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Governmental Debt, Interest Policy, and Tax Stabilization in a Stochastic OLG Economy*

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Abstract

The paper analyzes the sustainability of governmental debt and its welfare properties in an overlapping generations economy with stochastic production and capital accumulation. In the absence of taxation, equilibria with positive debt generically converge to debtless equilibria which are typically inefficient. It is shown that this may be overcome by a tax on labor income which stabilizes the level of debt against unfavorable shocks. A long-run welfare criterion is formulated which measures consumer utility at the stabilized equilibrium. Based on this criterion, the welfare effects of different interest policies and alternative stabilization objectives are investigated. The results offer a simple explanation why empirical debt levels are high and typically yield a riskless return despite both fails to be optimal in the long run.

Keywords: OLG, governmental debt, interest policy, risk sharing, tax stabilization, stabilized equilibrium, long-run welfare.

JEL Classification: C62, E27, E62, H21, H63.

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Introduction

Most industrialized countries have large governmental debt. In the U.S., total outstanding debt amounted to a little less than 60% of GDP in 2002 and rose moderately to about 66% in 2007. Largely due to the gigantic fiscal stimuli in response to the recent economic crisis, the past three years have seen a dramatic increase of this ratio to more than 90% as of 2010. Similar figures apply for other countries suggesting that the sustainability of governmental debt is - or should be - a highly relevant issue for policy making.

From a theoretical perspective, it is well-known that an increase in governmental debt may stimulate aggregate demand in the short run but crowds out capital investment in the long run, cf. Elmendorf & Mankiw (1999). The latter effect is particularly important in overlapping generations (OLG) economies where the first welfare theorem need not hold and competitive equilibria may be inefficient due to an overaccumulation of capital. In such a situation, as first shown by Diamond (1965), introducing governmental debt leads to a welfare improvement by implementing a dynamically efficient allocation. Subsequent studies to investigate governmental debt in deterministic OLG economies may be found, e.g., in de la Croix & Michel (2002, Ch.4), Farmer (1986), and in Bullard & Russell (1999) for consumers with multiperiod lives.

There is a close relationship between the sustainability of governmental debt and the emergence of a bubble. The latter corresponds to an intrinsically worthless asset that is traded at a positive price such as fiat money or a private asset that does not pay dividends. The differences between debt and a bubble are thoroughly exhibited in de la Croix & Michel (2002, p.212). Starting with the work by Tirole (1985), a large body of the literature discusses the emergence of bubbles in deterministic OLG models. For examples see, e.g., in Bertocchi & Wang (1994), Kunieda (2008), or Michel & Wigniolle (2003). Due to the structural similarities between debt and a bubble, the results by Tirole (1985) also characterize sustainable levels of governmental debt in deterministic OLG models. In the absence of taxation, there exists a unique sustainable debt-to GDP ratio for which the economy converges to the golden-rule steady state with positive debt. Debt smaller than the critical level leads to an asymptotically debtless (and inefficient) situation while larger values imply an unsustainable situation in which debt grows without bound.

Starting with the work of Wang (1993), the literature has increasingly focused on OLG economies with aggregate risk due to random production shocks. It seems not yet known, however, how the previous deterministic findings carry over to a stochastic setting, i.e., under what conditions equilibria with positive debt exist and which debt levels are sustainable. A first approach in this direction is put forward in Bertocchi (1994), who analyzes possible equilibrium scenarios in an OLG model with riskless debt. If there is aggregate risk, another function of governmental debt is to provide a possibility of *risk-sharing* between generations. While, e.g., Bohn (1998) and Krüger & Kübler (2006) analyze the issue of intergenerational risk-sharing in the context of Social Security, a similar study for governmental debt seems not to have been conducted in the literature. If payments on outstanding debt are financed by issuing new debt to the next generation, the implied risk sharing is essentially determined by the extent to which interest payments on debt are indexed to risk. This motivates the question how different interest policies affect intergenerational risk-sharing and consumer welfare.

Following the previous motivation, the present paper studies the role of governmental debt in a stochastic OLG framework. Two issues are at the center of interest: 1. *Which* *levels of debt are sustainable and which level is optimal*? 2. *Which interest policy is favorable and induces optimal risk sharing between generations*? The main contributions of the paper are as follows. Firstly, we unveil the forward-recursive structure of equilibria and derive necessary and sufficient conditions for their existence together with an explicit characterization of sustainable levels of debt under arbitrary interest policies. Secondly, we provide a complete characterization of the long-run dynamic behavior of the model with and without tax stabilization of debt. Furthermore, we develop a long-run welfare criterion on the basis of which an optimal interest policy and an optimal stabilization objective can be selected. Based on this criterion we analyze the welfare effects of alternative debt policies and use numerical simulations to characterize optimal policies. The results offer a simple explanation why empirical debt levels are so high and typically yield a riskless return despite both fails to be optimal in the long run. Finally, our results shed light on the emergence of asset bubbles in stochastic OLG economies. The paper is organized as follows. Section 1 introduces the model. Section 2 analyzes equilibria when the return on debt coincides with the capital return. This structure is generalized in Section 3 which allows for general interest policies. Section 4 demonstrates how the level of debt can be stabilized by a labor income tax. The welfare properties of stabilized equilibria under different debt policies are investigated in Section 5. Section 6 concludes, all proofs are placed in the Mathematical Appendix.

1 The Model

The framework to be introduced in this section generalizes the stochastic overlapping generations model in Wang (1993) to include governmental debt and a tax system.

Population. The consumption sector consists of overlapping generations of homogeneous consumers who live for two periods. The index $j \in \{y, o\}$ identifies the young and old generation in each period. Abstracting from population growth, each generation consists of $N > 0$ consumers. A young consumer is endowed with one unit of labor time supplied inelastically to the labor market. Since old consumers are retired and do not supply labor, $L_t \equiv N$ denotes aggregate labor force at time $t \geq 0$. The old generation in period t owns the existing stock of capital K_t which they supply to the production process.

Production. A single representative firm employs labor and capital as inputs to produce a homogeneous consumption good. In addition, the production process in period t is subjected to an exogenous random production shock $\varepsilon_t \in \mathcal{E}$. The linear homogeneous technology is represented by the intensive form production function $f : \mathbb{R}_+ \times \mathcal{E} \longrightarrow \mathbb{R}_+$ which determines gross output Y_t (including depreciated capital) produced at time t as

$$Y_t = L_t f(K_t/L_t; \varepsilon_t). \tag{1}$$

The function f is assumed to be continuous and twice differentiable with respect to its first argument with continuous derivatives satisfying $f_{kk}(k;\varepsilon) < 0 < f_k(k;\varepsilon)$ for all $k > 0$ and $\varepsilon \in \mathcal{E}$ as well as the Inada conditions $\lim_{k\to 0} f_k(k;\varepsilon) = \infty$ and $\lim_{k\to\infty} f_k(k;\varepsilon) < 1$. The noise process $\{\varepsilon_t\}_{t\geq 0}$ consists of independent, identically distributed random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each ε_t is distributed according to the probability measure ν supported on $\mathcal{E} \subset [\varepsilon_{min}, \varepsilon_{max}] \subset \mathbb{R}_{++}$. The process is adapted to a suitable filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub σ-algebras of $\mathcal{F}$ such that each $\varepsilon_t : \Omega \longrightarrow \mathcal{E}$ is Borel-measurable with respect to $\mathcal{F}_t$. Let $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t]$ denote the expectations operator conditional on the information represented by $\mathcal{F}_t$. Throughout, the notion of an adapted stochastic process $\{\xi_t\}_{t\geq 0}$ taking values in some set $\Xi \subset \mathbb{R}^M$ refers to the probability space and the filtration defined. It implies that each random variable $\xi_t : \Omega \longrightarrow \Xi$ is Borel-measurable with respect to $\mathcal{F}_t$ and hence determined in period t. All equalities or inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely without further notice.[1]

Let $w^g_t > 0$ be the gross wage and $r_t > 0$ the capital return at time $t \geq 0$. Given capital $k_t := \frac{K_t}{N} > 0$ and $\varepsilon_t \in \mathcal{E}$, profit maximizing behavior of the firm implies that market clearing factor prices are determined by the respective marginal products, i.e.,

$$\begin{aligned} w^g_t &= \mathcal{W}(k_t;\varepsilon_t) := f(k_t;\varepsilon_t) - k_t f_k(k_t;\varepsilon_t) \quad (2)\\ r_t &= \mathcal{R}(k_t;\varepsilon_t) := f_k(k_t;\varepsilon_t). \quad (3)\end{aligned}$$

Government. The infinitely-lived government taxes consumers and issues debt to finance its deficit. For the purpose of this paper, debt may be thought of as a one-period lived bond which pays a (possibly random) return $r^\star_{t+1} > 0$ in $t+1$ per unit invested at time $t \geq 0$. In light of the empirical evidence reported in the introduction, negative debt will not be considered. Let $b_t \geq 0$ be the number of bonds per young consumer issued at time t and τ^y_t and τ^o_t be the lump sum taxes levied on the incomes of young and old consumers, respectively. Negative taxes are interpreted as subsidies on the income of the respective group. Abstracting from governmental consumption, debt evolves as

$$b_t = r^\star_t b_{t-1} - \tau^y_t - \tau^o_t, \quad t \geq 0. \quad (4)$$

Consumers. At time $t \geq 0$ a young consumer earns net labor income $w_t := w^g_t - \tau^y_t > 0$ to be consumed and invested. Let s_t and b_t be the investments in capital and bonds at time $t \geq 0$. These choices define current consumption

$$c^y_t = w_t - b_t - s_t \quad (5)$$

while next period's consumption is given by the random variable

$$c^o_{t+1} = b_t\, r^\star_{t+1} + s_t\, r_{t+1} - \tau^o_{t+1}. \quad (6)$$

Here the randomness enters through the uncertain returns on both investments and uncertain tax payments which are all treated as given random variables in the decision. Young consumers evaluate the expected utility of different consumption plans (c^y_t, c^o_{t+1}) defined by (5) and (6) according to the von-Neumann Morgenstern utility function

$$U(c^y, c^o) = u(c^y) + v(c^o). \quad (7)$$

Both functions u and v are C^2 with derivatives $z''(c) < 0 < z'(c)$ for $c > 0$ and satisfy

$$\lim_{c\to 0} z'(c) = \infty \quad \text{for} \quad z \in \{u, v\}. \quad (8)$$

Each young consumer chooses investment to maximize her expected lifetime utility. The decision problem reads:

$$\max_{b,s}\Big\{ u(w_t - b - s) + \mathbb{E}_t\big[v\big(r^\star_{t+1}\, b + r_{t+1}\, s - \tau^o_{t+1}\big)\big] \,\big|\, s \geq 0, b + s \leq w_t \Big\}. \quad (9)$$

[1] The underlying probability space may be constructed by defining $\Omega := \mathcal{E}^{\mathbb{N}_0}$ which is endowed with the product topology and the Borel-σ-algebra $\mathcal{F} := \mathcal{B}(\Omega)$ on which the product measure $\mathbb{P} := \otimes_{t\geq 0}\nu$ is defined. The sub-σ-algebra $\mathcal{F}_t$ is generated by the class of measurable rectangular sets $A = \prod_{n=0}^{\infty} A_n$ where each A_n is a Borel-measurable subset of $\mathcal{E}$ and $A_n = \mathcal{E}$ for $n > t$.

Note that no short-selling constraints on b are imposed at the individual level. The investment in capital s_t determines next period's capital stock (per labor force)

$$k_{t+1} = s_t. \tag{10}$$

Old consumers in period $t \geq 0$ consume the proceeds of their investments in bonds and capital made during the previous period - net of taxes - as defined by (6).

Equilibrium. Combining the assumptions of market clearing, individual optimality, and rational expectations yields the following definition of equilibrium.

Definition 1.1 *Given initial values $b_0 \geq 0$, $k_0 > 0$, and $\varepsilon_0 \in \mathcal{E}$, an equilibrium is an adapted process $\left\{w_t^g, r_t, r_t^\star, \tau_t^y, \tau_t^o, b_t, s_t, c_t^y, c_t^o, k_{t+1}\right\}_{t\geq 0}$ which satisfies for each $t \geq 0$:*

(i) Debt returns satisfy $r_t^\star > 0$ while $w_t^g > 0$ and $r_t > 0$ are determined by (2), (3).

(ii) Taxes satisfy $\tau_t^y < w_t^g$ and $\tau_t^o < b_t r_t^\star + k_t r_t$ while debt $b_t \geq 0$ evolves as in (4).

(iii) The pair (b_t, s_t) solves the decision problem (9) at the given wage, returns, and taxes while c_t^y, c_t^o, and k_{t+1} are determined by (5), (6), and (10).

Indeterminacy of fiscal policy. The following result shows that without further restrictions on taxes $\{\tau_t^y, \tau_t^o\}_{t\geq 0}$, any debt process is consistent with equilibrium. This is a straightforward generalization of the deterministic result in de la Croix & Michel (2002).

Lemma 1.1 *Let an interior allocation $\left\{s_t, c_t^y, c_t^o, k_{t+1}\right\}_{t\geq 0}$ and prices $\{w_t^g, r_t, r_t^\star\}_{t\geq 0}$ satisfy (2), (3), and (10), the feasibility condition $c_t^y + c_t^o + k_{t+1} = f(k_t, \varepsilon_t)$ for all $\varepsilon_t \in \mathcal{E}$ and the intertemporal efficiency condition $u'(c_t^y) = \mathbb{E}_t[r_{t+1} v'(c_{t+1}^o)] = \mathbb{E}_t[r_{t+1}^\star v'(c_{t+1}^o)]$ for all $t \geq 0$. Then, for any non-negative debt process $\{b_t\}_{t\geq 0}$ there is a feasible tax process $\{\tau_t^y, \tau_t^o\}_{t\geq 0}$ such that $\left\{w_t^g, r_t, r_t^\star, \tau_t^y, \tau_t^o, b_t, s_t, c_t^y, c_t^o, k_{t+1}\right\}_{t\geq 0}$ is an equilibrium.*

Lemma 1.1 shows that the sustainability of debt becomes irrelevant if unbounded taxation is possible. The reason for this result is simple: The government can directly set-off its payment obligations on outstanding debt by a corresponding tax on the incomes of old consumers who receive these payments. Thus, any level of debt can be sustained. Clearly, the previous result fails to hold if restrictions on τ_t^o are imposed. For this reason, and also to avoid time-consistency problems, the remainder confines attention to the case where $\tau_t^o \equiv 0$, i.e., there is no taxation of capital incomes.

2 Equilibria with Capital-Equivalent Debt

Capital-equivalent debt. The following two sections study existence and properties of equilibria in the absence of taxation ($\tau_t^y \equiv 0$) under different assumptions on the return on debt, i.e., on the process $\{r_t^\star\}_{t\geq 0}$. As a first scenario, suppose the government commits itself to paying the capital return on debt such that $r_t^\star \equiv r_t$ for each $t \geq 0$. This case will be called *capital-equivalent (CE) debt* and the remainder of this section studies the existence and properties of equilibria under this assumption.

Equilibrium structure. As a first step, we seek to unveil the recursive structure of equilibria by considering the temporary situation in an arbitrary period t. Let current capital $k_t > 0$ and the shock $\varepsilon_t \in \mathcal{E}$ be given which determine the net wage $w_t = w_t^g > 0$ and the return on capital and debt $r_t > 0$ according to (2) and (3). Current debt $b_t \geq 0$ corresponding to the supply of bonds then follows from its previous value b_{t-1} and (4). The number of bonds traded is therefore predetermined by the supply side. Since investments in debt and capital are perfect substitutes, the equilibrium problem for period t reduces to determining next period's capital stock $0 < k_{t+1} < w_t - b_t$. The latter must be chosen consistent with an optimal savings decision derived from (9) and rational, self-confirming expectations. Clearly, this requires $w_t > b_t$. Let $\mathbb{E}_\nu[\cdot]$ denote the expected value with respect to the distribution ν of next period's production shock. Combining (3) and (10) with the first order condition from (9), define $H(\cdot; w, b) :]\,0, w-b\,[\longrightarrow \mathbb{R}$,

$$H(k; w, b) := u'(w-b-k) - \mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'\big(\mathcal{R}(k;\cdot)(b+k)\big)\right]. \tag{11}$$

Then, given $w_t > b_t \geq 0$, the expectations-consistent solution k_{t+1} is determined by the condition $H(k_{t+1}; w_t, b_t) = 0$. Before establishing existence and uniqueness of such a zero in Lemma 2.1, we introduce a set of additional restrictions on f in (1) and v in (7) which will be used frequently. Here and in the sequel, we denote the elasticity of a differentiable function $h : \mathbb{D} \longrightarrow \mathbb{R}\backslash\{0\}$ as $E_h(x) := xh'(x)/h(x)$, $x \in \mathbb{D} \subset \mathbb{R}$.

$$(P1)\ E_{v'}(c) \geq -1\ \forall c > 0 \quad (P2)\ \lim_{c\to\infty} c\, v'(c) = \infty \quad (P3)\ E_{f_k}(k;\varepsilon) \geq -1\ \forall k > 0, \varepsilon \in \mathcal{E}.$$

While (P1) and (P3) are standard, (cf. de la Croix & Michel (2002) and Wang (1993)), (P2) is more restrictive as it excludes several popular parameterizations such as log utility. Examples satisfying (P1) and (P2) are power utility $v(c) = \theta^{-1}c^\theta$, $0 < \theta < 1$, or CES utility $v(c) = [1-\theta+\theta c^\beta]^{1/\beta}$, $0 < \theta < 1$, $\beta > 0$.

Lemma 2.1 *Let v satisfy (P1). Then, each $w > 0$ defines an upper bound $0 < b^{\max}(w) \leq w$ such that $H(\cdot; w, b)$ has a zero in $]0, w-b[$ if and only if $b < b^{\max}(w)$. This zero is unique and $w \longmapsto b^{\max}(w)$ continuous. If, in addition, (P2) holds, $b^{\max}(w) = w$.*

In the sequel we assume that (P1) holds. Then, Lemma 2.1 permits to define the set $\mathbb{V} := \{(w, b) \in \mathbb{R}^2_+ \,|w > 0, b < b^{\max}(w)\}$ and a mapping $\mathcal{K} : \mathbb{V} \longrightarrow \mathbb{R}_{++}$ which determines k_{t+1} as the unique zero of $H(\cdot; w_t, b_t)$. The next result establishes properties of this map.

Lemma 2.2 *Let v satisfy (P1). Then, $\mathcal{K}$ is C^1 on $\mathbb{V}$ (cf. Remark A.1) and the derivatives satisfy $0 < \partial_w\mathcal{K}(w, b) < -\partial_b\mathcal{K}(w, b) \leq 1$.*

Equilibrium dynamics. Combining the previous results with equations (2)–(4) and (10) defines a map $\Phi = (\Phi_w, \Phi_b) : \mathbb{V} \times \mathcal{E} \longrightarrow \mathbb{R}^2_+$ which determines the evolution of wages and debt under the exogenous noise process as

$$\begin{aligned} w_{t+1} &= \Phi_w(w_t, b_t; \varepsilon_{t+1}) := \mathcal{W}(\mathcal{K}(w_t, b_t); \varepsilon_{t+1}) && \text{(12a)} \\ b_{t+1} &= \Phi_b(w_t, b_t; \varepsilon_{t+1}) := \mathcal{R}(\mathcal{K}(w_t, b_t); \varepsilon_{t+1})b_t. && \text{(12b)} \end{aligned}$$

Given initial values $(w_0, b_0) \in \mathbb{V}$, the equilibrium process $\{w_t, b_t\}_{t\geq 0}$ is therefore generated by randomly mixing the family of mappings $\{\Phi(\cdot; \varepsilon)\}_{\varepsilon\in\mathcal{E}}$, i.e., the realization of next period's shock 'selects' a map that determines the next state from the current one. Structurally, this corresponds to a two-dimensional version of the one-dimensional dynamics in Wang (1993). The endogenous state variables $\{w_t, b_t\}_{t\geq 0}$ together with the exogenous noise process $\{\varepsilon_t\}_{t\geq 0}$ completely determine the other equilibrium variables of the model. Therefore, existence of a dynamic equilibrium is equivalent to determining $(w_0, b_0) \in \mathbb{V}$ such that the process generated by (12a,b) satisfies $(w_t, b_t) \in \mathbb{V}$ for all $t \geq 0$ under $\mathbb{P}$-almost all paths of the noise process. Since $b_0 = 0$ implies $b_t = 0$ for all $t > 0$, it is clear that a trivial equilibrium with zero debt exists for all $w_0 > 0$. In this case, the dynamics reduce to the evolution of wages defined by the map $\phi_0 : \mathbb{R}_{++} \times \mathcal{E} \longrightarrow \mathbb{R}_{++}$

$$w_{t+1} = \phi_0(w_t; \varepsilon_{t+1}) := \mathcal{W}(\mathcal{K}(w_t, 0); \varepsilon_{t+1}). \tag{13}$$

Similar to Tirole (1985), the steady state properties of (13) will play a crucial for the existence of non-trivial equilibria. The next assumption rules out multiplicity of steady states of ϕ_0.

