\cdot)}\rangle \\ &= \sum_m \frac{1}{2}\langle \hat{f}\overline{\phi_n}, e^{i\pi m(\cdot)}\rangle\langle \phi_n e^{-i\pi m(\cdot)}, \check{\psi}\rangle \\ &= \sum_m \frac{1}{2}\langle \hat{f}\overline{\phi}_n, e^{i\pi m(\cdot)}\rangle e^{-i\pi nm}\langle \check{\phi}_0(\cdot - \pi m)e^{in(\cdot)}, \psi\rangle \\ &= \sum_m a_{nm}\langle e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m), \psi\rangle \\ &= \langle \sum_m a_{nm}e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m), \psi\rangle. \end{aligned}$$

In the third equality, we develop $\phi_n\check{\psi}$ in a Fourier series. Note that its coefficients are rapidly decreasing, since $\phi_n\check{\psi} \in C_c^\infty(\mathbb{R})$. Hence, the sum can be taken out in the subsequent equality. In the last equality, the sum can be taken inside by dominated convergence, because $\|(a_{nm})_m\|_\infty \leqslant \|(a_{nm})_m\|_p < \infty$.

Step 4: According to step 3, for any $f \in E_p^\alpha$ and $n \in \mathbb{Z}$,

$$f * \check{\phi}_n = \sum_m a_{nm}e^{in(\cdot)}\check{\phi}_0(\cdot - \pi m).$$

Take the sum over $n \in \mathbb{Z}$ on both sides of this equation. The left hand side converges to f in E_p^α after proposition 5.2, and the right hand side converges to STf. This shows $f = STf$.

(2): Let $f \in E_\infty^\alpha$ be π-periodic.

$$\begin{aligned} (Tf)_{nm} &= \int_{\mathbb{R}} f(t)e^{-int}\overline{\phi_0}\check{}\,(t - \pi m)dt = \int_{\mathbb{R}} f(t - \pi m)e^{-int}\overline{\phi_0}\check{}\,(t - \pi m)dt \\ &= e^{-inm\pi}\int_{\mathbb{R}} f(t)e^{-int}\overline{\phi_0}\check{}\,(t)dt = (-1)^{nm}a_n. \end{aligned}$$

We show that

$$\sum_m (-1)^{nm}\check{\phi}_0(t-\pi m) = \begin{cases} \pi & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

For the case n even, note that $\sum_m \check{\phi}_0(t-\pi m)$ is a π-periodic function. Hence it suffices to check its Fourier coefficients.

$$\begin{aligned}\int_0^\pi \sum_m \check{\phi}_0(t-\pi m)e^{-2ikt}dt &= \sum_m \int_0^\pi \check{\phi}_0(t-\pi m)e^{-2ikt}dt \\ &= \sum_m \int_0^\pi \check{\phi}_0(t-\pi m)e^{-2ik(t-\pi m)}dt \\ &= \sum_m \int_{\mathbb{R}} \check{\phi}_0(t)e^{-2ikt}dt \\ &= \phi_0(2k) \\ &= \delta_{k=0}.\end{aligned}$$

For the case n odd, note that $\sum_m (-1)^m\check{\phi}_0(t-\pi m)$ is 2π-periodic. Similarly, we compute the Fourier coefficients

$$\begin{aligned}\int_0^{2\pi} \sum_m (-1)^m\check{\phi}_0(t-\pi m)e^{-ikt}dt &= \sum_m \int_0^{2\pi} \check{\phi}_0(t-2\pi m)e^{-ikt}dt - \sum_m \int_0^{2\pi} \check{\phi}_0(t-2\pi m-\pi)e^{-ikt}dt \\ &= \int_{\mathbb{R}} \check{\phi}_0(t)e^{-ikt}dt - \int_{\mathbb{R}} \check{\phi}_0(t)e^{-ikt}e^{-ik\pi}dt \\ &= \phi_0(k)(1-e^{-i\pi k}) \\ &= 0.\end{aligned}$$

This shows that STf is the Fourier series of f, and the “only if” part. For the “if” part, define $a_{nm} = (-1)^{nm}a_n$ for even n and $a_{nm} = 0$ for odd n. Then $f = S(a_{nm})$ and $\|f\|_{E_\infty^\alpha} \lesssim \|a_{nm}\|_{\ell_\alpha^1(\ell^\infty)} = \sum_{n\in 2\mathbb{Z}}\langle n\rangle^\alpha |a_n| \lesssim \sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha |a_{2n}| \lesssim \|f\|_{E_\infty^\alpha}$. □

Remark 5.4 *The decomposition $f(t) = \sum_{n,m} a_{nm}e^{int}\check{\phi}_0(t-\pi m)$ is not unique. Indeed, let $\overline{\phi_0^{(1)}} \geqslant \overline{\phi_0^{(2)}}$ be two different choices of $\overline{\phi_0}$ with corresponding $T^{(1)}, T^{(2)}$ as in the above proposition. Let $f \in E_p^\alpha$ such that $\hat{f}(t) > 0$ for $t \in [-1,1]$. Then*

$$(T^{(1)}f)_{00} = \frac{1}{2}\langle \hat{f}, \overline{\phi_0^{(1)}}\rangle > \frac{1}{2}\langle \hat{f}, \overline{\phi_0^{(2)}}\rangle = (T^{(2)}f)_{00}.$$

As an application of proposition 5.3, we compare the spaces E_∞^α with classical Besov spaces.

Proposition 5.5 *Let $\alpha \geqslant 0$.*

(1) *If $\beta \geqslant \alpha+1$, then $\mathcal{B}^\beta \hookrightarrow E_\infty^\alpha \hookrightarrow \mathcal{B}^\alpha$.*

(2) *If $\beta < \alpha+1$, then $\mathcal{B}^\beta \not\hookrightarrow E_\infty^\alpha$.*

(3) *If $\beta > \alpha+\frac{1}{2}$, then for any $f \in \mathcal{B}^\beta$ periodic or with compact support, $f \in E_\infty^\alpha$ and there exists $C > 0$ depending on the period or the length of the support such that $\|f\|_{E_\infty^\alpha} \leqslant C\|f\|_{\mathcal{B}^\beta}$.*

(4) *If $\beta < \alpha+\frac{1}{2}$, then there exists a periodic $f \in \mathcal{B}^\beta$ such that $f \notin E_\infty^\alpha$.*

Proof. (1): Let $(\phi_k)_k$ be an equidistant Fourier partition of unity and $(\psi_n)_n$ a dyadic one. For $n \in \mathbb{N}$, we let $A_n = \{k \in \mathbb{Z} : 2^n - 1 \leqslant k \leqslant 2^{n+1}\}$, $A_{-n} = -A_n$ and $A_0 = \{0\}$. Further we let $A_n' = A_{n-1} \cup A_n \cup A_{n+1}$. Then $\bigcup_{n\in\mathbb{Z}} A_n = \mathbb{Z}$ as a disjoint union, $\text{card}A_n' \lesssim 2^{|n|}$ and $\sum_{k\in A_n'} \phi_k \psi_n = \psi_n$. Further, for $k \in A_n'$, $\langle k \rangle \cong 2^{|n|}$ and for $k \in A_n$, $\widetilde{\psi}_n \phi_k = \phi_k$. Therefore,

$$\|f\|_{\mathcal{B}^\alpha} = \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \|f * \check{\psi}_n\|_\infty \leqslant \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} \sum_{k\in A_n'} \|f * \check{\psi}_n * \check{\phi}_k\|_\infty \lesssim \sum_k \langle k \rangle^\alpha \|f * \check{\phi}_k\|_\infty = \|f\|_{E^\alpha_\infty}.$$

In the other direction,

$$\|f\|_{E^\alpha_\infty} = \sum_{n\in\mathbb{Z}} \sum_{k\in A_n} \langle k \rangle^\alpha \|f * \check{\phi}_k * \widetilde{\psi}_n\check{}\,\|_\infty \leqslant \sum_{n\in\mathbb{Z}} 2^{|n|\alpha} 2^{|n|} \|f * \widetilde{\psi}_n\check{}\,\|_\infty \lesssim \|f\|_{\mathcal{B}^{\alpha+1}}.$$

(2): Since J_α in proposition 5.2 (6) is an isomorphism $\mathcal{B}^0 \to \mathcal{B}^\alpha$ and $E^0_\infty \to E^\alpha_\infty$ simultaneously, it suffices to consider the case $\alpha = 0$, $\beta < 1$. Let

$$f_N(t) = \sum \langle n \rangle^{-1} e^{int} \check{\phi}_0(t - \pi n),$$

where the sum ranges over all even n such that $1 \leqslant n \leqslant N$. For such an n fixed, we have in turn

$$\|f_N * \check{\phi}_n\|_\infty = \|\sum_{k\leqslant N \text{ even}} \langle k \rangle^{-1} (\phi_k \phi_n e^{i\pi n(\cdot)})\check{}\,\|_\infty = \langle n \rangle^{-1} \|(\phi_n^2)\check{}\,(t - \pi n)\|_\infty,$$

and the last factor is a constant. Thus, $\|f_N\|_{E^0_\infty} \geqslant \sum_{n=1}^N \|f_N * \check{\phi}_n\|_\infty \gtrsim \sum_{n\leqslant N \text{ even}} \langle n \rangle^{-1} \to \infty$ for $N \to \infty$. On the other hand, f_N is bounded in $\mathcal{B}^\beta$. Indeed, $\|f_N\|_{\mathcal{B}^\beta} \lesssim \|f_N\|_\infty + \|f_N'\|_\infty$ by [128]. But

$$\|f_N\|_\infty \leqslant \sup_{t\in\mathbb{R}} |\sum_{n=1}^N \langle n \rangle^{-1} e^{int} \check{\phi}_0(t - \pi n)| \leqslant \sup_{t\in\mathbb{R}} \sum_{n=1}^\infty |\check{\phi}_0(t - \pi n)| < \infty, \text{ and}$$

$$\|f_N'\|_\infty \leqslant \sup_{t\in\mathbb{R}} |\sum_{n=1}^N \langle n \rangle^{-1} n e^{int} \check{\phi}_0(t - \pi n)| + \sup_{t\in\mathbb{R}} |\sum_{n=1}^N \langle n \rangle^{-1} e^{int} (\check{\phi}_0)'(t - \pi n)|$$
$$\leqslant \sup_{t\in\mathbb{R}} \sum_{n=1}^\infty |\check{\phi}_0(t - \pi n)| + |(\check{\phi}_0)'(t - \pi n)| < \infty.$$

(3): Since J_α maps periodic functions to periodic functions, we can again assume $\alpha = 0$. By [128], any $f \in \mathcal{B}^\beta$ is β' Hölder continuous with $\beta' \in (\frac{1}{2}, \beta)$. Then by [74, p. 34], the Fourier coefficients a_n of f are absolutely summable. By proposition 5.2 (5), we can assume that f is π-periodic, and by proposition 5.3, $\|f\|_{E^0} \cong \sum_{n\in\mathbb{Z}} |a_n| < \infty$.

If f has compact support, say $[N\frac{\pi}{2}, (N+1)\frac{\pi}{2}]$ for some $N \in \mathbb{Z}$, then $g = \sum_{m\in\mathbb{Z}} f(\cdot - m\pi)$ is periodic, and by [128, p. 110], $\|g\|_{\mathcal{B}^\beta} \cong \|f\|_{\mathcal{B}^\beta}$. Thus, the first part yields $g \in E^\alpha_\infty$, and consequently $\|f\|_{E^\alpha_\infty} = \|g\varphi\|_{E^\alpha_\infty} \lesssim \|g\|_{E^\alpha_\infty} \|\varphi\|_{E^\alpha_\infty} < \infty$, where $\varphi \in C^\infty_c(\mathbb{R})$ is chosen appropriately.

(4): Again we can assume $\alpha = 0$. Choose a periodic $\frac{1}{2}$ Hölder continuous function f whose Fourier coefficients are not absolutely summable [74, p. 36]. By [128], $f \in \mathcal{B}^\beta$, and by proposition 5.3 (2), $f \notin E^0_\infty$. $\square$

5.3 The E_∞^α calculus

Throughout the rest of this chapter, we let iB be a generator of a c_0-group $U(t) = e^{itB}$ on a Banach space X. As remarked in section 2.2 from chapter 2, B is a strip-type operator and we can consider the extended holomorphic calculus of B. Boyadzhiev and deLaubenfels [15], see also [24, thm 2.4], have shown that if X is a Hilbert space, then an exponential growth $\|U(t)\| \lesssim e^{\omega|t|}$ can be characterized (up to ε) by the strip height ω of the bounded $H^\infty(\mathrm{Str}_\omega)$ functional calculus of B. This result has been extended by Kalton and Weis to spaces X with property (α), where the boundedness of $e^{-\omega|t|}U(t)$ is replaced by γ-boundedness [72, thm 6.8].

If we replace exponential growth by some polynomial growth, i.e. there exists $\alpha \geqslant 0$ such that $\|U(t)\| \lesssim \langle t\rangle^\alpha$, then it will turn out in this section that there is an analogous result in terms of an E_∞^α functional calculus.

Let us construct this calculus. We want to define $f(B)$ by making sense of the "Fourier inversion formula"

$$f(B)x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)U(t)x\,dt = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} \hat{f}(t)\phi_n(t)U(t)x\,dt,$$

where $(\phi_n)_n$ is an equidistant Fourier partition of unity. We shall see that E_∞^α is the natural function space for which the above sum converges.

The construction of the calculus is based on the operators of definition 5.6 below and relies on Haase's transference principle (see [53, thm 3.1] and [54, thm 3.2]).

For the Gaussian function spaces and γ-boundedness that we need in a general Banach space X, we refer to chapter 2.

Definition 5.6 *We assume that for a given $\alpha \geqslant 0$, $\|U(t)\| \lesssim \langle t\rangle^\alpha$ and that $\{U(t) : t \in [0,1]\}$ is γ-bounded. For $n \in \mathbb{Z}$, we let $\chi_n = \chi_{[n-2,n+1]}$. We define*

$$I_n : X \to \gamma(\mathbb{R}, X),\ x \mapsto \chi_n(-t)U(-t)x.$$

Since $U(t)$ is a group, $\{U(t) : t \in [n, n+1]\} = U(n) \circ \{U(t) : t \in [0,1]\}$, and therefore, the assumptions imply

$$\gamma(\{U(t) : t \in [n, n+1]\}) \leqslant \|U(n)\|\gamma(\{U(t) : t \in [0,1]\}) \lesssim \langle n\rangle^\alpha.$$

Thus, by lemma 2.7,

$$\|I_n x\| = \|\chi_n(-t)U(-t)x\|_{\gamma(\mathbb{R},X)} \leqslant \gamma(\{U(-t) : t \in [n-2, n+1]\})\|\chi_n\|_2\|x\| \lesssim \langle n\rangle^\alpha\|x\|.$$

For $f \in L^1(\mathbb{R})$, we let

$$T_f : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}),\ g \longmapsto f * g$$

and denote the adjoint of T_f with respect to the Banach space duality $\langle g, h\rangle = \int g(t)h(t)dt$ by T_f'. Then according to lemma 2.7 (5), for any $u \in \gamma(L^2(\mathbb{R}), X)$, we have

$$T_f^{\otimes}(u) := u \circ T_f' \text{ belongs to } \gamma(L^2(\mathbb{R}), X) \text{ and } \|T_f^{\otimes}u\|_{\gamma(L^2(\mathbb{R}),X)} \leqslant \|T_f\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\|u\|_{\gamma(L^2(\mathbb{R}),X)}.$$

Recall that for a function $g \in \gamma(\mathbb{R}, X)$, we denote u_g the operator in $\gamma(L^2(\mathbb{R}), X) \subset B(L^2(\mathbb{R}), X)$ defined by the identity $\langle u_g(h), x'\rangle = \int \langle g(t), x'\rangle h(t)dt$ for any $x' \in X'$ and $h \in L^2(\mathbb{R})$. We claim that for $g \in \gamma(\mathbb{R}, X)$,

$$f * g \text{ belongs to } \gamma(\mathbb{R}, X) \text{ and } T_f^{\otimes} u_g = u_{f*g}. \tag{5.8}$$

Indeed, it is clear that since $g \in P_2(\mathbb{R}, X)$, i.e. g is weakly L^2, also $f * g \in P_2(\mathbb{R}, X)$. The fact that $f * g \in \gamma(\mathbb{R}, X)$ and $T_f^{\otimes} u_g = u_{f*g}$ then follows immediately from the following identities:

$$\begin{aligned} \langle T_f^{\otimes} u_g(h), x'\rangle = \langle u_g(T_f' h), x'\rangle &= \int_{\mathbb{R}} \langle g(t), x'\rangle T_f' h(t)dt \\ &= \int_{\mathbb{R}} T_f[\langle g(\cdot), x'\rangle](t)h(t)dt = \int_{\mathbb{R}} \langle f * g(t), x'\rangle h(t)dt. \end{aligned}$$

In particular, 5.8 implies that

$$\|f * g\|_{\gamma(\mathbb{R},X)} \leqslant \|f * (\cdot)\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \|g\|_{\gamma(\mathbb{R},X)} = \|\hat{f}\|_\infty \|g\|_{\gamma(\mathbb{R},X)}.$$

Next, we put

$$P : \gamma(\mathbb{R}, X) \to X,\ f \mapsto \int_{\mathbb{R}} \chi_{[0,1]}(t)U(t)f(t)dt.$$

We have

$$\begin{aligned} \|Pf\| &= \|\int_{\mathbb{R}} \chi_{[0,1]}(t)U(t)f(t)dt\| \\ &\leqslant \sup_{\|x'\|\leqslant 1} \int_{\mathbb{R}} |\langle f(t), \chi_{[0,1]}(t)U(t)'x'\rangle|dt \\ &\leqslant \|f\|_{\gamma(\mathbb{R},X)} \sup_{\|x'\|\leqslant 1} \|\chi_{[0,1]}(t)U(t)'x'\|_{\gamma(\mathbb{R},X')} \\ &\leqslant \gamma(\{U(t) :\ t \in [0,1]\})\|f\|_{\gamma(\mathbb{R},X)}, \end{aligned}$$

according to lemma 2.7. We summarize:

$$\|I_n\|_{X\to\gamma(\mathbb{R},X)} \lesssim \langle n\rangle^\alpha,\ \|T_f\|_{\gamma(L^2(\mathbb{R}),X)\to\gamma(L^2(\mathbb{R}),X)} \leqslant \|\hat{f}\|_{L^\infty(\mathbb{R})},\ \|P\|_{\gamma(\mathbb{R},X)\to X} \lesssim 1.$$

In the next proposition, the E_∞^α calculus is constructed by means of I_n, P and T_f.

Proposition 5.7 *Let $\alpha \geqslant 0$. Assume that $\|U(t)\| \lesssim \langle t\rangle^\alpha$ and that $\{U(t) :\ t \in [0,1]\}$ is γ-bounded.*

(1) *For $f \in H_0^\infty(\mathrm{Str}_\omega)$ for some $\omega > 0$, we have $PT_{f*\check{\phi}_n} I_n = f * \check{\phi}_n(B)$.*

(2) *For every $f \in E_\infty^\alpha$, $\sum_{n\in\mathbb{Z}} PT_{f*\check{\phi}_n} I_n$ converges absolutely and the mapping*

$$\Psi : E_\infty^\alpha \to B(X),\ f \mapsto \sum_{n\in\mathbb{Z}} PT_{f*\check{\phi}_n} I_n \tag{5.9}$$

is bounded.

Proof. (1): For $f \in H_0^\infty(\mathrm{Str}_\omega)$ we have $|f(z)| \lesssim e^{-\varepsilon|\operatorname{Re} z|}$ for some $\varepsilon > 0$. Therefore, $\|f * \check{\phi}_n\|_{L^2(\mathbb{R})} \leqslant \|f\|_{L^2(\mathbb{R})} \|\check{\phi}_n\|_{L^1(\mathbb{R})} < \infty$. By the Cauchy integral formula, also $f^{(k)} \in H_0^\infty(\mathrm{Str}_\omega)$, and thus $f * \check{\phi}_n \in W_2^k$ for any $k \in \mathbb{N}$. Note that $\|\langle t\rangle^{-\alpha-1}U(t)x\|_{L^2(\mathbb{R})} \lesssim \|\langle t\rangle^{-1}\|_{L^2(\mathbb{R})}\|x\|$, so that B has a $W_2^{\alpha+1}$ calculus and consequently, by proposition 4.22 and remark 4.23 (2),

$$f * \check{\phi}_n(B)x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)\phi_n(t)U(t)xdt.$$

On the other hand, $\hat{f}(t)\phi_n(t) \in L^1(\mathbb{R})$ and for $g \in \gamma(\mathbb{R}, X)$, $(T_{f*\check{\phi}_n} g)(t) = \int_{\mathbb{R}} \hat{f}(s)\phi_n(s)g(t-s)ds$. Then (cf. [54, thm 3.2]) we have

$$\begin{aligned} PT_{f*\check{\phi}_n} I_n x &= \frac{1}{2\pi}\int_{\mathbb{R}} \chi_{[0,1]}(t)U(t)\int_{\mathbb{R}} \hat{f}(s)\phi_n(s)(I_n x)(t-s)dsdt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[0,1]}(t)U(t)\hat{f}(s)\phi_n(s)\chi_n(s-t)U(s-t)xdsdt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \chi_{[0,1]} * \chi_n(s)\hat{f}(s)\phi_n(s)U(s)xds \\ &= f * \check{\phi}_n(B)x, \end{aligned} \tag{5.10}$$

noting that $\chi_{[0,1]} * \chi_n(s) = 1$ for $s \in [n-1, n+1]$.

(2): Summarizing what we know from definition 5.6, we get

$$\sum_{n\in\mathbb{Z}} \|PT_{f*\check{\phi}_n} I_n\| \lesssim \sum_{n\in\mathbb{Z}} \|f * \check{\phi}_n\|_\infty \langle n\rangle^\alpha = \|f\|_{E^\alpha_\infty} < \infty.$$

This shows that $\sum_{n\in\mathbb{Z}} PT_{f*\check{\phi}_n} I_n$ converges absolutely and that Ψ is bounded. □

We have the following convergence property of the calculus Ψ in 5.9.

