Maximal functions, functional calculus, and generalized Triebel-Lizorkin spaces for sectorial operators

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Alexander Ullmann

aus Bad Segeberg

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Referent: HDoz. Dr. Peer Christian Kunstmann

Korreferent: Prof. Dr. Lutz Weis

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Introduction

In 1986, Alan McIntosh introduced in the fundamental paper *Operators which have an H_∞ functional calculus* ([McI86]) his notion of a bounded H^∞-calculus for sectorial operators: Let A be a sectorial operator in a complex Banach spaces X, i.e. the set of resolvents $\{\lambda R(\lambda, A) \mid \lambda \in \mathbb{C}\setminus\overline{\Sigma_\omega}\}$ is bounded for some $\omega \in (0, \pi)$, where $\Sigma_\omega := \{z \in \mathbb{C}\setminus\{0\} \mid |\arg(z)| < \omega\}$ denotes the open sector symmetric around the positive axis with half opening angle ω. Then based on the ideas of the Dunford functional calculus one defines

$$\varphi(A) := \frac{1}{2\pi i} \int_\Gamma \varphi(\lambda) R(\lambda, A)\, d\lambda \in L(X),$$

where Γ is the canonical parametrization of the oriented boundary of a suitable sector, and φ is of class H_0^∞, i.e. a bounded holomorphic function on a larger open sector that decays polynomially to 0 as z tends to 0 or ∞. Then $\varphi \mapsto \varphi(A)$ defines a functional calculus on H_0^∞, which can be naturally extended to the larger algebra of holomorphic functions with at most polynomial growth at 0 and ∞, where in this case the resulting operators are in general unbounded. In particular, $f(A)$ is defined for all bounded holomorphic functions $f \in H^\infty$. Now one central question is the following:

Is $f(A)$ bounded for any $f \in H^\infty$, and does an estimate $\|f(A)\| \lesssim \|f\|_\infty$ hold?

In this case, A is said to have a *bounded H^∞-calculus.* McIntosh was able to give various characterizations of the boundedness of the H^∞-calculus in the case that the underlying space is a Hilbert space. One of these is given in terms of so-called square functions and can be rewritten in the following form: A sectorial operator A in a Hilbert space X has a bounded H^∞-calculus if and only if the following norm equivalence holds for one (and then for all) $\varphi \in H_0^\infty$ with $\varphi \neq 0$:

$$\|x\|_X \approx \left(\int_0^\infty \|\varphi(tA)x\|_X^2 \, \frac{dt}{t} \right)^{1/2} \quad \text{for } x \in X. \tag{1}$$

This condition was motivated by well known concepts of square functions from harmonic analysis. Indeed, the methods McIntosh used were operator theoretic, but many of them are motivated by harmonic analysis. McIntosh himself says the following in his paper [McI86]:

The material in this paper has two heritages: One is operator theory [...]; the other is harmonic analysis [...],

and this thesis follows the same tradition.

The condition (1) has been generalized to other classes of spaces, in a first step to spaces $X = L^p$, $p \in (1, +\infty)$, where it takes the following form: A sectorial operator A in the space $X = L^p$ has a bounded H^∞-calculus if and only if the following norm equivalence holds for one (and then for all) $\varphi \in H_0^\infty$ with $\varphi \neq 0$:

$$\|x\|_X \approx \left\| \left(\int_0^\infty |\varphi(tA)x|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_X \quad \text{for } x \in X. \tag{2}$$

This has first been treated in [CDMY96]. Note that for $X = L^2$, the norm expressions in (1) and (2) coincide by Fubini. Again, the idea for (2) is based on methods from classical harmonic analysis in L^p, in particular Littlewood-Paley theory. Let us mention that this concept of characterizing the boundedness of the H^∞-calculus has finally been transferred to general Banach spaces by Nigel Kalton and Lutz Weis in [KW-1], cf. also [KW04] and [KKW06], where the square-function norms in (2) are replaced by more general square functions in terms of so called *Rademacher-norms* and in terms of γ-norms. Furthermore, square function estimates are used in various fields of analysis, e.g. questions of admissibility of certain operators for control systems have been treated in [LeM03] using square function norms of the form (1), and the related concept of $\mathcal{R}$-admissibility is treated in [LeM04] in terms of the square function norms in (2). Moreover, [KW-1] and the survey [LeM07] give a nice overview of different characterizations and applications for square functions and square function estimates.

In this thesis, we will concentrate on a certain class of Banach function spaces instead of general Banach spaces, so in particular, we have an additional lattice structure, and expressions as in (2) are still well defined. We note that this class of spaces covers the spaces L^p, where $p \in [1, +\infty)$, but also certain kinds of Lorentz-, Orlicz- and mixed L^pL^q-spaces. The central challenge we meet in this work is to change the power 2 in (2) to a power $s \in [1, +\infty]$. This leads to the following expressions:

$$\|x\|_{s,A,\varphi} := \left\| \left(\int_0^\infty |\varphi(tA)x|^s \, \frac{dt}{t} \right)^{1/s} \right\|_X \text{ if } s < +\infty, \text{ and } \|x\|_{\infty,A,\varphi} := \big\| \sup_{t>0} |\varphi(tA)x| \big\|_X. \quad (3)$$

Although starting from the same idea, i.e. generalizing the square function norms (2), we will use these two expressions for two different ideas:

In the first part of this thesis, we will study the terms $\sup_{t>0} |\varphi(tA)x|$. These are well known in classical situations and are referred to as maximal functions. In this context, the question arises naturally, if an estimate of the form

$$\big\| \sup_{t>0} |\varphi(tA)x| \big\|_X \lesssim \|x\|_X \quad \text{for } x \in X \tag{4}$$

holds. Here we will work more generally in vector-valued Banach function spaces $X(E)$ (e.g. vector-valued Lebesgue spaces $L^p(\Omega, E)$), where E is a Banach space. Given a sectorial operator A in $X(E)$ we ask for the validity of a maximal estimate

$$\big\| \sup_{z \in \Sigma_\delta} |\varphi(zA)x|_E \big\|_X \lesssim \|x\|_{X(E)} \quad \text{for } x \in X(E). \tag{5}$$

One important issue in this context is the *Banach principle*, which states that if the estimate (5) holds, then the set of all $x \in X(E)$ such that

$$(\varphi(zA)x)_{z \in \Sigma_\delta} \text{ converges pointwise a.e. if } \Sigma_\delta \ni z \to 0$$

is closed in $X(E)$. If e.g. $A = -\Delta$ is the Laplacian and $X = L^p(\mathbb{R}^d)$ with $p \in (1, +\infty)$ and $\varphi(z) = e^{-z}$, then (5) (for $\delta = 0$ and $\Sigma_0 := (0, \infty)$) reads as

$$\big\| \sup_{t>0} |h_t * u|_E \big\|_{L^p} \lesssim \|u\|_{L^p(E)} \quad \text{for } u \in L^p(E), \tag{6}$$

where h_t is the heat kernel in $\mathbb{R}^d$. Since convergence a.e. is clear if e.g. $u \in \mathcal{S}(\mathbb{R}^d, E)$, the validity of a maximal estimate (6) implies that

$$h_t * u \to u \;\text{ a.e. as } t \to 0 \;\text{ for all } u \in L^p(\mathbb{R}^d, E).$$

Actually we will show that the maximal estimate (6) holds in the case that $E = [E_0, E_1]_\theta$ is a complex interpolation space between a UMD-space E_0 and a Banach space E_1 for some $\theta \in (0,1)$. We note that the question about maximal estimates for semigroups has a long history in connection with the maximal ergodic theorem, cf. e.g. [St61], or [DS58], Chapter VIII.

In this thesis we will do a systematic treatment of maximal estimates according to (5) using concepts of functional calculi for sectorial operators in Banach function spaces, that also work in vector-valued spaces. We note in this place that the concept of using functional calculi methods for considering maximal estimates has already been regarded by Michael Cowling in [Co83], and a generalization to the vector-valued setting has been done e.g. in [Bl02] and [Ta09]. We note that in [Bl02], the considered operators are assumed to have bounded imaginary powers, and the underlying space has to be a UMD-space, whereas in [Ta09] generators of symmetric diffusion semigroups are treated, hence self-adjoint operators in L^2, and the underlying Banach spaces for the vector-valued maximal estimates are interpolation spaces $[H, E]_\theta$, where H is a Hilbert space and E is an UMD space. In both cases, we show that the assumptions can be weakened. We will give more details in the sequel when we discuss the structure of Chapter 2.

In the second part of this thesis we will pursue a different idea, considering the terms $\|x\|_{s,A,\varphi}$ for the whole scale of $s \in [1, +\infty]$. More generally, we consider the terms

$$\|x\|_{\theta,s,A,\varphi} := \left\| \left(\int_0^\infty |t^{-\theta}\varphi(tA)x|^s \,\frac{dt}{t} \right)^{1/s} \right\|_X \tag{7}$$

for each $\theta \in \mathbb{R}$ (with the usual modification if $s = +\infty$). These terms have an interesting interpretation if one considers the natural counterparts

$$|||x|||_{\theta,s,A,\varphi} := \left(\int_0^\infty \|t^{-\theta}\varphi(tA)x\|_X^s \,\frac{dt}{t} \right)^{1/s}. \tag{8}$$

The term (8) is well known to be the homogeneous part of the norm in real interpolation spaces $(X, D(A))_{\theta,s}$ if $\theta \in (0,1)$ for appropriate auxiliary functions φ, i.e. the norm in $(X, D(A))_{\theta,s}$ is equivalent to the norm $\|\cdot\|_X + |||\cdot|||_{\theta,s,A,\varphi}$, cf. e.g. [Ha06], Chapter 6. In particular, if $A = -\Delta$ in $X = L^p(\mathbb{R}^d)$ and $\theta \in (0,1)$, then $(X, D(A))_{\theta,s} = B^{2\theta}_{p,s}$ is a Besov space, whereas the norm $\|\cdot\|_X + \|\cdot\|_{\theta,s,A,\varphi}$ is equivalent to the norm in the Triebel-Lizorkin space $F^{2\theta}_{p,s}$, cf. [Tr83]. The usefulness of Besov spaces is widely known, since they are real interpolation spaces and hence e.g. occur as trace spaces in many applications in differential equations. Moreover, Besov and Triebel-Lizorkin spaces coincide in the case $p = s$, i.e. $B^{2\theta}_{p,p} = F^{2\theta}_{p,p}$, whereas in the case $s = 2$ the Triebel-Lizorkin spaces coincide by Littlewood-Paley theory with the Bessel potential spaces $H^{2\theta,p}(\mathbb{R}^d)$ if $p \in (1, +\infty)$. The case $s \neq 2$ has become of interest e.g. in connection with Navier-Stokes equations, cf. [KY04]. Moreover, vector-valued variants of Triebel-Lizorkin spaces have been recognized to occur naturally in characterizing the sharp temporal regularity of certain trace spaces in the theory of evolution equations, cf. [We02] and [We05], but we will not study vector-valued Triebel-Lizorkin spaces in this thesis.

Turning back to the general expressions (7) and (8), the idea naturally arises to define *generalized Triebel-Lizorkin spaces* for any sectorial operator A using the norm (7). Of course, the norm expression in (7) should be independent of the special function $\varphi \in H_0^\infty \backslash \{0\}$. To ensure this, we resort to the notion of $\mathcal{R}_s$-boundedness from [We01a]: A set $\mathcal{T}$ of linear operators in X is called $\mathcal{R}_s$-bounded if an estimate

$$\Big\| \Big(\sum_{j=1}^{n} |T_j x_j|^s \Big)^{1/s} \Big\|_X \lesssim \Big\| \Big(\sum_{j=1}^{n} |x_j|^s \Big)^{1/s} \Big\|_X \tag{9}$$

holds uniformly for all $T_j \in \mathcal{T}$, $x_j \in X$ and $n \in \mathbb{N}$ (with the usual modification if $s = +\infty$). If the Banach function space X fulfills a certain geometric property, then $\mathcal{R}_2$-boundedness is equivalent to $\mathcal{R}$-boundedness, and for the concept of $\mathcal{R}_s$-boundedness we show some basic results that are similar to corresponding results for $\mathcal{R}$-bounded sets of operators. Nevertheless, there are also considerable differences between the concepts of $\mathcal{R}_s$- and $\mathcal{R}$-boundedness. The most striking one is that even a single operator does not need to be $\mathcal{R}_s$-bounded if $s \neq 2$, cf. e.g. [Du01], Chapter 8.

In this manner, a sectorial operator is said to be $\mathcal{R}_s$-sectorial if the set of resolvents $\{\lambda R(\lambda, A) \mid \lambda \in \mathbb{C}\backslash\overline{\Sigma_\omega}\}$ is $\mathcal{R}_s$-bounded for some $\omega \in (0, \pi)$. If A is $\mathcal{R}_s$-sectorial, we will show that the norm expression in (7) is independent of φ, for φ within a suitable class of bounded holomorphic functions, in the sense of equivalent norms. Having this concept at hand, we can define *generalized Triebel-Lizorkin spaces for $\mathcal{R}_s$-sectorial operators*, which we will refer to as the *associated s-intermediate spaces*, via the norm expression (7):

$$X_{s,A}^\theta := \{x \in X \mid \|x\|_{\theta,s,A,\varphi} < +\infty\}, \quad \|x\|_{X_{s,A}^\theta} := \|x\|_X + \|x\|_{\theta,s,A,\varphi},$$

where $\varphi \neq 0$ is a suitable holomorphic function such that $z \mapsto z^{-\theta}\varphi(z)$ is an H_0^∞-function. Moreover we will define the *associated homogeneous s-intermediate spaces* $\dot{X}_{s,A}^\theta$ to be the completion of $X_{s,A}^\theta$ with respect to the norm $\|\cdot\|_{\theta,s,A,\varphi}$. In both cases, we will show that the norms are independent of φ with the above properties in the sense of equivalent norms. One main result of this thesis is that the "part" of A (which is defined by an abstract extrapolation argument) always has a bounded H^∞-calculus in the homogeneous spaces $\dot{X}_{s,A}^\theta, \theta \in \mathbb{R}$, and if A is invertible or has a bounded H^∞-calculus in X, then the part of A in the inhomogeneous spaces $X_{s,A}^\theta, \theta \geq 0$ has a bounded H^∞-calculus. This can be seen as a counterpart to Dore's Theorem (cf. [Do99], [Do01]), that states a similar result for the real interpolation spaces $(X, D(A))_{\theta,s}$ instead of the s-intermediate spaces.

Let us again have a short look at the case $s = 2$, from which we started. Then the homogeneous norm $\|\cdot\|_{\theta,2,A,\varphi}$ is a (classical) square function norm associated to A, and the corresponding spaces $\dot{X}_{2,A}^\theta$ have also been studied in the context of general Banach spaces (sharing some suitable geometric properties) by Nigel Kalton and Lutz Weis, cf. [KW-1], [KW-2]: the norm $\|\cdot\|_{\theta,2,A,\varphi}$ in a Banach function space X can be reformulated in terms of γ-norms, and it is e.g. well known that if A has a bounded H^∞-calculus, then $\dot{X}^\theta_{2,A}$ equals the homogeneous fractional domain $\dot{D}(A^\theta)$. However, many methods used in the case $s = 2$, in particular equivalence of square function norms to Rademacher- and γ-norms, break down in the case $s \neq 2$, and we have to develop new approaches in this situation.

Using methods from harmonic analysis we will show that wide classes of operators are $\mathcal{R}_s$-sectorial and indeed even have an $\mathcal{R}_s$-bounded H^∞-calculus, i.e. the set $\{f(A) \,|\, f \in \Sigma_\omega, \|f\|_\infty \leq 1\}$ is $\mathcal{R}_s$-bounded for some $\omega \in (0, \pi)$. Moreover, we establish comparison and perturbation results that enable us to show that for certain kinds of elliptic differential operators the generalized Triebel-Lizorkin spaces associated to these operators coincide with the classical Triebel-Lizorkin spaces. In particular, such operators have a bounded H^∞-calculus in classical Triebel-Lizorkin spaces. Again, we give a more detailed exposition below when we give an overview for Chapter 3.

This thesis is organized as follows: In **Chapter 1** we present notations and preliminaries. After introducing some notations in Section 1.1, we recall the notion of the functional calculus for sectorial operators and its extension to an operator-valued functional calculus in Sections 1.2 and 1.3. In Section 1.4 we describe the operator-valued version of the Mikhlin multiplier theorem in UMD-spaces, that relies on results from [We01b], [SW07] and [HHN02]. After this we give a brief summary of abstract interpolation functors, and further present the concrete concepts of real and complex interpolation spaces. In particular, we will introduce a multilinear version of the abstract Stein interpolation theorem for complex interpolation spaces due to [Vo92], which should be known, but seems not to be explicitly written down in the literature. Furthermore we give a detailed exposition of Banach function spaces, and finally a brief review on (classical) Besov and Triebel-Lizorkin spaces.

In **Chapter 2** we introduce the notion of maximal estimates for sets of linear operators in vector-valued Banach function spaces $X(E)$, and in particular the notion of a *bounded maximal function* for a sectorial operator A in $X(E)$. We say that A has a bounded $H_0^\infty(\Sigma_\sigma)$-maximal function, or shortly that A has a bounded H_0^∞-maximal function, if there is a $C > 0$ such that the maximal estimate

$$\Big\| \sup_{t>0} |\varphi(tA)x|_E \Big\|_X \leq C \, \|x\|_{X(E)} \tag{10}$$

holds for all $\varphi \in H_0^\infty(\Sigma_\sigma), x \in X(E)$. Actually (10) implies a more general maximal estimate: Define the maximal function

$$\mathcal{M}_{A,\omega}(x) := \sup \left\{ |\varphi(A)x|_E \;\middle|\; \varphi \in H_0^\infty(\Sigma_{\omega'}) \text{ for some } \omega' \in (\omega, \pi) \text{ with } \|\varphi\|_{L^1_*(\partial\Sigma_\omega)} \leq 1 \right\}$$

for all $x \in X(E)$, where $\|\varphi\|_{L^1_*(\partial\Sigma_\omega)} := \int_{\partial\Sigma_\omega} |\varphi(\lambda)| \frac{|d\lambda|}{|\lambda|}$. We will show that if A has a bounded $H_0^\infty(\Sigma_\sigma)$-maximal function, then also the maximal function $\mathcal{M}_{A,\omega}$ is bounded on $X(E)$ if $\omega > \sigma$.

In Section 2.2 we give examples for large classes of operators that have a bounded H_0^∞-maximal function, namely operators that have BIP, operators that satisfy one-sided square-function estimates, or operators which are generators of semigroups that satisfy suitable maximal estimates.

Moreover, we give an example for an operator without BIP which also has a bounded H_0^∞-maximal function, thus we show that our methods generalize results from [Bl02], where only operators which have BIP are considered.

In Section 2.3 we consider the maximal estimate (10) in the case that φ is in the larger class $\mathcal{E}(\Sigma_\sigma)$ of bounded holomorphic functions that have polynomial limits in 0, i.e. $f(z) - f(0) = O(z^\alpha)$ if $z \to 0$ for some $\alpha > 0$, and in ∞, i.e. $z \mapsto f(z^{-1})$ has a polynomial limit in 0. We will show that if A has a bounded H_0^∞-maximal function, then the maximal estimate (10) holds *for all* $\varphi \in \mathcal{E}(\Sigma_\sigma)$ if it holds *for some* $\varphi \in \mathcal{E}(\Sigma_\sigma)$ with $\varphi(0) \neq \varphi(\infty)$, and moreover under this assumption also the maximal estimate (5) holds for some $\delta > 0$.

In Section 2.4 we will present an interpolation result for maximal functions if A acts as a sectorial operator in spaces $X_j(E_j)$, $j = 0, 1$ with compatible resolvents. In the last Section 2.5 of this chapter we consider the special case of operators A^E in the vector-valued space $X(E)$ which arise as tensor extensions of operators A in the scalar space X. As an application we generalize a result by Robert J. Taggart from [Ta09] about maximal estimates for tensor extensions of symmetric diffusion semigroups in vector-valued space $L^p(\Omega, E)$.

In the last part, **Chapter 3**, we present the concept of $\mathcal{R}_s$-boundedness of linear operators, cf. (9) above, and, as already noted, we will use this concept to define $\mathcal{R}_s$-sectorial operators in the natural way. In Subsection 3.2.2 we will prove the important fact that the norm expressions in (3) are equivalent for all $\varphi \in H_0^\infty(\Sigma_\sigma)\backslash\{0\}$ if A is $\mathcal{R}_s$-sectorial of type smaller than σ. In Subsection 3.2.3 we introduce the notion of an $\mathcal{R}_s$-bounded H^∞-calculus: The operator A in X is said to have an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_\omega)$-calculus if the set

$$\{f(A) \mid f \in H^\infty(\Sigma_\omega), \|f\|_{\infty,\omega} \leq 1\}$$

is $\mathcal{R}_s$-bounded. We will show that if $f(A)$ is $\mathcal{R}_s$-bounded for each $f \in H^\infty(\Sigma_\sigma)$, then the sectorial operator A has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_\omega)$-calculus for all $\omega > \sigma$.

In Section 3.3 we introduce the associated s-intermediate spaces for an $\mathcal{R}_s$-sectorial operator A, namely the homogeneous spaces $\dot{X}^\theta_{s,A}, \theta \in \mathbb{R}$ and the inhomogeneous spaces $X^\theta_{s,A}, \theta \geq 0$. After presenting elementary properties of these spaces in Subsection 3.3.1 we will show in Subsection 3.3.2 that these spaces are indeed reasonable intermediate spaces for X and $D(A^m)$ if $\theta \in (0, m)$, and moreover we present results about real and complex interpolation of these spaces. In Subsection 3.3.3 we constitute the main theorem already mentioned (Theorem 3.3.23), which states that A has a bounded H^∞-calculus in the spaces $\dot{X}^\theta_{s,A}, \theta \in \mathbb{R}$, and under appropriate assumptions also in the spaces $X^\theta_{s,A}, \theta \geq 0$.

In Section 3.4 we present comparison and interpolation results for $\mathcal{R}_s$-sectorial operators. In both cases we can show that if C is a *comparable operator* or an *additive perturbation* (i.e. $C = A + B$ for some linear operator $B : X \supseteq D(A) \to X$) of an operator A with an $\mathcal{R}_s$-bounded H^∞-calculus, then under appropriate conditions also C has an $\mathcal{R}_s$-bounded H^∞-calculus. Moreover, and that is what we are more interested in, under similar conditions we can also show that the operator C has the same associated s-intermediate spaces as A, i.e. we have $\dot{X}^\theta_{s,C} \approx \dot{X}^\theta_{s,A}$ and $X^\theta_{s,C} \approx X^\theta_{s,A}$ for some range of θ. Together with the results of Section 3.3 this yields that these operators C have a bounded H^∞-calculus in the s-intermediate spaces associated to the operator A.

In Section 3.5 we will show that the negative A of a generator of an analytic semigroup T in the space $L^2(\Omega)$ over some space of homogeneous type Ω has an $\mathcal{R}_s$-bounded H^∞-calculus in the spaces $L^p(\Omega)$ for all $s, p \in (p_0, p_1)$, if the semigroup T satisfies a certain kind of weighted $p_0 \to p_1$ estimates, which are also referred to as off-diagonal or generalized Gaussian estimates. Thus we generalize a theorem by Sönke Blunck and Peer Kunstmann from [BK03], where it is shown that under similar conditions the operator A has a bounded H^∞-calculus in the spaces $L^p(\Omega)$ for all $p \in (p_0, p_1)$. As an application we will show that wide classes of differential operators, covering certain elliptic operators in divergence and non-divergence form and Schrödinger operators $-\Delta + V$ with appropriate potentials V, have an $\mathcal{R}_s$-bounded H^∞-calculus, thus are in particular $\mathcal{R}_s$-sectorial, and the associated s-intermediate spaces are well defined for these operators, and the theory from Section 3.3 can be applied.

We note that for Schrödinger operators with special potentials, a similar concept of generalized Triebel-Lizorkin spaces is introduced in [OZ06] and [Zh06]. There, the main issue is to show that the norm in those spaces is independet of the auxiliary function φ used in the definition of the norm. Nevertheless, the definition given there differs from ours and is closer to the original definition of Triebel-Lizorkin spaces, where auxiliary functions $\varphi \in C_c^\infty(\mathbb{R})$ are used to define the norm in those spaces. On the other hand, our concept is more general in the sense that we can also handle non-selfadjoint sectorial operators with non-real spectrum. Let us also mention that the case $s = 2$ in the framework of [OZ06] and [Zh06] is also covered by [Kr09], Chapter 4, where Littlewood-Paley decompositions associated to 0-sectorial operators are studied in general. We give more comments on this topic at the end of Subsection 3.5.2.

Finally, in the last Section 3.6 we will apply the results from Section 3.4 to identify the s-intermediate spaces associated to certain elliptic operators in non-divergence and in divergence form. We will show that these spaces coincide with the classical Triebel-Lizorkin spaces $F^s_{p,q}$ for a certain range of s if the top order coefficients of the differential operators satisfy appropriate regularity assumptions. In the case of non-divergence form operators this will be a Hölder-continuity condition, and in the case of divergence form operators this condition can be weakened to the assumption that the top order coefficients are bounded and uniformly continuous. So in particular, operators that satisfy these conditions have a bounded H^∞-calculus in the classical Triebel-Lizorkin spaces. For the case of non-divergence form operators with Hölder-continuous coefficients, such results are already indicated in [ES08] and [DSS09], where even more generally pseudodifferential operators are considered, but the details in the proof for the case of Triebel-Lizorkin spaces are left out. We note that our methods are operator theoretical and are totally different from the approaches in [ES08] and [DSS09], which are based on pseudodifferential calculus. Moreover, our results will not only show that the differential operators A_p (of order $2m$ in $X = L^p(\mathbb{R}^d)$) we consider have a bounded H^∞-calculus in classical Triebel-Lizorkin spaces, but also that the norm equivalences $\dot{X}^\theta_{\nu+A_p,q} \approx \dot{X}^\theta_{\nu+(-\Delta)^m,q}$ and $X^\theta_{\nu+A_p,q} \approx X^\theta_{\nu+(-\Delta)^m,q} \approx F^{2m\theta}_{p,q}$ hold for all $q \in (1, +\infty)$ and appropriate θ and $\nu > 0$, i.e. we can express the norm in the classical Triebel-Lizorkin spaces by the s-power function norms associated to the more general elliptic operator A_p instead of the Laplacian. These representations of the classical Triebel-Lizorkin spaces are new. The corresponding results for divergence operators are entirely new.

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Chapter 1

Notations and Preliminaries

1.1 Notations

We introduce some notations. For any set X, we denote the power set of X by $\mathrm{Pot}(X)$. If $(X, \leq)$ is a partially ordered set and $x, y \in X$, we let $x \wedge y := \inf\{x, y\}$ and $x \vee y := \sup\{x, y\}$, if the latter exist. If X, Y are normed spaces we let $\mathcal{L}(X, Y) := \mathrm{Hom}(X, Y)$ be the set of linear maps from X to Y and denote by $L(X, Y) := \{T \in \mathcal{L}(X, Y) \mid T \text{ bounded}\}$ the bounded linear operators from X to Y. We use the notation $X' := L(X, \mathbb{K})$ for the dual space. For the dual pairing (X, X') we will use the notation $\langle x, x' \rangle := \langle x, x' \rangle_{X \times X'} := x'(x)$ if $(x, x') \in X \times X'$. If X, Y are normed spaces with $X \subseteq Y$, or X is canonically identified with a subspace of Y, we write $X \hookrightarrow Y$ if the canonical inclusion map is continuous.

If Ω is a set and μ is a measure on some σ-algebra over Ω we say that (Ω, μ) is a measure space. Observe that the underlying σ-algebra can be recovered as the domain $D(\mu)$ of μ[1], but we will usually only consider the σ-algebra of μ-measurable subsets of Ω. Let (Ω, μ) be a measure space, E a Banach space and $p \in [1, +\infty]$, then we denote the usual Lebesgue-spaces by $\mathcal{L}^p(\mu, E)$, or a little improperly by $\mathcal{L}^p(\Omega, E)$. If it is clear what the underlying measure space (Ω, μ) is we just write $\mathcal{L}^p(E)$. The corresponding spaces of equivalence classes modulo the equivalence relation given by *pointwise equality up to a μ-nullset* are denoted by $L^p(\mu, E), L^p(\Omega, E)$ or $L^p(E)$, respectively. If μ is the counting measure, we will use the notations $\ell^p(\Omega, E)$ or $\ell^p(E)$ instead of $L^p(\Omega, E)$. Finally, if $E = \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ we will usually drop E in the notation and just write $\mathcal{L}^p(\mu), \mathcal{L}^p(\Omega), \mathcal{L}^p, L^p(\mu), L^p(\Omega), L^p, \ell^p(\Omega)$ or ℓ^p, respectively.

Let $d \in \mathbb{N}$ and E be a Banach space. We will use the common notation of multi-indices, i.e. if $\alpha \in \mathbb{N}_0^d$ we let $x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}$, $\partial^\alpha := \prod_{j=1}^d \partial_j^{\alpha_j}$ and $|\alpha| = \sum_{j=1}^d \alpha_j$, and for later use we already introduce the notations $D_j := \frac{1}{i}\partial_j$ and $D^\alpha := (-i)^{|\alpha|}\partial^\alpha$. If $u \in \mathcal{L}^1(\mathbb{R}^d, E)$ we define the Fourier transform of u by

$$\widehat{u}(\xi) := (\mathcal{F}f)(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx \quad \text{for all } \xi \in \mathbb{R}^d,$$

and for $u \in \mathcal{F}(\mathcal{L}^1(\mathbb{R}^d, E))$ we denote the inverse Fourier transform of u by $\check{u} := \mathcal{F}^{-1}(u)$. Let

[1] and in fact, also the set Ω can be recovered from μ by $\Omega = \bigcup D(\mu)$

$\mathcal{S}_d(E)$ be the space of E-valued Schwartz-functions on $\mathbb{R}^d$ an let $\mathcal{S}'_d(E) := L(\mathcal{S}_d, E)$ be the space of E-valued tempered distributions on $\mathbb{R}^d$. Then we will extend the notions of Fourier transforms and e.g. convolution to the space of tempered distributions as usual, cf. e.g. [Am95] for the vector-valued version of these standard notions.

Finally, in estimates we will sometimes drop constants and use the relation symbols $\lesssim$ and $\approx$. This is done if we only leave out universal constants (like e.g. natural numbers) or if it is clear on which other terms the constants depend. To be more concrete we use sometimes the symbols $\lesssim_M$ and $\approx_M$ to indicate that the dropped constants depend on the term M.

1.2 The functional calculus for sectorial operators and the H^∞-calculus

We give a short introduction to the functional calculus for sectorial operators, for details we refer to the standard literature as [Ha06] or [KW04].

Let X be a complex Banach space. For $\sigma \in (0, \pi]$ we define the open sector

$$\Sigma_\sigma := \{z \in \mathbb{C}\backslash(-\infty, 0] \mid |\arg(z)| < \sigma\},$$

where arg is the principal branch of the argument-function, and we let $\Sigma_0 := (0, +\infty)$. Moreover we define $\overline{\Sigma}_\sigma := \overline{\Sigma_\sigma}$ for all $\sigma \in [0, \pi]$.

Definition 1.2.1 (Sectorial operator, type of a sectorial operator)**.** Let $A : X \supseteq D(A) \to X$ be a linear operator. A is called a *sectorial operator of type* $\omega \in [0, \pi)$ if the spectrum $\sigma(A)$ is contained in the closed sector $\overline{\Sigma}_\omega$ and the set of operators $\{zR(z, A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}$ is bounded for all $\sigma \in (\omega, \pi)$. The infimum $\omega(A)$ over all such ω is called the *type of* A.

For the remaining section we fix some injective sectorial operator $A : X \supseteq D(A) \to X$ and $\sigma \in (\omega(A), \pi]$.

For $f : \Sigma_\sigma \to \mathbb{C}$ let $\|f\|_{\infty,\sigma} := \sup_{z\in\Sigma_\sigma} |f(z)|$, where we sometimes drop the index σ if there is no risk of confusion. We introduce the algebra $H^\infty(\Sigma_\sigma) := \{f : \Sigma_\sigma \to \mathbb{C} \mid f \text{ analytic}, \|f\|_{\infty,\sigma} < +\infty\}$ of bounded analytic functions on the sector Σ_σ and the subalgebra $H_0^\infty(\Sigma_\sigma)$ consisting of those $f \in H^\infty(\Sigma_\sigma)$ for which there exists an $\varepsilon > 0$ with $\sup_{z\in\Sigma_\sigma} \big((|z|^\varepsilon \vee |z|^{-\varepsilon})|f(z)|\big) < +\infty$.

Let $\omega \in (\omega(A), \sigma)$ and define the path of integration $\Gamma_\omega(t) := |t|e^{-\operatorname{sgn}(t)i\omega}$ for all $t \in \mathbb{R}$, then

$$\varphi \mapsto \varphi(A) := \frac{1}{2\pi i}\int_{\Gamma_\omega} \varphi(\lambda)R(\lambda, A)\, d\lambda \tag{1.2.1}$$

defines an algebra homomorphism $H_0^\infty(\Sigma_\sigma) \to L(X)$ that is independent of $\omega \in (\omega(A), \sigma)$ and only depends on the germ of φ on $\overline{\Sigma}_{\omega(A)}$.

By a standard extension procedure we obtain a functional calculus for all $f \in H^\infty(\Sigma_\sigma)$ and even for a larger class of holomorphic functions: we define $\rho(\lambda) := \lambda(1 + \lambda)^{-2}$ for $\lambda \in \mathbb{C}\backslash(-\infty, 0)$. Let

$$\mathfrak{B}(\Sigma_\sigma) := \{f : \Sigma_\sigma \to \mathbb{C} \mid z \mapsto \rho(z)^m f(z) \in H_0^\infty(\Sigma_\sigma) \text{ for some } m \in \mathbb{N}\}$$

be the algebra of analytic functions on the sector Σ_σ that are polynomially bounded at 0 and ∞. Then it is easy to check that $\rho^m(A) = \rho(A)^m = A^m(1+A)^{-2m}$, and $\rho(A)$ is an injective operator. Let $f \in \mathfrak{B}(\Sigma_\sigma)$ and choose $m \in \mathbb{N}$ such that $\rho^m f \in H_0^\infty(\Sigma_\sigma)$, then the operator $(\rho^m f)(A) \in L(X)$ is well defined by the functional calculus described above and we can define

$$f(A) := (\rho(A))^{-m}(\rho^m f)(A).$$

It can be shown that the definition of $f(A)$ is independent of $m \in \mathbb{N}$ such that $\rho^m f \in H_0^\infty(\Sigma_\sigma)$, and that $f \mapsto f(A)$ is an (abstract) functional calculus for A in the sense of [Ha06], Chapter 1.3.

As a special class of analytic functions f that still yield a nice representation formula and ensure that $f(A)$ is bounded we introduce the *extended Dunford-Riesz class*, which is defined by

$$\mathcal{E}(\Sigma_\sigma) := H_0^\infty(\Sigma_\sigma) \oplus \left\langle \frac{1}{1+\mathrm{id}_{\Sigma_\sigma}} \right\rangle_{\mathbb{C}} \oplus \langle \mathbb{1}_{\Sigma_\sigma} \rangle_{\mathbb{C}} \quad \text{and } \mathcal{E}_\omega := \bigcup_{\sigma'>\omega} \mathcal{E}(\Sigma_{\sigma'}) \quad \text{for any } \omega \in [0,\pi).$$

It can easily be shown that $\mathcal{E}(\Sigma_\sigma)$ is exactly the algebra of bounded analytic functions on Σ_σ that have finite polynomial limits in 0 and ∞. Here we say that f has a finite polynomial limit in 0, if there is an $a \in \mathbb{C}$ and $\alpha > 0$ such that $f(z) - a = O(|z|^\alpha)$ as $z \to 0$, and a finite polynomial limit in ∞, if the latter is true for $f(z^{-1})$. In this case, the values $f(0), f(\infty) \in \mathbb{C}$ are well defined. Moreover, by the mean value theorem, bounded holomorphic functions on Σ_σ that are either decaying to 0 or holomorphic in a neighborhood of 0 and ∞, respectively, belong to the class $\mathcal{E}(\Sigma_\sigma)$. For $f \in \mathcal{E}(\Sigma_\sigma)$ let $\varphi := f - \frac{f(0)-f(\infty)}{1+\mathrm{id}_{\Sigma_\sigma}} - f(\infty)\,\mathbb{1}_{\Sigma_\sigma}$ be the corresponding H_0^∞-function, then it is easily checked that

$$f(A) = \varphi(A) + (f(0) - f(\infty))\,(1+A)^{-1} + f(\infty)\,\mathrm{id}_X\,.$$

For details we refer to [Ha06], Section 2.2.

From now on we assume additionally that the operator A has dense domain and range[2]. Actually, this is not much loss of generality in our situation, because our main examples will be in reflexive spaces, and in this case sectorial operators always have dense domain, and they are injective if and only if they have dense range, cf. [Ha06], Proposition 2.1.1. An important issue in this context is the so-called *Convergence Lemma*, which we state in the following version, cf. [Ha06], Proposition 5.1.4.

Proposition 1.2.2 (Convergence Lemma). *Let $(f_n)_{n\in\mathbb{N}} \in (H^\infty(\Sigma_\sigma))^{\mathbb{N}}$ be a sequence such that the following assertions hold:*

(i) The pointwise limit $f_0(z) := \lim_{n\to\infty} f_n(z)$ exists for all $z \in \Sigma_\sigma$,

(ii) $\sup_{n\in\mathbb{N}} \|f_n\|_{\infty,\sigma} < +\infty$,

(iii) $f_n(A) \in L(X)$ for all $n \in \mathbb{N}$ and $M := \sup_{n\in\mathbb{N}} \|f_n(A)\| < +\infty$.

[2]Observe that the density of $R(A)$ already implies that A is injective by the sectoriality condition, cf. [Ha06], Proposition 2.1.1.

Then $f_0 \in H^\infty(\Sigma_\sigma)$ and $f_0(A) \in L(X)$ with $\|f_0(A)\| \le M$. Moreover, $f_n(A)x \to f_0(A)x$ if $n \to \infty$ for all $x \in X$.

We can now turn to the important notion of a *bounded H^∞-calculus.*

Definition 1.2.3 (Bounded H^∞-calculus)**.** Let $\sigma \in (\omega(A), \pi]$. The operator A is said to have a *bounded $H^\infty(\Sigma_\sigma)$-calculus* if

$$M_\sigma^\infty(A) := \sup\{\|f(A)\| \mid f \in H^\infty(\Sigma_\sigma), \|f\|_{\infty,\sigma} \le 1\} < +\infty.$$

Moreover, $\omega_{H^\infty}(A) := \inf\{\sigma \in (\omega(A), \pi] \mid M_\sigma^\infty(A) < +\infty\}$ is called the *H^∞-type* of A. If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\sigma > \omega(A)$, we also just say that A has a bounded H^∞-calculus.

The following characterization is an easy consequence of the convergence lemma and the closed graph theorem, cf. e.g. [KW04], Remark 9.11 and [Ha06], Proposition 5.3.4.

Remark 1.2.4. *The operator A has a bounded $H^\infty(\Sigma_\sigma)$-calculus if and only if there is a $C > 0$ such that*

$$\|\varphi(A)\| \le C\,\|\varphi\|_{\infty,\sigma} \quad \text{for all } \varphi \in H_0^\infty(\Sigma_\sigma),$$

and in this case $M_\sigma^\infty(A) \le C$.

1.3 $\mathcal{R}$-sectorial operators and the operator-valued H^∞-calculus

It has been shown in [KW01-a] that a bounded H^∞-calculus can be extended to an operator-valued H^∞-calculus for operator-valued functions with an $\mathcal{R}$-bounded range. Moreover, under an additional geometric assumption on the underlying Banach space this can be even extended to the stronger notion of an $\mathcal{R}$-bounded H^∞-calculus. We will use these tools frequently in this work, where we also are interested in controlling the involved constants, hence we will present a slightly more general version of the corresponding results from [KW01-a] and [KW04]. We will also introduce the notion of $\mathcal{R}$-sectorial operators.

Let X, Y be complex Banach spaces. Let $(r_j)_{j\in\mathbb{N}}$ be a *Rademacher-sequence*, i.e. a sequence of independent symmetric ± 1-valued random variables on some probability space (Ω, P), and let $\mathbb{E}$ denote the expectation with respect to the corresponding probability measure P. A standard example are the Rademacher functions $r_j(t) := \operatorname{sgn}(\sin(2^j\pi t))$ for all $t \in [0,1], j \in \mathbb{N}$ on the probability space $[0,1]$ endowed with the usual Lebesgue-measure. Observe that for any Banach space E and $p \in [1, +\infty)$ the expressions

$$\left(\mathbb{E}\Big|\sum_{j=1}^n r_j \otimes x_j\Big|_E^p\right)^{1/p} = \left(\frac{1}{2^n}\sum_{\sigma\in\{-1,1\}^n}\Big\|\sum_{j=1}^n \sigma_j x_j\Big\|_E^p\right)^{1/p} \tag{1.3.2}$$

for $x \in E^n, n \in \mathbb{N}$ do not depend on the special choice of the Rademacher-sequence $(r_j)_{j\in\mathbb{N}}$.

Definition 1.3.1 ($\mathcal{R}$-boundedness)**.** A set $\mathcal{T} \subseteq L(X,Y)$ is called $\mathcal{R}$-*bounded* if there exists a constant $C \in \mathbb{R}_{>0}$ such that for all $n \in \mathbb{N}, T \in \mathcal{T}^n$ and $x \in X^n$:

$$\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes T_j x_j\Big|_X \leq C\,\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes x_j\Big|_X. \tag{1.3.3}$$

In this case, the infimum over all such constants $C > 0$ is denoted by $\mathcal{R}(\mathcal{T})$ and called the $\mathcal{R}$-bound of $\mathcal{T}$.

A detailed exposition of the notion of $\mathcal{R}$-boundedness can be found e.g. in [CdPSW00] or in [KW04], Section 2.

Definition 1.3.2 ($\mathcal{R}$-sectorial operator, $\mathcal{R}$-type of an $\mathcal{R}$-sectorial operator)**.** Let $A : X \supseteq D(A) \to X$ be a linear operator. A is called an $\mathcal{R}$-*sectorial operator of* $\mathcal{R}$-*type* $\omega \in [0,\pi)$ if the spectrum $\sigma(A)$ is contained in the closed sector $\overline{\Sigma}_\omega$ and the set of operators $\{zR(z,A) \,|\, z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}$ is $\mathcal{R}$-bounded for all $\sigma \in (\omega,\pi)$. The infimum $\omega_{\mathcal{R}}(A)$ over all such ω is called the $\mathcal{R}$-*type of* A. In this case we define

$$M_{\mathcal{R},\sigma}(A) := \mathcal{R}(\{zR(z,A), AR(z,A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\})$$

for all $\sigma \in (\omega_{\mathcal{R}}(A),\pi)$. Observe that this set is indeed also $\mathcal{R}$-bounded, since

$$AR(z,A) = zR(z,A) - \mathrm{id}_X \quad \text{for all } z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma,$$

hence $M_{\mathcal{R},\sigma}(A) \leq \mathcal{R}(\{zR(z,A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}) + 1 \leq 2M_{\mathcal{R},\sigma}(A)$.

We will now turn to the notion of the operator-valued H^∞-calculus as presented e.g. in [KW01-a] or [KW04], Chapter 12. Let $\mathcal{A} \subseteq L(X)$ denote the subalgebra of all bounded operators that commute with resolvents of A. Then we define

$$RH^\infty(\Sigma_\sigma,\mathcal{A}) := \{F : \Sigma_\sigma \to \mathcal{A} \,|\, F \text{ is analytic and } F(\Sigma_\sigma) \text{ is } \mathcal{R}\text{-bounded}\}.$$

For each $F \in RH^\infty(\Sigma_\sigma,\mathcal{A})$ we define the norm $\|F\|_{RH^\infty,\sigma} := \mathcal{R}(F(\Sigma_\sigma))$. By $RH_0^\infty(\Sigma_\sigma,\mathcal{A})$ we denote the subspace of functions $F \in RH^\infty(\Sigma_\sigma,\mathcal{A})$ such that $\sup_{z\in\Sigma_\sigma}\big((|z|^\varepsilon \vee |z|^{-\varepsilon})\|F(z)\|\big) < +\infty$ for some $\varepsilon > 0$.

Then it can be shown that in the same manner as for scalar-valued analytic functions the mapping

$$F \mapsto \frac{1}{2\pi i}\int_{\Gamma_\omega} F(\lambda)R(\lambda,A)\,d\lambda$$

is independent of $\omega \in (\omega(A),\sigma)$ and defines a functional calculus $\Phi_A : RH_0^\infty(\Sigma_\sigma) \to L(X)$ for the operator A. If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus, it can be shown that the functional calculus Φ_A can be extended to the algebra $RH^\infty(\Sigma_{\sigma'})$ for all $\sigma' > \sigma$, cf. [KW01-a] Theorem 4.4 or [KW04], Theorem 12.7. In fact, a careful inspection of the proof of [KW04], Theorem 12.7 shows the following sharper version covering also the involved constants.

Theorem 1.3.3. *Assume that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus. Then for each $\omega > \sigma$ there is a constant $C_{\omega,\sigma} > 0$ independent of A such that*

$$\forall\, \sigma' \geq \omega\, \forall\, F \in RH^\infty(\Sigma_{\sigma'}) : \ \|F(A)\| \leq C_{\omega,\sigma} \cdot M_\sigma^\infty(A) \cdot \|F\|_{RH^\infty,\omega}.$$

As an application one can show that a bounded H^∞-calculus of the operator A implies an $\mathcal{R}$-bounded H^∞-calculus and even an $\mathcal{R}$-bounded RH^∞-calculus (in the sense of Theorem 1.3.5 below), if the Banach space X has the so-called property (α) introduced by Pisier in [Pi78].

Definition 1.3.4 (Property (α))**.** The Banach space X has property (α) if there is a constant $C_X > 0$ such that for all $n \in \mathbb{N}$, $\alpha = (\alpha_{jk})_{j,k=1}^n \in \mathbb{C}^{n\times n}$ with $|\alpha_{jk}| \leq 1$ for all $j,k \in \mathbb{N}_{\leq n}$ and $x = (x_{jk})_{j,k=1}^n \in X^{n\times n}$ we have

$$\mathbb{E}^{P\otimes P}\Big|(r_j \otimes r_k) \otimes \alpha_{jk} x_{jk}\Big|_X \leq C_X\, \mathbb{E}^{P\otimes P}\Big|(r_j \otimes r_k) \otimes x_{jk}\Big|_X, \tag{1.3.4}$$

where $\mathbb{E}^{P\otimes P}$ denotes the expectation on the probability space $(\Omega \times \Omega, P \otimes P)$.

If we work with the standard Rademacher functions, (1.3.4) can be rewritten as

$$\int_0^1 \int_0^1 \Big\|r_j(t) r_k(s) \alpha_{jk} x_{jk}\Big\|_X \,dt\,ds \leq C_X \int_0^1 \int_0^1 \Big\|r_j(t) r_k(s) x_{jk}\Big\|_X \,dt\,ds. \tag{1.3.5}$$

There are wide classes of Banach spaces which are known to have property (α). We just refer to Proposition 1.6.22 in Subsection 1.6.3 for the special case of q-concave Banach function spaces, which is a sufficiently large class of such spaces for this work.

Having this notions at hand, [KW01-a], Theorem 5.3 and its Corollary 5.4, or [KW04], Theorem 12.8 and Remark 12.10 show that if X has property (α) and A has a bounded H^∞-calculus, then A has also an $\mathcal{R}$-bounded H^∞-calculus, i.e. the set

$$\{f(A) \mid \|f\|_{\infty,\sigma'} \leq 1\}$$

is $\mathcal{R}$-bounded for all $\sigma' > \omega_{H^\infty}(A)$. In particular, if X has property (α) and A has a bounded H^∞-calculus, then A is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(A) \leq \omega_{H^\infty}(A)$. This assertion is still true under much weaker assumptions on the Banach space X, cf. [KW01-a].

In fact, even more is proven: under the same assumptions the operator-valued functional calculus is $\mathcal{R}$-bounded, and again, a careful inspection of the proofs yields the following theorem.

Theorem 1.3.5. *Assume that X has property (α) and A has a bounded $H^\infty(\Sigma_\sigma)$-calculus. Then for each $\omega > \sigma$ there is a constant $C_{\omega,\sigma} > 0$ independent of A such that for all $\mathcal{T} \subseteq L(X)$ the following holds:*

$$\forall\, \sigma' \geq \omega \ : \ \mathcal{R}\big(\{F(A) \mid F \in RH^\infty(\Sigma_{\sigma'}, \mathcal{A}), F(\Sigma_{\sigma'}) \subseteq \mathcal{T}\}\big) \leq C_{\omega,\sigma} \cdot M_\sigma^\infty(A) \cdot \mathcal{R}(\mathcal{T}). \tag{1.3.6}$$

If $\sigma' \geq \omega$ we can consider $H^\infty(\Sigma_{\sigma'})$ as a subspace of $RH^\infty(\Sigma_\omega, \mathcal{A})$ by the injection $f \mapsto \big(f|_{\Sigma_\omega}\big) \otimes \mathrm{id}_X$. Then for each subset $\mathcal{F} \subseteq H^\infty(\Sigma_{\sigma'})$ we have $\mathcal{R}\big(\bigcup_{f\in\mathcal{F}} f(\Sigma_\omega)\big) \leq 2\sup_{f\in\mathcal{F}} \|f\|_{\infty,\omega}$ by Kahane's contraction principle, hence we obtain the following special case of Theorem 1.3.5 (cf. also [KW04], Theorem 12.8 and Remark 12.10).

Corollary 1.3.6. *Assume that X has property (α) and A has a bounded $H^\infty(\Sigma_\sigma)$-calculus. Then for each $\omega > \sigma$ there is a constant $C_{\omega,\sigma} > 0$ independent of A such that the following holds:*

$$\forall\, \sigma' \geq \omega\, \forall\, \mathcal{F} \subseteq H^\infty(\Sigma_{\sigma'}) \;:\; \mathcal{R}(\{f(A) \mid f \in \mathcal{F}\}) \leq C_{\omega,\sigma} \cdot M_\sigma^\infty(A) \cdot \sup_{f\in\mathcal{F}} \|f\|_{\infty,\omega}.$$

In particular, A is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(A) \leq \sigma$, and for each $\omega > \sigma$ there is a constant $\widetilde{C}_{\omega,\sigma} > 0$ independent of A such that the following holds:

$$\forall\, \sigma' \geq \omega \;:\; M_{\mathcal{R},\sigma'}(A) \leq \widetilde{C}_{\omega,\sigma} \cdot M_\sigma^\infty(A).$$

1.4 UMD spaces and the operator-valued Mikhlin Theorem

In this work we will often use the operator-valued version of the Mikhlin Multiplier Theorem in UMD-spaces, which in the version on the real line is due to Lutz Weis, [We01b]. We will also use versions of this theorem in the space $\mathbb{R}^d$, which are presented in [KW04] or [SW07], cf. also [HHN02]. Beside the concept of UMD-spaces, which is briefly described in this section, we will also need the notion of $\mathcal{R}$-boundedness and the property (α) for Banach spaces, which have been introduced in the preceding section.

Let E, F be complex Banach spaces, $p \in (1, +\infty)$ and $d \in \mathbb{N}$.

Definition 1.4.1. Let $m \in L^\infty(\mathbb{R}^d, L(E,F))$. Then the operator

$$T_m : \mathcal{S}_d(E) \to \mathcal{S}_d(F), u \mapsto \mathcal{F}^{-1}(m \cdot \mathcal{F}u)$$

is called the *Fourier multiplier operator associated to m*. The function m is called an *L^p-Fourier-multiplier* if $T_m(\mathcal{S}_d(E)) \subseteq L^p(F)$ and there is a constant $C_p > 0$ such that $\|T_m u\|_p \leq C_p \|u\|_p$ for all $u \in \mathcal{S}_d(E)$.

In this case, T_m can be extended to a bounded operator $L^p(E) \to L^p(F)$, which we will also denote by T_m.

An elementary multiplier operator on the real line is given by the function $m := -i\,\mathrm{sgn}$. The associated Fourier multiplier operator is called the *Hilbert transform on E*. Indeed, the boundedness of the Hilbert transform on $L^p(E)$ is sufficient for the boundedness of a large class of multiplier operators. We have the following concrete representation of the Hilbert transform, cf. e.g. [Am95], Section III.4.3.

Definition/Proposition 1.4.2 (Hilbert-transform)**.** The vector-valued *Hilbert-Transform* $\mathcal{H}_E : \mathcal{S}(E) \to \mathcal{S}'(E)$ on E is the Fourier multiplier operator associated to the function $-i\,\mathrm{sgn}$. For all $f \in \mathcal{S}(E)$ it is given by

$$\mathcal{H}_E f(x) := \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \int_{|y|\geq\varepsilon} \frac{f(x-y)}{y}\, dy \quad \text{for all } x \in \mathbb{R}.$$

We are now in position to give the definition of an UMD-space. We will present the original definition involving vector-valued martingales - indeed, the notation UMD-space originates in the term *Unconditionality of Martingal Differences* - as well as the characterizations in a geometric manner on the one hand and in terms of boundedness of the vector valued Hilbert transform on the other hand. Hence we will first present the following expansive characterization result for the property UMD that will be defined subsequently. The proof can be found in the papers [Bu81], [Bu83] and [Bo83].

Theorem 1.4.3. *Let $p \in (1, +\infty)$. Then the following conditions on E are equivalent:*

(1) E has the property UMD_p, i.e.: there exists a constant $M_p(E) > 0$ such that for all E-valued martingales $u = (u_n)_{n\in\mathbb{N}}$, all $\varepsilon \in \{-1, 1\}^{\mathbb{N}}$ and all $n \in \mathbb{N}$ the following holds:

$$\left\| \sum_{k=1}^{n} \varepsilon_n (u_k - u_{k-1}) \right\|_p \leq M_p(E) \cdot \left\| \sum_{k=1}^{n} (u_k - u_{k-1}) \right\|_p \Big(= M_p(E) \, \|u_n\|_p \Big),$$

where $u_0 := 0$.

(2) E is ζ-convex, i.e.: there exists a symmetric biconvex function $\zeta : E \times E \to \mathbb{R}$ with $\zeta(0, 0) > 0$ and

$$\forall\, x, y \in E : \;\; \|x\| \leq 1 \leq \|y\| \;\Rightarrow\; \zeta(x, y) \leq \|x + y\|.$$

(3) E has the property $\mathcal{HT}_p$, i.e.: the vector-valued Hilbert-Transform $\mathcal{H}_E$ on $\mathcal{S}(E)$ can be extended to a bounded operator on $L^p(E)$, in other words: $\mathcal{H}_E(\mathcal{S}(E)) \subseteq L^p(E)$, and

$$\exists\, C_p > 0 \,\forall\, f \in \mathcal{S}(E) \,:\, \|\mathcal{H}_E f\|_p \leq C_p \, \|f\|_p.$$

In particular, since (2) does not depend on $p \in (1 + \infty)$, the properties $\mathrm{UMD}_p, \mathcal{HT}_p$ hold for some $p \in (1, +\infty)$ if and only if they hold for all $p \in (1, +\infty)$. The equivalent conditions of Theorem 1.4.3 lead to the following definition of UMD-spaces.

Definition 1.4.4 (UMD-space)**.** The space E is called an *UMD-space* if E satisfies the equivalent conditions (1)-(3) of Theorem 1.4.3.

We give some important examples of UMD-spaces: By Plancherel's Theorem every Hilbert space is a UMD-space, and moreover if E is a UMD-space, then also closed subspaces and quotients of E are UMD-spaces, and $L^p(\mu, E)$ is a UMD-space for all σ-finite measure spaces (Ω, μ) and $p \in (1, +\infty)$, cf. also [Am95], Theorem III.4.5.2. More information concerning UMD-spaces can be found e.g. in [Am95], Sections III.4.4 and III.4.5.

We can now cite the operator-valued version of the Mikhlin Multiplier Theorem on $\mathbb{R}$, cf. [We01b], Theorem 3.4 or [KW04], Theorem 3.12.

Theorem 1.4.5. *Let E, F be UMD-spaces and $m \in C^1(\mathbb{R}\backslash\{0\}, L(E, F))$ satisfy the following condition:*

The sets $M_1 := m(\mathbb{R}\backslash\{0\})$ *and* $M_2 := \{tm'(t) \,|\, t \in \mathbb{R}\backslash\{0\}\}$ *are* $\mathcal{R}$*-bounded.*

Then m *is an* L^p*-Fourier-multiplier, and* $\|T_m\| \le C_p \big(\mathcal{R}(M_1) \vee \mathcal{R}(M_2)\big)$ *with a constant* C_p *only depending on* E, F *and* p*.*

We will next formulate the multi-dimensional version of this theorem, cf. [KW04], Theorems 4.6 and 4.13.

Theorem 1.4.6. *Let* E, F *be UMD-spaces,* $m \in C^1(\mathbb{R}^d\backslash\{0\}, L(E, F))$ *and* $M > 0$*. Assume that one of the following conditions is fulfilled:*

(1) The set $\mathcal{T}_1 := \big\{|x|^{|\alpha|}\partial^\alpha m(x) \,:\, x \in \mathbb{R}^d\backslash\{0\}, \alpha \le (1, 1, \dots, 1)\big\}$ *is* $\mathcal{R}$*-bounded with* $\mathcal{R}(\mathcal{T}_1) \le M$*, or*

(2) E *and* F *have property* (α)*, and the set* $\mathcal{T}_2 := \big\{x^\alpha \partial^\alpha m(x) \,:\, x \in \mathbb{R}^d\backslash\{0\}, \alpha \le (1, 1, \dots, 1)\big\}$ *is* $\mathcal{R}$*-bounded with* $\mathcal{R}(\mathcal{T}_2) \le M$*.*

Then m *is an* L^p*-Fourier-multiplier, and* $\|T_m\| \le C_p \cdot M$ *with a constant* C_p *only depending on* E, F, d *and* p*.*

1.5 Interpolation of Banach spaces

We will give a short overview of the theory of interpolation spaces as we will use it in the sequel. We will restrict ourselves here to some basic definitions and elementary properties. For the proofs and more detailed expositions we refer to the standard literature, e.g. [BL76], [Lu09] or [Tr78].

1.5.1 Interpolation couples and interpolation functors

Definition 1.5.1. A pair $X = (X_0, X_1)$ of Banach spaces is said to be an *interpolation couple* if there is a separated topological vector space Z such that $X_0, X_1 \subseteq Z$ with continuous inclusion.

Let $((X_0, \|\cdot\|_0), (X_1, \|\cdot\|_1))$ be an interpolation couple, then the spaces $X_0 \cap X_1$ and $X_0 + X_1$ are well defined as subspaces of Z. Define $\|x\|_\cap := \|x\|_0 + \|x\|_1$ for each $x \in X_0 \cap X_1$ and $\|z\|_\Sigma := \inf\{\|x_0\|_0 + \|x_1\|_1 \mid (x_0, x_1) \in X_0 \times X_1, z = x_0 + x_1\}$ for each $z \in X_0 + X_1$, then $(X_0 \cap X_1, \|\cdot\|_\cap)$ and $(X_0 + X_1, \|\cdot\|_\Sigma)$ are Banach spaces with

$$X_0 \cap X_1 \hookrightarrow X_j \hookrightarrow X_0 + X_1 \quad \text{for } j = 0, 1 \text{ with continuous inclusions.}$$

Any Banach space E such that $X_0 \cap X_1 \hookrightarrow E \hookrightarrow X_0 + X_1$ with continuous inclusions is called an *intermediate space* between X_0 and X_1.

The interpolation couples form the objects of a category, where the morphisms are bounded linear operators $T : X_0 + X_1 \to Y_0 + Y_1$ such that $T(X_j) \subseteq Y_j$ and $T|_{X_j} \in L(X_j, Y_j)$ for $j = 0, 1$, where $(X_0, X_1), (Y_0, Y_1)$ are interpolation couples, and the composition is the usual composition of maps.

A functor $\mathcal{F}$ from the category of interpolation couples into the category of Banach spaces is called an interpolation functor, if the following assertions hold:

(1) $\mathcal{F}(X)$ is an intermediate space for all interpolation couples $X = (X_0, X_1)$,

(2) $\mathcal{F}(T) = T|_{\mathcal{F}(X)}$ for all interpolation couples $X = (X_0, X_1), Y = (Y_0, Y_1)$ and morphisms $T : X \to Y$.

An interpolation functor $\mathcal{F}$ is called exact of exponent $\theta \in [0, 1]$ if

$$\|T\|_{\mathcal{F}(X)\to\mathcal{F}(Y)} \le \|T\|^{1-\theta}_{X_0\to Y_0} \|T\|^{\theta}_{X_1\to Y_1}$$

for all interpolation couples $X = (X_0, X_1), Y = (Y_0, Y_1)$ and morphisms $T : X \to Y$.

We will now turn to a useful tool to determine interpolation spaces of some general interpolation couple by representing it as a retract of some other interpolation couple, for which the interpolation spaces are already known. Let X, Y be objects in a category, then Y is called a *retract* of X if there are morphisms $R : X \to Y$ and $S : Y \to X$ such that $R \circ S = \mathrm{id}_Y$. In this situation R is called a *retraction* and S a corresponding *coretraction*.

Proposition 1.5.2 (cf. [Tr78], 1.2.4 Theorem)**.** *Let $X = (X_0, X_1), Y = (Y_0, Y_1)$ be interpolation couples such that Y is a retract of X in the category of interpolation couples of Banach spaces. Let $R : X \to Y$ be a retraction and $S : Y \to X$ a corresponding coretraction. Let $\mathcal{F}$ be an arbitrary interpolation functor. Then $\mathcal{F}(R) = R|_{\mathcal{F}(X)} : \mathcal{F}(X) \to \mathcal{F}(Y)$ is a retraction with corresponding coretraction $\mathcal{F}(S)$. Moreover, $SR|_{\mathcal{F}(X)}$ is a projection on a complemented subspace E of $\mathcal{F}(X)$ such that $\mathcal{F}(S) : \mathcal{F}(Y) \to E$ is an isomorphism.*

1.5.2 Real interpolation spaces

Let $((X_0, \|\cdot\|_0), (X_1, \|\cdot\|_1))$ be an interpolation couple. For all $t > 0$ and $z \in X_0 + X_1$ define the K-functional

$$K(t, z) := \inf\{\|x_0\|_0 + t\|x_1\|_1 \mid (x_0, x_1) \in X_0 \times X_1, z = x_0 + x_1\}.$$

If $p \in [1, +\infty]$ and $f : (0, \infty) \to E$ is Lebesgue-measurable with values in some Banach space E we let

$$\|f\|_{L^p_*} := \|f\|_{L^p_*(E)} := \begin{cases} \left(\int_0^\infty \|f(t)\|_E^p \frac{dt}{t}\right)^{1/p} & \text{if } p < +\infty, \\ \sup_{t>0} \|f(t)\|_E & \text{if } p = +\infty, \end{cases}$$

and we define the spaces

$$L^p_*(E) := L^p((0, \infty), dt/t, E) := \{f : (0, \infty) \to E \mid f \text{ Lebesgue-measurable}, \|f\|_{L^p_*} < +\infty\}$$

(modulo the usual identification of functions that are equal up to a Lebesgue-Nullset). Moreover, for all $\theta \in (0, 1), p \in [1, +\infty]$ and $x \in X_0 + X_1$ let $\|x\|_{\theta,p} := \|t \mapsto t^{-\theta} K(t, x)\|_{L^p_*}$.

Definition 1.5.3 (Real interpolation spaces)**.** Let $\theta \in (0,1)$ and $p \in [1,+\infty]$. Then the real interpolation space $(X_0, X_1)_{\theta,p}$ is defined as

$$(X_0, X_1)_{\theta,p} := \{x \in X_0 + X_1 \mid \|x\|_{\theta,p} < +\infty\},$$

endowed with the norm $\|\cdot\|_{\theta,p}$.

It can be shown that the real interpolation method $(\cdot,\cdot\cdot)_{\theta,p}$ defines an exact interpolation functor of exponent θ, cf. [BL76], Theorem 3.1.2. In the following proposition we list some elementary properties of real interpolation spaces.

Proposition 1.5.4 (cf. [BL76], Theorem 3.4.1, [Lu09], Proposition 1.1.4, Corollary 1.1.7)**.** *Let $\theta, \theta_j \in (0,1)$ for $j = 0,1$ and $p, q \in [1,+\infty]$.*

(1) $(X_0, X_1)_{\theta,p} = (X_1, X_0)_{1-\theta,q}$ with equal norms.

(2) If $p \le q$, then $(X_0, X_1)_{\theta,p} \hookrightarrow (X_0, X_1)_{\theta,q}$.

(3) If $X_1 \hookrightarrow X_0$ and $\theta_0 < \theta_1$, then $(X_0, X_1)_{\theta_1,p} \hookrightarrow (X_0, X_1)_{\theta_0,q}$.

(4) There is a constant $c(\theta,p)$ such that $\|x\|_{\theta,p} \le c(\theta,p)\|x\|_0^{1-\theta}\|x\|_1^{\theta}$ for all $x \in X_0 \cap X_1$.

We will now turn to the fundamental Reiteration Theorem for the real interpolation method, where we follow the lines of [Lu09], Chapter 1.3. Let (X_0, X_1) be an interpolation couple and E an intermediate space.

Definition 1.5.5 (Classes J_θ, K_θ)**.** Let $\theta \in [0,1]$. E is said to be of class J_θ if there is a constant $c > 0$ such that

$$\|x\|_E \le c\,\|x\|_0^{1-\theta}\,\|x\|_1^{\theta} \text{ for all } x \in X_0 \cap X_1.$$

In this case we write $E \in J_\theta$, or $E \in J_\theta(X_0, X_1)$ if we want to refer explicitly to the underlying interpolation couple.

E is said to be of class K_θ if there is a constant $k > 0$ such that

$$K(t,x) \le k\,t^{\theta}\|x\|_E \text{ for all } x \in E, t > 0.$$

In this case we write $E \in K_\theta$, or $E \in K_\theta(X_0, X_1)$ if we want to refer explicitly to the underlying interpolation couple.

We have the following important characterizations in the case $\theta \in (0,1)$.

Proposition 1.5.6. *Let $\theta \in (0,1)$. Then E is of class J_θ if and only if $(X_0, X_1)_{\theta,1} \hookrightarrow E$, and E is of class K_θ if and only if $E \hookrightarrow (X_0, X_1)_{\theta,\infty}$.*

Theorem 1.5.7 (Reiteration Theorem)**.** *Let $0 \le \theta_0 < \theta_1 \le 1$ and $\delta \in (0,1)$, and let $\theta := (1-\delta)\theta_0 + \delta\theta_1$. Let E_j be intermediate spaces between X_0 and X_1 for $j = 0,1$.*

(1) If $E_j \in J_{\theta_j}(X_0, X_1)$ for $j = 0,1$, then $(X_0, X_1)_{\theta,p} \hookrightarrow (E_0, E_1)_{\delta,p}$ for all $p \in [1,+\infty]$.

(2) If $E_j \in K_{\theta_j}(X_0, X_1)$ for $j = 0, 1$, then $(E_0, E_1)_{\delta,p} \hookrightarrow (X_0, X_1)_{\theta,p}$ for all $p \in [1, +\infty]$.

Consequently, if $E_j \in J_{\theta_j}(X_0, X_1) \cap K_{\theta_j}(X_0, X_1)$ for $j = 0, 1$, then

$$(E_0, E_1)_{\delta,p} = (X_0, X_1)_{\theta,p}$$

with equivalent norms for all $p \in [1, +\infty]$.

At the end of this subsection we consider the interpolation couple $(X, D(A^\alpha))$ for $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, where X is a complex Banach space and $A : X \supseteq D(A) \to X$ is an injective sectorial operator. In this case the K-functional and hence the real interpolation space $(X, D(A^\alpha))_{\theta,p}$ can be described in terms of functions of the operator A. This has first been shown in [Ko67], we will cite a recent version in terms of the functional calculus for sectorial operators given by [Ha06], Theorem 6.5.3 a) and Corollary 6.5.5.

Theorem 1.5.8. *Let A be an injective sectorial operator in a complex Banach space X, $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$, $\sigma \in (\omega(A), \pi)$, $\theta \in (0, 1)$ and $p \in [1, +\infty]$. Let $\varphi \in H^\infty(\Sigma_\sigma)\backslash\{0\}$ such that $z \mapsto z^{-\theta\alpha}\varphi(z) \in H_0^\infty(\Sigma_\sigma)$.*

(1) We have

$$(X, D(A^\alpha))_{\theta,p} = \big\{x \in X \,|\, t \mapsto t^{-\theta\,\mathrm{Re}(\alpha)}\varphi(tA)x \in L_*^p(X)\big\},$$

and an equivalent norm on $(X, D(A^\alpha))_{\theta,p}$ is given by

$$x \mapsto \|x\|_X + \big\|t \mapsto t^{-\theta\,\mathrm{Re}(\alpha)}\varphi(tA)x\big\|_{L_*^p(X)} = \|x\|_X + \left(\int_0^\infty \big\|t^{-\theta\,\mathrm{Re}(\alpha)}\varphi(tA)x\big\|_X^p \frac{dt}{t}\right)^{1/p}.$$

(2) If additionally $A^{-1} \in L(X)$, then also

$$x \mapsto \big\|t \mapsto t^{-\theta\,\mathrm{Re}(\alpha)}\varphi(tA)x\big\|_{L_*^p(X)} = \left(\int_0^\infty \big\|t^{-\theta\,\mathrm{Re}(\alpha)}\varphi(tA)x\big\|_X^p \frac{dt}{t}\right)^{1/p}$$

defines an equivalent norm on $(X, D(A^\alpha))_{\theta,p}$.

1.5.3 Complex interpolation spaces and multilinear Stein interpolation

We will now turn to the complex interpolation method. Beside some standard definitions and properties we will prove a generalized multilinear version of the standard Stein interpolation method for analytic families of operators in complex interpolation spaces. For notations and proofs in this subsection we refer to [BL76] and [Vo92].

Let $X = ((X_0, \|\cdot\|_0), (X_1, \|\cdot\|_1))$ be an interpolation couple of complex Banach spaces. We define the strip

$$S := \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \in [0, 1]\}$$

and the function space

$$\mathcal{F}_0(X) := \{f : S \to X_0 + X_1 \,|\, f \text{ bounded, analytic on } \mathring{S}, f(j + i\cdot) \in C_0(\mathbb{R}, X_j) \text{ for } j = 0, 1\},$$

endowed with norm $\|f\|_{\mathcal{F}} := \sup\{\|f(j + it)\|_j \,|\, t \in \mathbb{R}, j \in \{0, 1\}\}$.

Definition 1.5.9 (Complex interpolation space)**.** Let $\theta \in [0,1]$. The complex interpolation space $[X_0, X_1]_\theta$ is defined as

$$[X_0, X_1]_\theta := \{f(\theta) \mid f \in \mathcal{F}_0(X)\},$$

endowed with the norm $x \mapsto \|x\|_\theta := \inf\{\|f\|_{\mathcal{F}} \mid f \in \mathcal{F}_0(X), f(\theta) = x\}$.

It can be shown that the complex interpolation method $[\cdot, \cdot\cdot]_\theta$ defines an exact interpolation functor of exponent θ, cf. [BL76] Theorem 4.1.2. In the following proposition we list some elementary properties of complex interpolation spaces.

Proposition 1.5.10 (cf. [BL76], Theorems 4.2.1, 4.7.1)**.** *Let $\theta, \theta_j \in [0,1]$ for $j = 0, 1$.*

(1) $[X_0, X_1]_\theta = [X_1, X_0]_{1-\theta}$ with equal norms.

(2) If $\theta \in (0,1)$, then $[X_0, X_0]_\theta = X_0$.

(3) If $X_1 \hookrightarrow X_0$ and $\theta_0 < \theta_1$, then $[X_0, X_1]_{\theta_1} \hookrightarrow [X_0, X_1]_{\theta_0}$.

(4) If $\theta \in (0,1)$, then $(X_0, X_1)_{\theta,1} \hookrightarrow [X_0, X_1]_\theta \hookrightarrow (X_0, X_1)_{\theta,\infty}$, i.e. $[X_0, X_1]_\theta$ is of class $J_\theta \cap K_\theta$.

We cite the following density property from [Vo92], Corollary 1.2: Let $D \leq X_0 \cap X_1$ be a dense subspace and define

$$\widetilde{\mathcal{F}}_0(X, D) := \operatorname{lin}\{z \mapsto e^{\delta z^2}\varphi(z)x \mid \delta > 0, \varphi \in \mathbb{A}(S), x \in D\} \leq \mathcal{F}_0(X),$$

where $\mathbb{A}(S)$ denotes the algebra of bounded continuous functions on S, analytic in $\mathring{S}$. Then

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{F}} \mid f \in \widetilde{\mathcal{F}}_0(X, D), f(\theta) = x\}. \tag{1.5.7}$$

We are now in position to prove the following multilinear version of the abstract interpolation [Vo92], Theorem 2.1, which generalizes classical Stein interpolation to the setting of abstract complex interpolation spaces. In fact, the proof given here is just a combination of the proof of [Vo92], Theorem 2.1 with the proof of [BL76] Theorem 4.4.1 about multilinear interpolation.

Theorem 1.5.11. *Let $m \in \mathbb{N}$ and $X^{(k)}, Y$ be interpolations pairs (of Banach spaces), and let D_k be a dense subspace $X_0^{(k)} \cap X_1^{(k)}$ for each $k \in \mathbb{N}_{\leq m}$. Let $(T(z))_{z \in S}$ be a family of multilinear mappings $T(z) : D \to Y_0 + Y_1$ where $D := \prod_{k=1}^{m} D_k$ with the following properties:*

(i) For all $x \in D$ the function $T(\cdot)x : S \to Y_0 + Y_1$ is continuous and bounded and analytic on $\mathring{S}$,

(ii) For all $x \in D$ and $j \in \{0,1\}$, the function $t \mapsto T(j+it)x \in Y_j$ is continuous, and

$$M_j := \sup\{\|T(j+it)x\|_{Y_j} \mid t \in \mathbb{R}, x \in D \text{ with } \|x_k\|_{X_j^{(k)}} \leq 1 \text{ for each } k \in \mathbb{N}_{\leq m}\} < \infty.$$

Then, for all $\theta \in (0,1)$ we have $T(\theta)D \subseteq [Y_0, Y_1]_\theta$, and

$$\forall\, x \in D \,:\, \|T(\theta)x\|_{[Y_0,Y_1]_\theta} \leq M_0^{1-\theta} M_1^{\theta} \prod_{k=1}^{m} \|x_k\|_{[X_0^{(k)}, X_1^{(k)}]_\theta}. \tag{1.5.8}$$

Proof. Let $\theta \in (0,1)$. W.l.o.g. we may assume $M_0 = M_1 = 1$, otherwise replacing T by $z \mapsto M_0^{z-1} M_1^z T(z)$. If $f_k \in \widetilde{\mathcal{F}}_0(X^{(k)}, D_k)$ for each $k \in \mathbb{N}_{\leq m}$, we clearly have

$$(z \mapsto T(z)(f_1(z), \dots, f_m(z))) \in \mathcal{F}_0(Y),$$

and this immediately implies $T(\theta)D \subseteq [Y_0, Y_1]_\theta$. Now let $x \in D$. If $f_k \in \widetilde{\mathcal{F}}_0(X^{(k)}, D_k)$ with $f_k(\theta) = x_k$ for each $k \in \mathbb{N}_{\leq m}$, we obtain

$$\|T(\theta)x\|_{[Y_0,Y_1]_\theta} \leq \|T(\cdot)(f_1(\cdot), \dots, f_m(\cdot))\|_{\mathcal{F}_0(Y)} \leq \prod_{k=1}^m \|f_k\|_{\mathcal{F}_0(X^{(k)})},$$

and taking the infimum on the right hand side we obtain the required estimate by the density result (1.5.7). □

1.6 Banach function spaces

1.6.1 Definition and elementary properties

For this chapter we refer to the standard references [BS88] Chapter 1 and [Za67] Chapter 15. Let (Ω, μ) be a σ-finite measure space. We fix a *μ-localizing sequence* $(\Omega_n)_{n\in\mathbb{N}}$, i.e. an increasing sequence of μ-measurable subsets such that $\mu(\Omega_n) < +\infty$ for all $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. A μ-measurable subset $M \subseteq \Omega$ will be called *$(\Omega_n)_{n\in\mathbb{N}}$-bounded* if $M \backslash \Omega_n$ is a μ-nullset for some $n \in \mathbb{N}$. We will use the terminology that a property for a μ-measurable function f on Ω holds *$(\Omega_n)_{n\in\mathbb{N}}$-locally* if it holds for $f|_M$ for all $(\Omega_n)_{n\in\mathbb{N}}$-bounded sets M. In particular we introduce the following notation:

If $f_n : \Omega \to \mathbb{K}, n \in \mathbb{N}_0$ are μ-measurable functions, we say that $f_n \to f_0$ *converges $(\Omega_n)_{n\in\mathbb{N}}$-locally in measure* for $n \to \infty$ if $f_n|_M \to f_0|_M$ in measure for $n \to \infty$ for all $(\Omega_n)_{n\in\mathbb{N}}$-bounded sets M, i.e.

$$\mu\big(\{\omega \in M \mid |f_n(\omega) - f_0(\omega)| \geq \varepsilon\}\big) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{for all } (\Omega_n)_{n\in\mathbb{N}}\text{-bounded sets } M, \varepsilon > 0.$$

We denote by $\mathcal{M}^*(\Omega, \mu) := \mathcal{M}^*(\mu)$ the class of μ-measurable extended scalar-valued (real or complex) functions on Ω, by $\mathcal{M}(\Omega, \mu) := \mathcal{M}(\mu)$ the space of μ-measurable scalar-valued functions on Ω, endowed with the topology of $(\Omega_n)_{n\in\mathbb{N}}$-local convergence in measure, and by $\mathcal{M}^+(\Omega, \mu) := \mathcal{M}^+(\mu)$ the cone of μ-measurable functions on Ω with values in $[0, +\infty]$. Furthermore let $M(\Omega, \mu) := M(\mu) := \{[f]_\mu \mid f \in \mathcal{M}(\mu)\}$ denote the corresponding space of equivalence-classes of functions by the equivalence relation given by *pointwise equality up to a μ-nullset*, analogously $M^*(\Omega, \mu) := M^*(\mu)$ and $M^+(\Omega, \mu) := M^+(\mu)$. Moreover we define the spaces

$$\begin{aligned} \mathcal{L}^1_{loc}(\Omega, \mu) &:= \{f \in \mathcal{M}(\mu) \mid \ f|_M \in \mathcal{L}^1(M) \text{ for all } (\Omega_n)_{n\in\mathbb{N}}\text{-bounded sets } M\}, \text{ and} \\ \mathcal{L}^\infty_c(\Omega, \mu) &:= \{f \in \mathcal{L}^\infty(\Omega) \mid \operatorname{supp} f \text{ is } (\Omega_n)_{n\in\mathbb{N}}\text{-bounded}\}, \end{aligned}$$

of locally integrable functions and essentially bounded functions with bounded support, respectively. The corresponding spaces of equivalence-classes modulo the relation of pointwise equality up to a μ-nullset will be notated as $L^1_{loc}(\Omega,\mu)$ and $L^\infty_c(\Omega,\mu)$, respectively. The space $L^1_{loc}(\Omega,\mu)$ will be endowed with the (locally convex) topology of convergence on $(\Omega_n)_{n\in\mathbb{N}}$-bounded sets.

Finally let $S(\Omega,\mu) := \{f \in \mathcal{L}^\infty_c(\Omega) \mid f(\Omega) \text{ is finite}\}$ be the space of step functions with bounded support. All these spaces depend of course in general on the special choice of the underlying μ-localizing sequence $(\Omega_n)_{n\in\mathbb{N}}$. Nevertheless we will suppress the explicit notation of the sequence $(\Omega_n)_{n\in\mathbb{N}}$ in the sequel but keep it in mind.

Definition 1.6.1 (Banach function norm, Banach function space (B.f.s.))**.** A map $\rho : \mathcal{M}^+(\mu) \to [0,+\infty]$ is called a *Banach function norm*, if for all $f, g, f_n \in \mathcal{M}^+(\mu), n \in \mathbb{N}$, all constants $\alpha > 0$ and all μ-measurable subsets M of Ω the following properties hold:

(B1) $\rho(f) = 0 \iff f = 0$ μ-a.e. , $\rho(\alpha f) = \alpha\rho(f)$ and $\rho(f+g) \le \rho(f)+\rho(g)$ (*norm properties*),

(B2) $0 \le g \le f$ μ-a.e. $\Rightarrow \rho(g) \le \rho(f)$ (*monotonicity*),

(B3) $0 \le f_n \nearrow f$ μ-a.e. $\Rightarrow \rho(f_n) \nearrow \rho(f)$ (*Fatou property*),

(B4) M bounded $\Rightarrow \rho(\mathbb{1}_M) < +\infty$,

(B5) M bounded $\Rightarrow \int_M f\,d\mu \le C_M\rho(f)$, where $C_M > 0$ is a constant independent of f.

If $\rho : \mathcal{M}^+(\mu) \to [0,+\infty]$ is a Banach function norm, let $\mathcal{X}(\rho) := \{f \in \mathcal{M}^*(\mu) \mid \rho(|f|) < +\infty\}$ and $X := X(\rho) := \{[f]_\mu \mid f \in \mathcal{X}\}$. Then $[f]_\mu \mapsto \|[f]_\mu\|_X := \rho(|f|)$ defines a norm on X that turns X into a Banach space, and $(X, \|\cdot\|_X)$ is called a *Banach function space*.

It is an immediate consequence of the definition that $X(\rho) \hookrightarrow L^1_{loc}(\Omega,\mu)$ by (B5), and (B3) implies that appropriate versions of the classical Fatou Lemma and monotone convergence theorem hold in X, for a detailed exposition cf. [BS88], Chapter 1.1.

Observe that our definition of a Banach function spaces is a little more general than the one given in [BS88] since conditions (B4), (B5) need only to hold on bounded sets in the sense as discussed at the beginning. In [BS88] conditions (B4), (B5) are formulated for the collections of all μ-measurable sets M of finite measure. Nevertheless, the proofs given in [BS88] also work in our situation, hence we will usually cite [BS88] as our standard reference. In [Za67] a more general notion of Banach function spaces, which are called Köthe spaces there, are considered.

Before going further we cite the most important examples of Banach function spaces in our sense. Observe that in (a)-(c) the μ-localizing sequence $(\Omega_n)_{n\in\mathbb{N}}$ can be chosen arbitrarily and will always lead to the same spaces. In fact, these classes of examples are also covered by the definition in [BS88]. We will see that the situation is different in example (d).

(a) L^p-spaces. Let $p \in [1,+\infty]$, then the usual L^p-space $X = L^p(\Omega,\mu)$ is a Banach function spaces with the Banach function norm $\rho_p(f) := \left(\int_\Omega f^p\,d\mu\right)^{1/p}$ if $p < +\infty$ and $\rho_p(f) :=$

$\operatorname{ess\,sup}_{\omega\in\Omega} f(\omega)$ if $p = +\infty$ for all $f \in \mathcal{M}^+(\mu)$, where ess sup denotes the essential supremum with respect to the measure μ. More generally, let $w : \Omega \to [0, +\infty)$ be a weight function such that $\{w \neq 0\}$ is a μ-nullset, then $\rho_{p;w}(f) := \left(\int_\Omega f^p w \, d\mu\right)^{1/p}$ (with the same identification as above if $p = +\infty$) defines a Banach function norm, and the corresponding Banach function space is the weighted L^p-space $X = L^p(\Omega, w d\mu)$.

(b) Lorentz spaces. For $f \in \mathcal{M}(\Omega, \mu)$ we define the distribution function $d_f(\lambda) := \mu(\{|f| > \lambda\})$ for all $\lambda \geq 0$, the decreasing rearrangement f^* of f by $f^*(t) := \inf\{\lambda \geq 0 \mid d_f(\lambda) \leq t\}$ and the maximal function of f^* as $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)\, ds$ for all $t \geq 0$. Let $p, q \in [1, +\infty]$ and define

$$\rho_q^p(f) := \begin{cases} \left(\displaystyle\int_0^\infty \big(t^{1/p} f^*(t)\big)^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 \leq q \leq p < +\infty, \\ \left(\displaystyle\int_0^\infty \big(t^{1/p} f^{**}(t)\big)^q \frac{dt}{t}\right)^{1/q} & \text{if } 1 < p < q < +\infty, \\ \operatorname{ess\,sup}_{t>0} \big(t^{1/p} f^{**}(t)\big) & \text{if } 1 < p \leq q = +\infty \end{cases}$$

for all $f \in \mathcal{M}^+(\mu)$, then ρ_q^p is a Banach function norm, cf. [BS88] Theorem II.4.3, II.4.6. The corresponding Banach function spaces $L^{p,q}(\Omega, \mu) := X(\rho_q^p)$ are the *Lorentz spaces*.

(c) Orlicz spaces. We use the definition from [BS88], Chapter IV.8. Let $\varphi : [0, +\infty) \to [0, +\infty]$ be increasing and left-continuous with $\varphi(0) = 0$ such that $\varphi(s) \in (0, +\infty)$ for some $s > 0$. Then the function

$$\Phi : [0, +\infty) \to [0, +\infty), t \mapsto \int_0^t \varphi(s)\, ds$$

is said to be a *Young's function*[3]. Define the corresponding *Luxemburg norm*

$$\rho_\Phi(f) := \inf\left\{ c > 0 \ : \ \int_\Omega \Phi\Big(\frac{f(\omega)}{c}\Big)\, d\mu(\omega) \leq 1 \right\}$$

for all $f \in \mathcal{M}^+(\mu)$ then ρ_Φ is a Banach function norm, cf. [BS88] Theorem IV.8.9. The corresponding Banach functions spaces $L^\Phi(\Omega, \mu) := X(\rho_\Phi)$ are the *Orlicz spaces*.

(d) Mixed spaces L^pL^q. Let (J, ν) be another σ-finite measure space with ν-localizing sequence $(J_n)_{n\in\mathbb{N}}$. Let $p, q \in [1, +\infty]$ and define

$$\rho_q^p(f) := \begin{cases} \left(\displaystyle\int_\Omega \left(\int_J f(\omega, t)^q \, d\nu(t)\right)^{p/q} d\mu(\omega)\right)^{1/p} & \text{if } 1 \leq p, q < +\infty, \\ \left(\displaystyle\int_\Omega \Big(\operatorname{ess\,sup}_{t\in J} f(\omega, t)\Big)^p d\mu\right)^{1/p} & \text{if } 1 \leq p < q = +\infty, \\ \operatorname{ess\,sup}_{\omega\in\Omega} \left(\displaystyle\int_J f(\omega, t)^q \, d\nu(t)\right)^{1/q} & \text{if } 1 \leq q < p = +\infty, \\ \operatorname{ess\,sup}_{\omega\in\Omega, t\in J} f(\omega, t) & \text{if } p = q = +\infty \end{cases}$$

[3] Note that in the literature the exact definition of a Young's function might differ in some details from the one given here; we have chosen a definition that is suitable to provide the desired property of a Banach function space.

for all $f \in \mathcal{M}^+(\mu \otimes \nu)$, then $\rho_{p,q}$ is a Banach function norm if we choose $(\Omega_n \times J_n)_{n\in\mathbb{N}}$ as a $\mu\otimes\nu$-localizing sequence. The corresponding Banach function spaces $L^pL^q(\Omega \times J, \mu \otimes \nu) := X(\rho_{p,q})$ are the *mixed $L^p(L^q)$-spaces.*

In contrast to the examples (a)-(c) the mixed spaces do substantially depend on the chosen $\mu \otimes \nu$-localizing sequence that defines the bounded subsets:

Consider the space $L^1L^2(\mathbb{R}\times\mathbb{R})$ endowed with the usual Lebesgue-measure, which we will simply notate as $|\cdot|$. Let $M_1 := \dot{\bigcup}_{n\in\mathbb{N}}[n-1,n) \times [0, 1/n^2)$, then

$$|M_1| = \sum_{n\in\mathbb{N}} |[n-1,n)| \cdot |[0,1/n^2)| = \sum_{n\in\mathbb{N}} \frac{1}{n^2} < +\infty,$$

but

$$\| \mathbb{1}_{M_1} \|_{L^1L^2} = \sum_{n\in\mathbb{N}} \| \mathbb{1}_{[n-1,n)} \|_{L^1} \cdot \| \mathbb{1}_{[0,1/n^2)} \|_{L^2} = \sum_{n\in\mathbb{N}} 1/n = +\infty,$$

hence (B4) would not be satisfied if we choose a localizing sequence that contains M_1.

If X is a Banach function space, the following inclusions hold: $S(\Omega,\mu) \hookrightarrow X \hookrightarrow M(\mu)$, where the second inclusion is continuous ([BS88], Theorem 1.4). This implies e.g. the property, that each convergent sequence $f \in X^{\mathbb{N}}$ contains a subsequence that converges μ-a.e., cf. [BS88], Theorem I.1.7 (vi). One can also show that X is a complete lattice, to be more precise:

Proposition 1.6.2. *Let X be a Banach function space. Then X is a complete sub-lattice of $M^*(\mu)$, and to every $F \subseteq X$ there is a countable subset $F_0 \in F$ such that* $\sup F = \sup F_0$.

This can be proven in the same manner as in the case $X = L^p$ as it is done in [DS58] Cor. IV.11.7, cf. also [Me-Ni91], Lemma 2.6.1 and the following discussion.

We will usually also need the following additional property:

(B6) If $f \in X$ and $(M_n)_{n\in\mathbb{N}}$ is a decreasing sequence of μ-measurable sets with $\mathbb{1}_{M_n} \to 0$ μ-a.e., then $\|f\,\mathbb{1}_{M_n}\|_X \to 0$ for $n \to \infty$ (*absolute continuity*).

