Global properties of kernels of transition semigroups
(Globale Eigenschaften der Kerne von Übergangshalbgruppen)

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Contents

Chapter 1

Introduction

The main objective of this work is to study the properties of the integral kernels of Markov semigroups associated with elliptic differential operator of second order with unbounded coefficients. We treat locally regular and uniformly elliptic coefficients, and focus on the unboundedness of diffusion and drift terms. The interest towards elliptic operators with unbounded coefficients comes both from the theory of partial differential equations and of Markov semigroups on $\mathbb{R}^N$ and has grown in the last years (see e.g. [BL07], [MPW02], [BKR09], [DL95]). The global properties of these problems differ significantly from the case of bounded coefficients and the case of Schrödinger operators. For instance, typically the associated semigroup on $C_b\left(\mathbb{R}^N\right)$ is not analytic if the drift term is dominant and it does not leave invariant $L^p\left(\mathbb{R}^N\right)$ or $C_0\left(\mathbb{R}^N\right)$ (see [BL07]). Moreover, in general the bounded solutions of the Cauchy problem

$$\begin{cases} \partial_t u(x,t) = Au\left(x,t\right), & x \in \mathbb{R}^N,\, t>0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

with $f \in C_b\left(\mathbb{R}^N\right)$ and

$$A = A_0 + F \cdot D - H,$$

where

$$A_0 = \sum_{i=1}^{N} D_i \left(\sum_{j=1}^{N} a_{ij} D_j \right) \qquad \text{and} \qquad F = \left(F_i\right)_{i=1,\dots,N},$$

are not unique. This means that in such case there is no maximum principle for bounded functions on $\mathbb{R}^N$ (see [BL07, Theorem 4.1.3]). One obtains a maximum principle and uniqueness in bounded functions if there is a Lyapunov function V for A. This means that $1 \leq V \in C^2\left(\mathbb{R}^N\right)$ satisfies $AV \leq KV$ for a constant K and $V\left(x\right) \to \infty$ as $|x| \to \infty$ (see Definition 1.4). A standard example are functions like $e^{\delta |x|^r}$, see Example 2.4 and Proposition 2.8. The existence of a Lyapunov function excludes cases where the drift points towards infinity too strongly, compare with Example 2.4. We will assume throughout that A possesses a Lyapunov function. The prototype of such problems is the Ornstein-Uhlenbeck operator, first studied in [DL95],

$$A_{OU} = \frac{1}{2} \operatorname{trace}\left(QD^2\right) + Bx \cdot D, \tag{1.2}$$

where Q and B are $N \times N$ matrixes such that Q is positive definite, $D^2 v$ is the Hessian matrix of $v \in C^2\left(\mathbb{R}^N\right)$ and Dv is the gradient of v.

In [MPW02] a semigroup $(T(t))_{t\geq 0}$ was constructed on $C_b(\mathbb{R}^N)$ with generator $\left(A, \hat{D}(A)\right)$, such that for each $t \geq 0$ and $f \in C_b(\mathbb{R}^N)$, $u(x,t) = T(t) f(x)$ is the solution of the Cauchy problem (1.1).

Moreover, there exists an integral kernel $0 < p = p(x,y,t) : \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty) \to \mathbb{R}$ such that

$$T(t) f(x) = \int_{\mathbb{R}^N} p(x,y,t) f(y)\, dy, \qquad t > 0,$$

and

$$T(t) f \to f \qquad \text{as } t \to 0 \text{ locally uniformly on } \mathbb{R}^N.$$

For example, for $A = \Delta$ we obtain the Gauß kernel

$$p(x,y,t) = \frac{1}{\sqrt{(4\pi t)^N}} \exp\left(-\frac{|x-y|^2}{4t}\right), \qquad (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty).$$

Also for the Ornstein-Uhlenbeck operator given in (1.2) the formula of p is known and is given by

$$p(x,y,t) = \frac{1}{\sqrt{(2\pi)^N \det Q_t}} e^{-\frac{1}{2}\left(e^{tB}x-y\right)^T Q_t^{-1}\left(e^{tB}x-y\right)},$$

where

$$Q_t = \int_0^t e^{sB} Q\, e^{sB^T}\, ds$$

(see e. g. [BL07, Chapter 9]). We remark that the Ornstein-Uhlenbeck operator given in (1.2) has a Lyapunov function $V(x) = |x|^2 + 1$ with $K = \operatorname{trace} Q + 2N \|B\|_\infty$ since

$$A_{OU} V(x) = \operatorname{trace} Q + 2Bx \cdot x \leq \left(\operatorname{trace} Q + 2N \|B\|_\infty\right) V(x).$$

If $H = 0$, we obtain a transition semigroup $(T(t))_{t\geq 0}$. In this case the kernel p is a transition density of a Markov process.

We see that if $f \geq 0$, then the solution $T(\cdot) f$ is also positive. As said above, in general, the bounded solution of the problem (1.1) is not unique. If $f \geq 0$, then $T(\cdot) f(\cdot)$ is the minimal positive solution among all bounded and positive solutions of the problem (1.1).

If A possesses a Lyapunov function, the integral $\int_{\mathbb{R}^N} p(x,y,t) V(y)\, dy$ is bounded, see Proposition 1.7. This fact will be crucial for our investigations. It was already exploited in [MPR06] and [LMPR]. We want to establish a global bound on the transition kernel p. In the well studied case of Schrödinger operators (where $F = 0$) one obtains bounds of Gaussian type (if the negative part of H is not too big), see [D89], [Ou95] and also [AMP08] for the case of dominating potential with $|F| \leq cV^{\frac{1}{2}}$. Such estimates already fail for the Ornstein-Uhlenbeck case, see the above formula. Example 2.4 further shows that we cannot expect a uniform decay as $|x| \to \infty$ (which holds in the Gaussian case). Thus we will focus on estimates in y. In the case of bounded coefficients one treats the lower order coefficients as pertubations, which is not possible if they are unbounded. Moreover, it is not clear how to use in our case the functional analytic methods developed for Schrödinger operators. So one needs new techniques to estimate p in our setting.

The case of bounded diffusion coefficients $(a_{ij})_{i,j=1,\ldots,N}$ was investigated in [MPR06] and [LMPR]. It was shown that under growth conditions for the drift F and potential H, namely,

$$\left(1 + |F|^2 + |H|\right)^{M+1} \leq V,$$

for $M > \frac{N}{2}$ and some Lyapunov function V for given $0 < a_0 < a < b < b_0 < \infty$, it holds

$$\sup_{(y,t)\in\mathbb{R}^N\times(a,b)} |p(x,y,t)| \le C\left(\int_{a_0}^{b_0}\int_{\mathbb{R}^N} p(x,y,t)\,V(y)\,dy + \frac{b_0-a_0}{(a-a_0)^{M+1}}\right) < \infty$$

for a constant $C = C\left(\lambda, M, N, \|a_{ij}\|_{C_b^1(\mathbb{R}^N)}\right) > 0$, where $\lambda > 0$ is the ellipticity constant given in (1.10).

Under stronger assumptions, the papers [MPR06] and [LMPR] also gives pointwise bounds on $|Dp|$ and $|D^2p|$, as well as bounds on Sobolev norms of p. But we point out that the proofs of these papers use the boundedness of the diffusion coefficients and their derivatives in a crucial way.

In this work we develop new methods in order to extend the results of [MPR06] and [LMPR] to unbounded diffusion coefficients.

Other related results are contained in the papers [BKR01] and [BKR09] under weaker regularity assumptions. However, here the kernel $p(x,y,t_0)$ at some initial time t_0 enters into the estimate. Observe that $p(x,y,t_0)$ is not known apriori and tends to the Dirac distribution as $t_0 \to 0$, so that the results in [BKR01] and [BKR09] are of a different nature than ours.

We also want to mention the case of densities ϱ of invariant measure for $(T(t))_{t\ge 0}$, i. e.

$$\int_{\mathbb{R}^N} T(t)\,f(x)\,\varrho(x)\,dx = \int_{\mathbb{R}^N} f(x)\,\varrho(x)\,dx \qquad \text{for all } t\in(0,\infty) \text{ and } f\in C_b\left(\mathbb{R}^N\right).$$

Here one obtains similar upper and matching lower bounds of ϱ under analogous assumptions also in the case of unbounded diffusion coefficients, see [MPR05] and [BKR06]. The starting point for the proofs is the fact that ϱ satisfies the elliptic equation $A^*\varrho = 0$ on $\mathbb{R}^N$, where A^* is the formal adjoint of A (see (1.12)). Similary, p satisfies the adjoint parabolic problem $\partial_t p(x,\cdot,\cdot) = A^*p(x,\cdot,\cdot)$ for each $x\in\mathbb{R}^N$. We stress that for the parabolic problem an initial condition at $t = t_0$ on p has to enter where $p(x,y,t_0)$ is either unknown ($t_0 > 0$) or singular ($t_0 = 0$). This makes the case of transition kernels much more difficult than that of invariant measures.

In this work we obtain similar results as in [MPR06] and [LMPR] without assuming that the diffusion coefficients $(a_{ij})_{i,j=1,\dots,N}$ and their derivatives are bounded. We will also assume that there exists a Lyapunov function V for the operator A that dominates the coefficients of A. Since a typical Lyapunov function is $e^{\delta|x|^r}$, the domination assumption is fulfilled for polynomially growing coefficients.

In Chapter 2 we study the pointwise boundedness of p and L^q-regularity of the gradient of p. We will assume that the coefficients $(a_{ij})_{i,j=1,\dots,N}$, $(F_i)_{i=1,\dots,N}$ and H of the operator A satisfy

$$\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + |F|^2 + |\operatorname{div} F + H|\right)^{M+1} \le V, \tag{1.3}$$

where V is a Lyapunov function and $M > \frac{N}{2}$. From Theorem 2.2 we will conclude that under this assumption it holds

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N} |p(x,y,t)| &\le C\left(1+\frac{1}{t^{M+1}}\right)\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^N} p(x,y,s)\,V(y)\,dyds \\
&\le \frac{C}{2}\left(t+\frac{1}{t^M}\right)V(x)\,e^{Kt}
\end{aligned}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and some constants $C = C(\lambda, M, N) > 0$ and $K > 0$. So we obtain a global boundedness of $p(x,\cdot,t)$ on $\mathbb{R}^N$ for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. (If moreover $AV \leq -g(V)$ for some convex function g given as in Proposition 1.8, we obtain the global boundedness of $p(\cdot,\cdot,t)$ on $\mathbb{R}^N \times \mathbb{R}^N$ for each $t \in (0,\infty)$, see Corollary 2.6.) In the proof of this statement we use Morrey's inequality

$$\|v\|_\infty^q \leq S \left(\int_{\mathbb{R}^N} |Dv(x)|^q \, dx + \int_{\mathbb{R}^N} |v(x)|^q \, dx \right)$$

for a constant $S = S(q,N) > 0$, where $q > N$. We will apply Morrey's inequality to the function $p^{1-\frac{\varepsilon}{2}}$ multiplied with time- and space-cut-off functions for some $\varepsilon \in \left(0, \frac{1}{2M}\right]$. For example, for the operator

$$A = \left(1 + |x|^2\right)^\alpha \Delta - |x|^{2\beta} x \cdot D - |x|^{2\theta+2}, \qquad 1 < \alpha < \beta,\, 0 < \theta,$$

we obtain that

$$p(x,y,t) \leq C_0 e^{-C_1\left(|x|^2+|y|^2\right)+C_2 t} t \qquad \text{for all } x,y,t \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0,\infty),$$

for each $t_0 > 0$ and for the constants $C_0, C_1, C_2 > 0$ depending on α, β and θ, where the constant C_0 depends additionally on t_0, see Example 2.7. We further show that

$$|Dp(x,\cdot,t)|^2 \in L^q\left(\mathbb{R}^N\right) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty) \text{ and } q \in [1,M].$$

Using the classical parabolic maximum principle, in Chapter 3 we obtain upper bounds of $|Dp|$ and $|D^2p|$. We will also give a condition on the coefficients of A under which the semigroup is differentiable in $C_b\left(\mathbb{R}^N\right)$.

In Chapter 4 we consider the case that $\operatorname{div} F + H$ is bounded from below. We will see that $p(x,\cdot,\cdot) \in L^q\left(\mathbb{R}^N \times (a,b)\right)$ for each $q \in [1,\infty)$, each $x \in \mathbb{R}^N$ and all $0 < a < b < \infty$. Here we can use a method discovered by John Nash for the case of bounded coefficients $(a_{ij})_{i,j=1,\ldots,N}$, $F = 0$ and $H = 0$, see [Na58]. Moreover, if the formal adjoint operator A^* of A also has a Lyapunov function, we obtain the global boundedness of $p(\cdot,\cdot,t)$ on $\mathbb{R}^N \times \mathbb{R}^N$ for each $t \in \mathbb{R}^N$.

1.1 Notation

For $x \in \mathbb{R}^N$, $|x|$ denotes the Euclidean norm. As regards function spaces, $L^q(\Omega)$ spaces, $1 \leq q < \infty$, are always meant with respect to the Lebesgue measure, unless otherwise specified, and are endowed with the usual norm

$$\|\psi\|_{L^q(\Omega)} = \left(\int_\Omega |\psi(y)|^q \, dy \right)^{\frac{1}{q}}$$

Moreover, $W^{k,q}(\Omega)$ is the Sobolev space of measurable functions in the open set $\Omega \subseteq \mathbb{R}^N$ which have q-summable weak derivatives up to order k, endowed with the usual norm

$$\|\psi\|_{W^{k,q}(\Omega)} = \left(\sum_{0 \leq |\beta| \leq k} \int_\Omega \left|D^\beta \psi(y)\right|^q \, dy \right)^{\frac{1}{q}}.$$

We will write $\|\cdot\|_q$ and $\|\cdot\|_{k,q}$ instead of $\|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{W^{k,q}(\Omega)}$ if $\Omega = \mathbb{R}^N$. We set $u \in W^{k,q}_{loc}(\Omega)$, if $\varphi u \in W^{k,q}(\Omega)$ for each $\varphi \in C_0^\infty(\Omega)$. For $0 \le a < b$, we write $Q(a,b)$ for $\mathbb{R}^N \times (a,b)$ and Q_T for $Q(0,T)$. For $0 < \alpha \le 1$ we denote by $C^{2+\alpha,1+\alpha/2}_{loc}(Q(a,b))$ the space of all functions u such that $u, \partial_t u, D_i u$ and $D_{ij}u$ are locally bounded and locally α-Hölder continuous. $B(x,R)$ denotes the open ball of $\mathbb{R}^N$ of radius R and centre x. If $u : \mathbb{R}^N \times J \to \mathbb{R}$, where $J \subset [0,\infty)$ is an interval, we use the notations

$$\begin{aligned} \partial_t u &= \frac{\partial u}{\partial t}, \qquad D_i u = \frac{\partial u}{\partial x_i}, \qquad D_{ij}u = D_i D_j u, \\ Du &= (D_1 u, ..., D_N u), \qquad D^2 u = (D_{ij}u)_{i,j=1,\dots,N} \end{aligned}$$

and

$$|Du|^2 = \sum_{i=1}^N |D_i u|^2, \qquad \left|D^2 u\right|^2 = \sum_{i,j=1}^N |D_{ij}u|^2, \qquad \left|D^3 u\right|^2 = \sum_{i,j,h=1}^N |D_{ijh}u|^2.$$

We set

$$D_{\max}(A) = \left\{u \in C_b(\mathbb{R}^N) \cap W^{2,q}_{loc}(\mathbb{R}^N) \text{ for all } 1 \le q < \infty : Au \in C_b(\mathbb{R}^N)\right\}. \tag{1.4}$$

We write $a(\xi,\nu)$ for $\sum_{i,j=1}^N a_{ij}(\cdot)\xi_i\nu_i$ and $\xi,\nu \in \mathbb{R}^N$. It then holds

$$|a(\xi,\nu)|^2 \le a(\xi,\xi)\, a(\nu,\nu) \qquad \text{for all } \xi,\nu \in \mathbb{R}^N. \tag{1.5}$$

We further set

$$\begin{aligned} |a|^2 &= \sum_{i,j=1}^N (a_{ij})^2, \qquad |Da|^2 = \sum_{i,j,h=1}^N (D_h a_{ij})^2, \\ \left|D^2 a\right|^2 &= \sum_{i,j,h,k=1}^N (D_{hk}a_{ij})^2, \qquad \left|D^3 a\right|^2 = \sum_{i,j,h,k,l=1}^N (D_{hkl}a_{ij})^2 \\ |F|^2 &= \sum_{i=1}^N F_i^2, \qquad |DF|^2 = \sum_{i,j=1}^N (D_i F_j)^2, \qquad \left|D^2 F\right|^2 = \sum_{i,h,k=1}^N (D_{hk}F_i)^2. \end{aligned}$$

Observe that

$$|a(\xi,\nu)| \le |a|\,|\xi|\,|\nu| \qquad \text{for all } \xi,\nu \in \mathbb{R}^N. \tag{1.6}$$

We now define a cut-off function η_n. Let $\eta \in C_c^2(\mathbb{R}^N)$ be such that $\eta(y) = 1$ if $|y| \le 1$, $\eta(y) = 0$ if $|y| \ge 2$ and $0 \le \eta \le 1$. For each $n \in \mathbb{N}$ we set $\eta_n(y) := \eta\left(\frac{y}{n}\right)$. Then $\eta_n|_{B(0,n)} = 1$, $\eta_n|_{\mathbb{R}^N \setminus B(0,2n)} = 0$ and $0 \le \eta_n \le 1$. Moreover, there exists a constant $L = L(N) > 0$ not depending on n such that

$$|D\eta_n(y)| \le \frac{L}{1+|y|} \qquad \text{and} \qquad \left|D^2\eta_n(y)\right| \le \frac{L}{1+|y|^2} \qquad \text{for } n \le |y| \le 2n. \tag{1.7}$$

1.2 Preliminaries

Let now A be a second order elliptic partial differential operator with real coefficients given by

$$A = \sum_{i,j=1}^N D_j(a_{ij}D_i) + \sum_{i=1}^N F_i D_i - H = A_0 + F \cdot D - H, \tag{1.8}$$

where $A_0 = \sum_{i,j=1}^{N} D_j (a_{ij} D_i)$ and $F = (F_i)_{i=1,...,N}$. We study the parabolic problem

$$\begin{cases} \partial_t u(x,t) = Au(x,t), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.9}$$

where $f \in C_b(\mathbb{R}^N, \mathbb{R})$ for $N \in \mathbb{N}$ is given.

We assume the following conditions on the coefficients of A which will be kept without further mentioning.

Condition 1.1.

(i) $a_{ij} \in C_{loc}^{3+\alpha}(\mathbb{R}^N, \mathbb{R})$, $F_i, H \in C_{loc}^{2+\alpha}(\mathbb{R}^N, \mathbb{R})$, $a_{ij} = a_{ji}$ *for all* $i, j = 1, ..., N$, $\inf_{x\in\mathbb{R}} H(x) = H_0 \in \mathbb{R}$.

(ii) There exists $\lambda > 0$ *such that*

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \qquad \text{for all } x, \xi \in \mathbb{R}^N. \tag{1.10}$$

(iii) $N \geq 2$.

Notice, that the diffusion coefficients $(a_{ij})_{i,j=1,...,N}$, the drift $F = (F_i)_{i=1,...,N}$ and the potential H are not assumed to be bounded in $\mathbb{R}^N$.

In [BL07, Section 2.2] the existence of a classical solution $u = u(x,t)$ of the problem (1.9) was proved, i.e.

$$u \in C\left(\mathbb{R}^N \times [0,\infty)\right) \cap C^{2,1}\left(\mathbb{R}^N \times (0,\infty)\right),$$

under the weaker assumption $H, F_i, a_{ij} \in C^\alpha(\mathbb{R}^N, \mathbb{R})$, $i, j = 1, ..., N$. The idea of the proof was to consider the Cauchy-Dirichlet problem

$$\begin{cases} \partial_t u_n(x,t) = Au_n(x,t), & x \in B(0,n),\ t > 0, \\ u_n(x,t) = 0, & x \in \partial B(0,n),\ t > 0, \\ u_n(x,0) = f(x), & x \in B(0,n), \end{cases} \tag{1.11}$$

in the Ball $B(0,n)$ for a given $f \in C_c\left(\mathbb{R}^N\right)$ and $n \in \mathbb{N}$ with $\operatorname{supp} f \subseteq B(0,n_0)$ and $n \geq n_0$. By classical results for parabolic Cauchy problems in bounded domains (e.g. [Fr64, Chapter III, §4]) we know that the problem (1.11) admits a unique solution

$$u_n \in C\left(\overline{B(0,n)} \times [0,\infty)\right) \cap C^{2+\alpha,1+\alpha/2}\left(B(0,n) \times (0,\infty)\right).$$

Moreover, Condition 1.1 implies the existence of the unique Green's function

$$0 < p_n = p_n(x,y,t) \in C\left(B(0,n) \times B(0,n) \times (0,\infty)\right)$$

such that for each fixed $x \in B(0,n)$ it holds

$$p_n(x,\cdot,\cdot) \in C^{2+\alpha,1+\alpha/2}\left(B(0,n) \times (t_1,t_2)\right)$$

and for each fixed $y \in B(0,n)$ it holds

$$p_n(\cdot,y,\cdot) \in C^{2+\alpha,1+\alpha/2}\left(B(0,n) \times (t_1,t_2)\right)$$

for all $0 < t_1 < t_2 < \infty$. Furthermore, for each fixed $y \in B(0,n)$ the function $p_n(\cdot, y, \cdot)$ satisfies

$$\partial_t p_n(x,y,t) = A p_n(x,y,t)$$

with respect to $(x,t) \in B(0,n) \times (0,\infty)$ and for each fixed $x \in B(0,n)$ it holds

$$\partial_t p_n(x,y,t) = A^* p_n(x,y,t)$$

with respect to $(y,t) \in B(0,n) \times (0,\infty)$, where

$$A^* = A_0 - F \cdot D - \operatorname{div} F - H \tag{1.12}$$

is the formal adjoint operator of A, such that

$$p_n^*(y,x,t) = p_n(x,y,t) \tag{1.13}$$

is the unique Green's function of the problem

$$\begin{cases} \partial_t v_n(y,t) = A^* v_n(y,t), & y \in B(0,n), \ t > 0, \\ v_n(y,t) = 0, & y \in \partial B(0,n), \ t > 0, \\ v_n(y,0) = f(y), & y \in B(0,n), \end{cases} \tag{1.14}$$

One can find the proof of these statements in [Fr64, Section III, §7]. The existence of $p_n^*(y,x,t) = p_n(x,y,t)$ holds also under weaker assumptions such as $a_{ij} \in C_{loc}^{2+\alpha}(\mathbb{R}^N)$, $F_i \in C_{loc}^{1+\alpha}(\mathbb{R}^N)$ and $H \in C_{loc}^{\alpha}(\mathbb{R}^N)$ for all $i,j = 1,...,N$. For the solution u_n of problem (1.11) we have

$$u_n(x,t) = \int_{B(0,n)} p_n(x,y,t) f(y)\, dy$$

and

$$\int_{B(0,n)} p_n(x,y,t) f(y)\, dy \to f(x) \qquad \text{as } t \to 0 \text{ for each } x \in B(0,n)$$

and for the solution v_n of problem (1.14) we have

$$v_n(y,t) = \int_{B(0,n)} p_n(x,y,t) f(x)\, dx$$

and

$$\int_{B(0,n)} p_n(x,y,t) f(x)\, dx \to f(y) \qquad \text{as } t \to 0 \text{ for each } y \in B(0,n).$$

In the language of semigroup theory, the operator $A_n = (A, D_n(A))$, where

$$D_n(A) = \left\{ u \in C_0(B(0,n)) \cap W^{2,q}(B(0,n)) \ \text{for all } 1 \le q < \infty : Au \in C\left(\overline{B(0,n)}\right) \right\},$$

generates an analytic semigroup $(T_n(t))_{t \ge 0}$ in the space $C\left(\overline{B(0,n)}\right)$ and, for every $f \in C\left(\overline{B(0,n)}\right)$,

$$u_n(x,t) = T_n(t) f(x) = \int_{\mathbb{R}^N} p_n(x,y,t) f(y)\, dy, \qquad (x,t) \in B(0,n) \times (0,\infty).$$

(See [Lu95, Corollary 3.1.21 (ii)].) In [BL07, Chapter 2], using the classical maximum principle, one obtains that the sequence (p_n) is increasing with respect to $n \in \mathbb{N}$. One sets

$$p(x,y,t) = \lim_{n\to\infty} p_n(x,y,t), \qquad \text{pointwise for } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty), \tag{1.15}$$

and defines the linear operator $T(t)$ in $C_b(\mathbb{R}^N)$, for any $t>0$, by setting

$$T(t) f(x) = \int_{\mathbb{R}^N} p(x,y,t) f(y)\, dy, \qquad (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Furthermore, in [BL07, Capter 2] (and in [MPW02] for the case $H=0$) was proved that the family $(T(t))_{t\geq 0}$ is a semigroup of linear operators in $C_b(\mathbb{R}^N)$. In general, $(T(t))_{t\geq 0}$ is not a strongly continuous semigroup in $C_b(\mathbb{R}^N)$. Nevertheless, $T(t)f$ tends to f as t tends to 0, uniformly on compact sets. If f vanishes at infinity, then, actually, $T(t)f$ tends to f as t tends to 0, uniformly in $\mathbb{R}^N$. But this does not mean that the restriction of $(T(t))_{t\geq 0}$ to $C_0(\mathbb{R}^N)$ is a strongly continuous semigroup, because, in general, $C_0(\mathbb{R}^N)$ is not invariant for $(T(t))_{t\geq 0}$ (see e.g. [BL07, Proposition 5.3.4]). Since, in general, the semigroup $(T(t))_{t\geq 0}$ is neither strongly continuous nor analytic, then the infinitesimal generator does not exist in the classical sense. This gap is filled introducing the concept of a "weak generator" $\hat{A}$ with domain $D(\hat{A}) \subset C_b(\mathbb{R}^N)$. In [BL07, Capter 2] the weak generator $\left(\hat{A}, D\left(\hat{A}\right)\right)$ was defined by

$$\begin{aligned} D(\hat{A}) \;\; = \;\; & \Big\{ f \in C_b(\mathbb{R}^N) : (x,t) \longmapsto \frac{T(t)f(x) - f(x)}{t} \text{ is bounded in } \mathbb{R}^N \times (0,1) \\ & \text{and } \frac{T(t)f - f}{t} \to g \in C_b(\mathbb{R}^N) \text{ pointwise as } t \to 0^+ \Big\} \end{aligned} \tag{1.16}$$

and for $f \in D(\hat{A})$ it holds

$$\hat{A} f = A f = \lim_{t\to 0} \frac{T(t)f - f}{t} \qquad \text{pointwise.}$$

We have $D(\hat{A}) \subseteq D_{\max}(A)$ and $D(\hat{A}) = D_{\max}(A)$ if and only if the problem (1.9) is uniquely solvable for each $f \in C_b(\mathbb{R}^N)$ in bounded functions. Moreover, $T(\cdot) f(\cdot)$ is for $f \geq 0$ the minimal solution among all positive solutions of the problem (1.9).

We remark that we can construct the semigroup $(T^*(t))_{t\geq 0}$ with weak generator $\left(\hat{A}^*, D(\hat{A}^*)\right)$, $D(\hat{A}^*) \subseteq D_{\max}(A^*)$ (see (1.12)) if there exists $H_0^* \in \mathbb{R}$ such that

$$H^*(x) = H(x) + \operatorname{div} F(x) \geq H_0^* \qquad \text{for each } x \in \mathbb{R}^N.$$

This fact follows again from [BL07, Chapter 2]. Combining (1.13) and (1.15) we obtain

$$p^*(x,y,t) = p(y,x,t) \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty). \tag{1.17}$$

We formulate the main properties of $(T(t))_{t\geq 0}$ in the following proposition. The proof can be found in [BL07, Chapter 2] and in [MPW02] for the case $H=0$.

Proposition 1.2. *For the semigroup $(T(t))_{t\geq 0}$ the following statements hold.*

(i) $\int_{\mathbb{R}^N} p(x,y,t)\,dy \le e^{-tH_0}$ *for all* $(x,t) \in \mathbb{R}^N \times (0,\infty)$.

(ii) $0 < p(x,y,t+s) = \int_{\mathbb{R}^N} p(x,z,t)\,p(z,y,s)\,dz$ *for all* $x,y \in \mathbb{R}^N$ *and* $s,t > 0$.

(iii) For each fixed $y \in \mathbb{R}^N$ *it holds* $\partial_t p(x,y,t) = Ap(x,y,t)$ *for all* $(x,t) \in \mathbb{R}^N \times (0,\infty)$.

(iv) $u(x,t) = \int_{\mathbb{R}^N} p(x,y,t)\,f(y)\,dy$ *solves Problem (1.9) for each* $f \in C_b(\mathbb{R}^N)$, $u \in C(\mathbb{R}^N \times [0,\infty)) \cap C_{loc}^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times (0,\infty))$ *and it holds*

$$|u(x,t)| \le e^{-H_0 t} \|f\|_\infty .$$

(v) For each $f \in D(\hat{A})$ *it holds*

$$\partial_t \int_{\mathbb{R}^N} p(x,y,t)f(y)dy = \int_{\mathbb{R}^N} p(x,y,t)Af(y)dy \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{1.18}$$

(vi) For any bounded Borel function $f \ge 0$ *with* $f \not\equiv 0$ *it holds*

$$\int_{\mathbb{R}^N} p(x,y,t)f(y)dy > 0 \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty) \tag{1.19}$$

*(*positivity*) and hence for any nonempty open set* $U \subset \mathbb{R}^N$ *and all* $(x,t) \in \mathbb{R}^N \times (0,\infty)$ *it holds* $T(t)\,\mathbb{1}_U(x) > 0$ *(*irreducibility*).*

(vii) For any bounded Borel function f *it holds* $T(t)f \in C_b(\mathbb{R}^N)$ *for each* $t \in (0,\infty)$ *(*strong Feller property*).*

(viii) For any $f \in C_0(\mathbb{R}^N)$, $T(t)f \to f$ *as* $t \to 0$ *in* $C_b(\mathbb{R}^N)$.

(ix) Let $(f_n) \subset C_b(\mathbb{R}^N)$ *be a bounded sequence converging pointwise to a function* $f \in C_b(\mathbb{R}^N)$. *Then* $T(\cdot)f_n \to T(\cdot)f$ *as* $n \to \infty$ *locally uniformly in* $(0,\infty) \times \mathbb{R}^N$.

Remark 1.3. a) Analogous to the proof of the statement (iii) one sees that for each fixed $x \in \mathbb{R}^N$ it holds $\partial_t p(x,y,t) = A^* p(x,y,t)$ for all $(y,t) \in \mathbb{R}^N \times (0,\infty)$.

b) From Condition 1.1 (i) it follows that for each fixed $x \in \mathbb{R}^N$, $D^\beta p(x,\cdot,\cdot)$, $\partial_t D^\gamma p(x,\cdot,\cdot) \in C_{loc}^\alpha(\mathbb{R}^N \times (0,\infty))$ for $0 \le |\beta| \le 3$, $0 \le |\gamma| \le 1$ and for each fixed $y \in \mathbb{R}^N$, $D^\beta p(\cdot,y,\cdot)$, $\partial_t D^\gamma p(\cdot,y,\cdot) \in C_{loc}^\alpha(\mathbb{R}^N \times (0,\infty))$ for $0 \le |\beta| \le 4$, $0 \le |\gamma| \le 2$ (see e. g. [Fr64, Chapter III, §5, Theorem 10])

We now give a definition of a Lyapunov function.

Definition 1.4. *We call a function* $1 \le V \in C^2(\mathbb{R}^N)$ Lyapunov function *for* A *if* $\lim_{|x|\to\infty} V(x) = \infty$ *and there exists a constant* $K > -H_0$ *such that it holds* $AV(x) \le KV(x)$ *for all* $x \in \mathbb{R}^N$.

Remark 1.5. The most important consequence of the existence of a Lyapunov function is the uniqueness of a bounded solution of Problem (1.9) for each $f \in C_b(\mathbb{R}^N)$ (see [BL07, Theorem 4.1.3]). The uniqueness implies immediately that if $H = 0$ on $\mathbb{R}^N$, then

$$\int_{\mathbb{R}^N} p(x,y,t)\,dy = 1 \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty), \tag{1.20}$$

since $u(x,t) = 1$ is the unique solution of Problem (1.9) with $f = 1$ and $H = 0$.

We now prove some properties of the Lyapunov functions.

Proposition 1.6. *Let V be a Lyapunov function for A such that $AV \leq KV$ for some $K > -H_0$. Then for each $M > 1$ the function $W = V^{\frac{1}{M}}$ is also a Lyapunov function for A such that*

$$AW \leq \frac{K - (M-1)H_0}{M} W \leq KW.$$

Proof. A simply computation gives

$$\begin{aligned} AW &= \frac{1}{M} V^{\frac{1}{M}-1} AV - \frac{M-1}{M} HV^{\frac{1}{M}} - \frac{M-1}{M^2} V^{\frac{1}{M}-2} a(DV, DV) \\ &\leq \frac{K - (M-1)H_0}{M} V^{\frac{1}{M}} \\ &= \frac{K - (M-1)H_0}{M} W. \end{aligned}$$

Since

$$K \geq \frac{K - (M-1)H_0}{M} > -H_0,$$

the statement follows. ■

The next two propositions were proved in [MPW02 (2)], [LMPR, Proposition 2.4] and [MPR06] for the case $H = 0$.

Proposition 1.7. *Let V be a Lyapunov function for A such that $AV \leq KV$ for some $K > -H_0$. Then, for every $t > 0$ and $x \in \mathbb{R}^N$, the functions $p(x, \cdot, t) V(\cdot)$ and $p(x, \cdot, t) |AV(\cdot)|$ are integrable. If we set*

$$\zeta(x,t) = \int_{\mathbb{R}^N} p(x,y,t) V(y)\, dy, \qquad \zeta(x,0) = V(x),$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$, the function ζ belongs to $C^{2,1}\left((0,\infty) \times \mathbb{R}^N\right) \cap C\left([0,\infty) \times \mathbb{R}^N\right)$ and satisfies the inequalities

$$\zeta(x,t) \leq e^{Kt} V(x) \tag{1.21}$$

and

$$\partial_t \zeta(x,t) \leq \int_{\mathbb{R}^N} p(x,y,t) AV(y)\, dy. \tag{1.22}$$

Proof. For $\alpha \geq 1$ we set $V_\alpha = V \wedge \alpha$ and

$$\zeta_\alpha(x,t) = \int_{\mathbb{R}^N} p(x,y,t) V_\alpha(y)\, dy = T(t) V_\alpha(x) \qquad \text{and} \qquad \zeta_\alpha(x,0) = V_\alpha(x),$$

for $(x,t) \in \mathbb{R}^N \times (0,\infty)$. For every $\varepsilon \in (0,1]$ let $\varphi_\varepsilon \in C^\infty(\mathbb{R})$ be such that $\varphi_\varepsilon \leq \alpha + \frac{\varepsilon}{2}$, $\varphi_\varepsilon(t) = t$ for $t \leq \alpha$, $\varphi_\varepsilon = \alpha + \frac{\varepsilon}{2}$ on $[\alpha + \varepsilon, \infty)$, $\varphi_\varepsilon' \geq 0$ and $\varphi_\varepsilon'' \leq 0$. Observe that $\varphi_\varepsilon(t) \to t \wedge \alpha$ and $\varphi_\varepsilon'(t) \to \mathbb{1}_{(-\infty,\alpha]}(t)$ pointwise as $\varepsilon \to 0$. The function $\varphi_\varepsilon \circ V$ belongs to $D_{\max}(A)$ since $1 \leq \varphi_\varepsilon(V) \leq \alpha + \frac{\varepsilon}{2}$ and $\varphi_\varepsilon(V(x)) = \alpha + \frac{\varepsilon}{2}$ for all sufficient large $x \in \mathbb{R}^N$. Proposition 1.2 (v) and the fact that $D\left(\hat{A}\right) = D_{\max}(A)$ yield

$$\partial_t \int_{\mathbb{R}^N} p(x,y,t) \varphi_\varepsilon(V(y))\, dy = \int_{\mathbb{R}^N} p(x,y,t) A(\varphi_\varepsilon(V(y)))\, dy$$

for $(x,t)\in\mathbb{R}^N\times(0,\infty)$. On the other hand,

$$\begin{aligned}A\left(\varphi_\varepsilon\left(V\left(y\right)\right)\right) &= \varphi_\varepsilon'\left(V\left(y\right)\right)AV\left(y\right)+\varphi_\varepsilon''\left(V\left(y\right)\right)a\left(DV\left(y\right),DV\left(y\right)\right)\\ &\quad +\left(H\left(y\right)-H_0\right)V\left(y\right)\varphi_\varepsilon'\left(V\left(y\right)\right)-\left(H\left(y\right)-H_0\right)\varphi_\varepsilon\left(V\left(y\right)\right)\\ &\quad +H_0V\left(y\right)\varphi_\varepsilon'\left(V\left(y\right)\right)-H_0\varphi_\varepsilon\left(V\left(y\right)\right).\end{aligned}$$

Since $\varphi_\varepsilon''\le 0$, it holds

$$\left(t\varphi_\varepsilon'\left(t\right)\right)'=t\varphi_\varepsilon''\left(t\right)+\varphi_\varepsilon'\left(t\right)\le\varphi_\varepsilon'\left(t\right)\qquad\text{for } t\ge 0. \tag{1.23}$$

Integrating (1.23) from 0 to $t>0$, we obtan

$$t\varphi_\varepsilon'\left(t\right)\le\varphi_\varepsilon\left(t\right)\qquad\text{for } t\ge 0. \tag{1.24}$$

Using the fact that

$$H\left(y\right)-H_0\ge 0\qquad\text{for each } y\in\mathbb{R}^N,$$

we conclude

$$\left(H\left(y\right)-H_0\right)V\left(y\right)\varphi_\varepsilon'\left(V\left(y\right)\right)-\left(H\left(y\right)-H_0\right)\varphi_\varepsilon\left(V\left(y\right)\right)\le 0.$$

We then have

$$A\left(\varphi_\varepsilon\left(V\left(y\right)\right)\right)\le\varphi_\varepsilon'\left(V\left(y\right)\right)AV\left(y\right)+H_0V\left(y\right)\varphi_\varepsilon'\left(V\left(y\right)\right)-H_0\varphi_\varepsilon\left(V\left(y\right)\right)$$

and thus

$$\begin{aligned}&\partial_t\left(e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)\varphi_\varepsilon\left(V\left(y\right)\right)dy\right)\\ &\quad =H_0e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)\varphi_\varepsilon\left(V\left(y\right)\right)dy+e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)A\left(\varphi_\varepsilon\left(V\left(y\right)\right)\right)dy\\ &\quad \le e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)\varphi_\varepsilon'\left(V\left(y\right)\right)AV\left(y\right)dy\\ &\qquad +H_0e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)V\left(y\right)\varphi_\varepsilon'\left(V\left(y\right)\right)dy\end{aligned} \tag{1.25}$$

for $(x,t)\in\mathbb{R}^N\times(0,\infty)$. Observe that $\varphi_\varepsilon\circ V\le\alpha+1$ and $\varphi_\varepsilon\circ V\to V_\alpha$ pointwise as $\varepsilon\to 0$. From Proposition 1.2 (ix) we deduce that $T\left(t\right)\left(\varphi_\varepsilon\circ V\right)\to\zeta_\alpha$ uniformly on compact subsets of $\mathbb{R}^N\times(0,\infty)$. The interior Schauder estimates (see e. g. [Fr64, Chapter III, Section 2, Theorem 5]) imply that $\partial_tT\left(t\right)\left(\varphi_\varepsilon\circ V\right)\to\partial_t\zeta_\alpha$ as $\varepsilon\to 0$ pointwise on $\mathbb{R}^N\times(0,\infty)$. From (1.24) we obtain

$$\begin{aligned}e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)\varphi_\varepsilon'\left(V\left(y\right)\right)AV\left(y\right)dy &\le Ke^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)\varphi_\varepsilon'\left(V\left(y\right)\right)V\left(y\right)dy\\ &\le \left|K\right|e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)\varphi_\varepsilon'\left(V\left(y\right)\right)V\left(y\right)dy\\ &\le \left|K\right|e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)\varphi_\varepsilon\left(V\left(y\right)\right)dy\\ &\le \left|K\right|\left(\alpha+1\right)\end{aligned}$$

and

$$H_0e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)V\left(y\right)\varphi_\varepsilon'\left(V\left(y\right)\right)dy\le\left|H_0\right|e^{H_0t}\int_{\mathbb{R}^N}p\left(x,y,t\right)V\left(y\right)\varphi_\varepsilon'\left(V\left(y\right)\right)dy$$

$$
\begin{aligned}
&\le |H_0| e^{H_0 t} \int_{\mathbb{R}^N} p(x,y,t) \varphi_\varepsilon(V(y))\, dy \\
&\le |H_0| (\alpha+1).
\end{aligned}
$$

Observe that $0 \le \varphi_\varepsilon'(V) \le 1\!\!1_{\left\{V \le \alpha + \frac{\varepsilon}{2}\right\}}$ for all $\varepsilon \in (0,1]$. Letting $\varepsilon \to 0$ in (1.25), the theorem of dominated convergence with majorante $\left(\alpha + \frac{\varepsilon}{2}\right)(|K| + |H_0|)\, p(x,y,t)$ thus yields

$$
\begin{aligned}
\partial_t \left(e^{H_0 t} \zeta_\alpha(x,t)\right) &= H_0 e^{H_0 t} \zeta_\alpha(x,t) + e^{H_0 t} \partial_t \zeta_\alpha(x,t) \\
&\le e^{H_0 t} \int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, 1\!\!1_{\{V \le \alpha\}}(y)\, dy \\
&\quad + H_0 e^{H_0 t} \int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, 1\!\!1_{\{V \le \alpha\}}(y)\, dy. \qquad (1.26)
\end{aligned}
$$

If $H_0 \ge 0$, then

$$
\begin{aligned}
H_0 e^{H_0 t} \int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, 1\!\!1_{\{V \le \alpha\}}(y)\, dy &\le H_0 e^{H_0 t} \int_{\mathbb{R}^N} p(x,y,t)\, V_\alpha(y)\, dy \\
&= H_0 e^{H_0 t} \zeta_\alpha
\end{aligned}
$$

and hence

$$
\partial_t \zeta_\alpha(x,t) \le \int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, 1\!\!1_{\{V \le \alpha\}}(y)\, dy.
$$

If $H_0 < 0$, then

$$
\partial_t \left(e^{H_0 t} \zeta_\alpha(x,t)\right) \le e^{H_0 t} \int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, 1\!\!1_{\{V \le \alpha\}}(y)\, dy.
$$

So we get

$$
\partial_t \left(e^{\min\{H_0,0\} t} \zeta_\alpha(x,t)\right) \le e^{\min\{H_0,0\} t} \int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, 1\!\!1_{\{V \le \alpha\}}(y)\, dy. \qquad (1.27)
$$

Hence, since $AV \le KV$, it follows

$$
\begin{aligned}
\partial_t \left(e^{\min\{H_0,0\} t} \zeta_\alpha(x,t)\right) &\le K e^{\min\{H_0,0\} t} \int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, 1\!\!1_{\{V \le \alpha\}}(y)\, dy \\
&\le |K|\, e^{\min\{H_0,0\} t} \zeta_\alpha(x,t).
\end{aligned}
$$

Gronwall's lemma now gives $e^{\min\{H_0,0\} t} \zeta_\alpha(x,t) \le e^{|K| t} V_\alpha(x)$. Letting $\alpha \to \infty$ we obtain by Fatou's lemma that $\zeta(x,t) \le e^{(|K| - \min\{H_0,0\}) t} V(x)$ so that V is integrable with respect to the measure $p(x,y,t)\, dy$. Thus $\zeta_\alpha(x,t) \to \zeta(x,t)$ as $\alpha \to \infty$ for all $(x,t) \in \mathbb{R}^N \times [0,\infty)$ by dominated convergence. The inequality $0 \le \zeta_\alpha \le \zeta$, the interior Schauder estimates (see e. g. [Fr64, Chapter III, Section 2, Theorem 5]) and Ascoli's theorem show that (ζ_α) is relatively compact in $C^{2,1}\left(\mathbb{R}^N \times (0,\infty)\right)$. Since $\zeta_\alpha \to \zeta$ pointwise as $\alpha \to \infty$, it follows that $\zeta \in C^{1,2}\left(\mathbb{R}^N \times (0,\infty)\right)$. Moreover, since $\zeta_\alpha(x,t) \le \zeta(x,t) \le e^{(|K| - \min\{H_0,0\}) t} V(x)$, we obtain

$$
V_\alpha(x) \le \liminf_{t \to 0} \zeta(x,t) \le \limsup_{t \to 0} \zeta(x,t) \le V(x).
$$

It follows that $\zeta(\cdot,t) \to V$ as $t \to 0$ pointwise. Set $E = \{y \in \mathbb{R}^N : AV(y) \ge 0\}$. It holds

$$
\int_E p(x,y,t)\, AV(y)\, dy \le K \int_E p(x,y,t)\, V(y)\, dy \le |K|\, \zeta(x,t) < \infty. \qquad (1.28)
$$

Moreover, letting $\alpha \to \infty$ in (1.27) we obtain

$$\partial_t \left(e^{\min\{H_0,0\}t} \zeta(x,t)\right) \le \liminf_{\alpha\to\infty} e^{\min\{H_0,0\}t} \int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, \mathbb{1}_{\{V\le\alpha\}}(y)\, dy.$$

This fact and (1.28) imply that $|AV|$ is integrable with respect to $p(x,\cdot,t)$, and so the above lim inf is a limit.
Letting $\alpha \to \infty$ in (1.26), we also obtain

$$\partial_t \zeta(x,t) \le \int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, dy$$

and hence

$$\partial_t \zeta(x,t) \le K \zeta(x,t).$$

Since $\zeta(x,0) = V(x)$, Gronwall's lemma yields

$$\zeta(x,t) \le e^{Kt} V(x) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

■

The next statement can be found in [LMPR, Proposition 2.5] for the case $H_0 \ge 0$.

