On the Existence of Breathers Solutions

In non-linear Klein-Gordon equations

with periodic coefficients

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1 Introduction

In this work, we are interested in so-called breather solutions for nonlinear wave equations. A breather is a solution $u = u(x,t) \in \mathbb{R}$, $(x, t \in \mathbb{R})$, which is periodic in time and is spatially localized, i.e.,

$$u(x,t) = u(x,t+T), \qquad x,t \in \mathbb{R} \tag{1.1}$$

$$|u(x,t)| < C\,\mathrm{e}^{-\beta x}, \qquad x,t \in \mathbb{R} \tag{1.2}$$

for a (minimal)-period $T > 0$ and real-valued constants $C, \beta > 0$. The author of [5] proved, that any perturbation of the sine-Gordon equation

$$\partial_x^2 u(x,t) = \partial_t^2 u(x,t) + \sin(u(x,t))$$

will destroy the special symmetry of this equation. We conclude that breather solutions are a singular occurence for the sine-Gordan equation only. In particular the nonlinear Klein-Gordon equation

$$\partial_x^2 u(x,t) = \partial_t^2 u(x,t) + u(x,t) - u^3(x,t) \tag{1.3}$$

does not have any breather solutions. From applications in physics and electrical engineering it is known that for this equation there is a NLS-approximation which gives a reason to rethink the non-existence of breather solutions.
The authors of [8] looked at the nonlinear Klein-Gordon equation and found that, although there does not exist a breather solution, there is a generalized breather which is periodic in time but not localized in space: there is a small „periodic-tail". Speaking geometrically, only the sine-Gordon equation with periodic boundary conditions w.r.t.

t permits that the low-dimensional unstable and stable manifold of the origin of the spatial dynamics system intersect. But what if we add more degrees of freedom to the nonlinear Klein-Gordon equation such that we can make both manifolds meet each other?

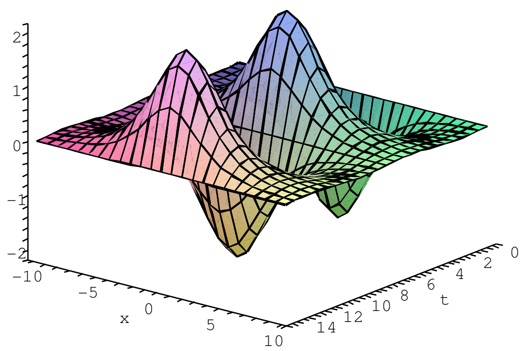

FIGURE 1.1: A depiction of one of the breathers of the sine-Gordon equation.

Motivated by the mathematical description of photonic band-gap materials we started looking at nonlinear Klein-Gordon equations with periodic coefficients, i.e.,

$$\partial_x^2 u(x,t) = s(x)\partial_t^2 u(x,t) + q(x)u(x,t) - r(x)u^3(x,t). \tag{1.4}$$

In this thesis we show that breathers exist for certain x-dependent periodic coefficients s, q, r, i.e. $s(x+1) = s(x)$, $q(x+1) = q(x)$ and $r(x+1) = r(x)$. Using the time-periodic boundary condition (1.1), this is equivalent to an countably infinite system of ODEs which can be reduced to a 2-dimensional center-manifold. The key to the application of center-manifold theory lies in the linear (Floquet-) spectrum which can be "tailored" with the help of the equation's (1.4) x-dependent coefficients. In the 2-dimensional center-manifold there exists a homoclinic orbit structurally stable to perturbations. This stability is due to the symmetry of the spatial coefficients and the consequential reversibility of the equation (1.4).

The method we use to prove a breather solution for (1.4) can also be applied to time-dependent nonlinear Klein-Gordon equation of the type

$$\partial_x^2 u(x,t) = \partial_t^2 (s \star u)(x,t) + u^3(x,t). \tag{1.5}$$

where $s(t)$ is a time-dependent coefficient and where "$\star$" denotes the convolution in time, see equation (3.2) for a definition. The coefficient s gives enough freedom to tailor the spectrum in such a way that a 2-dimensional center-manifold exists and that we can reduce the dynamics of this equation to the existence of a homoclinic solution.

The last chapter is both interesting in its own right and supplementary to the proof of the breather solution result of Chapter 2. It explains in detail how we chose the spatially periodic coefficients of equation (1.4) such that the (Floquet)-spectrum of (1.4) allows the application of the center-manifold theory.

The results of this thesis suggest that breather solutions in nonlinear wave equations are more common than we thought at first.

2 Breather solutions in Klein-Gordon equations

We consider the nonlinear, spatially periodic Klein-Gordon equation

$$s(x)\partial_t^2 u(x,t) = \partial_x^2 u(x,t) - q(x)u(x,t) + r(x)u^3(x,t), \tag{2.1}$$

where $u = u(x,t) \in \mathbb{R}$ with $x \in \mathbb{R}$, $t \in \mathbb{R}$ and a-periodic coefficients s, q and r, i.e.,

$$s(x) = s(x+a), \quad q(x) = q(x+a), \quad \text{and} \quad r(x) = r(x+a),$$

where w.l.o.g. we choose in the following $a = 1$. Then we prove the following theorem.

Theorem 2.1. *The equation* (2.1) *allows the existence of breather solutions, i.e. there exists a solution $u = u(x,t)$ with real-valued positive constants $\beta, C > 0$ such that*

$$\begin{aligned} |u(x,t)| &\le C\,\mathrm{e}^{-\beta|x|}, && \forall t \in \mathbb{R},\ x \in \mathbb{R}, \\ u(x,t) &= u(x, t + \frac{2\pi}{\omega_0}), && \forall x \in \mathbb{R}. \end{aligned} \tag{2.2}$$

There exists a breather solution for $\omega_0^2 = 2\pi\frac{13}{16}$, $\varepsilon \in (0, \varepsilon_0)$, $(\varepsilon_0 > 0)$, and the coefficients

$$\begin{aligned} s(x) &= \chi_{[0,6/13]} + 16\chi_{[6/13,7/13]} + \chi_{[7/13],1}(x \ mod\ 1) \\ q(x) &= (q_0 + q_1\varepsilon^2)s(x) \\ r(x) &= r_0 \end{aligned} \tag{2.3}$$

with $q_0 \in \mathbb{R}$, $(q_0 \approx 3.235)$ and $q_1, r_0 \in \{-1, +1\}$, which are determined in the proof.

Remark 2.1. *The breather solution is given in lowest order by*

$$\sup_{x\in\mathbb{R}} |u(x,t) - 2\varepsilon c_1 \mathrm{sech}(\varepsilon c_2 x) q_{11}(x)\sin(\omega_0 t)| \leq C\varepsilon^2 \tag{2.4}$$

with constants c_1, c_2 and a 2-periodic function q_{11} defined subsequently. See also Section 2.10 Remark 2.10.

Remark 2.2. *According to Theorem 2.1 and Remark 2.1 we have a family of breather solutions, where $\varepsilon \in (0, \varepsilon_0)$ takes the role of the parameter. The amplitude is $\mathcal{O}(\varepsilon)$ and the envelope modulates the underlying carrier-wave $q_{11}(x)\sin(\omega_0 t)$ on the spatial scale is $\mathcal{O}(\varepsilon^{-1})$. We use the big-O notation $\mathcal{O}$ throughout this work. It is defined for functions $h : \mathbb{R} \to \mathbb{R}$ by*

$$\mathcal{O}\big(h(x)\big) = \left\{ g : \mathbb{R} \to \mathbb{R} \ : \ g \neq 0, \ \lim_{x\to 0} \frac{h(x)}{g(x)} < \infty \right\}.$$

By convention we use the notation $g(x) = \mathcal{O}\big(h(x)\big)$ instead of the correct notation $g(x) \in \mathcal{O}\big(h(x)\big)$.

Remark 2.3. *The choice of the coefficients is not by chance. Chapter 4 shows a method to tailor the coefficients s and q to give an explicit form of the so-called discriminant from which the the Floquet-spectrum of* (2.1) *can be derived. Through this method we can apply an inverse analysis of the discriminant and can explicitly compute the coefficients s and q. However, this inverse analysis is only possible for coefficients with very few degrees of freedom. The general inverse problem, i.e. compute the coefficient when the Floquet-spectrum is given, is not solvable with the presented method.*

Remark 2.4. *Solutions of* (2.1) *can be approximated via the ansatz*

$$u(x,t) = \varepsilon A(\varepsilon(x - c_g t), \varepsilon^2 t) f_n(x,k)\, \mathrm{e}^{\mathrm{i}\omega_n(k)t} + \text{c.c.}$$

with $A(X,T) \in \mathbb{C}$, $c_g \in \mathbb{R}$ and $0 < \varepsilon \ll 1$ by an NLS-equation

$$\partial_T A = -\mathrm{i}\omega_n''(0)\partial_X^2 A + \mathrm{i}\gamma_n A|A|^2.$$

If $\omega_n''(0)\gamma_n < 0$ this equation possesses pulse solutions $A(X,T) = \tilde{A}(X)\,\mathrm{e}^{\mathrm{i}\tilde{\omega}T}$ of the form (2.4)*. In [3] an approximation result has been established that guarantees that solutions* *of (2.6) can be approximated on an $\mathcal{O}(\varepsilon^{-2})$ time-scale via the solutions of this NLS-equation.*

Due to the periodic coefficient we use Bloch-modes $f_n(x,k)$, where n is the number of the band, k is the Floquet-exponent and $\omega_n(k)$ gives the dispersion relation. Since we perturb off a band-edge, we usually consider $k = 0$ or $k = 1/2$. Then also $c_g = 0$.

For small spectral gaps there is one band edge where the associated NLS-equation possesses pulse solutions. For small spectral gaps we have $\gamma_n \approx \gamma_{n+1}$ but $\omega_n''(0) > 0$ and $\omega_{n+1}''(0) < 0$ or vice versa.

2.1 Breather solution: Construction

For the construction of the breather solution of the equation (2.1) we will use spatial dynamics, center-manifold theory and bifurcation theory. Motivated by [12] we write (2.1) as an evolutionary system w.r.t. $x \in \mathbb{R}$ in the phase space of $\frac{2\pi}{\omega_0}$-time periodic functions, i.e., we consider

$$\begin{aligned} \partial_x u &= v, \\ \partial_x v &= s(x)\partial_t^2 u + q(x)u - r(x)u^3. \end{aligned} \tag{2.5}$$

where we abbreviate $u(x,t)$, $v(x,t) \in \mathbb{R}$ with $u, v \in \mathbb{R}$ from now on, but we keep the x for the coefficients to emphasize the x-dependency. Due to the periodicity of s, q, and r w.r.t. x the system is non-autonomous. We use Floquet-theory to calculate the linear (Floquet-)spectrum, which describes the asymptotic behavior of the solutions

$$u_\omega(x,t) = p(x,\omega)\, \mathrm{e}^{k(\omega)x}\, \mathrm{e}^{\mathrm{i}\omega t}$$

of the linearized system of (2.5) with $\omega \in \mathbb{R}$, $k(\omega) \in \mathbb{C}$ and $p(x,\omega) = p(x+1,\omega)$ for all $\omega \in \mathbb{R}$. To emphasize the difference of the spatially periodic case and the spatially homogeneous case, i.e. $s, q, r =$const, where solutions of the linearized system are given by

$$u_\omega(x,t) = \mathrm{e}^{k(\omega)x}\, \mathrm{e}^{\mathrm{i}\omega t},$$

where the eigenvalues can be explicitly computed by the *dispersion relation*

$$k^2(\omega) = -s\omega^2 + q.$$

In the spatially periodic case there is a periodic, non-constant function $p(x, \omega)$ and a Floquet-exponent $k(\omega)$ which cannot be easily computed. We will show on the next few pages how one can calculate the Floquet-exponent $k(\omega)$.

Since we are in the space of $\frac{2\pi}{\omega_0}$-time periodic functions, there are countably many Floquet-exponents $k(\omega)|_{\omega=n\omega_0}$ for a fixed $\omega_0 \in \mathbb{R}$ indexed by $n \in \mathbb{Z}$. By using invariances of the equation 2.5 we can restrict ourselves to some invariant subspace of (2.5) which reduces the amount of Floquet-exponents by a factor 2, so that $k(\omega)|_{\omega=n\omega_0}$ will be indexed by $n \in \mathbb{N}_{\mathrm{odd}}$. We call $\mathbb{N}_{\mathrm{odd}}$ the set of all odd natural numbers. Then we will prove the following Lemma.

Lemma 2.1 (Property 1)**.** *Under the conditions of Theorem 2.1 in the invariant subspace defined in Section 2.3 the linearisation of the spatial dynamics system* (2.5) *with the coefficients* (2.3) *possesses only two Floquet exponents on the imaginary axis, which move off the axis as $\varepsilon > 0$ increases. The rest of the spectrum is uniformly bounded away from the imaginary axis for all sufficiently small $\varepsilon \geq 0$.*

This lemma allows the use of invariant manifold theory for periodic systems (see section 2.6) to reduce the infinite-dimensional system (2.5) to a two-dimensional system on the center-manifold associated with the two central eigenvalues. We then show that the reduced system has a homoclinic solution, i.e. a solution $U : \mathbb{R} \to \mathbb{R}$ with $U(x) \to 0$ for $|x| \to \infty$, which is structurally stable due to the reversibility of the spatial dynamics formulation, i.e. (2.5) is invariant under $(x, u, v) \mapsto (-x, u, -v)$. This is due to the symmetry of the coefficients

Lemma 2.2 (Property 2)**.** *The coefficients chosen above are even w.r.t. x, i.e.,*

$$s(x) = s(-x), \quad q(x) = q(-x), \quad \textit{and} \quad r(x) = r(-x).$$

for all ε.

2.2 Proof of the Theorem 2.1

As briefly explained in the previous section, we will prove Lemma 2.1 with the use of Floquet-theory and invariant subspaces. Then we will discuss the application of center-manifold theory to reduce the dynamics to a 2-dimensional system. There we will show that there is a homoclinic solution, i.e. a solution $U : \mathbb{R} \to \mathbb{R}$ with $U(x) \to 0$ with exponential decay for $|x| \to \infty$. From the center-manifold theory we know that the exponential decay of the homoclinic solution will carry over to the full system in the form of a spatially localized solution with exponential decay as $|x| \to \infty$. The Fourier-series in time then puts this spatially localized solution into a time-periodic frame, therefore completing the proof.

The proof consists of seven steps. First we discuss invariances of the system (2.5) such that we are able to restrict our solutions to a fitting invariant subspace, see Section 2.3. In this invariant subspace we go on to compute the Floquet-exponents. In section 2.4 we also explain why we had to use an invariant subspace. Section 2.5 and 2.6 talk about the center-manifold reduction and prove a modification of the center-manifold theorem. Sections 2.7 through 2.9 analyse the reduced two-dimensional system on the center-manifold. The two-dimensional system can be seen as a basic system which is known to have homoclinic solutions and a perturbation. We show that even under perturbations the basic system's homoclinic solution will persist. From there we conclude the proof of Theorem 2.1 with a summary of every step taken along the way. At the end we are able to give a first order approximation of the breather.

Preparations Since we are interested in time-periodic solutions of equation (2.1), i.e., $u(x, t + \frac{2\pi}{\omega_0}) = u(x, t)$ for all $x, t \in \mathbb{R}$ we use Fourier-series with respect to time leading to the system of countable many ODEs

$$\partial_x^2 u_m(x) = -s(x)m^2\omega_0^2 u_m(x) + q(x)u_m(x) - r(x)g_m(x), \qquad m \in \mathbb{Z}, \tag{2.6}$$

where

$$g_m(x) = \sum_{n\in\mathbb{Z}^3, |n|=m} u_{n_1}(x)u_{n_2}(x)u_{n_3}(x). \qquad m \in \mathbb{Z}. \tag{2.7}$$

In terms of the spatial dynamics formulation (2.5) we have the system

$$\begin{aligned} \partial_x u_m(x) &= v_m(x) \\ \partial_x v_m(x) &= -s(x)m^2\omega_0^2 u_m(x) + q(x)u_m(x) - r(x)g_m(x) \end{aligned} \qquad \text{for } m \in \mathbb{Z}. \tag{2.8}$$

2.3 Invariances

There is a number of linear subspaces invariant under the evolution of (2.6) resp. (2.8). These are as follows. The invariant subspace corresponding to real solutions of (2.8) is given by

$$U_{\mathbb{R}} := \{(u_m)_{m\in\mathbb{Z}} \;:\; u_m = \overline{u_{-m}}\}.$$

Since the system is invariant under the transform $S : (t, u, v) \mapsto (-t, -u, -v)$ also

$$U_{\text{odd}} = \{(u_m)_{m\in\mathbb{Z}} \;:\; u_m = -u_{-m}\}$$

is some invariant subspace. According to the fact that we have a cubic nonlinearity also

$$U_{\mathcal{O}} = \{(u_m)_{m\in\mathbb{Z}} \;:\; u_{2m} = 0\}$$

is an invariant subspace. Therefore the intersection of all these subspaces

$$U_{\mathbb{R}} \cap U_{\text{odd}} \cap U_{\mathcal{O}} = \{(u_m)_{m\in\mathbb{Z}} \;:\; \operatorname{Re} u_m = 0, u_{2m} = 0\} =: \hat{X}$$

is also invariant. In the following we restrict our analysis to those solutions of (2.1) whose Fourier-coefficients are in $\hat{X}$. Such solutions can be written as

$$u(x,t) = \sum_{m\in\mathbb{Z}_{\text{odd}}} \mathrm{i} u_m(x)\, \mathrm{e}^{\mathrm{i}m\omega t}$$

where $u_m \in \mathbb{R}$ and $u_m = u_{-m}$ satisfy a system of countable many ODEs

$$\begin{aligned} \partial_x^2 u_m(x) &= v_m(x) \\ \partial_x^2 v_m(x) &= -s(x)m^2\omega^2 u_m(x) + q(x)u_m(x) + r(x)g_m(x) \end{aligned} \tag{2.9}$$

but now with $m \in \mathbb{N}_{\text{odd}}$ and

$$g_m(x) = \sum_{n \in \mathbb{Z}^3, |n| = m} u_{n_1}(x) u_{n_2}(x) u_{n_3}(x). \qquad m \in \mathbb{N}_{\text{odd}}. \tag{2.10}$$

The tayloring of the coefficient and the restriction to the invariant subspace $\hat{X}$ are the most important steps in the proof. If we restrict the subspace of solutions to $\hat{X}$, we also restrict the spectrum to those Floquet-exponents that are off the imaginary axis, see Figure 2.2.

