First published in:

Schmalenbach Business Review ◆ Vol. 52 ◆ January 2000 ◆ pp. 26–44

Alexander Kempf/Marliese Uhrig-Homburg*

LIQUIDITY AND ITS IMPACT ON BOND PRICES

ABSTRACT

In this paper, we propose a theoretical continuous-time model to analyze the impact of liquidity on bond prices. This model prices illiquid bonds relative to liquid bonds and provides a testable theory of illiquidity induced price discounts. The model is tested using 1992–1994 data from bonds issued by the German government. These bonds define a market segment that is homogeneous in bankruptcy risk, taxes, age, and coupons, but the bonds differ with respect to their liquidity. The empirical findings suggest that bond prices not only depend on the dynamics of interest rates, but also on the liquidity of bonds. Therefore, bond liquidity should be used as an additional pricing factor. The findings of the out-of-sample test demonstrate the superiority of the model in comparison with traditional pricing models.

1 INTRODUCTION

One of the core results of asset pricing theory is the tradeoff between risk and expected return. Typically, risk is modeled as a price risk caused by only a few factors that influence all assets in common. An example is the *Capital Asset Pricing Model*[1] (CAPM) with the return of the market portfolio as the only factor. Another example is the general equilibrium model of *Cox/Ingersoll/Ross* (1985a), in which the factors do not have to be specified until the model is actually implemented.

However, these neoclassical models do not allow for individual factors that influence only a subset of assets. The liquidity of an asset is just such an individual factor. Its importance is well documented in the literature. Empirical studies suggest that the (risk-adjusted) average return of an asset increases with its illiquidity. This result is reported for both stock[2] and bond[3] markets.

* Prof. Dr. *Alexander Kempf,* Seminar für Finanzierung, Universität zu Köln, 50923 Köln, and Dr. *Marliese Uhrig-Homburg*, Lehrstuhl für Finanzierung, Universität Mannheim, 68131 Mannheim. We thank Dr. *Horst Bienert*, Prof. Dr. *Thomas Braun*, Prof. Dr. *Wolfgang Bühler*, Dr. *Dieter Hess*, Dr. *Stefan Jaschke*, Prof. Dr. *Steffen Rasch*, Prof. Dr. *Rainer Schöbel*, Dr. *Michael Schulze*, Prof. *Richard Stehle* Ph. D. and the participants of the annual meeting of the *Verein für Socialpolitik* 1997, the annual meeting of the *Deutsche Gesellschaft für Finanzwirtschaft* 1997, the *Financial Management Association* Meeting 1997, the *Southern Finance Association* Meeting 1997, and the annual meeting of the *Verband der Hochschullehrer* 1998 for helpful comments.

1 See, e. g., *Sharpe* (1964).

2 See, e. g., *Amihud/Mendelson* (1986), *Silber* (1991), *Kadlec/McConnell* (1994), and *Brennan/Subrahmanyam* (1996).

3 See, e. g., *Sarig/Warga* (1989), *Amihud/Mendelson* (1991), *Boudoukh/Whitelaw* (1991), *Warga* (1992), and *Kamara* (1994).

The additional return of illiquid assets compensates for the risks of holding illiquid assets. A typical risk for an investor who holds illiquid assets is that it might be impossible to sell the assets rapidly, making possible for an investor to run out of cash. A second form of liquidity risk is that selling an illiquid asset might cause the transaction price to drop. Furthermore, illiquid assets often cannot be used in the repurchase market.

Though the tradeoff between illiquidity and expected return has received growing attention in the empirical literature and from market participants, there are few theoretical models that analyze the impact of illiquidity on returns. Focusing on the stock market, *Amihud/Mendelson* (1986) show that because rational investors prefer liquid stocks that can be traded immediately at lower costs, the equilibrium expected return is higher for illiquid stocks than for liquid stocks. *Longstaff* (1995) argues that liquid stocks offer a valuable option to trade. Consequently, liquid stocks trade at higher prices than comparable illiquid stocks.

Models that analyze the impact of liquidity on bond prices are more closely related to our approach. *Grinblatt* (1995) develops a pricing model in which liquidity considerations generate interest-rate swap spreads. He argues that for liquidity reasons, the yield of a Treasury note is lower than a corresponding yield associated with the fixed rate of a swap. *Grinblatt* applies the valuation technique developed for derivatives and adds a liquidity factor to traditional models of the yield curve. Basically, he assumes that holding Treasury notes yields an additional continuous cash flow because of the liquidity of Treasuries. The motivation of this cash flow is that Treasuries earn an additional interest in the repurchase market.

On the theoretical side, our paper proposes an extension of *Grinblatt*'s (1995) approach. Our approach is similar to *Grinblatt* in the sense that we price fixed-income securities within a two-factor *Cox/Ingersoll/Ross*-type model [4]. These factors are the short-term interest rate and a second exogenous factor that accounts for securities' liquidity. Unlike *Grinblatt* (1995), who models the prices of liquid and illiquid securities endogenously, we price illiquid bonds relative to liquid ones. Thus, we can develop a testable model of the impact of illiquidity on bond prices that is not affected by pricing errors of perfectly liquid bonds. Therefore, we extend *Grinblatt's* model by introducing time-dependent parameters that change the non-market-fitting model of *Grinblatt* into a perfect fitting model for the liquid segment [5].

The empirical contribution of our paper is to analyze the impact of liquidity on bond prices for the German bond market. Not only do we measure this impact, but we are also able to extract determinants of this impact that can be used to forecast prices. An advantage of our empirical approach is that our model focuses on bond markets rather than swap spreads. The latter may reflect not only liquidity, but also default risk, while the Government segment of the bond market is essentially default-risk-free. Thus, this segment is ideally suited to separate the impact of liquidity from the impact of default risk.

4 See *Cox/Ingersoll/Ross* (1985 b).
5 *Hull/White* (1990) propose such a procedure.

The paper is organized as follows. In Section 2 we develop the theoretical model. Section 3 describes the empirical implementation. We present the empirical results in Section 4. Section 5 concludes.

2 Modeling the Liquidity Spread

In this section, we develop a simple continuous-time valuation framework for illiquid bonds that allows for both liquidity risk and interest-rate risk. Following the lines of *Grinblatt* (1995), we model two stochastic factors, the instantaneous risk-free rate of a liquid investment and an as yet unspecified factor that accounts for the price differences between liquid and illiquid bonds. In contrast to *Grinblatt,* we price illiquid bonds relative to liquid ones.