Proposition 1.8. *Let $g \in C^2([0,\infty), \mathbb{R})$ be a convex function such that $g(0) \le 0$, $\lim_{s\to\infty} g(s) = \infty$ and $1/g$ is integrable in a neighborhood of ∞ and V be a Lyapunov-function for A such that $AV \le -g(V)$. Then for each $t_0 > 0$ there exists a constant $C = C(t_0) > 0$ such that*

$$\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy \le e^{\max\{-H_0,0\}t} C \qquad \textit{for all } (x,t) \in \mathbb{R}^N \times [t_0,\infty).$$

Proof. Since g is convex, it follows that $g''(s) \ge 0$ for each $s \ge 0$. Then for each $s \ge 0$ we have

$$s g''(s) + g'(s) \ge g'(s)$$

and hence

$$(s g'(s))' \ge g'(s). \tag{1.29}$$

Integrating (1.29) from 0 to $s > 0$ we obtain

$$s g'(s) \ge g(s). \tag{1.30}$$

We investigate two cases: $H_0 \ge 0$ and $H_0 < 0$.
Let $H_0 \ge 0$. Then Proposition 1.2 (i) yields

$$1 - \int_{\mathbb{R}^N} p(x,y,t)\, dy \ge 0 \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty). \tag{1.31}$$

Let us prove that

$$\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy \ge g\left(\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right). \tag{1.32}$$

We set

$$s_0 = \int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy > 0 \qquad \text{for fixed } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

For every $y \in \mathbb{R}^N$ we then have

$$g(V(y)) \geq g(s_0) + g'(s_0)(V(y) - s_0) \tag{1.33}$$

(see [Ev97, Appendix B1, Theorem 1]) and therefore, multiplying by $p(x,y,t)$ and integrating, we get

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy &\geq g(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, dy + g'(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy \\ &\quad -s_0 g'(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, dy \\ &= g(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, dy + s_0 g'(s_0)\left(1 - \int_{\mathbb{R}^N} p(x,y,t)\, dy\right)\end{aligned}$$

With (1.30) and (1.31) it follows that

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy &\geq g(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, dy + g(s_0)\left(1 - \int_{\mathbb{R}^N} p(x,y,t)\, dy\right) \\ &= g(s_0) \\ &= g\left(\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right).\end{aligned}$$

Proposition 1.7 and the assumption $AV \leq -g(V)$ further yield

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right) &\leq \int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, dy \\ &\leq -\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy \\ &\leq -g\left(\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right).\end{aligned}$$

Therefore $\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy \leq z(x,t)$, where $z = z(x,t)$ is the solution of the ordinary Cauchy problem

$$\begin{cases} z' = -g(z), & t > 0, \\ z(x,0) = V(x), & \end{cases}$$

for each fixed $x \in \mathbb{R}^N$. Let $z_0 \in \mathbb{R}$ denote the greatest zero of g. If $V(x) = z_0$, then $z(x,t) = z_0$ for all $t > 0$. If $V(x) < z_0$, then $z(x,\cdot)$ is less than z_0. If $V(x) > z_0$, then $z(x,\cdot)$ is decreasing and greater than z_0. Let now $t \geq t_0 > 0$ and $V(x) > z_0$. It then holds $g(s) > 0$ for $s \in (z_0, \infty)$ and

$$0 < t_0 \leq t = -\int_{V(x)}^{z(x,t)} \frac{ds}{g(s)} \leq \int_{V(x)}^{\infty} \frac{ds}{g(s)} - \int_{V(x)}^{z(x,t)} \frac{ds}{g(s)} = \int_{z(x,t)}^{\infty} \frac{ds}{g(s)} < \infty.$$

Since $\frac{1}{g}$ is integrable in a neighborhood of ∞, there exists a unique $C_0 = C_0(t_0) \geq z(x,t) > z_0$ such that

$$t_0 = \int_{C_0(t_0)}^{\infty} \frac{ds}{g(s)}.$$

As a result, $C_0 \geq z(x,t)$ for all $t \geq t_0$ and $x \in \mathbb{R}^N$. So we obtain

$$\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy \leq C_0 \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [t_0, \infty).$$

Let now $H_0 < 0$. At first we show that

$$e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy \geq g\left(e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right).$$

From Proposition 1.2 (i) it follows that

$$e^{-H_0 t} - \int_{\mathbb{R}^N} p(x,y,t)\, dy \geq 0 \qquad \text{for all } (x,t)\in\mathbb{R}^N\times(0,\infty). \tag{1.34}$$

We set

$$s_0 = e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy > 0 \qquad \text{for fixed } (x,t)\in\mathbb{R}^N\times(0,\infty).$$

Multiplying (1.33) by $p(x,y,t)$ and integrating, we get

$$\begin{aligned}
\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy &\geq g(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, dy + g'(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy \\
&\quad - s_0 g'(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, dy \\
&= g(s_0)\int_{\mathbb{R}^N} p(x,y,t)\, dy + s_0 g'(s_0)\left(e^{-H_0 t} - \int_{\mathbb{R}^N} p(x,y,t)\, dy\right).
\end{aligned}$$

Using (1.30) and (1.34), we estimate

$$e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy \geq g\left(e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right).$$

We apply again Proposition 1.7. From (1.22) and the fact that $H_0 < 0$ it follows that

$$\begin{aligned}
\partial_t\left(e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right) &\leq e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, AV(y)\, dy \\
&\leq -e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, g(V(y))\, dy \\
&\leq -g\left(e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy\right).
\end{aligned}$$

Analogous as in the case $H_0 \geq 0$, we obtain that for each $t_0 > 0$ there exists a constant $C = C(t_0) > 0$ such that

$$e^{H_0 t}\int_{\mathbb{R}^N} p(x,y,t)\, V(y)\, dy \leq C_0 \qquad \text{for all } (x,t)\in\mathbb{R}^N\times[t_0,\infty)$$

and the statement follows. ∎

Remark 1.9. Under the conditions of Proposition 1.8 and if $H(x) = 0$ for each $x\in\mathbb{R}^N$, then the semigroup $(T(t))_{t\geq 0}$ is compact in $C_b\left(\mathbb{R}^N\right)$ (see e.g. [BL07, Theorem 5.1.5]).

Chapter 2

Sobolev regularity of the transition kernel

2.1 Global boundedness of the transition kernel

We fix an arbitrary $x \in \mathbb{R}^N$ and consider p as a function of $(y,t) \in \mathbb{R}^N \times (0,\infty)$.

Condition 2.1. *Assume that Condition 1.1 holds. There exist $K > \max\{0, H_0\}$ and $M > \frac{N}{2}$ such that $M \geq 2$, a function $1 \leq W \in C^2(\mathbb{R}^N)$ and a Lyapunov-function V with $AV \leq KV$ such that*

$$\frac{V}{W} \geq \left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + |F|^2 + |\operatorname{div} F + H| + \frac{a(DW,DW)}{W^2} + \left(\sum_{i=1}^{N} \frac{a(D(D_iW), D(D_iW))}{W^2}\right)^{\frac{1}{2}}\right)^{M+1}$$

on $\mathbb{R}^N$.

Theorem 2.2. *Assume that Condition 2.1 holds. Then we have*

$$W(y)\,p(x,y,t) \leq C \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^N} p(x,z,s)\,W(z)\left(\Psi(z) + \frac{1}{t^{M+1}}\right) dz ds$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$, where

$$\Psi = \left(1 + |Da|^2 + |F|^2 + |\operatorname{div} F + H| + \frac{a(DW,DW)}{W^2} + \sqrt{\sum_{i=1}^{N} \frac{a(D(D_iW), D(D_iW))}{W^2}}\right)^{M+1}$$

and $C = C(\lambda, N, M) > 0$.

Remark 2.3. We assume that Condition 2.1 holds and let $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Then Theorem 2.2 says that there exists a constant $C = C(\lambda, M, N) > 0$ such that

$$\|W(\cdot)\,p(x,\cdot,t)\|_\infty \leq C \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^N} p(x,y,s)\left(V(y) + \frac{1}{t^{M+1}} W(y)\right) dy ds. \qquad (2.1)$$

For possibly different constants $C = C(\lambda, M, N) > 0$ we obtain the following consequences.

a) Due to (1.21), for $W=1$ we obtain

$$\|p(x,\cdot,t)\|_\infty \le C\left(\frac{1}{K}V(x)\left(e^{Kt}-e^{\frac{K}{2}t}\right)+\frac{1}{2t^M}\right). \tag{2.2}$$

Hence for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,\infty)$ the function $p(x,\cdot,t)$ belongs to $L^q\left(\mathbb{R}^N\right)$ for each $q\in[1,\infty]$. Moreover, for each $x\in\mathbb{R}^N$ and all $0<t_1<t_2<\infty$ we have $p(x,\cdot,\cdot)\in L^q(Q(t_1,t_2))$ for each $q\in[1,\infty]$.

b) If W is also a Lyapunov-function such that $AW\le K_0W$ for some $K_0>0$, then we get

$$\|W(\cdot)\,p(x,\cdot,t)\|_\infty \le C\left(\frac{1}{K}\left(e^{Kt}-e^{\frac{K}{2}t}\right)V(x)+\frac{1}{K_0}\frac{1}{t^{M+1}}\left(e^{K_0t}-e^{\frac{K_0}{2}t}\right)W(x)\right).$$

c) Since $1\le W\le V$, in general we have

$$p(x,y,t)\le \frac{C}{K}\left(e^{Kt}-e^{\frac{K}{2}t}\right)\left(\frac{1}{t^{M+1}}+1\right)\frac{V(x)}{W(y)}, \tag{2.3}$$

for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,\infty)$.

d) If there exists a convex function $g\in C^2([0,\infty),\mathbb{R})$ such that $g(0)\le 0$, $\lim_{s\to\infty}g(s)=\infty$, $1/g$ is integrable in a neighborhood of ∞ and $AV\le -g(V)$ on $\mathbb{R}^N$, then Proposition 1.8 yields the boundedness of $\int_{\mathbb{R}^N}p(x,y,t)\,V(y)\,dy$ on $\mathbb{R}^N\times[\alpha,T]$ for all $0<\alpha<T<\infty$. From (2.1) and Proposition 1.8 we infer that for each $\alpha>0$ there exists a constant $C=C(\lambda,M,N,\alpha)>0$ such that

$$p(x,y,t)\le Cte^{\max\{-H_0,0\}t}\frac{1}{W(y)} \qquad \text{for all } (x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times[\alpha,\infty).$$

Since $W\ge 1$, we obtain the global boundedness of p on $\mathbb{R}^N\times\mathbb{R}^N\times[\alpha,T]$ for all $0<\alpha<T<\infty$.

Proof of Theorem 2.2. Let $x\in\mathbb{R}^N$ be fixed. We consider p as a function of $(y,t)\in\mathbb{R}^N\times(0,\infty)$. Further, let $0<\alpha<\infty$, and $\tau\in C^1(\mathbb{R})$ be such that $0\le\tau\le 1$, $\tau(t)=0$ for $0\le t\le\frac{\alpha}{2}$, $\tau(t)=1$ for $t\ge\alpha$ and $0\le\tau'\le\frac{4}{\alpha}$. Let $\varepsilon\in\left(0,\frac{1}{2M}\right]$ and set

$$\beta=\left(1-\frac{\varepsilon}{2}\right)(2M+2)>0 \qquad\text{and}\qquad \delta=\frac{\beta}{2}. \tag{2.4}$$

(In this proof we only need $\varepsilon=\frac{1}{2M}$, but for Proposition 2.11 below we also need $\varepsilon<\frac{1}{2M}$.) For $i\in\{1,...,N\}$ it holds

$$D_i\left(\tau^\delta\eta_n^\beta W^{1-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}\right)=\frac{2-\varepsilon}{2}\tau^\delta\eta_n^\beta W^{1-\frac{\varepsilon}{2}}p^{-\frac{\varepsilon}{2}}D_ip+\frac{2-\varepsilon}{2}\tau^\delta\eta_n^\beta W^{-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}D_iW$$
$$+\beta\tau^\delta\eta_n^{\beta-1}W^{1-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}D_i\eta_n. \tag{2.5}$$

We will use the Jensen's inequality

$$\left(\sum_{i=1}^J m_i\right)^q\le J^{q-1}\sum_{i=1}^J m_i^q, \qquad m_i\ge 0,\ q>1,\ J\in\mathbb{N},\ i=1,...,J.$$

We apply the Jensen's inequality to (2.5) with $J=3$ and $q=2$ and get

$$\begin{aligned}\left|D\left(\tau^{\delta}\eta_n^{\beta}W^{1-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}\right)\right|^2 &= \sum_{i=1}^{N}\left(D_i\left(\tau^{\delta}\eta_n^{\beta}W^{1-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}\right)\right)^2\\ &\leq 3\sum_{i=1}^{N}\Bigg(\left(\frac{2-\varepsilon}{2}\tau^{\delta}\eta_n^{\beta}W^{1-\frac{\varepsilon}{2}}p^{-\frac{\varepsilon}{2}}D_ip\right)^2+\left(\frac{2-\varepsilon}{2}\tau^{\delta}\eta_n^{\beta}W^{-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}D_iW\right)^2\\ &\quad+\left(\beta\tau^{\delta}\eta_n^{\beta-1}W^{1-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}D_i\eta_n\right)^2\Bigg)\\ &= 3\Bigg(\frac{(2-\varepsilon)^2}{4}\tau^{2\delta}\eta_n^{2\beta}W^{2-\varepsilon}p^{-\varepsilon}\left|Dp\right|^2+\frac{(2-\varepsilon)^2}{4}\tau^{2\delta}\eta_n^{2\beta}W^{-\varepsilon}p^{2-\varepsilon}\left|DW\right|^2\\ &\quad+\beta^2\tau^{2\delta}\eta_n^{2\beta-2}W^{2-\varepsilon}p^{2-\varepsilon}\left|D\eta_n\right|^2\Bigg). \end{aligned} \tag{2.6}$$

We apply the Jensen's inequality to (2.6) with $J=3$ and $q=M$. It follows

$$\begin{aligned}\left|D\left(\tau^{\delta}\eta_n^{\beta}W^{1-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}\right)\right|^{2M} &\leq \frac{3^{2M-1}(2-\varepsilon)^{2M}}{2^{2M}}\Bigg(\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{-\varepsilon M}\left|Dp\right|^{2M}\\ &\quad+\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M}\\ &\quad+\frac{2^{2M}\beta^{2M}}{(2-\varepsilon)^{2M}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\Bigg) \end{aligned} \tag{2.7}$$

Moreover since $2M>N$ and $\tau^{\delta}\eta_n^{\beta}W^{1-\frac{\varepsilon}{2}}p^{1-\frac{\varepsilon}{2}}\in W^{1,2M}\left(\mathbb{R}^N\right)$ (for each fixed $t>0$ and each $x\in\mathbb{R}^N$), Morrey's inequality (see [Ev97, Section 5.6.2, Theorem 4]) yields that there exists a constant $S=S(N,M)>0$ such that

$$\begin{aligned}&S\sup_{y\in\mathbb{R}^N}\left|\tau(t)^{\delta}\eta_n(y)^{\beta}W(y)^{1-\frac{\varepsilon}{2}}p(x,y,t)^{1-\frac{\varepsilon}{2}}\right|^{2M}\\ &\quad\leq\int_{\mathbb{R}^N}\Bigg(\left|D\left(\tau(t)^{\delta}\eta_n^{\beta}(y)W(y)^{1-\frac{\varepsilon}{2}}p(x,y,t)^{1-\frac{\varepsilon}{2}}\right)\right|^{2M}\\ &\quad\quad+\left(\tau(t)^{\delta}\eta_n^{\beta}(y)W(y)^{1-\frac{\varepsilon}{2}}p(x,y,t)^{1-\frac{\varepsilon}{2}}\right)^{2M}\Bigg)dy \end{aligned} \tag{2.8}$$

for all $(x,t)\in\mathbb{R}^N\times(0,\infty)$. Combining (2.7) and (2.8), we deduce

$$\begin{aligned}&\frac{2^{2M}S}{3^{2M-1}(2-\varepsilon)^{2M}}\sup_{y\in\mathbb{R}^N}\left|\tau(t)^{\delta}\eta_n(y)^{\beta}W(y)^{1-\frac{\varepsilon}{2}}p(x,y,t)^{1-\frac{\varepsilon}{2}}\right|^{2M}\\ &\quad\leq\int_{\mathbb{R}^N}\Bigg(\tau(t)^{2\delta M}\eta_n(y)^{2\beta M}W(y)^{2M-\varepsilon M}\frac{1}{p(x,y,t)^{\varepsilon M}}\left|Dp(x,y,t)\right|^{2M}\\ &\quad\quad+\tau(t)^{2\delta M}\eta_n(y)^{2\beta M}W(y)^{-\varepsilon M}p(x,y,t)^{2M-\varepsilon M}\left|DW(y)\right|^{2M}\\ &\quad\quad+\frac{2^{2M}\beta^{2M}}{(2-\varepsilon)^{2M}}\tau(t)^{2\delta M}\eta_n(y)^{2\beta M-2M}W(y)^{2M-\varepsilon M}p(x,y,t)^{2M-\varepsilon M}\left|D\eta_n(y)\right|^{2M}\\ &\quad\quad+\frac{2^{2M}}{3^{2M-1}(2-\varepsilon)^{2M}}\tau(t)^{2\delta M}\eta_n(y)^{2M\beta}W(y)^{2M-\varepsilon M}p(x,y,t)^{2M-\varepsilon M}\Bigg)dy. \end{aligned} \tag{2.9}$$

We set

$$\begin{aligned}\omega_n(x,y,t) &= \tau(t)^{2\delta M}\eta_n(y)^{2\beta M}W(y)^{2M-\varepsilon M}\frac{1}{p(x,y,t)^{\varepsilon M}}|Dp(x,y,t)|^{2M}\\ &+\tau(t)^{2\delta M}\eta_n(y)^{2\beta M}W(y)^{-\varepsilon M}p(x,y,t)^{2M-\varepsilon M}|DW(y)|^{2M}\\ &+\frac{2^{2M}\beta^{2M}}{(2-\varepsilon)^{2M}}\tau(t)^{2\delta M}\eta_n(y)^{2\beta M-2M}W(y)^{2M-\varepsilon M}p(x,y,t)^{2M-\varepsilon M}|D\eta_n(y)|^{2M}\\ &+\frac{2^{2M}}{3^{2M-1}(2-\varepsilon)^{2M}}\tau(t)^{2\delta M}\eta_n(y)^{2M\beta}W(y)^{2M-\varepsilon M}p(x,y,t)^{2M-\varepsilon M} \end{aligned} \tag{2.10}$$

for $(y,t)\in\mathbb{R}^N\times(0,\infty)$ and any fixed $x\in\mathbb{R}^N$. From (2.9) it then follows

$$\frac{2^{2M}S}{3^{2M-1}(2-\varepsilon)^{2M}}\sup_{y\in\mathbb{R}^N}\left|\tau(t)^{\delta}\eta_n(y)^{\beta}W(y)^{1-\frac{\varepsilon}{2}}p(x,y,t)^{1-\frac{\varepsilon}{2}}\right|^{2M}\le\int_{\mathbb{R}^N}\omega_n(x,y,t)\,dy \tag{2.11}$$

for all $(x,t)\in\mathbb{R}^N\times(0,\infty)$. Using

$$\partial_t p=\sum_{h,k=1}^N a_{hk}D_{hk}p+\sum_{h,k=1}^N D_k a_{hk}D_h p-\sum_{h=1}^N F_h D_h p-p(\operatorname{div}F+H) \tag{2.12}$$

and

$$\partial_t\left(|Dp|^2\right)=2\sum_{i=1}^N D_i p D_i(\partial_t p), \tag{2.13}$$

we compute

$$\begin{aligned}&\partial_t\left(\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\right)\\ &\quad=2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\\ &\qquad-\varepsilon M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}|Dp|^{2M}\partial_t p\\ &\qquad+M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M-2}\partial_t\left(|Dp|^2\right)\\ &\quad=2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\\ &\qquad-\varepsilon M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}|Dp|^{2M}\sum_{h,k=1}^N a_{hk}D_{hk}p\\ &\qquad-\varepsilon M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}|Dp|^{2M}\sum_{h,k=1}^N D_k a_{hk}D_h p\\ &\qquad+\varepsilon M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}|Dp|^{2M}\sum_{h=1}^N F_h D_h p\\ &\qquad+\varepsilon M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}(\operatorname{div}F+H)\end{aligned}$$

$$
\begin{aligned}
&+2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_i a_{hk}D_{hk}pD_ip\\
&+2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}a_{hk}D_{ihk}pD_ip\\
&+2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_{ik}a_{hk}D_ipD_hp\\
&+2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_k a_{hk}D_{ih}pD_ip\\
&-2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h=1}^{N}D_iF_hD_ipD_hp\\
&-2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h=1}^{N}F_hD_{ih}pD_ip\\
&-2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left(\operatorname{div}F+H\right)\\
&-2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}D_i\left(\operatorname{div}F+H\right)D_ip, \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t&\Big(\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M}\Big)\\
&=2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M}\\
&\quad+M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\partial_tp\\
&=2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M}\\
&\quad+M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\sum_{h,k=1}^{N}a_{hk}D_{hk}p\\
&\quad+M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\sum_{h,k=1}^{N}D_ka_{hk}D_hp\\
&\quad-M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\sum_{h=1}^{N}F_hD_hp\\
&\quad-M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M}\left(\operatorname{div}F+H\right), \qquad (2.15)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t&\left(\frac{2^{2M}\beta^{2M}}{\left(2-\varepsilon\right)^{2M}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\right)\\
&=\frac{2^{2M+1}M\beta^{2M}\delta}{\left(2-\varepsilon\right)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\\
&\quad+\frac{2^{2M}M\beta^{2M}}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\partial_tp
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{2^{2M+1}M\beta^{2M}\delta}{(2-\varepsilon)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\\
&+\frac{2^{2M}M\beta^{2M}}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\sum_{h,k=1}^{N}a_{hk}D_{hk}p\\
&+\frac{2^{2M}M\beta^{2M}}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\sum_{h,k=1}^{N}D_k a_{hk}D_{h}p\\
&-\frac{2^{2M}M\beta^{2M}}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\sum_{h=1}^{N}F_h D_{h}p\\
&-\frac{2^{2M}M\beta^{2M}}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\left(\operatorname{div}F+H\right),
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
&\partial_t\left(\frac{2^{2M}}{3^{2M-1}(2-\varepsilon)^{2M}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\right)\\
&\quad=\frac{2^{2M+1}M\delta}{3^{2M-1}(2-\varepsilon)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\\
&\qquad+\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\partial_t p\\
&\quad=\frac{2^{2M+1}M\delta}{3^{2M-1}(2-\varepsilon)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\\
&\qquad+\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\sum_{h,k=1}^{N}a_{hk}D_{hk}p\\
&\qquad+\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\sum_{h,k=1}^{N}D_k a_{hk}D_{h}p\\
&\qquad-\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\sum_{h=1}^{N}F_h D_{h}p\\
&\qquad-\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\operatorname{div}F+H\right).
\end{aligned}
\tag{2.17}
$$

Simplifying the sum of (2.14)-(2.17) and integrating over $\mathbb{R}^N$ with respect to y, we deduce

$$
\begin{aligned}
\partial_t\int_{\mathbb{R}^N}\omega_n dy=\int_{\mathbb{R}^N}\Bigg(&2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}a_{hk}D_{ihk}pD_ip\\
&+2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_{ik}a_{hk}D_ipD_hp\\
&-2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}D_i\left(\operatorname{div}F+H\right)D_ip
\end{aligned}
$$

$$
\begin{aligned}
& - 2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h=1}^{N}D_iF_hD_ipD_hp \\
& - \varepsilon M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}A_0p \\
& + \frac{2^{2M}M}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}A_0p \\
& + \frac{2^{2M}M\beta^{2M}}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}A_0p \\
& + M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}A_0p \\
& + \varepsilon M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}F\cdot Dp \\
& + 2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_ia_{hk}D_{hk}pD_ip \\
& + 2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_ka_{hk}D_{ih}pD_ip \\
& - 2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h=1}^{N}F_hD_{ih}pD_ip \\
& - \frac{2^{2M}M\beta^{2M}}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}F\cdot Dp \\
& - \frac{2^{2M}M}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}F\cdot Dp \\
& - M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}F\cdot Dp \\
& - M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left(\operatorname{div}F+H\right) \\
& + 2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M} \\
& - M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M}\left(\operatorname{div}F+H\right) \\
& + \frac{2^{2M+1}M\beta^{2M}\delta}{\left(2-\varepsilon\right)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M} \\
& + 2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M} \\
& + \frac{2^{2M+1}M\delta}{3^{2M-1}\left(2-\varepsilon\right)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M} \\
& - \frac{2^{2M}M}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2M\beta}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\operatorname{div}F+H\right) \\
& \left. - \frac{2^{2M}M\beta^{2M}}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\left(\operatorname{div}F+H\right)\right)dy.
\end{aligned}
$$

Integration by parts of the first 8 terms of the right hand side yields

$$
\begin{aligned}
\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right) = & \int_{\mathbb{R}^N}\Bigg(-\varepsilon M\left(\varepsilon M+1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right) \\
& -2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right) \\
& -4M\left(M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-4} \\
& \quad\cdot\sum_{h,k=1}^{N}a_{hk}\left(\sum_{i=1}^{N}D_{ih}pD_ip\right)\left(\sum_{i=1}^{N}D_{ik}pD_ip\right) \\
& -M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right) \\
& -\frac{2^{2M}M\left(2M-\varepsilon M-1\right)}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a\left(Dp,Dp\right) \\
& -\frac{2^{2M}\beta^{2M}M\left(2M-\varepsilon M-1\right)}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2} \\
& \quad\cdot\left|D\eta_n\right|^{2M}a\left(Dp,Dp\right) \\
& +4\varepsilon M^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}a_{hk}D_{ik}pD_ipD_hp \\
& -2M\left(2M-\varepsilon M\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2} \\
& \quad\cdot\sum_{i=1}^{N}D_ipa\left(D\left(D_ip\right),DW\right) \\
& -4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}D_ipa\left(D\left(D_ip\right),D\eta_n\right) \\
& -2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_ia_{hk}D_{ik}pD_hp \\
& -4M\left(M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-4} \\
& \quad\cdot\sum_{i,j,h,k=1}^{N}D_ia_{hk}D_{jk}pD_ipD_jpD_hp \\
& +2\varepsilon M^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_ia_{hk}D_ipD_hpD_kp \\
& -4M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2} \\
& \quad\cdot\sum_{i,h,k=1}^{N}D_ia_{hk}D_ipD_hpD_kW \\
& -4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i,h,k=1}^{N}D_ia_{hk}D_ipD_hpD_k\eta_n
\end{aligned}
$$

$$+2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-2}\left(\operatorname{div}F+H\right)\sum_{i=1}^{N}D_{ii}p$$

$$+4M\left(M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-4}\left(\operatorname{div}F+H\right)$$

$$\cdot\sum_{i,j=1}^{N}D_{ij}pD_ipD_jp$$

$$-\varepsilon M\left(2M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left(\operatorname{div}F+H\right)$$

$$+2M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}p^{1-\varepsilon M}\left|Dp\right|^{2M-2}\left(\operatorname{div}F+H\right)$$

$$\cdot\sum_{i=1}^{N}D_ipD_iW$$

$$+4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-2}\left(\operatorname{div}F+H\right)\sum_{i=1}^{N}D_ipD_i\eta_n$$

$$+2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}F\cdot Dp\sum_{i=1}^{N}D_{ii}p$$

$$+4M\left(M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-4}F\cdot Dp$$

$$\cdot\sum_{i,h=1}^{N}D_{ih}pD_ipD_hp$$

$$-\varepsilon M\left(2M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}F\cdot Dp$$

$$+2M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}F\cdot Dp\sum_{i=1}^{N}D_ipD_iW$$

$$+4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}F\cdot Dp\sum_{h=1}^{N}D_hpD_h\eta_n$$

$$-2M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M-2}$$

$$\cdot\sum_{i=1}^{N}a\left(Dp,D\left(D_iW\right)\right)D_iW$$

$$+\varepsilon M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M-1}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}a\left(DW,Dp\right)$$

$$-\frac{2^{2M}M^2}{3^{2M-1}\left(2-\varepsilon\right)^{2M-2}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}p^{2M-\varepsilon M-1}a\left(DW,Dp\right)$$

$$-2\beta M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M-1}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}a\left(D\eta_n,Dp\right)$$

$$-\frac{2^{2M+1}\beta M^2}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}a\left(D\eta_n,Dp\right)$$

$$-\frac{2^{2M+1}\beta^{2M}M^2}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M-2}$$

$$\cdot\sum_{i=1}^{N}a\left(Dp,D\left(D_i\eta_n\right)\right)D_i\eta_n$$

$$
\begin{aligned}
&-\frac{2^{2M}\beta^{2M}M^2}{(2-\varepsilon)^{2M-2}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M-1}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}a\left(DW,Dp\right)\\
&-\frac{2^{2M+1}\beta^{2M}\left(\beta-1\right)M^2}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M-1}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\\
&\quad\cdot\left|D\eta_n\right|^{2M}a\left(D\eta_n,Dp\right)\\
&+\varepsilon M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}a\left(DW,Dp\right)\\
&+2\varepsilon\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}a\left(D\eta_n,Dp\right)\\
&-\frac{2^{2M}\beta^{2M}M}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}F\cdot Dp\\
&-\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}F\cdot Dp\\
&+2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\\
&-M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}p^{2M-\varepsilon M-1}W^{-\varepsilon M}\left|DW\right|^{2M}F\cdot Dp\\
&-M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}p^{2M-\varepsilon M}W^{-\varepsilon M}\left|DW\right|^{2M}\left(\operatorname{div}F+H\right)\\
&+\frac{2^{2M+1}\beta^{2M}\delta M}{(2-\varepsilon)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\\
&-\frac{2^{2M}\beta^{2M}M}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\left(\operatorname{div}F+H\right)\\
&+\frac{2^{2M+1}\delta M}{3^{2M-1}(2-\varepsilon)^{2M}}\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\\
&-\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\operatorname{div}F+H\right)\\
&\left.+2\delta M\tau'\tau^{2\delta M-1}\eta_n^{2\beta M}p^{2M-\varepsilon M}W^{-\varepsilon M}\left|DW\right|^{2M}\right)dy.
\end{aligned}
$$

Employing (1.10), (1.5) and (1.6) we estimate

$$
\begin{aligned}
\partial_t\textstyle\int_{\mathbb{R}^N}\omega_n dy \leq\ & \int_{\mathbb{R}^N}\left(-\varepsilon M\left(\varepsilon M+1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)\right.\\
&-2M\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&-4M\left(M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-4}\\
&\quad\cdot\sum_{h,k=1}^{N}a_{hk}\left(\sum_{i=1}^{N}D_{ih}pD_ip\right)\left(\sum_{i=1}^{N}D_{ik}pD_ip\right)\\
&-M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right)\\
&-\frac{2^{2M}M\left(2M-\varepsilon M-1\right)}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a\left(Dp,Dp\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2^{2M}\beta^{2M}M\left(2M-\varepsilon M-1\right)}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}\\
&\qquad\cdot\left|D\eta_n\right|^{2M}a\left(Dp,Dp\right)\\
&+4\varepsilon M^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M-2}\sqrt{a\left(Dp,Dp\right)}\\
&\qquad\cdot\sqrt{\sum_{h,k=1}^{N}a_{hk}\left(\sum_{i=1}^{N}D_{ih}pD_ip\right)\left(\sum_{i=1}^{N}D_{ik}pD_ip\right)}\\
&+2M\left(2M-\varepsilon M\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\\
&\qquad\cdot\sum_{i=1}^{N}\sqrt{a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\sqrt{\left(D_ip\right)^2a\left(DW,DW\right)}\\
&+4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\\
&\qquad\cdot\sum_{i=1}^{N}\sqrt{a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\sqrt{\left(D_ip\right)^2a\left(D\eta_n,D\eta_n\right)}\\
&+\frac{9M^2}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-1}\left|D^2p\right|\left(\left|Da\right|+\left|F\right|\right)\\
&+\frac{9M^2}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-2}\left|D^2p\right|\left|\operatorname{div}F+H\right|\\
&+2M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}p^{1-\varepsilon M}\left|Dp\right|^{2M-1}\left|\operatorname{div}F+H\right|\left|DW\right|\\
&+4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-1}\left|\operatorname{div}F+H\right|\left|D\eta_n\right|\\
&+2\varepsilon M^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M+1}\left(\left|Da\right|+\left|F\right|\right)\\
&+\varepsilon M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}\\
&\qquad\cdot\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(DW,DW\right)}\\
&+2\varepsilon\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}\\
&+\frac{2^{2M}M^2}{3^{2M-1}\left(2-\varepsilon\right)^{2M-2}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}p^{2M-\varepsilon M-1}\\
&\qquad\cdot\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(DW,DW\right)}\\
&+\frac{2^{2M}M}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\sqrt{\frac{1}{\lambda}\left|F\right|^2}\sqrt{a\left(Dp,Dp\right)}\\
&+\frac{2^{2M+1}\beta M^2}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\\
&\qquad\cdot\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}\\
&+\frac{2^{2M+1}\beta^{2M}M^2}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M-2}\\
&\qquad\cdot\sum_{i=1}^{N}\sqrt{\left(D_i\eta_n\right)^2a\left(Dp,Dp\right)}\sqrt{a\left(D\left(D_i\eta_n\right),D\left(D_i\eta_n\right)\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2^{2M}\beta^{2M}M^2}{(2-\varepsilon)^{2M-2}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M-1}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\\
&\qquad\cdot\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(DW,DW\right)}\\
&+\frac{2^{2M+1}\beta^{2M}\left(\beta-1\right)M^2}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M-1}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\\
&\qquad\cdot\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}\\
&+\frac{2^{2M}\beta^{2M}M}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\\
&\qquad\cdot\sqrt{\frac{1}{\lambda}\left|F\right|^2}\sqrt{a\left(Dp,Dp\right)}\\
&+2\beta M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M-1}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\\
&\qquad\cdot\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}\\
&+M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}p^{2M-\varepsilon M-1}W^{-\varepsilon M}\left|DW\right|^{2M}\sqrt{\frac{1}{\lambda}\left|F\right|^2}\sqrt{a\left(Dp,Dp\right)}\\
&+2M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M-2}\\
&\qquad\cdot\sum_{i=1}^{N}\sqrt{\left(D_iW\right)^2a\left(Dp,Dp\right)}\sqrt{a\left(D\left(D_iW\right),D\left(D_iW\right)\right)}\\
&+\varepsilon M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M-1}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\\
&\qquad\cdot\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(DW,DW\right)}\\
&+M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}p^{2M-\varepsilon M}W^{-\varepsilon M}\left|\operatorname{div}F+H\right|\left|DW\right|^{2M}\\
&+\frac{2^{2M+3}\beta^{2M}\delta M}{(2-\varepsilon)^{2M}}\tau^{2\delta M-1}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|D\eta_n\right|^{2M}\frac{1}{\alpha}\\
&+\frac{2^{2M}\beta^{2M}M}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|\operatorname{div}F+H\right|\left|D\eta_n\right|^{2M}\\
&+\frac{2^{2M+3}\delta M}{3^{2M-1}(2-\varepsilon)^{2M}}\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{1}{\alpha}\\
&+\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|\operatorname{div}F+H\right|\\
&+8\delta M\tau^{2\delta M-1}\eta_n^{2\beta M}p^{2M-\varepsilon M}W^{-\varepsilon M}\left|DW\right|^{2M}\frac{1}{\alpha}\\
&+4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left(\left|Da\right|+\left|F\right|\right)\left|D\eta_n\right|\\
&+\varepsilon M\left(2M-1\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left|\operatorname{div}F+H\right|\\
&+4M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left(\left|Da\right|+\left|F\right|\right)\left|DW\right|\\
&+8\delta M\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\frac{1}{\alpha}\Bigg)dy. \qquad (2.18)
\end{aligned}
$$