2.4 Proof of Lemma 2.1

We want to compute the Floquet-exponents of the linearizations of equations (2.6)-(2.7) and (2.9)-(2.10) at the origin respectively and prove that there are only two Floquet-exponents near the imaginary axis and that all others are uniformly bounded away from the imaginary axis. In order to analyze the linear part

$$\begin{aligned} \partial_x u_m(x) &= v_m(x) \\ \partial_x v_m(x) &= -s(x) m^2 \omega_0^2 u_m(x) + (q_0 + q_1 \varepsilon^2) s(x) u_m(x) \end{aligned} \tag{2.11}$$

where $q(x) = (q_0 + q_1 \varepsilon^2) s(x)$, we substitute $m\omega_0 = \lambda$ and only look at the ODE

$$\begin{aligned} \dot{u}(x) &= v(x) \\ \dot{v}(x) &= -s(x) \lambda^2 u(x) + (q_0 + q_1 \varepsilon^2) s(x) u(x), \end{aligned} \tag{2.12}$$

where we can apply Floquet's theorem, see [6], to find its Floquet-exponents depending on $\lambda \in \mathbb{R}$.

Let $x \mapsto \Phi_{\lambda^2}(x; x_0)$ be the fundamental solution of the ODE (2.12) with $\Phi_{\lambda^2}(x_0; x_0) = I$ and $x_0 \in \mathbb{R}$. Then let $\rho_\pm$ be the eigenvalues of $C_{\lambda^2} := \Phi_{\lambda^2}(x_0 + 1; x_0)$, the so-called *monodromy matrix*. Floquet's Theorem says that the *Floquet-multipliers* are the eigenvalues of the monodromy matrix and are given by

$$\rho_\pm(\lambda^2) = \frac{1}{2} \operatorname{trace} C_{\lambda^2} \pm \frac{1}{2} \sqrt{(\operatorname{trace} C_{\lambda^2})^2 - 4}.$$

The trace of the monodromy matrix is called the *discriminant*. We use the abbreviation $D(\lambda^2) := \operatorname{trace} C_{\lambda^2} = \phi_1(x_0+1; x_0, \lambda^2) + \phi_2'(x_0+1; x_0, \lambda^2)$ where

$$\Phi_{\lambda^2}(x; x_0) = \begin{pmatrix} \phi_1(x; x_0, \lambda^2) & \phi_2(x; x_0, \lambda^2) \\ \phi_1'(x; x_0, \lambda^2) & \phi_2'(x; x_0, \lambda^2) \end{pmatrix}.$$

The discriminant depends on the parameter λ and so

$$\rho_\pm(\lambda^2) = \frac{1}{2} D(\lambda^2) \pm \frac{1}{2}\sqrt{(D(\lambda^2))^2 - 4}.$$

There exists a representation for solutions (see [6]), which gives a good connection to the autonomous case:

$$\Phi_{\lambda^2}(x; x_0) = P_{\lambda^2}(x; x_0)\, \mathrm{e}^{(x-x_0)M_{\lambda^2}} \tag{2.13}$$

where M is called the *exponential matrix*. Its eigenvalues are called the *Floquet-exponent* $k \in \mathbb{C}$ defined by $\rho_\pm(\lambda^2) = \mathrm{e}^{\pm k(\lambda^2)}$. The Floquet-exponent is not necessarily unique but can be chosen in such a way, that it is unique. We find that

S1a) if $|D(\lambda^2)| > 2$ then the Floquet-multipliers $\rho_\pm(\lambda^2)$ are real, i.e., $k \in \mathbb{R} \setminus \{0\}$. As a consequence Φ_{λ^2} shows exponential growth w.r.t. x.

S1b) if $|D(\lambda^2)| < 2$ then the Floquet-multipliers $\rho_\pm(\lambda^2)$ are on the complex unit circle, i.e., $k \in \mathrm{i}\mathbb{R}$. As a consequence Φ_{λ^2} is uniformly bounded w.r.t. x.

S2) if $|D(\lambda^2)| = 2$ then the Floquet-multipliers $\rho_\pm(\lambda^2)$ are $= -1$ or $= 1$. As a consequence Φ_{λ^2} has at most polynomial growth w.r.t. x.

Remark 2.5. *By the representation* (2.13) *it doesn't matter at which point x_0 we start. The discriminant is the same for each x_0.*

The discriminant of the equation (2.12) with the special choice of coefficients

$$s(x) = \chi_{[0,6/13]} + 16\chi_{[6/13,7/13]} + \chi_{[7/13],1}(x \bmod 1)$$
$$q(x) = (q_0 + q_1\varepsilon^2)s(x)$$

can be computed by the *Transfer-Matrix Method* or the *Fourier-Interface Method*, see Chapter 4, and yields

$$D(\lambda^2) = \frac{25}{8}\cos\left(\frac{16}{13}\sqrt{\lambda^2 + q_0 + q_1\varepsilon^2}\right) - \frac{9}{8}\cos\left(\frac{8}{13}\sqrt{\lambda^2 + q_0 + q_1\varepsilon^2}\right). \tag{2.14}$$

The graph of this discriminant and its corresponding *dispersion relation* is plotted in Figure 2.1. The dispersion relation simply plots the imaginary part of the Floquet-exponent $\pm k$ as a function of λ^2. The real part corresponds to exponential growing solutions, therefore we call an Floquet-exponent to lie in a *band-gap* if it has a non-vanishing real part. On the contrary, if the Floquet-exponent is purely imaginary, we say it lies in a *band*.

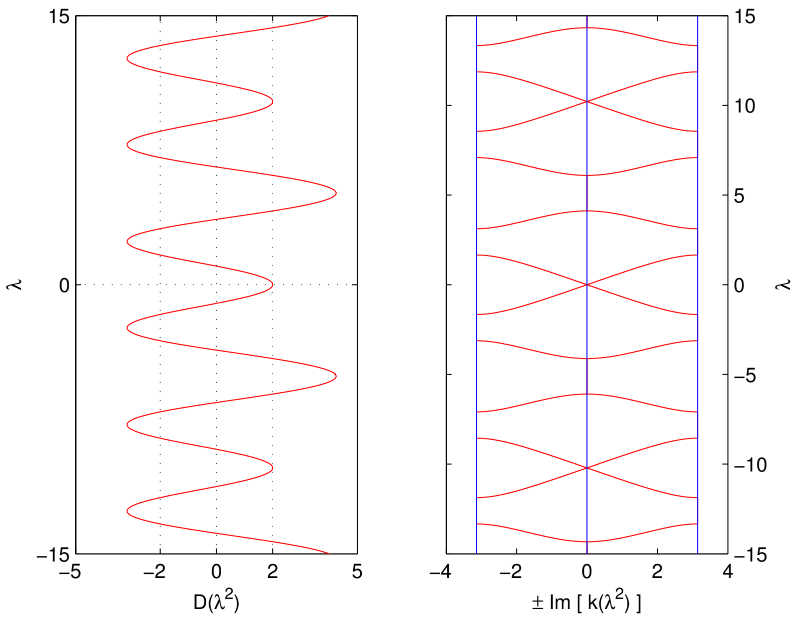

FIGURE 2.1: The figure shows the dispersion relation and the discriminant. When there are band-gaps then k is real.

The Floquet-spectrum We now choose

$$\omega_0^2 = 2\pi\frac{13}{16} \tag{2.15}$$

in equation (2.8) to be in phase with the leading term of the discriminant (2.14) in the following sense (with $q_0 = 0$ and $\varepsilon = 0$):

$$\cos\left(\frac{16}{13}\sqrt{\lambda^2 + q_0 + q_1\varepsilon^2}\right)\Big|_{\lambda = m\omega_0} = \cos(2\pi m)$$

For $q_0 = \varepsilon = 0$ we therefore get

$$D(m^2\omega_0^2) = \begin{cases} \frac{34}{8}, & m \in 2 + 4\mathbb{Z}, \\ 2, & m \in 4\mathbb{Z}, \\ \frac{25}{8}, & m \in 1 + 2\mathbb{Z}. \end{cases}$$

We choose q_0 in such a way that $D(\omega_0^2) = 2$ if $\varepsilon = 0$ and thus $D(\omega_0^2) = 2 + \mathcal{O}_+(\varepsilon^2)$ for $\varepsilon \to 0$.

Definition 2.1. *We define $g(x) = \mathcal{O}_+(h(x))$ for $x \to 0$ as*

$$\lim_{x\to 0} \frac{h(x)}{g(x)} = C_+$$

with $g(x), h(x) \geq 0$ for some constant $C_+ \geq 0$.

At this point we see why it is neccessary to restrict our analysis to the space $\hat{X}$, i.e. $\lambda^2 = m^2\omega_0^2$ with $m \in \mathbb{N}_{\text{odd}}$. Let $q_0 = \varepsilon = 0$ and look at the problems (2.6)-(2.7) and (2.9)-(2.10). We get Floquet-multipliers

$$\rho_\pm(m^2\omega_0^2) = \begin{cases} \frac{34}{16} \pm \frac{1}{2}\sqrt{(\frac{34}{8})^2 - 4}, & m \in 2 + 4\mathbb{Z}, \\ 1, & m \in 4\mathbb{Z}, \\ \frac{25}{16} \pm \frac{1}{2}\sqrt{(\frac{25}{8})^2 - 4}, & m \in 1 + 2\mathbb{Z}. \end{cases}$$

Therefore, for $q_0 = \varepsilon = 0$ there are five different Floquet-multipliers, four off the unit circle but one on the unit circle. The fact that there are infinitely many Floquet multipliers on the unit circle would prohibit the application of center-manifold theory. However, since we look for solutions in $\hat{X}$ we have $m \in \mathbb{N}_{\text{odd}}$ and therefore $\rho_\pm(m^2\omega_0^2) = \frac{25}{16} \pm \frac{1}{2}\sqrt{(\frac{25}{8})^2 - 4}$ are the only two Floquet-multipliers to be considered. See Figure 2.2.

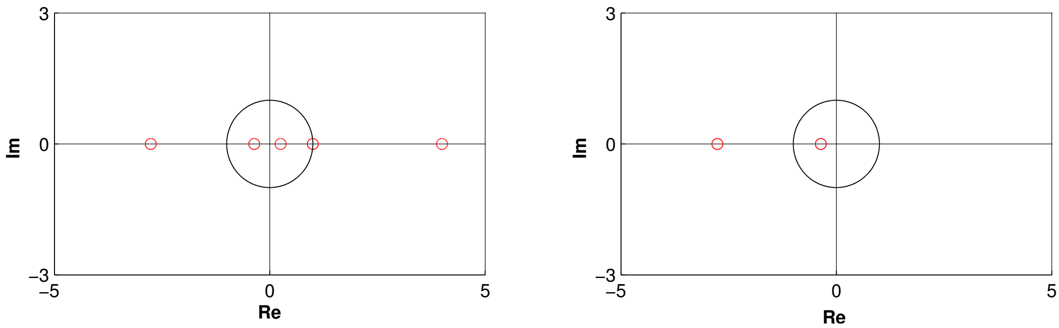

FIGURE 2.2: The Floquet-multipliers for the above choice of s, q, and ω.
LEFT: for $m \in \mathbb{Z}$. RIGHT: for $m \in \mathbb{N}$ odd.

For approximately $q_0 \approx 3.235$ the Floquet-multiplier for $m = 1$ is $\rho_\pm(\omega_0^2) = -1$. The two points $\frac{25}{16} \pm \frac{1}{2}\sqrt{(\frac{25}{8})^2 - 4}$ are now accumulation points, yet are only approached asymptotically by all other Floquet-multipliers $\rho_\pm(\omega^2 m^2)$ $(m > 1)$, which are still uniformly bounded away from the complex unit circle.

With the choice $q_1 = -1$ or $q_1 = 1$ we can make two Floquet-multipliers move from -1 off the unit circle for $\varepsilon > 0$. If q_1 is not set correctly, the Floquet-multipliers move from -1 along the imaginary unit circle. The value of q_1 sets the direction such that we bifurcate into the band-gap (instead of bifurcating further into the band) See Figure 2.3 for a depiction of the correct bifurcation.

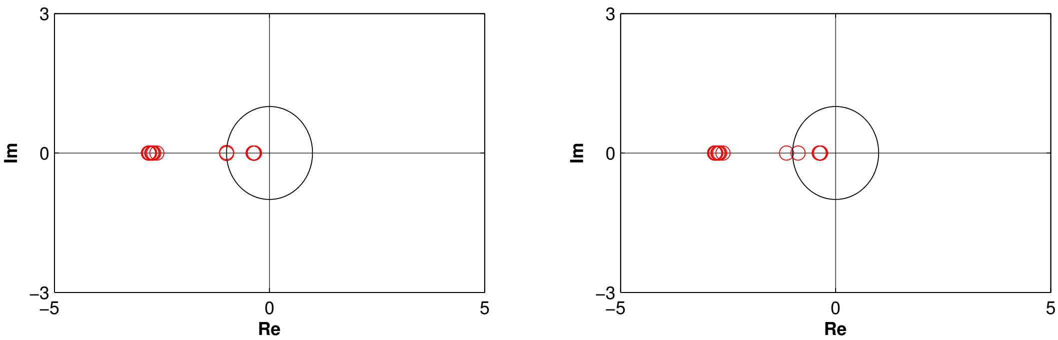

FIGURE 2.3: The distribution of Floquet-multipliers in $\hat{X}$ for the choice of $q = q_0$ with $\varepsilon = 0$ (left) and $\varepsilon > 0$ (right).

The Floquet-exponent is located on the imaginary axis if its corresponding Floquet-multiplier is located on the unit circle. All other Floquet-exponents are off the imaginary axis if their corresponding Floquet-multipliers are off the unit circle. Figure 2.4 shows the corresponding Floquet-exponents to the Floquet-multiplicators shown in Figure 2.3. The proof of Lemma 2.1 is therefore completed.

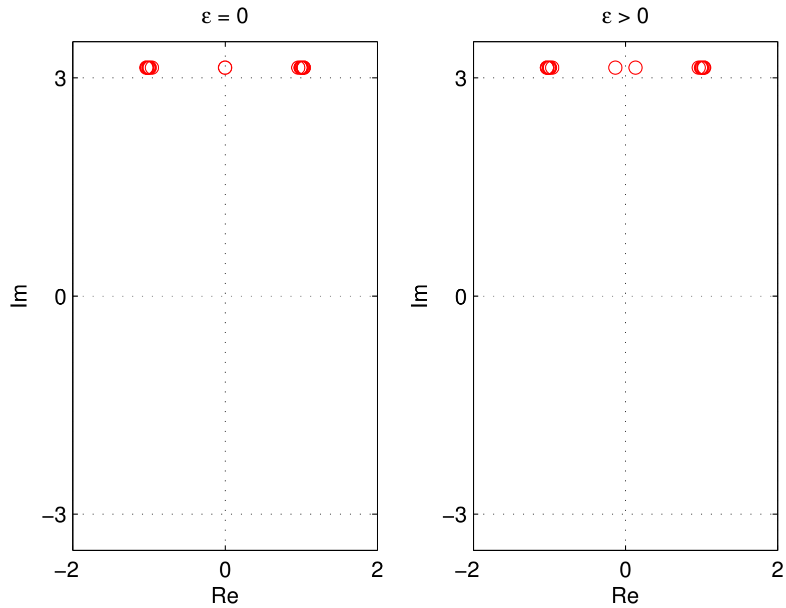

FIGURE 2.4: The distribution of Floquet-exponents in $\hat{X}$ for the choice of $q = q_0$ with $\varepsilon = 0$ (left) and $\varepsilon > 0$ (right).