The dynamics of the short rate r are given by

$$dr = \alpha(\gamma - r)dt + \sigma\sqrt{r}dz, \tag{1}$$

where α, γ, and σ are constants, and z is a standard Wiener process. This assumption is drawn from the term structure model proposed by *Cox/Ingersoll/Ross* (1985b). The specification captures many of the observed features of the short-rate behavior, such as mean-reversion, non-negative interest rates, and conditional volatility depending on the level of the short rate. In addition, the model is analytically tractable[6].

The second state variable x generates the liquidity-based discount of illiquid bonds. The state variable is assumed to follow the diffusion process

$$dx = \kappa(\theta - x)dt + \varsigma\sqrt{x}dw, \tag{2}$$

where κ, θ, and ς are constants, and w is a standard Wiener process. We consider x as a pay-off rate that must be paid continuously by the investor due to the illiquidity of the instrument. Any disadvantage of an illiquid investment in comparison with a liquid one would be included in this outflow. Examples are higher costs of immediacy and opportunity costs due to the lack of an active repurchase market.

The primary sources of these costs are not modeled explicitly here. Rather, we assume that the overall illiquidity costs can be approximated adequately by (2). This diffusion process aggregates the various stochastic events that are important for the actual costs of illiquidity at some point in time[7]. We assume costs that are changing stochastically over time to reflect varying repurchase rates and bid-ask spreads due to demand and supply changes. The square root process (2) precludes negative values and is therefore an appropriate assumption for illiqui-

6 See *Cox/Ingersoll/Ross* (1985 b).

7 See also *Grinblatt* (1995).

dity costs. In addition, the mean-reverting drift reflects that the costs of illiquidity are fluctuating randomly around some mean level θ.

In addition, we assume that the instantaneous correlation between *dz* and *dw* is zero. This assumption could easily be relaxed to allow for nonzero correlation between *dz* and *dw*, although the resulting pricing formula for illiquid bond prices would be more complex. Empirical validation of the model is needed to determine whether this additional complexity is warranted.

Apart from the costs of holding illiquid bonds, we assume perfect, frictionless, arbitrage-free markets in which securities trade in continuous time. Thus, we can express the bond prices as functions of the state variables. The state variables we consider are the short-term interest rate r, which influences all bond prices, and the illiquidity factor x, which influences only the prices of illiquid bonds.

In this framework, the price $L(r,t,T)$ of a liquid zero coupon bond with maturity in T is the solution to the partial differential equation

$$L_t + L_r[\alpha(\gamma - r) - \lambda\sigma r] + L_{rr}\frac{\sigma^2}{2}r = rL. \tag{3}$$

Subscripts denote partial derivatives. λ is a constant parameter related to the market price of interest-rate risk that depends on the preferences of the market participants.

A solution of this parabolic partial differential equation that satisfies the initial condition $L(r,T,T)$ for all r,T is given by *Cox/Ingersoll/Ross* (1985b) as

$$L(r,t,T) = A(t,T)e^{-B(t,T)r}, \tag{4}$$

with the functions $A(t,T)$ and $B(t,T)$ depending on the model parameters and residual maturity. For explicit expressions see *Cox/Ingersoll/Ross* (1985b), p. 393. The prices of coupon bonds can be easily obtained as portfolios of discount bonds.

The price of an *illiquid* interest-rate contingent claim depends not only on the short-term rate r but also on the factor x. It can be shown that the price $V(r,x,t,T)$ at time t of any *illiquid* discount bond with payoff at time T is the solution of the partial differential equation

$$V_t + V_r[\alpha(\gamma - r) - \lambda\sigma r] + V_x[\kappa(\theta - x) - \varphi\varsigma x] + V_{rr}\frac{\sigma^2}{2}r + V_{xx}\frac{\varsigma^2}{2}x = (r + x)V \tag{5}$$

with the initial condition $V(r,x,T,T) = 1$. λ and φ are the constant parameters related to the market price of interest-rate risk and liquidity risk, respectively[8]. As

8 In the following we refer to the parameters λ and φ shortly as market prices of interest-rate and liquidity risk, respectively.

the short-term rate, the liquidity factor is not a traded asset. Therefore, a second preference-dependent parameter, the market price of illiquidity risk φ, appears in the valuation equation. The basic difference between the valuation equation and standard two-factor term structure models is that the illiquid investment appreciates (adjusted for risk) at the instantaneous rate $(r+x)dt$, rather than at rdt, so an investor requires an additional instantaneous return of xdt due to the illiquidity of the bond.

Following a separation of variables approach, the partial differential equation (5) can be decomposed in two partial differential equations with only one state variable each. This leads to

$$V(r,x,t,T) = L(r,t,T)D(x,t,T). \tag{6}$$

Equation (6) has an intuitive (multiplicative) structure. The first term, $L(r,t,T)$ represents the value that the zero coupon bond would have if it were liquid. The second term, $D(x,t,T)$, adjusts for the illiquidity of the bond. This illiquidity function is expressed by a classical one-factor *Cox/Ingersoll/Ross*-model, the solution of which is given by

$$D(x,t,T) = G(t,T)e^{-H(t,T)x} \tag{7}$$

with

$$G(t,T) = \left[\frac{2\chi e^{(\kappa+\varphi\varsigma+\chi)\frac{T-t}{2}}}{(\kappa+\varphi\varsigma+\chi)(e^{\chi(T-t)}-1)+2\chi}\right]^{\frac{2\kappa\theta}{\varsigma^2}} \tag{8}$$

$$H(t,T) = \frac{2[e^{\chi(T-t)}-1]}{(\kappa+\varphi\varsigma+\chi)(e^{\chi(T-t)}-1)+2\chi} \tag{9}$$

$$\chi = \sqrt{(\kappa+\varphi\varsigma)^2+2\varsigma^2}. \tag{10}$$

The prices of illiquid coupon bonds can be obtained as portfolios of illiquid discount bonds.

Obviously, taking into account bond liquidity causes an increased complexity of the valuation model. To keep complexity at an acceptable level, we assume that an asset is considered either as liquid or illiquid, thus ignoring the real-world fuzziness of this feature. However, when an asset is classified as illiquid, we do not assume that this leads to a deterministic price reduction. Rather, the price of an illiquid asset depends on the stochastic state variable x.