We consider the positive terms on the right hand side of (2.18). Applying repeatedly Young's inequality and using $M \geq 2$ and (1.7), we estimate

$$
\begin{aligned}
4\varepsilon M^2 \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M+1}} \left|Dp\right|^{2M-2} \sqrt{a\left(Dp, Dp\right)} & \\
\cdot \sqrt{\sum_{h,k=1}^{N} a_{hk} \left(\sum_{i=1}^{N} D_{ih}pD_ip\right)\left(\sum_{i=1}^{N} D_{ik}pD_ip\right)} & \\
\leq 2\varepsilon^2 M^2 \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M+2}} \left|Dp\right|^{2M} a\left(Dp, Dp\right) & \\
+ 4M\left(M-1\right) \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-4} & \\
\cdot \sum_{h,k=1}^{N} a_{hk} \left(\sum_{i=1}^{N} D_{ih}pD_ip\right)\left(\sum_{i=1}^{N} D_{ik}pD_ip\right), &
\end{aligned}
$$

$$
\begin{aligned}
2M\left(2M-\varepsilon M\right) \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M-1} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} & \\
\cdot \sum_{i=1}^{N} \sqrt{a\left(D\left(D_ip\right), D\left(D_ip\right)\right)} \sqrt{\left(D_ip\right)^2 a\left(DW, DW\right)} & \\
\leq 2 \sum_{i=1}^{N} \sqrt{\frac{M}{2} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} a\left(D\left(D_ip\right), D\left(D_ip\right)\right)} & \\
\cdot \sqrt{2M^3 \left(2-\varepsilon\right)^2 \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M-2} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} \left(D_ip\right)^2 a\left(DW, DW\right)} & \\
\leq \frac{M}{2} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} \sum_{i=1}^{N} a\left(D\left(D_ip\right), D\left(D_ip\right)\right) & \\
+ 2M^3 \left(2-\varepsilon\right)^2 \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M} \frac{a\left(DW, DW\right)}{W^2}, &
\end{aligned}
$$

$$
\begin{aligned}
4\beta M^2 \tau^{2\delta M} \eta_n^{2\beta M-1} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} & \\
\cdot \sum_{i=1}^{N} \sqrt{a\left(D\left(D_ip\right), D\left(D_ip\right)\right)} \sqrt{\left(D_ip\right)^2 a\left(D\eta_n, D\eta_n\right)} & \\
\leq 2 \sum_{i=1}^{N} \sqrt{\frac{M}{2} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} a\left(D\left(D_ip\right), D\left(D_ip\right)\right)} & \\
\cdot \sqrt{8\beta^2 M^3 L^2 \tau^{2\delta M} \eta_n^{2\beta M-2} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} \left(D_ip\right)^2 \frac{|a|}{1+|y|^2} \mathbb{1}_{\{n \leq |y| \leq 2n\}}} & \\
\leq \frac{M}{2} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M-2} \sum_{i=1}^{N} a\left(D\left(D_ip\right), D\left(D_ip\right)\right) & \\
+ 8\beta^2 M^3 L^2 \tau^{2\delta M} \eta_n^{2\beta M-2} W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M}} \left|Dp\right|^{2M} \frac{|a|}{1+|y|^2} \mathbb{1}_{\{n \leq |y| \leq 2n\}}, &
\end{aligned}
$$

$$\frac{9M^2}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-1}\left|D^2p\right|\left(\left|Da\right|+\left|F\right|\right)$$

$$\begin{aligned}
&\leq 2\sqrt{\frac{M}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\\
&\quad\cdot\sqrt{\frac{81M^3}{8\lambda}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left(\left|Da\right|+\left|F\right|\right)^2}\\
&\leq \frac{M}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&\quad+\frac{81M^3}{8\lambda}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\left(\left|Da\right|+\left|F\right|\right)^2,
\end{aligned}$$

$$\frac{9M^2}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-2}\left|D^2p\right|\left|\operatorname{div}F+H\right|$$

$$\begin{aligned}
&\leq 2\sqrt{\frac{M}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\\
&\quad\cdot\sqrt{\frac{81M^3}{8\lambda}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2-\varepsilon M}\left|Dp\right|^{2M-2}\left|\operatorname{div}F+H\right|^2}\\
&\leq \frac{M}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&\quad+\frac{81M^3}{8\lambda}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2-\varepsilon M}\left|Dp\right|^{2M-2}\left|\operatorname{div}F+H\right|^2\\
&\leq \frac{M}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&\quad+\left(\frac{M+1}{M-1}\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)\right)^{\frac{M-1}{M+1}}\\
&\quad\cdot\left(\frac{3^{2M+2}\left(M-1\right)^{\frac{M-1}{2}}M^{\frac{M+5}{2}}}{8\varepsilon^{M-1}\lambda^M\left(M+1\right)^{\frac{M-1}{2}}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|\operatorname{div}F+H\right|^{M+1}\right)^{\frac{2}{M+1}}\\
&\leq \frac{M}{2}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M-2}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&\quad+\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)\\
&\quad+\frac{3^{2M+2}\left(M-1\right)^{\frac{M-1}{2}}M^{\frac{M+5}{2}}}{4\varepsilon^{M-1}\lambda^M\left(M+1\right)^{\frac{M+1}{2}}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|\operatorname{div}F+H\right|^{M+1},
\end{aligned}$$

$$2M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}p^{1-\varepsilon M}\left|Dp\right|^{2M-1}\left|\operatorname{div}F+H\right|\left|DW\right|$$

$$
\begin{aligned}
&\leq \left(\frac{2M+2}{2M-1}\frac{M^2\varepsilon^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)\right)^{\frac{2M-1}{2M+2}} \\
&\quad\cdot\left(\frac{2^{\frac{8M-1}{3}}\left(2-\varepsilon\right)^{\frac{2M+2}{3}}M^2\left(2M-1\right)^{\frac{2M-1}{3}}}{\varepsilon^{\frac{4M-2}{3}}\lambda^{\frac{2M-1}{3}}\left(2M+2\right)^{\frac{2M-1}{3}}}\tau^{2\delta M}\eta_n^{2\beta M}\right. \\
&\quad\left.\cdot W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\left|\operatorname{div}F+H\right|^{M+1}\right)^{\frac{2}{3}}\left(\frac{\left|DW\right|^{2M+2}}{W^{2M+2}}\right)^{\frac{1}{3}}\right)^{\frac{3}{2M+2}} \\
&\leq \frac{M^2\varepsilon^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right) \\
&\quad+\frac{3\cdot 2^{\frac{8M-1}{3}}\left(2-\varepsilon\right)^{\frac{2M+2}{3}}M^2\left(2M-1\right)^{\frac{2M-1}{3}}}{\varepsilon^{\frac{4M-2}{3}}\lambda^{\frac{2M-1}{3}}\left(2M+2\right)^{\frac{2M+2}{3}}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M} \\
&\quad\cdot\left(\left|\operatorname{div}F+H\right|^{M+1}\right)^{\frac{2}{3}}\left(\frac{\left|DW\right|^{2M+2}}{W^{2M+2}}\right)^{\frac{1}{3}},
\end{aligned}
$$

$$
\begin{aligned}
&4\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}p^{1-\varepsilon M}\left|Dp\right|^{2M-1}\left|\operatorname{div}F+H\right|\left|D\eta_n\right| \\
&\leq \left(\frac{2M+2}{2M-1}\frac{M^2\varepsilon^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)\right)^{\frac{2M-1}{2M+2}} \\
&\quad\cdot\left(\frac{2^{\frac{10M+1}{3}}\beta^{\frac{2M+2}{3}}M^2\left(2M-1\right)^{\frac{2M-1}{3}}L^{\frac{2M+2}{3}}}{\varepsilon^{\frac{4M-2}{3}}\lambda^{\frac{2M-1}{3}}\left(2M+2\right)^{\frac{2M-1}{3}}}\tau^{2\delta M}\eta_n^{2\beta M-\frac{2M+2}{3}}\right. \\
&\quad\left.\cdot W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{\left|\operatorname{div}F+H\right|^{\frac{2M+2}{3}}}{1+\left|y\right|^{\frac{2M+2}{3}}}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\right)^{\frac{3}{2M+2}} \\
&\leq \frac{M^2\varepsilon^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right) \\
&\quad+\frac{3\cdot 2^{\frac{10M+1}{3}}\beta^{\frac{2M+2}{3}}M^2\left(2M-1\right)^{\frac{2M-1}{3}}L^{\frac{2M+2}{3}}}{\varepsilon^{\frac{4M-2}{3}}\lambda^{\frac{2M-1}{3}}\left(2M+2\right)^{\frac{2M+2}{3}}}\tau^{2\delta M}\eta_n^{2\beta M-\frac{2M+2}{3}} \\
&\quad\cdot W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{\left|\operatorname{div}F+H\right|^{\frac{2M+2}{3}}}{1+\left|y\right|^{\frac{2M+2}{3}}}\mathbb{1}_{\{n\leq|y|\leq 2n\}} \\
&\leq \frac{M^2\varepsilon^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right) \\
&\quad+\left(\frac{3\cdot 2^{5M-1}M}{\varepsilon^{M-1}\lambda^{M}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|\operatorname{div}F+H\right|^{M+1}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\right)^{\frac{2}{3}} \\
&\quad\cdot\left(\frac{3\cdot\lambda\beta^{2M+2}ML^{2M+2}}{\varepsilon^{2M}}\tau^{2\delta M}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\right)^{\frac{1}{3}} \\
&\leq \frac{M^2\varepsilon^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right) \\
&\quad+\frac{2^{5M}M}{\varepsilon^{M-1}\lambda^{M}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left|\operatorname{div}F+H\right|^{M+1}\mathbb{1}_{\{n\leq|y|\leq 2n\}} \\
&\quad+\frac{\lambda\beta^{2M+2}ML^{2M+2}}{\varepsilon^{2M}}\tau^{2\delta M}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
\end{aligned}
$$

$$2\varepsilon M^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M+1}\left(\left|Da\right|+\left|F\right|\right)$$
$$\leq\left(\frac{2M+2}{2M+1}\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)\right)^{\frac{2M+1}{2M+2}}$$
$$\cdot\left(\frac{2^{6M+4}M^2\left(2M+1\right)^{2M+1}}{\varepsilon^{2M}\lambda^{2M+1}\left(M+1\right)^{2M+1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\left|Da\right|+\left|F\right|\right)^{2M+2}\right)^{\frac{1}{2M+2}}$$
$$\leq\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)$$
$$+\frac{2^{6M+3}M^2\left(2M+1\right)^{2M+1}}{\varepsilon^{2M}\lambda^{2M+1}\left(M+1\right)^{2M+2}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\left|Da\right|+\left|F\right|\right)^{2M+2},$$

$$\varepsilon M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(DW,DW\right)}$$
$$\leq2\sqrt{\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)}$$
$$\cdot\sqrt{2M^2\left(2-\varepsilon\right)^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\frac{a\left(DW,DW\right)}{W^2}}$$
$$\leq\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)$$
$$+2M^2\left(2-\varepsilon\right)^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\frac{a\left(DW,DW\right)}{W^2},$$

$$2\varepsilon\beta M^2\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+1}}\left|Dp\right|^{2M}\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}$$
$$\leq2\sqrt{\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)}$$
$$\cdot\sqrt{8M^2\beta^2L^2\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\frac{\left|a\right|}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq2n\}}}$$
$$\leq\frac{\varepsilon^2M^2}{8}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M}a\left(Dp,Dp\right)$$
$$+8M^2\beta^2L^2\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}\left|Dp\right|^{2M}\frac{\left|a\right|}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq2n\}},$$

$$\frac{2^{2M}M^2}{3^{2M-1}\left(2-\varepsilon\right)^{2M-2}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}p^{2M-\varepsilon M-1}\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(DW,DW\right)}$$
$$\leq2\sqrt{\frac{2^{2M-2}M\left(2M-\varepsilon M-1\right)}{3^{2M-1}\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a\left(Dp,Dp\right)}$$

$$\cdot\sqrt{\frac{2^{2M}M^3}{3^{2M-1}(2-\varepsilon)^{2M-3}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}}$$

$$\cdot\sqrt{p^{2M-\varepsilon M}\frac{a(DW,DW)}{W^2}}$$

$$\leq\frac{2^{2M-2}M(2M-\varepsilon M-1)}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a(Dp,Dp)$$

$$+\frac{2^{2M}M^3}{3^{2M-1}(2-\varepsilon)^{2M-3}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}$$

$$\cdot p^{2M-\varepsilon M}\frac{a(DW,DW)}{W^2},$$

$$\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\sqrt{\frac{1}{\lambda}|F|^2}\sqrt{a(Dp,Dp)}$$

$$\leq 2\sqrt{\frac{2^{2M-2}M(2M-\varepsilon M-1)}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a(Dp,Dp)}$$

$$\cdot\sqrt{\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)\lambda}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}|F|^2}$$

$$\leq\frac{2^{2M-2}M(2M-\varepsilon M-1)}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a(Dp,Dp)$$

$$+\frac{2^{2M}M}{3^{2M-1}(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)\lambda}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}|F|^2,$$

$$\frac{2^{2M+1}\beta M^2}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\sqrt{a(Dp,Dp)}\sqrt{a(D\eta_n,D\eta_n)}$$

$$\leq 2\sqrt{\frac{2^{2M-1}M(2M-\varepsilon M-1)}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a(Dp,Dp)}$$

$$\cdot\sqrt{\frac{2^{2M+1}M^3\beta^2L^2}{3^{2M-1}(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}}$$

$$\cdot\sqrt{p^{2M-\varepsilon M}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}}$$

$$\leq\frac{2^{2M-1}M(2M-\varepsilon M-1)}{3^{2M-1}(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}a(Dp,Dp)$$

$$+\frac{2^{2M+1}M^3\beta^2L^2}{3^{2M-1}(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}$$

$$\cdot p^{2M-\varepsilon M}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}},$$

$$\frac{2^{2M+1}\beta^{2M}M^2}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}|D\eta_n|^{2M-2}$$

$$
\begin{aligned}
&\cdot \sum_{i=1}^{N} \sqrt{(D_i\eta_n)^2\, a\,(Dp, Dp)}\sqrt{a\,(D\,(D_i\eta_n)\,, D\,(D_i\eta_n))} \\
\leq& 2\sum_{i=1}^{N} \sqrt{\frac{2^{2M-2}\beta^{2M} M\,(2M-\varepsilon M-1)}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M-2}} \\
&\cdot \sqrt{|D\eta_n|^{2M-2}\,(D_i\eta_n)^2\, a\,(Dp, Dp)} \\
&\cdot \sqrt{\frac{2^{2M}\beta^{2M} M^3}{(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M}} \\
&\cdot \sqrt{|D\eta_n|^{2M-2}\, a\,(D\,(D_i\eta_n)\,, D\,(D_i\eta_n))} \\
\leq& \frac{2^{2M-2}\beta^{2M} M\,(2M-\varepsilon M-1)}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M-2} \\
&\cdot |D\eta_n|^{2M}\, a\,(Dp, Dp) \\
&+ \frac{2^{2M}\beta^{2M} M^3 L^{2M}}{(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} \\
&\cdot p^{2M-\varepsilon M}\frac{|a|}{1+|y|^{2M+2}}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2^{2M}\beta^{2M} M^2}{(2-\varepsilon)^{2M-2}}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M-1} p^{2M-\varepsilon M-1}\,|D\eta_n|^{2M}\,\sqrt{a\,(Dp, Dp)}\sqrt{a\,(DW, DW)} \\
\leq& 2\sqrt{\frac{2^{2M-2}\beta^{2M} M\,(2M-\varepsilon M-1)}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M-2}\,|D\eta_n|^{2M}\, a\,(Dp, Dp)} \\
&\cdot \sqrt{\frac{2^{2M}\beta^{2M} M^3 L^{2M}}{(2-\varepsilon)^{2M-3}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M}} \\
&\cdot \sqrt{p^{2M-\varepsilon M}\frac{a\,(DW, DW)}{W^2}\frac{1}{1+|y|^{2M}}\mathbb{1}_{\{n\leq|y|\leq 2n\}}} \\
\leq& \frac{2^{2M-2}\beta^{2M} M\,(2M-\varepsilon M-1)}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M-2}\,|D\eta_n|^{2M}\, a\,(Dp, Dp) \\
&+ \frac{2^{2M}\beta^{2M} M^3 L^{2M}}{(2-\varepsilon)^{2M-3}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} \\
&\cdot p^{2M-\varepsilon M}\frac{a\,(DW, DW)}{W^2}\frac{1}{1+|y|^{2M}}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2^{2M+1}\beta^{2M}(\beta-1)\,M^2}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M-1} W^{2M-\varepsilon M} p^{2M-\varepsilon M-1}\,|D\eta_n|^{2M}\,\sqrt{a\,(Dp, Dp)}\sqrt{a\,(D\eta_n, D\eta_n)} \\
\leq& 2\sqrt{\frac{2^{2M-2}\beta^{2M} M\,(2M-\varepsilon M-1)}{(2-\varepsilon)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M-2}} \\
&\cdot \sqrt{|D\eta_n|^{2M}\, a\,(Dp, Dp)}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \sqrt{\frac{2^{2M+2}\beta^{2M}\left(\beta-1\right)^2 M^3 L^{2M+2}}{\left(2-\varepsilon\right)^{2M-1}\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}} \\
&\cdot \sqrt{p^{2M-\varepsilon M}\frac{|a|}{1+|y|^{2M+2}}\mathbb{1}_{\{n\le|y|\le 2n\}}} \\
\le&\frac{2^{2M-2}\beta^{2M}M\left(2M-\varepsilon M-1\right)}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2} \\
&\cdot \left|D\eta_n\right|^{2M}a\left(Dp,Dp\right) \\
&+\frac{2^{2M+2}\beta^{2M}\left(\beta-1\right)^2 M^3 L^{2M+2}}{\left(2-\varepsilon\right)^{2M-1}\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M} \\
&\cdot p^{2M-\varepsilon M}\frac{|a|}{1+|y|^{2M+2}}\mathbb{1}_{\{n\le|y|\le 2n\}},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{2^{2M}\beta^{2M}M}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-1}\left|D\eta_n\right|^{2M}\sqrt{\frac{1}{\lambda}\left|F\right|^2}\sqrt{a\left(Dp,Dp\right)} \\
\le&2\sqrt{\frac{2^{2M-2}\beta^{2M}M\left(2M-\varepsilon M-1\right)}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2}} \\
&\cdot \sqrt{\left|D\eta_n\right|^{2M}a\left(Dp,Dp\right)} \\
&\cdot \sqrt{\frac{2^{2M}\beta^{2M}ML^{2M}}{\lambda\left(2-\varepsilon\right)^{2M-1}\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}} \\
&\cdot \sqrt{p^{2M-\varepsilon M}\frac{\left|F\right|^2}{1+|y|^{2M}}\mathbb{1}_{\{n\le|y|\le 2n\}}} \\
\le&\frac{2^{2M-2}\beta^{2M}M\left(2M-\varepsilon M-1\right)}{\left(2-\varepsilon\right)^{2M-1}}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}p^{2M-\varepsilon M-2} \\
&\cdot \left|D\eta_n\right|^{2M}a\left(Dp,Dp\right) \\
&+\frac{2^{2M}\beta^{2M}ML^{2M}}{\lambda\left(2-\varepsilon\right)^{2M-1}\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M} \\
&\cdot p^{2M-\varepsilon M}\frac{\left|F\right|^2}{1+|y|^{2M}}\mathbb{1}_{\{n\le|y|\le 2n\}},
\end{aligned}
$$

$$
\begin{aligned}
&2\beta M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M-1}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)} \\
\le&2\sqrt{\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right)} \\
&\cdot \sqrt{\frac{4\beta^2 M^3\left(2-\varepsilon\right)L^2}{\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M-2}W^{-\varepsilon M}p^{2M-\varepsilon M}\left|DW\right|^{2M}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}} \\
\le&\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right) \\
&+\frac{4\beta^2 M^3\left(2-\varepsilon\right)L^2}{\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{\left|DW\right|^{2M}}{W^{2M}}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}},
\end{aligned}
$$

$$
\begin{aligned}
&M\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}p^{2M-\varepsilon M-1}W^{-\varepsilon M}\left|DW\right|^{2M}\sqrt{\frac{1}{\lambda}\left|F\right|^2}\sqrt{a\left(Dp,Dp\right)}\\
&\qquad\le 2\sqrt{\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right)}\\
&\qquad\quad\cdot\sqrt{\frac{M\left(2-\varepsilon\right)}{\lambda\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{\left|DW\right|^{2M}}{W^{2M}}\left|F\right|^2}\\
&\qquad\le\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right)\\
&\qquad\quad+\frac{M\left(2-\varepsilon\right)}{\lambda\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{\left|DW\right|^{2M}}{W^{2M}}\left|F\right|^2,
\end{aligned}
$$

$$
\begin{aligned}
&2M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-1}\left|DW\right|^{2M-2}\\
&\qquad\quad\cdot\sum_{i=1}^{N}\sqrt{\left(D_iW\right)^2a\left(Dp,Dp\right)}\sqrt{a\left(D\left(D_iW\right),D\left(D_iW\right)\right)}\\
&\qquad\le 2\sum_{i=1}^{N}\sqrt{\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M-2}}\\
&\qquad\quad\cdot\sqrt{\left(D_iW\right)^2a\left(Dp,Dp\right)}\\
&\qquad\quad\cdot\sqrt{\frac{4M^3\left(2-\varepsilon\right)}{\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{\left|DW\right|^{2M-2}}{W^{2M-2}}}\\
&\qquad\quad\cdot\sqrt{\frac{a\left(D\left(D_iW\right),D\left(D_iW\right)\right)}{W^2}}\\
&\qquad\le\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right)\\
&\qquad\quad+\frac{4M^3\left(2-\varepsilon\right)}{\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{\left|DW\right|^{2M-2}}{W^{2M-2}}\\
&\qquad\quad\cdot\sum_{i=1}^{N}\frac{a\left(D\left(D_iW\right),D\left(D_iW\right)\right)}{W^2},
\end{aligned}
$$

$$
\begin{aligned}
&\varepsilon M^2\left(2-\varepsilon\right)\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M-1}p^{2M-\varepsilon M-1}\left|DW\right|^{2M}\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(DW,DW\right)}\\
&\qquad\le 2\sqrt{\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right)}\\
&\qquad\quad\cdot\sqrt{\frac{\varepsilon^2M^3\left(2-\varepsilon\right)}{\lambda^M\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\frac{a\left(DW,DW\right)}{W^2}\right)^{M+1}}\\
&\qquad\le\frac{M\left(2-\varepsilon\right)\left(2M-\varepsilon M-1\right)}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{-\varepsilon M}p^{2M-\varepsilon M-2}\left|DW\right|^{2M}a\left(Dp,Dp\right)\\
&\qquad\quad+\frac{\varepsilon^2M^3\left(2-\varepsilon\right)}{\lambda^M\left(2M-\varepsilon M-1\right)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\frac{a\left(DW,DW\right)}{W^2}\right)^{M+1}.
\end{aligned}
$$

We then get

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^N} \omega_n dy \quad \leq \quad & \int_{\mathbb{R}^N} \Bigg(- \left(\varepsilon M - \varepsilon^2 M^2 - \frac{3\varepsilon^2 M^2}{4} \right) \tau^{2\delta M} \eta_n^{2\beta M} \\
& \quad \cdot W^{2M-\varepsilon M} \frac{1}{p^{\varepsilon M+2}} |Dp|^{2M} a(Dp, Dp) \\
& + M(2-\varepsilon) \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} |\operatorname{div} F + H| \frac{|DW|^{2M}}{W^{2M}} \\
& + \frac{2^{2M+3}\beta^{2M}\delta M L^{2M}}{(2-\varepsilon)^{2M}} \tau^{2\delta M-1} \eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} \\
& \quad \cdot \frac{1}{\alpha} \frac{1}{1+|y|^{2M}} \mathbb{1}_{\{n \leq |y| \leq 2n\}} \\
& + \frac{2^{2M}\beta^{2M} M L^{2M}}{(2-\varepsilon)^{2M-1}} \tau^{2\delta M} \eta_n^{2\beta M-2M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} \frac{|\operatorname{div} F + H|}{1+|y|^{2M}} \mathbb{1}_{\{n \leq |y| \leq 2n\}} \\
& + \frac{2^{2M+3}\delta M}{3^{2M-1}(2-\varepsilon)^{2M}} \tau^{2\delta M-1} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} \frac{1}{\alpha} \\
& + \frac{2^{2M} M}{3^{2M-1}(2-\varepsilon)^{2M-1}} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} |\operatorname{div} F + H| \\
& + \frac{3^{2M+2}(M-1)^{\frac{M-1}{2}} M^{\frac{M+5}{2}}}{4\varepsilon^{M-1}\lambda^M (M+1)^{\frac{M+1}{2}}} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} |\operatorname{div} F + H|^{M+1} \\
& + \frac{3 \cdot 2^{\frac{8M-1}{3}} (2-\varepsilon)^{\frac{2M+2}{3}} M^2 (2M-1)^{\frac{2M-1}{3}}}{\varepsilon^{\frac{4M-2}{3}} \lambda^{\frac{2M-1}{3}} (2M+2)^{\frac{2M+2}{3}}} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} \\
& \quad \cdot \left(|\operatorname{div} F + H|^{M+1} \right)^{\frac{2}{3}} \left(\frac{|DW|^{2M+2}}{W^{2M+2}} \right)^{\frac{1}{3}} \\
& + \frac{2^{5M} M}{\varepsilon^{M-1}\lambda^M} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} |\operatorname{div} F + H|^{M+1} \mathbb{1}_{\{n \leq |y| \leq 2n\}} \\
& + \frac{\lambda \beta^{2M+2} M L^{2M+2}}{\varepsilon^{2M}} \tau^{2\delta M} \eta_n^{2\beta M-2M-2} W^{2M-\varepsilon M} p^{2M-\varepsilon M} \mathbb{1}_{\{n \leq |y| \leq 2n\}} \\
& + \frac{2^{6M+3} M^2 (2M+1)^{2M+1}}{\varepsilon^{2M}\lambda^{2M+1}(M+1)^{2M+2}} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} (|Da| + |F|)^{2M+2} \\
& + 8\delta M \tau^{2\delta M-1} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} \frac{|DW|^{2M}}{W^{2M}} \frac{1}{\alpha} \\
& + \frac{2^{2M} M^3}{3^{2M-1}(2-\varepsilon)^{2M-3}(2M-\varepsilon M-1)} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} \\
& \quad \cdot p^{2M-\varepsilon M} \frac{a(DW, DW)}{W^2} \\
& + \frac{2^{2M} M}{3^{2M-1}(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)\lambda} \tau^{2\delta M} \eta_n^{2\beta M} W^{2M-\varepsilon M} p^{2M-\varepsilon M} |F|^2 \\
& + \frac{2^{2M+1} M^3 \beta^2 L^2}{3^{2M-1}(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)} \tau^{2\delta M} \eta_n^{2\beta M-2} W^{2M-\varepsilon M}
\end{aligned}$$

$$
\begin{aligned}
&\quad \cdot p^{2M-\varepsilon M}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{2^{2M}\beta^{2M}M^3L^{2M}}{(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}\\
&\quad \cdot p^{2M-\varepsilon M}\frac{|a|}{1+|y|^{2M+2}}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{2^{2M}\beta^{2M}M^3L^{2M}}{(2-\varepsilon)^{2M-3}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}\\
&\quad \cdot p^{2M-\varepsilon M}\frac{a(DW,DW)}{W^2}\frac{1}{1+|y|^{2M}}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{2^{2M+2}\beta^{2M}(\beta-1)^2M^3L^{2M+2}}{(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}\\
&\quad \cdot p^{2M-\varepsilon M}\frac{|a|}{1+|y|^{2M+2}}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{2^{2M}\beta^{2M}ML^{2M}}{\lambda(2-\varepsilon)^{2M-1}(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2M}W^{2M-\varepsilon M}\\
&\quad \cdot p^{2M-\varepsilon M}\frac{|F|^2}{1+|y|^{2M}}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{4\beta^2M^3(2-\varepsilon)L^2}{(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\\
&\quad \cdot\frac{|DW|^{2M}}{W^{2M}}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{M(2-\varepsilon)}{\lambda(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\frac{|DW|^{2M}}{W^{2M}}|F|^2\\
&+\frac{4M^3(2-\varepsilon)}{(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\\
&\quad \cdot\frac{|DW|^{2M-2}}{W^{2M-2}}\sum_{i=1}^{N}\frac{a(D(D_iW),D(D_iW))}{W^2}\\
&+\frac{\varepsilon^2M^3(2-\varepsilon)}{\lambda^M(2M-\varepsilon M-1)}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\left(\frac{a(DW,DW)}{W^2}\right)^{M+1}\\
&+8M^2\beta^2L^2\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+2M^2(2-\varepsilon)^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\frac{a(DW,DW)}{W^2}\\
&+4\beta M^2L\tau^{2\delta M}\eta_n^{2\beta M-1}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\frac{|Da|+|F|}{1+|y|}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+8\beta^2M^3L^2\tau^{2\delta M}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\varepsilon M(2M-1)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}|\operatorname{div}F+H|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{81M^3}{8\lambda}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}(|Da|+|F|)^2\\
&+4M^2(2-\varepsilon)\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M-1}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}(|Da|+|F|)|DW|\\
&+8\delta M\tau^{2\delta M-1}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\frac{1}{\alpha}\\
&+2M^3(2-\varepsilon)^2\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\frac{a(DW,DW)}{W^2}\Bigg)dy.
\end{aligned}
$$

We remark that

$$
0<2\delta M-1 \qquad \text{and} \qquad 0<2\beta M-2M-2<2\beta M-2
$$

so that it holds

$$
0\le\tau^{2\delta M}\le\tau^{2\delta M-1}\le 1 \qquad \text{and} \qquad 0\le\eta_n^{2\beta M}\le\eta_n^{2\beta M-2}\le\eta_n^{2\beta M-2M-2}\le 1.
$$

Using Young's inequality, (1.10) and the inequalities

$$
\frac{M(2-\varepsilon)}{2M-\varepsilon M-1}\le\frac{3}{2} \qquad \text{and} \qquad \varepsilon\le\frac{1}{2M},
$$

we infer that

$$
\begin{aligned}
\partial_t\int_{\mathbb{R}^N}\omega_n dy \;\le\; & \int_{\mathbb{R}^N}\Bigg[-\left(\varepsilon M-\varepsilon^2M^2-\frac{3\varepsilon^2M^2}{4}\right)\tau^{2\delta M}\eta_n^{2\beta M}\\
&\quad\cdot W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}|Dp|^{2M}a(Dp,Dp)\\
&+\frac{C}{\varepsilon^{2M}}\tau^{2\delta M-1}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M}\\
&\quad\cdot\Bigg[C_0\left(\frac{|a|}{1+|y|^2}+|F|^2+|\operatorname{div}F+H|+\frac{a(DW,DW)}{W^2}+1\right)^{M+1}\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{1}{\alpha^{M+1}}+\Bigg(|Da|^2+|F|^2+|\operatorname{div}F+H|\\
&+\frac{a(DW,DW)}{W^2}+\sqrt{\sum_{i=1}^N\frac{a(D(D_iW),D(D_iW))}{W^2}}+1\Bigg)^{M+1}\Bigg]\\
&+C\tau^{2\delta M-1}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}\\
&\quad\cdot\Bigg[C_0\left(\frac{|a|}{1+|y|^2}+\frac{|Da|+|F|}{1+|y|}\right)\mathbb{1}_{\{n\le|y|\le 2n\}}\\
&+\frac{1}{\alpha}+|Da|^2+|F|^2+|\operatorname{div}F+H|+\frac{a(DW,DW)}{W^2}\Bigg]\Bigg]dy, \qquad (2.19)
\end{aligned}
$$

for a constant $C_0=C_0(\alpha,\varepsilon,\lambda,M,N)>0$ and a constant $C=C(\lambda,M,N)>0$. Moreover, for arbitrary $U\ge 0$ it holds

$$
\tau^{2\delta M-1}\eta_n^{2\beta M-2}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M}}|Dp|^{2M}U
$$

$$
\begin{aligned}
&\leq \left(\frac{M+1}{M}\frac{\varepsilon^2 M^2}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M} a\left(Dp,Dp\right)\right)^{\frac{M}{M+1}} \\
&\quad\cdot\left(\frac{2^{2M}}{\varepsilon^{2M}\lambda^M M^M\left(M+1\right)^M}\tau^{2\delta M-M-1}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M}U^{M+1}\right)^{\frac{1}{M+1}} \\
&\leq \frac{\varepsilon^2 M^2}{4}\tau^{2\delta M}\eta_n^{2\beta M}W^{2M-\varepsilon M}\frac{1}{p^{\varepsilon M+2}}\left|Dp\right|^{2M} a\left(Dp,Dp\right) \\
&\quad+\frac{2^{2M}}{\varepsilon^{2M}\lambda^M M^M\left(M+1\right)^{M+1}}\tau^{2\delta M-M-1}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M}U^{M+1}.
\end{aligned}
\tag{2.20}
$$

From (2.19), (2.20) and the fact that $\varepsilon M-2\varepsilon^2M^2\geq 0$ it then follows

$$
\begin{aligned}
\partial_t\int_{\mathbb{R}^N}\omega_n dy \leq \int_{\mathbb{R}^N} &\frac{C}{\varepsilon^{2M}}\tau^{2\delta M-M-1}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M}p^{2M-\varepsilon M} \\
&\cdot\Bigg[C_0\Bigg(\frac{|a|}{1+|y|^2}+|Da|^2+|F|^2 \\
&\quad+|\operatorname{div}F+H|+\frac{a\left(DW,DW\right)}{W^2}+1\Bigg)^{M+1}\mathbb{1}_{\{n\leq|y|\leq 2n\}} \\
&\quad+\frac{1}{\alpha^{M+1}}+\Bigg(|Da|^2+|F|^2+|\operatorname{div}F+H|+\frac{a\left(DW,DW\right)}{W^2} \\
&\quad+\sqrt{\sum_{i=1}^N\frac{a\left(D\left(D_iW\right),D\left(D_iW\right)\right)}{W^2}}+1\Bigg)^{M+1}\Bigg]dy
\end{aligned}
\tag{2.21}
$$

for a constant $C_0=C_0\left(\alpha,\varepsilon,\lambda,M,N\right)>0$ and a constant $C=C\left(\lambda,M,N\right)>0$. Thus Condition 2.1 implies

$$
\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right)\leq C\int_{\mathbb{R}^N}\tau^{2\delta M-M-1}\eta_n^{2\beta M-2M-2}W^{2M-\varepsilon M-1}p^{2M-\varepsilon M-1}\,pV dy,
\tag{2.22}
$$

with a constant $C=C\left(\alpha,\varepsilon,\lambda,M,N\right)>0$. Moreover, from (2.22) and (2.4) we deduce

$$
\begin{aligned}
&\partial_t\left(\int_{\mathbb{R}^N}\omega_n\left(x,y,t\right)dy\right) \\
&\quad\leq C\int_{\mathbb{R}^N}\left(\left(\tau\left(t\right)^\delta\eta_n\left(y\right)^\beta W\left(y\right)^{1-\frac{\varepsilon}{2}}p\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right)^{2M}\right)^{1-\frac{1}{2M-\varepsilon M}}p\left(x,y,t\right)V\left(y\right)dy \\
&\quad\leq C\left(\sup_{y\in\mathbb{R}^N}\left|\tau\left(t\right)^\delta\eta_n\left(y\right)^\beta W\left(y\right)^{1-\frac{\varepsilon}{2}}p\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right|^{2M}\right)^{1-\frac{1}{2M-\varepsilon M}}\int_{\mathbb{R}^N}p\left(x,y,t\right)V\left(y\right)dy
\end{aligned}
$$

for all $(x,t)\in\mathbb{R}^N\times(0,\infty)$. From (2.11) it then follows

$$
\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right)\leq C\left(\int_{\mathbb{R}^N}\omega_n dy\right)^{1-\frac{1}{2M-\varepsilon M}}\int_{\mathbb{R}^N}pV dy
\tag{2.23}
$$

for a suitable constant $C = C(\alpha, \varepsilon, \lambda, M, N) > 0$. We remark that (1.19) and (2.10) yield

$$0 < \theta(x,t) := \int_{\mathbb{R}^N} \omega_1(x,y,t)\, dy \le \int_{\mathbb{R}^N} \omega_n(x,y,t)\, dy \tag{2.24}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^N$ and $t \ge \frac{\alpha}{2}$. Observe that $t \longmapsto \theta(x,t)$ is continuous for $t > \frac{\alpha}{2}$. Then from (2.23) it follows

$$\partial_t \left(\left(\int_{\mathbb{R}^N} \omega_n dy \right)^{\frac{1}{2M-\varepsilon M}} \right) \le C \int_{\mathbb{R}^N} pV dy$$

for a suitable constant $C = C(\alpha, \varepsilon, \lambda, M, N) > 0$. Integrating from $\frac{\alpha}{2}$ to $t \ge \alpha$, we compute

$$\int_{\mathbb{R}^N} \omega_n dy \le C \left(\int_{\frac{\alpha}{2}}^t \int_{\mathbb{R}^N} pV dyds \right)^{2M-\varepsilon M}$$

for a suitable constant $C = C(\alpha, \varepsilon, \lambda, M, N) > 0$ and since $\omega_n\left(x, y, \frac{\alpha}{2}\right) = 0$ for all $x, y \in \mathbb{R}^N$. Using (2.11) again, we deduce

$$\sup_{y \in \mathbb{R}^N} \left| \tau(t)^\delta \eta_n(y)^\beta W(y)^{1-\frac{\varepsilon}{2}} p(x,y,t)^{1-\frac{\varepsilon}{2}} \right|^{2M} \le C \left(\int_{\frac{\alpha}{2}}^t \int_{\mathbb{R}^N} p(x,y,s) V(y)\, dyds \right)^{2M-\varepsilon M}.$$

for a suitable constant $C = C(\alpha, \varepsilon, \lambda, M, N) > 0$. For $t \ge \alpha$ and $y \in \overline{B(0,n)}$ we get

$$|W(y) p(x,y,t)|^{2M-\varepsilon M} \le C \left(\int_{\frac{\alpha}{2}}^t \int_{\mathbb{R}^N} p(x,y,s) V(y)\, dyds \right)^{2M-\varepsilon M}$$

so that

$$\sup_{y \in \mathbb{R}^N} |W(y) p(x,y,t)| \le C \int_{\frac{\alpha}{2}}^t \int_{\mathbb{R}^N} p(x,y,s) V(y)\, dyds \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [\alpha, \infty).$$

for a suitable constant $C = C(\alpha, \varepsilon, \lambda, M, N) > 0$. Then (1.21) implies that

$$\sup_{y \in \mathbb{R}^N} |W(y) p(x,y,t)| \le \frac{C}{K} V(x) e^{Kt} \tag{2.25}$$

for $C = C(\alpha, \varepsilon, \lambda, M, N) > 0$ and all $(x,t) \in \mathbb{R}^N \times [\alpha, \infty)$. Since $\alpha > 0$ can be arbitrary close to 0, it follows that

$$\sup_{y \in \mathbb{R}^N} |W(y) p(x,y,t)| < \infty \tag{2.26}$$

for all $(x,t) \in \mathbb{R}^N \times (0, \infty)$. We remark that then for each fixed $x \in \mathbb{R}^N$ and each $t > 0$ dominated convergence theorem and (1.21) yield

$$\begin{aligned} &\int_{\mathbb{R}^N} W(y)^{2M-\varepsilon M-1} p(x,y,t)^{2M-\varepsilon M} V(y)\, \mathbb{1}_{\{n \le |y| \le 2n\}} dy \\ &\quad \le \left(\sup_{y \in \mathbb{R}^N} |W(y) p(x,y,t)| \right)^{2M-\varepsilon M-1} \int_{\mathbb{R}^N} p(x,y,t) V(y)\, \mathbb{1}_{\{n \le |y| \le 2n\}} dy \to 0 \qquad \text{as } n \to \infty. \end{aligned} \tag{2.27}$$

From (2.21) and Condition 2.1 it then follows

$$\partial_t \int_{\mathbb{R}^N} \omega_n(x,y,t)\, dy \le \int_{\mathbb{R}^N} \frac{C}{\varepsilon^{2M}} \tau(t)^{2\delta M - M - 1} \eta_n(y)^{2\beta M - 2M - 2} W(y)^{2M - \varepsilon M} p(x,y,t)^{2M - \varepsilon M} \cdot \left(\frac{1}{\alpha^{M+1}} + \Psi(y) \right) dy + \nu_n(x,t), \tag{2.28}$$

where

$$\Psi = \left(1 + |Da|^2 + |F|^2 + |\operatorname{div} F + H| + \frac{a(DW, DW)}{W^2} + \sqrt{\sum_{i=1}^{N} \frac{a(D(D_i W), D(D_i W))}{W^2}} \right)^{M+1},$$

$$\nu_n(x,t) = \frac{C_0}{\varepsilon^{2M}} \int_{\mathbb{R}^N} W(y)^{2M - \varepsilon M - 1} p(x,y,t)^{2M - \varepsilon M} V(y)\, \mathbb{1}_{\{n \le |y| \le 2n\}} dy \tag{2.29}$$

for constants $C = C(\lambda, M, N) > 0$ and $C_0 = C_0(\alpha, \varepsilon, \lambda, M, N) > 0$. Hence $0 \le \nu_n(x,t) \to 0$ as $n \to \infty$ for all fixed $(x,t) \in \mathbb{R}^N \times 0, \infty.$, As above, from (2.28) and (2.4) and further from (2.11) it follows