Proposition 5.8 *Assume that* $\|U(t)\| \lesssim \langle t\rangle^\alpha$ *and* $\{U(t) : t \in [0,1]\}$ *is* γ*-bounded. The mapping* $\Psi : E^\alpha_\infty \to B(X)$ *from proposition 5.7 has the following convergence property: If* $(f_k)_k \subset E^\alpha_\infty$ *with*

(1) $\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_k \|f_k * \check{\phi}_n\|_\infty < \infty$

(2) $f_k(t) \to f(t)$ *for all* $t \in \mathbb{R}$ *for some function* f,

then $f \in E^\alpha_\infty$ *and*

$$\Psi(f_k)x \to \Psi(f)x \quad (x \in X). \tag{5.11}$$

Note that this covers the H^∞ *calculus convergence lemma, proposition 2.5: If* $(f_k)_k \subset H^\infty(\Sigma_\omega)$ *is a bounded sequence for some* $\omega > 0$*, and* $f_k(z) \to f(z)$ *pointwise on* Str_ω*, then (1) and (2) above are satisfied for any* $\alpha \geqslant 0$.

Proof. We have $f*\check{\phi}_n(t) = \lim_k f_k*\check{\phi}_n(t)$ for any $t \in \mathbb{R}$ by dominated convergence. In particular, $|f * \check{\phi}_n(t)| \leqslant \sup_k |f_k * \check{\phi}_n(t)| \leqslant \sup_k \|f_k * \check{\phi}_n\|_\infty$. Thus, $f \in E^\alpha_\infty$. Furthermore, for any $N \in \mathbb{N}$,

$$\|\Psi(f)x - \Psi(f_k)x\| \leqslant \|\sum_{n\leqslant N} PT_{(f-f_k)*\check{\phi}_n} I_n x\| + \sum_{|n|>N} \|PT_{(f-f_k)*\check{\phi}_n} I_n x\|.$$

For the first sum, note that by lemma 2.7, $\|M_{(f-f_k)*\check{\phi}_n} g\|_{\gamma(\mathbb{R},X)} \to 0$ for all $g \in \gamma(\mathbb{R}, X)$, and thus also $\|T_{(f-f_k)*\check{\phi}_n} g\|_{\gamma(L^2(\mathbb{R}),X)} \to 0$ as $k \to \infty$. The second sum, we simply estimate by

$2\sum_{n\geqslant|N|}\langle n\rangle^\alpha \sup_k \|f_k*\check{\phi}_n\|$, which converges to 0 for $N\to\infty$. We have proved the convergence property 5.11.

It remains to show that this entails the H^∞ calculus convergence property. Let $f\in H^\infty(\mathrm{Str}_\omega)$ and $|\theta|<\omega$. By the Cauchy integral formula, $f*\check{\phi}_n(t)=f(\cdot-i\theta)*\check{\phi}_n(\cdot-i\theta)(t)$, so that $\|f*\check{\phi}_n\|_{L^\infty(\mathbb{R})}\leqslant\|f\|_{\infty,\theta}\|\check{\phi}_n(\cdot-i\theta)\|_{L^1(\mathbb{R})}$. But $|\check{\phi}_n(t+i\theta)|=|\check{\phi}_0(t+i\theta)|e^{n\theta}$, so choosing $\mathrm{sgn}(n\theta)\leqslant 0$, we get $\|f*\check{\phi}_n\|_{L^\infty(\mathbb{R})}\lesssim e^{-|n|\theta}\|f\|_{\infty,\theta}$. Now consider the sequence $(f_k)_k\subset H^\infty(\mathrm{Str}_\omega)$. Applying the above to $f=f_k$ gives

$$\sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha\sup_k\|f_k*\check{\phi}_n\|_\infty\leqslant\left(\sum_{n\in\mathbb{Z}}e^{-n|\theta|}\langle n\rangle^\alpha\right)\sup_k\|f_k\|_{\infty,\omega},\tag{5.12}$$

and the last sum is finite for any $\alpha\geqslant 0$. □

Remark 5.9 *From propositions 5.7 and 5.8, we can now deduce that 5.9 coincides with the H^∞ calculus for any $f\in H^\infty(\mathrm{Str}_\omega)$, and consequently, we write henceforth $f(B)$ instead of $\Psi(f)\quad(f\in E^\alpha_\infty)$:*

First, if $f\in H^\infty_0(\mathrm{Str}_\omega)$, then $\sum_{|n|\leqslant k}f*\check{\phi}_n\to f$ uniformly on $\mathbb{R}$ because for the Fourier inverse of it, we have $\sum_{|n|\leqslant k}\hat{f}\phi_n\to\hat{f}$ in $L^1(\mathbb{R})$ by dominated convergence. Also for $|\theta|<\omega$, $f*\check{\phi}_n(t+i\theta)=f_\theta*\check{\phi}_n(t)$, where $f_\theta(t)=f(t+i\theta)$, and $\sum_{|n|\leqslant k}f_\theta*\check{\phi}_n\to f_\theta$ in $L^1(\mathbb{R})$ uniformly for $|\theta|<\omega$, so that $\sum_{|n|\leqslant k}f*\check{\phi}_n\to f$ in $H^\infty(\mathrm{Str}_\omega)$.

Therefore, the continuity of the $H^\infty(\mathrm{Str}_\omega)$ calculus yields $f(B)=\lim_k\sum_{|n|\leqslant k}f*\check{\phi}_n(B)$. On the other hand, this equals $\Psi(f)$ by proposition 5.7 (1) and (2).

For general $f\in H^\infty(\mathrm{Str}_\omega)$, we apply again the convergence lemma from proposition 5.8 for both the H^∞ calculus and Ψ, choosing e.g. the sequence $f_k(z)=f(z)(e^z/(1+e^z)^2)^{1/k}\in H^\infty_0(\mathrm{Str}_\omega)$.

We are now ready to state the announced characterization of c_0-groups of polynomial growth. Note that in (2) below, the existence of a bounded E^α_∞ calculus can be stated independently of the construction with Ψ, but based on the H^∞ calculus:

If $\|f(B)\|\lesssim\|f\|_{E^\alpha_\infty}$ for all $f\in H^\infty(\mathrm{Str}_\omega)$ for some $\omega>0$, then by the density of $H^\infty(\mathrm{Str}_\omega)$ in E^α_∞ according to proposition 5.2, the same estimate holds for all $f\in E^\alpha_\infty$. This also matches definition 4.17.

Theorem 5.10 *Let B be a 0-strip-type operator such that iB generates a c_0-group $U(t)=e^{itB}$, and further $\alpha\geqslant 0$. Assume that $\{U(t):\ t\in[0,1]\}$ is γ-bounded. Then the following are equivalent.*

(1) *$\|U(t)\|\lesssim\langle t\rangle^\alpha$ and B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for some $\omega>0$.*

(2) *B has a bounded E^α_∞ calculus.*

Proof. (1) ⇒ (2): This follows from the above construction, definition 5.6 through remark 5.9.

(2) ⇒ (1): If B has a bounded E^α_∞ calculus, then it has also a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for any $\omega>0$, since $\|f\|_{E^\alpha_\infty}\leqslant C_\omega\|f\|_{\infty,\omega}$ by 5.12. Further, we have $U(t)=f_t(B)$ with $f_t(s)=e^{its}$, so it only remains to estimate $\|f_t\|_{E^\alpha_\infty}$. But $f_t*\check{\phi}_n(s)=e^{its}\phi_n(t)$, whence $\|f_t\|_{E^\alpha_\infty}=\sum_{n:\,|n-t|\leqslant 1}\langle n\rangle^\alpha|\phi_n(t)|\cong\langle t\rangle^\alpha$. □

Remark 5.11

(1) *Note that if the underlying Banach space X has property (α), then the above theorem can be stated without the assumption that $\{U(t) : t \in [0,1]\}$ is γ-bounded. Indeed, condition (2) still implies that B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus, which yields by property (α) that $\{U(t) : t \in [0,1]\}$ is γ-bounded [72, cor 6.6].*

(2) *The polynomial growth in condition (1) of the theorem does not guarantee the boundedness of the H^∞ calculus. A counterexample is the shift group $U(t)f(x) = f(x-t)$ on $L^p(\mathbb{R})$, which is even uniformly bounded. In [24, lem 5.3], it is shown that its generator does not have a bounded H^∞ calculus unless $p = 2$.*

(3) *Compare theorem 5.10 to proposition 4.18. In the latter, we have seen that B has a bounded $\mathcal{B}^\alpha$ calculus for some $\alpha > 0$, if and only if B has a bounded H^∞ calculus $\Psi_\omega : H^\infty(\mathrm{Str}_\omega) \to B(X)$ for any $\omega > 0$ and*

$$\|\Psi_\omega\| \lesssim \omega^{-\alpha}. \tag{5.13}$$

In theorem 5.10, we see that relaxing 5.13 to the boundedness of the operators e^{itB}, the functional calculus class $\mathcal{B}^\alpha$ reduces to E^α_∞.

5.4 Operator valued and R-bounded E^α_∞ calculus

Let A be a sectorial operator. In [81, thm 12.7] (see theorem 3.13 above), it is shown that if A has a bounded H^∞ calculus, then this calculus extends boundedly to a certain class of operator valued holomorphic functions.

We will show a similar result for 0-strip-type operators and the E^α_∞ calculus. As in the case of the H^∞ calculus, if the space X has property (α), this procedure can be used to obtain R-bounded families of the type $\{f(B) : f \in \tau\}$ with convenient $\tau \subset E^\alpha_\infty$.

Let $[B]' = \{T \in B(X) : TR(\lambda, B) = R(\lambda, B)T \,\forall\, \lambda \in \mathbb{C}\backslash\mathbb{R}\}$ denote the commutant set of B. Let further

$$H^\infty_0(\mathrm{Str}_\omega, [B]') = \{F : \mathrm{Str}_\omega \to [B]' \text{ analytic and bounded} : \|F(z)\| \lesssim e^{-\varepsilon|\operatorname{Re} z|} \text{ for some } \varepsilon > 0\}.$$

Similar to 3.6 in the sectorial case, we define an operator valued calculus by

$$H^\infty_0(\mathrm{Str}_\omega, [B]') \to B(X),\; F \mapsto F(B) = \frac{1}{2\pi i}\int_\Gamma F(\lambda)(\lambda - B)^{-1} d\lambda \tag{5.14}$$

for the usual contour Γ. Here the restriction to $[B]' \subset B(X)$ ensures the multiplicativity. We will see that under the polynomial growth condition for the group $U(t)$, this calculus extends to a bounded homomorphism on an operator valued version $\mathcal{E}^\alpha$ of the space E^α_∞. We define

$$\mathcal{E}^\alpha = \{F : \mathbb{R} \to [B]' \text{ strongly cont. and bounded}, \|F\|_{\mathcal{E}^\alpha} = \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \gamma(\{F * \check{\phi}_n(t) : t \in \mathbb{R}\}) < \infty\}$$

and equip it with the norm $\|F\|_{\mathcal{E}^\alpha}$.

At first, we record the following properties of $\mathcal{E}^\alpha$.

Proposition 5.12

(1) *$\mathcal{E}^\alpha$ is a Banach algebra.*

(2) *$\mathcal{E}^\alpha$ contains $H_0^\infty(\mathrm{Str}_\omega, [B]')$.*

(3) *For any $F \in \mathcal{E}^\alpha$, $n \in \mathbb{Z}$ and $\omega > 0$, $F * \check{\phi}_n$ is analytic and γ-bounded on Str_ω, and $\sum_{n=-N}^{M} F * \check{\phi}_n$ converges to F in $\mathcal{E}^\alpha$ $(N, M \to \infty)$.*

(4) *Two choices $(\phi_n)_n$ and $(\phi_n')_n$ of the Fourier partition give the same space $\mathcal{E}^\alpha$ with equivalent norms.*

Proof. (1): Recalling that

$$\gamma(\{G(t)H(t) :\ t \in \mathbb{R}\}) \leqslant \gamma(\{G(t)\})\gamma(\{H(t)\}) \tag{5.15}$$

for any operator families G and H, the proof becomes a copy of that of proposition 5.2.

(2): Let $F \in H_0^\infty(\mathrm{Str}_\omega, [B]')$. By the Cauchy formula, $F(z) = \frac{1}{2\pi i}\int_\Gamma F(\lambda)\frac{1}{\lambda - z}d\lambda$. Choosing a contour $\Gamma = \{\pm i\frac{\omega}{2} + t :\ t \in \mathbb{R}\}$ and restricting to $|\operatorname{Im} z| \leqslant \frac{\omega}{4}$, by proposition 2.6 (6), we see that $\gamma(\{F(z) :\ |\operatorname{Im} z| \leqslant \frac{\omega}{4}\}) < \infty$. Similarly, by the Cauchy integral formula for derivatives, we see that for any $k \in \mathbb{N}_0$,

$$\gamma(\{F^{(k)}(z) :\ |\operatorname{Im} z| \leqslant \frac{\omega}{4}\}) < \infty.$$

Then for $n \neq 0$,

$$\begin{aligned}\gamma(\{F * \check{\phi}_n(t) :\ t \in \mathbb{R}\})\langle n\rangle^\alpha &= \gamma(\{F^{(k)} * [\phi_n(s)s^{-k}]\check{\ }(t)\})\langle n\rangle^\alpha \\ &\leqslant \gamma(\{F^{(k)}(t)\})\|[\phi_n(s)s^{-k}]\check{\ }\|_1\langle n\rangle^\alpha,\end{aligned}$$

which is clearly summable over n if k is sufficiently large.

(3): Note that for $F * \check{\phi}_n$, and any $x \in X$ and $x' \in X'$,

$$\langle F * \check{\phi}_n(\cdot)x, x'\rangle = \int_{\mathbb{R}} \langle F(s)x, x'\rangle \check{\phi}_n(\cdot - s)ds.$$

By Morera's theorem, the boundedness of $\langle F(\cdot)x, x'\rangle$ and the fact that $\phi_n \in C_c^\infty$, we readily conclude that $F * \check{\phi}_n$ is analytic and γ-bounded. The convergence can be shown as in proposition 5.2, again using 5.15.

(4): This can also be shown as in the scalar case. □

Let us now construct the calculus $\Psi : \mathcal{E}^\alpha \to B(X)$, in a similar way as in the scalar case: we let $I_n : X \to \gamma(\mathbb{R}, X)$ and $P : \gamma(\mathbb{R}, X) \to X$ as in definition 5.6. We now construct the analogue of T_F from the scalar case. For an $F : \mathbb{R} \to B(X)$ such that $\{F(t) :\ t \in \mathbb{R}\}$ is γ-bounded, we put

$$M_F : \gamma(\mathbb{R}, X) \to \gamma(\mathbb{R}, X),\ g \mapsto F(t)g(t).$$

By lemma 2.7, $\|M_F g\| \leqslant \gamma\{F(t) : t \in \mathbb{R}\})\|g\|$. Let us "extend" M_F to $\gamma(L^2(\mathbb{R}), X)$: For $g \in \gamma(\mathbb{R}, X)$, denote $N_F u_g := u_{F(\cdot)g(\cdot)}$. Then $\|N_F u_g\|_{\gamma(L^2(\mathbb{R}),X)} \leqslant \|M_F\| \cdot \|u_g\|_{\gamma(L^2(\mathbb{R}),X)}$, and by density of $\{u_f : f \in \gamma(\mathbb{R}, X)\}$ in $\gamma(L^2(\mathbb{R}), X)$, N_F extends to a bounded operator

$$N_F : \gamma(L^2(\mathbb{R}), X) \to \gamma(L^2(\mathbb{R}), X).$$

Let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R})$ and denote $\mathcal{F}^\otimes$ the extension to $\gamma(L^2(\mathbb{R}), X)$ given as in lemma 2.7 (5). Further let

$$S_F = (\mathcal{F}^\otimes)^{-1} M_F \mathcal{F}^\otimes : \gamma(L^2(\mathbb{R}), X) \to \gamma(L^2(\mathbb{R}), X)$$

and

$$T_F g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{F}(t-s) g(s) ds \quad (g \in \gamma(\mathbb{R}, X) \cap L^1(\mathbb{R}, X),\ g \text{ comp. supp.}).$$

The remaining technicalities for the construction of the calculus are collected in the next lemma.

Lemma 5.13

(1) *Assume that* $F \in L^1(\mathbb{R}, B(X))$ *and* $\hat{F}(t) = \int_{\mathbb{R}} e^{-its} F(s) ds$ *has compact support. Then for any* $g \in \gamma(\mathbb{R}, X) \cap L^1(\mathbb{R}, X)$ *with compact support, we have*

$$T_F g \text{ belongs to } \gamma(\mathbb{R}, X) \text{ and } u_{T_F g} = S_F u_g. \tag{5.16}$$

(2) *Let* $(F_k)_k$ *be a sequence of functions* $\mathbb{R} \to B(X)$ *such that* $\sup_k \gamma(\{F_k(t) : t \in \mathbb{R}\}) < \infty$ *and* $F_k(t) \to 0$ *strongly for almost all* $t \in \mathbb{R}$. *Then for any* $\psi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and any* $R > 0$, $\gamma(\{F_k * \psi(t) : |t| \leqslant R\}) \to 0$. *Consequently, for any* $g \in \gamma(\mathbb{R}, X)$, $\|M_{F_k * \psi} g\|_{\gamma(\mathbb{R},X)} \to 0$.

Proof. (1): First note that $\{\hat{F}(t) : t \in \mathbb{R}\}$ is γ-bounded by proposition 2.6 (6). Then by lemma 2.7 (6) and the estimate

$$\int_{\mathbb{R}} \|\hat{F}(s) g(t-s)\|_{\gamma(\mathbb{R},X)} ds \leqslant C\gamma(\{\hat{F}(s) : s \in \operatorname{supp} \hat{F}\}) \|g\|_{\gamma(\mathbb{R},X)} < \infty,$$

we deduce immediately that $T_F g$ belongs to $\gamma(\mathbb{R}, X)$. If we let $\mathcal{F}_X : L^1(\mathbb{R}, X) \to C_0(\mathbb{R}, X)$ be the vector valued Fourier transform, i.e. $\mathcal{F}_X g(t) = \int_{\mathbb{R}} e^{-its} g(s) ds$, then we can express

$$\mathcal{F}_X T_F g(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-itr} \hat{F}(r-s) g(s) dr ds = \int_{\mathbb{R}} e^{-its} F(t) g(s) ds = F(t) \mathcal{F}_X g(t).$$

It is easy to check (see also [72, exa 4.9 b)]) that

$$u_{\mathcal{F}_X f} = \mathcal{F}^\otimes u_f$$

as soon as $f \in \gamma(\mathbb{R}, X) \cap L^1(\mathbb{R}, X)$. Applying this to both $f = T_F g$ and $f = g$, we deduce

$$\mathcal{F}^\otimes u_{T_F g} = u_{\mathcal{F}_X T_F g} = u_{F(\cdot)\mathcal{F}_X g(\cdot)} = N_F u_{\mathcal{F}_X g} = N_F \mathcal{F}^\otimes u_g.$$

Now the claim follows by applying $(\mathcal{F}^\otimes)^{-1}$ to both sides.

(2): Write $F_k * \psi(t) = \int_{|s|\leqslant C} F_k(s)\psi(t-s)ds + \int_{|s|\geqslant C} F_k(s)\psi(t-s)ds$. For the first integral, note that for any $x \in X$, $\chi_{[-C,C]}(\cdot)F_k(\cdot)x \to 0$ in $L^1(\mathbb{R},X)$ by assumption and dominated convergence. Since $\psi \in L^\infty(\mathbb{R})$, by proposition 2.6 (6)

$$\gamma(\{(\chi_{[-C,C]}F_k) * \psi(t) : |t| \leqslant R\}) \to 0 \text{ for } k \to \infty.$$

For the second integral, we appeal to proposition 2.6 (4), noting that

$$\sup_{|t|\leqslant R} \|\psi(t-\cdot)(1-\chi_{[-C,C]})\|_1 \to 0 \text{ for } C \to \infty,$$

and thus $\sup_{k,\,|t|\leqslant R} \gamma(\{(1-\chi_{[C,C]})F_k * \psi(t) : |t| \leqslant R\}) \to 0$. Then the rest follows from lemma 2.7. □

Now we define

$$\Psi : \mathcal{E}^\alpha \to [B]',\ F \mapsto \sum_{n\in\mathbb{Z}} PT_{F*\check{\phi}_n} I_n.$$

If $\|U(t)\| \lesssim \langle t\rangle^\alpha$ and $\gamma(\{U(t) : t \in [0,1]\}) < \infty$, then

$$\sum_{n\in\mathbb{Z}} \|PT_{F*\check{\phi}_n} I_n\| \leqslant \sum_{n\in\mathbb{Z}} \|P\|\gamma(\{F*\check{\phi}_n(t) : t\in\mathbb{R}\})\|I_n\| \lesssim \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \gamma(\{F*\check{\phi}_n(t) : t\in\mathbb{R}\}) = \|F\|_{\mathcal{E}^\alpha},$$

so that Ψ is well-defined and continuous.

Proposition 5.14 *Assume that* $\|U(t)\| \lesssim \langle t\rangle^\alpha$ *and that* $\{U(t) : t \in [0,1]\}$ *is* γ*-bounded.*

(1) *For* $F \in H_0^\infty(\text{Str}_\omega, [B]')$*, we have*

$$PT_{F*\check{\phi}_n} I_n x = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{F}(t)\phi_n(t)U(t)x dt = F(B)x \quad (x \in X,\ n \in \mathbb{Z}),$$

where $F(B)$ *refers to* 5.14.