A Banach function space that fulfills (B1)-(B6) will be called a Banach function space with *absolute continuous norm.* Observe that (B6) implies that the space of step functions $S(\Omega,\mu)$ as introduced above is dense in X, cf. [BS88], Theorem I.3.11. Moreover, property (B6) is equivalent to the *σ-order-continuity* of the lattice X, i.e.

$$\forall\,(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \,:\, \big(x_n \searrow 0 \text{ for } n\to\infty\big) \Rightarrow \inf_{n\in\mathbb{N}} \|x_n\| = 0,$$

and to the validity of Lebesgue's theorem, compare [BS88] Propositions 3.5,3.6.

In fact, the property (B6) implies also a version of Vitali's convergence theorem for X. For the following we assume that X has absolute continuous norm. We will need the following lemma, which can also be found in [BS88], Lemma I.3.4.

Lemma 1.6.3. *Let $M \subseteq \Omega$ be a bounded set. Then, to each $\varepsilon > 0$ there is a $\delta > 0$ such that for all μ-measurable $A \subseteq M$ we have*

$$\mu(A) < \delta \Rightarrow \| \mathbb{1}_A \|_X < \varepsilon.$$

Proof. Assume that the claim is false, then there is an $\varepsilon > 0$ and a sequence $(A_n)_{n\in\mathbb{N}}$ of μ-measurable subsets of M such that $\mu(A_n) < 2^{-n}$ and $\| \mathbb{1}_{A_n} \|_X \geq \varepsilon$ for all $n \in \mathbb{N}$. Then

$$\int_\Omega \sum_{n=1}^{\infty} \mathbb{1}_{A_n} \, d\mu = \sum_{n=1}^{\infty} \int_\Omega \mathbb{1}_{A_n} \, d\mu = \sum_{n=1}^{\infty} \mu(A_n) < +\infty$$

by the monotone convergence theorem, so we have $\sum\limits_{n=1}^{\infty} \mathbb{1}_{A_n} \in \mathcal{L}^1(\Omega)$ and hence in particular $\mathbb{1}_{A_n} \to 0$ μ-a.e. for $n \to \infty$. Since $x := \mathbb{1}_M \in X$, the absolute continuity leads to the contradiction

$$\varepsilon \leq \| \mathbb{1}_{A_n} \|_X = \|x \cdot \mathbb{1}_{A_n} \|_X \to 0 \quad \text{for } n \to \infty.$$

□

Lemma 1.6.4. *Let $x \in X$.*

(1) $\forall\, \varepsilon > 0 \,\exists\, M$ *μ-mb., bounded*: $\|x \, \mathbb{1}_{\Omega\setminus M} \|_X < \varepsilon$,

(2) $\forall\, \varepsilon > 0 \,\exists\, \delta > 0 \,\forall\, A$ *μ-mb.* : $\mu(A) < \delta \Rightarrow \|x \, \mathbb{1}_A \|_X < \varepsilon$.

Proof. Let $\varepsilon > 0$.

(1) Recall that $\Omega = \bigcup\limits_{n\in\mathbb{N}} \Omega_n$, where Ω_n are μ-measurable sets of finite measure with $\Omega_n \subseteq \Omega_{n+1}$ for all $n \in \mathbb{N}$. Hence $\mathbb{1}_{\Omega\setminus\Omega_n} \to 0$ μ-a.e. for $n \to \infty$ which implies $\|x \, \mathbb{1}_{\Omega\setminus\Omega_n} \|_X \to 0$ for $n \to \infty$, so we can choose $M := \Omega_n$ for a suitable large n.

(2) Choose $M \subseteq \Omega$ according to (1) with $\varepsilon/3$ in place of ε. W.l.o.g we can assume that $x \geq 0$. Let $x_n := x \wedge n$ for all $n \in \mathbb{N}$, then $x_n \to x$ in X for $n \to \infty$ by the Fatou property, hence we can choose $n \in \mathbb{N}$ such that $\|x_n - x\|_X < \varepsilon/3$. According to Lemma 1.6.3 we can now choose $\delta > 0$ such that for all μ-measurable $A \subseteq M$ we have

$$\mu(A) < \delta \Rightarrow \| \mathbb{1}_A \|_X < \frac{\varepsilon}{3n}.$$

Now let $A \subseteq \Omega$ be μ-measurable with $\mu(A) < \delta$, then in particular $\mu(A \cap M) < \delta$, hence

$$\begin{aligned} \|x \, \mathbb{1}_A \|_X &\leq \|x \, \mathbb{1}_{A\cap M} \|_X + \|x \, \mathbb{1}_{A\setminus M} \|_X \leq \|(x - x_n) \, \mathbb{1}_{A\cap M} \|_X + \|x_n \, \mathbb{1}_{A\cap M} \|_X + \|x \, \mathbb{1}_{A\setminus M} \|_X \\ &\leq \|x - x_n\|_X + n \, \| \mathbb{1}_{A\cap M} \|_X + \|x \, \mathbb{1}_{\Omega\setminus M} \|_X < \varepsilon. \end{aligned}$$

□

Proposition 1.6.5. *Let $x, x_n \in X$ for all $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(1) $x_n \to x$ *in* X *for* $n \to \infty$,

(2) *(a)* $x_n \to x$ *locally in measure,*

(b) $\forall\, \varepsilon > 0\, \exists\, M$ *μ-mb., bounded*: $\sup_{n\in\mathbb{N}} \|x_n\, \mathbb{1}_{\Omega\setminus M}\,\|_X < \varepsilon$,

(c) $\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \forall\, A$ *μ-mb.* : $\mu(A) < \delta \Rightarrow \sup\limits_{n\in\mathbb{N}} \|x_n\, \mathbb{1}_A\,\|_X < \varepsilon$.

Proof. (1)⇒(2): Assume (1), then we have already noted that (a) holds. For each μ-mb. $B \subseteq \Omega$ and $n \in \mathbb{N}$ we have

$$\|x_n\, \mathbb{1}_B\,\|_X \le \|(x_n - x)\, \mathbb{1}_B\,\|_X + \|x\, \mathbb{1}_B\,\|_X \le \|x_n - x\|_X + \|x\, \mathbb{1}_B\,\|_X,$$

hence (b) and (c) follow easily with Lemma 1.6.4 applied to x and finitely many of the x_n's for fixed $\varepsilon > 0$.

(2)⇒(1): We assume first that instead of (a) we even have μ-a.e. convergence $x_n \to x$ for $n \to \infty$. Let $\varepsilon > 0$, then choose $M \subseteq \Omega$ and $\delta > 0$ according to (b) and (c) for $\varepsilon/5$ in place of ε. By the Fatou property we than also have

$$\|x\, \mathbb{1}_{\Omega\setminus M}\,\|_X \le \liminf_{n\to\infty} \|x_n\, \mathbb{1}_{\Omega\setminus M}\,\|_X < \varepsilon/5 \quad \text{and} \quad \|x\, \mathbb{1}_A\,\|_X \le \liminf_{n\to\infty} \|x_n\, \mathbb{1}_A\,\|_X < \varepsilon/5 \tag{1.6.9}$$

for all μ-measurable A with $\mu(A) < \delta$. By Egoroff's theorem we can choose a μ-mb. $B \subseteq M$ with $\mu(M\setminus B) < \delta$ such that $(x_n - x)\, \mathbb{1}_B \to 0$ uniformly for $n \to \infty$. Then B is also bounded, hence we can choose $n_0 \in \mathbb{N}$ such that $\|(x_n - x)\, \mathbb{1}_B\,\|_\infty \cdot \|\, \mathbb{1}_B\,\|_X < \varepsilon/5$ for all $n \ge n_0$, hence

$$\begin{aligned}
\|x_n - x\|_X &\le \|(x_n - x)\, \mathbb{1}_B\,\|_X + \|(x_n - x)\, \mathbb{1}_{M\setminus B}\,\|_X + \|(x_n - x)\, \mathbb{1}_{\Omega\setminus M}\,\|_X \\
&\le \|(x_n - x)\, \mathbb{1}_B\,\|_\infty \cdot \|\, \mathbb{1}_B\,\|_X + \|x_n\, \mathbb{1}_{M\setminus B}\,\|_X + \|x\, \mathbb{1}_{M\setminus B}\,\|_X + \|x_n\, \mathbb{1}_{\Omega\setminus M}\,\|_X \\
&\quad + \|x\, \mathbb{1}_{\Omega\setminus M}\,\|_X \\
&< \varepsilon.
\end{aligned}$$

Now we consider the general case where only (a) holds. Let $(y_n)_{n\in\mathbb{N}}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$, then by standard arguments of measure theory we can choose a subsequence $(y_{n(k)})_{k\in\mathbb{N}}$ that converges μ-a.e. to x and has of course also properties (b) and (c), hence $y_{n(k)} \to x$ in X for $k \to \infty$ by what we have already proved. □

Corollary 1.6.6. *Let* $x, y, x_n, y_n \in X$ *for all* $n \in \mathbb{N}$ *such that*

(1) $x_n \to x$ *locally in measure for* $n \to \infty$,

(2) $|x_n| \le y_n$ *for all* $n \in \mathbb{N}$,

(3) $y_n \to y$ *in* X *for* $n \to \infty$.

Then $x_n \to x$ *in* X *for* $n \to \infty$.

Proof. Apply Proposition 1.6.5 (1)⇒(2) for $(y_n)_{n\in\mathbb{N}}$ and y, then the conditions (b) and (c) carry over to $(x_n)_{n\in\mathbb{N}}$ and x, hence 1.6.5 (2)⇒(1) yields the assertion. □

Now let E be a Banach space, where we always assume that E is non-trivial. We define the E-valued extension of the function space X as

$$X(E) := \{F \in M(\Omega, E) \mid |F|_E \in X\},$$

where the modulus is defined as $|F|_E := \|\widetilde{F}(\cdot)\|_E$ for some representative $\widetilde{F}$ of F. Letting $\|F\|_{X(E)} := \|\,|F|_E\,\|_X$ makes $X(E)$ a Banach space (cf. [Ca64]). If in addition X has absolute continuous norm, then the space of step functions $S(\Omega, \mu)$ is dense in X (cf. the remark made after introducing the property (B6)), hence $X \otimes E$ is dense in $X(E)$ in this case.

Assume now that X has absolute continuous norm and E is in addition a Banach function space over some σ-finite measure space (J, ν) with corresponding ν-localizing sequence $(J_n)_{n\in\mathbb{N}}$, and E has also absolute continuous norm. Then the natural embeddings and identification

$$X(E) \hookrightarrow M(\Omega, E) \hookrightarrow M(\Omega, M(J)) \cong M(\Omega \times J)$$

together with the density of the space of step functions show that $X(E)$ can be identified with a subspace $XE \hookrightarrow M(\Omega \times J)$, where for each $x \in M(\Omega \times J)$ the norm is given by $\|x\|_{XE} := \||\widetilde{x}(\cdot)|_E\|_X$ and $\widetilde{x} \in M(\Omega, M(J))$ corresponds to $x \in M(\Omega \times J)$.

In this case we will call XE a *mixed Banach function space*. Observe that properties (B1)-(B3) for XE follow easily from the corresponding properties of X, E, hence XE is a Banach lattice and has the Fatou property. Moreover, the sequence $(\Omega_n \times J_n)_{n\in\mathbb{N}}$ is $\mu \otimes \nu$-localizing, and with respect to this sequence properties (B4),(B5) for XE are an easy consequence of the corresponding properties for X and E. We will assume that the mixed space XE is endowed with the $\mu \otimes \nu$-localizing sequence $(\Omega_n \times J_n)_{n\in\mathbb{N}}$, then XE is again a Banach function space. Finally it is easily shown that the absolute continuity of the norm in both spaces X, E implies that also the mixed Banach function space XE has absolute continuous norm.

Observe that the construction of the space $X(E)$ works for real and complex Banach spaces E and thus makes $X(E)$ to a real or complex Banach space, respectively. In particular, the space $X(\mathbb{C})$ is well-defined. We will call the space $X(\mathbb{C})$ a *complex Banach function space*. We note that this notion of complex Banach function spaces is consistent with the abstract concepts of complexification of real Banach spaces and real Banach lattices as described in [Me-Ni91] Chapter II, §11 or [Sc74] Chapter 2.2. In the sequel we will just say that the space X is a complex Banach function space having in mind that $X = \widetilde{X}(\mathbb{C})$ for some (real) Banach function space $\widetilde{X}$. In this case, properties as (B6) for X are always understood as $\widetilde{X}$ having this property, and $X(E)$ denotes the space $\widetilde{X}(E)$ for any Banach space E.

We will encounter $X(E)$-valued integrable functions, for which we need the following proposition, that is well known for $X = L^p(\Omega, \mu)$, cf. [DS58], Chapter III.11:

Proposition 1.6.7. *Assume that X has absolute continuous norm. Let (J, ν) be a σ-finite measure-space and $F : J \to X(E)$ be an integrable function. Then there exists a $\nu \otimes \mu$-measurable* *function $f : J \times \Omega \to E$ with $[f(t,\cdot)]_\mu = F(t)$ for ν-a.e. $t \in J$ such that $f(\cdot,\omega)$ is integrable for μ-a.e. $\omega \in \Omega$ and the mapping*

$$\omega \mapsto \int_J f(t,\omega)\, d\nu(t)$$

is a representative of $\int_J F(t)\, d\nu(t)$.

We only want to give a sketch of the proof: If $\mu(\Omega) < +\infty$, then we have a continuous embedding $\iota : X(E) \hookrightarrow L^1(\Omega,\mu,E)$, so $\iota \circ F : J \to L^1(\Omega,\mu,E)$ is integrable and we can use the standard theory for this situation, cf. [DS58], Lemma III.11.16. The general case where (Ω,μ) is σ-finite can be reduced to the latter case by choosing a sequence $(M_n)_n \in \mathbb{N}$ of disjoint μ-measurable sets of finite measure and decompose F by the functions $t \mapsto F(t)|_{M_n}$. This procedure can be made precise in exactly the same way as in the proof of Thm. III.11.17 in [DS58], where one simply replaces the spaces $L^p(\Omega,\mu)$ by the general B.f.s X.

In the situation of Proposition 1.6.7 one obviously has that $|f|_E$ has the corresponding property for $|F|_{X(E)}$, hence we obtain a pointwise version of the triangle inequality for the integral:

$$\Big| \int_J F(t)\, d\nu(t) \Big|_E \le \int_J |f(t,\cdot)|_E\, d\nu(t) = \int_J |F(t)|_E\, d\nu(t). \tag{1.6.10}$$

Finally we remark the standard fact, that if $U \subseteq \mathbb{C}$ is open and $F : U \to X(E)$ is analytic, one can choose a version of F with analytic paths, i.e., there is a measurable function $f : U \times \Omega \to E$ such that $f(\cdot,w)$ is analytic for a.e. $w \in \Omega$, and for all $z \in U$ and $k \in \mathbb{N}^0$ we have $[\partial_z^k f(z,\cdot)]_\mu = F^{(k)}(z)$. This result goes back to Stein, cf. [St70], III.2 Lemma, a detailed exposition for $X = L^p$ can be found in [DH02], and the proof given there can easily be modified to work in our situation in the same way as it was already indicated above (i.e. since X locally embeds into L^1). Indeed, in the situation of Proposition 1.6.7 we will usually have analytic functions F, so we can choose the analytic version f such that the claim in Proposition 1.6.7 holds for this version.

1.6.2 Duality in Banach function spaces

We will now give a short summary of duality theory for Banach function spaces as we will use it.

Definition/Proposition 1.6.8 (cf. [BS88], Theorem 2.2)**.** Let $\rho : \mathcal{M}^+(\mu) \to [0,+\infty]$ be a Banach function norm. Then

$$\rho'(g) := \sup\left\{ \int_\Omega fg\, d\mu \;\Big|\; f \in \mathcal{M}^+(\mu), \rho(f) \le 1 \right\} \text{ for all } g \in \mathcal{M}^+(\mu)$$

is called the *associated norm* ρ'. Then ρ' is again a Banach function norm, and the space $X^\# := X(\rho')$ is called the *associated space* of X. In the literature, this space is also often denoted as the *Köthe dual* of X.

Observe that the norm in the associated space $X^\#$ is given by

$$\|x\|_{X^\#} = \sup\left\{ \int_\Omega |fg|\, d\mu \;\Big|\; f \in X, \|f\|_X \le 1 \right\} \text{ for all } g \in X^\#.$$

We have the validity of the following generalization of Hölder's inequality.

Proposition 1.6.9 (Hölder's inequality, cf. [BS88], Theorem 2.4)**.** *Let X be a Banach function space with associated space $X^\#$. If $f \in X$ and $g \in X^\#$, then fg is integrable and*

$$\int_\Omega |fg| \, d\mu \leq \|f\|_X \, \|g\|_{X^\#}.$$

This leads to the following dual representation of the norm in $X^\#$ (cf. [BS88], Lemma 2.8):

$$\|x\|_{X^\#} = \sup \left\{ \left| \int_\Omega fg \, d\mu \right| \; : \; f \in X, \|f\|_X \leq 1 \right\} \text{ for all } g \in X^\#. \tag{1.6.11}$$

Up to now we have only $X^\# \leq X'$ by the canonical isometric embedding $g \mapsto \int_\Omega g \cdot d\mu$, and it can easily be shown that $X^\#$ can be canonically isometrically identified with a norming subspace of X' (cf. [BS88], Theorem 2.9). We will have a closer look on the question when $X^\# = X'$ and the relation to reflexivity of X.

Theorem 1.6.10 (cf. [BS88], Corollaries 4.3 and 4.4)**.** *Let X be a Banach function space with associated space $X^\#$.*

(1) The dual space X' is canonically isometrically isomorphic to $X^\#$ if and only if X has absolute continuous norm.

(2) X is reflexive if and only if both spaces X and $X^\#$ have absolute continuous norm.

So if X is a Banach function space with absolute continuous norm, we will always identify the dual space X' with the associated space $X^\#$, which in turn will also be denoted by X'.

We will now turn to vector-valued Banach function spaces. Let X be a Banach function space with absolute continuous norm and E be a Banach space. Then it is clear that $X'(E') \hookrightarrow (X(E))'$ canonically by the dual pairing

$$\langle F, G \rangle_{X(E),X'(E')} := \int_\Omega \langle F(\omega), G(\omega) \rangle_{E,E'} \, d\mu(\omega),$$

and approximation with step functions shows easily that $X'(E')$ can be canonically isometrically identified with a norming subspace of $(X(E))'$. Recall that the Banach space E is said to have the *Radon-Nikodym property (RNP)* if one, respectively all of the following equivalent conditions hold:[4]

(1) For every finite measure space (J, Σ, ν) and for every ν-continuous vector measure $m : \Sigma \to E$ of bounded variation there exists $G \in L^1(J, \nu, X)$ such that

$$m(A) = \int_A G \, d\mu \quad \text{ for all } A \in \Sigma.$$

[4] Note that (1) is the usual definition as given in [DU77], and the equivalences are shown in [DU77], Chapter IV.3 and Corollary V.3.8.

(2) For every continuous vector measure $m : \mathcal{B}([0,1]) \to E$ of bounded variation there exists $G \in L^1([0,1], X)$ such that

$$m(A) = \int_A G(t)\, dt \quad \text{for all } A \in \mathcal{B}([0,1]),$$

where $\mathcal{B}([0,1])$ is the Borel-σ-algebra and dt the usual Lebesgue-measure on $[0,1]$.

(3) Every function $F : [0,1] \to E$ of bounded variation is differentiable a.e..

(4) Every absolutely continuous function $F : [0,1] \to E$ of bounded variation is differentiable a.e..

We have the following important standard classes of spaces that have (RNP).

Proposition 1.6.11. *Let E be a Banach space, then E has (RNP) if*

(1) E is separable and $E = F'$ for some Banach space F, or

(2) E is reflexive.

For the proof cf. [DU77], Theorem III.3.1 and Corollary III.3.4. Part (1) of Proposition 1.6.11 is also referred to as the Dunford-Pettis Theorem.

We can turn to the central theorem of this subsection.

Theorem 1.6.12 (cf. [GU72] Theorem 3.2, Corollary 3.4)**.** *Let X be a Banach function space with absolute continuous norm and E be a Banach space having (RNP). Then $X'(E')$ is canonically isometrically isomorphic to $(X(E))'$. Moreover, $X(E)$ is reflexive if and only if X and E are both reflexive.*

Observe that the additional requirements for the corresponding Theorem 3.2 in [GU72] are automatically fulfilled in our situation.

1.6.3 p-convexity and q-concavity

We present definitions and some basic results about p-convexity and q-concavity in Banach function spaces. Note that these concepts also make sense in the more general framework of Banach lattices, but we will only present the results in our special situation as we will use them in the sequel.

Definition 1.6.13 (p-convex/q-concave)**.** Let X be a Banach function space and $p, q \in [1, +\infty]$. Then X is called *p-convex* if there is a constant $M > 0$ such that

$$\Big\| \Big(\sum_{j=1}^n |x_j|^p \Big)^{1/p} \Big\|_X \le M \left(\sum_{j=1}^n \|x_j\|_X^p \right)^{1/p} \quad \text{for all } x \in X^n, n \in \mathbb{N} \tag{1.6.12}$$

in the case $p < +\infty$, and

$$\Big\| \sup_{j \in \mathbb{N}_{\leq n}} |x_j| \Big\|_X \leq M \cdot \sup_{j \in \mathbb{N}_{\leq n}} \|x_j\|_X \quad \text{for all } x \in X^n, n \in \mathbb{N} \tag{1.6.13}$$

in the case $p = +\infty$, respectively. If X is p-convex we define $M^{(p)}(X)$ as the infimum over all constants M such that (1.6.12) (or (1.6.13), respectively) holds. Moreover, X is called *q-concave* if there is a constant $M > 0$ such that

$$\Big(\sum_{j=1}^{n} \|x_j\|_X^q \Big)^{1/q} \leq M \Big\| \Big(\sum_{j=1}^{n} |x_j|^q \Big)^{1/q} \Big\|_X \quad \text{for all } x \in X^n, n \in \mathbb{N} \tag{1.6.14}$$

in the case $q < +\infty$, and

$$\sup_{j \in \mathbb{N}_{\leq n}} \|x_j\|_X \leq M \Big\| \sup_{j \in \mathbb{N}_{\leq n}} |x_j| \Big\|_X \quad \text{for all } x \in X^n, n \in \mathbb{N} \tag{1.6.15}$$

in the case $q = +\infty$, respectively. If X is q-concave we define $M_{(q)}(X)$ as the infimum over all constants M such that (1.6.14) (or (1.6.15), respectively) holds.

Let X be an arbitrary Banach function space, $n \in \mathbb{N}$ and $x \in X^n$, then we always have

$$\Big\| \sum_{j=1}^{n} |x_j| \Big\|_X \leq \Big(\sum_{j=1}^{n} \|x_j\|_X \Big)$$

and $|x_k| \leq \sup_{j \in \mathbb{N}_{\leq n}} |x_j|$, hence also $\sup_{k \in \mathbb{N}_{\leq n}} \|x_k\|_X \leq \big\| \sup_{j \in \mathbb{N}_{\leq n}} |x_j| \big\|_X$, so X is always 1-convex and ∞-concave with $M^{(1)}(X) = M_{(\infty)}(X) = 1$.

We have the following reformulation of p-convexity/q-concavity in terms of embeddings for vector-valued Banach function spaces: The Banach function space is p-convex (q-concave) if and only if there is a constant $M > 0$ such that the identity map $i_n : \ell^p_n(X) \to X(\ell^p_n)$ ($j_n : X(\ell^q_n) \to \ell^q_n(X)$) is bounded with $\|i_n\| \leq M$ ($\|j_n\| \leq M$) for all $n \in \mathbb{N}$. Since X has the Fatou property, we obtain for $p \in [1, +\infty], q \in [1, +\infty)$:

X is p-convex if and only if the identity map $\ell^p(X) \to X^{\mathbb{N}}$ induces a bounded map $I_p : \ell^p(X) \to X(\ell^p)$, and in this case $M^{(p)}(X) = \|I_p\|$. In the same manner X is q-concave if and only if the identity map $X(\ell^q) \to X^{\mathbb{N}}$ induces a bounded map $J_q : X(\ell^q) \to \ell^q(X)$, and in this case $M_{(q)}(X) = \|J_q\|$.

The above considerations can be found in [LT96] Definition II.1.d.3 and the following discussions.

The properties p-convex and q-concave are dual in the following sense, which is a special case of [LT96], Proposition II.1.d.4:

Proposition 1.6.14. *Let X be a Banach function space with absolute continuous norm and $p, q \in [1, +\infty]$ with corresponding dual exponents $p', q' \in [1, +\infty]$, i.e. $\frac{1}{r} + \frac{1}{r'} = 1$ for $r \in \{p, q\}$. Then*

(1) X is p-convex if and only if X' is p'-convex, and in this case $M^{(p)}(X) = M_{(p')}(X')$,

(2) X is q-concave if and only if X' is q'-convex, and in this case $M_{(q)}(X) = M^{(q')}(X')$.

Moreover we have the following important extension property, which one obtains from [LT96], Theorem II.1.f.7 and its preceding remarks:

Proposition 1.6.15. *Let X be a Banach function space and $p, q \in [1, +\infty]$.*

(1) If X is p-convex, then X is also s-convex for all $s \in [1, p]$,

(2) If X is q-concave, then X is also s-concave for all $s \in [q, +\infty]$.

Proposition 1.6.16. *Let X be a Banach function space, and assume that X is p-convex and q-concave for some $p, q \in (1, +\infty)$. Then X is reflexive.*

Proof. By Proposition 1.6.15 we can assume that $p \le 2 \le q$. Thus Theorem II.1.f.1 from [LT96] implies that X can be renormed equivalently so that X, endowed with the new norm, is uniformly convex, hence it is also reflexive, cf. e.g. [LT96] Theorem II.1.e.3. This implies that also X endowed with the original norm is reflexive. □

Let us now have a look at some examples.

Examples 1.6.17. Let $(\Omega, \mu), (J, \nu)$ be σ-finite measure spaces and $p, q \in [1, +\infty]$.

(a) The space $L^p(\Omega, \mu)$ is r-convex for all $r \in [1, p]$ and s-concave for all $s \in [q, +\infty]$.

(b) More general let $X := L^p L^q(\Omega \times J, \mu \otimes \nu)$. Then X is r-convex for all $r \in [1, p \wedge q]$ and s-concave for all $s \in [p \vee q, +\infty]$.

Example (a) is a trivial consequence of Proposition 1.6.15, and (b) is a special case of the following general result.

Proposition 1.6.18. *Let X, E be Banach function spaces and $p_0, p_1, q_0, q_1 \in [1, +\infty]$.*

(1) If X is p_0-convex and E is p_1-convex, then the mixed space XE is r-convex for all $r \in [1, p_0 \wedge p_1]$,

(2) If X is q_0-concave and E is q_1-concave, then the mixed space XE is s-concave for all $s \in [q_0 \vee q_1, +\infty]$.

Proof. (1) Let $r \in [1, p_0 \wedge p_1]$. By Proposition 1.6.15 both spaces X, E are r-convex, hence

$$\begin{aligned} \Big\| \Big(\sum_{j=1}^n |x_j|^r \Big)^{1/r} \Big\|_{XE} &= \Big\| \Big| \Big(\sum_{j=1}^n |x_j|^r \Big)^{1/r} \Big|_E \Big\|_X \le M^{(r)}(E) \Big\| \Big(\sum_{j=1}^n |x_j|_E^r \Big)^{1/r} \Big\|_X \\ &\le M^{(r)}(E) M^{(r)}(X) \Big(\sum_{j=1}^n \| |x_j|_E \|_X^r \Big)^{1/r} \\ &= M^{(r)}(E) M^{(r)}(X) \Big(\sum_{j=1}^n \|x_j\|_{XE}^r \Big)^{1/r} \end{aligned}$$

for all $x \in XE^n, n \in \mathbb{N}$ (with the usual modification if $r = +\infty$). The claim (2) can be proved in the same way. □

Finally we want to present a generalization of the Chintschin inequality in q-concave Banach function space. For this we first recall the classical Chintschin inequality and the Kahane inequality.

Let $(r_j)_{j\in\mathbb{N}}$ be a Rademacher sequence, cf. Section 1.3. Recall that for any Banach space E and $p \in [1, +\infty)$ the expressions

$$\Big(\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes x_j\Big|_E^p\Big)^{1/p} = \Big(\frac{1}{2^n}\sum_{\sigma\in\{-1,1\}^n}\Big\|\sum_{j=1}^{n}\sigma_j x_j\Big\|_E^p\Big)^{1/p} \tag{1.6.16}$$

for $x \in E^n, n \in \mathbb{N}$ do not depend on the special choice of the Rademacher-sequence $(r_j)_{j\in\mathbb{N}}$.

We will now turn to two classical inequalities for norm-expressions like (1.6.16).

Theorem 1.6.19 (Chintschin inequality, cf. [LT96] Theorem I.2.b.3)**.** *Let $p \in [1, +\infty)$. Then there are constants $A_p, B_p > 0$ such that*

$$A_p\Big(\sum_{j=1}^{n}|\alpha_j|^2\Big)^{1/2} \le \Big(\mathbb{E}\Big|\sum_{j=1}^{n}\alpha_j r_j\Big|^p\Big)^{1/p} \le B_p\Big(\sum_{j=1}^{n}|\alpha_j|^2\Big)^{1/2} \tag{1.6.17}$$

for all $(\alpha_j)_{j\in\mathbb{N}_{\le n}} \in \mathbb{C}^n$, $n \in \mathbb{N}$.

Theorem 1.6.20 (Kahane inequality, cf. [LT96] Theorem II.1.e.13)**.** *Let $p \in [1, +\infty)$. Then there is a constant $K_p > 0$ such that for any Banach space X the following inequality holds:*

$$\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes x_j\Big|_X \le \Big(\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes x_j\Big|_X^p\Big)^{1/p} \le K_p\,\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes x_j\Big|_X \tag{1.6.18}$$

for all $(x_j)_{j\in\mathbb{N}_{\le n}} \in E^n$, $n \in \mathbb{N}$.

This leads to the announced generalizations of the Chintschin inequality in q-concave Banach function spaces:

Proposition 1.6.21. *Let X be a Banach function space and $p \in [1, +\infty)$.*

(1) The following estimate holds for all $(x_j)_{j\in\mathbb{N}_{\le n}} \in X^n$, $n \in \mathbb{N}$:

$$A_1\Big\|\Big(\sum_{j=1}^{n}|x_j|^2\Big)^{1/2}\Big\|_X \le \Big(\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes x_j\Big|_X^p\Big)^{1/p}.$$

(2) Assume that X is in addition q-concave for some $q \in [1, +\infty)$, then

$$A_1\Big\|\Big(\sum_{j=1}^{n}|x_j|^2\Big)^{1/2}\Big\|_X \le \Big(\mathbb{E}\Big|\sum_{j=1}^{n} r_j \otimes x_j\Big|_X^p\Big)^{1/p} \le C_{p,q}\Big\|\Big(\sum_{j=1}^{n}|x_j|^2\Big)^{1/2}\Big\|_X$$

for all $(x_j)_{j\in\mathbb{N}_{\le n}} \in X^n$, $n \in \mathbb{N}$ with $C_{p,q} := K_p B_q M_{(q)}(X)$.

Proposition 1.6.21 is an easy consequence of the classical Chintschin and Kahane inequality, cf. [LT96], Theorem II.1.d.6 (i).

Finally we mention the following important property of q-concave Banach function spaces, which is a consequence of Proposition 1.6.21:

Proposition 1.6.22. *Let X be a Banach function space that is q-concave for some $q \in [1, +\infty)$, then X has property (α).*

1.6.4 Some approximation results

We recall the following well known fact.

Lemma 1.6.23. *Let E, F be Banach spaces, (J, μ) a σ-finite measure space, $S : J \to L(E, F)$ strongly measurable and $x : J \to E$ measurable. Then $t \mapsto S(t)x(t)$ defines a measurable mapping $S \cdot x : J \to F$.*

Proof. Since $x : J \to E$ is measurable, we can find a sequence of step function $(s_n)_{n\in\mathbb{N}}$ in $S(J) \otimes E$ such that $s_n(t) \to x(t)$ for ν-a.e. $t \in J$. For each $n \in \mathbb{N}$ we obtain a representation $s_n = \sum_{k=1}^{r_n} \mathbb{1}_{A_k} \otimes v_k$. Let $y_n(t) := S(t)s_n(t) = \sum_{k=1}^{r_n} \mathbb{1}_{A_k}(t)(S(t)v_k)$, then y_n is measurable since the mappings $S(\cdot)v_k$ are measurable, and since $S(t) \in L(E, F)$ we obtain $y_n(t) \to S(t)x(t)$ for a.e. $t \in J$ using the uniform boundedness principle, hence $S \cdot x$ is measurable as pointwise a.e. limit of measurable functions. □

We will often deal with convergent sequences of X-valued measurable functions. Then pointwise, the convergence in X yields a subsequence that converges μ-a.e.. We will need a refinement which states that the choice of the subsequence and nullsets can be done uniformly for the whole function.

Lemma 1.6.24. *Let (J, ν) be a σ-finite measure space and $x, x_n : J \to X$ for all $n \in \mathbb{N}$ be ν-measurable functions with $x_n(t) \to x(t)$ for ν-a.e. $t \in J$. Then one can choose a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ and a $(\nu \otimes \mu)$-nullset $N \subseteq J \times \Omega$ such that*

$$\forall\, (t, \omega) \in (J \times \Omega)\backslash N \,:\, x_{n_k}(t)(\omega) \to x(t)(\omega) \quad \textit{for } k \to \infty,$$

where $(t, \omega) \mapsto x(t)(\omega), x_n(t)(\omega)$ are any $(\nu \otimes \mu)$-measurable representatives, i.e. $x_{n_k}(\cdot)(\cdot\cdot) \to x(\cdot)(\cdot\cdot)$ converges $\lambda \otimes \mu$-a.e. for $n \to \infty$.

Proof. Let $y_n := x_n - x$ for all $n \in \mathbb{N}$. Write $\Omega = \bigcup_{m\in\mathbb{N}} \Omega_m$, where $(\Omega_m)_{m\in\mathbb{N}}$ is a corresponding μ-localizing sequence. By Egoroff's Theorem, we can find an increasing sequence $(J_m)_{m\in\mathbb{N}}$ of measurable subsets of J with finite measure, such that $N_J := J\backslash \bigcup_{m\in\mathbb{N}} J_m$ is a nullset and $y_n|_{J_m} \to 0$ uniformly, i.e. in the space $L^\infty(J_m, X)$. Hence, we can choose a subsequence such that $\sup_{t\in J_k} \|y_{n_k}(t)\|_X \le 2^{-k}$ for all $k \in \mathbb{N}$. Now fix $m \in \mathbb{N}$. Since the J_k are increasing, we obtain

$$\forall\, k \ge m \,:\, \sup_{t\in J_m} \|y_{n_k}(t)\|_X \le 2^{-k}.$$

Since $j_m : X|_{\Omega_m} \hookrightarrow L^1(\Omega_m)$ is bounded, we have

$$\begin{aligned}\int_{J_m}\int_{\Omega_m}\sum_{k=1}^N |y_{n_k}(t)|\,d\mu\,dt &= \sum_{k=1}^N \int_{J_m}\int_{\Omega_m} |y_{n_k}(t)|\,d\mu\,dt \le \sum_{k=1}^N |J_m| \sup_{t\in J_m} \left\| y_{n_k}(t)|_{\Omega_m}\right\|_{L^1(\Omega_m)} \\ &\le |J_m| \sum_{k=1}^N \sup_{t\in J_m} \|j_m\|\,\|y_{n_k}(t)\|_X \\ &\le |J_m|\,\|j_m\| \Big(\sum_{k=1}^{m-1} \sup_{t\in J_m} \|y_{n_k}(t)\|_X + \sum_{k=m}^{\infty} 2^{-k}\Big) =: C_m.\end{aligned}$$

Since $y_n|_{J_m} \to 0$ uniformly we have in particular $\sup_{n\in\mathbb{N}}\sup_{t\in J_m}\|y_n(t)\|_X < +\infty$, hence $C_m < +\infty$. By Beppo-Levi we have $\displaystyle\int_{J_m\times\Omega_m}\sum_{k=1}^{\infty}|y_{n_k}(t)|\,d\mu\,dt < +\infty$. In particular there is a nullset $N_m \subseteq J_m \times \Omega_m$ such that for all $(t,\omega) \in (J_m\times\Omega_m)\backslash N_m$ we have $\displaystyle\sum_{k=1}^{\infty}|y_{n_k}(t)(\omega)| < +\infty$, hence $y_{n_k}(t)(\omega) \to 0$ for $n \to \infty$.

So $N := (N_J \times \Omega) \cup \displaystyle\bigcup_{m\in\mathbb{N}} N_m$ is a nullset in $J \times \Omega$ with the desired property. □

Furthermore we will need the following approximation property:

Lemma 1.6.25. *Let E be a Banach function space over the σ-finite measure space (J,ν) and assume that X, E have absolute continuous norm. Let $x : J \to X$ be measurable with $\|x\|_{X(E)} < \infty$ via the usual identification $x \in \mathcal{M}(J,\mathcal{M}(\Omega)) \cong \mathcal{M}(J\times\Omega) \cong \mathcal{M}(\Omega,\mathcal{M}(J))$. Then one can find a sequence $(x_n)_{n\in\mathbb{N}}$ in $S(J,\nu)\otimes S(\Omega,\mu)$ with the following properties:*

1. *$x_n \to x$ pointwise $(\nu\otimes\mu)$-a.e. on $J\times\Omega$ for $n\to\infty$,*
2. *$x_n(t) \to x(t)$ in X for $n\to\infty$ for μ-a.e. $t \in J$,*
3. *$\liminf_{n\to\infty}\|x_n\|_{X(E)} \le \|x\|_{X(E)}$.*

Proof. We can find sequences $(s_n)_{n\in\mathbb{N}}, (s'_n)_{n\in\mathbb{N}} \in (S(J,\nu)\otimes S(\Omega,\mu))^{\mathbb{N}}$ such that $s_n(t) \to x(t)$ in X for a.e. $t \in J$ and $s'_n \to x$ in $X(E)$. By choosing a subsequence we can also assume w.l.o.g. $s'_n(\omega) \to x(\omega)$ in E for a.e. $\omega \in \Omega$. We now apply Lemma 1.6.24 to both sequences, and by possibly restricting again to suitable subsequences we can assume w.l.o.g. that also $s_n, s'_n \to x$ pointwise $(\nu\otimes\mu)$-a.e. on $J\times\Omega$.

Let $n \in \mathbb{N}$. Then we can find a common partition of rectangular sets

$$R_{n,j} \in \{I \times B \,|\, I \subseteq J, B \subseteq \Omega \text{ measurable and bounded}\},\ j = 1,\dots,r_n, n \in \mathbb{N}$$

for s_n, s'_n such that $s_n|_{R_{n,j}}$ und $s'_n|_{R_{n,j}}$ are constant. Now we write

$$\begin{aligned} s_n &= \operatorname{Re}(s_n)^+ - \operatorname{Re}(s_n)^- + i\operatorname{Im}(s_n)^+ - i\operatorname{Im}(s_n)^-, \quad \text{and} \\ s'_n &= \operatorname{Re}(s'_n)^+ - \operatorname{Re}(s'_n)^- + i\operatorname{Im}(s'_n)^+ - i\operatorname{Im}(s'_n)^- \end{aligned}$$

and define x_n by $\mathrm{Re}(x_n)^+ := \mathrm{Re}(s_n)^+ \wedge \mathrm{Re}(s_n')^+$, $\mathrm{Re}(x_n)^- := \mathrm{Re}(s_n)^- \wedge \mathrm{Re}(s_n')^-$, $\mathrm{Im}(x_n)^+ := \mathrm{Im}(s_n)^+ \wedge \mathrm{Im}(s_n')^+$, $\mathrm{Im}(x_n)^- := \mathrm{Im}(s_n)^- \wedge \mathrm{Im}(s_n')^-$.

Then each x_n is constant on the $R_{n,j}, j = 1, \ldots, r_n$, hence $x_n \in S(J,\nu) \otimes S(\Omega,\mu)$, and we obviously obtain the following properties:

1. $x_n \to x$ pointwise $(\nu \otimes \mu)$-a.e. on $J \times \Omega$ for $n \to \infty$,
2. $|x_n|_{\mathbb{C}} \leq |s_n|_{\mathbb{C}} \wedge |s_n'|_{\mathbb{C}}$ $(\nu \otimes \mu)$-a.e. on $J \times \Omega$.

In particular, we have $\|x_n(t)\|_X \leq \|s_n(t)\|_X$ for a.e. $t \in J$, hence $x_n(t) \to x(t)$ in X for $n \to \infty$ by Corollary 1.6.6. Moreover, the Fatou property of $X(E)$ yields

$$\liminf_{n\to\infty} \|x_n\|_{X(E))} \leq \| \liminf_{n\to\infty} |x_n|_{\mathbb{C}} \|_{X(E)} \leq \| \liminf_{n\to\infty} |s_n'|_{\mathbb{C}} \|_{X(E)} = \|x\|_{X(E)}.$$

□

1.6.5 H^∞-calculus in q-concave Banach function spaces

Let X be a Banach function spaces with absolute continuous norm such that X is q-concave for some $q < +\infty$, and let A be a sectorial operator in X. Then it is well known (at least in the case $X = L^p$) that A has a bounded H^∞-calculus if and only if A is $\mathcal{R}$-sectorial and satisfies suitable *square-function estimates*, which in a general Banach space have to be replaced by corresponding Rademacher-norms. For the applications in this work we will only need one implication, hence we will only cite this in detail.

Let $\sigma \in (0, \pi)$ and assume that the operator A has a bounded $H^\infty(\Sigma_\sigma)$-calculus. Observe that A is $\mathcal{R}$-sectorial by Corollary 1.3.6 since X has property (α) by Proposition 1.6.22. Then, for all $\sigma' > \sigma$ and $\varphi \in H_0^\infty(\Sigma_{\sigma'})\backslash\{0\}$ there is a constant $C > 0$ such that for all $x \in X$:

$$\frac{1}{C} \|x\|_X \leq \left\| \Big(\int_0^\infty |\varphi(tA)x|^2 \, \frac{dt}{t} \Big)^{1/2} \right\|_X \leq C \|x\|_X.$$

In the case $X = L^p$ this has been proven in [CDMY96], and in a more general context this can be found in [KKW06], [KW-1] and [KW-2]. To be more concrete, again a careful inspection of the proofs in the cited literature shows the following: For all $\omega > \sigma$ there is a constant $C_{\omega,\sigma} > 0$ independent of A such that

$$\forall\, x \in X \;:\; \left\| \Big(\int_0^\infty |\varphi(tA)x|^2 \, \frac{dt}{t} \Big)^{1/2} \right\|_X \leq C_{\omega,\sigma} \cdot M_\sigma^\infty(A) \cdot \|x\|_X.$$

1.7 Classical function spaces: Besov- and Triebel-Lizorkin spaces

We give a short description of the classical Besov-spaces $B^s_{p,q}$ and Triebel-Lizorkin spaces $F^s_{p,q}$ in terms of Littlewood-Paley-decompositions. Moreover we present equivalent descriptions involving the heat semigroup, which can be reformulated as representations of the norms in terms of the functional calculus for the Laplacian. For this section we refer to the standard literature

[Tr83], [Tr92] and [Tr78], and moreover to [BL76] for Besov spaces and to [FJW91] for additional material on the homogeneous spaces.

We fix $d \in \mathbb{N}$ in this section. If $\psi \in \mathcal{S}_d$ we use the standard notation $\psi(D)u := \mathcal{F}^{-1}(\psi \cdot \widehat{u}) = \check{\psi} * u$ for all $u \in \mathcal{S}'_d$.

Let $\psi \in \mathcal{S}_d$ be a Schwartz function having the following properties:

(1) $\operatorname{supp}(\psi) = \{\xi \in \mathbb{R}^d \mid 1/2 \leq |\xi| \leq 2\}$ and $\psi(\xi) > 0$ for all $\xi \in \mathbb{R}^d$ with $1/2 < |\xi| < 2$,

(2) $\displaystyle\sum_{j=-\infty}^{\infty} \psi(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^d \backslash \{0\}$.

We then define functions $\psi_j, \Psi \in \mathcal{S}_d$ for all $j \in \mathbb{Z}$ as

$$\psi_j(\xi) := \psi(2^{-j}\xi) \quad \text{and } \Psi(\xi) = 1 - \sum_{j=1}^{\infty} \psi_j(\xi) \text{ for all } \xi \in \mathbb{R}^d. \tag{1.7.19}$$

Let $s \in \mathbb{R}$ and $p, q \in (0, +\infty]$, then for each $u \in \mathcal{S}'_d$ we define

$$\begin{aligned}
\|u\|_{B^s_{p,q}} &:= \|\Psi(D)u\|_p + \|(2^{sj}\psi_j(D)u)_{j\in\mathbb{N}}\|_{\ell^q(L^p)} = \|\Psi(D)u\|_p + \Big(\sum_{j=1}^{\infty} \|2^{sj}\psi_j(D)u\|_p^q\Big)^{1/q}, \\
\|u\|_{F^s_{p,q}} &:= \|\Psi(D)u\|_p + \|(2^{sj}\psi_j(D)u)_{j\in\mathbb{N}}\|_{L^p(\ell^q)} = \|\Psi(D)u\|_p + \Big\|\Big(\sum_{j=1}^{\infty} |2^{sj}\psi_j(D)u|^q\Big)^{1/q}\Big\|_p, \\
\|u\|_{\dot{B}^s_{p,q}} &:= \|(2^{sj}\psi_j(D)u)_{j\in\mathbb{Z}}\|_{\ell^q(L^p)} = \Big(\sum_{j=-\infty}^{\infty} \|2^{sj}\psi_j(D)u\|_p^q\Big)^{1/q}, \\
\|u\|_{\dot{F}^s_{p,q}} &:= \|(2^{sj}\psi_j(D)u)_{j\in\mathbb{Z}}\|_{\ell^p(L^q)} = \Big\|\Big(\sum_{j=-\infty}^{\infty} |2^{sj}\psi_j(D)u|^q\Big)^{1/q}\Big\|_p
\end{aligned}$$

(with the usual modification if $p = +\infty$ or $q = +\infty$).

Definition 1.7.1 (The spaces $B^s_{p,q}$ and $F^s_{p,q}$)**.** Let $s \in \mathbb{R}$ and $p, q \in (0, +\infty]$, then we define the *inhomogeneous Besov space*

$$B^s_{p,q} := B^s_{p,q}(\mathbb{R}^d) := \big\{u \in \mathcal{S}'_d \mid \|u\|_{B^s_{p,q}} < +\infty\big\},$$

and in the case $p < +\infty$ we define the *homogeneous Triebel-Lizorkin space*

$$F^s_{p,q} := F^s_{p,q}(\mathbb{R}^d) := \big\{u \in \mathcal{S}'_d \mid \|u\|_{F^s_{p,q}} < +\infty\big\}.$$

It can be shown that the spaces $B^s_{p,q}, F^s_{p,q}$ are quasi-Banach spaces, and Banach spaces in the case $p, q \geq 1$, that do not depend on $\psi \in \mathcal{S}_d$ with (1), (2), and varying ψ with these properties leads to equivalent norms. Moreover, $\mathcal{S}_d \hookrightarrow B^s_{p,q} \hookrightarrow \mathcal{S}'_d$ and $\mathcal{S}_d \hookrightarrow F^s_{p,q} \hookrightarrow \mathcal{S}'_d$, and if $p, q < +\infty$, then $\mathcal{S}_d$ is dense in $B^s_{p,q}$ and $F^s_{p,q}$.

We will also introduce homogeneous variants of these spaces. For this let $\mathcal{P}_d$ be the set of all polynomial functions on $\mathbb{R}^d$ and

$$\begin{aligned} \mathcal{Z}_d :=^{\perp} \mathcal{P}_d &= \Big\{ u \in \mathcal{S}_d \, : \, \langle u, x^\alpha \rangle_{\mathcal{S}_d \times \mathcal{S}_d'} = 0 \text{ for all } \alpha \in \bigcup_{n\in\mathbb{N}} \mathbb{N}_0^n \Big\} \\ &= \Big\{ u \in \mathcal{S}_d \, : \, \partial^\alpha \widehat{u}(0) = 0 \text{ for all } \alpha \in \bigcup_{n\in\mathbb{N}} \mathbb{N}_0^n \Big\}, \end{aligned}$$

where x^α denotes the mapping $\xi \mapsto \xi^\alpha$ considered as an element of $\mathcal{S}_d'$. Then we have a canonical isomorphism $\mathcal{Z}_d' \cong \mathcal{S}_d'/\mathcal{P}_d$, i.e. $\mathcal{Z}_d'$ is the space of tempered distributions modulo polynomial functions. Many operations on $\mathcal{S}_d'$ can be transferred to $\mathcal{Z}_d'$, in particular if $\psi \in \mathcal{S}_d$, then it can be shown that $\psi(D)u := \check{\psi} * u$ is well-defined for $u \in \mathcal{Z}_d'$. Now the homogeneous spaces are defined as subspaces of $\mathcal{Z}_d'$ in the following way:

Definition 1.7.2 (The spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$)**.** Let $s \in \mathbb{R}$ and $p, q \in (0, +\infty]$, then we define the *homogeneous Besov space*

$$\dot{B}^s_{p,q} := \dot{B}^s_{p,q}(\mathbb{R}^d) := \big\{ u \in \mathcal{Z}_d' \mid \|u\|_{\dot{B}^s_{p,q}} < +\infty \big\},$$

and in the case $p < +\infty$ we define the *inhomogeneous Triebel-Lizorkin space*

$$\dot{F}^s_{p,q} := \dot{F}^s_{p,q}(\mathbb{R}^d) := \big\{ u \in \mathcal{Z}_d' \mid \|u\|_{\dot{F}^s_{p,q}} < +\infty \big\}.$$

Again it can be shown that the spaces $\dot{B}^s_{p,q}, \dot{F}^s_{p,q}$ are quasi-Banach spaces, and Banach spaces in the case $p, q \geq 1$, that do not depend on $\psi \in \mathcal{S}_d$ with (1), (2), and varying ψ with these properties leads to equivalent norms. Moreover, $\mathcal{Z}_d \hookrightarrow \dot{B}^s_{p,q} \hookrightarrow \mathcal{Z}_d'$ and $\mathcal{Z}_d \hookrightarrow \dot{F}^s_{p,q} \hookrightarrow \mathcal{Z}_d'$, , and if $p, q < +\infty$, then $\mathcal{Z}_d$ is dense in $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$, cf. [Tr83], Section 5.1.3.

These function spaces have been extensively investigated in the past, and there is an exhausting theory containing e.g. embedding and interpolation properties or multiplier theorems. We will not list all these assertions at this place but just refer to the standard literature cited above.

Note that various classes of classical function spaces appear in the scale of Besov and Triebel-Lizorkin spaces, we just give a short overview, details can be found e.g. in [Tr92], Chapter 1,2 and [Gr04], Chapter 6.

$\dot{F}^0_{p,2} = F^0_{p,2} = L^p$	$1 < p < \infty$	Lebesgue spaces
$F^m_{p,2} = W^{m,p}$	$1 < p < \infty, m \in \mathbb{N}_0$	Sobolev spaces
$F^s_{p,2} = H^{s,p}$	$1 < p < \infty, s \in \mathbb{R}$	Bessel potential or fractional Sobolev spaces
$\dot{F}^s_{p,2} = \dot{H}^{s,p}$	$1 < p < \infty, s \in \mathbb{R}$	Riesz potential spaces
$\dot{F}^0_{p,2} = H^p$	$0 < p < \infty$	Hardy spaces
$F^0_{p,2} = h^p$	$0 < p < \infty$	non-homogeneous or local Hardy spaces
$B^s_{p,p} = W^{s,p}$	$1 \leq p < \infty, s > 0, s \notin \mathbb{N}$	Sobolev-Slobodeckij spaces
$B^s_{p,q} = \Lambda^s_{p,q}$	$1 \leq p, q < \infty, q \neq \infty, s > 0$	classical Besov spaces
$B^s_{\infty,\infty} = C^s$	$s > 0, s \notin \mathbb{N}$	Hölder spaces
$B^s_{\infty,\infty} = \mathcal{C}^s$	$s > 0$	Zygmund spaces

Note that the case "$p = +\infty$" has been excluded in the definition of the Triebel-Lizorkin spaces. The reason is that in the case $p = +\infty$ the definition given above is not the "right one", e.g. it would lead to a norm depending on the auxiliary function φ. Nevertheless it is possible to define the spaces $F^s_{\infty,q}, \dot{F}^s_{\infty,q}$ if $1 < q < \infty$ by a modification of these definitions. In this case, one would obtain the additional identification $\dot{F}^0_{\infty,2} = BMO$, where BMO is the space of functions of Bounded Mean Oscillation. We refer to [Tr83], Sections 2.3.1, 2.3.4, and the additional literature cited there for this case.

For all the spaces $B^s_{p,q}, \dot{B}^s_{p,q}, F^s_{p,q}\dot{F}^s_{p,q}$ there a equivalence theorems for the norms that give rise to a much larger class of functions ψ that may be considered on the one hand, and give continuous counterparts of the norms on the other hand. A general treatment can be found in [Tr92], Chapter 2, and for the homogeneous norms in [Tr83], Section 5.2.3 and more detailed in [Tr82]. We will not present the most general known results here but just cite an equivalent norm expression in terms of the heat semigroup and hence in terms of the analytic functional calculus for the Laplacian.

For this purpose we fix $s \in \mathbb{R}$ and $p, q \in (0, +\infty]$, and choose some $m \in \mathbb{N}_{>s/2}$. Let $A := -\Delta$ be the negative of the Laplace operator in L^p, and denote by $(e^{-tA})_{t\geq 0}$ the heat semigroup. Let $\varphi(z) := z^m e^{-z}$, then the analytic functional calculus yields for all $t > 0$ and $u \in \mathcal{S}'$

$$\varphi(tA)u = (tA)^m e^{-tA}u = (-t)^m \Big(\frac{d}{dt}\Big)^m T(t)u. \tag{1.7.20}$$

With these notation we obtain the following reformulation of the characterizations theorems [Tr82], Corollaries 3.3, 3.4:

Theorem 1.7.3. *(1) The mapping* $u \mapsto \Big(\int_0^\infty \big\|t^{-s/2}\varphi(tA)u\big\|_p^q \frac{dt}{t}\Big)^{1/q}$ *(with the usual modification if $q = +\infty$) defines an equivalent quasi-norm on $\dot{B}^s_{p,q}$.*

(2) If $s > \max\{d(1/p - 1), 0\}$, then the mapping

$$u \mapsto \|u\|_p + \Big(\int_0^\infty \big\|t^{-s/2}\varphi(tA)u\big\|_p^q \frac{dt}{t}\Big)^{1/q}$$

(with the usual modification if $q = +\infty$) defines an equivalent quasi-norm on $B^s_{p,q}$.

(3) If $p < +\infty$, the mapping in $u \mapsto \Big\|\Big(\int_0^\infty \big|t^{-s/2}\varphi(tA)u\big|^q \frac{dt}{t}\Big)^{1/q}\Big\|_p$ *(with the usual modification if $q = +\infty$) defines an equivalent quasi-norm on $\dot{F}^s_{p,q}$.*

(4) If $p < +\infty$ and $s > \max\{d(1/p - 1), 0\}$, then the mapping

$$u \mapsto \|u\|_p + \Big\|\Big(\int_0^\infty \big|t^{-s/2}\varphi(tA)u\big|^q \frac{dt}{t}\Big)^{1/q}\Big\|_p$$

(with the usual modification if $q = +\infty$) defines an equivalent quasi-norm on $F^s_{p,q}$.

Observe that the equivalent norms in (1), (2) for the Besov spaces can also be seen from a different point of view, namely as equivalent norms in real interpolation spaces in accordance with Theorem 1.5.8. If e.g. $p \geq 1$ and $s \in (0, 2)$, then $B^s_{p,q} = (L^p, D(A^m))_{\theta,q}$ where $\theta := s/2$, and in this case (2) is just an application of Theorem 1.5.8. In contrast to this, the Triebel-Lizorkin norms in (3),(4) in general do not arise from real interpolation norms. This is due to the fact that real interpolation of Triebel-Lizorkin spaces leads in general to the scale of Besov spaces by the Reiteration Theorem 1.5.7, we refer to the standard literature given above. So this is one motivation to define norm expressions as in (3), (4) for more general sectorial operators A instead of $-\Delta$ and try to define associated spaces in terms of this norms, which would give rise to "generalized Triebel-Lizorkin spaces" associated to the operator A. This is in fact what we will do in Section 3.3, after we established the technical concepts of $\mathcal{R}_s$-boundedness in Sections 3.1 and $\mathcal{R}_s$-sectoriality in Section 3.2, which provide fundamental tools to deal with norm expressions as in (3), (4) for a general sectorial operator A.

Chapter 2

Maximal functions for sectorial operators

2.1 The H_0^∞-maximal function for sectorial operators

From now on we fix a Banach function space X over a σ-finite measure space (Ω,μ) with absolute continuous norm, a complex Banach space E and a sectorial operator A in $X(E)$ of type $\omega(A)$ with dense domain and range (cf. Section 1.2 for this notions). If $\omega(A)<\pi/2$, we denote by $(T_t)_{t\geq 0}$ or $(e^{-tA})_{t\geq 0}$ the bounded analytic semigroup generated by $-A$.

We will use the notations $H^\infty_{\omega,0} := \bigcup_{\sigma>\omega} H_0^\infty(\Sigma_\sigma)$ and $H^\infty_\omega := \bigcup_{\sigma>\omega} H^\infty(\Sigma_\sigma)$. Next we introduce some notations concerning maximal estimates.

Definition 2.1.1 (Maximal estimates for sets of operators)**.** Let $\mathcal{T}\subseteq L(X(E))$. We say the set $\mathcal{T}$ or the family $(T)_{T\in\mathcal{T}}$ satisfies a *maximal estimate* or *has a bounded maximal function*, if there is a constant $C>0$ such that for all $x\in X(E)$ it holds

$$\Big\| \sup_{T\in\mathcal{T}} |Tx|_E \Big\|_X \leq C\,\|x\|_{X(E)}.$$

We note that the supremum in the above definition is taken in the complete lattice $X\subseteq\mathcal{M}(\mu)$ in the sense of Proposition 1.6.2.

A standard application for maximal estimates is given by Banach's principle, which we cite in the following version:

Proposition 2.1.2 (Banach's principle)**.** *Let F be a Banach space and $(T_n)_{n\in\mathbb{N}}\in L(F,M(\mu,E))^{\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}}|T_nf|_E\in M(\mu)$ for all $f\in F$. Then the set*

$$F_0 := \{f\in F \mid (T_nf)_{n\in\mathbb{N}} \text{ converges pointwise } \mu\text{-a.e.}\}$$

is closed in $M(\mu,E)$.

A proof for $E = \mathbb{C}$ can be found e.g. in [BS88], Cor. 5.8 or a more general version in [DS58], Thm. IV.11.3., and the proofs given there easily extend to the vector-valued case. Note that Proposition 2.1.2 is usually applied in situations where it is already known that $(T_n f)_{n\in\mathbb{N}}$ converges pointwise μ-a.e. for a dense subset $D \subseteq X(E)$ of some Banach function space $X(E)$, and then yields the pointwise μ-a.e. convergence of $(T_n f)_{n\in\mathbb{N}}$ for all $f \in X(E)$.

We introduce more specific notations in the case that the sets of operators are induced by a single operator via the functional calculus.

Definition 2.1.3 (bounded f- and $H_0^\infty(\Sigma_\sigma)$-maximal function)**.** Let $\pi \geq \sigma > \omega(A)$. For $f \in H^\infty(\Sigma_\sigma)$ we say that A has a bounded f-maximal function, if $\{f(tA) \,|\, t > 0\} \subseteq L(X(E))$ and the family $(f(tA))_{t>0}$ satisfies a maximal estimate. If this is true for all $\varphi \in H_0^\infty(\Sigma_\sigma)$, we say that A has a bounded $H_0^\infty(\Sigma_\sigma)$-maximal function. In this case we define $\omega_M(A)$ as the infimum over all $\sigma > \omega(A)$ with that property and say shortly, the operator A has a bounded H_0^∞-maximal function.

Remark 2.1.4. *The property of having a bounded f-maximal function can actually be checked on finite sets: Let $x \in X$, then by Proposition 1.6.2 we can choose a countable set $J \subseteq (0,\infty)$ such that $\sup_{t>0} |\varphi(tA)x|_E = \sup_{t\in J} |\varphi(tA)x|_E$. Let $(t_k)_k \in \mathbb{N}$ be an enumeration of J, then since in X the monotone convergence theorem holds, we obtain*

$$\big\| \sup_{t>0} |\varphi(tA)x|_E \big\|_X \;=\; \big\| \sup_{k\in\mathbb{N}} |\varphi(t_kA)x|_E \big\|_X = \lim_{m\to\infty} \big\| \sup_{k\in\mathbb{N}_{\leq m}} |\varphi(t_kA)x|_E \big\|_X.$$

Obviously it is sufficient to check the condition on a dense subset of $X(E)$, so this shows that A has a bounded f-maximal function if and only if there is a $C > 0$ and a dense subspace $D \subseteq X(E)$ such that for all finite subsets $J \subseteq \mathbb{R}_{>0}$ and $x \in D$:

$$\big\| \sup_{t\in J} |f(tA)x|_E \big\|_X \leq C\|x\|_{X(E)}.$$

The denotation of having a bounded maximal function will become clear in the following, as in this case we will obtain more general uniform estimates in terms of the *s-maximal function* associated to A introduced next.

For $\pi \geq \sigma > \omega > 0$, $s \in [1,+\infty]$ and $f \in H^\infty(\Sigma_\sigma)$ we introduce the notation

$$\|f\|_{L^s_{*,\omega}} := \|f|_{\partial\Sigma_\omega}\|_{L^s_{*,\omega}} = \begin{cases} \left(\int_{\partial\Sigma_\omega} |f(\lambda)|^s \frac{|d\lambda|}{|\lambda|}\right)^{1/s} & \text{if } s < +\infty, \\ \sup_{\lambda\in\partial\Sigma_\omega} |f(\lambda)| & \text{if } s = +\infty \end{cases}$$

and the corresponding spaces $H^\infty_{\omega,s} := \bigcup_{\sigma>\omega}\{f \in H^\infty(\Sigma_\sigma) \mid \|f\|_{L^s_{*,\omega}} < +\infty\}$. Obviously we have $H^\infty_{\omega,0} \subseteq H^\infty_{\omega,s}$.

Definition 2.1.5 (s-maximal function for A)**.** Let $\pi > \omega > \omega(A)$ and $s \in [1,+\infty]$. Then we define

$$\mathcal{M}_{A,s,\omega}(x) := \sup\left\{ |\varphi(A)x|_E \;\middle|\; \varphi \in H^\infty_{\omega,0} \text{ with } \|\varphi\|_{L^{s'}_*(\partial\Sigma_\omega)} \leq 1 \right\} \qquad \text{for all } x \in X(E). \tag{2.1.1}$$

If $\mathcal{M}_{A,s,\omega}$ is bounded as a sublinear operator on $X(E)$, i.e. if there is a $C_\omega = C(s,\omega,A) > 0$ such that for all $x \in X(E)$ we have $\|\mathcal{M}_{A,s,\omega}(x)\|_X \leq C_\omega \|x\|_{X(E)}$, then we will say that A *has a bounded s-maximal function with respect to the angle ω*. In this case, the homogeneity in $\|f\|_{L_*^{s'}(\partial\Sigma_\omega)}$ leads to

$$\left\| \sup\{|f(A)x|_E \,|\, f \in H^\infty_{\omega,0}, \|f\|_{L_*^{s'}(\partial\Sigma_\omega)} \leq K\}\right\|_X \leq C_\omega\, K \,\|x\|_{X(E)} \qquad \text{for all } x \in X(E), K > 0.$$

Next, we define the auxiliary functions $\psi_{\alpha,\omega}(z) := \frac{z^\alpha}{e^{i\omega}-z}$ for all $\omega > 0, z \in \Sigma_\omega$ and $\alpha \in [0,1]$.

Lemma 2.1.6. *Let $\alpha \in [0,1]$, $\omega > \omega(A)$ and $\varphi \in H^\infty_{\omega,0}$. Then, for all $x \in X(E)$,*

$$|\varphi(A)x|_E \leq \frac{1}{2\pi} \|\varphi\|_{L_*^{s'}(\partial\Sigma_\omega)} \cdot \sum_{j\in\{-1,1\}} \left(\int_0^\infty |\psi_{\alpha,j\omega}(tA)x|_E^s \frac{dt}{t}\right)^{1/s}, \tag{2.1.2}$$

in case $s \neq +\infty$, and

$$|\varphi(A)x|_E \leq \frac{1}{2\pi} \|\varphi\|_{L_*^{1}(\partial\Sigma_\omega)} \cdot \sum_{j\in\{-1,1\}} \sup_{t>0} |\psi_{\alpha,j\omega}(tA)x|_E \tag{2.1.3}$$

in case $s = +\infty$, respectively.

Proof. Choose $\pi \geq \sigma > \omega$ with $\varphi \in H_0^\infty(\Sigma_\sigma)$. Let $x \in X(E)$, then

$$\varphi(A)x \;=\; \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \varphi(\lambda)\lambda R(\lambda,A)x\,\frac{d\lambda}{\lambda} = \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \varphi(\lambda)\Big(\lambda^{1-\alpha}A^\alpha R(\lambda,A)x\Big)\,\frac{d\lambda}{\lambda}.$$

If $\alpha \in (0,1)$ this is Proposition 4.2 in [KW01-a], if $\alpha = 0$ it is trivial and for $\alpha = 1$ use $\lambda R(\lambda,A) - AR(\lambda,A) = Id_X$ and Cauchy's Theorem. Hence $\varphi(A)x = \frac{1}{2\pi i}\int_{\partial\Sigma_\omega}\varphi(\lambda)G(\lambda)\,\frac{d\lambda}{\lambda}$, where $G(\lambda) := \psi\big(\frac{1}{\lambda}A\big)x$ with $\psi(z) := \psi_{0,\alpha}(z) = \frac{z^\alpha}{1-z}$ for all $\lambda \in \partial\Sigma_\omega, z \in \mathbb{C}\backslash\mathbb{R}$. In the case $s \neq +\infty$, we conclude with Hölders inequality

$$\begin{aligned} |\varphi(A)x|_E \;&\overset{(1)}{\leq}\; \frac{1}{2\pi}\int_{\partial\Sigma_\omega}|\varphi(\lambda)G(\lambda)|_E\,\frac{|d\lambda|}{|\lambda|} \leq \frac{1}{2\pi}\|\varphi\|_{L_*^{s'}(\partial\Sigma_\omega)}\left(\int_{\partial\Sigma_\omega}|G(\lambda)|_E^s\,\frac{|d\lambda|}{|\lambda|}\right)^{1/s} \\ &\overset{(2)}{=}\; \frac{1}{2\pi}\|\varphi\|_{L_*^{s'}(\partial\Sigma_\omega)}\left(\left(\int_0^\infty|\psi_{\alpha,\omega}(tA)x|_E^s\,\frac{dt}{t}\right)^{1/s} + \left(\int_0^\infty|\psi_{\alpha,-\omega}(tA)x|_E^s\,\frac{dt}{t}\right)^{1/s}\right). \end{aligned}$$

Here, (1) is true by (1.6.10) and (2) by plugging in the parametrization and the auxiliary function $\psi_{\omega,\alpha}$ defined above. This shows the statement. The modification for $s = +\infty$ is obvious. □

Lemma 2.1.6 provides a technical condition, namely a one-sided s-function estimate for the auxiliary functions $\psi_{\alpha,\pm\omega}$

$$\forall x \in X(E), j \in \{-1,1\} \;:\; \|\psi_{\alpha,j\omega}(\cdot\, A)x\|_{X(L_*^s(E))} \leq C_{s,\alpha,\omega}\,\|x\|_{X(E)}. \tag{$SE_{s,\alpha,\omega}$}$$

which is sufficient for a bounded s-maximal function:

Proposition 2.1.7. *Let $\alpha \in [0,1]$, $\omega > \omega(A)$ and $s \in [1,+\infty]$, and we assume there is a $C_{s,\alpha,\omega}$ such that $(SE_{s,\alpha,\omega})$ is fulfilled. Then A has a bounded s-maximal function with respect to the angle ω.* □

Proof. Define $\psi_j := \psi_{\alpha,(-1)^j\omega}$ for $j = 1, 2$. Let $\pi \geq \sigma > \omega$ and $M > 0$, then we obtain by Lemma 2.1.6 (with the usual modification in the case $s = +\infty$) the pointwise estimate

$$\sup\{|f(A)x| \mid f \in H^\infty_{\omega,0}, \|f\|_{L^{s'}_*(\partial\Sigma_\omega)} \leq M\} \leq \frac{1}{2\pi} \cdot M \sum_{j=1}^{2} \left(\int_0^\infty |\psi_j(tA)x|^s \frac{dt}{t} \right)^{1/s}.$$

Therefore the one-sided s-function estimate $(SE_{s,\alpha,\omega})$ leads to the norm estimate

$$\begin{aligned} &\left\| \sup\left\{ |f(A)x| \;\middle|\; f \in H^\infty_{\omega,0} \text{ with } \|f\|_{L^{s'}_*(\partial\Sigma_\omega)} \leq M \right\} \right\|_X \\ \leq\; &\frac{1}{2\pi} \cdot M \left(\left\| |\psi_1(\cdot\, A)x|_{L^s_*((0,+\infty))} \right\|_X + \left\| |\psi_2(\cdot\, A)x|_{L^s_*((0,+\infty))} \right\|_X \right) \leq \frac{C_{s,\alpha,\omega}}{\pi} \cdot M \|x\|. \end{aligned}$$

□

Remark 2.1.8. *It is an easy consequence of the convergence lemma, that in the situation of Proposition 2.1.7 we also have $f(A) \in L(X)$ for all $f \in H^\infty_{\omega,s'}$, and that in the definition of the s-maximal function we can take the supremum over all $f \in H^\infty_{\omega,s'}$ with $\|f\|_{L^{s'}_*(\partial\Sigma_\omega)} \leq 1$.*

Putting all together we obtain the following characterization of having a bounded maximal function.

Theorem 2.1.9. *Let $\omega, \sigma > \omega(A)$. Consider the following assertions:*

(1) $\exists\, C_1 > 0\, \forall\, \alpha \in (0,1)\, \forall\, x \in X(E) \,:\, \|\psi_{\alpha,\pm\omega}(tA)x\|_{X(L^\infty_*(E))} \leq C_1 \|x\|_{X(E)}$,

(2) $\exists\, C_2 > 0\, \exists\, \alpha \in [0,1]\, \exists s \in [1,+\infty]\, \forall\, x \in X(E) \,:\, \|\psi_{\alpha,\pm\omega}(tA)x\|_{X(L^s_*(E))} \leq C_2 \|x\|_{X(E)}$,

(3) $\mathcal{M}_{A,s,\omega}$ *is bounded for some* $s \in [1,+\infty]$,

(4) $\mathcal{M}_{A,\infty,\omega}$ *is bounded,*

(5) *A has a bounded $H_0^\infty(\Sigma_\sigma)$-maximal function.*

Then $(1) \Rightarrow (2) \Rightarrow (3)$ *and* $(1) \Rightarrow (4)$. *Moreover, if $\sigma > \omega$, then* $(3) \Rightarrow (5)$, *and if $\omega > \sigma$, then* $(5) \Rightarrow (1)$.

Proof. $(1) \Rightarrow (2)$ is trivial, and $(2) \Rightarrow (3)$ and $(1) \Rightarrow (4)$ are just Proposition 2.1.7. Now assume (3) and $\sigma > \omega$. Then (5) follows from the fact, that $\|\varphi(t\cdot)\|_{L^{s'}_*(\partial\Sigma_\omega)} = \|\varphi\|_{L^{s'}_*(\partial\Sigma_\omega)} =: M$ for all $t > 0$ and the boundedness of the s-maximal function. Now we assume (5) and $\omega > \sigma$. Then $\psi_{\alpha,\pm\omega} \in H_0^\infty(\Sigma_\sigma)$, hence (1) follows. □

The typical way to use Theorem 2.1.9 is the following: We will check condition (2) for all $\omega > \omega_0$ and obtain by (5) $\omega_M(A) \leq \omega_0$. Then the theorem asserts that for all $\omega > \omega_M(A)$ conditions (3) and (4) are fulfilled:

Corollary 2.1.10. *Let A have a bounded maximal function. Let $\pi \geq \sigma > \omega_M(A)$, $\varphi \in H_0^\infty(\Sigma_\sigma)$ and $0 \leq \delta < \sigma - \omega_M(A)$. Then $(\varphi(zA))_{z\in\overline{\Sigma}_\delta}$ satisfies a maximal estimate, i.e. there exists $C > 0$ such that*

$$\Big\| \sup_{z\in\overline{\Sigma}_\delta} |\varphi(zA)x|_E \Big\|_X \leq C\, \|x\|_{X(E)} \qquad \text{for all } x \in X(E). \tag{2.1.4}$$

Proof. Let $\sigma - \delta > \omega > \omega_M(A)$, then A has a bounded ∞-maximal function with respect to the angle ω. Choose $C_0 > 0, \beta > 0$ with $|\varphi(z)| \le C_0 \left(|z|^\beta \wedge |z|^{-\beta}\right)$ for all $z \in \Sigma_\sigma$. Let $z \in \overline{\Sigma_\delta}\backslash\{0\}$ (the case $z = 0$ can be neglected because $\varphi(0) = 0$), then $z = \tau w$ for some $\tau > 0, w \in \overline{\Sigma_\delta} \cap S^1$, and $w\lambda \in \Sigma_\sigma$ for all $\lambda \in \partial\Sigma_\omega$. We obtain

$$\begin{aligned}\int_{\partial\Sigma_\omega} |\varphi(z\lambda)| \frac{|d\lambda|}{|\lambda|} &= \sum_{j\in\{-1,1\}} \int_0^\infty |\varphi(\tau t w e^{ij\omega})| \frac{dt}{t} = \sum_{j\in\{-1,1\}} \int_0^\infty |\varphi(t w e^{ij\omega})| \frac{dt}{t} \\ &\le C_0^2 \sum_{j\in\{-1,1\}} \int_0^\infty (t^\beta \wedge t^{-\beta}) \frac{dt}{t} =: K < +\infty,\end{aligned}$$

i.e. $\|\varphi(z\cdot)\|_{L^1_*(\partial\Sigma_\omega)} \le K$. The boundedness of the ∞-maximal function yields

$$\Big\| \sup_{z\in\overline{\Sigma}_\delta} |\varphi(zA)x|_E \Big\|_X \le \Big\| \sup \left\{ |f(A)x|_E \mid f \in H_0^\infty, \|f\|_{L^1_*(\partial\Sigma_\omega)} \le K \right\} \Big\|_X \le C_\omega K \, \|x\|_{X(E)}.$$

□

Remark 2.1.11. *In fact, in the situation of Corollary 2.1.10 the constant C only depends on the constants C_0, β of the majorant $z \mapsto C_0 \left(|z|^\beta \wedge |z|^{-\beta}\right)$ of the H_0^∞-function φ, hence we obtain uniform estimates for any subset $\mathcal{F} \subseteq H_0^\infty(\Sigma_\sigma)$ which fulfills uniform estimates $|\varphi(z)| \le C_0 \left(|z|^\beta \wedge |z|^{-\beta}\right)$ for all $\varphi \in \mathcal{F}$ and some $C_0, \beta > 0$. Moreover, in the special case $\delta = 0$ we have the sharper estimate*

$$\Big\| \sup_{t>0} |\varphi(tA)x|_E \Big\|_X \le C_\omega \, \|\varphi\|_{L^1_*(\partial\Sigma_\omega)} \, \|x\|_{X(E)} \qquad \text{for all } x \in X(E) \tag{2.1.5}$$

and for all $\varphi \in H^\infty_{0,\omega_M(A)}$, and the constant C_ω is independent of φ.

Examples 2.1.12. Let A have a bounded H_0^∞-maximal function.

(1) Let $\alpha > 0$ and $\varphi(\lambda) := \frac{\lambda^\alpha}{(1+\lambda)^{2\alpha}}$ for all $\lambda \in \Sigma_\pi$, and let $0 \le \delta < \pi - \omega_M(A)$. Then $(\varphi(zA))_{z\in\overline{\Sigma}_\delta}$ satisfies a maximal estimate, i.e. there exists $C > 0$ such that for all $x \in X$:

$$\Big\| \sup_{z\in\Sigma_\delta} |z^\alpha A^\alpha (z+A)^{-2\alpha} x| \Big\|_X \le C \, \|x\|_X. \tag{2.1.6}$$

(2) We assume $\omega_M(A) < \pi/2$. Let $\alpha > 0$ and $\varphi(\lambda) := \lambda^\alpha e^{-\lambda}$ for all $\lambda \in \Sigma_{\pi/2}$, and let $0 \le \delta < \pi/2 - \omega_M(A)$. Then $(\varphi(zA))_{z\in\overline{\Sigma}_\delta}$ satisfies a maximal estimate, i.e. there exists $C > 0$ such that for all $x \in X$:

$$\Big\| \sup_{z\in\overline{\Sigma}_\delta} |z^\alpha A^\alpha T_z x| \Big\|_X \le C \, \|x\|_X. \tag{2.1.7}$$

Finally we present some standard persistence properties of having a bounded maximal function.

Proposition 2.1.13. *Let A have a bounded H_0^∞-maximal function.*

(1) If $r > 0$ and $\delta \in [0, \omega_M(A))$, then also $re^{i\delta}A$ has a bounded H_0^∞-maximal function with $\omega_M(re^{i\delta}A) \le \omega_M(A) + \delta$,

(2) Let $\alpha \in \mathbb{R}$ with $|\alpha| < \pi/\omega_M(A)$, then A^α has a bounded H_0^∞-maximal function and $\omega_M(A^\alpha) = |\alpha|\omega_M(A)$.

Proof. (1) is a consequence of Corollary 2.1.10.

(2) We first note that A^{-1} has a bounded maximal function with $\omega_M(A^{-1}) = \omega_M(A)$, this is a consequence of Theorem 2.1.9 and the simple fact that for $\alpha = 1/2$ and $\theta > \omega_M(A), t > 0$ we have $\psi_{\pm\theta}(tA^{-1}) = -e^{\mp i\theta}\psi_{\mp\theta}(t^{-1}A)$. Hence we may assume w.l.o.g. that $\alpha > 0$. Let $\sigma > \alpha\omega_M(A)$ and $\varphi \in H_0^\infty(\Sigma_\sigma)$. Define $\varphi_\alpha(z) := \varphi(z^\alpha)$, then $\varphi_\alpha \in H_0^\infty(\Sigma_{\sigma/\alpha})$, and by the composition rule for the functional calculus we have $\varphi(A^\alpha) = \varphi_\alpha(A)$, cf. e.g. [Ha06], Prop. 3.1.4. Since $\sigma/\alpha > \omega_M(A)$, the family $(\varphi(tA^\alpha))_{t>0} = (\varphi_\alpha(t^{1/\alpha}A))_{t>0}$ satisfies a maximal estimate, hence A^α has a bounded maximal function with $\omega_M(A^\alpha) \le \alpha\omega_M(A)$. If we apply this for A^α instead of A, we obtain the remaining estimate $\omega_M(A) = \omega_M((A^\alpha)^{\frac{1}{\alpha}}) \le \frac{1}{\alpha}\omega_M(A^\alpha)$. □

2.2 Examples of operators with a bounded H_0^∞-maximal function

In this section we present classes and examples of operators with a bounded maximal function. In particular we will give examples of operators without BIP that have a bounded maximal function. This shows that our techniques extend those used in [Bl02], where only operators with BIP are considered.

(a) Operators with bounded imaginary powers

Let $A \in \mathrm{BIP}(X(E))$, i.e. in this situation $A^{is} \in L(X(E))$ for all $s \in \mathbb{R}$, and hence $(A^{is})_{s\in\mathbb{R}}$ is a C_0-group[1] so there is $\omega \ge 0$ and $M_\omega \ge 1$ with

$$\|A^{is}\|_{X(E)} \le M\, e^{\omega|s|} \quad \text{für alle } s \in \mathbb{R}.$$

The infimum of all $\omega \ge 0$ for which there exists such an M_ω is denoted by $\omega_{BIP}(A)$. In the following we assume $\omega_{BIP}(A) < \pi$.

Let $\theta \in (-\pi, \pi)$ with $|\theta| < \pi - \omega_{BIP}(A)$ and $\omega_{BIP}(A) < \omega < \pi - |\theta|$, so $\delta := \pi - |\theta| - \omega > 0$. Moreover let $M_\omega \ge 1$ with $\|A^{is}\|_X \le M_\omega\, e^{\omega|s|}$ for all $s \in \mathbb{R}$. We will use the following representation formula which can be seen as a variant of the Mellin inversion formula:

$$r^{1/2}e^{i\theta/2}A^{1/2}(re^{i\theta} + A)^{-1}x \;=\; \frac{1}{2}\int_{-\infty}^{\infty} r^{-is}\frac{e^{\theta s}}{\cosh(\pi s)}A^{is}x\,ds. \tag{2.2.8}$$

We will give a derivation of (2.2.8) following the lines of [KW04], proof of 11.9.

[1] In this situation, this claim is equivalent to the usual definition in the general case, i.e. $(A^{is})_{s\in\mathbb{R}}$ is a C_0-group. This is due to the fact that if $x \in D(A) \cap R(A)$, then $s \mapsto A^{is}x =: T(s)x$ is bounded, and by density this yields that $s \mapsto T(s)x$ is measurable for each $x \in X$. Hence $(T(s))_{s\in\mathbb{R}}$ is a strongly measurable group, and it is well known that this implies strong continuity of $(T(s))_{s>0}$. For $s \le 0$ the group property leads to $T(s+h)x = T(1+h)T(s-1)x \to T(1)T(s-1)x = T(s)x$ for $x \in X$ if $h \to 0$, hence T is a C_0-group.

(i) The Balakrishnan representation yields for $y \in R(A)$, $x := A^{-1}y \in D(A)$ and $-1 < \mathrm{Re}(\alpha) < 0$:

$$A^\alpha y = A^{1+\alpha}x \overset{\text{Balakr.}}{=} \frac{\sin(\pi(1+\alpha))}{\pi}\int_0^\infty t^\alpha (t+A)^{-1}Ax\,dt = -\frac{\sin(\pi\alpha))}{\pi}\int_0^\infty t^\alpha(t+A)^{-1}y\,dt.$$

(ii) Let $x \in D(A)\cap R(A) \subseteq D(A^{1/2}) \cap D(A^{is}) = D(A^{is-1/2}A^{1/2})$, then

$$\begin{aligned} A^{is}x &= A^{is-1/2}A^{1/2}x \overset{(i)}{=} -\frac{\sin(\pi(is-1/2)))}{\pi}\int_0^\infty t^{is-1/2}(t+A)^{-1}A^{1/2}x\,dt \\ &= \frac{\cosh(\pi s)}{\pi}\int_0^\infty t^{is}t^{1/2}(t+A)^{-1}A^{1/2}x\,\frac{dt}{t}. \end{aligned}$$

(iii) Choose $\theta \in [0,\pi)$ with $|\theta| < \pi - \omega_{BIP}(A)$ and $\omega_{BIP}(A) < \omega < \pi - |\theta|$, so $\delta := \pi - |\theta| - \omega > 0$, and moreover $M_\omega \geq 1$ with $\|A^{is}\|_X \leq M\,e^{\omega|s|}$ for all $t \in \mathbb{R}$. We apply (ii) to $e^{-i\theta}A$ instead of A and obtain with the substitution $t = e^u, dt/t = du$:

$$\begin{aligned} \frac{\pi e^{\theta s}}{\cosh(\pi s)}A^{is}x &= \int_0^\infty t^{is}t^{1/2}e^{i\theta/2}(e^{i\theta}t+A)^{-1}A^{1/2}x\,\frac{dt}{t} \\ &= \int_{-\infty}^\infty e^{isu}\underbrace{e^{u/2}e^{i\theta/2}(e^{i\theta}e^u+A)^{-1}A^{1/2}x}_{F(u):=}\,du = \widehat{F}(-s). \end{aligned}$$

As in general $e^{|x|}/2 \leq \cosh(x) \leq e^{|x|}$ for all $x \in \mathbb{R}$, it follows that

$$\begin{aligned} \|\widehat{F}(s)\| &\leq \frac{\pi e^{|\theta s|}}{\cosh(\pi|s|)}\|A^{-is}x\| \leq 2\pi M_\omega\,e^{|\theta s|-\pi|s|}\,e^{\omega|s|}\,\|x\|_X = 2\pi M_\omega\,e^{-(\pi-|\theta|-\omega)|s|}\,\|x\|_X \\ &= 2\pi M_\omega\,e^{-\delta|s|}\,\|x\|_X \quad \text{for all } s \in \mathbb{R}, \end{aligned}$$

hence $\widehat{F} \in \mathcal{L}^1(\mathbb{R}, X)$. By Fourier inversion we obtain

$$e^{u/2}e^{i\theta/2}(e^{i\theta}e^u+A)^{-1}A^{1/2}x \;=\; F(u) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ius}\widehat{F}(s)\,ds = \frac{1}{2}\int_{-\infty}^\infty e^{ius}\frac{e^{-\theta s}}{\cosh(\pi s)}A^{-is}x\,ds.$$

Substituting $r = e^{u/2}$ and $s \mapsto -s$ in the integral yields (2.2.8) for all $x \in D(A)\cap R(A)$, and since $D(A)\cap R(A)$ is dense and both sides of equation (2.2.8) define bounded operators, this is also true for all $x \in X(E)$. $\square$

We now substitute $t = r^{-1}$ in formula (2.2.8) and obtain

$$|(tA)^{1/2}(e^{i\theta}+tA)^{-1}x|_E \leq \int_{\mathbb{R}} e^{-(\pi-|\theta|)|s|}\,|A^{-is}x|_E\,ds \quad \text{for all } x \in X(E), t > 0.$$

Since the right-hand side of the inequality above is independent of t,

$$\sup_{t>0}|(tA)^{1/2}(e^{i\theta}+tA)^{-1}x|_E \leq \int_{\mathbb{R}} e^{-(\pi-|\theta|)|s|}\,|A^{-is}x|_E\,ds.$$

Letting $\alpha := 1/2$ and $\psi_j := \psi_{\alpha,(-1)^j(\pi-|\theta|)}$, we have

$$\sup_{t>0}|\psi_j(tA)x|_E = \sup_{t>0}|(te^{(-1)^j(\pi-|\theta|)}A)^{1/2}(1-te^{(-1)^j(\pi-|\theta|}A)^{-1}x|_E \leq \int_{\mathbb{R}} e^{-(\pi-|\theta|)|s|}\,|A^{-is}x|_E\,ds,$$

and this leads finally to

$$\big\| \sup_{t>0} |\psi_j(tA)x|_E \|_X \le \int_{\mathbb{R}} e^{-(\pi-|\theta|)|s|} \, \|A^{-is}x\|_{X(E)} \, ds \le M_\omega \int_{\mathbb{R}} e^{-\delta|s|} \, ds \cdot \|x\|_{X(E)} = \frac{2M_\omega}{\delta} \, \|x\|_{X(E)}.$$

So we have proven:

Proposition 2.2.1. *Let $A \in \mathrm{BIP}(X(E))$ with $\omega_{BIP}(A) < \pi$. Then the operator A has a bounded H_0^∞-maximal function with $\omega_M(A) \le \omega_{BIP}(A)$.* □

Remark 2.2.2. *In recent years optimal BIP- and in fact H^∞-angles for certain classes of elliptic differential operators have been established also in the vector-valued setting of a space $L^p(\Omega, E)$, where E is a UMD-space, cf. e.g. [DHP03], [DDHPV04].*

As already mentioned above, this example shows in particular that in our framework we can handle the operators considered in [Bl02].

(b) Operators that satisfy a one-sided square function estimate

We want to draw attention to the case $s = 2$, then the condition (2) of Theorem 2.1.9 is fulfilled in particular if the operator A satisfies one-sided square function estimates, which are well known in classical situations in harmonic analysis. It is well known that if X is q-concave for some $q < +\infty$ and the operator A has a bounded H^∞-calculus then for all $\sigma > \omega_{H^\infty}(A)$ and $\varphi \in H_0^\infty(\Sigma_\sigma)\backslash\{0\}$ there is a constant $C > 0$ such that for all $x \in X$:

$$\frac{1}{C} \, \|x\|_X \le \Big\| \Big(\int_0^\infty |\varphi(tA)x|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_X \le C \, \|x\|_X,$$

cf. Subsection 1.6.5. Of course, this situation is already covered by (a). Nevertheless we want to mention that even in the Hilbert-space case $X = L^2$ there are examples of operators without an H^∞-calculus, hence without BIP, that satisfy only a one-sided square-function estimate

$$\Big\| \Big(\int_0^\infty |\varphi(tA)x|^2 \, \frac{dt}{t} \Big)^{1/2} \Big\|_X \le C \, \|x\|_X$$

for $\varphi \in H_0^\infty(\Sigma_\sigma)\backslash\{0\}$, cf. [LeM03], Section 5. So these operators fulfill condition $(SE_{s,\alpha,\omega})$ for $s = 2, \alpha = 1/2$ and $\omega > \sigma$ and hence have a bounded H_0^∞-maximal function.