Apart from technical details, the model presented so far corresponds to the approach of *Grinblatt* (1995). In the following, we take a different approach. Rather than determining the current values of liquid bonds endogenously, we are interested in pricing illiquid bonds relative to the prices of liquid bonds. Such an approach seeks to explain the price discount due to illiquidity given the prices of liquid bonds. Therefore, we need a model that is able to reproduce an arbitrary market yield curve within the liquid segment.

We follow the lines of *Hull/White* (1990) to turn the above non-market-fitting model into a model that exactly fits an exogenous term structure of liquid discount bonds. We do this by allowing for a time-dependent parameter $\lambda=\lambda(t)$ within the market price of interest-rate risk[9]. Given the parameters describing the short-rate process (1), we use a numerical procedure to determine $\lambda(t)$ in such a way that the model perfectly matches the current yield curve of liquid bonds. The basic structure of formula (4) still holds, although the time dependency involves more complex expressions for $A(t,T)$ and $B(t,T)$. These expressions must be solved numerically.

We obtain the values of illiquid bonds as a solution to the fundamental valuation equation (5), generalized through the time-dependent function $\lambda(t)$. The structure (6) of this solution and the valuation expressions (7) – (10) remain unchanged. As a result we obtain a valuation framework for illiquid instruments that takes as exogenous the initial term structure of interest rates generated by liquid bonds. This is the main difference between our approach and that of *Grinblatt* (1995).

3 Design of the Empirical Study

The empirical implementation of the model described in Section 2 requires four steps. First, we need a homogeneous market segment in which bond prices differ with respect to their liquidity [see Section 3.1]. Second, we must classify the bonds in the market segment according to their liquidity [see Section 3.2]. Third, we must test the empirical quality of the classification [see Section 3.3]. Fourth, we must estimate the parameters necessary for testing the model on the basis of the prices of bonds being classified as liquid and illiquid, respectively [see Section 3.4].

3.1 Data

Our model explains price differences between liquid and illiquid bonds. To test our model, we need bonds that differ only in their liquidity. To avoid return differences due to differences in default risk, we restrict ourselves to bonds issued by the German Government (BUND) and its state-operated funds (BAHN, POST). The bonds can be viewed as default-risk free. To form a homogeneous market segment, we further restrict the data set to non-callable straight bonds with fixed

9 Alternatively, we could choose to make the other model parameters time dependent. For economic and technical reasons, we choose to make the market price of interest-rate risk time dependent. See *Uhrig/Walter* (1996) for a discussion of this topic.

coupons, identical tax treatment, and initial time to maturity of ten years, which is the typical time to maturity of German Government bonds. The non-callable restriction avoids any bias caused by implicit call options. Restricting the analysis to bonds with identical initial maturity dates ensures that bonds with identical residual maturity are of the same age. Hence, we also avoid a possible bias due to effects of aging.

In total, 143 bonds satisfy the criteria for our sample period, January 1992 through December 1994. Using daily observations, 755 quotes are available for each bond. The bond prices used in this study result from the daily noon auction carried out at the Frankfurt Exchange, which is the largest in Germany. The data are provided by the *German Financial Data Base* in Mannheim.

3.2 Classification of Bonds

In our second step, the bonds described above are classified according to their liquidity. A straightforward approach is to estimate the liquidity of the bonds directly by using measures of liquidity, e. g., the bid-ask spread[10]. However, since bid-ask spreads are not available in our sample, our classification must be based on other observable indicators of liquidity.

For instance, *Elton/Green* (1998) use the trading volume of a bond as such an indicator. Our data set contains only information on volume traded at the Frankfurt Exchange. Since only a small part of total trading volume is carried out via an exchange in Germany, Frankfurt Exchange trading volume might not be a good proxy for total trading volume and liquidity. *Lassak* (1991) suggests measuring the liquidity of a bond by the number of trading days with zero trading volume for the bond and by the price jumps after a non-trading period. This approach is based on the idea that liquid bonds are traded more frequently and with smaller price changes than are illiquid bonds. However, this approach is inappropriate to our study, since nearly all the bonds in our sample are traded on every trading day. Therefore, the measure of *Lassak* is reduced to a measure of absolute price changes. Since our model should explain bond prices, measuring liquidity on the basis of absolute price changes would obviously lead to biased test results.

The classification used here follows *Warga* (1992), who reports that the age of a bond and the amount of bond issue[11] still available in the market are the best proxies for bond liquidity. Since the German government does not repurchase bonds, the amount outstanding equals the issue size of a bond. Therefore, we proxy bond liquidity with issue size.

To control for age effects, we only compare bonds that are issued within the same year. The bond with the maximum issue size of each year is classified as a liquid bond, the one with minimum issue size as an illiquid bond. All remaining bonds belong to a control group. Bonds with the same issue size are assigned to the

10 For a discussion of liquidity measures, see *Kempf* (1998).

11 See also *Fisher* (1959) and *Tanner/Kochin* (1971).

same group. Consequently, more than one bond of an issue year could be classified as liquid or illiquid.

Once a bond is classified as liquid, this classification persists for the entire life of the bond. In general, the liquidity of a bond reduces over time since a growing part of the issue is held by investors until maturity and is, therefore, not available for trading purposes. However, it seems reasonable to assume that the relative liquidity advantage of bonds with high issue size compared to those with low issue size remains. After summarizing the classification results in *Table 1* we empirically test this hypothesis.