(2) *If* $(F_k)_k \subset \mathcal{E}^\alpha$ *with* $\sum_{n\in\mathbb{Z}}\langle n\rangle^\alpha \sup_k \gamma(\{F_k * \check{\phi}_n(t) : t\in\mathbb{R}\}) < \infty$ *and* $F_k(t)x \to F(t)x$ *for any* $x \in X$ *and any* $t \in \mathbb{R}$*, then* $F \in \mathcal{E}^\alpha$ *and* $\Psi(F_k)x \to \Psi(F)x$ *for any* $x \in X$*.*

(3) *For* $F \in H^\infty(\text{Str}_\omega, [B]')$*,* $\Psi(F)$ *coincides with the operator valued* $H_0^\infty(\text{Str}_\omega, [B]')$ *calculus from* 5.14*.*

Proof. (1): To see the first equality, use lemma 5.13 (1) to express $T_{F*\check{\phi}_n} I_n x$ by the integral on the right hand side of 5.16. Then proceed as in 5.10 in the proof of the scalar proposition. The second equality can be shown as in the scalar case, see the proofs of proposition 4.22 and 5.7.

(2): Proceed as in the proof of proposition 5.8, using lemma 5.13 (2) instead of lemma 2.7.

(3): Let $F \in H_0^\infty(\text{Str}_\omega, [B]')$ and set $F_k = \sum_{|n|\leqslant k} F * \check{\phi}_k$. By (1), we have $\Psi(F) = \lim F_k(B)$.

Similar to remark 5.9, one shows that $\gamma(\{(F_k - F)(z) : z \in \text{Str}_{\omega/2}\}) \to 0$ for $k \to \infty$. The assumptions clearly imply that B has a bounded H^∞ calculus, so by the strip variant of [81, thm 12.7], we have $F(B)x = \lim_k F_k(B)x$ for any $x \in X$. □

Remark 5.15 *The bounded homomorphism $\Psi : \mathcal{E}^\alpha \to B(X)$ is uniquely determined by the two properties (2) and (3) in the above proposition.*

Indeed, if $F \in \mathcal{E}^\alpha$ is analytic and γ-bounded on a strip Str_ω, then $F_k(z) = F(z)(e^z/(1+e^z)^2)^{\frac{1}{k}}$ defines a sequence in $H^\infty_0(\mathrm{Str}_\omega)$ which approximates F in the sense of proposition 5.14: Argue as in the proof of proposition 5.8 to show that

$$\sum_{n\in\mathbb{Z}} \langle n \rangle^\alpha \sup_k \gamma(\{F_k * \check{\phi}_n(t) : \, t \in \mathbb{R}\}) \leqslant \left(\sum_{n\in\mathbb{Z}} e^{-n|\omega-\varepsilon|} \langle n\rangle^\alpha \right) \sup_k \gamma(\{F_k(z) : \, z \in \mathrm{Str}_\omega\}).$$

This fixes the value of $\Psi(F)$ for such F. For general $F \in \mathcal{E}^\alpha$, we appeal to proposition 5.12 (3).

Theorem 5.16 *Let iB be the generator of a c_0-group $U(t)$ and assume that $\{U(t) : \, t \in [0,1]\}$ is γ-bounded. Then the following conditions are equivalent:*

(1) $\|U(t)\| \lesssim \langle t \rangle^\alpha$.

(2) *B has a bounded E^α_∞ calculus.*

(3) *B has a bounded $\mathcal{E}^\alpha$ calculus (in the sense of remark 5.15).*

If in addition, X has property (α), then the following conditions are also equivalent to (1)-(3).

(4) B has a bounded E^α_∞ calculus and moreover, for any $\mathcal{G} \subset E^\alpha_\infty$ with

$$\sum_{n\in\mathbb{Z}} \langle n \rangle^\alpha \sup_{f\in\mathcal{G}} \|f * \check{\phi}_n\|_\infty = C < \infty,$$

$\{f(B) : \, f \subset \mathcal{G}\}$ is γ-bounded with constant $\lesssim C$.

(5) B has a bounded $\mathcal{E}^\alpha$ calculus and moreover, for any $\mathcal{G} \subset \mathcal{E}^\alpha$ with

$$\sum_{n\in\mathbb{Z}} \langle n \rangle^\alpha \gamma(\{F * \check{\phi}_n(t) : \, t \in \mathbb{R}, \, F \in \mathcal{G}\}) = C < \infty,$$

$\{F(B) : \, F \subset \mathcal{G}\}$ is γ-bounded with constant $\lesssim C$.

Proof. The construction of Ψ above shows $(1) \Rightarrow (3)$. By the mapping $f \mapsto F(\cdot) = f(\cdot) \cdot \mathrm{Id}_X$, any scalar function f yields an operator valued function, and clearly

$$\|F\|_{\mathcal{E}^\alpha} = \sum_n \langle n \rangle^\alpha \gamma(\{f(t) \, \mathrm{Id}_X * \check{\phi}_n : \, t \in \mathbb{R}\}) = \sum_n \langle n \rangle^\alpha \|f * \check{\phi}_n\|_\infty = \|f\|_{E^\alpha}.$$

Therefore, $(3) \Rightarrow (2)$ holds. The implication $(2) \Rightarrow (1)$ has been shown in theorem 5.10.

Assume now that X has property (α). We clearly have $(5) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1)$, so that only $(1) \Rightarrow (5)$ has to be shown.

Fix some $N \in \mathbb{N}$ and let $\tilde{X} = \mathrm{Gauss}_N(X)$. Put $\tilde{U}(t) = \mathrm{Id} \otimes U(t)$. Then $\tilde{U}(t)$ is a c_0-group on $\tilde{X}$ such that $\|\tilde{U}(t)\| = \|U(t)\| \lesssim \langle n \rangle^\alpha$. Also

$$\gamma(\{\tilde{U}(t) : \, t \in [0,1]\}) = \gamma(\{U(t) : \, t \in [0,1]\}) < \infty,$$

since for $y_k = \sum_n \gamma_n \otimes x_{n,k} \in \tilde{X}$ and $t_k \in [0,1]$, we have

$$\begin{aligned}
&\| \sum_k \gamma_k \otimes \tilde{U}(t_k) \left(\sum_n \gamma_n \otimes x_{nk} \right) \|_{\mathrm{Gauss}(\tilde{X})} = \| \sum_{k,n} \gamma_k \otimes \gamma_n \otimes U(t_k) x_{n,k} \|_{\mathrm{Gauss}(\mathrm{Gauss}_N(X))} \\
&= (\mathbb{E}_\omega \| \sum_k \gamma_k U(t_k) \left(\sum_n \gamma_n(\omega) x_{n,k} \right) \|^2_{\mathrm{Gauss}_N(X)})^{1/2} \\
&\leqslant \gamma(\{U(t) :\ t \in [0,1]\}) (\mathbb{E}_\omega \| \sum_k \gamma_k \left(\sum_n \gamma_n(\omega) x_{n,k} \right) \|^2_{\mathrm{Gauss}_N(X)})^{1/2} \\
&= \gamma(\{U(t) :\ t \in [0,1]\}) \| \sum_k \gamma_k \otimes y_k \|_{\mathrm{Gauss}(\tilde{X})}.
\end{aligned}$$

Therefore, $\tilde{B}$ with $\tilde{U}(t) = e^{it\tilde{B}}$ has a bounded $\tilde{\mathcal{E}^\alpha}$ calculus, where $\tilde{\mathcal{E}^\alpha}$ is the operator valued space associated to $\tilde{B}$. We consider $F_1, \ldots, F_N \in \mathcal{G}$. For $t \in \mathbb{R}$, put

$$\tilde{F}(t)(\sum_n \gamma_n \otimes x_n) = \sum_n \gamma_n \otimes F_n(t) x_n.$$

Then $\tilde{F}(t) \in [\tilde{B}]' \subset B(\tilde{X})$. Also $\tilde{F} \in \tilde{\mathcal{E}^\alpha}$, since

$$\begin{aligned}
\gamma(\{\tilde{F} * \check{\phi}_m(t) :\ t \in \mathbb{R}\}) &= \sup \| \sum_k \gamma_k \otimes \tilde{F} * \check{\phi}_m(t_k) y_k \|_{\mathrm{Gauss}(\tilde{X})} \\
&= \sup \| \sum_{k,n} \gamma_k \otimes \gamma_n \otimes F_n * \check{\phi}_m(t_k) x_{n,k} \|_{\mathrm{Gauss}(\tilde{X})} \\
&\lesssim \gamma(\{F * \check{\phi}_m(t) :\ t \in \mathbb{R},\ F \in \mathcal{G}\}),
\end{aligned}$$

and the last quantity is in $\ell^1(\mathbb{Z}, \langle n \rangle^\alpha)$ by assumption. Here, the supremum runs over all finite sums in k, $t_k \in \mathbb{R}$, $\sum_k \gamma_k \otimes y_k = \sum_k \gamma_k \otimes \gamma_n \otimes x_{n,k}$ of norm in $\mathrm{Gauss}(\mathrm{Gauss}(X))$ less than 1, and the last estimate follows from property (α). Using lemma 5.13, it is easy to check that $\tilde{F}(\tilde{B})(\sum_n \gamma_n \otimes x_n) = \sum_n \gamma_n \otimes F_n(B) x_n$. Now

$$\gamma(\{F_1(B), \ldots, F_N(B)\}) = \|\tilde{F}(\tilde{B})\|_{B(\mathrm{Gauss}(X))} \lesssim \|\tilde{F}\|_{\tilde{\mathcal{E}^\alpha}} \leqslant \sum_n \langle n \rangle^\alpha \gamma(\{F * \check{\phi}_n(t) :\ t \in \mathbb{R},\ F \in \mathcal{G}\}).$$

Since $\gamma(\{F(B) :\ F \in \mathcal{G}\})$ equals the supremum of $\gamma(\{F_1(B), \ldots, F_N(B)\})$ for all choices of N and F_n, (5) follows. □

In [73] (see also [81, thm 12.8]), the following R-boundedness result is shown for the H^∞ calculus.

Theorem 5.17 *Let A be a sectorial operator on a Banach space with property (α). Assume that for some angle $\omega \in (0, \pi)$, A has an $H^\infty(\mathrm{Str}_\omega)$ calculus. Then for any $\theta \in (\omega, \pi)$, we have*

$$R(\{f(A) :\ \|f\|_{\infty,\theta} \leqslant 1\}) < \infty.$$

In view of theorem 5.16, we have the following partial extension to 0-sectorial operators with polynomially bounded imaginary powers.

Corollary 5.18 *Let X be a Banach space with property (α) and A a 0-sectorial operator on X having an H^∞ calculus, such that $\|A^{it}\| \lesssim \langle t\rangle^\alpha$. Then there is a constant $C > 0$ such that for any $f : (0,\infty) \to \mathbb{C}$ with $f \circ \exp \in E_\infty^\alpha$, we have*

$$R(\{f(tA) : t > 0\}) \leqslant C\|f \circ \exp\|_{E_\infty^\alpha}.$$

Proof. Note that since X has property (α), any family $\tau \subset B(X)$ is γ-bounded iff it is R-bounded. Clearly, $B = \log(A)$ and $U(t) = A^{it}$ satisfy the assumptions of theorem 5.16. For $t > 0$, let $f_t(s) = f(te^s)$. By the implication (1) ⇒ (4), it only remains to show that

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \sup_{t>0} \|f_t * \check{\phi}_n\|_\infty \lesssim \|f_1\|_{E_\infty^\alpha}. \tag{5.17}$$

But $f_t * \check{\phi}_n(s) = f_1 * \check{\phi}_n(s + \log(t))$, so that clearly, for any $t > 0$, $\|f_t * \check{\phi}_n\|_\infty = \|f_1 * \check{\phi}_n\|_\infty$, and thus, the left hand side of 5.17 in fact equals $\|f_1\|_{E_\infty^\alpha}$. □

5.5 E_∞^α norms of special functions

In this section, we calculate the E_∞^α norms for some special functions. They correspond to semigroup operators generated by $A = e^B$, resolvents of A and variants of these.

By corollary 5.18, we will deduce R-boundedness results for semigroup and resolvent operators under the condition that A has (norm) polynomially bounded imaginary powers and a bounded H^∞ calculus. Furthermore, by the lower estimates for the E_∞^α norms, we shall see in the following section that the obtained bounds are optimal in the class of such A.

5.5.1 Analytic semigroups

We start with the operators

$$(rA)^\beta \exp(-re^{i\theta}A),$$

where A is 0-sectorial, $a = re^{i\theta}$ such that $r > 0$ and $|\theta| < \frac{\pi}{2}$, and $\beta \geqslant 0$. We have $(rA)^\beta \exp(-re^{i\theta}A)$ $= f_a(B)$, where $B = \log(A)$, and

$$f_a(t) = (ae^t)^\beta \exp(-ae^t).$$

Proposition 5.19 $\|f_a\|_{E_\infty^\alpha} \cong (\frac{\pi}{2} - |\theta|)^{-(\alpha+\beta+\frac{1}{2})}$.

For the lower estimate, we will make use of the following lemma.

Lemma 5.20 *Let $g \in L^1(\mathbb{R})$.*

(1) *For any $a, b \in \mathbb{R}$, $\|\check{g}\|_\infty = \|(e^{i(a+b(\cdot))}g)\check{}\|_\infty$.*

(2) *Assume that there exists an interval I of length strictly less than π such that for any $t \in \mathbb{R}$ with $g(t) \neq 0$ we have*

$$\arg(g(t)) \in I.$$

Then $\|\check{g}\|_\infty \cong \|g\|_1$, where the equivalence constants only depend on the length $|I| < \pi$.

Proof. The first part follows from $(e^{i(a+b(\cdot))}g)\check{} = e^{ia}\check{g}(\cdot + b)$.

In the second part, in any case $\|\check{g}\|_\infty \leqslant \|g\|_1$. By the first part, we can assume that I is centered around 0, so that $\operatorname{Re} g(t) \gtrsim_{|I|} |g(t)|$. Then

$$\|\check{g}\|_\infty \gtrsim |\check{g}(0)| = |\int_{\mathbb{R}} g(t)dt| \geqslant \operatorname{Re}\int_{\mathbb{R}} g(t)dt \gtrsim_{|I|} \int_{\mathbb{R}} |g(t)|dt = \|g\|_1.$$

□

Proof of proposition 5.19. We determine the Fourier transform of f_a. If $\beta > 0$, $f_a \in L^1(\mathbb{R})$ and

$$\hat{f}_a(t) = a^\beta \int_{\mathbb{R}} e^{-its}e^{\beta s}\exp(-ae^s)ds = a^\beta \int_0^\infty e^{-ax}x^{\beta - it}\frac{dx}{x} = a^{it}\Gamma(\beta - it).$$

If $\beta = 0$, then $f_a \in L^\infty(\mathbb{R})$ and an approximation argument yields that $\hat{f}_a(t)$ is given by the principal value of $a^{it}\Gamma(-it)$:

We have

$$\exp(-ae^t)\hat{} = \lim_{\beta\to 0+}(\exp(-ae^t)a^\beta e^{\beta t})\hat{} = \lim_{\beta \to 0+} a^{it}\Gamma(\beta - it) = a^{it}\Gamma(-it), \tag{5.18}$$

where equality and limits are in the space of tempered distributions $\mathcal{S}'(\mathbb{R})$. Indeed, the first equality holds because $\exp(-ae^t) = \lim_{\beta\to 0+}\exp(-ae^t)a^\beta e^{\beta t}$ locally uniformly and the expression $\exp(-ae^t)a^\beta e^{\beta t}$ is uniformly bounded for $\beta \leqslant 1$.

On the other hand, $\Gamma(-z) = -\frac{1}{z} + G(z)$ where G is analytic for $|\operatorname{Re} z| < 1$. Let $\phi \in \mathcal{S}(\mathbb{R})$. If $0 \notin \operatorname{supp}\phi$,

$$\int_{\mathbb{R}} \phi(t)a^{it}\Gamma(-it)dt = \lim_{\beta\to 0+}\int_{\mathbb{R}} \phi(t)a^{it}\Gamma(\beta - it)dt$$

by dominated convergence. If, say, $\operatorname{supp}\phi \subset (-\frac{1}{2}, \frac{1}{2})$, then

$$\int_{\mathbb{R}} \phi(t)G(-it)dt = \lim_{\beta\to 0+}\int_{\mathbb{R}} \phi(t)G(\beta - it)dt$$

again by dominated convergence, and

$$PV - \int_{\mathbb{R}} \phi(t)\frac{1}{-it}dt = -\int_{\mathbb{R}} \phi'(t)i\log|t|dt = -\lim_{\beta\to 0+}\int_{\mathbb{R}} \phi'(t)i\log(t+i\beta)dt = \lim_{\beta\to 0+}\int \phi(t)\frac{1}{\beta - it}dt,$$

since $|\log(t + i\beta)| \leqslant |\log|t|| + \frac{\pi}{2}$ for $\beta, |t| < \frac{1}{2}$, and this function is integrable. This shows the third equality in 5.18.

In [90], the following development of the Γ function is given:

$$\Gamma(\beta - it) = \sqrt{2\pi}e^{it-\beta}\exp((\beta - \frac{1}{2} - it)\log|\beta - it|)(1 + O(|t|^{-1})) \quad (|t| \to \infty).$$

Thus, we have

$$\begin{aligned}\hat{f}_a(t) &= a^{it}\Gamma(\beta - it)\\ &= \sqrt{2\pi}e^{-\beta}e^{it}r^{it}e^{-\theta t}\exp((\beta - \frac{1}{2} - it)\log|\beta - it|)(1 + O(|t|^{-1}))\\ &= (\sqrt{2\pi}e^{-\beta})(r^{it}e^{it})e^{-\theta t}|\beta - it|^{\beta - \frac{1}{2}}e^{-it\log|\beta - it|}e^{i(\beta - \frac{1}{2})\arg(\beta - it)}e^{t\arg(\beta - it)}(1 + O(|t|^{-1})).\end{aligned}$$

We split $\hat{f}_a = \hat{f}_{(1)} + \hat{f}_{(2)}$ according to the above summands $1 + O(|t|^{-1})$. For the term $\hat{f}_{(2)}$ corresponding to $O(|t|^{-1})$, we estimate for $n \neq 0$

$$\|f_{(2)} * \check{\phi}_n\|_\infty \leqslant \|\hat{f}_{(2)}\phi_n\|_1 \lesssim e^{-\theta n}|n|^{\beta-\frac{1}{2}}e^{-\frac{\pi}{2}|n|}|n|^{-1} \lesssim |n|^{\beta-\frac{3}{2}}e^{(\theta-\frac{\pi}{2})|n|}.$$

For the term $\|f_{(1)} * \check{\phi}_n\|_\infty$, we estimate from above and below.

$$\|f_{(1)} * \check{\phi}_n\|_\infty \cong \|(|\beta - it|^{\beta-\frac{1}{2}}e^{-\theta t}e^{t\arg(\beta-it)}[e^{-it\log|\beta-it|}e^{i(\beta-\frac{1}{2})\arg(\beta-it)}]\phi_n(t))\check{}\|_\infty.$$

We want to apply lemma 5.20 (2) to the last term and thus investigate the argument of the term in square brackets. Put $g(t) = -t\log|\beta - it|$. We use Taylor expansion around $t_0 = n$:

$$e^{ig(t)} = e^{ig(n)}e^{ig'(n)(t-n)}e^{i(t-n)^2 g''(\xi)/2}$$

for $t \in [n-1, n+1]$, with $\xi \in [n-1, n+1]$. In view of the first part of the lemma, we can omit the first two factors in our considerations, and by the second part, also the third one for large n : by a calculation, one checks $(t-n)^2 g''(\xi) \to 0$ for any $t \in [n-1, n+1]$, as $|n| \to \infty$.

Since $(\beta - \frac{1}{2})\arg(\beta - it) \to -\operatorname{sgn}(t)\frac{\pi}{2}(\beta - \frac{1}{2})$ for $|t| \to \infty$, the argument of $e^{i(\beta-\frac{1}{2})\arg(\beta-it)}$ is convergent for $|t| \to \infty$.