(c) Generators of analytic semigroups which satisfy a maximal estimate

Now we assume $\omega(A) \le \pi/2$ and that $-A$ generates a bounded C_0-semigroup $(T_t)_{t\ge 0}$. Further we assume that there is a constant $C > 0$ such that one of the following estimates holds for all $x \in X(E)$:

$$\big\| \sup_{t>0} \big|T_t x\big|_E \big\|_X \le C \, \|x\|_{X(E)}. \tag{2.2.9}$$

or

$$\Big\| \sup_{t>0} \Big| \frac{1}{t} \int_0^t T_s x \, ds \Big|_E \Big\|_X \le C \, \|x\|_{X(E)} \tag{2.2.10}$$

Proposition 2.2.3. *Under the above assumptions, the operator A has a bounded H_0^∞-maximal function in $X(E)$ with $\omega_M(A) \le \pi/2$.*

Proof. In both cases we use the representation formula of the resolvent as Laplace transform of the semigroup. Let $\pi \ge |\theta| > \pi/2, x \in X(E)$ and $t > 0, r := t^{-1}$. Let us first assume that (2.2.9) holds. Then using the Laplace-transform we obtain

$$R(e^{i\theta}, tA) \;=\; rR(re^{i\theta}, A) = -\int_0^\infty r e^{sre^{i\theta}} T_s x \, ds,$$

hence with $\alpha = 0$

$$|\psi_\theta(tA)x|_E \;\le\; \int_0^\infty \big| r e^{sre^{i\theta}} T_s x \big|_E \, ds \le \int_0^\infty r e^{-sr|\cos(\theta)|} \, ds \cdot \sup_{s>0} |T_s x|_E = \frac{1}{|\cos(\theta)|} \cdot \sup_{s>0} |T_s x|_E.$$

By our assumption we obtain the estimate

$$\Big\| \sup_{t>0} |\psi_{\pm\theta}(tA)x|_E \Big\|_X \le \frac{1}{|\cos(\theta)|} \, \| \sup_{s>0} |T_s x|_E \|_X \le \frac{C}{|\cos(\theta)|} \, \|x\|_{X(E)}.$$

Now we assume that (2.2.10) holds. Let $\mu := -re^{i\theta}$, then again by the Laplace-transform representation and partial integration it follows that

$$-R(e^{i\theta}, tA)x = \int_0^\infty r e^{-\mu s} T_s x \, ds = \underbrace{\Big[rse^{-\mu s} \, \frac{1}{s} \int_0^s T_\tau x \, d\tau \Big]_0^\infty}_{=0} + \int_0^\infty r\mu e^{-\mu s} \Big(\int_0^s T_\tau x \, d\tau \Big) \, ds,$$

hence

$$\begin{aligned} |R(e^{i\theta}, tA)x|_E \;&\le\; \int_0^\infty r^2 s e^{-\operatorname{Re}(\mu)s} \Big| \Big(\frac{1}{s} \int_0^s T_\tau x \, d\tau \Big) \Big|_E \, ds \\ &\le\; \int_0^\infty r^2 s e^{-sr|\cos(\theta)|} \, ds \cdot \sup_{s>0} \Big| \frac{1}{s} \int_0^s T_\tau x \, d\tau \Big|_E = \frac{1}{\cos^2(\theta)} \cdot \sup_{s>0} \Big| \frac{1}{s} \int_0^s T_\tau x \, d\tau \Big|_E. \end{aligned}$$

Similar as in the first case we obtain the estimate

$$\Big\| \sup_{t>0} |\psi_{\pm\theta}(tA)x|_E \Big\|_X \le \frac{1}{\cos^2(\theta)} \, \Big\| \sup_{s>0} \Big| \frac{1}{s} \int_0^s T_\tau x \, d\tau \Big|_E \Big\|_X \le \frac{C}{\cos^2(\theta)} \, \|x\|_{X(E)}.$$

This shows the claim in both cases. □

We will later see how we can obtain operators fulfilling the above assumptions, e.g. by using classical results from harmonic analysis and ergodic theory for operators in scalar-valued function spaces and then pass to tensor extensions in vector-valued spaces. Although we only get the rather "bad" angle $\pi/2$ with this method, we at least get a starting point for an angle that might be improved by interpolation.

(d) Another example for an operator without BIP, that has a bounded H_0^∞-maximal function

One such example is already cited in (b). Nevertheless we will construct another example which is taken from [KW04], Example 10.17. Let $p \in (1, +\infty)$ and $-1 < \gamma < p-1$, and define the weight function $w(x) := |x|^\gamma$ and the weighted space $X := L^p(\mathbb{R}, w(x)dx)$. We note that w is a so-called A_p-weight, cf. e.g. [Du01], Chapter 7. Define the operator $A := \exp(\frac{1}{i}\frac{d}{dx})$ as an unbounded Fourier-multiplier-operator in X, i.e. $Af := \mathcal{F}_\xi^{-1}(e^\xi \widehat{f}(\xi))$ for all $f \in D(A) := \{g \in X \mid \mathcal{F}_\xi^{-1}(e^\xi \widehat{f}(\xi)) \in X\}$. Then the following statements are true:

1. Since w is an A_p-weight, the Mikhlin-multiplier-theorem holds in X, cf. [KW79]. Hence the operator A is sectorial with $\omega(A) = 0$.
2. The imaginary powers are formally given by the translation operators $A^{is} f = f(s + \cdot)$, hence it can easily be shown that in the case $\gamma \neq 0$ the operator A does not have BIP.
3. A has a bounded H_0^∞-maximal function with $\omega_M(A) = 0$: Let $\alpha := 1/2$, $\omega \in (0, \pi]$ and $\psi := \psi_{\alpha, \pm\omega}$. Letting $t = e^r > 0$ we have
$$\psi(tA)f = \mathcal{F}_\xi^{-1}\left(\psi_r(\xi) \cdot \widehat{f}(\xi)\right) = \left(\mathcal{F}^{-1}\psi_r\right) * f,$$
with the symbol $\psi_r(\xi) = \frac{e^{\frac{r+\xi}{2}}}{e^{\pm i\omega} - e^{r+\xi}}$, hence $\psi_r(\xi) = \psi_0(r+\xi)$ for all $r \in \mathbb{R}$. This leads to an estimate
$$|\mathcal{F}^{-1}\psi_r(x)| \approx |e^{irx}\mathcal{F}^{-1}\psi_0(x)| = |\mathcal{F}^{-1}\psi_0(x)| \lesssim \frac{1}{1+|x|^2},$$
hence we obtain the pointwise estimate $\sup_{t>0} |\psi(tA)f(x)| \lesssim \|\mathcal{F}^{-1}\psi_0\|_{L^1(\mathbb{R})} \cdot \mathcal{M}|f|(x)$, where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Since w is an A_p-weight, the maximal operator $\mathcal{M}$ is bounded in X (cf. [GCRdF85], Theorem IV.2.8), hence
$$\|\sup_{t>0} |\psi(tA)f|\|_X \lesssim \|\mathcal{M}|f|\|_X \lesssim \|f\|_X.$$

2.3 Equivalence of maximal estimates for $f \in \mathcal{E}(\Sigma_\sigma)$

For arbitrary H^∞-functions f the family $(f(tA))_{t>0}$ need not fulfill maximal estimates in general. But there are nice transference properties between functions f in the *extended Dunford-Riesz class*. Recall that

$$\mathcal{E}(\Sigma_\sigma) = H_0^\infty(\Sigma_\sigma) \oplus \langle \frac{1}{1+\mathrm{id}_{\Sigma_\sigma}} \rangle_\mathbb{C} \oplus \langle \mathbb{1}_{\Sigma_\sigma} \rangle_\mathbb{C} \quad \text{and } \mathcal{E}_\omega := \bigcup_{\sigma > \omega} \mathcal{E}(\Sigma_\sigma) \quad \text{for all } \omega \in [0, \pi),$$

and that $\mathcal{E}(\Sigma_\sigma)$ is exactly the algebra of bounded analytic functions on Σ_σ having finite polynomial limits in 0 and ∞, hence in particular, by the mean value theorem, bounded holomorphic functions on Σ_σ that are either decaying to 0 or holomorphic in a neighborhood of 0 and ∞, respectively, belong to the class $\mathcal{E}(\Sigma_\sigma)$ (cf. Section 1.2). This is the typical situation we will be faced with.

Now we are in a position to formulate the equivalence theorem for bounded f-maximal functions if $f \in \mathcal{E}(\Sigma_\sigma)$:

Theorem 2.3.1. *Let A have a bounded H_0^∞-maximal function. The following assertions are equivalent:*

(1) There exists an $f \in \mathcal{E}_{\omega_M(A)}$ with $f(0) \neq f(\infty)$ such that A has a bounded f-maximal function.

(2) For all $\pi \geq \sigma > \omega_M(A)$, $g \in \mathcal{E}(\Sigma_\sigma)$ and $0 \leq \delta < \sigma - \omega_M(A)$, the family $(g(zA))_{z\in\Sigma_\delta}$ satisfies a maximal estimate.

The idea of the proof is simple: Every function in $\mathcal{E}$ can be decomposed into the sum of a H_0^∞-function and constants times the function $z \mapsto \frac{1}{1+z}$ and $\mathbb{1}$. Hence the main step is to handle these classes of functions separately. For H_0^∞-functions, this has been done in Corollary 2.1.10, so it remains to handle the function $z \mapsto \frac{1}{1+z}$. This is what we will do in the following lemma.

Lemma 2.3.2. *Let A have a bounded maximal function and assume that there is a constant $C_1 > 0$ such that for all $x \in X(E)$*

$$\Big\| \sup_{t>0} |t(t+A)^{-1}x|_E \Big\|_X \leq C_1 \, \|x\|_{X(E)}. \tag{2.3.11}$$

Then, to each $\omega' > \omega_M(A)$ there is a constant $C > 0$ such that

$$\Big\| \sup_{\lambda\notin\overline{\Sigma}_{\omega'}} |\lambda R(\lambda, A)x|_E \Big\|_X \leq C \, \|x\|_{X(E)} \qquad \text{for all } x \in X(E). \tag{2.3.12}$$

Proof. Choose $\omega' > \omega > \omega_M(A)$, then A has a bounded s-maximal function with respect to the angle ω for $s = +\infty$. Let $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_{\omega'}$, then $\lambda = te^{i\theta}$ for some $t > 0$, $\omega' < |\theta| \leq \pi$. By the resolvent equation we have

$$\begin{aligned} \lambda R(\lambda, A) - t(t+A)^{-1} &= (\lambda + t)A(\lambda - A)^{-1}(t+A)^{-1} = t(1+e^{i\theta})A(te^{i\theta} - A)^{-1}(t+A)^{-1} \\ &= (1+e^{i\theta})\, t^{-1}A\,(e^{i\theta} - t^{-1}A)^{-1}(1+t^{-1}A)^{-1} = \varphi_\theta(t^{-1}A), \end{aligned}$$

where $\varphi_\theta(z) := (1+e^{i\theta})\,\dfrac{z}{(e^{i\theta} - z)(1+z)}$. Fix $\omega' > \sigma > \omega$, then $\varphi_\theta \in H_0^\infty(\Sigma_\sigma)$ for all $\omega' < |\theta| \leq \pi$. Indeed one can find a constant $C_0 > 0$ such that for all $\omega' < |\theta| \leq \pi$ and $z \in \Sigma_\sigma$ we can estimate $|\varphi_\theta(z)| \leq C_0 \cdot \min\left\{|z|, \frac{1}{|1+z|}\right\}$, hence

$$\|\varphi_\theta\|_{L^1_*(\partial\Sigma_\omega)} \leq 2C \int_0^\infty \min\{t, 1/t\}\,\frac{dt}{t} =: M$$

By the translation invariance of the measure dt/t on $(0,\infty)$ we obtain $\|\varphi_\theta(t\cdot)\|_{L^1_*(\partial\Sigma_\omega)} \leq M$ for all $|\theta| > \omega', t > 0$.

By construction, to each $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_{\omega'}$ there are $t > 0$, $\omega' < |\theta| \leq \pi$ such that $\lambda R(\lambda, A) = t(t+A)^{-1} + \varphi_\theta(t^{-1}A)$, hence

$$\sup_{\lambda\notin\overline{\Sigma}_{\omega'}} |\lambda R(\lambda, A)x|_E \quad\leq\quad \sup_{t>0} |t(t+A)^{-1}x|_E + \sup_{\theta,t} |\varphi_\theta(tA)x|_E$$

So we have

$$\big\| \sup_{\lambda\notin\overline{\Sigma}_{\omega'}} |\lambda R(\lambda, A)x|_E \big\|_X \leq \big\| \sup_{t>0} |t(t+A)^{-1}x|_E \big\|_X + C_\omega M \, \|x\|_{X(E)} \leq (C_1 + C_\omega M) \, \|x\|_{X(E)},$$

where the constant C_ω is chosen by the boundedness of the maximal function of A with respect to the angle ω. □

Now we are in a position to prove Theorem 2.3.1.

Proof of Theorem 2.3.1. We only have to prove the non-trivial implication (1) ⇒ (2): Let $\sigma_0 \in (\omega_M(A), \pi]$ and $f \in \mathcal{E}(\Sigma_{\sigma_0})$ with $f(0) \neq f(+\infty)$ such that A has a bounded f-maximal function. Then we can decompose f as $f(z) = \varphi_0(z) + \frac{a}{1+z} + b$, where $a = f(0) - f(\infty) \neq 0$ and $\varphi_0 \in H_0^\infty(\Sigma_{\sigma_0})$, hence $\frac{1}{1+z} = \frac{1}{a}\big(f(z) - \varphi_0(z) - b\big)$. So we have

$$t(t+A)^{-1} = (1 + t^{-1}A)^{-1} = \frac{1}{a}\big(f(t^{-1}A) - \varphi_0(t^{-1}A) - b\,\mathrm{Id}_{X(E)}\big).$$

By our assumption the family $(f(tA))_{t>0}$ satisfies a maximal estimate, hence also $(t(t+A)^{-1})_{t>0}$ satisfies a maximal estimate.

Now let $\pi \geq \sigma > \omega_M(A)$, $g \in \mathcal{E}(\Sigma_\sigma)$ and $0 \leq \delta < \sigma - \omega_M(A)$ be arbitrary. Then again we have a decomposition $g(\zeta) = \varphi(\zeta) + \frac{c}{1+\zeta} + d$, hence for each $z \in \Sigma_\delta$ we have the representation $g(z\cdot) = \varphi(z\cdot) + \frac{c}{1+z\cdot} + d$, hence

$$g(zA) = \varphi(zA) + c(1 + zA)^{-1} + d\,\mathrm{Id}_{X(E)} = \varphi(zA) + c\lambda R(\lambda, A) + d\,\mathrm{Id}_{X(E)},$$

where $\lambda := -\frac{1}{z} \in \mathbb{C}\backslash\overline{\Sigma}_{\pi-\delta}$. Since $\omega' := \pi - \delta \geq \sigma - \delta > \omega_M(A)$, we can apply Lemma 2.3.2 together with Corollary 2.1.10 to obtain the claim. □

A simple application of Theorem 2.3.1 is the following:

Corollary 2.3.3. *We assume additionally that $\omega_M(A) < \pi/2$. Let $\alpha \in \mathbb{R}$ with $|\alpha| < \pi/\omega_M(A)$, $0 \leq \delta < \pi/2 - |\alpha|\omega_M(A)$ and $\omega' > |\alpha|\omega_M(A)$. Then the following assertions are equivalent:*

(1) There is a constant $C_1 > 0$ such that for all $x \in X(E)$:

$$\big\| \sup_{t>0} \Big| \frac{1}{t} \int_0^t e^{-sA}x \, ds \Big|_E \big\|_X \leq C_1 \, \|x\|_{X(E)}. \tag{2.3.13}$$

(2) There is a constant $C_2 > 0$ such that for all $x \in X(E)$:

$$\big\| \sup_{t>0} |e^{-tA}x|_E \big\|_X \leq C_2 \, \|x\|_{X(E)}. \tag{2.3.14}$$

(3) There is a constant $C_3 > 0$ such that for all $x \in X(E)$:

$$\big\| \sup_{t>0} |t(t+A)^{-1}x|_E \big\|_X \leq C_3 \, \|x\|_{X(E)}. \tag{2.3.15}$$

(4) There is a constant $C_4 > 0$ such that for all $x \in X(E)$:

$$\Big\| \sup_{|\arg z|<\delta} \Big| \frac{1}{z} \int_0^z e^{-\lambda A^\alpha} x \, d\lambda \Big|_E \Big\|_X \leq C_4 \, \|x\|_{X(E)}. \tag{2.3.16}$$

(5) There is a constant $C_5 > 0$ such that for all $x \in X(E)$:

$$\big\| \sup_{|\arg z|<\delta} |e^{-zA^\alpha} x|_E \big\|_X \leq C_5 \, \|x\|_{X(E)}. \tag{2.3.17}$$

(6) There is a constant $C_6 > 0$ such that for all $x \in X(E)$:

$$\big\| \sup_{\lambda \notin \overline{\Sigma_{\omega'}}} |\lambda R(\lambda, A^\alpha) x|_E \big\|_X \leq C_6 \, \|x\|_{X(E)}. \tag{2.3.18}$$

□

Proof. We assume first that $\alpha = 1$. Define $f_1(\lambda) := \frac{e^{-\lambda}-1}{-\lambda}$, $f_2(\lambda) := e^{-\lambda}$ and $f_3(\lambda) := \frac{1}{1+\lambda}$. Choose $\omega_M(A) + \delta < \sigma < \pi/2$, then $f_1, f_2 \in \mathcal{E}(\Sigma_\sigma)$, $f_3 \in \mathcal{E}(\Sigma_\pi)$ and $f_j(0) = 1 \neq 0 = f_j(\infty)$ for $j = 1, 2, 3$, and we have the representations

$$\frac{1}{z} \int_0^z e^{-\lambda A} x \, d\lambda = f_1(zA)x, \quad e^{-zA}x = f_2(zA)x, \quad \text{and } \lambda R(\lambda, A) = (1 - \lambda^{-1}A) = f_3(-\lambda^{-1}A)$$

for all $z \in \Sigma_\delta, \lambda \notin \overline{\Sigma}_{\omega'}$ and $x \in X(E)$. Hence we can apply Theorem 2.3.1 to obtain all stated equivalences.

Now assume $\alpha > 0$, then we first remark that also A^α has a bounded H_0^∞-maximal function by Proposition 2.1.13(2) with $\omega_M(A^\alpha) = \alpha\omega_M(A)$. Then the assertions (4), (5), (6) are equivalent by the same arguments given above, where we just replace A by A^α. Hence it is sufficient to show that (1) is equivalent to (1) with A^α in place of A. For this we define $f(\lambda) := \dfrac{1}{1+\lambda^\alpha}$ for all $\lambda \in \Sigma_{\pi-\alpha}$, then

$$t(t+A^\alpha)^{-1} = (1 + (t^{-1/\alpha}A)^\alpha)^{-1} = f(t^{-1/\alpha}A),$$

where in the last step we used again the composition rule for the functional calculus. Moreover, $f \in \mathcal{E}(\Sigma_{\pi-\alpha})$ with $f(0) = 1 \neq 0 = f(\infty)$, so the claim follows again from Theorem 2.3.1.

The remaining case $\alpha < 0$ can simply be reduced to the above case by considering $B := A^{-\alpha}$ and B^{-1} instead of A. Since the inversion $\lambda \mapsto \lambda^{-1}$ leaves all sectors invariant, and since by Proposition 2.1.13(2) we have $\omega_M(B^{-1}) = \omega_M(B)$, the claim also follows easily in this case. □

Remark 2.3.4. *Parts of the statement above can also be proved using integral representations, e.g. (2)⇒(3) via*

$$|t(t+A)^{-1}x|_E = \left|\int_0^\infty te^{-ts}T_s x\,ds\right|_E \leq \int_0^\infty te^{-ts}|T_s x|_E\,ds \leq \sup_{s>0}|T_s x|_E \cdot \underbrace{\int_0^\infty te^{-ts}\,ds}_{=1},$$

hence $\sup_{t>0}|t(t+A)^{-1}x|_E \leq \sup_{s>0}|T_s x|_E$. *Analogously, (6)⇒(2) can be proved using an integral representation of* T_t.

Combining Corollary 2.3.3 with Banach's Principle leads to the following standard applications, which in special situations are well known in semigroup and ergodic theory.

Corollary 2.3.5. *Let* $\omega_M(A) < \pi/2$ *and assume that the equivalent conditions of Corollary 2.3.3 are fulfilled.*

(1) If $T_t x \to x$ μ*-a.e. for* $t \to 0$ *for all* x *in a dense subset of* $X(E)$*, then the a.e. convergence holds for all* $x \in X(E)$.

(2) Assume that the semigroup $(T_t)_{t\geq 0}$ *is mean ergodic, i.e.* $\frac{1}{t}\int_0^t T_s x \to Px$ *for* $t \to +\infty$ *for all* $x \in X(E)$*, where* P *is the associated projection on* $\ker A$ *(for details cf. [DS58] Chapter VIII or [EN00] V.4). If* $\frac{1}{t}\int_0^t T_s x \to Px$ μ*-a.e. for* $t \to +\infty$ *for all* x *in a dense subset of* $X(E)$*, then the a.e. convergence holds for all* $x \in X(E)$.

□

Examples 2.3.6. We present some examples for the situation of Corollary 2.3.3 in the scalar-valued case. These are not new, but nevertheless we can give a different view with the methods described here. We will take a look on versions for vector-valued extensions in the examples in Section 2.5.

(1) Let $1 < p < +\infty$ and $X := L^p(\Omega)$. We assume that $\omega(A) < \pi/2$ and that the analytic contraction C_0-semigroup $(T_t)_{t\geq 0}$ generated by A in $L^p(\Omega)$ is positive. Then by [Fe98], Thm. 5.4.3 there is a constant $C > 0$ such that for all $x \in L^p(\Omega)$ the maximal estimate

$$\left\|\sup_{t>0}\left|\frac{1}{t}\int_0^t T_s x\,ds\right|\right\|_{L^p} \leq C\,\|x\|_{L^p} \tag{2.3.19}$$

holds. By the transference principle, A has a bounded H^∞-calculus with possibly $\omega_{H^\infty}(A) \geq \pi/2$ (cf. e.g. [KW04] Corollary 10.9 together with Theorem 4.2.1 in [Fe98]). Moreover, the angle $\omega_{\mathcal{R}}(A)$ of $\mathcal{R}$-sectoriality is strictly less then $\pi/2$, cf. [We01a] 4.d), and since the space $L^p(\Omega)$ has property (Δ), we have $\omega_{H^\infty}(A) = \omega_{\mathcal{R}}(A) < \pi/2$, cf. [KW01-a], Thm. 5.3). Hence we also have $\omega_M(A) < \pi/2$, so the equivalent conditions of Corollary 2.3.3 are fulfilled. Actually, the maximal estimate (2.3.19) is also a crucial tool in deriving $\omega_{\mathcal{R}}(A) < \pi/2$ by interpolation. In particular, in this situation the maximal estimate (2.3.19) implies for each $m \in \mathbb{N}_0$ also the maximal estimate

$$\left\|\sup_{t>0}\left|t^m\left(\frac{d}{dt}\right)^m T_t x\right|\right\|_{L^p} \leq C_m\,\|x\|_{L^p}, \tag{2.3.20}$$

since $(-t)^m\left(\frac{d}{dt}\right)^m T_t = f(tA)$, where $f(z) = z^m e^{-z}$. We note that this has very recently also been proven in [LeMX10], Corollary 4.2, as an application of corresponding maximal estimates for discrete semigroups, which are studied there in detail.

(2) Let again $1 < p < +\infty$, $X := L^p(\Omega)$ and $\omega(A) < \pi/2$. Instead of positivity we now assume that the analytic contraction C_0-semigroup $(T_t)_{t\geq 0}$ generated by A in $L^p(\Omega)$ is L^q-contractive for all $q \in [1, +\infty]$, i.e. $\|T_t x\|_{L^q} \leq \|x\|_{L^q}$ for all $x \in L^p(\Omega) \cap L^q(\Omega), t > 0$. Then by the classical Dunford-Hopf Ergodic Theorem (cf. [DS58], Theorem VIII.7.7), the estimate (2.3.19) holds for $(T_t)_{t\geq 0}$, hence we obtain with the same method as described in (1) that $\omega_M(A) = \omega_{H^\infty}(A) < \pi/2$, hence the equivalent conditions of Corollary 2.3.3 are also fulfilled in this situation.

(3) If $X = L^2(\Omega)$ and A is self-adjoint with $\sigma(A) \geq 0$, then of course $\omega_M(A) = \omega_{H^\infty}(A) = 0$. The question wether the equivalent conditions of Corollary 2.3.3 are fulfilled is investigated in [Ga81], proof of Theorem 1 and the observation 2) on page 17, where a characterization for the validity of (2.3.19) is formulated in terms of the projection-valued measure associated to A via the spectral theorem.

2.4 Interpolation of maximal functions

In this section we will derive a technique to interpolate the boundedness of maximal functions for consistent sectorial operators. For this we first give a different view on maximal estimates as the continuity of a suitable *linear* operator. If we consider the object $\varphi(\cdot A)x$ as an element of the space $X(L_*^\infty(E))$ (cf. Section 1.6), then $\|\sup_{t>0} |\varphi(tA)x|_E\|_X = \|\varphi(\cdot A)x\|_{X(L_*^\infty(E))}$, where we as usual do not distinguish $\varphi(\cdot A)x$ from the special version we choose. So we see that the family $(\varphi(tA))_{t>0}$ satisfies a maximal estimate if and only if $x \mapsto \varphi(\cdot A)x$ defines a bounded operator $X(E) \to X(L_*^\infty(E))$.

Now let X_0, X_1 be two Banach function spaces over the σ-finite measure space (Ω, μ), where at least one of them has absolute continuous norm, and E_0, E_1 be Banach spaces such that (E_0, E_1) is an interpolation couple. Then $(X_0(E_0), X_1(E_1))$ is an interpolation couple as well. For $\theta \in (0, 1)$ let $X_\theta := [X_0, X_1]_\theta$ and $E_\theta := [E_0, E_1]_\theta$ be the complex interpolation spaces.

Calderon defines in [Ca64] 13.5 an intermediate space $X^\theta := X_0^{1-\theta} X_1^\theta := \{f \in M(\mu) \mid \exists g_j \in X_j : |f| = |g_0|^{1-\theta}|g_1|^\theta\}$ endowed with the norm

$$\|f\|_{X_0^{1-\theta}X_1^\theta} := \inf\{\|g_0\|_{X_0}^{1-\theta} \cdot \|g_1\|_{X_1}^\theta \mid |f| = |g_0|^{1-\theta}|g_1|^\theta\}.$$

In fact, the definition given in [Ca64] is slightly different, but equivalent, cf. Remark 1.8 in [Pi79]. The space X^θ is consistent with the usual complex interpolation spaces X_θ in the following sense:

(Cal) $[X_0(E_0), X_1(E_1)]_\theta \subseteq X^\theta(E_\theta)$, the inclusion is norm-decreasing, and the spaces coincide with equal norm if X^θ has absolute continuous norm, cf. [Ca64] 13.6(i). Moreover it is shown in [KPS82], remark on p. 245, that X^θ has absolute continuous norm if at least on of the X_j has absolute continuous norm, which we have assumed. Thus we obtain $[X_0(E_0), X_1(E_1)]_\theta = X^\theta(E_\theta)$. Plugging in $E_0 = E_1 = \mathbb{C}$ yields $X^\theta = X_\theta$, then replacing E_j by $\ell^\infty_m(E_j)$ where $m \in \mathbb{N}$ leads to

$$[X_0(\ell^\infty_m(E_0)), X_1(\ell^\infty_m(E_1))]_\theta = X_\theta([\ell^\infty_m(E_0), \ell^\infty_m(E_1)]_\theta) = X_\theta(\ell^\infty_m(E_\theta)) \tag{2.4.21}$$

with equal norms. In the second step we used the general fact that with $L^\infty_0 := \overline{S_f(\Omega)}^{L^\infty}$, where $S_f(\Omega)$ is the space of step functions supported in a set of finite measure, we always have $[L^\infty(E_0), L^\infty_0(E_1)]_\theta = [L^\infty_0(E_0), L^\infty_0(E_1)]_\theta = L^\infty_0([E_0, E_1]_\theta)$ with equal norms, cf. [Tr78], 1.18.4 Remark 3 and the proof of Theorem 1.18.4.

For each $\theta \in [0,1]$ let A_θ be a sectorial operator in $X_\theta(E_\theta)$ such that the following consistency property is fulfilled:

$$(t + A_\theta)^{-1}x = (t + A_{\theta'})^{-1}x \quad \text{ for all } \theta, \theta' \in [0,1], x \in X_\theta(E_\theta) \cap X_{\theta'}(E_{\theta'}), t > 0.$$

Then by a connectedness argument we have that the resolvents of each two operators $A_\theta, A_{\theta'}$ coincide on the largest sector that is contained in both resolvent-sets, and if $\sigma > \max\{\omega(A_\theta), \omega(A_{\theta'})\}$ and $f \in H^\infty(\Sigma_\sigma)$ such that $f(A_j) \in L(X_j(E_j))$ for $j = 0, 1$, then $f(A_\theta)x = f(A_{\theta'})x$ for all $x \in X_\theta(E_\theta) \cap X_{\theta'}(E_{\theta'})$. Hence we will sometimes simply write $f(A)x$ instead of $f(A_\theta)x$ if $x \in X_\theta(E_\theta)$.

Now using abstract Stein interpolation (cf. [Vo92]) we obtain the following interpolation result for maximal estimates.

Proposition 2.4.1. *Let $\sigma \in (0, \pi]$, $f \in H^\infty(\Sigma_\sigma)$, and let $0 \le \delta_j < \sigma - \omega(A_j)$ such that the family $\big(f(zA_j)\big)_{z \in \overline{\Sigma}_{\delta_j} \setminus \{0\}}$ satisfies a maximal estimate in $X_j(E_j)$ for $j = 0, 1$. Then the family $\big(f(zA_\theta)\big)_{z \in \overline{\Sigma}_\delta \setminus \{0\}}$ satisfies a maximal estimate in $X_\theta(E_\theta)$, where $\delta = (1-\theta)\delta_0 + \theta\delta_1$.*

Proof. For $j = 0, 1$ choose $C_j > 0$ with $\big\| \sup_{z \in \overline{\Sigma}_{\delta_j} \setminus \{0\}} |f(zA_j)x|_{E_j} \big\|_{X_j} \le C_j \|x\|_{X_j(E_j)}$ for all $x \in X_j(E_j)$. We first make some observations. Fix $x \in D(A_\theta) \cap R(A_\theta)$ for a moment. Then, by Remark 2.1.4 and the Phragmén-Lindelöf-Theorem we have

$$\Big\| \sup_{z \in \overline{\Sigma}_{\delta_j} \setminus \{0\}} |f(zA)x|_{E_\theta} \Big\|_{X_\theta} = \lim_{m \to \infty} \Big\| \sup_{k \in \mathbb{N}_{\le m}} |f(t_k e^{\pm i\delta} A)x|_{E_\theta} \Big\|_{X_\theta},$$

where $(t_k)_{k \in \mathbb{N}}$ is any an enumeration of $\mathbb{Q}_{>0}$. Hence it is sufficient to estimate the norm of $(|f(t_k e^{\pm i\delta} A)x|_{E_\theta})_{k \in \mathbb{N}_{\le m}}$ with a constant independent of m. We note that the norm then is taken in $X_\theta(\ell^\infty_m(E_\theta))$, which is norm-isomorphic to $\ell^\infty_m(X_\theta(E_\theta))$, where the constants, of course, depend on m.

So fix a finite subset $\{t_1, \dots, t_m\} \subseteq \mathbb{R}_{>0}$, and define the strip $S := \{\lambda \in \mathbb{C} \,|\, 0 \le \mathrm{Re}(\lambda) \le 1\}$, $\delta(\lambda) := (1-\lambda)\delta_0 + \lambda\delta_1$ and $N(\lambda)x := \big(f(t_k e^{i\delta(\lambda)} A)x\big)_{k \in \mathbb{N}_{\le m}}$ for all $\lambda \in S$, $x \in X_0(E_0) \cap X_1(E_1)$. Define $D := \bigcap_{j \in \{0,1\}} D(A_j) \cap R(A_j)$, then D is dense in $X_0(E_0) \cap X_1(E_1)$: To see this we define standard approximation functions $\rho_n(z) := \frac{n}{n+z} - \frac{1}{1+nz}$, then $\rho_n(A_j) : X_j \to D(A_j) \cap R(A_j)$ is bijective and $\rho_n(A_j)x \to x$ in $X_j(E_j)$ for each $x \in X_j(E_j)$ and $j = 0, 1$ (cf. [KW04], Proposition 9.4 (b)). Hence, if $x \in X_0(E_0) \cap X_1(E_1)$, then $D \ni y_n := \rho_n(A_0)x = \rho_n(A_1)x \to x$ in $X_0(E_0) \cap X_1(E_1)$ for $n \to \infty$. For fixed $x \in D$ and $j = 0, 1$, the function

$$s \mapsto N(j + is)x = \big(f(t_k e^{s(\delta_0 - \delta_1)} e^{i\delta_j} A)x\big)_{k\in\mathbb{N}_{\le m}} \in X_j(\ell_m^\infty(E_j))$$

is continuous (since it is continuous considered as a mapping into $\ell_m^\infty(X_j(E_j))$) and bounded with

$$\|N(j + is)x\|_{X_j(\ell_m^\infty(E_j))} \le \Big\| \sup_{t>0} |f(te^{i\delta_j} A_j)x|_{E_j} \Big\|_{X_j} \le C_j \, \|x\|_{X_j(E_j)}.$$

Now choose $j \in \{0, 1\}$ such that $\delta_j = \max\{\delta_0, \delta_1\}$. Then $N(\cdot)x$ considered as a mapping $\mathring{S} \to X_j(\ell_m^\infty(E_j))$ is also continuous and even analytic, where we again use $X_j(\ell_m^\infty(E_j)) \cong_m \ell_m^\infty(X_j(E_j))$, hence it is also analytic considered as a mapping $\mathring{S} \to X_0(L_*^\infty(E_0)) + X_1(L_*^\infty(E_1))$. By abstract Stein interpolation we obtain

$$\|N(\theta)x\|_{X_\theta(\ell_m^\infty(E_\theta))} \le \underbrace{C_0^{1-\theta} C_1^\theta}_{=:C_\theta} \|x\|_{X_\theta(E_\theta)} \qquad \text{for all } x \in D,$$

hence (by density) $\| \sup_{k\in\mathbb{N}_{\le m}} |f(t_k e^{i\delta})Ax|_{E_\theta} \|_{X_\theta} \le C_\theta \, \|x\|_{X_\theta(E_\theta)}$ for all $x \in X_\theta(E_\theta)$. So the claim follows by the preceding discussion. $\square$

If we apply this lemma to the special functions $\psi(\lambda) = \frac{\lambda^{1/2}}{e^{\pm i\theta} - \lambda}$, then the simple fact that $\psi(e^{\pm i\delta}\lambda) = e^{\mp i\delta/2} \cdot \frac{\lambda^{1/2}}{e^{\pm i(\theta-\delta)} - \lambda}$ leads to an improvement of the angle of the interpolated maximal function.

Proposition 2.4.2. *Assume that the operators A_j have a bounded H_0^∞-maximal function in $X_j(E_j)$ for $j = 0, 1$. Then the operator A_θ in $X_\theta(E_\theta)$ has a bounded H_0^∞-maximal function with $\omega_M(A_\theta) \le (1 - \theta)\omega_M(A_0) + \theta\omega_M(A_1)$.*

Proof. Assume w.l.o.g. $\omega_M(A_1) > \omega_M(A_0)$. Let $\omega > (1 - \theta)\omega_M(A_0) + \theta\omega_M(A_1)$ and write $\omega = (1 - \theta)\omega_0 + \theta\omega_1$, where $\omega_j > \omega_M(A_j)$ and $\omega_1 > \omega_0$. Let $\delta_0 := \omega_1 - \omega_0 > 0$ and $\delta_1 := 0$, and choose $\sigma > 0$ such that $\max\{\omega_1 - (\omega_0 - \omega_M(A_0)), \omega_M(A_1)\} < \sigma < \omega_1$. Then $\delta_j < \sigma - \omega_M(A_j)$ for $j = 0, 1$, hence we can apply Lemma 2.4.1 to

$$\varphi(\lambda) := \frac{\lambda^{1/2}}{e^{\pm i\omega_1} - \lambda} \qquad \text{for all } \lambda \in \Sigma_\sigma$$

and obtain that $\big(\varphi(zA_\theta)\big)_{z\in\overline{\Sigma}_\delta\setminus\{0\}}$ satisfies a maximal estimate on $X_\theta(E_\theta)$, where $\delta = (1-\theta)(\omega_1 - \omega_0) = \omega_1 - \big((1 - \theta)\omega_0 + \theta\omega_1\big) = \omega_1 - \omega$. In particular, A has a bounded ψ-maximal function for

$$\psi(\lambda) := \varphi(e^{\pm i\delta}\lambda) = e^{\pm i\delta/2} \, \frac{\lambda^{1/2}}{e^{\mp i\omega} - \lambda}.$$

Hence $\omega_M(A) \le \omega$ by Theorem 2.1.9. $\square$

Combining Proposition 2.4.1 with Theorem 2.3.1 yields

Corollary 2.4.3. *Assume that the operators A_j have a bounded H_0^∞-maximal function in $X_j(E_j)$ for $j = 0, 1$. Assume that there exist $\sigma_0 > \max\{\omega_M(A_0), \omega_M(A_1)\}$ and $f_0 \in \mathcal{E}(\Sigma_\sigma)$ such that A_j has a bounded f-maximal function for $j = 0, 1$. Then, for every $\sigma > \max\{\omega_M(A_0), \omega_M(A_1)\}$ and $f \in \mathcal{E}(\Sigma_\sigma)$ the family $\big(f(zA_\theta)\big)_{z\in\Sigma_\delta}$ satisfies a maximal estimate on $X_\theta(E_\theta)$ for all*

$$0 \le \delta < \sigma - (1-\theta)\omega_M(A_0) - \theta\omega_M(A_1).$$

□

2.5 Maximal functions for tensor-extensions of A in vector-valued Banach function spaces

In concrete applications, it often happens that the operator in the vector-valued space $X(E)$ arises as a tensor extension of an operator A in the (scalar-valued) space X. We want to give an overview on how the developed theory applies to this special case.

Assume that X has absolute continuous norm and A is a sectorial operator in X. Since the algebraic tensor product $X \otimes E \subseteq X(E)$ contains all step functions, it is dense. We consider the algebraic tensor extension $A \otimes \mathrm{Id}_E$ on $X \otimes E$ and define the E-valued extension

$$A^E := \{(x, y) \in X(E) \times X(E) \,|\, \forall \varphi \in E' \,:\, \langle x, \varphi\rangle \in D(A) \text{ and } A\langle x, \varphi\rangle = \langle y, \varphi\rangle\}.$$

Proposition 2.5.1. *The following assertions are true:*

(1) A^E is densely defined and closed with $A \otimes \mathrm{Id}_E \subseteq A^E$,

(2) Let $\lambda \in \mathbb{C}$, then

$$\lambda \in \rho(A^E) \iff \lambda \in \rho(A) \text{ and } R(\lambda, A)^E \in L(X),$$

and in this case $R(\lambda, A)^E = \overline{R(\lambda, A) \otimes \mathrm{Id}_E} = R(\lambda, A^E)$,

(3) If $\rho(A^E) \neq \emptyset$, then $A^E = \overline{A \otimes \mathrm{Id}_E}$. In particular, if $D \subseteq D(A)$ is a core for A, then $D \otimes E$ is a core for A^E,

(4) If A^E is sectorial, $\sigma > \max\{\omega(A), \omega(A^E)\}$ and $f \in \mathfrak{B}(\Sigma_\sigma)$, then $f(A^E) = \overline{f(A) \otimes \mathrm{Id}_E}$.

Proof. Except for the implication $\Leftarrow$ in (2), this is proven for $X = L^p$ in [Ui98], Chapter 5, and the proof given there extends easily to the case of a Banach function space X. The remaining claim $\Leftarrow$ in (2) is an easy consequence of the definition. □

One problem is, that, only by the knowledge of A, there cannot be said too much about the vector-valued extension A^E as long as one has no further information on A^E. For this reason it is common to consider instead the bounded operators $R(\lambda, A)$ or e^{-tA}, if A is the generator of a C_0-semigroup. In particular, if $\max\{\omega(A), \omega(A^E)\} < \pi/2$, then $T_t^E = e^{-tA^E}$, and we can carry over maximal estimates for the semigroup. This method works particulary fine if the space X has nice tensor-extension properties, which we note in the next remark.

Remark 2.5.2. *(1) Let $X = L^2$ and H be a Hilbert space, then every bounded operator $T \in L(X)$ extends to a bounded operator T^H on $L^2(H)$ with $\|T^H\| = \|T\|$. If we apply this on resolvents, we obtain via Proposition 2.5.1 immediately the following result: If A is sectorial in L^2, then for each Hilbert space H the vector-valued extension A^H is sectorial as well with $\omega(A^H) = \omega(A)$.*

(2) Let $X = L^1$ and E be a Banach space, then every bounded operator $T \in L(X)$ extends to a bounded operator T^E on $L^1(E)$ with $\|T^E\| = \|T\|$. As in (1) we can conclude, that if A is sectorial in L^1, then for each Banach space E the vector-valued extension A^E is sectorial as well with $\omega(A^E) = \omega(A)$.

The first statement is a standard result in the theory of Hilbert spaces operators, cf. e.g. [We76], Satz 8.32, whereas (2) follows from the properties of the π- tensor product of Banach spaces, cf. e.g. [DF93], Proposition 3.2.

In the general case, one has no such nice extension properties, a prominent example for this situation is the Hilbert transform in L^p, $1 < p < +\infty$, which extends to a bounded operator on $L^p(E)$ if and only if E is a UMD-space. Nevertheless there are classes of operators that have nice extension properties, e.g. the class of *positive operators*, i.e. operators $S : X \to X$ such that $Sx \geq 0$ if $x \geq 0$ for all $x \in X$. Observe that positive operators on Banach function spaces are always bounded, cf. [Sc74] Theorem II.5.3. Moreover, positive operators always have bounded vector-valued extensions to any Banach space, and more generally, every operator dominated by a positive operator also has this extension property. This is the content of the following proposition, which is taken from [GCRdF85], Thm. V.1.12.. Since this proposition will also be important in subsequent chapters we give the full proof here.

Proposition 2.5.3. *Let $T \in L(X)$ be dominated by the positive operator $S : X \to X$, i.e. $|Tx| \leq S|x|$ for all $x \in X$. Then $T^E \in L(X(E))$ with $\|T^E\| \leq \|S\|$ and $|T^E F|_E \leq S|F|_E$ for all $F \in X(E)$.*

Proof. Assume first $F \in X \otimes E$, so there exist $m \in \mathbb{N}$ and $x \in X^m, v \in E^m$ such that $F = \sum_{k=1}^{m} x_k \otimes v_k$. Choose a countable subset $W \subseteq E'$ that is norming for the linear span $\langle \{v_k \,|\, k \in \mathbb{N}_{\leq m}\} \rangle$. Then for μ-a.e. $\omega \in \Omega$ the following holds:

$$\begin{aligned}
|T^E F|_E(\omega) &= \Big\| \sum_{k=1}^{m} (Tx_k)(\omega) \cdot v_k \Big\|_E = \sup_{\varphi \in W} \left| \varphi \left(\sum_{k=1}^{m} (Tx_k)(\omega) \cdot v_k \right) \right| \\
&= \sup_{\varphi \in W} \left| \sum_{k=1}^{m} (Tx_k)(\omega) \cdot \varphi(v_k) \right| = \sup_{\varphi \in W} \left| T \left(\sum_{k=1}^{m} \varphi(v_k)\, x_k \right)(\omega) \right| \\
&\leq S \left(\sup_{\varphi \in W} \left| \sum_{k=1}^{m} \varphi(v_k)\, x_k \right| \right)(\omega) = S \left(\sup_{\varphi \in W} \left| \varphi \circ \left(\sum_{k=1}^{m} x_k \otimes v_k \right) \right| \right)(\omega) \\
&= S \Big\| \sum_{k=1}^{m} x_k \otimes v_k(\cdot) \Big\|_E(\omega) = (S|F|_E)(\omega).
\end{aligned}$$

Now let $F_0 \in X(E)$ be arbitrary. Then one can choose a sequence $F \in (X \otimes E)^{\mathbb{N}}$ with $F_n \to F_0$ for $n \to \infty$. By possibly choosing a subsequence we can w.l.o.g. assume $F_n(\omega) \to F_0(\omega)$ and $(SF_n)(\omega) \to (SF_0)(\omega)$ for μ-a.e. $\omega \in \Omega$, hence we obtain

$$|T^E F_0|_E(\omega) = \lim_{n\to\infty} |T^E F_n|_E(\omega) \leq \lim_{n\to\infty} (S|F_n|_E)(\omega) = (S|F_0|_E)(\omega)$$

for μ-a.e. $\omega \in \Omega$. Hence $|T^E F_0|_E \in X$ and

$$\|T^E F_0\|_{X(E)} = \||T^E F_0|_E\|_X \leq \|S|F_0|_E\|_X \leq \|S\| \, \||F_0|_E\|_X = \|S\| \, \|F_0\|_{X(E)}.$$

□

We obtain immediately the following

Corollary 2.5.4. *Let $\mathcal{T} \subseteq L(X)$, and let $S : X \to X$ be a positive operator such that $|Tf| \leq S|f|$ for all $T \in \mathcal{T}, x \in X$. Then*

$$\Big\| \sup_{T\in\mathcal{T}} |T^E F|_E \Big\|_X \leq \|S\| \, \|F\|_{X(E)} \quad \textit{for all } F \in X(E).$$

Proof. Let $F \in X(E)$, then by Proposition 2.5.3 we have $\sup_{T\in\mathcal{T}} |T^E F|_E \leq S|F_E|$, hence also $\big\| \sup_{T\in\mathcal{T}} |T^E F|_E \big\|_X \leq \|S|F|_E\|_X \leq \|S\| \, \|F\|_{X(E)}$. □

Before we present more detailed examples, we will outline the general approach. We will usually consider an operator in two spaces $X_0(E_0)$ and $X_1(E_1)$. Although the following examples are just vector-valued extension of the examples in 2.3.6, we cannot just naively extend the angle $\omega_M(A)$ derived there. Instead of this, we will choose a "good" space $X_0(E_0)$, where we typically assume that E_0 is an UMD-space or even a Hilbert-space, such that the extended operator A^{E_0} has nice properties, as e.g. BIP or even an H^∞-calculus, which lead to an H_0^∞-maximal function with an angle $\omega_M(A^{E_0}) < \pi/2$. On the other end of the scala we choose an arbitrary space E_1 and get maximal estimates by extending the scalar estimates with the aid of positivity or domination. For this reason, we will assume that the operator A generates a C_0-semigroup $(T_t)_{t\geq 0}$ such that one of the following (scalar) maximal estimates hold:

$$\Big\| \sup_{t>0} \Big| \frac{1}{t} \int_0^t T_s x \, ds \Big| \Big\|_X \leq C \, \|x\|_X, \text{ or} \tag{2.5.22}$$

$$\Big\| \sup_{t>0} \big| T_t x \big| \Big\| \leq C \, \|x\|_X. \tag{2.5.23}$$

This estimate will be extended by Corollary 2.5.4 to the corresponding maximal estimates for A^{E_1}, hence by Proposition 2.2.3 we obtain that A^{E_1} has a bounded H_0^∞-maximal function with $\omega_M(A^{E_1}) \leq \pi/2$. Now assume that (E_0, E_1) is an interpolation couple, $\theta \in (0,1)$ and $E := [E_0, E_1]_\theta$ is the complex interpolation space. Then A^E has an H_0^∞-maximal function in $X_\theta(E)$ with $\omega_M(A^E) < \pi/2$, hence we can also apply Corollary 2.3.3 in these spaces.

Examples 2.5.5. (a) Generators of positive contraction semigroups in some L^p.

Let $X = L^p$ for some $1 < p < +\infty$ and A be a sectorial operator in X that generates a C_0-semigroup of positive contractions $(T_t)_{t\geq 0}$ on $L^p(\Omega)$. Then the estimate (2.5.22) always holds, cf. [Fe98], Theorem 5.4.3. Now let E be an arbitrary Banach space, then the semigroup and the maximal estimate (2.5.22) extend to $L^p(E)$ by positivity, hence the vector-valued extension A^E has a bounded maximal function with $\omega_M(A^E) \leq \pi/2$.

(b) The Laplace Operator.

Let $\Omega = \mathbb{R}^d$, $1 < p < +\infty$ and $A := -\Delta$ in $X := L^p := L^p(\mathbb{R}^d)$. Then A is the generator of an analytic positive contraction C_0-semigroup $(T_t)_{t\geq 0}$ in X, and the estimate (2.5.23) is fulfilled (this is obtain by estimating against the Hardy-Littlewood maximal function).[2] Now let E_0 be an UMD-space and E_1 be an arbitrary Banach space, then A^{E_0} has a bounded H^∞-calculus with $\omega_{H^\infty}(A^{E_0}) = 0$ (this is an easy consequence of the vector-valued Mikhlin multiplier theorem, cf. [KW04] Example 10.2 b)), hence $\omega_M(A^{E_0}) = \omega_{H^\infty}(A^{E_0}) = 0$. By the preceding arguments we have $\omega_M(A^{E_1}) \leq \pi/2$.

Now assume that (E_0, E_1) is an interpolation couple, $\theta \in (0,1)$ and $E := [E_0, E_1]_\theta$ is the complex interpolation space, then A^E has an H_0^∞-maximal function with $\omega_M(A^E) \leq \theta\pi/2 < \pi/2$.

Moreover it is well known that the semigroup $(e^{t\Delta})_{t\geq 0}$ is positive in L^p and $e^{t\Delta}x \leq Mx$ for all $x \in L^p$ with $x \geq 0$, where M is the Hardy-Littlewood maximal operator. Hence

$$\Big\| \sup_{t>0} |e^{t\Delta^E} x|_E \Big\|_p \leq \Big\| \sup_{t>0} e^{t\Delta^E} |x|_E \Big\|_p \leq \|M|x|_E\|_p \lesssim \|x\|_{L^p(E)} \quad \text{for all } x \in L^p(E),$$

since M is bounded in L^p. Hence we can apply Theorem 2.3.1 and obtain

Proposition 2.5.6. *Let (E_0, E_1) be an interpolation couple, where E_0 is an UMD-space. Let $\theta \in (0,1)$ and $E := [E_0, E_1]_\theta$. Then the negative Laplace operator $-\Delta^E$ in $L^p(E)$ has a bounded maximal function with $\omega_M(-\Delta^E) \leq \theta\pi/2$. Moreover, for all $\sigma > \theta\pi/2$, $\delta \in [0, \sigma - \theta\pi/2)$ and $f \in \mathcal{E}(\Sigma_\sigma)$ there is a constant $C > 0$ such that the following maximal estimate holds:*

$$\Big\| \sup_{z\in\Sigma_\delta} |f(-z\Delta^E)x|_E \Big\|_p \leq C\,\|x\|_{L^p(E)} \quad \textit{for all } x \in L^p(E). \tag{2.5.24}$$

□

In particular the assumptions of Corollary 2.3.3 are satisfied, hence $-\Delta^E$ in $L^p(E)$ fulfills the equivalent assertions (1)-(6) from Corollary 2.3.3.

(c) Generators of L^1-L^∞-contractive semigroups.

Now we assume more generally that A is a sectorial operator in L^2 that generates an L^1-L^∞-contractive semigroup, i.e. $(T_t)_{t\geq 0}$ is L^q-contractive for all $q \in [1, +\infty]$. Then we get the following generalization of [Ta09], Theorem 1.5:

[2] Of course, also (2.5.22) is fulfilled, this follows by example (a), on the other hand it is a simple consequence of (2.5.23) by the triangle inequality.

Proposition 2.5.7. *Let A be a sectorial operator in $L^2(\Omega)$ with $\omega_0 := \omega(A) < \pi/2$ such that the generated semigroup $(T(t))_{t\geq 0}$ is L^q-contractive for all $q \in [1, +\infty]$. Let (H, E) be an interpolation couple, where H is a Hilbert space and E an arbitrary Banach space, let $\theta \in (0, 1)$ and $Y := [H, E]_\theta$. Let $1 < p < +\infty$ such that $|2/p - 1| < \theta$ and $0 \leq \delta < \left(\frac{\pi}{2} - \omega_0\right)(1 - \theta)$. Then $(T(t))_{t\geq 0}$ extends to an analytic contraction semigroup $(T_p^Y(z))_{z\in\Sigma_\delta}$ on $L^p(Y)$, and there is a $C > 0$ such that*

$$\forall\, x \in L^p(Y) \,:\, \Big\| \sup_{z\in\Sigma_\delta} |T_p^Y(z)x|_Y \Big\|_{L^p} \leq C\, \|x\|_{L^p(Y)}. \tag{2.5.25}$$

We note that in [Ta09], Theorem 1.5, the assumptions are stronger: the Banach space E has to be an UMD-space, and moreover A is assumed to be self-adjoint.

On the other hand, the formulation of Theorem 1.5 from [Ta09] is a little more general in the sense that not only complex interpolation spaces $Y = [H, E]_\theta$ are considered, but more generally closed subquotients of the complex interpolation space $[H, E]_\theta$, i.e. quotient spaces $Y = Y_0/Y_1$, where $Y_1 \leq Y_0 \leq [H, E]_\theta$ are closed subspaces. In fact, this is a corollary, once the maximal estimate (2.5.25) is shown for the space $[H, E]_\theta$, and the proof only relies on the special situation that the semigroup T^Y in the vector-valued space is a tensor extension of a semigroup T in the scalar-valued space. Thus the same arguments also work in our situations, and in Proposition 2.5.7 we could also take Y just to be a Banach spaces isomorphic to a closed subquotient of $[H, E]_\theta$.

For the proof of Proposition 2.5.7 we will need the following density property.

Lemma 2.5.8. *Let X_0, X_1 be Banach function spaces over Ω and (E_0, E_1) be an interpolation couple. Then the set of step functions $S(\Omega, E_0 \cap E_1) \subseteq (X_0 \cap X_1) \otimes (E_0 \cap E_1)$ is dense in $X_0(E_0) \cap X_1(E_1)$.*

Proof. We will prove the claim in three steps.

1. Choose an increasing sequence $(\Omega_n)_{n\in\mathbb{N}}$ of measurable subsets of Ω of finite measure with $\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega$. Let $x \in X_0(E_0) \cap X_1(E_1)$ and define $x_n := x\, \mathbb{1}_{\Omega_n}$ for each $n \in \mathbb{N}$. Then $x_n \to x$ pointwise a.e. for $n \to \infty$ in both spaces $E_j, j = 0, 1$, and $|x_n|_{E_j} \leq |x|$ for all $n \in \mathbb{N}$, hence $x_n \to x$ for $n \to \infty$ in both spaces $X_j(E_j), j = 0, 1$. By this we have shown: the subspace of functions supported in a set of finite measure is dense in $X_0(E_0) \cap X_1(E_1)$.

2. Now let $x \in X_0(E_0) \cap X_1(E_1)$ where $\Omega_0 := \mathrm{supp}(x)$ has finite measure. We aim to define x_n in a way such that $|x_n|_{E_0} \vee |x_n|_{E_1} \leq C_n$ and $x_n \to x$ in both spaces $X_j(E_j), j = 0, 1$. That will imply that the subspace of functions supported in a set of finite measure and essentially bounded in $E_0 \cap E_1$ is dense. For this define

$$x_n := \left(\frac{n}{|x|_{E_0} \vee |x|_{E_1}} \wedge 1 \right) \cdot x\, \mathbb{1}_{\{|x|_{E_0} \vee |x|_{E_1} < +\infty\}} \quad \text{for all } n \in \mathbb{N}.$$

Then $|x_n|_{E_j} \leq |x|_{E_j} \wedge n$ for all $n \in \mathbb{N}, j = 0, 1$, and if $\omega \in \{|x|_{E_0} \vee |x|_{E_1} < +\infty\}$, then $x_n(\omega) = x(\omega)$ for all $n > \|x(\omega)\|_{E_0} \vee \|x(\omega)\|_{E_1}$, hence $x_n(\omega) \to x(\omega)$ for $n \to \infty$ in $E_0 \cap E_1$.

3. Now let finally $x \in X_0(E_0) \cap X_1(E_1) \cap L^\infty(E_0 \cap E_1)$, where $\Omega_0 := \operatorname{supp}(x)$ has finite measure. We can now construct step functions that approximate x pointwise a.e. as it is done in [Ca64], Section 33.6, p.171f, and a majorant is given by $\|x\|_{L^\infty(E_0 \cap E_1)} \cdot \mathbb{1}_{\Omega_0}$ for $j = 0, 1$. □

Proof of Proposition 2.5.7. First we note that the L^q-contractivity for all $q \in [1, +\infty]$ has two immediate consequences:

1. The analytic contraction C_0-semigroup $(T_2(t))_{t \geq 0}$ generated by $A_2 := A$ in $L^2(\Omega)$ can be extrapolated to a C_0-semigroup $(T_q(t))_{t \geq 0}$ of contractions in each $L^q(\Omega)$ for $1 \leq q < +\infty$, and the family of semigroups $(T_q)_{q \in [1,+\infty)}$ is consistent.

2. The semigroup $(T(t))_{t \geq 0}$ is dominated by a strongly measurable positive semigroup $(S_t)_{t \geq 0}$ which is contractive in each space $L^p, p \in [1, +\infty)$, i.e. $|T(t)x| \leq S_t|x|$ for all $x \in L^2, t > 0$ (cf. [Kr85] or [Ta09], Theorem 3.1 for a detailed exposition).

Hence, for each Banach space Z and $q \in [1, +\infty)$, the operator A_q has a well-defined sectorial tensor extension A_q^Z on $L^q(\Omega, Z)$ with $\omega(A_q^Z) \leq \pi/2$, since it is the generator of the strongly continuous contraction semigroup $T_q^Z = (T_q^Z(t))_{t \geq 0}$. Since $|2/p - 1| < \theta$, we can choose a $q \in (1, +\infty)$ with $\frac{1}{p} = (1-\theta)\frac{1}{2} + \theta\frac{1}{q}$. Then the resolvents of the operators A_2^H and A_q^E are consistent as well, since the space $(L^2 \cap L^q) \otimes (H \cap E)$ is dense in $L^2(\Omega, H) \cap L^q(\Omega, E)$.

The dominating semigroup $(S_t)_{t \geq 0}$ fulfills the maximal estimate (2.5.22) by the classical Dunford-Hopf ergodic theorem (cf. [DS58], Thm. VIII.7.7), hence by Corollary 2.5.4 this maximal estimate carries over to the vector-valued extension A_q^E, and then finally by Proposition 2.2.3 we obtain $\omega_M(A_q^E) \leq \pi/2$. On the other hand, by Remark 2.5.2 (1) the operator A_2^H is sectorial with $\omega(A_2^H) = \omega(A_2) < \pi/2$, and since A_2^H is the generator of an analytic C_0-semigroup of contractions in the Hilbert space $L^2(\Omega, H)$, by [KW04], Cor. 10.12 we obtain that A_2^H has an H^∞-calculus with optimal angle, hence $\omega_M(A_2^H) = \omega_{H^\infty}(A_2^H) = \omega(A_2^H) = \omega_0 < \pi/2$.

Now we can interpolate the H_0^∞-maximal function with Proposition 2.4.1 and obtain that the operator A_p^Y has a bounded maximal function with

$$\omega_M(A_p^Y) \leq (1-\theta)\omega_M(A_2^H) + \theta\omega_M(A_q^E) \leq (1-\theta)\omega_0 + \theta\pi/2 < \pi/2.$$

In particular, the semigroup T_p^Y is analytic, and by the same argument as given for A_q^E above we see that also A_p^Y satisfies the maximal estimate (2.2.9), hence the assumptions of Corollary 2.3.3 are fulfilled, and we obtain the maximal estimate (2.5.25), since

$$\delta < \left(\frac{\pi}{2} - \omega_0\right)(1-\theta) = \frac{\pi}{2} - (1-\theta)\omega_0 - \theta\frac{\pi}{2} \leq \frac{\pi}{2} - \omega_M(A_p^Y).$$

□

The class of spaces $[H, E]_\theta$ occurring in Proposition 2.5.7 are investigated by Pisier in [Pi79], where he calls these spaces θ-Hilbertian and asks for general characterizations of θ-Hilbertian spaces. A partial answer is given for Banach function spaces by [Pi79], Theorem 2.3, which states that a Banach lattice Y is order-isomorphic to a Banach function space $[H, E]_\theta$, where H, E are Banach function spaces and H is in addition a Hilbert space, if and only if Y is p-convex and p'-concave with corresponding constants $M^{(p)}(Y) = M_{(p')}(Y) = 1$, where $p > 1$ and $1/p = (1-\theta)/1 + \theta/2$ (observe that this implies $1 < p < 2 < p' < +\infty$). Further investigations of this topic can be found in [Pi08].

Chapter 3

$\mathcal{R}_s$-boundedness and $\mathcal{R}_s$-sectorial operators

3.1 $\mathcal{R}_s$-boundedness

In this section we introduce the notion of $\mathcal{R}_s$-boundedness, which is the central technical tool in this chapter. The concept of $\mathcal{R}_s$-boundedness in L^p-spaces is a subject of classical harmonic analysis, although it is not denoted in this way. It is mainly considered in the framework of vector-valued singular integrals, cf. e.g. the monographs [St93], [GCRdF85] or [Gr04]. At the end of this section we will give some examples that are based on results of classical harmonic analysis, in particular we will give more bibliographical references. The explicit notion of $\mathcal{R}_s$-boundedness in L^p-spaces was introduced bei Lutz Weis in [We01a], where also elementary properties are introduced. It was used there and in the sequel e.g. in [BK02] to show maximal regularity of certain sectorial operators. There the central fact is used that in L^p-spaces $\mathcal{R}_2$-boundedness is equivalent to $\mathcal{R}$-boundedness if $1 \leq p < +\infty$ (this is shown in a slightly more general version in Remark 3.1.7), which in turn is a central tool in dealing with the question of maximal regularity. In fact, many of the assertions we present in this section are already indicated in [We01a], or they are variants of corresponding assertions for $\mathcal{R}$-boundedness as shown in [KW04], Chapter 2.

In this section, let $(\Omega, \mu), (\widetilde{\Omega}, \widetilde{\mu})$ be σ-finite measure space and X, Y be complex Banach function spaces over (Ω, μ) and $(\widetilde{\Omega}, \widetilde{\mu})$, respectively, with absolute continuous norm, and let $s \in [1, +\infty]$. Our standard examples will be the spaces $L^p := L^p(\mu, \mathbb{C})$ with $p \in [1, +\infty)$. We note that the basic ideas and definitions presented in this chapter easily generalize to the more general setting of an abstract Banach lattice using the Krivine-calculus (cf. e.g. [LT96], Section II.1.d). Nevertheless, in view of our later applications we will need stronger assumptions on the Banach lattices which make it natural to consider Banach function spaces, and doing so we can avoid non-essential technical difficulties.

Definition 3.1.1 ($\mathcal{R}_s$-boundedness)**.** Let $\mathcal{T} \subseteq \mathcal{L}(X, Y)$. The set $\mathcal{T}$ is called $\mathcal{R}_s$*-bounded*, if there exists a constant $C \in \mathbb{R}_{>0}$, such that for all $n \in \mathbb{N}, T \in \mathcal{T}^n$ and $x \in X^n$:

$$\Big\| \Big(\sum_{j=1}^{n} |T_j x_j|^s \Big)^{1/s} \Big\|_Y \leq C \Big\| \Big(\sum_{j=1}^{n} |x_j|^s \Big)^{1/s} \Big\|_X, \quad \text{if } s < +\infty, \tag{3.1.1}$$

$$\Big\| \sup_{j \in \mathbb{N}_{\leq n}} |T_j x_j| \Big\|_Y \leq C \Big\| \sup_{j \in \mathbb{N}_{\leq n}} |x_j| \Big\|_X, \quad \text{if } s = \infty. \tag{3.1.2}$$

The infimum of all such bounds C is called the $\mathcal{R}_s$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}_s(\mathcal{T})$. If $T \in L(X, Y)$, we say that T is $\mathcal{R}_s$-bounded if $\{T\}$ is $\mathcal{R}_s$-bounded and let $\mathcal{R}_s(T) := \mathcal{R}_s(\{T\})$.

Observe that taking $n = 1$ in (3.1.1) immediately yields the following

Remark 3.1.2. *Let $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ be $\mathcal{R}_s$-bounded, then $\mathcal{T} \subseteq L(X, Y)$, and $\mathcal{T}$ is norm-bounded with $\sup_{T \in \mathcal{T}} \|T\| \leq \mathcal{R}_s(\mathcal{T})$.*

Moreover, the following is a direct consequence of the definition: If $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ and $C > 0$, then $\mathcal{T}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\mathcal{T}) \leq C$ if and only if $\mathcal{T}_0$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\mathcal{T}_s) \leq C$ for all finite subsets $\mathcal{T}_0 \subseteq \mathcal{T}$, and in this case

$$\mathcal{R}_s(\mathcal{T}) = \sup\{\mathcal{R}_s(\mathcal{T}_0) \,|\, \mathcal{T}_0 \subseteq \mathcal{T} \text{ finite}\}.$$

We note that in the sequel we will be faced with densely defined operators $A : X \supseteq D(A) \to Y$ such that $\overline{A}$ is $\mathcal{R}_s$-bounded. In this situation we will also simply say that A is $\mathcal{R}_s$-bounded and define $\mathcal{R}_s(A) := \mathcal{R}_s(\{A\}) := \mathcal{R}_s(\{\overline{A}\})$.

If one considers $x \in X^n$ as an element of $M(\Omega, \mathbb{C}^n)$, we have

$$\Big\| \Big(\sum_{j=1}^{n} |x_j|^s \Big)^{1/s} \Big\|_X = \|x\|_{X(\ell^s_n)} \quad \text{and} \; \Big\| \sup_{j \in \mathbb{N}_{\leq n}} |x_j|^s \Big\|_X = \|x\|_{X(\ell^\infty_n)},$$

respectively. So $T \in \mathcal{T}^n$ can be identified with the diagonal operator

$$\widetilde{T} : X^n \to Y^n, x \mapsto (T_j x_j)_{j \in \mathbb{N}_{\leq n}},$$

which can be considered as a bounded operator $X(\ell^s_n) \to Y(\ell^s_n)$. With this notation the set $\mathcal{T}$ is $\mathcal{R}_s$-bounded if and only if the set of operators

$$\{\widetilde{T} \,|\, T \in \mathcal{T}^n, n \in \mathbb{N}\} \subseteq \bigcup_{n \in \mathbb{N}} L\big(X(\ell^s_n), Y(\ell^s_n)\big)$$

is uniformly bounded.

Remark 3.1.3. *Since X, Y have the Fatou property, we can replace the finite sums in (3.1.1) in the definition of $\mathcal{R}_s$-boundedness by infinite series and the suprema in (3.1.2) by suprema over all $\mathbb{N}$. In particular, a single operator $T \in \mathcal{L}(X, Y)$ is $\mathcal{R}_s$-bounded if and only if the diagonal operator*

$$(x_n)_{n \in \mathbb{N}} \mapsto (T x_n)_{n \in \mathbb{N}}$$

induces a bounded operator $\widetilde{T}_s \in L(X(\ell^s), Y(\ell^s))$, and in this case $\mathcal{R}_s(T) = \|\widetilde{T}_s\|_{L(X(\ell^s), Y(\ell^s))}$.

Definition/Proposition 3.1.4. Let $\mathrm{R_sL}(X,Y) := \{T \in L(X,Y) \,|\, T \text{ is } \mathcal{R}_s\textit{-bounded}\}$. Then $\mathrm{R_sL}(X,Y)$, endowed with the norm $\mathcal{R}_s(\cdot)$, is a Banach space.

Proof. Let $(T_n)_{n\in\mathbb{N}} \in \mathrm{R_sL}(X,Y)^{\mathbb{N}}$ such that $\mathcal{R}_s(T_n - T_m) \to 0$ for $n, m \to \infty$. Since $\|\cdot\|_{L(X,Y)} \leq \mathcal{R}_s(\cdot)$ there is a $T_0 \in L(X,Y)$ such that $\|T_n - T_0\|_{L(X,Y)} \to 0$ for $n \to \infty$. We have to show that

(i) $T_0 \in \mathrm{R_sL}(X,Y)$,

(ii) $\mathcal{R}_s(T_n - T) \to 0$ for $n \to \infty$.

By our assumption we have $\|\widetilde{T_n - T_m}\|_{L(X(\ell^s),Y(\ell^s))} \to 0$ for $m, n \to \infty$ (we drop the lower index s for operators in this proof to hold notations concise), so in particular

$$C := \sup_{n\in\mathbb{N}} \|\widetilde{T_n}\|_{L(X(\ell^s),Y(\ell^s))} < +\infty.$$

(i) Let $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$, then for each $n \in \mathbb{N}$ we have

$$\Big\|\Big(\sum_{j=1}^m |T_n x_j|^s\Big)^{1/s}\Big\|_Y \leq C \Big\|\Big(\sum_{j=1}^m |x_j|^s\Big)^{1/s}\Big\|_X,$$

and moreover,

$$\begin{aligned}\Big\|\Big(\sum_{j=1}^m |T_n x_j|^s\Big)^{1/s} - \Big(\sum_{j=1}^m |T x_j|^s\Big)^{1/s}\Big\|_Y \quad &\leq \quad \Big\|\Big(\sum_{j=1}^m |(T_n - T)x_j|^s\Big)^{1/s}\Big\|_Y \leq \Big\|\sum_{j=1}^m |(T_n - T)x_j|\Big\|_Y \\ &\leq \quad \sum_{j=1}^m \|(T_n - T)x_j\|_Y \to 0 \quad \text{ if } n \to \infty,\end{aligned}$$

hence

$$\Big\|\Big(\sum_{j=1}^m |T x_j|^s\Big)^{1/s}\Big\|_Y = \lim_{n\to\infty} \Big\|\Big(\sum_{j=1}^m |T_n x_j|^s\Big)^{1/s}\Big\|_Y \leq C \Big\|\Big(\sum_{j=1}^n |x_j|^s\Big)^{1/s}\Big\|_X$$

(with the usual modifications if $s = +\infty$). So T is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(T) \leq C$.

(ii) We will again only consider the case $s < +\infty$, the case $s = +\infty$ follows easily by standard modifications. Let $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\|(\widetilde{T_n - T_k})\|_{L(X(\ell^s),Y(\ell^s))} < \varepsilon/3$ for all $k, n \geq n_0$. Let $n \geq n_0$ and $x \in X(\ell^s)$ with $\|x\|_{X(\ell^s)} \leq 1$. By the dominated convergence theorem we have

$$\Big\|\Big(\sum_{n=m}^\infty |x_n|^s\Big)^{1/s}\Big\|_X \to 0 \quad \text{ if } m \to \infty,$$

hence we can choose an $m \in \mathbb{N}_{\geq n_0}$ such that

$$2C\,\|\tilde{x}\|_{X(\ell^s)} = 2C\,\Big\|\Big(\sum_{n=m}^\infty |x_n|^s\Big)^{1/s}\Big\|_X < \varepsilon/3.$$

Define $\tilde{x} \in X(\ell^s)$ by

$$\tilde{x}_j := \begin{cases} 0 & \text{if } j < m, \\ x_n & \text{if } j \geq m. \end{cases}$$

Finally choose $k \in \mathbb{N}_{\geq n_0}$ such that $(m-1)\|(T_k - T)x_j\|_Y < \varepsilon/3$ for all $j = 1, \ldots, m$, then we obtain

$$\begin{aligned}
\|\widetilde{(T_n - T)}x\|_{Y(\ell^s)} &\leq \|\widetilde{(T_n - T_k)}x\|_{Y(\ell^s)} + \|\widetilde{(T_k - T)}x\|_{Y(\ell^s)} \\
&\leq \|\widetilde{(T_n - T_k)}\|_{L(X(\ell^s),Y(\ell^s))} \|x\|_{X(\ell^s)} + \Big\|\Big(\sum_{j=1}^{\infty} |(T_k - T)x_j|^s\Big)^{1/s}\Big\|_Y \\
&\leq \mathcal{R}_s(T_n - T_k) + \Big\|\Big(\sum_{j=1}^{m-1} |(T_k - T)x_j|^s\Big)^{1/s}\Big\|_Y + \Big\|\Big(\sum_{j=m}^{\infty} |(T_k - T)x_j|^s\Big)^{1/s}\Big\|_Y \\
&\leq \varepsilon/3 + \sum_{j=1}^{m-1} \|(T_k - T)x_j\|_Y + \big\|\widetilde{(T_k - T)}\tilde{x}\big\|_{Y(\ell^s)} \\
&\leq \varepsilon/3 + \varepsilon/3 + 2C\,\|\tilde{x}\|_{X(\ell^s)} \leq 2\varepsilon/3 + \varepsilon/3 = \varepsilon.
\end{aligned}$$

□

If the spaces X, Y have appropriate concavity and convexity properties, the norm-boundedness of a set of operators also implies the $\mathcal{R}_s$-boundedness for a certain range of s:

Remark 3.1.5. *Let $\mathcal{T} \subseteq L(X,Y)$. Let X be p-concave and Y be q-convex for some $1 \leq p \leq q \leq +\infty$, and let $\mathcal{T}$ be norm-bounded. Then $\mathcal{T}$ is $\mathcal{R}_s$-bounded for all $s \in [p,q]$. In particular, if $X = L^p, Y = L^q$ and $\mathcal{T}$ is norm-bounded, then $\mathcal{T}$ is $\mathcal{R}_s$-bounded for all $s \in [p,q]$.*

Proof. Let $s \in [p,q]$ and define $C := \sup_{T\in\mathcal{T}} \|T\| < +\infty$. Let $n \in \mathbb{N}, T \in \mathcal{T}^n$ and $x \in X^n$, then

$$\begin{aligned}
\|\widetilde{T}x\|_{Y(\ell^s_n)} &\leq M^{(s)}(Y)\,\|\widetilde{T}x\|_{\ell^s_n(Y)} = M^{(s)}(Y)\,\big\|(\|T_j x_j\|_Y)_j\big\|_{\ell^s_n} \leq M^{(s)}(Y)\,\big\|(C\,\|x_j\|_X)_j\big\|_{\ell^s_n} \\
&= M^{(s)}(Y)\,C\,\|x\|_{\ell^s_n(X)} \leq C\,M^{(s)}(Y) M_{(s)}(X) \cdot \|x\|_{X(\ell^s_n)}.
\end{aligned}$$

Here we used the fact that X is also s-concave and Y is s-convex since $p \leq s \leq q$, cf. Proposition 1.6.15, and the supplementary assertions follows from the fact that L^p is always p-concave and p-convex, cf. Example 1.6.17 (a). □

Moreover, the union of $\mathcal{R}_s$-bounded sets is again $\mathcal{R}_s$-bounded, if the $\mathcal{R}_s$-norms are summable:

Remark 3.1.6. *Let $\mathfrak{T} \subseteq \mathrm{Pot}(L(X,Y))$ such that $\sum_{\mathcal{T}\in\mathfrak{T}} \mathcal{R}_s(\mathcal{T}) < +\infty$, then also $\bigcup\mathfrak{T}$ is $\mathcal{R}_s$-bounded, and $\mathcal{R}_s(\bigcup\mathfrak{T}) \leq \sum_{\mathcal{T}\in\mathfrak{T}} \mathcal{R}_s(\mathcal{T})$.*

Proof. Define $C := \sum_{\mathcal{T}\in\mathfrak{T}} \mathcal{R}_s(\mathcal{T}) < +\infty$. Let $n \in \mathbb{N}, x \in X^n$ and $T \in \big(\bigcup\mathfrak{T}\big)^n$. Choose a finite subset $\mathfrak{T}_0 \subseteq \mathfrak{T}$ and an injective mapping $J : \mathfrak{T}_0 \to \mathrm{Pot}(\mathbb{N}_{\leq n})$ such that $J(\mathfrak{T}_0)$ is a partition of

$\mathbb{N}_{\leq n}$ and $T(J(\mathcal{T})) \subseteq \mathcal{T}$ for all $\mathcal{T} \in \mathfrak{T}_0$. Then the triangle inequality in $Y(\ell_n^s)$ yields

$$\begin{aligned}
\Big\| \Big(\sum_{j=1}^n |T_j x_j|^s \Big)^{1/s} \Big\|_Y &= \Big\| \Big(\sum_{\mathcal{T}\in\mathfrak{T}_0} \sum_{j\in J_{\mathcal{T}}} |T_j x_j|^s \Big)^{1/s} \Big\|_Y \leq \sum_{\mathcal{T}\in\mathfrak{T}_0} \Big\| \Big(\sum_{j\in J_{\mathcal{T}}} |T_j x_j|^s \Big)^{1/s} \Big\|_Y \\
&\leq \sum_{\mathcal{T}\in\mathfrak{T}_0} \mathcal{R}_s(\mathcal{T}) \Big\| \Big(\sum_{j\in J_{\mathcal{T}}} |x_j|^s \Big)^{1/s} \Big\|_X \leq C \Big\| \Big(\sum_{j=1}^n |x_j|^s \Big)^{1/s} \Big\|_X.
\end{aligned}$$

□

We recall the related definition of $\mathcal{R}$-boundedness from Section 1.3: A set $\mathcal{T} \subseteq L(X,Y)$ is called *$\mathcal{R}$-bounded*, if there exists a constant $C \in \mathbb{R}_{>0}$, such that for all $n \in \mathbb{N}, T \in \mathcal{T}^n$ and $x \in X^n$:

$$\mathbb{E}\Big| \sum_{j=1}^n r_j \otimes T_j x_j \Big|_X \leq C\, \mathbb{E}\Big| \sum_{j=1}^n r_j \otimes x_j \Big|_X, \tag{3.1.3}$$

where $(r_j)_{j\in\mathbb{N}}$ is any sequence of independent symmetric ±1-valued, i.e. Bernoulli-distributed, random variables on some probability space, and $\mathbb{E}$ denotes the expectation with respect to the corresponding probability measure. We usually choose the Rademacher functions $r_j(t) := \operatorname{sgn}\sin(2^j\pi t), j \in \mathbb{N}$ on $[0,1]$.

Then with Proposition 1.6.21 we obtain the following close relation between $\mathcal{R}$-boundedness and $\mathcal{R}_2$-boundedness as already indicated in the introduction of this section.

Remark 3.1.7. *Let $\mathcal{T} \subseteq L(X,Y)$.*

(1) If X is r-concave for some $r < +\infty$, then $\mathcal{R}$-boundedness of $\mathcal{T}$ implies that $\mathcal{T}$ is $\mathcal{R}_2$-bounded,

(2) If Y is r-concave for some $r < +\infty$, then $\mathcal{R}_2$-boundedness of $\mathcal{T}$ implies that $\mathcal{T}$ is $\mathcal{R}$-bounded.

In particular, if both X and Y are r-concave for some $r < +\infty$, then $\mathcal{T}$ is $\mathcal{R}$-bounded if and only if $\mathcal{T}$ is $\mathcal{R}_2$-bounded. □

We will now turn to some persistence properties of $\mathcal{R}_s$-boundedness that correspond to persistence properties of $\mathcal{R}$-boundedness, cf. e.g. [KW04], Section 2.

Proposition 3.1.8. *Let $\mathcal{S} \subseteq L(X,Y)$ be $\mathcal{R}_s$-bounded.*

(1) If $\mathcal{T} \subseteq L(X,Y)$ is $\mathcal{R}_s$-bounded, then the set $\mathcal{S}+\mathcal{T} := \{S+T \mid S \in \mathcal{S}, T \in \mathcal{T}\}$ is $\mathcal{R}_s$-bounded, and

$$\mathcal{R}_s(\mathcal{S}+\mathcal{T}) \leq \mathcal{R}_s(\mathcal{S}) + \mathcal{R}_s(\mathcal{T}).$$

(2) If V is another Banach function space and $\mathcal{T} \subseteq L(V,X)$ is $\mathcal{R}_s$-bounded, then the set $\mathcal{S}\mathcal{T} := \{ST \mid S \in \mathcal{S}, T \in \mathcal{T}\}$ is $\mathcal{R}_s$-bounded with

$$\mathcal{R}_s(\mathcal{S}\mathcal{T}) \leq \mathcal{R}_s(\mathcal{S}) \cdot \mathcal{R}_s(\mathcal{T}).$$

Proof. This follows immediately by properties of the norm in the spaces $X(\ell_n^s), Y(\ell_n^s), V(\ell_n^s)$. □

The following convexity property is an important tool and is again a variant of a corresponding result for $\mathcal{R}$-bounded operators, cf. e.g. [KW04], Theorem 2.13.

Proposition 3.1.9. *Let $\mathcal{T} \subseteq L(X,Y)$ be $\mathcal{R}_s$-bounded. Then the strong closure of the absolute convex hull $\overline{\mathrm{aco}}^s(\mathcal{T}) \subseteq L(X,Y)$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\overline{\mathrm{aco}}^s(\mathcal{T})) = \mathcal{R}_s(\mathcal{T})$.*

Proof. By similar arguments as used in the proof of Proposition 3.1.4 the strong closure $\overline{\mathcal{T}}^s$ is again $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\overline{\mathcal{T}}^s) \le \mathcal{R}_s(\mathcal{T})$, and we clearly have $\|(\alpha_j x_j)_j\|_{Y(\ell_n^s)} \le \|x\|_{X(\ell_n^s)}$ for $x = (x_j)_j \in X(\ell_n^s)$, $|\alpha_j| \le 1$, so it remains to show that the convex hull $\mathrm{co}(\mathcal{T})$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\mathrm{co}(\mathcal{T})) = \mathcal{R}_s(\mathcal{T})$. Let $S \in \mathrm{co}(\mathcal{T})^n = \mathrm{co}(\mathcal{T}^n)$, then $S = \sum_{k=1}^m \lambda_k T^{(k)}$ with suitable $T^{(k)} \in \mathcal{T}^n, \lambda_k \in [0,1]$ with $\sum_{k=1}^m \lambda_k = 1$. Hence

$$\begin{aligned}\|\widetilde{S}x\|_{Y(\ell_n^s)} &= \Big\|\sum_{k=1}^m \lambda_k \widetilde{T^{(k)}}x\Big\|_{Y(\ell_n^s)} \le \sum_{k=1}^m \lambda_k \|\widetilde{T^{(k)}}x\|_{Y(\ell_n^s)} \le \Big(\sum_{k=1}^m \lambda_k\Big)\, \mathcal{R}_s(\mathcal{T})\|x\|_{X(\ell_n^s)} \\ &= \mathcal{R}_s(\mathcal{T})\|x\|_{X(\ell_n^s)}.\end{aligned}$$

□

For every σ-finite measure space (J,ν) and strongly measurable $S : J \to L(X,Y)$ and $a \in L^1(J)$, define an operator $T_{a,S} \in L(X,Y)$ by

$$T_{a,S}x := \int_J a(t)S(t)x\,d\nu(t) \qquad \text{for all } x \in X.$$

By approximation, it follows that $T_{a,S} \in \overline{\mathrm{aco}}^s(S(J))$ if $\|a\|_{L^1} = 1$. In this situation, we get

Corollary 3.1.10. *Let $\mathcal{T} \subseteq L(X,Y)$ be $\mathcal{R}_s$-bounded, (J,ν) be a σ-finite measure space and $R > 0$. Then the set*

$$\mathcal{S} := \{T_{a,S} \,|\, S : J \to L(X,Y) \text{ strongly measurable with } S(J) \subseteq \mathcal{T}, a \in L_1(J) \text{ with } \|a\|_{L_1} \le R\}$$

is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\mathcal{S}) \le R\mathcal{R}_s(\mathcal{T})$. □

This is proven in [KW04], Corollary 2.14 for $\mathcal{R}$-boundedness, and using Proposition 3.1.9 the proof carries over to our situation. Again in view of the corresponding results in [KW04] we give some examples for an application of Corollary 3.1.10 that are taken from [KW04], Examples 2.15, 2.16 and can be proven in exactly the same way as it is done there using Corollary 3.1.10.

Example 3.1.11. Let $S : [0,+\infty) \to L(X,Y)$ be strongly continuous such that $\mathcal{S} := S([0,+\infty))$ is $\mathcal{R}_s$-bounded, and define the Laplace transforms

$$\widehat{S}(\lambda)x := \int_0^\infty e^{-\lambda t}S(t)x\,dt \quad \text{for all } x \in X, \mathrm{Re}(\lambda) > 0.$$

Then the set $\mathcal{T}_\omega := \{\lambda\widehat{S}(\lambda) \mid \lambda \in \Sigma_\omega\}$ is $\mathcal{R}_s$-bounded for all $\omega \in [0,\pi/2)$ with $\mathcal{R}_s(\mathcal{T}) \le \mathcal{R}_s(\mathcal{S})$.

Example 3.1.12. Let $\sigma \in (0,\pi]$ and $\omega \in [0,\sigma)$. Let $S : \Sigma_\sigma \to L(X,Y)$ be analytic such that $\mathcal{S} := S\big(\partial\Sigma_\omega\backslash\{0\}\big)$ is $\mathcal{R}_s$-bounded. Then

(a) $\mathcal{T} := S(\Sigma_\omega)$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\mathcal{T}) \leq \mathcal{R}_s(\mathcal{S})$,

(b) For each $\omega' \in [0, \omega)$ there is a constant $C_{\omega'} > 0$ (independent of S) such that $\mathcal{T}_{\omega'} := \{\lambda S'(\lambda) \mid \Sigma_{\omega'}\}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\mathcal{T}_{\omega'}) \leq C_{\omega'}\, \mathcal{R}_s(\mathcal{S})$.

We recall from Section 1.6 some notations: If $J \subseteq \mathbb{R}$ is an interval and $x : J \to X$ is λ-measurable, then by the continuous embedding $X \hookrightarrow M(\mu)$ we can find a $\lambda \otimes \mu$-measurable representative $\tilde{x} : J \times \Omega \to \mathbb{C}$ and hence identify x also with the measurable function $\omega \mapsto \tilde{x}(\cdot, \omega)$. In this manner we can e.g. deal with the question if $x \in X(L^s_*(J))$. If there is no risk of confusion, we will work with this identification in the sequel without explicitly mentioning it.

With the convergence and approximation claims from Subsection 1.6.4 we obtain the following continuous version of $\mathcal{R}_s$-boundedness, which is again a variant of a corresponding result for $\mathcal{R}$-boundedness from [We01a], Lemma 4 a).

Proposition 3.1.13. *Let $s \in [1, +\infty)$. Let $J \subseteq \mathbb{R}$ be a non-trivial interval and $S : J \to L(X, Y)$ be strongly measurable such that $S(J)$ is $\mathcal{R}_s$-bounded. Then for all measurable $x : J \to X$ we have*

$$\Big\| \Big(\int_J |S(t)x(t)|^s \, dt \Big)^{1/s} \Big\|_Y \leq \mathcal{R}_s(S(J)) \cdot \Big\| \Big(\int_J |x(t)|^s \, dt \Big)^{1/s} \Big\|_X . \tag{3.1.4}$$

In other words, the operator S extends to a continuous diagonal operator $(S(t))_{t \in J}$ from $X(L^s(J))$ to $Y(L^s(J))$.

