The Principles of Limit Absorption and Limit Amplitude for Periodic Operators

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Dipl. phys. Maria Barbara Radosz
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Referent: Prof. Dr. M. Plum
Korreferent: Prof. Dr. A. Kirsch

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Chapter 1

Introduction

1.1 Motivation: The principles of limit absorption, limit amplitude and radiation in homogeneous media.

An important problem in mathematical physics is the solution of the wave equation in a homogeneous medium

$$-c^2\Delta u + \frac{\partial^2 u}{\partial t^2} = f \tag{1.1}$$

on the whole of $\mathbb{R}^3$ where f is a function modeling an external force[1]. For simplicity we put $c = 1$ in this section. For time-harmonic external forces $f = e^{i\omega t}g$ one is often interested in stationary time-harmonic solutions of the form

$$u = e^{i\omega t}v.$$

Inserting u and f in (1.1) one arrives at the Helmholtz equation

$$-\Delta v - k^2 v = g \qquad (k = \omega). \tag{1.2}$$

So formally every solution of the Helmholtz equation gives rise to a time-harmonic solution of the wave equation. For a Hölder continuous compactly supported g the volume potential

[1] In the case of electromagnetic waves, f models a radiation source.

defined by

$$v^{\pm}(x) := \int_{\mathbb{R}^3} g(y) \frac{e^{\pm ik|x-y|}}{4\pi|x-y|}\, dy \quad (x \in \mathbb{R}^3)$$

is Hölder continuous and twice continuously differentiable, and for each choice of sign in the exponent it is a solution of (1.2) on $\mathbb{R}^3$. In mathematical physics, it is desirable to formulate problems involving the Helmholtz equation which admit unique solutions. On bounded domains this is achieved by imposing a boundary condition. But since we are looking for solutions on the unbounded set $\mathbb{R}^3$ we have to impose some "boundary condition at infinity". One could try to impose a decay condition like

$$v(x) \to 0 \text{ uniformly as } |x| \to \infty,$$

meaning that for all $\varepsilon > 0$ there exists a $r = r(\varepsilon)$ such that for all $x \in \mathbb{R}^3$ with $|x| > r$, $|v(x)| < \varepsilon$ holds. But it turns out that this decay condition is not sufficient to enforce unique solvability of the Helmholtz equation for every right hand side g and every value of k^2. Consider first the case $k^2 < 0$, i.e. $k = i\kappa$ for some $\kappa > 0$. The volume potentials

$$v_1(x) = \int_{\mathbb{R}^3} g(y) \frac{e^{-\kappa|x-y|}}{4\pi|x-y|}\, dy, \qquad v_2(x) = \int_{\mathbb{R}^3} g(y) \frac{e^{\kappa|x-y|}}{4\pi|x-y|}\, dy$$

are both solutions of (1.2), but only v_1 vanishes at infinity in the above sense. In this case, the uniform decay condition selects a decaying solution and seems to be suitable. But in the case $k^2 > 0$, i.e. putting $k = \kappa > 0$, we have the situation, that both potentials

$$v_1(x) = \int_{\mathbb{R}^3} g(y) \frac{e^{-i\kappa|x-y|}}{4\pi|x-y|}\, dy, \qquad v_2(x) = \int_{\mathbb{R}^3} g(y) \frac{e^{i\kappa|x-y|}}{4\pi|x-y|}\, dy$$

solve the Helmholtz equation (1.2) and tend uniformly to 0 at infinity. So the uniform decay condition does not work for all k^2. There is still a need for additional conditions at infinity to define the solution of (1.2) uniquely.

Now there are three well known possibilities to introduce additional physically motivated requirements on the solution of the Helmholtz equation, such that the solution becomes unique in these distinguished classes of solutions. These are the principles of radiation, limit absorption and limit amplitude, which turn out to be mutually equivalent in the case of homogeneous media.

The principle of radiation is often also called outgoing (Sommerfeld) radiation condition, in three space dimensions given by

$$v = O(r^{-1})$$

$$\frac{\partial v}{\partial r} + ikv \quad = \quad o(r^{-1}).$$

This condition is in particular fulfilled for a spherical wave, so it expresses the physical idea that every stationary wave field looks like a spherical wave at infinity provided the force term has compact support. Looking for solutions v satisfying (1.2) and the radiation condition selects v_1 and gives the desired uniqueness.

The principle of limit amplitude (of the first kind) tells to select the unique function v defined by the following process as the physical solution of (1.1). Consider the wave equation (1.1) with time harmonic right hand side $f = e^{i\omega t}g$ and study its solution as $t \to +\infty$. If the force term is time harmonic then for physical reasons the initial conditions (which we assume to be localized in some sense) become unimportant as $t \to +\infty$ and the solution u of the wave equation will perform forced oscillations with the frequency of the source. So as $t \to +\infty$ we expect the asymptotic behavior

$$u(x,t) \sim e^{i\omega t} v(x) \quad (t \to +\infty),$$

with a v that solves the Helmholtz equation. This unique v is then the limit amplitude solution of (1.2).

Finally the principle of limit absorption (of the first kind) is based on the idea that there is always a small positive damping in the physical system, and so one has to add a small absorption $\delta > 0$ to the wave equation and the Helmholtz equation for a more realistic physical situation. Then the damped (or absorptive) equations are

$$-\Delta u + \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = f \tag{1.3}$$

and

$$-\Delta v - (k^2 - i\delta\omega)v = g \qquad (k = \omega). \tag{1.4}$$

The unique limit absorption solution of (1.2) is given by the pointwise limit of the solutions v_δ of (1.4) as $\delta \to 0^+$.

The selected solution v is always the same for any of the three principles.

Since we speak of principles of limit amplitude and limit absorption of the first kind, this indicates that there are corresponding principles of the second kind.

For the limit absorption principle of the second kind equation (1.4) is replaced by

$$-\Delta \widetilde{v} - (k^2 + i\delta\omega)\widetilde{v} = g \qquad (k = \omega). \tag{1.5}$$

The limit absorption solution of (1.5) is the limit of solutions of this equation as $\delta \to 0^+$.

For the limit amplitude principle of second kind the right hand side f is written in the form $f = e^{-i\omega t} g$ and the asymptotic behavior of the wave equation reads now $u(x,t) \sim e^{-i\omega t}\widetilde{v}$ as $t \to +\infty$.

Third in row is the incoming (Sommerfeld) radiation condition, in three space dimensions given by

$$\begin{aligned} v &= O(r^{-1}) \\ \frac{\partial v}{\partial r} - ikv &= o(r^{-1}). \end{aligned}$$

The latter three principles select the same unique solution $\widetilde{v}$ of the Helmholtz-equation (1.2), namely $\widetilde{v} = v_2$. In general $v_1 \neq v_2$.

All facts in this section are taken from [1, 2, 3].

1.2 Limit absorption and amplitude principle, some other authors

The construction of physical solutions by the limiting absorption and limit amplitude principle has a long history (see [1, 2] and references to early papers therein). We briefly list some steps in the development of the theory of the principles of limit absorption and limit amplitude for various kinds of differential operators. The body of literature is very large. The following collection of results is not meant to be exhaustive, but rather we restrict ourselves to the most popular approaches to this field.

In [4] the limiting absorption principle is proved for the elliptic equation of the form

$$gu = -\sum_{k,l=1}^{m} \frac{\partial}{\partial x_k}\left(a_{kl}(x)\frac{\partial}{\partial x_l}\right)u + q(x)u = \lambda u + f(x)$$

for an exterior domain with bounded boundary Γ and boundary condition $u|_\Gamma = 0$. Furthermore, the differential operator g is supposed to coincide with

$$-\triangle + q \qquad (*)$$

outside a sphere and q and f decay sufficiently fast. For the proofs the space dimension is set equal to 3. $(*)$ allows to make use of the well-known Green's kernel for $-\triangle$. The method is based on a contradiction argument to show a certain a-priori bound in a weighted norm for solutions u_ϵ of the absorptive equation. Then a subsequence u_{ϵ_n} of $(u_\epsilon)_{\epsilon>0}$ converges[2] to a solution u of the equation with absorption parameter $\epsilon = 0$ in H^2-norm on every bounded interior subregion of the exterior domain and in $H^1(\Omega_\rho)$ for every ρ, where Ω_ρ is the part of the exterior domain that is contained in the sphere S_ρ. The limit element u satisfies an integral-form radiation condition

$$\lim_{\rho\to\infty}\int_{S_\rho}\left|\frac{\partial u}{\partial r}-i\sqrt{\lambda}u\right|^2\,d\sigma=0$$

which contributes uniqueness, and so for the whole sequence $u_\epsilon \to u$ holds in the above sense.

By similar methods a limiting absorption result is obtained in [4] for a region shaped like a semi-infinite cylindrical tube, where the operator g is simplified to $-\triangle$ outside a finite part of the domain and for right hand side is set to $f = 0$.

[5] extends the principles of radiation, limit amplitude and limit absorption to a class of hypoelliptic equations with constant coefficients and a class of elliptic equations that consist of a constant coefficient part of order $2m$ and a part of order $\leq 2m$ with variable compactly supported coefficients. The limiting absorption result is obtained by weak convergence of fundamental solutions of the absorptive equation and weak convergence of absorptive solutions.

In [6] the case of a magnetic Schrödinger operator

$$L=-\sum_{j=1}^{n}\left(\frac{\partial}{\partial x_j}+ib_j\right)^2+V$$

with decaying magnetic and electric potential is discussed. The limiting absorption principle is obtained as a limit of solutions of $(L-k^2-i\delta)u=f$, i.e. the limit

$$\lim_{\delta\to 0^+}(L-k^2-i\delta)^{-1}f$$

exists in a weighted L^2 space under the assumption of a unique continuation property for L. Moreover some sort of outgoing radiation condition is given in the form

$$\|\mathcal{D}u\|_{(-1+\varepsilon)/2,E_1}<\infty,$$

[2]First in local L^2-norm and then by an additional a-priori estimate in the H^2-norm.

where $\mathcal{D}u = (\mathcal{D}_1 u, \ldots, \mathcal{D}_n u)$, $\mathcal{D}_j u = (\partial_j + ib_j(x))u(x) + \frac{n-1}{2|x|}\frac{x_j}{|x|}u(x) - ik\frac{x_j}{|x|}u(x)$. The norm is a weighted L^2-norm with weight $(1+|x|)^{(-1+\varepsilon)/2}$ over the complement of the unit ball. Finally the absolute continuity for L on $(0,\infty)$ is shown using the limiting absorption method for L.

[7] (see also references therein) draws the limiting absorption principle for an operator $H = H_0 + V$ on a Hilbert space $\mathcal{H}$ from smoothness assumptions on the spectral family $E_0(\lambda)$ of H_0 and perturbation arguments on V. In particular, the spectral family $E_0(\lambda)$ is supposed to be weakly differentiable in λ with locally Hölder continuous derivative. For A_0 being the derivative of the spectral family associated to H_0 the following representation formula is given:

$$\lim_{\varepsilon\to 0}(H_0 - (\mu \pm i\varepsilon))^{-1} = \mathcal{P}\!\!\int_K \frac{A_0(\lambda)}{\lambda-\mu}\, d\lambda \pm i\pi A_0(\mu) + \int_{U\setminus K} \frac{A_0(\lambda)}{\lambda-\mu}\, d\lambda,$$

where $\mu \in K$ and $K \subset U$ are some open intervals (satisfying further conditions). For $H = H_0 + V$, the limit

$$\lim_{\varepsilon\to 0^+}(H - (\lambda \pm i\varepsilon))^{-1}$$

exists in uniform operator topology of $B(\mathcal{X}, \mathcal{X}^*_{H_0})$. Here $\mathcal{X}$ is a dense, continuously embedded subspace of $\mathcal{H}$ and $\mathcal{X}^*_{H_0} \subset \mathcal{X}^*$ is to be chosen suitably. The operator $V : \mathcal{X}^*_{H_0} \to \mathcal{X}$ is assumed to be compact. The method is then applied to the operator

$$H = -\frac{\partial^2}{\partial x_1^2} - x_1 + q(x_1) + T_{x'} + V(x), \quad (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$$

with twice continuously differentiable periodic q and self-adjoint operator $T_{x'}$ (semibounded from below) and V as above. The second application is the Schrödinger operator

$$H = -\triangle + V.$$

The absorptive resolvents converge in the operator norm topology between a weighted L^2 space and a suitably weighted Sobolev space. Finally some other classes of operators are treated.

In [8] the limiting absorption principle for an analytically fibered operator H_0 is derived in an abstract way from a so-called Mourre estimate for the commutator of H_0 with a conjugate operator A

$$\chi_\triangle(H_0)[H_0, iA]\chi_\triangle(H_0) \geq c_0 \chi_\triangle(H_0) + K,$$

where $\chi_{\triangle}(H_0)$ is the spectral projection on the interval $\triangle \subset \mathbb{R}$ and K is a compact operator. The limit

$$\lim_{\varepsilon \to 0}(H_0 - \lambda \pm i\varepsilon)^{-1}$$

for $\lambda \notin \sigma_p(H_0)$ exists as a bounded operator between suitable weighted spaces. This result is applied to perturbations of a periodic Schrödinger operator in [9].

The Mourre-estimate approach is applied to various differential operators and geometries. It can also be used to obtain resolvent estimates in Besov spaces (see [10, 11, 12, 13]).

Concerning periodic operators there is some interest coming from scattering theory in crystals. An important object in this field is the Green's function as the integral kernel of the resolvent. The periodicity of the medium suggests to use the Floquet-Bloch transform for a representation of the resolvent. In [14] (which also contains references to the application of the limiting absorption principle in scattering theory) this method is used to show the existence of the limit in $L^2_{\text{loc.}}(\mathbb{R}^d)$ of solutions

$$\lim_{\delta \to 0}(L - \lambda \mp i\delta)^{-1} f$$

for a second order elliptic operator L on $\mathbb{R}^d$, $d \geq 2$, with periodic coefficients and right hand side $f \in L^2(\mathbb{R}^d)$ with compact support. As a crucial requirement λ has to be close to the bottom of the spectrum of L. For the integral kernel of $(L - \lambda \mp i0)^{-1}$ an asymptotic formula for $|x| \to \infty$ is derived. The operator $\left(\frac{\partial}{\partial \lambda}\right)^m (L - \lambda \mp i0)^{-1}$ between the Besov spaces $B_{m+\frac{1}{2}}$ and $B^*_{m+\frac{1}{2}}$ is shown to be bounded.

The limit amplitude principle, in comparison to the limiting absorption principle, seems to play a subordinate role in the literature. A reason is that often one of the essential ingredients in the proof of the limit amplitude principle is the validity of the limiting absorption principle. Thus, many of the authors who study the limiting absorption principle, also derive versions of the limit amplitude principle (see [15, 16, 17, 18]).

1.3 Photonic crystals, periodic media

For reference to this section see [19, 20, 21].

A (3-dimensional) photonic crystal is a piece of optical material whose dielectric "constant" varies periodically with respect to some fixed spatial lattice. This discrete periodicity creates physical effects that are not present in homogeneous media, such as the existence of band gaps, i.e. frequency regions such that light at these frequencies cannot propagate in the crystal due to destructive interference. Not every periodic structure necessarily has band gaps, but photonic crystals with band gaps have already been fabricated. The periodicity of the dielectric constant with respect to some lattice is expressed by the relation $\varepsilon(r+R) = \varepsilon(r)$ for all $r \in \mathbb{R}^3$ and all lattice vectors R. Without loss of generality we take $\mathbb{Z}^3$ as the periodicity lattice, since any other 3-dimensional lattice can be produced from $\mathbb{Z}^3$ by a linear transformation. Since periodic functions can be expressed by their Fourier transform, a second lattice becomes important: the dual or reciprocal lattice. It is determined as the lattice build up from vectors q such that for all lattice vectors R there exists a $n \in \mathbb{Z}$ with $q \cdot R = n2\pi$. So the dual lattice of $\mathbb{Z}^3$ is $2\pi\mathbb{Z}^3$. Propagation of electromagnetic radiation is governed by the Maxwell equations. For time harmonic electric and magnetic fields E, H respectively, and the magnetic permeability $\mu = 1$ one can derive the following master equation for free propagation without a source

$$\nabla \times \left(\frac{1}{\varepsilon(r)} \nabla \times H(r) \right) = \frac{\omega^2}{c_0^2} H(r)$$

with divergence condition $\nabla \cdot H(r) = 0$, or alternatively

$$\frac{1}{\varepsilon(r)} \nabla \times (\nabla \times E(r)) = \frac{\omega^2}{c_0^2} E(r)$$

with divergence condition $\nabla \cdot (\varepsilon E) = 0$. To exploit the discrete periodicity of the medium the fields are expanded into Bloch modes. For illustration we focus on the H field. The Bloch modes H_{sk}, where s is a discrete index and $k \in \mathbb{R}^3$, are divergence free eigenfunctions of $\nabla \times \frac{1}{\varepsilon(r)} \nabla \times$ with eigenvalues $\frac{\omega_s(k)^2}{c_0^2}$ i.e.

$$\nabla \times \left(\frac{1}{\varepsilon(r)} \nabla \times H_{sk} \right) = \frac{\omega_s(k)^2}{c_0^2} H_{sk}.$$

Moreover, the H_{sk} are k-quasiperiodic, i.e. $H_{sk}(r+n) = e^{ink} H_{sk}(r)$ for any $n \in \mathbb{Z}^3$. k is the wave vector of H_{sk} and the functions ω_s are called band functions. Because of the k-quasiperiodicity, all information characterizing wave propagation in the crystal is contained in the set of Bloch modes and band functions with wave vector $k \in [-\pi, \pi]^3$. This set of wave vectors is called Brillouin zone (B or BZ).

Starting point for this work was the Green's function for the photonic crystals, which is frequently and successfully used in physics literature and given by the formula (here for the electric field E)

$$G_0(r,r';\omega) = \sum_s \int_{BZ} d^3k\, \frac{E^*_{sk}(r) \otimes E_{sk}(r')}{\omega^2_{sk}/c_0^2 - (\omega/c_0 + i0_+)^2}. \tag{1.6}$$

Clearly the Green's function is a formal limit of absorptive Green's functions

$$G_\delta(r,r';\omega) = \sum_s \int_{BZ} d^3k\, \frac{E^*_{sk}(r) \otimes E_{sk}(r')}{\omega^2_{sk}/c_0^2 - (\omega/c_0 + i\delta)^2}.$$

The solution of

$$\frac{1}{\varepsilon(r)} \nabla \times (\nabla \times E(r)) - \left(\frac{\omega}{c_0} + i\delta\right)^2 E(r) = f \tag{1.7}$$

for $f \in L^2$ is a L^2-function given by

$$E^\delta(r) = \int_{\mathbb{R}^3} G_\delta(r,r';\omega) f(r')\, dr'.$$

If $\omega \neq \omega_{sk}$ for all s,k, which means that ω is in a band gap of the crystal, there is no singularity in the denominator of the Green's function, when $\delta \to 0$. So E^δ converges in L^2 to some unique $E^0 \in L^2$ solving (1.7) with $\delta = 0$. The fast decaying solution[3] corresponds to the fact that there is no light propagation in the crystal. Also for ω in a band, i.e. if $\omega = \omega_{sk}$ for one or several s,k, the notation (1.6) is used in physics literature. But now it is not clear in which sense the limit $\delta \to 0$ is to be taken. Moreover, if the frequency ω lies in a band, propagation of radiation in the crystal is possible, i.e. one expects a solution that decays more slowly than a L^2-function (the decay depending on the space dimension) when $|r|$ goes to infinity. Moreover, the propagating radiation should be carried by modes that correspond to the frequency ω. The formal expression (1.6) provides no such information on propagating parts of the solution.

Motivated by these questions, we want to show in this work a limiting absorption principle for a class of periodic operators in order to define limiting absorption solutions rigorously in an appropriate sense. The representation formulas will reveal some more information on propagating and evanescent parts of the solutions. Having established a limiting absorption principle for periodic operators, it is also interesting to prove a limit amplitude principle. The kind of limiting absorption principle presented in this work admits an equivalent limit amplitude principle.

[3]If ω is in a band gap, one can show that the solution decays exponentially (see [22]).

1.4 The class of Floquet-Bloch decomposable operators

Let $d \geq 1$ be the space dimension and $M \geq 1$. $\Omega = [0,1]^d$ denotes the cell of periodicity and $B = [-\pi, \pi]^d$ the Brillouin zone. Let $\mathcal{J} \subseteq \mathbb{Z}$ be fixed subset of the integers (not necessarily bounded or semibounded). In this work we consider operators $\mathcal{L} : L^2(\mathbb{R}^d, \mathbb{C}^M) \supseteq D(\mathcal{L}) \to L^2(\mathbb{R}^d, \mathbb{C}^M)$ which fulfill the following five requirements

(1) $\mathcal{L}$ is densely defined, all compactly supported smooth functions are in the domain of $\mathcal{L}$, and $\mathcal{L}\varphi$ is a compactly supported smooth function provided φ is one[4].

(2) There exists a family of operators $(\mathcal{L}_k)_{k \in B}$,

$$\mathcal{L}_k : D(\mathcal{L}_k) \subseteq L^2(\Omega, \mathbb{C}^M) \to L^2(\Omega, \mathbb{C}^M),$$

a family of B-periodic[5] real valued Lipschitz functions $(\lambda_s)_{s \in \mathcal{J}}$ on B and for each $k \in B$ a $L^2(\Omega)$-orthonormal complete system

$$\{\psi_s(\cdot, k) : \ s \in \mathcal{J}\}$$

of k-quasiperiodic eigenfunctions of $\mathcal{L}_k$ on Ω with corresponding eigenvalues $\lambda_s(k)$, i.e.

$$\begin{aligned} \mathcal{L}_k \psi_s(\cdot, k) &= \lambda_s(k) \psi_s(\cdot, k) \quad (s \in \mathcal{J}, \ k \in B) \\ \psi_s(x + n, k) &= e^{ikn} \psi_s(x, k) \quad (n \in \mathbb{Z}^d, \ x \in \mathbb{R}^d), \end{aligned}$$

where $kn = \sum_{i=1}^d k_i n_i$ for $k = (k_1, \ldots, k_d), \ n = (n_1, \ldots, n_d)$. The functions λ_s are called the *band functions of* $\mathcal{L}$ and the $\psi_s(\cdot, k)$ are called the *Bloch waves of* $\mathcal{L}$.

(3) The mapping $k \mapsto \psi_s(\cdot, k) \in L^2(\Omega, \mathbb{C}^M)$ is measurable[6].

(4) For any fixed $k \in B$ the eigenspace of $\lambda_s(k)$ is finite dimensional and the sequence of eigenvalues $(\lambda_s(k))_{s \in \mathcal{J}}$ is ordered by magnitude and multiplicity (i.e. $\ldots \leq \lambda_s(k) \leq \lambda_{s'}(k) \leq \ldots$ if $s, s' \in \mathcal{J}, \ s \leq s'$). For any compact interval $I \subset \mathbb{R}$ there are only finitely many s with $I \cap \lambda_s(B) \neq \emptyset$.

---

[4] If $\mathcal{L}$ is a differential operator, for simplicity we assume the coefficients to be in C^∞.

[5] i.e. having the same values on opposite faces of B

[6] Since $L^2(\Omega, \mathbb{C}^M)$ is a separable Hilbert space, weak, strong and Borel-measurability are equivalent (see [23], Theorem IV.22).

(5) Let

$$D(\mathcal{L})=\{u\in L^2(\mathbb{R}^d,\mathbb{C}^M):\sum_{s\in\mathcal{J}}\int_B \lambda_s(k)\langle Uu(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(\cdot,k)\,dk \text{ converges in } L^2(\mathbb{R}^d,\mathbb{C}^M)\}.$$

(The series converges in $L^2(\mathbb{R}^d,\mathbb{C}^M)$ if

$$\sum_{s\in\mathcal{J}\cap\{n\in\mathbb{Z}:\ |n|\geq l\}}\int_B \lambda_s(k)\langle Uu(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)\,dk \overset{l\to\infty}{\longrightarrow} 0$$

in $L^2(\mathbb{R}^d,\mathbb{C}^M)$.)

For $u\in D(\mathcal{L})$ we have the following decomposition of $\mathcal{L}$ with respect to the Floquet-Bloch transform U (for definition of U, see section 5.1 in the appendix)

$$\begin{aligned}\mathcal{L}u(x) &= U^{-1}\mathcal{L}_k Uu(x)\\ &= \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \lambda_s(k)\langle Uu(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)\,dk \end{aligned}\tag{1.8}$$

where the series converges in $L^2(\mathbb{R}^d,\mathbb{C}^M)$.

In chapter 4 exclusively we make the following additional hypothesis:

(6) The spectrum of $\mathcal{L}$ is absolutely continuous and $\lambda_s>0$ for all $s\in\mathcal{J}$.

It follows that $\mathcal{L}$ is selfadjoint and

$$\sigma(\mathcal{L})=\bigcup_{s\in\mathcal{J}}\lambda_s(B).$$

The following representation is most important for this work. If $\lambda\notin\sigma(\mathcal{L})$, $f\in L^2(\mathbb{R}^d,\mathbb{C}^M)$ then

$$(\mathcal{L}-\lambda)^{-1}f(x)=\frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \frac{\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)}{\lambda_s(k)-\lambda}\,dk. \tag{1.9}$$

This framework fits to many operators in physical and mathematical applications. For example all self-adjoint elliptic differential operators with smooth periodic (with respect to some lattice) coefficients are Floquet-Bloch decomposable (see [24], which also contains more classes of operators compatible with the Floquet-Bloch transform). The first order Maxwell system with smooth periodic dielectric and magnetic constants as well as the corresponding second order Maxwell operator $\frac{1}{\varepsilon(x)}$curl curl are Floquet-Bloch decomposable using suitable operator domains (see [25]). Concerning the absolute continuity of the spectra of periodic operators, see [26] and [27].