Assumption 2.1 *For each $\varepsilon \in \mathcal{E}$, the map $\phi_0(\cdot; \varepsilon)$ possesses a unique fixed point $\bar{w}^0_\varepsilon > 0$ which is stable.*

Dynamic properties. From above's structure, it stands to reason that the existence and properties of equilibrium depend crucially on the dynamic properties of the mappings $(\Phi(\cdot; \varepsilon))_{\varepsilon\in\mathcal{E}}$ and whether these exhibit contractive or expansive behavior. We therefore begin by fixing a value $\varepsilon \in \mathcal{E}$ to study the dynamic properties of the single map $\Phi(\cdot; \varepsilon)$. In the sequel, define $\mathbb{V}_+ := \mathbb{V} \cap \mathbb{R}^2_{++}$ and let $\Phi^t(\cdot; \varepsilon) := \Phi(\cdot; \varepsilon) \circ \ldots \circ \Phi(\cdot; \varepsilon)$ denote the t-fold composition of $\Phi(\cdot; \varepsilon)$ for $t \geq 0$ where $\Phi^0(\cdot; \varepsilon) := \mathrm{id}_{\mathbb{V}}$. By Assumption 2.1, $\Phi(\cdot; \varepsilon)$ possesses a unique trivial steady state $(\bar{w}^0_\varepsilon, 0)$. The next result shows that the associated ex-post return $\mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon)$ determines whether $\Phi(\cdot; \varepsilon)$ displays stable - along a certain direction - or expansive behavior. In anticipation of this result, let $\mathcal{E}_s := \{\varepsilon \in \mathcal{E} \,|\, \mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon) < 1\}$ and $\mathcal{E}_x := \{\varepsilon \in \mathcal{E} \,|\, \mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon) > 1\}$. Since the case $\mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon) = 1$ is non-generic, $\mathcal{E}_0 := \mathcal{E} \backslash (\mathcal{E}_s \cup \mathcal{E}_x)$ is assumed to have measure zero, i.e., $\nu(\mathcal{E}_0) = 0$.[2]

Lemma 2.3 *Let (P1) and Assumption 2.1 be satisfied. Then, the following holds true:*

(i) For $\varepsilon \in \mathcal{E}^s$ the map $\Phi(\cdot; \varepsilon)$ possesses a unique non-trivial fixed point $(\bar{w}_\varepsilon, \bar{b}_\varepsilon) \in \mathbb{V}_+$. This fixed point is saddle-path stable, i.e., the Eigenvalues of the Jacobian matrix $D\Phi(\bar{w}_\varepsilon, \bar{b}_\varepsilon; \varepsilon)$ are real and satisfy $0 < |\lambda_1| < 1 < |\lambda_2|$.

(ii) For $\varepsilon \in \mathcal{E}^x$ the map $\Phi(\cdot; \varepsilon)$ is expansive, i.e., for each $(w, b) \in \mathbb{V}_+$ there exists a $t_0 \in \mathbb{N}$ such that $(w_{t_0}, b_{t_0}) := \Phi^{t_0}(w, b; \varepsilon) \notin \mathbb{V}$, that is, $w_{t_0} \leq b_{t_0}$.

If $\varepsilon \in \mathcal{E}^s$, (i) implies that the dynamics generated by $\Phi(\cdot; \varepsilon)$ converge to the non-trivial steady state only for certain initial values. These are defined by the *stable manifold*

$$\mathcal{M}_\varepsilon := \Big\{(w, b) \in \mathbb{V} \,|\, \Phi^n(w, b; \varepsilon) \in \mathbb{V}\ \forall n \geq 1 \wedge \lim_{n\to\infty} \Phi^n(w, b; \varepsilon) = (\bar{w}_\varepsilon, \bar{b}_\varepsilon)\Big\},\ \varepsilon \in \mathcal{E}^s. \tag{14}$$

The sets $\mathcal{M}_\varepsilon$ will play a key-role in the sequel. Note that $\mathcal{M}_\varepsilon$ is *self-supporting* under $\Phi(\cdot; \varepsilon)$, i.e., $\Phi(\mathcal{M}_\varepsilon; \varepsilon) \subset \mathcal{M}_\varepsilon$. Theorem A.1 in the appendix establishes existence of a C^1-map $\psi_\varepsilon : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ which is strictly increasing such that $\mathcal{M}_\varepsilon = \mathrm{graph}(\psi_\varepsilon)$, $\varepsilon \in \mathcal{E}^s$. Based on this representation, the next result shows that $\mathcal{M}_\varepsilon$ separates initial states which diverge from those which converge to the trivial steady state.

[2] If $\mathcal{E}$ is infinite, continuity of $\varepsilon \mapsto \mathcal{R}(\mathcal{K}(\bar{w}^0_\varepsilon, 0), \varepsilon)$ ensures (Borel-) measurability of $\mathcal{E}_s$, $\mathcal{E}_x$, and $\mathcal{E}_0$.

Lemma 2.4 *Under (P1) and Assumption 2.1, let $w > 0$ be arbitrary. Then, for each $\varepsilon \in \mathcal{E}^s$ the following holds:*

(i) $b < \psi_\varepsilon(w) \quad \Rightarrow \quad \Phi^t(w,b;\varepsilon) \in \mathbb{V} \ \forall t > 0 \quad \wedge \quad \lim_{t\to\infty} \Phi^t(w,b;\varepsilon) = (\bar{w}^0_\varepsilon, 0)$.

(ii) $b > \psi_\varepsilon(w) \quad \Rightarrow \quad \exists t_0 > 0$ *such that* $\Phi^{t_0}(w,b;\varepsilon) \notin \mathbb{V}$.

Geometrically, Lemma 2.4 implies that if (w,b) is below the curve $\mathcal{M}_\varepsilon$, the sequence $\Phi^t(w,b;\varepsilon)$ stays below $\mathcal{M}_\varepsilon$ for all $t \geq 0$ and converges to the trivial steady state with zero debt. Conversely, any state above $\mathcal{M}_\varepsilon$ stays above and leaves $\mathbb{V}$ in finite time.

Existence of equilibrium. Based on the dynamic properties of the involved mappings stated in Lemmata 2.3 and 2.4, we are now in a position to derive conditions for the existence of non-trivial equilibria. For simplicity, the following arguments assume that $\mathcal{E}$ is a finite set. A generalization, e.g., to distributions ν possessing a continuous density $d : [\varepsilon_{\min}, \varepsilon_{\max}] \longrightarrow \mathbb{R}_{++}$ seems straightforward. Let $w_0 := \mathcal{W}(k_0;\varepsilon_0) > 0$ be given. First observe from Lemma 2.3(ii) that if $\nu(\mathcal{E}^x) > 0$, any initial value in $\mathbb{V}_+$ will leave this set in finite time with positive probability. Hence, $\nu(\mathcal{E}^x) = 0$ is a necessary condition for non-trivial equilibria to exist. Note that this restriction typically implies that the trivial equilibrium is dynamically inefficient. For $w > 0$, let $b^{\text{crit}}(w) := \min_{\varepsilon\in\mathcal{E}^s}\{\psi_\varepsilon(w)\}$. By Lemma 2.4, $b_0 \leq b^{\text{crit}}(w_0)$ is also necessary for the existence of equilibrium. Sufficiency requires the following additional assumption.

Assumption 2.2 $b \leq b^{\text{crit}}(w)$ *implies* $\Phi_b(w,b;\varepsilon) \leq b^{\text{crit}}(\Phi_w(w,b;\varepsilon)) \ \forall w > 0$, $\varepsilon \in \mathcal{E}_s$.

Under Assumption 2.2, the curve $w \mapsto b^{\text{crit}}(w)$, $w > 0$ defines the maximum sustainable level of debt. Intuitively, in a stochastic setting sustainable levels must be chosen conservatively small to ensure that debt remains bounded under *all* possible shocks.
Combining Lemma 2.3 and 2.4 leads to the following theorem which includes the results of Tirole (1985) as a special case in which ν is degenerate and $\mathcal{E}^s = \mathcal{E} = \{\varepsilon\}$.

Theorem 2.1 *Under (P1) and Assumptions 2.1 and 2.2, let $\mathcal{E}$ be finite and $\nu(\mathcal{E}^x) = 0$. Then, any $b_0 \in]0, b^{\text{crit}}(w_0)]$ defines an equilibrium with debt $b_t > 0$ for all $t > 0$.*

Non-persistence of debt. While equilibria exist under the hypotheses of Theorem 2.1, the long-run level of debt generically converges to zero with probability one. Unlike the case in Tirole (1985), this holds even if $b_0 = b_0^{\text{crit}} := b^{\text{crit}}(w_0)$. Structurally, the reason is that positive *stable sets*, i.e., compact subsets $\mathbb{A} \subset \mathbb{V}_+$ which are self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ such that $\Phi(\mathbb{A};\varepsilon) \subset \mathbb{A}$ for all $\varepsilon \in \mathcal{E}$ typically fail to exist. To see this, note from Lemma 2.4 that $\mathbb{A} \subset \mathbb{V}_+$ closed and self-supporting under $\Phi(\cdot;\varepsilon)$ requires $\mathbb{A} \subset \mathcal{M}_\varepsilon$. Hence, positive stable sets are subsets of $\cap_{\varepsilon\in\mathcal{E}}\mathcal{M}_\varepsilon$ which is typically empty. Figure 1 illustrates these and the findings from Theorem 2.1 for the case with two shocks where $\mathcal{E} = \{\varepsilon, \varepsilon'\}$. The dotted arrow represents the case excluded by Assumption 2.2.

A final example shows, however, that stable sets may exist in non-generic situations. Let $U(c^y, c^o) = \ln c^y + \gamma c^o$, $\gamma > 0$ and $f(k;\varepsilon) = \varepsilon k^\alpha$, $0 < \alpha < 1$. Then, $b^{\max}(w) = \frac{\gamma}{1+\gamma} w$ such that $\mathbb{V} = \{(w,b) \in \mathbb{R}^2_+ | b < \frac{\gamma}{1+\gamma} w\}$. Furthermore, $\Phi_w(w,b;\varepsilon) = \varepsilon(1-\alpha)(\frac{\gamma}{1+\gamma} w - b)^\alpha$, $\Phi_b(w,b;\varepsilon) = \varepsilon\alpha(\frac{\gamma}{1+\gamma} w - b)^{\alpha-1} b$, and $\mathcal{E}^s \neq \emptyset$ if and only if $\bar{\zeta} := \frac{\gamma}{1+\gamma} - \frac{\alpha}{1-\alpha} > 0$.

Lemma 2.5 *For the previous parametrization, suppose $\bar{\zeta} > 0$. Then $\mathcal{E}^s = \mathcal{E}$ and the sets in (14) are independent of ε and of the form $\mathcal{M}_\varepsilon \equiv \mathcal{M} := \{(w,b) \in \mathbb{R}^2_{++} \,|\, b = \bar{\zeta} w\}$.*

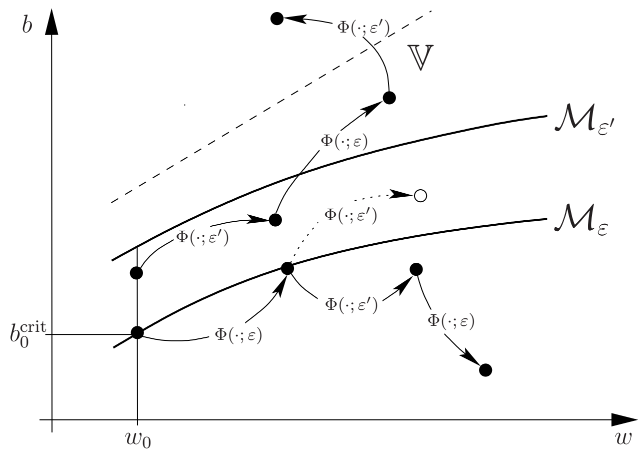


Figure 1: Equilibrium dynamics generated by mixing two saddle-path stable mappings.

The set $\mathcal{M} = \cap_{\varepsilon\in\mathcal{E}}\mathcal{M}_\varepsilon$ is thus self-supporting for the family $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$. Moreover, for any $(w_0, b_0) \in \mathcal{M}$ the dynamics converge to a compact subset of $\mathcal{M}$ defined by the non-trivial fixed points $((\bar{w}_\varepsilon, \bar{b}_\varepsilon))_{\varepsilon\in\mathcal{E}}$ of the mappings $(\Phi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ which is a stable set.

3 Equilibria with General Debt

Interest policies. Maintaining the assumption of no taxation ($\tau_t^y \equiv 0$), the present section extends the study of equilibria to arbitrary interest policies on debt. For simplicity, the remainder of the paper assumes that shocks in (1) are multiplicative, i.e., $f(k;\varepsilon) = \varepsilon g(k)$ where $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ inherits the properties of $f(\cdot;\varepsilon)$. While under the previous scenario the return on debt offered at time t would be $r^\star_{t+1} = \varepsilon_{t+1} g'(k_{t+1})$, the present section generalizes this structure by supposing that

$$r^\star_{t+1} = \mathcal{R}^\star_\vartheta(z_t;\varepsilon_{t+1}) := \vartheta(\varepsilon_{t+1})\, z_t, \quad t \geq 0. \tag{15}$$

The value $z_t > 0$ is determined in period t and $\vartheta : \mathcal{E} \longrightarrow \mathbb{R}_{++}$ is a time-invariant *interest policy* that determines the risk to which debt investments are subjected. Specifically, if $\vartheta \equiv \bar{\vartheta}$, debt is riskless while $\vartheta = \mathrm{id}_\mathcal{E}$ recovers the previous case with CE debt.[3]

Equilibrium structure. In the sequel we fix some interest policy ϑ and assume that in each period the return on debt is of the form (15). To derive the recursive equilibrium

[3] For each ϑ, the induced equilibrium is equivalent to an equilibrium with (sequentially) complete markets where the government issues contingent claims to finance its debt b_t in period t. To see this, suppose $\mathcal{E} = \{\varepsilon^1, \ldots, \varepsilon^N\}$ and let p_t^n be the price of an Arrow security traded at time t that pays one unit in $t+1$ iff $\varepsilon_{t+1} = \varepsilon^n$, $n = 1, \ldots, N$. The government issues a portfolio $a_t = (a_t^n)_{n=1,\ldots,N} \in \mathbb{R}_+^N$ of these securities such that $\sum_{n=1}^N a_t^n p_t^n = b_t$. Specifically, suppose the government chooses the supply of security n as $a_t^n = b_t z_t \vartheta(\varepsilon^n)$ for $n = 1, \ldots, N$ and some $z_t > 0$. For young consumers to be willing to buy these claims, prices must satisfy $p_t^n = \nu(\{\varepsilon^n\}) v'(a_t^n + \varepsilon^n g'(k_{t+1})k_{t+1})/u'(w_t - b_t - k_{t+1})$. Combining these conditions with the first order conditions for an expectations-consistent capital investment derived from (9) yields precisely the conditions (16a, b) derived below to determine z_t and k_{t+1}. Hence, this modified setup implies *the same equilibrium allocation*. Under the previous interpretation, the interest policy ϑ therefore determines the – time-invariant – mix of Arrow securities that the government issues. The arguments also extend to an infinite set $\mathcal{E}$. An interesting generalization would be to consider dynamic interest policies with state-dependent mixing policy ϑ.

structure of the economy, we proceed as in the previous section and consider an arbitrary period $t \geq 0$. Let current capital $k_t > 0$ and the shock $\varepsilon_t \in \mathcal{E}$ be given which determine the net wage $w_t = w_t^g > 0$ according to (2). Furthermore, given previous values $b_{t-1} \geq 0$ and $z_{t-1} > 0$, the current shock determines the realized debt return $r_t^\star = z_{t-1}\vartheta(\varepsilon_t)$ and current debt corresponding to the supply of bonds $b_t \geq 0$ follows from (4). Assuming that $w_t > b_t$, the equilibrium problem for period t is to determine an expectations-consistent capital stock k_{t+1} and a value $z_t > 0$. The latter determines the ex-ante debt return $r_{t+1}^\star$ offered at time t according to (15) and must be chosen such that young consumers are willing to absorb the predetermined supply of bonds. To achieve this, note that any solution $s > 0$ and $b \geq 0$ to (9) satisfies the corresponding first order conditions since there are no short-selling restrictions on debt. Using this and equations (3), (10), and (15), let $H_i^\vartheta(\cdot,\cdot;w,b) : \mathbb{R}_{++}\times]0, w-b[\longrightarrow \mathbb{R}$, $i \in \{1,2\}$,

$$
\begin{aligned}
H_1^\vartheta(z,k;w,b) &:= u'(w-b-k) - \mathbb{E}_\nu\big[\mathcal{R}(k;\cdot)v'\big(b\,\mathcal{R}_\vartheta^\star(z;\cdot) + k\mathcal{R}(k;\cdot)\big)\big] \qquad (16a)\\
H_2^\vartheta(z,k;w,b) &:= u'(w-b-k) - \mathbb{E}_\nu\big[\mathcal{R}_\vartheta^\star(z;\cdot)v'\big(b\,\mathcal{R}_\vartheta^\star(z;\cdot) + k\mathcal{R}(k;\cdot)\big)\big]. \qquad (16b)
\end{aligned}
$$

Then, given $w_t > b_t \geq 0$ the previous problem reduces to solving $H_1^\vartheta(z_t, k_{t+1}; w_t, b_t) = H_2^\vartheta(z_t, k_{t+1}; w_t, b_t) = 0$. Existence and uniqueness of such a solution is established next.

Lemma 3.1 *Let (P1)–(P3) hold and ϑ be continuous. Then, for each $w > b \geq 0$ there exist unique $z > 0$ and $0 < k < w - b$ which solve $H_1^\vartheta(z,k;w,b) = H_2^\vartheta(z,k;w,b) = 0$.*

In the sequel let $\mathbb{V} = \{(w,b) \in \mathbb{R}_+^2 \,|\, w > b\}$ denote the endogenous state space of feasible wage-debt combinations. By Lemma 3.1, there exist mappings $\mathcal{K}^\vartheta : \mathbb{V} \longrightarrow \mathbb{R}_{++}$ and $\mathcal{Z}^\vartheta : \mathbb{V} \longrightarrow \mathbb{R}_{++}$ which determine the values k_{t+1} and z_t as zeros of (16a,b) for each $(w_t, b_t) \in \mathbb{V}$. Before stating properties of these mappings in Lemma 3.2, we introduce the following additional restrictions on the elasticities of the utility function (7) and the production technology (1) which will be used subsequently.

(P4) $|E_{v'}(c)| = \theta \;\forall c > 0$ *(P5)* $|E_{u'}(c)| \leq 1 \;\forall c > 0$ *(P6)* $E_g(k) + |E_{g'}(k)| \leq 1 \;\forall k > 0$.

Under (P4), second period utility v exhibits constant relative risk aversion. Property (P5) is automatically satisfied if (P1) holds and $v(c) = \beta u(c)$, $\beta > 0$. Finally, (P6) is necessary and sufficient for the elasticity $E_g(k)$ to be a non-decreasing function of k, which holds, e.g., if g is Cobb-Douglas or CES with elasticity of substitution $\sigma \geq 1$.