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^N} \omega_n(x,y,t)\, dy \;\le\;& \frac{C}{\varepsilon^{2M}} \int_{\mathbb{R}^N} \left(\tau(t)^{\delta} \eta_n(y)^{\beta} W(y)^{1-\frac{\varepsilon}{2}} p(x,y,t)^{1-\frac{\varepsilon}{2}} \right)^{\frac{2M-\varepsilon M-1}{1-\frac{\varepsilon}{2}}} \\
& \cdot p(x,y,t) \left(W(y) \frac{1}{\alpha^{M+1}} + W(y)\Psi(y) \right) dy + \nu_n(x,t) \\
\le\;& \frac{C}{\varepsilon^{2M}} \left(\sup_{y \in \mathbb{R}^N} \left| \tau(t)^{\delta} \eta_n(y)^{\beta} W(y)^{1-\frac{\varepsilon}{2}} p(x,y,t)^{1-\frac{\varepsilon}{2}} \right|^{2M} \right)^{1-\frac{1}{2M-\varepsilon M}} \\
& \cdot \int_{\mathbb{R}^N} p(x,y,t) \left(W(y) \frac{1}{\alpha^{M+1}} + W(y)\Psi(y) \right) dy + \nu_n(x,t) \\
\le\;& \frac{C}{\varepsilon^{2M}} \left(\frac{3^{2M-1}(2-\varepsilon)^{2M}}{2^{2M} S} \int_{\mathbb{R}^N} \omega_n(x,y,t)\, dy \right)^{1-\frac{1}{2M-\varepsilon M}} \\
& \cdot \int_{\mathbb{R}^N} p(x,y,t) \left(W(y) \frac{1}{\alpha^{M+1}} + W(y)\Psi(y) \right) dy + \nu_n(x,t)
\end{aligned}$$

for $C = C(\lambda, M, N) > 0$. By means of (2.24) we conclude

$$\begin{aligned}
\partial_t \left(\left(\int_{\mathbb{R}^N} \omega_n(x,y,t)\, dy \right)^{\frac{1}{2M-\varepsilon M}} \right) \le & \frac{C}{\varepsilon^{2M}(2M - \varepsilon M)} \\
& \cdot \int_{\mathbb{R}^N} p(x,y,t) \left(W(y) \frac{1}{\alpha^{M+1}} + W(y)\Psi(y) \right) dy \\
& + \frac{1}{(2M - \varepsilon M)} \theta(x,t)^{\frac{1}{2M-\varepsilon M} - 1} \nu_n(x,t)
\end{aligned}$$

for $C = C(\lambda, M, N) > 0$. Since $\omega_n\left(x, y, \frac{\alpha}{2}\right) = 0$ for all $x, y \in \mathbb{R}^N$, integrating from $\frac{\alpha}{2}$ to $t \ge \alpha$, we observe

$$\left(\int_{\mathbb{R}^N} \omega_n(x,y,t)\, dy \right)^{\frac{1}{2M-\varepsilon M}} \le \frac{C}{\varepsilon^{2M}(2M - \varepsilon M)} \int_{\frac{\alpha}{2}}^{t} \int_{\mathbb{R}^N} p(x,y,s)$$

$$\cdot \left(W(y) \frac{1}{\alpha^{M+1}} + W(y)\Psi(y) \right) dyds$$

$$+ \frac{1}{(2M - \varepsilon M)} \int_{\frac{\alpha}{2}}^{t} \theta(x,s)^{\frac{1}{2M-\varepsilon M}-1} \nu_n(x,s)\, ds. \qquad (2.30)$$

for $C = C(\lambda, M, N) > 0$. Then we obtain with (2.11)

$$\sup_{y\in\mathbb{R}^N} \left| \eta_n(y)^{2M+2} W(y)\, p(x,y,t) \right|$$

$$\leq \frac{C}{\varepsilon^{2M}} \int_{\frac{\alpha}{2}}^{t} \int_{\mathbb{R}^N} p(x,y,s) \left(W(y) \frac{1}{\alpha^{M+1}} + W(y)\Psi(y) \right) dyds$$

$$+ \frac{1}{(2M - \varepsilon M)} \int_{\frac{\alpha}{2}}^{t} \theta(x,s)^{\frac{1}{2M-\varepsilon M}-1} \nu_n(x,s)\, ds. \qquad (2.31)$$

for a suitable constant $C = C(\lambda, M, N) > 0$ and all $(x,t) \in \mathbb{R}^N \times [\alpha, \infty)$. Observe that (2.25), (2.26) and (1.21) yield

$$\nu_n(x,s) \leq C_0 \left(V(x)\, e^{Ks} \right)^{2M-\varepsilon M} \qquad \text{for all } (x,s) \in \mathbb{R}^N \times \left[\frac{\alpha}{2}, \infty \right)$$

for a suitable constant $C_0 = C_0(\alpha, \varepsilon, K, \lambda, M, N) > 0$. As above, from (2.27), (2.29), (1.21) and dominated convergence theorem we conclude that for all fixed $(x,t) \in \mathbb{R}^N \times [\alpha, \infty)$ it holds

$$\frac{1}{(2M - \varepsilon M)} \int_{\frac{\alpha}{2}}^{t} \theta(x,s)^{\frac{1}{2M-\varepsilon M}-1} \nu_n(x,s)\, ds \to 0 \qquad \text{as } n \to \infty.$$

Letting $n \to \infty$ in (2.31) we get

$$\sup_{y\in\mathbb{R}^N} |W(y)\, p(x,y,t)| \leq \frac{C}{\varepsilon^{2M}} \int_{\frac{\alpha}{2}}^{t} \int_{\mathbb{R}^N} p(x,y,s) \left(W(y) \frac{1}{\alpha^{M+1}} + W(y)\Psi(y) \right) dyds \qquad (2.32)$$

for all $(x,t) \in \mathbb{R}^N \times [\alpha, \infty)$. Let $\alpha = t$ and $\varepsilon = \frac{1}{2M}$ in (2.32). We arrive at

$$W(y)\, p(x,y,t) \leq C \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^N} p(x,z,s)\, W(z) \left(\Psi(z) + \frac{1}{t^{M+1}} \right) dzds$$

for a constant $C = C(\lambda, M, N) > 0$ and all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$. ∎

Example 2.4. We consider the operator A defined by

$$A = \left(1 + |x|^2\right)^{\alpha} \Delta - |x|^{2\beta}\, x \cdot D, \qquad 0 < \alpha < \beta,\ \beta \geq 1.$$

In this case we have

$$a_{ij}(x) = \delta_{ij} \left(1 + |x|^2\right)^{\alpha}, \qquad F_i = -\left(2\alpha \left(1 + |x|^2\right)^{\alpha-1} + |x|^{2\beta} \right) x_i$$

and

$$H(x) = 0.$$

Now let $\delta, C > 0$. Then for $V(x) = Ce^{\delta|x|^2}$ it holds

$$\begin{aligned} AV(x) &= 2\delta Ce^{\delta|x|^2}\left(-|x|^{2\beta+2} + N\left(1+|x|^2\right)^{\alpha} + 2\delta\left(1+|x|^2\right)^{\alpha}|x|^2\right) \\ &\leq 2\delta Ce^{\delta|x|^2}\left(-|x|^{2\beta+2} + 2^{\alpha}(N+2\delta)|x|^{2\alpha+2} + 2^{\alpha}(N+2\delta)\right) \\ &\leq KV(x), \end{aligned} \tag{2.33}$$

where

$$K = 2^{\alpha+1}\delta(N+2\delta)\left(\frac{2^{\frac{\alpha(\alpha+1)}{\beta-\alpha}}(N+2\delta)^{\frac{\alpha+1}{\beta-\alpha}}(\beta-\alpha)(\alpha+1)^{\frac{\alpha+1}{\beta-\alpha}}}{(\beta+1)^{\frac{\beta+1}{\beta-\alpha}}} + 1\right). \tag{2.34}$$

The function $V(x) = Ce^{\delta|x|^2}$ is thus a Lyapunov function for A for all $\delta, C > 0$.
We remark that If $0 < \alpha = \beta < 1$ then $V(x) = Ce^{\delta|x|^2}$ is a Lyapunov function only for $\delta \in \left(0, \frac{1}{2}\right)$ and all $C > 0$ and it holds

$$\begin{aligned} AV(x) &= 2\delta Ce^{\delta|x|^2}\left(-|x|^{2\alpha+2} + N\left(1+|x|^2\right)^{\alpha} + 2\delta\left(1+|x|^2\right)^{\alpha}|x|^2\right) \\ &\leq 2\delta Ce^{\delta|x|^2}\left(-|x|^{2\alpha+2} + N + N|x|^{2\alpha} + 2\delta|x|^2 + 2\delta|x|^{2\alpha+2}\right) \leq K_1V(x), \end{aligned}$$

where

$$K_1 = \frac{2^{\alpha+1}N^{\alpha+1}\delta\alpha^{\alpha}}{(1-2\delta)^{\alpha}(\alpha+1)^{\alpha+1}} + \frac{2^{\frac{2(\alpha+1)}{\alpha}}\delta^{\frac{2\alpha+1}{\alpha}}\alpha}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}(1-2\delta)^{\frac{1}{\alpha}}} + N.$$

If $1 \leq \alpha = \beta$, then $V(x) = Ce^{\delta|x|^2}$ is a Lyapunov function only for $\delta \in (0, 2^{-\alpha-1})$ and all $C > 0$ and it holds

$$\begin{aligned} AV(x) &= 2\delta Ce^{\delta|x|^2}\left(-|x|^{2\alpha+2} + N\left(1+|x|^2\right)^{\alpha} + 2\delta\left(1+|x|^2\right)^{\alpha}|x|^2\right) \\ &\leq 2\delta Ce^{\delta|x|^2}\left(-|x|^{2\alpha+2} + 2^{\alpha+1}\delta|x|^{2\alpha+2} + 2^{\alpha-1}N|x|^{2\alpha} + 2^{\alpha-1}N + 2^{\alpha+1}\delta\right) \\ &\leq K_2V(x), \end{aligned}$$

where

$$K_2 = \frac{2^{\alpha^2}N^{\alpha+1}\alpha^{\alpha}\delta}{(1-2^{\alpha+1}\delta)^{\alpha}(\alpha+1)^{\alpha+1}} + 2^{\alpha}N\delta + 2^{\alpha+2}\delta^2.$$

We return to the case $0 < \alpha < \beta$. Furthermore, for all $0 < \gamma < \delta < \infty$ there exists a constant $C > 0$ such that $V(x) = Ce^{\delta|x|^2}$ and $W(x) = e^{\gamma|x|^2}$ satisfy Condition 2.1 for each $M > \dfrac{N}{2}$ since the coefficients only grow polynomially. From (2.3) it then follows that for each $M > \frac{N}{2}$ such that $M \geq 2$ there exists a constant $C = C(\lambda, M, N, \alpha, \beta, \delta, \gamma) > 0$ such that it holds

$$p(x,y,t) \leq C\left(e^{Kt} - e^{\frac{K}{2}t}\right)\left(\frac{1}{t^{M+1}} + 1\right)\frac{e^{\delta|x|^2}}{e^{\gamma|y|^2}}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$.
Moreover, in this special case we can obtain a better estimate of p by a more direct estimate, see also Corollary 2.6 below. Let $W(x) = e^{\gamma|x|^2}$ and $V_0(x) = C_0e^{\delta|x|^2}$ for $0 < \gamma < \delta < \infty$ and $C_0 = C(\alpha, \beta, \delta, \gamma) \geq e$ such that

$$W \leq W\Psi \leq V_0.$$

From (2.33) we conclude

$$\begin{aligned} AV_0(x) &\leq -C_1 e^{\delta|x|^2}|x|^{2\beta+2}+C_2 \\ &= -C_1 e^{\delta|x|^2}\left(\frac{1}{\delta}\log e^{\delta|x|^2}\right)^{\beta+1}+C_2 \\ &= -C_3 V_0(x)(\log V_0(x))^{\beta+1}+C_2 \\ &\leq -C_3 V_0(x)(\log V_0(x))^2+C_2 V_0(x), \end{aligned}$$

where $C_1, C_2, C_3 > 0$ depend on α, β, δ and γ. We set

$$g(s) = C_3 s(\log s)^2 - C_2 s, \qquad s \geq 1.$$

Then,

$$AV_0(x) \leq -g(V_0(x)).$$

We remark that g is convex on $[1,\infty)$. From the fact that $\int_{\mathbb{R}^N} p(x,y,t) = 1$ for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ (since $H = 0$ and there exists a Lyapunov function for A, see (1.20)), (1.22) and Jensen's inequality (see [El09, VI, 1.3]) we deduce

$$\begin{aligned} \partial_t\left(\int_{\mathbb{R}^N} p(x,y,t)V_0(y)\,dy\right) &\leq \int_{\mathbb{R}^N} p(x,y,t)AV_0(y)\,dy \\ &\leq -\int_{\mathbb{R}^N} p(x,y,t)g(V_0(y))\,dy \\ &\leq -g\left(\int_{\mathbb{R}^N} p(x,y,t)V_0(y)\,dy\right). \end{aligned}$$

Thus for each fixed $x \in \mathbb{R}^N$ we have

$$\partial_t\left(\log\left(\int_{\mathbb{R}^N} p(x,y,t)V_0(y)\,dy\right)\right) \leq -C_3\left(\log\left(\int_{\mathbb{R}^N} p(x,y,t)V_0(y)\,dy\right)\right)^2 + C_2.$$

We set

$$\zeta(t) = \log\left(\int_{\mathbb{R}^N} p(x,y,t)V_0(y)\,dy\right) \geq \log\left(\int_{\mathbb{R}^N} p(x,y,t)e\,dy\right) = 1,$$

using (1.20). Then

$$\partial_t\zeta \leq -C_3\zeta^2 + C_2, \qquad \zeta(0) = \log V_0(x) \geq 1.$$

We then have

$$\partial_t\left(e^{-C_2t}\zeta\right) \leq -C_3\left(e^{-C_2t}\zeta\right)^2 e^{C_2t}$$

and hence

$$\partial_t\left(\frac{1}{e^{-C_2t}\zeta}\right) \geq C_3 e^{C_2t}. \tag{2.35}$$

Let now $0 < t_0 < \infty$ and $\tau \in C^\infty(\mathbb{R})$ be such that $0 \leq \tau \leq 1$, $\tau(t) = 0$ for $0 \leq t \leq \frac{t_0}{2}$, $\tau(t) = 1$ for $\tau \geq t_0$ and $\tau' \geq 0$. We multiply (2.35) by τ and get

$$\partial_t\left(\tau\frac{1}{e^{-C_2t}\zeta}\right) \geq C_3\tau e^{C_2t} + \tau'\frac{1}{e^{-C_2t}\zeta} \geq C_3\tau e^{C_2t}. \tag{2.36}$$

Integrating (2.36) from 0 to $t > t_0$ we obtain

$$\frac{1}{e^{-C_2 t}\zeta} \geq C_3 \int_0^t \tau(s) e^{C_2 s} ds \geq C_3 \int_{t_0}^t e^{C_2 s} ds = \frac{C_3}{C_2}\left(e^{C_2 t} - e^{C_2 t_0}\right).$$

We then have

$$\zeta \leq \frac{C_2 e^{C_2 t}}{C_3\left(e^{C_2 t} - e^{C_2 t_0}\right)}.$$

So, it follows

$$\int_{\mathbb{R}^N} p(x,y,t) V_0(y)\, dy \leq \exp\left(\frac{C_2 e^{C_2 t}}{C_3\left(e^{C_2 t} - e^{C_2 t_0}\right)}\right) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (t_0, \infty).$$

Setting $t_0 = \frac{t}{2}$, we then deduce

$$\int_{\mathbb{R}^N} p(x,y,t) V_0(y)\, dy \leq \exp\left(\frac{C_2 e^{C_2 t}}{C_3\left(e^{C_2 t} - e^{C_2 \frac{t}{2}}\right)}\right) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0, \infty).$$

We observe for $t > 0$

$$\begin{aligned}
\sup_{y\in\mathbb{R}^N} |W(y) p(x,y,t)| &\leq C \int_{\frac{t}{2}}^t \int_{\mathbb{R}^N} p(x,z,s) W(z) \left(\Psi(z) + \frac{1}{t^{M+1}}\right) dz ds \\
&\leq C\left(1 + \frac{1}{t^{M+1}}\right) \int_{\frac{t}{2}}^t \int_{\mathbb{R}^N} p(x,z,s) V_0(z)\, dz ds \\
&\leq C\left(1 + \frac{1}{t^{M+1}}\right) \int_{\frac{t}{2}}^t \exp\left(\frac{C_2 e^{C_2 s}}{C_3\left(e^{C_2 s} - e^{C_2 \frac{s}{2}}\right)}\right) ds \\
&\leq \frac{C}{2}\left(t + \frac{1}{t^M}\right) \exp\left(\frac{C_2 e^{C_2 \frac{t}{4}}}{C_3\left(e^{C_2 \frac{t}{4}} - 1\right)}\right) \\
&= \frac{C}{2}\left(t + \frac{1}{t^M}\right) e^{\frac{C_2}{C_3}} \exp\left(\frac{C_2}{C_3\left(e^{C_2 \frac{t}{4}} - 1\right)}\right).
\end{aligned}$$

Since $\delta > \gamma$ can be chosen arbitrary, we set $\delta = 2\gamma$. Therefore, for the operator A we deduce that for each $\gamma > 0$ there exist constants $C_1, C_2, C_3 > 0$ depending only on α and β such that

$$p(x,y,t) \leq C_1 \left(t + \frac{1}{t^M}\right) \exp\left(\frac{C_2}{\left(e^{C_3 t} - 1\right)}\right) e^{-\gamma|y|^2} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty).$$

Similar estimates for the case of bounded coefficients $(a_{ij})_{i,j=1,\ldots,N}$ one can find in [LMPR] and in [MPR06] in the case $H = 0$.
It also follows that for each $t_0 > 0$ and each $\gamma > 0$ there exists a constant $C = C(\alpha, \beta, \gamma, t_0) > 0$ such that

$$p(x,y,t) \leq Cte^{-\gamma|y|^2} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0, \infty). \tag{2.37}$$

We remark that the formal adjoint A^* of A has the form

$$A^* = \left(1 + |x|^2\right)^\alpha \Delta + \left(4\alpha\left(1 + |x|^2\right)^{\alpha-1} + |x|^{2\beta}\right) x \cdot D$$

$$+2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right)+(N+2\beta)|x|^{2\beta}$$

so that

$$A^*=A_0^*+F^*\cdot D-H^*$$

with

$$H^*(x)=\operatorname{div}F(x)=\begin{array}{l}-2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right)\\ -(N+2\beta)|x|^{2\beta}.\end{array}$$

We see that H^* is not bounded from below ($H_0^*=-\infty$) so that $\int_{\mathbb{R}^N}p(x,y,t)\,dx$ need not be finite and hence the above methods are not appliable for the estimation of $p(\cdot,y,\cdot)$ for a fixed $y\in\mathbb{R}^N$.
Moreover, Remark 1.9 implies that the semigroup $(T(t))_{t\geq0}$ is compact. From [BL07, Proposition 5.3.4] it then follows that for each $t>0$, $C_0\left(\mathbb{R}^N\right)$ is not invariant under $T(t)$. Then the estimate of the form $p(x,y,t)\leq e^{-\delta|x|^\gamma}\varphi(y)\tau(t)$ for $\delta,\gamma>0$, $0<\varphi\in C\left(\mathbb{R}^N\right)$, $0<\tau\in C(0,\infty)$ is not possible. In fact, if $f\in C_c\left(\mathbb{R}^N\right)$, then

$$u(x,t)=\int_{\mathbb{R}^N}p(x,y,t)f(y)\,dy,\qquad (x,t)\in\mathbb{R}^N\times(0,\infty),$$

is the unique solution of (1.9). Hence, if $p(x,y,t)\leq e^{-\delta|x|^\gamma}\varphi(y)\tau(t)$, then there exists a constant $C>0$ such that

$$|u(x,t)|\leq C\tau(t)e^{-\delta|x|^\gamma}.$$

Thus for each $t>0$ the function $u(\cdot,t)$ belongs to $C_0\left(\mathbb{R}^N\right)$ so that $T(t)\left(C_c\left(\mathbb{R}^N\right)\right)\subseteq C_0\left(\mathbb{R}^N\right)$. This is a contradiction to the compactness of the semigroup $(T(t))_{t\geq0}$.

Remark 2.5. In general we see that if $\operatorname{div}F+H\leq-\gamma$ for some $\gamma>0$, then

$$A^*\mathbb{1}(x)=-\operatorname{div}F(x)-H(x)\geq\gamma=\gamma\mathbb{1}(x)$$

and hence there is no Lyapunov function for the operator A^* (see [BL07, Proposition 4.2.1]).

Corollary 2.6. *Assume that Condition 2.1 holds and there exists a convex differentiable function $g:[0,\infty)\to\mathbb{R}$ such that $g(0)\leq0$, $\lim_{s\to\infty}g(s)=\infty$, $1/g$ is integrable in a neighborhood of ∞ and $AV\leq-g(V)$. Then for each $t_0>0$ there exists a constant $C=C(\lambda,M,N,t_0)>0$ such that*

$$p(x,y,t)\leq Ce^{\min\{-H_0,0\}t}t\frac{1}{W(y)}\qquad \textit{for all } (x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times[t_0,\infty).$$

Proof. From Theorem 2.2 we deduce that

$$W(y)p(x,y,t)\leq C\left(1+\frac{1}{t^{M+1}}\right)\int_{\frac{t}{2}}^t\int_{\mathbb{R}^N}p(x,z,s)V(z)\,dzds \tag{2.38}$$

with $C=C(\lambda,N,M)>0$ for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times(0,\infty)$. Let $t_0>0$. Proposition 1.8 yields the existence of a constant $C_0=C_0(t_0)>0$ such that

$$\int_{\mathbb{R}^N}p(x,z,s)V(z)\,dz\leq e^{\min\{-H_0,0\}t}C_0\qquad\text{for all } (x,s)\in\mathbb{R}^N\times\left[\frac{t_0}{2},\infty\right). \tag{2.39}$$

Letting $t \geq t_0$, we obtain from (2.38) and (2.39)

$$W(y)\,p(x,y,t) \leq \frac{CC_0}{2} e^{\min\{-H_0,0\}t}\left(1+\frac{1}{t^{M+1}}\right)t \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0,\infty)$$

that is for each $t_0 > 0$ there exists a constant $C = C(\lambda, M, N, t_0) > 0$ such that

$$p(x,y,t) \leq C e^{\min\{-H_0,0\}t}\left(1+\frac{1}{t^{M+1}}\right)t\frac{1}{W(y)} \leq C e^{\min\{-H_0,0\}t}\left(1+\frac{1}{t_0^{M+1}}\right)t\frac{1}{W(y)}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0,\infty)$.

■

Example 2.7. We consider the operator A defined by

$$A = \left(1+|x|^2\right)^{\alpha}\Delta - |x|^{2\beta}\,x\cdot D - |x|^{2\theta+2}, \qquad 0<\alpha<\beta<\theta,\ \beta \geq 1.$$

Analogous as in Example 2.4 we observe that for all $C, \delta > 0$ the function $V(x) = Ce^{\delta|x|^2}$ is a Lyapunov function for the operator A such that $AV \leq KV$ with K given in (2.34). Let now $\gamma > 0$ and $\delta > \gamma$ be such that $W(x) = e^{\gamma|x|^2}$ and $V(x) = e^{\delta\left(|x|^2+1\right)}$ satisfy Condition 2.1. Analogous as in Example 2.4 we calculate

$$\begin{aligned} AV(x) &= 2\delta e^{\delta\left(|x|^2+1\right)}\left(-\frac{1}{2\delta}|x|^{2\theta+2} - |x|^{2\beta+2} + N\left(1+|x|^2\right)^{\alpha} + 2\delta\left(1+|x|^2\right)^{\alpha}|x|^2\right) \\ &\leq 2\delta e^{\delta\left(|x|^2+1\right)}\left(-\frac{1}{2\delta}|x|^{2\theta+2} + K\right). \end{aligned}$$

Using

$$-|x|^{2\theta+2} \leq -2^{-\theta}\left(|x|^2+1\right)^{\theta+1} + 1, \tag{2.40}$$

we obtain

$$\begin{aligned} AV(x) &\leq 2\delta e^{\delta\left(|x|^2+1\right)}\left(-\frac{1}{\delta 2^{\theta+1}}\left(|x|^2+1\right)^{\theta+1} + \frac{1}{2\delta} + K\right) \\ &= -\frac{1}{\delta^{\theta+1}2^{\theta}} e^{\delta\left(|x|^2+1\right)}\left(\left(\log e^{\delta\left(|x|^2+1\right)}\right)^{\theta+1} - (1+2\delta K)\,\delta^{\theta+1}2^{\theta}\right) \\ &= -\frac{1}{\delta^{\theta+1}2^{\theta}} V(x)\left((\log V(x))^{\theta+1} - (1+2\delta K)\,\delta^{\theta+1}2^{\theta}\right). \end{aligned}$$

We further define the function $g_0 : \left[e^{\delta},\infty\right) \to \mathbb{R}$ by

$$g_0(s) = \frac{1}{\delta^{\theta+1}2^{\theta}} s\left((\log s)^{\theta+1} - (1+2\delta K)\,\delta^{\theta+1}2^{\theta}\right).$$

Observe that g_0 is convex on $\left[e^{\delta},\infty\right)$. We extend g_0 to $g : [0,\infty) \to \mathbb{R}$ such that g is convex, $g(0) \leq 0$ and $g(s) = g_0(s)$ for $s \in \left[e^{\delta},\infty\right)$. Moreover, $g(s) \to \infty$ as $s \to \infty$, $\frac{1}{g}$ is integrable in a neighborhood of ∞ and $AV(x) \leq -g(V(x))$ for each $x \in \mathbb{R}^N$. From Corollary 2.6 and the fact that $H_0 = 0$ it then follows that for each $t_0 > 0$ there exists a constant $C = C(\lambda, M, N, \alpha, \beta, \theta, \gamma, \delta, t_0) > 0$ such that

$$p(x,y,t) \leq Cte^{-\gamma|y|^2} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0,\infty). \tag{2.41}$$

We further have

$$\begin{aligned} A^* &= \left(1+|x|^2\right)^{\alpha}\Delta+\left(4\alpha\left(1+|x|^2\right)^{\alpha-1}+|x|^{2\beta}\right)x\cdot D \\ &\quad +2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right)+(N+2\beta)|x|^{2\beta}-|x|^{2\theta+2}. \end{aligned}$$

We remark that in this case it holds

$$\begin{aligned} H^*(x) &= -2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right)-(N+2\beta)|x|^{2\beta}+|x|^{2\theta+2} \\ &\geq -2^{\alpha-1}\alpha(N+2\alpha-2)-2^{\frac{(\theta+2)(\alpha-1)}{\theta-\alpha+2}}\alpha^{\frac{\theta+1}{\theta+2-\alpha}}(N+2\alpha-2)^{\frac{\theta+1}{\theta+2-\alpha}} \\ &\quad -2^{\frac{\beta}{\theta-\beta+1}}(N+2\beta)^{\frac{\theta+1}{\theta-\beta+1}}. \end{aligned}$$

Thus H^* is bounded from below and there exists $H_0^*=\inf_{x\in\mathbb{R}^N}H^*(x)\in(-\infty,0)$. We recall that in this case the transition kernel p^* of the semigroup $(T^*(t))_{t\geq 0}$ is given by (1.17). For $\delta, C>0$ and $V(x)=Ce^{\delta|x|^2}$ we calculate

$$\begin{aligned} A^*V(x) &= V(x)\Bigg(\left(1+|x|^2\right)^{\alpha}\left(2\delta N+4\delta^2|x|^2\right)+\left(4\alpha\left(1+|x|^2\right)^{\alpha-1}+|x|^{2\beta}\right)2\delta|x|^2 \\ &\quad +2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right)+(N+2\beta)|x|^{2\beta}-|x|^{2\theta+2}\Bigg) \\ &\leq V(x)\left(-\frac{1}{2}|x|^{2\theta+2}+K\right) \qquad (2.42) \\ &\leq KV(x) \end{aligned}$$

for some $K=K(\alpha,\beta,\theta,\delta)>0$. Therefore, $V(x)=Ce^{\delta|x|^2}$ is for all $\delta, C>0$ a Lyapunov-Function for A^*. Let $0<\gamma<\delta<\infty$ be so that $W(x)=e^{\gamma|x|^2}$ and $V(x)=e^{\delta\left(|x|^2+1\right)}$ satisfy

$$\begin{aligned} \frac{V}{W} &\geq \Bigg(1+\frac{|a|}{1+|y|^2}+|Da|^2+|F|^2+|H| \\ &\quad +\frac{a(DW,DW)}{W^2}+\left(\sum_{i=1}^{N}\frac{a\left(D\left(D_iW\right),D\left(D_iW\right)\right)}{W^2}\right)^{\frac{1}{2}}\Bigg)^{M+1}, \end{aligned}$$

namely Condition 2.1 for the adjoint operator A^*. From (2.42) and using (2.40) we deduce

$$A^*V(x)\leq-\frac{1}{(2\delta)^{\theta+1}}V(x)\left((\log V(x))^{\theta+1}-2^{\theta}\delta^{\theta+1}(1+2K)\right).$$

Setting $g(s)=\frac{1}{(2\delta)^{\theta+1}}s\left((\log s)^{\theta+1}-2^{\theta}\delta^{\theta+1}(1+2K)\right)$ for $s\in\left[e^{\delta},\infty\right)$, g is convex and differentiable on $[0,\infty)$ such that $g(0)\leq 0$, we observe that

$$A^*V(x)\leq -g(V(x)) \qquad \text{for each } x\in\mathbb{R}^N.$$

Moreover, $g(s)\to\infty$ as $s\to\infty$ and $\frac{1}{g}$ is integrable in a neighborhood of ∞. Corollary 2.6 implies that for each $t_0>0$ there exists a constant $C=C=C(\lambda,M,N,\alpha,\beta,\theta,\gamma,\delta,t_0)>0$ such that

$$p(x,y,t)\leq Ce^{-H_0^*t}te^{-\gamma|x|^2} \qquad \text{for all } (x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times[t_0,\infty).$$

Multiplying this estimate with (2.41) we obtain a constant $\sigma > 0$ such that for each $t_0 > 0$ there exists a constant $C = C(\lambda, M, N, \alpha, \beta, \theta, \sigma, \delta, t_0) > 0$ such that

$$p(x,y,t) \le Ce^{-\frac{\sigma}{2}\left(|x|^2+|y|^2\right)-\frac{H_0^*}{2}t}t \qquad \text{for all } x,y,t \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0,\infty), \tag{2.43}$$

where

$$H_0^* = \inf_{x\in\mathbb{R}^N} (\operatorname{div} F(x) + H(x)) < 0.$$

We now consider operators with a Lyapunov function $e^{\delta|x|^r}$ for $\delta > 0$ and $r > 2$. It holds

$$\begin{aligned} A\left(e^{\delta|x|^r}\right) = \delta r e^{\delta|x|^r}\Bigg(& |x|^{r-2}(r-2+\delta r|x|^r)\frac{a(x,x)}{|x|^2} + |x|^{r-2}\sum_{i=1}^N a_{ii} + |x|^{r-2}\sum_{i,j=1}^N D_j a_{ij} x_i \\ & + |x|^{r-1} F\cdot\frac{x}{|x|} - \frac{1}{\delta r}H\Bigg). \end{aligned} \tag{2.44}$$

Here we extend $\frac{a(x,x)}{|x|^2}$ and $\frac{x}{|x|}$ by 0 for $x = 0$. We see that if

$$|x|^{r-2}(r-2+\delta r|x|^r)\frac{a(x,x)}{|x|^2} + |x|^{r-2}\sum_{i=1}^N a_{ii} + |x|^{r-2}\sum_{i,j=1}^N D_j a_{ij} x_i + |x|^{r-1} F\cdot\frac{x}{|x|} - \frac{1}{\delta r}H$$

is bounded from above on $\mathbb{R}^N$, then $V(x) = e^{\delta|x|^r}$ is a Lyapunov function for A. We state a condition under which the transition kernel $p = p(x,y,t)$ decreases exponentially in $y \in \mathbb{R}^N$. A similar result one can find in [MPR06] for the case $(a_{ij})_{i,j=1,\dots,N} \in C_b^1(\mathbb{R}^N)$.

Proposition 2.8. *Assume that for the operator A defined in (1.8) it holds*

$$\left(\frac{r-2}{|x|^r} + \delta r\right)\frac{a(x,x)}{|x|^2} + \frac{1}{|x|^r}\sum_{i=1}^N a_{ii} + \frac{1}{|x|^r}\sum_{i,j=1}^N D_j a_{ij} x_i + |x|^{1-r} F\cdot\frac{x}{|x|} - \frac{1}{\delta r|x|^{2r-2}}H \le -C_0$$

for each $x \in \mathbb{R}^N \setminus B(0,R)$ and for some $R > 0$, $r > 2$, $\delta > 0$ and $C_0 > 0$. Then $V(x) = Ce^{\delta|x|^r}$ is a Lyapunov function for A for each constant $C > 0$. Further, assume that

$$|a| + |Da| + |F| + |\operatorname{div} F + H|$$

grows only polynomially. Then for each $M > \frac{N}{2}$ and each $0 < \gamma < \delta$ there exists a constant $C > 0$ such that it holds

$$p(x,y,t) \le Ce^{-\gamma|y|^r}\left(t + \frac{1}{t^M}\right)\exp\left\{\max\{-H_0, 0\}t + \left(\frac{t}{2}\right)^{-\frac{r}{r-2}}\right\} \tag{2.45}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$.

Proof. Since the coefficients of A grow only polynomially, for each $M > \frac{N}{2}$ and each $0 < \gamma < \delta$ there exists a constant $C \ge 1$, such that $V(x) = Ce^{\delta|x|^r}$ and $W(x) = e^{\gamma|x|^r}$ satisfy the inequality in Condition 2.1. From (2.44) and for $x \in \mathbb{R}^N \setminus B(0,R)$ it follows

$$AV(x) = \delta r V(x)|x|^{2r-2}\left(\left(\frac{r-2}{|x|^r} + \delta r\right)\frac{a(x,x)}{|x|^2} + \frac{1}{|x|^r}\sum_{i=1}^N a_{ii} + \frac{1}{|x|^r}\sum_{i,j=1}^N D_j a_{ij} x_i\right.$$

$$
\begin{aligned}
&\quad + |x|^{1-r} F \cdot \frac{x}{|x|} - \frac{1}{\delta r |x|^{2r-2}} H\Big) \\
\leq &\quad -\delta r C_0 V(x) |x|^{2r-2} \\
= &\quad -\delta^{-\frac{r-2}{r}} r C_0 V(x) (\log V(x) - \log C)^{2-\frac{2}{r}} \\
< &\quad 0.
\end{aligned}
$$

For $x \in B(0, R)$ we have

$$
AV(x) \leq \delta r C e^{\delta R^r} \Bigg(R^{r-2} \left(r - 2 + \delta r R^r + R^{r-2}\sqrt{N} \right) |a| + \sqrt{N} R^{r-1} |Da| + R^{r-1} |F| + \frac{1}{\delta r} |H_0| \Bigg).
$$

Thus there exist constants $C_1 = C_1(\delta, r, C_0) > 0$ and $C_2 = C_2\Big(\delta, r, R, N, H_0, \left(\|a_{ij}\|_{C^1(B(0,R))}\right)_{i,j=1,\ldots,N}\Big) > 0$ such that

$$
AV(x) \leq -\left(C_1 V(x) (\log V(x) - \log C)^{2-\frac{2}{r}} - C_2 \right) \qquad \text{for all } x \in \mathbb{R}^N. \tag{2.46}
$$

Moreover, V is a Lyapunov function for A with $AV \leq KV$ for some $K > \max\{-H_0, C_2\}$. It then follows from (2.1)

$$
e^{\gamma |y|^r} p(x, y, t) \leq C_3 \left(1 + \frac{1}{t^{M+1}} \right) \int_{\frac{t}{2}}^{t} \int_{\mathbb{R}^N} p(x, z, s) V(z)\, dy ds \tag{2.47}
$$

for all $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$ and some $C_3 = C_3(\lambda, M, N) > 0$. We further set

$$
g(s) = C_1 s (\log s - \log C)^{2-\frac{2}{r}} - C_2, \qquad s \geq C = V(0).
$$

We ramark that g is convex on $[C, \infty)$. From (2.46) we deduce that

$$
AV(x) \leq -g(V(x)) \qquad \text{for all } x \in \mathbb{R}^N.
$$

From the proof of Proposition 1.8 we obtain that $e^{\min\{H_0,0\}t} \int_{\mathbb{R}^N} p(x, y, t) V(y)\, dy \leq z(x, t)$, where $z = z(x, t)$ is the solution of the ordinary Cauchy problem

$$
\begin{cases} z' = -g(z), & t > 0, \\ z(x, 0) = V(x), \end{cases}
$$

for each fixed $x \in \mathbb{R}^N$. Let $z_0 \in \mathbb{R}$ denote the greatest zero of g. If $z(x, t) \leq 2z_0$, we have simply to choose a suitable constant in (2.45). If $z(x, t) \geq 2z_0$ (and thus $V(x) \geq z(x, t) \geq 2z_0$), then $g(s) > 0$ for all $s \in [z(x, t), \infty)$ and $t > 0$ and we obtain

$$
t = -\int_{V(x)}^{z(x,t)} \frac{ds}{g(s)} \leq \int_{z(x,t)}^{\infty} \frac{ds}{g(s)} = \int_{z(x,t)}^{\infty} \frac{ds}{C_1 s (\log s - \log C)^{2-\frac{2}{r}} - C_2}.
$$

We set $C_4 = \frac{g(2z_0)+C_2}{g(2z_0)}$. Then

$$
\frac{1}{C_1 s (\log s - \log C)^{2-\frac{2}{r}} - C_2} \leq \frac{C_4}{C_1 s (\log s - \log C)^{2-\frac{2}{r}}} \qquad \text{for all } s \in [z(x, t), \infty).
$$

It then follows that

$$t \leq \int_{z(x,t)}^{\infty} \frac{C_4 ds}{C_1 s \left(\log s - \log C\right)^{2-\frac{2}{r}}} = \frac{C_4}{C_1}\frac{r}{r-2}\frac{1}{\left(\log\left(\frac{z(x,t)}{C}\right)\right)^{\frac{r-2}{r}}}$$

and hence

$$z\left(x,t\right) \leq C_5 e^{t^{-\frac{r}{r-2}}}$$

for a suitable constant $C_5 > 0$. We can assume that $C_5 \geq 2z_0$. Thus we conclude that

$$\int_{\mathbb{R}^N} p\left(x,y,t\right) V\left(y\right) dy \leq C_5 e^{\max\{-H_0,0\}t} e^{t^{-\frac{r}{r-2}}} \qquad \text{for all } \left(x,t\right) \in \mathbb{R}^N \times \left(0,\infty\right). \quad (2.48)$$

Using (2.47), we observe that

$$p\left(x,y,t\right) \leq C_3 e^{-\gamma|y|^r}\left(1+\frac{1}{t^{M+1}}\right)\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^N} p\left(x,z,s\right) V\left(z\right) dyds \qquad (2.49)$$

Combining (2.48) and (2.49), we get

$$\begin{aligned} p\left(x,y,t\right) &\leq C_3 e^{-\gamma|y|^r}\left(1+\frac{1}{t^{M+1}}\right)\int_{\frac{t}{2}}^{t} C_5 e^{\max\{-H_0,0\}s} e^{s^{-\frac{r}{r-2}}} ds \\ &\leq C_6 e^{-\gamma|y|^r}\left(t+\frac{1}{t^M}\right)\exp\left\{\max\left\{-H_0,0\right\}t+\left(\frac{t}{2}\right)^{-\frac{r}{r-2}}\right\} \end{aligned}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$ and a suitable constant $C_6 > 0$. ∎

Corollary 2.9. *Under the assumptions of Proposition 2.8, the operator $T(t) : L^1\left(\mathbb{R}^N\right) \to L^\infty\left(\mathbb{R}^N\right)$ is bounded for each $t > 0$.*

2.2 The L^q-regularity of the gradient of the transition kernel

To study the L^q-regularity of gradient we specialize in Condition 2.1 to the case $W = 1$.

Condition 2.10. *We assume that Condition 1.1 holds. There exist $K > 0$, $M > \frac{N}{2}$ such that $M \geq 2$ and a Lyapunov function V with $AV \leq KV$, such that*

$$\left(1+\frac{|a|}{1+|y|^2}+|Da|^2+|F|^2+|\text{div}\, F+H|\right)^{M+1} \leq V.$$

Remark 2.3 a) yields boundedness of $p(x,\cdot,\cdot)$ on $Q(a,b)$ for all $0 < a < b < \infty$ under the above condition.
We first state a preliminary result which follows from the proof of Theorem 2.2.

Proposition 2.11. *Assume that Condition 2.10 holds and let $\varepsilon \in \left(0, \frac{1}{2M}\right]$. We then have*

$$\left|D\left(p\left(x,\cdot,t\right)^{1-\frac{\varepsilon}{2}}\right)\right|^2 \in L^M\left(\mathbb{R}^N\right) \qquad \textit{for all } \left(x,t\right) \in \mathbb{R}^N \times \left(0,\infty\right),$$

$$\left|D\left(p\left(x,\cdot,\cdot\right)^{1-\frac{\varepsilon}{2}}\right)\right|^{2}\in L^{M}\left(Q\left(a,b\right)\right)\qquad \textit{for each } x\in\mathbb{R}^{N} \textit{ and all } 0<a<b<\infty.$$

Moreover, we have

$$\int_{\mathbb{R}^N}\left|Dp\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right|^{2M}dy\leq\frac{C}{\varepsilon^{2M(2M-\varepsilon M)}}\left(\frac{1}{2t^{M}}+\frac{1}{K}\left(e^{Kt}-e^{K\frac{t}{2}}\right)V\left(x\right)\right)^{2M-\varepsilon M}$$

and

$$\int_{Q(a,b)}\left|D\left(p\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right)\right|^{2M}dydt\leq\frac{C}{\varepsilon^{2M(2M-\varepsilon M)}}\int_{a}^{b}\left(\frac{1}{2t^{M}}+\frac{1}{K}\left(e^{Kt}-e^{K\frac{t}{2}}\right)V\left(x\right)\right)^{2M-\varepsilon M}dt,$$

where $C=C\left(\lambda,M,N\right)>0$.