Thus, noting that for $|t| \geqslant 1$, $|\beta - it|^{\beta-\frac{1}{2}} \cong |t|^{\beta-\frac{1}{2}}$ and $e^{t(\arg(\beta-it))} \cong e^{-\frac{\pi}{2}|t|}$, by lemma 5.20 (2), there is some $n_0 \in \mathbb{N}$ such that for $|n| \geqslant n_0$

$$\|f_{(1)} * \check{\phi}_n\|_\infty \cong \||t|^{\beta-\frac{1}{2}}e^{-\theta t}e^{-\frac{\pi}{2}|t|}\phi_n(t)\|_1 \cong |n|^{\beta-\frac{1}{2}}e^{(\operatorname{sgn}(n)\theta-\frac{\pi}{2})|n|}.$$

Hence, there exists $n_1 \in \mathbb{N}$ such that for all $|n| \geqslant n_1$, $\|f_a * \check{\phi}_n\|_\infty \cong \|f_{(1)} * \check{\phi}_n\|_\infty$ and consequently

$$\begin{aligned}
\|f_a\|_{E^\alpha_\infty} &\cong \|f_a\|_\infty + \sum_{|n|\geqslant n_1} \langle n\rangle^\alpha \|f_a * \check{\phi}_n\|_\infty \\
&\cong (\frac{|a|}{\operatorname{Re} a})^\beta + \sum_{n=n_1}^{\infty} n^{\alpha+\beta-\frac{1}{2}}e^{(|\theta|-\frac{\pi}{2})|n|} \\
&\cong (\frac{\pi}{2} - |\theta|)^{-\beta} + \int_{n_1}^{\infty} t^{\alpha+\beta+\frac{1}{2}}e^{(|\theta|-\frac{\pi}{2})t}\frac{dt}{t} \\
&\cong (\frac{\pi}{2} - |\theta|)^{-(\alpha+\beta+\frac{1}{2})}.
\end{aligned}$$

□

Combining the above proposition with corollary 5.18, we get

Corollary 5.21 *Let A be a 0-sectorial operator on a space X with property (α). Assume that A has a bounded H^∞ calculus and $\|A^{it}\| \lesssim \langle t\rangle^\alpha$ for some $\alpha \geqslant 0$. Then for any $\beta \geqslant 0$, there exists $C > 0$ such that*

$$R(\{(tA)^\beta \exp(-te^{i\theta}A) : t > 0\}) \leqslant C(\frac{\pi}{2} - |\theta|)^{-(\alpha+\beta+\frac{1}{2})}.$$

5.5.2 Resolvents

We consider the operators

$$\lambda^\beta A^\gamma(\lambda + A^\delta)^{-1},$$

where $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi$ and real β, γ, δ. To obtain bounded operators, natural restrictions are $\delta > 0$ and $\gamma \in [0, \delta]$. Further, for homogeneity reasons $\lambda \rightsquigarrow s\lambda$, $s > 0$, we assume

$$\beta + \frac{\gamma}{\delta} = 1.$$

We have $\lambda^\beta A^\gamma(\lambda + A^\delta)^{-1} = f(B)$ with

$$f(t) = f_{\beta,\gamma,\delta,\lambda}(t) = \lambda^\beta e^{\gamma t}(\lambda + e^{\delta t})^{-1}. \tag{5.19}$$

Proposition 5.22

(1) *Fix $\delta = 1$. Then for any $\gamma \in [0, 1]$, $\|f_{\beta,\gamma,\delta,\lambda}\|_{E^\alpha_\infty} \cong (\pi - |\theta|)^{-(\alpha+1)}$ with equivalence constants independent of γ.*

(2) *Fix $\beta = 1$ and $\gamma = 0$. Assume that $\delta \geqslant 1$. Then*

$$\|f_{\beta,\gamma,\delta,\lambda}\|_{E^\alpha_\infty} \cong \begin{cases} \delta^\alpha(\pi - |\theta|)^{-(\alpha+1)} & \text{for } \alpha > 0 \\ \log(\delta) + (\pi - |\theta|)^{-1}\left(|\log(\pi - |\theta|)| + 1\right) & \text{for } \alpha = 0 \end{cases}.$$

Proof. We first show that for the restrictions given at the beginning of the subsection, i.e. $\lambda = re^{i\theta}$ with $r > 0$ and $|\theta| < \pi$, $\delta > 0$, $\gamma \in [0, \delta]$, $\beta + \frac{\gamma}{\delta} = 1$, the Fourier transform of f from 5.19 is given by

$$\hat{f}(s) = \frac{1}{\delta}\lambda^{is/\delta}\frac{\pi}{\sin(\frac{\pi\gamma}{\delta} + i\frac{\pi s}{\delta})} \tag{5.20}$$

(principle value at 0 for $\frac{\gamma}{\delta} \in \{0, 1\}$).

Start with the case $\delta = 1$, $\gamma \in (0, 1)$ and $\beta = 1-\gamma$, so that $f(t) = g(e^t)$ with $g(x) = \lambda^\beta x^\gamma(\lambda+x)^{-1}$. Then with the short hand notation $z = \gamma + is$,

$$\hat{f}(s) = \int_0^\infty \lambda^\beta x^z(\lambda+x)^{-1}\frac{dx}{x} = \lambda^{\beta-1}\int_0^\infty x^z(1+\frac{x}{\lambda})^{-1}\frac{dx}{x} \overset{(*)}{=} \lambda^{z+\beta-1}\int_0^\infty x^z(1+x)^{-1}\frac{dx}{x} = \frac{\lambda^{is}\pi}{\sin(\pi z)}.$$

In $(*)$, we have used Cauchy's integral theorem with the contour shift $x \rightsquigarrow x\lambda$ and for the last equality, we refer e.g. to [90].

With an approximation argument as in the proof of proposition 5.19, one can show that this extends for $\gamma \in \{0, 1\}$ to

$$\hat{f}(s) = PV - \lambda^{is}\frac{\pi}{\sin(\pi(\gamma + is))}.$$

The general case $\delta > 0$, $\gamma \in [0, \delta]$ follows then easily by a substitution.

(1): We want to estimate $\|[\hat{f}\phi_n]\check{\ }\|_\infty$ for an equidistant Fourier partition of unity $(\phi_n)_{n\in\mathbb{Z}}$. According to 5.20, we have with $\lambda = re^{i\theta}$,

$$\hat{f}(s) = r^{is}e^{-\theta s}\frac{\pi}{\sin(\pi\gamma + i\pi s)}.$$

Let us determine the modulus and the argument of the last factor for $|s| \geqslant 1$.

$$|\sin(\pi\gamma + i\pi s)| = |\sin(\pi\gamma)\cosh(\pi s) + i\cos(\pi\gamma)\sinh(\pi s)| \cong e^{\pi|s|}$$

If $\gamma = 0$ or $\gamma = 1$, then $\sin(\pi\gamma + i\pi s)$ is purely imaginary and for fixed sign of s, the argument of $\sin(\pi\gamma + i\pi s)$ is constant.

If $\gamma = \frac{1}{2}$, then $\sin(\pi\gamma + i\pi s)$ is real and for fixed sign of s, its argument is again constant.

If $\gamma \notin \{0, \frac{1}{2}, 1\}$, then $\operatorname{Re}\sin(\pi\gamma + i\pi s) = \sin(\pi\gamma)\cosh(\pi s) > 0$ and the sign of $\operatorname{Im}\sin(\pi\gamma + i\pi s) = \cos(\pi\gamma)\sinh(\pi s)$ only depends on the sign of s.

Therefore, for any $\gamma \in [0, 1]$ and $n \in \mathbb{Z}\backslash\{0, -1, 1\}$, $\hat{f}\phi_n$ satisfies the assumptions of lemma 5.20 (2) with $|I| = \frac{\pi}{2}$, and

$$\|[\hat{f}\phi_n]\check{}\|_\infty \cong \|\hat{f}\phi_n\|_1 \cong \|e^{-\theta s}e^{-\pi|s|}\phi_n(s)\|_1 \cong e^{-\theta n - \pi|n|}.$$

Further

$$\|f\|_\infty = \sup_{t>0} |\lambda^{1-\gamma}t^\gamma(\lambda + t)^{-1}| = \sup_{t>0} |(t/\lambda)^\gamma(1 + t/\lambda)^{-1}| = \sup_{t>0} |t^\gamma(1 + e^{-i\theta}t)^{-1}| \cong (\pi - |\theta|)^{-1}.$$

Thus,

$$\|f\|_{E^\alpha_\infty} \cong \|f\|_\infty + \sum_{|n|\geqslant 2} \langle n\rangle^\alpha \|[\hat{f}\phi_n]\check{}\|_\infty \cong (\pi - |\theta|)^{-1} + \sum_{|n|\geqslant 2} \langle n\rangle^\alpha e^{-\theta n - \pi|n|} \cong (\pi - |\theta|)^{-(\alpha+1)}.$$

(2): We have

$$\hat{f}(s) = \frac{1}{\delta}\pi r^{is/\delta}e^{-\theta s/\delta}\frac{\pi}{i\sinh(\frac{\pi s}{\delta})}.$$

Note that

$$\left[\sinh(\frac{\pi s}{\delta})\right]^{-1} \cong \begin{cases} \frac{\delta}{\pi s} & \text{for } \left|\frac{\pi s}{\delta}\right| \leqslant 1 \\ e^{-\pi|s|/\delta} & \text{for } \left|\frac{\pi s}{\delta}\right| > 1 \end{cases}.$$

Then for $|n| \geqslant 2$, by lemma 5.20,

$$\|f * \check{\phi}_n\|_\infty \cong \|\hat{f}\phi_n\|_1 \cong e^{-\theta n/\delta}\frac{1}{\delta}\begin{cases} \frac{\delta}{\pi|n|} & \text{for } |n| \leqslant \frac{\delta}{\pi} \\ e^{-\pi|n|/\delta} & \text{for } |n| > \frac{\delta}{\pi} \end{cases}.$$

Further, it is easy to see that $\|f\|_\infty \cong (\pi - |\theta|)^{-1}$. Thus, for $\delta > 2\pi$,

$$\begin{aligned}
\|f\|_{E^\alpha_\infty} &\cong \|f\|_\infty + \sum_{|n|\geqslant 2} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_\infty \\
&\cong (\pi - |\theta|)^{-1} + \sum_{|n|\geqslant 2, |n|\leqslant \frac{\delta}{\pi}} \langle n\rangle^\alpha e^{-\theta n/\delta} \frac{1}{\delta} \frac{\delta}{\pi |n|} + \sum_{|n|\geqslant 2, |n| > \frac{\delta}{\pi}} \langle n\rangle^\alpha e^{-\theta n/\delta} \frac{1}{\delta} e^{-\pi|n|/\delta} \\
&\cong (\pi - |\theta|)^{-1} + \sum_{2\leqslant n\leqslant \frac{\delta}{\pi}} n^\alpha e^{|\theta| n/\delta} \frac{1}{n} + \sum_{n\geqslant \max(2, \frac{\delta}{\pi})} n^\alpha e^{|\theta| n/\delta - \pi n/\delta} \frac{1}{\delta} \\
&\cong (\pi - |\theta|)^{-1} + \int_2^{\frac{\delta}{\pi}} x^\alpha e^{|\theta| x/\delta} \frac{dx}{x} + \int_{\max(2, \frac{\delta}{\pi})}^\infty x^\alpha e^{-(\pi - |\theta|)x/\delta} \frac{dx}{\delta} \\
&\cong (\pi - |\theta|)^{-1} + \delta^\alpha \int_{\frac{2}{\delta}}^{\frac{1}{\pi}} x^\alpha e^{|\theta| x} \frac{dx}{x} + \delta^\alpha (\pi - |\theta|)^{-(\alpha+1)} \int_{(\pi - |\theta|)\max(\frac{1}{\pi}, \frac{2}{\delta})}^\infty x^\alpha e^{-x} \frac{dx}{x} \\
&\cong (\pi - |\theta|)^{-1} + \delta^\alpha \cdot \left\{ \begin{array}{ll} 1 & \text{for } \alpha > 0 \\ \log\delta & \text{for } \alpha = 0 \end{array} \right\} + \delta^\alpha (\pi - |\theta|)^{-(\alpha+1)} \cdot \left\{ \begin{array}{ll} 1 & \text{for } \alpha > 0 \\ |\log(\pi - |\theta|)| + 1 & \text{for } \alpha = 0 \end{array} \right\} \\
&\cong \begin{cases} \delta^\alpha (\pi - |\theta|)^{-(\alpha+1)} & \text{for } \alpha > 0 \\ \log(\delta) + \big((\pi - |\theta|)^{-1} |\log(\pi - |\theta|)| + 1\big) & \text{for } \alpha = 0 \end{cases}.
\end{aligned}$$

If $\delta \leqslant 2\pi$, then the above sum $\sum_{|n|\geqslant 2,\, |n|\leqslant \frac{\delta}{\pi}}$ and its subsequent modifications vanish, and we get the same final estimate as before. □

The immediate consequence of corollary 5.18 and proposition 5.22 for 0-sectorial operators is the following.

Corollary 5.23 *Let A be a 0-sectorial operator on a space X with property (α). Assume that A has a bounded H^∞ calculus and $\|A^{it}\| \lesssim \langle t\rangle^\alpha$ for some $\alpha \geqslant 0$.*

(1) *There exists $C > 0$ such that for any $\gamma \in [0, 1]$ and $\theta \in [0, \pi)$, we have*

$$R(\{\lambda^{1-\gamma} A^\gamma (\lambda + A)^{-1} : |\arg\lambda| = \theta\}) \leqslant C(\pi - \theta)^{-(\alpha+1)}.$$

(2) *There exists $C > 0$ such that for any $\delta \geqslant 1$ and $\theta \in [0, \pi)$, we have*

$$R(\{\lambda(\lambda + A^\delta)^{-1} : |\arg\lambda| = \theta\}) \leqslant C \begin{cases} \delta^\alpha (\pi - \theta)^{-(\alpha+1)} & \text{for } \alpha > 0 \\ \log(\delta) + (\pi - \theta)^{-1}(|\log(\pi - \theta)| + 1) & \text{for } \alpha = 0. \end{cases}$$

5.6 Applications to the Mihlin calculus and extremal examples

Let A be a 0-sectorial operator on a space X and B a 0-strip-type operator such that $A^{it} = e^{itB}$. Up to now we know that the polynomial growth of e^{itB} is equivalent to the E^α_∞ calculus of B (theorem 5.10). Proposition 5.3 shows that the norm $\|f\|_{E^\alpha_\infty}$ is related to summability of Fourier coefficients of f. In contrast, the norms $\|f\|_{\mathcal{B}^\alpha}$ and $\|f\|_{\mathcal{M}^\alpha}$ of the Besov and Mihlin spaces are essentially given by the uniform boundedness of derivatives of f (see 4.10). This is often easier to determine.

Therefore, it is desirable to investigate the connection between the Besov (Mihlin) calculus and polynomially bounded c_0-groups (imaginary powers of 0-sectorial operators).

By the embedding of Besov spaces into E^α_∞ in proposition 5.5 (1), a 0-sectorial operator A with $\gamma(\{A^{it} : t \in [0,1]\}) < \infty$ and $\|A^{it}\| \lesssim \langle t \rangle^\alpha$ has a $\mathcal{M}^{\alpha+1}$ calculus. Conversely, if A has a $\mathcal{M}^\alpha$ calculus, then $\|A^{it}\| \lesssim \langle t \rangle^\alpha$.

We will see in proposition 5.28 that the gap $\alpha \rightsquigarrow \alpha + 1$ between the imaginary powers and the Mihlin calculus cannot be narrowed if the underlying space X is arbitrary. However, theorem 4.74 shows that conditions on the type and cotype of X can improve on the gap:

If X has property (α), the (possibly trivial) type p and cotype q, then any A with bounded H^∞ calculus satisfying $\|A^{it}\| \lesssim \langle t \rangle^\alpha$ has a bounded $\mathcal{M}^\beta$ calculus for any $\beta > \alpha + \max(\frac{1}{p} - \frac{1}{q}, \frac{1}{2})$.

If X is a Hilbert space, then this can also be proved using the Paley-Littlewood decomposition from theorem 4.53 and the local part of the Besov embedding proposition 5.5 (3).

In this section, we will show that the above gap between α and β is optimal, if $\frac{1}{p} - \frac{1}{q} \geqslant \frac{1}{2}$. The examples are multiplication operators $B_X g(t) = tg(t)$ on $X = E^\alpha_q$ and $X = E^\alpha_0$.

In the latter case, we will also see that the bounds for semigroup and resolvent operators from corollaries 5.21 and 5.23 are optimal within the class of polynomially bounded imaginary powers.

Let us start with some observations on the multiplication operator B_p and geometric properties on its underlying space.

Proposition 5.24 *Let* $1 \leqslant p \leqslant \infty$ *and* $\alpha \geqslant 0$. *If* $p < \infty$, *we let* $X = E^\alpha_p$, *and if* $p = \infty$, *we let* $X = E^\alpha_0 = E^\alpha_\infty \cap C_0(\mathbb{R})$. *Consider the group* $(U_p(t))_{t\in\mathbb{R}}$ *defined by*

$$U_p(t) : X \to X,\ g \mapsto e^{it(\cdot)} g.$$

Then $(U_p(t))_{t\in\mathbb{R}}$ *is a* c_0*-group with* $\|U_p(t)\| \cong \langle t \rangle^\alpha$. *The associated* 0*-strip-type operator* B_p *has a* E^α_∞ *calculus which is given by*

$$f(B_p)g = fg \quad (f \in E^\alpha_\infty,\ g \in X).$$

Proof. Since $e^{it(\cdot)} \in E^\alpha_\infty$ with $\|e^{it(\cdot)}\|_{E^\alpha_\infty} \cong \langle t \rangle^\alpha$ according to the proof of theorem 5.10, proposition 5.2 yields $\|U_p(t)\|_{B(X)} \lesssim \langle t \rangle^\alpha$. This estimate is also optimal. Indeed, we have $(e^{it(\cdot)} g) * \check{\phi}_n(s) = e^{-its}[g * \phi_n(\cdot - t)\check{\ }](s)$. Thus if $\hat{g}$ has its support in $[-1,1]$, then $\|U_p(t)g\|_X \cong \langle t \rangle^\alpha \|g\|_X$. It is clear that $t \mapsto U_p(t)$ is a group. It is further strongly continuous. Indeed, for any $g \in X$ and $n \in \mathbb{Z}$, $\|(e^{it(\cdot)}g - g) * \check{\phi}_n\|_p \leqslant \|\check{\phi}_n\|_1 \|(e^{it(\cdot)} - 1)g\|_p \to 0$ as $t \to 0$ by dominated convergence for $p < \infty$, and by the fact that $\lim_{|t|\to\infty} g(t) = 0$ for the case $X = E^\alpha_0$. Now the strong continuity follows from

$$\sum_{n\in\mathbb{Z}} \langle n \rangle^\alpha \sup_{|t|\leqslant 1} \|(U_p(t)g) * \check{\phi}_n\|_p = \sum_{n\in\mathbb{Z}} \langle n \rangle^\alpha \sup_{|t|\leqslant 1} \|g * \phi_n(\cdot + t)\check{\ }\|_p < \infty.$$

Denote iB_p the generator of $U_p(t)$. As mentioned in chapter 2 section 2.2, B_p is a 0-strip-type operator. For $f \in H_0^\infty(\mathrm{Str}_\omega)$ for some $\omega > 0$ and $g \in X$, we have

$$\begin{aligned} f(B)g &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)U_p(t)g dt \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(t)e^{it(\cdot)}g dt \\ &= fg. \end{aligned}$$

In particular, $\|f(B)\| \cong \|f\|_{E^\alpha_\infty} \lesssim \|f\|_{\infty,\omega}$, so that B has an H^∞ calculus. Theorem 5.10 implies that B has a E^α_∞ calculus. Then by an approximation argument, using proposition 5.8, $f(B)g = fg$ for any $f \in E^\alpha_\infty$ and $g \in X$. □

Proposition 5.25 *For $2 \leqslant p < \infty$, the space E^α_p has no better type than 1, and has cotype p and property (α).*

Proof. Proposition 5.3 yields an isomorphic embedding $T : E^\alpha_p \hookrightarrow \ell^1(\langle n\rangle^\alpha, \ell^p)$. It is well-known (see e.g. [109, 30]) that the latter space has cotype p and property (α) and that both properties are inherited by subspaces and isomorphic spaces. Finally, E^α_p has no better type than 1 since it contains an isomorphic copy of ℓ^1. Indeed, for $n \in 2\mathbb{Z}$, let $f_n \in E^\alpha_p$ such that $f_n * \check{\phi}_m = 0$ for $m \in 2\mathbb{Z}$, $m \neq n$, and further $\|f_n * \check{\phi}_n\|_p = \langle n\rangle^{-\alpha}$, and $\|f_n\|_{E^\alpha_p} \lesssim 1$. Then $\Psi : \ell^1(2\mathbb{Z}) \hookrightarrow E^\alpha_p$, $e_n \mapsto f_n$ is an isomorphic embedding: $\|\Psi(\sum_n \alpha_n e_n)\|_{E^\alpha_p} \leqslant \sum_n |\alpha_n|\,\|f_n\|_{E^\alpha_p} \lesssim \|\sum_n \alpha_n e_n\|_1$ and

$$\|\Psi(\sum_n \alpha_n e_n)\|_{E^\alpha_p} \geqslant \sum_{m\in 2\mathbb{Z}} \langle m\rangle^\alpha \|\Psi(\sum_n \alpha_n e_n) * \check{\phi}_m\|_p = \sum_{m \in 2\mathbb{Z}} \langle m\rangle^\alpha \|\alpha_m f_m * \check{\phi}_m\|_p = \sum_m |\alpha_m|.$$

□

Theorem 5.26 *Let X be a Banach space with property (α). Let further B be a 0-strip-type operator on X having an H^∞ calculus such that $\|e^{itB}\| \lesssim \langle t\rangle^\alpha$ for some $\alpha \geqslant 0$.*

Then B has a $\mathcal{B}^\beta$ calculus for

$$\beta > \alpha + \max\Big(\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}, \frac{1}{2}\Big).$$

On the other hand, for $2 \leqslant p < \infty$, the multiplication operator B_p on $X = E^\alpha_p$ from proposition 5.24 does not have a $\mathcal{B}^\beta$ calculus for any

$$\beta < \alpha + \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}.$$

Proof. The positive result for $\beta > \alpha + \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$ is proved in theorem 4.74. Note that even the stronger $\mathcal{W}^\beta_r$ calculus is proved for $r \in [1, 2]$ such that $\frac{1}{r} > \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$, and that moreover, this calculus is R-bounded.

It remains to show the negative result for the multiplication operator B_p and

$$\beta < \alpha + \frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X} = \alpha + 1 - \frac{1}{p}.$$

By proposition 5.24, it suffices to show that

$$\|fg\|_{E_p^\alpha} \not\lesssim \|f\|_{\mathcal{B}^\beta}\|g\|_{E_p^\alpha}.$$

Let

$$f(t) = \sum_{n=-N}^{N} \langle n\rangle^{-\alpha-\frac{1}{p'}+\varepsilon_1} e^{int}\phi(t-Kn)$$

and

$$g(t) = \sum_{n=-N}^{N}\sum_{m=-N}^{N} \langle n\rangle^{-\alpha-1-\varepsilon_2}\langle m\rangle^{-\frac{1}{p}-\varepsilon_3} e^{int}\phi(t-Km),$$

where $N, K \in \mathbb{N}$, $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ and ϕ is a function such that $\hat{\phi} \in C_0^\infty(\mathbb{R})\backslash\{0\}$ and $(\hat{\phi}*\hat{\phi})\phi_0 = \hat{\phi}*\hat{\phi}$. Here we choose $(\phi_n)_n$, the partition of unity in the definition of E_p^α, such that $\phi_0(t) = 1$ in a neighborhood of 0. Now the outline of the proof is as follows.

(1) $\sup_{K,N}\|f\|_{\mathcal{B}^\beta} < \infty$ for $\varepsilon_1 < \alpha + 1 - \frac{1}{p} - \beta$ fixed.

(2) $\sup_{K,N}\|g\|_{E_p^\alpha} < \infty$ for $\varepsilon_2, \varepsilon_3 > 0$ fixed.