Lemma 3.2 *Let (P1)–(P3) hold and ϑ be continuous. Then, $\mathcal{K}^\vartheta$ and $\mathcal{Z}^\vartheta$ are C^1 on $\mathbb{V}$ (cf. Remark A.1). Moreover, the following holds for all $(w,b) \in \mathbb{V}$:*

(i) The derivatives of $\mathcal{K}^\vartheta$ satisfy $0 < \partial_w \mathcal{K}^\vartheta(w,b) < -\partial_b \mathcal{K}^\vartheta(w,b)$.

(ii) If, in addition, (P4) holds, then $0 < -\partial_w \mathcal{Z}^\vartheta(w,b) < \partial_b \mathcal{Z}^\vartheta(w,b)$.[4]

Equilibrium dynamics. Unless stated otherwise, the remainder of the paper assumes that (P1)–(P4) hold. Then, by the previous results and (2), (4), and (15), the evolution of wages and debt under the exogenous shocks are given by $\Phi^\vartheta = (\Phi_w^\vartheta, \Phi_b^\vartheta) : \mathbb{V}\times\mathcal{E} \longrightarrow \mathbb{R}_+^2$

$$
\begin{aligned}
w_{t+1} &= \Phi_w^\vartheta(w_t, b_t; \varepsilon_{t+1}) := \mathcal{W}(\mathcal{K}^\vartheta(w_t,b_t), \varepsilon_{t+1}) \qquad (17a)\\
b_{t+1} &= \Phi_b^\vartheta(w_t, b_t; \varepsilon_{t+1}) := \vartheta(\varepsilon_{t+1})\mathcal{Z}^\vartheta(w_t,b_t)b_t. \qquad (17b)
\end{aligned}
$$

[4] Numerical experiments with utility functions v not satisfying (P4) have throughout displayed the same properties of $\mathcal{Z}^\vartheta$ as in Lemma 3.2(ii) suggesting that this restriction could probably be relaxed.

Thus, equilibria are generated by randomly mixing the mappings $(\Phi^\vartheta(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ and exist if and only if $(w_t, b_t) \in \mathbb{V}$ $\mathbb{P}$-a.s. for all $t \geq 0$. Note that for $b = 0$, the dynamics (17a,b) are independent of ϑ and governed by the map ϕ^0 in (13). In the sequel, the following slightly stronger version of Assumption 2.1 will be employed. The additional restriction is sufficient but far from necessary to obtain the uniqueness assertion in Lemma 3.3(i).

Assumption 3.1 *For each $\varepsilon \in \mathcal{E}$, the map $\phi_0(\cdot;\varepsilon)$ from (13) possesses a unique fixed point $\bar{w}^0_\varepsilon > 0$ which is stable. Moreover, the corresponding steady state capital stock $\bar{k}^0_\varepsilon := \mathcal{K}(\bar{w}^0_\varepsilon, 0)$ satisfies $E_g(\bar{k}^0_\varepsilon) \leq \frac{1}{2}$.*

Dynamic properties. Proceeding as above, we fix a value $\varepsilon \in \mathcal{E}$ to study the dynamic properties of a single map $\Phi^\vartheta(\cdot;\varepsilon)$. Under Assumption 2.1, $\Phi^\vartheta(\cdot;\varepsilon)$ possesses a unique trivial steady state $(\bar{w}^0_\varepsilon, 0)$ which is stable and independent of ϑ. Similar to the previous section, the dynamic behavior of $\Phi^\vartheta(\cdot;\varepsilon)$ is determined by the ex-post debt return at the associated trivial steady state. By (16a,b), this return is given by $\vartheta(\varepsilon)\bar{z}^0_\varepsilon$ where

$$\bar{z}^0_\varepsilon := \mathcal{Z}^\vartheta(\bar{w}^0_\varepsilon, 0) = \frac{\mathbb{E}_\nu[\mathcal{R}(\bar{k}^0_\varepsilon;\cdot)v'(\bar{k}^0_\varepsilon\mathcal{R}(\bar{k}^0_\varepsilon;\cdot))]}{\mathbb{E}_\nu[\vartheta(\cdot)v'(\bar{k}^0_\varepsilon\mathcal{R}(\bar{k}^0_\varepsilon;\cdot))]}. \tag{18}$$

Using (18), let $\mathcal{E}^\vartheta_s := \{\varepsilon \in \mathcal{E} \mid \bar{z}^0_\varepsilon\vartheta(\varepsilon) < 1\}$ and $\mathcal{E}^\vartheta_x := \{\varepsilon \in \mathcal{E} \mid \bar{z}^0_\varepsilon\vartheta(\varepsilon) > 1\}$ assuming again that $\mathcal{E}^\vartheta_0 = \mathcal{E}\backslash(\mathcal{E}^\vartheta_s \cup \mathcal{E}^\vartheta_x)$ satisfies $\nu(\mathcal{E}^\vartheta_0) = 0$. The next result extends Lemma 2.3 to the case with general interest policies. The proof draws on ideas from Galor (1992).

Lemma 3.3 *Under Assumption 3.1 and (P1)–(P6), the following holds for any ϑ:*

(i) For $\varepsilon \in \mathcal{E}^\vartheta_s$ the map $\Phi^\vartheta(\cdot;\varepsilon)$ has a unique non-trivial steady state $(\bar{w}^\vartheta_\varepsilon, \bar{b}^\vartheta_\varepsilon) \in \mathbb{V}_+$. This steady state is saddle-path stable.

(ii) For $\varepsilon \in \mathcal{E}^\vartheta_x$ the mapping $\Phi^\vartheta(\cdot;\varepsilon)$ is expansive.

Lemma 3.3(i) permits to define for each $\varepsilon \in \mathcal{E}^\vartheta_s$ the associated stable manifold

$$\mathcal{M}^\vartheta_\varepsilon := \Big\{(w,b) \in \mathbb{V} | (\Phi^\vartheta)^n(w,b;\varepsilon) \in \mathbb{V}\ \forall n \geq 1 \wedge \lim_{n\to\infty}(\Phi^\vartheta)^n(w,b;\varepsilon) = (\bar{w}^\vartheta_\varepsilon, \bar{b}^\vartheta_\varepsilon)\Big\}. \tag{19}$$

By Theorem A.1, there exists a C^1-map $\psi^\vartheta_\varepsilon : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ such that $\mathcal{M}^\vartheta_\varepsilon = \mathrm{graph}(\psi^\vartheta_\varepsilon)$. Moreover, Lemma 2.4 is shown in the appendix to extend to the present setup as well.

Properties of equilibria. Assuming that the hypotheses of Lemma 3.3 hold, it follows that all findings from the previous section including the existence result from Theorem 2.1 and the non-persistence of debt carry over to the present case with general interest policies. Specifically, under the restriction imposed in Assumption 2.2 (which is shown in Lemma 3.4 to automatically hold under riskless debt), equilibria exist iff $\nu(\mathcal{E}^\vartheta_x) = 0$ and $b_0 \leq b_0^{\mathrm{crit}} := \min_{\varepsilon\in\mathcal{E}^\vartheta_s}\{\psi^\vartheta_\varepsilon(w_0)\}$ but are generically asymptotically debtless with probability one. Again, the reason is that positive stable sets $\mathbb{A} \subset \mathbb{V}_+$ fail to exist.

Lemma 3.3 also provides important insights concerning the discussion in Bertocchi (1994) about stable sets under safe debt. Referring to the cases discussed there, it shows that steady states which are asymptotically stable and would give rise to stable sets with positive debt do not exist. In fact, using the arguments of the previous section, the following lemma implies that positive stable sets can never exist under riskless debt.

Lemma 3.4 *Under the hypotheses of Lemma 3.3, suppose $\vartheta \equiv \bar{\vartheta} > 0$. Then, for all $\varepsilon, \varepsilon' \in \mathcal{E}^{\bar{\vartheta}}_s$ it holds that $\varepsilon \neq \varepsilon'$ implies $\mathcal{M}^\vartheta_\varepsilon \cap \mathcal{M}^\vartheta_{\varepsilon'} = \emptyset$. Moreover, Assumption 2.2 holds.*

4 Tax-Stabilization of Debt

Stabilization objective. In the deterministic case where $\mathcal{E} = \{\varepsilon\}$ and $\mathcal{E}_x^\vartheta = \emptyset$, the results by Tirole (1985) uniquely determine the long-run optimal level of debt by the condition $(w_0, b_0) \in \mathcal{M}_\varepsilon^\vartheta$ for which the dynamics converge to the golden rule steady state $(\bar{w}_\varepsilon, \bar{b}_\varepsilon) \in \mathcal{M}_\varepsilon^\vartheta$. To analyze the long-run welfare effects of debt with non-degenerate shocks, it seems natural to extend the golden rule concept by measuring consumer welfare at some stationary solution of the state dynamics. The latter corresponds to an invariant probability distribution on $\mathbb{V}$ which extends the deterministic concept of a steady state. As argued above, however, even if $\mathcal{E}_x^\vartheta = \emptyset$, stable subsets of $\mathbb{V}_+$ – which can be associated with invariant distributions, cf. Wang (1993) – generically fail to exist and equilibria are asymptotically debtless and hence independent of ϑ. Therefore, neither the optimum quantity of debt nor the risk-sharing effects of different interest policies can be analyzed. The present section investigates whether this may be overcome by a tax on labor income which stabilizes debt against unfavorable shocks. Using the scenario from Section 3, the idea is to choose a subset $\mathbb{A} \subset \mathbb{V}$ and design a tax policy which keeps the state in $\mathbb{A}$ for all times and under all shocks. The set $\mathbb{A}$ will be referred to as a *stabilization objective.* Note that we permit $\mathcal{E}_x^\vartheta \neq \emptyset$, i.e., some – or all– mappings $\Phi^\vartheta(\cdot;\varepsilon)$ may be expansive.

Assumption 4.1 *The stabilization objective $\mathbb{A} \subset \mathbb{V}$ satisfies the following:*

(i) There is a map $\chi_\mathbb{A} : \mathbb{R}_{++} \longrightarrow \mathbb{R}_+$ with continuous derivative $0 \leq \chi'_\mathbb{A} < 1$ and an open interval $\mathbb{W}_\mathbb{A} =]\underline{w}_\mathbb{A}, \infty[\subset \mathbb{R}_{++}$ such that $\mathbb{A} = \{(w, \chi_\mathbb{A}(w)) \,|\, w \in \mathbb{W}_\mathbb{A}\}$.

(ii) The family $(\Phi^\vartheta(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$ maps $\mathbb{A}$ into the set $\mathbb{V}_\mathbb{A} := \{(w,b) \in \mathbb{V} \,|\, w - b > d_\mathbb{A}\} \subset \mathbb{V}$ where $d_\mathbb{A} := \inf\{w - \chi_\mathbb{A}(w) \,|w \in \mathbb{W}_\mathbb{A}\} \geq 0$. That is, $\Phi^\vartheta(\mathbb{A};\varepsilon) \subset \mathbb{V}_\mathbb{A}$ for all $\varepsilon \in \mathcal{E}$.

Assumption 4.1(i) restricts the stabilization objective to smooth, one-dimensional sets. This will allow us to obtain a unique stabilization policy. The value $d_\mathbb{A}$ in (ii) represents the minimal distance between $\mathbb{A}$ and the boundary of $\mathbb{V}$ which increases with $\underline{w}_\mathbb{A}$, cf. Figure 2. Assumption 4.1(ii) therefore embodies a sustainability constraint on $\mathbb{A}$ by requiring successors of states in $\mathbb{A}$ to retain the safeguard distance $d_\mathbb{A}$ to the boundary under all shocks. Note that a minimal choice such as $\underline{w}_\mathbb{A} = \bar{b}$ in Case 1 and $\underline{w}_\mathbb{A} = 0$ in Cases 2 and 3 studied below – each implying $d_\mathbb{A} = 0$ and $\mathbb{V}_\mathbb{A} = \mathbb{V}$ – will typically violate this condition if, as in Case 3, $\chi_\mathbb{A}$ is too close to the boundary of $\mathbb{V}$ for w close to $\underline{w}_\mathbb{A}$. The general structure from Assumption 4.1(i) covers the following three special cases:

Case 1: $\chi_\mathbb{A}(w) \equiv \bar{b}$. This objective stabilizes debt at a constant level $\bar{b} \geq 0$. It is the case is studied, e.g., in Diamond (1965). Note that $\mathbb{A} \subset \mathbb{V}$ requires $\underline{w}_\mathbb{A} \geq \bar{b}$.

Case 2: $\chi_\mathbb{A}(w) = \beta w$. This policy chooses a value $\beta \in [0, 1[$ to keep the debt-to net wage ratio constant. The objective is studied, e.g., in Bohn (1998, p.11) and is similar to a constant debt-to output ratio as in de la Croix & Michel (2002).

Case 3: $\chi_\mathbb{A}(w) = \psi_\varepsilon^\vartheta(w)$. Assuming $\mathcal{E}_s^\vartheta \neq \emptyset$, this policy chooses a reference shock $\varepsilon^{\text{ref}} \in \mathcal{E}_s^\vartheta$ to stabilize the state along the stable manifold $\mathcal{M}_{\varepsilon^{\text{ref}}}^\vartheta$ from (19). Since $\mathcal{M}_{\varepsilon^{\text{ref}}}^\vartheta$ is self-supporting under $\Phi^\vartheta(\cdot;\varepsilon^{\text{ref}})$, i.e., $\Phi^\vartheta(\mathcal{M}_{\varepsilon^{\text{ref}}}^\vartheta;\varepsilon^{\text{ref}}) \subset \mathcal{M}_{\varepsilon^{\text{ref}}}^\vartheta$, stabilization taxes are zero whenever the reference shock occurs and, by a continuity argument, small for shocks close to this value. Thus, the objective seems particularly promising to keep stabilization taxes small. In particular, taxes are uniformly zero if $\mathcal{E}_x^\vartheta = \emptyset$ and the sets $\mathcal{M}_\varepsilon^\vartheta$ are independent of ε, as in the example of Section 2.

By Theorem A.1(iii) and Lemma 3.2, for $\varepsilon \in \mathcal{E}_s^\vartheta$ the map $\psi_\varepsilon^\vartheta$ defining $\mathcal{M}_\varepsilon^\vartheta$ is strictly increasing with derivative $\psi_\varepsilon^{\vartheta\prime}(w) \le -\partial_w \mathcal{K}^\vartheta(w,b)/\partial_b \mathcal{K}^\vartheta(w,b) < 1$, $w > 0$, $b = \psi_\varepsilon^\vartheta(w)$. Hence, the restrictions on $\chi'_{\mathbb{A}}$ from Assumption 4.1 are indeed satisfied in all three cases.

Tax policy. In the sequel, let a debt policy $\pi = (\vartheta, \mathbb{A})$ consisting of some interest policy ϑ and a stabilization objective $\mathbb{A} \subset \mathbb{V}$ satisfying Assumption 4.1 be given. As a first step, we seek to establish existence of a tax policy such that $(w_t, b_t) \in \mathbb{A}$ for all t with probability one. Consider an arbitrary period $t \ge 0$. Let $w_t^g > 0$ be the gross wage defined by (2) and denote by $b_t^g := r_t^\star b_{t-1} \ge 0$ the given outstanding payments on previous debt. Assume that $(w_t^g, b_t^g) \in \mathbb{V}_{\mathbb{A}}$. By Assumption 4.1(ii), this holds if $(w_{t-1}, b_{t-1}) \in \mathbb{A}$. Let $\tau_t := \tau_t^y$ be the tax on labor income to be determined. Each choice $\tau_t \le b_t^g$ defines net labor income $w_t = w_t^g - \tau_t$ and current debt $b_t = b_t^g - \tau_t$ corresponding to the number of bonds issued in period t. If $\tau_t > 0$, the tax revenues are used to pay down part of the outstanding debt. If $\tau_t < 0$, young consumers receive a subsidy on their wage income financed by issuing additional debt. The following result permits to uniquely determine the value τ_t such that $(w_t, b_t) = (w_t^g - \tau_t, b_t^g - \tau_t) \in \mathbb{A}$.

Lemma 4.1 *In addition to Assumption 4.1, suppose* $\lim_{w\to\infty} \chi'_{\mathbb{A}}(w) \neq 1$*. Then, for all* $(w,b) \in \mathbb{V}_{\mathbb{A}}$ *there is a unique* τ *such that* $(w-\tau, b-\tau) \in \mathbb{A}$*.*

Stabilized dynamics. Under the hypotheses of Lemma 4.1 there is a map $\mathcal{T}_{\mathbb{A}} : \mathbb{V}_{\mathbb{A}} \longrightarrow \mathbb{R}$ which determines $\tau = \mathcal{T}_{\mathbb{A}}(w,b)$ for each $(w,b) \in \mathbb{V}_{\mathbb{A}}$ such that $(w-\tau, b-\tau) \in \mathbb{A}$. Specifically, $\mathcal{T}_{\mathbb{A}}(w,b) = b - \bar{b}$ in Case 1 and $\mathcal{T}_{\mathbb{A}}(w,b) = \frac{1}{1-\beta}(b - \beta w)$ in Case 2. In particular, any initial state in $\mathbb{V}_{\mathbb{A}}$ can be tax-adjusted to lie in $\mathbb{A}$. Thus, for $(w_0, b_0) \in \mathbb{A}$, the stabilized dynamics derived from (17a,b) are given by $\Psi^\pi = (\Psi_w^\pi, \Psi_b^\pi) : \mathbb{A} \times \mathcal{E} \longrightarrow \mathbb{A}$

$$w_{t+1} = \Psi_w^\pi(w_t, b_t; \varepsilon_{t+1}) := \Phi_w^\vartheta(w_t, b_t; \varepsilon_{t+1}) - \mathcal{T}_{\mathbb{A}}(\Phi^\vartheta(w_t, b_t; \varepsilon_{t+1})) \qquad (20a)$$

$$b_{t+1} = \Psi_b^\pi(w_t, b_t; \varepsilon_{t+1}) := \Phi_b^\vartheta(w_t, b_t; \varepsilon_{t+1}) - \mathcal{T}_{\mathbb{A}}(\Phi^\vartheta(w_t, b_t; \varepsilon_{t+1})). \qquad (20b)$$

Figure 2 illustrates Assumption 4.1 and the stabilized dynamics. Since $b_t = \chi_{\mathbb{A}}(w_t)$ for

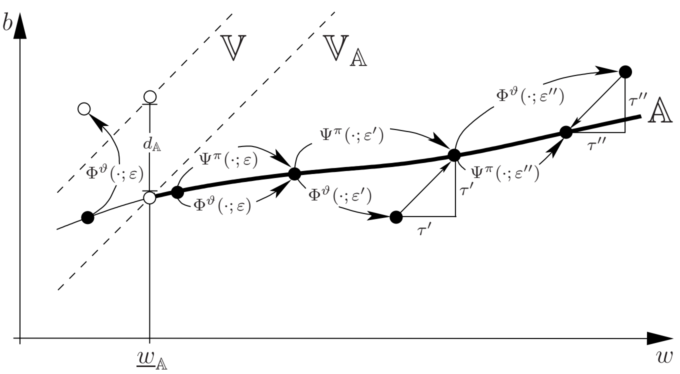


Figure 2: Stabilized dynamics on the set $\mathbb{A}$.

all t, the system (20a,b) is equivalent to the one-dimensional system $\phi^\pi : \mathbb{W}_{\mathbb{A}} \times \mathcal{E} \longrightarrow \mathbb{W}_{\mathbb{A}}$

$$w_{t+1} = \phi^\pi(w_t; \varepsilon_{t+1}) := \Phi_w^\vartheta(w_t, \chi_{\mathbb{A}}(w_t); \varepsilon_{t+1}) - \mathcal{T}_{\mathbb{A}}(\Phi^\vartheta(w_t, \chi_{\mathbb{A}}(w_t); \varepsilon_{t+1})). \qquad (21)$$

To characterize the stabilized dynamics, Lemma 4.2 establishes properties of the map ϕ^π using the following additional restrictions. In (P8) we let $\zeta_\vartheta : \mathcal{E} \longrightarrow \mathbb{R}_{++}$, $\zeta_\vartheta(\varepsilon) := \frac{\vartheta(\varepsilon)}{\varepsilon}$.