The reason for introducing the abstract class of Floquet-decomposable operators instead of working for example with a concrete elliptic differential operator is that the limit absorption and limit amplitude principle result from the Bloch-wave expansion of the operator and its resolvent (1.9) and not from other properties of the operator such as ellipticity or the order.

1.5 Contents and scope of this work

In this work we will prove a variant of limit absorption and limit amplitude principle for operators $\mathcal{L}$ as described in section 1.4. The principles work for any dimension including $d = 1$, though applying the Floquet-Bloch transform with respect to just one of the variables stronger results can be derived using methods of complex analysis, in particular by deforming the integral over B in the representation formula for the resolvent into the complex plane to avoid singularities (see e.g. [28]). Notice that for a d-dimensional periodicity lattice, it is also possible to perform the Floquet-Bloch transform with respect to less than d variables, i.e. to exploit just a part of the periodicity and ignore the rest. For example consider the s-dimensional Floquet-Bloch transform

$$U_s f(x_1, \dots, x_d, k) = \frac{1}{\sqrt{|B_s|}} \sum_{n \in \mathbb{Z}^s} f((x_1, \dots, x_s) - n, x_{s+1}, \dots, x_d) e^{ink}$$

with $B_s = [-\pi, \pi]^s$, $k = (k_1, \dots, k_s)$, $x_{s+1}, \dots, x_d$ are regarded as parameters. So the notion of dimensionality refers rather to the number of variables, which are involved in the Floquet-Bloch transform than to the dimension of the space $\mathbb{R}^d$. In this work we will exploit the full given periodicity, i.e. $U_s = U_d = U$.

Unlike other authors ([15, 7, 29]) we will not exploit Hölder continuity of the resolvent, of the derivative of the spectral family or of the solution. The only smoothness assumption is the Lipschitz condition on the band functions λ_s. We will see that this implies that (with $A(\lambda) = \frac{d}{d\lambda} E(\lambda)$ denoting the derivative of the spectral family $E(\lambda)$ associated with $\mathcal{L}$) for any $f \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ the mapping $\mu \mapsto A(\mu) f$ is is at least in L^2 for μ in an interval of regular (in the sense of definition 3.6) values of the band functions (regular frequencies), but it is not clear whether it is Hölder continuous. Thus we will develop a method, which uses only the L^2-property. We will work directly with the Bloch waves ψ_s and not with full projections on eigenspaces. But since no more smoothness of the Bloch waves is needed than to be measurable in k, multiple eigenvalues will not cause more problems than simple ones. The cost for doing without stronger smoothness assumptions is that we will not show convergence of absorptive resolvents with respect to the operator norm (hence there are no resolvent estimates in this work), neither we will try to extend the resolvent analytically across the real line. Instead we will show the *convergence of solutions* u_δ of absorptive equations

$$(\mathcal{L} - (\lambda + i\delta))u = f \tag{1.10}$$

in a suitable sense. The right hand side $f = f(\lambda, \cdot) \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ can explicitly depend on λ as long as the dependence is Lipschitz. If f is not constant in λ, for technical reasons we demand f to have compact support in $\mathbb{R}^d$. If f is constant in λ we can do without compact support and just assume $f \in L^2(\mathbb{R}^d, \mathbb{C}^M)$. We would like to mention that we do not use or derive any sort of (integral- or differential-form) radiation condition in this work.

It is the character of this work, that the results are obtained in a very direct way, i.e. by direct computations and estimations.

The key to our method to prove the principles of limit absorption and limit amplitude for the class of operators introduced in 1.4 is to regard the solution u_δ of the absorptive equation

$$(\mathcal{L} - \lambda - i\delta)u = f$$

as a function in the variables $x \in \mathbb{R}^d$ *and* $\lambda \in \mathbb{R}$.

In chapter 2 a distributional formulation of the limit absorption principle is formulated rigorously. We will show, that for all test functions $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{C}^M)$, $u_\delta[\varphi]$, where u_δ is regarded as a distribution in x and λ, converges to some $u^\pm[\varphi]$ as $\delta \to 0^\pm$, defining for each choice of sign a distribution $u^\pm$ being a distributional solution of $(\mathcal{L} - \lambda - i0^\pm)u = f$, i.e.

$$u^\pm[(\mathcal{L} - \lambda)\varphi] = f[\varphi] \quad (\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{C}^M)).$$

We then call u^- the distributional limit absorption solution of the first kind and u^+ the distributional limit absorption solution of the second kind in analogy to section 1.1. The distributional formulation has the great advantage that absolutely no restrictions on the range of λ are needed. Moreover we gain a semi-explicit representation formula for the limit distributions.

In chapter 3 we strive to establish a limit absorption principle for $\mathcal{L}$ in a function space. To this end, we have to make restrictions on the range of λ. We demand λ to be regular in some sense, i.e. the band functions λ_s have - in some sense - non-vanishing derivatives on the level set corresponding to λ. The physical interpretation of regularity of a frequency λ is that all Bloch waves $\psi_s(\cdot, k)$ with band functions crossing the frequency λ have non vanishing group velocity, and thus propagate in space. Then for an interval $I_0 \subset \mathbb{R}$ of regular frequencies and a suitable weight function w, we show the convergence of solutions $u_\delta = u_\delta(\lambda, x) \in L^p(I_0, L^2(\mathbb{R}^d; w(x)dx); d\lambda)$ of the absorptive equation

$$(\mathcal{L} - \lambda - i\delta)u = f$$

to some function $u^\pm(\lambda, x)$ in the space $L^p(I_0, L^2(\mathbb{R}^d; w(x)dx); d\lambda)$ as $\delta \to 0^\pm$. For each choice of sign, the function $u^\pm$ solves the limit equation $(\mathcal{L} - \lambda)u = f$ distributionally, i.e.

$$u^\pm[(\mathcal{L} - \lambda)\varphi] = f[\varphi] \quad (\varphi \in C_0^\infty(I_0 \times \mathbb{R}^d, \mathbb{C}^M)).$$

An important step to show the convergence is to realize that the inverse Floquet-Bloch transform can be restricted to operate on functions defined on the product of the periodicity cell and a level set of a band function. The restricted inverse Floquet-Bloch transform takes functions which are in L^2 on the product of the periodicity cell Ω and the level set of λ_s under consideration to functions on the whole $\mathbb{R}^d$, which are in a weighted L^2-space. Not only the existence of the limit is shown, but also a representation formula in terms of the Bloch waves is given. The structure of the limit reveals parts of the solution with different decay behavior, which are expected from a physical point of view.

In chapter 4 we show a limit amplitude principle for $\mathcal{L}$. Our starting point for this chapter is the exposition of the limit amplitude principle in the book [30] (Chapter XVII.B.§4.) which works for all frequencies $\lambda = \omega^2 \notin \sigma(\mathcal{L})$. We extend these results to regular frequencies ω of $\mathcal{L}^{1/2}$, for which $\omega^2 \in \sigma(\mathcal{L})$ holds. The assumption (6) in section 1.4 is made in order to use some results from [30]. Then by properties of the Hilbert transform and its Fourier multiplier we will show, that for the solution of the wave equation $u(x,t)$ with time periodic right hand side

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \mathcal{L}u = e^{i\omega t} g, \\ u(0) = u^0, \\ \frac{\partial u}{\partial t}(0) = u^1, \end{cases} \tag{1.11}$$

the following asymptotic relation

$$u \sim e^{i\omega t} v^-$$

holds in some sense as $t \to +\infty$, where v^- is the limit absorption solution of the first kind of the corresponding Helmholtz equation. To achieve this asymptotic behavior, the solution $u(t, \cdot)$ of the Cauchy problem (1.11) for any fixed t is regarded as an element of $L^2(I, L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx); d\omega)$, where $I \subset (0, \infty)$ is an interval of regular frequencies of $\mathcal{L}^{1/2}$ and w is a suitable weight function. Furthermore the well-known representation of the solution u of the wave equation

$$\begin{aligned} u(t) &= \cos(\mathcal{L}^{1/2}t)u^0 + \mathcal{L}^{-1/2}\sin(\mathcal{L}^{1/2}t)u^1 \\ &\quad + \int_0^t \mathcal{L}^{-1/2}\sin\left(\mathcal{L}^{1/2}(t-\sigma)\right) e^{i\omega\sigma} g \, d\sigma, \end{aligned}$$

with initial conditions u^0, u^1 and a time harmonic source $e^{i\omega t}g$ will be extremely helpful. The principles of limit amplitude and limit absorption presented in this work are equivalent, in the sense that they select the same physical solution v of the problem

$$(\mathcal{L} - \lambda)v = g$$

for regular $\lambda \in \sigma(\mathcal{L})$.

Chapter 2

Distributional limiting absorption principle

In this chapter we will state and prove a limiting absorption principle for distributional solutions of the Helmholtz-like equation

$$(\mathcal{L} - \lambda)u = f \tag{2.1}$$

on the entire $\mathbb{R}^d$, $d \geq 1$, for the class of Floquet-decomposable operators introduced in section 1.4. The fact that the absorptive solutions u_δ of

$$(\mathcal{L} - \lambda - i\delta)u = f$$

converge for $\delta \to 0^\pm$ in distributional sense with respect to the frequency variable and the spatial variables to limit distributions which are a distributional solutions of (2.1) will enable us to avoid regularity conditions on the range of frequencies. Thus the range of λ may even include band intersections and overlaps, local extrema of the bands, saddle point frequencies or eigenvalues of $\mathcal{L}$. We will regard the absorptive solution as a distribution in the variables x and λ and decompose it via the Floquet-Bloch transform. Then, based on a well known theorem about Cauchy principal value integrals, we will guess from its structure the limit distributions, prove the crucial semi-norm estimates as well as the convergence and the solution property.

2.1 Distributional formulation of the problem.

Throughout this chapter we make the following assumptions on the right hand side f.

- $\begin{cases} f : \mathbb{R} \times \mathbb{R}^d & \to & \mathbb{C}^M \\ \quad (\lambda, x) & \mapsto & f(\lambda, x) \end{cases}$
- $\exists\, \mathcal{K} \subseteq \mathbb{R}^d, \mathcal{K}$ compact $: \forall \lambda \in \mathbb{R} : \text{supp} f(\lambda, \cdot) \subseteq \mathcal{K}$
- $\forall \lambda \in \mathbb{R} : f(\lambda, \cdot) \in L^2(\mathbb{R}^d, \mathbb{C}^M)$
- $\lambda \mapsto f(\lambda, \cdot) \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ is locally Lipschitz continuous with local Lipschitz constant

  $$\text{Lip}(f)_{\mathfrak{C}} = \inf \Big\{ C > 0 : \|f(x, \cdot) - f(y, \cdot)\|_{L^2(\mathbb{R}^d, \mathbb{C}^M)} \leq C|x - y|,\ x, y \in \mathfrak{C} \Big\}$$

  for any compact $\mathfrak{C} \subseteq \mathbb{R}$.

Thus, f may explicitly depend on λ. In the most common case when f is constant in λ, the assumption on the compact support of f can be dropped, which will become apparent in the estimations below.

Consider the distributional version of the problem (2.1) in the variables x and λ.

Definition 2.1 (Distributional solution). *With a given f as above, a distribution u acting on test functions $\varphi = \varphi(\lambda, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{C}^M)$ is called a* distributional solution of (2.1) *if for all $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{C}^M)$*

$$u[(\mathcal{L} - \lambda)\varphi] = \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(\lambda, x)\varphi(\lambda, x)\, dx\, d\lambda. \tag{2.2}$$

In the right integrand the product of $f(\lambda, x)$ with $\varphi(\lambda, x)$ is the standard inner product in $\mathbb{C}^M$, which will be used without mentioning from now on. For $\delta \neq 0$ we have, because of the self-adjointness of $\mathcal{L}$, that $\lambda + i\delta \notin \sigma(\mathcal{L})$ and so we have a L^2-solution u_δ of (2.1) with fixed λ given by the Floquet-Bloch decomposition of $(\mathcal{L} - (\lambda + i\delta))^{-1} f$:

$$u_\delta(\lambda, x) = \frac{1}{\sqrt{|B|}} \sum_{s \in \mathcal{J}} \int_B \frac{\langle (Uf)(\lambda, \cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega, \mathbb{C}^M)}}{\lambda_s(k) - (\lambda + i\delta)} \psi_s(x, k)\, dk. \tag{2.3}$$

We can interpret u_δ as a distribution in the variables λ and x acting on test functions $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d, \mathbb{C}^M)$ by

$$u_\delta[\varphi] = \int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_B \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}}{\lambda_s(k)-(\lambda+i\delta)} \psi_s(x,k)\, dk \;\; \varphi(\lambda,x)\, dx\, d\lambda. \tag{2.4}$$

Note that the integral is well defined since $u_\delta \in L^2_{\text{loc.}}(\mathbb{R}\times\mathbb{R}^d, \mathbb{C}^M; d\lambda\otimes dx)$. Then u_δ is a distributional solution of the problem

$$(P_\delta) \qquad u[(\mathcal{L}-(\lambda+i\delta))\varphi] = \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(\lambda,x)\varphi(\lambda,x)\, dx\, d\lambda.$$

Now we are interested in the limit of $u_\delta[\varphi]$ as $\delta\to 0$. It will turn out that the limits are different whether $\delta\to 0$ from above or from below. We will treat both possibilities simultaneously.

Notation: To keep the notation simpler, we will suppress the target space $\mathbb{C}^M$ in the notation in the proofs. Furthermore

$$\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(\Omega,\mathbb{C}^M)}.$$

2.2 Determination of the limit distribution $u^\pm = \lim_{\delta\to 0^\pm} u_\delta$

The aim is now to show, that the limits

$$\lim_{\delta\to 0^\pm} u_\delta$$

exist in distributional sense and solve the problem (P_0).

Interchanging the order of integrations in (2.4), which will be justified by the estimations in proof of theorem 2.8, we see

$$u_\delta[\varphi] = \frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \int_{\mathbb{R}} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}}{\lambda_s(k)-(\lambda+i\delta)} \left(\int_{\mathbb{R}^d} \psi_s(x,k)\varphi(\lambda,x)\, dx\right) d\lambda\, dk. \tag{2.5}$$

The integral over $\mathbb{R}^d$ can be reduced to an integral over the periodicity cell, by splitting $\mathbb{R}^d$ into translates of Ω, using the Floquet-Bloch transform and the k-quasiperiodicity of the Bloch waves ψ_s. This is done in the next

Lemma 2.2. $\int_{\mathbb{R}^d} \varphi(\lambda, x)\psi_s(x,k)\,dx = \sqrt{|B|}\;\overline{\langle U\overline{\varphi}(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle}_{L^2(\Omega,\mathbb{C}^M)}$

Proof.

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(\lambda, x)\psi_s(x,k)\,dx \\
=\;& \sum_{j\in\mathbb{Z}^d}\int_\Omega \varphi(\lambda, x-j)\psi_s(x-j,k)\,dx = \sum_{j\in\mathbb{Z}^d}\int_\Omega e^{-ikj}\varphi(\lambda,x-j)\psi_s(x,k)\,dx \\
=\;& \sum_{j\in\mathbb{Z}^d}\int_\Omega \overline{e^{ikj}}\varphi(\lambda,x-j)\psi_s(x,k)\,dx = \int_\Omega \psi_s(x,k)\overline{\sum_{j\in\mathbb{Z}^d} e^{ikj}\overline{\varphi(\lambda,x-j)}}\,dx \\
=\;& \sqrt{|B|}\;\int_\Omega \psi_s(x,k)\overline{U\overline{\varphi}(\lambda,x,k)}\,dx = \sqrt{|B|}\;\langle \psi_s(\cdot,k), U\overline{\varphi}(\lambda,\cdot,k)\rangle_{L^2(\Omega)},
\end{aligned}$$

where we have used the k-quasiperiodicity of $\psi_s(\cdot,k)$. □

Since the the expression in the numerator in (2.5) will appear very often, we define

Definition 2.3.

$$\Phi_{s,k}(\lambda) := \sqrt{|B|}\;\overline{\langle U\overline{\varphi}(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle}_{L^2(\Omega,\mathbb{C}^M)}\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}.$$

Now we would like to apply theorem 5.6 from the appendix on every term in the series (2.5). Then the limit for each term, i.e. for fixed s and k, would be

$$\mathcal{P}\!\!\int_{\mathbb{R}} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k)-\lambda}\,d\lambda \pm i\pi\Phi_{s,k}(\lambda_s(k)),$$

provided $\Phi_{s,k}$ satisfies the requirements of theorem 5.6. This is indeed true by the following

Lemma 2.4. *$\Phi_{s,k}$ is locally Lipschitz continuous with a Lipschitz constant Lip$(\Phi_{s,k})_{\mathfrak{C}}$ depending on φ and f, but not on k and s.*

Proof. Let supp$(\varphi) \subseteq K_1 \times K_2 \subseteq \mathbb{R}\times\mathbb{R}^d$, s,k fixed. Let $\lambda, \widetilde{\lambda}$ lie in a compact set $\mathfrak{C}\subseteq\mathbb{R}$. First we use the definition of $\Phi_{s,k}$ (skipping the subscripts), insert cross terms and use the triangle inequality:

$$\begin{aligned}
\frac{1}{\sqrt{|B|}}|\Phi_{s,k}(\lambda)-\Phi_{s,k}(\widetilde{\lambda})| \;=\;& \Big|\overline{\langle U\overline{\varphi}(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle}\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle \\
& - \overline{\langle U\overline{\varphi}(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle}\langle Uf(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle\Big|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \Big|\overline{\langle U\overline{\varphi}(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle}\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle \\
&\quad - \overline{\langle U\overline{\varphi}(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle}\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle\Big| \\
&+\Big|\overline{\langle U\overline{\varphi}(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle}\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle \\
&\quad - \overline{\langle U\overline{\varphi}(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle}\langle Uf(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle\Big|.
\end{aligned}
$$

Then we rearrange and use linearity of U and Cauchy-Schwarz inequality on the inner products, noting that $\|\psi_s(\cdot,k)\|_{L^2(\Omega)} = 1$:

$$
\begin{aligned}
& \frac{1}{\sqrt{|B|}}|\Phi_{s,k}(\lambda) - \Phi_{s,k}(\widetilde{\lambda})| \\
\leq\ & |\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle||\langle U\overline{\varphi}(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle - \langle U\overline{\varphi}(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle| \\
& +|\langle U\overline{\varphi}(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle||\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle - \langle Uf(\widetilde{\lambda},\cdot,k),\psi_s(\cdot,k)\rangle| \\
\leq\ & \|Uf(\lambda,\cdot,k)\|_{L^2(\Omega)} \left\|U\overline{\varphi}(\lambda,\cdot,k) - U\overline{\varphi}(\widetilde{\lambda},\cdot,k)\right\|_{L^2(\Omega)} \\
& + \left\|U\overline{\varphi}(\widetilde{\lambda},\cdot,k)\right\|_{L^2(\Omega)} \left\|Uf(\lambda,\cdot,k) - Uf(\widetilde{\lambda},\cdot,k)\right\|_{L^2(\Omega)}.
\end{aligned}
$$

Next we use the estimate from lemma 5.3 in the appendix, noting that f and φ have compact support with respect to the variable x:

$$
\begin{aligned}
& \frac{1}{\sqrt{|B|}}|\Phi_{s,k}(\lambda) - \Phi_{s,k}(\widetilde{\lambda})| \\
\leq\ & \frac{C(\text{supp}(f),\text{supp}(\varphi))}{|B|}\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)} \left\|\varphi(\lambda,\cdot) - \varphi(\widetilde{\lambda},\cdot)\right\|_{L^2(\mathbb{R}^d)} \\
& +\frac{C(\text{supp}(\varphi),\text{supp}(f))}{|B|}\left\|\varphi(\widetilde{\lambda},\cdot)\right\|_{L^2(\mathbb{R}^d)}\left\|f(\lambda,\cdot) - f(\widetilde{\lambda},\cdot)\right\|_{L^2(\mathbb{R}^d)} \\
\leq\ & \frac{C(\text{supp}(f),\text{supp}(\varphi))}{|B|}\sup_{\lambda\in\mathfrak{C}}\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\,\text{Lip}(\varphi)_{\mathfrak{C}}|\lambda - \widetilde{\lambda}| \qquad (2.6)\\
& +\frac{C(\text{supp}(f),\text{supp}(\varphi))}{|B|}\sup_{\widetilde{\lambda}\in\mathfrak{C}}\left\|\varphi(\widetilde{\lambda},\cdot)\right\|_{L^2(\mathbb{R}^d)}\text{Lip}(f)_{\mathfrak{C}}|\lambda - \widetilde{\lambda}| \\
\leq\ & C(\mathfrak{C},\varphi,f)|\lambda - \widetilde{\lambda}|.
\end{aligned}
$$

In the second last step we use that $\lambda \mapsto \varphi(\lambda,\cdot) \in L^2(\mathbb{R}^d)$ and $\lambda \mapsto f(\lambda,\cdot) \in L^2(\mathbb{R}^d)$ are locally Lipschitz and hence are also locally bounded. □

By now we have a clear idea how the limit distribution should look like and so we make the following definition.

Definition 2.5. *For $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{C}^M)$ define the distributions $u^\pm$ by*

$$u^\pm[\varphi] := \frac{1}{\sqrt{|B|}} \int_B \left[\sum_{s \in \mathcal{J}} \mathcal{P}\!\!\int_{\mathbb{R}} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k) - \lambda} \, d\lambda \pm \sum_{s \in \mathcal{J}} i\pi \Phi_{s,k}(\lambda_s(k)) \right] dk. \tag{2.7}$$

In the proof of the next theorem we will show that the sum

$$\sum_{s \in \mathcal{J}} \mathcal{P}\!\!\int_{\mathbb{R}} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k) - \lambda} \, d\lambda$$

converges absolutely and that for each choice of sign $u^\pm$ is indeed a distribution.

Theorem 2.6. *For all compact $K = K_1 \times K_2 \subseteq \mathbb{R} \times \mathbb{R}^d$ there exist $C_1, C_2 \geq 0$ depending only on K, f and other system constants such that for all $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{C}^M)$ with support in K*

$$|u^\pm[\varphi]| \leq C_1 \left\|\varphi\right\|_\infty + C_2 \|\frac{\partial}{\partial \lambda} \varphi\|_\infty$$

holds.

Proof. Applying lemma 5.5 from the appendix to $\Phi_{s,k}$ we obtain ($\Phi_{s,k}$ is Lipschitz by lemma 2.4), noting that the support of $\Phi_{s,k}$ is contained in K_1:

$$\begin{aligned}
\left| \mathcal{P}\!\!\int_{\mathbb{R}} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k) - \lambda} \, d\lambda \right| &= \left| \int\limits_{|\lambda_s(k) - \lambda| > \epsilon} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k) - \lambda} \, d\lambda + \mathcal{P} \int\limits_{\lambda_s(k) - \epsilon}^{\lambda_s(k) + \epsilon} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k) - \lambda} \, d\lambda \right| \\
&= \left| \int\limits_{|\lambda_s(k) - \lambda| > \epsilon} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k) - \lambda} \, d\lambda + \int\limits_{\lambda_s(k) - \epsilon}^{\lambda_s(k) + \epsilon} \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k) - \lambda} \, d\lambda \right| \\
&\leq \int\limits_{|\lambda_s(k) - \lambda| > \epsilon} \frac{|\Phi_{s,k}(\lambda)|}{|\lambda_s(k) - \lambda|} \, d\lambda + \int\limits_{\lambda_s(k) - \epsilon}^{\lambda_s(k) + \epsilon} \left| \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k) - \lambda} \right| \, d\lambda \\
&\leq \frac{1}{\epsilon} \int_{K_1} |\Phi_{s,k}(\lambda)| \, d\lambda + \int\limits_{\lambda_s(k) - \epsilon}^{\lambda_s(k) + \epsilon} \left| \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k) - \lambda} \right| d\lambda.
\end{aligned}$$

We use this estimate to estimate the first part of $u^\pm$ (see definition 2.7) containing the principal value integral:

$$\left| \frac{1}{\sqrt{|B|}} \int_B \sum_{s \in \mathcal{J}} \mathcal{P}\!\!\int_{\mathbb{R}} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k) - \lambda} \, d\lambda \, dk \right|$$

$$\leq \frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \frac{1}{\epsilon} \int_{K_1} |\Phi_{s,k}(\lambda)|\, d\lambda\, dk \tag{2.8}$$
$$+\frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \int_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \left| \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k) - \lambda} \right| d\lambda\, dk.$$

Now we take care of the first summand of (2.8). In the following estimations the steps marked with $(\star)$ are justified by Fubini's theorem on the sum and the two integrals (see [31], Section 1.4, Theorem 1 for a suitable version of Fubini's theorem).