Table 1: Classification Results

Year of issue	Number of liquid bonds (Issue size in million DM)	Number of illiquid bonds (Issue size in million DM)	Number of remaining bonds	Coupon range of liquid bonds	Coupon range of illiquid bonds
1982	8 (1,600)	2 (850)	4	7.75% – 9.75%	8% – 9.5%
1983	8 (1,600)	1 (800)	3	7.5% – 8.25%	8.5%
1984	8 (2,000)	2 (875)	2	7% – 8.25%	7.25% – 8.25%
1985	5 (2,500)	1 (800)	6	6.5% – 7.5%	7%
1986	2 (4,000)	1 (850)	6	5.75% – 6.5%	6.375%
1987	7 (4,000)	1 (900)	8	5.5% – 6.75%	6.25%
1988	1 (5,000)	3 (2,000)	7	6.375%	6.25% – 6.625%
1989	1 (5,000)	1 (2,000)	7	6.5%	7%
1990	1 (17,000)	1 (2,000)	7	9%	9%
1991	1 (18,000)	2 (3,000)	5	8.25%	8.375% – 8.5%
1992	1 (19,000)	1 (4,000)	10	8%	7.5%
1993	1 (16,000)	2 (5,000)	8	6.5%	6.125% – 6.25%
1994	3 (10,000)	1 (5,000)	4	6.75% – 7.5%	7.5%
Sum	47	19	77	–	–

Table 1 shows the number of issues by group and issue year. The corresponding issue size is given in brackets (in million DM). The bonds issued in 1982 determine the short end of the yield curve in 1992, which is the beginning of our testing period.

The issue size increased over the research period, especially after the German unification in 1989. However, the issue size varied considerably within each year. Except for 1982, liquid bonds have at least twice the issue size of illiquid bonds in our sample. Both groups contain issues from the German government (*BUND*) and issues from the state-operated funds (*BAHN* and *POST*). The majority of liquid bonds are *BUNDs*. Illiquid bonds are often issued from *BAHN* and *POST*. It is often argued that *BUNDs* are more popular for foreign investors, which leads to yield differences between *BUND* and *BAHN* or *POST*. If this is the case, the higher number of investors should indeed make *BUNDs* more liquid than *BAHN* or *POST* bonds[12]. Thus, the argument of different degrees of popularity provides another explanation for liquidity differences.

Return differences could also be due to coupon differences between liquid and illiquid bonds. Since our data do not contain pairs of bonds with identical issue dates and coupon sizes, but different issue sizes, we cannot completely rule out this possibility. However, *Table 1* shows that there are no systematic coupon differences between liquid and illiquid bonds [see columns 5 and 6 in *Table 1*]. *Table 1* suggests that our portfolios of liquid and illiquid bonds do not show systematic tax-induced return differences.

3.3 *Empirical Quality of the Classification*

Having presented our classification scheme, we now test whether issue size is a reasonable proxy for the liquidity of a bond. Our test is based on the assumption that ceteris paribus, liquid bonds have higher prices than illiquid ones. If this assumption holds for the German bond market, the bonds classified as liquid should have higher prices than the corresponding bonds in the remaining group. The prices of the latter bonds should be above the prices of the illiquid bonds. This joint hypothesis (illiquidity yields to price reductions and issue size is a reasonable proxy for liquidity) is tested below.

Unfortunately, the question of whether issue size is an adequate proxy for liquidity cannot be analyzed separately. Therefore, a rejection of the joint hypothesis could either be traced back to issue size being an inappropriate liquidity measure or to the lack of a price reduction due to illiquidity. However, in our case, this well-known problem of joint hypothesis should be of minor importance. As the discussion in Section 1 shows, there is strong empirical evidence that illiquidity causes price reductions. Therefore, we can safely assume that a possible rejection of the joint hypothesis indicates that issue size is in fact an inappropriate liquidity proxy.

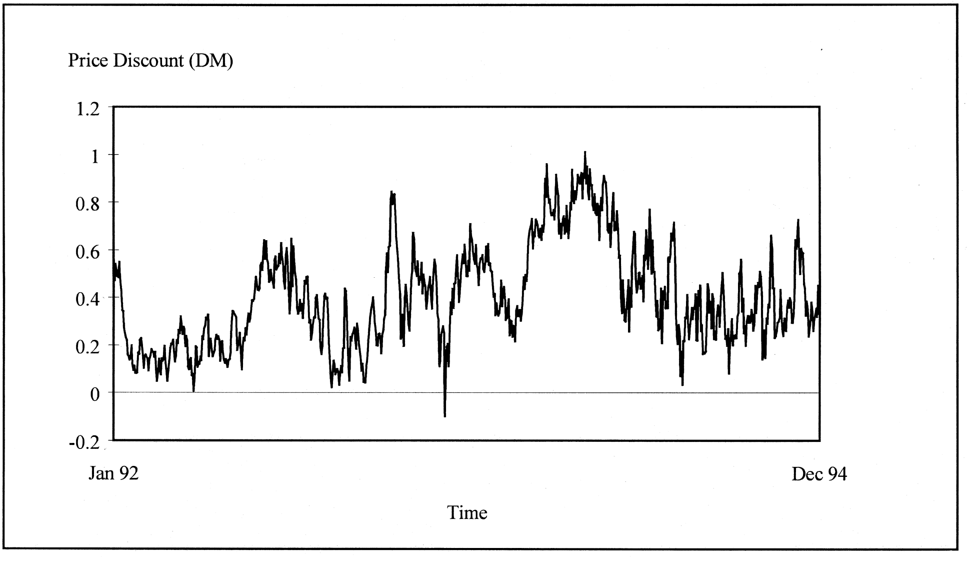
To test this hypothesis, we use only liquid bonds to estimate the term structure of interest rates for each of the 755 days. To find discount factors that explain the observed market prices of coupon bonds, we apply a two-step procedure. In the first step, we use a quadratic linear programming approach to determine a discrete discount function. This results in term structure estimates that can very accurately explain observed bond prices. However, this approach fully transfers noise in the data to the term structure of interest rates. Therefore, to obtain a smooth yield curve, we approximate the corresponding term structure of interest rates in a second step, using a cubic spline function consisting of ten polynomials[13]. Thus, we are able to obtain term structures that explain observed prices of liquid coupon bonds with an average absolute pricing error of only DM 0.001.

12 For example, *Demsetz* (1968) finds a positive effect of the number of investors on the liquidity of an asset.

13 For details, see *Uhrig/Walter* (1997).

Given the term structure of interest rates within the liquid segment, we are able to determine the theoretical prices the bonds would have if they were liquid. We compare these theoretical prices with the actual market prices of the illiquid bonds. *Figure 1* shows the mean price discounts due to illiquidity for each day of the sample period.

Figure 1: Liquidity Discount Over Time



Several results are worth noting. First, *Figure 1* shows that, on average, illiquid bonds trade at a discount compared to liquid bonds. Second, there is no time trend in the liquidity discount[14].