(P7) $\chi'_{\mathbb{A}}(w) \le \dfrac{\partial_w \Phi_w^\vartheta(w,b;\varepsilon) - \partial_w \Phi_b^\vartheta(w,b;\varepsilon)}{\partial_b \Phi_b^\vartheta(w,b;\varepsilon) - \partial_b \Phi_w^\vartheta(w,b;\varepsilon)}$ $\forall (w,b,\varepsilon) \in \mathbb{A} \times \mathcal{E}$ *(P8)* ζ_ϑ is non-increasing.

Lemma 4.2 *Under (P1)–(P4) and the hypotheses of Lemma 4.1, the following holds:*

(i) $\phi^\pi(\cdot;\varepsilon) : \mathbb{W}_{\mathbb{A}} \longrightarrow \mathbb{W}_{\mathbb{A}}$ *is weakly increasing for all* $\varepsilon \in \mathcal{E}$ *if* $(\vartheta, \mathbb{A})$ *satisfies (P7).*

(ii) $\phi^\pi(w;\cdot) : \mathcal{E} \longrightarrow \mathbb{W}_{\mathbb{A}}$ *is strictly increasing for all* $w \in \mathbb{W}_{\mathbb{A}}$ *if* ϑ *satisfies (P8).*

Since $0 < \partial_w \Phi_w^\vartheta(w,b;\varepsilon) < -\partial_b \Phi_w^\vartheta(w,b;\varepsilon)$ and $0 < -\partial_w \Phi_b^\vartheta(w,b;\varepsilon) < \partial_b \Phi_b^\vartheta(w,b;\varepsilon)$ by Lemma 3.2, (P7) strengthens the restriction $\chi'_{\mathbb{A}} < 1$. It ensures that debt does not increase too fast along $\mathbb{A}$ in the sense that $w \mapsto \Phi_w^\vartheta(w,\chi_{\mathbb{A}}(w),\varepsilon) - \Phi_b^\vartheta(w,\chi_{\mathbb{A}}(w),\varepsilon)$ is increasing for all $w \in \mathbb{W}_{\mathbb{A}}$ and $\varepsilon \in \mathcal{E}$. It is clear that (P7) always holds in Case 1 and in Case 2 if β is not too large. A sufficient condition for Case 3 is stated next.

Lemma 4.3 *Let* $\chi_{\mathbb{A}} = \psi^\vartheta_{\varepsilon_{\mathrm{ref}}}$ *for* $\varepsilon_{\mathrm{ref}} \in \mathcal{E}^\vartheta_s$*. If* $\chi'_{\mathbb{A}}(w) \leq \frac{\varepsilon\,\vartheta(\varepsilon_{\mathrm{ref}})}{\vartheta(\varepsilon)\,\varepsilon_{\mathrm{ref}}}$ *for all* $(w,\varepsilon) \in \mathbb{W}_{\mathbb{A}} \times \mathcal{E}$*, then (P7) holds.*

The class $\vartheta_\lambda(\varepsilon) = \lambda\varepsilon + (1-\lambda)\bar{\varepsilon}$, $\varepsilon \in \mathcal{E}$, $\lambda \in [0,1]$, $\bar{\varepsilon} := \mathbb{E}_\nu[\varepsilon_t]$ studied in Section 5 satisfies the condition in Lemma 4.3 directly for $\lambda = 1$ (CE) and for all $\lambda \in [0,1]$ if $\chi'_{\mathbb{A}}(w) \leq \frac{\varepsilon_{\min}}{\varepsilon_{\max}}$ for all $w \in \mathbb{W}_{\mathbb{A}}$, i.e., if $[\varepsilon_{\min}, \varepsilon_{\max}]$ is not too large. Clearly, (P8) holds for all $\lambda \leq 1$.
The following final result of this section establishes conditions under which a unique invariant distribution of the dynamics (21) exists. This provides the basis for studying the long-run welfare effects of debt as motivated above. For a formal definition of the employed concepts, the reader is referred to Brock & Mirman (1972) and Wang (1993).

Theorem 4.1 *Let* ϕ^π *satisfy the monotonicity properties stated in Lemma 4.2. Suppose (a) there exists* $\varepsilon_0 \in \mathcal{E}$ *such that* $\phi^\pi(\cdot;\varepsilon_0)$ *possesses a unique fixed point which is stable (b)* $\lim_{w\to\infty} \phi^\pi(w;\varepsilon_{max})/w < 1 < \lim_{w\to\underline{w}_{\mathbb{A}}} \phi^\pi(w;\varepsilon_{min})/w$*. Then, the following holds:*

(i) There exists a unique stable set $\overline{\mathbb{W}}^\pi \subset \mathbb{W}_{\mathbb{A}}$ *for the family* $\phi^\pi = (\phi^\pi(\cdot;\varepsilon))_{\varepsilon\in\mathcal{E}}$*.*

(ii) There exists a unique invariant distribution μ^π *of the dynamical system (21) which is supported on* $\overline{\mathbb{W}}^\pi$ *and which is stable in the weak convergence sense.*

Condition (a) holds directly in Case 3 (for $\varepsilon_0 = \varepsilon_{\mathrm{ref}}$). If $\chi_{\mathbb{A}} \equiv 0$, (a) holds under Assumption 3.1 (as $\phi^\pi = \phi^0$) which also ensures that (a) holds in Cases 1 and 2 for $\bar{b}$ and β sufficiently small by the implicit function theorem. As (a) implies $\lim_{w\to\infty} \phi^\pi(w;\varepsilon_0)/w < 1 < \lim_{w\to\underline{w}_{\mathbb{A}}} \phi^\pi(w;\varepsilon_0)/w$, (b) generally holds if the range of shocks is not too large.

5 Optimal Debt Policies

The present section studies the welfare effects of alternative debt policies and uses the result from Theorem 4.1 to develop a long-run welfare criterion. For simplicity, consider the class of interest policies $\vartheta_\lambda(\varepsilon) := \lambda\varepsilon + (1-\lambda)\bar{\varepsilon}$, $\varepsilon \in \mathcal{E}$ permitting to gradually increase the risk on debt investments by increasing λ. For $\lambda = 0$, debt is riskless while $\lambda = 1$ implies capital-equivalent debt. By abuse of notation, write $\mathcal{K}(w,b,\lambda) := \mathcal{K}^{\vartheta_\lambda}(w,b)$, etc.

Interim welfare. Consider first the lifetime utility of a generation conditional on their net income $w > 0$, current debt $b \geq 0$ and the interest policy ϑ_λ, $\lambda \in [0,1]$. Let $c^y(w,b,\lambda) := w - b - \mathcal{K}(w,b,\lambda)$ and $c^o(w,b,\lambda,\varepsilon') := b\,\vartheta_\lambda(\varepsilon')\mathcal{Z}(w,b,\lambda) + \mathcal{K}(w,b,\lambda)\mathcal{R}(\mathcal{K}(w,b,\lambda),\varepsilon')$ denote first and planned second period consumption. These define *interim utility*

$$V(w,b,\lambda) := u(c^y(w,b,\lambda)) + \mathbb{E}_\nu[v(c^o(w,b,\lambda,\cdot))]. \tag{22}$$

Theorem 5.1 *Under (P1)–(P4), the following holds for each $w > 0$:*

(i) The map $b \mapsto V(w, b, \lambda)$ is strictly increasing on $[0, w[$ for all $\lambda \in [0, 1]$.

(ii) The map $\lambda \mapsto V(w, b, \lambda)$ is strictly decreasing on $[0, 1]$ for all $b > 0$.

Theorem 5.1 shows two key properties. Firstly, at the interim stage, young consumers benefit from any additional increase in current debt not exceeding their net income. The intuition is that higher debt investment could one-for one replace capital investment leaving current consumption invariant while increasing second-period consumption due to increased returns on both investments. Secondly, consumers dislike debt indexed to risk since any increase in λ decreases the possibility to diversify risk. Thus, a policy involving low and risky debt would never be supported by current generations.

Long-run welfare. The interim perspective clearly fails to take into account the capital accumulation process and the debt burden that future generations will have to bear. To develop a criterion which incorporates these effects, let $\lambda \in [0, 1]$ and a stabilization objective $\mathbb{A} \subset \mathbb{V}$ be given. Assuming that the hypotheses of Theorem 4.1 are satisfied, the choice of policy $\pi = (\vartheta_\lambda, \mathbb{A})$ yields a random variable (w^π, b^π) whose distribution on $\mathbb{V}$ is defined by μ^π. The associated long-run expected utility then takes the form

$$U(\lambda, \mathbb{A}) := \int_{\mathbb{W}^\pi} V(w, \chi_{\mathbb{A}}(w), \lambda)\mu^\pi(dw). \tag{23}$$

The value $U(\pi)$ can be interpreted as the interim utility that generations attain on average under policy π. Note that the interest policy affects utility directly at the interim stage and, in combination with $\mathbb{A}$, through its impact on the long-run distribution. The latter incorporates the trade-off between higher current debt and lower future incomes. Note that for $\chi_{\mathbb{A}} \equiv 0$, (23) yields the long-run utility at the trivial equilibrium which is independent of λ and provides a natural reference point for any welfare analysis of debt.

Simulation results. Unlike the interim welfare effects in Theorem 5.1, a theoretical characterization of the invariant distribution μ^π depending on policy π seems not possible. For this reason, the remainder presents results from a numerical simulation study which quantifies the long-run welfare effects and further properties of alternative policies. Consider the scenario from Section 3 with CRRA utilities $u(c) = c^\theta$, $v(c) = \gamma u(c)$, CES technology $g(k) = [1 - A + Ak^\alpha]^{\frac{1}{\alpha}}$, and three shocks $\mathcal{E} = \{\varepsilon^{\min}, \varepsilon^{\text{med}}, \varepsilon^{\max}\}$ drawn with probabilities $p^{\min}$, p^{med}, and $p^{\max}$. For the values listed in Table 1, $\mathcal{E}_s^{\vartheta_\lambda} \equiv \mathcal{E}$ implying that the trivial equilibrium is dynamically inefficient. All of the following results were found to be robust against parameter changes for which this continues to hold.[5]

Parameter	Value	Parameter	Value	Parameter	Value
$\varepsilon^{\min}$	.9	$\varepsilon^{\max}$	1.1	A, α, θ	.5
ε^{med}	1	$p^{\min}$, p^{med}	1/3	γ	1

Table 1: Parameter set used in the simulations.

[5] All simulations iterate the system for $T = 35$ periods. For this length, convergence of expected utilities and other variables computed as averages of $N = 5000$ different noise paths is established. To verify the numerical results, the reader is invited to download the simulation data and the $C++$ simulation files from my website `http://www.marten-hillebrand.de/research/TC/TC.htm`.

The study compares the three stabilization objectives from Section 4 under different values for λ. For each scenario, Theorem 4.1 is verified to hold and an optimal stabilization policy is computed. This amounts to determining an optimal debt level $\bar{b}^\star(\lambda) \geq 0$ in Case 1, an optimal debt-to wage ratio $\beta^\star(\lambda) \in [0,1[$ in Case 2, and an optimal reference shock $\varepsilon^\star_{\text{ref}}(\lambda) \in \mathcal{E}_s^{\vartheta_\lambda}$ in Case 3. These values turn out to be uniquely determined and imply a similar debt-to net income ratio of $\approx 16.5\%$ on average in each case. Table 2 reports the associated increases in utility (23) relative to the trivial equilibrium.

$\lambda =$	0 (safe debt)	0.25	0.5	0.75	1 (CE debt)
Case 1:	.820%	.823%	.826%	.827%	.828%
Case 2:	.805%	.809%	.812%	.815%	.815%
Case 3:	.816%	.820%	.823%	.825%	.826%

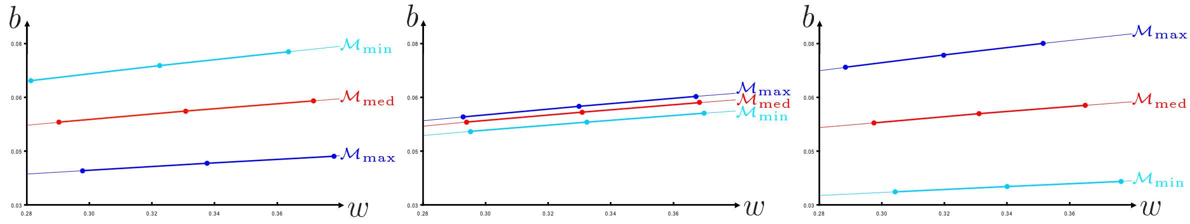
Table 2: Long-run welfare increase under different debt policies.

Each policy yields a positive welfare gain which is throughout highest in Case 1, closely followed by Case 3 and least under Case 2. Interestingly, welfare increases monotonically with λ in each case which shows that the negative effect of risk indexation at the interim stage is overcompensated by the corresponding impact on the long-run distribution. Intuitively, a riskless debt return shifts risk from old to young (cf. Bohn (1998)) while CE debt is essentially risk-neutral which seems favorable according to the previous result. Observe, however, that the associated welfare gain is rather small ($\leq .01\%$ in each case) compared to the overall increase. Thus, determining the optimal stabilization objective seems more important than the interest policy. With reference to the introduction, this indicates that the crowding-out effect of debt dominates the risk-sharing effect. Krüger & Kübler (2006) note a similar observation in the context of Social Security.
The interest policy, however, crucially affects the size of stabilization taxes. This is shown in Table 3 which displays absolute taxes $|\tau_t|$ relative to gross income w_t^g.

$\lambda =$	0 (safe debt)	0.25	0.5	0.75	1 (CE debt)
Case 1:	.534%	.58%	.72%	1.00%	1.27%
Case 2:	1.65%	1.31%	1.12%	.97%	.82%
Case 3:	.532%	.22%	.08%	.38%	.68%

Table 3: Average absolute stabilization taxes as percentages of gross income.

In Case 1, taxes are least for $\lambda = 0$ and increase monotonically with λ while the converse holds in Case 2. This seems intuitive because under safe debt, the level b_t^g becomes independent of production risk ε_t while under CE debt this is true of the gross-debt to wage ratio b_t^g/w_t^g. Moreover, taxes are least in Case 3 confirming our earlier suspicion that taxes are small if the inherent stabilizing forces of the dynamical system are exploited. Under this stabilization objective, a unique $\lambda^\star \in]0,1[$ can be determined for which taxes become minimal ($\approx .08\%$ if $\lambda^\star = .5$ and even $.04\%$ if $\lambda^\star = .4$).
To provide some intuition for this last result, Figure 3 portrays the location of the stable manifolds (19) in the state space along which stabilization takes place in Case 3. The bold sections represent the support of the invariant distribution which is bordered by the (smallest and largest) fixed points of $\Psi^\pi(\cdot;\varepsilon)$ respectively $\phi^\pi(\cdot;\varepsilon)$ which are also



(a) $\lambda = 0$ (riskless) (b) $\lambda = 0.5$ (c) $\lambda = 1$ (CE)

Figure 3: Stable manifolds $\mathcal{M}_{\min} := \mathcal{M}^{\vartheta_\lambda}_{\varepsilon_{\min}}$, $\mathcal{M}_{\text{med}} := \mathcal{M}^{\vartheta_\lambda}_{\varepsilon_{\text{med}}}$, $\mathcal{M}_{\max} := \mathcal{M}^{\vartheta_\lambda}_{\varepsilon_{\max}}$.

depicted. Intuitively, if the shock $\varepsilon_t = \varepsilon$ occurs at time t, taxes τ_t are large (in absolute value) if the previous state (w_{t-1}, b_{t-1}) is far away from the set $\mathcal{M}^{\vartheta}_{\varepsilon}$ and small for (w_{t-1}, b_{t-1}) close to $\mathcal{M}^{\vartheta}_{\varepsilon}$. As a consequence, taxes are least in Figure 3(b) where the stable manifolds are close together. Ideally, they would coincide as in the example of Section 2 and there would be no need for stabilization. Albeit this can not be achieved in the present case, the interest policy can be chosen as in Figure 3(b) such that taxes become negligible and debt is 'nearly' persistent resembling the bubble in Tirole (1985).

6 Conclusions

The results of this paper suggest that any sustainable debt policy must be accompanied by a tax policy which stabilizes debt against unfavorable random shocks. Based on this insight, a welfare criterion was suggested which measures long-run consumer welfare at the stabilized equilibrium permitting to simultaneously determine an optimal stabilization objective and an optimal interest policy. For a situation where the debtless equilibrium is inefficient, numerical findings indicate that the long-run optimal policy involves moderate levels of debt with returns fully indexed to production risk. The analysis also revealed that such a policy is never in the interest of current generations who prefer large and riskless debt. This conflict might explain why many countries have large debt and offer a riskless return despite both fails to be optimal in the long run.
Since unstabilized equilibria were shown to be asymptotically debtless, the findings of this paper also suggest that persistent asset bubbles as studied in Tirole (1985) can generically not occur in stochastic OLG models even if the trivial equilibrium is inefficient. In this regard, several deterministic studies (e.g., Kunieda (2008)) have introduced credit market frictions to explain the emergence of bubbles in OLG models where the bubbleless equilibrium is dynamically efficient. Such imperfections could also explain existence of equilibria with debt in situations where the trivial equilibrium is efficient.

A Mathematical Appendix

A.1 Proof of Lemma 1.1

For $t \geq 0$, define taxes $\tau^y_t := w^g_t - c^y_t - k_{t+1} - b_t$ and $\tau^o_t := b_{t-1} r^\star_t + k_t r_t - c^o_t$ which are feasible in the sense of Definition 1.1(ii). Using the corresponding expressions for c^y_t

and c_t^o together with (2) and (3) in the aggregate feasibility condition shows that debt evolves according to equation (4). Since Definition 1.1(i) is satisfied by assumption, it remains to show that (b_t, s_t) solves (9). Since $s_t > 0$ and there are no short-sale constraints, it suffices to show that the first-order conditions are satisfied. This follows from the intertemporal efficiency condition and (10) by direct substitution. ■

A.2 Proof of Lemma 2.1

Given $w > b \geq 0$, let $c^o(k, b, \varepsilon) := \mathcal{R}(k; \varepsilon)(b + k)$. By (P1), the derivative[6] satisfies

$$\begin{aligned} \partial_k H(k; w, b) &= -u''(w - b - k) - \mathbb{E}_\nu \left[\mathcal{R}(k; \varepsilon)^2 v''\big(c^o(k, b, \cdot)\big)\right] \\ &\quad -\mathbb{E}_\nu \Big[f_{kk}(k; \cdot)\big(v'\big(c^o(k, b, \cdot)\big) + c^o(k, b, \cdot) v''\big(c^o(k, b, \cdot)\big)\big)\Big] > 0. \end{aligned} \tag{A.1}$$

Thus, $H(\cdot; w, b)$ is strictly increasing and can have at most one zero in $]0, w - b[$. The arguments of Wang (1993) imply existence of a zero for $b = 0$ which is unique by (A.1). Since $\partial_k H(k; w, 0) > 0$, a zero exists also for $b > 0$ sufficiently small by the implicit function theorem (IFT). Let

$$b^{\max}(w) := \sup\Big\{b \in]0,\ w[\ |\ H(k; w, b) = 0 \ \text{ for some } k \in]0, w - b[\Big\}. \tag{A.2}$$

Note that $b^{\max}(w)$ being the supremum of a non-empty set bounded by w is well-defined for all $w > 0$ and the map $w \mapsto b^{\max}(w)$ is continuous since H is continuous. We claim that H has a zero for each $b \in]0, b^{\max}(w)[$. By contradiction, suppose this fails to hold for some $0 < b' < b^{\max}(w)$. As $\lim_{k \to w - b'} H(k; w, b') = \infty$ by (8), $H(k; w, b') > 0$ for all $0 < k < w - b'$. The derivative with respect to b satisfies

$$\partial_b H(k; w, b) = -u''(w - b - k) - \mathbb{E}_\nu \left[\mathcal{R}(k; \cdot)^2 v''\big(c^o(k, b, \cdot)\big)\right] > 0. \tag{A.3}$$

Let $b'' > b'$. By (A.3), $H(k; w, b'') > H(k; w, b') > 0$ for all $0 < k < w - b'' < w - b'$. Hence, $H(\cdot; w, b'')$ has no zero for any $b'' > b'$. But then $b^{\max}(w) \leq b'$, a contradiction. Finally, note that $\lim_{k \to 0} c^o(k, b, \varepsilon) \geq \lim_{k \to 0} b\mathcal{R}(k; \varepsilon) = \infty$ for each $\varepsilon \in \mathcal{E}$ which implies

$$\lim_{k \to 0} H(k; w, b) = u'(w - b) - \lim_{k \to 0} \left(\frac{1}{b + k} \mathbb{E}_\nu \left[c^o(k, b, \cdot) v'\big(c^o(k, b, \cdot)\big)\right]\right) = -\infty \tag{A.4}$$

if (P3) holds. In this case, a zero exists for all $b < w$, i.e., $b^{\max}(w) = w$. ■

A.3 Proof of Lemma 2.2

Using (A.1) and (A.3), the partial derivatives of H defined in (11) satisfy

$$0 < -\partial_w H(k; w, b) = -u''(w - b - k) < \partial_b H(k; w, b) \leq \partial_k H(k; w, b)$$

where the last inequality holds due to (P1). Thus, by the implicit function theorem,

$$0 < \partial_w \mathcal{K}(w, b) = \frac{-\partial_w H(k; w, b)}{\partial_k H(k; w, b)} < -\partial_b \mathcal{K}(w, b) = \frac{\partial_b H(k; w, b)}{\partial_k H(k; w, b)} \leq 1.$$ ■

[6] Recall that interchanging differentiation with the expectations operator $\mathbb{E}_\nu[\cdot]$ is legitimate whenever the integrand is continuously differentiable and integration is over a compact set.