(3) $\|fg\|_{E_p^\alpha} \to \infty$ for $K = K(N)$ an appropriate sequence, $N \to \infty$ and $\varepsilon_3 < \varepsilon_1$ fixed.

(1): Due to the almost equivalence of the Hölder norm and the Besov norm [128]

$$\|f\|_{C^{\beta_1}} \lesssim \|f\|_{\mathcal{B}^{\beta_2}} \lesssim \|f\|_{C^{\beta_3}},$$

where $\beta_1 < \beta_2 < \beta_3$, it suffices to estimate the β-Hölder norm of f. For integer $l \leqslant \beta < \alpha + \frac{1}{p'}$, we have

$$\begin{aligned} |f^{(l)}(t)| &= |\sum_{n=-N}^{N} \langle n\rangle^{-\alpha-\frac{1}{p'}+\varepsilon_1} \sum_{k=0}^{l}\binom{l}{k} e^{int}(in)^{l-k}\phi^{(k)}(t-Kn)| \\ &\lesssim \sum_{k=0}^{l}\sum_{n=-N}^{N}\langle n\rangle^{-\alpha-\frac{1}{p'}+\varepsilon_1+l}|\phi^{(k)}(t-Kn)|, \end{aligned} \tag{5.21}$$

which can be uniformly estimated for $\varepsilon_1 < \alpha + \frac{1}{p'} - \beta$, because $\sup_{t,K,N}\sum_{-N}^{N}|\phi^{(k)}(t-Kn)| < \infty$. If β is non-integer, we need to estimate $\sup_{t\in\mathbb{R}} h_\gamma(f^{(l)})(t)$ to show (1), where $l \in \mathbb{N}_0$, $\gamma = \beta - l \in (0,1)$ and

$$h_\gamma(f^{(l)})(t) = \sup_{|h|\in(0,1)} \frac{|f^{(l)}(t+h) - f^{(l)}(t)|}{|h|^\gamma}.$$

In view of the development of $f^{(l)}$ in 5.21, it suffices to estimate $\sup_{t\in\mathbb{R}} h_\gamma(f_0)(t)$ instead of $\sup_{t\in\mathbb{R}} h_\gamma(f^{(l)})(t)$, where

$$f_0(t) = \sum_{n=-N}^{N}\langle n\rangle^{-\alpha-\frac{1}{p'}+\varepsilon_1} n^{l-k} e^{int}\phi^{(k)}(t-Kn)$$

for $k = 0, \ldots, l$. Using the "product rule"

$$h_\gamma(f_1 f_2)(t) \leqslant h_\gamma(f_1)(t)\sup_{|h|\leqslant 1}|f_2(t+h)| + |f_1(t)|h_\gamma(f_2)(t)$$

and $h_\gamma(f_2)(t) \leqslant \sup_{|h|\leqslant 1} |f_2'(t+h)|$ for $f_1(t) = e^{int}$ and $f_2(t) = \phi^{(k)}(t - Kn)$, we get

$$h_\gamma(f_0)(t) \leqslant \sum_{n=-N}^{N} \langle n\rangle^{-\alpha-\frac{1}{p'}+\varepsilon_1} |n|^{l-k} \left(h_\gamma(e^{in(\cdot)})(t) \sup_{|h|\leqslant 1} |f_2(t+h)| + \sup_{|h|\leqslant 1} |f_2'(t+h)| \right).$$

Since for any $k \in \mathbb{N}_0$, $\sup_{t,K,N} \sum_{n=-N}^{N} \sup_{|h|\leqslant 1} |\phi^{(k)}(t+h-Kn)| < \infty$ and $\langle n\rangle^{-\alpha-\frac{1}{p'}+\varepsilon_1}|n|^{l-k} \leqslant \langle n\rangle^{-\gamma}$, we are reduced to show that $\sup_t h_\gamma(e^{in(\cdot)})(t) \lesssim |n|^\gamma$. This in turn can easily be verified.

(2): Proceeding as in the proof of proposition 5.3, one shows that

$$\|g\|_{E_p^\alpha} \lesssim \|(\langle n\rangle^{-\alpha-1-\varepsilon_2} \langle m\rangle^{-\frac{1}{p}-\varepsilon_3})_{nm}\|_{\ell^1(\langle n\rangle^\alpha,\ell^p)},$$

where the estimate does not depend on K. But

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \langle n\rangle^{-\alpha-1-\varepsilon_2} \left(\sum_{m\in\mathbb{Z}} \langle m\rangle^{-1-\varepsilon_3 p} \right)^{1/p} < \infty.$$

(3): We have

$$\begin{aligned} fg(t) &= \sum_{n,m,k} \chi_M(m)\chi_N(k)\chi_N(n-k)\langle k\rangle^{-\alpha-\frac{1}{p'}+\varepsilon_1} \langle n-k\rangle^{-\alpha-1-\varepsilon_2} \langle m\rangle^{-\frac{1}{p}-\varepsilon_3} e^{int}\phi(t-Kk)\phi(t-Km) \\ &= \sum_{n,m,k:\, m=k} + \sum_{n,m,k:\, m\neq k} \\ &=: h_1 + h_2. \end{aligned}$$

with $\chi_N(l) = 1$ for $|l| \leqslant N$ and 0 else. We will show that

$$\text{for } \varepsilon_1 > \varepsilon_3 \text{ and } K_0 \text{ sufficiently large, } \inf_{K\geqslant K_0} \|h_1\|_{E_p^\alpha} \to \infty \text{ as } N \to \infty, \tag{5.22}$$

whereas for a convenient sequence $K = K(N)$,

$$\sup_{K=K(N),N\in\mathbb{N}} \|h_2\|_{E_p^\alpha} < \infty. \tag{5.23}$$

Clearly, 5.22 and 5.23 give (3). The support condition on $\hat{\phi} * \hat{\phi} = 2\pi(\phi^2)\hat{}$ implies that for any $n \in \mathbb{Z}$, $\phi_n \cdot (e^{in_0(\cdot)}\phi^2)\hat{} = \delta_{nn_0}(e^{in_0(\cdot)}\phi^2)\hat{}$ and hence, $h_1 * \check{\phi}_n$ equals the summand n of h_1 :

$$h_1 * \check{\phi}_n(t) = e^{int} \sum_m a_m \phi^2(t-Km)$$

where we have put $a_m = \chi_N(m)\chi_N(n-m)\langle m\rangle^{-\alpha-1+\varepsilon_1-\varepsilon_3}\langle n-m\rangle^{-\alpha-1-\varepsilon_2}$. We claim that

$$\|h_1 * \check{\phi}_n\|_p \gtrsim \|(a_m)_m\|_p \tag{5.24}$$

uniformly for $K \geqslant K_0$ and proceed by interpolation between $p = 1$ and $p = \infty$. For $p = 1$,

$$\|h_1 * \check{\phi}_n\|_1 \geqslant \int_{\mathbb{R}} |a_{m_t}\phi^2(t-Km_t)|dt - \int_{\mathbb{R}} \sum_{m\neq m_t} |a_m\phi^2(t-Km)|dt,$$

where m_t is the unique integer such that $t \in [K(m_t - \frac{1}{2}), K(m_t + \frac{1}{2}))$. The first integral already dominates $\|(a_m)_m\|_1$:

$$\int_{\mathbb{R}} |a_{m_t}\phi^2(t - Km_t)|dt = \sum_{m\in\mathbb{Z}} \int_{K(m-\frac{1}{2})}^{K(m+\frac{1}{2})} |a_m\phi^2(t - Km)|dt \gtrsim \sum_{m\in\mathbb{Z}} |a_m|$$

uniformly in K, since $0 \neq \int_{K(m-\frac{1}{2})}^{K(m+\frac{1}{2})} |\phi^2(t - Km)|dt = \int_{-K/2}^{K/2} |\phi^2(t)|dt \to \|\phi^2\|_1 \neq 0$ for $K \to \infty$. The second integral gets small compared to $\|(a_m)_m\|_1$ for K sufficiently large:

$$\begin{aligned}\int_{\mathbb{R}} \sum_{m\neq m_t} |a_m\phi^2(t - Km)|dt &= \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}\setminus[K(m-\frac{1}{2}),K(m+\frac{1}{2})]} |a_m\phi^2(t - Km)|dt \\ &= \sum_{m\in\mathbb{Z}} |a_m| \int_{\mathbb{R}\setminus[-K/2,K/2]} |\phi^2(t)|dt.\end{aligned}$$

For $p = \infty$, we argue similarly:

$$\begin{aligned}\|h_1 * \check{\phi}_n\|_\infty &\geqslant \sup_{t\in\mathbb{R}} |a_{m_t}\phi^2(t - Km_t)| - \sup_{t\in\mathbb{R}} \sum_{m\neq m_t} |a_m\phi^2(t - Km)| \\ &\geqslant \|(a_m)_m\|_\infty \sup_{|t|\leqslant K/2} |\phi^2(t)| - \|(a_m)_m\|_\infty \sup_{t\in\mathbb{R}} \sum_{m\neq m_t} |\phi^2(t - Km)|.\end{aligned}$$

Note that $\sum_{m\neq m_t} |\phi^2(t - Km)| \leqslant \sup_{|t|\leqslant K/2} \sum_{m\neq 0} |\phi^2(t - Km)| \to 0$ for $K \to \infty$, since ϕ^2 is rapidly decreasing. Thus, 5.24 is shown. Therefore,

$$\begin{aligned}\inf_{K\geqslant K_0} \|h_1\|_{E_p^\alpha} &= \inf_{K\geqslant K_0} \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|h_1 * \check{\phi}_n\|_p \\ &\gtrsim \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \|(a_m)_m\|_p \\ &= \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \left(\sum_{m\in\mathbb{Z}} \chi_N(m)\chi_N(n - m)\langle m\rangle^{(-\alpha-1+\varepsilon_1-\varepsilon_3)p}\langle n - m\rangle^{(-\alpha-1-\varepsilon_2)p} \right)^{1/p} \\ &\geqslant \sum_{n\in\mathbb{Z}} \langle n\rangle^\alpha \chi_N(n)\langle n\rangle^{-\alpha-1+\varepsilon_1-\varepsilon_3} \\ &\to \infty \quad (N \to \infty),\end{aligned}$$

since $\varepsilon_1 > \varepsilon_3$, so that 5.22 is shown.

As a first step towards 5.23, note that for any $j, l \in \mathbb{N}_0$

$$\sup_{m\neq k} \|\phi^{(l)}(t - Km)\phi^{(j)}(t - Kk)\|_p \to 0 \text{ as } K \to \infty. \tag{5.25}$$

Indeed, split the p norm into the cases $|t - Km| \leqslant K/2$ and $|t - Km| > K/2$. Since $m \neq k$, $|t - Km| \leqslant K/2$ implies $|t - Kk| \geqslant K/2$ and

$$\begin{aligned}\|\phi^{(l)}(t - Km)\phi^{(j)}(t - Kk)\|_p^p &\leqslant \sup_{|t-Kk|\geqslant K/2} |\phi^{(j)}(t - Kk)|^p \|\phi^{(l)}(t - Km)\|_p^p \\ &\quad + \sup_{|t-Km|>K/2} |\phi^{(l)}(t - Km)|^p \|\phi^{(j)}(t - Kk)\|_p^p \\ &\to 0 \quad (K \to \infty)\end{aligned}$$

as the derivatives of ϕ are rapidly decreasing. The sums over m, k, n in the definition of h_2 are all finite, whence 5.25 implies that for any $l \in \mathbb{N}_0$, $\|h_2^{(l)}\|_p \to 0$ for $K \to \infty$ and N fixed. Choosing now a convenient sequence $K = K(N)$, we get $\sup_{N\in\mathbb{N}} \|h_2^{(l)}\|_p < \infty$. As in proposition 5.5, one can show that for $l \geqslant \alpha + 1$,

$$\|h_2\|_{E_p^\alpha} \lesssim \|h_2^{(0)}\|_p + \ldots + \|h_2^{(l)}\|_p,$$

so that 5.23 follows. □

Remark 5.27 (1) *The second part of the theorem could also be stated as follows: The pointwise multiplication $E_\infty^\alpha \cdot E_p^\alpha$ maps to E_p^α, whereas*

$$\mathcal{B}^\beta \cdot E_p^\alpha \to E_p^\alpha \text{ only if } \beta \geqslant \alpha + 1 - \frac{1}{p}.$$

(2) *If the difference $\frac{1}{\operatorname{type} X} - \frac{1}{\operatorname{cotype} X}$ is less than $\frac{1}{2}$, then it is not clear what the optimal order β_0 for the Besov calculus in theorem 5.26 is. Theorem 5.26 only yields the estimate $\beta_0 - \alpha \in [0, \frac{1}{2}]$.*

For example, if iB is the generator of a uniformly bounded c_0-group on a Hilbert space, then by the transference principle of Coifman and Weiss, B has a bounded $H^\infty(\mathrm{Str}_\omega)$ calculus for any $\omega > 0$ and the norm of this calculus is independent of ω. Thus, by proposition 4.18, $\beta_0 - \alpha = 0$ in this case.

We now turn to the multiplication operator B_∞ on E_0^α. This is an extremal example in the following sense.

Proposition 5.28 *Let $\alpha \geqslant 0$. Let B_∞ be the multiplication operator on $X = E_0^\alpha$ as in proposition 5.24. Then*

$$\|f(B_\infty)\| \cong \|f\|_{E_\infty^\alpha} \quad (f \in E_\infty^\alpha).$$

In particular, if C is a further 0-strip-type operator on some Banach space Y also having a bounded E_∞^α calculus, then

$$\|f(C)\|_{B(Y)} \lesssim \|f(B_\infty)\|_{B(X)} \quad (f \in E_\infty^\alpha).$$

Further, B_∞ has a $\mathcal{B}^\beta$ calculus if and only if

$$\beta \geqslant \alpha + 1.$$

Proof. In view of proposition 5.5, all we have to show is

$$\|f\|_{E_\infty^\alpha} \cong \sup\{\|fg\|_{E_\infty^\alpha} : g \in E_0^\alpha, \|g\|_{E_0^\alpha} \leqslant 1\}.$$

The inequality "$\gtrsim$" follows from the fact that E_∞^α is a Banach algebra. For the other estimate, consider a sequence χ_k in E_0^α with the properties $\chi_k(t) \to 1$ for any $t \in \mathbb{R}$ and $\sup_{k\in\mathbb{N}} \|\chi_k\|_{E_\infty^\alpha} < \infty$ (e.g. $\chi_k(t) = \chi(\frac{t}{k})$ for some $\chi \in C_0^\infty(\mathbb{R})$ such that $\chi(0) = 1$). Then for any $n \in \mathbb{Z}$ and $t \in \mathbb{R}$, by dominated convergence $(f\chi_k)*\check{\phi}_n(t) \to f*\check{\phi}_n(t)$, so that $\limsup_k \|(f\chi_k)*\check{\phi}_n\|_\infty \geqslant \|f*\check{\phi}_n\|_\infty$. Thus,

$$\sup_k \|f\chi_k\|_{E_\infty^\alpha} = \sup_{k,N} \sum_{|n|\leqslant N} \langle n\rangle^\alpha \|(f\chi_k) * \check{\phi}_n\|_\infty \geqslant \sup_N \sum_{|n|\leqslant N} \langle n\rangle^\alpha \|f * \check{\phi}_n\|_\infty = \|f\|_{E_\infty^\alpha}.$$

□

Finally, we resume the results on semigroup and resolvent type operators.

Corollary 5.29 *Let A be a 0-sectorial operator on some Banach space X such that $\gamma(\{A^{it} : t \in [0,1]\}) < \infty$ and $\|A^{it}\| \lesssim \langle t \rangle^\alpha$ for some $\alpha \geqslant 0$.*

Then for $\beta \geqslant 0,\ \delta > 0,\ \gamma \in [0,\delta], \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ \omega \in (-\pi, \pi)$:

(1) $\sup_{r>0} \|r^\beta A^\beta \exp(-e^{i\theta} r A)\| \lesssim (\frac{\pi}{2} - |\theta|)^{-(\alpha+\beta+\frac{1}{2})}$,

(2) $\sup_{r>0} \|r^{1-\gamma} A^\gamma (re^{i\omega} + A)^{-1}\| \lesssim (\pi - |\omega|)^{-(\alpha+1)}$,

(3) $\sup_{r>0} \|r(re^{i\omega} + A^\delta)^{-1}\| \lesssim \begin{cases} \delta^\alpha (\pi - |\omega|)^{-(\alpha+1)} & \text{for } \alpha > 0 \\ \log(\delta) + (\pi - |\omega|)^{-1}(|\log(\pi - |\omega|)| + 1) & \text{for } \alpha = 0. \end{cases}$

We point out two particular cases: Firstly, if $A = \log(B_\infty)$ on $X = E_0^\alpha$, then the "$\lesssim$" above may be replaced by "$\cong$". Secondly, if X has property (α), then uniform norm bounds may be replaced by R-bounds.

Proof. The norm bound results follow as in corollaries 5.21 and 5.23 from theorem 5.10 and the norm estimates in propositions 5.19 and 5.22.

The case $A = \log(B_\infty)$ follows from the lower estimates in propositions 5.19 and 5.22, and the last statement is the content of corollaries 5.21 and 5.23. □

6 Analyticity angle for non-commutative diffusion semigroups

6.1 Introduction

The spectral theory of (generators of) diffusion semigroups (T_t) on commutative (i.e. classical) L^p-spaces has been studied in a series of articles [5, 102, 92, 134, 77, 78]. Here we follow Stein's classical work [121] and mean by the term diffusion the fact that T_t is contractive as an operator $L^p \to L^p$ for all $p \in [1, \infty]$ and self-adjoint on L^2 (see 6.5 for the exact definition). Such a semigroup has an analytic extension on L^2 to the right half plane. Then it follows from a version of Stein's complex interpolation [133] that there is an analytic extension on L^p to a sector in the complex plane, symmetric to the real axis and with half opening angle

$$\frac{\pi}{2} - \pi\left|\frac{1}{p} - \frac{1}{2}\right|.$$

In [92], it is shown with a different method that this angle can be enlarged.

Theorem 6.1 *[92, cor 3.2] Let $(T_t)_{t \geqslant 0}$ be a diffusion semigroup on some σ-finite measure space, i.e.*

(1) $\|T_t : L^p \to L^p\| \leqslant 1$ *for all* $t \geqslant 0$ *and* $1 \leqslant p \leqslant \infty$,

(2) T_t *is self-adjoint on* L^2.

(3) $t \mapsto T_t$ *is strongly continuous on* L^p *for* $p < \infty$ *and* w^**-continuous for* $p = \infty$.

Assume further that $T_t f \geqslant 0$ for any $f \in L^\infty$, $f \geqslant 0$. Then T_t has an analytic contractive extension on L^p to the sector

$$\left\{ z \in \mathbb{C}^* : \, |\arg z| < \frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}} \right\}. \tag{6.1}$$

This result is optimal. In fact, there is already a strikingly simple example on a two-dimensional space with this angle (see example 6.7).
Generators of diffusion semigroups have a bounded H^∞ functional calculus on L^p for any $1 < p < \infty$. This follows from [25]. In [80] the H^∞ calculus angle of these operators was improved, using theorem 6.1. In the more recent past, besides vector valued spaces $L^p(\Omega, X)$ (see for example [62]), the attention turned to diffusion semigroups on non-commutative spaces $L^p(M)$ associated to a von Neumann algebra ([65], see also [66, 67]).

In this chapter, we consider non-commutative semigroups which are families (T_t) of operators acting on $L^p(M, \tau)$ for all $1 \leqslant p \leqslant \infty$. Under reasonable hypotheses, we obtain the same sector 6.1 as in the commutative case. Our method works for hyperfinite von Neumann algebras

M and for semi-commutative semigroups on $L^\infty(\Omega)\overline{\otimes}N$, where N is a QWEP von Neumann algebra.

Our assumptions are as follows. The operators T_t are completely contractive. In the commutative case, an operator $T : L^\infty \to L^\infty$ is completely contractive iff it is contractive, so that our assumption then reduces to the classical setting. The positivity assumption in theorem 6.1 is replaced by a certain property $(\mathcal{P})$, see definition 6.9. If for example the semigroup consists of complete positive operators, this property is satisfied. In the commutative case, an operator $T : L^\infty \to L^\infty$ satisfies $(\mathcal{P})$ if and only if T is contractive and extends to a self-adjoint operator on L^2 (proposition 6.16). In particular, we get theorem 6.1 without the positivity assumption, see corollary 6.17.

Note that our method (and also that of [92]) does not use the semigroup property in an extensive way. Theorem 6.14 gives a result on the numerical range for a single operator T instead of a semigroup. Furthermore, one could state theorem 6.14 for an operator acting on L^p for a single value of p, by replacing $(\mathcal{P})$ by some condition for an operator $L^p \to L^p$.

In section 6.2, we introduce non-commutative L^p-spaces and mention their properties that we need and give some examples. In section 6.3, completely positive and completely bounded maps are developed as far as needed for our considerations. The diffusion semigroups are then defined in section 6.4 and a basic guiding example is discussed. Section 6.5 contains the main theorems, and sections 6.6 and 6.7 are devoted to examples of diffusion semigroups to which our method applies.

6.2 Background on von Neumann algebras and non-commutative L^p-spaces

Throughout this chapter, we denote M a von Neumann algebra (see e.g. [124] for the definition) and assume that there is a semifinite, normal, faithful (s.n.f.) trace τ on M. The following examples for (M, τ) will frequently occur.

Examples of von Neumann algebras and definitions

1. For every $n \in \mathbb{N}$, we have the algebra of matrices $M_n = B(\ell^2_n) = \mathbb{C}^{n\times n}$, equipped with the common trace $\tau = \operatorname{tr}$. Note that every finite dimensional von Neumann algebra has a representation as a direct sum

$$M = M_{n_1} \oplus \ldots \oplus M_{n_K}$$

with $\tau(x_1 \oplus \ldots \oplus x_K) = \sum_{k=1}^K \lambda_k \operatorname{tr}(x_k)$ for some $K \in \mathbb{N}$ and $\lambda_k > 0$. Further,

$$\|x_1 \oplus \ldots \oplus x_K\|_M = \sup_{k=1}^{K} \|x_k\|_{M_{n_k}}.$$

2. M is called hyperfinite if there exists a net of finite dimensional $*$-subalgebras A_α which are directed by inclusion, such that $\bigcup A_\alpha$ is w^*-dense in M. If τ is a s.n.f. trace then one can always choose A_α such that $\tau|_{A_\alpha}$ is finite [110, chap 3].