Proof. Let ω_n be as in (2.10) of the proof of Theorem 2.2 with $W=1$. Let $t\geq\alpha>0$. We recall estimate (2.30) saying that

$$\begin{aligned}\left(\int_{\mathbb{R}^N}\omega_n\left(x,y,t\right)dy\right)^{\frac{1}{2M-\varepsilon M}}\leq&\frac{C}{\varepsilon^{2M}\left(2M-\varepsilon M\right)}\int_{\frac{\alpha}{2}}^{t}\int_{\mathbb{R}^N}p\left(x,y,s\right)\\&\cdot\left(\frac{1}{\alpha^{M+1}}+V\left(y\right)\right)dyds\\&+\frac{1}{\left(2M-\varepsilon M\right)}\int_{\frac{\alpha}{2}}^{t}\theta\left(x,s\right)^{\frac{1}{2M-\varepsilon M}-1}\nu_n\left(x,s\right)ds.\end{aligned}$$

where $\nu_n=\nu_n\left(x,t\right)\to0$ as $n\to\infty$ for all $(x,t)\in\mathbb{R}^N\times(0,\infty)$, ν_n is locally bounded and $C=C\left(\lambda,M,N\right)>0$ is a constant. We remark that for $t\geq\alpha$ it holds

$$\begin{aligned}\int_{\mathbb{R}^N}\eta_n\left(y\right)^{2\beta M}\left|Dp\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right|^{2M}dy&\leq\frac{2^{2M}}{\left(2-\varepsilon\right)^{2M}}\int_{\mathbb{R}^N}\eta_n\left(y\right)^{2\beta M}\left|Dp\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right|^{2M}dy\\&\leq\int_{\mathbb{R}^N}\omega_n\left(x,y,t\right)dy.\end{aligned}$$

Hence there exists a constant $C=C\left(\lambda,M,N\right)>0$ such that

$$\begin{aligned}\int_{\mathbb{R}^N}\eta_n\left(y\right)^{2\beta M}\left|Dp\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right|^{2M}dy&\leq\int_{\mathbb{R}^N}\omega_n\left(x,y,t\right)dy\\\leq&\frac{C}{\varepsilon^{2M(2M-\varepsilon M)}}\left(\int_{\frac{\alpha}{2}}^{t}\int_{\mathbb{R}^N}p\left(x,y,s\right)\left(\frac{1}{\alpha^{M+1}}+V\left(y\right)\right)dyds\right.\\&\left.+\frac{1}{\left(2M-\varepsilon M\right)}\int_{\frac{\alpha}{2}}^{t}\theta\left(x,s\right)^{\frac{1}{2M-\varepsilon M}-1}\nu_n\left(x,s\right)ds\right)^{2M-\varepsilon M}.\end{aligned}$$

Hier $\theta\left(x,t\right)>0$ for all $(x,t)\in\mathbb{R}^N\times\left(\frac{\alpha}{2},\infty\right)$ and continuous. Fatou's lemma yields

$$\begin{aligned}\int_{\mathbb{R}^N}\left|Dp\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right|^{2M}dy&=\int_{\mathbb{R}^N}\lim_{n\to\infty}\eta_n\left(y\right)^{2\beta M}\left|Dp\left(x,y,t\right)^{1-\frac{\varepsilon}{2}}\right|^{2M}dy\\&\leq\liminf_{n\to\infty}\int_{\mathbb{R}^N}\omega_n\left(x,y,t\right)dy\end{aligned}$$

for all $(x,t) \in \mathbb{R}^N \times [\alpha,\infty)$. Using (1.21) and Lebesgue's convergence theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \left| Dp(x,y,t)^{1-\frac{\varepsilon}{2}} \right|^{2M} dy &\leq \frac{C}{\varepsilon^{2M(2M-\varepsilon M)}} \Bigg(\left(\int_{\frac{\alpha}{2}}^{t} \left(\frac{1}{\alpha^{M+1}} + e^{Ks} V(x) \right) ds \right. \\ &\quad \left. + \frac{1}{(2M-\varepsilon M)} \lim_{n\to\infty} \int_{\frac{\alpha}{2}}^{t} \theta(x,s)^{\frac{1}{2M-\varepsilon M}-1} \nu_n(x,s)\, ds \right)^{2M-\varepsilon M} \\ &= \frac{C}{\varepsilon^{2M(2M-\varepsilon M)}} \left(\frac{1}{\alpha^{M+1}} \left(t - \frac{\alpha}{2} \right) + \frac{1}{K} \left(e^{Kt} - e^{K\frac{\alpha}{2}} \right) V(x) \right)^{2M-\varepsilon M}.\end{aligned}$$

Letting $\alpha = t$, we get

$$\int_{\mathbb{R}^N} \left| Dp(x,y,t)^{1-\frac{\varepsilon}{2}} \right|^{2M} dy \leq \frac{C}{\varepsilon^{2M(2M-\varepsilon M)}} \left(\frac{1}{2t^M} + \frac{1}{K} \left(e^{Kt} - e^{K\frac{t}{2}} \right) V(x) \right)^{2M-\varepsilon M}$$

for $C = C(\lambda, M, N) > 0$ and all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Hence for all $0 < a < b < \infty$ it follows

$$\int_{Q(a,b)} \left| D\left(p(x,y,t)^{1-\frac{\varepsilon}{2}} \right) \right|^{2M} dydt \leq \frac{C}{\varepsilon^{2M(2M-\varepsilon M)}} \int_a^b \left(\frac{1}{2t^M} + \frac{1}{K} \left(e^{Kt} - e^{K\frac{t}{2}} \right) V(x) \right)^{2M-\varepsilon M} dt.$$

∎

We now investigate the L^q-regularity of the gradient of p.

Theorem 2.12. *Under Condition 2.10 it holds*

$$\left| D\left(p(x,\cdot,\cdot)^{\frac{\beta+1}{2}} \right) \right| \in L^2(Q(a,b))$$

and more precisely

$$\begin{aligned}&\int_{Q(a,b)} \left| D\left(p(x,y,t)^{\frac{\beta+1}{2}} \right) \right|^2 dydt \\ &\quad \leq \|p(x,\cdot,\cdot)\|^{\beta}_{L^\infty(Q(a,b))} \frac{\beta+1}{2\lambda\beta} \Bigg[\max\{-H_0, 0\} e^{-H_0 b}(b-a) + e^{-H_0 a} \\ &\quad\quad + \beta \left(\int_a^b e^{-H_0 t} dt \right)^{\frac{M}{M+1}} \left(\int_{Q(a,b)} p(x,y,t) \left| \operatorname{div} F(y) + H(y) \right|^{M+1} dydt \right)^{\frac{1}{M+1}} \Bigg]\end{aligned}$$

for all $x \in \mathbb{R}^N$, $\beta > 0$ and $0 < a < b < \infty$.

Remark 2.13. Observe that under Condition 2.10 it holds

$$\begin{aligned}\int_{\mathbb{R}^N} p(x,y,t) \left| \operatorname{div} F(y) + H(y) \right|^{M+1} dy &\leq \int_{\mathbb{R}^N} p(x,y,t) V(y)\, dy \\ &\leq e^{Kt} V(x)\end{aligned}$$

for all $(x,t) \in \mathbb{R}^N \times [0,\infty)$. If additionally $AV \leq -g(V)$ holds for a function g given as in Proposition 1.8, then we obtain from Proposition 1.8 and Remark 2.3 d) for the case $H_0 \geq 0$

$$\int_{Q(a,b)} \left| D\left(p(x,y,t)^{\frac{\beta+1}{2}} \right) \right|^2 dydt \leq Ct^{\beta} \frac{\beta+1}{2\lambda\beta} \left[\beta(b-a) + e^{-H_0 a} \right]$$

and for the case $H_0 < 0$

$$\int_{Q(a,b)} \left| D\left(p(x,y,t)^{\frac{\beta+1}{2}}\right)\right|^2 dydt \leq Ct^\beta e^{-\beta H_0 t} \frac{\beta+1}{2\lambda\beta}\left[(\beta - H_0)(b-a)e^{-H_0 b} + e^{-H_0 a}\right]$$

for a suitable constant $C = C(\lambda, M, N, a) > 0$.

Proof of Theorem 2.12. Let $\beta > 0$. For fixed $x \in \mathbb{R}^N$ it then holds

$$\beta(\beta-1)p^{\beta-2}a(Dp,Dp) = -\partial_t\left(p^\beta\right) + A_0\left(p^\beta\right) - F\cdot D\left(p^\beta\right) - \beta p^\beta(\operatorname{div}F + H) \quad (2.50)$$

with respect to $(y,t) \in \mathbb{R}^N \times (0,\infty)$. We multiply (2.50) by $p\eta_n^2$ and integrate over $Q(a,b)$ for $0 < a < b < \infty$. It then follows

$$\begin{aligned}\int_{Q(a,b)} \beta(\beta-1)\eta_n^2 p^{\beta-1}a(Dp,Dp)\,dydt &= -\int_{Q(a,b)} \frac{\beta}{\beta+1}\partial_t\left(\eta_n^2 p^{\beta+1}\right)dydt\\ &\quad+\int_{Q(a,b)} \eta_n^2 pA_0\left(p^\beta\right)dydt\\ &\quad-\int_{Q(a,b)} \frac{\beta}{\beta+1}\eta_n^2 F\cdot D\left(p^{\beta+1}\right)dydt\\ &\quad-\int_{Q(a,b)} \beta\eta_n^2 p^{\beta+1}(\operatorname{div}F+H)\,dydt.\end{aligned}$$

Integration by parts gives

$$\begin{aligned}\int_{Q(a,b)} \beta\eta_n^2 p^{\beta-1}a(Dp,Dp)\,dydt &= -\int_{Q(a,b)} \frac{1}{\beta+1}\partial_t\left(\eta_n^2 p^{\beta+1}\right)dydt\\ &\quad-\int_{Q(a,b)} 2\eta_n p^\beta a(D\eta_n, Dp)\,dydt\\ &\quad+\int_{Q(a,b)} \frac{2}{\beta+1}\eta_n p^{\beta+1}F\cdot D\eta_n dydt\\ &\quad-\int_{Q(a,b)} \frac{\beta}{\beta+1}\eta_n^2 p^{\beta+1}(\operatorname{div}F+H)\,dydt\\ &\quad-\int_{Q(a,b)} \frac{1}{\beta+1}\eta_n^2 p^{\beta+1}H dydt. \qquad (2.51)\end{aligned}$$

We further have

$$\begin{aligned}-\int_{Q(a,b)} \frac{1}{\beta+1}\partial_t\left(\eta_n^2 p^{\beta+1}\right)dydt &= \int_{\mathbb{R}^N} \frac{1}{\beta+1}\eta_n^2\left[p^{\beta+1}\right]_{t=b}^{t=a}dydt\\ &\leq \frac{\|p(x,\cdot,\cdot)\|^\beta_{L^\infty(Q(a,b))}e^{-H_0 a}}{\beta+1},\end{aligned}$$

since $\int_{\mathbb{R}^N} p(x,y,a)\,dy \leq e^{-H_0 a}$ (see Proposition 1.2). Moreover, (1.5) yields

$$-\int_{Q(a,b)} 2\eta_n p^\beta a(D\eta_n, Dp)\,dydt$$

$$
\begin{aligned}
&\le \int_{Q(a,b)} 2\sqrt{\frac{\beta}{2}\eta_n^2 p^{\beta-1} a\left(Dp,Dp\right)}\sqrt{\frac{2}{\beta}p^{\beta+1}a\left(D\eta_n,D\eta_n\right)}dydt \\
&\le \int_{Q(a,b)} \frac{\beta}{2}\eta_n^2 p^{\beta-1} a\left(Dp,Dp\right)dydt + \int_{Q(a,b)} \frac{2}{\beta}p^{\beta+1}a\left(D\eta_n,D\eta_n\right)dydt \\
&\le \int_{Q(a,b)} \frac{\beta}{2}\eta_n^2 p^{\beta-1} a\left(Dp,Dp\right)dydt \\
&\quad + \frac{2L^2\left\|p\left(x,\cdot,\cdot\right)\right\|^{\beta}_{L^\infty(Q(a,b))}}{\beta}\int_a^b\int_{\mathbb{R}^N} p\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}dydt.
\end{aligned}
$$

We further have

$$
\int_{Q(a,b)} \frac{2}{\beta+1}\eta_n p^{\beta+1}F\cdot D\eta_n dy \le \frac{2L\left\|p\left(x,\cdot,\cdot\right)\right\|^{\beta}_{L^\infty(Q(a,b))}}{n\left(\beta+1\right)}\int_a^b\int_{\mathbb{R}^N} p\left|F\right|\mathbb{1}_{\{n\le|y|\le 2n\}}dydt.
$$

Hölder's inequality implies

$$
\begin{aligned}
&-\int_{Q(a,b)} \frac{\beta}{\beta+1}\eta_n^2 p^{\beta+1}\left(\operatorname{div}F+H\right)dydt \\
&\qquad\le \int_{Q(a,b)} \frac{\beta}{\beta+1}\eta_n^2 p^{\beta+1}\left|\operatorname{div}F+H\right|dydt \\
&\qquad\le \frac{\beta}{\beta+1}\int_{Q(a,b)}\left(p^{\beta\frac{M+1}{M}+1}\right)^{\frac{M}{M+1}}\left(p\left|\operatorname{div}F+H\right|^{M+1}\right)^{\frac{1}{M+1}}dydt \\
&\qquad\le \frac{\beta}{\beta+1}\left(\int_{Q(a,b)} p^{\beta\frac{M+1}{M}+1}dydt\right)^{\frac{M}{M+1}}\left(\int_{Q(a,b)} p\left|\operatorname{div}F+H\right|^{M+1}dydt\right)^{\frac{1}{M+1}} \\
&\qquad\le \frac{\beta\left\|p\left(x,\cdot,\cdot\right)\right\|^{\beta}_{L^\infty(Q(a,b))}}{\beta+1}\left(\int_a^b e^{-H_0 t}dt\right)^{\frac{M}{M+1}}\left(\int_{Q(a,b)} p\left|\operatorname{div}F+H\right|^{M+1}dydt\right)^{\frac{1}{M+1}}.
\end{aligned}
$$

Finally, it holds

$$
\begin{aligned}
-\int_{Q(a,b)} \frac{1}{\beta+1}\eta_n^2 p^{\beta+1}Hdydt &\le -\frac{H_0}{\beta+1}\int_{Q(a,b)} p^{\beta+1}dydt \\
&\le \max\left\{-H_0,0\right\}\left\|p\left(x,\cdot,\cdot\right)\right\|^{\beta}_{L^\infty(Q(a,b))}e^{-H_0 b}\frac{b-a}{\beta+1}.
\end{aligned}
$$

Combining these estimates with (2.51), we conclude

$$
\begin{aligned}
&\int_{Q(a,b)} \frac{\beta}{2}\eta_n^2 p^{\beta-1}a\left(Dp,Dp\right)dydt \\
&\qquad\le \frac{\left\|p\left(x,\cdot,\cdot\right)\right\|^{\beta}_{L^\infty(Q(a,b))}e^{-H_0 a}}{\beta+1} \\
&\qquad\quad + \frac{2L^2\left\|p\left(x,\cdot,\cdot\right)\right\|^{\beta}_{\infty}}{\beta}\int_a^b\int_{\mathbb{R}^N} p\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}dydt \\
&\qquad\quad + \frac{2L\left\|p\left(x,\cdot,\cdot\right)\right\|^{\beta}_{L^\infty(Q(a,b))}}{n\left(\beta+1\right)}\int_a^b\int_{\mathbb{R}^N} p\left|F\right|\mathbb{1}_{\{n\le|y|\le 2n\}}dydt
\end{aligned}
$$

$$+\frac{\beta\left\|p\left(x,\cdot,\cdot\right)\right\|_{L^{\infty}(Q(a,b))}^{\beta}}{\beta+1}\left(\int_a^b e^{-H_0t}dt\right)^{\frac{M}{M+1}}$$
$$\cdot\left(\int_{Q(a,b)}p\left|\operatorname{div}F+H\right|^{M+1}dydt\right)^{\frac{1}{M+1}}$$
$$+\max\left\{-H_0,0\right\}\left\|p\left(x,\cdot,\cdot\right)\right\|_{L^{\infty}(Q(a,b))}^{\beta}e^{-H_0b}\frac{b-a}{\beta+1}.$$

Due to Proposition 1.6, $V^{\frac{1}{M+1}}$ is also a Lypunov function for A. From (1.21) and Condition 2.10 we conclude that for all fixed $(x,t)\in\mathbb{R}^N\times(0,\infty)$

$$\begin{aligned}\int_a^b\int_{\mathbb{R}^N}p\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}dy &\le \int_a^b\int_{\mathbb{R}^N}pV^{\frac{1}{M+1}}dy\\ &\le V\left(x\right)^{\frac{1}{M+1}}\int_a^b e^{\frac{K-MH_0}{M+1}t}dt<\infty\end{aligned}$$

and

$$\begin{aligned}\int_a^b\int_{\mathbb{R}^N}p\left|F\right|\mathbb{1}_{\{n\le|y|\le 2n\}}dy &\le \int_a^b\int_{\mathbb{R}^N}pV^{\frac{1}{M+1}}dy\\ &\le V\left(x\right)^{\frac{1}{M+1}}\int_a^b e^{\frac{K-MH_0}{M+1}t}dt<\infty.\end{aligned}$$

Lebesgue's convergence theorem with majorante $pV^{\frac{1}{M+1}}$ then yields

$$\frac{2L^2\left\|p\left(x,\cdot,\cdot\right)\right\|_{L^{\infty}(Q(a,b))}^{\beta}}{\beta}\int_a^b\int_{\mathbb{R}^N}p\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}dydt\to 0$$

and

$$\frac{2L\left\|p\left(x,\cdot,\cdot\right)\right\|_{L^{\infty}(Q(a,b))}^{\beta}}{n\left(\beta+1\right)}\int_a^b\int_{\{n\le|y|\le 2n\}}p\left|F\right|dydt\to 0$$

as $n\to\infty$ for each fixed $x\in\mathbb{R}^N$. Letting $n\to\infty$ and using Fatou's lemma we deduce that

$$\begin{aligned}&\int_{Q(a,b)}\frac{\beta\lambda}{2}p^{\beta-1}\left|Dp\right|^2dydt\\ &\quad\le\int_{Q(a,b)}\frac{\beta}{2}p^{\beta-1}a\left(Dp,Dp\right)dydt\\ &\quad\le\frac{\left\|p\left(x,\cdot,\cdot\right)\right\|_{L^{\infty}(Q(a,b))}^{\beta}}{\beta+1}\\ &\qquad\cdot\Bigg(\max\left\{-H_0,0\right\}e^{-H_0b}\left(b-a\right)\\ &\qquad\quad+e^{-H_0a}+\beta\left(\int_a^b e^{-H_0t}dt\right)^{\frac{M}{M+1}}\left(\int_{Q(a,b)}p\left|\operatorname{div}F+H\right|^{M+1}dydt\right)^{\frac{1}{M+1}}\Bigg).\end{aligned}$$

Using

$$\frac{\left(\beta+1\right)^2}{4}p^{\beta-1}\left|Dp\right|^2=\left|Dp^{\frac{\beta+1}{2}}\right|^2,$$

we observe

$$\begin{aligned}\int_{Q(a,b)} \left|Dp^{\frac{\beta+1}{2}}\right|^2 dydt &\leq \|p(x,\cdot,\cdot)\|^{\beta}_{L^\infty(Q(a,b))} \frac{\beta+1}{2\lambda\beta} \\ &\cdot \Bigg(\max\{-H_0, 0\} e^{-H_0 b} (b-a)(\beta+1) + e^{-H_0 a} \\ &+\beta \left(\int_a^b e^{-H_0 t} dt\right)^{\frac{M}{M+1}} \left(\int_{Q(a,b)} p\,|\operatorname{div} F + H|^{M+1} dydt\right)^{\frac{1}{M+1}} \Bigg).\end{aligned}$$

Because of $|\operatorname{div} F + H|^{M+1} \leq V$, from (1.21) it follows that

$$\int_{Q(a,b)} \left|D\left(p(x,y,t)^{\frac{\beta+1}{2}}\right)\right|^2 dydt < \infty$$

for all $(x,t) \in \mathbb{R}^N \times (a,b)$. ■

Corollary 2.14. *Under condition 2.10 for each $\varepsilon \in \left(0, \frac{1}{2M}\right]$ it holds*

$$\left|D\left(p(x,\cdot,\cdot)^{1-\frac{\varepsilon}{2}}\right)\right|^2 \in L^q(Q(a,b)) \qquad \text{for each } q \in [1, M]$$

for each $x \in \mathbb{R}^N$ and all $0 < a < b < \infty$.

Proof. With $\beta = 1-\varepsilon$ in Theorem 2.12 yields

$$\left|D\left(p(x,\cdot,\cdot)^{1-\frac{\varepsilon}{2}}\right)\right|^2 \in L^1(Q(a,b)) \qquad \text{for every } x \in \mathbb{R}^N.$$

The statement then follows from Proposition 2.11. ■

We further show that under Condition 2.10 we have $|D(p(x,\cdot,\cdot))|^2 \in L^q(Q(a,b))$ for each fixed $x \in \mathbb{R}^N$, each $q \in [1, M]$ and all $0 < a < b < \infty$.

Corollary 2.15. *Under condition 2.10 it holds*

$$|Dp(x,\cdot,\cdot)|^2 \in L^q(Q(a,b)) \qquad \text{for each } q \in [1, M]$$

for each $x \in \mathbb{R}^N$ and all $0 < a < b < \infty$.

Proof. Let $x \in \mathbb{R}^N$, $0 < a < b < \infty$, $\varepsilon \in \left(0, \frac{1}{2M}\right]$ and $q \in [1, M]$. It then holds

$$\begin{aligned}\int_{Q(a,b)} |Dp(x,y,t)|^{2q} dydt &= \frac{2^q}{(2-\varepsilon)^q} \int_{Q(a,b)} \left|D\left(p(x,y,t)^{1-\frac{\varepsilon}{2}}\right)\right|^{2q} p(x,y,t)^{\varepsilon q} dydt \\ &\leq \frac{2^q}{(2-\varepsilon)^q} \sup_{(y,t)\in Q(a,b)} |p(x,y,t)|^{\varepsilon q} \int_{Q(a,b)} \left|D\left(p(x,y,t)^{1-\frac{\varepsilon}{2}}\right)\right|^{2q} dydt.\end{aligned}$$

The statement follows from Remark 2.3 a) and Corollary 2.14. ■

To obtain the L^q-regularity of $|Dp(x,\cdot,t)|^2$ for all fixed $(x,t) \in \mathbb{R}^N \times (0,\infty)$ we need the following corollary.

Corollary 2.16. *Under condition 2.10 for each* $\varepsilon \in \left(0, \frac{1}{2M}\right]$ *it holds*

$$\left|D\left(p(x,\cdot,t)^{1-\frac{\varepsilon}{2}}\right)\right|^2 \in L^q\left(\mathbb{R}^N\right) \qquad \text{for each } q \in [1, M]$$

and all $(x,t) \in \mathbb{R}^N \times (0,\infty)$.

Proof. Let $x \in \mathbb{R}^N$ be fixed. We consider p as a function of $(y,t) \in \mathbb{R}^N \times (0,\infty)$. Further let τ be given as in the proof of Theorem 2.2. For $\delta, \frac{\beta}{2} > 2$ and $\varepsilon \in \left(0, \frac{1}{2M}\right]$ we set

$$\omega_n(x,y,t) = \tau(t)^\delta \eta_n(y)^\beta \frac{1}{p(x,y,t)^\varepsilon} |Dp(x,y,t)|^2 .$$

As in (2.14) we compute

$$\begin{aligned}
\partial_t \omega_n = & -\varepsilon\tau^\delta\eta_n^\beta \frac{1}{p^{\varepsilon+1}}|Dp|^2 A_0 p + 2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h,k=1}^N a_{hk}D_{ihk}pD_ip \\
& + 2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h,k=1}^N D_{ik}a_{hk}D_ipD_hp - 2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h=1}^N D_iF_hD_ipD_hp \\
& - 2\tau^\delta\eta_n^\beta p^{1-\varepsilon}\sum_{i=1}^N D_i(\operatorname{div} F + H)D_ip \\
& + \varepsilon\tau^\delta\eta_n^\beta\frac{1}{p^{\varepsilon+1}}|Dp|^2 F\cdot Dp - 2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h=1}^N F_hD_{ih}pD_ip \\
& + 2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h,k=1}^N D_ia_{hk}D_{hk}pD_ip + 2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h,k=1}^N D_ka_{hk}D_{ih}pD_ip \\
& + \delta\tau'\tau^{\delta-1}\eta_n^\beta\frac{1}{p^\varepsilon}|Dp|^2 - (2-\varepsilon)\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}|Dp|^2(\operatorname{div} F + H) \qquad (2.52)
\end{aligned}$$

and hence

$$\begin{aligned}
\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right) = & \int_{\mathbb{R}^N}\Bigg(-\varepsilon\tau^\delta\eta_n^\beta\frac{1}{p^{\varepsilon+1}}|Dp|^2 A_0p \\
& +2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h,k=1}^N a_{hk}D_{ihk}pD_ip \\
& +2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h,k=1}^N D_{ik}a_{hk}D_ipD_hp \\
& -2\tau^\delta\eta_n^\beta\frac{1}{p^\varepsilon}\sum_{i,h=1}^N D_iF_hD_ipD_hp \\
& -2\tau^\delta\eta_n^\beta p^{1-\varepsilon}\sum_{i=1}^N D_i(\operatorname{div} F + H)D_ip \\
& +\varepsilon\tau^\delta\eta_n^\beta\frac{1}{p^{\varepsilon+1}}|Dp|^2 F\cdot Dp
\end{aligned}$$

$$
\begin{aligned}
&-2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i,h=1}^{N}F_hD_{ih}pD_ip\\
&+2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i,h,k=1}^{N}D_ia_{hk}D_{hk}pD_ip\\
&+2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i,h,k=1}^{N}D_ka_{hk}D_{ih}pD_ip\\
&+\delta\tau'\tau^{\delta-1}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\\
&\left.-\left(2-\varepsilon\right)\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(\operatorname{div}F+H\right)\right)dy. \qquad (2.53)
\end{aligned}
$$

Integration by parts of the first five terms of right hand side of (2.53) yields

$$
\begin{aligned}
\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right) = \int_{\mathbb{R}^N}\Bigg(&-2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&-\varepsilon\left(\varepsilon+1\right)\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2a\left(Dp,Dp\right)\\
&+4\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+1}}\sum_{i=1}^{N}D_ipa\left(D\left(D_ip\right),Dp\right)\\
&-2\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}D_ipa\left(D\left(D_ip\right),D\eta_n\right)\\
&-2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i,h,k=1}^{N}D_ia_{hk}D_{ik}pD_hp\\
&+2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}F\cdot Dp\sum_{i=1}^{N}D_{ii}p\\
&+2\tau^{\delta}\eta_n^{\beta}p^{1-\varepsilon}\left(\operatorname{div}F+H\right)\sum_{i=1}^{N}D_{ii}p\\
&+2\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+1}}\sum_{i,h,k=1}^{N}D_ia_{hk}D_ipD_hpD_kp\\
&+\varepsilon\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^2a\left(D\eta_n,Dp\right)\\
&-\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^2F\cdot Dp\\
&+2\beta\tau^{\delta}\eta_n^{\beta-1}p^{1-\varepsilon}\left(\operatorname{div}F+H\right)\sum_{i=1}^{N}D_ipD_i\eta_n\\
&-2\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}\sum_{i,h,k=1}^{N}D_ia_{hk}D_ipD_hpD_k\eta_n
\end{aligned}
$$

$$+2\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}F\cdot Dp\sum_{i=1}^{N}D_ipD_i\eta_n$$
$$-\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}|Dp|^2(\operatorname{div}F+H)$$
$$\left.+\delta\tau'\tau^{\delta-1}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}|Dp|^2\right)dy.$$

Using (1.5), (1.6) and Cauchy-Schwarz inequality, we can estimate

$$\begin{aligned}
\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right) \leq\ & \int_{\mathbb{R}^N}\left(-2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\right.\\
&-\varepsilon\left(\varepsilon+1\right)\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}|Dp|^2 a\left(Dp,Dp\right)\\
&+4\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+1}}\sum_{i=1}^{N}\sqrt{a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\sqrt{\left(D_ip\right)^2 a\left(Dp,Dp\right)}\\
&+2\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}\sqrt{a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\sqrt{\left(D_ip\right)^2 a\left(D\eta_n,D\eta_n\right)}\\
&+2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|D^2p\right||Dp||Da|\\
&+2\sqrt{N}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|D^2p\right||Dp||F|\\
&+2\sqrt{N}\tau^{\delta}\eta_n^{\beta}p^{1-\varepsilon}\left|D^2p\right||\operatorname{div}F+H|\\
&+2\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+1}}|Dp|^3\left(|Da|+|F|\right)\\
&+\varepsilon\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon+1}}|Dp|^2\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}\\
&+2\beta\tau^{\delta}\eta_n^{\beta-1}p^{1-\varepsilon}|Dp||\operatorname{div}F+H||D\eta_n|\\
&+2\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}|Dp|^2\left(|Da|+|F|\right)|D\eta_n|\\
&+\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}|Dp|^2|\operatorname{div}F+H|\\
&\left.+4\delta\tau^{\delta-1}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}|Dp|^2\frac{1}{\alpha}\right)dy. \qquad (2.54)
\end{aligned}$$

We consider the positive terms on the right hand side of (2.54). Analogously as in the proof of Theorem 2.2, using repeatedly Young's inequality, (1.10) and (1.7), we estimate

$$\begin{aligned}
&4\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+1}}\sum_{i=1}^{N}\sqrt{a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\sqrt{\left(D_ip\right)^2 a\left(Dp,Dp\right)}\\
&\quad\leq 2\sum_{i=1}^{N}\sqrt{\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\\
&\qquad\cdot\sqrt{4\varepsilon^2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left(D_ip\right)^2 a\left(Dp,Dp\right)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right) \\
&\quad + 4\varepsilon^2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp, Dp\right),
\end{aligned}
$$

$$
\begin{aligned}
&2\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}\sqrt{a\left(D\left(D_i p\right), D\left(D_i p\right)\right)}\sqrt{\left(D_i p\right)^2 a\left(D\eta_n, D\eta_n\right)} \\
&\quad\leq 2\sum_{i=1}^{N}\sqrt{\frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}a\left(D\left(D_i p\right), D\left(D_i p\right)\right)} \\
&\qquad\cdot\sqrt{4\beta^2 L^2\tau^{\delta}\eta_n^{\beta-2}\frac{1}{p^{\varepsilon}}\left(D_i p\right)^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}} \\
&\quad\leq \frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right) \\
&\qquad + 4\beta^2 L^2\tau^{\delta}\eta_n^{\beta-2}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
\end{aligned}
$$

$$
\begin{aligned}
&2\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|D^2 p\right|\left|Dp\right|\left|Da\right| \\
&\quad\leq 2\sqrt{\frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right)}\sqrt{\frac{4}{\lambda}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left|Da\right|^2} \\
&\quad\leq \frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right) + \frac{4}{\lambda}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left|Da\right|^2,
\end{aligned}
$$

$$
\begin{aligned}
&2\sqrt{N}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|D^2 p\right|\left|Dp\right|\left|F\right| \\
&\quad\leq 2\sqrt{\frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right)}\sqrt{\frac{4N}{\lambda}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left|F\right|^2} \\
&\quad\leq \frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right) + \frac{4N}{\lambda}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left|F\right|^2,
\end{aligned}
$$

$$
\begin{aligned}
&2\sqrt{N}\tau^{\delta}\eta_n^{\beta}p^{1-\varepsilon}\left|D^2 p\right|\left|\operatorname{div} F + H\right| \\
&\quad\leq 2\sqrt{\frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right)}\sqrt{\frac{4N}{\lambda}\tau^{\delta}\eta_n^{\beta}p^{2-\varepsilon}\left|\operatorname{div} F + H\right|^2} \\
&\quad\leq \frac{1}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N} a\left(D\left(D_i p\right), D\left(D_i p\right)\right) + \frac{4N}{\lambda}\tau^{\delta}\eta_n^{\beta}p^{2-\varepsilon}\left|\operatorname{div} F + H\right|^2,
\end{aligned}
$$

$$2\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^3\left(\left|Da\right|+\left|F\right|\right)$$

$$\leq 2\sqrt{\frac{\varepsilon^2}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)}\sqrt{\frac{8}{\lambda}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(\left|Da\right|^2+\left|F\right|^2\right)}$$

$$\leq \frac{\varepsilon^2}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)+\frac{8}{\lambda}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(\left|Da\right|^2+\left|F\right|^2\right),$$

$$\varepsilon\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^2\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}$$

$$\leq 2\sqrt{\frac{\varepsilon^2}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)}\sqrt{\beta^2L^2\tau^{\delta}\eta_n^{\beta-2}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{\left|a\right|}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}}$$

$$\leq \frac{\varepsilon^2}{4}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)+\beta^2L^2\tau^{\delta}\eta_n^{\beta-2}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{\left|a\right|}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}},$$

$$2\beta\tau^{\delta}\eta_n^{\beta-1}p^{1-\varepsilon}\left|Dp\right|\left|\operatorname{div}F+H\right|\left|D\eta_n\right|$$

$$\leq 2\sqrt{\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2}\sqrt{\beta^2L^2\tau^{\delta}\eta_n^{\beta-2}p^{2-\varepsilon}\frac{\left|\operatorname{div}F+H\right|^2}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}}$$

$$\leq \tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2+\beta^2L^2\tau^{\delta}\eta_n^{\beta-2}p^{2-\varepsilon}\frac{\left|\operatorname{div}F+H\right|^2}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}},$$

$$2\beta\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(\left|Da\right|+\left|F\right|\right)\left|D\eta_n\right|\leq 2\beta L\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{\left|Da\right|+\left|F\right|}{1+\left|y\right|}\mathbb{1}_{\{n\leq|y|\leq 2n\}}.$$

We then get

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right) \leq \int_{\mathbb{R}^N}\Bigg(&-\left(\varepsilon-3\varepsilon^2-\frac{\varepsilon^2}{2}\right)\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)\\
&+5\beta^2L^2\tau^{\delta}\eta_n^{\beta-2}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{\left|a\right|}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\\
&+\frac{8N}{\lambda}\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(\left|Da\right|^2+\left|F\right|^2\right)\\
&+2\beta L\tau^{\delta}\eta_n^{\beta-1}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{\left|Da\right|+\left|F\right|}{1+\left|y\right|}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\\
&+\varepsilon\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left|\operatorname{div}F+H\right|+4\delta\tau^{\delta-1}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{1}{\alpha}\\
&+\tau^{\delta}\eta_n^{\beta}\frac{1}{p^{\varepsilon}}\left|Dp\right|^2+\frac{4N}{\lambda}\tau^{\delta}\eta_n^{\beta}p^{2-\varepsilon}\left|\operatorname{div}F+H\right|^2\\
&+\beta^2L^2\tau^{\delta}\eta_n^{\beta-2}p^{2-\varepsilon}\frac{\left|\operatorname{div}F+H\right|^2}{1+\left|y\right|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\Bigg)dy\end{aligned}$$

and hence

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right) &\le \int_{\mathbb{R}^N}\Bigg(-\left(\varepsilon-3\varepsilon^2-\frac{\varepsilon^2}{2}\right)\tau^\delta\eta_n^\beta\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)\\ &\quad +C\tau^{\delta-1}\eta_n^{\beta-2}\frac{1}{p^\varepsilon}\left|Dp\right|^2\Bigg(\left(\frac{|a|}{1+|y|^2}+\frac{|Da|+|F|}{1+|y|}\right)\mathbb{1}_{\{n\le|y|\le 2n\}}\\ &\quad +|Da|^2+|F|^2+|\operatorname{div}F+H|+1+\frac{1}{\alpha}\Bigg)\\ &\quad +C\tau^\delta\eta_n^{\beta-2}p^{2-\varepsilon}\left(|\operatorname{div}F+H|^2+\frac{|\operatorname{div}F+H|^2}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}\right)\Bigg)dy,\end{aligned} \tag{2.55}$$

with a constant $C=C\left(\lambda,\varepsilon,\beta,\delta,M,N\right)>0$. Moreover, for arbitrary $U\ge 0$ it holds

$$\begin{aligned}\tau^{\delta-1}\eta_n^{\beta-2}\frac{1}{p^\varepsilon}\left|Dp\right|^2U &\le 2\sqrt{\frac{\varepsilon^2}{2}\tau^\delta\eta_n^\beta\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)}\sqrt{\frac{1}{2\varepsilon^2\lambda}\tau^{\delta-2}\eta_n^{\beta-4}p^{2-\varepsilon}U^2}\\ &\le \frac{\varepsilon^2}{2}\tau^\delta\eta_n^\beta\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)+\frac{1}{2\varepsilon^2\lambda}\tau^{\delta-2}\eta_n^{\beta-4}p^{2-\varepsilon}U^2.\end{aligned}$$

From (2.55) we then deduce

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}^N}\omega_n dy\right) &\le \int_{\mathbb{R}^N}\Bigg(-\left(\varepsilon-4\varepsilon^2\right)\tau^\delta\eta_n^\beta\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)\\ &\quad +C\tau^{\delta-2}\eta_n^{\beta-4}p^{2-\varepsilon}\Bigg(\frac{|a|}{1+|y|^2}+|Da|^2+|F|^2\\ &\quad +|\operatorname{div}F+H|+1\Bigg)^2\Bigg)dy,\end{aligned}$$

with a constant $C=C\left(\lambda,\varepsilon,\beta,\delta,M,N,\alpha\right)>0$. Because of Condition 2.10 and the fact that $\varepsilon-4\varepsilon^2\ge 0$, it follows

$$\begin{aligned}\partial_t\left(\int_{\mathbb{R}^N}\omega_n\left(x,y,t\right)dy\right) &\le C\int_{\mathbb{R}^N}\tau\left(t\right)^{\delta-2}\eta_n\left(y\right)^{\beta-4}p\left(x,y,t\right)^{2-\varepsilon}V\left(y\right)^{\frac{2}{M+1}}dy\\ &\le C\sup_{y\in\mathbb{R}^N}\left|p\left(x,y,t\right)\right|^{1-\varepsilon}\int_{\mathbb{R}^N}p\left(x,y,t\right)V\left(y\right)^{\frac{2}{M+1}}dy\end{aligned}$$

with a constant $C=C\left(\lambda,\varepsilon,\beta,\delta,M,N,\alpha\right)>0$ and for all $(x,t)\in\mathbb{R}^N\times(0,\infty)$. Moreover, (2.2), Proposition 1.6 and (1.21) yield

$$\partial_t\left(\int_{\mathbb{R}^N}\omega_n\left(x,y,t\right)dy\right)\le C\left(\frac{1}{K}V\left(x\right)\left(e^{Kt}-e^{\frac{K}{2}t}\right)+\frac{1}{2t^M}\right)^{1-\varepsilon}e^{\frac{2K-(M-1)H_0}{M+1}t}V\left(x\right)^{\frac{2}{M+1}} \tag{2.56}$$

with a constant $C=C\left(\lambda,\varepsilon,\beta,\delta,M,N,\alpha\right)>0$ and for all $(x,t)\in\mathbb{R}^N\times(0,\infty)$. Integrating (2.56) from $\frac{\alpha}{2}$ to $t\ge\alpha$ we deduce

$$\int_{\mathbb{R}^N}\eta_n\left(y\right)^\beta\frac{1}{p\left(x,y,t\right)^\varepsilon}\left|Dp\left(x,y,t\right)\right|^2dy$$

$$\leq CV(x)^{\frac{2}{M+1}} \int_{\frac{\alpha}{2}}^{t} \left(\frac{1}{K} V(x) \left(e^{Ks} - e^{\frac{K}{2}s} \right) + \frac{1}{2s^M} \right)^{1-\varepsilon} e^{\frac{2K-(M-1)H_0}{M+1}s} ds.$$

Letting $n \to \infty$, Fatou's lemma implies

$$\int_{\mathbb{R}^N} \frac{1}{p(x,y,t)^{\varepsilon}} |Dp(x,y,t)|^2 dy = \frac{4}{(2-\varepsilon)^2} \int_{\mathbb{R}^N} \left| D\left(p(x,y,t)^{1-\frac{\varepsilon}{2}} \right) \right|^2 dy < \infty$$

for all $(x,t) \in \mathbb{R}^N \times [\alpha, \infty)$. Since $\alpha > 0$ was arbitrary, we conclude that

$$\int_{\mathbb{R}^N} \left| D\left(p(x,y,t)^{1-\frac{\varepsilon}{2}} \right) \right|^2 dy < \infty$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. The statement then follows from Proposition 2.11. ■

Remark 2.17. From above we deduce that under condition 2.10 for all $\varepsilon \in \left(0, \frac{1}{2M}\right]$ and $\alpha > 0$ it holds

$$\int_{\mathbb{R}^N} \frac{|Dp(x,y,t)|^2}{p(x,y,t)^{\varepsilon}} dy \leq C\left(1 + \frac{1}{\alpha^2}\right) \int_{\frac{\alpha}{2}}^{t} \sup_{z\in\mathbb{R}^N} |p(x,z,s)|^{1-\varepsilon} \int_{\mathbb{R}^N} p(x,y,s) V(y) dyds$$

for a constant $C = C(\lambda, \varepsilon, M, N) > 0$ and all $t \geq \alpha$.