On average, the illiquid bonds trade at a significant discount of DM 0.40 per DM 100 nominal value (t-value[15] = 7.14)[16]. This result supports our joint hypothesis that illiquid bonds have a higher yield than liquid bonds, and that issue size is a reasonable proxy for liquidity.

The average discount cannot indicate whether liquidity differences between bonds with high and low issue size persist during the life of the bond, or if they disappear while the bond is maturing. Therefore, we repeat the above analysis, grouping the bonds in time-to-maturity segments with length of one year. For each illi-

14 We used an Augmented-Dickey-Fuller-Test with time trend to check this visual impression. We found no significant trend. The liquidity discount fluctuated around a constant, estimated as 0.024. The mean-reversion parameter was highly significant.

15 All significance tests in this paper are based upon the estimation procedure proposed by *Newey/West* (1987) which allows for autocorrelation and heteroscedasticity in the residuals.

16 The remaing bonds trade at a significant discount of DM 0.10 per DM 100 nominal value (t-value = 3.62).

quid bond with a residual maturity between zero and one year, we determine the price it should have if it were liquid. This price is compared to its market price. The remaining nine maturity segments are treated in a similar manner. *Table 2* summarizes the results.

Table 2: Liquidity Discount for Different Maturity Segments

Maturity Segment (t)	Mean Discount (DM)	t-Statistics (Level of Significance)	F-Statistics (Level of Significance)
1	0.035	1.94 (0.053)	–
2	0.108	9.34 (< 0.001)	11.56 (< 0.001)
3	0.087	7.17 (< 0.001)	1.61 (0.205)
4	0.206	16.71 (< 0.001)	47.34 (< 0.001)
5	0.122	11.10 (< 0.001)	26.05 (< 0.001)
6	0.350	30.37 (< 0.001)	205.72 (< 0.001)
7	0.386	34.47 (< 0.001)	4.91 (0,027)
8	0.805	58.22 (< 0.001)	555.81 (< 0.001)
9	0.799	68.83 (< 0.001)	0.12 (0.726)
10	0.998	87.86 (< 0.001)	151.43 (< 0.001)

Explanation: The maturity segment t includes all bonds with a residual maturity between t−1 and t years. The mean discount is the mean price difference between the theoretical price of comparable liquid bonds and the market prices of the illiquid bonds of the respective segment. The *t*-statistic corresponds to the hypothesis that the mean discount is zero. The F-statistic corresponds to the hypothesis, that the mean discount in segment t is equal to the discount in segment t−1.

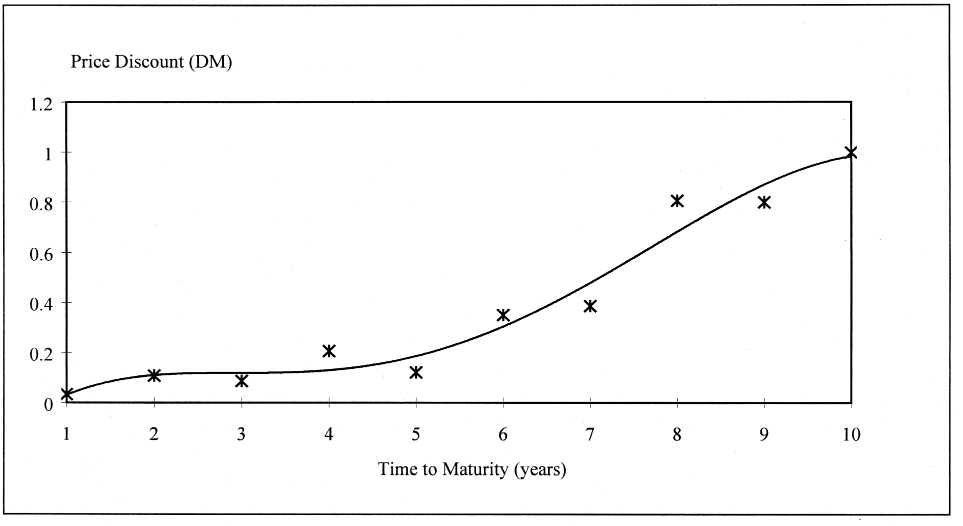
The liquidity discount in *Table 2* is significant for all maturity segments. Thus, liquidity differences between illiquid and liquid bonds persist while the bonds are maturing.

The table shows that the discounts tend to increase with the bonds' maturity. The smallest discount of DM 0.035 is obtained for the maturity segment between zero and one year. The maximum of almost DM 1 per DM 100 nominal value is obtained for the segment between nine and ten years. The hypothesis that all ten estimated liquidity discounts are equal is rejected on a 1% level (F-statistic = 825).

Table 2 also shows F-statistics for the hypothesis that neighboring maturity segments result in identical price discounts. For most comparisons, the liquidity discount is significantly higher in the longer-maturity segment.

Figure 2 illustrates the results. The mean liquidity discounts as a function of residual maturity [in DM per DM 100 nominal value] are approximated with a fifth-order polynomial. The figure also graphs the corresponding mean values presented in *Table 2*.

Figure 2: Liquidity Discount as a Function of Residual Maturity



Summing up the (model-independent) results obtained so far, two results are worth noting. First, issue size turns out to be a reasonable proxy for liquidity. Second, liquidity differences persist during the life of a bond.