Proof. Let $C := \mathcal{R}_s(S(J))$. We first consider $x \in S(J, \nu) \otimes S(\Omega, \mu)$. Let $x = \sum_{k=1}^d \alpha_k \, \mathbb{1}_{I_k} \otimes \mathbb{1}_{A_k}$ be a disjoint representation of x, where $I_k \subseteq J$ and $A_k \subseteq \Omega$ are bounded sets and the I_k are intervals such that $J_0 := \bigcup_{k=1}^d I_k$ is a finite interval. For each $n \in \mathbb{N}$ let $\mathcal{D}_n := \{D_{n,j} \mid j \in \mathbb{N}_{\leq 2^n}\}$ be a "dyadic" decomposition of J_0, i.e each $D_{n,j}$ has length $\ell_n := |D_{n,j}| = \frac{|J_0|}{2^n}$ for all $j \in \mathbb{N}_{\leq 2^n}$. Moreover we define the approximation $S_n : J \to L(X, Y)$ by

$$S_n(\cdot)\xi := \mathbb{E}(S(\cdot)\xi | \mathcal{D}_n) = \sum_{j=1}^{2^n} \mathbb{1}_{D_{n,j}}(\cdot) \, S_{n,j}\xi, \quad \text{where } S_{n,j}\xi := \frac{1}{|D_{n,j}|} \int_{D_{n,j}} S(t)\xi \, dt$$

for all $\xi \in X$, $n \in \mathbb{N}$. Then in particular we have $S_{n,j} \in \overline{\operatorname{aco}}^s(S(J))$ for all $n \in \mathbb{N}, j \in \mathbb{N}_{\leq 2^n}$. We have the following estimate

$$\begin{aligned}
\Big\| \Big(\int_J |S_n(t)x(t)|^s \, dt \Big)^{1/s} \Big\|_Y &= \Big\| \Big(\sum_{j=1}^{2^n} \int_{D_{n,j}} \Big| \sum_{k=1}^d \alpha_k \, \mathbb{1}_{I_k}(t) (S_n(t) \, \mathbb{1}_{A_k}) \Big|^s \, dt \Big)^{1/s} \Big\|_Y \\
= \quad & \Big\| \Big(\sum_{j=1}^{2^n} \int_{D_{n,j}} \sum_{k=1}^d |\alpha_k|^s \, \mathbb{1}_{I_k}(t) \, |S_n(t) \, \mathbb{1}_{A_k}|^s \, dt \Big)^{1/s} \Big\|_Y \\
= \quad & \Big\| \Big(\sum_{k=1}^d \sum_{j=1}^{2^n} \int_{D_{n,j}} \mathbb{1}_{I_k}(t) \, |S_{n,j}(\alpha_k \, \mathbb{1}_{A_k})|^s \, dt \Big)^{1/s} \Big\|_Y \\
= \quad & \Big\| \Big(\sum_{k=1}^d \sum_{j=1}^{2^n} \underbrace{|D_{n,j} \cap I_k|}_{=: r_{jk}} \cdot |S_{n,j}(\alpha_k \, \mathbb{1}_{A_k})|^s \Big)^{1/s} \Big\|_Y = \Big\| \Big(\sum_{k=1}^d \sum_{j=1}^{2^n} \big| S_{n,j}(r_{jk}^{1/s} \alpha_k \, \mathbb{1}_{A_k}) \big|^s \Big)^{1/s} \Big\|_Y ,
\end{aligned}$$

where we used that we had chosen a disjoint representation. Since $\overline{\text{aco}}^s(S(J))$ is still $\mathcal{R}_s$-bounded by Proposition 3.1.9 with $\mathcal{R}_s$-constant not greater than C, we obtain

$$\Big\| \Big(\int_J |S_n(t)x(t)|^s \, dt \Big)^{1/s} \Big\|_Y \;\; \leq \;\; C \Big\| \Big(\sum_{k=1}^{d} \sum_{j=1}^{2^n} |r_{jk}^{1/s} \alpha_k \, \mathbb{1}_{A_k} |^s \Big)^{1/s} \Big\|_X .$$

Moreover the same calculations as above lead to

$$\begin{aligned} & \Big\| \Big(\sum_{k=1}^{d} \sum_{j=1}^{2^n} |r_{jk}^{1/s} \alpha_k \, \mathbb{1}_{A_k} |^s \Big)^{1/s} \Big\|_X = \Big\| \Big(\sum_{k=1}^{d} \Big(\sum_{j=1}^{2^n} |D_{n,j} \cap I_k| \Big) \cdot |\alpha_k \, \mathbb{1}_{A_k} |^s \Big)^{1/s} \Big\|_X \\ = \;\; & \Big\| \Big(\sum_{k=1}^{d} |I_k| \cdot |\alpha_k|^s \; \mathbb{1}_{A_k} \Big)^{1/s} \Big\|_X = \Big\| \Big(\sum_{k=1}^{d} \int_J \mathbb{1}_{I_k}(t) \cdot |\alpha_k|^s \; \mathbb{1}_{A_k} \; dt \Big)^{1/s} \Big\|_X \\ = \;\; & \Big\| \Big(\int_J \sum_{k=1}^{d} |\alpha_k|^s \, \mathbb{1}_{I_k}(t) \; \mathbb{1}_{A_k} \; dt \Big)^{1/s} \Big\|_X = \Big\| \Big(\int_J | \sum_{k=1}^{d} \alpha_k \, \mathbb{1}_{I_k}(t) \; \mathbb{1}_{A_k} |^s \, dt \Big)^{1/s} \Big\|_X \\ = \;\; & \Big\| \Big(\int_J |x(t)|^s \, dt \Big)^{1/s} \Big\|_X , \end{aligned}$$

hence putting all together we obtain

$$\Big\| \Big(\int_J |S_n(t)x(t)|^s \, dt \Big)^{1/s} \Big\|_Y \;\; \leq \;\; C \, \Big\| \Big(\int_J |x(t)|^s \, dt \Big)^{1/s} \Big\|_X .$$

Further we have

$$S_n(t)x(t) - S(t)x(t) = \sum_{k=1}^{d} \alpha_k \, \mathbb{1}_{I_k}(t) \left((S_n(t) - S(t)) \, \mathbb{1}_{A_k} \right),$$

hence

$$\begin{aligned} \Big(\int_J |S_n(t)x(t) - S(t)x(t)|^s \, dt \Big)^{1/s} \;\; &= \;\; \Big(\sum_{k=1}^{d} |\alpha_k|^s \int_J \mathbb{1}_{I_k}(t) \left| (S_n(t) - S(t)) \, \mathbb{1}_{A_k} \right|^s dt \Big)^{1/s} \\ &\leq \;\; \sum_{k=1}^{d} |\alpha_k| \Big(\int_J \mathbb{1}_{I_k}(t) \left| (S_n(t) - S(t)) \, \mathbb{1}_{A_k} \right|^s dt \Big)^{1/s} \\ &\leq \;\; \sum_{k=1}^{d} |\alpha_k| \Big(\int_{J_0} \left| S_n(t) \, \mathbb{1}_{A_k} - S(t) \, \mathbb{1}_{A_k} \right|^s dt \Big)^{1/s} \to 0 \end{aligned}$$

pointwise μ-a.e. for $n \to \infty$ (this can easily be seen by considering suitable representatives pointwise a.e. and using the well known fact that $\mathbb{E}(F|\mathcal{D}_n) \to \mathbb{E}(F|\mathcal{D}_\infty) = F$ pointwise a.e. and in $L^s\big(J_0, dt/|J_0|\big)$ for $F \in L^s\big(J_0, dt/|J_0|\big)$, where $\mathcal{D}_\infty$ is the σ-algebra generated by $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n$, cf. e.g. [Ka97], Theorem 6.23 in combination with Corollary 6.22 and an L^p-version of Lemma 5.5).

So the Fatou property of Y yields

$$\begin{aligned}\Big\|\Big(\int_J |S(t)x(t)|^s\,dt\Big)^{1/s}\Big\|_Y &= \Big\|\lim_{n\to\infty}\Big(\int_J |S_n(t)x(t)|^s\,dt\Big)^{1/s}\Big\|_Y \\ &\le \liminf_{n\to\infty}\Big\|\Big(\int_J |S_n(t)x(t)|^s\,dt\Big)^{1/s}\Big\|_Y \\ &\le C\cdot\Big\|\Big(\int_J |x(t)|^s\,dt\Big)^{1/s}\Big\|_X .\end{aligned}$$

Now let $x : J \to L(X,Y)$ be an arbitrary measurable function such that $\|x\|_{X(L^s(J))} < +\infty$. By Lemma 1.6.25 we can choose a sequence $(x_n)_{n\in\mathbb{N}} \in (S(J,\nu)\otimes S(\Omega,\mu))^{\mathbb{N}}$ with the following properties:

1. $x_n(t,\omega) \to x(t,\omega)$ for $n\to\infty$ for $(\lambda\otimes\mu)$-a.e. $(t,\omega)\in J\times\Omega$,
2. $x_n(t)\to x(t)$ in X for $n\to\infty$ for μ-a.e. $t\in J$,
3. $\liminf_{n\to\infty}\|x_n\|_{X(L^s(J))} \le \|x\|_{X(L^s(J))}$.

We apply Lemma 1.6.24 to the measurable functions $t\mapsto S(t)x(t), S(t)x_n(t)$ and obtain that we can w.l.o.g. (by possibly choosing a subsequence) assume that $(S(\cdot)x_n(\cdot))(\cdot\cdot) \to (S(\cdot)y(\cdot))(\cdot\cdot)$ $\lambda\otimes\mu$-a.e. for $n\to\infty$. Using again the Fatou property yields finally

$$\begin{aligned}&\Big\|\Big(\int_J |S(t)x(t)|^s\,dt\Big)^{1/s}\Big\|_Y = \Big\|\Big(\int_J \lim_{n\to\infty}|S(t)x_n(t)|^s\,dt\Big)^{1/s}\Big\|_Y \\ \le\ &\Big\|\liminf_{n\to\infty}\Big(\int_J |S(t)x_n(t)|^s\,dt\Big)^{1/s}\Big\|_Y \le \liminf_{n\to\infty}\Big\|\Big(\int_J |S(t)x_n(t)|^s\,dt\Big)^{1/s}\Big\|_Y \\ \le\ & C\liminf_{n\to\infty}\Big\|\Big(\int_J |x_n(t)|^s\,dt\Big)^{1/s}\Big\|_X \le C\cdot\Big\|\Big(\int_J |x(t)|^s\,dt\Big)^{1/s}\Big\|_X .\end{aligned}$$

□

In the case $s=2$ the converse conclusion of Proposition 3.1.4 is also true if X,Y are r-concave for some $r<+\infty$, as in this case $\mathcal{R}_s$-boundedness is equivalent to $\mathcal{R}$-boundedness, a proof can be found in [We01a], 4a).

We obtain an analogous result for $s=+\infty$, in this case we can of course drop the measurability assumptions on S.

Proposition 3.1.14. *Let J be a non-empty set and $S : J\to L(X,Y)$ such that $S(J)$ is $\mathcal{R}_\infty$-bounded. Then for all mappings $x : J\to X$ we have*

$$\big\|\sup_{t\in J}|S(t)x(t)|\big\|_Y \le \mathcal{R}_\infty(S(J))\cdot\big\|\sup_{t\in J}|x(t)|\big\|_X. \tag{3.1.5}$$

Proof. Let $C := \mathcal{R}_\infty(S(J))$, and let $x : J\to X$. Then by Proposition 1.6.2 we can choose a countable subset $J_0\in J$ such that $\sup_{t\in J}|S(t)x(t)| = \sup_{t\in J_0}|S(t)x(t)|$. Let $(t_j)_{k\in\mathbb{N}}$ be an enumeration of J_0, then for all $n\in\mathbb{N}$ we obtain

$$\big\|\sup_{j\in\mathbb{N}_{\le n}}|S(t_j)x(t_j)|\big\|_Y \le C\cdot\big\|\sup_{j\in\mathbb{N}_{\le n}}|x(t_j)|\big\|_X,$$

and the Fatou property yields

$$\begin{aligned}\big\|\sup_{t\in J}|S(t)x(t)|\big\|_Y &= \big\|\sup_{t\in J_0}|S(t)x(t)|\big\|_Y = \big\|\lim_{n\to\infty}\sup_{j\in\mathbb{N}_{\le n}}|S(t_j)x(t_j)|\big\|_Y \\ &\le \liminf_{n\to\infty}\big\|\sup_{j\in\mathbb{N}_{\le n}}|S(t_j)x(t_j)|\big\|_Y \le C\cdot\liminf_{n\to\infty}\big\|\sup_{j\in\mathbb{N}_{\le n}}|x(t_j)|\big\|_X \\ &\le C\cdot\big\|\sup_{j\in\mathbb{N}}|x(t_j)|\big\|_X \le C\cdot\big\|\sup_{t\in J}|x(t)|\big\|_X.\end{aligned}$$

□

Remark 3.1.15. *For later purpose we note again that there are, of course, also discrete versions of Propositions 3.1.13, 3.1.14 which follow immediately with the Fatou property. For, e.g., $s\in[1,+\infty)$ and $(S_j)_{j\in\mathbb{Z}}\in L(X,Y)^{\mathbb{Z}}$ such that $\mathcal{S}:=\{S_j\,|\,j\in\mathbb{Z}\}$ is $\mathcal{R}_s$-bounded, we have for all $(x_j)_{j\in\mathbb{Z}}\in X^{\mathbb{Z}}$:*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|S_jx_j|^s\Big)^{1/s}\Big\|_X \le \mathcal{R}_s(\mathcal{S})\cdot\Big\|\Big(\sum_{j\in\mathbb{Z}}|x_j|^s\Big)^{1/s}\Big\|_X. \tag{3.1.6}$$

Further standard methods to obtain $\mathcal{R}_s$-boundedness are by means of interpolation and duality. Recall that a set $\mathcal{T}$ of operators is $\mathcal{R}_s$-bounded if and only if the diagonal operators $\widetilde{T}$ for $T\in\mathcal{T}^n, n\in\mathbb{N}$ induce uniformly bounded operators from $X(\ell_n^s)$ to $Y(\ell_n^s)$. By complex interpolation we obtain $[X(\ell_n^{s_0}),X(\ell_n^{s_1})]_\theta = X(\ell_n^{s_\theta})$ (with equal norms) where $\frac{1}{s_\theta}=(1-\theta)\frac{1}{s_0}+\theta\frac{1}{s_1}$ (for more details cf. Section 2.4). This leads immediately to the following

Proposition 3.1.16. *Let $1\le s_0<s_1\le\infty$. If $\mathcal{T}\subseteq L(X,Y)$ is $\mathcal{R}_{s_j}$-bounded for $j=1,2$, then $\mathcal{T}$ is $\mathcal{R}_s$-bounded for all $s\in[s_0,s_1]$.* □

In the special case $X=Y=L^p$, a norm-bounded set $\mathcal{T}\subseteq X$ is always $\mathcal{R}_p$-bounded, as we have seen in Remark 3.1.5. As there are various results for $\mathcal{R}$-boundedness of operators, sometimes the following remark is helpful.

Corollary 3.1.17. *Let $X=L^p$ and $\mathcal{T}\subseteq L(X)$ be $\mathcal{R}$-bounded. Then $\mathcal{T}$ is $\mathcal{R}_s$-bounded for all $s\in[2\wedge p, 2\vee p]$.* □

In particular, finite operator sets in L^p are always $\mathcal{R}_s$-bounded for all $s\in[2\wedge p,2\vee p]$, since the latter is true for $\mathcal{R}$-boundedness. Hence it is noteworthy that in spite of these special cases, finitely many or even a single operator need not to be $\mathcal{R}_s$-bounded, even in the Hilbert space case.

Example 3.1.18. Consider $X=Y=L^2([0,1])$. Let $r_j(t):=\operatorname{sgn}(\sin(2^j\pi t))$ for all $t\in[0,1], j\in\mathbb{N}$ be the Rademacher functions. Then $(r_j)_{j\in\mathbb{N}}$ is an orthonormal system in X. Let us further define $I_j:=(2^{-j},2^{-j+1}]$, $f_j(t):=2^{j/2}\,\mathbb{1}_{I_j}$ for all $j\in\mathbb{N}$, then it is easily checked that $(f_j)_{j\in\mathbb{N}}$ is an orthonormal system in X as well. For all $n\in\mathbb{N}$, we have

$$\Big\|\Big(\sum_{j=1}^n|r_j|^s\Big)^{1/s}\Big\|_{L_2} = \Big\|\Big(\sum_{j=1}^n\mathbb{1}_{[0,1]}\Big)^{1/s}\Big\|_{L_2} = \big\|n^{1/s}\,\mathbb{1}_{[0,1]}\big\|_{L_2} = n^{1/s}.$$

Because the f_j have disjoint supports, we have on the other hand

$$\Big\|\Big(\sum_{j=1}^n |f_j|^s\Big)^{1/s}\Big\|_{L_2} = \Big(\int_0^1 \Big(\sum_{j=1}^n |f_j|^s\Big)^{2/s}\Big)^{1/2} = \Big(\sum_{j=1}^n \int_0^1 \big(|f_j|^s\big)^{2/s}\Big)^{1/2}$$
$$= \Big(\sum_{j=1}^n \int_0^1 |f_j|^2\Big)^{1/2} = \Big(\sum_{j=1}^n \|f_j\|_{L_2}^2\Big)^{1/2} = n^{1/2}.$$

As the $(r_j)_j, (f_j)_j$ are orthonormal sequences, we can construct operators T, S on X with the property $Tf_j = r_j$ and $Sr_j = f_j$ for all $j \in \mathbb{N}$ (for example by defining T, S on the f_j ,r_j respectively, an letting them 0 on the orthogonal complement). Then the above equalities show that T is not $\mathcal{R}_s$-bounded in case $s < 2$ and S is not $\mathcal{R}_s$-bounded in case $s > 2$.

We now have a look at duality. Recall that the dual space X' can be identified with the associated space $X^{\#}$ of X (cf. Subsection 1.6.2) since X has absolute continuous norm, and for $s \in [1, +\infty)$ we have in this sense $(X(\ell^s))' = X'(\ell^{s'})$ by Theorem 1.6.12 since ℓ^s has (RNP) if $s \in [1, +\infty)$. Moreover, for $T \in L(X, Y)$ we identify its dual operator T' with the corresponding operator $T' : Y^{\#} \to X^{\#}$. Then we obtain the following duality result.

Proposition 3.1.19. *Let $s \in [1, +\infty)$ and $\mathcal{T} \subseteq L(X)$ be $\mathcal{R}_s$-bounded. Then $\mathcal{T}' := \{T' \,|\, T \in \mathcal{T}\}$ is $\mathcal{R}_{s'}$-bounded in X'.* □

We note that in general the Banach function space X' does not have absolute continuous norm and hence does not fit in our framework. So if we use duality results like Proposition 3.1.19, we usually require X to be reflexive, so in turn X' has also absolute continuous norm, cf. Theorem 1.6.12.

We will give some more classical criteria to check $\mathcal{R}_s$-boundedness for concrete operators. Recall that a linear operator $S : X \to Y$ is called *positive*, if $x \geq 0$ implies $Sx \geq 0$ for all $x \in X$, and positive operators are always bounded, cf. Section 2.5. Then we obtain the following criterion for $\mathcal{R}_s$-boundedness, which we prove in detail for the sake of completeness.

Proposition 3.1.20. *Let $\mathcal{T} \subseteq L(X, Y)$ and $S : X \to Y$ be a positive operator that dominates $\mathcal{T}$, i.e. $|Tx| \leq S|x|$ for all $T \in \mathcal{T}, x \in X$. Then $\mathcal{T}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\mathcal{T}) \leq \|S\|$ for all $s \in [1, +\infty]$.*

Proof. Choose $r \in [1, +\infty]$ with $1/s + 1/r = 1$. Let $n \in \mathbb{N}$ and $T \in \mathcal{T}^n, x \in Y^n$. Then we have by duality

$$\Big(\sum_{j=1}^n |T_j x_j|^s\Big)^{1/s} = \sup_{\alpha \in B_{\ell_r^n}} \sum_{j=1}^n |\alpha_j T_j x_j| \leq \sup_{\alpha \in B_{\ell_r^n}} \sum_{j=1}^n S|\alpha_j x_j| = \sup_{\alpha \in B_{\ell_r^n}} S\Big(\sum_{j=1}^n |\alpha_j x_j|\Big)$$
$$\leq S\Big(\sup_{\alpha \in B_{\ell_r^n}} \sum_{j=1}^n |\alpha_j x_j|\Big) = S\Big(\sum_{j=1}^n |x_j|^s\Big)^{1/s},$$

hence

$$\Big\|\Big(\sum_{j=1}^n |T_j x_j|^s\Big)^{1/s}\Big\|_X \leq \Big\|S\Big(\sum_{j=1}^n |x_j|^s\Big)^{1/s}\Big\|_X \leq \|S\| \, \Big\|\Big(\sum_{j=1}^n |x_j|^s\Big)^{1/s}\Big\|_X$$

(with the usual modification if $s = +\infty$). □

In fact, the proof is based on a simplified version of the general result that positive operators $S : X \to Y$ always have bounded extensions $S \otimes \mathrm{Id}_E$ in the vector-valued spaces $X(E) \to Y(E)$ (cf. Proposition 2.5.3), and the obvious, but useful fact, that $\mathcal{R}_s$-boundedness is inherited by domination in the following sense.

Remark 3.1.21. *Let $\mathcal{T}, \mathcal{S} \subseteq L(X,Y)$ such that for all $T \in \mathcal{T}$ there is an $S \in \mathcal{S}$ such that $|Tx| \le |Sx|$ for all $x \in X$. Then $\mathcal{T}$ is $\mathcal{R}_s$-bounded if $\mathcal{S}$ is $\mathcal{R}_s$-bounded.* □

At the end of this section we have a glance at the concrete situation where $X = L^p(\Omega), Y = L^q(\Omega)$ based on classical Calderón-Zygmund theory. More involved examples in this framework will be given in Section 3.5.

We assume that (Ω, d) is a metric space and μ is a σ-finite regular Borel measure on Ω such that (Ω, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss, cf. [CW71], [CW77], i.e. there is a constant $C_1 \ge 1$ such that

$$\mu(B(x,2r)) \le C_1\, \mu(B(x,r)) \quad \text{for all } x \in \Omega, r > 0. \tag{3.1.7}$$

From (3.1.7) one can deduce the existence of some $D > 0$ and $C_D \ge 1$ such that

$$\mu(B(x,\lambda r)) \le C_D\, \lambda^D\, \mu(B(x,r)) \quad \text{for all } x \in \Omega, r > 0, \lambda \ge 1. \tag{3.1.8}$$

One central issue in this situation is the Fefferman-Stein-inequality, which states $\mathcal{R}_s$-boundedness of the (uncentered) Hardy-Littlewood maximal operator, which is defined as

$$(Mf)(x) := \sup \left\{ \frac{1}{\mu(B)} \int_B |f|\, d\mu \;\middle|\; B \subseteq \Omega \text{ is a ball with } x \in B \right\} \quad \text{for all } f \in L^1_{loc}(\Omega), x \in \Omega.$$

The following classical theorem holds.

Theorem 3.1.22 (Fefferman-Stein)**.** *Let $p \in (1, +\infty)$ and $s \in (1, +\infty]$, then the sublinear operator M is $\mathcal{R}_s$-bounded in $L^p(\Omega)$.*

This originates in the paper [FS71] for the case $\Omega = \mathbb{R}^d$. Alternative proofs for this special situation can be found in [GCRdF85], Corollary V.4.3 or in [Gr04], Theorem 4.4.6 . The generalization on spaces of homogeneous type can be found in [GLY07] , Theorem 1.2. We will give more details about the proof of Theorem 3.1.22 in the sequel after the following examples.

The $\mathcal{R}_s$-boundedness of the Hardy-Littlewood maximal operator yields a wide class of examples for $\mathcal{R}_s$-bounded sets of classical operators by well known uniform estimates against the maximal function and the simple fact that $\mathcal{R}_s$-boundedness is preserved under domination, cf. Proposition 3.1.20 above. One important fact is that in the classical case $\Omega = \mathbb{R}^D$ the dilations of any function that has a radial positive decreasing integrable majorant can be controlled by the Hardy-Littlewood maximal operator: For any function $u : \mathbb{R}^D \to \mathbb{C}$ we define its dilations $u_t(x) := t^{-D} u(x/t)$ for all $x \in \mathbb{R}^D, t > 0$. Then we obtain the following classical result.

Proposition 3.1.23. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a decreasing measurable function such that $r \mapsto \varphi(r)r^{D-1}$ is integrable. Let $\Phi \in L^1(\mathbb{R}^D)$ such that $|\Phi(x)| \le \varphi(|x|)$ for all $x \in \mathbb{R}^D, t > 0$. Then we have the pointwise estimate*

$$\sup_{t>0} |\Phi_t * f(x)| \le \|\varphi(|\cdot|)\|_{L^1(\mathbb{R}^D)} (Mf)(x) \quad \text{for all } x \in \mathbb{R}^D, f \in L^1_{loc}(\mathbb{R}^D). \tag{3.1.9}$$

Hence the set of convolution operators $\{\Phi_t * \cdot \mid t > 0\}$ is $\mathcal{R}_s$-bounded on $L^p(\mathbb{R}^D)$ for all $p \in (1,+\infty)$, $s \in (1,+\infty]$.

A proof may be found in many standard monographs about harmonic or Fourier analysis, e.g. in [Du01], Proposition 2.7 and Corollary 2.8. A concrete application is the following example.

Example 3.1.24. Let $D \in \mathbb{N}, p \in (1,\infty)$ and

$$\Delta : L^p(\mathbb{R}^D) \supseteq H^{p,2}(\mathbb{R}^D) \to L^p(\mathbb{R}^D), f \mapsto \Delta f = \sum_{j=1}^{D} \partial_j^2 f$$

be the Laplace operator, where $H^{p,2} = H^{p,2}(\mathbb{R}^D) = \{f \in L_p(\mathbb{R}^D) \mid \partial^\alpha f \in L^p \text{ for all } |\alpha| \le 2\}$. Then the set of operators $\mathcal{T} := \{t(t-\Delta)^{-1} \mid t > 0\}$ is $\mathcal{R}_s$-bounded in $L^p(\mathbb{R}^D)$ for all $s \in (1,+\infty]$.

Proof. Define $\psi_t(\xi) := t(t+|\xi|^2)^{-1}$ for all $\xi \in \mathbb{R}^D, t > 0$. Then $t(t-\Delta)^{-1}f = \mathcal{F}^{-1}\psi_t\mathcal{F}f$ for each $f \in \mathcal{S}$, where $\mathcal{F}$ is the Fourier transform. So the operator $t(t-\Delta)^{-1}$ is given by convolution with the kernel $\mathcal{F}^{-1}\psi_t$. Since $\psi_t = \psi_1(t^{-1/2}\cdot)$ we have $\mathcal{F}^{-1}\psi_t = \mathcal{F}^{-1}(\psi_1(r\cdot)) = r^{-D}(\mathcal{F}^{-1}\psi_1)(\cdot/r)$ with $r = t^{-1/2}$. Hence the set $\mathcal{T}$ equals the set of convolution operators $\{\Phi_r * \cdot \mid r > 0\}$ with $\Phi := \mathcal{F}^{-1}\psi_1$. A standard calculation (or using Laplace transform of the heat semigroup) shows that Φ is given by

$$\Phi(x) = \int_0^\infty (4\pi\tau)^{-D/2} e^{-\tau} e^{-|x|^2/4\tau}\, d\tau \quad \text{for all } x \in \mathbb{R}^D. \tag{3.1.10}$$

Hence $\Phi \in L_1$ with $\|\Phi\|_{L_1} = 1$, so by Young's inequality we have $\|\Phi_r * f\|_p \le \|f\|_p$ for all $f \in L^p$. This shows that the convolution operators $\Phi_r * \cdot\,, r > 0$ are uniformly bounded on L^p. Moreover, the radial function Φ fulfills the assumptions of Proposition 3.1.23, so this gives the claim. □

As announced above we will now show one possible way to prove Theorem 3.1.22, which is indeed the line of proof in the literature cited above. It is based on a result about extrapolating continuity on the L^p-scale for (singular) integral operators which have a kernel that fulfills the Hörmander condition, which is interesting in itself as well. We give a short sketch of this: Let E, F be Banach spaces, $K : \Omega \times \Omega \backslash \mathrm{Id}_\Omega \to L(E,F)$ be a locally integrable mapping. A linear operator $T : L^\infty_c(\Omega, E) \to L^1_{loc}(\Omega, F)$ is said to be associated with the kernel K if

$$Tf(x) = \int_\Omega K(x,\omega) f(\omega)\, d\mu(\omega)$$

for all $f \in L_c^\infty(\Omega, E)$ and $x \notin \operatorname{supp}(F)$. The kernel K is said to fulfill the Hörmander condition, if there is a constant $B > 0$ such that

$$\int_{d(\omega,y)>2d(y,z)} \|K(\omega, y) - K(\omega, z)\|_{L(E,F)}\, d\mu(\omega) \le B, \quad \text{and} \tag{3.1.11}$$
$$\int_{d(\omega,y)>2d(y,z)} \|K(y,\omega) - K(z,\omega)\|_{L(E,F)}\, d\mu(\omega) \le B \quad \text{for all } y, z \in \Omega.$$

Then we have the following result taken from [GLY07], Theorem 1.1, which is classical for $\Omega = \mathbb{R}^D$.

Proposition 3.1.25. *Let K fulfill the Hörmander condition (3.1.11) and let T be a bounded operator from $L^r(\Omega, E)$ to $L^r(\Omega, F)$ for some $r \in (1, \infty]$ with associated kernel K. Then T extends to a bounded operator from $L^p(\Omega, E)$ to $L^p(\Omega, F)$ for all $p \in (1, \infty)$, and we have an estimate*

$$\|Tf\|_p \le C_p \|F\|_p \quad \textit{for all } f \in L_p(\Omega, E), \tag{3.1.12}$$

where the constant C_p only depends on Ω, p and the constants $B, \|T\|_r$ associated to the kernel K and the operator T.

In fact, this proposition can be applied twice and is then self improving in the following sense (cf. [GLY07], Corollary 2.9):

Proposition 3.1.26. *Let K fulfill the Hörmander condition (3.1.11) and let T be a bounded operator from $L^r(\Omega, E)$ to $L^r(\Omega, F)$ for some $r \in (1, \infty]$ with associated kernel K. Then $T \otimes \mathrm{Id}_{\ell^q}$ extends to a bounded operator from $L^p(\Omega, \ell^q(E))$ to $L^p(\Omega, \ell^q(F))$ for all $p, q \in (1, \infty)$, and we have an estimate*

$$\Big\| \Big(\sum_{j\in\mathbb{N}} \|Tf_j\|_F^q \Big)^{1/q} \Big\|_p \le C_{p,q} \Big\| \Big(\sum_{j\in\mathbb{N}} \|Tf_j\|_E^q \Big)^{1/q} \Big\|_p \quad \textit{for all } (f_j)_{j\in\mathbb{N}} \in L^p(\Omega, \ell^q(E)), \tag{3.1.13}$$

where the constant $C_{p,q}$ only depends on Ω, p, q and the constants $B, \|T\|_r$ associated to the kernel K and the operator T.

In fact, Proposition 3.1.26 can easily be deduced from Proposition 3.1.25 by considering the vector-valued kernels $K(\cdot, \cdot\cdot) \otimes \mathrm{Id}_{\ell^q}$. If $E = F = \mathbb{C}$, Proposition 3.1.26 gives a classical criterion for $\mathcal{R}_s$-boundedness.

A well known and well studied class of operators that satisfy the assumptions of Proposition 3.1.25 (for $E = F = \mathbb{C}$) are the *Calderón-Zygmund operators.* These are bounded operators on $L^2(\Omega)$ that are associated to a so called *standard kernel*, which in particular satisfies the Hörmander condition. Since we will usually not deal explicitly with Calderón-Zygmund operators in this work, we will not go into detail but just refer to the standard literature as [St93], [GCRdF85], [Gr04] or [Du01].

As indicated above we will now give a sketch of how the proof of $\mathcal{R}_s$-boundedness of the Hardy-Littlewood maximal operator can be reduced to the above Proposition 3.1.26. In a first step we define the so called *centered Hardy-Littlewood maximal operator* by

$$(M_c f)(x) := \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| \, d\mu \quad \text{ for all } f \in L^1_{loc}(\Omega), x \in \Omega.$$

Then it is an easy consequence of the doubling property (3.1.8) that we have the pointwise estimate

$$(M_c f)(x) \leq (Mf)(x) \leq 3^D C_D \, (M_c f)(x) \quad \text{ for all } f \in L^1_{loc}(\Omega) \text{ and a.e. } x \in \Omega,$$

hence it is sufficient to consider M_c instead of M. We will only show the idea in the case $\Omega = \mathbb{R}^D$, the proof of the general case follows the same idea, but one has to make more involved approximations in the general space (Ω, μ, d). So we assume from now on that $\Omega = \mathbb{R}^D$, then $M_c f(x) = \sup_{r>0} \psi_r * |f|(x)$ for all $f \in L^1_{loc}(\Omega)$ and a.e. $x \in \Omega$, where $\psi := \frac{1}{\mu(B_1(0))} \mathbb{1}_{B_1(0)}$. Let $\varphi \in \mathcal{S}(\mathbb{R}^D)$ be positive and radially symmetric decreasing such that $\varphi|_{B_1(0)} \geq \frac{1}{\mu(B_1(0))}$, then $\varphi \geq \psi \geq 0$, hence

$$M_c f(x) \approx \sup_{t>0} \varphi_t * |f(x)| = \sup_{t \in \mathbb{Q}_{>0}} \varphi_t * |f(x)|$$

by Proposition 3.1.23 and the fact that $0 \leq \psi \leq \varphi$, and since φ is continuous. Moreover it is sufficient to consider only the term $\sup_{t\in F} |\varphi_t * f(x)|$ for any finite subset $F \subseteq \mathbb{Q}_{>0}$ and to show that we obtain uniform bounds that do not depend on F. Let us consider the maximal operator $M_{\varphi,F}$ defined by $M_{\varphi,F} f(x) := \sup_{t\in F} |\varphi_t * f(x)|$ for all $f \in L^1_{loc}(\mathbb{R}^D)$ and a.e. $x \in \mathbb{R}^D$. Then in the same way as in Chapter 2 the maximal operator $M_{\varphi,F}$ can be considered as a linear operator

$$M_{\varphi,F} : L^p(\mathbb{R}^D) \supseteq \mathcal{S}_D \to C_0(\mathbb{R}^D, \ell^\infty(F))$$

with associated convolution kernel $K_F(x) := (\varphi_t(x))_{x\in F} \in L(\mathbb{C}, \ell^\infty(F))$. Since $\varphi \in \mathcal{S}$, it is easily checked that the Hörmander condition for the kernel K_F is satisfied in the following simpler form for convolution operators:

$$\exists\, C > 0 \,\forall\, y \in \mathbb{R}^D \,:\, \int_{|x|>2|y|} \sup_{t\in F} |\varphi_t(x-y) - \varphi_t(x)| \, dx \leq \int_{|x|>2|y|} \sup_{t>0} |\varphi_t(x-y) - \varphi_t(x)| \, dx \leq C$$

Moreover, $M_{\varphi,F}$ acts obviously as a bounded operator $L^\infty(\mathbb{R}^D) \to L^\infty(\mathbb{R}^D, \ell^\infty(F))$, where the norm is controlled by $\|\varphi\|_\infty$. Thus Proposition 3.1.26 yields Theorem 3.1.22, since the norm bounds do only depend on φ, but not on F. The details for this approach in the case $\Omega = \mathbb{R}^D$ can be found e.g. in [GCRdF85], Section V.4, [Gr04] Theorem 4.6.6, or [Du01] Chapter 5, §6.7, and the general case is treated in [GLY07], Section 3.

Finally we note that there are many works concerning $\mathcal{R}_s$-boundedness (without using this terminology) in the framework of classical harmonic analysis. Indeed, Proposition 3.1.26 is an example for a modern version of such classical results. It is noteworthy that in the monograph

[GCRdF85] this topic is intensively studied in connection with some kind of weighted estimates, also involving Muckenhoupt weights. We note that nevertheless that notion of weighted estimates differs from the concept of weighted estimates as we will use it in Section 3.5. In fact, in the article [Ga90] the topics from the monograph [GCRdF85] are also considered in the more general framework of Banach function spaces, as we do it here as well.

3.2 $\mathcal{R}_s$-sectorial operators

Let X be a Banach function space with absolute continuous norm. In this section, $A : X \supseteq D(A) \to X$ will always denote a sectorial operator with type $\omega(A)$ and with dense range and domain, cf. Section 1.2.

To avoid technical difficulties in some situations, we assume additionally that the operator A is injective and has dense domain and range. Recall that the density of $R(A)$ already implies that A is injective, and if X is reflexive, then $D(A)$ is always dense, and $R(A)$ is dense if and only if A is injective. Actually, our assumption is not much loss of generality since in our situation the considered spaces are usually reflexive, and we are mostly interested in injective operators since we want to use the full strength of the general functional calculus.

3.2.1 Definition and elementary properties of $\mathcal{R}_s$-sectorial operators

Definition 3.2.1. Let $s \in [1, +\infty]$. The operator A is called $\mathcal{R}_s$*-sectorial*, if there exists an $\omega \in [0, \pi)$ such that $\sigma(A) \subseteq \overline{\Sigma}_\omega$ and the set $\{zR(z, A) \,|\, z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}$ is $\mathcal{R}_s$-bounded for each $\sigma \in (\omega, \pi)$. The infimum $\omega_{\mathcal{R}_s}(A)$ of all such ω is called the $\mathcal{R}_s$-type of A.

In this case we define

$$M_{s,\sigma}(A) := M_{\mathcal{R}_s,\sigma}(A) := \mathcal{R}_s(\{zR(z, A), AR(z, A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\})$$

for all $\sigma \in (\omega_{\mathcal{R}_s}(A), \pi)$. Observe that this set is indeed also $\mathcal{R}_s$-bounded, since

$$AR(z, A) = zR(z, A) - \mathrm{id}_X \quad \text{for all } z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma,$$

hence $M_{\mathcal{R}_s,\sigma}(A) \le \mathcal{R}_s(\{zR(z, A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}) + 1 \le 2M_{\mathcal{R}_s,\sigma}(A)$.

Recall our notation $\ell^s := \ell^s(\mathbb{N}, \mathbb{C})$. We will introduce a diagonal operator extension $\widetilde{A}_s$ of $A \otimes \mathrm{Id}_{\ell^s}$ – which turns out to be equal to the vector-valued extension A^{ℓ^s} as introduced in Section 2.5 if $s < +\infty$ – such that properties as $\mathcal{R}_s$-sectoriality and $\mathcal{R}_s$-boundedness of the H^∞-calculus can be expressed as "simple" sectoriality and boundedness of the H^∞-calculus for the single operator $\widetilde{A}_s$. Although this is a natural concept, we note that it is not always straightforward to check these correspondences. Indeed, while $\mathcal{R}_s$-sectoriality of A will turn out to imply sectoriality of $\widetilde{A}_s$ by definition, the converse is not clear, since $\mathcal{R}_s$-sectoriality of the *single operators* $\lambda R(\lambda, A)$ might not imply $\mathcal{R}_s$-sectoriality of *the set* $\{\lambda R(\lambda, A) \mid \lambda \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}$. Furthermore, the correspondence between $\mathcal{R}_s$-boundedness of the H^∞-calculus of A and boundedness of the H^∞-calculus of $\widetilde{A}_s$ is more involved, and in the case that $f(A)$ is $\mathcal{R}_s$-bounded for any $f \in H^\infty(\Sigma)_\sigma$, we will be able to show that indeed A has an $\mathcal{R}_s$-bounded H^∞-calculus under suitable assumptions. The latter will be done in Subsection 3.2.3.

Definition/Proposition 3.2.2. Let $s \in [1, +\infty]$ and assume that A is $\mathcal{R}_s$-sectorial. Define the diagonal operator

$$\widetilde{A}_s := \{((x_n)_{n\in\mathbb{N}}, (Ax_n)_{n\in\mathbb{N}}) \,|\, (x_n)_{n\in\mathbb{N}} \in X(\ell^s), x_n \in D(A) \text{ for all } n \in \mathbb{N} \text{ and } (Ax_n)_{n\in\mathbb{N}} \in X(\ell^s)\} \tag{3.2.1}$$

in $X(\ell^s)$. Then $\widetilde{A}_s$ is a sectorial operator in $X(\ell^s)$ with $\omega(\widetilde{A}_s) \le \omega_{\mathcal{R}_s}(A)$, and we have

$$\forall\, \lambda \in \mathbb{C}\backslash\overline{\Sigma}_{\omega_{\mathcal{R}_s}(A)}, x \in X(\ell^s) \;:\; R(\lambda, \widetilde{A}_s)x = (R(\lambda, A)x_n)_{n\in\mathbb{N}}. \tag{3.2.2}$$

Proof. Let $\sigma \in (\omega_{\mathcal{R}_s}(A), \pi)$. For all $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_\sigma =: S$ define the operator $R(\lambda) := \lambda R(\lambda, A)$, then $R(S)$ is $\mathcal{R}_s$-bounded, so let $M := \mathcal{R}_s(R(S))$. Moreover, for each $\lambda \in S$ and $x = (x_n)_{n\in\mathbb{N}} \in X(\ell^s)$ define

$$\widetilde{R}(\lambda)x := (R(\lambda)x_n)_{n\in\mathbb{N}}.$$

Then $\widetilde{R}(\lambda) \in L(X(\ell^s))$, and the operator set $\widetilde{R}(S)$ is uniformly bounded by M. Moreover it is an easy calculation that $\widetilde{R}(\lambda) = \lambda R(\lambda, \widetilde{A}_s)$, hence $\widetilde{A}_s$ is sectorial with $\omega(\widetilde{A}_s) \le \sigma$. □

The representation (3.2.2) of the resolvents of $\widetilde{A}_s$ implies that also the functional calculus for $\widetilde{A}_s$ is just given by diagonal operators with maximal domains. Recall that $\mathfrak{B}(\Sigma_\sigma)$ is the algebra of analytic functions on the sector Σ_σ that are polynomially bounded at 0 and ∞, cf. Section 1.2.

Lemma 3.2.3. *Let $s \in [1, +\infty]$. Assume that A is $\mathcal{R}_s$-sectorial and let $\sigma \in (\omega_{\mathcal{R}_s}(A), \pi]$ and $f \in \mathfrak{B}(\Sigma_\sigma)$. Then*

$$D(f(\widetilde{A}_s)) = \{(x_n)_{n\in\mathbb{N}} \in X(\ell^s) \,|\, x_n \in D(f(A)) \text{ for all } n \in \mathbb{N} \text{ and } (f(A)x_n)_{n\in\mathbb{N}} \in X(\ell^s)\}$$

and $f(\widetilde{A}_s)x = (f(A)x_n)_{n\in\mathbb{N}}$ for all $x = (x_n)_{n\in\mathbb{N}} \in D(f(\widetilde{A}_s))$.

Proof. Let first $\varphi \in H_0^\infty(\Sigma_\sigma)$. Let $\omega \in (\omega_{\mathcal{R}_s}(A), \sigma)$ and Γ be the usual parametrization of the boundary $\partial\Sigma_\omega$. Since the projections $\pi_k : X(\ell^s) \to X, (x_n)_{n\in\mathbb{N}} \mapsto x_k$ are continuous for all $k \in \mathbb{N}$, the representation (3.2.2) of the resolvents of $\widetilde{A}_s$ implies

$$\varphi(\widetilde{A}_s)x = \frac{1}{2\pi i}\int_\Gamma \varphi(z)R(z, \widetilde{A}_s)x\,dz = \left(\frac{1}{2\pi i}\int_\Gamma \varphi(z)R(z, A)x_n\,dz\right)_{n\in\mathbb{N}} = (\varphi(A)x_n)_{n\in\mathbb{N}}$$

for all $x = (x_n)_{n\in\mathbb{N}} \in X(\ell^s)$. Now consider the general case $f \in \mathfrak{B}(\Sigma_\sigma)$. Let $\rho(z) := z(1+z)^{-2}$ and choose $m \in \mathbb{N}_0$ such that $\varphi := \rho^m f \in H_0^\infty(X(\ell^s))$. Then

$$f(\widetilde{A}_s) = (\rho(\widetilde{A}_s))^{-m}(\rho^m f)(\widetilde{A}_s) = \big(\widetilde{A}_s(1+\widetilde{A}_s)^{-2}\big)^{-m}\varphi(\widetilde{A}_s) = \big((1+\widetilde{A}_s)^2\widetilde{A}_s^{-1}\big)^m\varphi(\widetilde{A}_s).$$

This yields the claim since $\big((1+\widetilde{A}_s)^2\widetilde{A}_s^{-1}\big)^m$ is a diagonal operator with maximal domain. □

Finally we observe that in the case $s < +\infty$ the diagonal operator $\widetilde{A}_s$ coincides with the vector-valued extension A^{ℓ^s} as defined in Section 2.5

Remark 3.2.4. *Let $s \in [1, +\infty)$. Then $\widetilde{A}_s = A^{\ell^s}$, where the latter operator is defined as in Section 2.5.*

Proof. So assume first that $(x, y) \in A^{\ell^s}$. Let $k \in \mathbb{N}$ and define $\varphi_k : \ell^s \to \mathbb{C}, z \mapsto z_k$, then $\varphi_k \in (\ell^s)'$, $\langle x, \varphi_k\rangle = x_k$ and $\langle y, \varphi_k\rangle = y_k$. Thus by definition of A^{ℓ^s} we obtain $x_k = \langle x, \varphi_k\rangle \in D(A)$ and $y_k = \langle y, \varphi_k\rangle = A\langle x, \varphi_k\rangle Ax_k$, hence $(Ax_n)_{n\in\mathbb{N}} = y \in X(\ell^s)$, so $x \in D(\widetilde{A}_s)$ and $A^{\ell^s}x = y = \widetilde{A}_s x$.

Now let conversely $x = (x_n)_{n\in\mathbb{N}} \in D(\widetilde{A}_s)$. Let $\varphi = (a_n)_{n\in\mathbb{N}} \in \ell^{s'} \cong (\ell^s)'$, then by Hölder's inequality (cf. Subsection 1.6.2)

$$\begin{aligned}\|\langle x, \varphi\rangle\|_X &= \Big\|\sum_{n=1}^{\infty} a_n x_n\Big\|_X \leq \Big\|\sum_{n=1}^{\infty} |a_n x_n|\Big\|_X \leq \Big\|\Big(\sum_{n=1}^{\infty} |x_n|^s\Big)^{1/s} \cdot \Big(\sum_{n=1}^{\infty} |a_n|^{s'}\Big)^{1/s'}\Big\|_X \\ &= \|a\|_{\ell^{s'}} \cdot \|x\|_{X(\ell^s)}\end{aligned}$$

and analogously $\|\langle \widetilde{A}_s x, \varphi\rangle\|_X = \Big\|\sum_{n=1}^{\infty} a_n\, Ax_n\Big\|_X \leq \Big\|\sum_{n=1}^{\infty} |a_n\, Ax_n|\Big\|_X \leq \|a\|_{\ell^{s'}} \cdot \|\widetilde{A}_s x\|_{X(\ell^s)}$.

Since A is closed, this implies $\langle x, \varphi\rangle = \sum_{n=1}^{\infty} a_n x_n \in D(A)$ and $A\langle x, \varphi\rangle = \sum_{n=1}^{\infty} a_n\, Ax_n = \langle \widetilde{A}_s x, \varphi\rangle$, i.e. $(x, \widetilde{A}_s x) \in A^{\ell^s}$. □

With Proposition 2.5.1 this immediately yields the following

Corollary 3.2.5. *Let $s \in [1, +\infty)$ and A be $\mathcal{R}_s$-sectorial, then the following statements hold.*

(1) $A \otimes \mathrm{Id}_{\ell^s} \subseteq \widetilde{A}_s$ *and* $\widetilde{A}_s = \overline{A \otimes \mathrm{Id}_E}$,

(2) If $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_{\omega_{\mathcal{R}_s(A)}}$, *then* $R(\lambda, \widetilde{A}_s) = \overline{R(\lambda, A) \otimes \mathrm{Id}_{\ell^s}} = \widetilde{R(\lambda, A)}_s$,

(3) If D is a core for A, then $D^{(\mathbb{N})} := \big\{(x_n)_n \in D^{\mathbb{N}} \mid \{n \in \mathbb{N} : x_n \neq 0\}$ *is finite*$\big\}$ *is a core for* $\widetilde{A}_s$,

(4) Let $\sigma > \omega_{\mathcal{R}_s}(A)$ *and* $f \in \mathfrak{B}(\Sigma_\sigma)$, *then* $f(\widetilde{A}_s) = \overline{f(A) \otimes \mathrm{Id}_{\ell^s}} = \widetilde{f(A)}_s$.

□

We have left out the case $s = +\infty$ in our considerations of Remark 3.2.4 above. In this case only the inclusion $A^{\ell^s} \subseteq \widetilde{A}_s$ is trivial, but it seems that the other inclusion $\widetilde{A}_\infty \subseteq A^{\ell^\infty}$ might fail, since the dual space of ℓ^∞ is "too large". Nevertheless, in the sequel it is sufficient to work with the operators $\widetilde{A}_s$ for $s \in [1, +\infty]$, hence we will not discuss this problem any further.

We will now turn to some elementary properties of $\mathcal{R}_s$-sectorial operators that are standard for sectorial operators or e.g. $\mathcal{R}$-sectorial operators. In fact, many properties can be proven analogously as it is done for $\mathcal{R}$-sectorial operators in [KW04], Chapter 2, hence we will often refer to the proofs given there.

The same arguments as for $\mathcal{R}$-sectorial operators show the following

Remark 3.2.6. *If the set $\{t(t+A)\,|\,t>0\}$ is $\mathcal{R}_s$-bounded, then A is $\mathcal{R}_s$-sectorial.* □

By Example 3.1.24 we obtain immediately that the Laplace operator is $\mathcal{R}_s$-sectorial in $L^p(\mathbb{R}^d)$ for all $p\in(1,+\infty), s\in(1,+\infty]$. We will show a more detailed assertion in Proposition 3.2.11 below, where we also state the $\mathcal{R}_s$-sectoriality angle.

Using the elementary functional calculus for H_0^∞-functions we can extend $\mathcal{R}_s$-boundedness to more general sets consisting of operators generated by functions of A.

Lemma 3.2.7. *Let $s\in[1,+\infty]$ and A be an $\mathcal{R}_s$-sectorial operator in X. Let $\sigma>\omega_{\mathcal{R}_s}(A)$ and $\mathcal{F}\subseteq H_0^\infty(\Sigma_\sigma)$ such that one can choose $C_0,\beta>0$ with $|\varphi(z)|\le C_0\left(|z|^\beta\wedge|z|^{-\beta}\right)$ for all $z\in\Sigma_\sigma,\varphi\in\mathcal{F}$. Then for each $0\le\delta<\sigma-\omega_{\mathcal{R}_s}$, the set*

$$\{\varphi(zA)\,|\,z\in\Sigma_\delta,\varphi\in\mathcal{F}\}$$

is $\mathcal{R}_s$-bounded. To be more precise, for each $\omega\in(\omega_{\mathcal{R}_s}(A),\sigma)$ we can choose a constant $C=C(\omega,C_0,\beta)$ such that the estimate

$$\mathcal{R}_s(\{\varphi(zA)\,|\,z\in\Sigma_\delta,\varphi\in\mathcal{F}\})\le C\,M_{s,\sigma}(A)$$

holds for all $\delta\in[0,\sigma-\omega)$.

Proof. Let $\sigma-\delta>\omega>\omega_M(A)$. Let $z\in\Sigma_\delta$, then $z=\tau w$ for some $\tau>0,w\in\Sigma_\delta\cap S^1$, and $w\lambda\in\Sigma_\sigma$ for all $\lambda\in\partial\Sigma_\omega$, hence

$$\begin{aligned}\int_{\partial\Sigma_\omega}|\varphi(z\lambda)|\,\frac{|d\lambda|}{|\lambda|} &= \sum_{j\in\{-1,1\}}\int_0^\infty|\varphi(\tau twe^{ij\omega})|\,\frac{dt}{t}=\sum_{j\in\{-1,1\}}\int_0^\infty|\varphi(twe^{ij\omega})|\,\frac{dt}{t}\\ &\le C_0\sum_{j\in\{-1,1\}}\int_0^\infty t^\beta\wedge t^{-\beta}\,\frac{dt}{t}=:M<+\infty,\end{aligned}$$

i.e. $\|\varphi(z\cdot)\|_{L^1\left(\partial\Sigma_\omega,|\frac{d\lambda}{\lambda}|\right)}\le M$ for all $z\in\Sigma_\delta,\varphi\in\mathcal{F}$. We have

$$\varphi(zA)=\frac{1}{2\pi i}\int_{\partial\Sigma_\omega}\varphi(z\lambda)R(\lambda,A)\,d\lambda=\frac{1}{2\pi i}\int_{\partial\Sigma_\omega}\varphi(z\lambda)\cdot\lambda R(\lambda,A)\,\frac{d\lambda}{\lambda},$$

hence $|\varphi(zA)x|\le\frac{1}{2\pi}\int_{\partial\Sigma_\omega}|\varphi(z\lambda)|\cdot|\lambda R(\lambda,A)x|\,|\frac{d\lambda}{\lambda}|$ for each $\varphi\in\mathcal{F},z\in\Sigma_\delta$ and $x\in X$. Since $\{\lambda R(\lambda,A)\,|\,\lambda\in\partial\Sigma_\omega\}$ is $\mathcal{R}_s$-bounded, the assertion follows with Corollary 3.1.10. □

In view of the functional calculus for the extended Dunford-Riesz class (cf. Section 1.2) we obtain the following slightly more general version.

Corollary 3.2.8. *Let $s\in[1,+\infty]$ and A be an $\mathcal{R}_s$-sectorial operator in X. Let $\sigma>\omega_{\mathcal{R}_s}(A)$ and $\mathcal{F}\subseteq\mathcal{E}(\Sigma_\sigma)$ such that there exists an $\varepsilon>0$ with*

$$M_0:=\sup_{f\in\mathcal{F}}\|f\|_{\infty,\sigma}<+\infty\qquad\text{and } C_0:=\sup_{f\in\mathcal{F}}\sup_{z\in\Sigma_\sigma}\left(|z|^\varepsilon\vee|z|^{-\varepsilon}\right)\left|f(z)-\frac{f(0)+f(\infty)z}{1+z}\right|<+\infty.\tag{3.2.3}$$

Let $0\le\delta<\sigma-\omega_{\mathcal{R}_s}$, then the set $\{f(zA)\,|\,z\in\Sigma_\delta,f\in\mathcal{F}\}$ is $\mathcal{R}_s$-bounded.

Proof. Let $f \in \mathcal{F}$, then by Section 1.2 we have a decomposition $f(\zeta) = \varphi_f(\zeta) + \frac{a}{1+\zeta} + b$, where $\varphi \in H_0^\infty(\Sigma_\sigma)$ and $b = f(\infty), a + b = f(0)$. Hence $|a|, |b| \leq 2M_0$ and

$$|\varphi_f(\zeta)| = \left| f(\zeta) - \frac{(a+b) + b\zeta}{1+\zeta} \right| \leq C_0 \, |z|^\varepsilon \wedge |z|^{-\varepsilon}.$$

For each $z \in \Sigma_\delta$ we obtain the representation $f(z\cdot) = \varphi_f(z\cdot) + \frac{a}{1+z\cdot} + b$, hence

$$f(zA) = \varphi_f(zA) + a(1+zA)^{-1} + b\,\mathrm{Id}_Y = \varphi_f(zA) + a\lambda R(\lambda, A) + b\,\mathrm{Id}_Y,$$

where $\lambda := -\frac{1}{z} \in \mathbb{C} \backslash \overline{\Sigma_{\pi-\delta}}$. Since $\omega' := \pi - \delta \geq \sigma - \delta > \omega_M(A)$, the set

$$\left\{ a\lambda R(\lambda, A) \;\middle|\; -\frac{1}{\lambda} \in \Sigma_\delta, |a| \leq 2M_0 \right\}$$

is $\mathcal{R}_s$-bounded, and Lemma 3.2.7 implies the $\mathcal{R}_s$-boundedness of $\{\varphi_f(zA) \,|\, z \in \Sigma_\delta, f \in \mathcal{F}\}$, hence also $\{f(zA) \,|\, z \in \Sigma_\delta, f \in \mathcal{F}\}$ is $\mathcal{R}_s$-bounded. □

A special case is the following corollary with just one function $f \in \mathcal{E}(\Sigma_\sigma)$.

Corollary 3.2.9. *Let $s \in [1, +\infty]$ and A be an $\mathcal{R}_s$-sectorial operator in X. Let $\sigma > \omega_{\mathcal{R}_s}(A)$, $0 \leq \delta < \sigma - \omega_{\mathcal{R}_s}$ and $f \in \mathcal{E}(\Sigma_\sigma)$. Then the set $\{f(zA) \,|\, z \in \Sigma_\delta\}$ is $\mathcal{R}_s$-bounded.* □

An immediate consequence is the following: If $\omega_{\mathcal{R}_s}(A) < \pi/2$, the generated analytic semigroup $(e^{-zA})_{z \in \Sigma_\delta}$ is $\mathcal{R}_s$-bounded for all $\delta \in [0, \pi/2 - \omega_{\mathcal{R}_s}(A))$. We will say in this case that A is $\mathcal{R}_s$-*analytic*. More characterizations of $\mathcal{R}_s$-analyticity are the content of the following proposition, which is an $\mathcal{R}_s$-bounded version of [KW04], Theorem 2.20, and again the proof given there carries over to our situation if we replace "$\mathcal{R}$-boundedness" by "$\mathcal{R}_s$-boundedness" and take into account Examples 3.1.11, 3.1.12. Moreover, note that assumption (4) is a modification of the corresponding assumption in [KW04], Theorem 2.20, but equivalent by Remark 3.1.6.

Proposition 3.2.10. *Assume $\omega(A) < \pi/2$. Then the following conditions are equivalent:*

(1) A is $\mathcal{R}_s$-analytic,

(2) A is $\mathcal{R}_s$-sectorial with $\omega_{\mathcal{R}_s}(A) < \pi/2$,

(3) The set $\{t^n(it + A)^{-n} \mid t \in \mathbb{R}\backslash\{0\}\}$ is $\mathcal{R}_s$-bounded for some $n \in \mathbb{N}$,

(4) The sets $\{e^{-tA} \mid t > 0\}, \{tAe^{-tA} \mid t > 0\}$ are $\mathcal{R}_s$-bounded.

We conclude this subsection with our most important example.

Proposition 3.2.11. *Let $d \in \mathbb{N}$ and $p \in (1, +\infty), s \in (1, \infty]$. Then the Laplace operator $-\Delta$ is $\mathcal{R}_s$-analytic in $L^p(\mathbb{R}^d)$, and if $s < +\infty$, then $\omega_{\mathcal{R}_s}(-\Delta) = 0$.*

Proof. We will show first that $-\Delta$ is $\mathcal{R}_s$-sectorial with $\omega_{\mathcal{R}_s}(-\Delta) < \pi/2$. By Proposition 3.2.10 it is sufficient to show that the sets $\{e^{t\Delta} \,|\, t > 0\}, \{t(-\Delta)e^{t\Delta} \,|\, t > 0\}$ are $\mathcal{R}_s$-bounded, which in turn we will show with Proposition 3.1.23 in the same way as we proved Example 3.1.24. Observe that $-\Delta$ generates the heat semigroup which is given by convolution with the kernel

$$h_t(x) := t^{-d/2}\,\varphi(|x|/\sqrt{t}) \quad \text{for all } x \in \mathbb{R}^d, t > 0,$$

where $\varphi(y) := (4\pi)^{-d/2}e^{-y^2/4}$ for all $y \geq 0$, i.e. $e^{t\Delta}f = h_t * f$ for all $f \in L^p, t > 0$. Since the function φ fulfills obviously the assumptions of Proposition 3.1.23 we obtain that the set $\{e^{t\Delta} \mid t > 0\}$ is $\mathcal{R}_s$-bounded.

Moreover, the operator $-t\Delta e^{t\Delta}$ is formally given by convolution with the kernel

$$\begin{aligned} k_t(x) &:= -t\frac{d}{dt}h_t(x) = -t\Big(-\frac{d}{2}t^{-d/2-1}\varphi(|x|/\sqrt{t}) + t^{-d/2}\varphi'(|x|/\sqrt{t})\cdot\Big(-\frac{1}{2}t^{-3/2}\Big)\Big) \\ &= t^{-d/2}\,\psi(|x|/\sqrt{t}) \quad \text{for all } x \in \mathbb{R}^d, t > 0, \end{aligned}$$

where $\psi(y) := \frac{d}{2}\varphi(y) + \frac{1}{2}y\varphi'(y)$ for all $y > 0$, and this formal calculation leads easily to the identity $t(-\Delta)e^{t\Delta}f = k_t * f$ for all $f \in L^p, t > 0$. The function ψ can obviously be dominated by a function $\widetilde{\varphi}$ that fulfills the assumptions of Proposition 3.1.23, hence we obtain that also the set $\{t(-\Delta)e^{t\Delta} \mid t > 0\}$ is $\mathcal{R}_s$-bounded. So the first claim follows with Proposition 3.2.10.

The claim on the angle will be proven in a more general form in Proposition 3.2.24 in Subsection 3.2.3. □

3.2.2 Equivalence of s-power function norms

We will now turn to the central estimates for a reasonable definition of the associated s-intermediate spaces for an $\mathcal{R}_s$-sectorial operator, which will be done in the next section. The following proposition is well known for $s = 2$ and $X = L^p$, cf. [LeM04], Theorem 1.1, and our proof follows the line of the proof in that case given in [LeM04].

Proposition 3.2.12. *Let $s \in [1, +\infty]$ and A be an $\mathcal{R}_s$-sectorial operator in X. Let $\sigma > \omega_{\mathcal{R}_s}(A)$ and $\varphi, \psi \in H_0^\infty(\Sigma_\sigma)\backslash\{0\}$. Then there is a constant $C > 0$ such that for all $f \in H^\infty(\Sigma_\sigma)$ and $x \in X$ we have*

$$\Big\|\Big(\int_0^\infty |f(A)\varphi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X \;\leq\; C\,\|f\|_{\infty,\sigma}\Big\|\Big(\int_0^\infty |\psi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X \tag{3.2.4}$$

(with the usual modification if $s = +\infty$).

Remark 3.2.13. *The norm expressions occurring in the estimate (3.2.4) will also be referred to as* s-power function norms.

Proof of Proposition 3.2.12. We will first show (3.2.4) for $f \in H_0^\infty(\Sigma_\sigma)$. Let $x \in X$ be such that $\|x\|_\psi := \|(\int_0^\infty |\psi(tA)x|^s\,\frac{dt}{t})^{1/s}\|_X < +\infty$. Let $\omega \in (\omega_{\mathcal{R}_s}(A), \sigma)$ and Γ be the parameterized contour of $\partial\Sigma_\omega$. Choose auxiliary functions $F, G \in H_0^\infty(\Sigma_\sigma)$ such that

$$\int_0^\infty F(t)G(t)\psi(t)\,\frac{dt}{t} = 1. \tag{3.2.5}$$

Then for each $H \in \{F, G\}$ we have

$$\sup_{t>0} \int_\Gamma |H(tz)| \frac{|dz|}{|z|} < +\infty \quad \text{and} \quad \sup_{z\in\partial\Sigma_\omega} \int_0^\infty |H(tz)| \frac{dt}{t} < +\infty,$$

since $|H(z)| \lesssim |z|^\varepsilon \wedge |z|^{-\varepsilon}$ for some $\varepsilon > 0$: from this, both claims follow by the translation invariance of the Haar measure dt/t on $(0,\infty)$, where for the first claim one just applies the parametrization of $\partial\Sigma_\omega$, and for the second claim we observe that for $z = re^{\pm i\omega}$:

$$\begin{aligned}\int_0^\infty |H(tre^{\pm i\omega})| \frac{dt}{t} &= \int_0^\infty |H(te^{\pm i\omega})| \frac{dt}{t} \lesssim \int_0^\infty |te^{\pm i\omega}|^\varepsilon \wedge |te^{\pm i\omega}|^{-\varepsilon} \frac{dt}{t} \\ &= \int_0^\infty t^\varepsilon \wedge t^{-\varepsilon} \frac{dt}{t} < +\infty.\end{aligned}$$

We will proceed in four steps, where the constants C_k wich occur in each step do not depend on x and f.

Step 1. For all $t > 0$ we have

$$S(t) := f(A)G(tA) = \frac{1}{2\pi i} \int_\Gamma f(z)G(tz)zR(z,A) \frac{dz}{z}. \tag{3.2.6}$$

Then

$$\|f(\cdot)G(t\cdot)\|_{L^1\left(\Gamma, |\frac{dz}{z}|\right)} = \int_\Gamma |f(z)G(tz)| \frac{|dz|}{|z|} \leq \Big(\sup_{r>0} \int_\Gamma |G(rz)| \frac{|dz|}{|z|}\Big) \cdot \|f\|_{\infty,\sigma}.$$

Since $\mathcal{S} := \{zR(z,A) \,|\, z \in \Gamma\}$ is $\mathcal{R}_s$-bounded by assumption, Corollary 3.1.10 yields that also $S((0,+\infty))$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(S((0,+\infty))) \leq C_1 \|f\|_{\infty,\sigma}$. Hence by Proposition 3.1.13 we obtain

$$\Big\|\Big(\int_0^\infty |f(A)G(tA)\psi(tA)x|^s \frac{dt}{t}\Big)^{1/s}\Big\|_X = \Big\|\Big(\int_0^\infty |S(t)\psi(tA)x|^s \frac{dt}{t}\Big)^{1/s}\Big\|_X \leq C_2 \cdot \|f\|_{\infty,\sigma} \cdot \|x\|_\psi. \tag{3.2.7}$$

(with the usual modification if $s = +\infty$ using Proposition 3.1.14.)

Step 2. Let $w(t) := S(t)\psi(tA)x$ for all $t > 0$ and $u(z) := \displaystyle\int_0^\infty F(tz)w(t) \frac{dt}{t}$. By choosing appropriate representatives, by Hölder's inequality and Fubini's theorem we have μ-a.e. for $s < +\infty$:

$$\begin{aligned}\int_\Gamma |u(z)|^s \frac{|dz|}{|z|} &\leq \int_\Gamma \Big(\int_0^\infty |F(tz)w(t)| \frac{dt}{t}\Big)^s \frac{|dz|}{|z|} \\ &= \int_\Gamma \Big(\int_0^\infty |F(tz)|^{1/s'} \cdot |F(tz)|^{1/s}|w(t)| \frac{dt}{t}\Big)^s \frac{|dz|}{|z|} \\ &\leq \int_\Gamma \Big(\int_0^\infty |F(tz)| \frac{dt}{t}\Big)^{s-1} \cdot \int_0^\infty |F(tz)||w(t)|^s \frac{dt}{t} \frac{|dz|}{|z|} \\ &\leq \sup_{z\in\Gamma} \Big(\int_0^\infty |F(tz)| \frac{dt}{t}\Big)^{s-1} \cdot \int_0^\infty \Big(\int_\Gamma |F(tz)| \frac{|dz|}{|z|}\Big)|w(t)|^s \frac{dt}{t} \\ &\leq \sup_{z\in\Gamma} \Big(\int_0^\infty |F(tz)| \frac{dt}{t}\Big)^{s-1} \cdot \Big(\sup_{t>0} \int_\Gamma |F(tz)| \frac{|dz|}{|z|}\Big) \cdot \int_0^\infty |w(t)|^s \frac{dt}{t},\end{aligned}$$

hence we obtain the estimate

$$\Big\|\Big(\int_\Gamma |u(z)|^s \frac{|dz|}{|z|}\Big)^{1/s}\Big\|_X \le C_3 \cdot \Big\|\Big(\int_0^\infty |w(t)|^s \frac{dt}{t}\Big)^{1/s}\Big\|_X \overset{(3.2.7)}{\le} C_4 \cdot \|f\|_{\infty,\sigma} \cdot \|x\|_\psi. \quad (3.2.8)$$

If $s = +\infty$, we obtain similarly

$$\sup_{z\in\Gamma} |u(z)| \le \Big(\sup_{z\in\Gamma} \int_0^\infty |F(tz)| \frac{dt}{t}\Big) \cdot \sup_{t>0} |w(t)|,$$

hence also

$$\big\|\sup_{z\in\Gamma} |u(z)|\big\|_X \le C_3 \cdot \big\|\sup_{t>0} |w(t)|\big\|_X \overset{(3.2.7)}{\le} C_4 \cdot \|f\|_{\infty,\sigma} \cdot \|x\|_\psi. \quad (3.2.9)$$

Step 3. Let $v(t) := \int_\Gamma \varphi(tz) zR(z,A)u(z) \frac{dz}{z}$ for all $t > 0$. Then again, with Hölder's inequality and Fubini's theorem we can conclude if $s < +\infty$:

$$\begin{aligned}
\int_0^\infty |v(t)|^s \frac{dt}{t} &\le \int_0^\infty \Big(\int_\Gamma |\varphi(tz)|\, |zR(z,A)u(z)| \frac{|dz|}{|z|}\Big)^s \frac{dt}{t}\\
&\le \int_0^\infty \Big(\int_\Gamma |\varphi(tz)| \frac{|dz|}{|z|}\Big)^{s-1} \int_\Gamma |\varphi(tz)||zR(z,A)u(z)|^s \frac{|dz|}{|z|} \frac{dt}{t}\\
&\le \sup_{t>0} \Big(\int_\Gamma |\varphi(tz)| \frac{|dz|}{|z|}\Big)^{s-1} \cdot \int_\Gamma \Big(\int_0^\infty |\varphi(tz)| \frac{dt}{t}\Big) |zR(z,A)u(z)|^s \frac{|dz|}{|z|}\\
&\le \sup_{t>0} \Big(\int_\Gamma |\varphi(tz)| \frac{|dz|}{|z|}\Big)^{s-1} \cdot \Big(\sup_{z\in\Gamma} \int_0^\infty |\varphi(tz)| \frac{dt}{t}\Big) \cdot \int_\Gamma |zR(z,A)u(z)|^s \frac{|dz|}{|z|}.
\end{aligned}$$

Using again $\mathcal{R}_s$-boundedness of $\{zR(z,A) \,|\, z \in \Gamma\}$ in the same way as in Step 1 we obtain

$$\begin{aligned}
\Big\|\Big(\int_0^\infty |v(t)|^s \frac{dt}{t}\Big)^{1/s}\Big\|_X &\le C_5 \cdot \Big\|\Big(\int_\Gamma |zR(z,A)u(z)|^s \frac{|dz|}{|z|}\Big)^{1/s}\Big\|_X\\
&\le C_5 \mathcal{R}_s(\mathcal{S}) \cdot \Big\|\Big(\int_\Gamma |u(z)|^s \frac{|dz|}{|z|}\Big)^{1/s}\Big\|_X\\
&\overset{(3.2.8)}{\le} C_6 \cdot \|f\|_{\infty,\sigma} \cdot \|x\|_\psi. \quad (3.2.10)
\end{aligned}$$

The analogous inequality holds also in the case $s = +\infty$, which can be shown in the same manner as in Step 2.

Step 4. By analytic continuation we have

$$\int_0^\infty F(tz)G(tz)\psi(tz) \frac{dt}{t} = 1 \quad \text{for all } z \in \Sigma_\sigma.$$

By the multiplicativity of the functional calculus (and Fubini) we obtain

$$f(A) = \int_0^\infty F(tA)G(tA)\psi(tA)f(A) \frac{dt}{t},$$

hence for all $\tau > 0$:

$$\begin{aligned} f(A)\varphi(\tau A)x &= \int_0^\infty \varphi(\tau A)F(tA)G(tA)\psi(tA)f(A)x\,\frac{dt}{t} \\ &= \int_0^\infty \Big(\frac{1}{2\pi i}\int_\Gamma \varphi(\tau z)F(tz)R(z,A)\,dz\Big)G(tA)\psi(tA)f(A)x\,\frac{dt}{t} \\ &= \frac{1}{2\pi i}\int_\Gamma \varphi(\tau z)R(z,A)\Big(\int_0^\infty F(tz)G(tA)\psi(tA)f(A)x\,\frac{dt}{t}\Big)\,dz \\ &= \frac{1}{2\pi i}\int_\Gamma \varphi(\tau z)R(z,A)u(z)\,dz = v(\tau). \end{aligned}$$

So with (3.2.10) the claim follows for $f \in H_0^\infty(\Sigma_\sigma)$.

Now let $f \in H^\infty(\Sigma_\sigma)$ be arbitrary. Let again $\rho_n(z) := \frac{n}{n+z} - \frac{1}{1+nz}$ for all $z \in \Sigma_\sigma, n \in \mathbb{N}$. Then $\rho_n \in H_0^\infty(\Sigma_\sigma)$ and $|\rho_n(z)| \le K_\sigma$ for all $z \in \Sigma_\sigma, n \in \mathbb{N}$ (where $K_\sigma := 2\,\mathrm{dist}(-1,\overline{\Sigma}_\sigma)$). Let $f_n := \rho_n \cdot f$ for all $n \in \mathbb{N}$, then $(f_n)_{n\in\mathbb{N}} \in (H_0^\infty(\Sigma_\sigma))^{\mathbb{N}}$ is a bounded sequence such that $|f_n(z)| \le K_\sigma \cdot |f(z)|$ and $f_n(A)\phi(tA)x \to f(A)\phi(tA)x$ in X for $n \to \infty$ and all $x \in X, t > 0$. Let $x \in X$, then by Lemma 1.6.24 we may assume w.l.o.g. that also $f_n(A)\phi(\cdot A)x \to f(A)\phi(\cdot A)x$ pointwise $\frac{dt}{t} \otimes \mu$-a.e. for $n \to \infty$, hence we obtain with the Fatou property (again, with the usual modification if $s = +\infty$):

$$\begin{aligned} &\Big\|\Big(\int_0^\infty |f(A)\varphi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X \le \liminf_{n\to\infty}\Big\|\Big(\int_0^\infty |f_n(A)\varphi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X \\ \le\;& C\,\liminf_{n\to\infty}\|f_n\|_{\infty,\sigma}\Big\|\Big(\int_0^\infty |\psi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X \le CK_\theta\,\|f\|_{\infty,\sigma}\Big\|\Big(\int_0^\infty |\psi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X. \end{aligned}$$

□

We now turn to a discrete version of Proposition 3.2.12. At present we do not know if such a discrete version (in the sense of Propositions 3.2.17, 3.2.18 below) will hold in full generality for all $\varphi, \psi \in H_0^\infty(\Sigma_\sigma)\backslash\{0\}$. In fact, the proof of Proposition 3.2.12 would also work in a discrete setting if a discrete analogon of formula (3.2.5) holds, i.e., if for any $\psi \in H_0^\infty(\Sigma_\sigma)\backslash\{0\}$ one can choose functions $F, G \in H_0^\infty(\Sigma_\sigma)$ such that

$$\forall\, t > 0 \,:\, \sum_{j\in\mathbb{Z}} F(2^j t)G(2^j t)\psi(2^j t) = 1. \tag{3.2.11}$$

Since we do not know if such a general formula holds, we will restrict ourselves to a suitable subclass of H_0^∞-functions which will lead to an equivalence between the continuous s-power function norms as in (3.2.4) to some corresponding discrete s-power function norms. The assumptions for this class of functions are rather technical and made to fit our needs, but they are not too restrictive: we will show that the standard H_0^∞-functions we usually use as concrete auxiliary functions belong to this subclass of H_0^∞.

Definition 3.2.14. Let $\sigma \in (0,\pi]$ and $\varphi \in H_0^\infty(\Sigma_\sigma)$ with $0 \notin \varphi(\Sigma_\sigma)$. We say that φ belongs to the class $\Phi_{\sigma,0}^\Sigma$ if the following property (named after *uniform in* $\mathcal{E}$) holds:

(UE) There exist $d \in \mathbb{Z}$ and a set of functions $\mathcal{F} \subseteq \mathcal{E}(\Sigma_\sigma)$ such that $\mathcal{F}$ fulfills condition (3.2.3) from Corollary 3.2.8 and

$$\{\varphi(2^j t\cdot)/\varphi(2^{j-d}\cdot), \varphi(2^{j+d}\cdot)/\varphi(2^j t\cdot) \mid j \in \mathbb{Z}, t \in [1,2]\} \subseteq \{f(r\cdot) \mid f \in \mathcal{F}, r > 0\}.$$

By Corollary 3.2.8 we obtain the following important issue:

Lemma 3.2.15. *Let $\sigma \in (0,\pi]$ and $\varphi \in \Phi^{\Sigma}_{\sigma,0}$. Choose $d \in \mathbb{Z}$ due to property (UE) above and define the operators $S_j(t) := \big(\varphi(2^{j+d}\cdot)/\varphi(2^j t\cdot)\big)(A)$ and $\overline{S}_j(t) := \big(\varphi(2^j t\cdot)/\varphi(2^{j-d}\cdot)\big)(A)$ for all $j \in \mathbb{Z}, t \in [0,1]$. Then the set*

$$\{S_j(t), \overline{S}_j(t) \mid j \in \mathbb{Z}, t \in [1,2]\}$$

is $\mathcal{R}_s$-bounded, and for all $j \in \mathbb{Z}, t \in [1,2]$ we have

$$\varphi(2^j tA)\, S_j(t) = S_j(t)\varphi(2^j tA) = \varphi(2^{j+d}A), \quad \textit{hence } S_j(t) = \varphi(2^j tA)^{-1}\varphi(2^{j+d}A). \tag{3.2.12}$$

and

$$\varphi(2^{j-d}A)\, \overline{S}_j(t) = \overline{S}_j(t)\varphi(2^{j-d}A) = \varphi(2^j tA) \quad \textit{hence } \overline{S}_j(t) = \varphi(2^{j-d}A)^{-1}\varphi(2^j tA), \tag{3.2.13}$$

Proof. The first statement follows immediately from the definition of the class $\Phi^{\Sigma}_{0,\sigma}$ and Corollary 3.2.8, and (3.2.12) and (3.2.13) follow from the multiplicativity property of the functional calculus in the class $\mathcal{E}(\Sigma_\sigma)$. □

We now turn to the most important examples of functions in $\Phi^{\Sigma}_{\sigma,0}$.

Examples 3.2.16. (1) Let $\sigma \in (0,\pi)$ and $m \in \mathbb{N}$ and define $\varphi(z) := \frac{z^m}{(1+z)^{2m}}$ for all $z \in \Sigma_\sigma$, then $\varphi \in \Phi^{\Sigma}_{0,\sigma}$.

Proof. We first consider the case $m = 1$, where we will show that the condition (UE) is fulfilled with $d = 0$. For $j \in \mathbb{Z}, t \in [1,2]$ and $z \in \Sigma_\sigma$ define

$$g_{j,t}(z) := \varphi(2^j tz)/\varphi(2^j z) = t\left(\frac{1+2^j z}{1+2^j tz}\right)^2 = t\left(\frac{1+t^{-1}2^j tz}{1+2^j tz}\right)^2.$$

With $r = 2^j t > 0$ and $\tau := t^{-1} \in [1/2,1]$ we obtain $g_{j,t}(z) = t \cdot f_\tau(rz)$ where $f_\tau(z) := \left(\frac{1+\tau z}{1+z}\right)^2$. We will show that $\mathcal{F} := \{t \cdot f_\tau \mid t, \tau \in [1/2,2]\}$ fulfills condition (3.2.3) from Corollary 3.2.8. So let $\tau \in [1/2,2]$, then $f_\tau(0) = 1$ and $f_\tau(\infty) = \tau^2$, and

$$\psi_\tau(z) \;:=\; f_\tau(z) - \frac{f_\tau(0) + f_\tau(\infty)z}{1+z} = \frac{(1+\tau z)^2 - (1+z)(1+\tau^2 z)}{(1+z)^2} = (2\tau - \tau^2 - 1)\,\frac{z}{(1+z)^2},$$

hence $|\psi_\tau(z)| \le |2\tau - \tau^2 - 1| \left|\frac{z}{(1+z)^2}\right| \le 9 \left|\frac{z}{(1+z)^2}\right|$. This shows that the uniform estimate (3.2.3) holds with $\varepsilon = 1$.

Now consider

$$h_{j,t}(z) := \varphi(2^j z)/\varphi(2^j tz) = t^{-1}\left(\frac{1+2^j tz}{1+2^j z}\right)^2,$$

then $h_{j,t}(z) = \tau f_t(2^j z)$ with the same notations as above, hence also $\{h_{j,t} \mid j \in \mathbb{Z}, t \in [1,2]\} \subseteq \{f(r\cdot) \mid f \in \mathcal{F}\}$.

This shows that condition (UE) is fulfilled with $d = 0$, hence $\varphi \in \Phi^{\Sigma}_{\sigma,0}$ for $m = 1$. The general case $m \in \mathbb{N}$ can be treated analogously, hence we omit the proof. □

(2) Let $\sigma \in (0, \pi/2)$ and $\alpha > 0$ and define $\varphi(z) := z^\alpha e^{-z}$ for all $z \in \Sigma_\sigma$, then $\varphi \in \Phi^{\Sigma}_{0,\sigma}$.

Proof. For $j \in \mathbb{Z}, t \in [1,2]$ and $z \in \Sigma_\sigma$ define

$$g_{j,t}(z) := \varphi(2^j tz)/\varphi(2^{j-1}z) = (2t)^\alpha\, e^{-(2t-1)2^{j-1}z},$$

then $g_{j,t}(z) = e^{-rz}$ with $r := (2t-1)2^{j-1} > 0$, hence $g_{j,t} \in \{\tau \cdot e^{-r\cdot} \mid \tau \in [1, 4^\alpha], r > 0\} =: \mathcal{F}$. Let

$$h_{j,t}(z) := \varphi(2^{j+1}z)/\varphi(2^j tz) = 2/t^\alpha\, e^{-(2-t)2^j z},$$

then also $h_{j,t} \in \mathcal{F}$. Since $\mathcal{F}$ clearly fulfills condition (3.2.3) from Corollary 3.2.8, this shows that condition (UE) is fulfilled with $d = -1$, hence $\varphi \in \Phi^{\Sigma}_{\sigma,0}$. □

We can now turn to the central equivalence of continuous and discrete versions of s-power function norms.

Proposition 3.2.17. *Let $s \in [1, +\infty]$ and A be an $\mathcal{R}_s$-sectorial operator in X. Let $\sigma > \omega_{\mathcal{R}_s(A)}$ and $\varphi \in \Phi^{\Sigma}_{\sigma,0}$. Then there is a constant $C > 0$ such that for all $x \in X$:*

$$C^{-1}\left\|\Big(\sum_{j\in\mathbb{Z}}|\varphi(2^j A)x|^s\Big)^{1/s}\right\|_X \le \left\|\Big(\int_0^\infty |\varphi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\right\|_X \le C\left\|\Big(\sum_{j\in\mathbb{Z}}|\varphi(2^j A)x|^s\Big)^{1/s}\right\|_X.$$

(with the usual modification if $s = +\infty$.)

Proof. We assume first that $s < +\infty$. Choose the integer $d \in \mathbb{Z}$ due to property (UE) of φ and define

$$S_j(t) := \big(\varphi(2^{j+d}\cdot)/\varphi(2^j t\cdot)\big)(A) = \varphi(2^j tA)^{-1}\varphi(2^{j+d}A)$$

and

$$\overline{S}_j(t) := \big(\varphi(2^{j+d}t\cdot)/\varphi(2^j\cdot)\big)(A) = \varphi(2^j A)^{-1}\varphi(2^{j+d}tA)$$

for all $j \in \mathbb{Z}, t \in [0,1]$. By Lemma 3.2.15 the set $\mathcal{S} := \{S_j(t), \overline{S}_j(t) \mid j \in \mathbb{Z}, t \in [1,2]\}$ is $\mathcal{R}_s$-bounded, so let $C := \mathcal{R}_s(\mathcal{S})$.

Let $x \in X$. Define

$$S(r) := \sum_{j\in\mathbb{Z}} \mathbb{1}_{[2^j,2^{j+1})}(r)\varphi(rA)^{-1}\varphi(2^{j+d}A) \quad \text{and } y(t) := \varphi(tA)x \quad \text{for all } t, r > 0,$$

then $y : (0, +\infty) \to X$ is measurable, $S : (0, +\infty) \to L(X)$ is strongly measurable and $S(2^j t) = \varphi(2^j tA)^{-1}\varphi(2^{j+d}A) = S_j(t)$ for all $t \in [1, 2)$ and $j \in \mathbb{Z}$. Moreover, $S((0, +\infty)) \subseteq \mathcal{S}$ is $\mathcal{R}_s$-bounded. By Proposition 3.1.13 we obtain

$$\begin{aligned}
&\Big\|\Big(\sum_{j\in\mathbb{Z}} |\varphi(2^j A)x|^s\Big)^{1/s}\Big\|_X = \Big\|\Big(\sum_{j\in\mathbb{Z}} |\varphi(2^{j+d}A)x|^s\Big)^{1/s}\Big\|_X = \left\|\left(\sum_{j\in\mathbb{Z}}\int_1^2 |\varphi(2^{j+d}A)x|^s\,dt\right)^{1/s}\right\|_X \\
=\;& \left\|\left(\sum_{j\in\mathbb{Z}}\int_1^2 |\varphi(2^j tA)S_j(t)x|^s\,dt\right)^{1/s}\right\|_X \approx \left\|\left(\sum_{j\in\mathbb{Z}}\int_1^2 |S(2^j t)y(2^j t)|^s\,\frac{dt}{t}\right)^{1/s}\right\|_X \\
=\;& \left\|\left(\sum_{j\in\mathbb{Z}}\int_{2^j}^{2^{j+1}} |S(t)y(t)|^s\,\frac{dt}{t}\right)^{1/s}\right\|_X = \left\|\left(\int_0^\infty |S(t)y(t)|^s\,\frac{dt}{t}\right)^{1/s}\right\|_X \\
=\;& \left\|\left(\int_0^\infty |S(t)(t^{-s}y(t))|^s\,dt\right)^{1/s}\right\|_X \le C\cdot\left\|\left(\int_0^\infty |(t^{-s}y(t))|^s\,dt\right)^{1/s}\right\|_X \\
=\;& C\cdot\left\|\left(\int_0^\infty |\varphi(tA)x|^s\,\frac{dt}{t}\right)^{1/s}\right\|_X.
\end{aligned}$$

We now turn to the inverse inequality. By the Fatou property we obtain in a first step

$$\begin{aligned}
&\Big\|\Big(\int_0^\infty |\varphi(tA)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X \approx \Big\|\Big(\sum_{j\in\mathbb{Z}}\int_1^2 |\varphi(2^j tA)x|^s\,dt\Big)^{1/s}\Big\|_X \\
=\;& \Big\|\Big(\sum_{j\in\mathbb{Z}}\int_1^2 |\varphi(2^{j+d}tA)x|^s\,dt\Big)^{1/s}\Big\|_X = \Big\|\Big(\sum_{j\in\mathbb{Z}}\int_1^2 |\overline{S}_j(t)\varphi(2^j A)x|^s\,dt\Big)^{1/s}\Big\|_X \\
=\;& \Big\|\Big(\lim_{N\to\infty}\sum_{j=-N}^{N}\int_1^2 |\overline{S}_j(t)\varphi(2^j A)x|^s\,dt\Big)^{1/s}\Big\|_X \le \liminf_{N\to\infty}\Big\|\Big(\sum_{j=-N}^{N}\int_1^2 |\overline{S}_j(t)\varphi(2^j A)x|^s\,dt\Big)^{1/s}\Big\|_X
\end{aligned}$$

Since $t \mapsto \overline{S}_j(t)$ is analytic, we can work with a version with analytic, hence in particular continuous, paths (cf. Subsection 1.6.1) and obtain

$$\int_1^2 |\overline{S}_j(t)\varphi(2^j A)x|^s\,dt = \lim_{\ell\to\infty}\frac{1}{\ell}\sum_{k=1}^{\ell} |\overline{S}_j\big(1+\frac{k}{\ell}\big)\varphi(2^j A)x|^s$$

μ-a.e. in Ω. Let $\overline{S}_{j,k}^{(\ell)} := \overline{S}_j\big(1+\frac{k}{\ell}\big)$ for all $j \in \mathbb{Z}, \ell \in \mathbb{N}$ and $k \in \mathbb{N}_{\le \ell}$, then using the Fatou property again leads to

$$\begin{aligned}
\Big\|\Big(\int_0^\infty |\varphi(tA)x|^s \frac{dt}{t}\Big)^{1/s}\Big\|_X &\le \liminf_{N\to\infty}\Big\|\Big(\lim_{\ell\to\infty}\sum_{j=-N}^{N}\frac{1}{\ell}\sum_{k=1}^{\ell}|\overline{S}^{(\ell)}_{j,k}\varphi(2^jA)x|^s\Big)^{1/s}\Big\|_X \\
&\le \liminf_{N\to\infty}\liminf_{\ell\to\infty}\ \ell^{-1/s}\Big\|\Big(\sum_{j=-N}^{N}\sum_{k=1}^{\ell}|\overline{S}^{(\ell)}_{j,k}\varphi(2^jA)x|^s\Big)^{1/s}\Big\|_X \\
&\le C\cdot\liminf_{N\to\infty}\liminf_{\ell\to\infty}\ \ell^{-1/s}\Big\|\Big(\sum_{j=-N}^{N}\sum_{k=1}^{\ell}|\varphi(2^jA)x|^s\Big)^{1/s}\Big\|_X \\
&= C\cdot\liminf_{N\to\infty}\Big\|\Big(\sum_{j=-N}^{N}|\varphi(2^jA)x|^s\Big)^{1/s}\Big\|_X \\
&\le C\cdot\Big\|\Big(\sum_{j=-\infty}^{\infty}|\varphi(2^jA)x|^s\Big)^{1/s}\Big\|_X.
\end{aligned}$$

Now let $s=+\infty$. Then we have trivially $\sup_{j\in\mathbb{Z}}|\varphi(2^jA)x|\le\sup_{t>0}|\varphi(tA)x|$ for all $x\in X$, hence we obtain the first inequality $\|\sup_{j\in\mathbb{Z}}|\varphi(2^jA)x|\|_X\le\|\sup_{t>0}|\varphi(tA)x|\|_X$ for all $x\in X$. For the second estimate we use the same notations as above and obtain by Proposition 3.1.14

$$\begin{aligned}
\|\sup_{t>0}|\varphi(tA)x|\|_X &= \|\sup_{j\in\mathbb{Z}}\sup_{t\in[1,2]}|\varphi(2^{j+d}tA)x|\|_X = \|\sup_{j\in\mathbb{Z}}\sup_{t\in[1,2]}|\overline{S}_j(t)\varphi(2^jA)x|\|_X \\
&= \|\sup_{(j,t)\in\mathbb{Z}\times[1,2]}|\overline{S}_j(t)\varphi(2^jA)x|\|_X \le C\cdot\|\sup_{(j,t)\in\mathbb{Z}\times[1,2]}|\varphi(2^jA)x|\|_X \\
&= C\cdot\|\sup_{j\in\mathbb{Z}}|\varphi(2^jA)x|\|_X.
\end{aligned}$$

□

If we combine Proposition 3.2.17 with Proposition 3.2.12 we obtain

Proposition 3.2.18. *Let $s\in[1,+\infty]$ and A be an $\mathcal{R}_s$-sectorial operator in X. Let $\sigma>\omega_{\mathcal{R}_s}(A)$ and $\varphi,\psi\in\Phi^{\Sigma}_{\sigma,0}$. Then there is a constant $C>0$ such that for all $f\in H^\infty(\Sigma_\sigma)$ and $x\in X$ we have*

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|f(A)\varphi(2^jA)x|^s\Big)^{1/s}\Big\|_X \le C\,\|f\|_{\infty,\sigma}\Big\|\Big(\sum_{j\in\mathbb{Z}}|\psi(2^jA)x|^s\Big)^{1/s}\Big\|_X \qquad (3.2.14)$$

(with the usual modification if $s=+\infty$). □

3.2.3 $\mathcal{R}_s$-bounded H^∞-calculus

For this subsection we fix some $s\in[1,+\infty]$.

Definition 3.2.19. Let $\sigma>\omega(A)$. We say that A has an *$\mathcal{R}_s$-bounded $H^\infty(\Sigma_\sigma)$-calculus* if the set

$$\{f(A)\,|\,f\in H^\infty(\Sigma_\sigma),\|f\|_\infty\le 1\}$$

is $\mathcal{R}_s$-bounded, which is equivalent to the existence of a constant $C > 0$ such that the estimate

$$\|(f_j(A)x_j)_j\|_{X(\ell^s)} \leq C \sup_{j\in\mathbb{N}} \|f_j\|_{\infty,\sigma} \cdot \|(x_j)_j\|_{X(\ell^s)}$$

holds for all $f_j \in H^\infty(\Sigma_\sigma)$ and $x_j \in X$, $j \in \mathbb{N}$. In this case we define

$$M^\infty_{s,\sigma}(A) := \mathcal{R}_s\big(\{f(A) \mid f \in H^\infty(\Sigma_\sigma), \|f\|_{\infty,\sigma} \leq 1\}\big).$$

Moreover,

$$\omega_{\mathcal{R}_s^\infty}(A) := \inf\{\sigma \in (\omega(A), \pi] \mid \ A \text{ has an } \mathcal{R}_s\text{-bounded } H^\infty(\Sigma_\sigma)\text{-calculus}\}$$

is called the $\mathcal{R}_s^\infty$-*type of* A, and in this situation we will also just say that A has an $\mathcal{R}_s$-bounded H^∞-calculus.