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \frac{1}{\epsilon} \int_{K_1} |\Phi_{s,k}(\lambda)|\, d\lambda\, dk \\
\overset{(\star)}{=}\;& \frac{1}{\epsilon} \frac{1}{\sqrt{|B|}} \int_{K_1} \int_B \sum_{s\in\mathcal{J}} |\sqrt{|B|}\, \overline{\langle U\overline{\varphi}(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle} \langle Uf(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle|\, dk\, d\lambda \\
=\;& \frac{1}{\epsilon} \int_{K_1} \int_B \sum_{s\in\mathcal{J}} |\langle U\overline{\varphi}(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle| |\langle Uf(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle|\, dk\, d\lambda \\
\leq\;& \frac{1}{\epsilon} \int_{K_1} \int_B \sqrt{\sum_{s\in\mathcal{J}} |\langle U\overline{\varphi}(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle|^2} \sqrt{\sum_{s\in\mathcal{J}} |\langle Uf(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle|^2}\, dk\, d\lambda.
\end{aligned}$$

Up to now we inserted the definition of $\Phi_{s,k}$ (see def. 2.3) and used the Cauchy-Schwarz inequality on the sum. Next we use the completeness of $\{\psi_s(\cdot,k) : s \in \mathcal{J}\}$ and then again the Cauchy-Schwarz inequality on the integral over B to obtain

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \frac{1}{\epsilon} \int_{K_1} |\Phi_{s,k}(\lambda)|\, d\lambda\, dk \\
\leq\;& \frac{1}{\epsilon} \int_{K_1} \int_B \|U\overline{\varphi}(\lambda,\cdot,k)\|_{L^2(\Omega)} \|Uf(\lambda,\cdot,k)\|_{L^2(\Omega)}\, dk\, d\lambda \\
\leq\;& \frac{1}{\epsilon} \int_{K_1} \left(\int_B \|U\overline{\varphi}(\lambda,\cdot,k)\|^2_{L^2(\Omega)}\, dk \right)^{1/2} \left(\int_B \|Uf(\lambda,\cdot,k)\|^2_{L^2(\Omega)}\, dk \right)^{1/2} d\lambda \\
=\;& \frac{1}{\epsilon} \int_{K_1} \|U\overline{\varphi}(\lambda,\cdot,\cdot)\|_{L^2(\Omega\times B)} \|Uf(\lambda,\cdot,k)\|_{L^2(\Omega\times B)}\, d\lambda.
\end{aligned}$$

The isometry property of U now implies:

$$\frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \frac{1}{\epsilon} \int_{K_1} |\Phi_{s,k}(\lambda)|\, d\lambda\, dk$$
$$\leq \frac{1}{\epsilon} \int_{K_1} \|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)} \|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\, d\lambda \tag{2.9}$$

$$\leq \frac{1}{\epsilon}\sqrt{|K_2|}\,\|\varphi\|_\infty \int_{K_1} \|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\,d\lambda < \infty.$$

Thus $(\star)$ is justified. In the last step we estimated the $L^2(\mathbb{R}^d)$ norm of φ by $\sqrt{|K_2|}\,\|\varphi\|_\infty$ and used that $\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ is locally Lipschitz in λ.

We are finished with the first summand and continue the estimations with the second summand of (2.8). Let

$$r = \max\{|s| : s \in \mathcal{J},\ \text{conv}(\lambda_s(B)-\epsilon, \lambda_s(B)+\epsilon) \cap K_1 \neq \emptyset\}$$

and

$$\mathfrak{C} = \bigcup_{|s|\leq r} \text{conv}(\lambda_s(B)-\epsilon, \lambda_s(B)+\epsilon) \cup K_1.$$

conv denotes the convex hull taken in $\mathbb{R}$. Clearly $\mathfrak{C}$ is compact. The sum over s in the second summand of (2.8) is in fact finite, since for $|s| \geq r+1$, $\Phi_{s,k}(\lambda_s(k)) = 0$ for all $k \in B$, because $\lambda_s(k) \notin \text{supp}(\Phi_{s,k})$. Moreover, if $|s| \geq r+1$, the interval $(\lambda_s(k)-\epsilon, \lambda_s(k)+\epsilon)$ has empty intersection with supp$(\Phi_{s,k})$ and so in that case $\Phi_{s,k}(\lambda) = 0$ for all $\lambda \in (\lambda_s(k)-\epsilon, \lambda_s(k)+\epsilon)$. Hence

$$\frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \int\limits_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \left| \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k)-\lambda} \right| d\lambda\, dk$$

$$= \frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J};\ |s|\leq r} \int\limits_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \left| \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k)-\lambda} \right| d\lambda\, dk$$

$$\overset{(\star)}{\leq} \frac{1}{\sqrt{|B|}} \int_{\mathfrak{C}}\int_B \sum_{s\in\mathcal{J};\ |s|\leq r} \left| \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k)-\lambda} \right| dk\ d\lambda.$$

The interchange of integrals marked with $(\star)$ will be justified at the end by Fubini's theorem. Next we use lemma 2.4 and the estimate (2.6) in its proof. $|\lambda_s(k)-\lambda|$ cancels out in the numerator and the denominator:

$$\frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \int\limits_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \left| \frac{\Phi_{s,k}(\lambda) - \Phi_{s,k}(\lambda_s(k))}{\lambda_s(k)-\lambda} \right| d\lambda\, dk$$

$$\leq \frac{C(\text{supp}(f), K_2)}{|B|} \int_{\mathfrak{C}}\int_B \sum_{s\in\mathcal{J};\ |s|\leq r} \Bigg(\sup_{\lambda\in\mathfrak{C}} \|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\, \text{Lip}(\varphi)_{\mathfrak{C}}$$

$$+\sup_{\lambda\in\mathfrak{C}} \|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)} \operatorname{Lip}(f)_{\mathfrak{C}}\Big) \; dk\, d\lambda.$$

Counting the summands and carrying out the integrals we obtain:

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}}\int_B \sum_{s\in\mathcal{J}} \int\limits_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \left|\frac{\Phi_{s,k}(\lambda)-\Phi_{s,k}(\lambda_s(k))}{\lambda_s(k)-\lambda}\right| d\lambda\, dk \\
\leq\;& (2r+1)C(\operatorname{supp}(f),K_2)|B||\mathfrak{C}|\frac{1}{|B|}\Big(\sup_{\lambda\in\mathfrak{C}}\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\operatorname{Lip}(\varphi)_{\mathfrak{C}} \\
&+\sup_{\lambda\in\mathfrak{C}}\|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\operatorname{Lip}(f)_{\mathfrak{C}}\Big) \qquad (2.10)\\
\leq\;& (2r+1)C(\operatorname{supp}(f),K_2)|\mathfrak{C}|\sqrt{|K_2|}(\sup_{\lambda\in\mathfrak{C}}\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\left\|\frac{\partial}{\partial\lambda}\varphi\right\|_\infty \\
&+\|\varphi\|_\infty\operatorname{Lip}(f)_{\mathfrak{C}}) < \infty.
\end{aligned}$$

Thus, $(\star)$ is justified. In the last step we estimated the $L^2(\mathbb{R}^d)$-norm of φ by $\sqrt{|K_2|}\,\|\varphi\|_\infty$ and used that the Lipschitz constant can be estimated by $\sqrt{|K_2|}\,\left\|\frac{\partial}{\partial\lambda}\varphi\right\|_\infty$.
It remains to estimate the $\int_B\sum_{s\in\mathcal{J}} i\pi\Phi_{s,k}(\lambda_s(k))\,dk$ part of $u^\pm$. First we note that due to the compact support of φ, for r as defined above we have that for all $x\in\mathbb{R}^d$, $s\in\mathcal{J}$, $|s|>r$ and $k\in B$: $\varphi(\lambda_s(k),x)=0$. Furthermore there exists an $a>0$ such that for all λ

$$\operatorname{supp}(\varphi(\lambda,\cdot)) \subseteq \bigcup_{n\in\mathbb{Z}^d,|n|\leq a} \Omega+n.$$

Finally we note that since $\lambda_s(B)$ is compact and $\lambda\mapsto\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ is locally Lipschitz we have that

$$\sup_{k\in B}\|f(\lambda_s(k),\cdot)\|_{L^2(\mathbb{R}^d)} < \infty. \qquad (2.11)$$

Now we can start the last estimation. First we use the triangle inequality, the definition of $\Phi_{s,k}$ (see definition 2.3) and lemma 2.2:

$$\begin{aligned}
I_{s,k} \;:=\;& \left|\pm\frac{1}{\sqrt{|B|}}\int_B\sum_{s\in\mathcal{J}} i\pi\Phi_{s,k}(\lambda_s(k))\,dk\right| \\
\leq\;& \frac{1}{\sqrt{|B|}}\int_B\sum_{s\in\mathcal{J};\,|s|\leq r}\pi\left|\Phi_{s,k}(\lambda_s(k))\right|\,dk \\
=\;& \frac{1}{\sqrt{|B|}}\int_B\sum_{s\in\mathcal{J};\,|s|\leq r}\pi\left|\int_{\mathbb{R}^d}\varphi(\lambda_s(k),x)\psi_s(x,k)dx\right|
\end{aligned}$$

$$\times \left|\langle Uf(\lambda_s(k),\cdot,k),\psi_s(\cdot,k)\rangle\right| dk.$$

Note that $\varphi(\lambda_s(k)) = 0$ for all $k \in B$ if $|s| \geq r+1$. Then we split the integral over $\mathbb{R}^d$ into integrals over translates of Ω, use the triangle inequality, the k-quasiperiodicity of $\psi_s(\cdot,k)$ and the Cauchy-Schwarz inequality on the inner product with Uf:

$$\begin{aligned} I_{s,k} \;&\leq\; \frac{1}{\sqrt{|B|}}\pi \int_B \sum_{s\in\mathcal{J};\ |s|\leq r} \left(\sum_{|n|\leq a} \int_\Omega |\varphi(\lambda_s(k), x+n)||\psi_s(x,k)|\, dx \right) \\ &\qquad \times \|Uf(\lambda_s(k),\cdot,k)\|_{L^2(\Omega)}\ dk. \end{aligned} \tag{2.12}$$

We estimate $|\varphi(\lambda_s(k), x+n)|$ by $\|\varphi\|_\infty$, use the Cauchy-Schwarz inequality on the integral over Ω and that $\|\psi_s(\cdot,k)\|_{L^2(\Omega)} = 1$ to obtain

$$I_{s,k} \leq \frac{1}{\sqrt{|B|}}\pi \|\varphi\|_\infty \int_B \sum_{s\in\mathcal{J};\ |s|\leq r} \sum_{|n|\leq a} \sqrt{|\Omega|}\, \|Uf(\lambda_s(k),\cdot,k)\|_{L^2(\Omega)}\ dk. \tag{2.13}$$

Counting the summands in the sum over n and using the estimate 5.3 from the appendix yields

$$\begin{aligned} I_{s,k} \;&\leq\; \frac{C(\mathrm{supp}(f))}{|B|}\pi\sqrt{|\Omega|}(2a+1)^d \|\varphi\|_\infty \int_B \sum_{s\in\mathcal{J};\ |s|\leq r} \|f(\lambda_s(k),\cdot)\|_{L^2(\mathbb{R}^d)}\, dk \\ &\leq\; C(\mathrm{supp}(f))\pi\sqrt{|\Omega|}(2a+1)^d \|\varphi\|_\infty \sum_{s\in\mathcal{J};\ |s|\leq r} \sup_{k\in B} \|f(\lambda_s(k),\cdot)\|_{L^2(\mathbb{R}^d)} \\ &<\; \infty. \end{aligned}$$

In the last step we estimated $\|f(\lambda_s(k),\cdot)\|_{L^2(\mathbb{R}^d)}$ by its supremum for $k \in B$ and integrated out the integral over B. From (2.11) follows now that the result is $< \infty$.

Collecting the results of the previous estimations we arrive at

$$|u^\pm[\varphi]| \leq C_1 \|\varphi\|_\infty + C_2 \left\| \frac{\partial}{\partial\lambda}\varphi \right\|_\infty,$$

where the constants can be chosen as follows

$$\begin{aligned} C_1 \;=\; & \frac{1}{\epsilon}\sqrt{|K_2|}\int_{K_1} \|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\, d\lambda \\ & +(2r+1)C(\mathrm{supp}(f),K_2)|\mathfrak{C}|\sqrt{|K_2|}\mathrm{Lip}(f)_{\mathfrak{C}} \\ & +C(\mathrm{supp}(f))\pi\sqrt{|\Omega|}(2a+1)^d \sum_{s\in\mathcal{J};\ |s|\leq r} \sup_{k\in B} \|f(\lambda_s(k),\cdot)\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

$$C_2 \;=\; (2r+1)C(\mathrm{supp}(f),K_2)|\mathfrak{C}|\sqrt{|K_2|}\sup_{\lambda\in\mathfrak{C}}\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}\,,$$

where a, r and $\mathfrak{C}$ are as defined above in this proof. □

Changing the proof of the previous theorem by using Hölder's inequality in the place (2.9), continuing the estimate (2.10) by estimating $\sup_{\lambda\in\mathfrak{C}}\|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ by $\sup_{\lambda\in\mathbb{R}}\|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ and continuing the estimate (2.12) with

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}}\pi\int_B\sum_{s\in\mathcal{J};\,|s|\le r}\left(\sum_{|n|\le a}\int_\Omega|\varphi(\lambda_s(k),x+n)||\psi_s(x,k)|\,dx\right)\\
&\qquad\times\|Uf(\lambda_s(k),\cdot,k)\|_{L^2(\Omega)}\;dk\\
\le\;& (2a+1)^{d/2}\sqrt{|B|}\pi\left(\sum_{s\in\mathcal{J};\,|s|\le r}\sup_{k\in B}\|Uf(\lambda_s(k),\cdot,k)\|^{q_2}_{L^2(\Omega)}\right)^{1/q_2}\\
&\qquad\times\left(\sum_{s\in\mathcal{J};\,|s|\le r}\sup_{k\in B}\|\varphi(\lambda_s(k),\cdot)\|^{q_1}_{L^2(\mathbb{R}^d)}\right)^{1/q_1}
\end{aligned}$$

we can state an alternative version of the previous theorem.

Corollary 2.7. *Let $p_1,,p_2,q_1,q_2\in[1,\infty]$ with $\frac{1}{p_1}+\frac{1}{p_2}=1$, $\frac{1}{q_1}+\frac{1}{q_2}=1$. For all compact $K=K_1\times K_2\subseteq\mathbb{R}\times\mathbb{R}^d$ there exist $C_1,C_2,C_3,C_4\ge 0$ depending only on K and f such that for all $\varphi\in C_0^\infty(\mathbb{R}\times\mathbb{R}^d,\mathbb{C}^M)$ with support in K*

$$|u^\pm[\varphi]|\le C_1\left\|\frac{\partial}{\partial\lambda}\varphi\right\|_\infty+C_2\left(\int_\mathbb{R}\|\varphi(\lambda,\cdot)\;d\lambda\|^{p_1}_{L^2(\mathbb{R}^d)}\right)^{1/p_1}$$

$$+C_3\sup_{\lambda\in\mathbb{R}}\|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}+C_4\left(\sum_{s\in\mathcal{J}}\sup_{k\in B}\|\varphi(\lambda_s(k),\cdot)\|^{q_1}_{L^2(\mathbb{R}^d)}\right)^{1/q_1},$$

where one can choose as constants

$$C_1=(2r+1)|\mathfrak{C}|C(\mathit{supp}(f),K_2)\sup_{\lambda\in\mathfrak{C}}\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$$

$$C_2=\frac{1}{\epsilon}\left(\int_{K_1}\|f(\lambda,\cdot)\|^{p_2}_{L^2(\mathbb{R}^d)}\,d\lambda\right)^{1/p_2}$$

$$C_3=(2r+1)|\mathfrak{C}|C(\mathit{supp}(f),K_2)\,\mathrm{Lip}(f)_\mathfrak{C}$$

$$C_4 = \sqrt{|B|}\pi(2a+1)^{d/2}\left(\sum_{s\in\mathcal{J};\ |s|\le r}\sup_{k\in B}\|Uf(\lambda_s(k),\cdot,k)\|_{L^2(\Omega)}^{q_2}\right)^{1/q_2}$$

with r, $\mathfrak{C}$ defined as in the proof of theorem 2.6.

2.3 The convergence of distributional solutions u_δ to $u^\pm$ as $\delta \longrightarrow 0^\pm$.

In this section we show the convergence of the distributional solutions $u_\delta \rightarrow u^\pm$ as $\delta \rightarrow 0^\pm$. The main tools will be Fubini's theorem and the Lebesgue dominated convergence theorem. Note that the Bloch waves $\psi_s(\cdot, k)$ depend on k in a measurable way by one of the hypotheses made in section 1.4.

Theorem 2.8. *For $\delta \neq 0$, $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d, \mathbb{C}^M)$, supp$(\varphi) \subseteq K_1 \times K_2 \subseteq \mathbb{R}\times\mathbb{R}^d$, let*

$$u_\delta[\varphi] =$$
$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left(\frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}}{\lambda_s(k)-(\lambda+i\delta)}\psi_s(x,k)\,dk\right)\varphi(\lambda,x)\,dx\,d\lambda.$$

Recall that u_δ solves

$$(P_\delta) \qquad u[(\mathcal{L}-(\lambda+i\delta))\varphi] = \int_{\mathbb{R}}\int_{\mathbb{R}^d} f(\lambda,x)\varphi(\lambda,x)\,dx\,d\lambda.$$

Then for $u^\pm$ from definition 2.5 we have

$$u_\delta[\varphi] \overset{\delta\to 0^\pm}{\longrightarrow} u^\pm[\varphi]$$

and $u^\pm$ solves (P_0).

Before we come to the proof of theorem 2.8, we show three lemmas.
For better readability we omit the arguments of Uf, ψ_s, f and φ and subscripts that indicate inner products in the proofs. Note that $\langle Uf, \psi_s\rangle$ always means $\langle Uf(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}$ and thus depends on λ and k throughout the rest of this chapter.

Lemma 2.9. *Let* λ *be fixed and* $\rho : \mathbb{R} \to \mathbb{C}$ *be a continuous function such that there exists a* $c > 0$ *with* $|\rho(\lambda_s(k))| \le c$ *for all* $s \in \mathcal{J}$ *and* $k \in B$ *and let* $\varphi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^d, \mathbb{C}^M)$ *with support contained in some compact set* $K_1 \times K_2$*. Then*

$$\begin{aligned} &\int_{\mathbb{R}^d} \sum_{s \in \mathcal{J}} \int_B \rho(\lambda_s(k))\varphi(\lambda, x)\psi_s(x,k)\langle Uf(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\, dk\, dx \\ =\ & \sum_{s \in \mathcal{J}} \int_B \int_{\mathbb{R}^d} \rho(\lambda_s(k))\varphi(\lambda, x)\psi_s(x,k)\langle Uf(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\, dx\, dk. \end{aligned}$$

Proof. Because of the boundedness of $\rho(\lambda_s(k))$ and the isometry property of the Floquet-Bloch transform U, the series

$$\sum_{s \in \mathcal{J}} \int_B \rho(\lambda_s(k))\varphi(\lambda,\cdot)\psi_s(\cdot,k)\langle Uf, \psi_s\rangle_{L^2(\Omega)}\, dk$$

converges in $L^2(\mathbb{R}^d)$. Hence by the continuity of the inner product in $L^2(\mathbb{R}^d)$ we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \sum_{s \in \mathcal{J}} \int_B \rho(\lambda_s(k))\varphi(\lambda, x)\psi_s(x,k)\langle Uf, \psi_s\rangle_{L^2(\Omega)}\, dk\, dx \\ =\ & \left\langle \sum_{s \in \mathcal{J}} \int_B \rho(\lambda_s(k))\psi_s(\cdot,k)\langle Uf, \psi_s\rangle_{L^2(\Omega)}\, dk, \overline{\varphi(\lambda,\cdot)} \right\rangle_{L^2(\mathbb{R}^d)} \\ =\ & \sum_{s \in \mathcal{J}} \left\langle \int_B \rho(\lambda_s(k))\psi_s(\cdot,k)\langle Uf, \psi_s\rangle_{L^2(\Omega)}\, dk, \overline{\varphi(\lambda,\cdot)} \right\rangle_{L^2(\mathbb{R}^d)} \\ =\ & \sum_{s \in \mathcal{J}} \int_{\mathbb{R}^d} \int_B \rho(\lambda_s(k))\psi_s(x,k)\langle Uf, \psi_s\rangle_{L^2(\Omega)}\varphi(\lambda, x)\, dk\, dx. \end{aligned}$$

It remains to swap $\int_{\mathbb{R}^d}$ and $\int_B$ for s fixed. Let $a \in \mathbb{N}$ be as in proof of theorem 2.6. Then we begin to split up the integral over $\mathbb{R}^d$ into integrals over translates of Ω, estimate $|\varphi|$ by its supremum, use that $|\rho(\lambda_s(k))| \le c$ and count the summands of the sum over n:

$$\begin{aligned} &\int_B \int_{\mathbb{R}^d} |\rho(\lambda_s(k))\varphi\psi_s\langle Uf, \psi_s\rangle|\ dk\, dx \\ =\ & \int_B \sum_{|n| \le a} \int_{\Omega+n} |\rho(\lambda_s(k))||\varphi||\psi_s||\langle Uf, \psi_s\rangle|\, dk\, dx \\ \le\ & (2a+1)^d \left\|\varphi\right\|_\infty c \int_B |\langle Uf, \psi_s\rangle| \int_\Omega 1 \cdot |\psi_s|\, dk\, dx. \end{aligned}$$

Next we use Cauchy-Schwarz inequality repeatedly to obtain:

$$\begin{aligned}
&\int_B \int_{\mathbb{R}^d} |\rho(\lambda_s(k))\varphi\psi_s\langle Uf,\psi_s\rangle| \; dk\, dx \\
\leq\ & (2a+1)^d \|\varphi\|_\infty c \int_B |\langle Uf,\psi_s\rangle| \sqrt{|\Omega|}\, \underbrace{\|\psi_s\|_{L^2(\Omega)}}_{=1} \; dk \\
\leq\ & (2a+1)^d \|\varphi\|_\infty c \int_B \|Uf\|_{L^2(\Omega)} \sqrt{|\Omega|}\; dk \\
\leq\ & (2a+1)^d \|\varphi\|_\infty c \|Uf\|_{L^2(\Omega\times B)} \sqrt{|B|}\sqrt{|\Omega|} \\
=\ & (2a+1)^d \|\varphi\|_\infty c \|f\|_{L^2(\mathbb{R}^d)} \sqrt{|B|}\sqrt{|\Omega|} < \infty.
\end{aligned}$$

The lemma follows now from Fubini's theorem.

□

Lemma 2.10. *Let $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d, \mathbb{C}^M)$ with support contained in some compact set $K_1 \times K_2$ and $\rho : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ be a continuous function such that there exists a $c > 0$ with $|\rho(\lambda_s(k),\lambda)| \leq c$ for all $s \in \mathcal{J}$, $k \in B$ and $\lambda \in K_1$. Then*

$$\int_{\mathbb{R}} \int_B \sum_{s\in\mathcal{J}} \left| \int_{\mathbb{R}^d} \rho(\lambda_s(k),\lambda)\varphi(\lambda,x)\psi_s(x,k)\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\, dx \right| dk\, d\lambda < \infty,$$

and so by Fubini's theorem one can change the order of $\int_{\mathbb{R}}\int_B\sum_s$ arbitrarily while keeping the integral over $\mathbb{R}^d$ in the innermost position.