A.4 Proof of Lemma 2.3

Let $\varepsilon \in \mathcal{E}$ be fixed. For brevity, we omit the subscript ε such that $\bar{w}^0 > 0$ denotes the trivial steady state. In addition, define $\bar{k}^0 := \mathcal{K}(\bar{w}^0, 0)$ and $\underline{w} := \mathcal{W}(0; \varepsilon) \geq 0$.

(i) Let $\varepsilon \in \mathcal{E}_s$. We determine unique values $\bar{k} > 0$ and $(\bar{w}, \bar{b}) \in \mathbb{V}_+$ solving $k = \mathcal{K}(w, b)$, $w = \mathcal{W}(k, \varepsilon)$, and $\mathcal{R}(k, \varepsilon) = 1$. Since $\lim_{k \to 0} \mathcal{R}(k, \varepsilon) = \infty$ and $\mathcal{R}(\bar{k}^0, \varepsilon) < 1$, the last condition has a solution $\bar{k} \in]0, \bar{k}^0[$ which is unique by strict concavity of $f(\cdot; \varepsilon)$ and determines $\bar{w} := \mathcal{W}(\bar{k}, \varepsilon) < \bar{w}^0$. Finally, we determine the value $\bar{b}$ as a solution to $\bar{w} = \mathcal{W}(\mathcal{K}(\bar{w}, b), \varepsilon)$. By Lemma 2.2, there can be at most one such solution. Using (13), uniqueness and stability of $\bar{w}^0$ imply $\phi_0(w; \varepsilon) > w$ for all $w \in]\underline{w}, \bar{w}^0[$. Hence, $\bar{w} < \bar{w}^0$ implies $\lim_{b \to 0} \mathcal{W}(\mathcal{K}(\bar{w}, b), \varepsilon) = \mathcal{W}(\mathcal{K}(\bar{w}, 0), \varepsilon) > \bar{w}$. Since $\lim_{b \to b^{\max}(\bar{w})} \mathcal{K}(\bar{w}, b) = 0$, $\lim_{b \to b^{\max}(\bar{w})} \mathcal{W}(\mathcal{K}(\bar{w}, b), \varepsilon) = \underline{w} < \bar{w}$ proving that a unique non-trivial steady state exists. The Jacobian at the steady state computes

$$\bar{J} := D\Phi(\bar{w}, \bar{b}; \varepsilon) = \begin{bmatrix} -\bar{k} f_{kk}(\bar{k}; \varepsilon) \partial_w \mathcal{K}(\bar{w}, \bar{b}) & -\bar{k} f_{kk}(\bar{k}; \varepsilon) \partial_b \mathcal{K}(\bar{w}, \bar{b}) \\ \bar{b} f_{kk}(\bar{k}; \varepsilon) \partial_w \mathcal{K}(\bar{w}, \bar{b}) & 1 + \bar{b} f_{kk}(\bar{k}; \varepsilon) \partial_b \mathcal{K}(\bar{w}, \bar{b}) \end{bmatrix}.$$

By Lemma 2.2, the determinant and trace satisfy $\det \bar{J} = -\bar{k} f_{kk}(\bar{k}; \varepsilon) \partial_w \mathcal{K}(\bar{w}, \bar{b}) > 0$ and $\mathrm{tr}\bar{J} = 1 + \det \bar{J} + \bar{b} f_{kk}(\bar{k}; \varepsilon) \partial_b \mathcal{K}(\bar{w}, \bar{b}) > 1 + \det \bar{J}$. The latter inequality implies $0 \leq (1 - \mathrm{tr}\bar{J}/2)^2 = 1 - \mathrm{tr}\bar{J} + (\mathrm{tr}\bar{J})^2/4 < -\det \bar{J} + (\mathrm{tr}\bar{J})^2/4$ ensuring real and distinct Eigenvalues of $\bar{J}$. By Galor (2007, p.88), these properties imply saddle-path stability.

(ii) Let $\varepsilon \in \mathcal{E}_x$. By contradiction, suppose there exists $(\tilde{w}, \tilde{b}) \in \mathbb{V}_+$ such that $(\tilde{w}_t, \tilde{b}_t) := \Phi^t(\tilde{w}, \tilde{b}; \varepsilon) \in \mathbb{V}$ for all $t \geq 0$. Use (13) to define $\hat{w}_t := \phi_0^t(\tilde{w}; \varepsilon)$, $t \geq 0$. Lemma 2.2 implies $\hat{w}_t > \tilde{w}_t \geq b^{\max}(\tilde{w}_t) > \tilde{b}_t > 0$ for all t. Since $\lim_{t \to \infty} \hat{w}_t = \bar{w}^0$ by Assumption 2.1, continuity of $\mathcal{R}(\cdot; \varepsilon)$ and $\mathcal{K}$ imply existence of $T > 0$ such that $\mathcal{R}(\mathcal{K}(\hat{w}_t, 0); \varepsilon) > 1$ for all $t \geq T$ implying $\tilde{b}_{t+1}/\tilde{b}_t = \mathcal{R}(\mathcal{K}(\tilde{w}_t, \tilde{b}_t); \varepsilon) > \mathcal{R}(\mathcal{K}(\hat{w}_t, 0); \varepsilon) > 1$. Hence, $\lim_{t \to \infty} \tilde{b}_t =: B$ exists where $\tilde{b}_T < B \leq \infty$. Suppose $B < \infty$. Then, $\lim_{t \to \infty} \mathcal{R}(\mathcal{K}(\tilde{w}_t, \tilde{b}_t); \varepsilon) = 1$, contradicting $\lim_{t \to \infty} \mathcal{R}(\mathcal{K}(\tilde{w}_t, \tilde{b}_t); \varepsilon) \geq \lim_{t \to \infty} \mathcal{R}(\mathcal{K}(\hat{w}_t, \tilde{b}_t); \varepsilon) = \mathcal{R}(\mathcal{K}(\bar{w}^0, B); \varepsilon) > \mathcal{R}(\mathcal{K}(\bar{w}^0, 0); \varepsilon) > 1$. Thus, $B = \infty$ which contradicts $\tilde{b}_t < \hat{w}_t$ for all t. ■

A.5 Properties of the Stable Manifold

This section establishes properties of the stable manifold $\mathcal{M}_\varepsilon^\vartheta$ in (19). Especially the first part draws heavily on results by Tirole (1985). A somewhat related analysis may be found in Galor (1992) from which several ideas are used. For a definition of manifolds, etc. the reader is referred to Villanacci et al. (2002). While the formal arguments adopt the setup and notation of Section 3, neither the multiplicative structure of f nor the additional assumptions (P2)–(P6) are used. Therefore, Theorem A.1 also applies for the scenario of Section 2 under the hypotheses of Lemma 2.3 where the stable manifold $\mathcal{M}_\varepsilon$ is defined as in (14) and the state space is the open set $\mathbb{V}_+ = \{(w, b) \in \mathbb{R}^2_{++} \,|b < b^{\max}(w)\}$.

Theorem A.1 *Given ϑ, let the hypotheses of Lemma 3.3 be satisfied. In addition, suppose (P9) $\lim_{c \to \infty} u'(c) = 0$. Then, for each $\varepsilon \in \mathcal{E}_s^\vartheta$ the following holds:*

(i) The set $\mathcal{M}_\varepsilon^\vartheta$ defined in (19) is the graph of a map $\psi_\varepsilon^\vartheta : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$.

(ii) The map $\psi_\varepsilon^\vartheta$ is C^1, strictly increasing, and satisfies $\lim_{w \to 0} \psi_\varepsilon^\vartheta(w) = 0$.

(iii) The derivative satisfies ${\psi_\varepsilon^\vartheta}'(w) \leq q(w) := -\frac{\partial_w \mathcal{K}^\vartheta(w, \psi_\varepsilon^\vartheta(w))}{\partial_b \mathcal{K}^\vartheta(w, \psi_\varepsilon^\vartheta(w))} < 1$ for all $w > 0$.

Proof: Fix ϑ and $\varepsilon \in \mathcal{E}_s^\vartheta$ and suppress these parameters writing $\Phi = \Phi^\vartheta(\cdot;\varepsilon)$, $\mathcal{M} = \mathcal{M}_\varepsilon^\vartheta$, etc. Thus, $(\bar{w}^0, 0)$ and $(\bar{w}, \bar{b})$ denote the unique trivial and non-trivial steady state of Φ, respectively. The following arguments employ Lemmata 3 to 11 in Tirole (1985).[7] Note that our setup corresponds to his no-rent case where $R = 0$ and $a_0 = b_0$.
(i) For $w_0 > 0$, let $\mathbb{B} := \{b \,|\Phi^n(w_0, b) \in \mathbb{V}\ \forall n \geq 1\}$, $\mathbb{B}_0 := \{b \in \mathbb{B}|\ \lim_{n\to\infty} \Phi^n(w_0, b) = (\bar{w}^0, 0)\}$, $\mathbb{B}_+ := \{b \in \mathbb{B}|\ \lim_{n\to\infty} \Phi^n(w_0, b) = (\bar{w}, \bar{b})\}$. By Tirole (1985), $\mathbb{B}$ is a convex set (Lemma 6) and right-closed (Lemma 10). Combined with his Lemma 4 implies that $\mathbb{B} =]0, \hat{b}_0]$ for some $\hat{b}_0 > 0$. Moreover, $\mathbb{B}_0$ is right open (Lemma 9), $\mathbb{B}_+$ is at most single-valued (Lemma 5) and $\mathbb{B} = \mathbb{B}_0 \cup \mathbb{B}_+$ (Lemma 3). Hence, $\mathbb{B}_+ = \{\hat{b}_0\}$. Since w_0 was arbitrary, this implies existence of a map $\psi : \mathbb{R}_{++} \longrightarrow \mathbb{R}_{++}$ such that $\mathcal{M} = \mathrm{graph}(\psi)$.
(ii) Tirole's Lemma 11 implies that ψ is strictly increasing. To establish smoothness of ψ, let $\underline{w} := \mathcal{W}(0;\varepsilon) \geq 0$, $\bar{w}^\infty := \lim_{k\to\infty} \mathcal{W}(k;\varepsilon) \leq \infty$ and $\mathbb{G} :=]\underline{w}, \bar{w}^\infty[\times \mathbb{R}_{++}$.
The remainder draws on the following auxiliary result.

Lemma A.1 *Under (P9), the map Φ defined in (17a,b) is a C^1-diffeomorphism between the sets $\mathbb{V}_+$ and $\mathbb{G}$.*

Proof of Lemma A.1.
Given some $(w', b') \in \mathbb{G}$ we determine a unique $(w, b) \in \mathbb{V}_+$ such that $\Phi(w, b) = (w', b')$. The condition $w' = \Phi_w(w, b)$ determines a unique $k' = \mathcal{K}(w, b)$ such that $w' = \mathcal{W}(k';\varepsilon)$. The value $z' = \mathcal{Z}(w, b)$ then follows from the first order conditions $\mathbb{E}_\nu[z'\vartheta(\cdot)v'(b'\vartheta(\cdot)/\vartheta(\varepsilon) + k'\mathcal{R}(k';\cdot))] = \mathbb{E}_\nu[\mathcal{R}(k';\cdot)v'(b'\vartheta(\cdot)/\vartheta(\varepsilon) + k'\mathcal{R}(k';\cdot))]$ from which $b = b'/(z'\vartheta(\varepsilon))$ can be inferred. Using (P9), w is the unique solution to $u'(w - b - k') = \mathbb{E}_\nu[z'\vartheta(\cdot)v'(b'\vartheta(\cdot)/\vartheta(\varepsilon) + k'\mathcal{R}(k';\cdot))]$. Hence, Φ^{-1} is a well-defined function. Φ is clearly C^1 by the IFT. To see that Φ^{-1} is C^1, it is straightforward to show from (17a,b) that the Jacobian $D\Phi(w, b)$ satisfies $\det D\Phi(w, b) > 0$ for each $(w, b) \in \mathbb{V}_+$. By the inverse function theorem, $D\Phi^{-1}(w', b') = [D\Phi(w, b)]^{-1}$ which is a continuous function. □

We first show that $\mathcal{M}$ is a one-dimensional C^1-manifold. By the *Stable Manifold Theorem* (cf. Nitecki (1971)), the locally stable set $\mathcal{M}^{\mathrm{loc}} := \{(w, b) \in \mathbb{V}_+ \,|\Phi^n(w, b) \in U\ \forall n \geq 1 \wedge \lim_{n\to\infty} \Phi^n(w, b) = (\bar{w}, \bar{b})\}$ is a one-dimensional manifold as smooth as Φ. Here $U \subset \mathbb{V}_+ \cap \mathbb{G}$ is an open neighborhood of $(\bar{w}, \bar{b})$. By Nitecki (1971, p.89) or Galor (1992, Definition 4, p.1371), the globally stable manifold obtains as $\mathcal{M} = \cup_{n\geq 0}\Phi^{-n}(\mathcal{M}^{\mathrm{loc}})$. Exploiting Lemma A.1, $\mathcal{M}$ inherits the smoothness of $\mathcal{M}^{\mathrm{loc}}$ and is thus a one-dimensional C^1-manifold. The same arguments are used in Galor (1992, Corollary 3, p.1371).
We show that ψ is continuous. Since $\mathcal{M}$ is C^1, there exists an open neighborhood $\mathcal{N} \subset \mathcal{M}$ of $\bar{x} := (\bar{w}, \bar{b})$, an open subset $\mathcal{U} \subset \mathbb{R}$ and a C^1-diffeomorphism $\varphi : \mathcal{N} \longrightarrow \mathcal{U}$. W.l.o.g., suppose $\mathcal{U}$ is an interval and $\mathcal{N} \subset \mathcal{M}^{\mathrm{loc}}$ (otherwise, choose an open interval $\tilde{\mathcal{U}} \subset \mathcal{U}$ containing $\varphi(\bar{x})$ small enough such that $\varphi^{-1}(\tilde{\mathcal{U}}) \subset \mathcal{M}^{\mathrm{loc}}$ and switch to $\tilde{\varphi} := \varphi_{|\tilde{\mathcal{N}}}$ where $\tilde{\mathcal{N}} := \varphi^{-1}(\tilde{\mathcal{U}})$). By Theorem I.4 in Dugundji (1970, p.108), $\mathcal{N} = \varphi^{-1}(\mathcal{U})$ being the image of an open and connected set under a homeomorphism is an open and connected subset of $\mathcal{M}$ containing $\bar{x}$. Let $x = (w, b) \in \mathcal{M}$ be arbitrary. By (19), $\lim_{n\to\infty} \Phi^n(x) = \bar{x}$ implying $\Phi^n(x) \in \mathcal{N}$ for n large enough, i.e., $x \in \Phi^{-n}(\mathcal{N})$. Thus, since x was arbitrary and $\mathcal{N} \subset \mathcal{M}^{\mathrm{loc}}$ we obtain $\mathcal{M} = \cup_{n\geq 0}\Phi^{-n}(\mathcal{N})$. Continuity of Φ^{-n} and Theorem I.4 in Dugundji (1970) imply that each $\Phi^{-n}(\mathcal{N})$ is a connected set containing $\bar{x}$. By Theorem I.5 in Dugundji (1970, p.108), $\mathcal{M}$ is a connected set implying continuity of ψ.
We show that ψ is C^1. Let $w_0 > 0$ be arbitrary. Since $\mathcal{M}$ is C^1, there exist an open neighborhood $\mathcal{V}_0 \subset \mathcal{M}$ of $x_0 := (w_0, \psi(w_0))$, an open set $\mathcal{U}_0 \subset \mathbb{R}$ and a C^1-diffeomorphism

[7] Previous versions of this paper contained alternative proofs which are available upon request.