We trivially have the following

Remark 3.2.20. *Let A have an $\mathcal{R}_s$-bounded H^∞-calculus, then A is also $\mathcal{R}_s$-sectorial with $\omega_{\mathcal{R}_s}(A) \leq \omega_{\mathcal{R}_s^\infty}(A)$.* □

We will now show how the property of A having an $\mathcal{R}_s$-bounded H^∞-calculus can be expressed in terms of the diagonal operator $\widetilde{A}_s$.

Lemma 3.2.21. *Let $\sigma, \sigma' > \omega(A)$. Consider the following assertions:*

(1) For each $f \in H^\infty(\Sigma_\sigma)$ the operator $f(A)$ is $\mathcal{R}_s$-bounded,

(2) The diagonal operator $\widetilde{A}_s$ is sectorial with $\omega(\widetilde{A}_s) < \sigma'$ and has a bounded $H^\infty(\Sigma_{\sigma'})$-calculus in $X(\ell^s)$.

Then (2) ⇒ (1) if $\sigma \geq \sigma'$, and (1) ⇒ (2) if $\sigma' > \sigma$. Moreover, if (1) holds there is a constant $C_\sigma > 0$ such that

$$\mathcal{R}_s(f(A)) \leq C_\sigma \cdot \|f\|_{\infty,\sigma} \quad \textit{for all } f \in H^\infty(\Sigma_{\sigma'})$$

for each $\sigma' > \sigma$.

Proof. It is trivial that (2) implies (1) if $\sigma \geq \sigma'$, so we assume $\sigma' > \sigma$ and that (1) holds. Observe that A has in particular a bounded $H^\infty(\Sigma_\sigma)$-calculus, hence

$$\Phi_A : H^\infty(\Sigma_\sigma) \to L(X), f \mapsto f(A)$$

is bounded. By (1) we have in addition $R(\Phi_A) \subseteq \mathrm{R_sL}(X) \hookrightarrow L(X)$ and $\mathrm{R_sL}(X)$ is a Banach space by Proposition 3.1.4, hence the Closed Graph Theorem implies that $\Phi_A : H^\infty(\Sigma_\sigma) \to \mathrm{R_sL}(X), f \mapsto f(A)$ is bounded, i.e. there is a constant $C_\sigma > 0$ such that

$$\mathcal{R}_s(f(A)) \leq C_\sigma \cdot \|f\|_{\infty,\sigma} \quad \text{for all } f \in H^\infty(\Sigma_\sigma). \tag{3.2.15}$$

Choose $\omega \in (\sigma, \sigma')$, then by (3.2.15) the set $\{\lambda R(\lambda, \widetilde{A}_s) \mid \lambda \in \mathbb{C}\backslash\overline{\Sigma}_\omega\}$ is bounded in the space $L(X(\ell^s))$, hence the diagonal operator $\widetilde{A}_s$ is sectorial with $\omega(\widetilde{A}_s) \leq \omega < \sigma'$, and again (3.2.15) implies that the diagonal operator $\widetilde{A}_s$ has a bounded $H^\infty(\Sigma_{\sigma'})$-calculus in $X(\ell^s)$. □

Observe that the restriction $\sigma' > \sigma$ in (1) $\Rightarrow$ (2) if of Lemma 3.2.21 is due to the fact that we do not assume A to be $\mathcal{R}_s$-sectorial. If we do this, we get the following slightly sharper condition, which can be proven in the same way.

Lemma 3.2.22. *Let A be an $\mathcal{R}_s$-sectorial operator and $\sigma > \omega_{\mathcal{R}_s}(A)$. Then the following conditions are equivalent:*

(1) For each $f \in H^\infty(\Sigma_\sigma)$ the operator $f(A)$ is $\mathcal{R}_s$-bounded,

(2) The diagonal operator $\widetilde{A}_s$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus in $X(\ell^s)$.

□

We will now show the important fact that in a suitable framework the $\mathcal{R}_s$-boundedness of the *single* operators $f(A)$ for $f \in H^\infty(\Sigma_\sigma)$ as in Lemma 3.2.21 (1) already implies an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_{\sigma'})$-calculus for all $\sigma' > \sigma$.

Proposition 3.2.23. *Let $\sigma, \sigma' > \omega(A)$ and $s \in [1, +\infty)$, and assume that X is r-concave for some $r < +\infty$. Consider the following assertions:*

(1) A has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_{\sigma'})$-calculus.

(2) For each $f \in H^\infty(\Sigma_\sigma)$ the operator $f(A)$ is $\mathcal{R}_s$-bounded,

(3) For each $\varphi \in H_0^\infty(\Sigma_\sigma)$ the operator $\varphi(A)$ is $\mathcal{R}_s$-bounded, and there is a constant $C > 0$ such that

$$\forall \varphi \in H_0^\infty(\Sigma_\sigma) \,:\, \mathcal{R}_s(\varphi(A)) \le C \,\|\varphi\|_{\infty,\sigma}$$

Then (1)$\Rightarrow$(3)$\Rightarrow$(2) if $\sigma \ge \sigma'$, and (2)$\Rightarrow$(1) if $\sigma' > \sigma$.

More precisely, if (2) holds, then for each $\omega > \sigma$ there is a constant $C_{\omega,\sigma} > 0$ independent of A such that

$$\forall \sigma' \ge \omega : M^\infty_{s,\sigma'}(A) \le C_{\omega,\sigma} \cdot \sup\{\mathcal{R}_s(f(A)) \mid f \in H^\infty(\Sigma_\sigma), \|f\|_{\infty,\sigma} \le 1\}. \tag{3.2.16}$$

Proof. We clearly have (1) $\Rightarrow$ (3) if $\sigma \ge \sigma'$, so we assume that (3) holds and show (2). For this let again $\rho_m(z) := \frac{m}{m+z} - \frac{1}{1+mz}$ for all $z \in \Sigma_\sigma, m \in \mathbb{N}$. Then $\rho_m \in H_0^\infty(\Sigma_\sigma)$ and $|\rho_m(z)| \le K_\sigma$ for all $z \in \Sigma_\sigma, m \in \mathbb{N}$ (where $K_\sigma := 2\,\mathrm{dist}(-1, \overline{\Sigma}_\sigma)$), and for all $z \in \Sigma_\sigma$ we have $\rho_m(z) \to 1$ for $m \to \infty$. Now let $f \in H^\infty(\Sigma_\sigma)$ with $\|f\|_{\infty,\sigma} \le 1$. Let $n \in \mathbb{N}$ and $(x_j)_{j\in\mathbb{N}_{\le n}} \in X(\ell^s_n)$ and $(y_j)_{j\in\mathbb{N}_{\le n}} \in X'(\ell^{s'}_n)$ with $\|(y_j)_{j\in\mathbb{N}_{\le n}}\|_{X'(\ell^{s'}_n)} = 1$. By the Convergence Lemma 1.2.2 we have $(\rho_m \cdot f)(A)x_j \to f(A)x_j$ in X as $m \to \infty$ for all $j \in \mathbb{N}_{\le n}$, hence we obtain

$$\begin{aligned}
&\left|\langle (f(A)x_j)_{j\in\mathbb{N}_{\le n}}, (y_j)_{j\in\mathbb{N}_{\le n}}\rangle_{X(\ell^s_n),X'(\ell^{s'}_n)}\right| \le \int_\Omega \sum_{j=1}^n |f(A)x_j \cdot y_j| \, d\mu \\
\le\ & \sum_{j=1}^n \langle |f(A)x_j|, |y_j|\rangle_{X\times X'} = \lim_{m\to\infty} \sum_{j=1}^n \langle |(\rho_m \cdot f)(A)x_j|, |y_j|\rangle_{X\times X'} \\
=\ & \lim_{m\to\infty} \left\langle \big(|(\rho_m \cdot f)(A)x_j|\big)_j, (|y_j|)_j\right\rangle_{X(\ell^s_n),X'(\ell^{s'}_n)} \le \lim_{m\to\infty} \left\|\big(|(\rho_m \cdot f)(A)x_j|\big)_j\right\|_{X(\ell^s_n)} \\
\le\ & C \lim_{m\to\infty} \|\rho_m \cdot f\|_{\infty,\sigma} \|(x_j)_j\|_{X(\ell^s_n)} \le K_\sigma C \, \|f\|_{\infty,\sigma} \, \|(x_j)_j\|_{X(\ell^s_n)}.
\end{aligned}$$

By duality this yields $\|(f(A)x_j)_j\|_{X(\ell^s_n)} \le K_\sigma C \|f\|_{\infty,\sigma} \|(x_j)_j\|_{X(\ell^s_n)}$, hence the operator $f(A)$ is $\mathcal{R}_s$-bounded.

Finally we assume that $\sigma' > \sigma$ and (2) holds, and we will show (1). Observe that by a similar argument as used above in the proof of (3)$\Rightarrow$(2) it is sufficient to show that the set $\{\varphi(A) \mid \varphi \in H_0^\infty(\Sigma_{\sigma'}), \|\varphi\|_{\infty,\sigma'} \le 1\}$ is $\mathcal{R}_s$-bounded.

Choose $\omega \in (\sigma, \sigma')$ and let $\omega' := \frac{1}{2}(\sigma+\omega)$, then $\sigma < \omega' < \omega$. We will show the estimate in (3.2.16), where it is sufficient to consider the case $\sigma' = \omega$, since trivially we have $M^\infty_{s,\sigma'}(A) \le M^\infty_{s,\omega}(A)$.

By Lemma 3.2.21, the diagonal operator $\widetilde{A}_s$ is sectorial with $\omega(\widetilde{A}_s) < \omega'$ and has a bounded $H^\infty(\Sigma_{\omega'})$-calculus in $X(\ell^s)$. Hence it has also an $RH^\infty(\Sigma_\omega)$-bounded functional calculus, i.e. a bounded functional calculus for operator valued functions $F : \Sigma_\omega \to \mathcal{A} \subseteq L(X(\ell^s))$ with $\mathcal{R}$-bounded range, where $\mathcal{A} \subseteq L(X(\ell^s))$ denotes the subalgebra of all bounded operators that commute with resolvents of $\widetilde{A}_s$, cf. Section 1.3 in the preliminaries for this notion. Moreover, by Theorem 1.3.3 we can choose a constant $C_{\omega,\omega'} > 0$ independent of $\widetilde{A}_s$ such that

$$\|F(\widetilde{A}_s)\|_{L(X(\ell^s))} \le C_{\omega,\omega'} \cdot M^\infty_{\omega'}(\widetilde{A}_s) \cdot \mathcal{R}(F(\Sigma_\omega)) \tag{3.2.17}$$

for all $F \in RH^\infty(\Sigma_\omega, \mathcal{A})$.

Now let $\varphi_n \in H^\infty(\Sigma_\omega)$ with $\|\varphi_n\|_{\infty,\omega} \le 1$ for all $n \in \mathbb{N}$. For each $\lambda \in \Sigma_\omega$ we define $F(\lambda) \in L(X(\ell^s))$ as the diagonal operator $(x_n)_{n\in\mathbb{N}} \mapsto (\varphi_n(\lambda)x_n)_{n\in\mathbb{N}}$, then $F : \Sigma_\omega \to L(X(\ell^s))$ is analytic. Moreover, $F(\Sigma_\omega)$ is $\mathcal{R}_2$-bounded in $L(X(\ell^s))$: Let $\lambda \in \Sigma_\omega^{\mathbb{N}}$ and $x_j = (x_j^{(n)})_{n\in\mathbb{N}} \in X(\ell^s)$ for all $j \in \mathbb{N}$. Then

$$|F(\lambda_j)x_j|_{\mathbb{C}} = \big(|\varphi_n(\lambda_j)x_j^{(n)}|\big)_{n\in\mathbb{N}} = \big(|\varphi_n(\lambda_j)|\,|x_j^{(n)}|\big)_{n\in\mathbb{N}} \le \big(|x_j^{(n)}|\big)_{n\in\mathbb{N}} = |x_j|_{\mathbb{C}}$$

in $X(\ell^s)$ for all $j \in \mathbb{N}$ (observe that for $x \in X(\ell^s) \cong X\ell^s$ we have the modulus $|x|_{\mathbb{C}} := (|x_n|_{\mathbb{C}})_{n\in\mathbb{N}} \in X(\ell^s)$, cf. the corresponding remarks about mixed Banach function spaces in Subsection 1.6.1), hence

$$\Big\|\Big(\sum_{j\in\mathbb{N}} |F(\lambda_j)x_j|^2_{\mathbb{C}}\Big)^{1/2}\Big\|_{X(\ell^s)} \le \Big\|\Big(\sum_{j\in\mathbb{N}} |x_j|^2_{\mathbb{C}}\Big)^{1/2}\Big\|_{X(\ell^s)}.$$

Since X is r-concave, the mixed Banach function space $X\ell^s \cong X(\ell^s)$ is $r \vee s$-concave by Proposition 1.6.18 with $r \vee s < +\infty$, so Remark 3.1.7 implies that the set $F(\Sigma_\omega)$ is also $\mathcal{R}$-bounded. Hence, the $RH^\infty(\Sigma_\omega)$-calculus of $\widetilde{A}_s$ yields boundedness of the operator $F(\widetilde{A}_s)$. If $x \in X(\ell^s)$ and Γ is the oriented boundary of the sector $\Sigma_{\omega'}$ (recall that $\omega' \in (\omega(\widetilde{A}_s), \omega)$), we obtain

$$F(\widetilde{A}_s)x = \int_\Gamma F(\lambda)R(\lambda,\widetilde{A}_s)x\,d\lambda = \Big(\int_\Gamma \varphi_n(\lambda)R(\lambda,A)x_n\,d\lambda\Big)_{n\in\mathbb{N}} = (\varphi_n(A)x_n)_{n\in\mathbb{N}}.$$

So boundedness of $F(\widetilde{A}_s)$ in $L(X(\ell^s))$ is just $\mathcal{R}_s$-boundedness of $\{\varphi_n(A) \,|\, n \in \mathbb{N}\}$ in $L(X)$, and by (3.2.17) we obtain

$$\mathcal{R}(\{\varphi_n(A) \,|\, n \in \mathbb{N}\}) = \|F(\widetilde{A}_s)\|_{L(X(\ell^s))} \le C_{\omega,\omega'} \cdot M^\infty_{\omega'}(\widetilde{A}_s) \cdot \mathcal{R}(F(\Sigma_\omega)) \lesssim C_{\omega,\omega'} \cdot M^\infty_{\omega'}(\widetilde{A}_s).$$

Since $M^\infty_{\omega'}(\widetilde{A}_s) \le \sup\{\mathcal{R}_s(f(A)) \mid f \in H^\infty(\Sigma_\sigma), \|f\|_{\infty,\sigma} \le 1\}$ this yields the estimate (3.2.16). $\square$

Again, we conclude with our standard example.

Proposition 3.2.24. *Let $d, m \in \mathbb{N}$ and $p, s \in (1, +\infty)$. Then the Laplace operator $A := (-\Delta)^m$ has an $\mathcal{R}_s$-bounded H^∞-calculus in $L^p(\mathbb{R}^d)$ with $\omega_{\mathcal{R}_s^\infty}\big((-\Delta)^m\big) = 0$.*

Proof. Observe that with the notations of Section 2.5 we just have to show that the operator A^{ℓ^s} has a bounded $H^\infty(\Sigma_\sigma)$-calculus in $L^p(\mathbb{R}^d, \ell^s)$ for each $\sigma > 0$. Since ℓ^s is a UMD-spaces, this is a consequence of the vector-valued Mikhlin Multiplier Theorem. This is shown in detail for $m = 1$ in [KW04], Example 10.2 b).

□

3.3 The associated s-intermediate spaces

In this section we fix $s \in [1, +\infty]$, and A will always denote an $\mathcal{R}_s$-sectorial operator in X with dense domain and dense range.

We now turn to the definition of the associated homogeneous and inhomogeneous s-intermediate spaces $\dot{X}^\theta_{s,A}, X^\theta_{s,A}$ which we will also refer to as generalized Triebel-Lizorkin spaces. This will be justified in Proposition 3.3.12, where we show that the s-intermediate spaces for the Laplace operator $A = -\Delta$ coincide with the classical Triebel-Lizorkin spaces, cf. also the remarks given at the end of Section 1.7. After the definition and some elementary properties in Subsection 3.3.1 we will show in Subsection 3.3.2 that the s-intermediate spaces are indeed intermediate spaces for the couple $(X, D(A^m))$, and we will explore real and complex interpolation of these spaces. In the last Subsection 3.3.3 we will present one of the main results of this work, where we show that the "part" of A (which has to be defined properly) always has a bounded H^∞-calculus in its scale of homogeneous s-intermediate spaces $\dot{X}^\theta_{s,A}, \theta \in \mathbb{R}$, and in the case that A is invertible, or A has a bounded H^∞-calculus in X, also in the inhomogeneous spaces $X^\theta_{s,A}$ if $\theta > 0$. This can be seen as a variant of Dore's Theorem that states that an invertible sectorial operator A in a Banach space X has a bounded H^∞-calculus in the scale of *real interpolation spaces* $(X, D(A))_{p,\theta}$ for $p \in [1, +\infty], \theta \in (0, 1)$, cf. [Do99]. A more general version is given in [Do01], and an extensive treatment using functional calculus is given in [Ha06], Chapter 6. Indeed, in the same way as classical Triebel-Lizorkin spaces are to a certain extent a natural counterpart to Besov spaces, the s-intermediate spaces are appropriate counterparts to the real interpolation spaces. Hence we will sometimes be able to use techniques similar to the one used in [Ha06] for real interpolation spaces.

3.3.1 Definition end elementary properties of the spaces $X^\theta_{s,A}$ and $\dot{X}^\theta_{s,A}$

For each $\sigma \in (0, \pi]$ and $\theta \in \mathbb{R}$ let

$$\Phi_{\sigma,\theta} := \{\varphi \in \mathcal{E}(\Sigma_\sigma)\backslash\{0\} \,|\, z \mapsto z^{-\theta}\varphi(z) \in H_0^\infty(\Sigma_\sigma)\}.$$

For the discrete counterparts we define the subset

$$\Phi^\Sigma_{\sigma,\theta} := \{\varphi \in \Phi_{\sigma,\theta} \,|\, z \mapsto z^{-\theta}\varphi(z) \in \Phi^\Sigma_{\sigma,0}\}.$$

Note that $\Phi_{\sigma,\theta} \hookrightarrow \Phi_{\sigma',\theta'}$ and $\Phi^{\Sigma}_{\sigma,\theta} \hookrightarrow \Phi^{\Sigma}_{\sigma',\theta'}$ for $\sigma \geq \sigma'$ and $\theta \geq \theta'$.

Definition 3.3.1. Let $\theta \in \mathbb{R}$, $s \in [1,+\infty]$ and $\sigma > \omega(A)$. Let $\varphi \in \Phi_{\sigma,\theta}$, then we define the corresponding s-power function norm as

$$\|x\|_{\theta,s,A,\varphi} := \Big\|\Big(\int_0^\infty |t^{-\theta}\varphi(tA)x|^s \,\frac{dt}{t}\Big)^{1/s}\Big\|_X \quad \text{for all } x \in X \tag{3.3.1}$$

(with the usual modification if $s = +\infty$). Moreover, for $\varphi \in \Phi^{\Sigma}_{\sigma,\theta}$ we define the corresponding discrete counterpart as

$$\|x\|^{\Sigma}_{\theta,s,A,\varphi} := \Big\|\Big(\sum_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^jA)x|^s\Big)^{1/s}\Big\|_X \quad \text{for all } x \in X \tag{3.3.2}$$

(with the usual modification if $s = +\infty$).

Finally we define the space

$$X^\theta_{s,A,\varphi} := \{x \in X \mid \|x\|_{\theta,s,A,\varphi} < +\infty\}.$$

We will show that $\|\cdot\|_{\theta,s,A,\varphi}$ defined by (3.3.1) actually defines a norm on $X^\theta_{s,A,\varphi}$: The mapping

$$J : X^\theta_{s,A,\varphi} \to X(L^s_*), x \mapsto \big(t^{-\theta}\varphi(tA)x\big)_{t>0}$$

is linear, and $\|x\|_{\theta,s,A,\varphi} = \|Jx\|_{X(L^s_*)}$ for all $x \in X^\theta_{s,A,\varphi}$ by definition, hence we only have to show that J is injective. This is a consequence of Proposition 3.2.12, but we will also give a direct argument (that is indeed also used in the proof of Proposition 3.2.12), which would also work without the assumed $\mathcal{R}_s$-sectoriality of A: Let $x \in X^\theta_{s,A,\varphi}$ with $Jx = 0$. Define $\rho(z) := z/(1+z)^2$ and $c := \int_0^\infty \rho(t)|\varphi(t)|^2 \,\frac{dt}{t} > 0$, and let $\psi := \frac{1}{c}\rho\overline{\varphi}$, then $\psi \in H^\infty_0(\Sigma_\sigma)$ and $\int_0^\infty \psi(t)\varphi(t)\,\frac{dt}{t} = \frac{1}{c}\int_0^\infty \rho(t)|\varphi(t)|^2\,\frac{dt}{t} = 1$. Since dt/t is a translation invariant measure on the multiplicative group $(0,\infty)$ this yields

$$\int_0^\infty \psi(tz)\varphi(tz)\,\frac{dt}{t} = 1 \tag{3.3.3}$$

for all $z \in (0,\infty)$, and by analytic continuation and the identity theorem for analytic functions, (3.3.3) is also true for all $z \in \Sigma_\sigma$. By the functional calculus we obtain

$$x = \int_0^\infty \psi(tA)\underbrace{\varphi(tA)x}_{=0}\,\frac{dt}{t} = 0.$$

By the preceding section we have the important issue that the s-power function norm $\|\cdot\|_{\theta,s,A,\varphi}$ does not depend on φ in the following sense:

Proposition 3.3.2. *Let $\theta \in \mathbb{R}$, $\sigma > \omega_{\mathcal{R}_s}(A)$ and $\varphi,\psi \in \Phi_{\sigma,\theta}$. Then there is a constant $C > 0$ such that for all $x \in D(A^\theta)$ and $f \in H^\infty(\Sigma_\sigma)$*

(1) $C^{-1}\,\|x\|_{\theta,s,A,\varphi} \leq \|x\|_{\theta,s,A,\psi} \leq C\,\|x\|_{\theta,s,A,\varphi}$,

(2) $\|f(A)x\|_{\theta,s,A,\varphi} \le C\,\|f\|_\infty \cdot \|x\|_{\theta,s,A,\varphi}$.

If $\varphi, \psi \in \Phi^{\Sigma}_{\sigma,\theta}$, *then we have in addition*

(3) $C^{-1}\,\|x\|_{\theta,s,A,\varphi} \le \|x\|^{\Sigma}_{\theta,s,A,\psi} \le C\,\|x\|_{\theta,s,A,\varphi}$,

and (1), (2) *also hold for the discrete counterparts* $\|\cdot\|^{\Sigma}_{\theta,s,A,\varphi}, \|\cdot\|^{\Sigma}_{\theta,s,A,\psi}$ *instead of* $\|\cdot\|_{\theta,s,A,\varphi}, \|\cdot\|_{\theta,s,A,\psi}$.

Proof. We apply Proposition 3.2.12 with $\widetilde{\varphi}(z) := z^{-\theta}\varphi(z)$ and $\widetilde{\psi}(z) := z^{-\theta}\psi(z)$ instead of φ, ψ and choose the constant $C > 0$ as given there. Let $x \in D(A^\theta)$ and $f \in H^\infty(\Sigma_\sigma)$, then:

$$\begin{aligned} \|f(A)x\|_{\theta,s,A,\varphi} &= \Big\|\Big(\int_0^\infty |t^{-\theta}\varphi(tA)f(A)x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X = \Big\|\Big(\int_0^\infty |f(A)\widetilde{\varphi}(tA)A^\theta x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X \\ &\le C\,\|f\|_\infty \cdot \Big\|\Big(\int_0^\infty |\widetilde{\psi}(tA)A^\theta x|^s\,\frac{dt}{t}\Big)^{1/s}\Big\|_X = C\,\|f\|_\infty \cdot \|x\|_{\theta,s,A,\psi} \end{aligned}$$

(with the usual modification if $s = +\infty$). This shows (1) and (2). In the same manner, (3) follows by Propositions 3.2.17 and 3.2.18. □

The central objects are now the following normed spaces:

(1) $X^\theta_{s,A,\varphi}$ endowed with the norm $\|\cdot\|_{X^\theta_{s,A,\varphi}} := \|\cdot\|_X + \|\cdot\|_{\theta,s,A,\varphi}$ if $\theta \ge 0$,

(2) $\dot{X}^\theta_{s,A,\varphi}$ as the completion of the space $X^\theta_{s,A,\varphi}$ endowed with the norm $\|\cdot\|_{\theta,s,A,\varphi}$.

We will see later in Proposition 3.3.5 (based, of course, on Proposition 3.3.2) that these spaces are independent of φ in the sense that varying $\varphi \in \Phi_{\sigma,\theta}$ leads to equivalent norms, hence we will later drop the φ in notation, and the space $X^\theta_{s,A}$ will be called the *associated inhomogeneous s-intermediate space*, and $\dot{X}^\theta_{s,A}$ the *associated homogeneous s-intermediate space*. Although the definition (1) would also make sense for $\theta < 0$, we leave out these spaces from our considerations, since they appear to be quite unnatural. In fact, even for $\theta = 0$ these spaces are delicate, since they are forced to be embedded into X, which might not be natural, if one looks at the concrete examples of classical Triebel-Lizorkin spaces.

As usual, the homogeneous space is somehow closer related to the operator A, but has a more complicated structure, since e.g. it is in general not embedded into X. Nevertheless many properties of the homogeneous spaces can easily be carried over to the inhomogeneous space; this is due to the fact that if A is invertible, then $X^\theta_{s,A} \cong \dot{X}^\theta_{s,A}$ for $\theta > 0$, as we will show in Proposition 3.3.11. Hence we will start with a detailed study of the homogeneous spaces.

We will show first the following important density property.

Proposition 3.3.3. *Let* $\theta \in \mathbb{R}$, $m \in \mathbb{N}$ *with* $m > |\theta|$ *and* $\sigma > \omega_{\mathcal{R}_s}(A)$. *Then* $D(A^m) \cap R(A^m)$ *is a dense subset in* $\dot{X}^\theta_{s,A,\varphi}$ *for all* $\varphi \in \Phi_{\sigma,\theta}$.

Proof. We will show first that $D(A^m) \cap R(A^m) \subseteq \dot{X}^\theta_{s,A,\varphi}$. Let $x \in D(A^m) \cap R(A^m) \subseteq D(A^\theta)$. Choose $\varepsilon > 0$ such that $\varepsilon < m - |\theta|$. Let $\varphi(z) := \frac{z^m}{(1+z)^{2m}}$ and $\psi_\pm(z) := \frac{z^{m-\theta\pm\varepsilon}}{(1+z)^{2m}}$, then $\varphi \in \Phi_{\sigma\theta}$ and $\psi \in H_0^\infty(\Sigma_\sigma)$, and

$$t^{-\theta}\varphi(tA)x = t^{\mp\varepsilon}(tA)^{-\theta\pm\varepsilon}\varphi(tA)x = t^{\mp\varepsilon}\psi_\pm(tA) \quad \text{for all } t > 0.$$

Hence we obtain (with the usual modification if $s = +\infty$)

$$\begin{aligned}
\|x\|_{\theta,s,A,\varphi} &= \left\| \left(\int_0^\infty |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \right)^{1/s} \right\| \\
&\lesssim \left\| \left(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \right)^{1/s} \right\| + \left\| \left(\int_1^\infty |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \right)^{1/s} \right\| \\
&\lesssim \left\| \left(\int_0^1 |t^{\varepsilon}\psi_-(tA)x|^s \, \frac{dt}{t} \right)^{1/s} \right\| + \left\| \left(\int_1^\infty |t^{-\varepsilon}\psi_+(tA)x|^s \, \frac{dt}{t} \right)^{1/s} \right\| \\
&\overset{(*)}{\lesssim} \left\| \left(\int_0^1 |t^{\varepsilon}x|^s \, \frac{dt}{t} \right)^{1/s} \right\| + \left\| \left(\int_1^\infty |t^{-\varepsilon}x|^s \, \frac{dt}{t} \right)^{1/s} \right\| = \frac{2}{s\varepsilon}\,\|x\|_X < +\infty,
\end{aligned}$$

where we used in $(*)$ that the operator set $\{\psi_\pm(tA) \,|\, t > 0\}$ is $\mathcal{R}_s$-bounded. This shows that $D(A^m) \cap R(A^m) \subseteq \dot{X}^\theta_{s,A,\varphi}$ for the special φ we have chosen, and by Proposition 3.3.2 this is also true for arbitrary $\varphi \in \Phi_{\sigma,\theta}$ since $D(A^m) \cap R(A^m) \subseteq D(A^\theta)$.

Now we define

$$\widetilde{X}^\theta_{s,A,\varphi} := \overline{D(A^m) \cap R(A^m)}^{\|\cdot\|_{X_{\theta,s,A,\varphi}}} \leq \dot{X}^\theta_{s,A,\varphi},$$

then by Proposition 3.3.2 all the spaces $\widetilde{X}^\theta_{s,A,\varphi}$, where $\varphi \in \Phi_{\omega_{\mathcal{R}_s}(A),\theta}$, coincide and have equivalent norms. Hence $D(A^m) \cap R(A^m)$ is dense in all $\widetilde{X}^\theta_{s,A,\varphi}, \varphi \in \Phi_{\omega_{\mathcal{R}_s}(A),\theta}$ if it is dense for some $\varphi \in \Phi_{\omega_{\mathcal{R}_s}(A),\theta}$, so we may assume that $\varphi(z) = \frac{z^m}{(1+z)^{2m}}$, hence $\varphi \in \Phi^\Sigma_{\sigma,\theta}$. Let $\widetilde{X}^{\theta,\Sigma}_{s,A,\varphi}$ be the completion of $D(A^m) \cap R(A^m)$ with respect to the norm $\|\cdot\|^\Sigma_{\theta,s,A,\varphi}$. Then again by Proposition 3.3.2 (3) we also have $\widetilde{X}^\theta_{s,A,\varphi} = \widetilde{X}^{\theta,\Sigma}_{s,A,\varphi}$ with equivalent norms, so it is enough to show that $X^\theta_{s,A,\varphi} \subseteq \widetilde{X}^{\theta,\Sigma}_{s,A,\varphi}$.

Let $x \in X^\theta_{s,A,\varphi}$ and define $T_n := n(n + A^{-1})^{-1} n(n + A)^{-1} = nA(1 + nA)^{-1}(1 + \frac{1}{n}A)^{-1}$ for all $n \in \mathbb{N}$. Let $x_n := T_n^m x \in D(A^m) \cap R(A^m)$ for all $n \in \mathbb{N}$, then it is well known that $x_n \to x$ in X for $n \to \infty$.

We consider first the case $s < +\infty$. Let $\varepsilon > 0$, then since $x \in X^\theta_{s,A,\varphi}$, we can choose $N \in \mathbb{N}$ such that

$$\left\| \Big(\sum_{|j| \geq N} |2^{-j\theta}\varphi(2^j A)x|^s \Big)^{1/s} \right\|_X < \varepsilon/2.$$

Let $K_N := \sum\limits_{|j| \leq N} 2^{-j\theta}\|\varphi(2^j A)x\|_X$, then we can choose $n_0 \in \mathbb{N}$ such that $K_N \cdot \|x_n - x\|_X < \varepsilon/2$

for all $n \geq n_0$. Let $n \geq n_0$, then

$$\begin{aligned}
\|x_n - x\|^{\Sigma}_{\theta,s,A,\varphi} &= \Big\| \Big(\sum_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^j A)(x_n - x)|^s \Big)^{1/s} \Big\|_X \\
&\leq \Big\| \Big(\sum_{|j|\leq N} |2^{-j\theta}\varphi(2^j A)(x_n - x)|^s \Big)^{1/s} \Big\|_X \\
&\quad + \Big\| \Big(\sum_{|j|\geq N} |(T_n^m - \mathrm{Id})2^{-j\theta}\varphi(2^j A)x|^s \Big)^{1/s} \Big\|_X \\
&\overset{(1)}{\lesssim}_m \Big\| \sum_{|j|\leq N} 2^{-j\theta}|\varphi(2^j A)(x_n - x)| \Big\|_X + \Big\| \Big(\sum_{|j|\geq N} |2^{-j\theta}\varphi(2^j A)x|^s \Big)^{1/s} \Big\|_X \\
&\leq \sum_{|j|\leq N} 2^{-j\theta}\|\varphi(2^j A)x\|_X \cdot \|x_n - x\|_X + \Big\| \Big(\sum_{|j|\geq N} |2^{-j\theta}\varphi(2^j A)x|^s \Big)^{1/s} \Big\|_X < \varepsilon
\end{aligned}$$

for all $n \geq n_0$, where we used in (1) that $\ell^1 \hookrightarrow \ell^s$ and that the operators $T_n^m, n \in \mathbb{N}$ are $\mathcal{R}_s$-bounded. So we have $\|x - x_n\|^{\Sigma}_{\theta,s,A,\varphi} \to 0$ for $n \to \infty$.

Now consider the case $s = +\infty$. Let $\varepsilon > 0$. Then again, since $x \in X^{\theta}_{\infty,A,\varphi}$ and X has the Fatou property, we can choose $N \in \mathbb{N}$ such that

$$\Big\| \sup_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^j A)x| - \sup_{|j|\leq N} |2^{-j\theta}\varphi(2^j A)x| \Big\|_X < \varepsilon/2,$$

and we can proceed as in the first case by using the estimate

$$\begin{aligned}
\|x - x_n\|_{\theta,\infty,A,\varphi} &= \Big\| \sup_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^j A)x| \Big\|_X \\
&\leq \Big\| \sup_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^j A)x| - \sup_{|j|\leq N} |2^{-j\theta}\varphi(2^j A)x| \Big\|_X + \Big\| \sup_{|j|\leq N} |2^{-j\theta}\varphi(2^j A)x| \Big\|_X.
\end{aligned}$$

□

Of course, Proposition 3.3.3 implies an analogous density property for the inhomogeneous spaces:

Corollary 3.3.4. *Let $\theta \geq 0$, $m \in \mathbb{N}_{>\theta}$ and $\sigma > \omega_{\mathcal{R}_s}(A)$. Then $D(A^m) \cap R(A^m)$ is a dense subset in $X^{\theta}_{s,A,\varphi}$ for all $\varphi \in \Phi_{\sigma,\theta}$.* □

With these density properties we can extend the norm estimates from Proposition 3.3.2 to the whole spaces $\dot{X}^{\theta}_{s,A,\varphi}, X^{\theta}_{s,A,\varphi}$.

Proposition 3.3.5. *Let $\theta \in \mathbb{R}$, $\sigma > \omega_{\mathcal{R}_s}(A)$ and $\varphi, \psi \in \Phi_{\sigma,\theta}$. Then there is a constant $C > 0$ such that for all $x \in X^{\theta}_{s,A,\varphi}$ and $f \in H^{\infty}(\Sigma_\sigma)$*

(1) $C^{-1}\|x\|_{\theta,s,A,\varphi} \leq \|x\|_{\theta,s,A,\psi} \leq C\|x\|_{\theta,s,A,\varphi}$,

(2) $\|f(A)x\|_{\theta,s,A,\varphi} \leq C\|f\|_\infty \cdot \|x\|_{\theta,s,A,\varphi}$.

In particular, for each $\varphi, \psi \in \Phi_{\sigma,\theta}$ the spaces $\dot{X}^{\theta}_{s,A,\varphi}$ and $\dot{X}^{\theta}_{s,A,\psi}$ have equivalent norms, and if $\varphi \in \Phi^{\Sigma}_{\sigma,\theta}$, then also $\|\cdot\|^{\Sigma}_{\theta,s,A,\varphi}$ is an equivalent norm on $\dot{X}^{\theta}_{s,A,\varphi}$, and (1), (2) also hold for the discrete counterpart $\|\cdot\|^{\Sigma}_{\theta,s,A,\varphi}$ instead of $\|\cdot\|_{\theta,s,A,\varphi}$.

Finally, if $\theta \geq 0$, then all statements are also true for the inhomogeneous spaces $X^{\theta}_{s,A,\varphi}$ with the inhomogeneous norms $\|\cdot\|_X + \|\cdot\|_{\theta,s,A,\varphi}$ and $\|\cdot\|_X + \|\cdot\|^{\Sigma}_{\theta,s,A,\varphi}$, respectively. □

Hence we will usually drop the φ and sometimes A in our notation of the spaces $\dot{X}^{\theta}_{s,A,\varphi}, X^{\theta}_{s,A,\varphi}$, if there is no risk of confusion. Moreover, if $\theta \in \mathbb{R}$, $\sigma > \omega_{\mathcal{R}_s}(A)$ and $\varphi \in \Phi_{\sigma,\theta}$ (or $\varphi \in \Phi^{\Sigma}_{\sigma,\theta}$, respectively), we will use the notation

$$\|x\|_{\theta,s} \approx \Big\| \Big(\int_0^{\infty} |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X \quad \Big(\|x\|_{\theta,s} \approx \Big\| \Big(\sum_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^j A)x| \Big)^{1/s} \Big\|_X \Big)$$

to indicate that $\|\cdot\|_{\theta,s}$ is any of the equivalent norms $\|\cdot\|_{\theta,s,\psi}$, $\psi \in \Phi_{\sigma,\theta}$ (or $\|\cdot\|^{\Sigma}_{\theta,s,\psi}$, $\psi \in \Phi^{\Sigma}_{\sigma,\theta}$, respectively).

If $\theta > 0$ and $\varphi \in \Phi_{\sigma,\theta}$ for some $\sigma > \omega_{\mathcal{R}_s}(A)$, we observe that by $\mathcal{R}_s$-boundedness of $\{\varphi(tA) \mid t > 0\}$ we have

$$\begin{aligned}
\|x\|_{\theta,s,\varphi} &\leq \Big\| \Big(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X + \Big\| \Big(\int_1^{\infty} |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X \\
&\leq \Big\| \Big(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X + \Big\| \Big(\int_1^{\infty} |t^{-\theta}x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X \\
&= \Big\| \Big(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X + \Big(\int_1^{\infty} |t^{-\theta}|^s \, \frac{dt}{t} \Big)^{1/s} \cdot \||x|\|_X \\
&= \Big\| \Big(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X + (\theta s)^{-1/s} \cdot \|x\|_X.
\end{aligned}$$

This leads to the following

Remark 3.3.6. *Let $\theta > 0$ and $\varphi \in \Phi_{\sigma,\theta}$ for some $\sigma > \omega_{\mathcal{R}_s}(A)$. Then*

$$X^{\theta}_{A,s} = \Big\{ x \in X \,\Big|\, \Big\| \Big(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X < +\infty \Big\},$$

and $x \mapsto \|x\|_X + \Big\| \Big(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X$ defines an equivalent norm on $X^{\theta}_{A,s}$. □

The next proposition describes some elementary embedding properties.

Proposition 3.3.7. *Let $\theta, \theta' \in \mathbb{R}$ and $r \in [1, +\infty]$. Then the following embeddings hold:*

(1) If $r \leq s$ and A is also $\mathcal{R}_r$-sectorial, then $\dot{X}^{\theta}_{r,A} \hookrightarrow \dot{X}^{\theta}_{s,A}$ and $X^{\theta}_{r,A} \hookrightarrow X^{\theta}_{s,A}$ if $\theta \geq 0$, respectively.

(2) If $\theta' \geq \theta > 0$, then $X^{\theta'}_{A,s} \hookrightarrow X^{\theta}_{A,s} \hookrightarrow X$.

Proof. (1) This follows immediately if we use the discrete norm representation in the spaces $\dot{X}^\theta_{r,A}, \dot{X}^\theta_{s,A}$ and the fact that $\ell^r \hookrightarrow \ell^s$.

(2) This is an immediate consequence of Remark 3.3.6. □

This is a first point that shows that the space $X^0_{A,s}$ does not really fit into the scale of the inhomogeneous spaces: We cannot prove a general embedding $X^\theta_{A,s} \hookrightarrow X^0_{A,s}$. Indeed, this would require an estimate of the kind

$$\Big\| \Big(\int_1^\infty |\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X \lesssim \Big\| \Big(\int_0^1 |t^{-\theta}\varphi(tA)x|^s \, \frac{dt}{t} \Big)^{1/s} \Big\|_X + \|x\|_X$$

if $\varphi \in \Phi_{\sigma,\theta} \cap H^\infty_0(\Sigma_\sigma)$ for some $\sigma > \omega_{\mathcal{R}_s}(A)$. It seems that this cannot be expected in general.

We will show now that also the homogeneous spaces $\dot{X}^\theta_{s,A}$ can be embedded into some natural extrapolation spaces associated to A. A suitable framework is the theory of abstract extrapolation spaces as it is developed in [Ha06], Chapter 6.3. We will give a short summary of those parts of the theory that are sufficient for our work.

We define the operator $J := A(1+A)^{-2} : X \to X$, then $JX = D(A) \cap R(A) \hookrightarrow X$, and J is a topological isomorphism. Now consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] & X \arrow[d, dash] \\ X \arrow[u, "J"'] \arrow[r, "J"'] & JX \end{tikzcd}$$

We rename some components and let $X_{(1)} := D(A) \cap R(A)$, then it becomes

$$\begin{tikzcd} X_{(-1)} \arrow[r, "J_{(-1)}"] & X \arrow[d, dash] \\ X \arrow[u, "\iota"'] \arrow[r, "J"'] & X_{(1)} \end{tikzcd}$$

Then $\iota : X \hookrightarrow X_{(-1)}$ is an embedding, hence we may view $X_{(-1)}$ as a proper superspace of X, the so-called *extrapolation space of order* 1. Since the diagram is commuting, we have moreover $J_{(-1)} \circ \iota = J$, hence after some identifications we can also write J instead of $J_{(-1)}$. This leads to the following diagram:

$$\begin{tikzcd} X_{(-1)} \arrow[r, "J"] \arrow[d, dash] & X \arrow[d, dash] \\ X \arrow[r, "J"'] \arrow[d, dash] & X_{(1)} \arrow[d, dash] \\ X_{(1)} \arrow[r, "J"'] & X_{(2)} \end{tikzcd}$$

Iterating this construction leads to a scale of spaces

$$X = X_0 \hookrightarrow X_{(-1)} \hookrightarrow X_{(-2)} \hookrightarrow \cdots \hookrightarrow X_{(-n)} \hookrightarrow ...$$

together with a family of (compatible) isometric isomorphisms $J : X_{(-n)} \to X_{(-n+1)}$. Finally, $U := \bigcup_{n\in\mathbb{N}} X_{(-n)}$ is called the *universal extrapolation space* corresponding to A. This space can be endowed with a notion of net-convergence in the following sense: Let $(x_\alpha)_{\alpha\in A}$ be a net in U and $y \in U$, then

$$x_\alpha \to y :\iff \exists\, n \in \mathbb{N}, \alpha_0 \in A :\ \big(\forall\, \alpha \in A_{\geq\alpha_0} :\ y, x_\alpha \in X_{-n}\big) \wedge\ \|x_\alpha - y\| \to 0.$$

Then the limit of a net in U is unique, and sum and scalar multiplication are "continuous" with respect to the so-defined notion of convergence. Since the operator J is defined on each space X_{-n}, $n \in \mathbb{N}$, it can be considered as a mapping $J : U \to U$, which then is obviously surjective, whence it is an algebraic isomorphism, continuous with respect to the notion of convergence defined above.

In fact, the construction of the space U and in particular the notion of convergence in U is only an ad-hoc construction, which is suitable to make formulations easier: For example, convergence in the space U is convergence in the space $X^{(-m)}$ for some $m \in \mathbb{N}$, and in the same manner arguments made in the space U always have to be understood to be made in the space $X^{(-m)}$ for some $m \in \mathbb{N}$.

Now the operator A can also be lifted to the whole scale of extrapolation spaces and the whole space U: We define

$$A_{(-1)} := J^{-1}AJ \quad \text{ with domain } D(A_{(-1)}) := J^{-1}D(A).$$

Then A is an injective sectorial operator in $X_{(-1)}$ that is isometrically similar to A. Moreover $X_{(1)} \subseteq D(A_{(-1)}) \subseteq X_{(-1)}$, and A is the part of $A_{(-1)}$, i.e.

$$A = A_{(-1)} \cap (X \times X) = \{(x, A_{(-1)}x) \,|\, x, A_{(-1)}x \in X\}.$$

Iterating this procedure leads to a sequence of isometrically similar sectorial operators $A_{(-n)}$ in $X_{(-n)}$ where $A_{(-n)}$ is the part of $A_{(-n-1)}$ in $X_{(-n)}$. Thus A can be considered as an operator on the whole space U.

We now take a short look at the functional calculus in this framework. Let $\sigma \in (\omega(A), \pi]$ and $f \in \mathfrak{B}(\Sigma_\sigma)$. Then the operator $f(A)$ can be considered as an operator in each $X_{(-n)}$, and we have consistency in the sense that $f(A_{(-n-1)})|_{X_{(-n)}} = f(A_{(-n)})$ for all $n \in \mathbb{N}$. To be more precise, if we choose $m \in \mathbb{N}$ such that $\rho^m f \in \mathcal{E}(\Sigma_\sigma)$, where $\rho(z) = z/(1+z)^2$, then $f(A) : X_{(-n)} \to X_{(-n-m)}$ is bounded for each $n \in \mathbb{N}$. Hence $f(A)$ can be considered as an operator on the whole space U, and we have the following important lemma.

Lemma 3.3.8 ([Ha06], Lemma 6.3.1)**.** *Let $\sigma \in (\omega(A), \pi)$ and $f \in \mathfrak{B}(\Sigma_\sigma)$. Then $D(f(A)) = \{x \in X \,|\, f(A)x \in X\}$, i.e., the operator $f(A)$ considered as an operator in X is the part in X of $f(A)$ considered as an operator in U.* □

Finally we define for later use for each $\alpha \in \mathbb{R}$ the *homogeneous fractional space*

$$\dot{X}_\alpha := A^{-\alpha}X \quad \text{endowed with the norm } \|\cdot\|_\alpha := \|\cdot\|_{\dot{X}_\alpha} := \|A^\alpha \cdot\|_X,$$

where $A^{-\alpha}$ has to be understood in the sense of Lemma 3.3.8 and its preceding remarks.

After this excursus we turn back to the theory of s-intermediate spaces. We can now give a concrete description of the homogeneous spaces as subspaces of the abstract extrapolation space.

Proposition 3.3.9. *Let $\theta \in \mathbb{R}$. Then*

$$\dot{X}^\theta_{s,A} \cong \{x \in U \,|\, \|x\|_{\theta,s,A} < +\infty\} \hookrightarrow A^{-\theta}X_{(-1)}$$

Proof. For brevity we drop the A in the notations of norms and spaces for this proof. Define the auxiliary space

$$\widetilde{X}^\theta_s := \{x \in U \,|\, \|x\|_{\theta,s} < +\infty\}, \quad \text{endowed with the norm } \|\cdot\|_{\theta,s}.$$

We will start by showing the embedding $\widetilde{X}^\theta_s \hookrightarrow A^{-\theta}X_{(-1)}$. Choose any $\sigma \in (\omega_{\mathcal{R}_s}(A), \pi]$ and $\varphi \in \Phi_{\sigma,\theta}$. With $\widetilde{\varphi}(z) := z^{-\theta}\varphi(z)$ we have $\widetilde{\varphi} \in \Phi_{\sigma,0}$ and

$$\widetilde{\varphi}(tA)A^\theta x = (tA)^{-\theta}\varphi(tA)A^\theta x = t^{-\theta}\varphi(tA)x \quad \text{for all } x \in D(A^\theta), t > 0,$$

hence A^θ is an isomorphism from $\widetilde{X}^\theta_s$ to $\widetilde{X}^0_s$ and we may assume w.l.o.g. that $\theta = 0$. Let $x \in U$ with $\|x\|_{0,s,\varphi} < +\infty$. We will now argue similar as in the proof of Proposition 3.2.12. We choose a function $\psi \in H^\infty_0(\Sigma_\sigma)$ such that $\int_0^\infty \varphi(t)\psi(t)\frac{dt}{t} = 1$ and conclude by the same techniques as in the proof of Proposition 3.2.12 that

$$\int_0^\infty \varphi(tA)\psi(tA)x\,\frac{dt}{t} = x \quad \text{in } U,$$

i.e. the integral is taken in the extrapolation space $X_{(-m)}$ for some $m \in \mathbb{N}$. Let $\rho(z) := z/(1+z)^2$, choose $\omega \in (\omega_{\mathcal{R}_s}(A), \sigma)$ and let Γ by the usual parametrization of $\partial\Sigma_\omega$. Using the functional calculus and Fubini-Tonelli yields

$$\begin{aligned}
\rho(A)x &= \int_0^\infty \rho(A)\varphi(tA)\psi(tA)x\,\frac{dt}{t} = \int_0^\infty \frac{1}{2\pi i}\int_\Gamma \rho(z)\psi(tz)zR(z,A)\varphi(tA)x\,\frac{dz}{z}\,\frac{dt}{t} \\
&= \frac{1}{2\pi i}\int_\Gamma \rho(z)\,zR(z,A)\,\underbrace{\left(\int_0^\infty \psi(tz)\varphi(tA)x\,\frac{dt}{t}\right)}_{=:u(z)}\,\frac{dz}{z}.
\end{aligned}$$

By Hölder's inequality we have

$$|u(z)| \;\le\; \underbrace{\left(\int_0^\infty |\psi(tz)|^{s'}\,\frac{dt}{t}\right)^{1/s'}}_{C(z):=} \cdot \left(\int_0^\infty |\varphi(tA)x|^s\,\frac{dt}{t}\right)^{1/s},$$

where $C := \sup_{z\in\Gamma} C(z) < +\infty$ since $\psi \in H_0^\infty(\Sigma_\sigma)$. Since also $\rho \in H_0^\infty(\Sigma_\sigma)$ and $M := \sup_{z\in\Sigma_\omega} \|zR(z,A)\| < +\infty$ we obtain $Jx = \rho(A)x \in X$, hence $x \in X_{(-1)}$, with

$$\begin{aligned} \|x\|_{X_{(-1)}} &= \|Jx\|_X = \|\rho(A)x\|_X \leq \frac{1}{2\pi}\int_\Gamma |\rho(z)|\, \|zR(z,A)\|_X \cdot \||u(z)|\|_X \frac{|dz|}{|z|} \\ &\lesssim \frac{CM}{2\pi}\int_\Gamma |\rho(z)| \frac{|dz|}{|z|} \cdot \left\| \left(\int_0^\infty |\varphi(tA)x|^s \frac{dt}{t}\right)^{1/s} \right\|_X \lesssim \|x\|_{X_s^0} \end{aligned}$$

as desired.

We will now show that $\big(\widetilde{X}_s^\theta, \|\cdot\|_{\theta,s}\big)$ is a Banach space. Again, we may assume w.l.o.g. that $\theta = 0$ and choose $\varphi(z) := z/(1+z)^2$ to calculate the norm in X_s^0. Let $(x_n)_{n\in\mathbb{N}} \in \big(\widetilde{X}_s^0\big)^{\mathbb{N}}$ be a Cauchy sequence. Then by the already proven embedding $\widetilde{X}_s^0 \hookrightarrow X_{(-1)}$ we can find an $x \in X_{(-1)}$ with $x_n \to x$ in $x \in X_{(-1)}$, hence also $\varphi(tA)x_n \to \varphi(tA)x$ in X for $n \to \infty$, since by the special choice of φ we have $\varphi(tA) \in L(X_{(-1)}, X)$. On the other hand, $(\varphi(tA)x)_{t>0}$ is a Cauchy sequence in the Banach space $X(L_*^s)$, hence we can find an $F \in X(L_*^s)$ with $\varphi(\cdot)Ax \to F$ in $X(L_*^s)$. By Lemma 1.6.24 we may assume w.l.o.g. by possibly choosing subsequences, that also $\varphi(\cdot A)x_n \to \varphi(\cdot A)x$ and $\varphi(\cdot A)x_n \to F$ pointwise a.e. for $n \to \infty$. Thus we obtain $\varphi(\cdot A)x = F \in X(L_*^s)$, hence $x \in \widetilde{X}_s^\theta$, and $\|x - x_n\|_{X_s^0} = \|\varphi(\cdot A)x_n - F\|_{X(L_*^s)} \to 0$ for $n \to \infty$.

Since $\widetilde{X}_s^\theta$ is a Banach space and trivially $X_s^\theta \subseteq \widetilde{X}_s^\theta$, we also obtain $\dot{X}_s^\theta \subseteq \widetilde{X}_s^\theta$, and it only remains to show the other inclusion $\widetilde{X}_s^\theta \subseteq \dot{X}_s^\theta$. But this can easily be seen by a density argument, since for sufficiently large $m \in \mathbb{N}$ we have again that $D(A^m) \cap R(A^m)$ is also dense in the space $\widetilde{X}_s^\theta$. This can be proven in the same way as it is done in the proof of Proposition 3.3.3. □

We want to show a sketch of another possible proof of the embedding $\dot{X}_{s,A}^0 \hookrightarrow X_{(-1)}$ where we use a corresponding result for the so called McIntosh-Yagi spaces from [Ha06], Proposition 6.4.1.:

With the notations of the above proof we obtain with [Ha06], Proposition 6.4.1 b) the estimate

$$\|x\|_{X_{(-1)}} \leq C \cdot \sup_{t>0} \|\varphi(tA)x\|_X.$$

Since A is sectorial, by similar arguments as used in the proof of Proposition 3.2.17 we obtain

$$\begin{aligned} \sup_{t>0} \|\varphi(tA)x\|_X &= \sup_{j\in\mathbb{Z}} \sup_{t\in[1,2]} \|\varphi(2^j tA)x\|_X \approx \sup_{j\in\mathbb{Z}} \|\varphi(2^j A)x\|_X \lesssim \|\sup_{j\in\mathbb{Z}} |\varphi(2^j A)x|\|_X \\ &\lesssim \left\| \left(\int_0^\infty |\varphi(tA)x|^s \frac{dt}{t}\right)^{1/s} \right\|_X. \end{aligned}$$

With the aid of Proposition 3.3.9 we can deduce a close relationship between the homogeneous and the inhomogeneous spaces:

Corollary 3.3.10. *Let $\theta \geq 0$, then $X_{s,A}^\theta = \dot{X}_{s,A}^\theta \cap X$ with equivalent norms.*

Proof. This follows immediately from Proposition 3.3.9 since

$$X_{s,A}^\theta = \{x \in X \mid \|x\|_{\theta,s,A} < +\infty\} = \dot{X}_{s,A}^\theta \cap X.$$

□

There are more relations between the homogeneous and the inhomogeneous spaces if A is invertible. Moreover, the inhomogeneous spaces do not change if A is replaced by $A+\varepsilon$ for some $\varepsilon > 0$. This is contained in the following

Proposition 3.3.11. *Let $\varepsilon, \theta > 0$.*

(1) If $A^{-1} \in L(X)$, then $\dot{X}^{\theta}_{s,A} \cong X^{\theta}_{s,A}$,

(2) $X^{\theta}_{s,\varepsilon+A} \cong X^{\theta}_{s,A}$.

In particular we have $X^{\theta}_{s,A} \cong \dot{X}^{\theta}_{s,\varepsilon+A}$.

Proof. (1) Assume that $A^{-1} \in L(X)$. Choose $\sigma \in (\omega_{\mathcal{R}_s(A)}, \pi)$ and $\varphi \in \Phi^{\Sigma}_{\sigma,\theta}$ and let $x \in X$. By [Ha06], Proposition 6.5.4 we obtain

$$\begin{aligned}\|x\|_X &\lesssim \left\|\left(t^{-\theta}\varphi(tA)x\right)_{t>0}\right\|_{L^{\infty}_*(X)} = \sup_{t>0}\left\|t^{-\theta}\varphi(tA)x\right\|_X \overset{(*)}{\lesssim} \sup_{j\in\mathbb{Z}}\left\||2^{-j\theta}\varphi(2^jA)x|\right\|_X \\ &\leq \left\|\sup_{j\in\mathbb{Z}}|2^{-j\theta}\varphi(2^jA)x|\right\|_X,\end{aligned}$$

where $(*)$ can be seen by analogous arguments as in the proof of Proposition 3.2.17 for the case $s = +\infty$, where in this case we just use the sectoriality of A. If $s < +\infty$ we can proceed with the embedding $\ell^s \hookrightarrow \ell^\infty$:

$$\|x\|_X \lesssim \left\|\sup_{j\in\mathbb{Z}}|2^{-j\theta}\varphi(2^jA)x|\right\|_X \leq \left\|\Big(\sum_{j\in\mathbb{Z}}|2^{-j\theta}\varphi(2^jA)x|\Big)^{1/s}\right\|_X \approx \|x\|_{\theta,s,A},$$

since $\varphi \in \Phi^{\Sigma}_{\sigma,\theta}$. So we obtain

$$\|x\|_{X^{\theta}_{s,A}} \approx \|x\|_X + \|x\|_{\theta,s,A} \lesssim \|x\|_{\theta,s,A},$$

hence $\|\cdot\|_{\theta,s,A}$ is an equivalent norm on the Banach space $X^{\theta}_{s,A}$ which implies $X^{\theta}_{s,A} \cong \dot{X}^{\theta}_{s,A}$.

(2) Choose $\sigma \in (\omega_{\mathcal{R}_s(A)}, \pi)$ and $m \in \mathbb{N}$ with $m-1 \leq \theta < m$ and define $\varphi(z) := \varphi_m(z) := z^m/(1+z)^m$, then $\varphi \in \Phi_{\sigma,\theta}$, and for all $t > 0$ we obtain

$$\varphi(t^{-1}A) = t^{-m}A^m(1+t^{-1}A)^{-m} = A^m(t+A)^{-m}.$$

We will first show the embedding $X^{\theta}_{s,A+\varepsilon} \hookrightarrow X^{\theta}_{s,A}$, so let $x \in X^{\theta}_{s,A+\varepsilon}$. Then by Remark 3.3.6 we have

$$\|x\|_{X^{\theta}_{s,A}} \approx \|x\|_X + \left\|\left(\int_0^1 |t^{-\theta}\varphi(tA)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X$$

and

$$\begin{aligned}\left\|\left(\int_0^1 |t^{-\theta}\varphi(tA)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X &= \left\|\left(\int_1^\infty |t^{\theta}\varphi(t^{-1}A)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X \\ &= \left\|\left(\int_1^\infty |t^{\theta}A^m(t+A)^{-m}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X \\ &= \left\|\left(\int_1^\infty |t^{\theta}S(t)(\varepsilon+A)^m(t+\varepsilon+A)^{-m}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\end{aligned}$$

with $S(t) := (t+\varepsilon+A)^m(\varepsilon+A)^{-m}A^m(t+A)^{-m}$, hence

$$\begin{aligned} S(t) &= \left[(t+\varepsilon+A)(\varepsilon+A)^{-1}A(t+A)^{-1}\right]^m = \left[A(\varepsilon+A)^{-1}(t+\varepsilon+A)(t+A)^{-1}\right]^m \\ &= \left[A(\varepsilon+A)^{-1}(1+\varepsilon(t+A)^{-1})\right]^m = \left[A(\varepsilon+A)^{-1}(1+\underbrace{\frac{\varepsilon}{t}}_{\in[0,\varepsilon]}\cdot t(t+A)^{-1})\right]^m. \end{aligned}$$

Since A is $\mathcal{R}_s$-sectorial, the range $S([1,\infty))$ is also $\mathcal{R}_s$-bounded, hence by Proposition 3.1.13

$$\begin{aligned} \left\|\left(\int_0^1 |t^{-\theta}\varphi(tA)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X &= \left\|\left(\int_1^\infty |t^{\theta}S(t)(\varepsilon+A)^m(t+\varepsilon+A)^{-m}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X \\ &\lesssim \left\|\left(\int_1^\infty |t^{\theta}(\varepsilon+A)^m(t+\varepsilon+A)^{-m}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X \\ &= \left\|\left(\int_0^1 |t^{-\theta}\varphi(t(A+\varepsilon))x|^s\frac{dt}{t}\right)^{1/s}\right\|_X, \end{aligned}$$

and we obtain

$$\begin{aligned} \|x\|_{X^\theta_{s,A}} &\approx \|x\|_X + \left\|\left(\int_0^1 |t^{-\theta}\varphi(tA)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X \lesssim \|x\|_X + \left\|\left(\int_0^1 |t^{-\theta}\varphi(t(A+\varepsilon))x|^s\frac{dt}{t}\right)^{1/s}\right\|_X \\ &\approx \|x\|_{X^\theta_{s,A+\varepsilon}}. \end{aligned}$$

We now show the reverse embedding $X^\theta_{s,A} \hookrightarrow X^\theta_{s,A+\varepsilon}$, so let $x \in X^\theta_{s,A}$. Then for all $t > 0$ we have

$$\begin{aligned} (A+\varepsilon)^m(t+\varepsilon+A)^{-m} &= \sum_{k=0}^m \binom{m}{k}\varepsilon^{m-k}\,A^k(t+\varepsilon+A)^{-m} \\ &= \underbrace{\sum_{k=0}^m \binom{m}{k}\varepsilon^{m-k}}_{=:a_k}\cdot\underbrace{t^{m-k}(t+A)^k\cdot(t+\varepsilon+A)^{-m}}_{=:S_k(t)}\cdot t^{-(m-k)}A^k(t+A)^{-k}, \end{aligned}$$

and

$$\begin{aligned} S_k(t) &= t^{m-k}(t+\varepsilon+A)^{-(m-k)}\cdot(t+A)^k(t+\varepsilon+A)^{-k} \\ &= \Big[\underbrace{\frac{t}{t+\varepsilon}}_{\in[0,1]}\cdot(t+\varepsilon)(t+\varepsilon+A)^{-1}\Big]^{m-k}\cdot\Big[\underbrace{\frac{t}{t+\varepsilon}}_{\in[0,1]}\cdot(t+\varepsilon)(t+\varepsilon+A)^{-1}+A(t+\varepsilon+A)^{-1}\Big]^k. \end{aligned}$$

This shows that also $\{S_k(t)\,|\,t>0\}$ is $\mathcal{R}_s$-bounded for each $k\in(\mathbb{N}_0)_{\le m}$, hence

$$\begin{aligned}
&\left\|\left(\int_0^1 |t^{-\theta}\varphi(t(A+\varepsilon))x|^s\frac{dt}{t}\right)^{1/s}\right\|_X = \left\|\left(\int_1^\infty |t^{\theta}\varphi(t^{-1}(A+\varepsilon))x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
= &\left\|\left(\int_1^\infty |t^{\theta}(A+\varepsilon)^m(t+\varepsilon+A)^{-m}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
= &\left\|\left(\int_1^\infty |t^{\theta}\sum_{k=0}^m a_k\cdot S_k(t)\cdot t^{-(m-k)}A^k(t+A)^{-k}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
\lesssim &\sum_{k=0}^m\left\|\left(\int_1^\infty |t^{\theta-(m-k)}\cdot S_k(t)\cdot A^k(t+A)^{-k}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
\lesssim &\sum_{k=0}^m\left\|\left(\int_1^\infty |t^{\theta-(m-k)}\cdot A^k(t+A)^{-k}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X = \sum_{k=0}^m\left\|\left(\int_0^1 |t^{-(\theta-k)}\varphi_{m-k}(tA)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
\lesssim &\sum_{k=0}^{m-2}\|x\|_{\theta-k,s,A}+\left\|\left(\int_0^1 |t^{-(\theta-(m-1))}\varphi_1(tA)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X+\left\|\left(\int_0^1 |t^{m-\theta}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X.
\end{aligned}$$

By the choice of m we have $\alpha:=m-\theta>0$ and $\beta:=\theta-(m-1)\in[0,1)$. If $m=1$, then $\beta=\theta\in(0,1)$, and we define $\delta:=\beta$. If $m\ge 2$, then $\beta\in[0,1\wedge\theta)$, hence we can choose $\delta\in(\beta,1\wedge\theta)$, and in both cases we obtain $\delta\in(0,1)\cap[\beta,\theta]$. Then we can continue the estimate to

$$\begin{aligned}
&\left\|\left(\int_0^1 |t^{-\theta}\varphi(t(A+\varepsilon))x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
\overset{\delta\ge\beta}{\lesssim} &\sum_{k=0}^{m-2}\|x\|_{\theta-k,s,A}+\left\|\left(\int_0^1 |t^{-\delta}\varphi_1(tA)x|^s\frac{dt}{t}\right)^{1/s}\right\|_X+\left\|\left(\int_0^1 |t^{\alpha}x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
\lesssim &\sum_{k=0}^{m-2}\|x\|_{\theta-k,s,A}+\|x\|_{\delta,s,A}+(\alpha s)^{-1}\,\|x\|_X,
\end{aligned}$$

where we used $\varphi_{m-k}\in\Phi_{\sigma,\theta-k}$ for $k\in(\mathbb{N}_0)_{\le m-2}$ and $\varphi_1\in\Phi_{\sigma,\delta}$. So we also have an estimate

$$\begin{aligned}
\|x\|_{X^\theta_{s,A+\varepsilon}} &\approx \|x\|_X+\left\|\left(\int_0^1 |t^{-\theta}\varphi(t(A+\varepsilon))x|^s\frac{dt}{t}\right)^{1/s}\right\|_X\\
&\lesssim \|x\|_X+\sum_{k=0}^{m-2}\|x\|_{\theta-k,s,A}+\|x\|_{\delta,s,A}.
\end{aligned}$$

By Proposition 3.3.7 we have embeddings $X^\theta_{s,A}\hookrightarrow X^{\theta-k}_{s,A}$ for all $k\in(\mathbb{N}_0)_{\le m-2}$ and $X^\theta_{s,A}\hookrightarrow X^\delta_{s,A}$ by our choice of δ, hence also

$$\|x\|_{X^\theta_{s,A+\varepsilon}} \lesssim \sum_{k=0}^{m-2}\|x\|_{X^{\theta-k}_{s,A}}+\|x\|_{X^\delta_{s,A}}\lesssim\|x\|_{X^\theta_{s,A}}.$$

□

Again, to the end of the subsection we consider our standard example, the Laplacian in the space $L^p(\mathbb{R}^d)$.

Proposition 3.3.12. *Let $m, d \in \mathbb{N}$ and $p, s \in (1, +\infty)$, and let $A := (-\Delta)^m$ be the m-th power of the Laplace operator in $L^p(\mathbb{R}^d)$ with domain $D(A) = W^{2m,p}(\mathbb{R}^d)$. Let $\theta \in \mathbb{R}$, then*

$$\dot{X}^\theta_{s,A} = \dot{F}^{2m\theta}_{p,s}(\mathbb{R}^d),$$

and if $\theta > 0$, then also

$$X^\theta_{s,A} = F^{2m\theta}_{p,s}(\mathbb{R}^d)$$

with equivalent norms.

Proof. Choose $\sigma \in (0, \pi/2)$ and $k \in \mathbb{N}_{>|\theta|}$, and define $\varphi(z) := z^k e^{-z^{1/m}}$ for all $z \in \Sigma_\sigma$. Then $\varphi \in \Phi^\Sigma_{\sigma,\theta}$, hence φ is suitable to calculate the norm in $\dot{X}^\theta_{s,A}$, and also in $X^\theta_{s,A}$ in the case $\theta > 0$. On the other hand, if $t > 0$ and $r := t^{1/m}$, then

$$\varphi(tA)u = t^k(-\Delta)^{km} e^{(t^{1/m}\Delta)} u = (-r\Delta)^{km} e^{r\Delta} u \quad \text{for all } u \in \mathcal{S}'_d.$$

Thus Theorem 1.7.3 shows that $\|\cdot\|_{\theta,s,A,\varphi}$ is also an equivalent norm for the homogeneous Triebel-Lizorkin space $\dot{F}^{2m\theta}_{p,s}$, and in the case $\theta > 0$, the norm $\|\cdot\|_{X^\theta_{s,A,\varphi}}$ is an equivalent norm for the inhomogeneous Triebel-Lizorkin space $F^{2m\theta}_{p,s}$. Hence the result for the inhomogeneous spaces is immediate, and for the homogeneous spaces it follows from density: it can be shown that $\mathcal{Z}_d \subseteq D(A^k) \cap R(A^k) \subseteq \dot{F}^{2m\theta}_{p,s}(\mathbb{R}^d)$, hence $D(A^k) \cap R(A^k)$ is a dense subspace of $\dot{F}^{2m\theta}_{p,s}(\mathbb{R}^d)$, and on the other hand it is also dense in $\dot{X}^\theta_{s,A}$ by Proposition 3.3.3. □

Let us finally mention that we can reformulate the well known fact that having a bounded H^∞-calculus is equivalent to square function estimates (cf. Subsection 1.6.5) in terms of the coincidence of X with the space $\dot{X}^0_{2,A}$:

Remark 3.3.13. *Assume that X is q-concave for some $q < +\infty$ and let A be an $\mathcal{R}_2$-sectorial operator in X. Then A has a bounded H^∞-calculus in X if and only if $X = \dot{X}^0_{2,A}$ with equivalent norms, and in this case $\omega_{H^\infty}(A) = \omega_{\mathcal{R}_2}(A)$.*

Proof. The "if"-part follows immediately from Proposition 3.2.12. Recall that in this situation $\mathcal{R}_2$-boundedness is equivalent to $\mathcal{R}$-boundedness by Remark 3.1.7, hence the other implication follows by the remarks in Subsection 1.6.5. The identity $\omega_{H^\infty}(A) = \omega_{\mathcal{R}_2}(A)$ follows again from Proposition 3.2.12 and Theorem 1.3.5 (cf. also its preceding remarks). □

3.3.2 The s-spaces as intermediate spaces and interpolation

We will show now that the s-spaces $X^\theta_{s,A}, \dot{X}^\theta_{s,A}$ defined in the previous subsection are "reasonable" intermediate space. We will start with the following connection with the real interpolation spaces $(X, D(A^m))_{\alpha,q}$.

Proposition 3.3.14. *Let $\alpha > \theta > 0$ and $1 \le p \le s \le q \le +\infty$.*

(1) If X is q-concave, then $X^\theta_{s,A} \hookrightarrow (X, D(A^\alpha))_{\theta/\alpha,q}$,

(2) If X is p-convex, then $(X, D(A^\alpha))_{\theta/\alpha,p} \hookrightarrow X^\theta_{s,A}$.

Proof. We will only proof (1), since the proof of (2) can be done similarly. Choose $\sigma \in (\omega_{\mathcal{R}_s(A)}, \pi)$ and $\varphi \in \Phi^{\Sigma}_{\sigma,\theta}$. Then Theorem 1.5.8, an equivalent norm in $(X, D(A^\alpha))_{\theta/\alpha,q}$ is given by

$$x \mapsto \|x\|_X + \left\|\left(t^{-\theta}\varphi(tA)x\right)_{t>0}\right\|_{L^q_*(X)}.$$

We consider only the case $q < +\infty$, the case $q = +\infty$ can be treated similarly as usual. By analogous arguments as in the proof of Proposition 3.2.17 using the sectoriality of A we obtain

$$\begin{aligned}
\left\|\left(t^{-\theta}\varphi(tA)x\right)_{t>0}\right\|_{L^q_*(X)} &= \left(\int_0^\infty \|t^{-\theta}\varphi(tA)x\|_X^q \frac{dt}{t}\right)^{1/q} \approx \Big(\sum_{j\in\mathbb{Z}} \|2^{-j\theta}\varphi(2^jA)x\|_X^q\Big)^{1/q} \\
&\overset{(*)}{\leq} M_{(q)}(X) \cdot \left\|\Big(\sum_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^jA)x|^q\Big)^{1/q}\right\|_X \\
&\leq M_{(q)}(X) \cdot \left\|\Big(\sum_{j\in\mathbb{Z}} |2^{-j\theta}\varphi(2^jA)x|^s\Big)^{1/s}\right\|_X \approx \|x\|_{\theta,s,A}.
\end{aligned}$$

where in (*) we used the Fatou-property and the q-concavity of X, and in the last inequality we used that $\ell^s \hookrightarrow \ell^q$. Hence we also obtain

$$\|x\|_{(X,D(A^\alpha))_{\theta/\alpha,q}} \approx \|x\|_X + \left\|\left(t^{-\theta}\varphi(tA)x\right)_{t>0}\right\|_{L^q_*(X)} \lesssim \|x\|_X + \|x\|_{\theta,s,A} \approx \|x\|_{X^\theta_{s,A}}.$$

$\square$

In particular, since X is always ∞-concave and 1-convex, we have trivially

Corollary 3.3.15. *Let $\alpha > \theta > 0$, then*

$$(X, D(A^\alpha))_{\theta/\alpha,1} \hookrightarrow X^\theta_{s,A} \hookrightarrow (X, D(A^\alpha))_{\theta/\alpha,\infty}. \tag{3.3.4}$$

$\square$

We can now use standard methods from the theory of real interpolation spaces, which lead to the following

Corollary 3.3.16. *(1) Let $\alpha > \theta > \beta > 0$, then $D(A^\alpha) \hookrightarrow X^\theta_{s,A} \hookrightarrow D(A^\beta)$.*

(2) Let $\theta_0 < \theta_1 < \alpha$ and $s_0, s_1 \in [1, +\infty]$ and $\delta \in (0,1)$, then

$$(X^{\theta_0}_{s_0,A}, X^{\theta_1}_{s_1,A})_{\delta,q} = (X, D(A^\alpha))_{\theta/\alpha,q} \quad \textit{with } \theta := (1-\delta)\theta_0 + \delta\theta_1. \tag{3.3.5}$$

Proof. (1) This is simply due to the fact that by real interpolation theory and Corollary 3.3.15 we have

$$D(A^\alpha) \hookrightarrow (X, D(A^\alpha))_{\theta/\alpha,1} \hookrightarrow X^\theta_{s,A} \hookrightarrow (X, D(A^\alpha))_{\theta/\alpha,\infty} \hookrightarrow (X, D(A^\alpha))_{\beta/\alpha,1} \hookrightarrow D(A^\beta).$$

For (2) we observe that equation (3.3.4) from Corollary 3.3.15 is equivalent to the fact that the spaces $X^{\theta_j}_{s,A}$ are in the class $J_{\theta_j/\alpha}(X, D(A^\alpha)) \cap K_{\theta_j/\alpha}(X, D(A^\alpha))$ for $j = 0, 1$, hence (2) follows from the reiteration theorem for real interpolation, Theorem 1.5.7. $\square$

We conclude this subsection by considering the complex interpolation of the s-spaces.

Proposition 3.3.17. *Let $s_0, s_1 \in (1,\infty)$ and assume that A has an $\mathcal{R}_{s_j}$-bounded H^∞-calculus with $\omega_0 := \omega_{\mathcal{R}^\infty_{s_0}}(A) \vee \omega_{\mathcal{R}^\infty_{s_1}}(A) < \pi/2$ for $j = 0,1$. Let $\theta_0, \theta_1 \in \mathbb{R}$ with*

$$|\theta_0|, |\theta_1|, |\theta_0 - \theta_1| < \frac{\pi/2}{\omega_0}. \tag{3.3.6}$$

Then for

$$\Big(\theta, \frac{1}{s}\Big) := (1-\alpha)\Big(\theta_0, \frac{1}{s_0}\Big) + \alpha\Big(\theta_1, \frac{1}{s_1}\Big)$$

we have $\big[\dot X^{\theta_0}_{s_0,A}, \dot X^{\theta_1}_{s_1,A}\big]_\alpha \cong \dot X^{\theta}_{s,A}$, and if $\theta_0, \theta_1 > 0$ we also have $\big[X^{\theta_0}_{s_0,A}, X^{\theta_1}_{s_1,A}\big]_\alpha \cong X^{\theta}_{s,A}$.

We remark that the restriction (3.3.6) for the interpolation indices θ_j has its reason in our method of proof. Up to now it seems not clear if this restriction is reasonable or just a matter of lack of technique in our method. Nevertheless, in the classical situation of Triebel-Lizorkin-spaces where $A = -\Delta$ we have $\omega_0 = 0$, hence complex interpolation works for all $\theta_j \in \mathbb{R}$ as it is also known from the classical results.

Proof of Proposition 3.3.17. First we observe that if we have proved the result for the homogeneous spaces, then by Proposition 3.3.11 we obtain the corresponding result for the inhomogeneous spaces, hence it is sufficient to consider the homogeneous spaces.

According to the assumption on θ_0, θ_1 we can choose $0 < \alpha < \beta < \frac{\pi/2}{\omega_0}$ such that $\theta_j \in (\alpha - \beta, \alpha)$ for $j = 0,1$. Furthermore fix some $\sigma \in \big(\omega_0, \frac{\pi/2}{\beta}\big)$.

We will show first that we can reduce to the case $\beta = 1$. If $\beta < 1$, we can obviously replace β by 1, so consider the case $\beta > 1$. Since A has an $\mathcal{R}_{s_j}$-bounded H^∞-calculus, the operator A^β also has an $\mathcal{R}_{s_j}$-bounded H^∞-calculus with

$$\omega_{\mathcal{R}^\infty_{s_j}}(A^\beta) \le \beta\omega_{\mathcal{R}^\infty_{s_j}}(A) \le \beta\omega_0 < \pi/2 \quad \text{for } j = 0,1.$$

Observe that $\dot X^{\beta\delta}_{s,A} \cong \dot X^{\delta}_{s,A^\beta}$ for $\delta \in \mathbb{R}$ canonically, since for $\varphi \in \Phi_{\beta\sigma,\delta}$ and $\psi(z) := \varphi(z^\beta)$ we have $\psi \in \Phi_{\sigma,\beta\delta}$ and

$$\begin{aligned}\|x\|_{\beta\delta,s,A,\psi} &= \bigg\|\Big(\int_0^\infty |t^{-\beta\delta}\psi(tA)x|^s \,\frac{dt}{t}\Big)^{1/s}\bigg\|_X = \bigg\|\Big(\int_0^\infty |t^{-\beta\delta}\varphi(t^\beta A^\beta)x|^s \,\frac{dt}{t}\Big)^{1/s}\bigg\|_X \\ &= \beta^{-1/s}\cdot\bigg\|\Big(\int_0^\infty |t^{-\delta}\varphi(tA^\beta)x|^s \,\frac{dt}{t}\Big)^{1/s}\bigg\|_X \approx \|x\|_{\delta,s,A^\beta,\varphi}.\end{aligned}$$

So since the above stated isomorphisms are canonical we can replace A by A^β and then β by 1.

We define the auxiliary spaces

$$\ell^{s,\theta} := \ell^{s,\theta}(\mathbb{Z}) := \Big\{(\alpha_j)_j \in \mathbb{C}^{\mathbb{Z}} \mid \|(\alpha_j)\|_{\ell^{s,\theta}} := \big\|(2^{-\theta j}\alpha_j)_j\big\|_{\ell^s} < \infty\Big\},$$

endowed with the weighted norm $\|\cdot\|_{\ell^{s,\theta}}$ for all $s \in [1,+\infty], \theta \in \mathbb{R}$. Then we have for all $s_0, s_1 \in [1,+\infty]$ and $\theta_0, \theta_1 \in \mathbb{R}$ and $\alpha \in (0,1)$

$$[X(\ell^{s_0,\theta_0}), X(\ell^{s_1,\theta_1})]_\alpha \cong X(\ell^{s,\theta}) \quad \text{where } \Big(\theta, \frac{1}{s}\Big) = (1-\alpha)\Big(\theta_0, \frac{1}{s_0}\Big) + \alpha\Big(\theta_1, \frac{1}{s_1}\Big), \tag{3.3.7}$$

cf. Section 2.4 and [BL76], Theorem 5.6.3.

The strategy is now to show that the homogeneous spaces are retracts of the spaces $X(\ell^{s,\theta})$ with canonical (co-)retractions and then use Proposition 1.5.2 and (3.3.7). For this we will construct a coretraction $J : \dot{X}^\theta_{s,A} \to X(\ell^{s,\theta})$ and a retraction $P : X(\ell^{s,\theta}) \to \dot{X}^\theta_{s,A}$, i.e. bounded operators such that $PJ = \mathrm{Id}_{\dot{X}^\theta_{s,A}}$, which are independent of $\theta \in (\alpha-1,\alpha)$ and $s \in [1,+\infty]$ such that A has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_s \le \omega_0$.

By the above reduction we have $\beta = 1$, hence $\alpha \in (0,1)$. We define auxiliary functions

$$\varphi(z) := -\frac{z^\alpha}{2+z}, \quad \psi(z) := \frac{z^{1-\alpha}}{1+z}, \quad \rho(z) := \frac{z^\alpha}{1+z}$$

and $f(z) := \varphi(z)\psi(z) = \frac{-z}{(1+z)(2+z)} = \frac{1}{1+z} - \frac{2}{2+z}$ for all $z \in \Sigma_\sigma$. For later use we mention the following easily proved estimate: If $a, b, c > 0$ and $g(z) := \frac{cz}{(1+az)(1+bz)}$, then

$$|g(z)| \lesssim_\sigma g(|z|) \le 2\,\frac{c}{a+b} \quad \text{for all } z \in \Sigma_\sigma. \tag{3.3.8}$$

We now define the operator $Jx := \big(\varphi(2^j A)x\big)_{j\in\mathbb{Z}}$ for all $x \in X$ and formally

$$P(y_j)_j := \sum_{j\in\mathbb{Z}} \psi(2^j A)y_j := \lim_{N\to\infty} \underbrace{\sum_{j=-N}^{N} \psi(2^j A)y_j}_{=:P_N(y_j)_j} \quad \text{for } (y_j)_j \in X^{\mathbb{Z}}.$$

Let $\theta \in (\alpha-1,\alpha)$ and $s \in [1,+\infty]$ such that A has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_s \le \omega_0$. We will show now that $J|_{\dot{X}^\theta_{s,A}} : \dot{X}^\theta_{s,A} \to X(\ell^{s,\theta})$ is a coretraction and $P|_{X(\ell^{s,\theta})} : X(\ell^{s,\theta}) \to \dot{X}^\theta_{s,A}$ is a corresponding retraction, i.e. we have to show that

(1) $J|_{\dot{X}^\theta_{s,A}} : \dot{X}^\theta_{s,A} \to X(\ell^{s,\theta})$ is bounded,

(2) $P|_{X(\ell^{s,\theta})} : X(\ell^{s,\theta}) \to \dot{X}^\theta_{s,A}$ is well-defined and bounded, and

(3) $PJx = x$ for all $x \in X^\theta_{s,A}$.

Ad (1): This is simply due to the fact that $\varphi \in \Phi^\Sigma_{\sigma,\theta}$ (this can be shown in exactly the same way as the calculations in Example 3.2.16 (1)).

Ad (2): We now use the function ρ to calculate the norm in the space $\dot{X}^\theta_{s,A}$, which is possible since also $\rho \in \Phi^\Sigma_{\sigma,\theta}$:

$$\begin{aligned}
\|P_N(y_j)_j\|_{\dot{X}^\theta_{s,A}} &= \left\| \Big(2^{-k\theta}\rho(2^kA) \sum_{j=-N}^{N} \psi(2^jA)y_j \Big)_k \right\|_{X(\ell^s)} \\
= \left\| \Big(\sum_{j=-N}^{N} 2^{-(k-j)\theta}\rho(2^kA)\, 2^{-j\theta}\psi(2^jA)y_j \Big)_k \right\|_{X(\ell^s)} &= \left\| \Big(\sum_{\ell=-k-N}^{-k+N} 2^{\ell\theta}\rho(2^kA)\psi(2^{k+\ell}A)y_{k+\ell} \Big)_k \right\|_{X(\ell^s)} \\
\leq \left\| \Big(\sum_{\ell\in\mathbb{Z}} |2^{\ell\theta}\rho(2^kA)\psi(2^{k+\ell}A)y_{k+\ell}| \Big)_k \right\|_{X(\ell^s)} &\leq \sum_{\ell\in\mathbb{Z}} \left\| \big(|2^{\ell\theta}\rho(2^kA)\psi(2^{k+\ell}A)y_{k+\ell}| \big)_k \right\|_{X(\ell^s)} \\
\lesssim \sum_{\ell\in\mathbb{Z}} \Big(\sup_{z\in\Sigma_\sigma} |2^{\ell\theta}\rho(2^kz)\psi(2^{k+\ell}z)| \Big) \left\| \big(y_{k+\ell}\big)_k \right\|_{X(\ell^s)} &\leq C \left\| \big(y_k\big)_k \right\|_{X(\ell^s)}
\end{aligned}$$

with $C := \sum_{\ell\in\mathbb{Z}} \Big(\sup_{z\in\Sigma_\sigma} |2^{\ell\theta}\rho(2^kz)\psi(2^{k+\ell}z)| \Big)$. So we only have to show that $C < +\infty$, because then with the Fatou property we obtain

$$\|P(y_j)_j\|_{\dot{X}^\theta_{s,A}} \leq \liminf_{N\to\infty} \|P_N(y_j)_j\|_{\dot{X}^\theta_{s,A}} \lesssim C \cdot \left\| \big(y_k\big)_k \right\|_{X(\ell^s)}.$$

For this let

$$g(z) := \rho(2^kz)\psi(2^{k+\ell}z) = -\frac{cz}{(1+az)(1+bc)},$$

where $a := 2^k, b := 2^{k+\ell}$ and $c := a^\alpha b^{1-\alpha} = 2^{\alpha k + (1-\alpha)(k+\ell)} = 2^{k+(1-\alpha)\ell}$, then by (3.3.8)

$$|g(z)| \lesssim \left(\frac{a+b}{c}\right)^{-1} = \left(2^{-(1-\alpha)\ell} + 2^{\alpha\ell}\right)^{-1} \leq 2^{-\alpha\ell} \wedge 2^{(1-\alpha)\ell}.$$

For $\ell \in \mathbb{N}_0$ this implies $2^{\theta\ell}|g(z)| \lesssim 2^{-(\alpha-\theta)\ell}$, and for $\ell \in -\mathbb{N}$ we have $2^{\theta\ell}|g(z)| \lesssim 2^{-(\theta+1-\alpha)|\ell|}$, hence altogether with $\delta := \min\{\alpha-\theta, (\theta+1-\alpha)\} > 0$:

$$\sup_{z\in\Sigma_\sigma} |2^{\ell\theta}\rho(2^kz)\psi(2^{k+\ell}z)| = \sup_{z\in\Sigma_\sigma} |2^{\ell\theta}g(z)| \lesssim 2^{-\delta|\ell|} \quad \text{for all } \ell \in \mathbb{Z},$$

so $C \lesssim \sum_{\ell\in\mathbb{Z}} 2^{-\delta|\ell|} < +\infty$ as desired.

Ad (3): Let $x \in X^\theta_{s,A}$, then

$$\begin{aligned}
\sum_{j=-N}^{N} \psi(2^jA)\varphi(2^jA)x &= \sum_{j=-N}^{N} f(2^jA)x = \sum_{j=-N}^{N} \Big(2^{-j}(2^{-j}+A)x - 2^{-(j-1)}(2^{-(j-1)}+A)x \Big) \\
&= 2^{-N}(2^{-N}+A)x - 2^{N+1}(2^{N+1}+A)x \overset{N\to\infty}{\longrightarrow} 0 - x = x \quad \text{in } \dot{X}^\theta_{s,A},
\end{aligned}$$

since the part of A in $\dot{X}^\theta_{s,A}$ is sectorial (cf. Lemma 3.3.22 in the following subsection).

Since the operators J, P are appropriate (co-)retractions, the claim follows by Proposition 1.5.2 together with (3.3.7). □

Let us finally mention the correspondence of the s-intermediate spaces for $\mathcal{R}_2$-sectorial operators with the so called Rademacher interpolation spaces $\langle X,Y\rangle_\theta$, which have been introduced in [KKW06], we refer also to [SW06] for the relationship with interpolation by the γ-method, and to [KW-2], where this interpolation method is studied in a general framework in connection with Euclidean structures. Then the same techniques as used in the proof of [KKW06], Theorem 7.4 show the following.

Remark 3.3.18. *Let X be q-concave and p-convex for some $1<p,q<+\infty$, and assume that A is $\mathcal{R}_2$-sectorial. Then $\dot{X}^\theta_{2,A} = \langle X, \dot{X}_1\rangle_\theta$ for all $\theta\in(0,1)$ with equivalent norms.*

In fact, the inclusion $\dot{X}^\theta_{2,A} \subseteq \langle X,\dot{X}_1\rangle_\theta$ can be shown by similar arguments as in the proof of [KKW06], Theorem 7.4, p. 782, and the other inclusion can be derived by means of duality with similar arguments as in the proof of [KKW06], Theorem 7.4, p. 783f.

3.3.3 The part of A in the s-intermediate spaces

For this subsection let $\theta\in\mathbb{R}$. We recall that we can extrapolate the operator A to an operator in the universal extrapolation space U such that A is sectorial in each extrapolation space $X_{(-m)}, m\in\mathbb{N}$.

Observe first that the operators A^α shift the scales of associated s-spaces in the following sense.