Proof. First we use lemma 2.2, $|\rho(\lambda_s(k),\lambda)| \leq c$ and apply Cauchy-Schwarz inequality on the sum over s:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_B \sum_{s\in\mathcal{J}} \left| \int_{\mathbb{R}^d} \rho(\lambda_s(k),\lambda)\varphi\psi_s\langle Uf,\psi_s\rangle\, dx \right| dk\, d\lambda \\
=\ & \int_{\mathbb{R}} \int_B \sum_{s\in\mathcal{J}} |\rho(\lambda_s(k),\lambda)| \left| \int_{\mathbb{R}^d} \varphi\psi_s\langle Uf,\psi_s\rangle\, dx \right| dk\, d\lambda \\
=\ & \sqrt{|B|} \int_{K_1} \int_B \sum_{s\in\mathcal{J}} |\rho(\lambda_s(k),\lambda)| |\overline{\langle U\overline{\varphi},\psi_s\rangle}| |\langle Uf,\psi_s\rangle|\, dk\, d\lambda \\
\leq\ & \sqrt{|B|} c \int_{K_1} \int_B \left(\sum_{s\in\mathcal{J}} |\overline{\langle U\overline{\varphi},\psi_s\rangle}|^2 \right)^{1/2} \left(\sum_{s\in\mathcal{J}} |\langle Uf,\psi_s\rangle|^2 \right)^{1/2} dk\, d\lambda.
\end{aligned}$$

Then we use Parseval's identity and afterwards the Cauchy-Schwarz inequality on the integral over B:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_B \sum_{s\in\mathcal{J}} \left| \int_{\mathbb{R}^d} \rho(\lambda_s(k),\lambda)\varphi\psi_s\langle Uf,\psi_s\rangle \, dx \right| \, dk\, d\lambda \\
\leq\ & \sqrt{|B|}c \int_{K_1} \int_B \|U\overline{\varphi}\|_{L^2(\Omega)} \|Uf\|_{L^2(\Omega)} \, dk\, d\lambda \\
\leq\ & \sqrt{|B|}c \int_{K_1} \left(\int_B \|U\overline{\varphi}\|^2_{L^2(\Omega)} \, dk \right)^{1/2} \left(\int_B \|Uf\|^2_{L^2(\Omega)} \, dk \right)^{1/2} d\lambda \\
\leq\ & \sqrt{|B|}\sqrt{|K_2|}\, \|\varphi\|_\infty \, c \int_{K_1} \|f\|_{L^2(\mathbb{R}^d)} \, d\lambda < \infty.
\end{aligned}$$

In the last step we used that U is an isometry and estimated the $L^2(\mathbb{R}^d)$-norm of φ by $\sqrt{|K_2|}\, \|\varphi\|_\infty$. Since $\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ is locally Lipschitz in λ, the result is $< \infty$. ☐

Lemma 2.11. *Let $\epsilon > \delta_0 > 0$, $s \in \mathcal{J}$ and $k \in B$. Then for all $0 < \delta < \delta_0$*

$$\left| \int_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \frac{1}{\lambda_s(k) - \lambda - i\delta} \, d\lambda \right| \leq \frac{\pi\epsilon}{\epsilon - \delta_0} + 2\pi.$$

Proof. We use the Cauchy integral formula

$$h(p) = \frac{1}{2\pi i} \int_\Gamma \frac{h(z)}{z-p} \, dz$$

with $h(z) = 1$, $p = \lambda_s(k) - i\delta$ and Γ being the path from $\lambda_s(k)+\epsilon$ to $\lambda_s(k)-\epsilon$ along the real line and then from $\lambda_s(k)-\epsilon$ back to $\lambda_s(k)+\epsilon$ in a half-circle in the lower complex half-plane with radius ϵ. Since $0 < \delta < \epsilon$, this path encloses $\lambda_s(k) - i\delta$. So

$$\begin{aligned}
\left| \int_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \frac{1}{\lambda_s(k) - \lambda - i\delta} \, d\lambda \right| &= \left| - \int_\pi^{2\pi} \frac{i\epsilon\, e^{i\theta}}{\epsilon\, e^{i\theta} + i\delta} \, d\theta - 2\pi i \right| \\
&\leq \int_\pi^{2\pi} \frac{\epsilon}{|\epsilon\, e^{i\theta} + i\delta|} \, d\theta + 2\pi \\
&= \int_\pi^{2\pi} \frac{\epsilon}{\sqrt{\epsilon^2 + \delta^2 + 2\delta\epsilon\sin\theta}} d\theta + 2\pi \\
&\leq \int_\pi^{2\pi} \frac{\epsilon}{\sqrt{\epsilon^2 + \delta^2 + 2\delta\epsilon(-1)}} d\theta + 2\pi \\
&= \int_\pi^{2\pi} \frac{\epsilon}{|\epsilon - \delta|} d\theta + 2\pi
\end{aligned}$$

$$\begin{aligned} &= \frac{\epsilon\pi}{\epsilon-\delta} + 2\pi \\ &\le \frac{\epsilon\pi}{\epsilon-\delta_0} + 2\pi. \end{aligned}$$

□

Proof of theorem 2.8. Recall that for $\lambda \in \mathbb{R}, \delta \neq 0$ we have $\lambda + i\delta \notin \sigma(\mathcal{L})$. Since $\sigma(\mathcal{L}) = \bigcup_{s\in\mathcal{J}} \lambda_s(B)$, we conclude that for all $s \in \mathcal{J}$, $k \in B$: $|\lambda_s(k) - \lambda - i\delta|^{-1} \le |\delta|^{-1}$ holds. Let $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d, \mathbb{C}^M)$ with support contained in some compact $K_1 \times K_2$. For the continuous function $\rho(r,s) = (r-s-i\delta)^{-1}$ we have that $|\rho(\lambda_s(k),\lambda)| \le |\delta|^{-1}$ for all $\lambda \in K_1$, $s \in \mathcal{J}$ and $k \in B$. From lemma 2.9 and 2.10 follows that

$$\begin{aligned} \sqrt{|B|}u_\delta[\varphi] &= \int_{\mathbb{R}}\int_{\mathbb{R}^d}\sum_s\int_B \frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta}\,dk\,dx\,d\lambda \\ &= \int_B\sum_s\int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta}\,dx\,d\lambda\,dk. \end{aligned} \tag{2.14}$$

We claim that for any fixed $\varphi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^d, \mathbb{C}^M)$

$$\lim_{\delta\to 0^\pm} u_\delta[\varphi] = u^\pm[\varphi]$$

holds. For this it is sufficient to show

$$\begin{aligned} &\lim_{\delta\to 0^\pm}\int_B\sum_s\int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta}\,dx\,d\lambda\,dk \\ = &\int_B\sum_s\lim_{\delta\to 0^\pm}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta}\,dx\,d\lambda\,dk, \end{aligned} \tag{2.15}$$

the claim follows then from lemma 2.4 and theorem 5.6. Let $K_1 \times K_2 \subseteq \mathbb{R}\times\mathbb{R}^d$ be compact and contain the support of φ. There exists a $l \in \mathbb{N}$ and an $\zeta > 0$ such that for all $k \in B$ and all $s \in \mathcal{J}$, $|s| \ge l+1$ (recall property (4) in section 1.4)

$$\max\{|\lambda| : \lambda \in K_1\} + \zeta < |\lambda_s(k)|. \tag{2.16}$$

Thus we can split the sum over s in (2.15) in the following way

$$\lim_{\delta\to 0^\pm}\int_B\sum_{s\in\mathcal{J}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\cdots$$

$$
\begin{aligned}
&= \lim_{\delta\to 0^\pm}\int_B \sum_{s\in\mathcal{J},\ |s|\le l}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\cdots + \lim_{\delta\to 0^\pm}\int_B \sum_{s\in\mathcal{J},\ |s|\ge l+1}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\cdots \\
&= \underbrace{\sum_{s\in\mathcal{J},\ |s|\le l}\lim_{\delta\to 0^\pm}\int_B\int_{\mathbb{R}}\int_{\mathbb{R}^d}\cdots}_{=(A)} + \underbrace{\lim_{\delta\to 0^\pm}\int_B \sum_{s\in\mathcal{J},\ |s|\ge l+1}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\cdots}_{=(B)}
\end{aligned}
$$

and treat (A) and (B) separately.

ad (A)**:** We want to show

$$
\lim_{\delta\to 0^\pm}\int_B\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta} = \int_B \lim_{\delta\to 0^\pm}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta}. \tag{2.17}
$$

Take $\Phi_{s,k}$ from definition 2.3 and define temporarily (we will overwrite the definition of g_δ in the next step of the proof)

$$
g_\delta(k) := \int_{\mathbb{R}}\frac{\Phi_{s,k}(\lambda)}{\lambda_s(k)-\lambda-i\delta}\,d\lambda. \tag{2.18}
$$

Then we can write

$$
\lim_{\delta\to 0^\pm}\int_B\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta} = \lim_{\delta\to 0^\pm}\int_B g_\delta(k)\,dk.
$$

Consider the case $\delta\to 0^+$. The case $\delta\to 0^-$ is proved the same way changing some signs. We want to use Lebesgue's dominated convergence theorem. By Theorem 5.6 g_δ converges a.e. in k to the function

$$
g(k) := \mathcal{P}\!\!\int_{\mathbb{R}}\frac{\Phi_{s,k}(\lambda)}{\lambda_s(k)-\lambda}\,d\lambda + i\pi\Phi_{s,k}(\lambda_s(k)).
$$

Now we seek an integrable bound for $|g_\delta|$. Let $\mathfrak{C}$ be as in the proof of theorem 2.6 and choose an $\epsilon>0$. First note that in the latest definition of g_δ (see (2.18)) $\mathbb{R}$ can be replaced by K_1. We insert a zero in the form $\Phi_{s,k}(\lambda_s(k))-\Phi_{s,k}(\lambda_s(k))$ in the numerator of the integrand and use the triangle inequality:

$$
\begin{aligned}
|g_\delta(k)| &= \left|\int_{K_1}\frac{\Phi_{s,k}(\lambda)}{\lambda_s(k)-\lambda-i\delta}\,d\lambda\right| \\
&\le \left|\int_{K_1}\frac{\Phi_{s,k}(\lambda_s(k))}{\lambda_s(k)-\lambda-i\delta}\,d\lambda\right| + \left|\int_{K_1}\frac{\Phi_{s,k}(\lambda)-\Phi_{s,k}(\lambda_s(k))}{\lambda_s(k)-\lambda-i\delta}\,d\lambda\right|
\end{aligned}
$$

Then we split the first integral into a part away from the point $\lambda_s(k)$ and a part close to this point

$$
\begin{aligned}
|g_\delta(k)| \;\leq\; & |\Phi_{s,k}(\lambda_s(k))| \int\limits_{K_1\setminus(\lambda_s(k)-\epsilon,\lambda_s(k)+\epsilon)} \frac{1}{\sqrt{(\lambda_s(k)-\lambda)^2+\delta^2}}\, d\lambda \\
& +|\Phi_{s,k}(\lambda_s(k))| \left| \int\limits_{\lambda_s(k)-\epsilon}^{\lambda_s(k)+\epsilon} \frac{1}{\lambda_s(k)-\lambda-i\delta}\, d\lambda \right| \\
& + \int_{K_1} \frac{|\Phi_{s,k}(\lambda)-\Phi_{s,k}(\lambda_s(k))|}{\sqrt{(\lambda_s(k)-\lambda)^2+\delta^2}}\, d\lambda.
\end{aligned}
$$

To estimate the first integral in this sum of three integrals, we use that the denominator is greater than ϵ and carry out the integral $d\lambda$. For the second integral we use lemma 2.11 and to estimate the third integral we use the Lipschitz property of $\Phi_{s,k}$ given in lemma 2.4. The constant $C(\mathfrak{C})$ there is independent of k. By now we have with some δ_0 as in lemma 2.11:

$$
\begin{aligned}
|g_\delta(k)| \;\leq\; & |K_1|\frac{1}{\epsilon}|\Phi_{s,k}(\lambda_s(k))| + |\Phi_{s,k}(\lambda_s(k))| \left(\frac{\pi\epsilon}{\epsilon-\delta_0}+2\pi\right) \\
& + \int_{K_1} \frac{C(\mathfrak{C})|\lambda-\lambda_s(k)|}{|\lambda_s(k)-\lambda|}\, d\lambda.
\end{aligned}
$$

We use that $\Phi_{s,k}$ is locally bounded to obtain

$$
|g_\delta(k)| \;\leq\; C(K_1,\mathfrak{C},\epsilon),
$$

where $C(K_1,\mathfrak{C},\epsilon)$ does not depend on k and $\delta \leq \delta_0$. By Lebesgue's dominated convergence theorem we can pull the limit inside the integral and get (2.17).

ad (B)**:** In this part we want to show

$$
\lim_{\delta\to 0^\pm} \int_B \sum_{s\in\mathcal{J},\,|s|\geq l+1} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta} = \int_B \sum_{s\in\mathcal{J},\,|s|\geq l+1} \lim_{\delta\to 0^\pm} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta}
$$

Define (for the second equality see lemma 2.2)

$$
g_\delta(s,k) := \frac{1}{\sqrt{|B|}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta} = \int_{\mathbb{R}} \frac{\overline{\langle U\overline{\varphi},\psi_s\rangle}\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda-i\delta} d\lambda.
$$

g_δ converges pointwise a.e. as $\delta\to 0^+$ to the function

$$
g(s,k) := \mathcal{P}\!\!\int_{\mathbb{R}} \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k)-\lambda}\, d\lambda + i\pi\Phi_{s,k}(\lambda_s(k)).
$$

Again we are looking for an integrable bound for $|g_\delta|$ to apply the Lebesgue dominated convergence theorem. We start the estimation with the triangle inequality and estimate the denominator by ζ from (2.16)

$$\begin{aligned} |g_\delta(s,k)| &= \left| \int_{\mathbb{R}} \frac{\overline{\langle U\overline{\varphi}, \psi_s\rangle} \langle Uf, \psi_s\rangle}{\lambda_s(k) - \lambda - i\delta} \, d\lambda \right| \\ &\leq \int_{K_1} \frac{|\langle U\overline{\varphi}, \psi_s\rangle| |\langle Uf, \psi_s\rangle|}{|\lambda_s(k) - \lambda - i\delta|} \, d\lambda \\ &\leq \frac{1}{\zeta} \int_{K_1} |\langle U\overline{\varphi}, \psi_s\rangle| |\langle Uf, \psi_s\rangle| \, d\lambda. \end{aligned}$$

The last expression is integrable as a function of s and k with respect to the measure $dk \otimes d\mathcal{H}^0(s)$ ($d\mathcal{H}^0$ denoting the counting measure), which can be seen from the following calculation:

$$\begin{aligned} & \int_B \sum_{s\in\mathcal{J}} \frac{1}{\zeta} \int_{K_1} |\langle U\overline{\varphi}, \psi_s\rangle| |\langle Uf, \psi_s\rangle| \, d\lambda \, dk \\ \overset{\text{lemma 2.10}}{=} & \frac{1}{\zeta} \int_{K_1} \int_B \sum_{s\in\mathcal{J}} |\langle U\overline{\varphi}, \psi_s\rangle| |\langle Uf, \psi_s\rangle| \, dk \, d\lambda \\ \leq \; & \frac{1}{\zeta} \int_{K_1} \int_B \left(\sum_{s\in\mathcal{J}} |\langle U\overline{\varphi}, \psi_s\rangle|^2 \right)^{1/2} \left(\sum_{s\in\mathcal{J}} |\langle Uf, \psi_s\rangle|^2 \right)^{1/2} dk \, d\lambda \\ = \; & \frac{1}{\zeta} \int_{K_1} \int_B \|U\overline{\varphi}\|_{L^2(\Omega)} \|Uf\|_{L^2(\Omega)} \; dk \, d\lambda \\ \leq \; & \frac{1}{\zeta} \int_{K_1} \|U\overline{\varphi}\|_{L^2(\Omega\times B)} \|Uf\|_{L^2(\Omega\times B)} \; d\lambda \\ = \; & \frac{1}{\zeta} \int_{K_1} \|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)} \|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)} \; d\lambda < \infty, \end{aligned}$$

since $\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ and $\|\varphi(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ are bounded in λ. By Lebesgue's dominated convergence theorem we get

$$\lim_{\delta\to 0^+} \int_B \sum_{s\in\mathcal{J};\ |s|\geq l+1} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf, \psi_s\rangle}{\lambda_s(k) - \lambda - i\delta} = \int_B \sum_{s\in\mathcal{J};\ |s|\geq l+1} \lim_{\delta\to 0^+} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\varphi\psi_s\langle Uf, \psi_s\rangle}{\lambda_s(k) - \lambda - i\delta}.$$

Altogether we obtain

$$\lim_{\delta\to 0} \int_B \sum_{s\in\mathcal{J}} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \cdots \;=\; (A) + (B)$$

$$
\begin{aligned}
&= \int_B \sum_{s\in\mathcal{J};\ |s|\le l} \lim_{\delta\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \cdots + \int_B \sum_{s\in\mathcal{J};\ |s|\ge l+1} \lim_{\delta\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \cdots \\
&= \int_B \sum_{s\in\mathcal{J}} \lim_{\delta\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \cdots ,
\end{aligned}
$$

which is (2.15). The solution property of $u^{\pm}$ follows easily from this convergence result, since $\mathcal{L}\varphi$ is also a test function like φ by hypothesis (1) made in section 1.4 and so

$$
\int_{\mathbb{R}}\int_{\mathbb{R}^d} f(\lambda,x)\varphi(\lambda,x)\,dx\,d\lambda = u_\delta[(\mathcal{L}-\lambda)\varphi] - i\delta u_\delta[\varphi] \to u^{\pm}[(\mathcal{L}-\lambda)\varphi]
$$

as $\delta \to 0^{\pm}$. □

Remark 2.12. *Using the lemmas 2.9 and 2.10 we interchange step by step the order of the integrations in*

$$
u_\delta[\varphi] = \int_{\mathbb{R}}\int_{\mathbb{R}^d}\left(\frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}}{\lambda_s(k)-(\lambda+i\delta)}\psi_s(x,k)\,dk\right)\ \varphi(\lambda,x)\,dx\,d\lambda,
$$

(see (2.14)*) such that we can pull inside the limit $\delta \to 0^{\pm}$. Trying to take a short cut using Fubini's theorem directly on all three integrations and the summation fails. The problem is to show*

$$
\frac{1}{\sqrt{|B|}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\sum_{s\in\mathcal{J}}\int_B\left|\frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}}{\lambda_s(k)-(\lambda+i\delta)}\psi_s(x,k)\varphi(\lambda,x)\right|\,dk\,dx\,d\lambda < \infty
$$

without further additional assumptions.

Making further assumptions on the growth rate of $|\lambda_s|$ as $|s| \to \infty$ makes this short cut possible. Then we can proceed as follows. Since the support of φ is contained in $K_1 \times K_2$ and we can find an $a \in \mathbb{N}$ such that

$$
K_2 \subset \bigcup_{n\in\mathbb{Z}^d;\ |n|\le a} \Omega + n.
$$

We begin with

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^d}\sum_{s\in\mathcal{J}}\int_B\left|\frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}}{\lambda_s(k)-(\lambda+i\delta)}\psi_s(x,k)\varphi(\lambda,x)\right|\,dk\,dx\,d\lambda \\
= &\sum_{n\in\mathbb{Z}^d;\ |n|\le a}\int_{\Omega+n}\int_{K_1}\sum_s\int_B\left|\frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}}{\lambda_s(k)-(\lambda+i\delta)}\psi_s(x,k)\varphi(\lambda,x)\right|\,dk\,dx\,d\lambda
\end{aligned}
$$

$$\leq \|\varphi\|_\infty \sum_{n\in\mathbb{Z}^d;\ |n|\leq a} \int_\Omega \int_{K_1} \sum_s \int_B \left| \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)}}{\lambda_s(k)-(\lambda+i\delta)} \psi_s(x,k) \right| dk\; dx\, d\lambda.$$

Here we used the k-quasiperiodicity of $\psi_s(\cdot,k)$ and estimated $|\varphi(\lambda,x)|$ by its supremum. Next we can get rid of the integral over Ω and count the summands in the sum over n. Interchanging the order of the summation over s and the integration over B is justified by the final outcome of the estimation. Thus an upper bound on the last expression is given by

$$(2a+1)^d \sqrt{|\Omega|}\, \|\varphi\|_\infty \int_{K_1} \int_B \sum_s \frac{|\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)}|}{|\lambda_s(k)-(\lambda+i\delta)|}\, dk\, d\lambda.$$

Now we use Cauchy-Schwarz inequality on the sum over s.

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^d} \sum_{s\in\mathcal{J}} \int_B \left| \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)}}{\lambda_s(k)-(\lambda+i\delta)} \psi_s(x,k)\varphi(\lambda,x) \right| dk\; dx\, d\lambda \\
&\leq (2a+1)^d \sqrt{|\Omega|}\, \|\varphi\|_\infty \int_{K_1} \int_B \sqrt{\sum_s |\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle|^2} \\
&\qquad \times \sqrt{\sum_s \frac{1}{|\lambda_s(k)-(\lambda+i\delta)|^2}}\, dk\, d\lambda \\
&= (2a+1)^d \sqrt{|\Omega|}\, \|\varphi\|_\infty \int_{K_1} \int_B \|Uf(\lambda,\cdot,k)\|_{L^2(\Omega)} \sqrt{\sum_s \frac{1}{|\lambda_s(k)-(\lambda+i\delta)|^2}}\, dk\, d\lambda,
\end{aligned}$$

where we used Parseval's identity. Cauchy-Schwarz inequality applied to the integral over B and the isometry property of U yield:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^d} \sum_{s\in\mathcal{J}} \int_B \left| \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)}}{\lambda_s(k)-(\lambda+i\delta)} \psi_s(x,k)\varphi(\lambda,x) \right| dk\; dx\, d\lambda \\
&\leq (2a+1)^d \sqrt{|\Omega|}\, \|\varphi\|_\infty \int_{K_1} \sqrt{\int_B \|Uf(\lambda,\cdot,k)\|^2_{L^2(\Omega)}\; dk} \\
&\qquad \times \sqrt{\int_B \sum_s \frac{1}{|\lambda_s(k)-(\lambda+i\delta)|^2}\, dk}\, d\lambda \\
&= (2a+1)^d \sqrt{|\Omega|}\, \|\varphi\|_\infty \int_{K_1} \|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d,\mathbb{C}^M)} \sqrt{\int_B \sum_s \frac{1}{|\lambda_s(k)-(\lambda+i\delta)|^2}\, dk}\, d\lambda.
\end{aligned}$$

Now we see that with a sufficient growth rate of $|\lambda_s|$ as $|s|\to\infty$ uniformly in k the series over s converges and one can carry out the integral over B. For partial differential operators, such sufficient growth conditions become hard to satisfy for high space dimensions. For example, *if $\mathcal{L} = -\triangle$ on $[0,1]^d$ then $\lambda_s \sim s^{2/d}$ (see [32], where the method is explained for $d = 2$ and $d = 3$). We see that in this case the above argument would only work if $d \leq 3$.*

Remark 2.13. *If f is independent of λ, we do not need the compact support of f. The proof of lemma 2.4 is even easier, noting that the support of φ is still compact. In the proof of theorem 2.6 we needed the compact support of f only to get rid of the Floquet- Bloch transform U in the expression $\|Uf(\lambda_s(k),\cdot,k)\|_{L^2(\Omega,\mathbb{C}^M)}$ in the right hand side of the inequality (2.13). If f is independent of λ we can use Cauchy-Schwarz inequality on the integral over B to estimate*

$$\int_B \|Uf(\cdot,k)\|_{L^2(\Omega,\mathbb{C}^M)}\; dk \leq \sqrt{|B|}\,\|f\|_{L^2(\mathbb{R}^d,\mathbb{C}^M)}$$

and do not need the compact support. In the proof of theorem 2.8 we did not need the compact support of f directly, but only that the function $\Phi_{s,k}$ was Lipschitz continuous. But this follows form lemma 2.4, which remains true if $f \in L^2(\mathbb{R}^d,\mathbb{C}^M)$ without compact support as mentioned earlier.

Remark 2.14. *If f is Lipschitz continuous but not constant in λ, then the compact support of f was a sufficient condition for the right hand side of (2.13) to be $< \infty$. In fact we could assume for f the weaker condition*

$$\int_B \|Uf(\lambda_s(k),\cdot,k)\|_{L^2(\Omega,\mathbb{C}^M)}\; dk < \infty$$

for all $s \in \mathcal{J}$ instead of assuming that supp$(f(\lambda,\cdot)) \subset \mathcal{K}$ for some compact $\mathcal{K} \subset \mathbb{R}^d$ for all $\lambda \in \mathbb{R}$.

Chapter 3

Limiting absorption principle for regular frequencies λ

In classical physics, a physical (field) quantity Q is modeled as a function of space, time and possibly other variables (for example frequency, temperature,...), i.e. $Q = Q(x, t, \ldots)$. Examples for physical quantities are the electric field, magnetic field, energy, momentum, a scalar or vector potential, the wave function. In quantum theory of fields a new perception of physical quantities came up. In this theory physical quantities are modeled by field operators, i.e. $Q = Q(x, t, \ldots)$ is an operator in a Hilbert space. Moreover, it was realized, that the field operators at a fixed point x have no meaning in a proper sense. This is so, since it is impossible to measure Q at a fixed point x. Instead, on can only measure average values

$$\int Q(x)\gamma(x)\,dx$$

with γ being a function of a certain class (see discussion of K.O. Friedrichs in [33]). In view of this interpretation we can give the results of the previous chapter a corresponding physical meaning.

In the last chapter we have established a distributional version of the limit absorption principle for Helmholtz-like equations on the whole $\mathbb{R}^d$. Because $u^{\pm}$, which models a physical quantity, was supposed to be a distributional solution of the Helmholtz equation

$$(\mathcal{L} - \lambda)u = f$$

in the sense of definition 2.1, we were able to integrate with respect to λ and control the singularity by means of Cauchy principal value integrals. Doing this we had no restrictions on the range of frequencies and only weak requirements on the right hand side f, but in exchange the distributional solution provides only coarse information of the behavior of the examined system which can be extracted by applying $u^{\pm}$ on test functions $\varphi = \varphi(x, \lambda)$. This corresponds to the idea of the measurement of an average value of $u^{\pm}$, where the support of the test function models the area in space and in the frequency variable where the average is taken.

One can also extract qualitative information concerning the behavior of $u^{\pm}$ in space and with respect to the frequency variable by employing families of test functions and comparing the response of $u^{\pm}$. For example, consider a family of test functions $\varphi_a(\lambda, x) = \varphi(\lambda+a, x)$ where the support of φ is concentrated around a certain (λ_0, x_0). Then the comparison of the values $u^{\pm}[\varphi_a]$ for different a indicates the behavior of $u^{\pm}$ around a point x_0 as the frequency varies with a. If one finds that $u^{\pm}[\varphi_a]$ has a great resonance around a certain a_0, this would indicate that the frequency $\lambda_0 + a_0$ - or rather a frequency somewhere nearby - is a frequency with special properties for the examined system. Similarly, the comparison of the values $u^{\pm}[\varphi_b]$ for a family $\varphi_b(\lambda, x) = \varphi(\lambda, x + b)$ of test functions where the support of φ is concentrated around some (λ_0, x_0) gives a rough impression of the spatial behavior of the solution $u^{\pm}$ for a fixed frequency interval with nonempty interior.

At this stage it is interesting from a mathematical point of view to ask for conditions such that the distribution $u^{\pm}$ can be represented by a function. A representation of physical quantities as functions is also favorable from a physical viewpoint, since it would coincide more with the common classical perception of these quantities mentioned at the beginning to have at any point a well defined value as a tuple of real or complex numbers.

For example, in classical physics the values of the magnetic field generated by a radiation source is supposed to exist as a triple of complex numbers in every point in space and at any time for a fixed frequency. However, the requirement for a function to have well defined values in every point is too restrictive for mathematical treatment, so this chapter we will find a representation of the distribution $u^{\pm}$ from the previous chapter as a function that is locally L^p with respect to the frequency variable λ and in a weighted L^2 space with respect to the spatial variable x.

To accomplish this task, it turns out that we have to restrict the range of frequencies λ to *regular* frequencies. Physically, these are frequencies for which all corresponding Bloch modes with band functions intersecting λ have non-vanishing group velocities. In studying radiation phenomena, it is natural to consider such frequencies, since Bloch modes with zero group velocity do not propagate in space.

As we will only - apart of section 3.2 - rely on the Lipschitz property of the band functions, we will first recall a suitable notion of regular frequencies. This requirement relates only to a certain condition on the derivatives of the band functions at the frequencies in question. In particular, it does not affect intersections of the band functions. This means that multiple eigenvalues $\lambda_s(k)$ of $\mathcal{L}_k$ cause no problems as long as the regularity condition is fulfilled for each of the intersecting band functions under consideration.