$\Gamma = (\Gamma_1, \Gamma_2) : \mathcal{U}_0 \longrightarrow \mathcal{V}_0$. Define $F := (\mathrm{id}_{\mathbb{R}_{++}}, \psi) : \mathbb{R}_{++} \longrightarrow \mathcal{M}$, $w \mapsto (w, \psi(w))$ which is a homeomorphism with inverse F^{-1} equal to the projection onto the first component which is C^∞. Thus, $\Gamma_1 = F^{-1} \circ \Gamma : \mathcal{U}_0 \longrightarrow \mathcal{W}_0 := F^{-1}(\mathcal{V}_0)$ is a C^1-homeomorphism (since both F^{-1} and Γ are, cf. Proposition 12 in Villanacci et al.(2002)). The strategy is to show that Γ_1 is even a C^1-diffeomorphism, i.e., Γ_1^{-1} is C^1. Suppose $\Gamma_1'(\tilde{u}) = 0$ for some $\tilde{u} \in \mathcal{U}_0$. Let $\tilde{w} := \Gamma_1(\tilde{u})$. Since $\Gamma_2 = \psi \circ \Gamma_1$ and $\frac{\psi(w)-\psi(\tilde{w})}{w-\tilde{w}}$ takes values in the unit interval for all $w > 0$[8], $\Gamma_2'(\tilde{u}) = \Gamma_1'(\tilde{u}) \lim_{w\to\tilde{w}}(\psi(w) - \psi(\tilde{w}))/(w - \tilde{w}) = 0$. Following Villanacci et al.(2002, p.39), let Ψ be a C^1-extension of Γ^{-1} to an open set in $\mathbb{R}^2$ containing $\mathcal{V}_0$, i.e., $\Psi_{|\mathcal{V}_0} = \Gamma^{-1}$. Then, $(\Psi \circ \Gamma)'(\tilde{u}) = \partial_1\Psi(\Gamma(\tilde{u}))\Gamma_1'(\tilde{u}) + \partial_2\Psi(\Gamma(\tilde{u}))\Gamma_2'(\tilde{u}) = 0$. On the other hand, $\Psi \circ \Gamma = \mathrm{id}_{\mathcal{U}_0}$ implying $(\Psi \circ \Gamma)'(\tilde{u}) = 1$ which is a contradiction.
Conclude that $\Gamma_1'(u) \neq 0$ for all $u \in \mathcal{U}_0$. Then, by the inverse function theorem $(\Gamma_1^{-1})'(w) = 1/\Gamma_1'(\Gamma_1^{-1}(w))$ for all $w \in \mathcal{W}_0$. Since Γ_1 is C^1 and Γ_1^{-1} continuous, $(\Gamma_1^{-1})'$ is a continuous function. Thus, Γ_1 is a C^1-diffeomorphism which implies that $F = \Gamma \circ \Gamma_1^{-1}$ restricted to $\mathcal{W}_0$ is a C^1 diffeomorphism. Thus, ψ is C^1 on $\mathcal{W}_0$ and, in particular, at w_0. Observing that $\mathcal{M} \subset \mathbb{V}_+$ implies $0 < \psi(w) < w$ for all $w > 0$ completes the proof of (ii).
(iii) Suppose $\psi'(\tilde{w}) > q(\tilde{w})$ for $\tilde{w} > 0$. Then, $\psi'(\tilde{w}) > -\partial_w\mathcal{Z}(\tilde{w}, \psi(\tilde{w}))/\partial_b\mathcal{Z}(\tilde{w}, \psi(\tilde{w}))$ by (A.15). By continuity, $\tilde{\Phi}_w(w) := \mathcal{W}(\mathcal{K}(w, \psi(w)); \varepsilon)$ is locally strictly decreasing while $w \mapsto \mathcal{Z}(w, \psi(w))$ and, using (ii) $\tilde{\Phi}_b(w) := \psi(w)\mathcal{Z}(w, \psi(w))\vartheta(\varepsilon)$ are strictly increasing, respectively around $\tilde{w}$. Let $\hat{w} > \tilde{w}$ be close to $\tilde{w}$. Set $\hat{b} := \psi(\hat{w}) > \tilde{b} := \psi(\tilde{w})$. Then, $(\hat{w}, \hat{b}), (\tilde{w}, \tilde{b}) \in \mathcal{M}$ and $\hat{w}_1 := \tilde{\Phi}_w(\hat{w}) = \Phi_w(\hat{w}, \hat{b}) < \Phi_w(\tilde{w}, \tilde{b}) = \tilde{\Phi}_w(\tilde{w}) =: \tilde{w}_1$ while $\hat{b}_1 := \tilde{\Phi}_b(\hat{w}) = \Phi_b(\hat{w}, \hat{b}) > \Phi_b(\tilde{w}, \tilde{b}) = \tilde{\Phi}_b(\tilde{w}) =: \tilde{b}_1$. But $\mathcal{M}$ being self-supporting under Φ implies $(\tilde{w}_1, \tilde{b}_1) \in \mathcal{M}$ and $(\hat{w}_1, \hat{b}_1) \in \mathcal{M}$. Therefore, $\tilde{b}_1 = \psi(\tilde{w}_1)$ and $\hat{b}_1 = \psi(\hat{w}_1)$ which contradicts that ψ is strictly increasing, proving the claim. ■

A.6 Proof of Lemma 2.4

Again we show the claim for the more general scenario of Section 3 under the hypotheses of Lemma 3.3. The claim of Lemma 2.4 follows from the preface in Section A.5.
Let ϑ be given and $\varepsilon \in \mathcal{E}_s^\vartheta$ be fixed. Dependence on these parameters will be suppressed.
(i) Given $w_0 > 0$, let $\hat{b}_0 := \psi(w_0)$ and define the sets $\mathbb{B}$, $\mathbb{B}_0$ and B_+ as in the proof of Theorem A.1(i). As shown there, $\mathbb{B}_0 =]0, \hat{b}_0[$ which proves (i).
(ii) Given $w_0 > 0$, let $\tilde{b}_0 > \psi(w_0) =: \hat{b}_0$ and suppose by way of contradiction that $(\tilde{w}_t, \tilde{b}_t) := \Phi^t(w_0, \tilde{b}_0) \in \mathbb{V}$ for all $t \geq 0$. Note that $(\hat{w}_t, \hat{b}_t) := \Phi^t(w_0, \hat{b}_0) \in \mathcal{M}$ for all $t \geq 0$ and $\lim_{t\to\infty}(\hat{w}_t, \hat{b}_t) = (\bar{w}, \bar{b})$. By Lemma 3.2 and an induction argument, $\hat{w}_t > \tilde{w}_t > \tilde{b}_t > \hat{b}_t > 0$ for all $t > 0$. Define $\beta_t := \tilde{b}_t/\hat{b}_t$ to observe that $\beta_0 > 1$ and $\beta_{t+1} = \beta_t\mathcal{Z}(\tilde{w}_t, \tilde{b}_t)/\mathcal{Z}(\hat{w}_t, \hat{b}_t) > \beta_t$ for all $t \geq 0$. Hence, $\lim_{t\to\infty} \beta_t = \bar{\beta} > 1$ and $\lim_{t\to\infty} \tilde{b}_t = \bar{\beta}\bar{b} =: \tilde{b}_\infty > \bar{b}$ exist. Since $\tilde{w}_t$ remains bounded, $(\tilde{w}_t, \tilde{b}_t) \in \mathbb{V}$ for all t only if $\tilde{b}_\infty < \infty$ which requires $\lim_{t\to\infty} \mathcal{Z}(\tilde{w}_t, \tilde{b}_t) = 1/\vartheta(\varepsilon)$. But, by the previous properties and Lemma 3.2, $\lim_{t\to\infty} \mathcal{Z}(\tilde{w}_t, \tilde{b}_t) \geq \lim_{t\to\infty} \mathcal{Z}(\hat{w}_t, \tilde{b}_t) = \mathcal{Z}(\bar{w}, \tilde{b}_\infty) > \mathcal{Z}(\bar{w}, \bar{b}) = 1/\vartheta(\varepsilon)$. ■

A.7 Proof of Lemma 2.5

For $t \geq 0$, let $\zeta_t := b_t/w_t$. Using Φ_w, Φ_b gives $\zeta_{t+1} = \phi(\zeta_t) := \frac{\alpha}{1-\alpha}[\frac{\gamma}{1+\gamma} - \zeta_t]^{-1}\zeta_t$, $t \geq 0$. The map ϕ has $\bar{\zeta}$ as its unique non-trivial fixed point which is unstable. Moreover,

[8] This follows from monotonicity of ψ and a slight modification of the contradiction argument in the proof of (iii) where $\psi'(\tilde{w})$ needs to be replaced by the difference quotient $\frac{\Delta b}{\Delta w} := \frac{\psi(w)-\psi(\tilde{w})}{w-\tilde{w}}$.

$\zeta_0 < \bar{\zeta}$ implies $\lim_{t\to\infty} \zeta_t = 0$ and $\zeta_0 > \bar{\zeta}$ implies that $\phi^{t_0}(\zeta_0) > \frac{\gamma}{1+\gamma}$ for finite t_0. Hence, $b_0 = \bar{\zeta} w_0$ is necessary for $(w_0, b_0) \in \mathcal{M}_\varepsilon$. Sufficiency follows from Theorem A.1(i). ■

A.8 Proof of Lemma 3.1

Given $(w, b) \in \mathbb{V}$, let $\bar{k} := w - b > 0$. The argument $c^o(z, k, b, \varepsilon) := b\, z\, \vartheta(\varepsilon) + k\mathcal{R}(k; \varepsilon)$ will be suppressed when convenient. Suppose $b = 0$. Then, H_1^ϑ is independent of z and ϑ and $H_1^\vartheta(z, k; w, 0) = H(k; w, 0)$ for all $k \in]0, \bar{k}[$ with H defined as in (11). Hence, existence of $k' \in]0, \bar{k}[$ to satisfy $H_1^\vartheta(z, k'; w, 0) = 0$ is due to Lemma 2.1. Using k' condition $H_2^\vartheta(z, k'; w, 0) = 0$ can be solved explicitly for $z > 0$ proving the case $b = 0$. Suppose $b > 0$. The strategy is to use (16b) to eliminate z reducing (16a) to a one-dimensional problem. Let $\hat{k} \in]0, \bar{k}[$ be arbitrary. We prove existence of a unique $\hat{z} > 0$ to satisfy $H_2^\vartheta(\hat{z}, \hat{k}; w, b) = 0$. Since $\lim_{z\to\infty} c^o(z, k, b, \varepsilon) = \infty$ for each $\varepsilon \in \mathcal{E}$, (P2) implies

$$\lim_{z\to\infty} z\,\vartheta(\varepsilon)\, v'(-) = b^{-1} \lim_{z\to\infty} c^o(z, \hat{k}, b, \varepsilon) v'(-) - b^{-1}\hat{k}\mathcal{R}(\hat{k}, \varepsilon) \lim_{z\to\infty} v'(-) = \infty.$$

This being true for all $\varepsilon \in \mathcal{E}$ implies $H_2^\vartheta(z, \hat{k}; w, b) < 0$ for z sufficiently large. Since $H_2^\vartheta(0, \hat{k}; w, b) = u'(w - b - \hat{k}) > 0$ this proves existence of $\hat{z}$. To show uniqueness, we prove that $z \mapsto H_2^\vartheta(z, k; w, b)$ is strictly decreasing for all $k \in]0, \bar{k}[$. By (P1),

$$\begin{aligned} \partial_z H_2^\vartheta(z, k; w, b) &= -\mathbb{E}_\nu \big[\vartheta(\cdot)\, v'\big(c^o(z, k, b, \cdot)\big) + b\, z\, \vartheta(\cdot)^2\, v''\big(c^o(z, k, b, \cdot)\big)\big] \\ &< -\mathbb{E}_\nu \big[\vartheta(\cdot)\big(v'\big(c^o(z, k, b, \cdot)\big) + c^o(z, k, b, \cdot) v''\big(c^o(z, k, b, \cdot)\big)\big)\big] \le 0. \end{aligned} \tag{A.5}$$

These results ensure the existence of a map $\hat{Z}(\cdot; w, b) :]0, \bar{k}[\longrightarrow \mathbb{R}_{++}$ which determines $\hat{z}$ for each $\hat{k} \in]0, \bar{k}[$ such that $H_2^\vartheta(\hat{z}, \hat{k}; w, b) = 0$. By equation (3) and (P3),

$$\partial_k H_2^\vartheta(z, k; w, b) = -u''(w - b - k) - \big(1 + E_{g'}(k)\big)\mathbb{E}_\nu \big[\mathcal{R}(k; \cdot)\, z\, \vartheta(\cdot) v''(-)\big] > 0. \tag{A.6}$$

Thus, by the implicit function theorem, $\hat{Z}(\cdot; w, b)$ is C^1 and strictly increasing since for each $k \in]0, \bar{k}[$, $\partial_k \hat{Z}(k; w, b) = -\partial_k H_2^\vartheta(\hat{z}, k; w, b)/\partial_z H_2^\vartheta(\hat{z}, k; w, b) > 0$, $\hat{z} = \hat{Z}(k; w, b)$. As a second step, let $\hat{H}_1(\cdot; w, b) :]0, \bar{k}[\longrightarrow \mathbb{R}$, $\hat{H}_1(k; w, b) := H_1^\vartheta(\hat{Z}(k; w, b), k; w, b)$. We determine a unique $k' \in]0, \bar{k}[$ that solves $\hat{H}_1(k'; w, b) = 0$. Since v' is strictly decreasing, $\mathcal{R}(k; \varepsilon) v'\big(b\, \hat{Z}(k; w, b)\, \vartheta(\varepsilon) + k\mathcal{R}(k; \varepsilon)\big) < \mathcal{R}(k; \varepsilon) v'\big(k\mathcal{R}(k; \varepsilon)\big)$ for all $\varepsilon \in \mathcal{E}$ and, therefore, $\hat{H}_1(k; w, b) > u'(w - b - k) - \mathbb{E}_\nu \big[\mathcal{R}(k; \cdot) v'\big(k\mathcal{R}(k; \cdot)\big)\big]$ for all $k \in]0, \bar{k}[$. Thus, by (8)

$$\lim_{k\to\bar{k}} \hat{H}_1(k; w, b) \ge \lim_{k\to\bar{k}} \Big(u'(w - b - k) - \mathbb{E}_\nu \big[\mathcal{R}(k; \cdot) v'\big(k\mathcal{R}(k; \cdot)\big)\big]\Big) = \infty.$$

Let $(k_n)_{n\ge1}$ be a sequence in $]0, w - b[$ with $\lim_{n\to\infty} k_n = 0$. Since $k \mapsto \hat{Z}(k; w, b)$ and, by (P3), $k \mapsto k\mathcal{R}(k; \varepsilon)$ are increasing, $c_n(\varepsilon) := b\, \hat{Z}(k_n; w, b)\, \vartheta(\varepsilon) + k_n \mathcal{R}(k_n, \varepsilon)$ is bounded from above for all $\varepsilon \in \mathcal{E}$ which implies $\lim_{n\to\infty} \mathcal{R}(k_n, \varepsilon)\, v'\big(c_n(\varepsilon)\big) = \infty$. This being true for all $\varepsilon \in \mathcal{E}$ gives $\lim_{n\to\infty} \mathbb{E}_\nu[\mathcal{R}(k_n, \cdot) v'(c_n(\cdot))] = \infty$ and $\lim_{n\to\infty} \hat{H}_1(k_n; w, b) = -\infty$. Since $(k_n)_{n\ge1}$ was arbitrary, $\lim_{k\to0} \hat{H}_1(k; w, b) = -\infty$. Combining both limits yields existence of a zero of $\hat{H}_1(\cdot; w, b)$. Finally, using (P2) the partial derivatives satisfy

$$\partial_k H_1^\vartheta(z, k; w, b) = -u''(-) - \mathbb{E}_\nu \big[f_{kk}(k; \cdot)\, v'(-) + (1 + E_{g'}(k))\mathcal{R}(k; \cdot)^2\, v''(-)\big] > 0 \tag{A.7}$$

$$\partial_z H_1^\vartheta(z, k; w, b) = -\mathbb{E}_\nu \big[\mathcal{R}(k, \cdot)\, b\, \vartheta(\cdot)\, v''(-)\big] > 0. \tag{A.8}$$

Combining (A.7) and (A.8) with the monotonicity of $\hat{Z}(\cdot; w, b)$ yields $\partial_k \hat{H}_1(k; w, b) = \partial_z H_1^\vartheta(z, k; w, b)\partial_k \hat{Z}(k; w, b) + \partial_k H_1^\vartheta(z, k; w, b) > 0$ where $z = \hat{Z}(k; w, b)$. Hence, k' is the unique zero of $\hat{H}_1(\cdot; w, b)$ on $]0, \bar{k}[$. Setting $z = \hat{Z}(k'; w, b)$ completes the proof. ■

A.9 Proof of Lemma 3.2

As in the previous proof, the argument $c^o(z,k,b,\varepsilon)$ defined as before is omitted when convenient. We preface the proof by the following technical result.

Lemma A.2 *For the scenario of Section 3, let (P1)–(P4) hold and ϑ be continuous. Then, for all $(w,b) \in \mathbb{V}$, $z := \mathcal{Z}^\vartheta(w,b)$ and $k := \mathcal{K}^\vartheta(w,b)$ the following holds:*

(a) $k\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right] = -b\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))z\vartheta(\cdot)|v''(-)|\right]$.

(b) $\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right] \geq 0 \geq \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))z\vartheta(\cdot)|v''(-)|\right]$.

Proof of Lemma A.2.
(a) By (16a,b), $0 = H_1^\vartheta(z,k;w,b) - H_2^\vartheta(z,k;w,b) = \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))v'(-)\right]$. Using that $v'(c) = \theta^{-1}c|v''(c)|$ for all $c = bz\vartheta(\varepsilon) + k\mathcal{R}(k,\varepsilon) > 0$ by (P4) yields (a).
(b) We have $\mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))^2|v''(-)|\right] \geq 0$ which can equivalently be written as $\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))|v''(-)|\right] \geq \mathbb{E}_\nu\left[z\vartheta(\cdot)(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))|v''(-)|\right]$. Since, by (a), the two sides are either both zero or have opposite signs, the claim follows. □

Let $(w,b) \in \mathbb{V}$ be arbitrary and set $z := \mathcal{Z}^\vartheta(w,b)$ and $k := \mathcal{K}^\vartheta(w,b)$ noting that $z > 0$ and $0 < k < w - b$. Write $H^\vartheta = (H_1^\vartheta, H_2^\vartheta)$ and $\xi = (z,k)$. The signs of the derivatives in (A.5), (A.6), (A.7), and (A.8) imply that the Jacobian matrix

$$D_\xi H^\vartheta(z,k;w,b) = \begin{bmatrix} \partial_z H_1^\vartheta(z,k;w,b) & \partial_k H_1^\vartheta(z,k;w,b) \\ \partial_z H_2^\vartheta(z,k;w,b) & \partial_k H_2^\vartheta(z,k;w,b) \end{bmatrix}.$$

has determinant $\det D_\xi H^\vartheta(z,k;w,b) > 0$ and is hence invertible. The inverse computes

$$[D_\xi H^\vartheta(z,k;w,b)]^{-1} = \frac{1}{\det D_\xi H^\vartheta(z,k;w,b)} \begin{bmatrix} \partial_k H_2^\vartheta(z,k;w,b) & -\partial_k H_1^\vartheta(z,k;w,b) \\ -\partial_z H_2^\vartheta(z,k;w,b) & \partial_z H_1^\vartheta(z,k;w,b) \end{bmatrix}. \tag{A.9}$$

The partial derivatives with respect to w and b take the form

$$\begin{aligned} \partial_w H_1^\vartheta(z,k;w,b) &= \partial_w H_2^\vartheta(z,k;w,b) = u''(w-b-k) < 0 && \text{(A.10)} \\ \partial_b H_1^\vartheta(z,k;w,b) &= -u''(w-b-k) - \mathbb{E}_\nu\big[\mathcal{R}(k;\cdot)\, z\,\vartheta(\cdot)v''\big(-\big)\big] > 0 && \text{(A.11)} \\ \partial_b H_2^\vartheta(z,k;w,b) &= -u''(w-b-k) - \mathbb{E}_\nu\big[(z\,\vartheta(\cdot))^2 v''\big(-\big)\big] > 0. && \text{(A.12)} \end{aligned}$$

By the implicit function theorem, omitting the arguments for notational convenience

$$\begin{aligned} \partial_w \mathcal{Z}^\vartheta(w,b) &= \frac{-\partial_w H_1^\vartheta[\partial_k H_2^\vartheta - \partial_k H_1^\vartheta]}{\det D_\xi H^\vartheta}, & \partial_b \mathcal{Z}^\vartheta(w,b) &= \frac{\partial_k H_1^\vartheta \partial_b H_2^\vartheta - \partial_k H_2^\vartheta \partial_b H_1^\vartheta}{\det D_\xi H^\vartheta} \\ \partial_w \mathcal{K}^\vartheta(w,b) &= \frac{-\partial_w H_1^\vartheta[\partial_z H_1^\vartheta - \partial_z H_2^\vartheta]}{\det D_\xi H^\vartheta}, & \partial_b \mathcal{K}^\vartheta(w,b) &= \frac{\partial_z H_2^\vartheta \partial_b H_1^\vartheta - \partial_z H_1^\vartheta \partial_b H_2^\vartheta}{\det D_\xi H^\vartheta}. \end{aligned}$$

(i) As $\det D_\xi H^\vartheta = \partial_z H_1^\vartheta\, \partial_k H_2^\vartheta - \partial_k H_1^\vartheta\, \partial_z H_2^\vartheta > 0$, $\partial_z H_2^\vartheta < 0 \leq \partial_z H_1^\vartheta$ by (A.5) and (A.8), and $0 < -\partial_w H_1^\vartheta < \partial_b H_i^\vartheta$, $i = 1,2$, it follows that

$$0 < \partial_w \mathcal{K}^\vartheta(w,b) = \frac{-\partial_w H_1^\vartheta[\partial_z H_1^\vartheta - \partial_z H_2^\vartheta]}{\det D_\xi H^\vartheta} < \frac{\partial_z H_1^\vartheta \partial_b H_2^\vartheta - \partial_z H_2^\vartheta \partial_b H_1^\vartheta}{\det D_\xi H^\vartheta} = -\partial_b \mathcal{K}^\vartheta(w,b).$$

(ii) If, in addition, (P4) holds, straightforward calculations and Lemma A.2 imply

$$\begin{aligned}
\partial_k H_1^\vartheta - \partial_k H_2^\vartheta &= \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))\mathcal{R}(k;\cdot)|v''(-)|\right](1 + E_{g'}(k)) \\
&\quad - \mathbb{E}_\nu\left[f_{kk}(k;\cdot)v'(-)\right] > 0 \qquad (A.13) \\
\partial_b H_1^\vartheta - \partial_b H_2^\vartheta &= \mathbb{E}_\nu\left[(\mathcal{R}(k;\cdot) - z\vartheta(\cdot))z\vartheta(\cdot)|v''(-)|\right] \le 0. \qquad (A.14)
\end{aligned}$$

By (A.10) and (A.13), $\partial_w \mathcal{Z}^\vartheta(w,b) < 0$. By (A.13) and (A.14), $\partial_b \mathcal{Z}^\vartheta(w,b) > 0$. Finally,

$$\partial_w \mathcal{K}^\vartheta(w,b)\partial_b \mathcal{Z}^\vartheta(w,b) - \partial_b \mathcal{K}^\vartheta(w,b)\partial_w \mathcal{Z}^\vartheta(w,b) = \frac{-\partial_w H_1^\vartheta}{\det D_\xi H^\vartheta}(\partial_b H_2^\vartheta - \partial_b H_1^\vartheta) \ge 0 \quad (A.15)$$

which follows from direct calculations and shows that $\frac{-\partial_w \mathcal{Z}^\vartheta(w,b)}{\partial_b \mathcal{Z}^\vartheta(w,b)} \le \frac{\partial_w \mathcal{K}^\vartheta(w,b)}{-\partial_b \mathcal{K}^\vartheta(w,b)} < 1$. ■

Remark A.1 *Since $\mathcal{Z}^\vartheta$ and $\mathcal{K}^\vartheta$ are well-defined and the matrix $D_\xi H^\vartheta(z,k;w,b)$ is non-singular also at any boundary point $(w,0) \in \mathbb{V}$, the implicit function theorem implies that the mappings $\mathcal{Z}^\vartheta$ and $\mathcal{K}^\vartheta$ can locally be extended to an open neighborhood around $(w,0)$. Hence, their derivatives are well-defined and continuous also on the boundary of $\mathbb{V}$ where $b = 0$ and Lemma 3.2 and also Lemma 2.2 indeed hold on the entire set $\mathbb{V}$.*

A.10 Proof of Lemma 3.3

(i) Let ϑ be given. For notational convenience, the shock $\varepsilon \in \mathcal{E}$ will subsequently be suppressed. With this convention, denote the trivial steady state as $\bar{w}^0 > 0$ and let $\underline{w}_k := \mathcal{W}(0;\varepsilon) \ge 0$. By the monotonicity of $\mathcal{K}^\vartheta$ (cf. Lemma 3.2) and $\mathcal{W}(\cdot;\varepsilon)$, any steady state $(\bar{w},\bar{b}) \in \mathbb{V}_+$ satisfies $\underline{w}_k < \bar{w} < \bar{w}^0$. Further results are collected in the next lemma.