Lemma 3.3.19. *Let $\alpha\in\mathbb{R}$. Then $A^\alpha X^\theta_{s,A}$ coincides with $X^{\theta-\alpha}_{s,A}$ in the set-theoretical sense, and the operator A^α (defined on U) induces a topological isomorphism*

$$A^\alpha : \dot{X}^\theta_{s,A} \to \dot{X}^{\theta-\alpha}_{s,A}.$$

If in addition $\theta>\alpha\vee 0$, then also the operator $(1+A)^\alpha$ induces an isomorphism

$$(1+A)^\alpha : X^\theta_{s,A} \to X^{\theta-\alpha}_{s,A}.$$

Proof. Choose $\sigma\in(\omega_{\mathcal{R}_s(A)},\pi)$ and $\varphi\in\Phi_{\sigma,\theta-\alpha}$ such that $\psi(z):=z^\alpha\varphi(z)$ defines a function in $\mathcal{E}(\Sigma_\sigma)$, then $\psi\in\Phi_{\sigma,\theta}$. Let $x\in X$, then

$$\begin{aligned}\|A^\alpha x\|_{\theta-\alpha,s,A} &\approx \left\|\left(\int_0^\infty |t^{-\theta+\alpha}\varphi(tA)A^\alpha x|^s\,\frac{dt}{t}\right)^{1/s}\right\|_X = \left\|\left(\int_0^\infty |t^{-\theta}(tA)^\alpha\varphi(tA)x|^s\,\frac{dt}{t}\right)^{1/s}\right\|_X\\ &= \left\|\left(\int_0^\infty |t^{-\theta}\psi(tA)x|^s\,\frac{dt}{t}\right)^{1/s}\right\|_X \approx \|x\|_{\theta,s,A},\end{aligned}$$

hence $A^\alpha x\in X^{\theta-\alpha}_{s,A} \iff x\in X^\theta_{s,A}$, and $A^\alpha : (X^\theta_{s,A},\|\cdot\|_{\theta,s,A,\psi}) \to (X^{\theta-\alpha}_{s,A},\|\cdot\|_{\theta,s,A,\varphi})$ is an isometric isomorphism. Since $X^{\theta-\gamma}_{s,A}$ is dense in $\dot{X}^{\theta-\gamma}_{s,A}$ for $\gamma\in\{0,\alpha\}$ this also yields $A^\alpha x\in\dot{X}^{\theta-\alpha}_{s,A} \iff x\in\dot{X}^\theta_{s,A}$ for all $x\in U$ and that A^α induces a topological isomorphism $A^\alpha:\dot{X}^\theta_{s,A}\to\dot{X}^{\theta-\alpha}_{s,A}$.

If in addition $\theta>\alpha\vee 0$, then $X^{\theta-\gamma}_{s,A}\cong\dot{X}^{\theta-\gamma}_{s,A+1}$ for $\gamma\in\{0,\alpha\}$ by Proposition 3.3.11, and we can apply the first part for $1+A$ instead of A. □

Definition 3.3.20. Let $\dot{A}_{\theta,s} := A_{\dot{X}^\theta_{A,s}}$ be the part of A in $\dot{X}^\theta_{A,s}$ and $A_{\theta,s} := A_{X^\theta_{A,s}}$ be the part of A in $X^\theta_{A,s}$ if $\theta \geq 0$, respectively.

Remark 3.3.21. *The spaces $\dot{X}^\theta_{s,A}$, and $X^\theta_{A,s}$ if $\theta \geq 0$, respectively, are invariant under resolvents of A, i.e. $R(\lambda, A)\dot{X}^\theta_{A,s} \subseteq \dot{X}^\theta_{A,s}$, and if $\theta \geq 0$ then $R(\lambda, A)X^\theta_{A,s,\varphi} \subseteq X^\theta_{A,s,\varphi}$, respectively, for all $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_\sigma$ and $\sigma \in (\omega_{\mathcal{R}_s(A)}, \pi)$. In fact, it is sufficient to show that $\lambda R(\lambda, A)X^\theta_{s,A} \subseteq X^\theta_{s,A}$. To see this we let $x \in X^\theta_{s,A}$ and choose $\sigma \in (\omega_{\mathcal{R}_s(A)}, \pi)$ and $\varphi \in \Phi_{\sigma,\theta}$, then*

$$\begin{aligned}
\|\lambda R(\lambda, A)x\|_{\theta,s,A} &\approx \left\|(\lambda R(\lambda, A))t^{-\theta}\varphi(tA)x)_{t>0}\right\|_{X(L^s_*)} \leq M_{\mathcal{R}_s,\sigma}(A)\left\|(t^{-\theta}\varphi(tA)x)_{t>0}\right\|_{X(L^s_*)} \\
&\approx \|x\|_{\theta,s,A}
\end{aligned}$$

for all $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_\sigma$, since the set $\{zR(z, A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}$ is $\mathcal{R}_s$-bounded. □

By Remark 3.3.21 we obtain the following elementary properties of the operators $\dot{A}_{\theta,s}, A_{\theta,s}$.

Lemma 3.3.22. *The operator $\dot{A}_{\theta,s}$ is an injective sectorial operator in $\dot{X}^\theta_{s,A}$ of type $\omega(\dot{A}_{\theta,s}) \leq \omega_{\mathcal{R}_s}(A)$ with $D(\dot{A}_{\theta,s}) = \dot{X}^\theta_{s,A} \cap \dot{X}^{\theta+1}_{s,A}$. If $\theta \geq 0$, the operator $A_{\theta,s}$ is an injective sectorial operator in $X^\theta_{s,A}$ of type $\omega(A_{\theta,s}) \leq \omega_{\mathcal{R}_s}(A)$ with $D(A_{\theta,s}) = X^{\theta+1}_{s,A}$.*

Moreover, if $m \in \mathbb{N}_{>|\theta|}$, then $D(A^m) \cap R(A^m)$ is a core of $\dot{A}_{\theta,s}$, and of $A_{\theta,s}$ in the case $\theta \geq 0$, respectively.

Proof. It is well know that the statements of Remark 3.3.21 imply the asserted sectoriality properties, so we only have to verify the statements concerning the domains. But this follows immediately from Lemma 3.3.19 with $\alpha = 1$. The final assertions follows from the approximation result that is also used in the proof of Proposition 3.3.3. □

Combining Lemma 3.3.22 with Proposition 3.3.5 we immediately obtain the following theorem, which is one of the main results of this work.

Theorem 3.3.23. *(1) The part $\dot{A}_{\theta,s}$ of A in $\dot{X}^\theta_{s,A}$ with domain $D(\dot{A}_{\theta,s}) = \dot{X}^\theta_{s,A} \cap \dot{X}^{\theta+1}_{s,A}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\dot{A}_{\theta,s}) \leq \omega_{\mathcal{R}_s}(A)$.*

(2) Let $\theta \geq 0$. If $A^{-1} \in L(X)$ or A has a bounded H^∞-calculus in X, then the part $A_{\theta,s}$ of A in $X^\theta_{s,A}$ with domain $D(A_{\theta,s}) = X^{\theta+1}_{s,A}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(A_{\theta,s}) \leq \omega_{\mathcal{R}_s}(A)$. □

As we already noted in the introduction of this section, Theorem 3.3.23 can be seen as a variant of Dore's Theorem that states that an invertible sectorial operator A in a Banach space X has a bounded H^∞-calculus in the scale of *real interpolation spaces* $(X, D(A))_{p,\theta}$ for $p \in [1, +\infty], \theta \in (0, 1)$, cf. [Do99] and [Do01], and also the extensive treatment using functional calculus which is given in [Ha06], Chapter 6.

3.4 Comparison and perturbation for $\mathcal{R}_s$-sectorial operators

In this section we consider perturbation and comparison results for $\mathcal{R}_s$-sectoriality, $\mathcal{R}_s$-bounded functional calculi and the associated s-intermediate spaces. One central task is, given an $\mathcal{R}_s$-sectorial operator A, to give sufficient conditions for a linear operator C such that C is $\mathcal{R}_s$-sectorial as well and one has $X^\theta_{s,A} \doteq X^\theta_{s,C}$. We will closely follow the lines of the corresponding comparison and perturbation results of [KW01-b], [KW04], Chapter 13 and [KKW06] for the H^∞-calculus. In fact, our assumptions are in many cases $\mathcal{R}_s$-versions of corresponding assumptions of comparison and perturbation results in the articles cited above. Since these are equivalent to usual boundedness assumptions for the vector-valued extensions $\widetilde{A}_s$ in the vector-valued space $X(\ell^s)$ it is not surprising that under our assumptions we will obtain similar results for an $\mathcal{R}_s$-bounded H^∞-calculus for A. Nevertheless, we are also interested in coincidence of the corresponding s-intermediate spaces, which will be done by suitable estimates of the s-power function norms. Here we will use techniques similar to the ones used in [KW04], Chapter 13, but the estimates we use will be more involved.

We fix $s \in [1, +\infty]$, and A will always denote an $\mathcal{R}_s$-sectorial operator in X with dense domain and dense range. Moreover we fix $\sigma \in (0, \pi)$. Recall the following notations:

$$M_{s,\sigma}(A) := M_{\mathcal{R}_s,\sigma}(A) := \mathcal{R}_s(\{zR(z,A), AR(z,A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\})$$

and

$$M^\infty_{s,\sigma}(A) = \mathcal{R}_s\big(\{f(A) \mid f \in H^\infty(\Sigma_\sigma), \|f\|_{\infty,\sigma} \le 1\}\big)$$

in the case that A has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_\sigma)$-calculus.

We start with an $\mathcal{R}_s$-version of a standard perturbation result for sectorial operators.

Proposition 3.4.1. *Assume $\sigma > \omega_{\mathcal{R}_s}(A)$ and let B be a linear operator in X with $D(B) \supseteq D(A)$ such that BA^{-1} extends to an $\mathcal{R}_s$-bounded operator on X with $a := \mathcal{R}_s(\overline{BA^{-1}}) < 1/M_{s,\sigma}(A)$. Then $A + B$ is again $\mathcal{R}_s$-sectorial with*

$$M_{s,\sigma}(A+B) \le \frac{M_{s,\sigma}(A)}{1 - aM_{s,\sigma}(A)} + 1.$$

In particular, we have $\omega_{\mathcal{R}_s}(A+B) \le \sigma$.

Proof. Since $BR(z,A) = BA^{-1}\,AR(z,A)$ for all $z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma$, we obtain that $\{BR(z,A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}$ is $\mathcal{R}_s$-bounded with

$$\mathcal{R}_s(\{BR(z,A) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}) \le aM_{s,\sigma}(A) < 1.$$

In particular, the operator $\mathrm{Id}_X - BR(z,A)$ is invertible and

$$zR(z,A+B) = zR(z,A)(\mathrm{Id}_X - BR(z,A))^{-1} = zR(z,A)\sum_{k=0}^\infty (BR(z,A))^k \in L(X) \quad (3.4.1)$$

for all $z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma$. This shows that $\sigma(A+B) \subseteq \overline{\Sigma_\sigma}$ and that $A+B$ is $\mathcal{R}_s$-sectorial with

$$\mathcal{R}_s(\{zR(z, A+B) \mid z \in \mathbb{C}\backslash\overline{\Sigma}_\sigma\}) \leq M_{s,\sigma}(A) \sum_{k=0}^{\infty}(aM_{s,\sigma}(A))^k = \frac{M_{s,\sigma}(A)}{1 - aM_{s,\sigma}(A)}.$$

□

We now turn to the comparison result, for which the following lemma states the central estimate.

Lemma 3.4.2. *Let $\alpha, \beta > 0$, and let A and B be $\mathcal{R}_s$-sectorial operators such that $D(A^\alpha) = D(B^\alpha)$ and $R(A^\beta) = R(B^\beta)$ and the operators $B^\alpha A^{-\alpha}$ and $B^{-\beta}A^\beta$ extend to $\mathcal{R}_s$-bounded operators on X. Then, for all $\theta \in (-\beta, \alpha)$ we have*

$$\dot{X}^\theta_{s,A} \hookrightarrow \dot{X}^\theta_{s,B}.$$

Proof. We define

$$M := \max\{M_{s,\sigma}(A), M_{s,\sigma}(B), \mathcal{R}_s(B^\alpha A^{-\alpha}), \mathcal{R}_s(B^{-\beta}A^\beta)\} < +\infty.$$

Choose some $\omega \in (\omega_{\mathcal{R}_s}(A) \vee \omega_{\mathcal{R}_s}(B), \pi)$. We define for all $a, b > 0$ the auxiliary functions

$$\psi_{a,b}(z) := \frac{z^a}{(1+z)^b},$$

then $\psi_{a,b} \in H^\infty_0(\Sigma_\omega)$ if $b > a > 0$ and $\psi_{a,b} \in \Phi_{\omega,\theta}$ if $b > a - \theta > 0$. Moreover we have

$$\forall\, t > 0 \,:\, \psi_{a,b}(t^{-1}A) = t^{b-a}A^a(t+A)^{-b}.$$

Let A, B be operators according to the assumptions. Define $T := \overline{B^\alpha A^{-\alpha}}$ and $S := \overline{B^{-\beta}A^\beta}$, then T, S are $\mathcal{R}_s$-bounded. Choose $a > \alpha + \beta$ and $n \in \mathbb{N}$ with $n > 3a$. Let $t > 0$, then we have

$$B^a(t+B)^{-n} \quad = \quad B^a(t+B)^{-n}(t+A)^n(t+A)^{-n} = \sum_{k=0}^{n}\binom{n}{k} B^a(t+B)^{-n}t^{n-k}A^k(t+A)^{-n},$$

hence

$$\psi_{a,n}(t^{-1}B) \quad = \quad t^{n-a}B^a(t+B)^{-n} = \sum_{k=0}^{n}\binom{n}{k} \underbrace{t^{n-a}B^a(t+B)^{-n}t^{n-k}A^k(t+A)^{-n}}_{S_k(t):=}.$$

Now let $\theta \in (-\beta, \alpha)$ and define $N := [|\theta|]$. We split up the sum as

$$\psi_{a,n}(t^{-1}B) \quad = \quad \sum_{k=0}^{N}\binom{n}{k}S_k(t) + \sum_{k=N+1}^{n-N-1}\binom{n}{k}S_k(t) + \sum_{k=n-N}^{n}\binom{n}{k}S_k(t)$$

and consider the three types of summands separately.

Case 1. For $k \in \mathbb{N}_0$ with $k \leq N \leq |\theta|$ we have

$$S_k(t) = t^{n-(a-\alpha)}B^{a-\alpha}(t+B)^{-n}Tt^{n-(k+\alpha)}A^{k+\alpha}(t+A)^{-n} = \psi_{a-\alpha,n}(t^{-1}B)T\psi_{k+\alpha,n}(t^{-1}A).$$

We have $n > a - \alpha > 0$, hence $\psi_{a-\alpha,n} \in H_0^\infty(\Sigma_\omega)$, and

$$0 < \alpha - \theta \le k + \alpha - \theta \le |\theta| - \theta + \alpha \le 2|\theta| + \alpha < n,$$

hence $\psi_{k+\alpha,n} \in \Phi_{\theta,n}$. So for all $x \in X^\theta_{s,A}$ we have by Proposition 3.1.13 and Lemma 3.2.7:

$$\left\|(t^\theta S_k(t)x)_{t>0}\right\|_{X(L^s_*)} \lesssim_M \left\|(t^\theta \psi_{k+\alpha,n}(t^{-1}A)x)_{t>0}\right\|_{X(L^s_*)} \approx \|x\|_{\theta,s,A}.$$

Case 2. Now consider the case $k \in \mathbb{N}_0$ with $N+1 \le k \le n-N-1$, hence $|\theta| < k < n - |\theta|$. Then

$$S_k(t) = t^{n-a}B^a(t+B)^{-n}t^{n-k}A^k(t+A)^{-n} = \psi_{a,n}(t^{-1}B)\psi_{k,n}(t^{-1}A).$$

We have $n > a > 0$, hence $\psi_{a,n} \in H_0^\infty(\Sigma_\omega)$, and

$$0 \le |\theta| - \theta < k - \theta < n - |\theta| - \theta \le n,$$

hence $\psi_{k,n} \in \Phi_{\theta,n}$. So for all $x \in X^\theta_{s,A}$ we have

$$\left\|(t^\theta S_k(t)x)_{t>0}\right\|_{X(L^s_*)} \lesssim_M \left\|(t^\theta \psi_{k,n}(t^{-1}A)x)_{t>0}\right\|_{X(L^s_*)} \approx \|x\|_{\theta,s,A}.$$

Case 3. Finally we consider the case $k \in \mathbb{N}_0$ with $n - N \le k \le n$, hence $n - |\theta| \le k \le n$. Then

$$S_k(t) = t^{n-(a+\beta)}B^{a+\beta}(t+B)^{-n}St^{n-(k-\beta)}A^{k-\beta}(t+A)^{-n} = \psi_{a+\beta,n}(t^{-1}B)S\psi_{k-\beta,n}(t^{-1}A).$$

We have $n > a + \beta > 0$, hence $\psi_{a+\beta,n} \in H_0^\infty(\Sigma_\omega)$, and

$$0 < n - \beta - |\theta| - \theta \le k - \beta - \theta \le n - (\beta - |\theta|) < n,$$

hence $\psi_{k-\beta,n} \in \Phi_{\theta,n}$. So for all $x \in X^\theta_{s,A}$ we have

$$\left\|(t^\theta S_k(t)x)_{t>0}\right\|_{X(L^s_*)} \lesssim_M \left\|(t^\theta \psi_{k-\beta,n}(t^{-1}A)x)_{t>0}\right\|_{X(L^s_*)} \approx \|x\|_{\theta,s,A}.$$

Now we put all cases together, then for all $x \in X^\theta_{s,A}$ we obtain by Proposition 3.1.13

$$\begin{aligned}
\|x\|_{\theta,s,B} &\approx \left\|\Big(\int_0^\infty |t^\theta \psi_{a,m}(t^{-1}B)x|^s \frac{dt}{t}\Big)^{1/s}\right\|_X = \left\|(t^\theta \psi_{a,m}(t^{-1}B)x)_{t>0}\right\|_{X(L^s_*)} \\
&= \left\|\Big(\sum_{k=0}^n \binom{n}{k} t^\theta S_k(t)x\Big)_{t>0}\right\|_{X(L^s_*)} \le \sum_{k=0}^n \binom{n}{k} \left\|\big(t^\theta S_k(t)x\big)_{t>0}\right\|_{X(L^s_*)} \\
&\lesssim_M \sum_{k=0}^n \binom{n}{k} \|x\|_{\theta,s,A}
\end{aligned}$$

(with the usual modifications if $s = +\infty$). □

Observe that in the proof of Lemma 3.4.2, the derivation of the estimates shows that the constants can be chosen independent of $\theta \in (-\beta, \alpha)$, since we only considered finitely many summands, and moreover can be estimated by a multiple of the constant M. Hence we obtain the following more detailed assertion about the norm of the embedding $\dot{X}^\theta_{s,A} \hookrightarrow \dot{X}^\theta_{s,B}$:

For all $\alpha, \beta, M > 0$ there is a constant $K > 0$ (only depending on the preceding quantities) with the following property: let A and B be $\mathcal{R}_s$-sectorial operators such that $D(A^\alpha) = D(B^\alpha)$ and $R(A^\beta) = R(B^\beta)$ and such that the operators $B^\alpha A^{-\alpha}$ and $B^{-\beta}A^\beta$ extend to $\mathcal{R}_s$-bounded operators on X, and

$$\max\{M_{s,\sigma}(A), M_{s,\sigma}(B), \mathcal{R}_s(B^\alpha A^{-\alpha}), \mathcal{R}_s(B^{-\beta}A^\beta)\} \leq M.$$

Then, for all $\theta \in (-\beta, \alpha), \omega \in (\omega_{\mathcal{R}_s}(A) \vee \omega_{\mathcal{R}_s}(B), \pi)$ and $\varphi \in \Phi_{\omega,\theta}$ there is a constant $C(\varphi)$ such that

$$\|x\|_{\theta,s,B,\varphi} \leq K \cdot C(\varphi) \cdot \|x\|_{\theta,s,A,\varphi} \quad \text{for all } x \in X^\theta_{s,A}.$$

Now Lemma 3.4.2 leads to the following comparison result.

Theorem 3.4.3. *Let A and B be $\mathcal{R}_s$-sectorial operators with $\sigma > \omega_{\mathcal{R}_s(A)} \vee \omega_{\mathcal{R}_s(B)}$ and assume that there are $\alpha_j, \beta_j > 0$ for $j = 1, 2$ such that*

(a) $D(A^{\alpha_j}) = D(B^{\alpha_j})$ and $R(A^{\beta_j}) = R(B^{\beta_j})$ for $j = 1, 2$,

(b) The operators $B^{\alpha_1}A^{-\alpha_1}, B^{-\beta_1}A^{\beta_1}$ and $A^{\alpha_2}B^{-\alpha_2}, A^{-\beta_2}B^{\beta_2}$ extend to $\mathcal{R}_s$-bounded operators on X.

Then

$$\dot{X}^\theta_{s,A} \cong \dot{X}^\theta_{s,B} \text{ for all } \theta \in (-(\beta_1 \wedge \beta_2), \alpha_1 \wedge \alpha_2). \tag{3.4.2}$$

In the same spirit as in the remarks to Lemma 3.4.2 and its proof we obtain in addition, that the norm equivalence constants do not depend on θ and the explicit operators A, B but only on the $\mathcal{R}_s$-sectoriality constants $M_{s,\sigma}(A)$ and $M_{s,\sigma}(B)$, the $\mathcal{R}_s$-norms of the operators in (b) and the auxiliary function φ that is used to determine the norms in the spaces $\dot{X}^\theta_{s,A}, \dot{X}^\theta_{s,B}$.

Proof of Theorem 3.4.3. This follows immediately from Lemma 3.4.2 by interchanging the roles of A and B. □

In view of the corresponding comparison theorem from [KKW06], Theorem 5.1, we also obtain a comparison result for an $\mathcal{R}_s$-bounded H^∞-calculus. We will apply [KKW06], Theorem 5.1 for the vector-valued extensions $\widetilde{A}_s, \widetilde{B}_s$ of A, B in the space $X(\ell^s)$, so we only have to ensure that these operators satisfy the assumptions of the latter theorem.

Theorem 3.4.4. *Let $s < +\infty$ and X be q-concave for some $q < +\infty$. Let A and B be $\mathcal{R}_s$-sectorial operators with $\sigma > \omega_{\mathcal{R}_s(A)} \vee \omega_{\mathcal{R}_s(B)}$ and assume that there are $\alpha_j, \beta_j > 0$ for $j = 1, 2$ such that*

(a) $D(A^{\alpha_j}) = D(B^{\alpha_j})$ and $R(A^{\beta_j}) = R(B^{\beta_j})$ for $j = 1, 2$,

(b) The operators $B^{\alpha_1}A^{-\alpha_1}, B^{-\beta_1}A^{\beta_1}$ and $A^{\alpha_2}B^{-\alpha_2}, A^{-\beta_2}B^{\beta_2}$ extend to $\mathcal{R}_s$-bounded operators on X.

Moreover assume that

(c) A has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_\sigma)$-calculus,

(d) The operator $\widetilde{B}_s$ is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(\widetilde{B}_s) \le \sigma$.

Then, for each $\sigma' > \sigma$ the operator B has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_{\sigma'})$-calculus. Moreover, for each $\omega > \sigma$ there is a constant $C_{\omega,\sigma}$ that only depends on ω, σ and the $\mathcal{R}_s$-norms of the operators in (b) such that the following estimate holds for all $\sigma' \ge \omega$:

$$M^\infty_{s,\sigma'}(B) \le C_{\omega,\sigma} \cdot M^\infty_{s,\sigma}(A) \cdot M_{\mathcal{R},\sigma}(\widetilde{B}_s). \tag{3.4.3}$$

Before we prove Theorem 3.4.4 let us have a closer look on the condition (d). Since B is $\mathcal{R}_s$-sectorial and $s < +\infty$, we already know that for each $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_\sigma$ the operator $R(\lambda, B) \otimes \mathrm{Id}_{\ell^s}$ has a bounded extension to the space $X(\ell^s)$. Moreover, a single bounded operator is always $\mathcal{R}$-bounded, hence in this setting $\mathcal{R}_2$-bounded, which means that we also obtain a bounded extension of the operator

$$(R(\lambda, B) \otimes \mathrm{Id}_{\ell^s}) \otimes \mathrm{Id}_{\ell^2} = R(\lambda, B) \otimes (\mathrm{Id}_{\ell^s} \otimes \mathrm{Id}_{\ell^2}) = R(\lambda, B) \otimes (\mathrm{Id}_{\ell^s \otimes \ell^2}) \subseteq R(\lambda, B) \otimes \mathrm{Id}_{\ell^s(\ell^2)}$$

in the space $X\ell^s(\ell^2) \cong X(\ell^s(\ell^2))$. So condition (d) just means that the set

$$\{\lambda\overline{R(\lambda, B) \otimes \mathrm{Id}_{\ell^s(\ell^2)}} \mid \lambda \in \mathbb{C}\backslash\Sigma_{\sigma'}\}$$

of tensor extensions of the operators $\lambda R(\lambda, B)$ is bounded in the space $X(\ell^s(\ell^2))$ for all $\sigma' > \sigma$.

Proof of Theorem 3.4.4. Let $\sigma' \ge \omega > \omega' > \omega'' > \sigma$. As announced before we consider the vector-valued extensions $\widetilde{A}_s, \widetilde{B}_s$ as operators in the space $X(\ell^s)$, then $\widetilde{A}_s, \widetilde{B}_s$ are sectorial operators with $\omega(\widetilde{A}_s), \omega(\widetilde{B}_s) < \omega''$, and $\widetilde{A}_s$ has an $H^\infty(\Sigma_{\omega''})$-calculus, cf. Proposition 3.2.23 and Lemma 3.2.21. Moreover, the space $X(\ell^s)$ is $q \vee s$-concave by Proposition 1.6.18, thus $X(\ell^s)$ has property (α) by Proposition 1.6.22. Hence $\widetilde{A}_s$ is also $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(\widetilde{A}_s) \le \omega_{H^\infty}(\widetilde{A}_s) \le \sigma$ by Corollary 1.3.6.

This shows that the conditions (a)–(d) are just the assumptions of the comparison theorem [KKW06], Theorem 5.1 for the operators $\widetilde{A}_s, \widetilde{B}_s$ (note that the restriction $|\alpha|, |\beta| < 3/2$ assumed there can be dropped by the same technique of proof as used in the proof of Lemma 3.4.2). Accordingly the operator $\widetilde{B}_s$ has a bounded $H^\infty(\Sigma_{\omega'})$-calculus in the space $X(\ell^s)$, hence B has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_{\sigma'})$-calculus in X by Proposition 3.2.23. Finally, the estimate (3.4.3) is a consequence of the proof of [KKW06], Theorem 5.1 and Proposition 3.2.23 and Corollary 1.3.6 (cf. also Section 1.3 for this kind of estimates). □

We now turn to perturbation theorems for $\mathcal{R}_s$-sectorial operators, where the main focus lies on the coincidence of s-intermediate spaces $X^\theta_{s,A}, \dot{X}^\theta_{s,A}$ of some $\mathcal{R}_s$-sectorial operator A with the s-intermediate spaces $X^\theta_{s,A+B}, \dot{X}^\theta_{s,A+B}$ of an additive perturbation of A under suitable conditions on B. They are extensions on corresponding perturbation theorems for the H^∞-calculus, cf. [KW04], Chapter 13 and [KKW06], Section 6. We will start with a perturbation result under rather weak assumptions, which in turn will only provide one inclusion for the s-intermediate spaces. Nevertheless, in this proposition we will already derive the representation (3.4.4) for resolvents of the perturbed operator, which is one central tool also in the subsequent perturbation theorems.

Proposition 3.4.5. *Let A be $\mathcal{R}_s$-sectorial with $\sigma > \omega_{\mathcal{R}_s}(A)$ and let $\delta := \pi - \sigma$ and $\pi > \omega > \sigma$. Define $\varepsilon := 1/(2M_{s,\sigma}(A))$, then there is a constant $C_{\omega,\sigma}$ only depending on σ, ω with the following property:*

Let $\alpha \in (0,1)$ and B be a linear operator in X such that

(a) $D(B) \supseteq D(A)$ and $\|\overline{BA^{-1}}\| \leq \varepsilon$,

(b) $B(D(A)) \subseteq R(A^{1-\alpha})$ and $L := \overline{A^{\alpha-1}BA^{-\alpha}}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(L) \leq \varepsilon$.

Then $A+B$ is again $\mathcal{R}_s$-sectorial with

$$(\lambda + A + B)^{-1} = (\lambda + A)^{-1} - A^{1-\alpha}(\lambda + A)^{-1}M(\lambda)A^{\alpha}(\lambda + A)^{-1}, \tag{3.4.4}$$

where $M(\lambda) := \sum_{k=0}^{\infty}(-LA(\lambda + A)^{-1})^k L \in L(X)$ for all $\lambda \in \Sigma_\delta$, and the set $\{M(\lambda) \,|\, \lambda \in \Sigma_\delta\}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\{M(\lambda) \,|\, \lambda \in \Sigma_\delta\}) < 2\varepsilon$. Moreover

$$\dot{X}^{\theta}_{s,A} \hookrightarrow \dot{X}^{\theta}_{s,A+B} \text{ for all } \theta \in (\alpha - 1, \alpha). \tag{3.4.5}$$

Again, the norm of the embedding map in (3.4.5) does not depend on the explicit operators A, B but only on ω, σ, α, the $\mathcal{R}_s$-sectoriality constant $M_{s,\sigma}(A)$ of A and the auxiliary function used to determine the norms in the spaces $\dot{X}^{\theta}_{s,A}, \dot{X}^{\theta}_{s,A+B}$. An analogous assertion will be true for all subsequent perturbation theorems in this section, hence we will not emphasize this fact in the sequel.

Proof of Proposition 3.4.5. We follow the lines of [KKW06] and will first derive a representation formula for the resolvents of $A+B$.

Let B be a linear operator in X having the stated properties (a) and (b). Then $\mathcal{R}_s(\{LA(\lambda + A)^{-1} \mid \lambda \in \Sigma_\delta\}) \leq \mathcal{R}_s(L)\, M_{s,\sigma}(A) \leq 1/2$. This implies that indeed

$$M(\lambda) = \sum_{k=0}^{\infty}(-LA(\lambda + A)^{-1})^k L \in L(X),$$

and the set $\{M(\lambda) \,|\, \lambda \in \Sigma_\delta\}$ is $\mathcal{R}_s$-bounded by Proposition 3.1.9 and Remark 3.1.7 with

$$\mathcal{R}_s(\{M(\lambda) \,|\, \lambda \in \Sigma_\delta\}) \leq \sum_{k=0}^{\infty}(1/2)^k \cdot \mathcal{R}_s(L) < 2\varepsilon.$$

For all $\lambda \in \Sigma_\delta$ we have

$$A^{1-\alpha}LA^{\alpha}(\lambda + A)^{-1} \supseteq A^{1-\alpha}A^{\alpha-1}BA^{-\alpha}A^{\alpha}(\lambda + A)^{-1} = A^{1-\alpha}A^{\alpha-1}B(\lambda + A)^{-1} = B(\lambda + A)^{-1},$$

hence $A^{1-\alpha}LA^{\alpha}(\lambda+A)^{-1} = B(\lambda+A)^{-1} \in L(X)$, since the operator on the right hand side is closed and defined on the whole space X. By the choice of $\varepsilon > 0$ we have the standard representation

$$(\lambda+A+B)^{-1} = (\lambda+A)^{-1} + \sum_{k=0}^{\infty}(\lambda+A)^{-1}(-B(\lambda+A)^{-1})^{k+1} \quad \text{for all } \lambda \in \Sigma_\delta, \tag{3.4.6}$$

cf. Proposition 3.4.1 and its proof. We claim that for all $\lambda \in \Sigma_\delta$ and $k \in \mathbb{N}_0$

$$(\lambda+A)^{-1}(-B(\lambda+A)^{-1})^{k+1} = -A^{1-\alpha}(\lambda+A)^{-1}(-LA(\lambda+A)^{-1})^k LA^{\alpha}(\lambda+A)^{-1}. \tag{3.4.7}$$

We prove (3.4.7) by induction. Let $\lambda \in \Sigma_\delta$. We have

$$(\lambda+A)^{-1}B(\lambda+A)^{-1} = (\lambda+A)^{-1}A^{1-\alpha}LA^{\alpha}(\lambda+A)^{-1} \supseteq A^{1-\alpha}(\lambda+A)^{-1}LA^{\alpha}(\lambda+A)^{-1} \in L(X),$$

hence $(\lambda+A)^{-1}\big(-B(\lambda+A)^{-1}\big) = -A^{1-\alpha}(\lambda+A)^{-1}LA^{\alpha}(\lambda+A)^{-1}$, which is the claim for $k=0$. Now assume that (3.4.7) holds for some $k \in \mathbb{N}_0$, then

$$\begin{aligned}
& (\lambda+A)^{-1}(-B(\lambda+A)^{-1})^{k+2} \\
= \;& -(\lambda+A)^{-1}(-B(\lambda+A)^{-1})^{k+1}B(\lambda+A)^{-1} \\
= \;& A^{1-\alpha}(\lambda+A)^{-1}(-LA(\lambda+A)^{-1})^k LA^{\alpha}((\lambda+A)^{-1}B(\lambda+A)^{-1}) \\
= \;& A^{1-\alpha}(\lambda+A)^{-1}(-LA(\lambda+A)^{-1})^k LA^{\alpha}A^{1-\alpha}(\lambda+A)^{-1}LA^{\alpha}(\lambda+A)^{-1} \\
= \;& A^{1-\alpha}(\lambda+A)^{-1}(-LA(\lambda+A)^{-1})^k LA^{1}(\lambda+A)^{-1}LA^{\alpha}(\lambda+A)^{-1} \\
= \;& -A^{1-\alpha}(\lambda+A)^{-1}(-LA(\lambda+A)^{-1})^{k+1}LA^{\alpha}(\lambda+A)^{-1},
\end{aligned}$$

and the claim is proved.

Plugging (3.4.7) into (3.4.6) yields

$$(\lambda+A+B)^{-1} = (\lambda+A)^{-1} - A^{1-\alpha}(\lambda+A)^{-1}M(\lambda)A^{\alpha}(\lambda+A)^{-1} \tag{3.4.8}$$

for all $\lambda \in \Sigma_\delta$. We define the auxiliary functions $\psi_{\nu,\beta}(z) := \frac{z^\beta}{\nu+z}$ and $\psi_\beta := \psi_{1,\beta}$ for all $\nu > 0, \beta \in (0,1)$. Then the representation

$$\begin{aligned}
\lambda(\lambda+A+B)^{-1} &= \lambda(\lambda+A)^{-1} - \lambda^{\alpha}A^{1-\alpha}(\lambda+A)^{-1}M(\lambda)\,\lambda^{1-\alpha}A^{\alpha}(\lambda+A)^{-1} \\
&= \lambda(\lambda+A)^{-1} - (\lambda^{-1}A)^{1-\alpha}(1+\lambda^{-1}A)^{-1}M(\lambda)\,(\lambda^{-1}A)^{\alpha}(1+\lambda^{-1}A)^{-1} \\
&= \lambda(\lambda+A)^{-1} - \psi_{1-\alpha}(\lambda^{-1}A)M(\lambda)\psi_{\alpha}(\lambda^{-1}A)
\end{aligned}$$

for all $\lambda \in \Sigma_\delta$ shows that $A+B$ is $\mathcal{R}_s$-sectorial with $\omega_{\mathcal{R}_s}(A+B) \leq \sigma$ and $M_{s,\sigma}(A+B) \leq C_{\alpha,\sigma}\, M_{s,\sigma}(A)$ for some constant $C_{\alpha,\sigma} > 0$.

Now let $\theta \in (\alpha-1,\alpha)$. We will use the representation (3.4.8) to estimate the s-power function-norms associated to A and $A+B$, respectively. For this we define

$$\psi(z) := \frac{1}{1+z} - \frac{2}{2+z} = \frac{-z}{(1+z)(2+z)}, \quad \text{then } \psi \in \Phi_{\sigma,\theta}.$$

Then

$$\begin{aligned}\psi(t^{-1}(A+B)) &= t(t+A+B)^{-1} - 2t(2t+A+B)^{-1}\\ &= t(t+A)^{-1} - \psi_{1-\alpha}(t^{-1}A)M(t)\psi_{1,\alpha}(t^{-1}A)\\ &\quad -\big(2t(2t+A)^{-1} - \psi_{1-\alpha}((2t)^{-1}A)M(2t)\psi_\alpha((2t)^{-1}A)\big)\\ &= \psi(t^{-1}A) - \psi_{1-\alpha}(t^{-1}A)M(t)\psi_{1,\alpha}(t^{-1}A) + 2\psi_{2,1-\alpha}(t^{-1}A)M(2t)\psi_{2,\alpha}(t^{-1}A).\end{aligned}$$

Since $\psi_{\nu,1-\alpha} \in H_0^\infty(\Sigma_\sigma)$ and $\psi_{\nu,1-\alpha} \in \Phi_{\sigma,\theta}$ this yields via Proposition 3.1.13

$$\begin{aligned}\|x\|_{\theta,s,A+B} &\approx \Big\|\Big(\int_0^\infty |t^\theta\psi(t^{-1}(A+B))x|^s \frac{dt}{t}\Big)^{1/s}\Big\|_X = \big\|(t^\theta\psi(t^{-1}(A+B))x)_{t>0}\big\|_{X(L^s_*)}\\ &\le \big\|\big(t^\theta\psi(t^{-1}A)x\big)_{t>0}\big\|_{X(L^s_*)} + 2\,\big\|\big(t^\theta\psi_{1-\alpha}(t^{-1}A)M(2t)\psi_{2,\alpha}(t^{-1}A)x\big)_{t>0}\big\|_{X(L^s_*)}\\ &\quad+\big\|\big(t^\theta\psi_{1-\alpha}(t^{-1}A)M(t)\psi_{1,\alpha}(t^{-1}A)x\big)_{t>0}\big\|_{X(L^s_*)}\\ &\le \big\|\big(t^\theta\psi(t^{-1}A)x\big)_{t>0}\big\|_{X(L^s_*)} + 2\,\big\|\big(t^\theta\psi_{2,\alpha}(t^{-1}A)x\big)_{t>0}\big\|_{X(L^s_*)}\\ &\quad+\big\|\big(t^\theta\psi_{1,\alpha}(t^{-1}A)x\big)_{t>0}\big\|_{X(L^s_*)}\\ &\approx \|x\|_{\theta,s,A}\end{aligned}$$

(with the usual modification if $s = +\infty$). □

We now turn to a more involved version of this perturbation theorem, where we assume X to be p-convex and q-concave, thus in particular reflexive, and $s \in (1,+\infty)$ to be in the reflexive range. In addition we assume that A has an $\mathcal{R}_s$-bounded H^∞-calculus. Under these assumptions we can show that we have not only a one-sided embedding but indeed coincidence of the s-intermediate spaces of A and $A+B$, and moreover also the operator $A+B$ has an $\mathcal{R}_s$-bounded H^∞-calculus.

Theorem 3.4.6. *Let $s \in (1,+\infty)$ and assume that X is p-convex and q-concave for some $p, q \in (1,+\infty)$. Let A be $\mathcal{R}_s$-sectorial with $\sigma > \omega_{\mathcal{R}_s}(A)$ and assume that A has in addition an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_\sigma)$-calculus. Let $\pi > \omega > \sigma$.*

Then there is an $\varepsilon > 0$ only depending on σ, ω and $M^\infty_{s,\sigma}(A)$ and a constant $C_{\omega,\sigma} > 0$ independent of A with the following property: If $\alpha \in (0,1)$ and B is a linear operator in X with the following properties:

(a) $D(B) \supseteq D(A)$ and $\|\overline{BA^{-1}}\| \le \varepsilon$,

(b) $B(D(A)) \subseteq R(A^{1-\alpha})$ and $\overline{A^{\alpha-1}BA^{-\alpha}}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\overline{A^{\alpha-1}BA^{-\alpha}}) \le \varepsilon$,

then the following assertions hold:

(1) $A+B$ has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_{\sigma'})$-calculus, and we have an estimate

$$M^\infty_{s,\sigma'}(A+B) \le C_{\omega,\sigma}\, M^\infty_{s,\sigma}(A) \tag{3.4.9}$$

for each $\sigma' \ge \omega$. Moreover

$$\dot X^\theta_{s,A} \cong \dot X^\theta_{s,A+B} \text{ for all } \theta \in (\alpha-1,\alpha). \tag{3.4.10}$$

(2) If in addition to (a),(b) the operator $\overline{BA^{-1}}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\overline{BA^{-1}}) < \varepsilon$, then we have

$$\dot{X}^\theta_{s,A} \cong \dot{X}^\theta_{s,A+B} \text{ for all } \theta \in (\alpha - 1, 1). \tag{3.4.11}$$

Before we turn to the proof of Theorem 3.4.6 we make some comments.

1. The conditions $q, s < +\infty$ will be used to ensure that the mixed space $X(\ell^s)$ is $q \vee s$-concave with $q \vee s < +\infty$, hence X has property (α), and $\mathcal{R}$-boundedness in $L(X)$ is equivalent to $\mathcal{R}_2$-boundedness. Since we will also work with the dual operator A' in the dual space X', we assume additionally $p, s > 1$ so that also $X'(\ell^{s'})$ is $p' \vee s'$-concave with $p' \vee s' < +\infty$.

2. Observe that the domain and range conditions in (a),(b) ensure that the occurring operators are densely defined, so indeed these operators extend to ($\mathcal{R}_s$-)bounded operators if and only if their closures are ($\mathcal{R}_s$-)bounded operators.

3. Note that in the assumption (a) we require the operator $\overline{BA^{-1}}$ only to be bounded (with small norm) and do not need $\mathcal{R}_s$-boundedness, as one might conjecture. This is due to the fact that the crucial estimates depend only on the representation (3.4.4) of the resolvent, where (beside functions of A) only the operator $\overline{A^{\alpha-1}BA^{-\alpha}}$ occurs instead of $\overline{BA^{-1}}$. Nevertheless, if we assume $\overline{BA^{-1}}$ to be $\mathcal{R}_s$-bounded with sufficiently small norm, then we obtain an improvement for the upper bound of the range of indices θ for which $\dot{X}^\theta_{s,A} \cong \dot{X}^\theta_{s,A+B}$ holds.

4. We will split up the proof and put the main work into the separated Lemma 3.4.7 below, since we will use this part of the proof also in Theorem 3.4.8, which is a variant of this perturbation theorem.

Proof of Theorem 3.4.6. Let $\delta_0 := \pi - \sigma$ and $L := \overline{A^{\alpha-1}BA^{-\alpha}}$. We will choose $0 < \varepsilon < 1/(2M_{s,\sigma}(A))$, then from Proposition 3.4.5 we obtain the representation

$$(\lambda + A + B)^{-1} = (\lambda + A)^{-1} - A^{1-\alpha}(\lambda + A)^{-1} M(\lambda) A^\alpha (\lambda + A)^{-1}, \tag{3.4.12}$$

where $M(\lambda) := \sum_{k=0}^\infty (-LA(\lambda + A)^{-1})^k L \in L(X)$ for all $\lambda \in \Sigma_{\delta_0}$, and that the set $\{M(\lambda) \,|\, \lambda \in \Sigma_{\delta_0}\}$ is $\mathcal{R}_s$-bounded with $\mathcal{R}_s(\{M(\lambda) \,|\, \lambda \in \Sigma_{\delta_0}\}) \leq 1/M_{s,\sigma}(A)$.

Let $\omega' := \sigma + \frac{1}{3}(\sigma - \omega)$ and $\omega'' := \sigma + \frac{2}{3}(\sigma - \omega)$, then $\sigma < \omega' < \omega'' < \omega$. Observe that by Definition/Proposition 3.2.2 and Proposition 3.2.23 the diagonal operator $\widetilde{A}_s : X(\ell^s) \supseteq D(\widetilde{A}_s) \to X(\ell^s), (y_j)_j \mapsto (Ay_j)_j$ is sectorial with $\omega(\widetilde{A}_s) \leq \sigma$ and has a bounded $H^\infty(\Sigma_{\omega'})$-calculus. Moreover, the mixed Banach function space $X(\ell^s)$ is $q \vee s$-concave and $q \vee s < +\infty$, hence $X(\ell^s)$ has property (α) by Proposition 1.6.22. By Corollary 1.3.6 the diagonal operator $\widetilde{A}_s$ is also R-sectorial with $\omega_{\mathcal{R}}(\widetilde{A}_s) \leq \omega'$ and we can choose a constant $C_{\omega,\sigma} \geq 1$ (independent of A) such that the following estimate holds for each $\sigma' \geq \omega$:

$$M_{\mathcal{R},\sigma'}(\widetilde{A}_s) \quad \leq \quad C_{\omega,\sigma} M^\infty_{s,\sigma}(A). \tag{3.4.13}$$

Let $\delta := \pi - \omega < \delta_0$. By choosing $\varepsilon := 1/2C_{\omega,\sigma}(M^\infty_{s,\sigma}(A) \vee M_{s,\sigma}(A))$ we obtain that the operator set $\{\widetilde{M(\lambda)}_s \mid \lambda \in \Sigma_\delta\}$ is also $\mathcal{R}$-bounded in $X(\ell^s)$ with

$$\mathcal{R}(\{\widetilde{M(\lambda)}_s \mid \lambda \in \Sigma_\delta\}) \leq \sum_{k=0}^{\infty}(1/2)^k \cdot \mathcal{R}(\widetilde{L}_s) = 2\mathcal{R}_s(L) < 2\varepsilon \leq 1/C_{\omega,\sigma}M^\infty_{s,\sigma}(A).$$

Together with the representation formula (3.4.12) for the resolvent of $A+B$, the assumptions of the following Lemma 3.4.7 are fulfilled with $C = A + B$, hence (1) follows from Lemma 3.4.7.

If the additional assumptions of (2) are fulfilled we let $K := \overline{BA^{-1}}$. Then

$$(A+B)A^{-1} = AA^{-1} + BA^{-1} \subseteq I + K,$$

hence $\overline{(A+B)^1A^{-1}}$ is $\mathcal{R}_s$-bounded. Moreover $I + K$ is invertible with

$$A(A+B)^{-1} \subseteq (I+K)^{-1} = \sum_{k=0}^{\infty}(-K)^k,$$

hence also $\overline{A^1(A+B)^{-1}}$ is $\mathcal{R}_s$-bounded. Combining this with (3.4.15) from the following Lemma 3.4.7 with $C = A + B$, the assertion of (2) follows by the Comparison Theorem 3.4.3. □

Before we go further, we have a closer look on the assertion on $\varepsilon > 0$ in connection with the preceding proof of Theorem 3.4.6. At first glance it might be surprising that we cannot choose an $\varepsilon > 0$ *independent* of $\omega > \sigma$ such that $A + B$ has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_{\mathcal{R}_s^\infty}(A + B) \leq \sigma$. This is due to the fact that we needed the $\mathcal{R}$-sectoriality of the operator $\widetilde{A}_s$ with $\omega_{\mathcal{R}}(\widetilde{A}_s) < \omega$, which from the assumptions we could only derive if $\omega > \sigma$, and then the constant $C_{\omega,\sigma}$ from (3.4.13) enters into the definition of ε. Hence, if we make the following additional assumption on A:

The diagonal operator $\widetilde{A}_s$ in $X(\ell^s)$ is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(\widetilde{A}_s) < \sigma$,

then the proof shows that we can choose $\varepsilon := \frac{1}{2}M_{\mathcal{R},\sigma}(\widetilde{A}_s)^{-1}$ independently of $\omega > \sigma$ and obtain that $A + B$ has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_{\mathcal{R}_s^\infty}(A + B) \leq \sigma$, if B satisfies conditions (a),(b) from Theorem 3.4.6.

We will now turn to the announced lemma that will conclude the proof of Theorem 3.4.6. As already mentioned we follow the line of proof from [KW04] and [KKW06], and the following lemma is a variant of [KW04] Lemma 13.6 and the corresponding Lemma 6.2 from [KKW06], that is adapted to our setting.

Lemma 3.4.7. *Let $s \in (1, +\infty)$ and X be p-convex and q-concave for some $p, q \in (1, +\infty)$ and $\alpha \in (0, 1)$, and assume that A has an $\mathcal{R}_s$-bounded H^∞-calculus. Let $\delta \in (0, \pi - \omega_{\mathcal{R}_s^\infty}(A))$ and let C be an $\mathcal{R}_s$-sectorial operator in X, and assume there is a family $(M(\lambda))_{\lambda\in\Sigma_\delta} \in L(X)^{\Sigma_\delta}$ such that $M(\Sigma_\delta)$ is $\mathcal{R}_s$-bounded and we have a representation*

$$(\lambda + C)^{-1} = (\lambda + A)^{-1} - A^{1-\alpha}(\lambda + A)^{-1}M(\lambda)A^\alpha(\lambda + A)^{-1} \quad \textit{for all } \lambda \in \Sigma_\delta. \qquad (3.4.14)$$

Finally assume that the family $(\widetilde{M(\lambda)}_s)_{\lambda\in\Sigma_\delta}$ *of the extended diagonal operators* $\widetilde{M(\lambda)}_s$ *in the space* $X(\ell^s)$ *is* $\mathcal{R}$*-bounded as a family of operators in* $X(\ell^s)$*. Then the following statements hold:*

$$D(C^\theta) = D(A^\theta), \text{ and } A^\theta C^{-\theta} \text{ and } C^\theta A^{-\theta} \text{ are } \mathcal{R}_s\text{-bounded for all } \theta \in (\alpha-1,\alpha). \tag{3.4.15}$$

Moreover, C *has an* $\mathcal{R}_s$*-bounded* $H^\infty(\Sigma_\sigma)$*-calculus, where* $\sigma := \pi-\delta$*, and*

$$\dot{X}^\theta_{s,A} \cong \dot{X}^\theta_{s,C} \text{ for all } \theta \in (\alpha-1,\alpha). \tag{3.4.16}$$

Furthermore, for all $\omega > \sigma$ *there is a constant* $C_{\omega,\sigma} > 0$ *independent of* A, C *such that for all* $\sigma' \geq \omega$ *we have an estimate*

$$M^\infty_{s,\sigma'}(C) \leq C_{\omega,\sigma}\, \mathcal{R}(\{\widetilde{M(\lambda)}_s \mid \lambda\in\Sigma_\delta\}) \cdot M^\infty_{s,\sigma}(A)^2. \tag{3.4.17}$$

Proof. To have suitable representation formulas for the fractional powers, we break up the claim (3.4.15) into four parts:

(1) $D(C^\theta) \subseteq D(A^\theta)$ and $A^\theta C^{-\theta}$ is $\mathcal{R}_s$-bounded for all $\theta\in(0,\alpha)$,

(1)' $R(A^\theta) \supseteq R(C^\theta)$ and $A^{-\theta}C^\theta$ is $\mathcal{R}_s$-bounded for all $\theta\in(0,1-\alpha)$.

(2) $D(C^\theta) \supseteq D(A^\theta)$ and $C^\theta A^{-\theta}$ is $\mathcal{R}_s$-bounded for all $\theta\in(0,\alpha)$,

(2)' $R(A^\theta) \subseteq R(C^\theta)$ and $C^{-\theta}A^\theta$ is $\mathcal{R}_s$-bounded for all $\theta\in(0,1-\alpha)$.

We will start by showing (1) in all details, so let $\theta\in(0,\alpha)$. Let $x\in R(C)$ and $x'\in D(A')\subseteq D\big((A')^\theta\big)$. Observe that $R(C)$ is a core for $C^{-\theta}$ and $D(A')$ is a core for $(A')^\theta$, respectively. Observe that X is reflexive by Proposition 1.6.16, so we can use duality methods, where as usual we identify the dual space X' with the associated space of X, cf. Subsection 1.6.2.

By the Balakrishnan representation formula for fractional powers (cf. e.g. [MS01], Section 7.2 or [Ha06], Proposition 3.2.1 d)) we have

$$C^{-\theta}x = \underbrace{\frac{\sin(\pi\theta)}{\pi}}_{=:c_\theta} \int_0^\infty t^{-\theta}(t+C)^{-1}x\,dt,$$

hence

$$\begin{aligned}\langle C^{-\theta}x, (A^\theta)'x'\rangle_\Omega &= \frac{\sin(\pi\theta)}{\pi}\int_0^\infty t^{-\theta}\,\langle (t+C)^{-1}x, (A^\theta)'x'\rangle_\Omega\,dt \\ &= \lim_{r\to 0}\lim_{R\to\infty} \underbrace{c_\theta \int_r^R t^{-\theta}\,\langle (t+C)^{-1}x, (A^\theta)'x'\rangle_\Omega\,dt}_{=:I^\theta_{r,R}(x,x')}\,.\end{aligned}$$

Let $0 < r \le R < +\infty$, then

$$\begin{aligned}
I^{\theta}_{r,R}(x,x') &= c_\theta \int_r^R t^{-\theta}\, \langle (t+A)^{-1}x, (A^\theta)'x'\rangle_\Omega\, dt \\
&\quad -c_\theta \int_r^R t^{-\theta}\, \langle A^{1-(\alpha-\theta)}(t+A)^{-1}M(t)A^\alpha(t+A)^{-1}x, x'\rangle_\Omega\, dt \\
&= \left\langle c_\theta \int_r^R t^{-\theta}\, (t+A)^{-1}x\, dt, (A^\theta)'x' \right\rangle_\Omega \\
&\quad -c_\theta \int_r^R t^{-\theta}\, \langle M(t)A^\alpha(t+A)^{-1}x, (A')^{1-(\alpha-\theta)}(t+A')^{-1}x'\rangle_\Omega\, dt \\
&= \left\langle c_\theta \int_r^R t^{-\theta}\, (t+A)^{-1}x\, dt, (A^\theta)'x' \right\rangle_\Omega - c_\theta \int_r^R \langle M(t)\psi_\alpha(t^{-1}A)x, \psi_\beta(t^{-1}A')x'\rangle_\Omega \,\frac{dt}{t}
\end{aligned}$$

where $\beta := 1-(\alpha-\theta) \in (0,1)$ since $\theta \in (\alpha-1,\alpha)$. Now let $n \in \mathbb{N}$, $(x_j)_{j\in\mathbb{N}_{\le n}} \in R(C)^n$ and $(x'_j)_{j\in\mathbb{N}_{\le n}} \in D(A')^n$, then

$$\begin{aligned}
\sum_{j=1}^n I^{\theta}_{r,R}(x_j,x'_j) &= \sum_{j=1}^n \left\langle c_\theta \int_r^R t^{-\theta}\, (t+A)^{-1}x_j\, dt, (A^\theta)'x'_j \right\rangle_\Omega \\
&\quad -c_\theta \sum_{j=1}^n \int_r^R \langle M(t)\psi_\alpha(t^{-1}A)x_j, \psi_\beta(t^{-1}A')x'_j\rangle_\Omega \,\frac{dt}{t}.
\end{aligned}$$

We will estimate the absolute value of the latter integral and show in particular that the limit for $r \to 0, R \to \infty$ exists; this will imply that also the first integral converges $r \to 0, R \to \infty$ and becomes

$$\sum_{j=1}^n \left\langle c_\theta \int_0^\infty t^{-\theta}\, (t+A)^{-1}x_j\, dt, (A^\theta)'x'_j \right\rangle_\Omega = \sum_{j=1}^n \langle A^{-\theta}x_j, (A^\theta)'x'_j\rangle_\Omega = \sum_{j=1}^n \langle x_j, x'_j\rangle_\Omega.$$

To handle the latter term we use Hölder inequality in the spaces $X(\ell^s_n(L^2_*(r,R)))$, $X'(\ell^{s'}_n(L^2_*(r,R)))$, cf. Proposition 1.6.9, and obtain

$$\begin{aligned}
&\left| \sum_{j=1}^n \int_r^R \langle M(t)\psi_\alpha(t^{-1}A)x_j, \psi_\beta(t^{-1}A')x'_j\rangle_\Omega \,\frac{dt}{t} \right| \\
\le\;& \sum_{j=1}^n \int_r^R \int_\Omega |M(t^{-1})\psi_\alpha(tA)x_j|\, |\psi_\beta(tA')x'_j|\, d\mu \,\frac{dt}{t} \\
=\;& \int_\Omega \sum_{j=1}^n \int_r^R |M(t^{-1})\psi_\alpha(tA)x_j|\, |\psi_\beta(tA')x'_j| \,\frac{dt}{t}\, d\mu \\
\le\;& \left\| \left(\int_r^R |M(t^{-1})\psi_\alpha(tA)x_j|^2 \,\frac{dt}{t}\right)^{1/2} \right\|_{X(\ell^s_n)} \cdot \left\| \left(\int_r^R |\psi_\beta(tA')x'_j|^2 \,\frac{dt}{t}\right)^{1/2} \right\|_{X'(\ell^{s'}_n)}.
\end{aligned}$$

We now use that by our assumption the family $(\widetilde{M(\lambda)}_s)_{\lambda\in\Sigma_\delta}$ is $\mathcal{R}$-bounded, which in this situation is equivalent to $\mathcal{R}_2$-boundedness, cf. Remark 3.1.7. Moreover, the diagonal operator $\widetilde{A}_{s,n}$: $X(\ell^s_n) \supseteq D(A)^n \to X(\ell^s_n), (y_j)_j \mapsto (Ay_j)_j$ has a bounded H^∞-calculus by Proposition 3.2.23, hence the norm in $X(\ell_n^s)$ is equivalent to square function norms associated with the operator $\widetilde{A}_{s,n}$, where the equivalence constants can be estimated in terms of $M_{s,\sigma}^\infty(A)$, cf. Subsection 1.6.5. Hence we obtain

$$\begin{aligned}
&\Big\| \Big(\int_r^R |M(t^{-1})\psi_\alpha(tA)x_j|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{X(\ell_n^s)} \\
\leq\ & \mathcal{R}_2\{\overline{M(t)\otimes \mathrm{Id}_{\ell_n^s}} \mid t>0\} \cdot \Big\| \Big(\int_0^\infty |\psi_\alpha(tA)x_j|^2 \frac{dt}{t} \Big)^{1/2} \Big\|_{X(\ell_n^s)} \\
\lesssim\ & \mathcal{R}\{\widetilde{M(\lambda)}_s \mid \lambda\in\Sigma_\delta\} \cdot M_{s,\sigma}^\infty(A) \cdot \|(x_j)_j\|_{X(\ell_n^s)}.
\end{aligned}$$

We can apply the same arguments to the second factor: By our assumptions we have $s' < +\infty$, and X' is p'-concave with $p' < +\infty$ by Proposition 1.6.14, hence A' inherits by dualization the corresponding properties of A, i.e. it has an $\mathcal{R}_{s'}$-bounded H^∞-calculus in $X'(\ell_n^{s'})$. Hence we obtain

$$\begin{aligned}
&\sum_{j=1}^n \int_r^R \big|\langle M(t)\psi_\alpha(t^{-1}A)x_j, \psi_\beta(t^{-1}A')x_j'\rangle_\Omega\big| \frac{dt}{t} \\
\lesssim\ & \mathcal{R}\{\widetilde{M(\lambda)}_s \mid \lambda\in\Sigma_\delta\} \cdot M_{s,\sigma}^\infty(A)^2 \cdot \|(x_j)_j\|_{X(\ell_n^s)} \cdot \|(x_j')_j\|_{X'(\ell_n^{s'})}.
\end{aligned}$$

As announced before we can now conclude that the limits for $r\to 0, R\to\infty$ exist, and putting all together yields

$$\sum_{j=1}^n \big|\langle C^{-\theta}x_j, (A^\theta)'x_j'\rangle_\Omega\big| \ \lesssim\ \mathcal{R}\{\widetilde{M(\lambda)}_s \mid \lambda\in\Sigma_\delta\} \cdot M_{s,\sigma}^\infty(A)^2 \cdot \|(x_j)_j\|_{X(\ell_n^s)} \cdot \|(x_j')_j\|_{X'(\ell_n^{s'})}.$$

By duality this provides $C^{-\theta}x_j \in D(A^\theta)$, and we have the estimate

$$\begin{aligned}
\|(A^\theta C^{-\theta}x_j)_j\|_{X(\ell_n^s)} &= \sup\Big\{ \Big| \Big\langle C^{-\theta}(x_j)_j, (A^\theta)'(x_j')_j \Big\rangle_{X(\ell_n^s)\times X'(\ell_n^{s'})} \Big| \ :\ \|(x_j')_j\|_{X'(\ell_n^{s'})} \leq 1 \Big\} \\
&\lesssim \mathcal{R}\{\widetilde{M(\lambda)}_s \mid \lambda\in\Sigma_\delta\} \cdot M_{s,\sigma}^\infty(A)^2 \cdot \|(x_j)_j\|_{X(\ell_n^s)}
\end{aligned}$$

uniformly in $n\in\mathbb{N}$, hence the operator $A^\theta C^{-\theta}$ extends to an $\mathcal{R}_s$-bounded operator on X.

The other cases can be treated similar, so we will only give short sketches: For the proof of (1)' let $\theta\in(0,1-\alpha)$ and $x\in D(C)$, then another variant of the Balakrishnan representation formula for fractional powers (cf. e.g. [MS01], Theorem 5.2.1 or [Ha06], Proposition 3.1.12) yields

$$\begin{aligned}
C^\theta x &= c_\theta \int_0^\infty t^\theta C(t+C)^{-1}x \frac{dt}{t} = c_\theta \int_0^\infty t^\theta (I - t(t+C)^{-1})x \frac{dt}{t} \\
&= \lim_{r\to 0, R\to\infty} \Big(c_\theta \int_r^R t^\theta x \frac{dt}{t} - c_\theta \int_r^R t^\theta t(t+C)^{-1}x \frac{dt}{t} \Big)
\end{aligned}$$

hence for all $x' \in R((A')^\theta)$

$$\begin{aligned}
\langle C^\theta x, (A^{-\theta})'x'\rangle &= \lim_{\substack{r\to 0\\ R\to\infty}} \left(c_\theta \int_{1/R}^R \langle t^\theta x, (A^{-\theta})'x'\rangle \frac{dt}{t} - c_\theta \int_r^R t^\theta \langle t(t+C)^{-1}x, (A^{-\theta})'x'\rangle \frac{dt}{t}\right)\\
&= \lim_{\substack{r\to 0\\ R\to\infty}} \left(c_\theta \int_r^R \langle t^\theta x, (A^{-\theta})'x'\rangle \frac{dt}{t} - c_\theta \int_r^R t^\theta \langle t(t+A)^{-1}x, (A^{-\theta})'x'\rangle \frac{dt}{t}\right.\\
&\qquad \left. + c_\theta \int_r^R t^{1+\theta} \langle A^{-\theta}A^{1-\alpha}(t+A)^{-1}M(t)A^\alpha(t+A)^{-1}x, x'\rangle \frac{dt}{t}\right)\\
&= \lim_{\substack{r\to 0\\ R\to\infty}} \left(c_\theta \int_r^R t^\theta \langle A(t+A)^{-1}x, (A^{-\theta})'x'\rangle \frac{dt}{t}\right.\\
&\qquad \left. + c_\theta \int_r^R t^{1+\theta} \langle M(t)A^\alpha(t+A)^{-1}x, (A')^{1-\alpha-\theta}(t+A')^{-1}x'\rangle \frac{dt}{t}\right)\\
&= \lim_{\substack{r\to 0\\ R\to\infty}} \left(c_\theta \int_r^R t^\theta \langle A(t+A)^{-1}x, (A^{-\theta})'x'\rangle \frac{dt}{t}\right.\\
&\qquad \left. + c_\theta \int_r^R \langle M(t)\psi_\alpha(t^{-1}A)x, \psi_\beta(t^{-1}A')x'\rangle \frac{dt}{t}\right),
\end{aligned}$$

where $\beta := 1-\alpha-\theta \in (0,1)$ since $\theta \in (0, 1-\alpha)$. From now on we can proceed as in the proof of (1) and obtain that the operator $A^{-\theta}C^\theta$ extends to an $\mathcal{R}_s$-bounded operator on X.

We now turn to equations (2),(2)', where we just use the same arguments as above for the dual operators. Observe that (3.4.14) implies

$$(\lambda + C')^{-1} = (\lambda + A')^{-1} - (A')^\alpha(\lambda + A')^{-1}M(\lambda)'(A')^{1-\alpha}(\lambda + A')^{-1} \quad \text{for all } \lambda \in \Sigma_\delta. \tag{3.4.18}$$

For (2)' let $\theta \in (0, 1-\alpha)$. Let $x \in D(A)$ and $x' \in R\big((C')^\theta\big)$, then again by the Balakrishnan representation formula for fractional powers we have

$$(C')^{-\theta}x' = c_\theta, \int_0^\infty t^{-\theta}(t+C')^{-1}x'\,dt,$$

hence

$$\langle A^\theta x, (C')^{-\theta}x'\rangle_\Omega = \lim_{r\to 0}\lim_{R\to\infty} \underbrace{c_\theta \int_r^R t^{-\theta}\, \langle A^\theta x, (t+C')^{-1}x'\rangle_\Omega \, dt}_{=:I^\theta_{r,R}(x,x')},$$

and for $0 < r \le R < +\infty$

$$\begin{aligned} I^\theta_{r,R}(x,x') &= c_\theta \int_r^R t^{-\theta}\,\langle A^\theta x, (t+A')^{-1}x'\rangle_\Omega\,dt \\ &\quad - c_\theta \int_r^R t^{-\theta}\,\langle x, (A')^{\alpha+\theta}(t+A')^{-1}M(t)'(A')^{1-\alpha}(t+A')^{-1}x'\rangle_\Omega\,dt \\ &= \left\langle A^\theta x\,, c_\theta \int_r^R t^{-\theta}(t+A')^{-1}x'\right\rangle_\Omega \\ &\quad - c_\theta \int_r^R t^{-\theta}\,\langle A^{\alpha+\theta}(t+A)^{-1}x, M(t)'(A')^{1-\alpha}(t+A')^{-1}x'\rangle_\Omega\,dt \\ &= \left\langle A^\theta x\,, c_\theta \int_r^R t^{-\theta}(t+A')^{-1}x'\right\rangle_\Omega - c_\theta \int_r^R \langle \psi_\beta(t^{-1}A)x, \psi_{1-\alpha}(t^{-1}A')x'\rangle_\Omega \,\frac{dt}{t} \end{aligned}$$

where $\beta := \alpha+\theta \in (0,1)$ since $\theta \in (0, 1-\alpha)$ and we can proceed as in the proof of (1). Finally, in the same manner (2) can be proved analogously to (1)' using again the dual resolvents (3.4.18).

For the claim concerning the $\mathcal{R}_s$-bounded H^∞-calculus of the operator C, we can argue in the same way as in the proof of Theorem 3.4.4:

We consider again the vector-valued extensions $\widetilde{A}_s, \widetilde{C}_s$ as operators in the space $X(\ell^s)$, then $\widetilde{A}_s$ is a sectorial operator having an $H^\infty(\Sigma_\sigma)$-calculus, and $\widetilde{C}_s$ is a perturbation of the operator $\widetilde{A}_s$ in the sense of Lemma 6.2 from [KKW06] or Lemma 13.6 from [KW04], respectively. This implies that the operator $\widetilde{C}_s$ in the space $X(\ell^s)$ has an $H^\infty(\Sigma_{\sigma'})$-calculus for each $\sigma' > \sigma$, i.e. C has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_{\sigma'})$-calculus for all $\sigma' > \sigma$ by Proposition 3.2.23. The norm estimate (3.4.17) follows from a careful inspection of the proof of [KKW06] or Lemma 6.2.

□

We now present a variant of Theorem 3.4.6 where the perturbation operator need not be an operator in X but is an operator $B : \dot{X}_\alpha \to \dot{X}_{\alpha-1}$. This situation appears e.g. in perturbation of boundary conditions of differential operators, or when differential operators in divergence form are considered, as we will do it in Subsection 3.6.2. A first version of this kind of theorem in the context of maximal regularity can be found in [KW01-b], Theorem 8. In that paper examples are given how to apply this theorem to perturbation of boundary conditions of differential operators. An extended version that deals with perturbation of the H^∞-calculus is [KKW06], Theorem 6.6. In fact, we will intensively use the ideas and proofs given in [KKW06].

Recall that the fractional spaces $\dot{X}_\alpha$ and the (universal) extrapolated operator A have been introduced in Subsection 3.3.1. In particular, the operator A^α acts as an isometry $\dot{X}_\alpha \to X$ and $A^{-\alpha}$ acts as an isometry $X \to \dot{X}_\alpha$ for $\alpha > 0$.

Theorem 3.4.8. *Let $s \in (1,+\infty)$ and X be p-convex and q-concave for some $p, q \in (1,+\infty)$, and assume that A has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_\sigma)$-calculus. Let $\alpha \in (0,1)$ and $\pi > \omega > \sigma$. Then there is a constant $C_{\omega,\sigma} > 0$ independent of A, and an $\varepsilon > 0$ only depending on ω, σ and $M^\infty_{s,\sigma}(A)$ such that if $B : \dot{X}_\alpha \to \dot{X}_{\alpha-1}$ is a bounded linear operator and*

$$L := A^{\alpha-1}BA^{-\alpha} \text{ is an } \mathcal{R}_s\text{-bounded operator on } X \text{ with } \mathcal{R}_s(L) < \varepsilon, \tag{3.4.19}$$

then there exists a unique sectorial operator C in X whose resolvents are consistent with those of $\dot{A}_{\alpha-1} + B$ in $\dot{X}_{\alpha-1}$. Moreover, the following assertions hold:

(1) The operator C has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_{\sigma'})$-calculus for all $\sigma' \geq \omega$, and

$$M^\infty_{s,\sigma'}(C) \leq C_{\omega,\sigma}\, M^\infty_{s,\sigma}(A). \tag{3.4.20}$$

(2) $\dot{X}^\theta_{s,A} \cong \dot{X}^\theta_{s,C}$ for all $\theta \in (\alpha-1,\alpha)$.

Remark 3.4.9. *Again, if we make the additional assumption that*

the diagonal operator $\widetilde{A}_s$ in $X(\ell^s)$ is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(\widetilde{A}_s) < \sigma$,

we can choose $\varepsilon > 0$ (not depending on $\omega > \sigma$) such that C in Theorem 3.4.8 has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_{\mathcal{R}_s^\infty}(C) \leq \sigma$ if B satisfies the assumptions of Theorem 3.4.8. □

Proof of Theorem 3.4.8. This can be deduced from the corresponding theorem and the line of proof in [KKW06], Theorem 6.6, using similar arguments as in the proof of Theorem 3.4.6. In a first step, by the same arguments as given in the proof of Theorem 3.4.6 we can conclude that the diagonal operator $\widetilde{A}_s$ is $\mathcal{R}$-sectorial, and in particular also A is $\mathcal{R}$-sectorial. Let $\omega' \in (\sigma,\omega)$ and $\delta' := \pi - \omega'$. Then the assumptions of [KKW06], Theorem 6.6 are fulfilled, so the proof of [KKW06], Theorem 6.6 implies the existence of a unique sectorial operator C in X whose resolvents are consistent with those of $\dot{A}_{\alpha-1} + B$ in $\dot{X}_{\alpha-1}$ with

$$(\lambda + C)^{-1} = (\lambda + A)^{-1} - A^{1-\alpha}(\lambda + A)^{-1} M(\lambda) A^\alpha (\lambda + A)^{-1} \quad \text{for all } \lambda \in \Sigma_{\delta'},$$

where $M(\lambda) := \sum_{k=0}^\infty (-LA(\lambda + A)^{-1})^k L \in L(X)$. Choosing $\varepsilon > 0$ as in the proof of Theorem 3.4.6 yields that $M(\Sigma_{\delta'})$ is $\mathcal{R}_s$-bounded and moreover the set of diagonal operator extensions $\{\widetilde{M(\lambda)}_s \mid \lambda \in \Sigma_{\delta'}\}$ in the space $X(\ell^s)$ is $\mathcal{R}$-bounded with $\mathcal{R}(\{\widetilde{M(\lambda)}_s \mid \lambda \in \Sigma_{\delta'}\}) \lesssim_\omega M^\infty_{s,\sigma}(A)$. Hence all assertions of Theorem 3.4.8 follow from Lemma 3.4.7. □

Finally we turn to a version of the Perturbation Theorem 3.4.6 that will also give norm equivalence for higher order associated spaces. This is the central theorem we will use in the application on differential operators in non-divergence form in Subsection 3.6.1. In fact, the line of proof gives the idea that under similar assumptions as in Theorem 3.4.10 we can also obtain norm equivalence of the associated spaces for arbitrary order. Nevertheless we will only consider perturbation and corresponding norm-equivalences up to the order 2, this will be sufficient for our application in Subsection 3.6.1 to differential operators.

This theorem is a combination of the corresponding theorem for perturbation of the H^∞-calculus in [KKW06], Theorem 6.1 (as already worked with earlier), and [DDHPV04], Theorem 3.2. In fact, we will use a representation formula for resolvents of the perturbed operator that goes back to Jan Prüß and is presented in the proof of [DDHPV04], Theorem 3.2.

Theorem 3.4.10. *Let $s \in (1,+\infty)$ and X be p-convex and q-concave for some $p,q \in (1,+\infty)$, and assume that A has an $\mathcal{R}_s$-bounded H^∞-calculus. Let $\alpha,\beta \in (0,1)$ and $\pi > \omega > \sigma > \omega_{\mathcal{R}_s^\infty}(A)$. Then there is a constant $C_{\omega,\sigma} > 0$ independent of A and an $\varepsilon > 0$ only depending on ω,σ and $M^\infty_{s,\sigma}(A)$ such that if B is a linear operator in X with the following properties:*

(a) $D(B) \supseteq D(A)$ *and* $\mathcal{R}_s(\overline{BA^{-1}}) \leq \varepsilon$,

(b) $B(D(A)) \subseteq R(A^\beta)$, *and* $\overline{A^{-\beta}BA^{\beta-1}}$ *is* $\mathcal{R}_s$*-bounded with* $\mathcal{R}_s(\overline{A^{-\beta}BA^{\beta-1}}) \leq \varepsilon$,

(c) $B(D(A^2)) \subseteq D(A^\alpha)$, *and* $\overline{A^\alpha BA^{-1-\alpha}}$ *is* $\mathcal{R}_s$*-bounded with* $\mathcal{R}_s(\overline{A^\alpha BA^{-1-\alpha}}) \leq \varepsilon$,

then the operator $A+B$ *has an* $\mathcal{R}_s$*-bounded* $H^\infty(\Sigma_{\sigma'})$*-calculus for all* $\sigma' > \omega$, *and*

$$M^\infty_{s,\sigma'}(A+B) \leq C_{\omega,\sigma}\, M^\infty_{s,\sigma}(A). \tag{3.4.21}$$

Moreover

$$\dot X^\theta_{s,A} \cong \dot X^\theta_{s,A+B} \text{ for all } \theta \in (-\beta, 1+\alpha \wedge (1-\beta)). \tag{3.4.22}$$

Discussion of the assumptions in Theorem 3.4.10:

1. Let us note first that in comparison to the formulation of Theorem 3.4.6, the exponent β from (b) is a substitute for $1-\alpha$ in the formulation of (b) in Theorem 3.4.6. Observe that for given $\alpha \in (0,1)$ one obtains the largest range of θ such that the norm equivalence $\dot X^\theta_{s,A} \cong \dot X^\theta_{s,A+B}$ holds, namely the interval $(-\beta, 2-\beta)$ of length 2, if one can choose $\beta \geq 1-\alpha$. On the other hand, the largest possible upper bound for admissible θ, namely $1+\alpha$, is obtained if one can choose $\beta \leq 1-\alpha$.

2. The mapping condition on the domains in (c) ensures that the operator $A^\alpha BA^{-1-\alpha}$ is densely defined.

3. Furthermore, the assumptions in Theorem 3.4.10 are rather strong compared with the assumptions for perturbation of the H^∞-calculus. In this case one would either assume (a) and (b) or (a) and (c). The first combination matches with the perturbation theorem by Kunstmann and Weis (cf. [KW01-b], [KW04], [KKW06]) and the second one with the earlier perturbation result of Jan Prüß as it can be found in [DDHPV04]. Indeed, this second result can also be deduced from the first one by complex interpolation, this is also shown in the literature cited above. Of course, (a) and (c) are also in this situation already sufficient for $A+B$ to have an $\mathcal{R}_s$-bounded H^∞-calculus, this can be proven with the aid of the diagonal operator extensions in the spaces $X(\ell^s)$ as it has also been done in the proofs before. Nevertheless the crucial point we are interested in are the norm equivalences $\dot X^\theta_{s,A} \cong \dot X^\theta_{s,A+B}$ for the wider range $\theta \in \big(-\beta, (1+\alpha) \wedge (2-\beta)\big)$, and this makes it reasonable that both assumptions might be necessary. Moreover, in our concrete application to differential operators in Subsection 3.6.1, both conditions (b) and (c) are (under suitable regularity assumptions on the coefficients) equivalent by duality, hence this result is a suitable tool.

4. Again, if we make the additional assumption that

   the diagonal operator $\widetilde{A}_s$ in $X(\ell^s)$ is $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(\widetilde{A}_s) < \sigma$, we can choose $\varepsilon > 0$ (not depending on $\omega > \sigma$) such that $A+B$ in Theorem 3.4.10 has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_{\mathcal{R}_s^\infty}(A+B) \le \sigma$ if B satisfies the assumptions of Theorem 3.4.10.

Proof of Theorem 3.4.10. In view of Theorem 3.4.6 and its proof we obtain immediately that if we choose $\varepsilon > 0$ properly, then $C := A+B$ has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_{\mathcal{R}_s^\infty}(C) \le \omega_{\mathcal{R}_s^\infty}(A)$ and that $\dot{X}^\theta_{s,A} \cong \dot{X}^\theta_{s,C}$ for all $\theta \in (-\beta, 1)$. Moreover we get from Lemma 3.4.7 the $\mathcal{R}_s$-boundedness of the interchanging fractional power operators, i.e.

$$D(C^\theta) = D(A^\theta), \text{ and } A^\theta C^{-\theta} \text{ and } C^\theta A^{-\theta} \text{ are } \mathcal{R}_s\text{-bounded for all } \theta \in (-\beta, 1-\beta) \quad (3.4.23)$$

We only have to show that $\dot{X}^\theta_{s,A} \cong \dot{X}^\theta_{s,C}$ for all $\theta \in [1, 1+\alpha \wedge (1-\beta))$, and in view of (3.4.23) and the comparison theorem it is sufficient to show for all $\delta \in (0, \alpha \wedge (1-\beta))$:

$$D(C^{1+\delta}) = D(A^{1+\delta}), \text{ and } C^{1+\delta}A^{-1-\delta} \text{ and } A^{1+\delta}C^{-1-\delta} \text{ are } \mathcal{R}_s\text{-bounded} \quad (3.4.24)$$

We show first that the assumed $\mathcal{R}_s$-boundedness of the operators $\overline{BA^{-1}}$ and $L := \overline{A^\alpha BA^{-1-\alpha}}$ also implies the $\mathcal{R}_s$-boundedness of the operators $\overline{A^\delta BA^{-1-\delta}}$ and the existence of some $M_0 \ge 1$ with $\mathcal{R}_s(\overline{A^\delta BA^{-1-\delta}}) < M_0\varepsilon_0$ for all $\delta \in (0,\alpha)$ by complex interpolation: Let $K := \overline{BA^{-1}}$, then $\overline{A^\delta BA^{-1-\delta}} = \overline{A^\delta KA^{-\delta}}$. Now fix some $n \in \mathbb{N}$ and let $\widetilde{A}$ and $\widetilde{K}$ be the diagonal operator extensions of A and K in the space $X(\ell^s_n)$, then $\widetilde{K}$ is a bounded operator, and the operator $\widetilde{A}$ is sectorial and has a bounded H^∞-calculus in $X(\ell^s_n)$, with bounds that are independent of $n \in \mathbb{N}$. In particular, the operator $\widetilde{A}$ has bounded imaginary powers, hence we can choose constants $M \ge 1, \omega \in \mathbb{R}$ such that

$$\|\widetilde{A}^{it}x\|_{X(\ell^s_n)} \le M\, e^{\omega|t|} \cdot \|x\|_{X(\ell^s_n)} \qquad \text{for all } x \in X(\ell^s_n), t \in \mathbb{R}.$$

Let

$$T(z)(x_k)_k := (e^{z^2} A^{\alpha z} K A^{-\alpha z} x_k)_k$$

for all $(x_k)_k \in \widetilde{D} := D(A^2)^n \cap R(A^2)^n$ and $z \in S := \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in [0,1]\}$ (observe that

$$KA^{-\alpha z}x = BA^{-1-\alpha u}A^{-i\alpha t}Ax \in D(A^\alpha) \hookrightarrow D(A^{\alpha u}) \hookrightarrow D(A^{\alpha z})$$

for each $x \in D(A^2) \cap R(A^2)$ and $z = u + it \in S$). For fixed $x = (x_k)_k \in \widetilde{D}$ we have in each component

$$u_k(z) := (T(z)x)_k = e^{z^2}[A^{\alpha z}(1+A)^{-\alpha}]\,[(1+A)^\alpha KA^{-\alpha z}]x_k,$$

where the operators $z \mapsto e^{z^2}A^{\alpha z}(1+A)^{-\alpha} = e^{z^2}A^{i\alpha \operatorname{Im}(z)}\,[A^{\alpha \operatorname{Re}(z)}(1+A)^{-\alpha}]$ are uniformly bounded, hence $u_k : S \to X$ is continuous and bounded, and u_k is analytic on the open strip $\mathring{S}$. Moreover,

$$\|T(j+it)x\|_{X(\ell^s_n)} \le M^2 e^{1-t^2-2\omega t}\,\varepsilon_0 \cdot \|x\|_{X(\ell^s_n)} \le M_0\varepsilon_0 \cdot \|x\|_{X(\ell^s_n)}$$

for some constant $M_0 \ge 1$ and for all $x \in \widetilde{D}, j \in \{0,1\}$. By abstract Stein interpolation we can conclude

$$\|(A^\delta KA^{-\delta}x_k)_k\|_{X(\ell^s_n)} = \|T(\delta/\alpha)x\|_{X(\ell^s_n)} \le M_0\varepsilon_0 \cdot \|x\|_{X(\ell^s_n)}$$

for all $x = (x_k)_k \in X(\ell_s^n)$, hence $A^\delta BA^{-1-\delta}$ extends to an $\mathcal{R}_s$-bounded operator with $\mathcal{R}_s(\overline{A^\delta BA^{-1-\delta}}) < M_0\varepsilon_0$.

We now fix some $\delta \in (0, \alpha)$. Then

$$C^{1+\delta}A^{-1-\delta} = C^\delta CA^{-1-\delta} \supseteq \left[C^\delta A^{-\delta}\right]\left[A^\delta CA^{-1-\delta}\right],$$

and the operator $C^\delta A^{-\delta}$ is $\mathcal{R}_s$-bounded by (3.4.23) since $\delta \in (0, 1-\beta)$, and moreover

$$A^\delta CA^{-1-\delta} \supseteq A^\delta AA^{-1-\delta} + A^\delta BA^{-1-\delta} = 1 + A^\delta BA^{-1-\delta},$$

and the second operator extends to an $\mathcal{R}_s$-bounded operator by the preceding interpolation result. This shows that $D(A^{1+\delta}) \subseteq D(C^{1+\delta})$ and that the operator $C^{1+\delta}A^{-1-\delta}$ extends to an $\mathcal{R}_s$-bounded operator on X.

It remains to show that also the operator $A^{1+\delta}C^{-1-\delta}$ is $\mathcal{R}_s$-bounded. For this we use the following representation formula of the resolvent that is taken from [DDHPV04] and can be derived in a similar way as the corresponding representation formula in Theorem 3.4.5:

$$\begin{aligned}(t+C)^{-1} &= (I+K)\Big((t+A)^{-1} + \sum_{k=0}^{\infty}(-A(t+A)^{-1}K)^{k+1}(t+A)^{-1}\Big)(I+K)^{-1}\\ &= (I+K)\Big((t+A)^{-1} + A^{1-\alpha}(t+A)^{-1}M(t)A^{\alpha}(t+A)^{-1}\Big)(I+K)^{-1}\end{aligned}$$

where $M(t) := -\sum_{k=0}^{\infty}(-LA(t+A)^{-1})^k L$. Observe that $I+K, (I+K)^{-1}$ are $\mathcal{R}_s$-bounded extensions of the operator CA^{-1} and AC^{-1}, respectively.
Now let $x \in R(C^2)$ and $x' \in D((A^\delta)')$, then we use again the Balakrishnan representation formula and the additivity of fractional powers to obtain

$$AC^{-1-\delta}x = (I+K)^{-1}C^{-\delta}x = \underbrace{\frac{\sin(\pi\delta)}{\pi}}_{=:c_\delta}\int_0^\infty t^{-\delta}\,(I+K)^{-1}(t+C)^{-1}x\,dt,$$

hence

$$\begin{aligned}\langle AC^{-1-\delta}x, (A^\delta)'x'\rangle_\Omega &= c_\delta\int_0^\infty t^{-\delta}\,\langle\,(I+K)^{-1}(t+C)^{-1}x, (A^\delta)'x'\rangle_\Omega\,dt\\ &= \lim_{r\to 0}\lim_{R\to\infty}\underbrace{c_\theta\int_r^R t^{-\delta}\,\langle(I+K)^{-1}(t+C)^{-1}x, (A^\delta)'x'\rangle_\Omega\,dt}_{=:I^\delta_{r,R}(x,x')}\,.\end{aligned}$$

Let $0 < r \le R < +\infty$ and $T := (I+K)^{-1}$, then

$$\begin{aligned}
I_{r,R}^{\delta}(x,x') &= c_\delta \int_r^R t^{-\delta} \langle (t+A)^{-1}Tx, (A^\delta)'x' \rangle_\Omega \, dt \\
&\quad + c_\delta \int_r^R t^{-\delta} \langle A^{1-(\alpha-\delta)}(t+A)^{-1}M(t)A^\alpha(t+A)^{-1}Tx, x' \rangle_\Omega \, dt \\
&= \left\langle c_\delta \int_r^R t^{-\delta} (t+A)^{-1}Tx \, dt, (A^\delta)'x' \right\rangle_\Omega \\
&\quad + c_\delta \int_r^R t^{-\delta} \langle M(t)A^\alpha(t+A)^{-1}Tx, (A')^{1-(\alpha-\delta)}(t+A')^{-1}x' \rangle_\Omega \, dt \\
&= \left\langle c_\delta \int_r^R t^{-\delta} (t+A)^{-1}Tx \, dt, (A^\theta)'x' \right\rangle_\Omega \\
&\quad + c_\delta \int_r^R \langle M(t)\psi_\alpha(t^{-1}A)Tx, \psi_\beta(t^{-1}A')x' \rangle_\Omega \, \frac{dt}{t}
\end{aligned}$$

where $\beta := 1-(\alpha-\delta) \in (0,1)$ since $\delta \in (0,\alpha)$. Now let $n \in \mathbb{N}$, $(x_j)_{j\in\mathbb{N}_{\le n}} \in R(C)^n$ and $(x_j')_{j\in\mathbb{N}_{\le n}} \in D\big((A')^\delta\big)^n$, then

$$\begin{aligned}
\sum_{j=1}^n I_{r,R}^{\delta}(x_j,x_j') &= \sum_{j=1}^n \left\langle c_\delta \int_r^R t^{-\delta} (t+A)^{-1}Tx_j \, dt, (A^\delta)'x_j' \right\rangle_\Omega \\
&\quad + c_\delta \sum_{j=1}^n \int_r^R \langle M(t)\psi_\alpha(t^{-1}A)Tx_j, \psi_\beta(t^{-1}A')x_j' \rangle_\Omega \, \frac{dt}{t}. \qquad (3.4.25)
\end{aligned}$$

In the same way as in the proof of Lemma 3.4.7 we can handle the second integral with Hölder's inequality in the spaces $X(\ell_n^s(L_*^2(r,R))), X'(\ell_n^{s'}(L_*^2(r,R)))$ and obtain

$$\begin{aligned}
&\left| \sum_{j=1}^n \int_r^R \langle M(t)\psi_\alpha(t^{-1}A)Tx_j, \psi_\beta(t^{-1}A')x_j' \rangle_\Omega \, \frac{dt}{t} \right| \\
\le\ & \left\| \left(\int_r^R |M(t^{-1})\psi_\alpha(tA)Tx_j|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{X(\ell_n^s)} \cdot \left\| \left(\int_r^R |\psi_\beta(tA')x_j'|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{X'(\ell_n^{s'})},
\end{aligned}$$

and using the fact that the diagonal operator $\widetilde{A} : X(\ell_n^s) \supseteq D(A)^n \to X(\ell_n^s), (y_j)_j \mapsto (Ay_j)_j$ has a bounded H^∞-calculus, the same arguments as in the proof of Lemma 3.4.7 lead to

$$\begin{aligned}
&\left\| \left(\int_r^R |M(t^{-1})\psi_\alpha(tA)Tx_j|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{X(\ell_n^s)} \\
\le\ & \mathcal{R}\{M(t) \otimes \mathrm{Id}_{\ell_n^s} \mid t > 0\} \cdot \left\| \left(\int_0^\infty |\psi_\alpha(tA)Tx_j|^2 \, \frac{dt}{t} \right)^{1/2} \right\|_{X(\ell_n^s)} \lesssim \|(Tx_j)_j\|_{X(\ell_n^s)} \\
\le\ & \mathcal{R}_s(T)\, \|(x_j)_j\|_{X(\ell_n^s)}.
\end{aligned}$$

We can apply the same arguments to the second factor, since A' inherits by dualization the corresponding properties of A, i.e. it has an $\mathcal{R}_{s'}$-bounded H^∞-calculus in $X'(\ell_n^{s'})$. Thus we obtain

$$\sum_{j=1}^{n} \int_r^R \left|\langle M(t)\psi_\alpha(t^{-1}A)Tx_j, \psi_\beta(t^{-1}A')x'_j\rangle_\Omega\right| \frac{dt}{t} \quad \lesssim \quad \|(x_j)_j\|_{X(\ell_n^s)} \cdot \|(x'_j)_j\|_{X'(\ell_n^{s'})}.$$

Hence we can conclude that the limits for $r \to 0, R \to \infty$ in (3.4.25) exist, and putting all together yields

$$\sum_{j=1}^{n} \left|\langle AC^{-1-\delta}x_j, (A^\delta)'x'_j\rangle_\Omega\right| \lesssim \|(x_j)_j\|_{X(\ell_n^s)} \cdot \|(x'_j)_j\|_{X'(\ell_n^{s'})}. \tag{3.4.26}$$

By duality this provides $C^{-1-\delta}x_j \in D(A^\delta)$, and we have the estimate

$$\begin{aligned} \|A^{1+\delta}C^{-1-\delta}(x_j)_j\|_{X(\ell_n^s)} &= \|A^\delta AC^{-1-\delta}(x_j)_j\|_{X(\ell_n^s)} \\ &= \sup\left\{\left|\left\langle C^{-\delta}(x_j)_j, (A^\delta)'(x'_j)_j\right\rangle_{X(\ell_n^s)\times X'(\ell_n^{s'})}\right| \;:\; \|(x'_j)_j\|_{X'(\ell_n^{s'})} \le 1\right\} \\ &\lesssim \|(x_j)_j\|_{X(\ell_n^s)} \end{aligned}$$

uniformly in $n \in \mathbb{N}$, hence the operator $A^{1+\delta}C^{-1-\delta}$ extends to an $\mathcal{R}_s$-bounded operator on X, and the proof is finished. $\square$

3.5 Weighted estimates and $\mathcal{R}_s$-boundedness in L^p

In this section we will show that negatives of generators of analytic semigroups always have an $\mathcal{R}_s$-bounded H^∞-calculus in an appropriate scale of L^p-spaces if the semigroup satisfies suitable *generalized Gaussian estimates*. This result is similar to the corresponding one in [BK03], where under slightly more general assumptions it is shown that such operators have a bounded H^∞-calculus in the same scale of L^p-spaces. We will apply our result to show that large classes of differential operators have an $\mathcal{R}_s$-bounded H^∞-calculus. Thus the theory developed in Section 3.3 can be applied to these operators.