In one space dimension it is known that almost all frequencies λ are regular, since the band functions can be chosen analytically (see [34]). In higher dimensions, a sufficient condition that almost all frequencies are regular is that the band functions can be build up as restrictions from sufficiently smooth functions defined on overlapping domains.

To find a representing function for $u^{\pm}$, we will investigate the restriction of the inverse Floquet-Bloch transform to level sets of the band functions at regular frequencies. The restricted inverse Floquet-Bloch transform takes L^2-functions on the product of the periodicity cell and the level set to weighted L^2-functions on the whole space.

After that we study the properties of the level set integral

$$h_s(x) = \frac{1}{\sqrt{|B|}} \int_{\{k \in B : \lambda_s(k) = \tau\}} \frac{\langle Uf, \psi_s \rangle \psi_s(x,k)}{|\nabla \lambda_s|} \, d\mathcal{H}^{d-1}(k) \quad (x \in \mathbb{R}^d),$$

which is nothing else than the inverse Floquet-Bloch transform restricted to the level set $\{k \in B : \lambda_s(k) = \tau\}$[1] applied to the integrand. It follows that h_s lies in a weighted L^2-space on $\mathbb{R}^d$ in the variable x. Such integrals arise, when the Corarea formula is applied on the integral over B in the Floquet-Bloch representation of the resolvent

$$(\mathcal{L} - \lambda)^{-1} f = \frac{1}{\sqrt{|B|}} \sum_{s \in \mathcal{J}} \int_B \frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega, \mathbb{C}^M)} \psi_s(x,k)}{\lambda_s(k) - \lambda} \, dk.$$

The limiting absorption principle for $\mathcal{L}$ in the space $L^p(I_0, L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx); d\lambda)$, where I_0 is an interval of regular frequencies, $p \in (1, \infty)$ and w a suitable weight function, is shown by the following steps. First, the integral over B in the representation of the absorptive resolvent

$$(\mathcal{L} - \lambda - i\delta)^{-1} f = \frac{1}{\sqrt{|B|}} \sum_{s \in \mathcal{J}} \int_B \frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega, \mathbb{C}^M)} \psi_s(x,k)}{\lambda_s(k) - \lambda - i\delta} \, dk$$

[1] This restriction is understood in the same way as the restriction of the Fourier transform to submanifolds in [35].

is split into a two parts: one, where the integration over k avoids singularities in the denominator for $\lambda \in I_0$ and a part where we possibly have $\lambda_s(k) = \lambda \in I_0$. The first part causes no problems when $\delta \to 0^{\pm}$. The second part is rewritten by the Coarea formula. For this step it is necessary that we restrict to regular frequencies. Finally we will see that the second part converges in $L^p(I_0, L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx); d\lambda)$ as $\delta \to 0^{\pm}$ to a sum of terms of the form

$$-\pi H h_s(\lambda, x) \pm i\pi T h_s(\lambda, x),$$

where H is a variant of the Hilbert transform and T is a sort of point evaluation operator. This representation has a similar structure as the distributional solution $u^{\pm}$, which consists of a part involving a Cauchy principal value integral and a point evaluation part.

Convention: Vectors, gradients and coordinate tuples are always to be regarded as columns, though mostly written as rows to save space. So, if M is a matrix, and $(k_1, \dots, k_d)$ a vector, then $M(k_1, \dots, k_d)$ means

$$M \begin{pmatrix} k_1 \\ \vdots \\ k_d \end{pmatrix} = \begin{pmatrix} m_{11} & \dots & m_{1d} \\ \vdots & \ddots & \vdots \\ m_{d1} & \dots & m_{dd} \end{pmatrix} \begin{pmatrix} k_1 \\ \vdots \\ k_d \end{pmatrix}.$$

Consequently, the transpose $(k_1, \dots, k_d)^t$ is a row.

Furthermore we will often skip the target space $\mathbb{C}^M$ in the subscripts of norms and inner products.

3.1 The notion of regular λ.

Usually a value λ of a function h is called a regular value if for any element k in the domain of h with the property $h(k) = \lambda$ the derivative $Dh(k)$ is not zero. Since we will only rely on the Lipschitz continuity of the band functions we recall a notion of derivative that is suited for Lipschitz functions. By Rademacher's theorem Lipschitz functions possess classical derivatives almost everywhere, but we will need a notion of derivative, that is defined everywhere on B since we will deal with level sets having d-dimensional measure zero. Based on that notion we will introduce regular frequencies, i.e. regular values of the band functions λ_s. For derivatives for Lipschitz functions, it will be convenient to use the concept of the generalized Jacobian (generalized gradient) from [36].

Definition 3.1 (Generalized Jacobian)**.** *The* generalized Jacobian *of a function* $f : \mathbb{R}^m \to \mathbb{R}^n$ *at a point* $x \in \mathbb{R}^m$*, denoted by* $\partial f(x)$*, is the convex hull of all matrices* m *of the form*

$$m = \lim_{i \to \infty} Df(x_i),$$

where x_i *converges to* x *and* f *is differentiable at* x_i *for all* i*.* $\partial f(x)$ *is said to be* of maximal rank *if every* $m \in \partial f(x)$ *is of maximal rank. Partial generalized Jacobians*

$$\partial_y f(x, y)$$

are defined in the same way.

We will also need the upper semicontinuity of the generalized gradient (see [37]).

Theorem 3.2 (Upper[2] semicontinuity)**.** *If* v_i *and* x_i *are sequences tending to* v *and* x *respectively, and if* v_i *belongs to* $\partial f(x_i)$ *for each* i*, then* v *belongs to* $\partial f(x)$*.*

For Lipschitz functions an inverse mapping theorem holds, which was proved in [36].

Theorem 3.3 (Inverse mapping theorem for Lipschitz functions)**.** *Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be Lipschitz in a neighborhood of* $x_0 \in \mathbb{R}^n$ *and let* $\partial f(x_0)$ *be of maximal rank. Then there exist neighborhoods* U *and* V *of* x_0 *and* $f(x_0)$ *respectively, and a Lipschitz function* $g : V \to \mathbb{R}^n$*, such that*

$$\begin{array}{llcll} (a) & g(f(u)) & = & u & \text{for every } u \in U \\ (b) & f(g(v)) & = & v & \text{for every } v \in V. \end{array}$$

From this theorem it is easy - at least in the case of generalized *gradients*, i.e. if f is taking its values in $\mathbb{R}$ - to show an implicit function theorem for Lipschitz functions:

Theorem 3.4 (Implicit function theorem for Lipschitz functions)**.** *Let* $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ *be Lipschitz in a neighborhood of* (x_0, y_0)*,* $f(x_0, y_0) = 0$ *and let* $\partial_y f(x_0, y_0)$ *be of maximal rank. Then there exist neighborhoods* $U \subseteq \mathbb{R}^m$ *and* $V \subseteq \mathbb{R}$ *of* x_0 *and* y_0 *respectively, and a Lipschitz function* $g : U \to V$ *such that*

$$f(x, y) = 0 \quad \Leftrightarrow \quad y = g(x), \qquad ((x, y) \in U \times V).$$

[2]Informally this theorem states "$\lim_{i \to \infty} \partial f(x_i) \subseteq \partial f(x)$ if $x_i \to x,\ i \to \infty$".

We will use the implicit function theorem for Lipschitz functions to describe level sets of the band functions at regular frequencies by finitely many Lipschitz parametrizations. Now we introduce the notions of regular points and values.

Definition 3.5 (Regular point)**.** *$k \in B$ is called a* regular point of λ_s *if* $0 \notin \partial\lambda_s(k)$. *A point that is not regular is called* singular.

Definition 3.6 (Regular value)**.** *$\lambda \in \mathbb{R}$ is called a* regular value of λ_s *if* $\lambda_s^{-1}(\{\lambda\}) = \emptyset$ *or* $\lambda_s^{-1}(\{\lambda\})$ *consists only of regular points of λ_s. The set of regular values of λ_s is denoted by $\mathcal{R}_s$. We also define $\mathcal{R}$, the set of* regular values*, by*

$$\mathcal{R} = \{\lambda \in \mathbb{R} : \lambda \textit{ is a regular value of } \lambda_s \textit{ for all } s\}.$$

A value that is not regular is called singular.

3.2 A criterion on the band functions, such that almost all λ are regular

Assumptions: In addition to the assumptions on the band functions we have made in chapter 1, suppose for this section that for each $s \in \mathcal{J} \subseteq \mathbb{Z}$ there exists a subset X_s of B of dimension $\leq (d-1)$ and sets $B_{s,1}, \ldots, B_{s,N}$ $(N = N(s))$ which are relatively open in B such that

$$B \setminus X_s = B_{s,1} \cup \ldots \cup B_{s,N}$$

and such that λ_s is analytic on $B \setminus X_s$.[3] Furthermore assume that there exist C^{d+N-1}-continuations of every $\lambda_s|_{B_{s,j}}$ to some open (in $\mathbb{R}^d$) sets $A_{s,j}$ containing $B_{s,j}$. These extensions will be denoted by abuse of notation by $\lambda_s|_{A_{s,j}}$. Note that such extensions do not exist by the analyticity of the band functions on $B_{s,j}$ and the global Lipschitz property only. For example, the function defined by $h(x) = x^2 \sin \frac{1}{x}$ for $x > 0$ and $h(x) = 0$ for $x \leq 0$ is analytic on $(-\infty, 0)$ and $(0, \infty)$, Lipschitz continuous everywhere, but there is no C^1-extension of h from $(0, \infty)$ to $x < 0$.

In our situation we have the following

[3] The existence of such $B_{s,j}$ and X_s (satisfying further conditions) was shown by C.H. Wilcox for Schrödinger operators with periodic coefficients in [38].

Proposition 3.7.

(a) $\partial\lambda_s(k) = \{\nabla\lambda_s(k)\}$ *if* $k \in B \setminus X_s$

(b) $\partial\lambda_s(k) =$ *conv*$\{\nabla\lambda_s|_{A_{s,j_1}}(k), \ldots, \nabla\lambda_s|_{A_{s,j_r}}(k)\}$ *if* $k \in \partial B_{s,j_1} \cap \ldots \cap \partial B_{s,j_r}$ *and* $k \notin \partial B_{s,p}$ *for all other* p.

Lemma 3.8. *Let* $r \geq 1$ *be fixed and let* $h_i : \mathbb{R}^d \supseteq A_i \to \mathbb{R}$*,* $i \in \{1, \ldots, r\}$*, be* C^{d+r-1}*-functions on open sets* A_i*. For any* $J = \{j_1, \ldots, j_{|J|}\} \subseteq \{1, \ldots, r\}$ *such that* $\bigcap_{j \in J} A_j \neq \emptyset$ *define the function*

$$g_J : \left(\bigcap_{j \in J} A_j\right) \times \mathbb{R}^{|J|-1} \to \mathbb{R}$$

by

$$g_J(k, \alpha_1, \ldots, \alpha_{|J|-1}) := \sum_{\nu=1}^{|J|-1} \alpha_\nu h_{j_\nu}(k) + \left(1 - \sum_{\nu=1}^{|J|-1} \alpha_\nu\right) h_{j_{|J|}}(k).$$

In particular, g_J *is a* $C^{d+|J|-1}$*-function. Consider the set*

$$\begin{aligned} \mathfrak{A}_J &= \Big\{\lambda \in \mathbb{R} : \exists\, k_0 \in \mathbb{R}^d \text{ such that } h_j(k_0) = \lambda \text{ for } j \in J \\ &\qquad \text{and } 0 \in \text{conv}\{\nabla h_j(k_0) : j \in J\}\Big\} \end{aligned}$$

and the set

$$\begin{aligned} \mathfrak{B}_J = \Big\{&\lambda \in \mathbb{R} : \exists\, k_0 \in \mathbb{R}^d\ \exists\, \alpha_1, \ldots, \alpha_{|J|-1} \in [0,1] \text{ such that } 1 - \sum_{j=1}^{|J|-1} \alpha_j \in [0,1] \\ &g_J(k_0, \alpha_1, \ldots, \alpha_{|J|-1}) = \lambda \text{ and } \nabla g_J(k_0, \alpha_1, \ldots, \alpha_{|J|-1}) = 0\Big\} \end{aligned}$$

Then $\mathfrak{A}_J \subseteq \mathfrak{B}_J$ *and* $\mathcal{L}^1(\mathfrak{B}_J) = 0$*, where* $\mathcal{L}^1$ *denotes the 1-dim Lebesgue measure.*

Proof. Let $J = \{j_1, \ldots, j_{|J|}\} \subseteq \{1, \ldots, r\}$ and $\lambda \in \mathfrak{A}_J$. So there exist $k_0 \in \mathbb{R}^d$ and $\alpha_1, \ldots, \alpha_{|J|-1} \in [0,1]$ such that $1 - \sum_{\nu=1}^{|J|-1} \alpha_\nu \in [0,1]$, $h_j(k_0) = \lambda$ for all $j \in J$ and $0 = \sum_{\nu=1}^{|J|-1} \alpha_\nu \nabla h_{j_\nu}(k_0) + \left(1 - \sum_{\nu=1}^{|J|-1} \alpha_\nu\right) \nabla h_{j_{|J|}}(k_0)$. For this k_0 and $\alpha_1, \ldots, \alpha_{|J|-1}$ we have clearly

$$g_J(k_0, \alpha_1, \ldots, \alpha_{|J|-1}) = \sum_{\nu=1}^{|J|-1} \alpha_\nu h_{j_\nu}(k_0) + \left(1 - \sum_{\nu=1}^{|J|-1} \alpha_\nu\right) h_{j_{|J|}}(k_0)$$

$$
\begin{aligned}
&= \sum_{\nu=1}^{|J|-1} \alpha_\nu \lambda + \left(1 - \sum_{\nu=1}^{|J|-1} \alpha_\nu\right) \lambda \\
&= \lambda
\end{aligned}
$$

and

$$
\begin{aligned}
& \nabla g_J(k_0, \alpha_1, \ldots, \alpha_{|J|-1}) \\
= \; & \Big(\sum_{\nu=1}^{|J|-1} \alpha_\nu \nabla h_{j_\nu}(k_0) + \left(1 - \sum_{\nu=1}^{|J|-1} \alpha_\nu\right) \nabla h_{j_{|J|}}(k_0), \\
& h_{j_1}(k_0) - h_{j_{|J|}}(k_0), \ldots, h_{j_{|J|-1}}(k_0) - h_{j_{|J|}}(k_0) \Big) \\
= \; & (0, 0, \ldots, 0)
\end{aligned}
$$

so we conclude $\lambda \in \mathfrak{B}_J$. Since $\mathfrak{B}_J$ is a subset of the singular values of the $C^{d+|J|-1}$-function g_J we can use theorem 3.4.3 from [39] (Sard's theorem) to obtain

$$
\mathcal{H}^{0+(d+|J|-1-0)/(d+|J|-1)}(\mathfrak{B}_J) = \mathcal{L}^1(\mathfrak{B}_J) = 0,
$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. □

Lemma 3.9. *For band functions satisfying the assumptions from the beginning of this section, the set of singular values is a closed set of Lebesgue measure zero.*

Proof. Let $(\lambda^{(j)})_{j\in\mathbb{N}}$ be a sequence of singular values of λ_s and $\lambda^{(j)} \overset{j\to\infty}{\longrightarrow} \lambda \in \mathbb{R}$. We want to show that the limit λ is a singular value of λ_s. Since $\lambda^{(j)}$ is singular there exists a $k^{(j)} \in B$ with $\lambda_s(k^{(j)}) = \lambda^{(j)}$ and $0 \in \partial\lambda_s(k^{(j)})$. We can assume $\lim\limits_{j\to\infty} k^{(j)} = k \in B$ (note that B is compact). Since λ_s is continuous we have $\lambda_s(k) = \lambda$.
Case 1: $k \in B_{s,i}$ for some $i \in \{1, \ldots, N\}$. Then we may assume $k^{(j)} \in B_{s,i}$ and so $\partial\lambda_s(k^{(j)}) = \nabla\lambda_s(k^{(j)}) = 0$. Since $\nabla\lambda_s$ is continuous on $B_{s,i}$ we have $0 = \lim\limits_{j\to\infty} \nabla\lambda_s(k^{(j)}) = \nabla\lambda_s(k) \in \partial\lambda_s(k)$, and so λ is a singular value of λ_s.
Case 2: $k \in \partial B_{s,i_1} \cap \ldots \cap \partial B_{s,i_r}$ for some $i_1, \ldots, i_r \in \{1, \ldots, N\}$. Without loss of generality we may assume $r = 2$. If there exists a subsequence of $k^{(j)}$ that lies as a whole in one of the $B_{s,i}$, $i \in \{i_1, \ldots, i_r\}$ then the proof goes like in case 1 using the continuously differentiable extension of λ_s on $A_{s,i}$. Otherwise there exists a subsequence $k^{(j)} \in \partial B_{s,i_1} \cap \partial B_{s,i_2}$ converging to k. We denote this subsequence again by $k^{(j)}$. Since $k^{(j)}$ is singular there exist $\alpha_1^{(j)}, \alpha_2^{(j)} \geq 0$, $\alpha_1^{(j)} + \alpha_2^{(j)} = 1$ such that $0 = \alpha_1^{(j)} \nabla\lambda_s|_{A_{s,i_1}}(k^{(j)}) + \alpha_2^{(j)} \nabla\lambda_s|_{A_{s,i_2}}(k^{(j)})$.

Once again we pass to a subsequence such that $(\alpha_1^{(j)}, \alpha_2^{(j)})$ converges to some (α_1, α_2) with $\alpha_1, \alpha_2 \geq 0,\ \alpha_1 + \alpha_2 = 1$. Then for the final subsequence we obtain due to continuity $0 = \alpha_1^{(j)} \nabla\lambda_s|_{A_{s,i_1}}(k^{(j)}) + \alpha_2^{(j)} \nabla\lambda_s|_{A_{s,i_2}}(k^{(j)}) \overset{j\to\infty}{\longrightarrow} \alpha_1 \nabla\lambda_s|_{A_{s,i_1}}(k) + \alpha_2 \nabla\lambda_s|_{A_{s,i_2}}(k)$ which means $0 \in \partial\lambda_s(k)$. So k is singular and $\lambda_s(k) = \lambda$ is a singular value of λ_s.

This means that the set of singular values of λ_s is closed. We apply lemma 3.8 to the collection of functions $\lambda_s|_{A_1}, \dots, \lambda_s|_{A_N}$. So the set of singular values of a band function λ_s has Lebesgue measure zero (note proposition 3.7). Now the set of all singular values is the countable union of the sets of singular values of all band functions. So it has Lebesgue measure zero. It is closed, since for every frequency λ there are only finitely many band functions whose ranges contain λ. □

Remark 3.10. *The openness result for the set of regular values is true in general and does not need the special extension property of the band functions. This can be seen as follows. For a sequence $\lambda^{(j)} \in \mathbb{R}$ of singular values converging to some $\lambda \in \mathbb{R}$ we find a sequence of singular $k^{(j)} \in B$, without loss of generality converging to some $k \in B$ with $\lambda_s(k^{(j)}) = \lambda^{(j)}$. By upper semicontinuity of the generalized gradient, we obtain $0 \in \partial\lambda_s(k)$. Since $\lambda_s(k) = \lambda$ by continuity of λ_s, λ is singular.*

3.3 A theorem on the inverse Floquet-Bloch transform restricted to level sets at regular λ

Level sets of Lipschitz functions can be very irregular. In general one cannot even expect a level set of a C^1-function to be rectifiable (see [40]). In this section we investigate *regular level sets* of Lipschitz functions, i.e. level sets of regular values. It turns out, that regular level sets can be locally described as graphs of Lipschitz functions. This property will enable us to define properly a restriction of the Floquet-Bloch transform to regular level sets.

The next lemma shows that the generalized gradients $\partial\lambda_s(k)$ are bounded away from zero, when k runs through all $k \in B$ with the property that $\lambda_s(k) \in I$, and I is an interval compactly embedded in the set of regular values of λ_s. The lemma will be needed to verify the hypothesis for the application of the Coarea formula.

Lemma 3.11. *Consider the band function λ_s and define the* level set of λ_s at λ *by*

$$B_\lambda^s = \{k \in B : \lambda_s(k) = \lambda\}.$$

Let $\Lambda \subset\subset \mathcal{R}_s$. Then

$$\inf\left\{\min_{m\in\partial\lambda_s(k)}|m| : k \in \bigcup_{\lambda\in\overline{\Lambda}} B^s_\lambda\right\} > 0 \qquad (*).$$

Proof. First notice that for any $k \in B$, $\min\limits_{m\in\partial\lambda_s(k)}|m|$ exists, since $\partial\lambda_s(k)$ is a nonempty compact convex subset in $\mathbb{R}^d$ (see [36]). Now suppose that the infimum in $(*)$ is 0. Then there exists a sequence $k^{(j)} \in \bigcup_{\lambda\in\overline{\Lambda}} B^s_\lambda$ such that $k^{(j)} \in B^s_{\lambda^{(j)}}$ for some regular $\lambda^{(j)} \in \overline{\Lambda}$ and $0 < \min\limits_{m\in\partial\lambda_s(k^{(j)})}|m| \overset{j\to\infty}{\longrightarrow} 0$. Because $\bigcup_{\lambda\in\overline{\Lambda}} B^s_\lambda$ is compact we may assume $k^{(j)} \overset{j\to\infty}{\longrightarrow} \kappa \in \bigcup_{\lambda\in\overline{\Lambda}} B^s_\lambda$. Hence, κ is a regular point.

For each j the minimum $\min_{m\in\partial\lambda_s(k^{(j)})}|m|$ is attained in some $m_j \in \partial\lambda_s(k^{(j)})$. Since $m_j \to 0$ and $k^{(j)} \to \kappa$ we obtain $0 \in \partial\lambda_s(\kappa)$ by upper semicontinuity of the generalized gradient (see theorem 3.2) in contradiction to the regularity of κ. □

Lemma 3.12. *Let $\lambda_s : B \to \mathbb{R}$ be Lipschitz, λ a regular value of λ_s and $k \in B^s_\lambda$. If $m_d \geq c > 0$ for some fixed c for all $(m_1, \ldots, m_d) \in \partial\lambda_s(k)$, then $0 \notin \partial_{k_d}\lambda_s(k)$. Here, $\partial_{k_d}\lambda_s(k)$ is the partial generalized gradient with respect to the d-th component, i.e.*

$$\partial_{k_d}\lambda_s(k) = \mathit{conv}\left\{\lim_{x_j\to k}\frac{\partial\lambda_s}{\partial k_d}(x_j)\right\}.$$

For this convex hull all sequences $(x_j)_{j\in\mathbb{N}}$ are taken into account that converge to k and $\frac{\partial\lambda_s}{\partial k_d}(x_j)$ exists for all $j \in \mathbb{N}$ (according to definition 3.1).

Proof. Since in the definition of the generalized gradient of a real valued function one can as well constrain the sequences $(x_j)_{j\in\mathbb{N}}$ to lie in the complement of a fixed set with measure 0 (see [37]), we constrain the sequences $(x_j)_{j\in\mathbb{N}}$ to the subset of $\mathbb{R}^d$ where all partial derivatives of λ_s exist, i.e. to the set of all $x \in \mathbb{R}^d$ such that the full gradient $\nabla\lambda_s(x)$ exists. Suppose w.l.o.g. $d = 2$ and hence that for all $m = (m_1, m_2) \in \partial\lambda_s(k)$, $m_2 \geq c > 0$ holds. Assume $0 \in \partial_{k_2}\lambda_s(k) = [a_2, b_2] \subset \mathbb{R}$. Since a_2 and b_2 are extremal points there exist sequences $x_j, y_j \to k$ such that

$$\frac{\partial}{\partial k_2}\lambda_s(x_j) \to a_2,\ \frac{\partial}{\partial k_2}\lambda_s(y_j) \to b_2,\ j \to \infty$$

and $0 = \alpha a_2 + \beta b_2$ for some $\alpha, \beta \in [0, 1]$, $\alpha + \beta = 1$. For those sequences we find subsequences, denoted again by x_j, y_j to avoid double indices, and $a_1, b_1 \in \mathbb{R}$ such that

$$\frac{\partial}{\partial k_1}\lambda_s(x_j) \to a_1,\ \frac{\partial}{\partial k_1}\lambda_s(y_j) \to b_1,\ j \to \infty.$$

Note here that $\frac{\partial}{\partial k_1}\lambda_s$ is bounded by the Lipschitz constant of λ_s. Then for the subsequences we have

$$\nabla\lambda_s(x_j) \to (a_1, a_2),\ \nabla\lambda_s(y_j) \to (b_1, b_2),\ j \to \infty.$$

So by definition 3.1,

$$\partial\lambda_s(k) \ni \alpha(a_1, a_2) + \beta(b_1, b_2) = (\alpha a_1 + \beta b_1, 0),$$

a contradiction since the 2nd component of every element in $\partial\lambda_s(k)$ is > 0. The proof for an arbitrary $d \geq 1$ is clear now. $\square$

Remark 3.13. *Since the band functions are periodic with respect to B we can identify B with the torus $\mathbb{R}^d/2\pi\mathbb{Z}^d$ and also regard the level sets B_τ^s as subsets of the torus. For a matrix $M \in \mathbb{R}^{d\times d}$ we also may interpret MB as the torus $M\mathbb{R}^d/2\pi M\mathbb{Z}^d$ We do this in order to avoid the distinction of the two cases whether k lies in the interior of B or $k \in \partial B$.*

Definition 3.14. *Let $\mathfrak{M}$ be the class of all subsets $\mathcal{M}$ of B that have Hausdorff dimension $\mathcal{H}_{dim}(\mathcal{M}) = d-1$ and such that there exist finitely many invertible matrices $D_i \in \mathbb{R}^{d\times d}$ together with Lipschitz parametrizations $\phi_1, \ldots, \phi_r$, $\phi_i : U_i' \to U_i''$, where $U_i' \subset \mathbb{R}^{d-1}$, $U_i'' \subset \mathbb{R}$ and $D_i(U_i' \times U_i'')$ is open in the torus D_iB, with*

$$\mathcal{M} \subseteq \bigcup_i D_i(U_i' \times U_i'')$$

and such that for all $i \in \{1, \ldots, r\}$

$$k_d = \phi_i(k_1 \ldots, k_{d-1}) \Leftrightarrow D_i(k_1, \ldots, k_d) \in \mathcal{M} \cap D_i(U_i' \times U_i'')$$

holds.