Remark 2.6. *The Floquet-diagram (Figure 2.3) is actually the spectrum of the linear monodromy operator* $C = \bigoplus_{m \in \mathbb{N}_{odd}} C_m$ *where* C_m *is the monodromy matrix of each ODE of Equation* (2.8)*. Hence,*

$$\sigma(C) = \bigcup_{m \in \mathbb{N}_{odd}} \{\rho_\pm^m\}$$

where $\rho_\pm^m$ *are the eigenvalues of* C_m*. The Floquet-spectrum (Figure 2.4) is the generalisation of the spectrum of linear autonomous operators. In the next chapter we introduce „rotational coordinates" and transform the linear operator* $A(x)$ *(see definition next section) into an autonomous linear operator* M*. Its eigenvalues are the Floquet-exponents, and therefore its spectrum is given by*

$$\sigma(M) = \bigcup_{m \in \mathbb{N}_{odd}} \{\pm k_m\}$$

with $k_m = k(m^2\omega_0^2)$*.*

2.5 Rotational Coordinates

In the last chapter we discussed the spectrum of M by calculating the discriminant. In this section we apply a change of coordinates to the system (2.9) which preserves its reversibility and takes the rotational nature of the periodic equation into account such that the tranformed system has an autonomous linear part. Therefore we speak of *rotational coordinates*. With the system in new coordinates the center-manifold theory can be modified to encompass an x-periodic nonlinear part.
We consider the spatial dynamics formulation (2.9)

$$\begin{aligned} \partial_x u_m &= v_m, \\ \partial_x v_m &= -s(x)m^2\omega_0^2 u_m(x) + (q_0 + q_1\varepsilon^2)s(x)u_m(x) + r(x)g_m(x) \end{aligned} \tag{2.16}$$

for $m \in \mathbb{N}_{\text{odd}}$. The equation (2.16) can be written in a more compact form

$$\partial_x U_m = A_m U_m + S_\varepsilon U_m + N_m[(U_m)_m], \qquad m \in \mathbb{N}_{\text{odd}} \tag{2.17}$$

with

$$\begin{aligned} U_m &= (u_m, v_m), \\ A_m(x)U_m(x) &= \begin{pmatrix} 0 & 1 \\ -s(x)(m^2\omega_0^2 + q_0) & 0 \end{pmatrix} U_m, \\ S_\varepsilon(x)U_m &= \begin{pmatrix} 0 & 0 \\ \varepsilon^2 q_1 s(x) & 0 \end{pmatrix} U_m \\ N_m[(U_m)](x) &= \begin{pmatrix} 0 \\ r(x)g_m(x) \end{pmatrix}. \end{aligned}$$

For convenience we write $U = (U_m)_m = (u_m, v_m)_m = (u_m, v_m)_{m\in\mathbb{N}_{\text{odd}}} = (U_m)_{m\in\mathbb{N}_{\text{odd}}}$ from here on.

The reversibility: In Section 2.9 we need the reversibility of (2.6) in order to prove the persistence of the homoclinic solution with respect to higher order perturbations. Therefore we define the *reversibility operator* R by

$$R_m(u_m, v_m) = (u_m, -v_m).$$

The system is *reversible*, i.e., invariant under $(x,u,v)\mapsto(-x,u,-v)$, which implies that with $U_m(x)=(u_m,v_m)(x)$ also $V_m(x)=R_mU_m(-x)$ is a solution. Furthermore we define the *reversibility map*

$$T_m[U_m](x) := R_mU_m(-x)$$

The reversibility property holds for an equation

$$\dot U = F(x,U)$$

if

$$RF(x,U) = -F(-x,RU).$$

This holds for our system as well since

$$\begin{aligned}
\dot V_m(x) &= -R_m\dot U_m(-x)\\
&= -R_mA(-x)U_m(-x) - R_mN_m(-x,(U_j)_j(-x))\\
&= -R_mA(-x)R_mR_mU_m(-x) + N_m(-x,(R_jU_j)_j(-x))\\
&= A(-x)V_m(x) + N_m(-x,(V_j)_j(x))\\
&= A(x)V_m(x) + N_m(x,(V_j)_j(x)).
\end{aligned}$$

The last step is due to the symmetry of the coefficients. In the following arguments the *fixed space of reversibility* plays a major role. It is given by

$$R_{fix} = \{U = RU\} = \{(u_m,0)_m\}.$$

The change of coordinates Due to the theorem of Floquet the solutions of

$$\partial_x U_m = A_m(x)U_m$$

are given by

$$U_m(x,x_0) = P_m(x,x_0)\,\mathrm{e}^{(x-x_0)M_m}U_m(x_0,x_0)$$

with

$$P_m(x,x_0) = P_m(x+1,x_0), \quad M_m\in\mathbb{R}^{2\times 2}.$$

Since all Floquet-multipliers have a negative real part and a vanishing imaginary part (see Figure 2.3), the associated Floquet-exponents, the eigenvalues of $M = \bigoplus_{m\in\mathbb{N}_{\mathrm{odd}}} M_m$, are of the form $\alpha \pm i\pi$ with $\alpha \in \mathbb{R}$, see Figure 2.5.

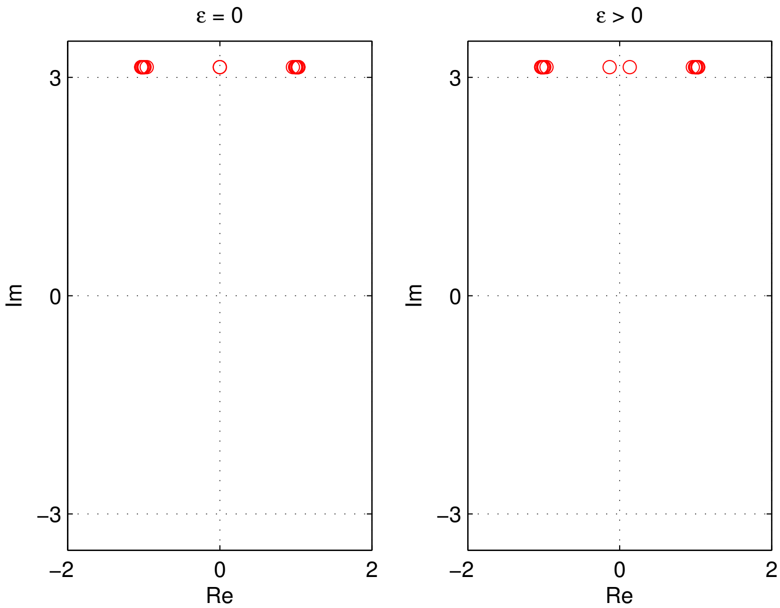

FIGURE 2.5: All Floquet-multipliers are of the form $\alpha \pm i\pi$ with $\alpha \in \mathbb{R}$.

In order to have real-valued Floquet-exponents we apply Floquet's theorem for 2-periodic functions, i.e. the solutions of $\partial_x U_m = A_m(x)U_m$ are given by

$$U_m(x, x_0) = Q_m(x, x_0)\, \mathrm{e}^{(x-x_0)B_m} U_m(x_0, x_0)$$

with $Q_m(x, x_0) = Q_m(x + 2, x_0)$ and $B_m \in \mathbb{R}^{2\times 2}$.

Preserving the reversibility In order to make the linear part of the system (2.17) autonomous we could make a change of variables $U_m(x, x_0) = Q_m(x, x_0)V_m(x, x_0)$. However, this choice would destroy the reversibility. Instead we use a slightly modified version of this change of coordinates. We write

$$\begin{aligned}
U_m(x, x_0) &= Q_m(x, x_0)\, \mathrm{e}^{(x-x_0)B_m} U_m(x_0, x_0) \\
&= Q_m(x, x_0) S_m^{-1}\, \mathrm{e}^{(x-x_0)J_m} S_m U_m(x_0, x_0) \\
&= \tilde{Q}_m(x, x_0)\, \mathrm{e}^{(x-x_0)J_m} V_m(x_0, x_0)
\end{aligned}$$

such that $V_m(x, x_0)$ defined by

$$U_m(x, x_0) = \tilde{Q}_m(x, x_0) V_m(x, x_0) \tag{2.18}$$

satisfies the autonomous ODE $\partial_x V_m = J_m V_m$. Next we want to show that (2.18) preserves the reversibility. Let λ_{1m} and λ_{2m} be the eigenvalues of B_m.

Case S1: Assume first that $\lambda_{1m} \neq \lambda_{2m}$. Then the solutions of $\partial_x U_m = A_m(x) U_m$ can also be written as the linear combination of two linearly independent solutions ψ_{1m} and ψ_{2m}:

$$U_m(x) = c_{1m}\psi_{1m}(x) + c_{2m}\psi_{2m}(x) = c_{1m}\,\mathrm{e}^{\lambda_{1m}x}\phi_{1m}(x) + c_{2m}\,\mathrm{e}^{\lambda_{2m}x}\phi_{2m}(x)$$

with 2-periodic ϕ_{jm}, $j = 1, 2$, here and in the following. Since the system is reversible, $x \mapsto \mathrm{e}^{-\lambda_{1m}x} R\phi_{1m}(-x)$ is also a solution if $x \mapsto \mathrm{e}^{\lambda_{1m}x}\phi_{1m}(x)$ is a solution. Hence we define the second fundamental solution

$$\psi_{2m}(x) = \mathrm{e}^{\lambda_{2m}x}\phi_{2m}(x) = \mathrm{e}^{-\lambda_{1m}x} R\phi_{1m}(-x).$$

such that $\lambda_{2m} = -\lambda_{1m}$ and $\phi_{2m}(x) = R\phi_{1m}(-x)$. We introduce the new variable $V_m(x) = (\tilde{u}_m, \tilde{v}_m)(x)$ by

$$U_m(x) = \tilde{u}_m(x)\phi_{1m}(x) + \tilde{v}_m(x)\phi_{2m}(x) = (\phi_{1m}(x), \phi_{2m}(x)) \begin{pmatrix} \tilde{u}_m(x) \\ \tilde{v}_m(x) \end{pmatrix}$$

where by construction $\partial_x V_m(x) = J_m V_m(x)$ with $J_m = \mathrm{diag}(\lambda_{1m}, \lambda_{2m})$. Therefore, the above change of variables (2.18) for $x_0 = 0$ and the last change of variables coincide and the new system is still reversible in case $x_0 = 0$ w.r.t. the transformed reversibility operator $\tilde{R}_m$ defined through

$$\tilde{R}_m \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} = \begin{pmatrix} \tilde{v}_m \\ \tilde{u}_m \end{pmatrix}.$$

Case S2: Next assume that we have a Jordan block. Then

$$U_m(x) = c_{1m}\psi_{1m}(x) + c_{2m}\psi_{2m}(x) = c_{1m}\,\mathrm{e}^{\lambda_m x}\phi_{1m}(x) + c_{2m}(\,\mathrm{e}^{\lambda_m x} x\phi_{1m}(x) + \,\mathrm{e}^{\lambda_m x}\phi_{2m}(x)).$$

As a result of the reversibility the eigenvalues necessarily fulfil $\lambda_m = 0$, $\phi_{1m}(x) = R\phi_{1m}(-x)$ and $\phi_{2m}(x) = -R\phi_{2m}(-x)$. We introduce the new variable $V_m(x) = (\tilde{u}_m, \tilde{v}_m)(x)$ by

$$U_m(x) = \tilde{u}_m(x)\phi_{1m}(x) + \tilde{v}_m(x)\phi_{2m}(x) = (\phi_{1m}(x), \phi_{2m}(x)) \begin{pmatrix} \tilde{u}_m(x) \\ \tilde{u}_m(x) \end{pmatrix}$$

where by construction $\partial_x V_m(x) = J_m V_m(x)$, where $J_m = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case the representation of the reversibility operator is preserved, i.e.

$$\tilde{R}_m \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} = \begin{pmatrix} \tilde{v}_m \\ -\tilde{v}_m \end{pmatrix}.$$

Remark 2.7. *We could also choose* $J_m = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. *The change of coordinates would then be given by interchanging* ϕ_{1m} *and* ϕ_{2m}. *The reversibility operator then is*

$$\tilde{R} \begin{pmatrix} \tilde{u}_m \\ \tilde{v}_m \end{pmatrix} = \begin{pmatrix} -\tilde{v}_m \\ \tilde{u}_m \end{pmatrix}.$$

The old reversibility operator R_m and the new reversibility operator $\tilde{R}_m$ (valid for both cases) are conjugated w.r.t. the transform $U_m = \tilde{Q}_m V_m$ defined by $\tilde{Q}_m = (\phi_{1m}\ \phi_{2m})$, i.e.

$$R_m \tilde{Q}_m(x) = \tilde{Q}_m(-x)\tilde{R}_m$$

which implies $R_m \tilde{Q}_m^{-1}(-x) = \tilde{Q}_m^{-1}(x)\tilde{R}_m$.
For the operator of reversibility for the full system (2.17)

$$R = \bigoplus_{m \in \mathbb{N}_{\mathrm{odd}}} R_m$$

and

$$\tilde{R} = \bigoplus_{m \in \mathbb{N}_{\mathrm{odd}}} \tilde{R}_m,$$

the reversibility of the old nonlinearity N means $RN(U) = -N(RU)$. The reversibility of the transformed nonlinearity $\tilde{N}(x, V(x)) := \tilde{Q}^{-1}(x)N(\tilde{Q}(x)V(x))$ means $\tilde{R}\tilde{N}(x, V) =$

$-\tilde{N}(-x,\tilde{R}V)$. This holds according to

$$\begin{aligned}\tilde{R}\tilde{N}(x,V(x)) &= \tilde{R}\tilde{Q}^{-1}(x)N(\tilde{Q}(x)V(x)) \\ &= \tilde{Q}^{-1}(-x)RN(\tilde{Q}(x)V(x)) = -\tilde{Q}^{-1}(-x)N(R\tilde{Q}(x)V(x)) \\ &= -\tilde{Q}^{-1}(-x)N(\tilde{Q}(-x)\tilde{R}V(x)) = -\tilde{N}(-x,\tilde{R}V(x)).\end{aligned}$$

The reversibility property carries over for S_ε in an analogous way. Although reversibility is only preserved for $x_0 = 0$, we keep x_0 in order to make some subsequent arguments clearer.

The reversible change of coordinates With $U_m(x,x_0) = \tilde{Q}_m(x,x_0)V_m(x,x_0)$ we find

$$\partial_x V_m(x,x_0) = J_m V_m(x,x_0) + F_{m,\varepsilon}\big(x,x_0,(V_j)_j\,(x,x_0)\big) \tag{2.19}$$

with

$$\begin{aligned}F_{m,\varepsilon}\big(x,x_0,(V_j)_j\,(x,x_0)\big) = \quad & \tilde{Q}_m(x,x_0)^{-1}S_\varepsilon(x)\tilde{Q}_m(x,x_0)V_m(x,x_0) \\ & + (\tilde{Q}_m(x,x_0))^{-1}N_m(x,(\tilde{Q}_jV_j)_j\,(x,x_0)).\end{aligned} \tag{2.20}$$

We find by construction that J_1 has one Jordan block of size 2 with associated eigenvalue 0 (Case S2). All other J_m with $m \geq 3$ possess one positive and one negative eigenvalue which are uniformly bounded away from the imaginary axis w.r.t. m (Case S1), i.e. (2.19) has the spectral picture plotted in the right panel of Figure 2.5. Moreover, system (2.19) is reversible w.r.t. the transformed reversibility operator

$$\tilde{R} = \bigoplus_{m\in\mathbb{N}_{\mathrm{odd}}} \tilde{R}_m$$

if $x_0 = 0$, since we have

$$\begin{aligned}J_m &= -\tilde{R}_m J_m \tilde{R}_m \\ F_{m,\varepsilon}\left(x,0,(V_j)_j\,(x)\right) &= -\tilde{R}_m F_{m,\varepsilon}\left(-x,0,(\tilde{R}_jV_j)_j\,(x)\right).\end{aligned} \tag{2.21}$$

The change of coordinates is bounded in the sense of Lemma 2.3. In particular, $\tilde{Q}_m^{-1}S_\varepsilon(x)\tilde{Q}_m$ is only of order ε^2 and is uniformly bounded for all m.

Lemma 2.3. *Let* $\tilde{Q}_m = (\phi_1\ \phi_2) = \begin{pmatrix} q_{11,m} & q_{12,m} \\ q_{21,m} & q_{22,m} \end{pmatrix}$. *Then there exists a* $C > 0$ *such that for all* $m \in \mathbb{N}_{odd}$ *we have* $\sup_{x\in[0,2]}(|q_{11,m}(x)|+|q_{12,m}(x)|) < C$ *and* $\sup_{x\in[0,2]} |(\tilde{Q}_m(x))^{-1}| < C$.