Corollary 2.18. *Under condition 2.10 it holds*

$$|Dp(x,\cdot,t)|^2 \in L^q\left(\mathbb{R}^N\right) \qquad \textit{for each } q \in [1,M]$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$.

Proof. Let $(x,t) \in \mathbb{R}^N \times (0,\infty)$ be fixed, $\varepsilon \in \left(0, \frac{1}{2M}\right]$ and $q \in [1,M]$. It then holds

$$\begin{aligned} \int_{\mathbb{R}^N} |Dp(x,y,t)|^{2q} dy &= \left(\frac{2}{2-\varepsilon}\right)^q \int_{\mathbb{R}^N} \left| D\left(p(x,y,t)^{1-\frac{\varepsilon}{2}} \right) \right|^{2q} p(x,y,t)^{\varepsilon q} dy \\ &\leq \left(\frac{2}{2-\varepsilon}\right)^q \sup_{y\in\mathbb{R}^N} |p(x,y,t)|^{\varepsilon q} \int_{\mathbb{R}^N} \left| D\left(p(x,y,t)^{1-\frac{\varepsilon}{2}} \right) \right|^{2q} dy. \end{aligned}$$

The statement follows from Remark 2.3 a) and Corollary 2.16. ■

Example 2.19. We consider the operator

$$A = \left(1 + |x|^2\right)^{\alpha} \Delta - |x|^{2\beta} x \cdot D, \qquad 1 < \alpha < \beta,\ \beta \geq 1,$$

from Example 2.4. For each $\delta > 0$ there exist constants $C, K > 0$ such that $V(x) = Ce^{\delta|x|^2}$ is a Lyapunov function vor the operator A such that $AV \leq KV$ and V satisfies Condition 2.10. Corollaries 2.15 and 2.18 then yield for each $q \in [1,M]$

$$|Dp(x,\cdot,\cdot)|^2 \in L^q(Q(a,b)) \qquad \text{for each } x \in \mathbb{R}^N \text{ and all } 0 < a < b < \infty$$

and

$$|Dp(x,\cdot,t)|^2 \in L^q\left(\mathbb{R}^N\right) \qquad \text{for all } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

Chapter 3

Pointwise bounds of the derivatives of the transition kernel

In this chapter we apply the parabolic maximum principle (see e.g. [Kr96, Chapter 8]) to estimate the derivatives of p. We will use the fact that $D_i p(x,\cdot,\cdot)$, $D_{ij} p(x,\cdot,\cdot)$, $D_{ijh} p(x,\cdot,\cdot)$, $\partial_t p(x,\cdot,\cdot)$, $\partial_t D_i p(x,\cdot,\cdot) \in C^{\alpha}_{loc}\left(\mathbb{R}^N \times (0,\infty)\right)$ for each fixed $x \in \mathbb{R}^N$ (see Remark 1.3 b)). In Section 3.2 we additionally assume that $F_i \in C^{3+\alpha}_{loc}\left(\mathbb{R}^N\right)$. It implies that $D_{ijhk} p(x,\cdot,\cdot)$, $\partial_t D_i p(x,\cdot,\cdot) \in C^{\alpha}_{loc}\left(\mathbb{R}^N \times (0,\infty)\right)$.

3.1 Pointwise bounds on gradient

Condition 3.1. *There exist $K > 0$, $\varepsilon \in \left(0, \frac{1}{4}\right] \cap \left(0, \frac{4}{N+2}\right)$ and a Lyapunov function V with $AV \le KV$, such that*

$$\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + \left|D^2 a\right| + |F|^2 + |DF| + |\operatorname{div} F + H| + |D(\operatorname{div} F + H)| + |H|\right)^{\frac{2}{\varepsilon}} \le V.$$

Theorem 3.2. *Under Condition 3.1 for all $0 < \alpha < T < \infty$ it holds*

$$|Dp(x,y,t)|^2 \le e^{\beta t} p(x,y,t)^{\varepsilon} V(y)^{\varepsilon} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [\alpha, T],$$

where

$$\beta = K + C\left(1 + \frac{1}{\alpha^2}\right) \sup_{(y,t)\in\mathbb{R}^N \times \left[\frac{\alpha}{2}, T\right]} |p(x,y,t)|^{2-\varepsilon}, \tag{3.1}$$

the constant $C > 0$ depends only on λ, ε and N and K is given by Condition 3.1.

Remark 3.3. We set $M = \frac{2}{\varepsilon} - 1$. Then $M > \dfrac{N}{2}$, $M \ge 2$ and it holds

$$\begin{aligned}
&\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + |F|^2 + |\operatorname{div} F + H|\right)^{M+1} \\
= \; &\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + |F|^2 + |\operatorname{div} F + H|\right)^{\frac{2}{\varepsilon}} \\
\le \; &\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + \left|D^2 a\right| + |F|^2 + |DF| + |\operatorname{div} F + H| + |D(\operatorname{div} F + H)| + |H|\right)^{\frac{2}{\varepsilon}}
\end{aligned}$$

$$\leq \; V.$$

So Condition 3.1 implies Condition 2.1 with $W = 1$ and hence Condition 2.10. Theorem 2.2 thus yields boundedness of $p(x,\cdot,\cdot)$ on $\mathbb{R}^N \times (a,b)$ for each $x \in \mathbb{R}^N$ and all $0 < a < b < \infty$. Since the boundedness of p is necessary for the proof of the Theorem 3.2, we assumed that $\varepsilon < \frac{4}{N+2}$. Moreover, Remark 2.3 a) implies that for all $0 < \alpha < T < \infty$ it holds

$$|Dp(x,y,t)|^2 \leq Ce^{\beta t}\left(\frac{1}{K}V(x)\left(e^{Kt} - e^{\frac{K}{2}t}\right) + \frac{1}{2t^M}\right)^{\varepsilon} V(y)^{\varepsilon}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [\alpha, T]$ for a constant $C = C(\lambda, M, N) > 0$, where the constant β is given in (3.1).

Proof of Theorem 3.2. Let $x \in \mathbb{R}^N$ be fixed. We consider p as a function of $(y,t) \in \mathbb{R}^N \times (0,\infty)$. Further τ be given as in the proof of Theorem 2.2 for some $\alpha > 0$. For fixed $x \in \mathbb{R}^N$ we set

$$\omega_n(x,y,t) = \tau(t)^2 \eta_n(y)^4 \frac{1}{p(x,y,t)^{\varepsilon}} |Dp(x,y,t)|^2 \qquad \text{for } (y,t) \in \mathbb{R}^N \times (0,\infty).$$

Using (2.52) with $\delta = 2$ and $\beta = 4$, we obtain

$$\begin{aligned}
\partial_t \omega_n = &- \varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} |Dp|^2 A_0 p + 2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} \sum_{i,h,k=1}^N a_{hk} D_{ihk} p D_i p \\
&+ 2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} \sum_{i,h,k=1}^N D_{ik} a_{hk} D_i p D_h p - 2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} \sum_{i,h=1}^N D_i F_h D_i p D_h p \\
&- 2\tau^2\eta_n^4 p^{1-\varepsilon} \sum_{i=1}^N D_i (\operatorname{div} F + H) D_i p \\
&+ \varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} |Dp|^2 F \cdot Dp - 2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} \sum_{i,h=1}^N F_h D_{ih} p D_i p \\
&+ 2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} \sum_{i,h,k=1}^N D_i a_{hk} D_{hk} p D_i p + 2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} \sum_{i,h,k=1}^N D_k a_{hk} D_{ih} p D_i p \\
&+ 2\tau'\tau\eta_n^4 \frac{1}{p^{\varepsilon}} |Dp|^2 - (2-\varepsilon)\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} |Dp|^2 (\operatorname{div} F + H).
\end{aligned} \tag{3.2}$$

Further, we have

$$\begin{aligned}
-A_0\omega_n - F \cdot D\omega = &-12\tau^2\eta_n^2 \frac{1}{p^{\varepsilon}} |Dp|^2 a(D\eta_n, D\eta_n) + 8\varepsilon\tau^2\eta_n^3 \frac{1}{p^{\varepsilon+1}} |Dp|^2 a(Dp, D\eta_n) \\
&-4\tau^2\eta_n^3 \frac{1}{p^{\varepsilon}} |Dp|^2 \sum_{h,k=1}^N a_{hk} D_{hk}\eta_n - \varepsilon(\varepsilon+1)\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a(Dp, Dp) \\
&+\varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} |Dp|^2 A_0 p - 4\tau^2\eta_n^3 \frac{1}{p^{\varepsilon}} |Dp|^2 \sum_{h,k=1}^N D_k a_{hk} D_h \eta_n \\
&-2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon}} \sum_{i,h,k=1}^N D_k a_{hk} D_{ih} p D_i p - 16\tau^2\eta_n^3 \frac{1}{p^{\varepsilon}} \sum_{i,h,k=1}^N a_{hk} D_{ih} p D_i p D_k \eta_n
\end{aligned}$$

$$+4\varepsilon\tau^2\eta_n^4\frac{1}{p^{\varepsilon+1}}\sum_{i,h,k=1}^{N}a_{hk}D_{ih}pD_ipD_kp-2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i,h,k=1}^{N}a_{hk}D_{ihk}pD_ip$$
$$-2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)-4\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2F\cdot D\eta_n$$
$$+\varepsilon\tau^2\eta_n^4\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^2F\cdot Dp-2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i,h=1}^{N}F_hD_{ih}pD_ip. \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$\begin{aligned}
\partial_t\omega_n - A_0\omega_n - F\cdot D\omega_n + H\omega_n
&= -\varepsilon\left(\varepsilon+1\right)\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2a\left(Dp,Dp\right)-2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&+4\varepsilon\tau^2\eta_n^4\frac{1}{p^{\varepsilon+1}}\sum_{i=1}^{N}D_ipa\left(D\left(D_ip\right),Dp\right)-4\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i,h=1}^{N}F_hD_{ih}pD_ip\\
&+2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i,h,k=1}^{N}D_ia_{hk}D_{hk}pD_ip-16\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}D_ipa\left(D\left(D_ip\right),D\eta_n\right)\\
&+2\varepsilon\tau^2\eta_n^4\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^2F\cdot Dp+8\varepsilon\tau^2\eta_n^3\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^2a\left(Dp,D\eta_n\right)\\
&+2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i,h,k=1}^{N}D_{ik}a_{hk}D_ipD_hp-2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i,h=1}^{N}D_iF_hD_ipD_hp\\
&-\left(2-\varepsilon\right)\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(\operatorname{div}F+H\right)+2\tau'\tau\eta_n^4\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\\
&-4\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\sum_{h,k=1}^{N}D_ka_{hk}D_h\eta_n-12\tau^2\eta_n^2\frac{1}{p^{\varepsilon}}\left|Dp\right|^2a\left(D\eta_n,D\eta_n\right)\\
&-4\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\sum_{h,k=1}^{N}a_{hk}D_{hk}\eta_n-4\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2F\cdot D\eta_n\\
&-2\tau^2\eta_n^4p^{1-\varepsilon}\sum_{i=1}^{N}D_i\left(\operatorname{div}F+H\right)D_ip+\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\left|Dp\right|^2H.
\end{aligned} \tag{3.4}$$

Using (1.5), (1.6) and the Cauchy-Schwarz inequality, we estimate

$$\begin{aligned}
\partial_t\omega_n - A_0\omega_n - F\cdot D\omega_n + H\omega_n
&\le -\varepsilon\left(\varepsilon+1\right)\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2a\left(Dp,Dp\right)-2\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}a\left(D\left(D_ip\right),D\left(D_ip\right)\right)\\
&+4\varepsilon\tau^2\eta_n^4\frac{1}{p^{\varepsilon+1}}\sum_{i=1}^{N}\sqrt{a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\sqrt{\left(D_ip\right)^2a\left(Dp,Dp\right)}\\
&+4\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\left|D^2p\right|\left|Dp\right|\left(\left|F\right|+\left|Da\right|\right)\\
&+16\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\sum_{i=1}^{N}\sqrt{a\left(D\left(D_ip\right),D\left(D_ip\right)\right)}\sqrt{\left(D_ip\right)^2a\left(D\eta_n,D\eta_n\right)}
\end{aligned}$$

$$+ 2\varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} |Dp|^3 |F| + 8\varepsilon\tau^2\eta_n^3 \frac{1}{p^{\varepsilon+1}} |Dp|^2 \sqrt{a(Dp,Dp)}\sqrt{a(D\eta_n, D\eta_n)}$$
$$+ 2\sqrt{N}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \left(\left|D^2 a\right| + |DF| + |\operatorname{div} F + H|\right)$$
$$+ 8\tau\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \frac{1}{\alpha} + 4\sqrt{N}\tau^2\eta_n^3 \frac{1}{p^\varepsilon} |Dp|^2 (|F| + |Da|) |D\eta_n|$$
$$- 12\tau^2\eta_n^2 \frac{1}{p^\varepsilon} |Dp|^2 a(D\eta_n, D\eta_n) + 4\tau^2\eta_n^3 \frac{1}{p^\varepsilon} |Dp|^2 |a| \left|D^2\eta_n\right|$$
$$+ 2\tau^2\eta_n^4 p^{1-\varepsilon} |Dp| |D(\operatorname{div} F + H)| + \tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 |H|. \tag{3.5}$$

We consider the positive terms on the right hand side of (3.5). Applying repeatedly Young's inequality and using (1.7), we estimate

$$4\varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} \sum_{i=1}^N \sqrt{a(D(D_ip), D(D_ip))}\sqrt{(D_ip)^2 a(Dp,Dp)}$$
$$\leq 2\sum_{i=1}^N \sqrt{\tau^2\eta_n^4 \frac{1}{p^\varepsilon} a(D(D_ip), D(D_ip))}\sqrt{4\varepsilon^2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} (D_ip)^2 a(Dp,Dp)}$$
$$\leq \tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a(D(D_ip), D(D_ip)) + 4\varepsilon^2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a(Dp,Dp),$$

$$4\tau^2\eta_n^4 \frac{1}{p^\varepsilon} \left|D^2 p\right| |Dp| (|F| + |Da|)$$
$$\leq 2\sqrt{\frac{1}{2}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a(D(D_ip), D(D_ip))}\sqrt{\frac{16}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \left(|F|^2 + |Da|^2\right)}$$
$$\leq \frac{1}{2}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a(D(D_ip), D(D_ip)) + \frac{16}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \left(|F|^2 + |Da|^2\right),$$

$$16\tau^2\eta_n^3 \frac{1}{p^\varepsilon} \sum_{i=1}^N \sqrt{a(D(D_ip), D(D_ip))}\sqrt{(D_ip)^2 a(D\eta_n, D\eta_n)}$$
$$\leq 2\sum_{i=1}^N \sqrt{\frac{1}{2}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} a(D(D_ip), D(D_ip))}\sqrt{128\tau^2\eta_n^2 \frac{1}{p^\varepsilon} (D_ip)^2 a(D\eta_n, D\eta_n)}$$
$$\leq \frac{1}{2}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a(D(D_ip), D(D_ip)) + 128L^2\tau^2\eta_n^2 \frac{1}{p^\varepsilon} \left|D^2 p\right| \frac{|a|}{1+|y|^2} \mathbb{1}_{\{n\leq|y|\leq 2n\}},$$

$$2\varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} |Dp|^3 |F|$$
$$\leq 2\sqrt{\frac{\varepsilon^2}{4}\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a(Dp,Dp)}\sqrt{\frac{4}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 |F|^2}$$
$$\leq \frac{\varepsilon^2}{4}\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a(Dp,Dp) + \frac{4}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 |F|^2,$$

$$
\begin{aligned}
&8\varepsilon\tau^2\eta_n^3\frac{1}{p^{\varepsilon+1}}\left|Dp\right|^2\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}\\
\leq\ & 2\sqrt{\frac{\varepsilon^2}{4}\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)}\sqrt{64\tau^2\eta_n^2\frac{1}{p^{\varepsilon}}\left|Dp\right|^2 a\left(D\eta_n,D\eta_n\right)}\\
\leq\ & \frac{\varepsilon^2}{4}\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)+52L^2\tau^2\eta_n^2\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\\
&+12\tau^2\eta_n^2\frac{1}{p^{\varepsilon}}\left|Dp\right|^2 a\left(D\eta_n,D\eta_n\right),
\end{aligned}
$$

$$
4\sqrt{N}\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(|F|+|Da|\right)\left|D\eta_n\right|\leq 4L\sqrt{N}\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{|F|+|Da|}{1+|y|}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
$$

$$
4\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2|a|\left|D^2\eta_n\right|\leq 4L\tau^2\eta_n^3\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
$$

$$
\begin{aligned}
2\tau^2\eta_n^4 p^{1-\varepsilon}\left|Dp\right|\left|D\left(\operatorname{div}F+H\right)\right| &\leq 2\sqrt{\tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\left|Dp\right|^2}\sqrt{\tau^2\eta_n^4 p^{2-\varepsilon}\left|D\left(\operatorname{div}F+H\right)\right|^2}\\
&\leq \tau^2\eta_n^4\frac{1}{p^{\varepsilon}}\left|Dp\right|^2+\tau^2\eta_n^4 p^{2-\varepsilon}\left|D\left(\operatorname{div}F+H\right)\right|^2.
\end{aligned}
$$

Since

$$
0\leq\tau^2\leq\tau\leq 1\qquad\text{and}\qquad 0\leq\eta_n^4\leq\eta_n^3\leq\eta_n^2\leq 1,
$$

we obtain

$$
\begin{aligned}
\partial_t\omega_n - A_0\omega_n &- F\cdot D\omega_n + H\omega_n\\
&\leq -\left(\varepsilon-3\varepsilon^2-\frac{\varepsilon^2}{2}\right)\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)\\
&\quad + C\tau\eta_n^2\frac{1}{p^{\varepsilon}}\left|Dp\right|^2\left(\frac{1}{\alpha}+1+\frac{|a|}{1+|y|^2}+|Da|^2+\left|D^2a\right|\right.\\
&\qquad\left. +|F|^2+|DF|+\left|\operatorname{div}F+H\right|+|H|\right)\\
&\quad +\tau^2\eta_n^4 p^{2-\varepsilon}\left|D\left(\operatorname{div}F+H\right)\right|^2 \qquad (3.6)
\end{aligned}
$$

for a constant $C=C\left(\lambda,\varepsilon,N\right)>0$. Further, for each $U\geq 0$, it holds

$$
\begin{aligned}
C\tau\eta_n^2\frac{1}{p^{\varepsilon}}\left|Dp\right|^2 U &\leq 2\sqrt{\frac{\varepsilon^2}{2}\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)}\sqrt{\frac{C^2}{2\varepsilon^2\lambda}p^{2-\varepsilon}U^2}\\
&\leq \frac{\varepsilon^2}{2}\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}\left|Dp\right|^2 a\left(Dp,Dp\right)+\frac{C^2}{2\varepsilon^2\lambda}p^{2-\varepsilon}U^2.
\end{aligned}
$$

From (3.6) it then follows

$$
\partial_t\omega_n - A_0\omega_n - F\cdot D\omega_n + H\omega_n
$$

$$\leq -\left(\varepsilon - 4\varepsilon^2\right)\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a(Dp, Dp)$$
$$+ \frac{C}{\varepsilon^2} p^{2-\varepsilon} \left(\frac{1}{\alpha^2} + \left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + \left|D^2 a\right| \right.\right.$$
$$\left.\left. + |F|^2 + |DF| + |\operatorname{div} F + H| + |D(\operatorname{div} F + H)| + |H| \right)^2 \right).$$

Hence, using Condition 3.1 and the fact that $\varepsilon - 4\varepsilon^2 \geq 0$, we deduce

$$\partial_t \omega_n - A_0 \omega_n - F \cdot D\omega_n + H\omega_n \leq C\left(1 + \frac{1}{\alpha^2}\right) p^{2-\varepsilon} V^{\varepsilon} \tag{3.7}$$

for a constant $C = C(\lambda, \varepsilon, N) > 0$. Since $p(x, \cdot, \cdot)$ is bounded for all $T > \alpha$ and $x \in \mathbb{R}^N$ on $\mathbb{R}^N \times \left[\frac{\alpha}{2}, T\right]$, estimate (3.7) leads to

$$\begin{aligned} &\partial_t \omega_n(x, y, t) - A_0 \omega_n(x, y, t) - F(y) \cdot D\omega_n(x, y, t) + H(y)\omega_n(x, y, t) \\ &\qquad \leq C\left(1 + \frac{1}{\alpha^2}\right) V(y)^{\varepsilon} \sup_{(z,s)\in\mathbb{R}^N\times\left[\frac{\alpha}{2}, T\right]} |p(x, z, s)|^{2-\varepsilon} \end{aligned} \tag{3.8}$$

for a constant $C = C(\lambda, \varepsilon, N) > 0$. Let now $\beta > K$. From Proposition 1.6 we obtain

$$A_0(V^{\varepsilon}) + F \cdot D(V^{\varepsilon}) - HV^{\varepsilon} \leq KV^{\varepsilon}.$$

It then follows

$$\begin{aligned} &\partial_t\left(-e^{\beta t}V^{\varepsilon}\right) - A_0\left(-e^{\beta t}V^{\varepsilon}\right) - F \cdot D\left(-e^{\beta t}V^{\varepsilon}\right) + H\left(-e^{\beta t}V^{\varepsilon}\right) \\ &\qquad = -\beta e^{\beta t}V^{\varepsilon} + e^{\beta t}A(V^{\varepsilon}) \leq -(\beta - K)V^{\varepsilon}. \end{aligned} \tag{3.9}$$

Estimate (3.8) then implies

$$\begin{aligned} &\partial_t\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) - A_0\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) - F \cdot D\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) + H\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) \\ &\qquad \leq -\left(\beta - K - C\left(1 + \frac{1}{\alpha^2}\right) \sup_{(z,s)\in\mathbb{R}^N\times\left[\frac{\alpha}{2}, T\right]} |p(x, z, s)|^{2-\varepsilon}\right) V^{\varepsilon} \end{aligned}$$

for $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times \left[\frac{\alpha}{2}, T\right]$. Set

$$\beta(x, \alpha, T) = \beta = K + C\left(1 + \frac{1}{\alpha^2}\right) \sup_{(y,t)\in\mathbb{R}^N\times\left[\frac{\alpha}{2}, T\right]} |p(x, y, t)|^{2-\varepsilon}.$$

It then follows

$$\partial_t\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) - A_0\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) - F \cdot D\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) + H\left(\omega_n - e^{\beta t}V^{\varepsilon}\right) \leq 0.$$

Observe that

$$\omega_n - e^{\beta t}V^{\varepsilon} \leq 0 \qquad \text{for } t \in \left[0, \frac{\alpha}{2}\right] \text{ and for } |y| = 2n.$$

The parabolic maximum principle (see e. g. [Kr96, Chapter 8]) thus yields

$$0 \leq \omega_n \leq e^{\beta t}V^{\varepsilon} \qquad \text{on } B(0, 2n) \times [0, T].$$

Letting $n \to \infty$, we conclude that

$$|Dp(x,y,t)|^2 \leq e^{\beta t} p(x,y,t)^{\varepsilon} V(y)^{\varepsilon}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [\alpha, T]$, for

$$\beta = K + C\left(1 + \frac{1}{\alpha^2}\right) \sup_{(y,t)\in\mathbb{R}^N \times \left[\frac{\alpha}{2}, T\right]} |p(x,y,t)|^{2-\varepsilon}$$

and the constant $C = C(\lambda, \varepsilon, N) > 0$. ■

We now combine the results from Theorem 3.2 with (2.2).

Example 3.4. We consider again the operator

$$A = \left(1 + |x|^2\right)^{\alpha} \Delta - |x|^{2\beta} x \cdot D, \qquad 0 < \alpha < \beta,\ \beta \geq 1,$$

from Example 2.4. Then for each $\varepsilon \in \left(0, \frac{1}{4}\right] \cap \left(0, \frac{4}{N+2}\right)$ we can find $0 < \gamma < \delta < \infty$ and $C_1, C_2 > 0$, such that $W(x) = C_1 e^{\gamma |x|^2}$ satisfies Condition 3.1 and $V(x) = C_2 e^{\delta |x|^2}$ and W satisfy Condition 2.1 by Example 2.4. So Theorem 3.2 yields

$$\int_{\mathbb{R}^N} |Dp(x,y,t)|^{\frac{2}{\varepsilon}} dy \leq e^{\frac{\beta}{\varepsilon} t} \int_{\mathbb{R}^N} p(x,y,t) V(y) \, dy$$

so that

$$|Dp(x,\cdot,t)|^2 \in L^q\left(\mathbb{R}^N\right) \qquad \text{for each } q \in \left[\max\left\{4, \frac{N}{4} + \frac{1}{2}\right\}, \infty\right)$$

and for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Combining this result with Corollary 2.18, we deduce that

$$|Dp(x,\cdot,t)|^2 \in L^q\left(\mathbb{R}^N\right) \qquad \text{for each } q \in [1,\infty)$$

and for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$. Moreover, from (2.37) it follows that for all $t_0 > 0$ there exists a constant $C = C(\alpha, \beta, \gamma, t_0) > 0$ such that

$$p(x,y,t) W(y) \leq Ct \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0, \infty)$$

and thus Theorem 3.2 yields

$$|Dp(x,y,t)| \leq e^{\frac{\beta}{2} t} p(x,y,t)^{\frac{\varepsilon}{2}} W(y)^{\frac{\varepsilon}{2}} \leq C e^{\frac{\beta}{2} t} t^{\frac{\varepsilon}{2}}, \tag{3.10}$$

for a constant $C = C(\lambda, N, \alpha, \beta, \gamma, t_0) > 0$. That is $|Dp(x,\cdot,t)|$ is bounded for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ since $t_0 > 0$ can be arbitrary close to 0. So we get

$$|Dp(x,\cdot,t)|^2 \in L^q\left(\mathbb{R}^N\right) \qquad \text{for each } q \in [1,\infty].$$

Further, from Example 2.4 and Proposition 1.8 we infer that for each $t_0 > 0$ there exists a constant $C_0 = C_0(t_0) > 0$ such that

$$\int_{\mathbb{R}^N} p(x,y,t) V(y) \, dy \leq C_0.$$

Moreover, Remark 2.17 yields for some $\varepsilon' \in \left(0, \frac{1}{2M}\right]$, where a constant $M > \frac{N}{2}$ such that $M \geq 2$ is given as in Corollary 2.16,

$$\int_{\mathbb{R}^N} \frac{|Dp(x,y,t)|^2}{p(x,y,t)^{\varepsilon'}} dy \leq C_1\left(1 + \frac{1}{t_0^2}\right) \int_{\frac{t_0}{2}}^{t} \sup_{z\in\mathbb{R}^N} |p(x,z,s)|^{1-\varepsilon'} \int_{\mathbb{R}^N} p(x,y,s) V(y) \, dyds$$

$$\leq \; C_2 t^{2-\varepsilon'}$$

for suitable constants $C_1, C_2 > 0$ and all $t \geq t_0$. Thus it follows

$$\begin{aligned}\int_{\mathbb{R}^N} |Dp(x,y,t)|^2 \, dy &= \int_{\mathbb{R}^N} \frac{|Dp(x,y,t)|^2}{p(x,y,t)^{\varepsilon'}} p(x,y,t)^{\varepsilon'} \, dy \\ &\leq \sup_{z\in\mathbb{R}^N} |p(x,z,t)|^{\varepsilon'} \int_{\mathbb{R}^N} \frac{|Dp(x,y,t)|^2}{p(x,y,t)^{\varepsilon'}} dy \\ &\leq Ct^2\end{aligned}$$

for a constant $C = C(\lambda, N, \alpha, \beta, \gamma, t_0) > 0$ and all $t \geq t_0$. From (3.10) we obtain

$$\int_{\mathbb{R}^N} |Dp(x,y,t)|^{2q} \, dy \leq C_1 C_2^{q-1} e^{\beta(q-1)t} t^{\varepsilon(q-1)+2}$$

for constants $C_1 = C_1(\lambda, N, \alpha, \beta, \gamma, t_0,) > 0$, $C_2 = C_2(\lambda, N, \alpha, \beta, \gamma, t_0,) > 0$, all $t \geq t_0$ and all $q \geq 1$.

3.2 Pointwise bounds on second derivatives

Condition 3.5. *Assume that Condition 1.1 holds. Let also $F_i \in C^{3+\alpha}(\mathbb{R}^N)$. There is a function $U \in C^2(\mathbb{R}^N)$ such that*

$$1 + \frac{|a|}{1+|y|^2} + |Da|^2 + \left|D^2 a\right| + |F|^2 + |DF| + |\operatorname{div} F + H| + |H| \leq U.$$

Moreover, there exists a Lyapunov-function Q such that $AQ \leq KQ$ for some $K > \max\{0, H_0\}$ and

$$\begin{aligned}&1 + \left|D^3 a\right|^2 + \left|D^2 F\right|^2 + |D(\operatorname{div} F + H)|^2 + \left|D^2(\operatorname{div} F + H)\right|^2 \\ &+ U^2 + \frac{(A_0 U + F \cdot DU)^2}{U^2} + \frac{(a(DU,DU))^2}{U^4} \leq \frac{Q^\varepsilon}{U}\end{aligned}$$

for some $\varepsilon \in \left(0, \frac{1}{4}\right] \cap \left(0, \frac{4}{N+2}\right)$.

Remark 3.6. We set $M = \frac{2}{\varepsilon} - 1$. Then $M > \frac{N}{2}$ and Condition 3.5 implies that

$$\begin{aligned}&\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + |F|^2 + |\operatorname{div} F|\right)^{M+1} \\ = \; &\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + |F|^2 + |\operatorname{div} F|\right)^{\frac{2}{\varepsilon}} \\ \leq \; &\left(1 + \frac{|a|}{1+|y|^2} + |Da|^2 + \left|D^2 a\right| + |F|^2 + |DF| + |\operatorname{div} F + H|\right)^{\frac{2}{\varepsilon}} \\ \leq \; &U^{\frac{2}{\varepsilon}} \leq Q.\end{aligned}$$

Therefore, Q satisfies condition 2.1 with $W = 1$ and $V = Q$ if Condition 3.5 holds. Theorem 2.2 thus yields the boundedness of $p(x,\cdot,\cdot)$ on $\mathbb{R}^N \times (a,b)$ for each $x \in \mathbb{R}^N$ and all $0 < a < b < \infty$ and with (2.2) it holds

$$\sup_{y\in\mathbb{R}^N} p(x,y,t) \leq C\left(\frac{1}{K}\left(e^{Kt} - e^{\frac{K}{2}t}\right) + \frac{1}{2t^{\frac{2}{\varepsilon}-1}}\right) Q(x) \tag{3.11}$$

with a constant $C = C(\lambda, \varepsilon, N) > 0$.

Theorem 3.7. *Under condition 3.5 for all $0 < \alpha < T < \infty$ it holds*

$$\left|D^2 p(x,y,t)\right|^2 + |Dp(x,y,t)|^2 \leq e^{\beta t} p(x,y,t)^{\varepsilon} Q(y)^{\varepsilon}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [\alpha, T]$, where

$$\beta = K + \left(1 + \frac{1}{\alpha}\right)^3 \frac{C}{\varepsilon^2} \sup_{(y,t)\in\mathbb{R}^N\times\left[\frac{\alpha}{2},T\right]} |p(x,y,t)|^{2-\varepsilon}$$

and the constant $C > 0$ depends only on N and λ, where $K > \max\{0, H_0\}$ is given as in Condition 3.5.

Proof. Let $x \in \mathbb{R}^N$ be fixed. We consider p as a function of $(y,t) \in \mathbb{R}^N \times (0,\infty)$. Further let τ be given as in the proof of Theorem 2.2. We remark that $D_{ijk}p$ and $D_{ijkh}p$ exist and are continuous (see Remark 1.3). For fixed $x \in \mathbb{R}^N$ we have

$$\partial_t\left(\tau^3\eta_n^6\left|D^2p\right|^2\right) = 3\tau'\tau^2\eta_n^6\left|D^2p\right|^2 + 2\tau^3\eta_n^6\sum_{i,j=1}^N D_{ij}pD_{ij}(\partial_t p).$$

Equation (2.12) yields

$$\begin{aligned}
D_{ij}(\partial_t p) &= \sum_{h,k=1}^N D_{ij}a_{hk}D_{hk}p + \sum_{h,k=1}^N D_i a_{hk}D_{jhk}p + \sum_{h,k=1}^N D_j a_{hk}D_{ihk}p \\
&\quad + \sum_{h,k=1}^N a_{hk}D_{ijhk}p + \sum_{h,k=1}^N D_{ijk}a_{hk}D_h p + \sum_{h,k=1}^N D_{ik}a_{hk}D_{jh}p \\
&\quad + \sum_{h,k=1}^N D_{jk}a_{hk}D_{ih}p + \sum_{h,k=1}^N D_k a_{hk}D_{ijh}p - \sum_{h=1}^N D_{ij}F_h D_h p \\
&\quad - \sum_{h=1}^N D_i F_h D_{jh}p - \sum_{h=1}^N D_j F_h D_{ih}p - \sum_{h=1}^N F_h D_{ijh}p \\
&\quad - D_j(\operatorname{div} F + H) D_i p - (\operatorname{div} F + H) D_{ij}p \\
&\quad - D_i(\operatorname{div} F + H) D_j p - pD_{ij}(\operatorname{div} F + H).
\end{aligned}$$

It thus follows

$$\partial_t\left(\tau^3\eta_n^6\left|D^2p\right|^2\right) = 2\tau^3\eta_n^6\sum_{i,j,h,k=1}^N a_{hk}D_{ijhk}pD_{ij}p + 4\tau^3\eta_n^6\sum_{i,j,h,k=1}^N D_j a_{hk}D_{ihk}pD_{ij}p$$

$$
\begin{aligned}
&+2\tau^3\eta_n^6 \sum_{i,j,h,k=1}^{N} D_k a_{hk} D_{ijh}p D_{ij}p - 2\tau^3\eta_n^6 \sum_{i,j,h=1}^{N} F_h D_{ijh}p D_{ij}p \\
&+2\tau^3\eta_n^6 \sum_{i,j,h,k=1}^{N} D_{ij} a_{hk} D_{ij}p D_{hk}p - 4\tau^3\eta_n^6 \sum_{i,j,h=1}^{N} D_j F_h D_{ij}p D_{ih}p \\
&+4\tau^3\eta_n^6 \sum_{i,j,h,k=1}^{N} D_{jk} a_{hk} D_{ij}p D_{ih}p + 2\tau^3\eta_n^6 \sum_{i,j,h,k=1}^{N} D_{ijk} a_{hk} D_{ij}p D_h p \\
&-2\tau^3\eta_n^6 \sum_{i,j,h=1}^{N} D_{ij} F_h D_{ij}p D_h p - 4\tau^3\eta_n^6 \sum_{i,j=1}^{N} D_j \left(\operatorname{div} F + H\right) D_{ij}p D_i p \\
&-2\tau^3\eta_n^6 p \sum_{i,j=1}^{N} D_{ij} \left(\operatorname{div} F + H\right) D_{ij}p - 2\tau^3\eta_n^6 \left|D^2 p\right|^2 \left(\operatorname{div} F + H\right) \\
&+3\tau'\tau^2\eta_n^6 \left|D^2 p\right|^2 .
\end{aligned}
$$

On the other hand, we have

$$
D_h\left(\tau^3\eta_n^6 \left|D^2 p\right|^2\right) = 6\tau^3\eta_n^5 \left|D^2 p\right|^2 D_h\eta_n + 2\tau^3\eta_n^6 \sum_{i,j=1}^{N} D_{ijh}p D_{ij}p,
$$

$$
\begin{aligned}
D_{hk}\left(\tau^3\eta_n^6 \left|D^2 p\right|^2\right) &= 30\tau^3\eta_n^4 \left|D^2 p\right|^2 D_h\eta_n D_k\eta_n + 12\tau^3\eta_n^5 \sum_{i,j=1}^{N} D_{ijk}p D_{ij}p D_h\eta_n \\
&\quad +6\tau^3\eta_n^5 \left|D^2 p\right|^2 D_{hk}\eta_n + 12\tau^3\eta_n^5 \sum_{i,j=1}^{N} D_{ijh}p D_{ij}p D_k\eta_n \\
&\quad +2\tau^3\eta_n^6 \sum_{i,j=1}^{N} D_{ijhk}p D_{ij}p + 2\tau^3\eta_n^6 \sum_{i,j=1}^{N} D_{ijh}p D_{ijk}p
\end{aligned}
$$

for all $h,k \in \{1,...,N\}$. Hence,

$$
\begin{aligned}
A_0\left(\tau^3\eta_n^6 \left|D^2 p\right|^2\right) &= 30\tau^3\eta_n^4 \left|D^2 p\right|^2 a\left(D\eta_n, D\eta_n\right) + 12\tau^3\eta_n^5 \sum_{i,j,h,k=1}^{N} a_{hk} D_{ijk}p D_{ij}p D_h\eta_n \\
&\quad +6\tau^3\eta_n^5 \left|D^2 p\right|^2 \sum_{h,k=1}^{N} a_{hk} D_{hk}\eta_n + 12\tau^3\eta_n^5 \sum_{i,j,h,k=1}^{N} a_{hk} D_{ijh}p D_{ij}p D_k\eta_n \\
&\quad +2\tau^3\eta_n^6 \sum_{i,j,h,k=1}^{N} a_{hk} D_{ijhk}p D_{ij}p + 2\tau^3\eta_n^6 \sum_{i,j=1}^{N} a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right) \\
&\quad +6\tau^3\eta_n^5 \left|D^2 p\right|^2 \sum_{h,k=1}^{N} D_k a_{hk} D_h\eta_n + 2\tau^3\eta_n^6 \sum_{i,j,h,k=1}^{N} D_k a_{hk} D_{ijh}p D_{ij}p
\end{aligned}
$$

We then compute

$$
\partial_t\left(\tau^3\eta_n^6 \left|D^2 p\right|^2\right) - A_0\left(\tau^3\eta_n^6 \left|D^2 p\right|^2\right) - F\cdot D\left(\tau^3\eta_n^6 \left|D^2 p\right|^2\right)
$$

$$
\begin{aligned}
= & -2\tau^3\eta_n^6 \sum_{i,j=1}^N a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right) + 4\tau^3\eta_n^6 \sum_{i,j,h,k=1}^N D_j a_{hk} D_{ihk}p D_{ij}p \\
& - 4\tau^3\eta_n^6 \sum_{i,j,h=1}^N F_h D_{ijh}p D_{ij}p - 12\tau^3\eta_n^5 \sum_{i,j,h,k=1}^N a_{hk} D_{ijh}p D_{ij}p D_k\eta_n \\
& + 2\tau^3\eta_n^6 \sum_{i,j,h,k=1}^N D_{ij}a_{hk} D_{ij}p D_{hk}p - 4\tau^3\eta_n^6 \sum_{i,j,h=1}^N D_j F_h D_{ij}p D_{ih}p \\
& + 4\tau^3\eta_n^6 \sum_{i,j,h,k=1}^N D_{jk}a_{hk} D_{ij}p D_{ih}p + 2\tau^3\eta_n^6 \sum_{i,j,h,k=1}^N D_{ijk}a_{hk} D_{ij}p D_h p \\
& - 2\tau^3\eta_n^6 \sum_{i,j,h=1}^N D_{ij}F_h D_{ij}p D_h p - 4\tau^3\eta_n^6 \sum_{i,j=1}^N D_j\left(\operatorname{div} F + H\right) D_{ij}p D_i p \\
& - 2\tau^3\eta_n^6 p \sum_{i,j=1}^N D_{ij}\left(\operatorname{div} F + H\right) D_{ij}p - 12\tau^3\eta_n^5 \sum_{i,j,h,k=1}^N a_{hk} D_{ijk}p D_{ij}p D_h\eta_n \\
& - 2\tau^3\eta_n^6 \left|D^2p\right|^2 \left(\operatorname{div} F + H\right) + 3\tau'\tau^2\eta_n^6 \left|D^2p\right|^2 - 30\tau^3\eta_n^4 \left|D^2p\right|^2 a\left(D\eta_n, D\eta_n\right) \\
& - 6\tau^3\eta_n^5 \left|D^2p\right|^2 F\cdot D\eta_n - 6\tau^3\eta_n^5 \left|D^2p\right|^2 \sum_{h,k=1}^N a_{hk} D_{hk}\eta_n \\
& - 6\tau^3\eta_n^5 \left|D^2p\right|^2 \sum_{h,k=1}^N D_k a_{hk} D_h\eta_n.
\end{aligned} \tag{3.12}
$$