Let us show this under the additional assumption that the net can be chosen as a sequence $(A_n)_n$. This is the case in most examples. Let $(q_k)_k$ be a sequence of orthogonal projections in M such that $\tau(q_k) < \infty$, $\mathrm{Im}(q_k) \subseteq \mathrm{Im}(q_{k+1})$ and $q_k \to 1$ in the strong operator topology. Such a sequence exists since τ is semifinite. Put now $p_k = q_k - q_{k-1}$ and $A_n' = \bigoplus_{k \leqslant n} p_k A_n p_k$. Then $(A_n')_n$ has the desired properties, i.e. $\tau|_{A_n'}$ is finite.
3. Let (Ω, μ) be a σ-finite measure space. Then $M = L^\infty(\Omega) = L^\infty(\Omega, \mu)$ is a von Neumann algebra with the s.n.f. trace $\tau(f) = \int f d\mu$. M is hyperfinite: Indeed, we explicitly give a net of finite dimensional $*$-subalgebras. We call a finite collection $\{A_1, \ldots, A_n\}$ of pairwise disjoint measurable subsets of Ω such that $0 < \mu(A_k) < \infty$ a semi-partition. Let $\mathcal{A}$ be the set of all semi-partitions. $\mathcal{A}$ is directed by: $\{A_1, \ldots, A_n\} \prec \{B_1, \ldots, B_m\}$ iff any A_k is the union of some of the $B_k's$. For $\alpha = \{A_1, \ldots, A_n\} \in \mathcal{A}$, put $M_\alpha := \{\sum_{k=1}^n c_k \chi_{A_k} : c_k \in \mathbb{C}\} \subset M$. Clearly, $\tau|_{M_\alpha}$ is finite and for any $x \in L^\infty(\Omega)$ and $y \in L^1(\Omega)$, $\int x_\alpha y d\mu \to \int x y d\mu$, which is the w^*-density.
4. If $M \subset B(H)$ and $N \subset B(K)$ is a further von Neumann algebra with s.n.f. trace σ, then $N \overline{\otimes} M$ defined as the w^*-closure of $N \otimes M$ in $B(K \otimes_2 H)$ is again a von Neumann algebra. $(\sigma \otimes \tau)(x \otimes y) := \sigma(x)\tau(y)$ can be extended to a s.n.f. trace on $N \overline{\otimes} M$. We will use this fact for the cases $N = M_n$ as in 1 and $N = L^\infty(\Omega)$ as in 3. $L^\infty(\Omega) \overline{\otimes} M$ can be naturally identified with the space of w^*-measurable, essentially bounded functions $\Omega \to M$, see [8, p. 40-41].

For $1 \leqslant p < \infty$, the non-commutative L^p-spaces $L^p(M) = L^p(M, \tau)$ are defined as follows. If S_+ is the set of all positive $x \in M$ (i.e. $x = x^*$ and $\sigma(x) \subset [0, \infty)$) such that $\tau(x) < \infty$ and S is its linear span, then $L^p(M)$ is the completion of S with respect to the norm $\|x\|_p = \tau(|x|^p)^{1/p}$. It can also be described as a space of unbounded operators x affiliated to M in a certain sense such that $\tau(|x|^p)^{1/p} < \infty$, where the domain of τ is extended to all of $L^1(M)$. One sets $L^\infty(M) = M$. As for the commutative (i.e. usual) L^p-spaces, one has: $L^p(M)' = L^q(M)$ via the duality $(x, y) \mapsto \tau(xy)$, for $1 \leqslant p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. We denote this duality from now on by $\langle x, y \rangle$. The Hölder inequality holds in the form $\|x\|_{L^p(M)} = \sup\{|\langle x, y\rangle| : \|y\|_{L^q(M)} \leqslant 1\}$. The space $L^2(M)$ is a Hilbert space with respect to the scalar product $(x, y) \mapsto \langle x, y^* \rangle$. For $1 \leqslant p, q \leqslant \infty$, $(L^p(M), L^q(M))$ is, in the sense of complex interpolation [6], a compatible couple of spaces such that $(L^p(M), L^q(M))_\theta = L^r(M)$ with $\frac{1}{r} = \frac{\theta}{q} + \frac{1-\theta}{p}$.

See [125, 114] for further reference on non-commutative L^p-spaces. Examples which will appear are:

Examples of non-commutative L^p-spaces
1. For $(M, \tau) = (M_n, \mathrm{tr})$, we write $S_n^p = L^p(M_n)$. More generally, if H is a Hilbert space and tr the usual trace on $B(H)$, then $S^p(H) = L^p(B(H), \mathrm{tr})$. If $H = \ell^2$, then we write $S^p = S^p(H)$.
2. If M is finite dimensional and $(M, \tau) = (M_{n_1}, \lambda_1 \mathrm{tr}) \oplus \ldots \oplus (M_{n_K}, \lambda_K \mathrm{tr})$, then for $x = x_1 \oplus \ldots \oplus x_K$, $\|x\|_{L^p(M)}^p = \sum_k \lambda_k \|x_k\|_{S_n^p}^p$.
3. If (Ω, μ) is a σ-finite measure space and $M = L^\infty(\Omega)$, then $L^p(\Omega) = L^p(M)$, $1 \leqslant p \leqslant \infty$.
4. If $M = L^\infty(\Omega)$ and N is a further von Neumann algebra with s.n.f. trace σ, then $L^p(M \overline{\otimes} N)$ is naturally isometric to the Bochner space $L^p(\Omega, L^p(N))$ for $1 \leqslant p < \infty$.

Finally, the following notion of a dual element will play an eminent role.

Definition 6.2 *Let $1 < p < \infty$ and $q = \frac{p}{p-1}$ the conjugate number. Let $x \in L^p(M)$. Then x has a polar decomposition $x = u|x|$ with $u \in M$ unitary and $|x| = (x^*x)^{1/2}$. The dual element of x is defined as $\widehat{x} = |x|^{p-1}u^*$.*

Lemma 6.3 *The above defined $\widehat{x}$ is the unique element in $L^q(M)$ with:*

(1) $\langle x, \widehat{x}\rangle = \|x\|_p^p$.

(2) $\|x\|_p^p = \|\widehat{x}\|_q^q$.

Further, the (in non trivial cases nonlinear) mapping

$$\widehat{\ }: \begin{cases} L^p(M) & \longrightarrow L^q(M) \\ x & \longmapsto \widehat{x} \end{cases}$$

is norm-continuous.

Proof. It is plain that $\widehat{x}$ satisfies the claimed properties. On the other hand, it is well known that $L^p(M)$ is uniformly smooth, which implies uniqueness. To see the continuity of $x \mapsto \widehat{x}$, let $x, x_1, x_2, \ldots \in L^p(M)$ such that $x_n \to x$. We can exclude the trivial case $x = 0$. Since $\|\widehat{x_n}\|_q = \|x_n\|_p^{p/q}$ is bounded, the Banach-Alaoglu theorem gives a weak limit point of $(\widehat{x_n})_n$. We show that any such limit point y equals $\widehat{x}$, which implies that $\widehat{x}$ is the weak limit of $\widehat{x_n}$. Since $L^p(M)$ is uniformly convex and $\|\widehat{x_n}\|_q \to \|\widehat{x}\|_q$, it will follow that $\|\widehat{x_n} - \widehat{x}\|_q \to 0$.

Let $y = \text{w-}\lim_k \widehat{x_{n_k}}$. We have

$$\langle x, y\rangle = \lim_k \langle x, \widehat{x_{n_k}}\rangle = \lim_k \langle x_{n_k}, \widehat{x_{n_k}}\rangle + \langle x - x_{n_k}, \widehat{x_{n_k}}\rangle = \|x\|_p^p + 0.$$

This shows that y satisfies (1) of the lemma, and that $\|y\|_q \geqslant \|x\|_p^{p-1}$. On the other hand, $\|y\|_q \leqslant \limsup_k \|\widehat{x_{n_k}}\|_q = \|\widehat{x}\|_q = \|x\|_p^{p-1}$, so that y satisfies (2). By uniqueness of $\widehat{x}$, $y = \widehat{x}$. □

6.3 Operators between non-commutative L^p-spaces

For $n \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$, we denote by $S_n^p(L^p(M))$ the space $S_n^p \otimes L^p(M) = \{(x_{ij})_{ij} : i, j = 1, \ldots, n,\ x_{ij} \in L^p(M)\}$ equipped with the norm of $L^p(M_n \otimes M, \operatorname{tr} \otimes \tau)$ [110, chap 1]. Let $T : L^p(M) \to L^p(N)$ be a linear mapping, where N is a further von Neumann algebra with s.n.f. trace σ. Following [110, lem 1.7], we call T completely bounded if

$$\|T\|_{cb} := \sup_n \| \operatorname{Id}_{S_n^p} \otimes T :\ (x_{ij})_{ij} \mapsto (Tx_{ij})_{ij} \|_{B(S_n^p(L^p(M)), S_n^p(L^p(N)))} < \infty, \tag{6.2}$$

and completely contractive if this quantity is less than 1. Clearly, $\|T\| \leqslant \|T\|_{cb}$. If $T : L^p(M) \to L^p(N)$ is completely bounded for $p = 1$ and $p = \infty$, then by complex interpolation,

$$\|T : L^q(M) \to L^q(N)\|_{cb} \leqslant \|T : L^1(M) \to L^1(N)\|_{cb}^{1/q} \|T : M \to N\|_{cb}^{1-1/q}$$

for any $1 < q < \infty$. $T : L^p(M) \to L^p(N)$ is called positive if $Tx \geqslant 0$ for all $x \geqslant 0$. T is called completely positive, if $I_{S_n^p} \otimes T \in B(S_n^p(L^p(M)), S_n^p(L^p(N))) = B(L^p(M_n \otimes M), L^p(M_n \otimes N))$

is positive for all $n \in \mathbb{N}$. In the case $M = N = M_n$, Choi showed in [20] the following characterization:

$$T : M_n \to M_n \text{ completely positive } \iff \exists a_1, \ldots, a_N \in M_n : Tx = \sum_{k=1}^{N} a_k^* x a_k. \tag{6.3}$$

Assume that $T \in B(L^p(M), L^p(N)))$ for some $1 \leqslant p \leqslant \infty$ and that T is w^*-continuous if $p = \infty$. In view of the duality $L^p(M)' = L^q(M)$ for $1 \leqslant p < \infty$, $q = \frac{p}{p-1}$, the operator $T' : L^q(N) \to L^q(M)$ is defined. If $p = \infty$, we denote $T' : L^1(M) \to L^1(M)$ the pre-adjoint operator.

Lemma 6.4 *Let N, σ, p, q, T be as above.*

(1) *If $T : L^p(M) \to L^p(N)$ is (completely) positive, then $T' : L^q(N) \to L^q(M)$ is (completely) positive also.*

(2) *If T is completely bounded, then T' is also completely bounded, with the same cb-norm.*

Proof. T is positive if and only if $\sigma((Ta)b) \geqslant 0$ for all positive $a \in L^p(M)$ and $b \in L^q(N)$. On the other hand, $\sigma((Ta)b) = \tau(a(T'b))$, so that the positivity part follows. The complete positivity part is then a consequence of $(\mathrm{Id}_{S_n^p} \otimes T)' = \mathrm{Id}_{S_n^q} \otimes T'$. This also gives the complete boundedness statement, in view of 6.2. □

6.4 Non-commutative diffusion semigroups

Let $T : M \to M$ be a w^*-continuous operator with $\|T\|_{M \to M} \leqslant 1$. Assume that

$$\text{for } x, y \in M \cap L^1(M),\ \langle Tx, y^* \rangle = \langle x, (Ty)^* \rangle. \tag{6.4}$$

We call a T with this property self-adjoint. Then by the Hölder inequality, $T|_{M \cap L^1(M)}$ extends to a contraction $T_1 : L^1(M) \to L^1(M)$ and by complex interpolation, also to T_p with

$$\|T_p : L^p(M) \to L^p(M)\| \leqslant 1 \quad (1 \leqslant p \leqslant \infty).$$

Since T is w^*-continuous, 6.4 yields that $T_1 = T'(\cdot^*)^*$. Clearly, $T_2 : L^2(M) \to L^2(M)$ is self-adjoint in the classical sense. If $T : M \to M$ is in addition completely contractive, then by lemma 6.4, $T_1 = T'(\cdot^*)^* : L^1(M) \to L^1(M)$ is also completely contractive, and hence T_p also.

The following notion of a (non-commutative) diffusion semigroup has been defined in [65] and generalizes Stein's setting in [121].

Definition 6.5 *Let $(T_t)_{t \geqslant 0}$ be a family of completely contractive operators of the above type. (T_t) is called a diffusion semigroup (on M) if*

$$T_0 = I_M \text{ and } T_t T_s = T_{t+s} \text{ for } t, s \geqslant 0. \tag{6.5}$$

$$T_t x \to x \text{ as } t \to 0 \text{ in the } w^* \text{ topology.} \tag{6.6}$$

Clearly, for $1 \leqslant p < \infty$, $(T_{t,p})$ is a semigroup on $L^p(M)$ and by [27, prop 1.23], 6.6 implies that $(T_{t,p})$ is strongly continuous. Examples of such diffusion semigroups are given in [65, chap 8,9,10] and will be discussed in section 6.6.

It is shown in [65, chap 5] - using the functional calculus for self-adjoint operators and a version of Stein's complex interpolation - that for $(T_{t,p})$, there exists an analytic and contractive extension to a sector $S(\frac{\pi}{2} - \pi|\frac{1}{p} - \frac{1}{2}|)$, where we put

$$S(\omega) = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \omega\}.$$

This means that there exists an analytic function $S(\frac{\pi}{2} - \pi|\frac{1}{p} - \frac{1}{2}|) \to B(L^p(M))$, $z \mapsto S_z$ such that $S_t = T_{t,p}$ for $t > 0$ and $\|S_z\|_{B(L^p(M))} \leqslant 1$. The major question of this chapter is:

Given a diffusion semigroup (T_t) on M and $1 < p < \infty$, what is the optimal $\omega_p > 0$ such that $T_{t,p}$ has an analytic and contractive extension to $S(\omega_p)$?

This question and related ones have been studied in the commutative case in [5, 92, 102, 134, 77, 78, 18, 19].
In the rest of this section, let us work out the candidate for ω_p. First recall the following characterization.

Proposition 6.6 *Let $(T_{t,p})$ be a c_0-semigroup on $L^p(M)$ for some $1 < p < \infty$. Denote A_p its generator. Fix some $\omega \in (0, \frac{\pi}{2})$. Then the following are equivalent.*

(1) $-\langle A_p x, \widehat{x}\rangle \in \overline{S(\frac{\pi}{2} - \omega)}$ *for all* $x \in D(A_p)$.

(2) $(T_{t,p})$ *has an analytic and contractive extension to* $S(\omega)$.

The first condition is obviously verified if

$$\langle (I - T_{t,p})x, \widehat{x}\rangle \in \overline{S(\frac{\pi}{2} - \omega)} \text{ for all } x \in L^p(M) \text{ and } t > 0.$$

Proof. See for example [49, thm 5.9] □

The following easy example already gives a good insight into what we can expect.

Example 6.7 *Let M be the commutative 2-dimensional von Neumann algebra ℓ_2^∞ with trace $\tau((a, b)) = a + b$. We consider $T_t = e^{tA}$ with*

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = -e \otimes e,$$

where $e = (1, -1)$. Then $A^n = -2A^{n-1} = (-2)^{n-1}A$ for $n \geqslant 2$. Hence

$$T_t = I_M - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-2t)^n}{n!} A = \frac{1}{2}\begin{pmatrix} 1 + e^{-2t} & 1 - e^{-2t} \\ 1 - e^{-2t} & 1 + e^{-2t} \end{pmatrix}.$$

Since this matrix is self-adjoint and $|1+\frac{1}{2}e^{-2t}|+|1-\frac{1}{2}e^{-2t}| = 1$ *for all* $t \geqslant 0$, (T_t) *is indeed a diffusion semigroup. Now fix some* $1 < p < \infty$ *and let* $x = (a,b) \in \ell_2^p$. *Then* $\widehat{x} = (\widehat{a}, \widehat{b}) = (\overline{a}|a|^{p-2}, \overline{b}|b|^{p-2})$ *and*

$$-\langle Ax, \widehat{x}\rangle = (a-b)(\widehat{a}-\widehat{b}).$$

To answer our angle question, in view of the preceding proposition, we are supposed to determine the smallest sector containing this quantity for arbitrary $a, b \in \mathbb{C}$. *The solution is the following proposition which appears in [92, lem 2.2].*

Proposition 6.8 *Let* $1 < p < \infty$, $a, b \in \mathbb{C}$. *Then for* $\omega_p = \arctan \frac{|p-2|}{2\sqrt{p-1}}$,

$$(a-b)(\widehat{a}-\widehat{b}) = |a|^p + |b|^p - a\overline{b}|b|^{p-2} - b\overline{a}|a|^{p-2} \in \overline{S(\omega_p)}.$$

Further, this result is optimal, i.e. the statement is false for any $\omega < \omega_p$.

Proof. The fact that $z = (a-b)(\widehat{a}-\widehat{b}) \in \overline{S(\omega_p)}$ has been shown in [92, lem 2.2]. We show the optimality for the convenience of the reader. Let $b = 1$ and $a = re^{i\phi}$ with $r \neq 1$. Then $z = r^p + 1 - re^{i\phi} - r^{p-1}e^{-i\phi}$, so that

$$\begin{aligned} \operatorname{Im} z &= -r\sin\phi + r^{p-1}\sin\phi, \\ \operatorname{Re} z &= r^p + 1 - r\cos\phi - r^{p-1}\cos\phi, \end{aligned}$$

whence

$$\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)^2 = \frac{(r^{p-1}-r)^2(1-\cos^2\phi)}{(r^p+1-r\cos\phi - r^{p-1}\cos\phi)^2}.$$

Maximizing this expression in ϕ, i.e. choosing $\cos\phi = \frac{r(r^{p-2}+1)}{(r^p+1)} < 1$ gives

$$\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)^2 = \frac{r^2(r^{p-2}-1)^2}{(r^2-1)(r^{2p-2}-1)}.$$

The limit for $r \to 1$ of this expression is $\frac{(p-2)^2}{4(p-1)}$, so that $|\arg z| \to \frac{|p-2|}{2\sqrt{p-1}}$. □

From now on, write

$$\Sigma_p = \overline{S(\arctan \tfrac{|p-2|}{2\sqrt{p-1}})}, \quad \Sigma_p' = S(\tfrac{\pi}{2} - \arctan \tfrac{|p-2|}{2\sqrt{p-1}}).$$

In view of the above, Σ_p is our candidate for the sector which supports the numerical range of $-A_p$, and we are looking for diffusion semigroups (T_t) such that:

For every $1 < p < \infty$, $T_{t,p}$ has an analytic and contractive extension $\Sigma_p' \to B(L^p(M))$. (6.7)

6.5 The angle theorem

We begin with some notation. If $A, B, C, D \in B(L^p(M))$, we denote

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} : L^p(M_2 \otimes M) \to L^p(M_2 \otimes M), \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} A(a) & B(b) \\ C(c) & D(d) \end{pmatrix}.$$

For $p = 2$, this operator is self-adjoint if and only if A, B, C, D are all self-adjoint.

The key notion to establish the theorem for the analytic extension of a diffusion semigroup is the following one.

Definition 6.9 *Let $T : M \to M$ be a w^*-continuous operator. Denote $T_* : M \to M, T_*(x) = T(x^*)^*$. Then we say that T satisfies $(\mathcal{P})$ if there exist $S_1, S_2 : M \to M$ such that*

$$W := \begin{pmatrix} S_1 & T \\ T_* & S_2 \end{pmatrix} : M_2 \otimes M \to M_2 \otimes M$$

is completely positive, completely contractive and self-adjoint.

Note that a completely positive linear mapping between von Neumann algebras is completely contractive iff the image of the unity has norm less than 1 [107, prop 3.6].

Hence we can replace the complete contractivity in definition 6.9 by the assumption

$$\left\| W \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| \leqslant 1. \tag{6.8}$$

Remark 6.10 *1. A T satisfying $(\mathcal{P})$ is necessarily completely contractive and self-adjoint. Indeed, W is self-adjoint iff S_1, S_2 and T are. Further, it is well-known that the complete positivity of W implies that T is completely contractive.*

2. On the other hand, if T is completely contractive, self-adjoint and in addition completely positive, then it satisfies $(\mathcal{P})$. Just take $S_1 = S_2 = T$, and note that $T_* = T$. Then $W = \begin{pmatrix} T & T \\ T & T \end{pmatrix}$ is again completely positive, completely contractive on $M_2(M)$ and self-adjoint.