The matrix D_i describes a suitable local change of coordinates. In these new coordinates, the piece of $\mathcal{M}$ lying in $D_i(U_i' \times U_i'')$ is the graph of a Lipschitz function.

Proposition 3.15. *Let B_λ^s be a level set of a Lipschitz function λ_s at a regular value $\lambda \in \mathcal{R}_s$. Then $B_\lambda^s \in \mathfrak{M}$ and there exist invertible matrices $\widetilde{D}_i$ with integer entries such that the matrices D_i from definition 3.14 can be chosen as $D_i = \left(\widetilde{D}_i^t\right)^{-1}$, $\widetilde{D}_i^t$ denoting the transpose of $\widetilde{D}_i$.*

Proof. First we give some preparatory arguments, which will help us to construct the parametrizations via an implicit function theorem for a suitable function defined later in Step 3.

Step 1. Let $k \in B^s_\lambda$. Consider the generalized gradient $\partial\lambda_s(k)$. Since λ is regular, $0 \notin \partial\lambda_s(k)$. So, since $\partial\lambda_s(k)$ is compact and convex, there exists a $(d-1)$-dimensional affine hyperplane H separating 0 and $\partial\lambda_s(k)$. After projecting 0 and $\partial\lambda_s(k)$ orthogonally onto H we find a $(d-1)$-dimensional closed ball C in H around the orthogonal projection of 0 on H containing the projection of $\partial\lambda_s(k)$. Then we see that $\partial\lambda_s(k)$ is contained in the double cone $\mathcal{C} = \mathbb{R} \cdot \mathrm{conv}\{0, C\}$ where the convex hull is taken in $\mathbb{R}^d$. Let $H' = \langle h_1, \dots, h_{d-1}\rangle$ be the parallel hyperplane to H through 0. Since the lattice directions of the lattice $\mathbb{Z}^d$ are dense in the $(d-1)$-dim sphere S^{d-1}, we can perturb the h_i to $\widetilde{h}_i \in \mathbb{Z}^d$ such that for the hyperplane $\widetilde{H}$ spanned by $\widetilde{h}_i,\ i = 1, \dots, d-1$, $\langle \widetilde{h}_1, \dots, \widetilde{h}_{d-1}\rangle \cap \mathcal{C} = 0$ still holds and $\{\widetilde{h}_1, \dots, \widetilde{h}_{d-1}\}$ is linearly independent. Now we can choose a $\widetilde{h}_d \in (\mathbb{R}_{>0} \cdot \mathrm{conv}\{\{0\} \cup C\}) \cap \mathbb{Z}^d$ such that $\{\widetilde{h}_1, \dots, \widetilde{h}_d\}$ is a basis of $\mathbb{R}^d$. Let $\widetilde{D}$ be the matrix whose columns are the $\widetilde{h}_i$. Then, if $x = \mu_1 e_1 + \ldots + \mu_d e_d = \widetilde{m}_1 \widetilde{h}_1 + \ldots + \widetilde{m}_d \widetilde{h}_d$, where $\{e_1, \dots, e_d\}$ is the standard basis of $\mathbb{R}^d$, we have the following relation between the coordinate tuples in the two bases

$$(\mu_1, \dots, \mu_d) = \widetilde{D}(\widetilde{m}_1, \dots, \widetilde{m}_d).$$

Furthermore, from the construction of the basis $\{\widetilde{h}_1, \dots, \widetilde{h}_d\}$, it follows that $\widetilde{m}_d \geq c > 0$ for some c if $(\widetilde{m}_1, \dots, \widetilde{m}_d) = \widetilde{D}^{-1}(\mu_1, \dots, \mu_d)$ and $(\mu_1, \dots, \mu_d) \in \partial\lambda_s(k)$. Define

$$D := \left(\widetilde{D}^t\right)^{-1}.$$

Step 2. Observe that $k \in B^s_\lambda \Leftrightarrow \lambda_s(k) = \lambda \Leftrightarrow \lambda_s(DD^{-1}k) = \lambda \Leftrightarrow \widetilde{\lambda}_s(D^{-1}k) = \lambda$ where $\widetilde{\lambda}_s := \lambda_s \circ D$. From the definition of the generalized gradient it follows that

$$\partial\widetilde{\lambda}_s(D^{-1}k) = D^t \partial\lambda_s(k).$$

Let $m = (m_1, \dots, m_d) \in \partial\widetilde{\lambda}_s(D^{-1}k)$. Then $m = D^t\mu = \widetilde{D}^{-1}\mu$ for some $\mu \in \partial\lambda_s(k)$. From step 1. follows that there exists a c such that for all $m = (m_1, \dots, m_d) \in \partial\widetilde{\lambda}_s(D^{-1}k)$ the d-th component satisfies $m_d \geq c > 0$. So by lemma 3.12, $0 \notin \partial_{k_d}\widetilde{\lambda}_s(D^{-1}k)$.

Step 3. Since $\lambda \in \mathcal{R}_s$ and $\mathcal{R}_s$ is open, there exists an interval $I \subset \mathcal{R}_s$ with $\lambda \in I$ and $\overline{I} \subset \mathcal{R}_s$. Let $(\tau_0, k_0) = (\tau_0, k_{0,1}, \dots, k_{0,d}) \in \overline{I} \times \lambda_s^{-1}(\overline{I})$ be fixed. Define $\widetilde{F}(\tau, k) := \widetilde{\lambda}_s(k) - \tau$ on $\mathbb{R} \times D^{-1}B$. Then

$$\widetilde{F}(\tau, k) = 0 \Leftrightarrow \widetilde{\lambda}_s(k) = \tau \Leftrightarrow \lambda_s(Dk) = \tau \Leftrightarrow k \in D^{-1}B^s_\tau.$$

Thus $\widetilde{F}(\tau_0, D^{-1}k_0) = 0$. Since τ_0 is regular, we have $0 \notin D^t \partial\lambda_s(k_0) = \partial\widetilde{\lambda}_s(D^{-1}k_0)$ and, by step 2 and lemma 3.12, for the generalized partial gradient of $\widetilde{\lambda}_s$ with respect to the d-th component $0 \notin \partial_{k_d}\widetilde{\lambda}_s(D^{-1}k_0) = \partial_{k_d}\widetilde{F}(\tau_0, D^{-1}k_0)$ holds. Let $\Pi_{1,\ldots,d-1} : \mathbb{R}^d \to \mathbb{R}^{d-1}$, $\Pi_{1,\ldots,d-1}(k_1, \ldots, k_{d-1}, k_d) = (k_1, \ldots, k_{d-1})$ the projection on the first $d-1$ components. By the implicit function theorem for Lipschitz functions 3.4 there exist neighborhoods $U \times U' \subset \mathbb{R} \times \Pi_{1,\ldots,d-1}(D^{-1}B)$ and $U'' \subset \mathbb{R}$ of $(\tau_0, (D^{-1}k_0)_1, \ldots, (D^{-1}k_0)_{d-1})$ and $(D^{-1}k_0)_d$ respectively, where $(D^{-1}k_0)_i$ is the i-th component of the vector $D^{-1}k_0$, $U' \times U''$ open in $D^{-1}B$, and a Lipschitz function $\phi_{(\tau_0,k_0)} : U \times U' \to U''$ such that

$$\widetilde{F}(\tau, k) = 0 \quad (\tau, k_1, \ldots, k_d) \in (U \times U') \times U''$$

$$\Leftrightarrow k_d = \phi_{(\tau_0,k_0)}(\tau, k_1, \ldots, k_{d-1}) \quad (\tau, k_1, \ldots, k_{d-1}) \in U \times U'.$$

Furthermore, for all $(\tau, k_1, \ldots, k_{d-1}) \in U \times U'$ we have

$$D(k_1, \ldots, k_{d-1}, \phi_{(\tau_0,k_0)}(\tau, k_1, \ldots, k_{d-1})) \in B^s_\tau.$$

Since $\overline{I} \times \lambda_s^{-1}(\overline{I})$ is compact, only finitely many invertible matrices D_i together with Lipschitz parametrizations $\phi_1, \ldots, \phi_r$, with $\phi_i : U_i \times U'_i \to U''_i$ where $U_i \times U'_i \subset \mathbb{R} \times D_i^{-1}B$ is a neighborhood of $(\tau_i, (D^{-1}k_i)_1, \ldots, (D^{-1}k_i)_{d-1})$, $U''_i \subset \mathbb{R}$ a neighborhood of $(D^{-1}k_i)_d$ for some $(\tau_i, k_i) \in \overline{I} \times \lambda_s^{-1}(\overline{I})$ such that $U'_i \times U''_i$ is open in $D_i^{-1}B$ are needed to have

$$\overline{I} \times \lambda_s^{-1}(\overline{I}) \subseteq \bigcup_{i=1}^{r} U_i \times D_i(U'_i \times U''_i).$$

Now $B^s_\lambda \subseteq \bigcup_i D_i(U'_i \times U''_i)$ and for all $i \in \{1, \ldots, r\}$

$$k_d = \phi_i(\lambda, k_1, \ldots, k_{d-1}) \Leftrightarrow D_i(k_1, \ldots, k_d) \in B^s_\lambda \cap D_i(U'_i \times U''_i).$$

In the following text we will skip the λ-argument in the parametrization ϕ_i. □

Definition 3.16. *Let $\mathcal{M}$ be a regular level set of λ_s. Define*

$$U_{\mathcal{M}}^{-1}h = \frac{1}{\sqrt{|B|}} \int_{\mathcal{M}} h(\cdot, k)\, d\mathcal{H}^{d-1}(k) \tag{3.1}$$

for $h \in (L^1(\Omega \times \mathcal{M}, \mathbb{C}^M); dx \otimes d\mathcal{H}^{d-1}(k))$, where in the right hand side of the formula h is extended to $\mathbb{R}^d$ by the k-quasiperiodicity condition $h(x+n, k) = e^{ik\cdot n}h(x,k)$, $n \in \mathbb{Z}^d$.

Remark 3.17. *Comparing* (3.1) *with the inverse Floquet-Bloch transform* U^{-1} *in section 5.1 in the appendix, we can interpret* (3.1) *as a sort of restriction of* U^{-1} *on* $\mathcal{M}$*. Analogously one can define* $U^{-1}_{\mathcal{M}\cap\mathcal{U}}$*, where* $\mathcal{U} \subseteq B$ *is open.*

Remark 3.18. *Since* $\mathcal{M} \in \mathfrak{M}$ *can be described by a finite number of Lipschitz graphs,* $\mathcal{H}^{d-1}(\mathcal{M}) < \infty$ *holds (see [31], section 3.3.4). In particular,* $\mathcal{M}$ *is* σ*-finite with respect to* $\mathcal{H}^{d-1}$ *and by Fubini's theorem* (3.1) *exists for almost all* $x \in \mathbb{R}^d$*. Furthermore, by the Lipschitz continuity of the parametrizations* ϕ_i*, for an interval* $I \subset\subset \mathcal{R}$ *there exist a constant* C*, such that for all* $\tau \in I$*,* $\mathcal{H}^{d-1}(B^s_\tau) \leq C$ *holds.*

Theorem 3.19. *Let* $\mathcal{M} = B^s_\lambda$ *be a regular level set of* λ_s *with Lipschitz parametrizations* $\phi_i : U'_i \to U''_i$ *and matrices* $D_i = \left(\widetilde{D}^t_i\right)^{-1}$ *chosen according to proposition 3.15. Let* w *be any strictly positive weight function* w *satisfying the following two conditions.*

(i) There exists a function $\widetilde{w}$ *such that for all* $\widetilde{D}_i$

$$w(\widetilde{D}_i(x_1, \ldots, x_d)) \leq \widetilde{w}(x_d)$$

(ii) For the function $\widetilde{w}$ *from (i)*

$$\sum_{s\in\mathbb{Z}} \|\widetilde{w}\|_{L^\infty([0,1]+s)} < \infty$$

holds.

Then, for $h \in L^2(\Omega \times \mathcal{M}, \mathbb{C}^M; dx \otimes d\mathcal{H}^{d-1}(k))$ *we have* $U^{-1}_{\mathcal{M}} h \in L^2(\mathbb{R}^d, \mathbb{C}^M; w)$ *and*

$$\left\|U^{-1}_{\mathcal{M}} h\right\|_{L^2(\mathbb{R}^d,\mathbb{C}^M;w)} \leq C \left\|h\right\|_{L^2(\Omega\times\mathcal{M},\mathbb{C}^M;dx\otimes d\mathcal{H}^{d-1}(k))} \tag{3.2}$$

with $C > 0$ *independent of* h*.*
As an example for w *take* $w(x_1, \ldots, x_d) = (1 + |x_1| + \ldots + |x_d|)^{-\alpha}$ $(\alpha > 1)$ *with* $\widetilde{w}(x_d) = \left(1 + \frac{|x_d|}{\max_i \|\widetilde{D}_i^{-1}\|_1}\right)^{-\alpha}$*, where* $\|\cdot\|_1$ *denotes a matrix-norm that is compatible with the* $|\cdot|_1$ *vector norm (i.e.* $|Mx|_1 \leq \|M\|_1 |x|_1$*).*
The constant C *in* (3.2) *depends on the parametrizations* ϕ_i *and* $\widetilde{w}$*, but see remark 3.20.*

Proof. The basic idea of the proof is similar to the corresponding proof for the Fourier transform, which can be found in [35]. Yet the details are more intricate, since we have to take into account the relations of the periodicity lattice and the reciprocal lattice. The Lipschitz parametrizations are used to flatten $\mathcal{M}$ locally. The weight function is used to separate a $(d-1)$-dimensional integral that can be estimated after using the flat $(d-1)$-dimensional Floquet-Bloch transform with its isometry quality.

For the proof let $B_n := [-\pi, \pi]^n$, $\Omega_n := [0,1]^n$ denote the n-dimensional Brillouin zone and cell of periodicity. Consider one of the parametrizations and drop the index i: $\phi : U' \to U''$. The accompanying matrix is $D = (\widetilde{D}^t)^{-1}$ according to proposition 3.15 (recall $\widetilde{D} \in \mathbb{Z}^{d\times d}$). Then

$$D(k_1, \dots, k_d) \in \mathcal{M} \cap D(U' \times U'') \Leftrightarrow k_d = \phi(k_1, \dots, k_{d-1}).$$

We will need the following function for the application of the Area formula. Define

$$\begin{array}{rcl} \vartheta : U' & \to & \mathcal{M} \cap D(U' \times U'') \\ (k_1, \dots, k_{d-1}) & \mapsto & D(k_1, \dots, k_{d-1}, \phi(k_1, \dots, k_{d-1})). \end{array}$$

The Area formula from section 5.4 in the appendix yields

$$\int_{\mathcal{M}\cap D(U'\times U'')} h(x,k)\, d\mathcal{H}^{d-1}(k) \quad = \quad \int_{U'} h(x, \vartheta(z)) J\vartheta(z)\, dz. \tag{3.3}$$

$J\vartheta$ is the Jacobian (in the sense of [31], section 3.2.2) of ϑ and it is given by

$$J\vartheta(z) = \sqrt{1 + |\nabla\phi(z)|^2}.$$

Since ϕ is Lipschitz there exist a constant $C(\phi) > 0$ such that for almost all $(k_1, \dots, k_{d-1}) \in U'$

$$1 \le J\vartheta(k_1, \dots, k_{d-1}) \le C(\phi). \tag{3.4}$$

Now we start the essential estimations. Let $\widetilde{x} = (x_1, \dots, x_{d-1})$, $\widetilde{k} = (k_1, \dots, k_{d-1})$. First we employ the definitions of the weighted L^2-norm and $U^{-1}_{\mathcal{M}\cap D(U'\times U'')}$:

$$\begin{aligned} \sqrt{|B|}\left\|U^{-1}_{\mathcal{M}\cap D(U'\times U'')} h\right\|^2_{L^2(\mathbb{R}^d; w)} &= \sqrt{|B|}\int_{\mathbb{R}^d} w(x) |U^{-1}_{\mathcal{M}\cap D(U'\times U'')} h(x)|^2\, dx \\ &= \int_{\mathbb{R}^d} w(x) \left| \int_{\mathcal{M}\cap D(U'\times U'')} h(x,k)\, d\mathcal{H}^{d-1}(k)\right|^2 dx. \end{aligned}$$

Then we insert the Area-formula result (3.3) and perform a change of variables $x \to \widetilde{D}x$ in the integral over $\mathbb{R}^d$:

$$\sqrt{|B|}\left\|U^{-1}_{\mathcal{M}\cap D(U'\times U'')} h\right\|^2_{L^2(\mathbb{R}^d; w)} = \int_{\mathbb{R}^d} w(x) \left| \int_{U'} h(x, \vartheta(\widetilde{k})) J\vartheta(\widetilde{k})\, d\widetilde{k}\right|^2 dx$$

$$= \int_{\mathbb{R}^d} w(\widetilde{D}x) \left| \int_{U'} h(\widetilde{D}x, \vartheta(\widetilde{k})) J\vartheta(\widetilde{k})\, d\widetilde{k} \right|^2 |\det \widetilde{D}|\, dx.$$

We use the property (i) of the weight function w and define an auxiliary function $\widetilde{h}(x_d)$:

$$\begin{aligned} &\sqrt{|B|} \left\| U^{-1}_{\mathcal{M}\cap D(U'\times U'')} h \right\|^2_{L^2(\mathbb{R}^d;w)} \\ \leq\; & |\det \widetilde{D}| \int_{\mathbb{R}} \widetilde{w}(x_d) \underbrace{\int_{\mathbb{R}^{d-1}} \left| \int_{U'} h(\widetilde{D}(\widetilde{x}, x_d), \vartheta(\widetilde{k})) J\vartheta(\widetilde{k})\, d\widetilde{k} \right|^2 d\widetilde{x}}_{=:\widetilde{h}(x_d)}\, dx_d \qquad (3.5) \\ =\; & |\det \widetilde{D}| \int_{\mathbb{R}} \widetilde{w}(x_d) \widetilde{h}(x_d)\, dx_d. \end{aligned}$$

The next step is to split up the integral over $\mathbb{R}$ into integrals over translates of Ω_1 and use Hölder's inequality:

$$\begin{aligned} \sqrt{|B|} \left\| U^{-1}_{\mathcal{M}\cap D(U'\times U'')} h \right\|^2_{L^2(\mathbb{R}^d;w)} &\leq |\det \widetilde{D}| \sum_{s\in\mathbb{Z}} \int_{\Omega_1} \widetilde{w}(x_d+s) \widetilde{h}(x_d+s)\, dx_d \qquad (3.6) \\ &\leq |\det \widetilde{D}| \sum_{s\in\mathbb{Z}} \|\widetilde{w}\|_{L^\infty(\Omega_1+s)} \left\| \widetilde{h}(\cdot+s) \right\|_{L^1(\Omega_1)}. \end{aligned}$$

We will now estimate $\widetilde{h}(x_d+s)$ before continuing the estimation. Define

$$\widehat{h}_{s,x_d}(\widetilde{x}, \widetilde{k}) := \chi_{U'}(\widetilde{k}) h(\widetilde{D}(\widetilde{x}, x_d+s), \vartheta(\widetilde{k})) J\vartheta(\widetilde{k}).$$

Then we make the following calculation, where we split up the integral over $\mathbb{R}^{d-1}$ into integrals over translates of $B_{d-1} \subseteq \mathbb{R}^{d-1}$:

$$\begin{aligned} \widetilde{h}(x_d+s) &= \int_{\mathbb{R}^{d-1}} \left| \int_{U'} h(\widetilde{D}(\widetilde{x}, x_d+s), \vartheta(\widetilde{k})) J\vartheta(\widetilde{k})\, d\widetilde{k} \right|^2 d\widetilde{x} \qquad (3.7) \\ &= \int_{\mathbb{R}^{d-1}} \left| \int_{\mathbb{R}^{d-1}} \widehat{h}_{s,x_d}(\widetilde{x}, \widetilde{k})\, d\widetilde{k} \right|^2 d\widetilde{x} \\ &= \int_{\mathbb{R}^{d-1}} \left| \sum_{j\in\mathbb{Z}^{d-1}} \int_{B_{d-1}} \widehat{h}_{s,x_d}(\widetilde{x}, \widetilde{k}+2\pi j)\, d\widetilde{k} \right|^2 d\widetilde{x} \\ &= |B_{d-1}| \int_{\mathbb{R}^{d-1}} \left| \frac{1}{\sqrt{|B_{d-1}|}} \int_{B_{d-1}} \sum_{j\in\mathbb{Z}^{d-1}} \widehat{h}_{s,x_d}(\widetilde{x}, \widetilde{k}+2\pi j)\, d\widetilde{k} \right|^2 d\widetilde{x}. \end{aligned}$$

Note here that the sum over j is finite due to the boundedness of U'. Define $\widehat{h}^j_{s,x_d}(\widetilde{x},\widetilde{k}) := \widehat{h}_{s,x_d}(\widetilde{x},\widetilde{k}+2\pi j)$. In order to interpret the integral over B_{d-1} as the $(d-1)$-dimensional inverse Floquet-Bloch transform of $\sum_{j\in\mathbb{Z}^{d-1}} \widehat{h}^j_{s,x_d}(\widetilde{x},\widetilde{k})$ we have to check, that $\widehat{h}^j_{s,x_d}$ is $\widetilde{k}$-quasiperiodic. Let $\widetilde{n}\in\mathbb{Z}^{d-1}$ and $n' := (\widetilde{n},0)\in\mathbb{Z}^d$.

$$\begin{aligned}
& \widehat{h}^j_{s,x_d}(\widetilde{x}+\widetilde{n},\widetilde{k}) \\
= \;& \widehat{h}_{s,x_d}(\widetilde{x}+\widetilde{n},\widetilde{k}+2\pi j) \\
= \;& \chi_{U'}(\widetilde{k}+2\pi j) h(\widetilde{D}(\widetilde{x}+\widetilde{n},x_d+s),\vartheta(\widetilde{k}+2\pi j))J\vartheta(\widetilde{k}+2\pi j) \\
= \;& \chi_{U'}(\widetilde{k}+2\pi j) h(\widetilde{D}(\widetilde{x},x_d+s)+\widetilde{D}(\widetilde{n},0),\vartheta(\widetilde{k}+2\pi j))J\vartheta(\widetilde{k}+2\pi j).
\end{aligned}$$

Since $h(x,k)$ is k-quasiperiodic and $\widetilde{D}$ has integer entries, we obtain

$$\begin{aligned}
& \widehat{h}^j_{s,x_d}(\widetilde{x}+\widetilde{n},\widetilde{k}) \\
= \;& \exp\left(i\widetilde{D}(\widetilde{n},0)\cdot\vartheta(\widetilde{k}+2\pi j)\right)\chi_{U'}(\widetilde{k}+2\pi j) \\
& \quad \times h(\widetilde{D}(\widetilde{x},x_d+s),\vartheta(\widetilde{k}+2\pi j))J\vartheta(\widetilde{k}+2\pi j) \\
= \;& \exp\left(i(\widetilde{n},0)^t\widetilde{D}^t\vartheta(\widetilde{k}+2\pi j)\right)\widehat{h}_{s,x_d}(\widetilde{x},\widetilde{k}+2\pi j) \\
= \;& \exp\left(i(\widetilde{n},0)^t\widetilde{D}^t D(\widetilde{k}+2\pi j,\phi(\widetilde{k}+2\pi j))\right)\widehat{h}_{s,x_d}(\widetilde{x},\widetilde{k}+2\pi j) \\
= \;& e^{i\widetilde{n}\cdot(\widetilde{k}+2\pi j)}\,\widehat{h}_{s,x_d}(\widetilde{x},\widetilde{k}+2\pi j) \\
= \;& e^{i\widetilde{n}\cdot\widetilde{k}}\,\widehat{h}^j_{s,x_d}(\widetilde{x},\widetilde{k}),
\end{aligned}$$

where we have used that $D=(\widetilde{D}^t)^{-1}$. So we can continue the calculation (3.7), remarking that the sum is in fact finite because of the boundedness of U'. Thus the number of summands is less than a constant $C(U')$. In the following calculation we make use of the isometry property of the $(d-1)$-dimensional Floquet-Bloch transform $U_{B_{d-1}}$:

$$\begin{aligned}
\widetilde{h}(x_d+s) &= |B_{d-1}|\int_{\mathbb{R}^{d-1}}\left|\frac{1}{\sqrt{|B_{d-1}|}}\int_{B_{d-1}}\sum_{j\in\mathbb{Z}^{d-1}}\widehat{h}^j_{s,x_d}(\widetilde{x},\widetilde{k})\,d\widetilde{k}\right|^2 d\widetilde{x} \\
&= |B_{d-1}|\int_{\mathbb{R}^{d-1}}\left|U^{-1}_{B_{d-1}}\sum_{j\in\mathbb{Z}^{d-1}}\widehat{h}^j_{s,x_d}(\widetilde{x})\right|^2 d\widetilde{x} \\
&= |B_{d-1}|\left\|U^{-1}_{B_{d-1}}\sum_{j\in\mathbb{Z}^{d-1}}\widehat{h}^j_{s,x_d}\right\|^2_{L^2(\mathbb{R}^{d-1})}
\end{aligned}$$

$$
\begin{aligned}
&= |B_{d-1}| \left\| U_{B_{d-1}} U_{B_{d-1}}^{-1} \sum_{j \in \mathbb{Z}^{d-1}} \widehat{h}_{s,x_d}^{j} \right\|_{L^2(\Omega_{d-1} \times B_{d-1})}^2 \\
&= |B_{d-1}| \left\| \sum_{j \in \mathbb{Z}^{d-1}} \widehat{h}_{s,x_d}^{j} \right\|_{L^2(\Omega_{d-1} \times B_{d-1})}^2 .
\end{aligned}
$$