Proof. By explicitly solving for initial conditions $\phi_1(0) = 1$, $\phi_1'(0) = 0$ we see that $\phi_1(x) = \mathcal{O}(1)$ and $\phi_1'(x) = \mathcal{O}(m)$ as $m \to \infty$. Asymptotically only Case S1 is relevant, so we know $\phi_2(x) = R\phi_1(-x)$, hence $\phi_2(x) = \mathcal{O}(1)$ and $\phi_2'(x) = \mathcal{O}(m)$ as $m \to \infty$. Thus the lemma is proven since

$$\tilde{Q}_m = (\phi_1\ \phi_2).$$

□

Remark 2.8. *The direct application of the center-manifold theory to the equation* (2.17) *is not possible since the linear solution operator is not smoothing. The reason is as follows. The linear solution operator is given by*

$$\Phi(x, x_0) = \bigoplus_{m\in\mathbb{N}_{odd}} \Phi_m(x; x_0)$$

where each $\Phi_m(x; x_0)$ *is the fundamental system of*

$$\dot{U}_m = A_m(x)U_m$$

given by (2.13)*. By the proof of Lemma 2.3 the fundamental solution is* $\Phi_m(x; x_0) = \mathcal{O}(m)$ *for* $m \to \infty$*. From this it follows that*

$$\Phi(x; x_0) : \ell^2(\mathbb{R}^2; 1) \longrightarrow \ell^2(\mathbb{R}^2; 0)$$

where

$$\ell^2(\mathbb{R}^2; \sigma) = \left\{U = (u_m, v_m)_m \in \mathbb{R}^2 \ : \ \|U\|_{\ell^2(\mathbb{R}^2;\sigma)} < \infty\right\}$$

with the norm

$$\|U\|_{\ell^2(\mathbb{R}^2;\sigma)} = \sqrt{\sum_{m\in\mathbb{N}_{odd}} m^{2\sigma}|U_m|^2}.$$

By

2.6 The Center-Manifold Reduction

In Section 2.5 we changed the coordinates of equation (2.17) such that we have an autonomous linear part with eigenvalues according to Lemma 2.1. The next step is to apply center-manifold theory to the system in rotational coordinates (2.19).
We consider the system in rotational coordinates

$$\partial_x V_m(x, x_0) = J_m V_m(x, xa_0) + F_{m,\varepsilon}\big(x, x_0, (V_j)_j\,(x, x_0)\big). \tag{2.22}$$

We will compute the center-manifold for $x_0 = 0$ and then will use the flow of equation (2.22) to apply it to arbitrary starting point $x_0 \in \mathbb{R}$.
The center-manifold reduction will be done in the phase space

$$\ell^1(\mathbb{R}^2;\sigma) = \{V :\ m \mapsto V_m \in \mathbb{R}^2\ :\ m \in \mathbb{N}_{\mathrm{odd}},\ \|V\|_{\ell^1(\mathbb{R}^2;\sigma)} < \infty\}$$

where $\|V\|_{\ell^1(\mathbb{R}^2;\sigma)} = \sum_{m\in\mathbb{N}_{\mathrm{odd}}} m^\sigma |V_m|$. For the application of the center-manifold theory we use a cut-off function on the nonlinearity F to get a bounded nonlinearity:

$$\check{F}_{m,\varepsilon}(x, V) = F_{m,\varepsilon}(x, V)\chi(\|V\|_{\ell^1(\mathbb{R}^2;\sigma)}/\delta)$$

for a small but fixed $\delta > 0$, where χ is a C_0^∞-function with values in $[0, 1]$ satisfying $\chi(r) = 1$ for $r \leq 1$, $\chi(r) = 0$ for $r \geq 2$. Since in N_m only the first coordinate of U occurs, after the transforms $U_m = \tilde{Q}_m V_m$ only $q_{11,m}$ and $q_{12,m}$ occur in the transformed nonlinearity. Since both are uniformly bounded, since the same is true for $\tilde{Q}_m^{-1}$ according to Lemma 2.3, and since $\ell^1(\mathbb{R}^2;\sigma)$ is closed under convolutions, $(\check{F}_{m,\varepsilon}(U))_{m\in\mathbb{N}_{odd}}$ is Lipschitz continuous with Lipschitz constant proportional to δ^2 for $\delta \to 0$. The magnitude of the Lipschitz constant follows from the cut-off function and the fact that $\check{F}$ does not contain any quadratic terms. Moreover, the cut-off function does not change the reversibility property, hence $\check{F}$ is still reversible.
Next, we define projections $P_{s,m}$ and $P_{u,m}$ on the stable and unstable eigenspaces of each of the matrices J_m which are uniformly bounded in $\mathbb{R}^{2\times 2}$ w.r.t. m. With that we define $P_u = \bigoplus_m P_{u,m}$ and $P_s = \bigoplus_m P_{s,m}$ as the projections on the unstable or stable

eigenspace of equation (2.22). The projection on the center-eigenspace is denoted by $P_cV = P_1V = V_1$. The center-eigenspace E_c is therefore given by

$$E_c = \{(V_1, 0, 0, \dots) \in \ell^1(\mathbb{R}^2, \sigma) \mid V_1 \in \mathbb{R}^2\}$$

Furthermore, we introduce $V_m^{(s)}(x, x_0) = P_{m,s}V_m(x, x_0)$ and $V_m^{(u)}(x, x_0) = P_{m,u}V_m(x, x_0)$ for $m \geq 3$. Then we consider all such solutions $V(x) = (V_m)_m(x)$ of equation (2.22) which are element of the following space for small but fixed $\eta > 0$

$$Y_\eta = \{V \in C^0(\mathbb{R} \times \mathbb{R}, \ell^1(\mathbb{R}^2; \sigma)) \mid \sup_{x \in \mathbb{R}} \mathrm{e}^{-\eta|x|} \|V(x)\|_{\ell^1(\mathbb{R}^2;\sigma)} < \infty\}$$

where $V = (V_m)_m$ and $\|V\|_{\ell^1(\mathbb{R}^2;\sigma)} = \sum_{m \in \mathbb{N}_{\mathrm{odd}}} m^\sigma |V_m|$. According to [15] $V(x) \in Y_\eta$ is a solution of (2.22) if and only if the following equation holds,

$$\begin{aligned} V_1(x) &= \mathrm{e}^{xJ_1} V_1(0) + \int_0^x \mathrm{e}^{(x-\xi)J_1} \check{F}_{1,\varepsilon}(\xi, 0, (V_j)_j(\xi)))\, \mathrm{d}\xi, \\ V_m^{(s)}(x) &= \int_{-\infty}^x P_{m,s}\, \mathrm{e}^{(x-\xi)J_m} \check{F}_{m,\varepsilon}(\xi, 0, (V_j)_j(\xi))\, \mathrm{d}\xi, \\ V_m^{(u)}(x) &= -\int_x^\infty P_{m,u}\, \mathrm{e}^{(x-\xi)J_m} \check{F}_{m,\varepsilon}(\xi, 0, (V_j)_j(\xi))\, \mathrm{d}\xi. \end{aligned} \tag{2.23}$$

We then describe this equation abstractly as

$$V(x) = SV_c + KG(V)$$

where

$$\begin{aligned} V_c &= P_1 V(0) \in E_c, \\ SV_c &= \mathrm{e}^{xJ} V_c, \\ G(V)(x) &= \check{F}_\varepsilon(x, 0, V(x)) \end{aligned}$$

and

$$(KV)(x) = \int_0^x \mathrm{e}^{(x-\xi)J} P_1 V(\xi)\, \mathrm{d}\xi + \int_{-\infty}^x \mathrm{e}^{(x-\xi)J} P_s V(\xi)\, \mathrm{d}\xi - \int_x^\infty \mathrm{e}^{(x-\xi)J} P_u V(\xi)\, \mathrm{d}\xi.$$

Due to the asymptotics of the discriminant we have the estimates for an arbitrary but fixed $\beta > \eta$ and a $C > 0$ independent of $m \in \mathbb{N}_{\mathrm{odd}}$

$$\begin{aligned} \|\mathrm{e}^{J_1 x}\| &\leq C\, \mathrm{e}^{\eta|x|/2} \quad , \qquad \forall x \in \mathbb{R}, \\ \sup_m \|\mathrm{e}^{J_m x} P_{m,s}\| &\leq C\, \mathrm{e}^{-\beta x} \quad , \qquad \forall x \geq 0, \\ \sup_m \|\mathrm{e}^{J_m x} P_{m,u}\| &\leq C\, \mathrm{e}^{\beta x} \quad , \qquad \forall x \leq 0. \end{aligned}$$

Hence (see [15]) the map $\mathbb{I} - K \circ G$ has an inverse $\Psi : Y_\eta \to Y_\eta$. The solution can now be described as

$$V(x) = \Psi_\varepsilon(SV_c)(x)$$

with $P_c V(0) = V_c$. Because of the spectral gap, the cut-off function and all estimates are $\mathcal{O}(1)$ for $\varepsilon \to 0$, the size of the center-manifold will also be $\mathcal{O}(1)$ for $\varepsilon \to 0$. We define the graph of the center-manifold by a mapping h from the central subspace to the hyperbolic subspace by

$$h(0, V_1, \varepsilon) = P_h \Psi_\varepsilon(SI_1V_1)(0)$$

where $P_h = P_u + P_s$ is the projection onto the hyperbolic subspace and $I_1 : \mathbb{R}^2 \to E_c$ is the inclusion mapping. Then we have

Theorem 2.2. *For all $n \in \mathbb{N}$ there exist $\varepsilon_0 > 0$ and $\delta_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the spatial dynamics formulation in rotational coordinates* (2.22) *possesses a two-dimensional invariant manifold*

$$W_{c,\varepsilon}(0) = \{V^* \in \ell^1(\mathbb{R}^2; \sigma) \mid (0, V_3^*, V_5^*, \dots) = h(0, V_1^*, \varepsilon)\}$$

tangential to the center space

$$E_c = \{(V_1^*, 0, 0, \dots) \mid V_1^* \in \mathbb{R}^2\}$$

with

$$h(0, \cdot, \cdot) \in C^n\big(\{V_1^* \in \mathbb{R}^2 \mid \|V_1^*\| \le \delta_0\} \times [0, \varepsilon_0], \ell^1(\mathbb{R}^2; \sigma)\big)$$

The center-manifold $W_{c,\varepsilon}(0)$ has been constructed for the starting point $x_0 = 0$. The other center-manifolds are easily constructed by the evolution operator $\mathcal{S}_{x,x_0}$ of equation (2.22) defined by $\mathcal{S}_{x,x_0}V_0 = V(x, x_0, V_0)$ with $x, x_1 \in \mathbb{R}$ and $V_0 \in \ell^1(\mathbb{R}^2, \sigma)$. Then

$$W_{c,\varepsilon}(x_0) = \mathcal{S}_{x_0,0} W_{c,\varepsilon}(0)$$

Therefore we define the reduction function $h(x, \cdot, \varepsilon)$ for $W_{c,\varepsilon}(x)$ by

$$V(x) = \mathcal{S}_{x,0}V(0) = \mathcal{S}_{x,0}\big(V_1(0) \oplus h(0, V_1(0), \varepsilon)\big) = V_1(x) \oplus h(x, V_1(x), \varepsilon)$$

Remark 2.9. *Note that $W_{c,\varepsilon}(x_0)$ is not smooth w.r.t. x_0 due to the jumps in the coefficient function $s = s(x)$ w.r.t. x. More essential for our purposes is however the smoothness of the manifold for fixed x_0 as a function of V_1.*

On the reversibility The reduction mapping carries over the reversibility property. From equation (2.23) we can calculate the identity $\tilde{R}V_- = S\tilde{R}V_c + KG_\varepsilon(\tilde{R}V_-)$ where $V_-(x) = V(-x)$. Hence it is $\tilde{R}V(-x) = \Psi_\varepsilon(S\tilde{R}V_c)(x)$ and we conclude $\tilde{R}_h h(x, V_1, \varepsilon) = h(-x, \tilde{R}_1 V_1, \varepsilon)$ with $\tilde{R}_h = \bigoplus_{m\in\{3,5,7,\ldots\}} \tilde{R}_m$. Since with $x \mapsto V(x)$ being a solution, also $x \mapsto \tilde{R}V(-x) = \tilde{R}_1 V_1(-x) \oplus \tilde{R}_h h(-x, V_1(-x), \varepsilon)$ is a solution on the center-manifold, we can conclude that $\tilde{R}_h h(-x, V_1(-x), \varepsilon) = h(x, \tilde{R}_1 V_1(x), \varepsilon)$ by the following consideration:

$$\begin{aligned}
\tilde{R}V(-x) &= \tilde{R}_1 V_1(-x) \oplus \tilde{R}_h h(-x, V_1(-x), \varepsilon) \\
&= \tilde{R}\mathcal{S}_{-x,0} V(0) \\
&= \mathcal{S}_{-x,0} \tilde{R}V(0) \\
&= \mathcal{S}_{-x,0}\big(\tilde{R}_1 V_1(0) \oplus \tilde{R}_h h(0, V_1(0), \varepsilon)\big) \\
&= \mathcal{S}_{-x,0}\big(\tilde{R}_1 V_1(0) \oplus h((0)_-, \tilde{R}_1 V_1(0), \varepsilon)\big) \\
&= \tilde{R}_1 V_1(-x) \oplus h(x, \tilde{R}_1 V_1(-x), \varepsilon).
\end{aligned}$$

From this we find that

$$\begin{aligned}
\tilde{R}_1 F_{1,\varepsilon}\left(x, 0, V_1(x) \oplus h(x, V_1(x), \varepsilon)\right) &= -F_{1,\varepsilon}\left(-x, \tilde{R}_1 V_1(x) \oplus \tilde{R}_h h(x, V_1(x), \varepsilon)\right) \\
&= -F_{1,\varepsilon}\left(-x, \tilde{R}_1 V_1(x) \oplus h(-x, \tilde{R}_1 V_1(x), \varepsilon)\right)
\end{aligned}$$

As a consequence all small bounded solutions can be found on the center-manifold and the reduced system on the center-manifold is given by

$$\partial_x V_1(x) = J_1 V_1(x) + F_{1,\varepsilon}\big(x, 0, V_1(x) \oplus h(x, V_1(x), \varepsilon)\big). \tag{2.24}$$

Since the center-manifold reduction preserves reversibility the reduced system (2.24) is still reversible w.r.t. the transformed reversibility operator $\tilde{R}_1$.

2.7 Properties of the reduced system

From the multiple-scale analysis (see Remark 2.4) we derive a formal approximation of the solution, with a envelope-modulated carrier-wave

$$u(x,t) = \varepsilon A(\varepsilon(x-c_g t), \varepsilon^2 t) f_n(x,k)\, \mathrm{e}^{\mathrm{i}\omega_n(k)t} + c.c..$$

The envelope has an amplitude in $\mathcal{O}(\varepsilon)$ scaling. Motivated by this scaling we introduce

$$\begin{aligned} \tilde{u}_1(x) &= \varepsilon A(x) \\ \tilde{v}_1(x) &= \varepsilon^2 B(x). \end{aligned} \tag{2.25}$$

Then we have the following result.

Theorem 2.3. *Equation* (2.24) *together with the scaling* (2.25) *will read*

$$\begin{aligned} \frac{d}{dX} A &= \varepsilon B + \mathcal{O}(\varepsilon^2) \\ \frac{d}{dX} B &= \varepsilon s_1(x) A - \varepsilon s_3(x) A^3 + \mathcal{O}(\varepsilon^2) \end{aligned} \tag{2.26}$$

where $s_j(x+2) = s_j(x)$ *for* $j = 1,3$ *and* $s_j(x) \geq 0$ $j = 1,3$ *for all* $x \in \mathbb{R}$. *Furthermore* $s_j(x) > 0$, $j = 1,3$ *on a set with positive measure.*

Proof. By definition we have

$$U_1 = \tilde{Q}_1 V_1 = \tilde{Q}_1 \begin{pmatrix} \varepsilon A \\ \varepsilon^2 B \end{pmatrix} = \varepsilon \begin{pmatrix} q_{11} A + \varepsilon q_{12} B \\ q_{21} A + \varepsilon q_{22} B \end{pmatrix}$$

The center-manifold is tangential to the center-eigenspace, therefore

$$U_j(x) = \tilde{Q}_j(x) h_j(x, V_1(x), \varepsilon) = \mathcal{O}(\|V_1(x)\|^3) = \mathcal{O}(\varepsilon^3)$$

therefore for all $j = 3,5,7,\ldots$ the influence of U_j in the nonlinearity N_1 is small and we only need to look at U_1. Hence we have

$$N_1 \left(\tilde{Q}_1(x) V_1(x) \oplus (\tilde{Q}_j(x) h_j(x, V_1(x), \varepsilon))_j \right) = \begin{pmatrix} 0 \\ \varepsilon^3 r(x) (q_{11})^3 A^3 \end{pmatrix} + \mathcal{O}(\varepsilon^4)$$

which implies

$$\tilde{Q}_1^{-1} N_1(\tilde{Q}_1 V_1 \oplus (\tilde{Q}_j h_j)_j) = \begin{pmatrix} 0 \\ \varepsilon^3 r(x) \frac{(q_{11})^4}{\det \tilde{Q}_1} A^3 \end{pmatrix} + \text{h.o.t.}$$

Since the Wronskian equals 1 and is constant for all $x \in \mathbb{R}$ the determinant $\det \tilde{Q}_1 = \det S_1^{-1} = 1/\det S_1$. The last equation simplifies to

$$\tilde{Q}_1^{-1} N_1(\tilde{Q}_1 V_1 \oplus (\tilde{Q}_j h_j)_j) = \begin{pmatrix} 0 \\ \varepsilon^3 r(x) \det(S_1)(q_{11})^4 A^3 \end{pmatrix} + \text{h.o.t.}$$

This concludes the nonlinear part of the equation. The linear part is

$$\tilde{J}_{1,\varepsilon} V_1 = J_1 V_1 + \tilde{Q}_1^{-1} S_\varepsilon(x) \tilde{Q}_1 V_1 = \begin{pmatrix} \varepsilon^2 B \\ \varepsilon^3 q_1 \det(S_1) s(x) (q_{11})^2 A \end{pmatrix} + \text{h.o.t.}$$