Now let $\varepsilon \in \left(0, \frac{1}{4}\right] \cap \left(0, \frac{4}{N+2}\right)$ be as in condition 3.5. Observe that

$$
\partial_t\left(\frac{1}{p^\varepsilon}\right) - \left(A_0 + F\cdot D\right)\left(\frac{1}{p^\varepsilon}\right) = 2\varepsilon\frac{1}{p^{\varepsilon+1}}F\cdot Dp + \varepsilon\frac{1}{p^\varepsilon}\left(\operatorname{div} F + H\right) - \varepsilon\left(\varepsilon+1\right)\frac{1}{p^{\varepsilon+2}}a\left(Dp, Dp\right) \tag{3.13}
$$

and

$$
\partial_t\left(uv\right) - \left(A_0 + F\cdot D\right)\left(uv\right) = u\left(\partial_t v - \left(A_0 + F\cdot D\right)v\right) + v\left(\partial_t u - \left(A_0 + F\cdot D\right)u\right) - 2a\left(Du, Dv\right) \tag{3.14}
$$

for each $u, v \in C^{2,1}\left(\mathbb{R}^N \times (0,\infty)\right)$. From (3.12), (3.13) and (3.14) we deduce

$$
\begin{aligned}
\partial_t & \left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right) - A_0\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right) - F\cdot D\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right) \\
&= -2\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j=1}^N a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right) \\
&\quad - \varepsilon\left(\varepsilon+1\right)\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp, Dp\right) \\
&\quad + 4\varepsilon\tau^3\eta_n^6\frac{1}{p^{\varepsilon+1}}\sum_{i,j,h,k=1}^N D_{ij}p\, a\left(D\left(D_{ij}p\right), Dp\right)
\end{aligned}
$$

$$
\begin{aligned}
&+4\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j,h,k=1}^{N} D_j a_{hk} D_{ihk}p D_{ij}p \\
&-4\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j,h=1}^{N} F_h D_{ijh}p D_{ij}p \\
&-24\tau^3\eta_n^5\frac{1}{p^\varepsilon}\sum_{i,j,h,k=1}^{N} D_{ij}p a\left(D\left(D_{ij}p\right), D\eta_n\right) \\
&+12\varepsilon\eta_n^5\tau^3\left|D^2p\right|^2\frac{1}{p^{\varepsilon+1}}a\left(Dp, D\eta_n\right) \\
&+2\varepsilon\tau^3\eta_n^6\frac{1}{p^{\varepsilon+1}}\left|D^2p\right|^2 F\cdot Dp \\
&+2\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j,h,k=1}^{N} D_{ij}a_{hk} D_{ij}p D_{hk}p \\
&-4\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j,h=1}^{N} D_j F_h D_{ij}p D_{ih}p \\
&+4\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j,h,k=1}^{N} D_{jk}a_{hk} D_{ij}p D_{ih}p \\
&+2\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j,h,k=1}^{N} D_{ijk}a_{hk} D_{ij}p D_h p \\
&-2\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j,h=1}^{N} D_{ij}F_h D_{ij}p D_h p \\
&-4\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j=1}^{N} D_j\left(\operatorname{div}F+H\right)D_{ij}p D_i p \\
&-2\tau^3\eta_n^6 p^{1-\varepsilon}\sum_{i,j=1}^{N} D_{ij}\left(\operatorname{div}F+H\right)D_{ij}p \\
&-\left(2-\varepsilon\right)\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left(\operatorname{div}F+H\right) \\
&+3\tau'\tau^2\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2 \\
&-30\tau^3\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2 a\left(D\eta_n, D\eta_n\right) \\
&-6\tau^3\eta_n^5\left|D^2p\right|^2\frac{1}{p^\varepsilon}F\cdot D\eta_n \\
&-6\tau^3\eta_n^5\frac{1}{p^\varepsilon}\left|D^2p\right|^2\sum_{h,k=1}^{N} a_{hk}D_{hk}\eta_n \\
&-6\tau^3\eta_n^5\frac{1}{p^\varepsilon}\left|D^2p\right|^2\sum_{h,k=1}^{N} D_k a_{hk} D_h\eta_n
\end{aligned}
$$

and hence

$$\partial_t\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)-A_0\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)-F\cdot D\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)$$

$$\begin{aligned}\leq &-2\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j=1}^N a\left(D\left(D_{ij}p\right),D\left(D_{ij}p\right)\right)\\ &-\varepsilon\left(\varepsilon+1\right)\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2a\left(Dp,Dp\right)\\ &-30\tau^3\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2a\left(D\eta_n,D\eta_n\right)\\ &+4\varepsilon\tau^3\eta_n^6\frac{1}{p^{\varepsilon+1}}\sum_{i,j=1}^N\sqrt{a\left(D\left(D_{ij}p\right),D\left(D_{ij}p\right)\right)}\sqrt{\left(D_{ij}p\right)^2a\left(Dp,Dp\right)}\\ &+4\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^3p\right|\left|D^2p\right|\left(\left|Da\right|+\left|F\right|\right)\\ &+24\tau^3\eta_n^5\frac{1}{p^\varepsilon}\sum_{i,j=1}^N\sqrt{a\left(D\left(D_{ij}p\right),D\left(D_{ij}p\right)\right)}\sqrt{\left(D_{ij}p\right)^2a\left(D\eta_n,D\eta_n\right)}\\ &+12\varepsilon\eta_n^5\tau^3\left|D^2p\right|^2\frac{1}{p^{\varepsilon+1}}\sqrt{a\left(Dp,Dp\right)}\sqrt{a\left(D\eta_n,D\eta_n\right)}\\ &+2\varepsilon\tau^3\eta_n^6\frac{1}{p^{\varepsilon+1}}\left|D^2p\right|^2\left|Dp\right|\left|F\right|\\ &+4\sqrt{N}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|\left|Dp\right|\left(\left|D^3a\right|+\left|D^2F\right|+\left|D\left(\operatorname{div}F+H\right)\right|\right)\\ &+2\tau^3\eta_n^6p^{1-\varepsilon}\left|D^2p\right|\left|D^2\left(\operatorname{div}F+H\right)\right|\\ &+6\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left(\left|D^2a\right|+\left|DF\right|\right)\\ &+\left(2-\varepsilon\right)\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left|\operatorname{div}F+H\right|\\ &+12\tau^2\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\frac{1}{\alpha}\\ &+6\tau^3\eta_n^5\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left|F\right|\left|D\eta_n\right|\\ &+6\tau^3\eta_n^5\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left|a\right|\left|D^2\eta_n\right|\\ &+6\sqrt{N}\tau^3\eta_n^5\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left|Da\right|\left|D\eta_n\right|\end{aligned}\tag{3.15}$$

We consider the positive terms of the right side of (3.15). Using repeatedly the Young's inequality, (1.7), (1.5) and (1.6), we obtain

$$\begin{aligned}&4\varepsilon\tau^3\eta_n^6\frac{1}{p^{\varepsilon+1}}\sum_{i,j=1}^N\sqrt{a\left(D\left(D_{ij}p\right),D\left(D_{ij}p\right)\right)}\sqrt{\left(D_{ij}p\right)^2a\left(Dp,Dp\right)}\\ &\qquad=2\sum_{i,j=1}^N\sqrt{\tau^3\eta_n^6\frac{1}{p^\varepsilon}a\left(D\left(D_{ij}p\right),D\left(D_{ij}p\right)\right)}\sqrt{4\varepsilon^2\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left(D_{ij}p\right)^2a\left(Dp,Dp\right)}\end{aligned}$$

$$\leq \tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j=1}^{N} a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right) + 4\varepsilon^2\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp, Dp\right),$$

$$4\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^3p\right|\left|D^2p\right|\left(|Da| + |F|\right)$$
$$\leq 2\sqrt{\frac{1}{2}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j=1}^{N} a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right)}\sqrt{\frac{16}{\lambda}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left(|Da|^2 + |F|^2\right)}$$
$$\leq \frac{1}{2}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j=1}^{N} a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right) + \frac{16}{\lambda}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left(|Da|^2 + |F|^2\right),$$

$$24\tau^3\eta_n^5\frac{1}{p^\varepsilon}\sum_{i,j=1}^{N}\sqrt{a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right)}\sqrt{\left(D_{ij}p\right)^2 a\left(D\eta_n, D\eta_n\right)}$$
$$\leq 2\sum_{i,j=1}^{N}\sqrt{\frac{1}{2}\tau^3\eta_n^6\frac{1}{p^\varepsilon}a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right)}$$
$$\cdot\sqrt{288\tau^3\eta_n^4\frac{1}{p^\varepsilon}\left(D_{ij}p\right)^2 a\left(D\eta_n, D\eta_n\right)}$$
$$\leq \frac{1}{2}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\sum_{i,j=1}^{N} a\left(D\left(D_{ij}p\right), D\left(D_{ij}p\right)\right) + 258L^2\tau^3\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}$$
$$+ 30\tau^3\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2 a\left(D\eta_n, D\eta_n\right),$$

$$12\varepsilon\tau^3\eta_n^5\frac{1}{p^{\varepsilon+1}}\left|D^2p\right|^2\sqrt{a\left(Dp, Dp\right)}\sqrt{a\left(D\eta_n, D\eta_n\right)}$$
$$\leq 2\sqrt{\frac{\varepsilon^2}{4}\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp, Dp\right)}\sqrt{144L^2\tau^3\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}}$$
$$\leq \frac{\varepsilon^2}{4}\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp, Dp\right) + 144L^2\tau^3\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}},$$

$$2\varepsilon\tau^3\eta_n^6\frac{1}{p^{\varepsilon+1}}\left|D^2p\right|^2|Dp||F|$$
$$\leq 2\sqrt{\frac{\varepsilon^2}{4}\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp, Dp\right)}\sqrt{\frac{4}{\lambda}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2|F|^2}$$
$$\leq \frac{\varepsilon^2}{4}\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp, Dp\right) + \frac{4}{\lambda}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2|F|^2,$$

$$4\sqrt{N}\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right||Dp|\left(\left|D^3a\right| + \left|D^2F\right| + \left|D\left(\operatorname{div}F + H\right)\right|\right)$$

$$
\begin{aligned}
&\leq 2\sqrt{\frac{\varepsilon^2}{2}\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp,Dp\right)}\\
&\qquad\cdot\sqrt{\frac{8N}{\varepsilon^2\lambda}\tau^3\eta_n^6p^{2-\varepsilon}\left(\left|D^3a\right|+\left|D^2F\right|+\left|D\left(\operatorname{div}F+H\right)\right|\right)^2}\\
&\leq \frac{\varepsilon^2}{2}\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp,Dp\right)\\
&\quad+\frac{8N}{\varepsilon^2\lambda}\tau^3\eta_n^6p^{2-\varepsilon}\left(\left|D^3a\right|+\left|D^2F\right|+\left|D\left(\operatorname{div}F+H\right)\right|\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
2\tau^3\eta_n^6p^{1-\varepsilon}\left|D^2p\right|\left|D^2\left(\operatorname{div}F+H\right)\right| &\leq 2\sqrt{\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2}\sqrt{\tau^3\eta_n^6p^{2-\varepsilon}\left|D^2\left(\operatorname{div}F+H\right)\right|^2}\\
&\leq \tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2+\tau^3\eta_n^6p^{2-\varepsilon}\left|D^2\left(\operatorname{div}F+H\right)\right|^2,
\end{aligned}
$$

$$
6\tau^3\eta_n^5\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2|F|\left|D\eta_n\right|\leq 6L\tau^3\eta_n^5\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\frac{|F|}{1+|y|}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
$$

$$
6\tau^3\eta_n^5\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2|a|\left|D^2\eta_n\right|\leq 6L\tau^3\eta_n^5\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}},
$$

$$
6\sqrt{N}\tau^3\eta_n^5\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2|Da|\left|D\eta_n\right|\leq 6L\sqrt{N}\tau^3\eta_n^5\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\frac{|Da|}{1+|y|}\mathbb{1}_{\{n\leq|y|\leq 2n\}}.
$$

So we obtain

$$
\begin{aligned}
&\partial_t\left(\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\right)-A_0\left(\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\right)-F\cdot D\left(\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\right)\\
&\qquad\leq-\left(\varepsilon-4\varepsilon^2\right)\tau^3\eta_n^6\frac{1}{p^{\varepsilon+2}}\left|D^2p\right|^2 a\left(Dp,Dp\right)\\
&\qquad\quad+\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\\
&\qquad\quad+\frac{20}{\lambda}\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\left(|Da|^2+|F|^2\right)\\
&\qquad\quad+6\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\left(\left|D^2a\right|+|DF|\right)\\
&\qquad\quad+\left(2-\varepsilon\right)\tau^3\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\left|\operatorname{div}F+H\right|\\
&\qquad\quad+12\tau^2\eta_n^6\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\frac{1}{\alpha}\\
&\qquad\quad+6L\sqrt{N}\tau^3\eta_n^5\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\left(\frac{|a|}{1+|y|^2}+\frac{|Da|}{1+|y|}+\frac{|F|}{1+|y|}\right)\mathbb{1}_{\{n\leq|y|\leq 2n\}}\\
&\qquad\quad+402L^2\tau^3\eta_n^4\frac{1}{p^{\varepsilon}}\left|D^2p\right|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\leq|y|\leq 2n\}}\\
&\qquad\quad+\frac{8N}{\varepsilon^2\lambda}\tau^3\eta_n^6p^{2-\varepsilon}\left(\left|D^3a\right|+\left|D^2F\right|+\left|D\left(\operatorname{div}F+H\right)\right|\right)^2\\
&\qquad\quad+\tau^3\eta_n^6p^{2-\varepsilon}\left|D^2\left(\operatorname{div}F+H\right)\right|^2
\end{aligned}
$$

and thus

$$\partial_t\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)-A_0\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)-F\cdot D\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)+\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\ H$$
$$\leq C\tau^2\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2\left(\frac{1}{\alpha}+1+\frac{|a|}{1+|y|^2}+|Da|^2+\left|D^2a\right|\right.$$
$$\left.+|F|^2+|DF|+|\operatorname{div}F+H|+|H|\right)$$
$$+\frac{C}{\varepsilon^2}\tau^3\eta_n^6p^{2-\varepsilon}\left(\left|D^3a\right|+\left|D^2F\right|+|D(\operatorname{div}F+H)|+\left|D^2(\operatorname{div}F+H)\right|\right)^2$$

for a constant $C=C(\lambda,N)>0$. Now Condition 3.5 yields

$$\partial_t\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)-A_0\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)-F\cdot D\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)+\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\ H$$
$$=\partial_t\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)-A\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2\right)$$
$$\leq C_1\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right|^2U$$
$$+\frac{C_1}{\varepsilon^2}\tau^3\eta_n^6p^{2-\varepsilon}\left(\left|D^3a\right|+\left|D^2F\right|+|D(\operatorname{div}F+H)|+\left|D^2(\operatorname{div}F+H)\right|\right)^2 \tag{3.16}$$

for a constant $C_1=C_1(\lambda,N)$, where we may assume that $C_1\geq\frac{\lambda}{4}>0$. Equation (3.5) further gives

$$\partial_t\left(\tau^2\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2\right)-A\left(\tau^2\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2\right)$$
$$\leq-\varepsilon(\varepsilon+1)\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}|Dp|^2a(Dp,Dp)$$
$$-2\tau^2\eta_n^4\frac{1}{p^\varepsilon}\sum_{i=1}^N a(D(D_ip),D(D_ip))$$
$$+4\varepsilon\tau^2\eta_n^4\frac{1}{p^{\varepsilon+1}}\sum_{i=1}^N\sqrt{a(D(D_ip),D(D_ip))}\sqrt{(D_ip)^2a(Dp,Dp)}$$
$$+4\tau^2\eta_n^4\frac{1}{p^\varepsilon}\left|D^2p\right||Dp|(|F|+|Da|)$$
$$+16\tau^2\eta_n^3\frac{1}{p^\varepsilon}\sum_{i=1}^N\sqrt{a(D(D_ip),D(D_ip))}\sqrt{(D_ip)^2a(D\eta_n,D\eta_n)}$$
$$+2\varepsilon\tau^2\eta_n^4\frac{1}{p^{\varepsilon+1}}|Dp|^3|F|$$
$$+8\varepsilon\tau^2\eta_n^3\frac{1}{p^{\varepsilon+1}}|Dp|^2\sqrt{a(Dp,Dp)}\sqrt{a(D\eta_n,D\eta_n)}$$
$$+2\sqrt{N}\tau^2\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2\left(\left|D^2a\right|+|DF|+|\operatorname{div}F+H|\right)$$
$$+8\tau\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2\frac{1}{\alpha}$$

$$
\begin{aligned}
&+ 4\sqrt{N}\tau^2\eta_n^3 \frac{1}{p^\varepsilon} |Dp|^2 (|F| + |Da|) |D\eta_n| \\
&- 12\tau^2\eta_n^2 \frac{1}{p^\varepsilon} |Dp|^2 a (D\eta_n, D\eta_n) \\
&+ 4\tau^2\eta_n^3 \frac{1}{p^\varepsilon} |Dp|^2 |a| \left|D^2\eta_n\right| \\
&+ 2\tau^2\eta_n^4 p^{1-\varepsilon} |Dp| |D (\operatorname{div} F + H)| \\
&+ \tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 |H| . \qquad (3.17)
\end{aligned}
$$

We consider the positive terms of the right side of (3.17). Using repeatedly the Young's inequality, (1.7), (1.5) and (1.6), we obtain

$$
\begin{aligned}
4\varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} \sum_{i=1}^N & \sqrt{a (D (D_i p), D (D_i p))} \sqrt{(D_i p)^2 a (Dp, Dp)} \\
&\leq 2 \sum_{i=1}^N \sqrt{\tau^2\eta_n^4 \frac{1}{p^\varepsilon} a (D (D_i p), D (D_i p))} \sqrt{4\varepsilon^2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} (D_i p)^2 a (Dp, Dp)} \\
&\leq \tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a (D (D_i p), D (D_i p)) + 4\varepsilon^2\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a (Dp, Dp) ,
\end{aligned}
$$

$$
\begin{aligned}
4\tau^2\eta_n^4 \frac{1}{p^\varepsilon} & \left|D^2 p\right| |Dp| (|F| + |Da|) \\
&\leq 2 \sqrt{\frac{1}{4}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a (D (D_i p), D (D_i p))} \sqrt{\frac{32}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \left(|F|^2 + |Da|^2\right)} \\
&\leq \frac{1}{4}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a (D (D_i p), D (D_i p)) + \frac{32}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \left(|F|^2 + |Da|^2\right) ,
\end{aligned}
$$

$$
\begin{aligned}
16\tau^2\eta_n^3 \frac{1}{p^\varepsilon} \sum_{i=1}^N & \sqrt{a (D (D_i p), D (D_i p))} \sqrt{(D_i p)^2 a (D\eta_n, D\eta_n)} \\
&\leq 2 \sum_{i=1}^N \sqrt{\frac{1}{4}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} a (D (D_i p), D (D_i p))} \sqrt{256\tau^2\eta_n^2 \frac{1}{p^\varepsilon} (D_i p)^2 a (D\eta_n, D\eta_n)} \\
&\leq \frac{1}{4}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N a (D (D_i p), D (D_i p)) + 244L^2\tau^2\eta_n^2 \frac{1}{p^\varepsilon} |Dp|^2 \frac{|a|}{1 + |y|^2} \mathbb{1}_{\{n\leq|y|\leq 2n\}} \\
&\quad + 12\tau^2\eta_n^2 \frac{1}{p^\varepsilon} |Dp|^2 a (D\eta_n, D\eta_n) ,
\end{aligned}
$$

$$
\begin{aligned}
2\varepsilon\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+1}} |Dp|^3 |F| &\leq 2 \sqrt{\frac{\varepsilon^2}{4}\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a (Dp, Dp)} \sqrt{\frac{4}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 |F|^2} \\
&\leq \frac{\varepsilon^2}{4}\tau^2\eta_n^4 \frac{1}{p^{\varepsilon+2}} |Dp|^2 a (Dp, Dp) + \frac{4}{\lambda}\tau^2\eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 |F|^2 ,
\end{aligned}
$$

$$8\varepsilon\tau^2\eta_n^3\frac{1}{p^{\varepsilon+1}}|Dp|^2\sqrt{a(Dp,Dp)}\sqrt{a(D\eta_n,D\eta_n)}$$
$$\le 2\sqrt{\frac{\varepsilon^2}{4}\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}|Dp|^2 a(Dp,Dp)}\sqrt{64L^2\tau^2\eta_n^2\frac{1}{p^\varepsilon}|Dp|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}}}$$
$$\le \frac{\varepsilon^2}{4}\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}|Dp|^2 a(Dp,Dp)+64L^2\tau^2\eta_n^2\frac{1}{p^\varepsilon}|Dp|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}},$$

$$4\sqrt{N}\tau^2\eta_n^3\frac{1}{p^\varepsilon}|Dp|^2(|F|+|Da|)|D\eta_n|\le 4L\sqrt{N}\tau^2\eta_n^3\frac{1}{p^\varepsilon}|Dp|^2\frac{|F|+|Da|}{1+|y|}\mathbb{1}_{\{n\le|y|\le 2n\}},$$

$$4\tau^2\eta_n^3\frac{1}{p^\varepsilon}|Dp|^2|a|\left|D^2\eta_n\right|\le 4L\tau^2\eta_n^3\frac{1}{p^\varepsilon}|Dp|^2\frac{|a|}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}},$$

$$\begin{aligned}2\tau^2\eta_n^4p^{1-\varepsilon}|Dp||D(\operatorname{div}F+H)| &\le 2\sqrt{\tau^2\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2}\sqrt{\tau^2\eta_n^4p^{2-\varepsilon}|D(\operatorname{div}F+H)|^2}\\ &\le \tau^2\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2+\tau^2\eta_n^4p^{2-\varepsilon}|D(\operatorname{div}F+H)|^2.\end{aligned}$$

It then follows

$$\begin{aligned}&\partial_t\left(\tau^2\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2\right)-A\left(\tau^2\eta_n^4\frac{1}{p^\varepsilon}|Dp|^2\right)\\
&\le-\left(\varepsilon-3\varepsilon^2-\frac{\varepsilon^2}{2}\right)\tau^2\eta_n^4\frac{1}{p^{\varepsilon+2}}|Dp|^2a(Dp,Dp)\\
&\quad-\frac{1}{2}\tau^2\eta_n^4\frac{1}{p^\varepsilon}\sum_{i=1}^N a(D(D_ip),D(D_ip))\\
&\quad+C\tau\eta_n^2\frac{1}{p^\varepsilon}|Dp|^2\left(\frac{1}{\alpha}+1+\frac{|a|}{1+|y|^2}+|Da|^2+\left|D^2a\right|\right.\\
&\quad\left.+|F|^2+|DF|+|\operatorname{div}F+H|+|H|\right)\\
&\quad+\tau^2\eta_n^4p^{2-\varepsilon}|D(\operatorname{div}F+H)|^2.\end{aligned}$$

with a constant $C=C(\lambda,N)>0$. Using also (3.14), we conclude that

$$\begin{aligned}&\partial_t\left(\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right)-A\left(\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right)\\
&\le-\left(\varepsilon-3\varepsilon^2-\frac{\varepsilon^2}{2}\right)\tau^2\eta_n^4U\frac{1}{p^{\varepsilon+2}}|Dp|^2a(Dp,Dp)\\
&\quad-\frac{1}{2}\tau^2\eta_n^4U\frac{1}{p^\varepsilon}\sum_{i=1}^N a(D(D_ip),D(D_ip))\\
&\quad+C\tau\eta_n^2U\frac{1}{p^\varepsilon}|Dp|^2\left(\frac{1}{\alpha}+1+\frac{|a|}{1+|y|^2}+|Da|^2+\left|D^2a\right|\right.\end{aligned}$$

$$
\begin{aligned}
&+ |F|^2 + |DF| + |\operatorname{div} F + H| + |H| \Big) \\
&+ \tau^2 \eta_n^4 U p^{2-\varepsilon} \left| D \left(\operatorname{div} F + H \right) \right|^2 \\
&- \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \left(A_0 U + F \cdot DU \right) \\
&- 4 \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N D_i p a \left(D \left(D_i p \right), DU \right) \\
&+ 2 \varepsilon \tau^2 \eta_n^4 \frac{1}{p^{\varepsilon+1}} |Dp|^2 a \left(Dp, DU \right) \\
&- 8 \tau^2 \eta_n^3 \frac{1}{p^\varepsilon} |Dp|^2 a \left(D\eta_n, DU \right). \qquad (3.18)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
-4\tau^2 \eta_n^4 \frac{1}{p^\varepsilon} \sum_{i=1}^N & D_i p a \left(D \left(D_i p \right), DU \right) \\
\leq & 2 \sum_{i=1}^N \sqrt{\frac{1}{4} \tau^2 \eta_n^4 U \frac{1}{p^\varepsilon} a \left(D \left(D_i p \right), D \left(D_i p \right) \right)} \sqrt{16 \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} \left(D_i p \right)^2 \frac{a \left(DU, DU \right)}{U}} \\
\leq & \frac{1}{4} \tau^2 \eta_n^4 U \frac{1}{p^\varepsilon} \sum_{i=1}^N a \left(D \left(D_i p \right), D \left(D_i p \right) \right) + 16 \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \frac{a \left(DU, DU \right)}{U},
\end{aligned}
$$

$$
\begin{aligned}
2\varepsilon \tau^2 \eta_n^4 \frac{1}{p^{\varepsilon+1}} & |Dp|^2 a \left(DU, Dp \right) \\
\leq & 2 \sqrt{\frac{\varepsilon^2}{4} \tau^2 \eta_n^4 U \frac{1}{p^{\varepsilon+2}} |Dp|^2 a \left(Dp, Dp \right)} \sqrt{4 \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \frac{a \left(DU, DU \right)}{U}} \\
\leq & \frac{\varepsilon^2}{4} \tau^2 \eta_n^4 U \frac{1}{p^{\varepsilon+2}} |Dp|^2 a \left(Dp, Dp \right) + 4 \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \frac{a \left(DU, DU \right)}{U},
\end{aligned}
$$

$$
\begin{aligned}
-8\tau^2 \eta_n^3 \frac{1}{p^\varepsilon} & |Dp|^2 a \left(DU, D\eta_n \right) \\
\leq & 2 \sqrt{4 \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \frac{a \left(DU, DU \right)}{U}} \sqrt{4 L^2 \tau^2 \eta_n^2 U \frac{1}{p^\varepsilon} |Dp|^2 \frac{|a|}{1 + |y|^2}} \\
\leq & 4 \tau^2 \eta_n^4 \frac{1}{p^\varepsilon} |Dp|^2 \frac{a \left(DU, DU \right)}{U} + 4 L^2 \tau^2 \eta_n^2 U \frac{1}{p^\varepsilon} |Dp|^2 \frac{|a|}{1 + |y|^2}.
\end{aligned}
$$

Now, (3.18) yields

$$
\begin{aligned}
\partial_t \left(\tau^2 \eta_n^4 U \frac{1}{p^\varepsilon} |Dp|^2 \right) & - A \left(\tau^2 \eta_n^4 U \frac{1}{p^\varepsilon} |Dp|^2 \right) \\
& \leq - \left(\varepsilon - 3\varepsilon^2 - \frac{\varepsilon^2}{4} \right) \tau^2 \eta_n^4 U \frac{1}{p^{\varepsilon+2}} |Dp|^2 a \left(Dp, Dp \right) \\
& \quad - \frac{1}{4} \tau^2 \eta_n^4 U \frac{1}{p^\varepsilon} \sum_{i=1}^N a \left(D \left(D_i p \right), D \left(D_i p \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+ C\tau\eta_n^2 U\frac{1}{p^\varepsilon}|Dp|^2\left(\frac{1}{\alpha}+1+\frac{|a|}{1+|y|^2}+|Da|^2+\left|D^2a\right|\right.\\
&+|F|^2+|DF|+|\operatorname{div}F+H|+|H|\\
&\left.+\left|\frac{A_0U+F\cdot DU}{U}\right|+\frac{a\left(DU,DU\right)}{U^2}\right)\\
&+\tau^2\eta_n^4 Up^{2-\varepsilon}\left|D\left(\operatorname{div}F+H\right)\right|^2. \qquad (3.19)
\end{aligned}
$$

with a constant $C = C(\lambda, N) > 0$. Analogous as in (2.20) we deduce from (3.19), the fact that $\varepsilon - 4\varepsilon^2 \geq 0$ and Condition 3.5

$$
\begin{aligned}
\partial_t\left(\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right) - A\left(\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right)
&\leq -\frac{\lambda}{4}\tau^2\eta_n^4U\frac{1}{p^\varepsilon}\left|D^2p\right|^2\\
&+\frac{C}{\varepsilon^2}Up^{2-\varepsilon}\left(\frac{1}{\alpha^2}+\left|D\left(\operatorname{div}F+H\right)\right|^2+U^2\right.\\
&\left.+\left|\frac{A_0U+F\cdot DU}{U}\right|^2+\frac{a\left(DU,DU\right)^2}{U^4}\right). \qquad (3.20)
\end{aligned}
$$

with a constant $C = C(\lambda, N) > 0$. We further set $C_2 = \frac{4C_1}{\lambda}$, where the constant C_1 is given by (3.16). We remark that $C_2 \geq 1$. Then (3.20) yields

$$
\begin{aligned}
\partial_t\left(C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right) - A\left(C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right)
&\leq -C_1\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4U\frac{1}{p^\varepsilon}\left|D^2p\right|^2\\
&+\left(1+\frac{1}{\alpha}\right)\frac{C}{\varepsilon^2}Up^{2-\varepsilon}\left(\frac{1}{\alpha^2}+\left|D\left(\operatorname{div}F+H\right)\right|^2+U^2\right.\\
&\left.+\left|\frac{A_0U+F\cdot DU}{U}\right|^2+\frac{a\left(DU,DU\right)^2}{U^4}\right) \qquad (3.21)
\end{aligned}
$$

with a constant $C = C(\lambda, N) > 0$. We now combine (3.16) with (3.21) and Condition 3.5. Let $0 < \alpha < T < \infty$. It follows

$$
\begin{aligned}
&\partial_t\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2+C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right)\\
&-A\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2+C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4U\frac{1}{p^\varepsilon}|Dp|^2\right)\\
&\leq\left(1+\frac{1}{\alpha}\right)^3\frac{C}{\varepsilon^2}Up^{2-\varepsilon}\left(1+\left|D^3a\right|^2+\left|D^2F\right|^2\right.\\
&+\left|D\left(\operatorname{div}F+H\right)\right|^2+\left|D^2\left(\operatorname{div}F+H\right)\right|^2\\
&\left.+\frac{\left(A_0U+F\cdot DU\right)^2}{U^2}+\frac{a\left(DU,DU\right)^2}{U^4}\right)\\
&\leq\left(1+\frac{1}{\alpha}\right)^3\frac{C_3}{\varepsilon^2}\left\|p\left(x,\cdot,\cdot\right)\right\|_{L^\infty\left(Q\left(\frac{\alpha}{2},T\right)\right)}^{2-\varepsilon}Q^\varepsilon \qquad (3.22)
\end{aligned}
$$

with a constant $C_3 = C_3(\lambda, N) > 0$. Let now

$$\beta(x,\alpha,T) = \beta = K + \left(1+\frac{1}{\alpha}\right)^3 \frac{C_3}{\varepsilon^2} \sup_{(y,t)\in\mathbb{R}^N\times\left[\frac{\alpha}{2},T\right]} |p(x,y,t)|^{2-\varepsilon},$$

where K is given by Condition 3.5. Using (3.9) for the Lyapunov function Q, we deduce from (3.22)

$$\begin{aligned}
&\partial_t\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2 + C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4 U\frac{1}{p^\varepsilon}|Dp|^2 - e^{\beta t}Q^\varepsilon\right)\\
&-A\left(\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2 + C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4 U\frac{1}{p^\varepsilon}|Dp|^2 - e^{\beta t}Q^\varepsilon\right)\\
&\quad\le\left(1+\frac{1}{\alpha}\right)^3\frac{C_3}{\varepsilon^2}\|p(x,\cdot,\cdot)\|^{2-\varepsilon}_{L^\infty\left(Q\left(\frac{\alpha}{2},T\right)\right)}Q^\varepsilon - (\beta-K)Q^\varepsilon\\
&\quad=0.
\end{aligned}$$

Observe that

$$\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2 + C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4 U\frac{1}{p^\varepsilon}|Dp|^2 - e^{\beta t}Q^\varepsilon \le 0$$

for $t\in\left[0,\frac{\alpha}{2}\right]$ and for $|y| = 2n$. The parabolic maximum principle thus yields

$$\tau^3\eta_n^6\frac{1}{p^\varepsilon}\left|D^2p\right|^2 + C_2\left(1+\frac{1}{\alpha}\right)\tau^2\eta_n^4 U\frac{1}{p^\varepsilon}|Dp|^2 - e^{\beta t}Q^\varepsilon \le 0$$

on $B(0,2n)\times[0,T]$. Hence, letting $n\to\infty$, we obtain

$$\frac{1}{p(x,y,t)^\varepsilon}\left|D^2p(x,y,t)\right|^2 + C_2\left(1+\frac{1}{\alpha}\right)U(y)\frac{1}{p(x,y,t)^\varepsilon}|Dp(x,y,t)|^2 \le e^{\beta t}Q(y)^\varepsilon$$

for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times[\alpha,T]$. Since $C_2\left(1+\frac{1}{\alpha}\right)U(y)\ge 1$ for each $y\in\mathbb{R}^N$, we conclude that

$$\left|D^2p(x,y,t)\right|^2 + |Dp(x,y,t)|^2 \le e^{\beta t}p(x,y,t)^\varepsilon Q(y)^\varepsilon$$

for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}^N\times[\alpha,T]$. ∎

Corollary 3.8. *Assume that Condition 3.5 holds. We then have*

$$\left|D^2p(x,\cdot,t)\right|\in L^{\frac{2}{\varepsilon}}\left(\mathbb{R}^N\right) \qquad \text{for all } (x,t)\in\mathbb{R}^N\times(0,\infty),$$

$$\left|D^2p(x,\cdot,\cdot)\right|\in L^{\frac{2}{\varepsilon}}(Q(\alpha,T)) \qquad \text{for each } x\in\mathbb{R}^N \text{ and all } 0<\alpha<T<\infty$$

and it holds

$$\int_{\mathbb{R}^N}\left(\left|D^2p(x,y,t)\right|^{\frac{2}{\varepsilon}} + |Dp(x,y,t)|^{\frac{2}{\varepsilon}}\right)dy \le 2e^{\left(\frac{\beta}{\varepsilon}+K\right)t}Q(x), \tag{3.23}$$

where β is as in Theorem 3.7 and ε, K and Q satisfy Condition 3.5.

Proof. The assertion follows directly from Theorem 3.7 and (1.21). ∎

Corollary 3.9. *Assume that Condition 3.5 and* $\varepsilon < \frac{2}{N}$ *holds. Then we have*

$$|Dp(x,\cdot,\cdot)| \in L^{\infty}(Q(\alpha,T)) \qquad \text{for each } x \in \mathbb{R}^N \text{ and all } 0<\alpha<T<\infty$$

and it holds

$$|Dp(x,y,t)|^2 \le Ce^{(\beta+\varepsilon K)t} Q(x)^{\varepsilon}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [\alpha,T]$*, a constant* $C = C(\varepsilon,N) > 0$ *and* β*,* K *from Theorem 3.7.*

Proof. Since $\frac{2}{\varepsilon} > N$, from Morrey's inequality and (3.23) we conclude

$$\begin{aligned} & \| |Dp(x,\cdot,t)| \|_{\infty}^{\frac{2}{\varepsilon}} + \|p(x,\cdot,t)\|_{\infty}^{\frac{2}{\varepsilon}} \\ \le\ & C_1 \int_{\mathbb{R}^N} \left(\left|D^2 p(x,y,t)\right|^{\frac{2}{\varepsilon}} + |Dp(x,y,t)|^{\frac{2}{\varepsilon}} + p(x,y,t)^{\frac{2}{\varepsilon}} \right) dy \\ \le\ & C_1 e^{\left(\frac{\beta}{\varepsilon}+K\right)t} Q(x) + C_1 \|p(x,\cdot,t)\|_{\infty}^{\frac{2-\varepsilon}{\varepsilon}} \\ \le\ & C_1 e^{\left(\frac{\beta}{\varepsilon}+K\right)t} Q(x) + \|p(x,\cdot,t)\|_{\infty}^{\frac{2}{\varepsilon}} + C_2, \end{aligned}$$

where the constants $C_1, C_2 > 0$ depends only on N and ε. Theorefore we have

$$\| |Dp(x,\cdot,t)| \|_{\infty}^2 \le Ce^{(\beta+\varepsilon K)t} Q(x)^{\varepsilon},$$

with a constant $C = C(\varepsilon,N) > 0$. ■

Example 3.10. We consider again the operator A from Example 2.4 defined by

$$A = \left(1+|x|^2\right)^{\alpha} \Delta - |x|^{2\beta} x \cdot D, \qquad 0<\alpha<\beta,\ \beta \ge 1.$$

In Example 2.4 it was proved that for each $t_0 > 0$ and each $\gamma > 0$ there exists a constant $C = C(\alpha,\beta,\gamma,t_0) > 0$ such that

$$p(x,y,t) \le Cte^{-\gamma|y|^2} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0,\infty). \tag{3.24}$$

Furthermore, the function $x \longmapsto Ce^{\delta|x|^2}$ is a Lyapunov function for A for all $C, \delta > 0$. Since the coefficients of A and their relevant derivatives grow only polynomially, for each $\delta > 0$ and each $\varepsilon \in \left(0, \frac{1}{4}\right] \cap \left(0, \frac{4}{N+2}\right)$ there exists $C = C(\alpha,\beta,\delta,\varepsilon) > 0$ such that $Q(x) = Ce^{\delta|x|^2}$ satisfies Condition 3.5 with $U(x) = C_1 e^{\frac{\varepsilon\delta}{4}|x|^2}$. Theorem 3.7 then implies that for all $0 < t_0 < T < \infty$ it holds

$$\left|D_y^2 p(x,y,t)\right|^2 + |D_y p(x,y,t)|^2 \le e^{\beta_0 t} p(x,y,t)^{\varepsilon} Ce^{\delta\varepsilon|y|^2} \tag{3.25}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0,T]$, where

$$\beta_0 = K + \left(1+\frac{1}{t_0}\right)^3 \frac{C_0}{\varepsilon^2} \sup_{(z,s)\in\mathbb{R}^N \times \left[\frac{t_0}{2},T\right]} |p(x,z,s)|^{2-\varepsilon},$$

K is given as in Condition 3.5 and $C_0 = C_0(\lambda,N) > 0$. We combine (3.24) and (3.25). Setting $\gamma = \frac{\delta}{2}$, we obtain

$$\left|D_y^2 p(x,y,t)\right|^2 + |D_y p(x,y,t)|^2 \le C_1 t^{\varepsilon} e^{\beta_0 t} e^{-\delta\varepsilon|y|^2}$$

for a constant $C_1 = C_1(\alpha, \beta, \delta, t_0, \lambda, \varepsilon, N) > 0$. From Example 2.4 we obtain the formal adjoint operator

$$\begin{aligned} A^* &= \left(1+|x|^2\right)^\alpha \Delta + \left(4\alpha\left(1+|x|^2\right)^{\alpha-1} + |x|^{2\beta}\right) x \cdot D \\ &\quad +2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right) + (N+2\beta)|x|^{2\beta}. \end{aligned}$$

Since $\partial_t p = A^* p$ (for each fixed $x \in \mathbb{R}^N$), it follows from above that

$$\begin{aligned} |\partial_t p(x,y,t)| &\leq \sqrt{N}\left(1+|x|^2\right)^\alpha \left|D^2 p(x,y,t)\right| + \left(4\alpha\left(1+|x|^2\right)^{\alpha-1} + |x|^{2\beta}\right)|x|\,|Dp(x,y,t)| \\ &\quad + \left(2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right) + (N+2\beta)|x|^{2\beta}\right) p(x,y,t) \\ &\leq C_2\Bigg(\sqrt{N}\left(1+|x|^2\right)^\alpha + 4\alpha\left(1+|x|^2\right)^{\alpha-1}|x| + |x|^{2\beta+1} \\ &\quad +2\alpha\left(1+|x|^2\right)^{\alpha-2}\left(N+(N+2\alpha-2)|x|^2\right) + (N+2\beta)|x|^{2\beta}\Bigg) t e^{\frac{\beta_0}{2}t} e^{-\frac{\delta\varepsilon}{2}|y|^2} \end{aligned}$$

for a constant $C_2 = C_2(\alpha, \beta, \delta, t_0, \lambda, \varepsilon, N) > 0$. Thus there exist $\theta \in \left(0, \frac{\delta\varepsilon}{2}\right)$ such that

$$|\partial_t p(x,y,t)| \leq C_3 t e^{\frac{\beta_0}{2}t} e^{-\theta|y|^2} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times [t_0, T]$$

and a constant $C_3 = C_3(\alpha, \beta, \delta, \theta, t_0, \lambda, \varepsilon, N) > 0$. Let now $f \in C_b\left(\mathbb{R}^N\right)$. It holds

$$T(t) f(x) = \int_{\mathbb{R}^N} p(x,y,t) f(y)\, dy \qquad \text{on } \mathbb{R}^N \text{ for each } t > 0.$$

For each $t \geq t_0$ and $x \in \mathbb{R}^N$ we have

$$\|\partial_t p(\cdot, y, t) f(y)\|_\infty \leq C_4 e^{-\theta|y|^2} \|f\|_\infty$$

for a constant $C_4 = C_4(\alpha, \beta, \delta, \theta, t_0, \lambda, \varepsilon, N, T) > 0$. The dominated convergence theorem implies that $t \longmapsto T(t) f \in C_b\left(\mathbb{R}^N\right)$ is differentiable in $C_b\left(\mathbb{R}^N\right)$.

Analogously we obtain the following result.