3. Assume that $(M, \tau) = (M_n, \mathrm{tr})$. It is well-known that $T : M_n \to M_n$ is completely contractive if and only if there exist $a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ such that $Tx = \sum_{k=1}^N a_k x b_k$ and

$$\|\sum_{k=1}^N a_k a_k^*\| \leqslant 1, \quad \|\sum_{k=1}^N b_k^* b_k\| \leqslant 1.$$

$T : x \mapsto \sum_k a_k x b_k$ is self-adjoint if and only if $\sum_k a_k x b_k = \sum_k a_k^* x b_k^*$ for all x. On the other hand, T satisfies $(\mathcal{P})$ if and only if

$$\exists\, a_1, \ldots, a_N, b_1, \ldots, b_N \in M_n \text{ self-adjoint} : \; Tx = \sum_{k=1}^N a_k x b_k, \; \sum_k a_k^2 \leqslant 1, \; \sum_k b_k^2 \leqslant 1. \tag{6.9}$$

Indeed, if 6.9 is satisfied, then put $S_1x = \sum_k a_kxa_k$ *and* $S_2x = \sum_k b_kxb_k$*. Then*

$$\begin{pmatrix} S_1 & T \\ T_* & S_2 \end{pmatrix} x = \sum_k c_kxc_k$$

with $c_k = \begin{pmatrix} a_k & 0 \\ 0 & b_k \end{pmatrix}$*. Property* $(\mathcal{P})$ *follows, since* c_k *is self-adjoint and* $\sum c_k^2 \leqslant 1$*.*

Conversely, if $(\mathcal{P})$ *is satisfied, then by the complete positivity of* W *and 6.3, there exist* $c_1, \ldots, c_N \in M_{2n}$ *such that* $Wx = \sum_k c_k^*xc_k$*. Since* W *is self-adjoint,* $\sum_k c_k^*xc_k = \sum_k c_kxc_k^*$ *and consequently,*

$$Wx = \sum_k \left(\frac{c_k + c_k^*}{2}\right) x \left(\frac{c_k + c_k^*}{2}\right) + \sum_k \left(\frac{c_k - c_k^*}{2i}\right) x \left(\frac{c_k - c_k^*}{2i}\right).$$

We have $\|W(1)\| \leqslant 1$*, so replacing* $(c_1, \ldots, c_N)$ *by* $((c_1 + c_1^*)/2, \ldots, (c_N + c_N^*)/2, (c_1 - c_1^*)/(2i),$ $\ldots, (c_N - c_N^*)/(2i))$*, one can assume that the* c_k'*s are self-adjoint and* $\|\sum_k c_k^2\| \leqslant 1$*. Write* $c_k = \begin{pmatrix} a_k & d_k \\ d_k^* & b_k \end{pmatrix}$*. By definition of* W*,* $Tx = \sum_k a_kxb_k$*. Further,* $\|\sum_k a_k^2\|, \|\sum_k b_k^2\| \leqslant \|\sum_k c_k^2\|$*, so that* a_k, b_k *match 6.9.*

4. The property $(\mathcal{P})$ *is connected to the definition of decomposable maps.* $T : M \to M$ *is by definition decomposable* ($\|T\|_{dec} \leqslant 1$) *if* S_1 *and* S_2 *exist such that* $W = \begin{pmatrix} S_1 & T \\ T_* & S_2 \end{pmatrix}$ *is completely positive (and contractive) [113, p. 130]. One has* $\|T\|_{dec} \leqslant 1$ *for all complete contractions* $T : M \to M$ *iff* M *is hyperfinite [50]. In general, the assumptions* $\|T\|_{dec} \leqslant 1$ *and* T *self-adjoint do not imply* $(\mathcal{P})$*, see the example below. However, we will see in section 6.6 that this holds true in some special cases.*

Example 6.11 *Parts 1 and 2 of the preceding remark lead to the question if the property* $(\mathcal{P})$ *is equivalent to complete contractivity and self-adjointness. But in general,* $(\mathcal{P})$ *is strictly stronger. Indeed, there is a self-adjoint and complete contractive* T *which does not satisfy* $(\mathcal{P})$*. I am grateful to Éric Ricard for showing me the following example. The operator space theory used here goes beyond what is explained in section 6.2, see for example [113, 40].*

Let $n \in \mathbb{N}$ *and* $(E_{ij})_{0 \leqslant i,j \leqslant n}$ *be the canonical basis of* M_{n+1}*. Define* $T : M_{n+1} \to M_{n+1}$ *by*

$$Tx = \sum_{i=1}^n E_{i0}xE_{i0} + \sum_{i=1}^n E_{0i}xE_{0i}.$$

Then T *is self-adjoint and by writing*

$$Tx = \sum_{i=1}^n (n^{1/4}E_{i0})x(n^{-1/4}E_{i0}) + \sum_{i=1}^n (n^{-1/4}E_{0i})x(n^{1/4}E_{0i}),$$

one sees that $\|T\|_{cb} \leqslant \sqrt{n}$ *(cf. remark 6.10.3 above). Now assume that* $a_1, \ldots, a_N, b_1, \ldots, b_N \in M_{n+1}$ *are self-adjoint such that* $Tx = \sum_{k=1}^N a_kxb_k$*. We will show that*

$$\|\sum_k a_k^2\|^{1/2}\|\sum_k b_k^2\|^{1/2} \geqslant n, \tag{6.10}$$

so that for $n \geqslant 2$, the self-adjoint completely contractive operator $\frac{1}{\sqrt{n}}T$ does not satisfy $(\mathcal{P})$.

We denote R_N and C_N the row and column operator space of dimension N [113, p. 21]. Further, $R_N \cap C_N$ is equipped with the operator space structure

$$\|(x_{ij})\| = \max\{\|(x_{ij})\|_{M_n(R_N)}, \|(x_{ij})\|_{M_n(C_N)}\}$$

and $R_N + C_N$ is the operator space dual of $R_N \cap C_N$ [113, p. 55,194]. Then for any operator space X and any $x_1, \dots, x_N \in X$,

$$\|\sum_{k=1}^{N} e_k \otimes x_k : X^* \to R_N \cap C_N\|_{cb} = \max\left\{\|\sum_k x_k x_k^*\|^{1/2}, \|\sum_k x_k^* x_k\|^{1/2}\right\}.$$

Let

$$\alpha = \sum_{k=1}^{N} a_k \otimes e_k : \ell_N^2 \to M_{n+1}, \quad \beta = \sum_{k=1}^{N} e_k \otimes b_k : S_{n+1}^1 \to \ell_N^2.$$

Here, $(e_k)_{1 \leqslant k \leqslant N}$ is the canonical basis of ℓ_N^2, $a_k \otimes e_k$ maps x to $\langle x, e_k\rangle_{\ell_N^2} a_k$ and $e_k \otimes b_k$ maps x to $\operatorname{tr}(x b_k) e_k$. Then

$$\|\alpha : R_N + C_N \to M_{n+1}\|_{cb} = \|\sum_k a_k^2\|^{1/2},$$

$$\|\beta : S_{n+1}^1 \to R_N \cap C_N\|_{cb} = \|\sum_k b_k^2\|^{1/2} \text{ and}$$

$$\alpha\beta(x) = (\sum_k E_{k0} \otimes E_{k0} + \sum_k E_{0k} \otimes E_{0k})(x) \text{ for any } x \in M_{n+1}.$$

Let us denote $C_n \oplus_\infty R_n \subset M_{n+1}$ the subspace spanned by $\{E_{i0}, E_{0i} : 1 \leqslant i \leqslant n\}$. In the same manner, we regard this space as $R_n \oplus_1 C_n \subset S_{n+1}^1$. If $J : R_n \oplus_1 C_n \to C_n \oplus_\infty R_n$ is the identity, then $\alpha\beta$ is obtained by projecting canonically S_{n+1}^1 to $R_n \oplus_1 C_n$, then applying J and finally injecting $C_n \oplus_\infty R_n$ into M_{n+1}. Denote $\hat{\alpha} = p\alpha$ and $\hat{\beta} = \beta j$, where p is the natural projection of M_{n+1} onto $C_n \subset C_n \oplus_\infty R_n \subset M_{n+1}$ and j is the embedding of $R_n \subset R_n \oplus_1 C_n \subset S_{n+1}^1$ into S_{n+1}^1. Then one obtains the following commuting diagram

$$\begin{array}{ccc} R_N \cap C_N & \xrightarrow{Id} & R_N + C_N \\ \uparrow \hat{\beta} & & \downarrow \hat{\alpha} \\ R_n & \xrightarrow[Id]{} & C_n \end{array}$$

with

$$\|\hat{\alpha} : R_N + C_N \to C_n\|_{cb} \leqslant \|\sum_k a_k^2\|^{1/2},$$

$$\|\hat{\beta} : R_n \to R_N \cap C_N\|_{cb} \leqslant \|\sum_k b_k^2\|^{1/2}.$$

According to the factorization $I_{\ell^2_N} = \hat{\alpha}\hat{\beta}$ we have $n \leqslant \|\hat{\beta}\|_{HS}\|\hat{\alpha}\|_{HS}$. But the Hilbert-Schmidt norm of any $\gamma : \ell^2_{m_1} \to \ell^2_{m_2}$ equals the cb-norm of $\gamma : R_{m_1} \to C_{m_2}$ [113, p. 21], so

$$\begin{aligned} n &\leqslant \|\hat{\beta} : R_n \to C_N\|_{cb} \|\hat{\alpha} : R_N \to C_n\|_{cb} \\ &\leqslant \|\hat{\beta} : R_n \to C_N \cap R_N\|_{cb} \|\hat{\alpha} : R_N + C_N \to C_n\|_{cb} \\ &\leqslant \| \sum_k b_k^2 \|^{1/2} \| \sum_k a_k^2 \|^{1/2}. \end{aligned}$$

This shows 6.10.

Now the matrix version of the main theorem reads as follows.

Theorem 6.12 *Let $n \in \mathbb{N}$ and $T : M_n \to M_n$ satisfy $(\mathcal{P})$. Fix some $p \in (1, \infty)$. Then for any $x \in S^p_n$,*

$$\langle (I-T)x, \widehat{x} \rangle \in \Sigma_p.$$

Proof. Use remark 6.10 and write $Tx = \sum_{k=1}^m a_k x b_k$ with a_k, b_k as in 6.9. Decompose

$$x = udv,$$

with $u, v \in M_n$ unitaries and d a diagonal matrix with non-negative diagonal entries $d_1, \ldots, d_n$. Then $\widehat{x} = v^* d^{p-1} u^*$. For simplifying the calculation, we write $g_k = u^* a_k u$ and $h_k = v b_k v^*$.

$$\begin{aligned} \langle (I-T)x, \widehat{x} \rangle &= \operatorname{tr}(d^p - \sum_k g_k d h_k d^{p-1}) \\ &= \sum_{r=1}^n d_r^p - \sum_{k,r,s} g_{k,rs} d_s h_{k,sr} d_r^{p-1}. \end{aligned}$$

Write $c_{rs} := \sum_k g_{k,rs} h_{k,sr}$. Since g_k and h_k are self-adjoint, $c_{rs} = \overline{c_{sr}}$. Thus, the above expression equals

$$\begin{aligned} &\sum_r d_r^p - \frac{1}{2} \sum_{r,s} c_{rs} d_s d_r^{p-1} - \frac{1}{2} \sum_{r,s} \overline{c_{rs}} d_r d_s^{p-1} \\ =&\frac{1}{2} \left\{ \sum_r d_r^p (1 - \sum_s |c_{rs}|) + \sum_s d_s^p (1 - \sum_r |c_{rs}|) + \sum_{r,s} \left(d_r^p |c_{rs}| + d_s^p |c_{rs}| - c_{rs} d_s d_r^{p-1} - \overline{c_{rs}} d_r d_s^{p-1} \right) \right\}. \end{aligned}$$

The expression in round brackets of the last double sum is a term $(a-b)(\widehat{a}-\widehat{b})$ as in proposition 6.8, putting

$$a = d_r |c_{rs}|^{1/p} \text{ and } b = d_s |c_{rs}|^{1/p} \frac{c_{rs}}{|c_{rs}|}.$$

Since Σ_p is closed under addition,

$$\sum_{r,s} d_r^p |c_{rs}| + d_s^p |c_{rs}| - c_{rs} d_s d_r^{p-1} - \overline{c_{rs}} d_r d_s^{p-1} \in \Sigma_p.$$

Moreover, it now suffices to show that

$$1 - \sum_s |c_{rs}| \geqslant 0,\ 1 - \sum_r |c_{rs}| \geqslant 0. \tag{6.11}$$

First we use Cauchy-Schwarz:

$$\left(\sum_s |c_{rs}|\right)^2 = \left(\sum_s |\sum_k g_{k,rs}h_{k,sr}|\right)^2 \leqslant \left(\sum_s \sum_k |g_{k,rs}|^2\right)\left(\sum_s \sum_k |h_{k,sr}|^2\right).$$

We estimate the first factor:

$$\sum_s \sum_k |(u^* a_k u)_{rs}|^2 = \sum_s \sum_k (u^* a_k u)_{sr}(u^* a_k u)_{rs} = \sum_k (u^* a_k^2 u)_{rr} \leqslant 1,$$

where we use the assumption $\sum_k a_k^2 \leqslant 1$ in the last inequality. In the same way, one estimates the second factor, which gives the first estimate in 6.11. The second estimate in 6.11 follows at once, since $|c_{rs}| = |c_{sr}|$. □

Our next goal is to extend the theorem to hyperfinite von Neumann algebras instead of M_n by a limit process. The following lemma contains the necessary information how the property $(\mathcal{P})$ and the dual element behave when passing from a "small" von Neumann algebra N to a "big" von Neumann algebra $\mathcal{N}$ and vice versa.

Lemma 6.13 *Let (N, σ) and $(\mathcal{N}, \tilde{\sigma})$ be two von Neumann algebras with s.n.f. trace. Assume that there exist $J : N \to \mathcal{N}$ and $Q : \mathcal{N} \to N$ with the following properties:*

(1) *J and Q are completely positive,*

(2) *J and Q are (completely) contractive,*

(3) $QJ = \mathrm{Id}_N$,

(4) *$\langle Jx, y\rangle = \langle x, Qy\rangle$ for all $x \in L^1(N) \cap N$ and $y \in L^1(\mathcal{N}) \cap \mathcal{N}$.*

Then J and Q extend to complete contractions $J_p : L^p(N) \to L^p(\mathcal{N})$ and $Q_p : L^p(\mathcal{N}) \to L^p(N)$ for any $1 \leqslant p \leqslant \infty$. Furthermore, the following holds.

(1) *If $T : \mathcal{N} \to \mathcal{N}$ satisfies $(\mathcal{P})$, then also $QTJ : N \to N$ does. For all $1 < p < \infty$ and $x \in L^p(N)$, $\langle (\mathrm{Id}_{L^p(\mathcal{N})} - T_p)J_p(x), \widehat{J_p(x)}\rangle = \langle (\mathrm{Id}_{L^p(N)} - Q_pT_pJ_p)x, \widehat{x}\rangle$.*

(2) *If $T : N \to N$ satisfies $(\mathcal{P})$, then also $JTQ : \mathcal{N} \to \mathcal{N}$ does. For all $1 < p < \infty$ and $x \in L^p(N)$, $\langle (\mathrm{Id}_{L^p(N)} - T_p)x, \widehat{x}\rangle = \langle (\mathrm{Id}_{L^p(\mathcal{N})} - J_pT_pQ_p)J_p(x), \widehat{J_p(x)}\rangle$.*

Proof. The completely contractive extensions J_p and Q_p follow from assumption 4 by Hölder's inequality and complex interpolation, as in the beginning of section 6.4.
(1) Let W be an extension of T according to the definition of $(\mathcal{P})$. Then $\widetilde{W} = \widetilde{Q}W\widetilde{J}$ is an appropriate extension of QTJ, where $\widetilde{Q} = \mathrm{Id}_{M_2} \otimes Q$ and $\widetilde{J} = \mathrm{Id}_{M_2} \otimes J$. Indeed, since J and Q are completely positive, also $\widetilde{J}$ and $\widetilde{Q}$ are, and therefore $\widetilde{W}$ is. As J and Q are completely contractive, $\widetilde{J}$, $\widetilde{Q}$, and thus $\widetilde{W}$ are contractive. By 6.8, $\widetilde{W}$ is completely contractive. It is plain to check the self-adjointness of $\widetilde{W}$. Just note that by the positivity of $\widetilde{Q}$, $\widetilde{Q}(x^*) = [\widetilde{Q}(x)]^*$, so $\widetilde{Q}_2$ is the adjoint of $\widetilde{J}_2$ in the Hilbert space sense.

For the second part, note that by approximation, $\langle Q_p x, y\rangle = \langle x, J_q y\rangle$ for any $x \in L^p(\mathcal{N})$, $y \in L^q(N)$ and $1 \leqslant p, q \leqslant \infty$ conjugated exponents. Also assumption 3 extends to $Q_pJ_p = \mathrm{Id}_{L^p(N)}$

for all $1 \leqslant p \leqslant \infty$. Now the assertion follows if we know that $\widehat{J_p(x)} = J_q(\widehat{x})$ for any $x \in L^p(N)$. We check the two determining properties of the dual element.

$$\langle J_q(\widehat{x}), J_p(x)\rangle = \langle Q_q J_q(\widehat{x}), x\rangle = \langle \widehat{x}, x\rangle = \|x\|_p^p.$$

Further,

$$\|J_q(\widehat{x})\|_q^q = \|\widehat{x}\|_q^q = \|x\|_p^p = \|J_p(x)\|_p^p = \|\widehat{J_p(x)}\|_q^q.$$

Here, we have used that J_q (and J_p) is an isometry. This follows from $Q_q J_q = \mathrm{Id}_{L^q(N)}$ and the contractivity of J_q.
(2) Put $\widetilde{W} = \widetilde{J} W \widetilde{Q}$. The rest of the proof is very similar to that of (1). □

Theorem 6.14 *Let M be a hyperfinite von Neumann algebra and $T : M \to M$ satisfy $(\mathcal{P})$. Then for all $1 < p < \infty$ and $x \in L^p(M)$,*

$$\langle (\mathrm{Id}_{L^p(M)} - T_p)x, \widehat{x}\rangle \in \Sigma_p.$$

Proof. 1st case: M finite dimensional. Then there exist $K \in \mathbb{N}$, $\lambda_1, \ldots, \lambda_K > 0$ and $n_1, \ldots, n_K \in \mathbb{N}$ such that (M, τ) has a representation as a direct sum

$$(M, \tau) = (M_{n_1}, \lambda_1 \operatorname{tr}) \oplus \ldots \oplus (M_{n_K}, \lambda_K \operatorname{tr}).$$

We want to apply (2) of lemma 6.13.
Assume for a moment that $\lambda_1, \ldots, \lambda_K \in \mathbb{N}$. Take $N = M$ and $\mathcal{N} = M_m$ with $m = \sum_{k=1}^K \lambda_k n_k$, endowed with the standard trace tr. Put

$$J : M \to M_m, \quad J(x_1 \oplus x_2 \oplus \ldots \oplus x_K) = \begin{pmatrix} x_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & x_1 & & & & \\ & & & \ddots & & & \\ & & & & x_K & & \\ & & & & & \ddots & \\ 0 & & & & & & x_K \end{pmatrix}.$$

Here, the multiplicity of the $x_k's$ on the diagonal of the big matrix is λ_k. Let $Q : M_m \to M$ be defined by $\langle Jx, y\rangle = \langle x, Qy\rangle$. J is completely positive by its simple structure, and then Q also is by lemma 6.4. J is a contraction, since $\|J(x)\| = \max_k \|x_k\| = \|x\|$. Q is a contraction, since $\|x\|_{L^1(M)} = \sum_k \lambda_k \|x_k\|_{S^1_{n_k}} = \|J(x)\|_{S^1_m}$. Finally, the identity $QJ = I_M$ is easy to check, so that the assumptions of lemma 6.13 are satisfied, and

$$\langle (\mathrm{Id}_{L^p(M)} - T_p)x, \widehat{x}\rangle = \langle (\mathrm{Id}_{M_m} - JTQ)Jx, \widehat{Jx}\rangle \overset{\text{thm } 6.12}{\in} \Sigma_p.$$

Assume now that $\lambda_k \in \mathbb{Q}_+$. Then let $t \in \mathbb{N}$ be the common denominator of the $\lambda_k's$. Put $m = t \sum_k \lambda_k n_k$ and $\mathcal{N} = M_m$ with the trace $t^{-1} \cdot \operatorname{tr}$. Use the same J as before, the multiplicity of the $a_k's$ being now $t\lambda_k$. We appeal again to lemma 6.13 (2). Note that the theorem 6.12 is also valid with the modified trace $t^{-1} \cdot \operatorname{tr}$.

The general case $\lambda_k \in \mathbb{R}_+^*$ follows by rational approximation: Let $x \in L^p(M,\tau)$ and $y = y_1 \oplus \ldots \oplus y_K := [(I_{L^p(M)} - T_p)x] \cdot \hat{x} \in L^1(M,\tau) = L^1(M_{n_1}, \lambda_1 \operatorname{tr}) \oplus_1 \ldots \oplus_1 L^1(M_{n_K}, \lambda_K \operatorname{tr})$. Then

$$\langle (I_{L^p(M)} - T_p)x, \hat{x} \rangle = \tau(y) = \sum_{k=1}^{K} \lambda_k \operatorname{tr}(y_k).$$

We already know that this quantity belongs to Σ_p for any $\lambda_1, \ldots, \lambda_K \in \mathbb{Q}_+$. Thus for $\lambda_1, \ldots, \lambda_K \in \mathbb{R}_+^*$, it belongs to $\overline{\Sigma_p} = \Sigma_p$, too.
2nd case: M is hyperfinite. There exists a net M_α of finite dimensional subalgebras of the kind as in the 1st case. Further, for every α, there exists $J_\alpha : M_\alpha \to M$ satisfying the assumptions of lemma 6.13. For every $1 < p < \infty$ and every $x \in L^p(M)$, $J_{\alpha,p} Q_{\alpha,p} x \overset{\alpha}{\to} x$ in $L^p(M)$ ([110, thm 3.4 and rem] and [124, p. 332]). Now (1) of lemma 6.13 yields that $Q_\alpha T J_\alpha$ is an operator as in the 1st case of the proof. Therefore for any $x \in L^p(M)$, by lemma 6.13

$$\begin{aligned} \langle (\operatorname{Id}_{L^p(M)} - T_p)x, \widehat{x} \rangle &= \lim_\alpha \langle (I - T_p) J_{\alpha,p} Q_{\alpha,p} x, (\widehat{J_{\alpha,p} Q_{\alpha,p}} x) \rangle \\ &= \lim_\alpha \langle (I - Q_{\alpha,p} T_p J_{\alpha,p}) Q_{\alpha,p} x, (\widehat{Q_{\alpha,p} x}) \rangle \\ &\in \Sigma_p. \end{aligned}$$

□

The following theorem now answers our question in section 6.4. In addition, the contractivity in 6.7 can be extended to complete contractivity.