The rescaled equation is then given by

$$\begin{pmatrix} \dot{A} \\ \dot{B} \end{pmatrix} = \varepsilon \begin{pmatrix} B \\ s(x) q_1 \det(S_1) q_{11}^2(x) A + r(x) \det(S_1) q_{11}^4(x) A^3 \end{pmatrix} + \mathcal{O}(\varepsilon^3)$$

We set

$$\begin{aligned} r(x) &= r_0 = -1/\det(S_1), \\ s_1(x) &= s(x)(q_{11}(x))^2, \\ q_1 &= 1/\det(S_1), \\ s_3(x) &= (q_{11}(x))^4. \end{aligned} \tag{2.27}$$

The coefficients s, r, q are from the original Klein-Gordon equation (2.1). Both s_1, s_3 are 2-periodic functions. Since q_{11} is continuous and $q_{11}(x_0, x_0) = 1$, $q_{11} > 0$ on a set with positive measure, which means that s_1 and s_3 are also positive on a set of positve measure. □

2.8 Averaging Argument

We use *averaging* as discussed in [11] to analyze the dynamics of equation (2.26) which we will rewrite abstractly to

$$\partial_x \mathbf{a} = \varepsilon \mathbf{F}(x, \mathbf{a}) + \varepsilon^2 \mathbf{G}(x, \mathbf{a}). \tag{2.28}$$

with $\mathbf{a} = (A, B)$ and

$$\mathbf{F}(x, \mathbf{a}) = \varepsilon \begin{pmatrix} B \\ s_1(x)A - s_3(x)A^3 \end{pmatrix}.$$

Then equation (2.28) can be transformed according to [11] to the following equation

$$\partial_x \mathbf{A} = \varepsilon \bar{\mathbf{F}}(\mathbf{A}) + \varepsilon^2 \mathbf{H}(x, \varepsilon, \mathbf{A}), \tag{2.29}$$

with the averaged part of the equation given by

$$\partial_x \mathbf{A}_{\mathrm{avg}} = \varepsilon \bar{\mathbf{F}}(\mathbf{A}_{\mathrm{avg}}) \tag{2.30}$$

with

$$\bar{\mathbf{F}}(\cdot) = \frac{1}{2} \int_0^2 \mathbf{F}(x, \cdot) \mathrm{d}x = \begin{pmatrix} \tilde{B} \\ \bar{s}_1 \tilde{A} - \bar{s}_3 \tilde{A}^3 \end{pmatrix}, \tag{2.31}$$

where $\bar{s}_j = \frac{1}{2} \int_0^2 s_j(x) \mathrm{d}x$, $j = 1, 3$. By the fact that $s_j(x) > 0$ on a set of positive measure we conclude that $\bar{s}_j > 0$, $j = 1, 3$. From the choice of coefficients we note that we have set $r(x) = r_0 = 1/\det(S_1)$ for all x.

The solution to equation (2.30) is computed by rescaling $\mathbf{A}(x) = \tilde{\mathbf{A}}(\varepsilon x) = \tilde{\mathbf{A}}(X)$, so we have

$$\partial_X \tilde{\mathbf{A}}_{\mathrm{avg}} = \bar{\mathbf{F}}(\tilde{\mathbf{A}}_{\mathrm{avg}})$$

with

$$\begin{aligned} \partial_X \tilde{A} &= \tilde{B}, \\ \partial_X \tilde{B} &= \bar{s}_1 \tilde{A} - \bar{s}_3 \tilde{A}^3, \end{aligned}$$

This system has a *homoclinic orbit* which is explicitly given by

$$\tilde{\mathbf{A}}_{\mathrm{avg/hom}}(X, X_0) = \begin{pmatrix} \tilde{A}_{hom}(X, X_0) \\ \tilde{B}_{hom}(X, X_0) \end{pmatrix}$$

with

$$\tilde{A}_{hom}(X, X_0) = \pm \sqrt{\frac{2\bar{s}_1}{\bar{s}_3}} \operatorname{sech}\left(\sqrt{s_1}(X - X_0)\right), \quad \tilde{B}_{hom}(X, X_0) = \partial_X \tilde{A}_{hom}(X, X_0)$$

Undoing the rescaling we have the solution of equation (2.30)

$$\mathbf{A}_{\mathrm{avg/hom}}(x, x_0) = \begin{pmatrix} \pm\sqrt{\frac{2\bar{s}_1}{\bar{s}_3}} \operatorname{sech}\left(\varepsilon\sqrt{s_1}(x - x_0)\right) \\ \mp\sqrt{\frac{2(\bar{s}_1)^2}{\bar{s}_3}} \tanh\left(\varepsilon\sqrt{s_1}(x - x_0)\right) \operatorname{sech}\left(\varepsilon\sqrt{s_1}(x - x_0)\right) \end{pmatrix}$$

2.9 Persistence Proof

In this section we prove the existence of the homoclinic solution in equation (2.29) based on the existence of a homoclinic solution of equation of (2.30). Since both equations only differ in higher order terms ($\mathcal{O}(\varepsilon^2)$), we speak of the persistence of the homoclinic solution.

The homoclinic orbit $\mathbf{A}_{\text{avg/hom}}$ lies in the intersection of the stable manifold and the unstable manifold of system (2.30). In general, if higher order terms are added, the intersection will break up and the perturbed stable manifold and the unstable manifold will no longer intersect. In reversible systems the situation is different. The persistence of the homoclinic solution is established basically in two steps. First, by proving a transversal intersection of the stable manifold with the fixed space of reversibility for the unperturbed system. Second, by arguing that this transversal intersection will remain even in the perturbed system. This results in the homoclinic orbit, for $x \in [0, \infty)$. Applying the reversibility operator R to this part of the solution, also results in the homoclinic orbit, for $x \in (-\infty, 0]$.

The actual persistence proof consists of three steps:

i) Beyond other things in [11, Theorem 4.1.1] the following is shown

   Lemma 2.4. *There exists a C^r-change of coordinates $\mathbf{A} = \mathbf{a} + \varepsilon w(\mathbf{a}, x, \varepsilon)$ under which* (2.28) *becomes* (2.29)

$$\partial_x \mathbf{A} = \varepsilon \bar{\mathbf{F}}(\mathbf{A}) + \varepsilon^2 \mathbf{H}(x, \varepsilon, \mathbf{A})$$

   where $\mathbf{H}$ is of period 2 w.r.t. x.

   Hence in an $\mathcal{O}(1)$-neighborhood the stable manifold $\overline{W}_s$ of the averaged system (2.30) and the stable manifold W_s of the full system (2.28) resp. (2.29) are $\mathcal{O}(\varepsilon)$-close together.

ii) In addition to the statement in Lemma 2.4, in [11, Theorem 4.1.1] it is shown

Lemma 2.5. *If* $\mathbf{A}_{avg}(x)$ *and* $\mathbf{A}(x)$ *are solutions of* (2.29) *and* (2.30) *with* $|\mathbf{A}_{avg}(0)-\mathbf{A}(0)| = \mathcal{O}(\varepsilon)$*, then* $|\mathbf{A}_{avg}(x) - \mathbf{A}(x)| = \mathcal{O}(\varepsilon)$ *on a scale* $\mathcal{O}(1/\varepsilon)$*.*

By applying the approximation result from Lemma 2.5 shows that the stable manifold $\overline{W}_s$ of the averaged system (2.30) and the stable manifold W_s of the full system (2.28) resp. (2.29) are $\mathcal{O}(\varepsilon)$-close together on a scale $\mathcal{O}(1/\varepsilon)$. Hence, $\mathcal{O}(\varepsilon)$-close to the intersection point of the averaged system (2.30) with the fixed space of reversibility there is an intersection point of the full system (2.28) resp. (2.29). See Figure 2.9. As a consequence we have a solution $\mathbf{a}(x)$ of (2.28) for $x \in [0, \infty)$ which satisfies $\lim_{x\to\infty} \mathbf{a}(x) = 0$ and $\mathbf{a}(0) \in \{B = 0\}$.

iii) Finally, we use the reversibility of the reduced system (2.24) resp. (2.28). It allows us to extend $V_1(x)$ for $x \in [0, \infty)$ by $V_1(-x) = RV_1(x)$ to $x \in \mathbb{R}$. In response, we constructed a homoclinic solution to the origin for (2.24) and as a consequence of the exact center-manifold reduction finally one for the original system (2.6).

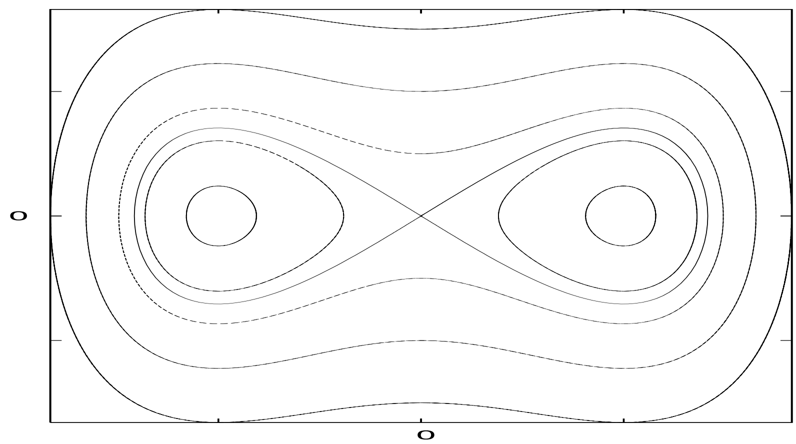

FIGURE 2.5: The combination of local estimate for the difference from (i) with the approximation result from ii). The dotted/full line is the stable manifold of the averaged system (2.30)/full system (2.28).

2.10 Lowest Order Approximation

In this section we want to summarize the steps of the last section in order to give an approximation result which independently affirms the NLS-approximation of Remark 2.4.

The solution of (2.30) and (2.29) are related by

$$\mathbf{A}(x) = \mathbf{A}_{\mathrm{avg/hom}} + \varepsilon \mathbf{R}(x)$$

with a remainder smaller than the homoclinic solution. By Lemma 2.4 the change of coordinates $\mathbf{A} = \mathbf{a} + \varepsilon w(\mathbf{a}, x, \varepsilon)$ has an inverse which we shall call

$$\mathbf{a} = \mathbf{A} + \varepsilon W(\mathbf{A}, x, \varepsilon),$$

so we get

$$\mathbf{a}(x) = \mathbf{A}_{\mathrm{avg/hom}}(x) + \mathcal{O}(\varepsilon).$$

Undoing the rescaling of (2.25) we have

$$\begin{aligned} V_1(x) &= \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \mathbf{a}(x) \\ &= \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \mathbf{A}_{\mathrm{avg/hom}}(x) + \begin{pmatrix} \mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon^3) \end{pmatrix}. \end{aligned}$$

We move from rotational coordinates back into the original Fourier-space

$$\begin{aligned} U_1(x) =& \tilde{Q}_1(x) V_1(x) = \tilde{Q}_1(x) \mathbf{A}_{\mathrm{avg/hom}}(x) + \begin{pmatrix} \mathcal{O}(\varepsilon^2) \\ \mathcal{O}(\varepsilon^3) \end{pmatrix} \\ =& \begin{pmatrix} \varepsilon q_{11}(x) A_{\mathrm{avg/hom}}(x) + \mathcal{O}(\varepsilon^2) \\ \varepsilon q_{21}(x) A_{\mathrm{avg/hom}}(x) + \varepsilon^2 q_{22}(x) B_{\mathrm{avg/hom}} + \mathcal{O}(\varepsilon^3) \end{pmatrix}. \end{aligned}$$

Since we only need u_1 for an approximation, we get from the last statement

$$u_1(x) = \pm \varepsilon q_{11}(x) \sqrt{\frac{2\bar{s}_1}{\bar{s}_3}} \operatorname{sech}(\varepsilon \sqrt{\bar{s}_1} x) + \mathcal{O}(\varepsilon^2)$$

which corresponds to the original solution in the following way

$$u(x,t) = \pm 2\varepsilon q_{11}(x)\sqrt{\frac{2\bar{s}_1}{\bar{s}_3}}\operatorname{sech}(\varepsilon\sqrt{\bar{s}_1}x)\sin(\omega_0 t) + \mathcal{O}(\varepsilon^2). \tag{2.32}$$

From the last section the coefficients $\bar{s}_1, \bar{s}_3$ are all given by

$$\bar{s}_1 = \frac{1}{2}\int_0^2 s(x)q_{11}^2(x)\mathrm{d}x$$

and

$$\bar{s}_3 = \frac{1}{2}\int_0^2 q_{11}^4(x)\mathrm{d}x.$$

Remark 2.10. *In Remark 2.1 we have stated that in the lowest order the breather is given by*

$$u(x,t) = 2\varepsilon c_1 \operatorname{sech}(\varepsilon c_2 x) q_{11}(x)\sin(\omega_0 t) + \mathcal{O}(\varepsilon^2) \tag{2.33}$$

with constants

$$\begin{aligned} c_1 &= \sqrt{\frac{2\bar{s}_1}{\bar{s}_3}} \\ c_2 &= \sqrt{\bar{s}_1} \end{aligned} \tag{2.34}$$

From the approximation result (2.32) and the coefficients $\bar{s}_1$ and $\bar{s}_3$ we only need the knowledge of $q_{11}(x)$. It is a 2-periodic function given by the periodic part of the linear solution of

$$y'' = -(\omega_0^2 - q_0)s(x)y. \tag{2.35}$$

Its fundamental solution is given by $\Phi_1(x;0) = \Phi_1(x+2;2)$, with $\Phi(0;0) = \mathbb{I}_2$ which can be decomposed according to Floquet's Theorem

$$\Phi_1(x;0) = \tilde{Q}_1(x;0)\,\mathrm{e}^{xJ_1}$$

and

$$Q_1(x;0) = \begin{pmatrix} q_{11}(x) & q_{12}(x) \\ q_{21}(x) & q_{22}(x) \end{pmatrix}.$$

This concludes the chapter. We have shown that there exists a breather solution and that its approximation matches the expected approximation from the multiple scale analysis of Remark 2.4.

QED.

3 Breather Solutions in Time-dependent Wave Equations

In the last chapter we proved the existence of breather solutions in the spatially periodic nonlinear Klein-Gordon equation. The question arises if the linear spectrum can be tailored by other means, for example by also using time-dependent coefficients.
In this chapter we first show that a time-dependent but spatially homogeneous coefficient is enough to repeat the steps of Chapter 2 and to prove the existence of a breather solution. Then we generalize the idea to a slightly more physical nonlinear wave equation with time- and space-dependent coefficients.

3.1 The Spatially Homogeneous Case

We show that for a certain choice of time-dependent coefficients in the nonlinear Klein-Gordon equation all eigenvalues except of two are bounded away from the imaginary axis. The two eigenvalues close to the imaginary axis can be moved with a small change in the temporal frequency ω. A center-manifold reduction is then possible and admits a homoclinic solution in the reduced system. In summary

Theorem 3.1. *Consider the equation*

$$\partial_x^2 u = \partial_t^2 \big(s \star u\big) + u - u^3 \tag{3.1}$$

with

$$(s \star u)(x,t) := \int_{-\infty}^{\infty} s(t-\tau) u(x,\tau)\,\mathrm{d}\tau$$

and periodic boundary conditions *in time:*

$$u(x,t) = u(x,t+T), \quad x,t \in \mathbb{R},$$

where $T = \frac{2\pi}{\omega}$*. Let* $s(t)$ *be the coefficient of equation* (3.1) *whose Fourier-transform* $\hat{s}$ *is given by Figure 3.1. Then for* $\omega = 1 - \varepsilon^2$ *(*$\varepsilon \ll 1$*) there exists a breather solution, i.e. there exists constants* $C, \beta > 0$ *such that*

$$|u(x,t)| < C\,\mathrm{e}^{-\beta|x|}$$

holds.

Remark 3.1. *We will discuss an extension to this theorem in the next section: the coefficient* s *may also be periodic in space, i.e.* $s(x,t) = s(x+1,t)$ *for all* $x,t \in \mathbb{R}$.

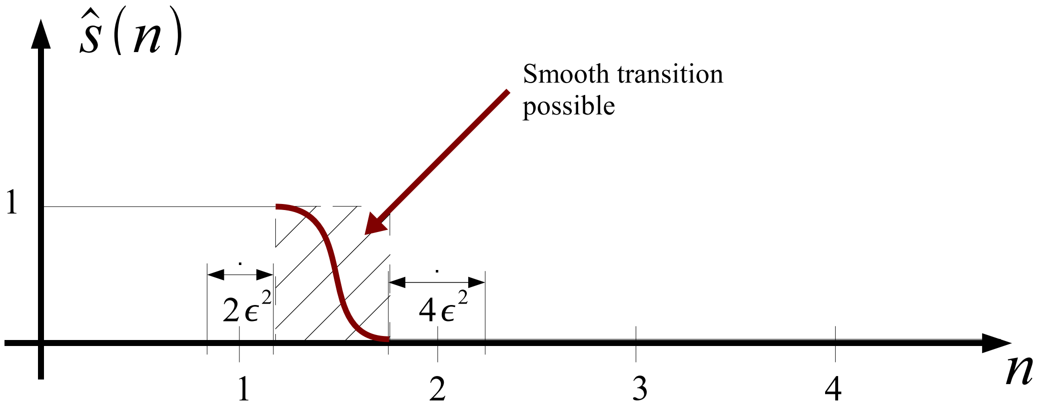

FIGURE 3.1: The (smooth) function $\hat{s}$.