3.5.1 Weighted estimates and $\mathcal{R}_s$-bounded H^∞-calculus

We will use the general framework of spaces of homogeneous type, that was already introduced in Section 3.1. Let (Ω_0, d) be a metric space and μ a σ-finite regular Borel measure on Ω_0 such that (Ω_0, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss (cf. [CW71], [CW77]). Recall from Section 3.1 that this means there is a constant $C_1 \geq 1$ such that

$$\mu(B_{\Omega_0}(x, 2r)) \leq C_1\, \mu(B_{\Omega_0}(x, r)) \quad \text{for all } x \in \Omega_0, r > 0. \tag{3.5.1}$$

For abbreviation we will write $|B| := \mu(B)$ for μ-measurable sets $B \subseteq \Omega_0$ and $\rho B_{\Omega_0}(x, r) := B_{\Omega_0}(x, r\rho)$ for all $r, \rho \geq 0$ and $x \in \Omega_0$.

From (3.5.1) one can deduce the existence of some $D > 0$ and $C_D \geq 1$ such that

$$|B_{\Omega_0}(x, \lambda r)| \leq C_D\, \lambda^D\, |B_{\Omega_0}(x, r)| \quad \text{for all } x \in \Omega_0, r > 0, \lambda \geq 1. \tag{3.5.2}$$

We will usually use this in the following more general form, which follows from the simple fact, that $B_{\Omega_0}(x, r) \subseteq B_{\Omega_0}\big(y, r + d(x, y)\big)$:

$$\forall\, x, y \in \Omega_0\, \forall\, r \geq \rho > 0 \,:\, |B_{\Omega_0}(x, r)| \leq C_D \Big(\frac{r + d(x, y)}{\rho}\Big)^D |B_{\Omega_0}(y, \rho)|. \tag{3.5.3}$$

Moreover we fix a μ-localizing sequence $(\Omega_n)_{n\in\mathbb{N}}$ for (Ω_0, μ) with the additional property

$$\operatorname{diam}(\Omega_n) := \sup\{d(x, y) \mid x, y \in \Omega_n\} < +\infty \text{ for all } n \in \mathbb{N}$$

and define the spaces $L^\infty_c(\Omega_0, \mu), L^1_{loc}(\Omega_0, \mu)$ with respect to this fixed sequence $(\Omega_n)_{n\in\mathbb{N}}$ according to Subsection 1.6.1.

Finally we fix a measurable $\Omega \subseteq \Omega_0$ with $|\Omega| > 0$ and Banach spaces E, F. In this section, we will again use the abbreviation $L^p := L^p(\Omega)$ for the scalar-valued L^p-spaces and $L^p(E) := L^p(\Omega, E)$ for the vector-valued L^p-spaces. The set Ω will be endowed with the induced Borel-measure from (Ω_0, μ) and the localizing sequence $(\Omega \cap \Omega_n)_{n\in\mathbb{N}}$. We will use the notation $B(x, r) := B_{\Omega_0}(x, r) \cap \Omega$ for open balls in Ω, and for later use we define the annulus $A_k(x, r) := B(x, (k+1)r) \backslash B(x, kr)$ for all $k \in \mathbb{N}_0$ and $x \in \Omega_0, r > 0$. Note that by (3.5.2) we have

$$|A_k(x, r)| \leq C_D (1 + k)^D |B_{\Omega_0}(x, r)| \text{ for all } k \in \mathbb{N}_0, x \in \Omega_0, r > 0.$$

The subset $\Omega \subseteq \Omega_0$ will be used for a slightly more general formulation of the main theorems of this section. Nevertheless, in all lemmata and proofs we will concentrate on the case $\Omega = \Omega_0$, wich is justified by the following reason: each function $f \in L^1_{loc}(\Omega, E)$ can be identified with the zero extension $J_0 f := f \cup \big((\Omega_0 \backslash \Omega) \times \{0_E\}\big) \in L^1_{loc}(\Omega_0, E)$, and vice versa a function $g \in L^1_{loc}(\Omega_0, E)$ can be identified with the function $P_0 g := g|_\Omega \in L^1_{loc}(\Omega, E)$. In the same way each operator $T : L^\infty_c(\Omega, E) \to L^1_{loc}(\Omega, F)$ induces a corresponding operator $T_0 := J_0 T P_0 : L^\infty_c(\Omega_0, E) \to L^1_{loc}(\Omega_0, F)$. We will see that all assumptions we will make in the following assertions on operators in spaces over Ω will naturally carry over to the corresponding operator T_0 in spaces over Ω_0, hence in the proofs it will always be sufficient to consider the case $\Omega = \Omega_0$.

The main assumptions in this section will be generalized Gaussian estimates, also referred to as weighted or off-diagonal estimates. Roughly speaking these are estimates that generalize classical Gaussian kernel estimates on the one hand, and on the other hand they are a technical tool to formulate a substitute for the Hörmander condition on integral operators, which will give a weak (q_0, q_0)-criterion for non-integral operators in the sense of Proposition 3.5.11. To get a better insight we present some basic facts about estimates of this kind and their connection to classical kernel estimates. For a family $(S_t)_{t \in J}$ of operators $L^\infty_c(E) \to L^1_{loc}(F)$ we will consider estimates of the form

$$\| \mathbb{1}_{B(x,r_t)} S_t \mathbb{1}_{A_k(x,r_t)} \|_{p \to q} \le |B_{\Omega_0}(x, r_t)|^{\frac{1}{q} - \frac{1}{p}} g(k) \tag{3.5.4}$$

and their dual version

$$\| \mathbb{1}_{A_k(x,r_t)} S_t \mathbb{1}_{B(x,r_t)} \|_{p \to q} \le |B_{\Omega_0}(x, r_t)|^{\frac{1}{q} - \frac{1}{p}} g(k) \tag{3.5.5}$$

for all $x \in \Omega_0$ and $t \in J, k \in \mathbb{N}_0$ and some $(r_t)_{t \in J} \in (\mathbb{R}_{>0})^J$, where $g : [0, \infty) \to [0, \infty)$ is non-increasing and $1 \le p \le q \le +\infty$. Note that these estimates can be rewritten by explicitly writing out the integral norms, e.g. (3.5.5) can be rewritten as

$$\left(\frac{1}{|B_{\Omega_0}(x, r_t)|} \int_{A_k(x,r_t)} |S_t f|_F^q \, d\mu \right)^{1/q} \le g(k) \left(\frac{1}{|B_{\Omega_0}(x, r_t)|} \int_{B(x,r_t)} |f|_E^p \, d\mu \right)^{1/p} \tag{3.5.6}$$

for all $f \in L^\infty_c(E)$ with $\operatorname{supp}(f) \subseteq B(x, r_t)$. Moreover, estimates like (3.5.4), (3.5.5) also imply corresponding estimates for $\tilde{p}, \tilde{q} \in [1, +\infty]$ instead of p, q if $p \le \tilde{p} \le \tilde{q} \le q$, by maybe some waste of decay in the function g. Since we will use this fact later for the estimate (3.5.5) we give more details in that situation:

Lemma 3.5.1. *Let $1 \le p \le q \le +\infty$ and $g : [0, \infty) \to [0, \infty)$ be a non-increasing function. Let $(S_t)_{t \in J}$ be a family of operators $L^\infty_c(E) \to L^1_{loc}(F)$ which satisfies the estimate (3.5.5) for all $x \in \Omega_0$ and $t \in J, k \in \mathbb{N}_0$ and some $(r_t)_{t \in J} \in (\mathbb{R}_{>0})^J$. Let $\tilde{p}, \tilde{q} \in [p, q]$ such that $p \le \tilde{p} \le \tilde{q} \le q$. Then the estimate*

$$\| \mathbb{1}_{A_k(x,r_t)} S_t \mathbb{1}_{B(x,r_t)} \|_{\tilde{p} \to \tilde{q}} \le |B_{\Omega_0}(x, r_t)|^{\frac{1}{\tilde{q}} - \frac{1}{\tilde{p}}} \widetilde{g}(k) \tag{3.5.7}$$

holds for all $x \in \Omega_0$ and $t \in J, k \in \mathbb{N}_0$, where $\widetilde{g}(k) := C_D^{\frac{1}{\tilde{q}} - \frac{1}{q}} g(k)(1 + k)^{D(\frac{1}{\tilde{q}} - \frac{1}{q})}$.

Proof. We only consider the case $q < +\infty$, the modification for the remaining cases is obvious. Let $x \in \Omega_0$ and $t \in J, k \in \mathbb{N}_0$. For brevity let $B_0 := B_{\Omega_0}(x, r_t)$, $B := B(x, r_t)$ and $A := A_k(x, r_t)$, then $|B| \leq |B_0|$ and $|A| \leq C_D(1+k)^D |B_0|$. By Hölder's inequality, we obtain

$$\begin{aligned}
&\Big(\frac{1}{|B_0|}\int_A |S_t f|_F^{\tilde{q}}\, d\mu\Big)^{1/\tilde{q}} = \Big(\frac{|A|}{|B_0|}\Big)^{1/\tilde{q}}\Big(\frac{1}{|A|}\int_A |S_t f|_F^{\tilde{q}}\, d\mu\Big)^{1/\tilde{q}}\\
\leq\ &\Big(\frac{|A|}{|B_0|}\Big)^{1/\tilde{q}}\Big(\frac{1}{|A|}\int_A |S_t f|_F^{q}\, d\mu\Big)^{1/q} = \Big(\frac{|A|}{|B_0|}\Big)^{1/\tilde{q}-1/q}\Big(\frac{1}{|B_0|}\int_A |S_t f|_F^{q}\, d\mu\Big)^{1/q}\\
\leq\ & g(k)\Big(\frac{|A|}{|B_0|}\Big)^{1/\tilde{q}-1/q}\Big(\frac{1}{|B_0|}\int_B |f|_E^{p}\, d\mu\Big)^{1/p}\\
\leq\ & g(k) C_D^{\frac{1}{\tilde{q}}-\frac{1}{q}}(1+k)^{D(1/\tilde{q}-1/q)}\Big(\frac{1}{|B_0|}\int_B |f|_E^{\tilde{p}}\, d\mu\Big)^{1/\tilde{p}}
\end{aligned}$$

for all $f \in L_c^\infty(E)$ with $\operatorname{supp}(f) \subseteq B(x, r_t)$. □

We cite two results from [BK02] and [Ku02] to get a better understanding for this kind of estimates. The first result connects the estimate (3.5.4) with classical kernel estimates for $(p, q) = (1, +\infty)$ if the S_t are integral operators.

Lemma 3.5.2. *Let $(S_t)_{t\in J}$ be a family of linear integral operators $L^1(E) \to L^\infty(F)$ with kernels $k_t \in L^\infty(\Omega \times \Omega, L(E, F))$*[1] *and $(r_t)_{t\in J} \in (0, \infty)^J$. Then an estimate of the form*

$$\| \mathbb{1}_{B(x,r_t)}\, S_t\, \mathbb{1}_{A_k(x,r_t)} \|_{1\to\infty} \leq |B(x, r_t)|^{-1}\, g(k) \quad \text{for all } x \in \Omega_0, k \in \mathbb{R}_{\geq 0}, t \in J, \tag{3.5.8}$$

where $g : [0, \infty) \to [0, \infty)$ is non-increasing, is equivalent to an estimate of the form

$$\|k_t(x, y)\|_{L(E,F)} \leq |B(x, r_t)|^{-1}\, h\Big(\frac{d(x, y)}{r_t}\Big) \quad \text{for all } x, y \in \Omega_0, t \in J \tag{3.5.9}$$

where $h : [0, \infty) \to [0, \infty)$ is non-increasing. Moreover, if (3.5.8) holds, one can take $h := g$, and if (3.5.9) holds, one can take $g(t) := 2^D C_D h((t-1) \vee 0)$.

This lemma can be found in [BK02], Proposition 2.9 or [Ku02] Proposition 2.2. for the scalar-valued case, and the proof given there generalizes to the vector-valued case, since also then the identity $\|S_k\|_{1\to\infty} = \|k\|_\infty$ holds for all kernels $k \in L^\infty(\Omega \times \Omega, L(E, F))$ and associated integral operators S_k.

On the other hand, estimates of the above form can be compared with a symmetrized version with indicator functions of balls on both sides, and with exponential weights if the function g has a sufficiently fast decay.

Lemma 3.5.3. *Let $(S_t)_{t\in J}$ be a family of linear operators $S_t : L_c^\infty(E) \to L_{loc}^1(F)$ and $(r_t)_{t\in J} \in (\mathbb{R}_{>0})^J$. Then an estimate*

$$\| \mathbb{1}_{B(x,r_t)}\, S_t\, \mathbb{1}_{A_k(x,r_t)} \|_{p\to q} \leq C_{\kappa_1} |B(x, r_t)|^{\frac{1}{q}-\frac{1}{p}} (1+k)^{-\kappa_1} \text{ for all } x \in \Omega_0, k \in \mathbb{N}_0, t \in J \tag{3.5.10}$$

[1] i.e. $S_t f(x) := \int_\Omega k_t(x, y) f(y)\, d\mu(y)$ for $f \in L^1(E)$ and a.e. $x \in \Omega$

for all $\kappa_1 > 0$ is equivalent to an estimate

$$\| \mathbb{1}_{B(x,r_t)} S_t \mathbb{1}_{B(y,r_t)} \|_{p\to q} \leq C_{\kappa_2} |B(x,r_t)|^{\frac{1}{q}} |B(y,r_t)|^{-\frac{1}{p}} \Big(1 + \frac{d(x,y)}{r_t}\Big)^{-\kappa_2} \tag{3.5.11}$$

for all $x \in \Omega_0, k \in \mathbb{N}_0, t \in J$ and all $\kappa_2 > 0$. More precisely, if (3.5.10) holds for some κ_1, then (3.5.11) holds with $\kappa_2 = \kappa_1$, and if (3.5.11) holds for some $\kappa_2 > 2D$, then (3.5.10) holds with $\kappa_1 = \kappa_2 - 2D$.

This lemma is a special case of [Ku02], Proposition 2.6. Note that in the applications one usually has not only polynomial but even exponential decay in the weighted norm estimates, which of course also imply these kinds of estimates, cf. e.g. [Ku02], Proposition 2.6. Moreover, we can replace the exponents $1/p, -1/q$ in (3.5.11) by any $\alpha, \beta \in \mathbb{R}$ such that $\alpha+\beta = 1/p-1/q$ by maybe some waste of decay, i.e. one has to change κ_2 in this case. This is a simple consequence of (3.5.3).

Let us finally mention that in the classical situation Gaussian bounds are often only derived for *real times*. But in this case one can also derive corresponding estimates for complex times by eventually loosing some decay, and moreover the sector on which complex time estimates can be established depends on the decay on the underlying real time estimates. We will not go into detail but just refer to the exposition in [Ku02], Sections 2.4, 2.5.

The following theorem is the main result of this section.

Theorem 3.5.4. *Let $1 \leq p_0 < 2 < p_1 \leq +\infty$ and $\omega_0 \in (0, \pi/2)$. Let A be a sectorial operator in $L^2(\Omega)$ such that A has a bounded H^∞-calculus in $L^2(\Omega)$ with $\omega_{H^\infty}(A) \leq \omega_0$. Assume that the generated semigroup $T_\lambda := e^{-\lambda A}$ satisfies the following weighted norm estimates for each $\theta > \omega_0$:*

$$\| \mathbb{1}_{A_k(x,|\lambda|^{1/m})} T_\lambda \mathbb{1}_{B(x,|\lambda|^{1/m})} \|_{p_0\to p_1} \leq C_\theta |B(x,|\lambda|^{1/m})|^{\frac{1}{p_1}-\frac{1}{p_0}} (1+k)^{-\kappa_\theta}, \tag{3.5.12}$$

$$\| \mathbb{1}_{B(x,|\lambda|^{1/m})} T_\lambda \mathbb{1}_{A_k(x,|\lambda|^{1/m})} \|_{p_0\to p_1} \leq C_\theta |B(x,|\lambda|^{1/m})|^{\frac{1}{p_1}-\frac{1}{p_0}} (1+k)^{-\kappa_\theta} \tag{3.5.13}$$

for all $x \in \Omega_0, k \in \mathbb{N}_0, \lambda \in \Sigma_{\pi/2-\theta}$ and some constants $m > 0, \kappa_\theta > D\Big(\frac{1}{p_0} - \frac{1}{p_1} + \frac{1}{p_0 \wedge p_1'}\Big) + 1$ and $C_\theta > 0$. Then, for all $p, s \in (p_0, p_1)$ and $\omega > \omega_0$ the operator A has an $\mathcal{R}_s$-bounded $H^\infty(\Sigma_\omega)$-calculus in $L^p(\Omega)$.

To be more exact we can reformulate the statement of Theorem 3.5.4 in the following way:

> *For all $p \in (p_0, p_1)$, the semigroup T induces a consistent C_0-semigroup T_p on $L^p(\Omega)$ with generator $-A_p$, and for all $s \in (p_0, p_1)$ the operator A_p has an $\mathcal{R}_s$-bounded H^∞-calculus with $\omega_{\mathcal{R}_s^\infty}(A_p) \leq \omega_0$.*

The extrapolation procedure is well known, cf. e.g [Ou05] for extrapolation of semigroups in the classical situation of L^1-L^∞-contractivity or classical Gaussian bounds, and [BK02], where semigroups are considered that satisfy the assumptions of Theorem 3.5.4. Thus we will not go into further details for this.

3.5.2 Applications to differential operators

Before we turn to the proof of Theorem 3.5.4 we will give some examples of classes of operators that are well known to satisfy generalized $p_0 \to p_1$ Gaussian estimates for some $p_0 < 2 < p_1$ on the one hand and have a bounded H^∞-calculus in L^2 on the other hand. So in these cases, the assumptions of Theorem 3.5.4 are fulfilled and hence the considered operators also have an $\mathcal{R}_s$-bounded H^∞-calculus in L^p for all $p, s \in (p_0, p_1)$. In fact, in most situations much stronger weighted estimates than required are known, typically in the symmetric form (3.5.11) for arbitrary $\kappa_2 > 0$, i.e. with decay faster than every polynomial, or even exponential, i.e. "classical" Gaussian bounds.

In particular, the differential operators in the following examples turn out to be $\mathcal{R}_s$-sectorial in L^p for all $p, s \in (p_0, p_1)$ with appropriate p_0, p_1. Thus for these operators, the s-intermediate spaces are well defined, and the norms are independent of the auxiliary function chosen to determine the s-power function norms associated to the operator. Furthermore we can apply the theory developed in Section 3.3 to these operators. Nevertheless, we do not determine the associated s-intermediate spaces explicitly in this section. That will be done for uniformly elliptic operators in divergence and non-divergence form under stronger assumptions on the top order coefficients in the subsequent Section 3.6, where we will show that the s-intermediate spaces associated to certain classes of elliptic operators coincide with the classical Triebel-Lizorkin spaces $F^\alpha_{p,s}$ at least for some range of α.

(a) Elliptic operators in divergence form.

There are many contributions to Gaussian estimates for elliptic operators in divergence form, cf. e.g. [Da89], [Da97-2], [Ou05] and the literature cited there. If $\Omega \subseteq \mathbb{R}^D$ is a region, an elliptic operator on Ω in divergence form is formally given as

$$\mathcal{A}u = \sum_{|\alpha|,|\beta| \le m} (-1)^{|\beta|} \partial^\beta (a_{\alpha\beta} \partial^\alpha u), \tag{3.5.14}$$

with coefficients $a_{\alpha\beta} \in L^\infty(\Omega, \mathbb{C})$. To be more exact, the realization A_2 of the operator $\mathcal{A}$ in $L^2(\Omega) := L^2(\Omega, \mathbb{C})$ (with Dirichlet boundary conditions) is defined as the operator associated to the form

$$\mathfrak{a}(u, v) := \int \sum_{|\alpha|,|\beta| \le m} a_{\alpha\beta}(x) \partial^\alpha u(x) \overline{\partial^\beta v(x)}\, dx \quad \text{ for all } u, v \in W^{m,2}_0(\Omega), \tag{3.5.15}$$

where we have to impose appropriate ellipticity conditions for the principal part to ensure that A_2 is well-defined. For simplicity we restrict ourselves to the case $\Omega = \mathbb{R}^D$ and to homogeneous operators without lower order terms, i.e.

$$\mathfrak{a}(u, v) = \int \sum_{|\alpha|,|\beta| = m} a_{\alpha\beta}(x) \partial^\alpha u(x) \overline{\partial^\beta v(x)}\, dx \quad \text{ for all } u, v \in W^{m,2}(\mathbb{R}^D) \tag{3.5.16}$$

with coefficients $a_{\alpha\beta} \in L^\infty(\mathbb{R}^D, \mathbb{C})$. Note that additional lower order terms can be treated by perturbation arguments and will usually lead to the same results cited in the sequel for $\mathcal{A} + \nu$

instead of $\mathcal{A}$ and some sufficiently large $\nu \geq 0$.

We assume the form $\mathfrak{a}$ to be sectorial, i.e.

$$|\operatorname{Im} \mathfrak{a}(u,u)| \leq \tan(\psi) \operatorname{Re} \mathfrak{a}(u,u) \quad \text{for all } u \in W^{m,2}(\mathbb{R}^D) \tag{3.5.17}$$

for some $\psi \in [0, \pi/2)$, and to satisfy an ellipticity condition in the form that the following Garding's inequality

$$\operatorname{Re} \mathfrak{a}(u,u) \geq \eta \, \|(-\Delta)^{m/2} u\|_2^2 \quad \text{for all } u \in W^{m,2}(\mathbb{R}^D) \tag{3.5.18}$$

holds for some $\eta > 0$. Observe that in the case $m = 1$, the conditions (3.5.17) and (3.5.18) are consequences of the following *uniformly strong ellipticity condition*:

$$\operatorname{Re} \sum_{j,k=1}^{D} a_{jk}(x) \xi_j \overline{\xi_k} \geq \eta \, |\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^D, x \in \mathbb{R}^D, \tag{3.5.19}$$

with $a_{jk} := a_{e_j e_k}$, where maybe $\mathcal{A}$ has to be replaced by $\mathcal{A} + \nu$ for some constant $\nu \geq 0$.

In this situation $\mathfrak{a}$ is a closed sectorial form, hence the associated operator A_2 in $L^2(\mathbb{R}^D)$ is sectorial and has a bounded H^∞ calculus with $\omega_{H^\infty(A_2)} \leq \psi$, so the assumptions of Theorem 3.5.4 are fulfilled if the generated semigroup satisfies generalized Gaussian estimates, which is true in various cases. We take a closer look on some special situations.

(i) In the case $m = 1$ we can formally write $\mathcal{A}u = -\operatorname{div}(a \nabla u)$, where $a = (a_{jk})_{j,k=1}^D$. Assume that a is real-valued and symmetric, i.e. $a_{jk} = a_{kj} : \mathbb{R}^D \to \mathbb{R}$ for all $j,k \in \mathbb{N}_{\leq D}$, then the associated operator A_2 in $L^2(\mathbb{R}^D)$ is selfadjoint, and the semigroup generated by $-A_2$ has a kernel k_t that satisfies classical Gaussian bounds in the following sense: for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$0 \leq k_t(x,y) \leq C_\varepsilon \, t^{-D/2} \exp\left(- \frac{(x-y)^2}{4t(1+\varepsilon)\|a\|_\infty} \right) \quad \text{for all } x, y \in \mathbb{R}^D, t > 0.$$

This result is nowadays classical and we refer to the standard literature as [Da89], Corollary 3.2.8, where also the case of general regions $\Omega \subseteq \mathbb{R}^D$ is considered. Hence the operator $\mathcal{A}$ has an $\mathcal{R}_s$-bounded H^∞-calculus in $L^p(\mathbb{R}^D)$ for all $s, p \in (0, +\infty)$ in this case.

(ii) We consider again the case $m = 1$, so $\mathcal{A}u = -\operatorname{div}(a \nabla u)$, but now we admit $a : \mathbb{R}^D \to \mathbb{C}^{D \times D}$ to be complex valued and also drop the symmetry condition. In this setting, things are rigorously different than in the real symmetric situation, we refer to [AMT98] for an comprehensive treatment of this case. First of all, we usually have to consider $\nu + \mathcal{A}$ for some $\nu \geq 0$ to obtain Gaussian bounds, even in the absence of lower order terms. Furthermore, it is no more true that $\nu + \mathcal{A}$ has classical Gaussian bounds in any dimension D for some $\nu \geq 0$, if we assume no more regularity on the coefficients a. Nevertheless, in the case $D \leq 2$ Gaussian bounds for $\nu + \mathcal{A}$ are obtained without any further assumptions for some $\nu \geq 0$, whereas for $D \geq 3$ there are examples of operators that have no Gaussian bounds, cf. [HMM10]. In [Au96] it is shown that $\nu + \mathcal{A}$ satisfies Gaussian estimates for some $\nu \geq 0$ in the case $D \geq 3$ if the coefficients a are assumed to be uniformly continuous. So in these cases the operator $\nu + \mathcal{A}$ has an $\mathcal{R}_s$-bounded H^∞-calculus in $L^p(\mathbb{R}^D)$ for all $s, p \in (1, +\infty)$.

(iii) We will now consider the general case $m \in \mathbb{N}$, where for simplicity we assume $D \geq 3$. Then by (ii) it is already clear that we cannot expect $\mathcal{A}$ to satisfy classical Gaussian estimates without any further regularity assumptions for all $D \geq 3$. We cite a result without any additional regularity assumptions from [Ku02], Section 4.1, cf. also [Da97-2] for earlier results in this direction. Assume $D \neq 2m$ and let $p_1 := \frac{2D}{D-2m}$ if $D > 2m$ and $p_1 := +\infty$ if $D < 2m$ and $p_0 := p_1'$. Then there is an $\nu \geq 0$ such that the semigroup generated by the operator $-(\nu + A_2)$ in $L^2(\mathbb{R}^D)$ satisfies Gaussian bounds of the form (3.5.12), (3.5.13) for all $\kappa_\theta > 0$ and some $\theta \in (0, \pi/2)$. Hence the operator $\nu + \mathcal{A}$ has an $\mathcal{R}_s$-bounded H^∞-calculus in $L^p(\mathbb{R}^D)$ for all $s, p \in (p_0, p_1)$ in this case. In fact, more is known in this situation: It can be shown that this result is optimal in the sense that for all $r \notin [p_0, p_1]$ one can find an operator $\mathcal{A}$ of the above form such that the generated semigroup does not extend to $L^r(\mathbb{R}^D)$, cf. [Da97-1], and also the case $D = 2m$ has been treated. More references can be found in the more detailed exposition in [KW04], Chapter 8 and 14 and the corresponding notes.

(b) Elliptic operators in non-divergence form.

Although the notion of Gaussian bounds is by natural reasons strongly connected with elliptic operators in divergence form, there are also results on Gaussian bounds for elliptic operators in non-divergence form. We just cite one recent result due to Peer Kunstmann explicitly:

Let A_q be the realization in $L^q(\mathbb{R}^D) := L^q(\mathbb{R}^D, \mathbb{C})$ of the differential operator

$$\mathcal{A} := \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha$$

with $D(A_q) = W^{2m,q}(\mathbb{R}^D)$, where

(I) $a_\alpha \in L^\infty(\mathbb{R}^D, \mathbb{C})$ for all $|\alpha| \leq 2m$, and there are $\omega \in (0, \pi/2), \eta > 0$ such that for all $x, \xi \in \mathbb{R}^D$:

$$\sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha \in \Sigma_\omega, \quad \text{and} \left| \sum_{|\alpha|=2m} a_\alpha(x) \xi^\alpha \right| \geq \eta \, |\xi|^{2m}.$$

(II) a_α is bounded and uniformly continuous, i.e. $a_\alpha \in BUC(\mathbb{R}^D, \mathbb{C})$ if $|\alpha| = 2m$, or

(II)' a_α is of vanishing mean oscillation, i.e. $a_\alpha \in VMO(\mathbb{R}^D, \mathbb{C})$ if $|\alpha| = 2m$.

Then in [Ku08], Section 6.1 the following result is shown:

Proposition 3.5.5. *Assume that (I) and either (II) or (II)' holds. There are constants $\nu \geq 0$ and $\delta \in (0, \pi/2)$ such that $-(\nu + A_q)$ generates an analytic semigroup $(T(z))_{z \in \Sigma_\delta})$ in $L^q(\mathbb{R}^D)$ for all $q \in (1, +\infty)$. Moreover, the semigroup T satisfies the following Gaussian estimate*

$$\| \mathbb{1}_{B\left(x, |z|^{\frac{1}{2m}}\right)} T(z) \, \mathbb{1}_{B\left(y, |z|^{\frac{1}{2m}}\right)} \|_{p \to \infty} \leq C \, |z|^{-\frac{D}{2mp}} \exp\Big(- b\Big(\frac{|x-y|^{2m}}{|z|}\Big)^{\frac{1}{2m-1}} \Big) \tag{3.5.20}$$

for all $z \in \Sigma_\delta$ *for some* $C, b > 0$ *for any* $p > 1$.

Observe that the Gaussian estimate (3.5.20) implies the estimates (3.5.12), (3.5.13) in the assumptions of Theorem 3.5.4 for all $1 < p_0 < 2 < p_1 < +\infty$. Moreover, it is well known that $\nu + A_2$ and even $\nu + A_q$ for $q \in (1, +\infty)$ has a bounded H^∞-calculus for some $\nu \geq 0$ if the coefficients of the principle part are BUC, cf. [DS97], Theorem 6.1, and the same is true in the case (II)' under the additional assumption $m = 1$ and $a_\alpha = 0$ if $|\alpha| \neq 2$, cf. [DY02].

(c) Schrödinger operators with singular potentials.

Let us finally have a short look on Schrödinger operators, i.e. $\mathcal{A} = -\frac{1}{2}\Delta + V$, where Δ is the Laplace operator and the potential $V : \mathbb{R}^D \to \mathbb{R}$ is a measurable function. For simplicity we restrict ourselves to the case $D \geq 3$. We let $V^+ := V \vee 0$ and $V^- := -(V \wedge 0)$ so that $V = V^+ - V^-$. We assume that $V^+ \in L^1_{loc}(\mathbb{R}^D)$ and V^- is in the Pseudo-Kato class, cf. [KPS81]. Then the operator $\mathcal{A}_0 := -\frac{1}{2}\Delta + V^+$ is associated to the form

$$\mathfrak{a}_0(u, v) := \frac{1}{2}\int \nabla u(x)\overline{\nabla v(x)}\, dx + \int V^+(x)u(x)\overline{v(x)}\, dx \qquad \text{for all } u, v \in D(\mathfrak{a}_0),$$

where $D(\mathfrak{a}_0) := \{u \in W^{1,2}(\mathbb{R}^D) \mid \int V^+(x)|u(x)|^2\, dx < +\infty\}$, and it is well known that the realization of $\mathcal{A}_0$ in $L^2(\mathbb{R}^D)$ is a self-adjoint operator, and its generated semigroup is dominated by the heat semigroup, hence it satisfies classical Gaussian bounds. The realization A_2 of $\mathcal{A}$ in $L^2(\mathbb{R}^D)$ is defined as the operator associated to the form perturbation $(-\frac{1}{2}\Delta + V^+) - V^- = \mathcal{A}_0 - V^-$. Since this form is symmetric, the associated operator A_2 is self-adjoint and semi-bounded from below under certain assumptions on V^- (cf. e.g. [KPS81], Section 4 or [BS91]), so in that case a suitable translate of A_2 has a bounded H^∞-calculus in $L^2(\mathbb{R}^D)$, and in our sense the only task is to ask for (generalized) Gaussian estimates.

(i) Recall that $V^+ \in L^1_{loc}(\mathbb{R}^D)$. If in addition V^- is in the Kato class (cf. e.g. [Si82]), then it is shown in the comprehensive paper [Si82] that A_2 is selfadjoint (this goes indeed back to Kato, [Ka73]), and the generated semigroup has a kernel k_t that satisfies classical Gaussian bounds in the following sense: there is a $\nu \geq 0$ such that for all $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|k_t(x, y)| \leq C_\varepsilon\, t^{-d/2}\, e^{\nu t} \exp\left(-\frac{(x-y)^2}{2t(1+\varepsilon)}\right) \quad \text{for all } x, y \in \mathbb{R}^D, t > 0.$$

So the shifted operator $\nu + A$ has a bounded H^∞-calculus in $L^2(\mathbb{R}^D)$ and moreover satisfies weighted $1 \to \infty$ estimates. A more general treatment of this class of operators can be found in [Ou06].

(ii) One situation of more general potentials is described in [BK03], where V satisfies weaker assumptions such that A_2 still generates a semigroup, but in general no $1 \to \infty$ weighted estimates hold for the semigroup, a typical example is $V(x) = -\frac{c}{|x|^2}$ for a certain range of $c > 0$, cf. e.g. [KPS81], Section 5. Nevertheless, the semigroup in this case still satisfies weighted $p \to p'$ estimates for some $p > 1$, hence the theory of this section can be applied. For more details we refer to the article [BK03] and the literature cited there.

Note again, that this shows in particular that the associated s-intermediate spaces for Schrödinger operators according to Example (c) are well defined and independent of the auxiliary function chosen to determine the associated s-power function norms. For Schrödinger operators, a similar concept is developed in [OZ06] and [Zh06], where also generalized Triebel-Lizorkin spaces associated to Schrödinger operators are defined and studied. The definition given there differs from our definition and is closer to the original definition of Triebel-Lizorkin spaces, cf. Section 1.7. In particular, the fact that the considered Schrödinger operators are self-adjoint is essential, since the auxiliary functions used to define the spaces are in the class $C_c^\infty(\mathbb{R})$. On the other hand, our concept is more general in the sense that we can handle also non-self-adjoint sectorial operators, as we considered in Examples (a) and (b). Nevertheless, although we do not study this here, it seems reasonable to conjecture that the generalized Triebel-Lizorkin spaces introduced in [OZ06] for Schrödinger operators coincide with our notion of s-intermediate spaces, at least, if the negative part of the potential is in the Kato class. A proof might be based on suitable modifications of the methods in [Kr09], Chapter 4.4, where only the case $s = 2$ is treated in connection with Littlewood-Paley decompositions.

3.5.3 Proof of Theorem 3.5.4

We now turn to the proof of Theorem 3.5.4. We will use the following technical tool: For all $p \in [1, +\infty], r > 0$ we define

$$N_{p,r}f(y) := \| \mathbb{1}_{B(y,r)}\, f\|_{L^p\left(\Omega_0, \frac{dx}{|B_{\Omega_0}(y,r)|}, E\right)} = \|f|_{B(y,r)}\|_{L^p\left(B(y,r), \frac{dx}{|B_{\Omega_0}(y,r)|}, E\right)} \quad \text{for all } y \in \Omega_0$$

if $f \in L^p_{\mathrm{loc}}(\Omega_0, E)$. Moreover the Hardy-Littlewood p-maximal operator M_p is defined by

$$M_p f(x) := \sup_{r>0} N_{p,r} f(x) \quad \text{for all } x \in \Omega_0$$

if $f \in L^p_{\mathrm{loc}}(\Omega_0, E)$. Then an easy consequence of Theorem 3.1.22 is the following generalization:

Theorem 3.5.6. *Let $p \in [1, \infty)$, then the Hardy-Littlewood p-maximal operator M_p is $\mathcal{R}_s$-bounded on $L^q(\Omega_0)$ for all $q, s \in (p, \infty)$.* □

We have the following norm equivalence for the operators $N_{p,r}$.

Lemma 3.5.7. *Assume $\Omega = \Omega_0$. For each $p \in [1, +\infty]$ there is a constant $c_p > 0$ such that*

$$c_p^{-1}\|f\|_p \le \|N_{p,r}f\|_p \le c_p\|f\|_p \tag{3.5.21}$$

for all $r > 0$ and $f \in L^p_{loc}(\Omega_0, E)$.

Proof. The statement is trivial for $p = +\infty$, so we assume $p < +\infty$. Let $f \in L^p_{\mathrm{loc}}(\Omega_0, E)$, then

$$\|N_{p,r}f\|_p^p \;=\; \int_{\Omega_0} \frac{1}{|B(y,r)|} \int_{B_r(y)} \|f(x)\|^p \, dx\, dy.$$

For all $y \in \Omega_0, x \in B_r(y)$ we have $C_d^{-1} \leq \frac{|B(x,r)|}{|B(y,r)|} \leq C_d$ and $\mathbb{1}_{B_r(y)}(x) = \mathbb{1}_{B_r(x)}(y)$, hence by Fubini's theorem

$$\begin{aligned} \|N_{p,r}f\|_p^p &\approx \int_{\Omega_0} \frac{1}{|B(x,r)|} \int_{\Omega_0} \mathbb{1}_{B_r(y)}(x) \|f(x)\|^p \, dx \, dy \\ &= \int_{\Omega_0} \Big(\frac{1}{|B(x,r)|} \int_{\Omega_0} \mathbb{1}_{B_r(x)}(y) \, dy \Big) \|f(x)\|^p \, dx = \int_{\Omega_0} \|f(x)\|^p \, dx = \|f\|_p^p. \end{aligned}$$

□

We will now derive an important pointwise estimate $N_{q,r}(Sf) \lesssim M_p f(x)$ for operators S that satisfy suitable $p \to q$-estimates. This originates in [BK02], Lemma 2.6, and a more involved version can be found in [Ku08], Proposition 2.3.

Lemma 3.5.8. *Assume $\Omega = \Omega_0$. Let $p \in [1, \infty)$ and $\delta > \frac{D}{p} + \frac{1}{p'}$. Then there is a constant $C_0 = C_0(p, \delta, D, C_D) > 0$ with the following property: If $q \in (1, \infty], r, C_1 > 0$, and $S : L_c^\infty(E) \to L_{loc}^1(F)$ is a linear operator satisfying the weighted estimate*

$$\| \mathbb{1}_B \, S \, \mathbb{1}_{(k+1)B \setminus kB} \|_{p \to q} \leq C_1 \, |B|^{\frac{1}{q} - \frac{1}{p}} \, (1+k)^{-\delta} \tag{3.5.22}$$

for all balls B of radius r, then $N_{q,r}(Sf)(x) \leq C_0 C_1 \, M_p f(x)$ for μ-a.e. $x \in \Omega_0, f \in L_c^\infty(E)$.

Proof. Let q, r, C_1, S be as in the assumption and let $x \in \Omega_0$ and $f \in L^p(E) \cap L^q(E)$. Define $B := B(x,r)$, then

$$\begin{aligned} N_{q,r}(Sf)(x) &= |B|^{-1/q} \, \| \mathbb{1}_B \, Sf\|_q \leq |B|^{-1/q} \sum_{k=0}^\infty \| \mathbb{1}_B \, S \, \mathbb{1}_{(k+1)B \setminus kB} \, f\|_q \\ &\leq C_1 \, |B|^{-1/p} \sum_{k=0}^\infty (1+k)^{-\delta} \, \|f\|_{L^p((k+1)B \setminus kB), E)} \end{aligned}$$

(with the usual modifications if $q = +\infty$). Consider first the case $p = 1$. Then

$$\begin{aligned} N_{q,r}(Sf)(x) &\leq C_1 \, |B|^{-1} \sum_{k=0}^\infty (1+k)^{-\delta} \int_{(k+1)B \setminus kB} |f|_E \, d\mu \\ &= C_1 \, |B|^{-1} \sum_{k=0}^\infty (1+k)^{-\delta} \Big(\int_{(k+1)B} |f|_E \, d\mu - \int_{kB} |f|_E \, d\mu \Big) \\ &= C_1 \, |B|^{-1} \sum_{k=1}^\infty (k^{-\delta} - (1+k)^{-\delta}) \, \|f\|_{L^1(kB,E)} \\ &\leq C_1 \sum_{k=1}^\infty \frac{|kB|}{|B|} (k^{-\delta} - (1+k)^{-\delta}) \cdot (M_p f)(x) \\ &\leq C_1 C_D \sum_{k=1}^\infty k^D (k^{-\delta} - (1+k)^{-\delta}) \cdot (M_p f)(x) \leq \delta C_1 C_D \Big(\sum_{k=1}^\infty k^{D-\delta-1} \Big) \cdot (M_p f)(x). \end{aligned}$$

Since $D - \delta - 1 > 0$ by assumption the assertions follows in this case.

Now assume $p > 1$. Let $\gamma := \frac{D-1}{pp'}$, then

$$\alpha := \delta - \gamma p' > \frac{D}{p} + \frac{1}{p'} - \frac{D-1}{p} > 1 \quad \text{and} \quad \beta := \delta + \gamma p - D > \frac{D}{p} + \frac{1}{p'} + \frac{D-1}{p'} - D > 0,$$

hence

$$K := \Big(\sum_{k=0}^{\infty} (1+k)^{-\alpha} \Big)^{1/p'} < +\infty.$$

Since $\delta = (\delta/p' + \gamma) + (\delta/p' - \gamma)$, Hölder's inequality yields

$$\begin{aligned} N_{q,r}(Sf)(x) &\leq C_1 \Big(\sum_{k=0}^{\infty} (1+k)^{-\delta+\gamma p'} \Big)^{1/p'} \cdot \Big(|B|^{-1} \sum_{k=0}^{\infty} (1+k)^{-\delta-\gamma p} \, \|f\|^p_{L^p((k+1)B \setminus kB, E)} \Big)^{1/p} \\ &= C_1 K \Big(|B|^{-1} \sum_{k=0}^{\infty} (1+k)^{-D-\beta} \left(\|f\|^p_{L^p((k+1)B,E)} - \|f\|^p_{L^p(kB,E)} \right) \Big)^{1/p} \\ &= C_1 K \Big(|B|^{-1} \sum_{k=1}^{\infty} (k^{-D-\beta} - (1+k)^{-D-\beta}) \, \|f\|^p_{L^p(kB,E)} \Big)^{1/p} \\ &\leq C_D^{1/p} C_1 K \Big(\sum_{k=1}^{\infty} k^D (k^{-D-\beta} - (1+k)^{-D-\beta}) \, |kB|^{-1} \|f\|^p_{L^p(kB,E)} \Big)^{1/p} \\ &\leq C_D^{1/p} C_1 K \Big((D+\beta) \sum_{k=1}^{\infty} k^{-1-\beta} \Big)^{1/p} \cdot M_p f(x). \end{aligned}$$

$\square$

Although we will not use it explicitly, for sake of completeness we cite the following result that weighted norm estimates imply $\mathcal{R}_s$-boundedness from [Ku08], Theorem 2.2.

Proposition 3.5.9. *Assume $\Omega = \Omega_0$. Let $1 \leq q_0 \leq q_1 \leq +\infty$ and $\delta > \frac{D}{q_0} + \frac{1}{q_0'}$, and assume that $(S(t))_{t \in J}$ is a family of linear operators $S(t) : L^\infty_c \to L^1_{loc}$ satisfying the weighted estimate*

$$\| \mathbb{1}_{B(x,r_t)} \, S(t) \, \mathbb{1}_{A_k(x,r_t)} \|_{q_0 \to q_1} \leq C \, |B(x,r_t)|^{\frac{1}{q_1} - \frac{1}{q_0}} \, (1+k)^{-\delta} \tag{3.5.23}$$

for all $x \in \Omega, t \in J, k \in \mathbb{N}_0$ and some $C > 0$, where $(r_t)_{t \in J} \in (\mathbb{R}_{>0})^J$ is some family of radii. Then the set $S(J)$ of operators extends to a set of $\mathcal{R}_s$-bounded operators on L^p for all $(p,s) \in (q_0,q_1) \times [q_0,q_1] \cup \{(q_0,q_0),(q_1,q_1)\}$.

Proposition 3.5.9 is for $q_0 < p, s < q_1$ an easy consequence of Lemma 3.5.8 above, cf. also [BK02], Corollary 2.7. The more general version presented here is taken from [Ku08], Theorem 2.2, and is based on a more involved version of Lemma 3.5.8 we leave out here since we will not use it in the sequel.

We will now continue to provide technical tools for the proof of the main Theorem 3.5.4. Next we show how the weighted estimate (3.5.12) can be generalized to arbitrary radii:

Lemma 3.5.10. *Assume $\Omega = \Omega_0$. Let $1 \le p < 2 < q \le +\infty$, and let $(T_\rho)_{\rho>0}$ be a family of bounded operators in $L^2(\Omega)$ such that there exists a constant $\kappa > D/q' + 1$ such that*

$$\| \mathbb{1}_{(k+1)B_\rho \setminus kB\rho} T_\rho \mathbb{1}_{B_\rho} \|_{p\to q} \le C|B_\rho|^{\frac{1}{q}-\frac{1}{p}}(1+k)^{-\kappa} \tag{3.5.24}$$

for all $k \in \mathbb{N}_0, \rho > 0$ and balls B_ρ of radius $\rho > 0$. Let $m \in \mathbb{N}$, $\beta := \frac{1}{m}\big(\kappa - \frac{D}{q'} - 1\big) > 0$ and $\gamma := \kappa/m$, and define the weight function $w(t) := t^\beta(1+t)^{\gamma-\beta}$ for $t > 0$. Then

$$\| \mathbb{1}_{(k+1)B_r \setminus kB_r} T_\rho \mathbb{1}_{B_r} \|_{p\to q} \le c\, w(\rho^m/r^m) \cdot |B_r|^{\frac{1}{q}-\frac{1}{p}}(1+k)^{-\kappa},$$

for all $k \ge 2, \rho > 0$ and balls B_r of radius $r > 0$ with some constant $c > 0$.

Before we prove Lemma 3.5.10 we make the following observation: Let $x, y \in \Omega_0$ and $r > \rho > 0$.

($*$) For fixed $y \in B_r(x)$ the annulus $(k+1)B_r(x)\setminus kB_r(x)$ can be covered as

$$(k+1)B_r(x)\setminus kB_r(x) \subseteq (\ell+1)B_\rho(y)\setminus dB_\rho(y) = \bigcup_{\nu=\ell}^{d} (\nu+1)B_\rho(y)\setminus \nu B_\rho(y)$$

where $d := \lfloor (k+2)r/\rho \rfloor$ and $\ell := \lfloor (k-1)r/\rho \rfloor$, and we have

$$d - \ell \lesssim r/\rho, \quad \text{and for } k \ge 2: \ (1+\ell)^{-\kappa} \lesssim (r/\rho)^{-\kappa}(1+k)^{-\kappa}. \tag{3.5.25}$$

Proof of ($*$): Let $z \in (k+1)B_r(x)\setminus kB_r(x)$, then

$$d(y,z) \le d(y,x) + d(x,z) \le (k+2)r = (k+2)r/\rho \cdot \rho \le (\ell+1)\rho$$

and

$$d(y,z) \ge d(x,z) - d(y,x) \ge (k-1)r = (k-1)r/\rho \cdot \rho \ge d\rho,$$

hence $z \in (\ell+1)B_\rho(y)\setminus dB_\rho(y)$. Moreover by definition we obtain

$$d - \ell \le (k+2)r/\rho - \big((k-1)r/\rho - 1\big) = 3r/\rho + 1 \lesssim r/\rho$$

and for $k \ge 2$:

$$(1+\ell)^{-\kappa} \le \big((k-1)r/\rho\big)^{-\kappa} \lesssim \big((k+1)r/\rho\big)^{-\kappa} = (r/\rho)^{-\kappa}(1+k)^{-\kappa}. \quad \square$$

Proof of Lemma 3.5.10: Let $x \in \Omega_0, k \ge 2$ and $r, \rho > 0$. We first assume that $r > \rho$. We will work in the dual situation, so let $f \in L^{q'}(\Omega)$, then:

$$\begin{aligned}
&\| \mathbb{1}_{B_r(x)} T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} f\|_{p'} \overset{\text{Lemma 3.5.7}}{\lesssim} \|N_{p',\rho}(\mathbb{1}_{B_r(x)} T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} f)\|_{p'} \\
\le\ & \| \mathbb{1}_{B(x,r+\rho)} N_{p',\rho}(T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} f)\|_{p'} \\
\le\ & \| \mathbb{1}_{B(x,r+\rho)} \|_{p'} \cdot \sup_{y\in B(x,r+\rho)} N_{p',\rho}(T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} f)(y) \\
\lesssim\ & | \mathbb{1}_{B(x,r)} |^{1/p'} \cdot \sup_{y\in B(x,r+\rho)} N_{p',\rho}(T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} f)(y).
\end{aligned}$$

For fixed $y \in B(x, r+\rho)$ we have by $(*)$:

$$N_{p',\rho}(T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} f)(y) \lesssim |B(y,\rho)|^{-1/p'} \sum_{\nu=\ell}^{d} \| \mathbb{1}_{B(y,\rho)} T'_\rho \mathbb{1}_{(\nu+1)B_\rho(y)\setminus \nu B_\rho(y)} f)\|_{p'}$$
$$\begin{aligned}
&\lesssim |B(y,\rho)|^{-1/p'} \cdot (d-\ell+1) \cdot |B_\rho(y)|^{-(\frac{1}{q'}-\frac{1}{p'})}(1+\ell)^{-\kappa} \cdot \|f\|_{q'} \\
&\lesssim (r/\rho) \cdot |B_\rho(y)|^{-\frac{1}{q'}}(1+k)^{-\kappa}(r/\rho)^{-\kappa} \cdot \|f\|_{q'} \\
&\lesssim (r/\rho)^{1-\kappa} \cdot (r/\rho)^{D/q'} |B_r(x)|^{-\frac{1}{q'}}(1+k)^{-\kappa} \cdot \|f\|_{q'}.
\end{aligned}$$

Hence we obtain

$$\| \mathbb{1}_{B_r(x)} T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} f\|_{p'} \lesssim (\rho/r)^{m\beta} \cdot |B_r(x)|^{\frac{1}{p'}-\frac{1}{q'}}(1+k)^{-\kappa} \cdot \|f\|_{q'}.$$

By dualization this yields

$$\begin{aligned}
\| \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} T_\rho \mathbb{1}_{B_r(x)} \|_{p\to q} &= \| \mathbb{1}_{B_r(x)} T'_\rho \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} \|_{q'\to p'} \\
&\lesssim (r/\rho)^{m\beta} \cdot |B_r(x)|^{\frac{1}{q}-\frac{1}{p}}(1+k)^{-\kappa}.
\end{aligned}$$

Next we assume $r < \rho$, in this case define $\ell := \lfloor kr/\rho \rfloor$. If $z \in (k+1)B_r(x)\setminus kB_r(x)$, then

$$d(x,z) \le (k+1)r \le (kr/\rho)\cdot\rho + \rho \le (\ell+2)\rho, \quad \text{and } d(x,z) \ge (kr/\rho)\cdot\rho \ge \ell\rho,$$

hence

$$(k+1)B_r(x)\setminus kB_r(x) \subseteq \big((\ell+2)B_\rho(x)\setminus(\ell+1)B_\rho(x)\big) \cup \big((\ell+1)B_\rho(x)\setminus \ell B_\rho(x)\big).$$

So we get

$$\| \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} T_\rho \mathbb{1}_{B_r(x)} \|_{p\to q} \le \sum_{j=0}^{1} \| \mathbb{1}_{(\ell+j+1)B_\rho(x)\setminus(\ell+j)B_\rho(x)} T_\rho \mathbb{1}_{B_\rho(x)} \|_{p\to q}$$
$$\begin{aligned}
&\lesssim \sum_{j=0}^{1} |B_\rho(x)|^{1/q-1/p}(1+\ell+j)^{-\kappa} \le 2|B_r(x)|^{1/q-1/p}(1+\ell)^{-\kappa} \\
&\le 2|B_r(x)|^{1/q-1/p}(kr/\rho)^{-\kappa} \lesssim (\rho/r)^{m\gamma} \cdot |B_r(x)|^{1/q-1/p}(1+k)^{-\kappa}.
\end{aligned}$$

Putting both parts together we obtain

$$\| \mathbb{1}_{(k+1)B_r(x)\setminus kB_r(x)} T_\rho \mathbb{1}_{B_r(x)} \|_{p\to q} \lesssim w(|\lambda|^m/r) \cdot |B_r(x)|^{1/q-1/p}(1+k)^{-\kappa}.$$

□

The proof of Theorem 3.5.4 will essentially follow the lines in [BK03] with modifications as used in [Ku08]; a similar approach is sketched in [Au07], Section 6.1. Thus, the keystone for the proof is the following weak type (q_0, q_0)-criterion, which is a vector-valued version of [Ku08], Theorem 5.4.

Proposition 3.5.11. *Let E, F be Banach spaces, $1 \le q_0 < q \le \infty$, and let $T : L^q(\Omega, E) \to L^q(\Omega, F)$ be a bounded linear operator. Suppose that there is a family $(S_r)_{r>0}$ of uniformly bounded linear operators in $L^q(\Omega, E)$ such that the following weighted norm estimates hold:*

$$\| \mathbb{1}_{A_k(x,r)} S_r \mathbb{1}_{B(x,r)} \|_{L^{q_0}(E) \to L^q(E)} \le |B_{\Omega_0}(x,r)|^{\frac{1}{q} - \frac{1}{q_0}} h(k) \text{ for all } k \in \mathbb{N}_0, \quad (3.5.26)$$

$$\| \mathbb{1}_{A_k(x,r)} T(I - S_r) \mathbb{1}_{B(x,r)} \|_{L^{q_0}(E) \to L^q(F)} \le |B_{\Omega_0}(x,r)|^{\frac{1}{q} - \frac{1}{q_0}} h(k) \text{ for all } k \ge k_0, \quad (3.5.27)$$

for all $x \in \Omega_0, r > 0$ and some constant $k_0 \in \mathbb{N}$, where the sequence h satisfies $h(k) \le c_\delta (k+1)^{-\delta}$ for some constants $c_\delta > 0$ and $\delta > \frac{1}{q} + \frac{D}{q'}$. Then T is of weak type (q_0, q_0) and bounded as an operator $L^p(E) \to L^p(F)$ for all $p \in (q_0, q]$, where the norm $\|T\|_{L(L^p(E), L^p(F))}$ depends only on the involved constants and the sequence h, but not on the operator itself.

We will reproduce a proof that is due to Peer Kunstmann for the scalar-valued case and easily generalizes to the vector-valued case. We will start with a Calderon-Zygmund decomposition in the vector-valued spaces $L^p(E)$.

Lemma 3.5.12 (L^p-Calderon-Zygmund decomposition)**.** *Assume $\Omega = \Omega_0$ and let $p \in [1, \infty)$. Then there exists constants $C_p, A_p > 0$ such that for all $f \in L^p(E)$ an $\alpha > 0$, we find a μ-measurable function g and a countable index-set J and a family $(b_j)_{j \in J}$ of μ-measurable functions with disjoint supports and $(B_j)_{j \in J}$ of balls such that*

(i) $f(x) = g(x) + \sum_{j \in J} b_j(x)$ *for μ-a.e. $x \in \Omega_0$,*

(ii) $\|g\|_\infty \le C_p \, \alpha$,

(iii) $\operatorname{supp}(b_j) \subseteq B_j$ *for all $j \in J$ and* $|\{j \in J : x \in B_j\}| \le A_p$ *for all $x \in \Omega_0$,*

(iv) $\|b_j\|_p \le C_p \, \alpha |B_j|^{1/p}$ *for all $j \in J$,*

(v) $\Big(\sum_{j \in J} |B_j| \Big)^{1/p} \le C \, \alpha^{-1} \, \|f\|_p$,

(vi) $\|g\|_p \le C_p \, \|f\|_p$.

This lemma is proved in [BK03], Theorem 3.1 and Remark 3.2, for the scalar-valued case, and the proof given there extends immediately to the vector-valued case if one simply replaces the modulus $|f|$ by the norm-modulus $|f|_E$ in the proof.

Proof of Proposition 3.5.11. The remarks given at the beginning of this section imply that we can w.l.o.g. assume $\Omega = \Omega_0$. We first observe that by Lemma 3.5.8 the assumptions (3.5.26) and (3.5.27) imply the pointwise estimates

$$N_{q_0',r}(S_r' f)(x) \le C_0 \, M_{q'} f(x) \quad \text{and} \quad N_{q_0',r}((T(I - S_r))' f)(x) \le C_0 \, M_{q'} f(x), \quad (3.5.28)$$

where the former holds for μ-a.e. $x \in \Omega$ and all $f \in L^{q_0}(E) \cap L^q(E)$ and the latter holds for μ-a.e. $x \in \Omega$ and all $f \in L^{q_0}(E) \cap L^q(E)$ with $\operatorname{supp}(f) \cap B(x, k_0 r) = \emptyset$.

Moreover we only have to show that T is of weak type (q_0, q_0), the remaining strong estimates can then be obtained with a vector-valued version of the Marcinkiewicz interpolation theorem, cf. e.g. [GLY07], Lemma 2.5, and the comments given there.

Let $\alpha > 0$ and $f \in L^{q_0}(E) \cap L^q(E)$. Then choose an L^{q_0}-Calderon-Zygmund decomposition according to Lemma 3.5.12, i.e. we choose μ-measurable functions g and $(b_j)_{j\in J}$ for some countable index-set J with disjoint supports and a family of balls $B_j = B(x_j, r_j), j \in J$ such that (i)-(ví) from Lemma 3.5.12 hold. Then formally

$$Tf = Tg + \sum_{j\in J} TS_{2r_j} b_j + \sum_{j\in J} T(I - S_{2r_j})b_j =: Tg + h_1 + h_2 \tag{3.5.29}$$

We first treat the "good" part g:

$$\begin{aligned} |\{x \in \Omega \mid \|Tg(x)\|_F > \alpha\} &\le \alpha^{-q_0} \int_\Omega |Tg|_F^q \, d\mu \le \|T\|_{q\to q}^q \, \alpha^{-q} \int_\Omega |g|_E^q \, d\mu \\ &\le \|T\|_{q\to q}^q \, \alpha^{-q_0} \, C_{q_0}^{q-q_0} \, \|g\|_{q_0}^{q_0} \le \|T\|_{q\to q}^q \, C_{q_0}^q \left(\frac{\|f\|_{q_0}}{\alpha}\right)^{q_0}. \end{aligned}$$

To estimate the term h_1 and justify the representation (3.5.29) it is sufficient to show that

$$(S_{2r_j} b_j)_{j\in J} \text{ is summable in } L^q(E) \text{ and } \Big\| \sum_{j\in J} S_{2r_j} b_j \Big\|_q^q \le C\alpha^q \Big\| \sum_{j\in J} \mathbb{1}_{B_j} \Big\|_q^q, \tag{3.5.30}$$

because then by the L^q-boundedness of T and the properties of the L^{q_0}-Calderon-Zygmund decomposition

$$\begin{aligned} |\{x \in \Omega \mid \|h_1(x)\|_F > \alpha\} &\le \|T\|_{q\to q}^q \, \alpha^{-q} \Big\| \sum_{j\in J} S_{2r_j} b_j \Big\|_q^q \le C \|T\|_{q\to q}^q \Big\| \sum_{j\in J} \mathbb{1}_{B_j} \Big\|_q^q \\ &\le C \|T\|_{q\to q}^q \sum_{j\in J} |B_j| \le C C_{q_0}^{q_0} \|T\|_{q\to q}^q \left(\frac{\|f\|_{q_0}}{\alpha}\right)^{q_0}. \end{aligned}$$

For the proof of (3.5.30) we take $\phi \in L^{q'}(E')$ with $\|\phi\|_{q'} = 1$, and obtain

$$\begin{aligned} |\langle S_{2r_j} b_j, \phi\rangle| &= |\langle b_j, S'_{2r_j}\phi\rangle| \le C_{q_0} \alpha |B_j|^{1/q_0} \cdot \left(\int_{B_j} |S'_{2r_j}\phi|_{E'}^{q_0'} \, d\mu \right)^{1/q_0'} \\ &= C_{q_0} \alpha |B_j| \cdot N_{q_0', r_j}(S'_{2r_j}\phi)(x_j) \le 3C_D \, C_{q_0} \alpha \cdot \int_{B_j} N_{q_0', 2r_j}(S'_{2r_j}\phi) \, d\mu \\ &\le \underbrace{3C_D \, C_{q_0} C_0}_{=:c_0} \, \alpha \cdot \int_{B_j} M_{q'}(\phi) \, d\mu. \end{aligned}$$

Let $E := \bigcup_{j\in J} B_j$, then for any finite subset $J_0 \subseteq J$ we obtain

$$\begin{aligned} \Big| \Big\langle \sum_{j\in J_0} S_{2r_j} b_j, \phi \Big\rangle \Big| &\le c_0 \, \alpha \cdot \sum_{j\in J_0} \int_{B_j} M_{q'}(\phi) \, d\mu = c_0 \, \alpha \cdot \int_E \sum_{j\in J_0} \mathbb{1}_{B_j} \, M_{q'}(\phi) \, d\mu \\ &\le c_0 A_{q_0} \, \alpha \cdot \int_E M_{q'}(\phi) \, d\mu = c_0 A_{q_0} \, \alpha \cdot \int_E \left(M|\phi|_{E'}^{q'} \right)^{1/q'} d\mu. \end{aligned}$$

An inequality by Kolmogorov (cf. [GCRdF85], Lemma V.2.8, p. 485) states

$$\int_E g^{1/q'} \, d\mu \le q \, |E|^{1/q} \Big(\sup_{t>0} \big(t \cdot |\{x \in \Omega \mid g(x) > t\}|\big) \Big)^{1/q'}$$

for each measurable function $g \geq 0$, hence we obtain together with the weak $(1,1)$-boundedness of the maximal operator M

$$\begin{aligned} \Big|\Big\langle \sum_{j\in J_0} S_{2r_j} b_j, \phi \Big\rangle\Big| &\leq c_0 A_{q_0} q\, \alpha \cdot |E|^{1/q} \Big(\sup_{t>0} \big(t \cdot |\{x \in \Omega \mid M|\phi|_{E'}^{q'} > t\}|\big)\Big)^{1/q'} \\ &\lesssim \alpha \cdot \Big\| \sum_{j\in J} \mathbb{1}_{B_j} \Big\|_q \|\phi\|_{q'}. \end{aligned}$$

Since $L^{q'}(E')$ is a norming subspace of $(L^q(E))'$ this yields (3.5.30). The term h_2 can be treated similarly: let $F := \bigcup_{j\in J} B(x_j, 2k_0 r_j)$, then

$$|\{x \in \Omega \ : \ \|h_2(x)\|_F > \alpha\} \leq |F| + \Big|\Big\{x \in \Omega \ : \ \Big\| \sum_{j\in J} \mathbb{1}_{F^c}(x) T(I - S_{2r_j}) b_j(x))\Big\|_F > \alpha\Big\},$$

where $|F| \leq C_D C_{q_0} (2k_0)^D \cdot \alpha^{-q_0} \|f\|_{q_0}^{q_0}$ by (v) from Lemma 3.5.12. Now we can use the same argument as above for $\mathbb{1}_{F^c} T(I - S_{2r_j})$ in place of S_{2r_j}. □

If we apply Proposition 3.5.11 for the Banach spaces $E = F = \ell^s$ and tensor extensions $T \otimes \mathrm{Id}_{\ell^s}$ of operators in a scalar valued space $L^q(\Omega)$, we can derive the following corresponding criterion for $\mathcal{R}_s$-boundedness.

Corollary 3.5.13. *Let $\Omega \subseteq \Omega_0$ be a measurable subset, $1 \leq q_0 < q \leq \infty$, and let $\mathcal{T} \subseteq L(L^q)$ be a set of bounded linear operators. Suppose that there is a family $(S_r)_{r>0}$ of uniformly bounded linear operators in L^q such that the following weighted norm estimates hold:*

$$\| \mathbb{1}_{(k+1)B\setminus kB} S_r \mathbb{1}_B \|_{q_0\to q} \leq |B|^{\frac{1}{q} - \frac{1}{q_0}} h(k) \quad \text{for all } k \in \mathbb{N}_0, \tag{3.5.31}$$

$$\| \mathbb{1}_{(k+1)B\setminus kB} T(I - S_r) \mathbb{1}_B \|_{q_0\to q} \leq |B|^{\frac{1}{q} - \frac{1}{q_0}} h(k) \quad \text{for all } k \geq k_0, \tag{3.5.32}$$

for all balls B of radius r, $T \in \mathcal{T}$ and some constant $k_0 \in \mathbb{N}$, where the sequence h satisfies $h(k) \leq c_\delta (k+1)^{-\delta}$ for some constants $c_\delta > 0$ and $\delta > \frac{1}{q} + \frac{D}{q'}$. Then $\mathcal{T}$ is $\mathcal{R}_s$-bounded in L^p for all $p \in (q_0, q]$ and $s \in [p, q]$.

Proof. Note first that if we apply Proposition 3.5.11 for the scalar-valued operators we obtain that $\mathcal{T}$ is uniformly bounded in each L^p and hence also $\mathcal{R}_p$-bounded in each L^p for $p \in (q_0, q]$. Let $p \in (q_0, q]$ and $s \in [p, q]$, and let $n \in \mathbb{N}$. We define the operators $\widetilde{T} := T \otimes I_{\ell_n^s}$ and $S_r := S_r \otimes I_{\ell_n^s}$ for all $T \in \mathcal{T}, r > 0$. Then $\{\widetilde{T} \ : \ T \in \mathcal{T}\}$ and $\{\widetilde{S}_r \ : \ r > 0\}$ are uniformly bounded in $L^s(\ell_n^s)$.

Now let B be a ball of radius r, $k \in \mathbb{N}_0$, $C := (k+1)B\setminus kB$ and $f_j \in L^{q_1}$ for all $j \in \mathbb{N}$. Then we obtain with $g_j := \mathbb{1}_C S_r f_j$:

$$\| \mathbb{1}_{(k+1)B\setminus kB} \widetilde{S}_r \mathbb{1}_B (f_j)_j \|_{L^s(\ell^s)} = \|(g_j)_j\|_{L^s(\ell^s)} = \|(g_j)_j\|_{\ell^s(L^s)} \tag{3.5.33}$$

$$= \Big(\sum_{j\in\mathbb{N}} \|g_j\|_s^s\Big)^{1/s} \leq |B|^{\frac{1}{p} - \frac{1}{s}} h(k) \cdot \Big(\sum_{j\in\mathbb{N}} \|f_j\|_p^s\Big)^{1/s} \tag{3.5.34}$$

$$\leq |B|^{\frac{1}{p} - \frac{1}{s}} h(k) \cdot \|(f_j)_{j\in\mathbb{N}}\|_{L^p(\ell^s)}. \tag{3.5.35}$$

In the last estimate we used that $L^p(\ell^s) \hookrightarrow \ell^s(L^p)$, since $p \le s$. By the same argument we obtain

$$\| \mathbb{1}_{(k+1)B\setminus kB}\, \widetilde{T(I-S_r)}\, \mathbb{1}_B \|_{L^p(\ell^s)\to L^s(\ell^s)} \le |B|^{\frac{1}{p}-\frac{1}{s}} h(k)$$

for all $k \ge k_0$. This shows that the assumptions of Proposition 3.5.11 are fulfilled with $E = F = \ell^s_n$ and $(q_0, q) = (p, s)$, hence $\{\widetilde{T} \;:\; T \in \mathcal{T}\}$ is uniformly bounded in $L^r(\ell^s_n)$ for all $r \in (p, s)$, which just is the $\mathcal{R}_s$-boundedness of $\mathcal{T}$ in L^r. Since $p \in (q_0, q]$ was arbitrary this shows the claim. $\square$

We will also need the following technical lemma which is a special case of Lemma 3.7 in [BK03]:

Lemma 3.5.14. *Let $N \in \mathbb{N}, b > 0$ and $0 < \beta, \gamma < N$, and let $w(t) := t^\beta (1+t)^{\gamma-\beta}$ for $t > 0$. Then*

$$\int_0^\infty \Big(\int_0^\infty e^{-bts}(1 \wedge s^N)\, ds \Big) w(t)\, dt < +\infty.$$

Proof. We split up the integral as

$$\begin{aligned}
&\int_0^\infty \Big(\int_0^\infty e^{-bts}(1 \wedge s^N)\, ds \Big) w(t)\, dt \\
= \;& \int_0^1 \Big(\int_0^1 e^{-bts} s^N\, ds \Big) w(t)\, dt + \int_0^1 \Big(\int_1^\infty e^{-bts}\, ds \Big) w(t)\, dt \\
&+ \int_1^\infty \Big(\int_0^1 e^{-bts} s^N\, ds \Big) w(t)\, dt + \int_1^\infty \Big(\int_1^\infty e^{-bts}\, ds \Big) w(t)\, dt \\
=: \;& I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Then clearly $I_1 < +\infty$. Moreover we have

$$\begin{aligned}
I_2 &= \frac{1}{b} \int_0^1 e^{-bt} w(t)\, \frac{dt}{t} \lesssim \int_0^1 e^{-bt} t^\beta\, \frac{dt}{t} < +\infty\,, \text{and} \\
I_4 &= \frac{1}{b} \int_1^\infty e^{-bt} w(t)\, \frac{dt}{t} \lesssim \int_1^\infty e^{-bt} t^\gamma\, \frac{dt}{t} < +\infty
\end{aligned}$$

since $\beta, b > 0$. Finally

$$\int_0^1 e^{-bts} s^N\, ds = \frac{N!}{(bt)^{N+1}}\, e^{-bt} \sum_{j=N+1}^\infty \frac{(bt)^j}{j!} \lesssim 1 \wedge t^{-N-1},$$

so $\gamma - N < 0$ yields

$$I_3 \lesssim \int_1^\infty 1 \wedge t^{-N-1} w(t)\, dt \lesssim \int_1^\infty t^{\gamma-N} w(t)\, \frac{dt}{t} < +\infty.$$

$\square$

We can now turn to the proof of Theorem 3.5.4.

Proof of Theorem 3.5.4. Again, by the remarks given at the beginning of this section we can w.l.o.g. assume $\Omega = \Omega_0$. Let $q_0, q \in [p_0, p_1]$ with $q_0 < q$ and $\omega \in (\omega_0, \pi/2)$, and define $\kappa := \kappa_0 - D\Big(\frac{1}{p_0} - \frac{1}{p_1}\Big) > \frac{D}{p_0 \wedge p_1'} + 1$. We fix some $\theta_0 \in (\omega_0, \omega)$, then by the assumptions of Theorem 3.5.4 and by Lemma 3.5.1 the weighted estimate

$$\| \mathbb{1}_{A_k(x,|\lambda|^{1/m})} T_\lambda \mathbb{1}_{B(x,|\lambda|^{1/m})} \|_{q_0 \to q} \le C_0 |B(x, |\lambda|^{1/m})|^{\frac{1}{q} - \frac{1}{q_0}} (1+k)^{-\kappa} \tag{3.5.36}$$

holds for all $x \in \Omega_0, k \in \mathbb{N}_0$ and $\lambda \in \Sigma_{\pi/2-\theta_0}$ for some constant $C_0 > 0$.

Let $\varphi \in H_0^\infty(\Sigma_\omega)$ with $\|\varphi\|_{\infty,\omega} \le 1$ and define $T := \varphi(A)$. Let $N \in \mathbb{N}$, whose size will be specified later, and define

$$S_r := I - (I - e^{-r^m A})^N = \sum_{k=1}^{N} \binom{N}{k} (-1)^{k+1} e^{-k r^m A} \qquad \text{for all } r > 0.$$

Then the family $(S_r)_{r>0}$ is uniformly bounded in L^q. Since

$$\kappa > \frac{D}{p_1'} + 1 \ge \frac{D}{q'} + 1 > \frac{D}{q'} + \frac{1}{q} \tag{3.5.37}$$

we can apply Lemma 3.5.10 and obtain that $(S_r)_{r>0}$ also satisfies the assumption (3.5.26) of Proposition 3.5.11, or (3.5.31) of Corollary 3.5.13, respectively, with $q_0 = p_0$. Now the key step is to check (3.5.32). After this is done, we can derive the full statement of the theorem if we apply Proposition 3.5.11 and Corollary 3.5.13 in various steps and do some standard duality and approximation arguments.

We define the integration paths $\Gamma_\sigma^\pm : (0, \infty) \to \mathbb{C}, t \mapsto t e^{\pm i\sigma}$. Let $\theta \in (\theta_0, \omega)$ and choose $\delta > 0$ with $\pi/2 - \theta < \delta < \pi/2 - \theta_0$. Let $r > 0$ and B be a ball of radius r, $k \in \mathbb{N}$ and let $C := (k+1)B \backslash kB$. We define $\psi(z) := \varphi(z)(1 - e^{-r^m z})^N$, then $\psi \in H_0^\infty(\Sigma_\omega)$ and

$$|\psi(z)| \lesssim \|\varphi\|_\infty \cdot (1 \wedge (r^m z)^N).$$

We use the following variant of the Laplace transform representation of the resolvent:

$$R(z, A) = -\int_{\Gamma_\delta^\pm} e^{z\lambda} T_\lambda \, d\lambda \qquad \text{if } z \in (0, +\infty) \cdot e^{\pm i\theta}.$$

This leads to

$$\begin{aligned}
T(I - S_r) &= \psi(A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} \psi(z) R(z, A) \, dz \\
&= -\frac{1}{2\pi i} \int_{\Gamma_\theta^+} \psi(z) R(z, A) \, dz + \frac{1}{2\pi i} \int_{\Gamma_\theta^-} \psi(z) R(z, A) \, dz \\
&= \frac{1}{2\pi i} \int_{\Gamma_\theta^+} \psi(z) \int_{\Gamma_\delta^+} e^{z\lambda} T_\lambda \, d\lambda \, dz - \frac{1}{2\pi i} \int_{\Gamma_\theta^-} \psi(z) \int_{\Gamma_\delta^-} e^{z\lambda} T_\lambda \, d\lambda \, dz \\
&= \frac{1}{2\pi i} \int_{\Gamma_\delta^+} \Big(\int_{\Gamma_\theta^+} e^{z\lambda} \psi(z) \, dz \Big) T_\lambda \, d\lambda - \frac{1}{2\pi i} \int_{\Gamma_\delta^-} \Big(\int_{\Gamma_\theta^-} e^{z\lambda} \psi(z) \, dz \Big) T_\lambda \, d\lambda,
\end{aligned}$$

hence

$$\mathbb{1}_C\, T(I-S_r)\, \mathbb{1}_B = \frac{1}{2\pi i}\Big(\int_{\Gamma_\delta^+} \Big(\int_{\Gamma_\theta^+} e^{z\lambda}\psi(z)\, dz \Big)\, \mathbb{1}_C\, T_\lambda\, \mathbb{1}_B\; d\lambda - \int_{\Gamma_\delta^-} \Big(\int_{\Gamma_\theta^-} e^{z\lambda}\psi(z)\, dz \Big)\, \mathbb{1}_C\, T_\lambda\, \mathbb{1}_B\; d\lambda \Big).$$

Let $b := -\cos(\theta+\delta) > 0$, and choose $\beta, \gamma > 0$ and the weight function w according to Lemma 3.5.10. Then Minkowski's inequality and the weighted norm inequality from Lemma 3.5.10 yield

$$\begin{aligned}
&\|\, \mathbb{1}_C\, T(I-S_r)\, \mathbb{1}_B \,\|_{q_0\to q} \\
\le\;& \frac{1}{2\pi} \int_{\Gamma_\delta^+} \Big(\int_{\Gamma_\theta^+} e^{\mathrm{Re}(z\lambda)} |\psi(z)|\, d|z| \Big) \|\, \mathbb{1}_C\, T_\lambda\, \mathbb{1}_B \,\|_{q_0\to q}\, d|\lambda| \\
&+ \frac{1}{2\pi} \int_{\Gamma_\delta^-} \Big(\int_{\Gamma_\theta^-} e^{\mathrm{Re}(z\lambda)} |\psi(z)|\, d|z| \Big) \|\, \mathbb{1}_C\, T_\lambda\, \mathbb{1}_B \,\|_{q_0\to q}\, d|\lambda| \\
\lesssim\;& \int_{\Gamma_\delta^+} \Big(\int_{\Gamma_\theta^+} e^{-b|z||\lambda|} (1 \wedge (r^m|z|)^N)\, d|z| \Big) w(|\lambda|/r^m)\, d|\lambda| \cdot |B|^{\frac{1}{q}-\frac{1}{q_0}} (1+k)^{-\kappa} \cdot \|\varphi\|_\infty \\
&+ \int_{\Gamma_\delta^-} \Big(\int_{\Gamma_\theta^-} e^{-b|z||\lambda|} (1 \wedge (r^m|z|)^N)\, d|z| \Big) w(|\lambda|/r^m)\, d|\lambda| \cdot |B|^{\frac{1}{q}-\frac{1}{q_0}} (1+k)^{-\kappa} \cdot \|\varphi\|_\infty \\
\lesssim\;& \Big(\int_0^\infty \Big(\int_0^\infty e^{-bts} (1 \wedge s^N)\, ds \Big) w(t)\, dt \Big) \cdot |B|^{\frac{1}{q}-\frac{1}{q_0}} (1+k)^{-\kappa} \cdot \|\varphi\|_\infty \\
\lesssim\;& |B|^{\frac{1}{q}-\frac{1}{q_0}} (1+k)^{-\kappa} \cdot \|\varphi\|_\infty,
\end{aligned}$$

since the latter integral is finite by Lemma 3.5.14 if we choose $N > \gamma$.

We can now apply Proposition 3.5.11 with $q_0 = p_0, q = 2$ to obtain that T is bounded in L^p for all $p \in (p_0, 2]$. On the other hand, assumption (3.5.13) is just the same as (3.5.12) for the dual operator A' with the pairing (p_1', p_0') in place of (p_0, p_1). Observe that the condition on κ_0 is the same for this pairing, since

$$D\Big(\frac{1}{p_0} - \frac{1}{p_1} + \frac{1}{p_0 \wedge p_1'}\Big) + 1 = D\Big(\frac{1}{p_1'} - \frac{1}{p_0'} + \frac{1}{p_1' \wedge (p_0')'}\Big) + 1,$$

thus also in the dual situation we have the appropriate estimate on $\kappa := \frac{D}{p_0 \wedge p_1'} + 1$ to apply Lemma 3.5.10 and Proposition 3.5.11, namely $\kappa > \frac{D}{p_0} + 1 = \frac{D}{(p_0')'} + 1 \ge \frac{D}{(p_0')'} + \frac{1}{p_0'}$. Using the same arguments for A' in place of A yields that also T' is bounded for all $p' \in (p_1', 2]$, hence T is bounded in L^p for all $p \in (p_0, p_1)$.

Now we can take any $q \in (p_0, p_1)$ as a new starting point, i.e. we consider T as an operator in L^q with corresponding $p_0 \to q$ weighted estimates, and use Corollary 3.5.13 to get $\mathcal{R}_s$-boundedness of T in L^p for all $p \in (p_0, p_1)$ and $s \in [p, p_1)$. Then we again dualize and obtain the $\mathcal{R}_{s'}$-boundedness of T' in $L^{p'}$ for all $p' \in (p_1', p_0')$ and $s' \in [p', p_0')$, hence the $\mathcal{R}_s$-boundedness of T in L^p for all $p \in (p_0, p_1)$ and $s \in (p_0, p]$.

We use finally that the constants did not explicitly depend on $\varphi \in H_0^\infty(\Sigma_\omega)$ with $\|\varphi\|_{\infty,\omega} \le 1$, so the general assertion follows from Proposition 3.2.23. $\square$

3.6 The s-intermediate spaces for differential operators and Triebel-Lizorkin spaces

In this section we consider differential operators of order $2m$ in the spaces $X := L^p(\mathbb{R}^d, \mathbb{C}^N)$. Hence we fix $N, m, d \in \mathbb{N}$ and $p, q \in (1, \infty)$. In order to be consistent with the notation of classical Triebel-Lizorkin spaces we will use the terminology *$\mathcal{R}_q$-boundedness* instead of $\mathcal{R}_s$-boundedness in this section. Moreover, all function spaces will be defined on the whole space $\mathbb{R}^d$, hence we will simply write $\mathcal{F}$ instead of $\mathcal{F}(\mathbb{R}^d, \mathbb{C}^N)$ for any function space $\mathcal{F}(\mathbb{R}^d, \mathbb{C}^N)$, where e.g. $\mathcal{F} \in \{F^s_{p,q}, B^s_{p,q}, W^{s,p}, H^{s,p}, L^p\}$.

We will show that certain classes of elliptic operators have an $\mathcal{R}_q$-bounded H^∞-calculus, but the central issue in this section is to show that these operators have the same q-intermediate spaces as the Laplace operator. So we first recall the following theorem concerning only the Laplace operator, which is just a compilation of Proposition 3.2.24 and Proposition 3.3.12 and an application of Theorem 3.3.23 together with the well known fact that $(-\Delta)^m$ has a bounded H^∞-calculus in X with $\omega_{H^\infty}((-\Delta)^m) = 0$.

Theorem 3.6.1. *Let $A := (-\Delta)^m$ with $D(A) := W^{2m,p}$ in X. Then A has an $\mathcal{R}_q$-bounded H^∞-calculus with $\omega_{\mathcal{R}_q^\infty}(A) = 0$, and for all $\theta \in \mathbb{R}$ we have*

$$\dot{X}^\theta_{q,A} = \dot{F}^{2m\theta}_{p,q}, \text{ and } X^\theta_{q,A} = F^{2m\theta}_{p,q} \text{ if } \theta > 0.$$

In particular, $(-\Delta)^m$ has a bounded $H^\infty(\Sigma_\sigma)$ calculus for all $\sigma > 0$ in the spaces $\dot{F}^s_{p,q}$ for all $s \in \mathbb{R}$, and in the space $F^s_{p,q}$ for all $s > 0$.

So if we show that a differential operator $\mathcal{A}$ has the same q-intermediate spaces as the Laplace operator, Theorem 3.3.23 yields that $\mathcal{A}$ has a bounded H^∞-calculus in the Triebel-Lizorkin spaces $\dot{F}^s_{p,q}$, $s \in \mathbb{R}$, and in $F^s_{p,q}$ if $s > 0$.

3.6.1 Elliptic differential operators in non-divergence form on $\mathbb{R}^d$

We start with elliptic operators in non-divergence form. We will use the notion of (M, ω_0)-elliptic operators as it was introduced in [AHS94], cf. also [DS97]. The structure of the proof of the main theorem of this section follows the line of [KW04], Chapters 6 and 13, where under the same hypotheses it is shown that elliptic differential operators are $\mathcal{R}$-sectorial and have a bounded H^∞-calculus in the space L^p. In fact, this result goes back to [AHS94] for Hölder continuous coefficients and [DS97] for BUC-coefficients.

We consider the differential operator

$$\mathcal{A} := \mathcal{A}(x, D) \quad := \quad \sum_{|\alpha| \le 2m} a_\alpha(x) D^\alpha \tag{3.6.1}$$

of order $2m$ with measurable coefficients $a_\alpha : \mathbb{R}^d \to \mathbb{C}^{N \times N}$. Then $\mathcal{A}(x, \xi) = \sum_{|\alpha| \le 2m} a_\alpha(x) \xi^\alpha$, where $x, \xi \in \mathbb{R}^d$, is the symbol of $\mathcal{A}$, i.e. formally $\mathcal{A}$ acts as

$$\mathcal{A}(x, D)u(x) = \sum_{|\alpha| \le 2m} a_\alpha(x)(D^\alpha u)(x) = \mathcal{F}_\xi^{-1}\big(\mathcal{A}(x,\xi)\mathcal{F}(\xi)\big)(x).$$

Moreover we define the principle part of $\mathcal{A}$ as $\mathcal{A}_\pi(x, D) := \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha$.

Let $M > 0$ and $\omega_0 \in [0, \pi)$. The differential operator $\mathcal{A}$ as in (3.6.1) is called (M, ω_0)-elliptic if the following uniform ellipticity conditions on the principle symbol hold:

$$\sum_{|\alpha|=2m} \|a_\alpha\|_\infty \le M, \tag{3.6.2}$$

$$\sigma(\mathcal{A}_\pi(x,\sigma)) \subseteq \overline{\Sigma}_{\omega_0} \backslash \{0\} \text{ and } |(\mathcal{A}_\pi(x,\sigma))^{-1}| \le M \quad \text{for all } x \in \mathbb{R}^d, \sigma \in S^{d-1}. \tag{3.6.3}$$

Observe that in the case $N = 1$ the above assumption 3.6.3 is equivalent to the usual uniform ellipticity conditions $\mathcal{A}_\pi(x,\xi) \in \overline{\Sigma}_{\omega_0}$ and $|\mathcal{A}_\pi(x,\xi)| \ge \frac{1}{M}|\xi|^{2m}$ for all $x, \xi \in \mathbb{R}^d$. Moreover we will use the following additional boundedness property for the lower order terms:

$$\sum_{|\alpha| \le 2m-1} \|a_\alpha\|_\infty \le M. \tag{3.6.4}$$

If $\mathcal{A}$ is an (M, ω_0)-elliptic operator, then we define the L^p-realization A_p of $\mathcal{A}$ as

$$A_p u \;:=\; \sum_{|\alpha|=2m} a_\alpha(D^\alpha u) \quad \text{for all } u \in D(A_p) := W^{2m,p}.$$

If we assume in addition $N = 1$ and that the highest order coefficients are uniformly continuous, or $m = 1$ and the operator is homogeneous with VMO coefficients, we can conclude with the results of Section 3.5, Example (b) that an appropriate translate of A_p has an $\mathcal{R}_q$-bounded H^∞-calculus if $\omega_0 < \pi/2$ via generalized Gaussian estimates for the generated analytic semigroup:

Proposition 3.6.2. *Assume $N = 1$. Let $\omega_0 \in [0, \pi/2), M > 0$ and A be an (M, ω_0)-elliptic operator. Assume that either*

(I) $a_\alpha \in BUC$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = 2m$, or

(II) $m = 1$, $a_\alpha \in VMO$ for all $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = 2$ and $a_\alpha = 0$ if $|\alpha| < 2$.

Then the L^p-realization $\nu + A_p$ of $\mathcal{A}$ has an $\mathcal{R}_q$-bounded H^∞-calculus for some $\nu \ge 0$.

Nevertheless, we are more interested in identifying the associated q-intermediate spaces for the operators A_p. For this we will make stronger assumptions, namely we will assume the coefficients in the principal part to be Hölder-continuous. Then we obtain the following theorem, which is one of the main results of this section.

Theorem 3.6.3. *Let $\omega_0 \in [0, \pi), M > 0, \gamma \in (0, 2m)$ and $\sigma > \omega_0$. Then there is $\nu \ge 0$ such that for each L^p-realization A_p of an (M, ω_0)-elliptic operator*

$$\mathcal{A}(x, D) = \sum_{|\alpha| \le 2m} a_\alpha(x)\, D^\alpha,$$

where all $a_\alpha : \mathbb{R}^d \to \mathbb{C}^{N \times N}$ are measurable, the conditions (3.6.2), (3.6.3) and (3.6.4) are fulfilled and $a_\alpha \in C^\gamma(\mathbb{R}^d, \mathbb{C}^{N \times N})$ if $|\alpha| = 2m$, the following assertions hold:

(1) $\nu + A_p$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\sigma)$-calculus.

(2) For all $\theta \in (-\frac{\gamma\wedge 1}{2m}, 1)$ we have $\dot{X}^\theta_{\nu+A_p,q} \approx \dot{X}^\theta_{\nu+(-\Delta)^m,q}$, and $X^\theta_{\nu+A_p,q} \approx X^\theta_{\nu+(-\Delta)^m,q} \approx F^{2m\theta}_{p,q}$ if $\theta > 0$, respectively.

(3) For all $s \in (0, 2m)$ the part $\nu + A_{p,q,s}$ of the differential operator $\nu + A_p$ in the space $F^s_{p,q}$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus.

(4) If in addition $a_\alpha \in C^\gamma(\mathbb{R}^d, \mathbb{C}^{N\times N})$ for all $|\alpha| \le 2m$, then $D(\nu + A_{p,q,s}) = F^{s+2m}_{p,q}$ for all $s \in (0, \gamma)$.

In all cases the bounds and equivalency constants do not depend on the explicit operator $\mathcal{A}$ but only on the constants $N, \omega_0, M, \gamma, \sigma$.

It is clear that (3) follows by Theorem 3.3.23 once (2) has been established. Note that in general for $s \in (0, 2m)$ the domain of the operator $\nu + A_{p,q,s}$ in $F^s_{p,q}$ is the space $X^{1+s/2m}_{q,\nu+A_p}$, which need not coincide with the space $F^{s+2m}_{p,q}$. This is due to the fact that if the lower order coefficients are only assumed to be in L^∞ they are in general not pointwise multipliers in the space $F^s_{p,q}$. However, if the additional assumption of (4) holds, then the coefficients of the lower order terms are pointwise multipliers, and we obtain the "right" domain $D(\nu + A_{p,q,s}) = F^{s+2m}_{p,q}$ in that case. If there are lower order terms, it is necessary that the coefficients a_α are pointwise multipliers in $F^s_{p,q}$ (at least in the case $\alpha = 0$) for this identity to hold, so this shows the importance of the knowledge of pointwise multipliers in $F^s_{p,q}$ when considering differential operators in Triebel-Lizorkin spaces. We refer to [Si93] and [JL01] for some general results about multiplication of functions in Triebel-Lizorkin spaces, and to [Si99] for abstract characterizations of pointwise multipliers in the space $F^s_{p,q}$. Finally let us mention the recent paper [DM06], where known sufficient conditions for pointwise multipliers in the space $F^s_{p,q}$ have been improved.