Next we use Cauchy-Schwarz inequality on the sum over j and insert the definition of $\widehat{h}_{s,x_d}^{j}$ and $\widehat{h}_{s,x_d}$:

$$
\begin{aligned}
&\widetilde{h}(x_d + s) \\
&= |B_{d-1}| \int_{\Omega_{d-1}} \int_{B_{d-1}} \left| \sum_{j \in \mathbb{Z}^{d-1}} \widehat{h}_{s,x_d}^{j}(\widetilde{x}, \widetilde{k}) \right|^2 d\widetilde{k}\, d\widetilde{x} \\
&\leq |B_{d-1}| C(U') \int_{\Omega_{d-1}} \int_{B_{d-1}} \sum_{j \in \mathbb{Z}^{d-1}} \left| \widehat{h}_{s,x_d}^{j}(\widetilde{x}, \widetilde{k}) \right|^2 d\widetilde{k}\, d\widetilde{x} \\
&= |B_{d-1}| C(U') \int_{\Omega_{d-1}} \sum_{j \in \mathbb{Z}^{d-1}} \int_{B_{d-1} + 2\pi j} \left| \widehat{h}_{s,x_d}(\widetilde{x}, \widetilde{k}) \right|^2 d\widetilde{k}\, d\widetilde{x} \\
&= |B_{d-1}| C(U') \int_{\Omega_{d-1}} \sum_{j \in \mathbb{Z}^{d-1}} \int_{B_{d-1} + 2\pi j} \chi_{U'}(\widetilde{k}) \left| h(\widetilde{D}(\widetilde{x}, x_d + s), \vartheta(\widetilde{k})) J\vartheta(\widetilde{k}) \right|^2 d\widetilde{k}\, d\widetilde{x}.
\end{aligned}
$$

Then we use the quasiperiodicity of h and that the modulus of e^{it} is 1 for any real t:

$$
\begin{aligned}
&\widetilde{h}(x_d + s) \\
&\leq |B_{d-1}| C(U') \int_{\Omega_{d-1}} \sum_{j \in \mathbb{Z}^{d-1}} \int_{B_{d-1} + 2\pi j} \chi_{U'}(\widetilde{k}) \left| h(\widetilde{D}(\widetilde{x}, x_d + s), \vartheta(\widetilde{k})) J\vartheta(\widetilde{k}) \right|^2 d\widetilde{k}\, d\widetilde{x} \\
&= C(U') |B_{d-1}| \int_{\Omega_{d-1}} \int_{U'} \left| e^{i(0,\ldots,0,s)^t \widetilde{D}^t \vartheta(\widetilde{k})} h(\widetilde{D}(\widetilde{x}, x_d), \vartheta(\widetilde{k})) J\vartheta(\widetilde{k}) \right|^2 d\widetilde{k}\, d\widetilde{x} \\
&= |B_{d-1}| C(U') \int_{\Omega_{d-1}} \int_{U'} \left| h(\widetilde{D}(\widetilde{x}, x_d), \vartheta(\widetilde{k})) J\vartheta(\widetilde{k}) \right|^2 d\widetilde{k}\, d\widetilde{x}.
\end{aligned}
$$

The estimate (3.4) and the Area-formula result (3.3) imply:

$$
\begin{aligned}
&\widetilde{h}(x_d + s) \\
&\leq |B_{d-1}| C(U', \phi) \int_{\Omega_{d-1}} \int_{U'} \left| h(\widetilde{D}(\widetilde{x}, x_d), \vartheta(\widetilde{k})) \right|^2 J\vartheta(\widetilde{k})\, d\widetilde{k}\, d\widetilde{x} \\
&= |B_{d-1}| C(U', \phi) \int_{\Omega_{d-1}} \int_{\mathcal{M} \cap D(U' \times U'')} \left| h(\widetilde{D}(\widetilde{x}, x_d), k) \right|^2 d\mathcal{H}^{d-1}(k)\, d\widetilde{x}.
\end{aligned}
$$

Now we can continue the estimation (3.6) of the weighted norm of $U_M^{-1}h$. $\widetilde{h}(x_d+s)$ is estimated by an expression independent of s (see the above formula) which we can pull out of the sum over s. The remaining sum is by the assumption (ii) for the weight function w less than a constant $C(\widetilde{w})$:

$$\begin{aligned}
&\sqrt{|B|}\left\|U^{-1}_{\mathcal{M}\cap D(U'\times U'')}h\right\|^2_{L^2(\mathbb{R}^d;w)}\\
\leq\;& |\det\widetilde{D}|\sum_{s\in\mathbb{Z}}\|\widetilde{w}\|_{L^\infty(\Omega_1+s)}\left\|\widetilde{h}(\cdot+s)\right\|_{L^1(\Omega_1)}\\
=\;& |\det\widetilde{D}|\sum_{s\in\mathbb{Z}}\|\widetilde{w}\|_{L^\infty(\Omega_1+s)}\int_{\Omega_1}\widetilde{h}(x_d+s)\,dx_d\\
\leq\;& |B_{d-1}|C(U',\phi,\widetilde{w})|\det\widetilde{D}|\\
&\times\int_{\Omega_1}\int_{\Omega_{d-1}}\int_{\mathcal{M}\cap D(U'\times U'')}\left|h(\widetilde{D}(\widetilde{x},x_d),k)\right|^2\,d\mathcal{H}^{d-1}(k)\,d\widetilde{x}\,dx_d\\
=\;& |B_{d-1}|C(U',\phi,\widetilde{w})|\det\widetilde{D}|\int_\Omega\int_{\mathcal{M}\cap D(U'\times U'')}\left|h(\widetilde{D}x,k)\right|^2\,d\mathcal{H}^{d-1}(k)\,dx.
\end{aligned}$$

Again we change the variables $\widetilde{D}x\to x$ and afterwards split up the integration over $\widetilde{D}\Omega$ into integrals over translates of $\Omega\subseteq\mathbb{R}^d$:

$$\begin{aligned}
&\sqrt{|B|}\left\|U^{-1}_{\mathcal{M}\cap D(U'\times U'')}h\right\|^2_{L^2(\mathbb{R}^d;w)}\\
\leq\;& |B_{d-1}|C(U',\phi,\widetilde{w})\int_{\widetilde{D}\Omega}\int_{\mathcal{M}\cap D(U'\times U'')}|h(x,k)|^2\,d\mathcal{H}^{d-1}(k)\,dx\\
\leq\;& |B_{d-1}|C(U',\phi,\widetilde{w})\int_{\mathbb{R}^d}\int_{\mathcal{M}\cap D(U'\times U'')}\chi_{\widetilde{D}\Omega}(x)\,|h(x,k)|^2\,d\mathcal{H}^{d-1}(k)\,dx\\
\leq\;& |B_{d-1}|C(U',\phi,\widetilde{w})\sum_{j\in\mathbb{Z}^d}\int_\Omega\int_{\mathcal{M}\cap D(U'\times U'')}\chi_{\widetilde{D}\Omega}(x+j)\,|h(x+j,k)|^2\,d\mathcal{H}^{d-1}(k)\,dx
\end{aligned}$$

Finally we make use of the quasiperiodicity of h:

$$\begin{aligned}
&\sqrt{|B|}\left\|U^{-1}_{\mathcal{M}\cap D(U'\times U'')}h\right\|^2_{L^2(\mathbb{R}^d;w)}\\
\leq\;& |B_{d-1}|C(U',\phi,\widetilde{w})\sum_{j\in\mathbb{Z}^d}\int_\Omega\int_{\mathcal{M}\cap D(U'\times U'')}\chi_{\widetilde{D}\Omega}(x+j)\left|e^{ijk}h(x,k)\right|^2\,d\mathcal{H}^{d-1}(k)\,dx\\
\leq\;& |B_{d-1}|C(U',\phi,\widetilde{w},\widetilde{D})\int_\Omega\int_{\mathcal{M}\cap D(U'\times U'')}|h(x,k)|^2\,d\mathcal{H}^{d-1}(k)\,dx\\
=\;& |B_{d-1}|C(U',\phi,\widetilde{w},\widetilde{D})\,\|h\|^2_{L^2(\Omega\times\mathcal{M}\cap D(U'\times U'');dx\otimes d\mathcal{H}^{d-1}(k))}\,.
\end{aligned}$$

Note that the sum over j is finite, because $\widetilde{D}\Omega$ is bounded. Thus $C(\widetilde{D})$ is a constant bounding the number of summands in this sum.

Now it remains to put the estimates on the pieces $\mathcal{M} \cap D_i(U_i' \times U_i'')$ together to obtain (3.2). The constant C in (3.2) is to be chosen greater than the maximum of the constants $C_i = C_i(U_i', \phi_i, \widetilde{w}, \widetilde{D}_i)$ times a constant that depends on the total number of the parametrizations ϕ_i. $\square$

Remark 3.20. *From the proof of proposition 3.15 follows by compactness reasons that for any interval $I \subset\subset \mathcal{R}_s$ of regular values of λ_s one can use a finite set of ϕ_i, U_i', U_i'', D_i and $\widetilde{D}_i$ to describe all level sets B_τ^s ($\tau \in I$). Thus we obtain a corollary of theorem 3.19:*

Corollary 3.21. *Let $I \subset\subset \mathcal{R}_s$ be an interval of regular values of λ_s. Then for any weight function w with properties as in theorem 3.19 there exists a constant $C = C(I)$ such that for all $\tau \in I$ and $h \in L^2(\Omega \times B_\tau^s, \mathbb{C}^M; dx \otimes d\mathcal{H}^{d-1}(k))$ we have $U_{B_\tau^s}^{-1} h \in L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx)$ and*

$$\left\| U_{B_\tau^s}^{-1} h \right\|_{L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx)} \leq C \, \|h\|_{L^2(\Omega \times B_\tau^s, \mathbb{C}^M; dx \otimes d\mathcal{H}^{d-1}(k))} \, .$$

A suitable constant $C = C(I)$ can be computed from the properties of $\widetilde{w}$ and the finite set of the $\phi_i, U_i', D_i, \widetilde{D}_i$ that is used to describe all the level sets B_τ^s ($\tau \in I$), and it is independent of τ and h.

3.4 Representation of the distributional limiting absorption solution for regular λ.

Recall that $\mathcal{R}$ denotes the open set of regular values (see definition 3.6).

Theorem 3.22. *Let $\varphi \in C_0^\infty(\mathcal{R} \times \mathbb{R}^d, \mathbb{C}^M)$, $\mathcal{L}$ and f as in section 1.4 and 2.1. Then for every $\epsilon > 0$, there exist locally integrable functions $a_\epsilon, d^\pm : \mathcal{R} \times \mathbb{R}^d \to \mathbb{C}^M$ such that for almost all $\lambda \in \mathcal{R}$ we have $a_\epsilon(\lambda, \cdot) \in L^2(\mathbb{R}^d, \mathbb{C}^M)$, $d^\pm(\lambda, \cdot) \in L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx)$ with a weight function as in theorem 3.19, and for the distributional limiting absorption solutions $u^\pm$ of $(\mathcal{L} - \lambda)u = f$ from chapter 2 definition 2.5 the following representation holds*

$$u^\pm[\varphi] = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \int_{\mathbb{R}^d} a_\epsilon(\lambda, x) \varphi(\lambda, x) \, dx \, d\lambda + \int_{\mathbb{R}} \int_{\mathbb{R}^d} d^\pm(\lambda, x) \varphi(\lambda, x) \, dx \, d\lambda,$$

where the limit $\lim_{\epsilon \to 0} \ldots$ exists and is equal to the principal value part of $u^\pm[\varphi]$ (see definition 2.5). The functions $a_\epsilon, d^\pm$ can be chosen as follows:

$$a_\epsilon(\lambda, x) \; = \; \frac{1}{\sqrt{|B|}} \int_B \sum_s \chi_{\{k: |\lambda_s(k) - \lambda| > \epsilon\}}(k) \frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k) \rangle_{L^2(\Omega, \mathbb{C}^M)}}{\lambda_s(k) - \lambda} \psi_s(x, k) \, dk$$

$$d^{\pm}(\lambda,x) = \frac{(\pm i\pi)}{\sqrt{|B|}}\sum_s \int_{B_\lambda^s} \frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)}{|\nabla\lambda_s(k)|}\, d\mathcal{H}^{d-1}(k).$$

Note that $\nabla\lambda_s$ exists almost everywhere and is bounded away from 0 by a positive constant depending on the support of φ (see Lemma 3.11).

Proof. Each of the distributions $u^{\pm}$ consists of two parts. The first part containing the Cauchy principal value integral can be manipulated as follows using

$$\Phi_{s,k}(\lambda) = \int_{\mathbb{R}^d} \varphi(\lambda,x)\psi_s(x,k)\,\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\, dx$$

from definition 2.3 and lemma 2.2 in chapter 2.

$$\begin{aligned}
& \frac{1}{\sqrt{|B|}}\int_B \sum_{s\in\mathcal{J}} \mathcal{P}\!\int_0^\infty \frac{\Phi_{s,k}(\lambda)}{\lambda_s(k)-\lambda}\, d\lambda\, dk \\
= \;& \frac{1}{\sqrt{|B|}}\lim_{\epsilon\to 0}\int_B \chi_{\{k:|\lambda_s(k)-\lambda|>\epsilon\}}(k) \\
& \times \sum_{s\in\mathcal{J}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\varphi(\lambda,x)\psi_s(x,k)\, dx\, \frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}}{\lambda_s(k)-\lambda}\, d\lambda\, dk \\
= \;& \lim_{\epsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Bigg(\frac{1}{\sqrt{|B|}}\int_B \sum_{s\in\mathcal{J}} \chi_{\{k:|\lambda_s(k)-\lambda|>\epsilon\}}(k) \\
& \times \frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)}{\lambda_s(k)-\lambda}\, dk\Bigg)\varphi(\lambda,x)\, dx\, d\lambda.
\end{aligned}$$

The point evaluation part is subjected to the Coarea formula 5.11, applied to the integral over B. Notice that $\varphi \in C_0^\infty(\mathcal{R}\times\mathbb{R}^d,\mathbb{C}^M)$ and so the support of φ with respect to λ is contained in the set of regular values. Also the sum over s is actually finite because of the finite support of φ.

$$\begin{aligned}
& \frac{1}{\sqrt{|B|}}\int_B \pm i\pi \sum_{s\in\mathcal{J}}\Phi_{s,k}(\lambda_s(k))\, dk \\
= \;& \frac{(\pm i\pi)}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_{\mathbb{R}}\int_{B_\lambda^s}\int_{\mathbb{R}^d}\varphi(\lambda_s(k),x)\psi_s(x,k)\, dx\, \frac{\langle Uf(\lambda_s(k),\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}}{|\nabla\lambda_s(k)|}\, d\mathcal{H}^{d-1}(k)\, d\lambda \\
= \;& \frac{(\pm i\pi)}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_{\mathbb{R}}\int_{B_\lambda^s}\int_{\mathbb{R}^d}\varphi(\lambda,x)\psi_s(x,k)\, dx \frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}}{|\nabla\lambda_s(k)|}\, d\mathcal{H}^{d-1}(k)\, d\lambda \\
= \;& \int_{\mathbb{R}}\int_{\mathbb{R}^d}\Bigg(\frac{(\pm i\pi)}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_{B_\lambda^s}\frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)}{|\nabla\lambda_s(k)|}\, d\mathcal{H}^{d-1}(k)\Bigg)\varphi(\lambda,x)\, dx\, d\lambda.
\end{aligned}$$

Changing the order of integrations can be justified as in the proof of theorem 2.8 in chapter 2. □

3.5 Properties of integrals over regular level sets

In this section we study integrals of the function

$$g(\lambda, x, k) := \frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x, k)}{|\nabla\lambda_s(k)|}$$

over level sets of regular values of λ_s for a fixed band index s. Let $I_0 \subset\subset I \setminus \partial I$ and $I \subset\subset \mathcal{R} \subseteq \mathcal{R}_s$. For $\lambda, \tau \in I$ define for the $L^2(\Omega \times B^s_\tau, \mathbb{C}^M; dx \otimes d\mathcal{H}^{d-1}(k))$-function $g(\lambda, \cdot, \cdot)$, which is extended in x to $\mathbb{R}^d$ by k-quasiperiodicity

$$g(\lambda, x + n, k) = e^{ikn} g(\lambda, x, k), \quad (n \in \mathbb{Z}^d),$$

the level set integral h_s of g by

$$h_s(\lambda, \tau, x) := \frac{1}{\sqrt{|B|}} \int_{B^s_\tau} g(\lambda, x, k)\, d\mathcal{H}^{d-1}(k) = U^{-1}_{B^s_\tau} g(\lambda, x). \tag{3.8}$$

We put $h_s(\lambda, \tau, \cdot) = 0$ if λ or τ is not in I. To remind of this fact, we will sometimes insert an characteristic function χ_I in front of h_s. We are interested in the integrability properties of h_s.

Lemma 3.23. $h_s \in L^2(I \times I \times \mathbb{R}^d, \mathbb{C}^M; d\lambda \otimes d\tau \otimes w(x)dx)$.

Proof. By lemma 3.11 there exists a $q > 0$ such that $|\nabla\lambda_s(k)| \geq q > 0$ for all k belonging to the level sets B^s_τ with $\tau \in I$. First we write out the definition of the norm and use corollary 3.21:

$$\begin{aligned}
&\|h_s\|^2_{L^2(I\times I\times\mathbb{R}^d;d\lambda\otimes d\tau\otimes w(x)dx)} \\
= &\int_I \int_I \left\| U^{-1}_{B^s_\tau} g(\lambda, \cdot)\right\|^2_{L^2(\mathbb{R}^d;w)} d\tau\, d\lambda \\
\leq &\; C(I) \int_I \int_I \|g(\lambda, \cdot, \cdot)\|^2_{L^2(\Omega\times B^s_\tau)} d\tau\, d\lambda \\
= &\; C(I) \int_I \int_I \left\| \frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)}\psi_s(x, k)}{|\nabla\lambda_s(k)|}\right\|^2_{L^2(\Omega\times B^s_\tau)} d\tau\, d\lambda.
\end{aligned}$$

Then, using the definition of the $L^2(\Omega \times B^s_\tau)$-norm, we integrate out the integral over Ω and estimate one of the $|\nabla\lambda_s(k)|$ in the denominator by q:

$$\begin{aligned}
&\|h_s\|^2_{L^2(I\times I\times\mathbb{R}^d;d\lambda\otimes d\tau\otimes w(x)dx)} \\
\leq\ & \frac{1}{q}C(I)\int_I\int_I\int_{B^s_\tau}\frac{|\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}|^2}{|\nabla\lambda_s(k)|}\,d\mathcal{H}^{d-1}(k)\,d\tau\,d\lambda.
\end{aligned}$$

Again we use the Coarea formula and the isometry property of U:

$$\begin{aligned}
\|h_s\|^2_{L^2(I\times I\times\mathbb{R}^d;d\lambda\otimes d\tau\otimes w(x)dx)} &\leq \frac{1}{q}C(I)\int_I\int_B|\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}|^2\,dk\,d\lambda \\
&\leq \frac{1}{q}C(I)\int_I\|f(\lambda,\cdot)\|^2_{L^2(\mathbb{R}^d)}\,d\lambda<\infty.
\end{aligned}$$

□

Lemma 3.24. *Let $\varepsilon > 0$ be fixed and h_s be as in* (3.8)*, then for the truncated Hilbert transform (see appendix 5.6)*

$$H^{(\varepsilon)}h_s(\lambda,x)=\int_{|\lambda-\tau|>\varepsilon}\frac{\chi_I(\tau)h_s(\lambda,\tau,x)}{\lambda-\tau}\,d\tau\in L^2(I\times\mathbb{R}^d,\mathbb{C}^M;d\lambda\otimes dx)$$

holds without the weight function w.

Proof. Writing out the norm and the definition of $H^{(\varepsilon)}$ and h_s we obtain:

$$\begin{aligned}
&\left\|H^{(\varepsilon)}h_s\right\|^2_{L^2(I\times\mathbb{R}^d)} \\
=\ & \int_{\mathbb{R}^d}\int_I|H^{(\varepsilon)}h_s(\lambda,x)|^2\,d\lambda\,dx \\
=\ & \frac{1}{\sqrt{|B|}}\int_{\mathbb{R}^d}\int_I\left|\int_{|\lambda-\tau|>\varepsilon}\frac{\chi_I(\tau)\int_{B^s_\tau}g(\lambda,x,k)\,d\mathcal{H}^{d-1}(k)}{\lambda-\tau}\,d\tau\right|^2\,d\lambda\,dx.
\end{aligned}$$

Then we apply the Coarea Formula to the integrals $\int_{|\lambda-\tau|>\varepsilon}$ and $\int_{B^s_\tau}$:

$$\begin{aligned}
&\left\|H^{(\varepsilon)}h_s\right\|^2_{L^2(I\times\mathbb{R}^d)} \\
=\ & \frac{1}{\sqrt{|B|}}\int_{\mathbb{R}^d}\int_I\left|\int_{\{k\in B:|\lambda_s(k)-\lambda|>\varepsilon\}}\frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{\lambda-\lambda_s(k)}\,dk\right|^2\,d\lambda\,dx \\
=\ & \frac{1}{\sqrt{|B|}}\int_I\int_{\mathbb{R}^d}\left|\int_B\chi_{\{k\in B:|\lambda_s(k)-\lambda|>\varepsilon\}}(k)\frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{\lambda-\lambda_s(k)}\,dk\right|^2\,dx\,d\lambda.
\end{aligned}$$

We write the integral over B as an inverse Floquet-Bloch transform, which is possible, since the integrand is k quasiperiodic. Afterwards we use that U is an isometry:

$$\begin{aligned}
&\left\|H^{(\varepsilon)}h_s\right\|^2_{L^2(I\times\mathbb{R}^d)} \\
=\ & \frac{1}{\sqrt{|B|}}\int_I \left\| U^{-1}\chi_{\{k\in B:|\lambda_s(k)-\lambda|>\varepsilon\}}(k)\frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{\lambda-\lambda_s(k)}\,dk\right\|^2_{L^2(\mathbb{R}^d)} d\lambda \\
=\ & \frac{1}{\sqrt{|B|}}\int_I \left\| \chi_{\{k\in B:|\lambda_s(k)-\lambda|>\varepsilon\}}(k)\frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{\lambda-\lambda_s(k)}\right\|^2_{L^2(\Omega\times B;dx\otimes dk)} d\lambda.
\end{aligned}$$

In the next step we estimate the denominator by ε, integrate out the integral over Ω in the $L^2(\Omega\times B)$-norm and add terms with the remaining $s\in\mathcal{J}$ to complete the sum over s:

$$\begin{aligned}
&\left\|H^{(\varepsilon)}h_s\right\|^2_{L^2(I\times\mathbb{R}^d)} \\
\le\ & \frac{1}{\sqrt{|B|}}\frac{1}{\varepsilon^2}\int_I \left\|\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)\right\|^2_{L^2(\Omega\times B,dx\otimes dk)} d\lambda \\
\le\ & \frac{1}{\sqrt{|B|}}\frac{1}{\varepsilon^2}\int_I\int_B\sum_{s\in\mathcal{J}} |\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}|^2\,dk\,d\lambda.
\end{aligned}$$

Finally we use Parseval's identity on the sum $\sum_{s\in\mathcal{J}}$ and the isometry property of U:

$$\begin{aligned}
\left\|H^{(\varepsilon)}h_s\right\|^2_{L^2(I\times\mathbb{R}^d)} &\le \frac{1}{\sqrt{|B|}}\frac{1}{\varepsilon^2}\int_I \|Uf(\lambda,\cdot,\cdot)\|^2_{L^2(\Omega\times B)}\,d\lambda \\
&= \frac{1}{\sqrt{|B|}}\frac{1}{\varepsilon^2}\int_I \|f(\lambda,\cdot)\|^2_{L^2(\mathbb{R}^d)}\,d\lambda<\infty.
\end{aligned}$$

□

Definition 3.25. *We define*

$$\mathcal{B} := L^2(\mathbb{R}^d,\mathbb{C}^M;w(x)dx) \tag{3.9}$$

with inner product

$$\langle u,v\rangle_{\mathcal{B}} := \int_{\mathbb{R}^d} u(x)\overline{v(x)}w(x)\,dx.$$

Lemma 3.26. *The mapping*

$$h_s:\lambda\mapsto \underbrace{\Big(\tau\mapsto \underbrace{h_s(\lambda,\tau,\cdot)}_{\in\mathcal{B}}\Big)}_{\in L^\infty(I,\mathcal{B})}$$

is Lipschitz continuous for $\lambda\in I$ with values in $L^\infty(I,\mathcal{B})$.