Remark 3.2. *The coefficient* $s(t)$ *will be chosen in such a way that the general Cauchy-problem will become ill-posed. In this instance it means that only the initial value problem with respect to the space variable* x *is well posed whereas the initial value problem w.r.t.* t *is not defined.*

In fact, in physics or electrical engineering it is customary to interchange the role of space and time: the initial condition will be the "insertion" of a signal at a certain input point $x = 0$*,*

$$input(t) := u(x,t)|_{x=0}$$

and the output at point $x = L$ will be also measured w.r.t. time

$$output(t) := u(x,t)|_{x=L}\,.$$

Remark 3.3. *In causal systems we assume that the influence of $s(t)$ on $u(\cdot,t)$ can only come from the present and the past, but never from the future. As a result $s(t) = 0$ for all $t < 0$. Mathematically the principle of causality induces the „Kramers-Kronig"-relation, which states that the Fourier-transform of $s(t)$ must have nonvanishing imaginary parts (see [24] and [25]). We ignore the restrictions of causality in our analysis.*

3.1.1 Proof of Theorem 3.1

To prove Theorem 3.1 we use the periodic boundary conditions in time to display the solution as a Fourier-series

$$u(x,t) = \sum_{n\in\mathbb{Z}} u_n(x)\,\mathrm{e}^{\mathrm{i}n\omega t}.$$

Then the equation becomes the system of equations

$$\partial_x^2 u_n = -n^2\omega^2\hat{s}(n\omega)u_n + u_n - g_n, \quad n \in \mathbb{Z}.$$

with

$$g_n(x) := \sum_{j\in\mathbb{Z}^3, |j|=n} u_{j_1}(x)u_{j_2}(x)u_{j_3}(x). \qquad n \in \mathbb{Z}.$$

This equation has a dispersive term which gives us a gap near the origin – we will use this one and only gap to create eigenvalues far away from the imaginary axis. The *dispersion relation* can be written down explicitly by using $u_n(x) = C_n\,\mathrm{e}^{\pm k_n(\omega)x}$:

$$k_n^2(\omega) = 1 - n^2\omega^2\hat{s}(n\omega).$$

If $k_n^2(\omega)$ is negative $u_n(x)$ will be bounded, otherwise it will have exponential character. In order to prove the theorem we need a coefficient $\hat{s}$ such that the critical modes $n = \pm 1$ have very small exponential growth and all other modes have distinctively larger exponential character. Such a coefficient can be found. It will be defined the following way: for $|r| < 1 + \varepsilon^2$ we set $\hat{s}(r) = 1$ and for $|r| > 2 - 2\varepsilon^2$ we set $\hat{s}(r) = 0$, see Figure 3.1. The bifurcating parameter is ε which changes the time frequency $\omega = 1 - \varepsilon^2$.

Lemma 3.1. *For all $\varepsilon \ll 1$, the coefficient $\hat{s}$ chosen as above has the properties:*

- $k_{\pm 1}^2(1-\varepsilon^2) = \mathcal{O}_+(\varepsilon^2)$ *(where $\mathcal{O}_+ \geq 0$) and*
- $k_{\pm n}^2(1-\varepsilon^2) = 1$ *for all $n = 2, 3, \ldots$.*
- *The convolution is defined and bounded: $\hat{s} \in C^k(\mathbb{R})$, valid for all $k \in \mathbb{N}$, carries over to decay of s faster than any polynomial, thus the integral $(s \star u)(x,t) := \int_{-\infty}^{\infty} s(t-\tau)u(x,\tau)\,\mathrm{d}\tau$ exists.*

We then use spatial dynamics:

$$\begin{aligned} \partial u_n &= v_n \\ \partial v_n &= k_n^2(\omega)u_n - g_n \end{aligned}$$

The same invariances on the phase-space apply as in the previous chapter (see section 2.3) and we can restrict our analysis to $\hat{X}$.

$$U_{\mathbb{R}} \cap U_{\text{odd}} \cap U_{\mathcal{O}} = \{(u_m)_{m\in\mathbb{Z}} \ : \ \operatorname{Re} u_m = 0, u_{2m} = 0\} =: \hat{X}$$

Therefore we now have for $n \in \mathbb{N}_{\text{odd}}$

$$\begin{aligned} \partial u_n &= v_n \\ \partial v_n &= \big(1 - n^2\omega^2\hat{s}(n\omega)\big)u_n - g_n \end{aligned}$$

For $\omega = 1 - \varepsilon^2$ we have for $n = 1$

$$\begin{aligned} \partial u_1 &= v_1 \\ \partial v_1 &= \big(2\varepsilon^2 - \varepsilon^4\big)u_1 - g_1 \end{aligned}$$

and for $n > 1$

$$\begin{aligned} \partial u_n &= v_n \\ \partial v_n &= u_n - g_n \end{aligned}$$

Thus the center-manifold theory can be applied (see [15]). From there we eventually derive the ODE

$$\begin{aligned} \dot{A} &= B \\ \dot{B} &= 2A - CA^3 + \mathcal{O}(\varepsilon) \end{aligned}$$

for $u_1(x) = \varepsilon A(\varepsilon x)$ and $v_1(x) = \varepsilon^2 B(\varepsilon x)$ and a constant $C > 0$. In the unperturbed state, i.e. $\varepsilon = 0$, this ODE has two homoclinic solutions

$$u_{1,\text{hom}}(x) = \pm\sqrt{\frac{4}{C}}\operatorname{sech}\left(\sqrt{2}\varepsilon x\right)$$

The proof now follows analogously to the case of Chapter 2. By reversibility this homoclinic solution preserves when the higher order terms are added. Therefore there is a breather solution in the original equation (3.1).

3.2 The Spatially Periodic Case

Theorem 3.1 can be extended to spatially periodic coefficients $s(x,t) = s(x+1,t)$. We consider a modified nonlinear Klein-Gordon equation:

$$\partial_x^2 u(x,t) = \partial_t^2 (s \star u)(x,t) + u^3(x,t). \tag{3.2}$$

with $(s \star u)(x,t) = \int_{-\infty}^{\infty} s(x, t-\tau)u(x,\tau)\,\mathrm{d}\tau$.

Remark 3.4. *The difference to the nonlinear Klein-Gordon equation* (3.1) *is the lack of a „dispersive" term $+u$. The nonlinear wave equation which one derives from Maxwell's Equations for 1D periodic, nonlinear and linearly polarized materials is very close to equation* (3.2) *(See the appendix for the derivation).*

In order to find out the spectrum of the linearized equation of equation (3.2) we switch into frequency-domain by applying a Fourier-transform to (3.2):

$$\partial_x^2 u_n(x) = -\hat{s}(x, n\omega)n^2\omega^2 u_n(x) - g_n(x), \qquad n \in \mathbb{Z}. \tag{3.3}$$

We choose a separable time-space coefficient

$$\hat{s}(x, n\omega) = \hat{f}(n\omega)\hat{s}_1(x)$$

such that we consider the equation

$$\partial_x^2 u_n(x) = -n^2\omega^2 \hat{f}(n\omega)\hat{s}_1(x)u_n(x) - g_n(x), \qquad n \in \mathbb{Z}. \tag{3.4}$$

Linearizing around the zero-solutions gives us

$$\partial_x^2 u_n(x) = -n^2\omega^2 \hat{f}(n\omega)\hat{s}_1(x)u_n(x), \qquad n \in \mathbb{Z}, \tag{3.5}$$

which is now completely decoupled in terms of the n's. In the situation of the linearized equation (3.5) a „simple" dispersion relation will not show the asymptotic behavior of solutions of the linearized equation w.r.t. x. As a result of the periodicity of the coefficient $\hat{s}_1(x) = \hat{s}_1(x+1)$ we compute the linear (Floquet-) spectrum of (3.5) by the use of Floquet's Theorem. To discuss the equation (3.5) we use the help of the following ODE

$$y'' = -\gamma(\lambda)s(x)y \tag{3.6}$$

with periodic s and arbitrary γ. The Floquet-multipliers are easily computed by

$$\rho_{\pm}(\lambda) = \frac{1}{2}D(\gamma(\lambda)) \pm \frac{1}{2}\sqrt{D(\gamma(\lambda)) - 4}$$

where D is the discriminant of

$$y'' = -s(x)\lambda y.$$

Therefore the disciminant of (3.6) is simply given by

$$\tilde{D} = D \circ \gamma.$$

In terms of equation (3.5) we have $\gamma(\lambda) = \lambda^2 \hat{f}(\lambda)$, where we substituted $\lambda = n\omega$ for convenience. Therefore

$$\hat{f}(\lambda) = \frac{\gamma(\lambda)}{\lambda^2}.$$

In order to control and to simplify the computation of the Floquet-multipliers we want $\hat{f}$ to be symmetric. Additionally $\hat{f}$ should not have any singularities. With this in mind we decide to set

$$\gamma(\lambda) = \begin{cases} \lambda^2, & \lambda^2 < M \\ M, & \lambda^2 \geq M \end{cases}$$

with M to be the first maximum or minimum of D with $|D(M)| > 2$. This condition loosely states that $\hat{s}_1(x)$ must be chosen such that there is at least one band-gap. From the choice of γ the following holds

$$\hat{f}(\lambda) = \begin{cases} 1, & \lambda^2 < M \\ \frac{M}{\lambda^2}, & \lambda^2 \geq c \end{cases} \tag{3.7}$$

see Figure 3.3. Accordingly the discriminant is

$$\tilde{D}(\lambda) = D(\lambda^2 \hat{f}(\lambda)).$$

An example of the resulting discriminant is shown in Figure 3.2.

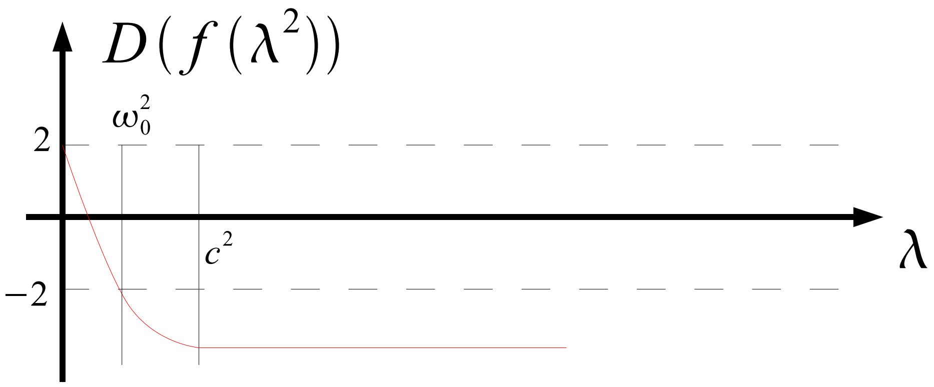

FIGURE 3.2: The discriminant $D(f(\lambda^2))$.

Remark 3.5. *The downside of the choice of $\hat{f}$ is the discontinuity at $\lambda = M$. Then the convolution $(s \star u)(x,t)$ might not exist as a smooth function. However, there are also generalizations to this choice of $\hat{f}$ which are C^∞.*

Let z be chosen to be the last zero of $D(\lambda)^2 - 4$ before the value M is reached. In other words, let $z_1 \leq z_2 \leq z_3 \leq \ldots$ be the zeros of $D(\lambda)^2 - 4$. Then $z = z_i$ with $z_i < M < z_{i+1}$. In terms of the time frequency ω we set the bifurcation point ω_0 to be the square root of this zero:

$$\omega_0^2 = z. \tag{3.8}$$

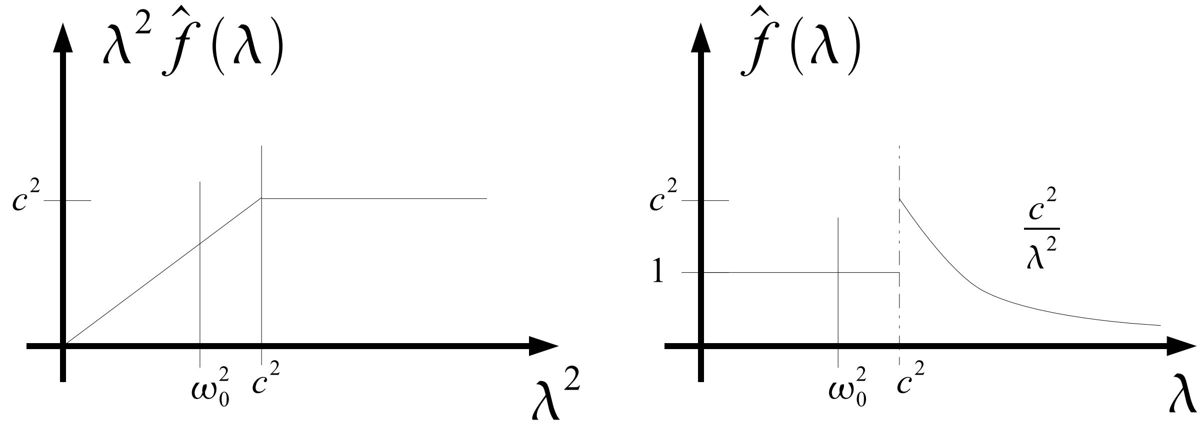

FIGURE 3.3: The functions $\lambda^2 \hat{f}(\lambda)$ and $\hat{f}(\lambda)$.

With $\omega = \omega_0 + \varepsilon^2$ we can calculate the discriminant for $n = \pm 1$

$$D\big((\omega_0+\varepsilon^2)^2\hat{f}(\omega_0+\varepsilon^2)\big)^2 - 4 = D\big((\omega_0+\varepsilon^2)^2\big)^2 - 4 = D\big(z + \mathcal{O}_+(\varepsilon^2)\big)^2 - 4 = \mathcal{O}_+(\varepsilon^4)$$

and for $|n| > 1$

$$D\big(n^2(\omega_0+\varepsilon^2)\hat{f}(n(\omega_0+\varepsilon))\big) = D(M).$$

Therefore we have:

$$\rho_\pm(\omega_0+\varepsilon^2) = \mathcal{O}_+(\varepsilon^2)$$

and for $|n| > 1$

$$\rho_\pm\left(n(\omega_0+\varepsilon^2)\right) = \frac{D(M) \pm \sqrt{D(M)^2 - 4}}{2}.$$

The Floquet-exponents are given by the formula

$$\rho_\pm\left(n(\omega_0+\varepsilon^2)\right) = \mathrm{e}^{k_{\pm n}(\omega_0+\varepsilon^2)}$$

see 2.4 and formula (2.13) for details. In summary, the equation

$$\begin{aligned} &\partial_x u_n(x) = v_n \\ &\partial_x v_n(x) = -\hat{s}_1(x)\hat{f}(n\omega_0+\varepsilon^2)n^2(\omega_0+\varepsilon^2)^2 u_n(x) \qquad n \in \mathbb{Z}. \end{aligned} \tag{3.9}$$

has solutions

$$u_n(x) = p_n(x;\omega_0+\varepsilon^2)\,\mathrm{e}^{k_n(\omega_0+\varepsilon)x}$$

with 1-periodic p_n and Floquet-exponents $k_{\pm 1}(\omega_0 + \varepsilon) = \mathcal{O}_+(\varepsilon^2)$ and $k_n(\omega_0 + \varepsilon) = \mathcal{O}(1)$ ($|n| > 0$) as $\varepsilon \to 0$.

Since the same invariances as discussed in Chapter 2 Section 2.3 apply, we can restrict the analysis to the invariant subspace $\hat{X}$. Then we will make a change of coordinates into rotational coordinates (as discussed in Chapter 2 Section 2.5). The resulting equation will be (abstractly)

$$\dot{U}_n(x) = B_n U_n(x) + \tilde{N}_n\big[(U_m)_{m\in\mathbb{Z}}\big](x), \qquad n \in \mathbb{N}_{\mathrm{odd}}$$

with eigenvalues $k_n(\omega_0 + \varepsilon^2)$. Therefore, we have an eigenvalue at $n = 1$ which is close to the imaginary axis. All other eigenvalues are uniformly bounded away from that eigenvalue. We can now perform the center-manifold reduction as described in Section 2.6. The proof of a homoclinic solution for $n = 1$ is analogous to Section 2.7 and following. Therefore we have proven the following theorem.

Theorem 3.2. *We consider the equation*

$$\partial_x^2 u(x,t) = \partial_t^2 (s \star u)(x,t) + u^3(x,t). \tag{3.10}$$

which corresponds in Fourier-space to

$$\partial_x^2 u_n(x) = -\hat{s}(x, n\omega) n^2 \omega^2 u_n(x) - g_n(x), \qquad n \in \mathbb{Z}. \tag{3.11}$$

We choose

$$\hat{s}(x, n\omega) = \hat{s}_1(x) \hat{f}(n\omega)$$

where $\hat{f}$ is a function defined by Figure 3.3, and $\hat{s}_1(x) = \hat{s}_1(x+1)$ is a periodic function with at least one band-gap.

Then there exist a ω_0 such that for $0 < \varepsilon \ll 1$ there exists a breather solution for $\omega = \omega_0 + \varepsilon^2$.