Lemma A.3 *Assumption 3.1 and the hypotheses of Lemma 3.3 imply the following:*

(a) $w > \mathcal{W}(\mathcal{K}^\vartheta(w,0);\varepsilon)$ for all $w \in]\underline{w}_k, \bar{w}^0[$.

(b) $\mathcal{W}(k;\varepsilon) \ge k\mathcal{R}(k;\varepsilon)$ for all $0 < k \le \bar{k}^0 := \mathcal{K}^\vartheta(\bar{w}^0, 0)$.

(c) For any sequence $(w_n,b_n)_{n\ge 0}$ in $\mathbb{V}$, $\lim_{n\to\infty}(w_n - b_n) = 0$ implies $\lim_{n\to\infty} \mathcal{Z}^\vartheta(w_n,b_n) = \infty$.

Proof of Lemma A.3
(a) By uniqueness of $\bar{w}^0$, $w \ne \mathcal{W}(\mathcal{K}^\vartheta(w,0);\varepsilon)\ \forall w \in]\underline{w}_k, \bar{w}^0[$. Stability implies the claim.
(b) By (2) and (3), the claim is equivalent to $E_g(k) \le \frac{1}{2}$ for all $k \in]0, \bar{k}^0]$. By Assumption 3.1, $E_g(\bar{k}^0) \le \frac{1}{2}$. The derivative computes $E_g'(k) = g'(k)/g(k)[1 - E_g(k) - |E_{g'}(k)|]$ and is non-negative by (P6) implying that E_g is non-decreasing from which the claim follows.
(c) Given $(w,b) \in \mathbb{V}$, let $z := \mathcal{Z}^\vartheta(w,b)$, $k := \mathcal{K}^\vartheta(w,b)$, and $c^o(z,k,b;\varepsilon)$ as in the previous proofs. By (16a,b), $\mathbb{E}_\nu\left[\mathcal{R}(k;\cdot)v'(c^o(z,k,b;\cdot))\right] = \mathbb{E}_\nu\left[z\vartheta(\cdot)\,v'(c^o(z,k,b;\cdot))\right]$. This requires $z\vartheta(\tilde{\varepsilon}) \ge \mathcal{R}(k;\tilde{\varepsilon}) = \tilde{\varepsilon}g'(k)$ for some $\tilde{\varepsilon} \in \mathcal{E}$. Setting $\zeta := \min_\varepsilon\{\varepsilon/\vartheta(\varepsilon) \mid \varepsilon \in \mathcal{E}\} > 0$ (which is well-defined by continuity of ϑ and compactness of $\mathcal{E}$) gives $\mathcal{Z}^\vartheta(w,b) \ge \zeta g'(\mathcal{K}^\vartheta(w,b))$ for all $(w,b) \in \mathbb{V}$. Since $\lim_{n\to\infty} \mathcal{K}(w_n,b_n) = 0$ for any sequence $(w_n,b_n)_{n\ge 0}$ in $\mathbb{V}$ with $\lim_{n\to\infty}(w_n - b_n) = 0$, this implies $\lim_{n\to\infty} \mathcal{Z}^\vartheta(w_n,b_n) \ge \zeta g'(\mathcal{K}^\vartheta(w_n,b_n)) = \infty$. □

(i) *Existence.* Define $H_w : \mathbb{V} \longrightarrow \mathbb{R}$, $H_w(w,b) := w - \mathcal{W}(\mathcal{K}^\vartheta(w,b),\varepsilon)$ and the so-called w-isocline $\mathbb{H}_w := \{(w,b) \in \mathbb{V} \mid H_w(w,b) = 0, w \in]\underline{w}_k, \bar{w}^0[\}$. Any interior steady state satisfies $(\bar{w},\bar{b}) \in \mathbb{H}_w$. Given any $\hat{w} \in]\underline{w}_k, \bar{w}^0[$ we claim there exists a unique $\hat{b} \in]0, \hat{w}[$

such that $H_w(\hat{w},\hat{b})=0$. By Lemma A.3(a), $\lim_{b\to 0} H_w(\hat{w},b)=\hat{w}-\mathcal{W}(\mathcal{K}^\vartheta(\hat{w},0),\varepsilon)<0$ and $\lim_{b\to\hat{w}}\mathcal{K}^\vartheta(\hat{w},b)=0$ gives $\lim_{b\to\hat{w}} H_w(\hat{w},b)=\hat{w}-\underline{w}_k>0$ implying existence of $\hat{b}$. Uniqueness follows from Lemma 3.2(i) due to which $H_w(w;\cdot)$ is strictly increasing.
This result permits to define a map $h_w:]\underline{w}_k,\bar{w}^0[\longrightarrow\mathbb{R}_{++}$ such that $\mathbb{H}_w=\text{graph}(h_w)$. By the implicit function theorem, h_w is C^1 with derivative

$$h_w'(w)=-\frac{\partial_w H_w(w,b)}{\partial_b H_w(w,b)}=-\frac{1+\varepsilon kg''(k)\partial_w\mathcal{K}^\vartheta(w,b)}{\varepsilon kg''(k)\partial_b\mathcal{K}^\vartheta(w,b)},\ b=h_w(w),k=\mathcal{K}^\vartheta(w,b). \tag{A.16}$$

Finally, since $H_w(\bar{w}^0,0)=0$ and $\lim_{w\searrow\underline{w}_k} H_w(w,\underline{w}_k)=0$, continuity of H_w implies the boundary behavior $\lim_{w\to\bar{w}^0} h_w(w)=0$ and $\lim_{w\searrow\underline{w}_k} h_w(w)=\underline{w}_k\geq 0$.
Analogously, let $H_b:\mathbb{V}\longrightarrow\mathbb{R}$, $H_b(w,b):=\mathcal{Z}^\vartheta(w,b)-1/\vartheta(\varepsilon)$. For $b=0$, $\varepsilon\in\mathcal{E}_s^\vartheta$ implies $\lim_{w\to\bar{w}^0} H_b(w,0)=\mathcal{Z}^\vartheta(\bar{w}^0,0)-1/\vartheta(\varepsilon)<0$. By Lemma A.3(c), $\lim_{w\to 0} H_b(w,0)=\infty$. As $w\mapsto\mathcal{Z}^\vartheta(w,0)$ is strictly decreasing by Lemma 3.2(ii), a unique $\underline{w}_z\in]0,\bar{w}^0[$ satisfying $H_b(\underline{w}_z,0)=0$ exists. Define the b-isocline $\mathbb{H}_b:=\{(w,b)\in\mathbb{V}\,|\,H_b(w,b)=0, w\in]\underline{w}_z,\bar{w}^0[\}$. Any interior steady state satisfies $(\bar{w},\bar{b})\in\mathbb{H}_b$. Given $\hat{w}\in]\underline{w}_z,\bar{w}^0[$ we again claim there exists a unique $\hat{b}\in]0,\hat{w}[$ such that $H_b(\hat{w},\hat{b})=0$. By Lemma 3.2(ii), $\lim_{b\to 0} H_b(\hat{w},b)=\mathcal{Z}^\vartheta(\hat{w},0)-1/\vartheta(\varepsilon)<\mathcal{Z}^\vartheta(\underline{w}_z,0)-1/\vartheta(\varepsilon)=0$. Lemma A.3(c) yields $\lim_{b\to\hat{w}} H_b(\hat{w},b)=\infty$ implying existence of $\hat{b}$. Uniqueness follows from monotonicity of $H_b(w;\cdot)$ due to Lemma 3.2(ii). Analogously, this result permits to define a map $h_b:]\underline{w}_z,\bar{w}^0[\longrightarrow\mathbb{R}_{++}$ such that $\mathbb{H}_b=\text{graph}(h_b)$. By the implicit function theorem, h_b is C^1 with derivative

$$h_b'(w)=-\frac{\partial_w H_b(w,b)}{\partial_b H_b(w,b)}=-\frac{\partial_w\mathcal{Z}(w,b)}{\partial_b\mathcal{Z}(w,b)}>0,\quad b=h_b(w). \tag{A.17}$$

Recall that $H_b(\bar{w}^0,0)<0$. By Lemma A.3(c), there exists a unique value $\bar{b}^0\in]0,\bar{w}^0[$ satisfying $H_b(\bar{w}^0,\bar{b}^0)=0$. Hence, $H_b(\underline{w}_z,0)=H_b(\bar{w}^0,\bar{b}^0)=0$. By continuity of H_b, this implies the boundary behavior $\lim_{w\to\bar{w}^0} h_b(w)=\bar{b}^0>0$ and $\lim_{w\to\underline{w}_z} h_b(w)=0$.
Set $\underline{w}:=\max\{\underline{w}_k,\underline{w}_z\}>0$ and define $\Delta:]\underline{w},\bar{w}^0[\longrightarrow\mathbb{R}$, $\Delta(w):=h_w(w)-h_b(w)$. Since $(\bar{w},\bar{b})\in\mathbb{V}$ is an interior steady state iff $(\bar{w},\bar{b})\in\mathbb{H}_w\cap\mathbb{H}_b$, steady state values $\bar{w}$ are zeros of Δ while $\bar{b}=h_w(\bar{w})$. By the boundary behavior derived above, $\lim_{w\to\bar{w}^0}\Delta(w)=-\bar{b}^0<0$. Let $\underline{w}_k>\underline{w}_z$. Then, $\lim_{w\to\underline{w}}\Delta(w)=\underline{w}_k-h_b(\underline{w}_k)>0$ since $h_b(w)<w$ for $w>\underline{w}_z$. If $\underline{w}_k=\underline{w}_z$, then $\lim_{w\to\underline{w}}\Delta(w)=\underline{w}_k>0$. Finally, let $\underline{w}_z>\underline{w}_k$. Then $\lim_{w\to\underline{w}}\Delta(w)=h_w(\underline{w}_z)>0$. In either case, $\lim_{w\to\underline{w}}\Delta(w)>0$ and a zero exists.

Uniqueness. Let $(\bar{w},\bar{b})\gg 0$ be an interior steady state. We show that $\Delta'(\bar{w})<0$ implying uniqueness by continuity of Δ'. Let $\bar{k}:=\mathcal{K}^\vartheta(\bar{w},\bar{b})<\bar{k}^0$ and $\bar{z}:=\mathcal{Z}^\vartheta(\bar{w},\bar{b})>0$. By (A.16) and (A.17),

$$\Delta'(\bar{w})=-\frac{\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b})+\varepsilon\bar{k}g''(\bar{k})\left[\partial_w\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b})-\partial_b\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_w\mathcal{Z}^\vartheta(\bar{w},\bar{b})\right]}{\varepsilon\bar{k}g''(\bar{k})\partial_b\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b})}. \tag{A.18}$$

Since the denominator is strictly positive by Lemma 3.2, it suffices to show that the numerator is strictly positive as well. Using (A.15) and the definition of $\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b})$ from Lemma 3.2 and recalling that $\det D_\xi H^\vartheta>0$, this is equivalent to showing that

$$M:=\partial_k H_1\partial_b H_2-\partial_k H_2\partial_b H_1-\varepsilon\bar{k}g''(\bar{k})\partial_w H_1(\partial_b H_2-\partial_b H_1)>0 \tag{A.19}$$

where the respective arguments have been omitted for convenience. In what follows, let $M_1:=\mathbb{E}_\nu[\bar{z}\vartheta(\cdot)\,|v'(-)|]=\mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)\,|v'(-)|]>0$, $M_2:=\mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)^2\,|v''(-)|]>0$,

$M_3 := \mathbb{E}_\nu[(\bar{z}\vartheta(\cdot))^2\,|v''(-)|] > 0$ and $M_4 := \mathbb{E}_\nu[\mathcal{R}(\bar{k};\cdot)\,\bar{z}\vartheta(\cdot)\,|v''(-)|] > 0$. Using the functional forms of the derivatives from (A.5)–(A.8), and (A.10)–(A.12), tedious but straightforward calculations reveal that M can be written as $M = A + B + C$ where

$$\begin{aligned} A &:= \; |u''(-)|\Big[-\tfrac{g''(\bar{k})}{g'(\bar{k})}M_1 + m(M_3 - M_4) + (1 + E_{g'}(\bar{k}))(M_2 - M_4)\Big] \\ m &:= \; 1 + \varepsilon\bar{k}g''(\bar{k}),\ B := -\tfrac{g''(\bar{k})}{g'(\bar{k})}M_1M_3,\ C := (1 + E_{g'}(\bar{k}))\Big[M_2M_3 - \big(M_4\big)^2\Big]. \end{aligned}$$

By Lemma A.2(b), $M_2 \geq M_4$ and $M_3 \geq M_4$ which implies $C \geq 0$ by (P3). Obviously, $B > 0$. Suppose $m \geq 0$. Then, $A > 0$ by (P3) which implies $M > 0$. Conversely, suppose $m < 0$. Then $-mM_4 > 0$. By (P5), $M_1 = u'(\bar{w} - \bar{b} - \bar{k}) \geq (\bar{w} - \bar{b} - \bar{k})|u''(\bar{w} - \bar{b} - \bar{k})|$ which implies $B \geq -\frac{g''(\bar{k})}{g'(\bar{k})}(\bar{w} - \bar{b} - \bar{k})|u''(-)|M_3$. By (P3), $(1 + E_{g'}(\bar{k}))(M_2 - M_4) \geq 0$. Finally $M_1 = \theta^{-1}(\bar{k}M_2 + \bar{b}M_3)$ by (P4) implying $M_1 > \bar{b}M_3$ by (P1). Combining the four inequalities derived gives finally the result

$$A + B > |u''(-)|M_3\Big[(1 + E_{g'}(\bar{k})) - \frac{g''(\bar{k})}{g'(\bar{k})}(\bar{w} - \varepsilon\bar{k}g'(\bar{k}))\Big].$$

Both terms in brackets are non-negative due to (P3) and Lemma A.3(b), respectively. Hence, $M > 0$ also in this case, proving uniqueness of the steady state.

Stability. The argument is similar to the one in Lemma 2.3. Computing the determinant and trace of the Jacobian $\bar{J}$ at the steady state gives, using Lemma 3.2 and (A.15)

$$\det\bar{J} = -\varepsilon\bar{k}g''(\bar{k})\Big[\partial_w\mathcal{K}^\vartheta(\bar{w},\bar{b}) + \frac{\bar{b}}{\bar{z}}\Big(\partial_w\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b}) - \partial_b\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_w\mathcal{Z}^\vartheta(\bar{w},\bar{b})\Big)\Big] > 0$$

$$\mathrm{tr}\bar{J} = 1 + \det\bar{J} + \frac{\bar{b}}{\bar{z}}\Big[\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b}) + \varepsilon\bar{k}g''(\bar{k})\,\big(\partial_w\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_b\mathcal{Z}^\vartheta(\bar{w},\bar{b}) - \partial_b\mathcal{K}^\vartheta(\bar{w},\bar{b})\partial_w\mathcal{Z}^\vartheta(\bar{w},\bar{b})\big)\Big].$$

As shown before, the numerator in (A.18) is positive which implies $\mathrm{tr}\bar{J} > 1 + \det\bar{J}$. The same reasoning as in the proof of Lemma 2.3 gives the claim.