Because of its importance we extract the result concerning the bounded H^∞-calculus by combining (3) and (4) of Theorem 3.6.3 as a separated theorem:

Theorem 3.6.4. *Let $\omega_0 \in [0, \pi), M > 0, \gamma \in (0, 2m)$ and $\sigma > \omega_0$. Then there is $\nu \ge 0$ with the following property: Let*

$$\mathcal{A}(x, D) = \sum_{|\alpha|\le 2m} a_\alpha(x)\, D^\alpha$$

be an (M, ω_0)-elliptic operator, where $a_\alpha \in C^\gamma(\mathbb{R}^d, \mathbb{C}^{N\times N})$ for all $|\alpha| \le 2m$ and the ellipticity conditions (3.6.2), (3.6.3), and moreover (3.6.4) are fulfilled. Then the realization $\nu + A_{p,q,s}$ of the differential operator $\nu + \mathcal{A}$ in the space $F^s_{p,q}$ with domain $D(\nu + A_{p,q,s}) := F^{s+2m}_{p,q}$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus for all $s \in (0, \gamma)$. □

One can already find results about bounded H^∞-calculus in Triebel-Lizorkin spaces in the literature, even for the more general class of pseudodifferential operators under mild regularity assumptions. Such results, also under the condition of Hölder continuous symbols in the principal part, are indicated in [ES08] and more directly in [DSS09]. Nevertheless, these results are in full details proven in some other scale of spaces like Sobolev- and Hölder spaces, and it is only suggested how the proofs can be adapted to the situation of operators in Triebel-Lizorkin spaces. Moreover, the techniques used in [DSS09] are based on methods for pseudodifferential operators, whereas our methods are operator theoretical, based on the comparison and perturbation theorems from Section 3.4. Finally, our results will not only show that the operators we consider have a bounded H^∞-calculus in classical Triebel-Lizorkin spaces, but also the norm equivalences $\dot{X}^\theta_{\nu+A_p,q} \approx \dot{X}^\theta_{\nu+(-\Delta)^m,q}$ and $X^\theta_{\nu+A_p,q} \approx X^\theta_{\nu+(-\Delta)^m,q} \approx F^{2m\theta}_{p,q}$ for appropriate θ, i.e. we can express the norm in the classical Triebel-Lizorkin spaces by the s-power function norms associated to more general elliptic operators instead of the Laplacian. These results are new.

The proof of Theorem 3.6.3 will be done in the classical 3-step method: We start by considering constant coefficient operators, which can be handled with the Comparison Theorems 3.4.3 and 3.4.4. In the second step we will consider small perturbations of operators with constant coefficient, where we use the Perturbation Theorem 3.4.6 for the assertions (1)-(3) and Theorem 3.4.10 for the assertion (4). The general case will be done in the third step using a localization procedure.

Step I. Constant Coefficients.

We start with homogeneous elliptic operators with constant coefficients. In this case it is no surprise that the assertions of Theorem 3.6.3 can be improved in the following way:

Theorem 3.6.5. *Let $\omega_0 \in [0,\pi), M > 0$ and A_p be the L^p-realization of a homogeneous (M,ω_0)-elliptic operator $\mathcal{A}$ with constant coefficients, i.e. $\mathcal{A} = \sum\limits_{|\alpha|=2m} a_\alpha D^\alpha$ where $a_\alpha \in \mathbb{C}^{N\times N}$. Then the following assertions hold:*

(1) For each $\sigma > \omega_0$ the operator A_p has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\sigma)$-calculus, and the $\mathcal{R}_q^\infty$-constant $M^\infty_{s,\sigma}$ does not depend on the explicit operator $\mathcal{A}$ but only on the constants M, ω_0.

(2) For all $\theta \in \mathbb{R}$ we have $\dot{X}^\theta_{A_p,q} \approx \dot{X}^\theta_{(-\Delta)^m,q} \approx \dot{F}^{2m\theta}_{p,q}$, and $X^\theta_{A_p,q} \approx X^\theta_{(-\Delta)^m,q} \approx F^{2m\theta}_{p,q}$ if $\theta > 0$, respectively, and the equivalence constants only depend on the constants M, ω_0, but not on the explicit operator A_p and can be chosen uniformly if $|\theta| \le \alpha$ for some fixed $\alpha > 0$, if one chooses a fixed auxiliary function to calculate the norms.

(3) Let $s \in \mathbb{R}$. The part (via extrapolation) $\dot{A}_{p,q,s}$ of the differential operator A_p in the space $\dot{F}^s_{p,q}$ with domain $D(\dot{A}_{p,q,s}) = \dot{F}^{s+2m}_{p,q} \cap \dot{F}^s_{p,q}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\dot{A}_{p,q,s}) \le \omega_0$.

(4) Let $s > 0$. The realization $A_{p,q,s}$ of the differential operator A_p in the space $F^s_{p,q}$ with domain $D(A_{p,q,s}) = F^{s+2m}_{p,q}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(A_{p,q,s}) \le \omega_0$.

Although we have no references for the explicit assertions (1),(3) and (4) of Theorem 3.6.5, we are sure that these results may already be seen as to be known. For example, since operators with constant coefficients are in particular pseudodifferential operators with smooth symbols, the results (3),(4) can be seen as special cases of the results from [ES08], [DSS09] in the version for Triebel-Lizorkin spaces indicated there. In this context we refer to [Tr92], Chapter 6 for the treatment of pseudodifferential operators in Triebel-Lizorkin spaces. On the other hand, resolvents of operators with constant coefficients can be handled within the theory of Calderón-Zygmund operators, which are also well-behaved in Triebel-Lizorkin spaces, cf. e.g. [FTW88], or by means of classical multiplier theorems, cf. e.g. [Tr83], Sections 2.3.7, 2.6.6 and 5.2.2. Both methods also work in vector-valued spaces, so this also indicates that (1) is true. In fact, our proof is also based on the operator-valued version of the Mikhlin multiplier theorem.

Proof of Theorem 3.6.5. For easier notation we will write $a(\xi) := \sum_{|\alpha|=2m} a_\alpha \xi^\alpha$ for the symbol of the differential operator $\mathcal{A} = \sum_{|\alpha|=2m} a_\alpha D^\alpha$ and accordingly $\mathcal{A} = a(D)$ in this proof.

We will first show that A_p is $\mathcal{R}_q$-sectorial with $\omega_{\mathcal{R}_q}(A_p) \le \omega_0$. Let $\omega_1 \in (\omega_0, \pi)$ be fixed. Let $I := [-\pi, -\omega_1] \cup [\omega_1, \pi], \Gamma := S^d := \{(\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^d \,:\, |\sigma|^2 + |\xi|^2 = 1\}$, and define

$$K \;:=\; \Big\{(a_\alpha)_{|\alpha|=2m} \,:\, a_\alpha \in \mathbb{C}^{N\times N}, \text{ and } (a_\alpha)_{\alpha=2m}, \, \mathcal{A} := \sum_{|\alpha|=2m} a_\alpha D^\alpha \text{ satisfy } (3.6.2), (3.6.3)\Big\}$$

then it is not hard to see that $K \times I \times \Gamma$ is compact. Moreover define

$$m_{(a_\alpha)_\alpha,\omega}(\sigma,\xi) := e^{i\omega}|\sigma|^{2m}(e^{i\omega}|\sigma|^{2m} - a(\xi))^{-1} \text{ for all } ((a_\alpha)_\alpha, \omega) \in K \times I, \, \sigma \in \mathbb{R}\backslash\{0\}, \xi \in \mathbb{R}^d\backslash\{0\},$$

then each function $m_{(a_\alpha)_\alpha,\omega}$ is homogeneous of degree 0, i.e. $m_{(a_\alpha)_\alpha,\omega}(tv) = m_{(a_\alpha)_\alpha,\omega}(v)$ for all $v \in (\mathbb{R}^d\backslash\{0\}) \times (\mathbb{R}^d\backslash\{0\}), t > 0$, and it is an easy consequence that the set of multiplier functions

$$\mathcal{M} := \big\{m_{(a_\alpha)_\alpha,\omega}(\sigma,\cdot) \mid ((a_\alpha)_\alpha, \omega, \sigma) \in K \times I \times (\mathbb{R}^d\backslash\{0\})\big\}$$

satisfies Mikhlin's condition uniformly, i.e.

$$\begin{aligned} c &:= \sup\big\{|\xi|^{|\beta|} D_\xi^\beta m_{(a_\alpha)_\alpha,\omega}(\sigma,\xi) \mid \sigma \in \mathbb{R}\backslash\{0\}, \xi \in \mathbb{R}^d\backslash\{0\}, \omega \in I, (a_\alpha)_\alpha \in K, \beta \le (1,\dots,1)\big\} \\ &= \sup\big\{|\xi|^{|\beta|} D_\xi^\beta m_{(a_\alpha)_\alpha,\omega}(\sigma,\xi) \mid (\sigma,\xi) \in \Gamma, \omega \in I, (a_\alpha)_\alpha \in K, \beta \in \mathbb{N}_0^d \text{ with } \beta \le (1,\dots,1)\big\} \\ &< +\infty. \end{aligned}$$

(All the above assertions can be found in detail in the proof of [KW04], Theorem 6.2).

Now fix some $\sigma \in \mathbb{R}\backslash\{0\}, \omega \in I$ and $(a_\alpha)_\alpha \in K$ and let $m := m_{(a_\alpha)_\alpha,\omega}(\sigma,\cdot)$. We will apply the operator valued version of Mikhlin's theorem, Theorem 1.4.6, in the spaces $L^p(\ell^2_N(\ell^q)) \cong_N L^p(\ell^q(\mathbb{C}^N))$ for the multiplier function

$$M(\xi)(u_j)_j := (m(\xi)u_j)_j \quad \text{ for } (u_j)_j \in \ell^q(\mathbb{C}^N).$$

Observe first that $\ell^q(\mathbb{C}^N)$ is a UMD-space with property (α) since $q \in (1, +\infty)$. So we have to show that the set

$$\mathcal{T} := \{|\xi|^{|\beta|} D_\xi^\beta M \mid \xi \in \mathbb{R}^d\backslash\{0\}, \beta \in \mathbb{N}_0^d \text{ with } \beta \le (1,\dots,1)\}$$

is $\mathcal{R}$-bounded in $L(L^p(\ell^q(\mathbb{C}^N)))$. By Remark 3.1.7 this is equivalent to $\mathcal{R}_2$-boundedness of $\mathcal{T}$ in $L(L^p(\ell^q(\mathbb{C}^N)))$, hence to boundedness of the corresponding diagonal operators in the spaces $L(L^p(\ell^q(\ell^2_N(\ell^2)))) \cong L(L^p(\ell^q(\ell^2(\mathbb{C}^N))))$. Let $(\xi_k)_{k\in\mathbb{N}} \in (\mathbb{R}^d\backslash\{0\})^{\mathbb{N}}$ and $\beta \in (\mathbb{N}_0^d)^{\mathbb{N}}$ with $\beta_k \le$

$(1,\dots,1)$ for all $k \in \mathbb{N}$ and define $F_k(\xi) := |\xi|^{|\beta_k|} D_\xi^{\beta_k} M(\xi)$ for abbreviation. Let $(u_j^{(k)})_{j,k} \in L^p(\ell^q \ell^2(\mathbb{C}^N)) \cong L^p(\ell^q(\ell^2(\mathbb{C}^N)))$, then

$$\begin{aligned}\Big\| \Big(\sum_{k=1}^{\infty} \big| \big(F_k(\xi_k)(u_j^{(k)})_j\big)_j \big|^2 \Big)^{1/2} \Big\|_{L^p(\ell^q)} &\leq \Big\| \Big(\sum_{k=1}^{\infty} \underbrace{|\xi_k|^{2|\beta|} |D_\xi^\beta m(\xi_k)|^2}_{\leq c^2} \cdot |u_j^{(k)}|^2 \Big)^{1/2} \Big\|_{L^p(\ell^q)} \\ &\leq c \cdot \Big\| \Big(\sum_{k=1}^{\infty} |u_j^{(k)}|^2 \Big)^{1/2} \Big\|_{L^p(\ell^q)}.\end{aligned}$$

This shows that indeed $\mathcal{T}$ is $\mathcal{R}_2$-bounded with $\mathcal{R}_2(\mathcal{T}) \leq c$, hence by the Mikhlin theorem the set of Fourier multiplier operators $\{m(D) \mid m \in \mathcal{M}\}$ is $\mathcal{R}_q$-bounded in L^p. Since for $((a_\alpha)_\alpha, \omega, \sigma) \in K \times I \times (\mathbb{R}^d \backslash \{0\})$ formally

$$m_{(a_\alpha)_\alpha,\omega}(\sigma, D) = e^{i\omega}|\sigma|^{2m}(e^{i\omega}|\sigma|^{2m} - a(D))^{-1} = \lambda R(\lambda, a(D))$$

with $\lambda = e^{i\omega}|\sigma|^{2m}$ we have proven that A_p is $\mathcal{R}_q$-sectorial with $\omega_{\mathcal{R}_q}(A_p) \leq \omega_0$.

Recall that this means that the corresponding tensor extension $\widetilde{A_p} = (A_p)^{\ell^q}$ is a sectorial operator in $L^p(\ell^q)$ with $\omega(\widetilde{A_p}) \leq \omega_0$. In fact, we can extend the above arguments to show that $\widetilde{A_p}$ is even $\mathcal{R}$-sectorial with $\omega_{\mathcal{R}}(\widetilde{A_p}) \leq \omega_0$. For this we consider the multiplier function

$$M(\xi)(u_{jk})_{jk} := (m(\xi)u_{jk})_{jk} \text{ for all } (u_{jk})_{jk} \in X(\ell^q \ell^2(\mathbb{C}^N))$$

in the larger space $L^p(\ell^2_N(\ell^q(\ell^2))) \cong_N L^p(\ell^q \ell^2(\mathbb{C}^N))$. Then the same arguments as above show that M satisfies Mikhlin's condition and hence the operators $\lambda R(\lambda, a(D))$ also have bounded tensor extensions to $\ell^q(\ell^2)$, uniformly bounded in $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_{\omega_1}$.

Using this fact we can show in a second step that each (M, ω_0)-elliptic operator has even an $\mathcal{R}_q$-bounded H^∞-calculus with $\omega_{\mathcal{R}_q}(A_p) \leq \omega_0$. For this let $\mathcal{A}$ be an (M, ω_0)-elliptic operator and $\alpha > 0$. Then the operators $A_p^{\pm\alpha}(-\Delta)^{\mp m\alpha}$ and $(-\Delta)^{\mp m\alpha}A_p^{\pm\alpha}$ have the symbols $\xi \mapsto (\xi^{-2m}a(\xi))^{\pm\alpha}$, which are homogeneous of degree 0 and C^∞ on $\mathbb{R}^d\backslash\{0\}$. Similar as in the first step we obtain with the operator valued version of Mikhlin's theorem that the operators $A_p^{\pm\alpha}(-\Delta)^{\mp m\alpha}$ and $(-\Delta)^{\mp m\alpha}A_p^{\pm\alpha}$ are $\mathcal{R}_q$-bounded, where the $\mathcal{R}_q$-norms depend only on the constants M, ω_0, α.

So we have shown that the assumptions of the Comparison Theorems 3.4.3 and 3.4.4 are satisfied, and together with Theorem 3.6.1 we can conclude that (1) and (2) hold. Then (3) follows immediately from (1) and (2) by Theorem 3.3.23, and also (4) follows by Theorem 3.3.23, since A_p has in particular a bounded H^∞-calculus in L^p. □

Step II. Small perturbations and lower order terms.

This is the crucial step: we consider "small" perturbations of operators with constant coefficients, where for later purposes we already admit lower order terms. So in this step we assume that the coefficients in the principal part of the (M,ω_0)-elliptic differential operator $\mathcal{A}$ are of the form $a_\alpha(x) = a_\alpha^0 + a_\alpha^1(x)$ for all $x \in \mathbb{R}^d$ and $|\alpha| = 2m$, where $a_\alpha^0 \in \mathbb{C}^{N\times N}$ is constant, hence $\mathcal{A}$ is of the form

$$\mathcal{A}(x,D) = \sum_{|\alpha|=2m} (a_\alpha^0 + a_\alpha^1(x))D^\alpha + \sum_{|\alpha|<2m} a_\alpha(x)D^\alpha,$$

where $a_\alpha^1 \in C^\gamma$ for some $\gamma \in (0,2m)$ if $|\alpha| = 2m$. We will show that the assumptions of the Perturbation Theorem 3.4.6 (2) are satisfied for a translate $\nu + \mathcal{A}_0$ of the operator $\mathcal{A}_0 := \sum_{|\alpha|=2m} a_\alpha^0 D^\alpha$ and the perturbation

$$\mathcal{B} := \sum_{|\alpha|=2m} a_\alpha^1(x)D^\alpha + \sum_{|\alpha|<2m} a_\alpha(x)D^\alpha,$$

if one chooses $\nu \geq 0$ sufficiently large and $\sum_{|\alpha|=2m} \|a_\alpha^1\|_{C^\gamma}$ sufficiently small, and that the additional assumption (c) of the Perturbation Theorem 3.4.10 holds if in addition $a_\alpha \in C^\gamma$ for all $|\alpha| < 2m$.

Let us get more concrete. The realizations of $\mathcal{A}, \mathcal{A}_0, \mathcal{B}$ in L^p with domain $W^{2m,p}$ will be denoted by A, A_0, B, respectively. We will start by showing the following for any $\alpha \in \big(1 - \frac{1}{2m}(1\wedge\gamma), 1\big]$ and $\nu \geq 1$:

(i) $D(B) \supseteq D(\nu + A_0)$, and $B(D(\nu + A_0)) \subseteq R((\nu + A_0)^{1-\alpha})$,

(ii) $\mathcal{R}_q\big((\nu + A_0)^{\alpha-1}B(\nu + A_0)^{-\alpha}\big) \lesssim_{M,\omega_0} \sum_{|\beta|=2m} \|a_\beta^1\|_{C^\gamma} + \nu^{-1/2m}$.

Note that (i) and (ii) imply that the assumptions of the Perturbation Theorem 3.4.6 can be satisfied if we choose $\nu \geq 1$ large enough and $\sum_{|\beta|=2m} \|a_\beta^1\|_{C^\gamma}$ sufficiently small.

Let us turn to the proof of (i) and (ii). Note that $D(B) \supseteq D(\nu + A_0)$ holds trivially by definition, and $B(D(\nu + A_0)) \subseteq R((\nu + A_0)^{1-\alpha})$ is true since the operator $(\nu + A_0)^{1-\alpha}$ is surjective onto L^p, so (i) holds. Observe now that by the same technique as in Step I we can resort to the case $A_0 = (-\Delta)^m$ since

$$\begin{aligned}(\nu + A_0)^{\alpha-1}B(\nu + A_0)^{-\alpha} &= \big[(\nu + A_0)^{\alpha-1}(\nu + (-\Delta)^m)^{1-\alpha}\big] \\ &\quad \cdot(\nu + (-\Delta)^m)^{\alpha-1}B(\nu + (-\Delta)^m)^{-\alpha}\,\big[(\nu + (-\Delta)^m)^{\alpha}(\nu + A_0)^{-\alpha}\big],\end{aligned}$$

and the operators $(\nu + (-\Delta)^m)^{\pm\delta}(\nu + A_0)^{\mp\delta}$, $(\nu + A_0)^{\mp\delta}(\nu + (-\Delta)^m)^{\pm\delta}$ extend to $\mathcal{R}_q$-bounded operators with $\mathcal{R}_q$-norms only depending on M, ω_0 but not on $\nu > 0$. This can be seen by the formal representations

$$\begin{aligned}(\nu + (-\Delta)^m)^{\delta}(\nu + A_0)^{-\delta} &= \Big(\nu(\nu + A_0)^{-1} + \big[(-\Delta)^m A_0^{-1}\big]\, A_0(\nu + A_0)^{-1}\Big)^{\delta}, \qquad (3.6.5)\\ (\nu + (-\Delta)^m)^{-\delta}(\nu + A_0)^{\delta} &= \Big(\nu(\nu + (-\Delta)^m)^{-1} + \big[A_0(-\Delta)^{-m}\big]\,(-\Delta)^m(\nu + (-\Delta)^m)^{-1}\Big)^{\delta}\end{aligned}$$

and again using the operator valued Mikhlin Theorem in the usual way as before.

Let $R_j := D_j(-\Delta)^{-1/2}$ be the j-th Riesz transform and as usual $R^\beta := R_1^{\beta_1}\cdots R_d^{\beta_d}$ for $\beta \in \mathbb{N}_0^d$, then formally

$$\begin{aligned}
&(\nu+(-\Delta)^m)^{\alpha-1}B(\nu+(-\Delta)^m)^{-\alpha}\\
\supseteq\ &\sum_{|\beta|=2m}(\nu+(-\Delta)^m)^{\alpha-1}a_\beta^1(x)(\nu+(-\Delta)^m)^{1-\alpha}\left[(-\Delta)^m(\nu+(-\Delta)^m)^{-1}R^\beta\right]\\
&+\sum_{k=0}^{2m-1}\sum_{|\beta|=k}\nu^{\frac{k}{2m}-1}\left[\nu^{1-\alpha}(\nu+(-\Delta)^m)^{\alpha-1}\right]a_\beta(x)\left[\nu^{\alpha-\frac{k}{2m}}\big((-\Delta)^m\big)^{\frac{k}{2m}}(\nu+(-\Delta)^m)^{-\alpha}R^\beta\right].
\end{aligned}$$

Observe that the Riesz transforms and the terms $\nu^\delta\big((-\Delta)^m\big)^\varepsilon(\nu+(-\Delta)^m)^{-\delta-\varepsilon}$ with $\delta,\varepsilon\geq 0$ are $\mathcal{R}_q$-bounded by some universal constants only depending on m by the operator valued version of Mikhlin's theorem, hence we obtain for $\nu\geq 1$

$$\begin{aligned}
&\mathcal{R}_q((\nu+(-\Delta)^m)^{\alpha-1}B(\nu+(-\Delta)^m)^{-\alpha})\\
\lesssim\ &\sum_{|\beta|=2m}\mathcal{R}_q((\nu+(-\Delta)^m)^{\alpha-1}a_\beta^1(x)(\nu+(-\Delta)^m)^{1-\alpha})+\nu^{-1/2m}\sum_{|\beta|<2m}\|a_\beta\|_\infty\\
\leq\ &\sum_{|\beta|=2m}\mathcal{R}_q((\nu+(-\Delta)^m)^{\alpha-1}a_\beta^1(x)(\nu+(-\Delta)^m)^{1-\alpha})+M\cdot\nu^{-1/2m}
\end{aligned}$$

In particular, for $\alpha=1$ we obtain

$$\mathcal{R}_q(B(\nu+(-\Delta)^m)^{-1})\ \ \lesssim_{M,\omega_0}\ \sum_{|\beta|=2m}\|a_\beta^1\|_\infty+\nu^{-1/2m}.$$

So it remains to show that the operators $(\nu+(-\Delta)^m)^{\alpha-1}a_\beta^1(x)(\nu+(-\Delta)^m)^{1-\alpha}$ are $\mathcal{R}_q$-bounded, and that the $\mathcal{R}_q$-norms can be controlled in terms of the Hölder-norms $\|a_\beta^1\|_{C^\gamma}$ if $\alpha\neq 1$.

This will be done by means of complex interpolation, hence we consider as a preparation the complex interpolation of Hölder-Zygmund spaces. For all $s>0$ let C^s be the Hölder space of order s, in particular

$$C^k = C_b^k = \{u\in C_b\,|\,u\ \ k\text{-times continuous differentiable, }\partial^\beta u\in C_b\text{ for all }|\beta|\leq k\}$$

is the space of k-times continuous differentiable functions with bounded derivatives up to order k. We define the Zygmund spaces in terms of Besov spaces

$$\mathcal{C}^s := \mathcal{C}^s(\mathbb{R}^d,\mathbb{C}^N) := B^s_{\infty,\infty}(\mathbb{R}^d,\mathbb{C}^N)\quad\text{ if } s\in\mathbb{R}.$$

Then it is well known that $\mathcal{C}^s = C^s$ if $s>0$ with $s\notin\mathbb{N}$, cf. e.g [Tr78], Section 2.7 and [Tr92], Section 2.6.5 and Chapter 1, and the literature given there. Moreover we define the so-called "little" Hölder spaces $c^s := \overline{C_b^\infty}^{\mathcal{C}^s}$ for all $s>0$. Then for all $0\leq s_0<s_1$ and $\theta\in(0,1)$ the following assertion holds for all $\varepsilon\in(0,\theta)$ and $\sigma>s:=(1-\theta)s_0+\theta s_1$:

$$C^\sigma\hookrightarrow c^s\hookrightarrow\mathcal{C}^s=(C^{s_0},C^{s_1})_{\theta,\infty}\hookrightarrow(C^{s_0},C^{s_1})_{\theta-\varepsilon,1}\hookrightarrow[C^{s_0},C^{s_1}]_{\theta-\varepsilon}. \tag{3.6.6}$$

Cf. e.g. [Lu95], Chapters 0 and 1 for the first inclusion and the first identity, and the other inclusions follow by Propositions 1.5.4 and 1.5.10.

Moreover $\mathcal{C}^s \hookrightarrow C_b^{2m}$ if $s > 2m$, and in particular $\mathcal{C}^s \hookrightarrow L^\infty$ if $s > 0$.

We are now in position to prove the following proposition that is the central tool for the perturbation argument in this step.

Proposition 3.6.6. *Let $\tau \in (0,1)$. Then there is a constant $C = C(d,N,p,q,\tau)$ such that for all $\nu \geq 1, \delta \in (0,\tau)$ and $a \in C^{2m\tau}(\mathbb{R}^d, \mathbb{C}^{N\times N})$ the following assertion holds: The operators*

$$(\nu + (-\Delta)^m)^{\mp\delta} a(x)(\nu + (-\Delta)^m)^{\pm\delta}$$

extend to $\mathcal{R}_q$-bounded operators in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, and

$$\mathcal{R}_q((\nu + (-\Delta)^m)^{\mp\delta} a(x)(\nu + (-\Delta)^m)^{\pm\delta}) \leq C \, \|a\|_{C^{2m\tau}}.$$

Proof. We proceed in several steps : We will first show that the "endpoint operators" $(\nu + (-\Delta)^m)^{-0} a(x)(\nu + (-\Delta)^m)^0 = a(x)$ and $(\nu + (-\Delta)^m)^{-1} a(x)(\nu + (-\Delta)^m)$ are $\mathcal{R}_q$-bounded if a is bounded or in C_b^{2m}, respectively. Then we will obtain the general result with multilinear interpolation, where we have to jiggle the endpoints a little to get into the scale of Zygmund-spaces.

Let M_a be the multiplication operator with the function $a \in L^\infty(\mathbb{R}^d, \mathbb{C}^{N\times N})$. First it is clear that M_a is $\mathcal{R}_q$-bounded in L^p with norm $\|a\|_\infty$. So we turn to the case $a \in C_b^{2m}(\mathbb{R}^d, \mathbb{C}^{N\times N})$. Then

$$\begin{aligned}
(\nu + (-\Delta)^m)(a \cdot u) &= \nu a \cdot u + (-1)^m \Big(\sum_{j=1}^d \partial_j^2\Big)^m (a \cdot u) = \nu a \cdot u + (-1)^m \sum_{\substack{k \in \mathbb{N}_0^d \\ |k|=m}} \binom{m}{k} \partial^{2k}(a \cdot u) \\
&= \nu a \cdot u + (-1)^m \sum_{|k|=m} \binom{m}{k} \sum_{\beta \leq 2k} \binom{2k}{\beta} \partial^{2k-\beta} a \cdot \partial^\beta u \\
&= \nu a \cdot u + a \cdot ((-\Delta)^m u) + \sum_{|k|=m} \sum_{\substack{\beta \leq 2k \\ \beta \neq 2k}} \underbrace{(-1)^m \binom{m}{k}\binom{2k}{\beta}}_{=:\kappa_{k,\beta}} \partial^{2k-\beta} a \cdot \partial^\beta u \\
&= a \cdot (\nu + (-\Delta)^m) u + \sum_{|k|=m} \sum_{\substack{\beta \leq 2k \\ \beta \neq 2k}} \kappa_{k,\beta} \, \partial^{2k-\beta} a \cdot \partial^\beta u \\
&= a \cdot (\nu + (-\Delta)^m) u \\
&\quad + \sum_{|k|=m} \sum_{\substack{\beta \leq 2k \\ \beta \neq 2k}} \kappa_{k,\beta} \, \partial^{2k-\beta} a \cdot \partial^\beta (\nu + (-\Delta)^m)^{-1} (\nu + (-\Delta)^m) u
\end{aligned}$$

for all $u \in C^{2m}(\mathbb{R}^d, \mathbb{C}^N)$, i.e.

$$(\nu + (-\Delta)^m) M_a \, u \;=\; \Big(M_a + \sum_{|k|=m} \sum_{\substack{\beta \leq 2k \\ \beta \neq 2k}} \kappa_{k,\beta} \, (\partial^{2k-\beta} a) \, \partial^\beta (\nu + (-\Delta)^m)^{-1} \Big) (\nu + (-\Delta)^m) u$$

for all $u \in C^{2m}(\mathbb{R}^d, \mathbb{C}^N)$, hence

$$(\nu + (-\Delta)^m)a(x)(\nu + (-\Delta)^m)^{-1} \quad \subseteq \quad M_a + \sum_{|k|=m} \sum_{\substack{\beta \leq 2k \\ \beta \neq 2k}} \kappa_{k,\beta} (\partial^{2k-\beta} a)\, \partial^\beta (\nu + (-\Delta)^m)^{-1}.$$

For all $k \in \mathbb{N}_0^d$ with $|k| = m$ and $\beta \in \mathbb{N}_0^d$ with $\beta \leq 2k, \beta \neq 2k$ we have

$$\partial^\beta (\nu + (-\Delta)^m)^{-1} \quad = \quad i^{|\beta|}\, \nu^{\frac{|\beta|}{2m}-1} \cdot \left[\nu(\nu + (-\Delta)^m)^{-1}\right]^{1-\frac{|\beta|}{2m}} \left[(-\Delta)^m(\nu + (-\Delta)^m)^{-1}\right]^{|\beta|/2m} R^\beta$$

where again $R = (R_1, \dots, R_d)$ is the vector of Riesz-transforms. Since multiplication operators are always $\mathcal{R}_q$-bounded and the Riesz-transforms and terms involving $(-\Delta)^m$ are $\mathcal{R}_q$-bounded by the operator valued Mikhlin Theorem, this yields

$$\mathcal{R}_q((\nu+(-\Delta)^m)a(x)(\nu+(-\Delta)^m)^{-1}) \lesssim \|a\|_\infty + \nu^{-1/2m} \cdot \sum_{|k|=m} \sum_{\substack{\beta \leq 2k \\ \beta \neq 2k}} \|\partial^{2k-\beta} a\|_\infty \lesssim \|a\|_{C^{2m}}. \tag{3.6.7}$$

This shows that $(\nu + (-\Delta)^m)a(x)(\nu + (-\Delta)^m)^{-1}$ is $\mathcal{R}_q$-bounded in L^p and the norm can be estimated by $\|a\|_{C_b^{2m}}$. Since this is true for all $p, q \in (1, \infty)$, we also obtain that the dual operator

$$\left((\nu + (-\Delta)^m)a(x)(\nu + (-\Delta)^m)^{-1}\right)' \supseteq (\nu + (-\Delta)^m)^{-1} a(x)(\nu + (-\Delta)^m)$$

is $\mathcal{R}_q$-bounded in L^p, i.e. the operator $(\nu+(-\Delta)^m)^{-1}a(x)(\nu+(-\Delta)^m)$ extends to an $\mathcal{R}_q$-bounded operator for all $p, q \in (1, \infty)$, and the $\mathcal{R}_q$-bound can be estimated by $\|a\|_{C_b^{2m}}$.

We now turn to the interpolation argument. Let $\varepsilon \in (0, \tau - \delta)$. We fix $n \in \mathbb{N}$ for a moment and consider the interpolation couples

$$(C^{2\varepsilon}(\mathbb{R}^d, \mathbb{C}^{N\times N}), C^{2(m+\varepsilon)}(\mathbb{R}^d, \mathbb{C}^{N\times N})) \quad \text{ and } (L^p(\ell_n^q(\mathbb{C}^N)), L^p(\ell_n^q(\mathbb{C}^N)))$$

with the dense subspaces $C^{2(m+\varepsilon)}(\mathbb{R}^d, \mathbb{C}^{N\times N}) \hookrightarrow C_b^{2m}(\mathbb{R}^d, \mathbb{C}^N)$ and $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)^n$, respectively. Define

$$T(z)(a, u) := e^{z^2}(\nu+(-\Delta)^m)^{-z} a(x)(\nu+(-\Delta)^m)^z u = \left(e^{z^2}(\nu+(-\Delta)^m)^{-z} a(x)(\nu+(-\Delta)^m)^z u_j\right)_j$$

for all $z \in S := \{\zeta \in \mathbb{C} \mid \operatorname{Re}(\zeta) \in [0, 1]\}$, $a \in C_b^{2m}(\mathbb{R}^d, \mathbb{C}^{N\times N})$ and $u \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)^n$. Then $T(z)$ is a bilinear operator for each $z \in S$, and for fixed $(a, u) \in C_b^{2m}(\mathbb{R}^d, \mathbb{C}^{N\times N}) \times \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N)$ we have

$$\begin{aligned} F(z) := T(z)(a, u) \quad &= \quad e^{z^2}(\nu + (-\Delta)^m)^{-z} a(x)(\nu + (-\Delta)^m)^z u \\ &= \quad e^{z^2}(\nu + (-\Delta)^m)^{-z} g(z) \in L^p(\ell_n^q(\mathbb{C}^N)), \end{aligned}$$

where $g(z) := a \cdot (\nu + (-\Delta)^m)^z u \in L^p(\ell_n^q(\mathbb{C}^N))$. Then the mapping $z \mapsto (\nu + (-\Delta)^m)^{-z}$ is analytic on $\{\operatorname{Re} z > 0\}$, and $z \mapsto g(z)$ is continuous on S and analytic in $\mathring{S}$ since $u \in \mathcal{S}(\mathbb{R}^d, \mathbb{C}^N) \subseteq D((-\Delta)^{2m}) \cap R((-\Delta)^{2m})$. Since the operator $\nu + (-\Delta)^m$ has an $\mathcal{R}_q$-bounded H^∞-calculus, it has also $\mathcal{R}_q$-bounded imaginary powers, and we have an estimate

$$\mathcal{R}_q((\nu + (-\Delta)^m)^{it}) \leq c_p(1 + |t|)^d \quad \text{for all } t \in \mathbb{R} \tag{3.6.8}$$

for some constant $c_p > 0$ that does not depend on $\nu \geq 1$; indeed, this is a direct consequence of the Mikhlin multiplier theorem, cf. Theorem 1.4.6. Then the representation

$$F(z) = (\nu + (-\Delta)^m)^{-it}(\nu + (-\Delta)^m)^{-s} e^{z^2} g(z) \quad \text{if } z = s + it \in S$$

shows that F is also continuous on S, and since

$$\|F(z)\|_{L^p} \leq e^{1-\mathrm{Im}(z)^2} \, \|(\nu + (-\Delta)^m)^{-z} g(z)\|_{L^p}$$

if $z \in S$ we see immediately that F is bounded on S.

Moreover we have

$$T(j+it)(a,u) = e^{j^2} e^{2jit} e^{-t^2} (\nu+(-\Delta)^m)^{-it}(\nu+(-\Delta)^m)^{-j} a(x)(\nu+(-\Delta)^m)^j(\nu+(-\Delta)^m)^{it} u.$$

Again using estimate (3.6.8) we obtain

$$\begin{aligned}
&\|T(j+it)(a,u)\|_{L^p(\ell^q_n(\mathbb{C}^N))} \\
\leq\ & c_p^2 e^{1-t^2}(1+|t|)^{2d}\, \mathcal{R}_q((\nu + (-\Delta)^m)^{-j} a(x)(\nu + (-\Delta)^m)^j)\, \|u\|_{L^p(\ell^q_n(\mathbb{C}^N))} \\
\leq\ & C_p\, \mathcal{R}_q((\nu + (-\Delta)^m)^{-j} a(x)(\nu + (-\Delta)^m)^j)\, \|u\|_{L^p(\ell^q_n(\mathbb{C}^N))}.
\end{aligned}$$

In the first part we have shown that if $j \in \{0, 1\}$ and

$$a \in C^{2(m+\varepsilon)}(\mathbb{R}^d, \mathbb{C}^{N\times N}) \hookrightarrow C_b^{2m}(\mathbb{R}^d, \mathbb{C}^{N\times N}) \hookrightarrow L^\infty(\mathbb{R}^d, \mathbb{C}^{N\times N}),$$

then the operator $(\nu+(-\Delta)^m)^{-j}a(x)(\nu+(-\Delta)^m)^j$ is $\mathcal{R}_q$-bounded in the space $L^p(\mathbb{R}^d, \mathbb{C}^N)$ with

$$\mathcal{R}_q\big((\nu + (-\Delta)^m)^{-j} a(x)(\nu + (-\Delta)^m)^j\big) \lesssim \|a\|_{C^{2mj}} \lesssim \|a\|_{C^{2(mj+\varepsilon)}},$$

i.e.

$$\|T(j+it)(a,u)\|_{L^p(\ell^q_n(\mathbb{C}^N))} \quad \lesssim \quad C_p\, \|a\|_{C^{2(jm+\varepsilon)}}\, \|u\|_{L^p(\ell^q_n(\mathbb{C}^N))}.$$

Now let $\theta := \delta$ and choose $\delta + \frac{\varepsilon}{m} < \widetilde{\tau} < \tau$, and let $\widetilde{\theta} := \widetilde{\tau} - \frac{\varepsilon}{m} > \theta$, then $(1-\widetilde{\theta})2\varepsilon + \widetilde{\theta}(2m+2\varepsilon) = 2m\widetilde{\tau}$. By bilinear Stein interpolation (cf. Proposition 2.4.1) we obtain

$$\begin{aligned}
\|(\nu + (-\Delta)^m)^{-\delta} a(x)(\nu + (-\Delta)^m)^\delta u\|_{L^p(\ell^q_n(\mathbb{C}^N))} &= \|T(\theta)(a,u)\|_{L^p(\ell^q_n(\mathbb{C}^N))} \\
&\lesssim C_p\, \|a\|_{[C^{2\varepsilon}, C^{2(m+\varepsilon)}]_\theta}\, \|u\|_{L^p(\ell^q_n(\mathbb{C}^N))},
\end{aligned}$$

and by the embeddings

$$c^{2m\widetilde{\tau}} \hookrightarrow \mathcal{C}^{2m\widetilde{\tau}} = (C^{2\varepsilon}, C^{2(m+\varepsilon)})_{\widetilde{\theta},\infty} \hookrightarrow (C^{2\varepsilon}, C^{2(m+\varepsilon)})_{\theta,1} \hookrightarrow [C^{2\varepsilon}, C^{2(m+\varepsilon)}]_\theta$$

(cf. (3.6.6)) we obtain

$$\|(\nu + (-\Delta)^m)^{-\delta} a(x)(\nu + (-\Delta)^m)^\delta u\|_{L^p(\ell^q_n(\mathbb{C}^N))} \quad \lesssim \quad C_p\, \|a\|_{c^{2m\widetilde{\tau}}}\, \|u\|_{L^p(\ell^q_n(\mathbb{C}^N))}.$$

Note that in all cases the constants in the estimates do not depend on $n \in \mathbb{N}$. Hence, by density, the operator $(\nu + (-\Delta)^m)^{-\delta} a(x)(\nu + (-\Delta)^m)^\delta$ extends to an $\mathcal{R}_q$-bounded operator in $L^p(\mathbb{R}^d, \mathbb{C}^N)$ for all $a \in c^{2m\widetilde{\tau}}$, and the $\mathcal{R}_q$-norm can be estimated in terms of the Hölder-norm

$\|a\|_{c^{2m\tilde{\tau}}}$. Finally we have $C^{2m\tau} \hookrightarrow c^{2m\tilde{\tau}}$, hence we obtain the same result for all $a \in C^{2m\tau}$, i.e. the operator $(\nu+(-\Delta)^m)^{-\delta}a(x)(\nu+(-\Delta)^m)^{\delta}$ extends to an $\mathcal{R}_q$-bounded operator in $L^p(\mathbb{R}^d, \mathbb{C}^N)$, and the $\mathcal{R}_q$-norm can be estimated in terms of the Hölder-norm $\|a\|_{C^{2m\tau}}$. By dualization we can also conclude the same estimates for the dual operators, i.e. the operator

$$(\nu + (-\Delta)^m)^{\delta} a(x)(\nu + (-\Delta)^m)^{-\delta} \subseteq \overline{\left((\nu + (-\Delta)^m)^{-\delta} a(x)(\nu + (-\Delta)^m)^{\delta}\right)'}$$

extends to an $\mathcal{R}_{q'}$-bounded in $L^{p'}$, and also in this case the $\mathcal{R}_q$-norm can be estimated in terms of the Hölder-norm $\|a\|_{C^{2m\tau}}$. Since this is true for all $p, q \in (1, +\infty)$ we have finished the proof. $\square$

We can now also finish the proof of the estimate (ii): Recall that it was only left to show that the operators $(\nu + (-\Delta)^m)^{\alpha-1} a^1_\beta(x)(\nu + (-\Delta)^m)^{1-\alpha}$ are $\mathcal{R}_q$-bounded if $|\beta| = 2m, \alpha \neq 1$, and that the $\mathcal{R}_q$-norms can be controlled in terms of the Hölder-norms $\|a^1_\beta\|_\gamma$. But this is an immediate consequence of Proposition 3.6.6, since by our choice we have $0 < 1 - \alpha < \gamma/2m$.

As already mentioned above, this shows that the assumptions of the Perturbation Theorem 3.4.6 can be satisfied if we choose $\nu \geq 1$ sufficiently large and $\sum_{|\beta|=2m} \|a^1_\beta\|_\gamma$ sufficiently small.

Let us now assume that in addition $a_\alpha \in C^\gamma$ for all $|\alpha| < 2m$. We show that in this case the following additional assertions to (i), (ii) hold: Let $\alpha \in \left(0, \frac{1}{2m}(\gamma \wedge 1)\right)$, and $\nu \geq 1$, then

(iii) $\mathcal{R}_q\left((\nu + A_0)^{-\alpha} B(\nu + A_0)^{\alpha-1}\right) \lesssim_{M,\omega_0} \sum_{|\beta|=2m} \|a^1_\beta\|_{C^\gamma} + \nu^{-1/2m}$,

(iv) $B(D((\nu + A_0))^2) \subseteq D((\nu + A_0)^\alpha)$,

(v) $\mathcal{R}_q\left((\nu + A_0)^{\alpha} B(\nu + A_0)^{-\alpha-1}\right) \lesssim_{M,\omega_0} \sum_{|\beta|=2m} \|a^1_\beta\|_{C^\gamma} + \nu^{-1/2m}$.

Observe that (iii) ist just a reformulation of (ii) with $1 - \alpha$ in place of α, and moreover $\alpha \wedge (1 - \alpha) = \alpha$, since $\alpha < \frac{1}{2}$. Hence in this case also the assumptions of the Perturbation Theorem 3.4.10 can be satisfied if we choose $\nu \geq 1$ large enough and $\sum_{|\beta|=2m} \|a^1_\beta\|_{C^\gamma}$ sufficiently small, and in particular we will then obtain $\dot{X}^\theta_{q,A_0} = \dot{X}^\theta_{q,A}$ for any $\theta \in (-\alpha, 1 + \alpha)$.

Let us first have a look on (iv): We have $D((\nu + A_0)^2) = D((1 + (-\Delta)^m))^2 = H^{4m,p}$, and for each $u \in H^{4m,p} \subseteq D(B)$ we have

$$Bu = \sum_{|\beta|=2m} a^1_\beta(x) D^\beta u(x) + \sum_{|\beta|<2m} a_\beta(x) D^\beta u(x).$$

For each $|\beta| \leq 2m$ we have $v_\beta := D^\beta u \in H^{2m,p} \subseteq H^{2m\alpha,p}$. Since $0 < 2m\alpha < \gamma$ and $a^1_\beta, a_\beta \in C^\gamma$, the functions a^1_β, a_β are pointwise multipliers in $H^{2m\alpha}$, cf. e.g. [Tr83], 2.8.2 Theorem and Corollary. In fact, this result is also contained in Proposition 3.6.6. So we obtain

$$Bu = \sum_{|\beta|=2m} a^1_\beta(x) v_\beta(x) + \sum_{|\beta|<2m} a_\beta(x) v_\beta(x) \in H^{2m\alpha,p} = R((1 + (-\Delta)^m)^\alpha) = R((\nu + A_0)^\alpha),$$

and thus (iv) is proven. Let us finally turn to the assertions (v). By the same arguments as used above we may assume $A_0 = (-\Delta)^m$, and the same calculations with α instead of $\alpha - 1$ show that

$$\begin{aligned}&\mathcal{R}_q((\nu + (-\Delta)^m)^\alpha B(\nu + (-\Delta)^m)^{-\alpha-1})\\ \lesssim_{M,\omega_0}\ &\sum_{|\beta|=2m} \mathcal{R}_q((\nu + (-\Delta)^m)^\alpha a^1_\beta(x)(\nu + (-\Delta)^m)^{-\alpha}) + M \cdot \nu^{-1/2m}.\end{aligned}$$

Thus (v) follows immediately from Proposition 3.6.6, since by our choice we have $0 < \alpha < \gamma/2m$.

Step III. Localization and the general case.

We now turn to the general case via localization, i.e. in this step we present the proof of Theorem 3.6.3, based on the local perturbation results in Step II. So let $\mathcal{A}$ be a general (M, ω_0)-elliptic operator that fulfills the conditions of Theorem 3.6.3 and let $A := A_p$ for abbreviation. Obviously it is sufficient to prove the assertions of Theorem 3.6.3 for any $\widetilde{\gamma} \in (0, \gamma)\backslash\mathbb{N}$ instead of γ, so we also fix a $\widetilde{\gamma} \in (0, \gamma)\backslash\mathbb{N}$.

Since the coefficients of the principal part are Hölder continuous, hence in particular BUC, we can choose $r > 0$ with respect to some $\varepsilon_0 \in (0, 1)$ that will be specified later such that

$$\forall\, x, y \in \mathbb{R}^n \,:\, |x - y| < 2r\sqrt{d} \Rightarrow \sum_{|\beta|=2m} |a_\beta(x) - a_\beta(y)| < \varepsilon_0. \tag{3.6.9}$$

Let $\Gamma := r\mathbb{Z}^d$, $Q := (-r, r)^d$ and $Q_\ell := \ell + Q$ for all $\ell \in \Gamma$. We define the relation $k \bowtie \ell :\iff Q_k \cap Q_\ell \neq \emptyset$ for all $k, \ell \in \Gamma$ and define the neighborhood $V_\ell := \{k \in \Gamma \,|\, k \bowtie \ell\}$ for all $\ell \in \Gamma$. Then $V_\ell = \ell + V_0$ for each $\ell \in \Gamma$, and the number $N_0 := |\{k \in \Gamma \,|\, k \bowtie \ell\}| = |V_0|$ is finite and independent of $\ell \in \Gamma$ and $r > 0$. Further we choose $\psi, \rho \in C_c^\infty(Q)$ with $0 \le \psi, \rho \le 1$, $\rho|_{\mathrm{supp}(\psi)} = 1$ and

$$\sum_{\ell \in \Gamma} \psi_l^2(x) = 1 \quad \text{for all } x \in \mathbb{R}^d,$$

where $\psi_\ell := \psi(\cdot - \ell)$. Let

$$\mathbb{X} := \ell^p(\Gamma, X) \cong \ell^p L^p(\Gamma \times \mathbb{R}^d, \mathbb{C}^N) \cong L^p \ell^p \ell^2(\mathbb{R}^d \times \Gamma \times \mathbb{N}_{\le N}),$$

then we can and will consider $\mathbb{X} \cong L^p\ell^p\ell^2_N \cong \ell^p L^p \ell^2_N$ as a Banach function space (cf. the corresponding remarks in Subsection 1.6.1). We define the operators

$$J : X \to \mathbb{X}, u \mapsto (\psi_\ell u)_\ell, \quad P : \mathbb{X} \to X, (u_\ell)_\ell \mapsto \sum_\ell \psi_\ell u_\ell.$$

Then it is easily seen that P, J are bounded and $PJ = \mathrm{Id}_X$, hence P is a retraction and J a corresponding coretraction. Moreover the operators P, J are evidently positive and hence $\mathcal{R}_q$-bounded.

We will now construct an operator $\mathbb{A}$ in $\mathbb{X}$ of the form $\mathbb{A} = \mathbb{A}_0 + \mathbb{B}$ associated to the operator A in the sense that $JA \subseteq \mathbb{A}J$, where $\mathbb{A}_0$ is a diagonal operator of (M, ω_0)-elliptic operators with constant coefficients, and $\mathbb{B}$ is a perturbation operator in the space $\mathbb{X}$, such that the assumptions of the Perturbation Theorem 3.4.6, or Theorem 3.4.10, respectively, are satisfied.

For this, we first define

$$A_\ell^0 u := \sum_{|\beta|=2m} a_\beta(\ell) D^\beta u \quad \text{ for all } u \in W^{2m,p}, \ell \in \Gamma,$$

and $\mathbb{A}_0(u_\ell)_\ell := (A_\ell^0 u_\ell)_\ell$ for all $(u_\ell)_\ell \in D(\mathbb{A}_0) := \mathbb{X}^{2m} := \ell^p(\Gamma, W^{2m,p})$. Then $\mathbb{A}_0$ is a diagonal operator which consists of (M, ω_0)-elliptic differential operators with constant coefficients. Hence it is an easy consequence of the first step that the operator $\mathbb{A}_0$ has an $\mathcal{R}_q$-bounded H^∞-calculus with $\omega_{\mathcal{R}_q^\infty}(\mathbb{A}_0) \leq \omega_0$.

In the next step we construct the perturbation operator $\mathbb{B}$. For this purpose we let

$$A_\ell^1(u) \quad := \quad \sum_{|\beta|=2m} (a_\beta(x) - a_\beta(\ell))\rho_\ell(x)\, D^\beta u \quad \text{ for all } u \in W^{2m,p}, \ell \in \Gamma.$$

Observe that the functions $x \mapsto (a_\beta(x) - a_\beta(\ell))\rho_\ell(x)$ are also γ-Hölder continuous, and we have an estimate $\|(a_\beta - a_\beta(\ell))\rho_\ell\|_{C^\gamma} \leq C_\rho \cdot \|a\|_{C^\gamma}$ for some constant $C_\rho > 0$ depending on the auxiliary function ρ. Now define $\theta := \widetilde{\gamma}/\gamma \in (0,1)$, then

$$C^{\widetilde{\gamma}} = \mathcal{C}^{\widetilde{\gamma}} = (C^0, C^\gamma)_{\theta,\infty},$$

cf. the remarks preceding Proposition 3.6.6. By Proposition 1.5.4 (4) we can choose a constant $c(\theta, p)$ such that

$$\begin{aligned} \|(a_\beta - a_\beta(\ell))\rho_\ell\|_{C^{\widetilde{\gamma}}} &\leq c(\theta,p)\, \|(a_\beta - a_\beta(\ell))\rho_\ell\|_\infty^{1-\theta}\, \|(a_\beta - a_\beta(\ell))\rho_\ell\|_{C^\gamma}^\theta \\ &\leq c(\theta,p)\, \varepsilon_0^{1-\theta}\, (C_\rho \cdot \|a_\beta\|_{C^\gamma})^\theta \leq \underbrace{\max_{|\beta|=2m} \left(c(\theta,p) C_\rho^\theta \|a_\beta\|_{C^\gamma}^\theta \right)}_{=:K} \cdot \varepsilon_0. \end{aligned}$$

Further we define the operator of lower order terms

$$A_{\text{low}}(u) := \sum_{|\beta|<2m} a_\beta(x) D^\alpha u \quad \text{ for all } u \in W^{2m,p},$$

then we have

$$\psi_\ell A u \quad = \quad A\psi_\ell u + \underbrace{(\psi_\ell A - A\psi_\ell)}_{=:C_\ell} u = (A_\ell^0 + A_\ell^1)\psi_\ell u + A_{\text{low}}\psi_\ell u + \sum_{k \bowtie \ell} C_\ell \psi_k \psi_k u$$

for all $u \in W^{2m,p}, \ell \in \Gamma$. Observe that C_ℓ is a differential operator of order $\leq 2m-1$, where the L^∞-norm of the coefficients can be controlled by $C_\psi\, M$ with some constant C_ψ only depending on the auxiliary function ψ. We now define the perturbation operator as

$$\mathbb{B}(u_\ell)_\ell \quad := \quad \left(A_\ell^1 u_\ell + A_{\text{low}} u_\ell + \sum_{k \bowtie \ell} C_\ell \psi_k u_k \right)_\ell \quad \text{ for all } (u_\ell)_\ell \in D(\mathbb{B}) := \mathbb{X}^{2m}.$$

Finally we let $\mathbb{A} := \mathbb{A}_0 + \mathbb{B}$ with $D(\mathbb{A}) = \mathbb{X}^{2m}$, then by construction we have $JA \subseteq \mathbb{A}J$.

We will show now that $\mathbb{A} = \mathbb{A}_0 + \mathbb{B}$ is a "small" perturbation of $\mathbb{A}_0$ in the sense of the Perturbation Theorem 3.4.6, or Theorem 3.4.10 in the case that in addition $a_\beta \in C^\gamma$ for all $|\beta| < 2m$, respectively. Recall that $\mathbb{A}_0$ is a diagonal operator of (M, ω_0)-elliptic operators with constant coefficients and has an $\mathcal{R}_q$-bounded H^∞-calculus with $\omega_{\mathcal{R}_q^\infty}(\mathbb{A}_0) \leq \omega_0$. The operator $\mathbb{B}$ is an "almost-diagonal" operator in the following sense: in each row and column there are at most N_0 entries not equal to 0. Furthermore, each component of $(A^1_\ell)_\ell$ is a homogeneous operator of order $2m$ with $\widetilde{\gamma}$-Hölder continuous coefficients, where the $\widetilde{\gamma}$-Hölder norm of the coefficients can be estimated by $K\varepsilon_0$.

Now let $\nu \geq 1$ and $\alpha \in (1 - \frac{1}{2m}(1 \wedge \widetilde{\gamma}), 1]$, then we have:

$$\begin{aligned} & (\nu + \mathbb{A}_0)^{\alpha-1}\mathbb{B}(\nu + \mathbb{A}_0)^{-\alpha}(u_\ell)_\ell \\ = & \left((\nu + A^0_\ell)^{\alpha-1}(A^1_\ell + A_{\text{low}})(\nu + A^0_\ell)^{-\alpha}u_\ell + \sum_{k \bowtie \ell} (\nu + A^0_\ell)^{\alpha-1} C_\ell \psi_k (\nu + A^0_k)^{-\alpha} u_k \right)_\ell \end{aligned}$$

In each component we have a finite sum of operators of the kind considered in Step II, where the quantity of the non-zero summands in each component is not larger than N_0. Hence we can apply the results (i), (ii) from Step II (uniformly) in each component, and we obtain that $D(\mathbb{B}) \supseteq D(\nu + \mathbb{A}_0)$ and $\mathbb{B}(D(\nu + \mathbb{A}_0)) \subseteq R((\nu + \mathbb{A}_0)^{1-\alpha})$; Moreover, we can w.l.o.g. replace A^0_ℓ by $(-\Delta)^m$ for all $\ell \in \Gamma$ (cf. the corresponding remarks in Step II), and then obtain

$$\begin{aligned} & \mathcal{R}_q\big((\nu + \mathbb{A}_0)^{\alpha-1}\mathbb{B}(\nu + \mathbb{A}_0)^{-\alpha}\big) \\ \lesssim & \sup_\ell \sup_{k \bowtie \ell} \big((\nu + (-\Delta)^m)^{\alpha-1}(A^1_\ell + A_{\text{low}} + C_\ell \psi_k)(\nu + (-\Delta)^m)^{-\alpha}\big). \end{aligned}$$

Thus (ii) from Step II yields

$$\mathcal{R}_q\big((\nu + \mathbb{A}_0)^{\alpha-1}\mathbb{B}(\nu + \mathbb{A}_0)^{-\alpha}\big) \quad \lesssim_{M,\omega_0,N_0} \quad K\varepsilon_0 + \nu^{-1/2m}.$$

This shows that we can ensure the assumptions of the Perturbation Theorem 3.4.6 (2) if we choose ε_0 sufficiently small and $\nu \geq 1$ sufficiently large. Note that in the same manner the assumptions of the Perturbation Theorem 3.4.10 can be fulfilled if additionally $a_\beta \in C^\gamma$ for all $|\beta| < 2m$, using (iii)-(v) from Step II.

So in the sequel we assume that $\varepsilon_0 > 0, \nu \geq 1$ are chosen in a way such that the assumptions of Theorem 3.4.6 (2), or even Theorem 3.4.10 in the case that $a_\beta \in C^\gamma$ for all $|\beta| < 2m$, are fulfilled for the operators $\mathbb{A}_0, \mathbb{B}$ in $\mathbb{X}$. Thus we obtain that $\mathbb{A} = \mathbb{A}_0 + \mathbb{B}$ has an $\mathcal{R}_q$-bounded H^∞-calculus with $\omega_{\mathcal{R}_q^\infty} \leq \omega_0$. We define $\vartheta := 1 + \frac{\widetilde{\gamma}}{2m}$ in the case that $a_\beta \in C^\gamma$ for all $|\beta| < 2m$, and $\vartheta := 1$ otherwise.

Then Theorem 3.4.6 (2), or Theorem 3.4.10 , respectively, implies that

$$\dot{\mathbb{X}}^\theta_{s,\nu+\mathbb{A}_0} \cong \dot{\mathbb{X}}^\theta_{s,\nu+\mathbb{A}} \text{ for all } \theta \in (-\tfrac{\widetilde{\gamma}\wedge 1}{2m}, \vartheta). \tag{3.6.10}$$

Now we will transfer the properties of the operator $\mathbb{A}$ in $\mathbb{X}$ to the operator A in $X = L^p$. W.l.o.g. we assume $\nu = 0$ by possibly replacing $a_0(\cdot)$ with $a_0(\cdot) + \nu$ for some $\nu \geq 0$. Consequently w.l.o.g. we can assume $A^{-1} \in L(X)$, since in our framework we always assume that $\nu \geq 1 > 0$. Fix

$\sigma > \omega > \omega_0$. Recall that we have already shown that the operator $\mathbb{A}$ is sectorial with $\omega(\mathbb{A}) \le \omega_0$. Moreover, the operator A in X is also sectorial with $\omega(A) \le \omega_0$, this can be extracted from the results proven here up to now, on the other hand this has been shown e.g. in [KW04], Chapter 6, even under weaker assumptions.

We will now compare the resolvents of A and $\mathbb{A}$, respectively. The relation $\mathbb{A}J \supseteq JA$ implies

$$\begin{aligned} JR(\lambda, A) &\supseteq (\lambda - \mathbb{A})^{-1}(\lambda J - \mathbb{A}J)(\lambda - A)^{-1} \supseteq (\lambda - \mathbb{A})^{-1}(\lambda J - JA)(\lambda - A)^{-1} \\ &= (\lambda - \mathbb{A})^{-1} J(\lambda - A)(\lambda - A)^{-1} = R(\lambda, \mathbb{A})J \in L(X, \mathbb{X}) \end{aligned}$$

for all $\lambda \in \mathbb{C}\backslash\overline{\Sigma}_{\omega_0}$. This yields for all $\varphi \in H_0^\infty(\Sigma_\sigma)$ the identity

$$J\varphi(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \varphi(\lambda) JR(\lambda, A)\, d\lambda = \frac{1}{2\pi i}\int_{\partial\Sigma_\omega} \varphi(\lambda) R(\lambda, \mathbb{A})J\, d\lambda = \varphi(\mathbb{A})J,$$

hence $\varphi(A) = PJ\varphi(A) = P\varphi(\mathbb{A})J$. Since $\mathbb{A}$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\sigma)$-calculus we obtain

$$\begin{aligned} &\mathcal{R}_q\big(\{\varphi(A) \mid \varphi \in H_0^\infty(\Sigma_\sigma), \|\varphi\|_{\sigma,\infty} \le 1\}\big) \\ \le\; &\mathcal{R}_q(P)\mathcal{R}_q(J) \cdot \mathcal{R}_q\big(\{\varphi(\mathbb{A}) \mid \varphi \in H_0^\infty(\Sigma_\sigma), \|\varphi\|_{\sigma,\infty} \le 1\}\big) < +\infty, \end{aligned}$$

hence also A has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\sigma)$-calculus by Proposition 3.2.23.

Now choose the special function $\varphi(z) := z^2/(1+z)^4$, then $\varphi \in H_0^\infty(\Sigma_\sigma)$, and φ is suitable to calculate the norm in the q-intermediate spaces $\dot{X}^\theta_{q,\mathbb{A}}, \dot{X}^\theta_{q,A}$ for $\theta \in (-2, 2)$, and for all $u \in \dot{X}^\theta_{q,A}$ we obtain

$$\|Ju\|_{\dot{\mathbb{X}}^\theta_{q,\mathbb{A}}} = \big\|t^{-\theta}\varphi(t\mathbb{A})Ju\big\|_{\mathbb{X}(L^q_*)} = \big\|t^{-\theta}J\varphi(tA)u\big\|_{\mathbb{X}(L^q_*)} \lesssim \big\|t^{-\theta}\varphi(tA)u\big\|_{X(L^q_*)} = \|u\|_{\dot{X}^\theta_{q,A}},$$

and

$$\begin{aligned} \|u\|_{\dot{X}^\theta_{q,A}} &= \big\|t^{-\theta}\varphi(tA)u\big\|_{X(L^q_*)} = \big\|t^{-\theta}PJ\varphi(tA)u\big\|_{X(L^q_*)} = \big\|t^{-\theta}P\varphi(t\mathbb{A})Ju\big\|_{X(L^q_*)} \\ &\lesssim \big\|t^{-\theta}\varphi(t\mathbb{A})Ju\big\|_{\mathbb{X}(L^q_*)} = \|Ju\|_{\dot{\mathbb{X}}^\theta_{s,\mathbb{A}}}, \end{aligned}$$

i.e.

$$\|Ju\|_{\dot{\mathbb{X}}^\theta_{q,\mathbb{A}}} \cong \|u\|_{\dot{X}^\theta_{q,A}}. \tag{3.6.11}$$

We can now finish the proof of Theorem 3.6.3 by showing the norm equivalence $\dot{X}^\theta_{q,A} \approx \dot{X}^\theta_{q,\nu+(-\Delta)^m}$ for any $\theta \in (-\frac{\widetilde{\gamma}\wedge 1}{2m}, \vartheta)$. Note that the remarks following Theorem 3.6.3 illustrate how the assertions of (3) and (4) can be concluded from this.

We apply the same procedure as above in addition to the Laplace-Operator $(-\Delta)^m$ instead of A and obtain an analogous representation

$$J(\mu + (-\Delta)^m) \subseteq \widetilde{\mathbb{A}}J = (\widetilde{\mathbb{A}}_0 + \widetilde{\mathbb{B}})J$$

for some $\mu \geq 0$ and operators $\widetilde{\mathbb{A}}_0, \widetilde{\mathbb{B}}$ of the same type as the operators $\mathbb{A}_0, \mathbb{B}$, respectively. This yields for all $\theta \in (-\frac{\widetilde{\gamma}\wedge 1}{2m}, \vartheta)$ the following norm equivalences:

$$\begin{aligned} \|u\|_{\theta,q,\mu+A} &\overset{(a)}{\cong} \|Ju\|_{\theta,q,\mu+\mathbb{A}} \overset{(b)}{\cong} \|Ju\|_{\theta,q,\mu+\mathbb{A}_0} \overset{(c)}{\cong} \|Ju\|_{\theta,q,\mu+\widetilde{\mathbb{A}}_0} \overset{\widetilde{(b)}}{\cong} \|Ju\|_{\theta,q,\mu+\widetilde{\mathbb{A}}} \\ &\overset{\widetilde{(a)}}{\cong} \|u\|_{\theta,q,\mu+(-\Delta)^m}. \end{aligned} \tag{3.6.12}$$

Here $(a), \widetilde{(a)}$ are the norm equivalences (3.6.11) and $(b), \widetilde{(b)}$ follow from the perturbation argument (3.6.10) in the space $\mathbb{X}$, applied to the operators $\mathbb{A}$ and $\widetilde{\mathbb{A}}$, respectively. The identity (c) follows from the comparison theorem using again the representation (3.6.5) from Step II and the vector-valued Mikhlin Theorem. Thus with these conclusions we have also finished the proof of Theorem 3.6.3.

3.6.2 Elliptic differential operators of 2$^{\text{nd}}$ order in divergence form on $\mathbb{R}^d$

With some modifications we can apply the same techniques as in the preceding subsection to operators in divergence form, but now the regularity assumptions on the principal part can be weakened. These operators can be handled by the same approach as above: First consider operators with constant coefficients, then do a local perturbation argument, and finally general operators can be reduced to the latter case by a localization procedure. In fact, the first step is the same as in Subsection 3.6.1, since differential operators in divergence form with constant coefficients are also differential operators in non-divergence form. The crucial result is again the second step, the perturbation argument for differential operators in divergence form, which in this situation is based on the Perturbation Theorem 3.4.8. Having done this, the final localization procedure can be done analogously to Subsection 3.6.1.

Let again $p, q \in (1, +\infty)$ and $d \in \mathbb{N}$, and define $X := L^p := L^p(\mathbb{R}^d, \mathbb{C})$. In this subsection we will use the notations $H^{\gamma,p} := H^{\gamma,p}(\mathbb{R}^d)$ and $H^\gamma := H^{\gamma,2}$, $\gamma \in \mathbb{R}$ for the Bessel potential spaces. For simplicity, we will only consider differential operators in divergence form of second order, i.e. operators that are formally (!) given by

$$\begin{aligned} \mathcal{A}u &= -\sum_{j,k=1}^{n} \partial_j(a_{jk}(x)\partial_k u) + \sum_{j=1}^{n} \big(-\partial_j(b_j(x)u) + c_j \partial_j u \big) + d(x)u \\ &= -\operatorname{div}(a(x)\nabla u + b(x)u) + c(x)\cdot \nabla u + d(x)u. \end{aligned} \tag{3.6.13}$$

We impose the usual *strong uniform ellipticity* condition on the top order coefficients of $\mathcal{A}$:

$$\forall\, \xi \in \mathbb{R}^d \,:\, \operatorname{Re}\big(a(x)\xi\cdot\xi\big) = \sum_{j,k=1}^{n} \operatorname{Re}\big(a_{jk}(x)\xi_j\xi_k\big) \geq \delta|\xi|^2 \tag{3.6.14}$$

for some $\delta > 0$. Moreover, we make the following regularity and boundedness assumptions on the coefficients:

$$\begin{aligned} &a(\cdot) \in BUC(\mathbb{R}^d, \mathbb{C}^{d\times d}), \;\; \text{and } b(\cdot), c(\cdot) \in L^\infty(\mathbb{R}^d, \mathbb{C}^d), d(\cdot) \in L^\infty(\mathbb{R}^d), \\ &\|a\|_\infty, \|b\|_\infty, \|c\|_\infty, \|d\|_\infty \leq M \end{aligned} \tag{3.6.15}$$

for some $M > 0$. Then the operator $\mathcal{A}$ is formally associated to the form $\mathfrak{a}$ defined by

$$\mathfrak{a}(u, v) := \langle (a\nabla u + bu) \,|\, \nabla v\rangle + \langle c\nabla u + du \,|\, v\rangle \quad \text{for all } u, v \in H^1. \tag{3.6.16}$$

The following remark is well known, cf. e.g. [Ou05], Chapter 1 and 4, and [KW04], Chapter 11:

Remark 3.6.7. *Let $M, \delta > 0$. Then there are $\nu_0 \geq 0$ and $\omega_0 \in (0, \pi/2)$ with the following property: If $\mathfrak{a}$ is a form given by (3.6.16), where the coefficients satisfy (3.6.14) and (3.6.15), then the operator $\nu_0 + A_2$ in L^2 associated to the form $\nu_0 + \mathfrak{a}$ is sectorial with $\omega(\nu_0 + A_2) \leq \omega_0$ and has a bounded H^∞-calculus with $\omega_{H^\infty}(\nu_0 + A_2) = \omega(\nu_0 + A_2)$.*

Thus the operator $-(\nu_0 + A_2)$ generates an analytic semigroup $(T_2(t))_{t\geq 0}$ in L^2. If this semigroup can be extended to L^p, which is the case if $T_2(t)|_{L^p}$ is bounded in L^p for all $t > 0$, then the negative of the generator of the semigroup T_p, defined by the bounded extensions $T_p(t) := \overline{T_2(t)|_{L^p}}$ for all $t > 0$, is called the realization of the differential operator $\nu_0 + \mathcal{A}$ in L^p and denoted by $\nu_0 + A_p$. So the usual way to study divergence form operators in L^p is to check if the semigroup T_2 is bounded in L^p. To do this, standard tools are (generalized) Gaussian estimates for the semigroup, cf. Section 3.5. In the situation described here this has been done in [Au96], where it is shown that the semigroup $T_2(t) = e^{-t(\nu_0 + A_2)}$ has Gaussian bounds[2], so we already know that these operators have an $\mathcal{R}_q$-bounded H^∞-calculus in each space L^p, cf. also Section 3.5, Example (a).

Nevertheless, we will use a different approach, since again we are also interested in the associated s-intermediate spaces. Thus we will use more direct methods applying perturbation arguments to constant coefficients operators in L^p. For this we will use the Perturbation Theorem 3.4.8 with $\alpha = 1/2$. Observe that Theorem 3.4.8 will only yield an abstract operator $\nu_0 + C_p$ with the properties described there, and we will have to justify that this operator indeed equals the operator $\nu_0 + A_p$. We give in advance a short description how this will be done: On the one hand, the operators $\nu_0 + A_p$, $p \in (1, +\infty)$ (constructed via form methods and extrapolation as described above) have consistent resolvents. On the other hand, also the operators $\nu_0 + C_p$, $p \in (1, +\infty)$ have consistent resolvents, since they are constructed by the same formula in all spaces L^p (we will give more details for this argument in the proof of Theorem 3.6.8). Hence it will be sufficient to show that the operators $\nu_0 + A_2$ and $\nu_0 + C_2$ have consistent resolvents, and this will be a direct consequence of the Perturbation Theorem 3.4.8, since both operators are the part of the same "lifted" operator $\nu_0 + \widetilde{A}_2 : H^1 \to H^{-1}, u \mapsto \nu_0 u + \mathcal{A}u$.

We note that our approach is closely related to the approach in [KW04], Chapter 13, and [KKW06], Section 9, Example (b): there it is shown that the operator $1 + \mathcal{A}$ has a bounded H^∞-calculus in the spaces $H^{s,p}$ for any $s \in (-1, 1)$ if $\mathcal{A}$ has no lower order terms, and if $a : \mathbb{R}^d \to \mathbb{C}^{d\times d}$ is Hölder-continuous, then $1 + \mathcal{A}$ also has a bounded H^∞-calculus in $H^{-1,p}$. These results are based on similar perturbation methods as we will use here. Let us also note that in [Mi05] it is shown that one can choose $\nu \geq \nu_0$ such that the operator $\nu + A_p$ is sectorial, and this is done

[2] In [Au96] this in only proven for homogeneous operators without lower order terms, but such terms can be handled by perturbation arguments.

via considering the operator $\widetilde{A}_p : H^{1,p} \to H^{-1,p}$.

We can now formulate the main theorem of this section.

Theorem 3.6.8. *Let $M, \delta > 0$ and choose $\nu_0 \geq 0, \omega_0 \in (0, \pi/2)$ depending on M, δ according to Remark 3.6.7. Then for all $\sigma \in (\omega_0, \pi/2)$ there is a $\nu > \nu_0$ with the following property:*

If $\mathcal{A}$ is a differential operator of the form (3.6.13) that fulfills (3.6.14) and (3.6.15), then the following assertions hold:

(1) $\nu + A_p$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\sigma)$-calculus.

(2) For all $\theta \in (-1/2, 1/2)$ we have $\dot{X}^\theta_{\nu+A_p,q} \approx \dot{X}^\theta_{\nu-\Delta,q}$, and $X^\theta_{\nu+A_p,q} \approx X^\theta_{\nu-\Delta,q} \approx F^{2\theta}_{p,q}$ if $\theta \in (0, 1/2)$, respectively.

(3) For all $s \in (0,1)$ the part $\nu + A_{p,q,s}$ of the differential operator $\nu + A_p$ in the space $F^s_{p,q}$ has a bounded $H^\infty(\Sigma_\sigma)$-calculus.

Proof. Let $\mathcal{A}$ be a differential operator of the form (3.6.13) that fulfills (3.6.14) and (3.6.15). W.l.o.g. we may assume $\nu_0 = 0$ by maybe replacing d by $d + \nu_0$. We will again do the proof in three steps analogously to the proof of Theorem 3.6.3 in the preceding subsection. So we start with the case that $\mathcal{A} = -\operatorname{div}(a_0 \nabla)$, where $a_0 \in \mathbb{C}^{d\times d}$ is constant. In this case, all the statements of Theorem 3.6.8 follow from Theorem 3.6.5, since $\mathcal{A}$ is $(\widetilde{M}, \omega_0)$-elliptic with some constant $\widetilde{M} > 0$ depending on $M, \delta > 0$.

So in the next step we assume that $a(x) = a^0 + a^1(x)$, where $a^0 \in \mathbb{C}^{d\times d}$ is constant. We will show that the assumptions of the Perturbation Theorem 3.4.8 are satisfied for $\alpha = 1/2$ with a translate $\nu + \mathcal{A}_0$ for some $\nu > 0$ of the constant coefficients operator $\mathcal{A}_0 := -\operatorname{div}(a_0 \cdot \nabla)$ and the perturbation

$$\mathcal{B} := -\operatorname{div}(a^1(x) \cdot \nabla) - \operatorname{div}(b(x) \cdot) + c(x) \cdot \nabla + d(x),$$

if the norm $\|a^1\|_\infty$ is sufficiently small. Let A_0 be the realization of $\mathcal{A}_0$ in X with $D(A_0) := H^{2,p}$, and define $Bu := \mathcal{B}u$ (in the sense of distributions) for all $u \in D(B) := H^{1,p}$. Note that since $\nu > 0$, the spaces $H^{\pm 1,p}$ can be identified with the homogeneous fractional spaces $\dot{X}_{\pm 1/2}$ associated to the operator $\nu + A_0$ in X, cf. [Ha06], Section 8.3, so B is an operator $B : \dot{X}_{1/2} \to \dot{X}_{-1/2}$. We will show the following estimate for all $\nu \geq 1$:

(P) $\mathcal{R}_q((\nu + A_0)^{-1/2} B (\nu + A_0)^{-1/2}) \lesssim_{M,\delta} \|a^1\|_\infty + \nu^{-1/2}$.

Note that in (P) we consider the operator $(\nu + A_0)^{-1/2}$ as an operator $(\nu + A_0)^{-1/2} : X \to \dot{X}_{-1/2}$, and also as an operator $(\nu + A_0)^{-1/2} : \dot{X}_{1/2} \to X$, cf. Subsection 3.3.1, where the concept of considering the operator A_0 as an universal operator in the whole scale of extrapolation spaces is briefly presented.

Furthermore, (P) implies that the assumptions of the Perturbation Theorem 3.4.8 can be fulfilled if we choose $\nu \geq 1$ sufficiently large and $\|a^1\|_\infty$ sufficiently small. Before we turn to the proof of

(P) we have a closer look on the assertions that the Perturbation Theorem 3.4.8 yields in this case:

So we assume for a moment that (P) holds, and that we have chosen $\nu \geq 1$ sufficiently large and $\|a^1\|_\infty$ sufficiently small such that via (P) the assumptions of the Perturbation Theorem 3.4.8 are fulfilled. Observe that we can define another realization $\nu + \widetilde{A}_p$ of the formal operator $\nu + \mathcal{A}$ as a bounded operator

$$\nu + \widetilde{A}_p : H^{1,p} \to H^{-1,p}, u \mapsto -\operatorname{div}(a \cdot \nabla u + b \cdot u) + c \cdot \nabla u + (\nu + d(\cdot)) \cdot u,$$

where div is considered as the divergence in the distributional sense. Since

$$\dot{X}_{1/2} = H^{1,p} \hookrightarrow L^p = X \hookrightarrow H^{-1,p} = \dot{X}_{-1/2},$$

Theorem 3.4.8 yields in this situation that the part $\nu + C_p := \nu + (\widetilde{A}_p)_{L^p}$ of $\nu + \widetilde{A}_p$ in $X = L^p$ (instead of $\nu + A_p$) fulfills (1) and (2) from Theorem 3.6.8. Moreover, the resolvents of the operators $\nu + C_p$, $p \in (1, +\infty)$ are given by the identity

$$(\lambda+\nu+C_p)^{-1} = (\lambda+\nu+A_0)^{-1} - \left[(\nu+A_0)^{1/2}(\lambda+\nu+A_0)^{-1}\right] M(\lambda) \left[(\nu+A_0)^{1/2}(\lambda+\nu+A_0)^{-1}\right] \tag{3.6.17}$$

for all $\lambda > 0$, where

$$M(\lambda) := \sum_{k=0}^{\infty} \left(-L[(\nu+A_0)(\lambda+\nu+A_0)^{-1}]\right)^k L, \text{ and } L := (\nu+A_0)^{-1/2} B (\nu+A_0)^{-1/2}, \tag{3.6.18}$$

cf. the proof of Theorem 3.4.8. Observe that all bounded operators in (3.6.17), (3.6.18) are consistent in the spaces L^p for $p \in (1, +\infty)$, hence the resolvents of the operators $\nu + C_p$, $p \in (1, +\infty)$ are consistent. Moreover, in the special case $p = 2$ the operator $\nu + A_2$ equals the part $\nu + C_2$ of $\nu + \widetilde{A}_2$ in L^2 by definition of the operator associated to a form. Since also the resolvents of the operators $\nu + A_p$, $p \in (1, +\infty)$ (constructed via form methods and extrapolation as described above) are consistent, this yields $\nu + C_p = \nu + A_p$ in L^p for all $p \in (1, +\infty)$.

We will now turn to the proof of (P). We define the Riesz transforms associated to the operator A_0 by $R_j := A_0^{-1/2} \partial_j$ for all $j \in \mathbb{N}_{\leq d}$. By the same arguments as we used in Subsection 3.6.1, Step I, the operators R_j are $\mathcal{R}_q$-bounded, and the $\mathcal{R}_q$-bounds depend only on M, δ. For all $\nu \geq 1$

we have the following identity:

$$\begin{aligned}
& (\nu + A_0)^{-1/2} B (\nu + A_0)^{-1/2} \\
= \; & -(\nu + A_0)^{-1/2} \operatorname{div}\big(a^1(x) \cdot \nabla (\nu + A_0)^{-1/2}\big) - (\nu + A_0)^{-1/2} \operatorname{div}\big(b(x)(\nu + A_0)^{-1/2}\big) \\
& +(\nu + A_0)^{-1/2} c(x) \cdot \nabla (\nu + A_0)^{-1/2} + (\nu + A_0)^{-1/2} d(x) (\nu + A_0)^{-1/2} \\
= \; & -A_0^{1/2} (\nu + A_0)^{-1/2} \Bigg(\sum_{j,k=1}^n R_j a^1_{jk}(x) R_k \Bigg) A_0^{1/2} (\nu + A_0)^{-1/2} \\
& -\nu^{-1/2} \cdot A_0^{1/2} (\nu + A_0)^{-1/2} \Bigg(\sum_{j=1}^n R_j b_j(x) \Bigg) \nu^{1/2} (\nu + A_0)^{-1/2} \\
& +\nu^{-1/2} \cdot \nu^{1/2} (\nu + A_0)^{-1/2} \Bigg(\sum_{j=1}^n c_j(x) R_j \Bigg) A_0^{1/2} (\nu + A_0)^{-1/2} \\
& +\nu^{-1} \cdot \nu^{1/2} (\nu + A_0)^{-1/2} d(x) \nu^{1/2} (\nu + A_0)^{-1/2}.
\end{aligned}$$

Observe that all occurring bounded operators are $\mathcal{R}_q$-bounded in X, and with $K := \mathcal{R}_q(\{t^{1/2}(t+A_0)^{-1/2}, A_0^{1/2}(t + A_0)^{-1/2} \,|\, t > 0\})$ we obtain

$$\mathcal{R}_q((\nu + A_0)^{-1/2} B (\nu + A_0)^{-1/2}) \quad \lesssim_{M,\delta} \quad K^2 (d^2 \|a^1\|_\infty + 2dM\nu^{-1/2} + M\nu^{-1}),$$

hence (P) is proven.

In the final step we do a localization procedure analogously to Step III in Subsection 3.6.1. We will only sketch this part of the proof since it is very similar to the proof for non-divergence form operators, where now we use the estimate (P) for the local perturbation result. So again, we let $\varepsilon_0 \in (0, 1)$, which will be specified later, and choose $r > 0$ according to (3.6.9) from Step III in Subsection 3.6.1. Let $\Gamma, N_0, \psi, \rho, Q, (Q_\ell)_{\ell\in\Gamma}$ and $J, P, \mathbb{X}$ be as in Step III from Subsection 3.6.1. We define

$$A^0_\ell u := -\operatorname{div}(a(\ell)\nabla u) \quad \text{ for all } u \in H^{2,p}, \ell \in \Gamma,$$

and $\mathbb{A}_0(u_\ell)_\ell := (A^0_\ell u_\ell)_\ell$ for all $(u_\ell)_\ell \in D(\mathbb{A}_0) := \mathbb{X}^2 := \ell^p(\Gamma, H^{2,p})$. Then $\mathbb{A}_0 : \mathbb{X} \supseteq D(\mathbb{A}_0) \to \mathbb{X}$ is a diagonal operator which consists of (M, ω_0)-elliptic differential operators with constant coefficients.

We will now turn to the construction of the perturbation operator $\mathbb{B}$. Define the spaces $\mathbb{X}^{\pm 1} := \ell^p(\Gamma, H^{\pm 1,p})$, then for each $\nu > 0$ the spaces $\mathbb{X}^{\pm 1}$ can be identified with the homogeneous spaces $\dot{\mathbb{X}}_{\pm 1/2}$ associated to the operator $\nu + \mathbb{A}_0$, and we have canonical embeddings

$$\dot{\mathbb{X}}_{1/2} \hookrightarrow \mathbb{X} \hookrightarrow \dot{\mathbb{X}}_{-1/2}.$$

Furthermore, for all $u \in H^{1,p}$ we define

$$A^1_\ell(u) \quad := \quad -\operatorname{div}\big((a(x) - a(\ell)\rho(x))\nabla u\big)$$

and the lower order terms operator

$$A_{\text{low}}(u) := -\operatorname{div}\big(b(x)u\big) + c(x)\nabla u + d(x)u,$$

as operators $H^{1,p} \to H^{-1,p}$. Then for all $u \in H^{1,p}$ we have

$$\psi_\ell A u = A\psi_\ell u + \underbrace{(\psi_\ell A - A\psi_\ell)}_{=:C_\ell} u = (A_\ell^0 + A_\ell^1)\psi_\ell u + A_{\mathrm{low}}\psi_\ell u + \sum_{k \bowtie \ell} C_\ell \psi_k \psi_k u,$$

where

$$\begin{aligned}
C_\ell u &= -\sum_{j,k=1}^n \psi_\ell \partial_j (a_{jk}(x)\partial_k u) + \sum_{j=1}^n \psi_\ell\big(-\partial_j(b_j(x)u) + c_j\partial_j u\big) + \psi_\ell d(x)u \\
&\quad + \sum_{j,k=1}^n \partial_j(a_{jk}(x)\partial_k(\psi_\ell u)) - \sum_{j=1}^n \big(-\partial_j(b_j(x)u\psi_\ell) + c_j\partial_j(u\psi_\ell)\big) - d(x)u\psi_\ell \\
&= \sum_{j,k=1}^n \big(\partial_{jk}\psi_\ell \cdot a_{jk}(x)u + \partial_k\psi_\ell \cdot \partial_j(a_{jk}(x)u) + \partial_j\psi_\ell \cdot a_{jk}(x)\partial_k u\big) \\
&\quad + \sum_{j=1}^n \partial_j\psi_\ell\,((b_j(x) - c_j(x))u \\
&= \sum_{j,k=1}^n \partial_k\psi_\ell \cdot \partial_j(a_{jk}(x)u) + \sum_{j,k=1}^n \partial_j\psi_\ell \cdot a_{jk}(x)\partial_k u \\
&\quad + \sum_{j=1}^n \left(\partial_j\psi_\ell\,((b_j(x) - c_j(x)) + \sum_{k=1}^n \partial_{jk}\psi_\ell \cdot a_{jk}(x)\right) u \\
&= \nabla\psi_\ell \cdot \operatorname{div}(ua(x)) + a(x)\nabla u \cdot \nabla\psi_\ell + \big(\nabla\psi_\ell \cdot (b(x) - c(x)) + \operatorname{tr}(\nabla^2\psi_\ell a(x))\big)\, u.
\end{aligned}$$

Thus C_ℓ is an operator of order less than or equal to 1. We now define the perturbation operator by

$$\mathbb{B}(u_\ell)_\ell := \left(A_\ell^1 u_\ell + A_{\mathrm{low}} u_\ell + \sum_{k \bowtie \ell} C_\ell \psi_k u_k\right)_\ell$$

for all $(u_\ell)_\ell \in D(\mathbb{B}) := \mathbb{X}^1$. Then $\mathbb{B} : \mathbb{X}^1 \to \mathbb{X}^{-1}$ is an almost diagonal operator in the same sense as in Step III of Subsection 3.6.1, where the top order coefficients are bounded:

$$\|(a - a(\ell))\rho\|_\infty \le \|(a - a(\ell))|_{Q_\ell}\|_\infty < \varepsilon_0.$$

For each $\nu > 0$ we obtain

$$\begin{aligned}
&(\nu + \mathbb{A}_0)^{-1/2}\mathbb{B}(\nu + \mathbb{A}_0)^{-1/2}(u_\ell)_\ell \\
= &\left((\nu + A_\ell^0)^{-1/2}(A_\ell^1 + A_{\mathrm{low}})(\nu + A_\ell^0)^{-1/2}u_\ell + (\nu + A_\ell^0)^{-1/2}\sum_{k \bowtie \ell} C_\ell\psi_k(\nu + A_k^0)^{-1/2}u_k\right)_\ell.
\end{aligned}$$

Hence in each component we have a finite sum of operators of the type we considered in the second step, and together with (P) this yields

$$\begin{aligned}
&\mathcal{R}_s\big((\nu + \mathbb{A}_0)^{-1/2}\mathbb{B}(\nu + \mathbb{A}_0)^{-1/2}\big) \\
\le &\sup_\ell \sup_{k \bowtie \ell} \mathcal{R}_s\big((\nu + A_\ell^0)^{-1/2}(A_\ell^1 + A_{\mathrm{low}} + C_\ell\psi_k)(\nu + A_\ell^0)^{-1/2}\big) \lesssim_{M,\delta,N_0} \varepsilon_0 + \nu^{-1/2}.
\end{aligned}$$

This shows that we can ensure the assumptions of the Perturbation Theorem 3.4.8 if we choose $\varepsilon_0 > 0$ sufficiently small and $\nu \geq 1$ sufficiently large. So assume that $\varepsilon_0 > 0, \nu \geq 1$ are chosen appropriately according to the assumptions of the Perturbation Theorem 3.4.8. Define the operator

$$\widetilde{\mathbb{A}} : \mathbb{X}^1 \to \mathbb{X}^{-1}, (u_\ell)_\ell \mapsto \mathbb{A}_0(u_\ell)_\ell + \mathbb{B}(u_\ell)_\ell,$$

then by Theorem 3.4.8 the part $\nu + \mathbb{A}$ of $\nu + \widetilde{\mathbb{A}}$ in $\mathbb{X}$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\sigma)$-calculus, and moreover

$$\mathbb{X}^\theta_{s,\nu+\mathbb{A}_0} \cong \mathbb{X}^\theta_{s,\nu+\mathbb{A}} \quad \text{for all } \theta \in (-1/2, 1/2). \tag{3.6.19}$$

In the same manner we define the lifted operators

$$\widetilde{A} : H^{1,p} \to H^{-1,p}, u \mapsto \mathcal{A}u, \quad \text{and } \widetilde{J} : H^{-1,p} \to \mathbb{X}^{-1}, u \mapsto (\psi_\ell u)_\ell,$$

then by construction we have $\widetilde{J}\widetilde{A} \subseteq \widetilde{\mathbb{A}}\widetilde{J}$, and furthermore $JA \subseteq \mathbb{A}J$, where A is the part of $\widetilde{A}$ in X. In the same way as it is done in Step III in Subsection 3.6.1, we can conclude that $\nu + A$ has an $\mathcal{R}_q$-bounded $H^\infty(\Sigma_\sigma)$-calculus in X, and that $\dot{X}^\theta_{\nu+A,q} \approx \dot{X}^\theta_{\nu-\Delta,q}$ for all $\theta \in (-1/2, 1/2)$. Moreover, using similar arguments as in Step II, we can conclude that $\nu + A = \nu + A_p$. So the assertions of Theorem 3.6.8 follow by Theorem 3.6.1 and Theorem 3.3.23. □

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