Remark 3.6. *The result can be generalized to a broader range of equations. The non-linear part of equation (3.2) can be made more complex:*

$$\partial_x^2 u(x,t) = \partial_t^2 \left((s \star u)(x,t) - (r \star u^3)(x,t)\right).$$

in which case we get

$$\partial_x^2 u_n(x) = -\hat{s}(x, n\omega) n^2 \omega^2 u_n(x) - n^2 \omega^2 r(x, n\omega) g_n(x), \qquad n \in \mathbb{Z}.$$

Then the nonlinear part

$$-n^2 \omega^2 \hat{r}(x, n\omega) g_n(x)$$

is bounded in n *if* $r(x, n\omega) = \mathcal{O}(1/n^2)$. *The center-manifold theory can also be applied in this situation.*

4 Fourier-Interface Method

In Chapter 2 we claimed in Remark 2.3 that the choice of the periodic coefficients given by

$$s(x) = \chi_{[0,6/13]} + 16\chi_{[6/13,7/13]} + \chi_{[7/13],1}(x \bmod 1)$$
$$q(x) = (q_0 + q_1\varepsilon^2)s(x)$$
$$r(x) = r_0$$

is not a choice by chance but is based on a method to compute the discriminant of step-functions explicitly. For the ODE

$$u'' = -\lambda^2 s(x)u \tag{4.1}$$

with periodic $s(x+a) = s(x)$ the linear asymptotics are determined by the Floquet-multiplier which can be calculated by the discriminant. We explore a way of calculating the discriminant by transforming this ODE into a wave-equation. With the use of explicit solution formulas and the wave-equation's characteristics we can calculate the transformed discriminant.

We also deal with the inverse problem: given a discriminant what is the coefficient s? As it will turn out only a few examples permit a well-posed inverse problem. For general step-functions it is generally insolvable.

4.1 The Method

We show how to compute the discriminant of the following ordinary differential equation

$$u''(x;\lambda) + \lambda^2 s(x)u(x;\lambda) = 0, \tag{4.2}$$

with initial conditions

$$\begin{aligned} \phi_1(0;\lambda) &= 1, \quad \phi_2(0;\lambda) = 0, \\ \phi_1'(0;\lambda) &= 0, \quad \phi_2'(0;\lambda) = 1. \end{aligned} \tag{4.3}$$

The growth of this ODE's solutions is determined by the Floquet-multiplier

$$\rho_\pm(\lambda^2) = \frac{D(\lambda^2)}{2} \pm \frac{1}{2}\sqrt{D(\lambda^2)^2 - 4}$$

with the discriminant $D(\lambda^2) = \phi_1(a;\lambda) + \phi_2'(a;\lambda)$. Applying Fourier-transform to ODE (4.2) w.r.t. λ yields

$$\partial_x^2 \hat{u} = s(x)\partial_\mu^2 \hat{u} \tag{4.4}$$

where $\hat{u}(x,\mu) = \int_{\mathbb{R}} u(x;\lambda)\, \mathrm{e}^{-\mathrm{i}\mu\lambda} \mathrm{d}\lambda$. The initial conditions translate to

$$\begin{aligned} \hat{\phi}_1(0,\mu) &= \delta(\mu), & \hat{\phi}_2(0,\mu) &= 0, \\ \partial_x\hat{\phi}_1(0,\mu) &= 0, & \partial_x\hat{\phi}_2(0,\mu) &= \delta(\mu), \end{aligned} \tag{4.5}$$

where δ is the *Dirac-delta distribution*. The Fourier-transform of the discriminant is then $\hat{D}(\mu) = \hat{\phi}_1(a,\mu) + \partial_x\hat{\phi}_2(a,\mu)$. We will only consider step-functions. In this case we can solve the equation (4.4) explicitly on each constant part of s. In detail, if the characteristic function is given by

$$\chi_M(x) = \begin{cases} 1, & x \in M \\ 0, & x \notin M \end{cases}$$

then we have for all functions s, which satisfy the following representation

$$s(x) = \sum_{n=1}^{N} s_n^2 \chi_{[a_{n-1},a_n)}(x),$$

the explicit solution of the linear wave equation

$$\partial_x^2 \hat{u} = s_n^2 \partial_\mu^2 \hat{u}, \quad x \in [a_{n-1}, a_n)$$

for each $n = 1, \dots, N$. The solution is given by

$$\begin{aligned} \hat{u}(x,\mu) &= \frac{1}{2}\big(f_{n-1}(\mu + s_n x) + f_{n-1}(\mu - s_n x)\big) + \frac{1}{2s_n}\int_{\mu - s_n x}^{\mu + s_n x} g_{n-1}(\nu)\mathrm{d}\nu \\ \partial_x\hat{u}(x,\mu) &= \frac{1}{2}\big(g_{n-1}(\mu + s_n x) + g_{n-1}(\mu - s_n x)\big) + \frac{s_n}{2}\big(f_{n-1}'(\mu + s_n x) - f_{n-1}'(\mu - s_n x)\big) \end{aligned} \tag{4.6}$$

with the initial conditions

$$\begin{aligned} f_n(\mu) &= \hat{u}(a_n, \mu), \quad n = 0, \ldots, N-1 \\ g_n(\mu) &= \partial_x \hat{u}(a_n, \mu), \quad n = 0, \ldots, N-1. \end{aligned} \tag{4.7}$$

Thus two initial value problems must be solved independently. For $\hat{\phi}_1$ they are

$$\begin{aligned} f_{1,0}(\mu) &= \delta(\mu), \\ g_{1,0}(\mu) &= 0. \end{aligned} \tag{4.8}$$

For $\hat{\phi}_2$ the initial conditions are

$$\begin{aligned} f_{2,0}(\mu) &= 0, \\ g_{2,0}(\mu) &= \delta(\mu). \end{aligned} \tag{4.9}$$

To explain the Fourier-Interface Method we start with the constant case, i.e. the situation with $N = 1$ where $s(x) \equiv s^2 \in \mathbb{R}$ is a constant. In this case the solution formula (4.6) can be directly applied. The following solution has straight lines as characteristics originating from the origin to $\pm sa$. We have

$$\begin{aligned} \hat{\phi}_1(x, \mu) &= \frac{1}{2}\big(\delta(\mu + sx) + \delta(\mu - sx)\big) \\ \partial_x \hat{\phi}_2(x, \mu) &= \frac{1}{2}\big(\delta(\mu + sx) + \delta(\mu - sx)\big). \end{aligned}$$

After the evolution to the point $x = a$ we stop. Now we have the transformed discriminant

$$\hat{D}(\mu) = \hat{\phi}_1(a, \mu) + \partial_x \hat{\phi}_2(a, \mu) = \delta(\mu + sa) + \delta(\mu - sa)$$

and recover it by an inverse Fourier-transform:

$$D(\lambda^2) = 2\cos(as\lambda).$$

There is a geometrical way of looking at this result: starting from a top-down view, there is a Dirac-delta at $x = 0$ at the position $\mu = 0$. Then two characteristics emanate from this starting Dirac with a slope of $\pm s$. At $x = a$ these Dirac-deltas will be moved to $\mu = \pm sa$ with mass each $\frac{1}{2}$. See figure 4.1.

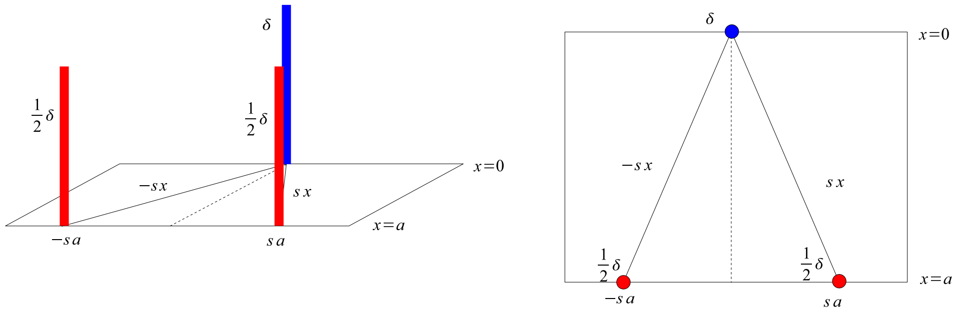

FIGURE 4.1: Evolution of a Dirac-delta distribution by a linear wave equation. LEFT: Isometric view. RIGHT: Top-down view.

Now we consider the case where $N > 1$. The main difference now are the interfaces between two layers. Each characteristic splits up into two characteristics at an interface. The characteristics after the split-up have slopes according to the layer's coefficient s_n where $n = 1, \dots, N$ denotes the layer. The masses of the Dirac-delta distributions split up with a preference to the direction of propagation.

For $\hat{\phi}_1$ the mass for the Dirac-delta which moves in the direction of propagation will have the following fraction of the original mass

$$\frac{1}{2}(1 + \frac{s_n}{s_{n+1}})$$

and the Dirac-delta in the opposite direction of propagation will have the following fraction of the original mass

$$\frac{1}{2}(1 - \frac{s_n}{s_{n+1}}).$$

See figure 4.2 for an illustration of this behavior. The evolution of $\partial_x \hat{\phi}_2$ is very semilar to the evolution of $\hat{\phi}_1$. The fraction of the original mass is

$$\frac{1}{2}(1 + \frac{s_{n+1}}{s_n})$$

for the direction of propagation and

$$\frac{1}{2}(1 - \frac{s_{n+1}}{s_n})$$

in the opposite direction. The basic rule also says that at the splitting the total mass of the splitted Dirac-distributions is conserved:

$$\frac{1}{2}(1 + \frac{s_n}{s_{n+1}}) + \frac{1}{2}(1 - \frac{s_n}{s_{n+1}}) = 1$$

and

$$\frac{1}{2}(1+\frac{s_{n+1}}{s_n})+\frac{1}{2}(1-\frac{s_{n+1}}{s_n})=1.$$

The basis-rule for propagation can be seen if $\delta(\mu+s_n x)|_{x=a_n}$ is used as initial condition in the solution formula (4.6). Please note, that there are no reflections, since the x-derivative is proportional to $+s_n$.

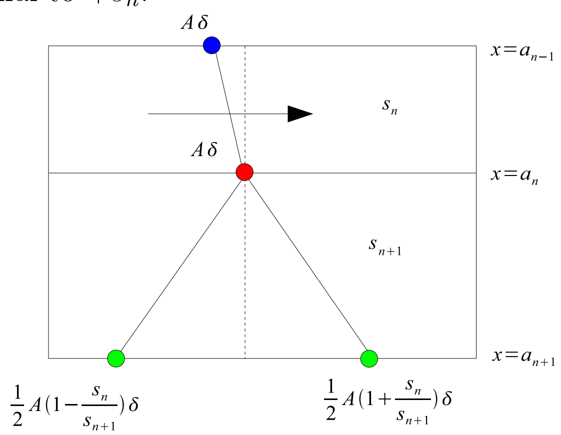

FIGURE 4.2: Basic rule when a "Dirac-ray" hits an interface.

The basic rule can be condensed in a general solution formula:

$$\hat{\phi}_1(a,\mu)=\frac{1}{2^N}\sum_{M\in\{-,+\}^N}\prod_{i=1}^{N-1}\left(1+M_iM_{i+1}\frac{s_i}{s_{i+1}}\right)\delta\left(\mu+\sum_{i=1}^{N}M_is_i(a_i-a_{i-1})\right)$$

and

$$\partial_x\hat{\phi}_2(a,\mu)=\frac{1}{2^N}\sum_{M\in\{-,+\}^N}\prod_{i=1}^{N-1}\left(1+M_iM_{i+1}\frac{s_{i+1}}{s_i}\right)\delta\left(\mu+\sum_{i=1}^{N}M_is_i(a_i-a_{i-1})\right)$$

where $a_0=0$ and $a_N=a$. The discriminant is then given by

$$\hat{D}(\mu)=\sum_{M\in\{-,+\}^N}S_M\;\delta\left(\mu+\sum_{i=1}^{N}M_is_i(a_i-a_{i-1})\right)$$

with the masses

$$S_M=\frac{1}{2^N}\left(\prod_{i=1}^{N-1}\left(1+M_iM_{i+1}\frac{s_i}{s_{i+1}}\right)+\prod_{i=1}^{N-1}\left(1+M_iM_{i+1}\frac{s_{i+1}}{s_i}\right)\right)$$

The proof of this statement is by induction.

Example 4.1. *Consider the two-step function* $s(x) = s_1^2\chi_{[0,a_1)} + s_2^2\chi_{[a_1,a]}$. *Then look at the evolution of the "Dirac-ray" in Figure 4.3. The solution of the linear wave equation is*

$$\begin{aligned}\hat{\phi}_1(a,\mu) =& \frac{1}{4}(1+\frac{s_1}{s_2})\delta(\mu+s_1a_1+s_2(a-a_1))\\ &+\frac{1}{4}(1+\frac{s_1}{s_2})\delta(\mu-s_1a_1-s_2(a-a_1))\\ &+\frac{1}{4}(1-\frac{s_1}{s_2})\delta(\mu+s_1a_1-s_2(a-a_1))\\ &+\frac{1}{4}(1-\frac{s_1}{s_2})\delta(\mu-s_1a_1+s_2(a-a_1)).\end{aligned}$$

and

$$\begin{aligned}\partial_x\hat{\phi}_2(a,\mu) =& \frac{1}{4}(1+\frac{s_2}{s_1})\delta(\mu+s_1a_1+s_2(a-a_1))\\ &+\frac{1}{4}(1+\frac{s_2}{s_1})\delta(\mu-s_1a_1-s_2(a-a_1))\\ &+\frac{1}{4}(1-\frac{s_2}{s_1})\delta(\mu+s_1a_1-s_2(a-a_1))\\ &+\frac{1}{4}(1-\frac{s_2}{s_1})\delta(\mu-s_1a_1+s_2(a-a_1)).\end{aligned}$$

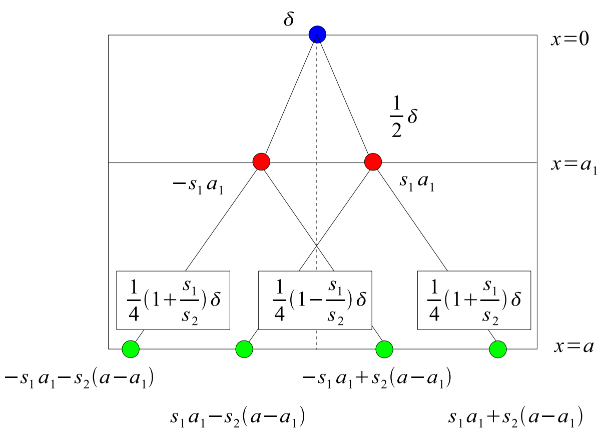

FIGURE 4.3: *The evolution of* $\hat{\phi}_1$ *with a two-step function.*

The corresponding image 4.3 quickly displays the result. The resulting discriminant is then

$$\begin{aligned}D(\lambda^2) =& \frac{1}{4}(2+\frac{s_1}{s_2}+\frac{s_2}{s_1})\cos\big((s_1a_1+s_2(a-a_1))\lambda\big)\\ &+\frac{1}{4}(2-\frac{s_1}{s_2}-\frac{s_2}{s_1})\cos\big((s_1a_1-s_2(a-a_1))\lambda\big).\end{aligned}$$

This concludes the example.

4.2 The Result

In summary of the Fourier-Interface Method it is possible to give a method to compute an explicit formula for the discriminant of the ODE (4.2) when the periodic coefficient is a step function:

$$s(x) = \sum_{n=1}^{N} s_n^2 \chi_{[a_{n-1},a_n)}(x) \tag{4.10}$$

where $N = 1, 2, 3, \dots$ is the number of steps. The individual steps are defined by a_n and the step value is s_n^2. Then we get the following result.