3.4 Parameter Estimation

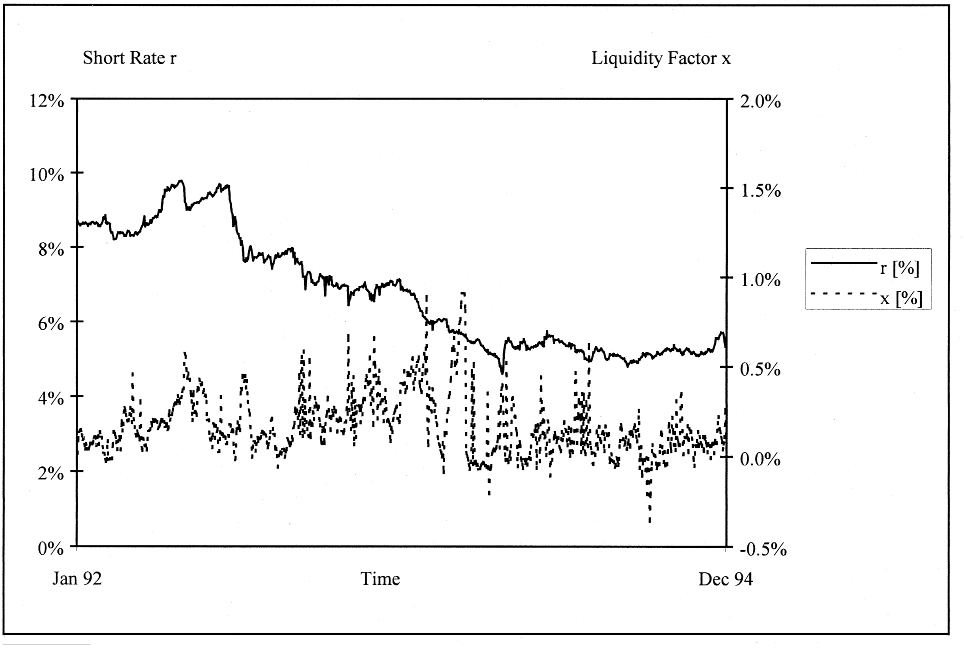
The aim of our theoretical model is to explain the price differences between liquid and illiquid bonds. According to this model, the prices of illiquid bonds depend on two stochastic factors, the instantaneous interest rate r and the instantaneous liquidity factor x. To test the model, we must estimate the parameters of the stochastic processes (1) and (2). In this section, we present the estimation procedure and the resulting parameters.

In the first step, we generate time-series of the instantaneous interest rate $r(t)$ and liquidity factor $x(t)$. Since neither the instantaneous interest rate nor the liquidity factor are directly observable, we must estimate them from observable variables. We calculate the short-term interest rate $r(t)$ of liquid bonds by extrapolating the yield to maturity of two liquid bonds with one payment day outstanding. As an alternative, we could use money-market rates. However, these might reflect default risk present in this market segment. In addition, we would have to assess the liquidity of the segment. Our procedure avoids both problems.

We determine the liquidity factor in a similar way. First, we calculate the yields of the two smallest-issue-size bonds with one payment day outstanding. In addition, we take the corresponding yields from the term structure of liquid bonds. If the illiquid bonds were liquid, then they should offer these corresponding theoretical yields. We take the difference between actual and theoretical yield to maturity as a proxy of the unobservable instantaneous liquidity spread $x(t)$. Thus, we assume that the instantaneous liquidity spread equals the liquidity spread derived from two illiquid bonds with time to maturity of less than one year.

This procedure leads to two time series of instantaneous interest rates $r(t)$ and instantaneous liquidity spreads $x(t)$. Each series covers the research period 1992–1994 and has 755 observations. The series are shown in *Figure 3*.

Figure 3: Time Series of the Short Rate (r) and the Liquidity Factor (x)



The average instantaneous interest rate is 6.77%, and the average instantaneous liquidity spread is 0.17%. The liquidity spread is different from zero at the 1% level with a t-statistic value of 7.45. However, we note that the liquidity spread is not uniformly positive through time. This contradicts our theoretical model and might be due to an approximation error in the procedure used to determine $x(t)$. A more detailed analysis indicates that short-term price variations of a few illiquid bonds are responsible for many of the negative values. The negative values are excluded in the estimation procedure presented below.

We now turn to the estimation of the parameters underlying the x- and the r-processes. We use a Euler scheme to discretize the stochastic processes (1) and (2), and determine the parameters of the discretized version by means of a maximum likelihood method. To estimate the parameters of the interest-rate process, we use the entire time series of 755 observations. For the process parameters of the instantaneous liquidity spread, we exclude the 99 that have negative values, leaving 656 observations. *Table 3* summarizes the estimation results.

Table 3: Parameter Estimates

	Long-Term Mean	Mean-Reversion Parameter	Volatility Parameter
Interest-Rate Process	$\gamma = 0.05144$	$\alpha = 0.6192$	$\sigma = 0.0495$
Liquidity Process	$\theta = 0.0018$	$\kappa = 77.8$	$\varsigma = 0.6883$

Explanation: This table contains the parameter estimates for the discretized version of the processes (1) and (2).

The parameter results for the interest-rate process are in line with those reported for other research periods in the German bond market.[17]

A comparison of the parameters for the interest-rate process and the liquidity factor shows quite different time-series properties. In particular, the strong mean-reversion (κ) and high volatility (ς) of the liquidity process are striking. Cross-correlation analysis between the changes in the interest rate and changes in liquidity spread shows a slightly negative contemporaneous correlation (-0.19), but no lagged correlation. Although the cross correlation turns out to be statistically significant, we interpret the value of -0.19 as low enough to be ignored in our theoretical modeling.

4 Empirical Tests

To provide insights into the empirical validity of the proposed model, we use a two-step approach. In Section 4.1 we perform in-sample tests to analyze how well the model fits observed price discounts due to liquidity differences. However, in-sample tests have a major drawback. They provide no information on the most relevant topic for investors, namely, whether the model can be used to improve investors' price forecasts. To test the forecasting ability of our model, we add an out-of-sample test in Section 4.2.

4.1 In-Sample Tests

The solution of the fundamental pricing equation (5) provides us with the theoretical prices of illiquid bonds. Given the current term structure of liquid bonds and the parameters of the liquidity process (2), there is only one unknown parameter in the model, which is the market price of liquidity risk, φ. This parameter reflects the unobservable preferences of market participants. We estimate the parameter in the sample by minimizing the sum of squared differences between observed prices and theoretical prices within a one-day period.

The average absolute price difference between market prices and theoretical prices of illiquid bonds is DM 0.2224 per DM 100 nominal value. Based on both

17 See *Walter* (1996), p. 155.

illiquid and liquid bonds, the average absolute price difference between market prices and theoretical prices reduces to DM 0.1394. This reduction results from the fact that our model explains prices of liquid bonds almost perfectly.

To provide a useful point of reference for the results obtained in our model, we estimate two alternative benchmark models. The first model is a one-factor *Cox/Ingersoll/Ross*-type model. We use it to explain only illiquid bonds, i. e., we assume that prices of illiquid bonds are determined solely by a single stochastic factor, which is described by (1). Given this assumption, the theoretical prices of illiquid bonds are given by (4). This benchmark is based on the following idea: A bond trader, who is aware of the price differences between liquid and illiquid bonds, could use two different valuation models, one for liquid and one for illiquid bonds. We test whether such a procedure results in lower pricing errors for illiquid bonds than does our model.