(ii) Replacing $\mathcal{R}(\mathcal{K}(w,b);\varepsilon)$ by $\vartheta(\varepsilon)\mathcal{Z}^\vartheta(w,b)$ and using Lemma 3.2 the proof is identical to the one of Lemma 2.3(ii). ■

A.11 Proof of Lemma 3.4

Let $\vartheta(\varepsilon) \equiv \bar{\vartheta} > 0$. We claim that for all $w > 0$ and $\varepsilon, \varepsilon' \in \mathcal{E}_s^{\bar{\vartheta}}$: $\varepsilon < \varepsilon' \Rightarrow \psi_\varepsilon^{\bar{\vartheta}}(w) < \psi_{\varepsilon'}^{\bar{\vartheta}}(w)$. By contradiction, suppose $\varepsilon < \varepsilon'$ but $b_0 := \psi_\varepsilon^{\bar{\vartheta}}(w) \geq \psi_{\varepsilon'}^{\bar{\vartheta}}(w) =: b_0'$ for some $w > 0$. By Lemma 3.2 and (17a,b), it is straightforward to show that the sequences $\{w_t, b_t\}_{t\geq 0}$ and $\{w_t', b_t'\}_{t\geq 0}$ defined as $(w_t, b_t) := \Phi^{\bar{\vartheta}}(w_{t-1}, b_{t-1};\varepsilon)$ and $(w_t', b_t') := \Phi^{\bar{\vartheta}}(w_{t-1}', b_{t-1}';\varepsilon')$ (where $w_0 = w_0' = w$) satisfy $w_t < w_t'$ and $b_t \geq b_t'$ for all $t > 0$. Thus, the steady states $(\bar{w}_\varepsilon^{\bar{\vartheta}}, \bar{b}_\varepsilon^{\bar{\vartheta}}) = \lim_{t\to\infty}(w_t, b_t)$ and $(\bar{w}_{\varepsilon'}^{\bar{\vartheta}}, \bar{b}_{\varepsilon'}^{\bar{\vartheta}}) = \lim_{t\to\infty}(w_t', b_t')$ satisfy $\bar{w}_\varepsilon^{\bar{\vartheta}} \leq \bar{w}_{\varepsilon'}^{\bar{\vartheta}}$ and $\bar{b}_\varepsilon^{\bar{\vartheta}} \geq \bar{b}_{\varepsilon'}^{\bar{\vartheta}}$. By Lemma 3.2(ii), however, the steady state property $\mathcal{Z}^{\bar{\vartheta}}(\bar{w}_\varepsilon^{\bar{\vartheta}}, \bar{b}_\varepsilon^{\bar{\vartheta}}) = \mathcal{Z}^{\bar{\vartheta}}(\bar{w}_{\varepsilon'}^{\bar{\vartheta}}, \bar{b}_{\varepsilon'}^{\bar{\vartheta}}) = \frac{1}{\bar{\vartheta}}$ can only be satisfied if $(\bar{w}_\varepsilon^{\bar{\vartheta}}, \bar{b}_\varepsilon^{\bar{\vartheta}}) = (\bar{w}_{\varepsilon'}^{\bar{\vartheta}}, \bar{b}_{\varepsilon'}^{\bar{\vartheta}})$ implying $\mathcal{K}^{\bar{\vartheta}}(\bar{w}_\varepsilon^{\bar{\vartheta}}, \bar{b}_\varepsilon^{\bar{\vartheta}}) = \mathcal{K}^{\bar{\vartheta}}(\bar{w}_{\varepsilon'}^{\bar{\vartheta}}, \bar{b}_{\varepsilon'}^{\bar{\vartheta}}) =: \bar{k}$. But this contradicts $\bar{w}_\varepsilon^{\bar{\vartheta}} = \mathcal{W}(\bar{k},\varepsilon) < \mathcal{W}(\bar{k},\varepsilon') = \bar{w}_{\varepsilon'}^{\bar{\vartheta}}$, proving the claim. Thus, $\mathcal{M}_\varepsilon^{\bar{\vartheta}} \bigcap \mathcal{M}_{\varepsilon'}^{\bar{\vartheta}} = \emptyset$. To see that the restriction from Assumption 2.2 is satisfied, suppose w.l.o.g. that $\mathcal{E}_s^{\bar{\vartheta}} = \mathcal{E}$. Then, by the previous result $b^{\bar{\vartheta}\,\mathrm{crit}}(w) = \min_{\varepsilon\in\mathcal{E}}\{\psi_\varepsilon^{\bar{\vartheta}}(w)\} = \psi_{\varepsilon_{\min}}^{\bar{\vartheta}}(w)$ for all $w > 0$. Using this, $\vartheta \equiv \bar{\vartheta}$, and the properties of Φ^ϑ and $\psi_{\varepsilon_{\min}}^{\bar{\vartheta}}$ respectively $\mathcal{M}_{\varepsilon_{\min}}^{\bar{\vartheta}}$, $b \leq b^{\bar{\vartheta}\,\mathrm{crit}}(w)$ implies $\Phi_b^{\bar{\vartheta}}(w,b;\varepsilon) = \Phi_b^{\bar{\vartheta}}(w,b;\varepsilon_{\min}) \leq \Phi_b^{\bar{\vartheta}}(w,b^{\bar{\vartheta}\,\mathrm{crit}}(w);\varepsilon_{\min}) = \psi_{\varepsilon_{\min}}^{\bar{\vartheta}}(\Phi_w^{\bar{\vartheta}}(w,b^{\bar{\vartheta}\,\mathrm{crit}}(w);\varepsilon_{\min})) \leq \psi_{\varepsilon_{\min}}^{\bar{\vartheta}}(\Phi_w^{\bar{\vartheta}}(w,b;\varepsilon_{\min})) \leq \psi_{\varepsilon_{\min}}^{\bar{\vartheta}}(\Phi_w^{\bar{\vartheta}}(w,b;\varepsilon)) = b^{\bar{\vartheta}\,\mathrm{crit}}(\Phi_w^{\bar{\vartheta}}(w,b;\varepsilon))$ for all $\varepsilon \in \mathcal{E}$. ■

A.12 Proof of Lemma 4.1

Given $(w, b) \in \mathbb{V}_{\mathbb{A}}$ we determine a unique $\tau < w - \underline{w}_{\mathbb{A}}$ such that $H(\tau; w, b) = 0$ where

$$H(\tau; w, b) := b - \tau - \chi_{\mathbb{A}}(w - \tau). \tag{A.20}$$

Let $\tau' := w - \underline{w}_{\mathbb{A}} \leq w$. As $(w, b) \in \mathbb{V}_{\mathbb{A}}$ and $d_{\mathbb{A}} = \lim_{w \to \underline{w}_{\mathbb{A}}}(w - \chi_{\mathbb{A}}(w))$ from Assumption 4.1, $\lim_{\tau \to \tau'} H(\tau; w, b) = b - w + d_{\mathbb{A}} < 0$. Furthermore, $\lim_{\tau \to -\infty} H(\tau; w, b) = b + \lim_{\tau \to \infty} \tau[1 - \chi_{\mathbb{A}}(w + \tau)/\tau]$. If $\lim_{\tau \to \infty} \chi_{\mathbb{A}}(w + \tau) < \infty$, then $\lim_{\tau \to -\infty} H(\tau; w, b) = \infty$. If $\lim_{\tau \to \infty} \chi_{\mathbb{A}}(w + \tau) = \infty$, then $\lim_{\tau \to \infty}[1 - \chi_{\mathbb{A}}(w + \tau)/\tau] = 1 - \lim_{\tau \to \infty} \chi'_{\mathbb{A}}(w + \tau) > 0$ by hypothesis and l'Hopital's rule. Again, $\lim_{\tau \to -\infty} H(\tau; w, b) = \infty$, which implies existence. Uniqueness follows from $\partial_\tau H(\tau; w, b) = -1 + \chi'_{\mathbb{A}}(w - \tau) < 0$ for all $\tau < w$. ■

A.13 Proof of Lemma 4.2

Using (A.20), the implicit function theorem implies that for all $(w, b) \in \mathbb{V}_{\mathbb{A}}$

$$\partial_w \mathcal{T}_{\mathbb{A}}(w, b) = -\frac{\chi'_{\mathbb{A}}(w - \tau)}{1 - \chi'_{\mathbb{A}}(w - \tau)} = 1 - \partial_b \mathcal{T}_{\mathbb{A}}(w, b) < 0, \qquad \tau = \mathcal{T}_{\mathbb{A}}(w, b). \tag{A.21}$$

(i) Let $w \in \mathbb{W}_{\mathbb{A}}$ and $\varepsilon \in \mathcal{E}$ be arbitrary. Using (A.21) the derivative of (21) computes

$$\partial_w \phi^\pi(w; \varepsilon) = (1 - \partial_w \mathcal{T}_{\mathbb{A}}(w, b))\Big[\partial_w \Phi^\vartheta_w + \chi'_{\mathbb{A}}(w)\partial_b \Phi^\vartheta_w - \partial_w \Phi^\vartheta_b - \chi'_{\mathbb{A}}(w)\partial_b \Phi^\vartheta_b\Big]. \tag{A.22}$$

Using Lemma 3.2, the bracketed term is non-negative under (P7) proving (i) by (A.21).
(ii) Let $w \in \mathbb{W}_{\mathbb{A}}$ be given and $\varepsilon > \varepsilon'$. We show that $\phi^\pi(w; \varepsilon) > \phi^\pi(w; \varepsilon')$. Set $(w', b') := \Phi^\vartheta(w, \chi_{\mathbb{A}}(w); \varepsilon') \in \mathbb{V}_{\mathbb{A}}$ and let $\tilde{\phi}(\gamma, \delta) := \gamma w' - \mathcal{T}_{\mathbb{A}}(\gamma w', \gamma\delta b')$ which is well-defined for all $(\gamma, \delta) \in \mathbb{R}^2_{++}$ such that $(\gamma w', \gamma\delta b') \in \mathbb{V}_{\mathbb{A}}$. Using (A.21), the partial derivatives satisfy

$$\partial_\gamma \tilde{\phi}(\gamma, \delta) = [1 - \partial_w \mathcal{T}_{\mathbb{A}}(-)](w' - \delta b') > 0 \geq -[1 - \partial_w \mathcal{T}_{\mathbb{A}}(-)]\gamma b' = \partial_\delta \tilde{\phi}(\gamma, \delta) \tag{A.23}$$

for all $\gamma > 0$ and $\delta \leq 1$. Set $\gamma' := \frac{\varepsilon}{\varepsilon'} > 1$ and $\delta' := \zeta_\vartheta(\varepsilon)/\zeta_\vartheta(\varepsilon')$. By (P8) $\delta' \leq 1$. Then, (A.23) implies $\phi^\pi(w; \varepsilon') = \tilde{\phi}(1, 1) \leq \tilde{\phi}(1, \delta') < \tilde{\phi}(\gamma', \delta') = \phi^\pi(w; \varepsilon)$ proving the claim. ■

A.14 Proof of Lemma 4.3

Since shocks in (17a,b) are multiplicative and $\mathcal{M}^\vartheta_{\varepsilon_{\text{ref}}}$ is self-supporting under $\Phi^\vartheta(\cdot; \varepsilon_{\text{ref}})$,

$$\Phi^\vartheta_b(w, \chi_{\mathbb{A}}(w); \varepsilon) = \frac{\vartheta(\varepsilon)}{\vartheta(\varepsilon_{\text{ref}})} \chi_{\mathbb{A}}\Big(\frac{\varepsilon_{\text{ref}}}{\varepsilon} \Phi^\vartheta_w(w, \chi_{\mathbb{A}}(w); \varepsilon)\Big) \tag{A.24}$$

which holds for all $w \in \mathbb{W}_{\mathbb{A}}$ and $\varepsilon \in \mathcal{E}$. Differentiating (A.24) with respect to w gives

$$\partial_w \Phi^\vartheta_b + \chi'_{\mathbb{A}}(w)\partial_b \Phi^\vartheta_b = \Big[\partial_w \Phi^\vartheta_w + \chi'_{\mathbb{A}}(w)\partial_b \Phi^\vartheta_w\Big] \frac{\vartheta(\varepsilon)}{\vartheta(\varepsilon_{\text{ref}})} \frac{\varepsilon_{\text{ref}}}{\varepsilon} \chi'_{\mathbb{A}}\big(\Phi^\vartheta_w(w, \chi_{\mathbb{A}}(w); \varepsilon_{\text{ref}})\big). \tag{A.25}$$

Since $\partial_w \Phi^\vartheta_w + \chi'_{\mathbb{A}}(w)\partial_b \Phi^\vartheta_w \geq 0$ by Theorem A.1(iii), (A.25) implies (P7). ■

A.15 Proof of Theorem 4.1

First note that both limits in (b) are well-defined since ϕ^π is continuous and monotonic. By (a), $\phi^\pi(\cdot;\varepsilon_0)$ has a unique fixed point $\bar{w}_0^\pi \in \mathbb{W}_\mathbb{A}$. By stability, $\phi^\pi(w;\varepsilon_0) \gtreqqless w$ iff $w \lesseqqgtr \bar{w}_0^\pi$. Since $\varepsilon \mapsto \phi^\pi(w;\varepsilon)$ is strictly increasing, this implies $\phi^\pi(w;\varepsilon_{\min}) < w$ for all $w > \bar{w}_0^\pi$ and $\phi^\pi(w;\varepsilon_{\max}) > w$ for all $w < \bar{w}_0^\pi$. Hence, non-trivial fixed points of $\phi^\pi(\cdot;\varepsilon_{\min})$ can only exist in $]\underline{w}_\mathbb{A}, \bar{w}_0^\pi]$ and do exist if $\lim_{w\to\underline{w}_\mathbb{A}} \phi^\pi(w;\varepsilon_{min})/w > 1$ while non-trivial fixed points of $\phi^\pi(\cdot;\varepsilon_{\max})$ can only exist in $[\bar{w}_0^\pi, \infty[$ and do exist if $\lim_{w\to\infty} \phi^\pi(w;\varepsilon_{max})/w < 1$. In the terminology of Brock & Mirman (1972, p.500), ϕ^π possesses a *stable fixed-point configuration.* Defining $\underline{w}^\pi := \max\{w \in \mathbb{W}_\mathbb{A} \,|\, \phi^\pi(w;\varepsilon_{\min}) = w\} \le \bar{w}_0^\pi \le \bar{w}^\pi := \min\{w \in \mathbb{W}_\mathbb{A} \,|\, \phi^\pi(w;\varepsilon_{\max}) = w\}$, the set $\bar{\mathbb{W}}^\pi := [\underline{w}^\pi, \bar{w}^\pi]$ is the unique stable set of ϕ^π (defined as in Wang (1993, p.428)). The claim (ii) then follows from the results in Wang (1993). ■

A.16 Proof of Theorem 5.1

(i) Fix $\lambda \in [0,1]$. Using Lemma 3.2 and (16a,b), the partial derivative of (22) satisfies

$$\partial_b V(w,b,\lambda) = u'(w-b-k)\Big[\partial_b \mathcal{Z}(w,b,\lambda)b/z + E_{g'}(k)\partial_b \mathcal{K}(w,b,\lambda)\Big] > 0.$$

(ii) Fix $(w,b) \in \mathbb{V}_+$ and write $k_\lambda := \mathcal{K}(w,b,\lambda)$ and $z_\lambda := \mathcal{Z}(w,b,\lambda)$. Given $\lambda \in [0,1[$, let $M_1 := \mathbb{E}_\nu[\mathcal{R}(k_\lambda,\cdot)v'(-)]$, $M_2 := \mathbb{E}_\nu[\mathcal{R}(k_\lambda,\cdot)^2|v''(-)|]$, $M_3 := \mathbb{E}_\nu[(z_\lambda\vartheta_\lambda(\cdot))^2|v''(-)|]$, and $M_4 := \mathbb{E}_\nu[\mathcal{R}(k_\lambda,\cdot)z_\lambda\vartheta_\lambda(\cdot)|v''(-)|]$. Write the map $H_i^{\vartheta_\lambda}(z,k;w,b)$ from (16a,b) as $H_i(z,k,\lambda)$, $i=1,2$. The derivatives with respect to λ exist and satisfy

$$(1-\lambda)\partial_\lambda H_1 = b[z/g'(k)M_2 - M_4] \text{ and } (1-\lambda)\partial_\lambda H_2 = -(z/g'(k)[M_1 - bM_4] + M_1 - bM_3. \tag{A.26}$$

By Lemma 3.1 and the IFT, $\lambda \longmapsto (\mathcal{Z}(\lambda),\mathcal{K}(\lambda)) := (\mathcal{Z}^{\vartheta_\lambda}(w,b),\mathcal{K}^{\vartheta_\lambda}(w,b))$ is a C^1-map. Using (A.9) and the notation from the proof of Lemma 3.2, the derivatives compute

$$\begin{pmatrix} \partial_\lambda \mathcal{Z}(\lambda) \\ \partial_\lambda \mathcal{K}(\lambda) \end{pmatrix} = -[D_\xi H]^{-1} \begin{pmatrix} \partial_\lambda H_1 \\ \partial_\lambda H_2 \end{pmatrix} = \frac{1}{\det D_\xi H} \begin{pmatrix} \partial_k H_1 \partial_\lambda H_2 - \partial_k H_2 \partial_\lambda H_1 \\ \partial_z H_2 \partial_\lambda H_1 - \partial_z H_1 \partial_\lambda H_2 \end{pmatrix}. \tag{A.27}$$

Using that $\vartheta_\lambda = \lambda\vartheta_1 + (1-\lambda)\vartheta_0$ implies $d_\lambda\vartheta_\lambda = \vartheta_1 - \vartheta_0$, the derivative of (22) computes

$$\partial_\lambda V(w,b,\lambda) = \mathbb{E}_\nu[(A_\lambda\vartheta_1(\cdot) - B_\lambda\vartheta_0(\cdot))v'(-)] \tag{A.28}$$

where $A_\lambda := bz_\lambda + g'(k_\lambda)[\partial_\lambda \mathcal{Z} b/z_\lambda + E_{g'}(k_\lambda)\partial_\lambda \mathcal{K}]$ and $B_\lambda := bz_\lambda > 0$. Let $\lambda \in [0,1[$ be arbitrary. We show that $\partial_\lambda V < 0$. If $A_\lambda \le 0$, this follows immediately from (A.28), so suppose $A_\lambda > 0$. By (16a,b), $\mathbb{E}_\nu[(\mathcal{R}(k_\lambda;\cdot) - z_\lambda\vartheta_\lambda(\cdot))v'(-)] = 0$ which can be written as $\mathbb{E}_\nu[(\vartheta_1(\cdot)C_\lambda - \vartheta_0(\cdot))v'(-)] = 0$ where $C_\lambda := \frac{g'(k_\lambda)/z_\lambda - \lambda}{1-\lambda} > 0$. Exploiting (A.28), we show that $M := C_\lambda B_\lambda - A_\lambda > 0$. Solving this condition by using (A.26) and (A.5)– (A.8) in (A.27), tedious but straightforward calculations show that $M > 0$ if and only if

$$b|u''|\big[M_2 - M_4 + M_3 - M_4\big] + b\big[M_2M_3 - M_4^2\big] - \frac{g''(k_\lambda)}{g'(k_\lambda)}M_1\big[b(M_3 - M_4) - k(M_2 - M_4)\big] > 0.$$

Using Lemma A.2(b) and the fact that by (P4) and (16a,b) $b(M_3 - M_4) - k(M_2 - M_4) = bM_3 + kM_4 - (bM_4 + kM_2) = 0$, all bracketed terms are positive, proving the claim. ■

References

Bertocchi, G. (1994): "Safe Debt, Risky Capital", *Economica*, 61(244), 493–508.

Bertocchi, G. & Y. Wang (1994): "The Real Value of Money under Endogenous Beliefs", *Journal of Economic Theory*, 67, 205–222.

Bohn, H. (1998): "Risk Sharing in a Stochastic Overlapping Generations Economy", Working paper, UC Santa Barbara.

Brock, W. A. & L. J. Mirman (1972): "Optimal Growth and Uncertainty: The Discounted Case", *Journal of Economic Theory*, 4, 479–513.

Bullard, J. & S. Russell (1999): "An empirically plausible model of low interest rates and unbacked government debt", *Journal of Monetary Economics*, 44, 477–508.

de la Croix, D. & P. Michel (2002): *A Theory of Economic Growth - Dynamics and Policy in Overlapping Generations.* Cambridge University Press.

Diamond, P. (1965): "National Debt in a Neoclassical Growth Model", *American Economic Review*, 55(5), 1126–1150.

Dugundji, J. (1970): *Topology.* Allyn and Bacon Inc., Boston.

Elmendorf, D. & G. Mankiw (1999): "Government Debt", in *Handbook of Macroeconomics*, ed. by J. Taylor & M. Woodford, chap. 25. Elsevier Science B.V., Amsterdam a.o.

Farmer, R. (1986): "Deficits and Cycles", *Journal of Economic Theory*, 40(1), 77–88.

Galor, O. (1992): "A two-sector overlapping-generations model: A global characterization of the dynamical system", *Econometrica*, 60(6), 1351–1386.

——— (2007): *Discrete Dynamical Systems.* Springer, Berlin, a.o.

Krüger, D. & F. Kübler (2006): "Pareto-Improving Social Security Reform when Financial Markets Are Incomplete!?", *American Economic Review*, 96(3), 737–755.

Kunieda, T. (2008): "Asset bubbles and borrowing constraints", *Journal of Mathematical Economics*, 44, 112–131.

Michel, P. & B. Wigniolle (2003): "Temporary Bubbles", *Journal of Economic Theory*, 112, 173–183.

Nitecki, Z. (1971): *Differentiable Dynamics.* MIT Press, Cambridge, MA.

Tirole, J. (1985): "Asset Bubbles and Overlapping Generations", *Econometrica*, 53(6), 1499–1528.

Villanacci, A., L. Carosi, P. Benevieri & A. Battinelli (2002): *Differential Topology and General Equilibrium with Complete and Incomplete Markets.* Kluwer Academic Publishers, Boston.

Wang, Y. (1993): "Stationary Equilibria in an Overlapping Generations Economy with Stochastic Production", *Journal of Economic Theory*, 61(2), 423–435.

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