Theorem 4.1. *The discriminant of equation* (4.1) *under the assumption that s is a step function as defined in* (4.10) *is given by:*

$$D(\lambda^2) = 2 \sum_{\substack{M\in\{-,+\}^N \\ \omega_M \geq 0}} S_M \cos(\omega_M \lambda) \tag{4.11}$$

where $\omega_M := \sum_{i=1}^{N} M_i s_i (a_i - a_{i-1})$ for $M \in \{-,+\}^N$ and

$$S_M := \frac{1}{2^N} \prod_{i=1}^{N-1} \left(1 + M_i M_{i+1} \frac{s_i}{s_{i+1}}\right) + \frac{1}{2^N} \prod_{i=1}^{N-1} \left(1 + M_i M_{i+1} \frac{s_{i+1}}{s_i}\right)$$

also for $M \in \{-,+\}^N$. Additionally:

$$\sum_{\substack{M\in\{-,+\}^N \\ \omega_M \geq 0}} S_M = 1.$$

Remark 4.1. *Since the discriminant is defined by*

$$D(\lambda^2) = \phi_1(a;\lambda) + \phi_2'(a;\lambda)$$

where

$$\begin{aligned} \phi_1(0;\lambda) &= 1, \quad \phi_2(0;\lambda) = 0, \\ \phi_1'(0;\lambda) &= 0, \quad \phi_2'(0;\lambda) = 1, \end{aligned} \tag{4.12}$$

the theorem also gives the solution formulas for ϕ_1 and ϕ_2':

$$\phi_1(a;\lambda) = 2 \sum_{\substack{M\in\{-,+\}^N \\ \omega_M \geq 0}} \frac{1}{2^N} \prod_{i=1}^{N-1} \left(1 + M_i M_{i+1} \frac{s_i}{s_{i+1}}\right) \cos(\omega_M \lambda)$$

and

$$\phi_2'(a;\lambda) = 2 \sum_{\substack{M\in\{-,+\}^N \\ \omega_M \geq 0}} \frac{1}{2^N} \prod_{i=1}^{N-1} \left(1 + M_i M_{i+1} \frac{s_{i+1}}{s_i}\right) \cos(\omega_M \lambda)$$

Corollary 4.1. *Every step function with $s_i = s_{N-i+1}$ ($i = 1,\dots,N$, N is odd) has the property $\phi_1(a,\lambda^2) = \phi_2'(a,\lambda^2)$. It is therefore sufficient to look at $D(\lambda) = 2\phi_1(a,\lambda^2)$.*

Proof. Let $N = 2k+1$ for $k \in \mathbb{N}$ and $M \in \{-,+\}^{2k}$. The masses after the evolution through $N-1$ interfaces are

$$\begin{aligned} \frac{1}{2^N}\prod_{i=1}^{N-1}(1 + M_i M_{i+1}\frac{s_i}{s_{i+1}}) &= \frac{1}{2^N}\prod_{i=1}^{k}(1 + M_i M_{i+1}\frac{s_i}{s_{i+1}})\prod_{i=k}^{2k}(1 + M_i M_{i+1}\frac{s_i}{s_{i+1}}) \\ &= \frac{1}{2^N}\prod_{i=1}^{k}(1 + M_i M_{i+1}\frac{s_i}{s_{i+1}})\prod_{i=1}^{k}(1 + M_i M_{i+1}\frac{s_{i+1}}{s_i}) \end{aligned}$$

for ϕ_1 and

$$\frac{1}{2^N}\prod_{i=1}^{N-1}(1 + M_{i+1} M_i\frac{s_{i+1}}{s_i}) = \frac{1}{2^N}\prod_{i=1}^{k}(1 + M_{i+1} M_i\frac{s_{i+1}}{s_i})\prod_{i=1}^{k}(1 + M_{i+1} M_i\frac{s_i}{s_{i+1}})$$

for ϕ_2'. □

Corollary 4.2. *Let $s(x) = \sum_{n=1}^{3} s_n^2 \chi_{(a_{n-1},a_n]}$ be a symmetric 3-step function with $s_1 = s_3$ and $d_1 = d_3$ where $d_n := a_n - a_{n-1}$ for $n = 1,2,3$. Then there are two frequencies ω_0, ω_1. If we assume that the two frequencies relate by $\omega_0 = 2\omega_1$ the following invariance is true:*

$$3d_2 s_2 = 2d_1 s_1$$

Proof. For a 3-step function the discriminant has 4 frequencies $\omega_0,\omega_1,\omega_2,\omega_3$. However, two cancel each other. The first two frequencies are:

$$\omega_0 = 2d_1 s_1 + d_2 s_2$$

$$\omega_1 = 2d_1 s_1 - d_2 s_2$$

and the other two are

$$\omega_2 = d_1 s_1 + d_2 s_2 - d_1 s_1 = d_2 s_2$$

$$\omega_3 = d_1 s_1 - d_2 s_2 - d_1 s_1 = -d_2 s_2$$

with masses

$$S_2 = \frac{1}{8}(1+\frac{s_1}{s_2})(1-\frac{s_2}{s_1}),$$
$$S_3 = \frac{1}{8}(1-\frac{s_1}{s_2})(1+\frac{s_2}{s_1}).$$

Then the frequencies ω_2, ω_3 cancel each other since $S_2 + S_3 = 0$ and $\omega_2 = -\omega_3$. Under the assumption $\omega_0 = 2\omega_1$ the following holds:

$$\omega_0 = 2\omega_1 \Longleftrightarrow 2d_1s_1 + d_2s_2 = 4d_1s_1 - 2d_2s_2$$

hence

$$3d_2s_2 = 2d_1s_1.$$

□

Remark 4.2. *This invariance states that for any given temporal frequency ω_0 there can be found a symmetric 3-step function such that the second frequency has the given relation. This is particularly important for the discriminant and the resulting band structure. Only under these conditions* every *„odd" gap is open!*

With these corollaries we can quickly re-establish our coefficient of Chapter 2. The discriminant will be:

$$D(\lambda^2) = 2S_0 \cos(\omega_0\lambda) + 2S_1 \cos(\omega_1\lambda)$$

We set $s_1 = 1$, $s_2 = 4$ and $a = 1$. Therefore $6d_2 = d_1$. So we have the ratios $6 : 1 : 6$ and we may set $d_1 = \frac{6}{13}$ and $d_2 = \frac{1}{13}$. Then we get $S_0 = \frac{1}{8}(1 + \frac{1}{4})(1 + \frac{4}{1}) = \frac{25}{32}$ and $S_1 = \frac{1}{8}(1 - \frac{1}{4})(1 - \frac{4}{1}) = -\frac{9}{32}$ and the discriminant is

$$D(\lambda^2) = \frac{25}{16}\cos(\frac{16}{13}\lambda) - \frac{9}{16}\cos(\frac{8}{13}\lambda).$$

4.3 On the Inverse Problem

The inverse Problem is not well-posed. We consider an N-step function, which gives us $2N$-dimensions: parameters are $s_1, \ldots, s_N$ and $d_1, \ldots, d_N$. Yet the amount of distinct frequencies for an N-step function is 2^{N-1}. That means for the general inverse problem that step functions is not solvable for $N > 4$. However, for $N = 1, 2, 3, 4$ the inverse problem is solvable and standard techniques can be applied.

5 Appendix: Applications in Mathematical Physics

The original question to this work has been whether it is possible to find breather solutions in other equations than the sine-Gordon equation. This question is interesting in its own right. Additionally the questions are also interesting for applications in mathematical physics. This work was done under the supervision of the Research Training Group 1294 of the German Research Foundation (DFG)

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at the Karlruhe Institute of Technology (KIT). As such we discussed the possible application of this work towards physics. This chapter shows the ideas to bridge the gap between theoretical analysis and physical application.

5.1 A Physical Application: PGB-Materials

The term PGB-Materials stands for Photonic-Band-Gap-Material which is any material which has some sort of optical semi-conductor property. From the early discovery of semiconductors the so-called band-gap is a gap between two energy bands of the dispersion relation of a Schrödinger equation modelling the periodically structured lattice of atoms. A material that has a band-gap for photons is therefore called a PGB-Material. This roughly means that the PGB-Material prohibits the passing of light for some frequencies. During the late 1980ies the property was found to be useful as a way to manipulate

light. Two physicists stand out in the developement of a new field of research, namely Sajeev John (1987) and Eli Yablonovitch (1987). They found that photonic crystals are exceptionally good PGB-Materials. Photonic Crystals are two dielectrics which are ordered periodically in space, with lattice constant to be proportional to the frequency of light to be forbidden to pass the structure.
In this chapter we derive a nonlinear wave equation which describes the interaction of light with the photonic crystal in a 1D setting. From there we will discuss how Theorem 3.2 can be applied.

5.2 Maxwell's Equation and Derivation of the Wave Equation

We want to model a 1D photonic crystal using Maxwell's equations. We assume linear polarization along the z-axis (TEM-polarization) and direction of propagation along the x-axis. Maxwell's equations in SI units are given by

$$\nabla \times H = J + \partial_t D \qquad \text{(Amperè's law)} \tag{5.1}$$

$$\nabla \times E = -\partial_t B \qquad \text{(Faraday's law)} \tag{5.2}$$

$$\nabla \cdot B = 0 \tag{5.3}$$

$$\nabla \cdot D = \rho \qquad \text{(Gauss' law)} \tag{5.4}$$

The vectors $E, D, B, H, J \in \mathbb{R}^3$ depend on space $(x, y, z) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. The *electric field* is denoted by E, its reaction on material – the *electric displacement field* – is called D. The *magnetic field* is denoted by H and its reaction on material B is called the *magnetic induction field*. Furthermore there is the *electric current density* J and the *electric charge density* $\rho \in \mathbb{R}$. We have the important *material equations*

$$J = \sigma E \tag{5.5}$$

$$D = \varepsilon E \tag{5.6}$$

$$B = \mu H \tag{5.7}$$

Here σ is called the *specific conductivity*, ε is known as the *permittivity* (with nonlinear terms also the susceptibility) and μ is called the *magnetic permeability*. In the course of this chapter we make important assumptions to simplify the equation. First, we will not consider conductors, nor magnetic materials, therefore $\sigma = 0$ and $B = \mu_0 H$ with μ_0 being a scalar value. The interaction of light and material will solely be governed by ε. We make an phenomenological approach to the choice of the permittivity:

$$D_i = \varepsilon_0 E_i + \varepsilon_0 \chi^{(1)}_{ij}(E_j) + \varepsilon_0 \chi^{(3)}_{ijkl}(E_j, E_k, E_l) \tag{5.8}$$

where we use Einstein's summation convention. The tensor $\chi^{(1)}$ is linear in E_3 whereas $\chi^{(3)}$ is a multilinear form in each argument. Since we are only interested in 1D photonic crystals we assume that the tensors $\chi^{(1)}, \chi^{(3)}$ will only depend on the direction of propagation e_1 and time t. The equations (5.1)-(5.4) then are reduced to

$$\begin{aligned} \nabla \times H &= \varepsilon_0 \partial_t \left(E + \chi^{(1)}(E) + \chi^{(3)}(E,E,E)\right) && (5.9) \\ \nabla \times E &= -\mu_0 \partial_t H && (5.10) \\ \nabla \cdot B &= 0 && (5.11) \\ \nabla \cdot \left(E + \chi^{(1)}(E) + \chi^{(3)}(E,E,E)\right) &= 0 && (5.12) \end{aligned}$$

Furthermore, we will restrict the electric field to be linearly polarized along the e_3-direction and propagating only in e_1-direction, i.e.

$$E(x,y,z,t) = E_3(x,t)e_3.$$

Taking $\nabla\times$ (5.10) and inserting it into (5.9), taking into account (5.12), gives the nonlinear wave equation of *Maxwell-type*:

$$\begin{aligned} \partial_x \left(\chi^{(1)}_{13}(E_3) + \chi^{(3)}_{1333}(E_3,E_3,E_3)\right) &= 0 \\ \partial_t^2 \left(\chi^{(1)}_{23}(E_3) + \chi^{(3)}_{2333}(E_3,E_3,E_3)\right) &= 0 \\ \mu_0\varepsilon_0\partial_t^2 \left(E_z + \chi^{(1)}_{33}(E_3) + \chi^{(3)}_{3333}(E_3,E_3,E_3)\right) &= \partial_x^2 E_3 \end{aligned}$$

We shall simply set $\chi_{i3}^{(1)} = 0 = \chi_{i333}^{(3)}$ for $i = 1,2$ to finally arrive at the Maxwell-type wave equation

$$\partial_x^2 E_3 = \mu_0 \varepsilon_0 \partial_t^2 \Big(E_3 + \chi_{33}^{(1)}(E_3) + \chi_{3333}^{(3)}(E_3, E_3, E_3) \Big) \tag{5.13}$$

The permeability and susceptibility are chosen in either a time-independent way, such that

$$\chi_{33}^{(1)}(x,t,E_3) := \chi_1(x) E_3(x,t) \in \mathbb{R}, \quad \chi_{3333}^{(3)}(x,t,E_3,E_3,E_3) := \chi_3(x) E_3^3(x,t) \in \mathbb{R}$$

or in a time-dependent part

$$\chi_{33}^{(1)}(x,t,E_3) = \int_{-\infty}^{t} E(x,t-\tau)\chi_1(x,\tau)\mathrm{d}\tau =: \big(\chi_1 \star E_3\big)(x,t)$$

and

$$\begin{aligned} \chi_{3333}^{(3)}(x,t,E_3,E_3,E_3) &= \int_{-\infty}^{t} E_3^3(x,t-\tau)\chi_3(x,\tau)\mathrm{d}\tau \\ &=: \big(\chi_3 \star E_3^3\big)(x,t) \end{aligned}$$

The two equations of interest are now

$$\partial_x^2 E_3 = \mu_0 \varepsilon_0 \partial_t^2 \left(E_3 + \chi_1 E_3 + \chi_3 E_3^3 \right) \tag{5.14}$$

$$\partial_x^2 E_3 = \mu_0 \varepsilon_0 \partial_t^2 \left(E_3 + \chi_1 \star E_3 + \chi_3 \star E_3^3 \right) \tag{5.15}$$

These wave equations explain the nonlinear propagation of light in the 1D direction (linearly polarized) under certain material assumptions. In context of photonic crystals the permittivity and susceptibility are periodic in x with lattice constant a, i.e. $\chi_i(x,t) = \chi_i(x+a,t)$ for $i = 1,3$.

5.3 Additional Remarks

We want to apply Theorem 3.2 to equation (5.15)

$$\partial_x^2 E_3(x,t) = \mu_0 \varepsilon_0 \partial_t^2 \left(E_3(x,t) + \big(\chi_1 \star E_3\big)(x,t) + \big(\chi_3 \star E_3^3\big)(x,t) \right)$$

with periodic boundary conditions in time

$$E_3(x,t) = E_3(x,t+T), \qquad T = \frac{2\pi}{\omega}$$

in order to show how the result of Chapter 2 and 3 can be applied to physical applications. By using Fourier-series as an ansatz to the equation (5.15) we have the set of equations

$$\partial_x^2 E_3(x,n) = \mu_0\varepsilon_0 n^2\omega^2 \left(\big(1+\hat{\chi}_1(x,n\omega)\big)\hat{E}_3(x,n) + \hat{\chi}_3(x,n\omega)\hat{E}_3^3(x,n) \right), \qquad n \in \mathbb{Z}$$

According to Remark 3.6 we set $n^2\omega^2\hat{\chi}_3(x,n\omega)$ to be bounded in $n \in \mathbb{Z}$ so that the nonlinearity is bounded. From a physical point of view $\hat{\chi}_3$ can be assumed to adhere to the drude model: for high frequencies it is $\mathcal{O}(1/n^2)$, see e.g. [26]. Regarding the linear part of the equation we will set

$$1+\hat{\chi}_1(x,n\omega) = \hat{s}(x,n\omega) = \hat{s}_1(x)\hat{f}(n\omega) > 0.$$

with

$$\hat{f}(\lambda) = \begin{cases} 1, & \lambda^2 < M, \\ \frac{M}{\lambda^2}, & \lambda^2 \geq c., \end{cases}$$

which is chosen according to (3.7) to fit Theorem 3.2, and

$$\hat{s}_1(x) = 2\chi_{[0,6/13)}(x) + 16\chi_{[6/13,7/13)}(x) + 2\chi_{[7/13,1]}\ (x \bmod 1).$$

The discriminant of $\hat{s}_1$ is very similar to the one of s of Chapter 2 equation (2.3): every „odd" gap is open. In fact, we used Corollary 4.2 and its subsequent considerations for the derivation of the coefficient $\hat{s}_1$. Moreover, we chose it to be greater or equal than 2 for physical reasons, as outlined below in Remark 5.1. Then we are able to apply Theorem 3.2. The existing breather is given in lowest order by

$$E_3(x,t) = 2\varepsilon c_1 \operatorname{sech}(\varepsilon c_2 x) p_{11}(x)\sin(\omega_0 t) + \mathcal{O}(\varepsilon^2)$$

with constants c_1, $c_2 > 0$, the 1-periodic p_{11} and the critical frequency of light ω_0 (see (3.8) for the definition).

Remark 5.1 (Physical Limitations). *The application of Theorem 3.2 is physically restricted for two reasons. First, the coefficients $\hat{\chi}_j$, $j = 1, 3$, must be physically realizable or realistic. By definition*

$$\hat{\chi}_1(x, n\omega) = \hat{s}_1(x)\hat{f}(n\omega) - 1. \tag{5.16}$$

For $n = 1$ we may choose the $\hat{\chi}_1$ physically reasonable, i.e.

$$\hat{\chi}_1(x, \omega) = \hat{s}_1(x) - 1 > 0$$

for all $x \in \mathbb{R}$. By the definition of $\hat{f}$, the coefficient $\hat{\chi}_1$ will become negative as n tends to infinity, i.e.

$$\hat{\chi}_1(x, n\omega) = s_1(x)\hat{f}(n\omega) - 1 \longrightarrow -1, \qquad n \to \infty \tag{5.17}$$

In other words, for high frequencies the material becomes opaque. However, whether such a material may be constructed remains to be investigated. Second, any physical system uses the principle of causality. Therefore, both coefficients $\hat{\chi}_1$ and $\hat{\chi}_3$ are subject to causality which means

$$\chi_j(x, t) = 0, \quad t < 0, \qquad j = 1, 3.$$

Mathematically the principle of causality is described by the „Kramers-Kronig"-relation, which states that the Fourier-transform of $\chi_1(t)$ and $\chi_3(t)$ (namely $\hat{\chi}_1$ and $\hat{\chi}_3$) must have nonvanishing imaginary parts (see [24] and [25]). However, we may be able to declare that the imaginary parts of the coefficients are "small" and we drop them to get a real-valued equation. As a result, the breather result by Theorem 3.2 of equation (5.15) *may be seen as an approximation.*

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