To implement the one-factor *Cox/Ingersoll/Ross*-type model, we use the same methodology as before. We hold constant over the whole sample period the parameters that describe the movement of the short rate, and estimate the market price of interest-rate risk, λ, each day on the cross-sectional data of illiquid bonds. More precisely, we estimate λ by minimizing the sum of squared differences between the model and market prices for each day under consideration. As we do in our model, we define the fitting error as the mean absolute difference between the model and market prices of illiquid bonds for the benchmark model. The average fitting error is DM 0.7682. This value is considerably higher than the value of DM 0.2224 obtained in our theoretical model. The difference of DM 0.5458 is different from zero at the 1% level. The *t*-statistic is 3.03. In addition, for 629 of the 656 valuation days, our model results in a lower fitting error than does the benchmark model.

In interpreting these results, we note that the same number of parameters – only one – is determined for both models in the sample. We use this parameter to fit the level of market prices of illiquid bonds for the benchmark model. However, it is fitted essentially to the price differences between liquid and illiquid bonds in our model. This might be the reason for the superiority of our model, since it might be easier to explain price differences than price levels. Therefore, we use a second, more challenging benchmark.

Ideally, we should compare our model with an alternative theoretical model to explain price discounts due to illiquidity. Unfortunately, there is no such model in the literature. Therefore, we have to adopt a purely empirical benchmark model to explain prices of liquid and illiquid bonds. We use a cubic spline function with ten polynomials to fit a joint term structure of liquid and illiquid bonds. Therefore, the benchmark model uses all the coefficients of the cubic spline to fit all in-sample bond prices. In contrast, our theoretical model uses the cubic spline function to fit the liquid bonds only and one single additional parameter to explain price differences between illiquid and liquid bonds.

As goodness-of-fit criterion, we again use the average absolute difference between model and market values, but we now consider both liquid and illiquid bonds.

The fitting error for the benchmark model (spline approach) is DM 0.1678 per DM 100 face value. In comparison, our theoretical model results in an average absolute difference between model and market values for liquid and illiquid bonds of DM 0.1394. The difference between these values is significant at the 1% level. The *t*-statistic is 6.38. In 525 of the 656 valuation days, our model results in a fitting error that is below that obtained for the benchmark model.

Overall, the results of the two in-sample tests show that the proposed model is able to explain observed prices significantly better than standard *Cox/Ingersoll/ Ross* and spline models.

4.2 Out-of-Sample Test

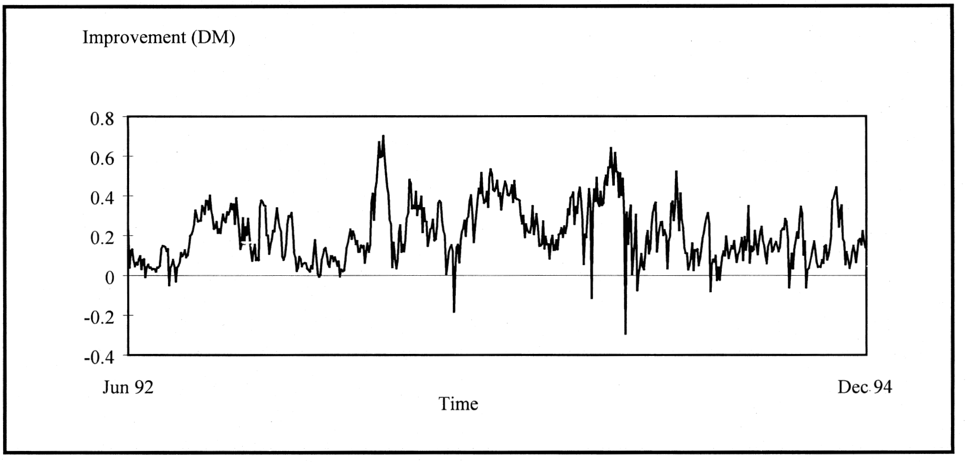
The economic relevance of a model can be tested only by out-of-sample tests. If a model improves our forecasting ability, then we can trade based on the model and increase our performance. Therefore, we test the one-day forecasting ability of our model.

As the benchmark, we use a model that forecasts the prices of illiquid bonds at day t based on the prices of liquid bonds at day t. Using information about liquid bonds from time t instead of $t-1$ ensures that our test results reflect only forecasting errors for illiquid bonds, but not forecasting errors for liquid ones.

For each day t, we estimate the term structure of liquid bonds, using the spline approach described above. Then we calculate the differences between the actual prices of illiquid bonds and prices implied by the current term structure of liquid bonds. The mean absolute difference is estimated as DM 0.461 per DM 100 nominal value. This result can be interpreted as follows: If an investor predicted prices of illiquid bonds for time t knowing the prices of liquid bonds at time t, the average forecast error would amount to DM 0.461.

In addition to the term structure of liquid bonds at time t, our model uses information on illiquid bonds available the day before, i. e., at time $t-1$. To determine our model's forecast ability, we proceed as follows: First, we estimate the parameters of the liquidity process (2) for each day t-1 from the previous 100 days using the method detailed in Section 3.4. Therefore, we use the first 100 observations of x only as input variables for estimating process parameters, and the number of observations on which our out-of-sample test is based is reduced. Second, the market price of liquidity risk, φ, is determined from the prices of illiquid bonds of day $t-1$. Third, we use the parameters of the spread process and the one-day lagged parameter estimate for φ to calculate the variable $D(x,t,T)$ from equations (7) through (10). Given the prices of liquid bonds, $L(r,t,T)$, we are thus able to determine our one-day price forecast, $V(r,x,t,T)$, of illiquid bonds from equation (6). We compare this forecast with the actual prices of illiquid bonds at time t. The mean absolute price difference is DM 0.253 per DM 100 nominal value. *Figure 4* provides daily differences between the pricing errors of the benchmark model and the pricing errors of our model.

Figure 4: Improvement of the Forecast



In comparison with the cubic-spline forecast, our model results in an average absolute forecast error that is DM 0.208 smaller. This difference is significantly different from zero at the 1% level. The *t*-statistic value is 8.407. Our model beats the benchmark model for 97% of all trading days.