Corollary 3.11. *Assume that for the operator A defined in (1.8) it holds*

$$\left(\frac{r-2}{|x|^r} + \delta r\right)\frac{a(x,x)}{|x|^2} + \frac{1}{|x|^r}\sum_{i=1}^N a_{ii} + \frac{1}{|x|^r}\sum_{i,j=1}^N D_j a_{ij} x_i + |x|^{1-r} F \cdot \frac{x}{|x|} - \frac{1}{\delta r |x|^{2r-2}} H \leq -C_0$$

for each $x \in \mathbb{R}^N \setminus B(0,R)$ for some $R > 0$, $r > 2$, $\delta > 0$ and $C_0 > 0$. Further, assume that

$$\begin{aligned} &|a| + |Da| + \left|D^2 a\right| + \left|D^3 a\right| + |F| + |DF| + \left|D^2 F\right| \\ &\quad + |D(\operatorname{div} F + H)| + \left|D^2(\operatorname{div} F + H)\right| + |H| \end{aligned}$$

grows only polynomially on $\mathbb{R}^N$. Then the semigroup $(T(t))_{t\geq 0}$ is differentiable in $C_b\left(\mathbb{R}^N\right)$.

Proof. We set $\varepsilon = \frac{1}{N+2}$ and fix some $\gamma \in \left(0, \frac{\delta}{2}\right)$. Since the coefficients of A and their relevant derivatives grow only polynomially, there exist constants $C_1, C_2 > 0$ such that $U = C_1 e^{\frac{\varepsilon\gamma}{4}|y|^r}$ and $Q = C_2 e^{\gamma|y|^r}$ satisfy Condition 3.5. Theorem 3.7 implies that for all $0 < \alpha < T < \infty$ it holds

$$\left|D_y^2 p\left(x,y,t\right)\right|^2 + \left|D_y p\left(x,y,t\right)\right|^2 \leq e^{\beta t} p\left(x,y,t\right)^{\varepsilon} Q\left(y\right)^{\varepsilon}$$

for all $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times [\alpha, T]$, where β is given as in Theorem 3.7. Moreover, for each fixed $x \in \mathbb{R}^N$ and $(y, t) \in \mathbb{R}^N \times [\alpha, T]$ we observe that

$$\begin{aligned} |\partial_t p| &\leq |a| \left|D^2 p\right| + \sqrt{N} \left|Da\right| \left|Dp\right| + |F| \left|Dp\right| + |H|\, p \\ &\leq |a|\, e^{\frac{\beta}{2}t} p^{\frac{\varepsilon}{2}} Q^{\frac{\varepsilon}{2}} + \sqrt{N} \left|Da\right| e^{\frac{\beta}{2}t} p^{\frac{\varepsilon}{2}} Q^{\frac{\varepsilon}{2}} + |F|\, e^{\frac{\beta}{2}t} p^{\frac{\varepsilon}{2}} Q^{\frac{\varepsilon}{2}} + |H|\, p. \end{aligned}$$

Using the polynomial growth of $|a| + |Da| + |F| + |H|$, we conclude that there exists a constant $C_3 > 0$ such that

$$|\partial_t p| \leq C_3 e^{\frac{\beta}{2}t} Q^{\frac{3\varepsilon}{4}} \left(p^{\frac{\varepsilon}{2}} + p\right) = C_2 C_3 e^{\frac{\beta}{2}t} e^{\frac{3\varepsilon\gamma}{4}|y|^r} \left(p^{\frac{\varepsilon}{2}} + p\right)$$

Furthermore, Proposition 2.8 says that for each $M > \frac{N}{2}$ and each $0 < \gamma_0 < \delta$ there exists a constant $C_4 > 0$ such that it holds

$$p\left(x,y,t\right) \leq C_4 e^{-\gamma_0 |y|^r} \left(t + \frac{1}{t^M}\right) \exp\left\{\max\left\{-H_0, 0\right\} t + \left(\frac{t}{2}\right)^{-\frac{r}{r-2}}\right\}$$

for all $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, \infty)$. Fix $\gamma_0 \in \left(\frac{3\delta}{4}, \delta\right)$. It then follows

$$|\partial_t p| \leq C_5 e^{\frac{\beta}{2}t} \left(t^{\frac{\varepsilon}{2}} e^{\max\left\{-\frac{H_0\varepsilon}{2}, 0\right\}t} + t e^{\max\{-H_0, 0\}t}\right) e^{-\frac{\varepsilon}{4}(2\gamma_0 - 3\gamma)|y|^r}$$

for all $(x, y, t) \in \mathbb{R}^N \times \mathbb{R} \times [\alpha, T]$ and some constant $C_5 > 0$. Thus there exists a constant $C_6 > 0$ and $\theta \in \left(0, \frac{3\varepsilon\delta}{8}\right)$ such that

$$|\partial_t p| \leq C_6 e^{-\theta|y|^r} \qquad \text{for all } (x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times [\alpha, T].$$

Recall that

$$T\left(t\right) f = \int_{\mathbb{R}^N} p\left(\cdot, y, t\right) f\left(y\right) dy \qquad \text{on } \mathbb{R}^N \text{ for each } t > 0.$$

Since

$$\left\|\partial_t p\left(\cdot, y, t\right) f\left(y\right)\right\|_\infty \leq C_6 e^{-\theta|y|^r} \left\|f\right\|_\infty \qquad \text{for each } f \in C_b\left(\mathbb{R}^N\right),$$

the dominated convergence theorem implies that $t \longmapsto T(t) f \in C_b\left(\mathbb{R}^N\right)$ is differentiable in $C_b\left(\mathbb{R}^N\right)$.

■

Chapter 4

The case of outward pointing drift

In this chapter we treat the case of an "outward pointing" drift coefficient F (i. e., $\operatorname{div} F$ is bounded from below). Here we can obtain a very explicit estimate of the L^q norm of p.

Condition 4.1. *We assume that Condition 1.1 holds and that there exist constants $M > N+2$, $K, K_1, K_2 > -H_0$ and Lyapunov functions V and W such that $W \leq V$,*

$$AV \leq KV, \qquad AW \leq K_1 W, \qquad \lambda \Delta V + F \cdot DV - HV \leq K_2 V$$

and

$$1 + |F|^M + |H|^M \leq W$$

on $\mathbb{R}^N$. Furthermore, we assume that $N \geq 3$ and there exists a constant $\gamma \in \mathbb{R}$ such that

$$\gamma = \inf_{x \in \mathbb{R}^N} \left(\operatorname{div} F(x) + H(x) \right).$$

Observe that for each $m \in \mathbb{N}$ there is a $n \in \mathbb{N}$ such that $|x| \geq n$ implies that $V(x) \geq m$.

Theorem 4.2. *Assume that Condition 4.1 holds. We then obtain*

$$\left(\int_{\mathbb{R}^N} p(x,y,t)^q \, dy \right)^{\frac{1}{q}} \leq \left(\frac{(q-1)N}{2\lambda S} \right)^{\frac{(q-1)N}{2q}} t^{-\frac{(q-1)N}{2q}} e^{-\left(\frac{H_0}{q} + \frac{q-1}{q} \min\{H_0, \gamma\} \right) t}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and each $q \in [2,\infty)$, where the constant $S > 0$ depends only on N.

Proof. Let ϕ be a function in $C_c^\infty(\mathbb{R})$ satisfying $\phi(s) = 1$ if $|s| \leq 1$, $\phi(s) = 0$ if $|s| \geq 2$, $0 \leq \phi \leq 1$ and $\phi'(s) \leq 0$ if $s \geq 0$. For each $m \in \mathbb{N}$ we define ϕ_m by

$$\phi_m(x) = \phi\left(\frac{V(x)}{m} \right), \qquad \text{for } x \in \mathbb{R}^N,$$

where V satisfies Condition 4.1. Observe that $\phi_m \in C_c^2(\mathbb{R}^N)$. For $i,j \in \{1,...,N\}$ and $m \in \mathbb{N}$ we define $a_{ij}^{(m)}$ by

$$a_{ij}^{(m)} = \phi_m a_{ij} + \lambda (1 - \phi_m) \delta_{ij}, \tag{4.1}$$

where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$. We remark that $a_{ij}^{(m)} \in C_{loc}^{2+\alpha}(\mathbb{R}^N) \cap C_b(\mathbb{R}^N)$ and introduce the approximating operators

$$A^{(m)} = \sum_{i=1}^N D_i \left(\sum_{j=1}^N a_{ij}^{(m)} D_j \right) + F \cdot D - H, \qquad m \in \mathbb{N}.$$

For each $m \in \mathbb{N}$ and all $x, \xi \in \mathbb{R}^N$ it holds

$$\begin{aligned} \sum_{i,j=1}^{N} a_{ij}^{(m)}(x)\, \xi_i \xi_j &= \phi_m \sum_{i,j=1}^{N} a_{ij} \xi_i \xi_j + \lambda (1-\phi_m) |\xi|^2 \\ &\geq \lambda \phi_m |\xi|^2 + \lambda (1-\phi_m) |\xi|^2 \\ &= \lambda |\xi|^2 . \end{aligned}$$

Moreover, we have

$$\begin{aligned} A^{(m)} V &= \phi_m A V + \frac{1}{m} \phi' \left(\frac{V}{m} \right) a(DV, DV) + \lambda (1-\phi_m) \Delta V + (1-\phi_m)(F \cdot DV - HV) \\ &\leq K \phi_m V + (1-\phi_m)(\lambda \Delta V + F \cdot DV - HV) \\ &\leq \max\{K, K_2\} V . \end{aligned}$$

Therefore, V is a Lyapunov function for $A^{(m)}$ for each $m \in \mathbb{N}$. Let $p_m = p_m(x, y, t)$ be the kernel of the semigroup generated by $\left(A^{(m)}, D_{\max}\left(A^{(m)}\right)\right)$ (see (1.4)). Then for each $f \in C_b\left(\mathbb{R}^N\right)$, the function

$$u(x,t) = \int_{\mathbb{R}^N} p_m(x, y, t) f(y)\, dy, \qquad (x,t) \in \mathbb{R}^N \times [0, \infty),$$

is the solution of the parabolic Cauchy problem

$$\begin{cases} \partial_t u(x,t) = A^{(m)} u(x,t), & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = f(x), & x \in \mathbb{R}^N. \end{cases}$$

Furthermore, Condition 4.1 and (1.21) imply that for all $0 < a < b < \infty$ and each $x \in \mathbb{R}^N$ it holds

$$\begin{aligned} \int_{Q(a,b)} p_m(x,y,t) \left(1 + |F(y)|^M + |H(y)|^M\right) dy dt &\leq \int_{Q(a,b)} p_m(x,y,t)\, W(y)\, dy dt \\ &\leq V(x) \int_a^b e^{\max\{K, K_2\} t} dt < \infty. \end{aligned}$$

Theorem 3.1 of [LMPR] the says that $p_m(x, \cdot, \cdot) \in L^\infty(Q(a,b))$ for all $0 < a < b < \infty$ and each $x \in \mathbb{R}^N$.
Let $f \in C_c^\infty\left(\mathbb{R}^N\right) \setminus \{0\}$ and for each $m \in \mathbb{N}$ we set

$$u_m(x,t) = \int_{\mathbb{R}^N} p_m(x,y,t) f(y)\, dy, \qquad (x,t) \in \mathbb{R}^N \times [0,\infty).$$

Choose $m_0 \in \mathbb{N}$ such that supp $f \subset B(0, m_0)$. There is a $n_0 \in \mathbb{N}$ such that $a_{ij}^{(m)} = a_{ij}$ on $B(0, m_0)$ for each $m > n_0$. It follows that for each $m > n_0$, the function u_m satisfies $\partial_t u_m = A u_m$ on $B(0, m_0) \times (0, \infty)$. Moreover, Proposition 1.2 (iv) yields

$$|u_m(x,t)| \leq e^{-H_0 t} \|f\|_\infty, \qquad \text{for all } (x,t) \in \mathbb{R}^N \times [0,\infty) \text{ and each } m > m_0.$$

Let $T > \delta > 0$, $r > 0$ be fixed. From [Fr64, Section III, Theorem 15] we conclude that there exists a subsequence $\left(u_{m_j}\right)$ of $(u_m)_{m > n_0}$ such that u_{m_k} is uniformly convergent in $C^{2,1}\left(\overline{B(0,r)}\times[\delta,T]\right)$ to some function $\widetilde{u}\in C^{2+\alpha,1+\alpha/2}\left(\overline{B(0,r)}\times[\delta,T]\right)$ such that $\partial_t\widetilde{u}=A\widetilde{u}$. Using an appropriate diagonal sequence (u_{m_k}) we can set

$$u(x,t)=\lim_{k\to\infty}u_{m_k}(x,t)\qquad\text{locally uniformly for }(x,t)\in\mathbb{R}^N\times(0,\infty).$$

Then $u\in C^{2,1}\left(\mathbb{R}^N\times(0,\infty)\right)$, $\partial_t u=Au$ and $|u(x,t)|\le e^{-H_0t}\|f\|_\infty$. Since $u_{m_k}(x,0)=f(x)$ and Proposition 1.2 (v), we have for all fixed $(x,t)\in\mathbb{R}^N\times(0,T]$

$$\begin{aligned}
|u(x,t)-f(x)| &= \left|\lim_{k\to\infty}u_{m_k}(x,t)-f(x)\right|\\
&= \left|\lim_{k\to\infty}\int_0^t\partial_s u_{m_k}(x,s)\,ds\right|\\
&= \left|\lim_{k\to\infty}\int_0^t\int_{\mathbb{R}^N}p_{m_k}(x,y,s)\,A^{(m_k)}f(y)\,dyds\right|\\
&\le \lim_{k\to\infty}\int_0^t e^{-H_0s}\left\|A^{(m_k)}f\right\|_\infty ds\\
&= \|Af\|_\infty\int_0^t e^{-H_0s}ds\\
&\le e^{|H_0|T}\|Af\|_\infty t.
\end{aligned}$$

It then follows that

$$u(x,0)=f(x)\qquad\text{for each }x\in\mathbb{R}^N.$$

Since A has a Lyapunov function, Remark 1.5 implies that u is the unique bounded solution of the parabolic problem (1.9)

$$\begin{cases}\partial_t u(x,t)=Au(x,t), & x\in\mathbb{R}^N,\ t>0,\\ u(x,0)=f(x), & x\in\mathbb{R}^N.\end{cases}$$

for each $f\in C_c^\infty\left(\mathbb{R}^N\right)$. So we conclude that

$$u(x,t)=\int_{\mathbb{R}^N}p(x,y,t)\,f(y)\,dy,\qquad(x,t)\in\mathbb{R}^N\times[0,\infty),$$

where p is the kernel of the semigroup $(T(t))_{t\ge0}$ generated by $(A,D_{\max}(A))$. Thus, for each $f\in C_c^\infty\left(\mathbb{R}^N\right)$ there exists a subsequence $\left(p_{m_j}\right)$ of (p_m) such that

$$\int_{\mathbb{R}^N}p(x,y,t)\,f(y)\,dy=\lim_{j\to\infty}\int_{\mathbb{R}^N}p_{m_j}(x,y,t)\,f(y)\,dy\qquad\text{pointwise}\tag{4.2}$$

on $\mathbb{R}^N\times(0,T]$. We now fix some arbitrary $m\in\mathbb{N}$. We remark that Proposition 1.2 is true for p_m. Let $\beta\ge1$ and $0<2t_1<t_2<\infty$. For $n\in\mathbb{N}$ we set

$$\zeta_n(x,t)=\int_{\mathbb{R}^N}\eta_n(y)^{2\beta}\,p_m(x,y,t)^{2\beta}\,dy,\qquad(x,t)\in\mathbb{R}^N\times[t_1,t_2].\tag{4.3}$$

We remark that for large $n\in\mathbb{N}$ it holds

$$0<\delta(x,t):=\int_{\mathbb{R}^N}p_m(x,y,t)\cdot\eta_1(y)^{2\beta}\,p_m(x,y,t)^{2\beta-1}\,dy\le\zeta_n(x,t)<\infty\tag{4.4}$$

for all $(x,t) \in \mathbb{R}^N \times [t_1, t_2]$ and all $m \in \mathbb{N}$. Moreover, δ is a continuous function. We set

$$A_0^{(m)} = \sum_{i=1}^{N} D_i \left(\sum_{j=1}^{N} a_{ij}^{(m)}(y) D_j \right).$$

Since

$$\begin{aligned} \partial_t p_m(x,y,t) &= A_0^{(m)} p_m(x,y,t) - F(y) \cdot D p_m(x,y,t) \\ &\quad - (\operatorname{div} F(y) + H(y)) p_m(x,y,t) \end{aligned}$$

for each fixed $x \in \mathbb{R}^N$ and all $(y,t) \in \mathbb{R}^N \times (0,\infty)$, it holds

$$\begin{aligned} \partial_t \zeta_n &= \int_{\mathbb{R}^N} 2\beta \eta_n^{2\beta} p_m^{2\beta-1} \partial_t p_m dy \\ &= \int_{\mathbb{R}^N} 2\beta \eta_n^{2\beta} p_m^{2\beta-1} A_0^{(m)} p_m dy - \int_{\mathbb{R}^N} 2\beta \eta_n^{2\beta} p_m^{2\beta-1} F \cdot D p_m dy \\ &\quad - \int_{\mathbb{R}^N} 2\beta \eta_n^{2\beta} p_m^{2\beta} (\operatorname{div} F + H) dy. \end{aligned}$$

We set

$$a^{(m)}(\xi, \nu) = \sum_{i,j=1}^{N} a_{ij}^{(m)} \xi_i \nu_j.$$

Integration by parts yields

$$\begin{aligned} -\partial_t \zeta_n &= \int_{\mathbb{R}^N} 2\beta(2\beta-1) \eta_n^{2\beta} p_m^{2\beta-2} a^{(m)}(Dp_m, Dp_m) dy \\ &\quad + \int_{\mathbb{R}^N} 4\beta^2 \eta_n^{2\beta-1} p_m^{2\beta-1} a^{(m)}(D\eta_n, Dp_m) dy \\ &\quad - \int_{\mathbb{R}^N} 2\beta \eta_n^{2\beta-1} p_m^{2\beta} F \cdot D\eta_n dy \\ &\quad + \int_{\mathbb{R}^N} \eta_n^{2\beta} p_m^{2\beta} ((2\beta-1) \operatorname{div} F + 2\beta H) dy. \end{aligned} \tag{4.5}$$

Moreover it holds

$$\begin{aligned} \int_{\mathbb{R}^N} 4\beta^2 \eta_n^{2\beta-1} p_m^{2\beta-1} a^{(m)}(D\eta_n, Dp_m) dy &= \int_{\mathbb{R}^N} 2a^{(m)}\left(D\left(\eta_n^\beta p_m^\beta\right), D\left(\eta_n^\beta p_m^\beta\right)\right) dy \\ &\quad - \int_{\mathbb{R}^N} 2\beta^2 \eta_n^{2\beta-2} p_m^{2\beta} a^{(m)}(D\eta_n, D\eta_n) dy \\ &\quad - \int_{\mathbb{R}^N} 2\beta^2 \eta_n^{2\beta} p_m^{2\beta-2} a^{(m)}(Dp_m, Dp_m) dy. \end{aligned}$$

Applying this identity to (4.5), we obtain

$$\begin{aligned} -\partial_t \zeta_n &= \int_{\mathbb{R}^N} 2\beta(\beta-1) \eta_n^{2\beta} p_m^{2\beta-2} a^{(m)}(Dp_m, Dp_m) dy \\ &\quad + \int_{\mathbb{R}^N} 2a^{(m)}\left(D\left(\eta_n^\beta p_m^\beta\right), D\left(\eta_n^\beta p_m^\beta\right)\right) dy \\ &\quad - \int_{\mathbb{R}^N} 2\beta^2 \eta_n^{2\beta-2} p_m^{2\beta} a^{(m)}(D\eta_n, D\eta_n) dy \end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^N} 2\beta\eta_n^{2\beta-1} p_m^{2\beta} F\cdot D\eta_n dy \\
&+\int_{\mathbb{R}^N} \eta_n^{2\beta} p_m^{2\beta}\left((2\beta-1)\operatorname{div} F+2\beta H\right) dy. \qquad (4.6)
\end{aligned}
$$

Observe that $p(x,y,t)\frac{|F(y)|}{1+|y|}$ is integrable in $y\in\mathbb{R}^N$ by (1.7), Proposition 1.6 and Condition 4.1. We then deduce

$$
\begin{aligned}
0 &\le \int_{\mathbb{R}^N} 2\beta^2\eta_n^{2\beta-2} p_m^{2\beta} a^{(m)}\left(D\eta_n, D\eta_n\right) dy \\
&\le \int_{\mathbb{R}^N} 2\beta^2\eta_n^{2\beta-2} p_m^{2\beta}\frac{C_m L^2}{1+|y|^2}\mathbb{1}_{\{n\le|y|\le 2n\}} dy \\
&\le 2\beta^2 C_m L^2\left\|p(x,\cdot,\cdot)\right\|_{L^\infty(Q(t_1,t_2))}^{2\beta-1}\int_{\mathbb{R}^N} p_m\mathbb{1}_{\{n\le|y|\le 2n\}} dy\longrightarrow 0 \qquad \text{as } n\to\infty, \quad (4.7)
\end{aligned}
$$

where $C_m=\left\|\sqrt{\sum_{i,j=1}^N\left(a_{ij}^{(m)}\right)^2}\right\|_\infty<\infty$, and

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} 2\beta\eta_n^{2\beta-1} p_m^{2\beta} F\cdot D\eta_n dy\right| \\
&\qquad\le 2\beta L\left\|p(x,\cdot,\cdot)\right\|_{L^\infty(Q(t_1,t_2))}^{2\beta-1}\int_{\mathbb{R}^N} p_m\frac{|F|}{1+|y|}\mathbb{1}_{\{n\le|y|\le 2n\}} dydy\longrightarrow 0 \qquad \text{as } n\to\infty
\end{aligned}
$$

for all fixed $(x,t)\in\mathbb{R}^N\times[t_1,t_2]$. Moreover, we have

$$
\int_{\mathbb{R}^N} 2\beta(\beta-1)\eta_n^{2\beta} p_m^{2\beta-2} a^{(m)}\left(Dp_m, Dp_m\right) dy\ge 0,
$$

$$
\int_{\mathbb{R}^N} 2a^{(m)}\left(D\left(\eta_n^\beta p_m^\beta\right), D\left(\eta_n^\beta p_m^\beta\right)\right) dy\ge 2\lambda\int_{\mathbb{R}^N}\left|D\left(\eta_n^\beta p_m^\beta\right)\right|^2 dy
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^N} \eta_n^{2\beta} p_m^{2\beta}\left((2\beta-1)\operatorname{div} F+2\beta H\right) dy &= \int_{\mathbb{R}^N} \eta_n^{2\beta} p_m^{2\beta}\left((2\beta-1)\left(\operatorname{div} F+H\right)+H\right) dy \\
&\ge \left((2\beta-1)\gamma+H_0\right)\int_{\mathbb{R}^N}\eta_n^{2\beta} p_m^{2\beta} dy \\
&= \left((2\beta-1)\gamma+H_0\right)\zeta_n.
\end{aligned}
$$

Hence, from (4.6) it follows

$$
-\partial_t\zeta_n\ge 2\lambda\int_{\mathbb{R}^N}\left|D\left(\eta_n^\beta p_m^\beta\right)\right|^2 dy+\left((2\beta-1)\gamma+H_0\right)\zeta_n-\nu_n, \qquad (4.8)
$$

where

$$
\begin{aligned}
\nu_n &= 2\beta^2 C_m L^2\left\|p(x,\cdot,\cdot)\right\|_{L^\infty(Q(t_1,t_2))}^{2\beta-1}\int_{\mathbb{R}^N} p_m\mathbb{1}_{\{n\le|y|\le 2n\}} dy \\
&\quad +2\beta L\left\|p(x,\cdot,\cdot)\right\|_{L^\infty(Q(t_1,t_2))}^{2\beta-1}\int_{\mathbb{R}^N} p_m\frac{|F|}{1+|y|}\mathbb{1}_{\{n\le|y|\le 2n\}} dydy \\
&\le 2\beta^2 C_m L^2\left\|p(x,\cdot,\cdot)\right\|_{L^\infty(Q(t_1,t_2))}^{2\beta-1} e^{-H_0 t}
\end{aligned}
$$

$$+2\beta L \left\| p\left(x,\cdot,\cdot\right) \right\|_{L^{\infty}(Q(t_1,t_2))}^{2\beta-1} e^{Kt} V\left(x\right)^{\frac{1}{M}}$$

for $(x,t) \in \mathbb{R}^N \times [t_1,t_2]$. Moreover, $0 \leq \nu_n = \nu_n(x,t) \to 0$ as $n \to \infty$ for all $(x,t) \in \mathbb{R}^N \times [t_1,t_2]$. Furthermore, the Gagliardo–Nirenberg–Sobolev inequality implies

$$\int_{\mathbb{R}^N} \left| D\left(\eta_n^{\beta} p_m^{\beta}\right)\right|^2 dy \geq S \left(\int_{\mathbb{R}^N} \left(\eta_n^{\beta} p_m^{\beta}\right)^{\frac{2N}{N-2}} dy \right)^{\frac{N-2}{N}} \tag{4.9}$$

for the Sobolev constant $S = S(N) > 0$. Since

$$0 < \int_{\mathbb{R}^N} \eta_1 p_m dy \leq \int_{\mathbb{R}^N} \eta_n p_m dy \leq \int_{\mathbb{R}^N} p_m dy \leq e^{-H_0 t},$$

it holds

$$e^{H_0 t} \leq \frac{1}{\int_{\mathbb{R}^N} \eta_n p_m dy} < \infty.$$

For $r > 1$, this fact leads to

$$\begin{aligned}
&\left(\int_{\mathbb{R}^N} \left(\eta_n^{\beta} p_m^{\beta}\right)^{\frac{2N}{N-2}} dy \right)^{\frac{1}{r}} \\
&\quad = \left(\int_{\mathbb{R}^N} \left(\left(\eta_n^{\beta} p_m^{\beta}\right)^{\frac{2N}{(N-2)r}} \right)^{r} dy \right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^N} \left(\left(\eta_n p_m\right)^{\frac{r-1}{r}} \right)^{\frac{r}{r-1}} dy \right)^{\frac{r-1}{r}} \\
&\quad \cdot \left(\frac{1}{\int_{\mathbb{R}^N} \eta_n p_m dy} \right)^{\frac{r-1}{r}} \\
&\quad \geq \left\| \left(\eta_n p_m\right)^{\frac{2\beta N}{(N-2)r}} \right\|_r \left\| \left(\eta_n p_m\right)^{\frac{r-1}{r}} \right\|_{\frac{r}{r-1}} e^{H_0 \frac{r-1}{r} t}.
\end{aligned}$$

Hölder's inequality then yields

$$\left(\int_{\mathbb{R}^N} \left(\eta_n^{\beta} p_m^{\beta}\right)^{\frac{2N}{N-2}} dy \right)^{\frac{1}{r}} \geq \left\| \left(\eta_n p_m\right)^{\frac{2\beta N}{(N-2)r} + \frac{r-1}{r}} \right\|_1 e^{H_0 \frac{r-1}{r} t}. \tag{4.10}$$

Choosing $r = \frac{2N\beta - N + 2}{2N\beta - N + 2 - 4\beta}$ in (4.10), we infer

$$\left(\int_{\mathbb{R}^N} \left(\eta_n^{\beta} p_m^{\beta}\right)^{\frac{2N}{N-2}} dy \right)^{\frac{2N\beta - N + 2 - 4\beta}{2N\beta - N + 2}} \geq \left\| \eta_n^{2\beta} p_m^{2\beta} \right\|_1 e^{\frac{4\beta H_0}{2N\beta - N + 2} t} = e^{\frac{4\beta H_0}{2N\beta - N + 2} t} \zeta_n$$

and hence

$$\left(\int_{\mathbb{R}^N} \left(\eta_n^{\beta} p_m^{\beta}\right)^{\frac{2N}{N-2}} dy \right)^{\frac{N-2}{N}} \geq e^{\frac{4\beta H_0}{(2\beta-1)N} t} \zeta_n^{1 + \frac{2}{(2\beta-1)N}}.$$

We combine the above inequality with (4.9) and arrive at

$$\int_{\mathbb{R}^N} \left| D\left(\eta_n^{\beta} p_m^{\beta}\right)\right|^2 dy \geq S e^{\frac{4\beta H_0}{(2\beta-1)N} t} \zeta_n^{1 + \frac{2}{(2\beta-1)N}}. \tag{4.11}$$

We set

$$\theta = (2\beta - 1)\gamma + H_0.$$

It then follows from (4.8)

$$-\partial_t \zeta_n \geq 2\lambda S e^{\frac{4\beta H_0}{(2\beta-1)N}t} \zeta_n^{1+\frac{2}{(2\beta-1)N}} + \theta \zeta_n - \nu_n$$

and hence

$$-\partial_t \left(e^{\theta t} \zeta_n\right) \geq 2\lambda S e^{\left(\frac{4\beta H_0}{(2\beta-1)N}+\theta\right)t} \zeta_n^{1+\frac{2}{(2\beta-1)N}} - e^{\theta t} \nu_n.$$

Taking into account (4.4), we conclude

$$\partial_t \left(\left(e^{\theta t} \zeta_n\right)^{-\frac{2}{(2\beta-1)N}} \right) \geq \frac{4\lambda S}{(2\beta-1)N} e^{\frac{2(H_0-\gamma)}{N}t} - \frac{2\delta^{-1-\frac{2}{(2\beta-1)N}}}{(2\beta-1)N} e^{-\frac{2\theta}{(2\beta-1)N}t} \nu_n. \tag{4.12}$$

Let now $\tau \in C^\infty(\mathbb{R})$ be such that $0 \leq \tau \leq 1$, $\tau(t) = 0$ for $0 \leq t \leq t_1$, $\tau(t) = 1$ for $t \geq 2t_1$ and $\tau' \geq 0$. We multiply (4.12) by τ and get

$$\begin{aligned}
\partial_t \left(\tau \left(e^{\theta t} \zeta_n\right)^{-\frac{2}{(2\beta-1)N}} \right) &\geq \frac{4\lambda S}{(2\beta-1)N} \tau e^{\frac{2(H_0-\gamma)}{N}t} - \frac{2\delta^{-1-\frac{2}{(2\beta-1)N}}}{(2\beta-1)N} e^{-\frac{2\theta}{(2\beta-1)N}t} \tau \nu_n \\
&\quad + \tau' \left(e^{\theta t} \zeta_n\right)^{-\frac{2}{(2\beta-1)N}} \\
&\geq \frac{4\lambda S}{(2\beta-1)N} \tau e^{\frac{2(H_0-\gamma)}{N}t} - \frac{2\delta^{-1-\frac{2}{(2\beta-1)N}}}{(2\beta-1)N} e^{-\frac{2\theta}{(2\beta-1)N}t} \nu_n.
\end{aligned}$$

Integration from t_1 to t for $t \in (2t_1, t_2]$ yields

$$\begin{aligned}
\left(e^{\theta t} \zeta_n\right)^{-\frac{2}{(2\beta-1)N}} &\geq \int_{t_1}^t \frac{4\lambda S}{(2\beta-1)N} \tau(s) e^{\frac{2(H_0-\gamma)}{N}s} ds - \int_{t_1}^t \frac{2\delta(x,s)^{-1-\frac{2}{(2\beta-1)N}}}{(2\beta-1)N} e^{-\frac{2\theta}{(2\beta-1)N}s} \nu_n(x,s)\, ds \\
&\geq \frac{4\lambda S}{(2\beta-1)N} \varphi(t) - \int_{t_1}^t \frac{2\delta(x,s)^{-1-\frac{2}{(2\beta-1)N}}}{(2\beta-1)N} e^{-\frac{2\theta}{(2\beta-1)N}s} \nu_n(x,s)\, ds
\end{aligned}$$

where φ is defined by

$$\varphi(t) = (t - 2t_1) e^{\min\left\{\frac{2(H_0-\gamma)}{N}, 0\right\}t}.$$

We remark that Lebesgue's theorem yields

$$-\int_{t_1}^t \frac{2\delta(x,s)^{-1-\frac{2}{(2\beta-1)N}}}{(2\beta-1)N} e^{-\frac{2\theta}{(2\beta-1)N}s} \nu_n(x,s)\, ds \longrightarrow 0 \qquad \text{as } n \to \infty$$

for all fixed $(x,t) \in \mathbb{R}^N \times (2t_1, t_2]$. For each $t \in (2t_1, t_2]$ and $x \in \mathbb{R}^N$ we can thus fix a $n_0 = n_0(x,t) \in \mathbb{N}$ such that

$$\frac{4\lambda S}{(2\beta-1)N} \varphi(t) - \int_{t_1}^t \frac{2\delta^{-1-\frac{2}{(2\beta-1)N}}}{(2\beta-1)N} e^{-\frac{2\theta}{(2\beta-1)N}s} \nu_n ds \geq \frac{2\lambda S}{(2\beta-1)N} \varphi(t)$$

for each $n \geq n_0$. Hence,

$$\zeta_n \leq \left(\frac{(2\beta-1)N}{2\lambda S} \right)^{\frac{(2\beta-1)N}{2}} (t - 2t_1)^{-\frac{(2\beta-1)N}{2}} e^{-H_0 t} e^{-(2\beta-1)\min\{H_0,\gamma\}t}$$

for all $(x,t) \in \mathbb{R}^N \times (2t_1, t_2]$. Letting $n \to \infty$, Fatou's lemma implies

$$\int_{\mathbb{R}^N} p_m(x,y,t)^{2\beta} dy \leq \left(\frac{(2\beta-1)N}{2\lambda S} \right)^{\frac{(2\beta-1)N}{2}} (t - 2t_1)^{-\frac{(2\beta-1)N}{2}} e^{-H_0 t} e^{-(2\beta-1)\min\{H_0,\gamma\}t}$$

for all $(x,t) \in \mathbb{R}^N \times (2t_1, t_2]$. Since $t_1 > 0$ can be arbitrary close to 0 and $t_2 > 2t_1$ can be arbitrary large, we deduce

$$\int_{\mathbb{R}^N} p_m(x,y,t)^{2\beta}\,dy \le \left(\frac{(2\beta-1)\,N}{2\lambda S}\right)^{\frac{(2\beta-1)N}{2}} t^{-\frac{(2\beta-1)N}{2}} e^{-H_0 t} e^{-(2\beta-1)\min\{H_0,\gamma\}t}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and all $m \in \mathbb{N}$. For $q = 2\beta$ we then observe

$$\left(\int_{\mathbb{R}^N} p_m(x,y,t)^q\,dy\right)^{\frac{1}{q}} \le \left(\frac{(q-1)\,N}{2\lambda S}\right)^{\frac{(q-1)N}{2q}} t^{-\frac{(q-1)N}{2q}} e^{-\left(\frac{H_0}{q}+\frac{q-1}{q}\min\{H_0,\gamma\}\right)t}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$ and each $m \in \mathbb{N}$.
From (4.2) for each $f \in C_c^\infty(\mathbb{R}^N)$ and each $q \in [2,\infty)$ it then follows

$$\begin{aligned}\left|\int_{\mathbb{R}^N} p(x,y,t)\,f(y)\,dy\right| &= \lim_{k\to\infty}\left|\int_{\mathbb{R}^N} p_{m_k}(x,y,t)\,f(y)\,dy\right| \\ &\le \left(\frac{(q-1)\,N}{2\lambda S}\right)^{\frac{(q-1)N}{2q}} t^{-\frac{(q-1)N}{2q}} e^{-\left(\frac{H_0}{q}+\frac{q-1}{q}\min\{H_0,\gamma\}\right)t}\left(\int_{\mathbb{R}^N} f(y)^{\frac{q}{q-1}}\,dy\right)^{\frac{q-1}{q}}\end{aligned}$$

using Hölder's inequality. The assertion follows since $L^{\frac{q}{q-1}}$ is the dual of L^q. ∎

Theorem 4.2 immediately yields the following statement.

Corollary 4.3. *Assume that Condition 4.1 holds. We then obtain*

$$\int_{\mathbb{R}^N} p(x,y,t)^2\,dy \le \left(\frac{N}{2\lambda S}\right)^{\frac{N}{2}} t^{-\frac{N}{2}} e^{-(H_0+\min\{H_0,\gamma\})t}$$

for all $(x,t) \in \mathbb{R}^N \times (0,\infty)$, where the constant $S > 0$ depends only on N.

Remark 4.4. The assumption $\inf_{x\in\mathbb{R}^N}(\operatorname{div} F(x) + H(x)) = \gamma \in \mathbb{R}$ implies the existence of the kernel p^* of the semigroup generated by $\left(A^*, D\left(\hat{A}^*\right)\right)$ (see (1.16)) and it holds

$$p^*(x,y,t) = p(y,x,t), \qquad (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty).$$

Corollary 4.5. *Under conditions of Theorem 4.2 assume that $F = 0$. It then holds*

$$p(x,y,t) \le \left(\frac{N}{\lambda S}\right)^{\frac{N}{2}} t^{-\frac{N}{2}} e^{-H_0 t} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty),$$

where the constant $S > 0$ depends only on N.

Proof. If $F = 0$, then it holds $A = A_0 - H = A^*$ and thus $p(x,y,t) = p(y,x,t)$. The Chapman–Kolmogorov equation (see Proposition 1.2 (ii)) and the fact that $\gamma = H_0$ yield

$$\begin{aligned} p(x,y,t) &= \int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right) p\left(z,y,\frac{t}{2}\right) dz \\ &\le \left(\int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} p\left(z,y,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}}\end{aligned}$$

$$= \left(\int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} p\left(y,z,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}}$$
$$\leq \left(\frac{N}{\lambda S}\right)^{\frac{N}{2}} t^{-\frac{N}{2}} e^{-H_0 t} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty).$$

∎

Example 4.6. We consider the operator

$$A = \left(1+|x|^2\right)^{\alpha} \Delta + 2\alpha \left(1+|x|^2\right)^{\alpha-1} x \cdot D - |x|^{2\theta+2}, \qquad 1 < \alpha < \theta.$$

In this case we have

$$A_0 = \left(1+|x|^2\right)^{\alpha} \Delta + 2\alpha \left(1+|x|^2\right)^{\alpha-1} x \cdot D \qquad \text{and} \qquad H(x) = |x|^{2\theta+2}$$

so that

$$A = A_0 - H.$$

The simple calculation shows that a function $V(x) = Ce^{\delta|x|^2}$ is a Lyapunov function for A and satisfies Condition 4.1 for all $\delta, C > 0$. Moreover, for each $\delta_0 > 0$ there exists $C_0 > 0$ such that the function $W(x) = C_0 e^{\delta_0|x|^2}$ satisfies Condition 4.1. Further, $\gamma = H_0 = 0$. Since in this case $\lambda = 1$, Corollary 4.5 yields

$$p(x,y,t) \leq \left(\frac{N}{S}\right)^{\frac{N}{2}} t^{-\frac{N}{2}} \qquad \text{for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty).$$

We further apply the above methods to the formal adjoint operator A^*. We recall from Chapter 1 that for $A = A_0 + F \cdot D - H$ we have

$$A^* = A_0 - F \cdot D - (\operatorname{div} F + H).$$

Condition 4.7. *We assume that Condition 4.1 holds. Moreover, there exist constants $K^* > -\gamma$ and $K_1^* > -\gamma$ and Lyapunov functions V^* and W^* for the operator $A^* = A_0 - F \cdot D - (\operatorname{div} F + H)$ such that*

$$A^* V^* \leq K^* V^*, \qquad A^* W^* \leq K_1^* W^*, \qquad \lambda \Delta V^* - F \cdot DV^* - (\operatorname{div} F + H) V^* \leq K^* V^*,$$

and

$$|F|^M + |\operatorname{div} F + H|^M \leq W^*$$

on $\mathbb{R}^N$, that is Condition 4.1 holds also for the adjoint operator A^* with

$$\gamma^* = \inf_{x \in \mathbb{R}^N} \left(-\operatorname{div} F(x) + (\operatorname{div} F(x) + H(x))\right) = H_0 \qquad \text{and} \qquad H_0^* = \gamma.$$

Corollary 4.8. *Under Condition 4.7 it holds*

$$p(x,y,t) \leq \left(\frac{N}{\lambda S}\right)^{\frac{N}{2}} t^{-\frac{N}{2}} e^{-\left(\frac{H_0+\gamma}{2} + \min\{H_0,\gamma\}\right)t}$$

for all $(x,y,t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$.

Proof. From Corollary 4.3 it follows that

$$\left(\int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \le \left(\frac{N}{\lambda S}\right)^{\frac{N}{4}} t^{-\frac{N}{4}} e^{-\frac{1}{4}(H_0+\min\{H_0,\gamma\})t}$$

and

$$\left(\int_{\mathbb{R}^N} p\left(z,y,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \le \left(\frac{N}{\lambda S}\right)^{\frac{N}{4}} t^{-\frac{N}{4}} e^{-\frac{1}{4}(\gamma+\min\{H_0,\gamma\})t}.$$

The Chapman–Kolmogorov equation (see Proposition 1.2 (ii)) then yields

$$\begin{aligned} p(x,y,t) &= \int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right) p\left(z,y,\frac{t}{2}\right) dz \\ &\le \left(\int_{\mathbb{R}^N} p\left(x,z,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} p\left(z,y,\frac{t}{2}\right)^2 dz\right)^{\frac{1}{2}} \\ &\le \left(\frac{N}{\lambda S}\right)^{\frac{N}{2}} t^{-\frac{N}{2}} e^{-\left(\frac{H_0+\gamma}{4}+\frac{\min\{H_0,\gamma\}}{2}\right)t}. \end{aligned}$$

■

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