Theorem 6.15 *Let M be a hyperfinite von Neumann algebra with s.n.f. trace τ and (T_t) a diffusion semigroup on (M,τ). Assume that for all $t > 0$, T_t satisfies $(\mathcal{P})$ (for example, T_t is completely positive). Then for all $1 < p < \infty$, $t \mapsto T_{t,p}$ has an analytic extension*

$$\Sigma_p' \to B(L^p(M)),\ z \mapsto T_{z,p}.$$

The operators $T_{z,p}$ are in addition completely contractive.

Proof. Proposition 6.6 together with theorem 6.14 gives the analytic extension and the contractivity. To show the complete contractivity, let $n \in \mathbb{N}$ and consider the space $N = M_n \otimes M$ with trace $\operatorname{tr} \otimes \tau$. Then $\tilde{T}_t := \operatorname{Id}_{M_n} \otimes T_t$ gives a diffusion semigroup on N. Further, $\tilde{T}_t$ inherits property $(\mathcal{P})$ from T_t : Indeed, if $W : M_2 \otimes M \to M_2 \otimes M$ is an "extension" of T_t as in the definition of $(\mathcal{P})$, then $I_{M_n} \otimes W : M_n \otimes (M_2 \otimes M) \cong M_2 \otimes (M_n \otimes M) \to M_n \otimes (M_2 \otimes M)$ is one of $\tilde{T}_t$. Let $\Sigma_p' \to B(L^p(N))$, $z \mapsto \tilde{T}_{z,p}$ be the analytic contractive extension of $\tilde{T}_{t,p}$. We claim that $\tilde{T}_{z,p} = \operatorname{Id}_{S_n^p} \otimes T_{z,p}$, where $T_{z,p}$ is the analytic extension of $T_{t,p}$. Indeed, by the equivalence of the norms $\|(x_{ij})_{ij}\|_{S_n^p(L^p(M))} \cong \sum_{ij} \|x_{ij}\|_{L^p(M)}$, one sees that $I_{S_n^p} \otimes T_z$ is analytic. Since $\tilde{T}_{z,p} = \operatorname{Id}_{S_n^p} \otimes T_{z,p}$ a priori for $z > 0$, the claim follows from the uniqueness theorem for analytic functions. Now the theorem follows from 6.2. □

6.6 Specific examples

We will now give some examples of diffusion semigroups (T_t) on hyperfinite von Neumann algebras which match the conditions of theorem 6.15. Recall that if for any $t > 0$, T_t is completely positive, then T_t satisfies $(\mathcal{P})$ and theorem 6.15 can be applied. In two specific cases to follow, the complete positivity is unnecessary.

6.6.1 Commutative case

We assume that $(M, \tau) = (L^\infty(\Omega), \mu)$ is a commutative von Neumann algebra. Then our definition 6.5 of a diffusion semigroup reduces to the classical one given in [121].
For any operator $T : L^\infty(\Omega) \to L^\infty(\Omega)$ or $T : L^1(\Omega) \to L^1(\Omega)$, $\|T\| = \|T\|_{cb}$. This is false in general for operators $T : L^p(\Omega) \to L^p(\Omega)$ with $1 < p < \infty$. The property $(\mathcal{P})$ has now a simple characterization.

Proposition 6.16 *A w^*-continuous operator $T : L^\infty(\Omega) \to L^\infty(\Omega)$ satisfies $(\mathcal{P})$ if and only if T is contractive and self-adjoint.*

Proof. The "only if" part follows from remark 6.10. For the "if" part, we assume that $L^\infty(\Omega) = \ell^\infty_n$ for some $n \in \mathbb{N}$. The general case can be deduced by an approximation argument as in theorem 6.14, using the semi-partitions of (Ω, μ) explained in section 6.2.

We identify T and T_* with matrices (t_{ij}) and $(\overline{t_{ij}})$. Since (t_{ij}) is self-adjoint, $(\overline{t_{ij}})$ and $(|t_{ij}|)$ are self-adjoint also. Hence

$$W = \begin{pmatrix} (|t_{ij}|) & (t_{ij}) \\ (\overline{t_{ji}}) & (|t_{ij}|) \end{pmatrix} : M_2(\ell^\infty_n) \to M_2(\ell^\infty_n)$$

is self-adjoint.

We show that W is completely positive. Let $\tilde{J} : \ell^\infty_n \hookrightarrow M_n$ be the embedding into the diagonal, $J = \mathrm{Id}_{M_2} \otimes \tilde{J} : M_2(\ell^\infty_n) \hookrightarrow M_2(M_n)$ and $P : M_2(M_n) \to M_2(\ell^\infty_n)$ its adjoint. Further, let $\phi_{ij} \in \mathbb{C}$ such that $t_{ij} = |t_{ij}|\phi_{ij}$. Denote $a_{ij} = \begin{pmatrix} \sqrt{|t_{ij}|}\phi_{ij} E_{ij} & 0 \\ 0 & \sqrt{|t_{ij}|} E_{ij} \end{pmatrix} \in M_2(M_n)$, where $(E_{ij})_{ij}$ is the canonical basis in M_n. Then $x \mapsto \sum_{i,j} a_{ij} x a_{ij}^*$ is completely positive by Choi's theorem 6.3. On the other hand, this mapping equals JWP. Indeed,

$$\begin{aligned}
&\sum_{i,j} \begin{pmatrix} \sqrt{|t_{ij}|}\phi_{ij} E_{ij} & 0 \\ 0 & \sqrt{|t_{ij}|} E_{ij} \end{pmatrix} \begin{pmatrix} x^{(11)} & x^{(12)} \\ x^{(21)} & x^{(22)} \end{pmatrix} \begin{pmatrix} \overline{\phi_{ij}} \sqrt{|t_{ij}|} E_{ji} & 0 \\ 0 & \sqrt{|t_{ij}|} E_{ji} \end{pmatrix} \\
&= \sum_{i,j} \begin{pmatrix} |t_{ij}| E_{ij} x^{(11)} E_{ji} & t_{ij} E_{ij} x^{(12)} E_{ji} \\ \overline{t_{ij}} E_{ij} x^{(21)} E_{ji} & |t_{ij}| E_{ij} x^{(22)} E_{ji} \end{pmatrix} \\
&= \sum_{i,j} \begin{pmatrix} |t_{ij}| x^{(11)}_{jj} E_{ii} & t_{ij} x^{(12)}_{jj} E_{ii} \\ \overline{t_{ij}} x^{(21)}_{jj} E_{ii} & |t_{ij}| x^{(22)}_{jj} E_{ii} \end{pmatrix} \\
&= JWPx.
\end{aligned}$$

Then $W = P(JWP)J$ is also completely positive.

As $\|T\|$ is given by $\sup_i \sum_j |t_{ij}|$, which does only depend on the absolute values of t_{ij}, we have $\|(|t_{ij}|)\| = \|T\|$. This implies $\left\| W \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\| \leqslant \|(|t_{ij}|)\| \leqslant 1$, and thus, W is completely contractive. □

As a corollary, we obtain [92, cor 3.2], but without the assumption of positivity.

Corollary 6.17 *Let (T_t) be a diffusion semigroup on $L^\infty(\Omega)$, i.e. the $T_{t,p}$ form consistent contractive c_0-semigroups on $L^p(\Omega)$ for $1 \leqslant p < \infty$ (w^*-continuous on $L^\infty(\Omega)$) such that $T_{t,2}$ are self-adjoint. Then for $1 < p < \infty$, $t \mapsto T_{t,p}$ has an analytic and contractive extension to*

$$\Sigma'_p = \left\{ z \in \mathbb{C}\backslash\{0\} : \, |\arg z| < \frac{\pi}{2} - \arctan \frac{|p-2|}{2\sqrt{p-1}} \right\}.$$

Proof. Recall that $L^\infty(\Omega)$ is a hyperfinite von Neumann algebra. By proposition 6.16, T_t satisfies $(\mathcal{P})$ for all $t > 0$, so that we can appeal to theorem 6.15. □

Remark 6.18 *In [80], [92, cor 3.2] is used to improve the angle of the H^∞ calculus of generators of commutative diffusion semigroups consisting of positive operators. With the above corollary, [80] gives the same angle improvement without the positivity assumption.*

6.6.2 Schur multipliers

A further example of non-commutative diffusion semigroups are the Schur multiplier semigroups, considered in [65, chap 8]. The underlying von Neumann algebra is $M = B(\ell^2(\mathbb{N})) = B(\ell^2)$, with the usual trace tr. We identify $B(\ell^2(\mathbb{N}))$ with some subspace of $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ in the usual way. Let $(t_{ij})_{ij} \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$. The Schur multiplier T associated with $(t_{ij})_{ij}$ is defined in the following way: If $x = (x_{ij})_{ij} \in B(\ell^2)$ then

$$Tx = (t_{ij}x_{ij})_{ij}. \tag{6.12}$$

Of course, it is not sure that $Tx \in B(\ell^2)$ nor that $T \in B(M)$. The following proposition characterizes, when the latter is the case. For a proof, see [107, cor 8.8].

Proposition 6.19 *Let T be given by 6.12. The following are equivalent.*

- *There exists a Hilbert space H and sequences $(x_i)_i$, $(y_i)_i \subset H$ such that $\sup_i \|x_i\| \leqslant 1$, $\sup_i \|y_i\| \leqslant 1$ and $t_{ij} = \langle x_i, y_j \rangle_H$.*
- *The Schur multiplier T is a bounded operator on M and $\|T\| \leqslant 1$.*
- *The Schur multiplier T is a completely bounded operator on M and $\|T\|_{cb} \leqslant 1$.*

Assume now that the conditions of the above proposition are satisfied. Then for $x, y \in S^1 \cap B(\ell^2)$, $\langle Tx, y^* \rangle = \operatorname{tr}(Txy^*) = \sum_{i,j=1}^\infty t_{ij}x_{ij}\overline{y_{ij}}$. Therefore, T is self-adjoint if and only if $t_{ij} \in \mathbb{R}$ for all $i, j \in \mathbb{N}$.

Lemma 6.20 *A Schur multiplier $T : B(\ell^2) \to B(\ell^2)$ satisfies $(\mathcal{P})$ if and only if T is contractive and self-adjoint.*

Proof. Only the "if" part has to be shown. Let $(x_i)_i, (y_i)_i \subset H$ be the sequences given as in proposition 6.19. By the self-adjointness of T, we know that $\langle x_i, y_j \rangle_H \in \mathbb{R}$. We may suppose that $\langle x_i, x_j \rangle_H$, $\langle y_i, y_j \rangle_H \in \mathbb{R}$.

Indeed, if this is not the case, let $(e_\gamma)_\gamma$ be an orthonormal basis of H and consider the $\mathbb{R}$-linear mapping

$$J : \begin{cases} H & \longrightarrow H \oplus_2 H \\ e_\gamma & \longmapsto e_\gamma \oplus 0 \\ ie_\gamma & \longmapsto 0 \oplus e_\gamma \end{cases} .$$

For $x \in H$, write $x = x_R + ix_I$, where x_R and x_I are in the real span of the $e_\gamma' s$. In the same manner, write $y = y_R + iy_I$. Then

$$\langle x, y \rangle_H = \langle x_R, y_R \rangle_H + \langle x_I, y_I \rangle_H + i \langle x_I, y_R \rangle_H - i \langle x_R, y_I \rangle_H,$$

so if $\langle x, y \rangle_H \in \mathbb{R}$, then $\langle J(x), J(y) \rangle_{H \oplus H} = \langle x, y \rangle_H$. Replace now x_i and y_i by $J(x_i)$ and $J(y_i)$. Then, we still have $t_{ij} = \langle J(x_i), J(y_j) \rangle$, and in addition $\langle x_i, x_j \rangle_H$, $\langle y_i, y_j \rangle_H \in \mathbb{R}$.

The operator W as in definition 6.9 that we will give in a moment acts on the space $M_2 \otimes B(\ell^2)$. We wish to consider Schur multipliers on this space and do this in virtue of the natural identification $M_2 \otimes B(\ell^2) \cong B(\ell^2(\mathbb{N} \times \{1,2\}))$. Note that T_* is the Schur multiplier associated with $(\langle y_i, x_j \rangle_H)_{ij}$. Further, by proposition 6.19, the Schur multipliers S_1 and S_2 associated with $(\langle x_i, x_j \rangle_H)_{ij}$ and $(\langle y_i, y_j \rangle)_{ij}$ are completely contractive. We put $W = \begin{pmatrix} S_1 & T \\ T_* & S_2 \end{pmatrix}$. This is a Schur multiplier on $M_2 \otimes B(\ell^2) \cong B(\ell^2(\mathbb{N} \times \{1,2\}))$ associated with the matrix

$$(\langle z_{(ik)}, z_{(jl)} \rangle_H)_{(ik),(jl) \in \mathbb{N} \times \{1,2\}},$$

where $z_{(ik)} = \begin{cases} x_i, & k = 1 \\ y_i, & k = 2 \end{cases}$. Therefore, W is completely positive [107, ex 8.7]. The (complete) contractivity of W is clear from proposition 6.19. Finally, as $\langle z_{(ik)}, z_{(jl)} \rangle_H \in \mathbb{R}$, W is self-adjoint. □

Now assume that (T_t) is a diffusion semigroup on M such that for any $t > 0$, T_t is a Schur multiplier associated to some $(t_{ij}^{(t)})_{ij} \in \mathbb{C}^{\mathbb{N} \times \mathbb{N}}$. For example, if H is a Hilbert space and $(\alpha_k)_{k \in \mathbb{N}}$ and $(\beta_k)_{k \in \mathbb{N}}$ are sequences in H, then the Schur multipliers T_t associated with $(e^{-t\|\alpha_i - \beta_j\|})_{ij}$ form such a diffusion semigroup [65, prop 8.17]. Then the above lemma and theorem 6.15 show that for any $1 < p < \infty$, $(T_{t,p})_{t>0}$ admits an analytic extension

$$\Sigma_p' \to B(L^p),\ z \mapsto T_{z,p}.$$

Further, by the uniqueness of analytic vector valued functions, $T_{z,p}$ is again a Schur multiplier for any $z \in \Sigma_p'$.

6.7 Semi-commutative diffusion semigroups

At the end, we give an example of a diffusion semigroup on a von Neumann algebra without the assumption of hyperfiniteness. Let (Ω, μ) be a measure space and (N, σ) a von Neumann algebra with s.n.f. trace. Suppose we are given a diffusion semigroup (T_t) on $L^\infty(\Omega)$. By the w^*-continuity of any T_t, we can define the contractions

$$T_t^N := T_t \overline{\otimes} I_N : L^\infty(\Omega)\overline{\otimes} N \to L^\infty(\Omega)\overline{\otimes} N.$$

(T_t^N) is a diffusion semigroup on $L^\infty(\Omega)\overline{\otimes} N$, and called semi-commutative diffusion semigroup.

Now assume that N has the QWEP property. This means that N is the quotient of a C^*-algebra having the weak expectation property (WEP) introduced in [84, 85]. It is unknown whether every von Neumann algebra has this property.

Recall the following notion of an ultraproduct of Banach spaces. Let $(X_\alpha)_{\alpha\in I}$ be a family of Banach spaces and $\mathcal{U}$ an ultrafilter on I. We will only need the case $X_\alpha = X$, a fixed Banach space. Consider the quotient space

$$\ell^\infty(I;X_\alpha) = \{(x_\alpha)_\alpha \in \prod_\alpha X_\alpha : \sup_\alpha \|x_\alpha\| < \infty\}$$

and the subspace

$$c_0(\mathcal{U};X_\alpha) = \{(x_\alpha)_\alpha \in \ell^\infty(I;X_\alpha) : \lim_{\mathcal{U}} \|x_\alpha\| = 0\}.$$

Then $\prod_{\mathcal{U}} X_\alpha = \ell^\infty(I;X_\alpha)/c_0(\mathcal{U};X_\alpha)$ is called an ultraproduct, see also [113, p. 59].

We will need a property of $L^p(N,\sigma)$ which appears in [64].

Proposition 6.21 *Let N be a von Neumann algebra with QWEP having a s.n.f. trace σ. Then there exists a Hilbert space H, an ultrafilter $\mathcal{U}$ on some index set I and an isometric embedding $J : L^p(N) \to \prod_{\mathcal{U}} S^p(H)$.*

The following proposition follows from [64, thm 2.10]. We include a simple proof for the convenience of the reader.

Proposition 6.22 *Let $1 < p < \infty$, $L^p(\Omega)$ be some commutative L^p-space and $T \in B(L^p(\Omega))$ be completely bounded. Let N be a von Neumann algebra with QWEP with a s.n.f. trace σ. Then $T \otimes \mathrm{Id}_{L^p(N)}$, initially defined on $L^p(\Omega)\otimes L^p(N)$, extends to $L^p(\Omega, L^p(N))$ and*

$$\|T\overline{\otimes}\,\mathrm{Id}_{L^p(N)} : L^p(\Omega, L^p(N)) \to L^p(\Omega, L^p(N))\| \leqslant \|T\|_{cb}.$$

Proof. By 6.2, for every $n \in \mathbb{N}$,

$$\|T \otimes \mathrm{Id}_{S_n^p} : L^p(\Omega, S_n^p) \to L^p(\Omega, S_n^p)\| \leqslant \|T\|_{cb}.$$

As in [110, prop 2.4], we deduce via a density argument that $\|T\overline{\otimes}\,\mathrm{Id}_{S^p(H)} : L^p(\Omega, S^p(H)) \to L^p(\Omega, S^p(H))\| \leqslant \|T\|_{cb}$.

Let $H, \mathcal{U}, I, J$ be as in proposition 6.21. We denote $(x_\alpha)_\alpha$ and $(f_\alpha)_\alpha$ elements of the ultraproduct spaces $\prod_{\mathcal{U}} S^p(H)$ and $\prod_{\mathcal{U}} L^p(\Omega, S^p(H))$. Consider the ultraproduct mapping

$$S : \prod_{\mathcal{U}} L^p(\Omega, S^p(H)) \to \prod_{\mathcal{U}} L^p(\Omega, S^p(H)),\ (f_\alpha)_\alpha \mapsto ((T\overline{\otimes}\,\mathrm{Id}_{S^p(H)})(f_\alpha))_\alpha.$$

Note that the space $L^p(\Omega, \prod_{\mathcal{U}} S^p(H))$ is isometrically embedded in $\prod_{\mathcal{U}} L^p(\Omega, S^p(H))$, via a mapping taking a step function $\sum_k f_k \otimes (x_{k,\alpha})_\alpha$ to the element $(\sum_k f_k \otimes x_{k,\alpha})_\alpha$. With this embedding, $S(L^p(\Omega, \prod_{\mathcal{U}} S^p(H))) \subset L^p(\Omega, \prod_{\mathcal{U}} S^p(H))$, and $\tilde{S} = S|_{L^p(\Omega, \prod_{\mathcal{U}} S^p(H))}$ is again a contraction, since $\|T\|_{cb} \leqslant 1$. Now use proposition 6.21 to restrict $\tilde{S}$ to $L^p(\Omega, L^p(N))$. This restriction equals $T \overline{\otimes} \mathrm{Id}_{L^p(N)}$, which is thus a contraction, as desired. □

Corollary 6.23 *Let $(T_t^N) = (T_t \overline{\otimes} I_N)$ be a semi-commutative diffusion semigroup as above. Then for $1 < p < \infty$, $t \mapsto T_{t,p}^N$ has an analytic and completely contractive extension to Σ'_p.*

Proof. By proposition 6.16, T_t satisfies $(\mathcal{P})$ and theorem 6.15 gives the completely contractive analytic extension $z \mapsto T_{z,p}$ on Σ'_p. Now appeal to proposition 6.22 to get the contractive operators $T_{z,p} \overline{\otimes} \mathrm{Id}_{L^p(N)}$. It is clear that the latter form an analytic extension of $T_{t,p}^N$. Replacing T_t by $I_{M_n} \otimes T_t$ in this argument gives the completely contractive result. □

Remark 6.24 *There is even a more general version of proposition 6.22, [64, thm 2.10]. From this, we deduce that if M is a hyperfinite von Neumann algebra with s.n.f. trace τ and $T : L^p(M) \to L^p(M)$ is completely contractive, then $T \overline{\otimes} \mathrm{Id}_{L^p(N)} : L^p(M \overline{\otimes} N) \to L^p(M \overline{\otimes} N)$ is completely contractive.*

With this generalization, one also gets the following result: If (T_t) is a diffusion semigroup on a hyperfinite von Neumann algebra such that T_t satisfies $(\mathcal{P})$ for all $t > 0$, then $T_t^N = T_t \overline{\otimes} \mathrm{Id}_N$ forms a diffusion semigroup and has an analytic and completely contractive extension to Σ'_p.

Corollary 6.23 allows us to generalize proposition 6.8, which was our starting observation, to the non-commutative case.

Corollary 6.25 *Let (N, σ) be a QWEP von Neumann algebra and $a, b \in L^p(N)$. Then*

$$\langle a - b, \widehat{a} - \widehat{b} \rangle = \|a\|_p^p + \|b\|_p^p - \mathrm{tr}(b|a|^{p-1}u_a) - \mathrm{tr}(a|b|^{p-1}u_b) \in \Sigma_p.$$

Here, $a = u_a|a|$ and $b = u_b|b|$ are the polar decompositions.

Proof. Let (T_t) be the diffusion semigroup on ℓ_2^∞ as in example 6.7, i.e. $T_t = e^{tA}$ with

$$A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Consider the semi-commutative semigroup $(T_t \overline{\otimes} \mathrm{Id}_N)$ with (bounded) generator $A_p = A \otimes \mathrm{Id}_{L^p(N)}$ on $L^p(\ell_2^\infty \otimes N)$ and define the element x in this space by $x = (a, b)$. Its dual element is given by $\widehat{x} = (\widehat{a}, \widehat{b})$. By corollary 6.23 and proposition 6.6,

$$\langle a - b, \widehat{a} - \widehat{b} \rangle = -\langle A_p x, \widehat{x} \rangle \in \Sigma_p.$$

□

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