This result suggests that the model proposed in this paper has economical relevance for investors, since it improves their ability to forecast prices of illiquid bonds. This improvement results from the fact that we can successfully forecast the liquidity factor x from its history. In essence, this means that equation (2) describes the evolution of the liquidity factor x well enough to draw valuable information from it.

5 Summary

In this paper, we propose a theoretical continuous-time model to analyze the impact of liquidity on bond prices. We use a market-fitting two-factor *Cox/Ingersoll/Ross*-type model with the short rate and a liquidity factor as stochastic state variables. This model prices illiquid bonds relative to liquid bonds and provides a testable theory of illiquidity induced price discounts.

The model is tested using 1992–1994 data from bonds issued by the German government. These bonds define a market segment that is homogeneous in bankruptcy risk, taxes, age, and coupons, but the bonds differ with respect to their liquidity. Therefore, the market segment we use is ideally suited to separate liquidity effects from other effects.

We provide three main empirical results. First, illiquid bonds trade at a significant discount of DM 0.40 per DM 100 nominal value as compared with liquid bonds. Second, the liquidity discount increases with maturity. Third, our model is able to explain price discounts due to illiquidity much better than can existing models. This result holds in the sample as well as out of the sample. In the sample, our model beats the two benchmark models for 96% and 80% of all trading days, respectively, and out of the sample as high as 97% of all days.

Summing up, our empirical findings suggest that bond prices not only depend on the dynamics of interest rates, but also on the liquidity of bonds. Therefore, bond liquidity should be used as an additional pricing factor. Our empirical results also show that our model can explain the price impact of liquidity. In particular, the findings of the out-of-sample test demonstrate the superiority of our model in comparison with traditional pricing models.

References

Amihud, Yakov/Mendelson, Haim (1986), Asset Prices and the Bid-Ask Spread, in: Journal of Financial Economics, Vol. 17, pp. 223–249.

Amihud, Yakov/Mendelson, Haim (1991), Liquidity, Maturity, and the Yields on U.S. Treasury Securities, in: The Journal of Finance, Vol. 46, pp. 1411–1425.

Boudoukh, Jacob/Whitelaw, Robert F. (1991), The Benchmark Effect in the Japanese Government Bond Market, in: The Journal of Fixed Income, September 1991, pp. 52–59.

Brennan, Michael J./Subrahmanyam, Avanidhar (1996), Market Microstructure and Asset Pricing: On the Compensation for Illiquidity in Stock Returns, in: Journal of Financial Economics, Vol. 41, pp. 441–464.

Cox, John/Ingersoll, Jonathan/Ross, Stephen A. (1985 a), An Intertemporal General Equilibrium Model of Asset Prices, in: Econometrica, Vol. 53, pp. 363–384.

Cox, John/Ingersoll, Jonathan/Ross, Stephen A. (1985 b), A Theory of the Term Structure of Interest Rates, in: Econometrica, Vol. 53, pp. 385–407.

Demsetz, Harold (1968), The Cost of Transacting, in: Quarterly Journal of Economics, Vol. 82, pp. 33–53.

Elton, Edwin J./Green, T. Clifton (1998), Tax and Liquidity Effects in Pricing Government Bonds, in: The Journal of Finance, Vol. 53, pp. 1533–1562.

Fisher, Lawrence (1959), Determinants of Risk Premiums on Corporate Bonds, in: The Journal of Political Economy, Vol. 67, pp. 217–237.

Grinblatt, Mark (1995), An Analytical Solution for Interest Rate Swap Spreads, Working Paper, UCLA.

Hull, John/White, Alan (1990), Pricing Interest-Rate-Derivative Securities, in: Review of Financial Studies, Vol. 3, pp. 573–592.

Kadlec, Gregory B./ McConnell, John J. (1994), The Effect of Market Segmentation and Illiquidity on Asset Prices: Evidence from Exchange Listings, in: The Journal of Finance, Vol. 49, pp. 611–636.

Kamara, Avraham (1994), Liquidity, Taxes, and Short-Term Treasury Yields, in: Journal of Financial and Quantitative Analysis, Vol. 29, pp. 403–417.

Kempf, Alexander (1998), Was messen Liquiditätsmaße?, in: Die Betriebwirtschaft, Vol. 58, pp. 299–311.

Lassak, Günter (1991), Liquidität am deutschen Rentenmarkt, in: Zeitschrift für Betriebswirtschaft, Vol. 61, pp. 75–86.

Longstaff, Francis A. (1995), How Much Can Marketability Affect Security Values?, in: The Journal of Finance, Vol. 46, pp. 1411–1425.

Newey, Whithney K./West, Kenneth D. (1987), A Simple Positive Semi-Definite, Heteroscedasticity and Autocorrelation Consistent Covariance Matrix, in: Econometrica, Vol. 55, pp. 703–708.

Sarig, Oded/Warga, Arthur (1989), Bond Price Data and Bond Market Liquidity, in: Journal of Financial and Quantitative Analysis, Vol. 24, pp. 367–378.

Sharpe, William F. (1964), Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk, in: The Journal of Finance, Vol. 19, pp. 425–442.

Silber, William L. (1991), Discounts on Restricted Stock: The Impact of Illiquidity on Stock Prices, in: Financial Analysts Journal, July-August 1991, pp. 60–64.

Tanner, J. Ernest/Kochin, Levis A. (1971), The Determinants of the Difference between Bid and Ask Prices on Government Bonds, in: The Journal of Business, Vol. 44, pp. 375–379.
Uhrig, Marliese/Walter, Ulrich (1996), A New Numerical Approach for Fitting the Initial Yield Curve, in: The Journal of Fixed Income, March 1996, pp. 82–90.
Uhrig, Marliese/Walter, Ulrich (1997), Ein neuer Ansatz zur Bestimmung der Zinsstruktur: Theorie und empirische Ergebnisse für den deutschen Rentenmarkt, in: Kredit und Kapital, Vol. 30, pp. 116–139.
Walter, Ulrich (1996), Die Bewertung von Zinsoptionen.
Warga, Arthur (1992), Bond Returns, Liquidity, and Missing Data, in: Journal of Financial and Quantitative Analysis, Vol. 27, pp. 605–617.