Proof. First we show that there exists a constant $C > 0$, that depends only on I and system constants, such that for all $\lambda \in I$, almost all $\tau \in I$, $\|h_s(\lambda, \tau, x)\|_{\mathcal{B}} \leq C < \infty$ holds. This shows that $h_s(\lambda, \cdot, \cdot) \in L^\infty(I, \mathcal{B})$. First we use the definition of h_s and the estimate from corollary 3.21:

$$\begin{aligned} \|h_s(\lambda, \tau, \cdot)\|_{\mathcal{B}}^2 &= \left\| U_{B_\tau^s}^{-1} \left(\frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)} \psi_s(x, k)}{|\nabla \lambda_s(k)|} \right) \right\|_{\mathcal{B}}^2 \\ &\leq C(I) \left\| \frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)} \psi_s(x, k)}{|\nabla \lambda_s(k)|} \right\|_{L^2(\Omega \times B_\tau^s; dx \otimes d\mathcal{H}^{d-1}(k))}^2 \end{aligned}$$

We integrate out the integral over Ω in the $L^2(\Omega \times B)$-norm:

$$\begin{aligned} \|h_s(\lambda, \tau, \cdot)\|_{\mathcal{B}}^2 &\leq C(I) \left\| \frac{\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)}}{|\nabla \lambda_s(k)|} \right\|_{L^2(B_\tau^s; d\mathcal{H}^{d-1}(k))}^2 \\ &= C(I) \int_{B_\tau^s} \frac{|\langle Uf(\lambda, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)}|^2}{|\nabla \lambda_s(k)|^2} \, d\mathcal{H}^{d-1}(k) \end{aligned}$$

Then we use that $|\nabla \lambda_s(k)|$ is bounded from below uniformly for all $\tau \in I$ by a positive constant by lemma 3.11. We absorb this constant into $C(I)$. Furthermore we use lemma 5.3 from the appendix:

$$\|h_s(\lambda, \tau, \cdot)\|_{\mathcal{B}}^2 \leq C(I) \frac{C(\text{supp}(f))}{|B|} \|f(\lambda, \cdot)\|_{L^2(\mathbb{R}^d)}^2 \int_{B_\tau^s} 1 \, d\mathcal{H}^{d-1}(k) < \infty$$

by remark 3.18. Now we show Lipschitz continuity of h_s with respect to λ by similar estimates. Let $\lambda, \widetilde{\lambda} \in I$. First we use corollary 3.21 and integrate out the integral over Ω in the emerging $L^2(\Omega \times B_\tau^s)$-norm, such that we obtain:

$$\begin{aligned} &\left\| h_s(\lambda, \cdot, \cdot) - h_s(\widetilde{\lambda}, \cdot, \cdot) \right\|_{L^\infty(I, \mathcal{B})}^2 \\ &= \sup_{\tau \in I} \left\| U_{B_\tau^s}^{-1} \left(\frac{\langle Uf(\lambda, \cdot, k) - Uf(\widetilde{\lambda}, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)} \psi_s(x, k)}{|\nabla \lambda_s(k)|} \right) \right\|_{\mathcal{B}}^2 \\ &\leq C(I) \sup_{\tau \in I} \left\| \frac{\langle Uf(\lambda, \cdot, k) - Uf(\widetilde{\lambda}, \cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)}}{|\nabla \lambda_s(k)|} \right\|_{L^2(B_\tau^s; d\mathcal{H}^{d-1}(k))}^2 \end{aligned}$$

Then we use lemma 3.11 to estimate the denominator and lemma 5.3 for the terms with Uf. All upcoming constants are absorbed into $C(I)$:

$$\left\| h_s(\lambda, \cdot, \cdot) - h_s(\widetilde{\lambda}, \cdot, \cdot) \right\|_{L^\infty(I, \mathcal{B})}^2 \leq \frac{C(\text{supp}(f))}{|B|} \left\| f(\lambda, \cdot) - f(\widetilde{\lambda}, \cdot) \right\|_{L^2(\mathbb{R}^d)}^2$$

$$\leq \; C(I)\frac{C(\mathrm{supp}(f))}{|B|}\mathrm{Lip}(f)_I^2|\lambda-\widetilde{\lambda}|^2.$$

In the last step we used the Lipschitz continuity of f with respect to λ. □

The level set integral h_s has a connection to the spectral family $\mu \mapsto E(\mu)$ associated with $\mathcal{L}$ for $\mu \in \mathcal{R}$. For simplicity, let $f \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ be independent of λ. Due to the Floquet-Bloch representation of $\mathcal{L}$

$$\mathcal{L}f(x) = \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \lambda_s(k)\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)\,dk$$

we find the following formula for $E(\mu)$

$$E(\mu)f(x) = \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \chi_{\{\kappa\in B:\lambda_s(\kappa)\leq\mu\}}(k)\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)\,dk.$$

Proposition 3.27. *Let $I \subset\subset \mathcal{R}$, $f \in L^2(\mathbb{R}^d, \mathbb{C}^M)$, $g \in \mathcal{B}$ with a weight function as in theorem 3.19 and $r := \max\{|s| : s \in \mathcal{J},\ \lambda_s(B)\cap I \neq \emptyset\}$. For the derivative $A(\mu)$ of the spectral family, which is defined by*

$$\langle A(\mu)f, g\rangle_{\mathcal{B}} := \frac{d}{d\mu}\langle E(\mu)f, g\rangle_{\mathcal{B}}$$

the following holds for almost all $\mu \in I$:

$$\begin{aligned} A(\mu)f &= \sum_{s\in\mathcal{J},|s|\leq r} h_s(\mu,\cdot) \\ &= \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J},|s|\leq r}\int_{B_\mu^s}\frac{\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(\cdot,k)}{|\nabla\lambda_s(k)|}\,d\mathcal{H}^{d-1}(k). \end{aligned}$$

Proof. For $(E(\mu+\nu)-E(\mu))f$ we find the following representation, noting that $|\nabla\lambda_s(k)| \geq q$ if $\mu \leq \lambda_s(k) \leq \mu+\nu$ and thus we are able to apply the Coarea formula.

$$\begin{aligned} &(E(\mu+\nu)-E(\mu))f \\ =\;& \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \chi_{\{\kappa\in B:\mu<\lambda_s(\kappa)\leq\mu+\nu\}}(k)\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(\cdot,k)\,dk \\ =\;& \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J};|s|\leq r}\int_\mu^{\mu+\nu}\int_{B_\tau^s}\frac{\langle Uf(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(\cdot,k)}{|\nabla\lambda_s(k)|}\,d\mathcal{H}^{d-1}(k)\,d\tau \end{aligned}$$

$$= \sum_{s\in\mathcal{J},|s|\le r} \int_\mu^{\mu+\nu} h_s(\tau,\cdot)\, d\tau.$$

We use this to calculate the derivative of $E(\mu)$:

$$\begin{aligned}
& \frac{1}{\nu}\left|\langle (E(\mu+\nu)-E(\mu))f - \sum_{s\in\mathcal{J};|s|\le r} h_s(\mu,\cdot)\nu, g\rangle_{\mathcal{B}}\right| \\
= & \frac{1}{\nu}\left|\langle \sum_{s\in\mathcal{J},|s|\le r} \int_\mu^{\mu+\nu} h_s(\tau,\cdot)\, d\tau - \sum_{s\in\mathcal{J};|s|\le r} h_s(\mu,\cdot)\nu, g\rangle_{\mathcal{B}}\right| \\
\le & \frac{1}{\nu}\sum_{s\in\mathcal{J},|s|\le r}\left|\langle \int_\mu^{\mu+\nu} h_s(\tau,\cdot)\, d\tau - h_s(\mu,\cdot)\nu, g\rangle_{\mathcal{B}}\right| \\
= & \frac{1}{\nu}\sum_{s\in\mathcal{J},|s|\le r}\left|\langle \int_\mu^{\mu+\nu} h_s(\tau,\cdot)\, d\tau - \int_\mu^{\mu+\nu} h_s(\mu,\cdot)\, d\tau, g\rangle_{\mathcal{B}}\right| \\
\le & \sum_{s\in\mathcal{J},|s|\le r}\frac{1}{\nu}\int_\mu^{\mu+\nu} |\langle h_s(\tau,\cdot)-h_s(\mu,\cdot), g\rangle_{\mathcal{B}}|\; d\tau
\end{aligned}$$

Now, since $\tau\mapsto\langle h_s(\tau,\cdot),g\rangle_{\mathcal{B}}$ is in $L^2(I)$ because $\tau\mapsto h_s(\tau,\cdot)$ is in $L^2(I,\mathcal{B})$ (lemma 3.23), by Lebesgue's Differentiation Theorem (see for example appendix in [31]) the last line goes to 0 as $\nu\longrightarrow 0$ for almost all $\mu\in I$. □

3.6 The limiting absorption principle for regular λ

Recall $\mathcal{B} := L^2(\mathbb{R}^d,\mathbb{C}^M; w(x)dx)$. We now investigate the limiting absorption principle for u_δ in the function space $L^p(I_0,\mathcal{B};d\lambda)$ for $p\in(1,\infty)$. For the whole section let $I_0\subset\subset I\setminus\partial I$, $I\subset\subset\mathcal{R}$, $V := \{k\in B : \lambda_s(k)\in I \text{ for some } s\in\mathcal{J}\}$, $r := \max\{|s|\in\mathcal{J} : \lambda_s(B)\cap I\neq\emptyset\}$, f as in chapter 2 section 2.1 and $\mathcal{L}$ as in chapter 1 section 1.4. Consider the absorptive distributional solution u_δ of

$$(P_\delta) \qquad (\mathcal{L}-(\lambda+i\delta))u = f$$

on $I_0\times\mathbb{R}^d$. Then for almost all $\lambda\in I_0$, u_δ can be represented as the unique following $L^2(\mathbb{R}^d)$-function of x using the Floquet-Bloch decomposition of $(\mathcal{L}-(\lambda+i\delta))^{-1}f$:

$$u_\delta(\lambda,x) = \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{\lambda_s(k)-(\lambda+i\delta)}\, dk$$

$$= \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k) - (\lambda + i\delta)}\, dk$$
$$+ \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{V} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k) - (\lambda + i\delta)}\, dk.$$

Recall that earlier we regarded $u_\delta(\lambda, x)$ as a distribution in x and λ and proved in chapter 2 convergence to a limit distribution $u^\pm$ as $\delta \to 0^\pm$. Then for regular frequencies in section 3.4 we found a representation formula for $u^\pm$. This formula encourages us to to seek a convergence result for $u_\delta(\lambda, x)$ in a space of functions in the variables x and λ where λ is confined to the set of regular frequencies.

Lemma 3.28. *Regarding u_δ as an element of $L^p(I_0, \mathcal{B}; d\lambda)$ we have the following convergence*

$$\frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k) - (\lambda + i\delta)}\, dk \tag{3.10}$$
$$\overset{\delta\to 0^\pm}{\longrightarrow} \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k) - \lambda}\, dk$$

in $L^p(I_0, \mathcal{B}; d\lambda)$.

Proof. First we observe that there exists an $\eta > 0$ such that for all $k \in B \setminus V$, all $\lambda \in I_0$ and all $\delta \in \mathbb{R}$, $|\lambda_s(k) - \lambda - i\delta|^{-1} \leq \eta$. Next, recall that that we can swap the summation and integration in (3.10) due to $L^2(\mathbb{R}^d)$-convergence of the sum using Lebesgue dominated convergence theorem. We study the convergence as $\delta \to 0^\pm$ for a fixed $\lambda \in I_0$:

$$\frac{1}{\sqrt{|B|}} \left\| \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{\langle Uf, \psi_s\rangle \psi_s(x,k)}{\lambda_s(k) - (\lambda + i\delta)} dk - \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{\langle Uf, \psi_s\rangle \psi_s(x,k)}{\lambda_s(k) - \lambda} dk \right\|^2_{L^2(\mathbb{R}^d)}$$
$$= \left\| \frac{1}{\sqrt{|B|}} \int_B \sum_{s\in\mathcal{J}} \chi_{B\setminus V} \left(\frac{\langle Uf, \psi_s\rangle}{\lambda_s(k) - (\lambda + i\delta)} - \frac{\langle Uf, \psi_s\rangle}{\lambda_s(k) - \lambda} \right) \psi_s(x,k) dk \right\|^2_{L^2(\mathbb{R}^d)}$$
$$= \left\| U^{-1} \chi_{B\setminus V} \sum_{s\in\mathcal{J}} \left(\frac{\langle Uf, \psi_s\rangle}{\lambda_s(k) - (\lambda + i\delta)} - \frac{\langle Uf, \psi_s\rangle}{\lambda_s(k) - \lambda} \right) \psi_s(x,k) \right\|^2_{L^2(\mathbb{R}^d)}$$
$$= \left\| \chi_{B\setminus V} \sum_{s\in\mathcal{J}} \left(\frac{\langle Uf, \psi_s\rangle}{\lambda_s(k) - (\lambda + i\delta)} - \frac{\langle Uf, \psi_s\rangle}{\lambda_s(k) - \lambda} \right) \psi_s(x,k) \right\|^2_{L^2(\Omega\times B)}.$$

Here we have used that U is an isometry. In the next step we write out the $L^2(B)$-norm and use that the $\psi_s(\cdot,k)$ are orthonormal in $L^2(\Omega)$:

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}}\left\|\sum_{s\in\mathcal{J}}\int_{B\setminus V}\frac{\langle Uf,\psi_s\rangle\psi_s(x,k)}{\lambda_s(k)-(\lambda+i\delta)}dk-\sum_{s\in\mathcal{J}}\int_{B\setminus V}\frac{\langle Uf,\psi_s\rangle\psi_s(x,k)}{\lambda_s(k)-\lambda}dk\right\|^2_{L^2(\mathbb{R}^d)}\\
=\;&\int_B\chi_{B\setminus V}\left\|\sum_{s\in\mathcal{J}}\left(\frac{\langle Uf,\psi_s\rangle}{\lambda_s(k)-(\lambda+i\delta)}-\frac{\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda}\right)\psi_s(x,k)\right\|^2_{L^2(\Omega)}dk\\
=\;&\int_B\chi_{B\setminus V}\sum_{s\in\mathcal{J}}\left\|\left(\frac{\langle Uf,\psi_s\rangle}{\lambda_s(k)-(\lambda+i\delta)}-\frac{\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda}\right)\psi_s(x,k)\right\|^2_{L^2(\Omega)}dk\\
=\;&\int_B\chi_{B\setminus V}\sum_{s\in\mathcal{J}}\left|\frac{\langle Uf,\psi_s\rangle}{\lambda_s(k)-(\lambda+i\delta)}-\frac{\langle Uf,\psi_s\rangle}{\lambda_s(k)-\lambda}\right|^2\underbrace{\|\psi_s(x,k)\|^2_{L^2(\Omega)}}_{=1}dk\\
=\;&\int_B\chi_{B\setminus V}\sum_{s\in\mathcal{J}}\frac{\delta^2|\langle Uf,\psi_s\rangle|^2}{|(\lambda_s(k)-\lambda-i\delta)(\lambda_s(k)-\lambda)|^2}dk.
\end{aligned}$$

Finally we estimate the denominator by η^4 and use Parseval's identity and the isometry property of U:

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}}\left\|\sum_{s\in\mathcal{J}}\int_{B\setminus V}\frac{\langle Uf,\psi_s\rangle\psi_s(x,k)}{\lambda_s(k)-(\lambda+i\delta)}dk-\sum_{s\in\mathcal{J}}\int_{B\setminus V}\frac{\langle Uf,\psi_s\rangle\psi_s(x,k)}{\lambda_s(k)-\lambda}dk\right\|^2_{L^2(\mathbb{R}^d)}\\
\leq\;&\eta^4\delta^2\int_B\|Uf(\lambda,\cdot,k)\|^2_{L^2(\Omega)}\;dk \qquad (3.11)\\
=\;&\eta^4\delta^2\,\|f(\lambda,\cdot)\|^2_{L^2(\mathbb{R}^d)}\overset{\delta\to0}{\longrightarrow}0.
\end{aligned}$$

To obtain the convergence in $L^p(I_0,\mathcal{B};d\lambda)$ integrate the pointwise estimate (3.11) $d\lambda$ over I_0. Note here that $\lambda\mapsto\|f(\lambda,\cdot)\|_{L^2(\mathbb{R}^d)}$ is locally Lipschitz in λ. □

Proposition 3.29. *Regarding u_δ as an element of $L^p(I_0,\mathcal{B};d\lambda)$ we have the convergence*

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_V\frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{\lambda_s(k)-(\lambda+i\delta)}\;dk\\
\overset{\delta\to0^\pm}{\longrightarrow}\;&-\pi\sum_{s\in\mathcal{J};\;|s|\leq r}(Hh_s)(\lambda,x)\mp i(Th_s)(\lambda,x)
\end{aligned}$$

in $L^p(I_0,\mathcal{B};d\lambda)$, where r is defined at the beginning of this section, H is the version of the

Hilbert transform from appendix 5.6, T is the trace operator on the diagonal from 5.5 and

$$h_s(\lambda,\tau,x) := \frac{1}{\sqrt{|B|}} \int_{B_\tau^s} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)} \psi_s(x,k)}{|\nabla \lambda_s(k)|} \, d\mathcal{H}^{d-1}(k).$$

We put $h_s(\lambda,\tau,\cdot) = 0$ if $\lambda \notin I$ or $\tau \notin I$.

Remark 3.30. *Informally, one could write the limit element as*

$$\pi \sum_{s\in\mathcal{J};\, |s|\le r} \mathcal{P}\!\!\int_I \frac{h_s(\lambda,\tau,x)}{\tau-\lambda} \, d\tau \pm i h_s(\lambda,\lambda,x)$$

to see the analogy with the distributional limit absorption solution: one part being a Cauchy principal value integral the other part being a point evaluation at λ.

Proof. Starting from the integral over V, the first step is to apply the Coarea formula 5.4 in each summand with respect to the band function λ_s. So we have to check the hypothesis of the Coarea formula $\operatorname{ess\,inf} |\nabla\lambda_s| > 0$. This is true because of lemma 3.11 since $\overline{I} \subset \mathcal{R}_s$. Now we apply the Coarea formula on the integral over V:

$$\begin{aligned}
&\frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_V \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k) - (\lambda + i\delta)} \, dk \\
= \;& \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J};\, |s|\le r} \int_I \int_{B_\tau^s} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{(\lambda_s(k) - (\lambda + i\delta))|\nabla\lambda_s(k)|} \, d\mathcal{H}^{d-1}(k)\, d\tau \\
= \;& \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J};\, |s|\le r} \int_I \int_{B_\tau^s} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{(\tau - (\lambda + i\delta))|\nabla\lambda_s(k)|} \, d\mathcal{H}^{d-1}(k)\, d\tau \\
= \;& \sum_{s\in\mathcal{J};\, |s|\le r} \int_I \frac{h_s(\lambda,\tau,x)}{\tau - \lambda - i\delta} \, d\tau,
\end{aligned}$$

with the level set integral

$$h_s(\lambda,\tau,x) := \frac{1}{\sqrt{|B|}} \int_{B_\tau^s} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{|\nabla \lambda_s(k)|} \, d\mathcal{H}^{d-1}(k).$$

We put $h_s(\lambda,\tau,\cdot) = 0$ if $\lambda \notin I$ or $\tau \notin I$. Since we are only interested in convergence on $I_0 \subset I$, we can put $Th_s(\lambda,\cdot) = 0$ and $Hh_s(\lambda,\cdot) = 0$ if $\lambda \notin I$. In the following calculations will remind the reader of this fact by inserting a characteristic function χ_I. We now focus on one of the summands.

$$\int_I \frac{h_s(\lambda,\tau,x)}{\tau-\lambda-i\delta} \, d\tau \;=\; \int_I \frac{(\tau-\lambda)}{(\tau-\lambda)^2+\delta^2} h_s(\lambda,\tau,x)\, d\tau + i \int_I \frac{\delta}{(\tau-\lambda)^2+\delta^2} h_s(\lambda,\tau,x)\, d\tau$$

$$
\begin{aligned}
&= -\pi \int_{\mathbb{R}} Q_\delta(\lambda-\tau)\chi_I(\tau)h_s(\lambda,\tau,x)\,d\tau \\
&\quad +i\pi \int_{\mathbb{R}} P_\delta(\lambda-\tau)\chi_I(\tau)h_s(\lambda,\tau,x)\,d\tau \\
&= -\pi Q_\delta * h_s(\lambda,x) + i\pi P_\delta * h_s(\lambda,x),
\end{aligned} \tag{3.12}
$$

with the Poisson kernel

$$P_\delta(\eta) = \frac{\delta}{\eta^2+\delta^2}$$

and the conjugate Poisson kernel

$$Q_\delta(\eta) = \frac{\eta}{\eta^2+\delta^2}.$$

Since P_δ is an approximate identity and h_s is uniformly continuous (Lipschitz) with respect to λ, we can use lemma 5.16 to obtain the convergence of the second term in (3.12):

$$\|P_\delta * h_s \mp Th_s\|_{L^p(I_0,\mathcal{B})} \stackrel{\delta\to 0^\pm}{\longrightarrow} 0. \tag{3.13}$$

Considering the first term in (3.12) first notice the following estimate for the operator G from 5.6 using the definition of G and Q_δ:

$$
\begin{aligned}
& \|Q_\delta * (h_s - Th_s)(\lambda,x) - Gh_s(\lambda,x)\|^p_{L^p(I,\mathcal{B};d\lambda)} \\
= \quad & \left\| \int_{\mathbb{R}} Q_\delta(\lambda-\tau)\big(h_s(\lambda,\tau,x)-Th_s(\tau,x)\big)d\tau - \frac{1}{\pi}\int_{\mathbb{R}} \frac{h_s(\lambda,\tau,x)-Th_s(\tau,x)}{\lambda-\tau}d\tau \right\|^p_{L^p(I,\mathcal{B})} \\
= \quad & \frac{1}{\pi}\left\| \int_{\mathbb{R}} \frac{(\lambda-\tau)^2}{(\lambda-\tau)^2+\delta^2}\cdot\frac{h_s(\lambda,\tau,x)-Th_s(\tau,x)}{\lambda-\tau}d\tau - \int_{\mathbb{R}} \frac{h_s(\lambda,\tau,x)-Th_s(\tau,x)}{\lambda-\tau}d\tau \right\|^p_{L^p(I,\mathcal{B})} \\
= \quad & \frac{1}{\pi}\left\| \int_{\mathbb{R}} \left(\frac{(\lambda-\tau)^2}{(\lambda-\tau)^2+\delta^2}-1\right)\chi_I(\tau)\frac{h_s(\lambda,\tau,x)-Th_s(\tau,x)}{\lambda-\tau}d\tau \right\|^p_{L^p(I,\mathcal{B})} \\
= \quad & \frac{1}{\pi}\int_I \left\| \int_{\mathbb{R}} \left(\frac{(\lambda-\tau)^2}{(\lambda-\tau)^2+\delta^2}-1\right)\chi_I(\tau)\frac{h_s(\lambda,\tau,x)-Th_s(\tau,x)}{\lambda-\tau}d\tau \right\|^p_{\mathcal{B}} d\lambda \\
\leq \quad & \frac{1}{\pi}\int_I \left(\int_{\mathbb{R}} \left|\frac{(\lambda-\tau)^2}{(\lambda-\tau)^2+\delta^2}-1\right|\chi_I(\tau)\left\|\frac{h_s(\lambda,\tau,x)-Th_s(\tau,x)}{\lambda-\tau}\right\|_{\mathcal{B}} d\tau\right)^p d\lambda \\
\leq \quad & \frac{1}{\pi}\int_I \left(\int_{\mathbb{R}} \left|\frac{(\lambda-\tau)^2}{(\lambda-\tau)^2+\delta^2}-1\right|\chi_I(\tau)\,\mathrm{Lip}(h_s)_I d\tau\right)^p d\lambda \qquad \text{(by lemma 5.17)} \\
\stackrel{\delta\to 0^\pm}{\longrightarrow} \quad & 0 \qquad \text{(by Lebesgue dominated convergence).}
\end{aligned}
$$

With the truncated Hilbert transform from section 5.6 we see that (note that $G^{(\delta)}h_s = H^{(\delta)}h_s - H^{(\delta)}Th_s$, see (5.8))

$$\begin{aligned}
& \left\|Q_\delta * h_s - H^{(\delta)}h_s\right\|_{L^p(I,\mathcal{B})} \\
\leq\ & \left\|Q_\delta * (h_s - Th_s) - H^{(\delta)}(h_s - Th_s)\right\|_{L^p(I,\mathcal{B})} + \left\|Q_\delta * Th_s - H^{(\delta)}Th_s\right\|_{L^p(I,\mathcal{B})} \\
=\ & \left\|Q_\delta * (h_s - Th_s) - G^{(\delta)}h_s\right\|_{L^p(I,\mathcal{B})} + \left\|Q_\delta * Th_s - H^{(\delta)}Th_s\right\|_{L^p(I,\mathcal{B})} \\
\overset{\delta\to 0^\pm}{\longrightarrow}\ & 0,
\end{aligned}$$

according to the last argument and theorem 4.1.5 in [41] and since $G^{(\delta)}h_s \overset{\delta\to 0^\pm}{\longrightarrow} Gh_s$ in $L^p(I,\mathcal{B})$. And since $H^{(\delta)}(Th_s) \overset{\delta\to 0^\pm}{\longrightarrow} H(Th_s)$ in $L^p(I,\mathcal{B})$, we conclude

$$\begin{aligned}
& \|Q_\delta * h_s - Hh_s\|_{L^p(I,\mathcal{B})} \\
\leq\ & \left\|Q_\delta * h_s - H^{(\delta)}h_s\right\|_{L^p(I,\mathcal{B})} - \left\|H^{(\delta)}h_s - Hh_s\right\|_{L^p(I,\mathcal{B})} \overset{\delta\to 0^\pm}{\longrightarrow} 0. \qquad (3.14)
\end{aligned}$$

□

Theorem 3.31 (Limiting absorption principle for regular frequencies)**.** *Let $\mathcal{L}$ and f be as in introduced in section 1.4 and 2.1. Let $I_0 \subset\subset I \setminus \partial I$, $I \subset\subset \mathcal{R}$ with $\mathcal{R}$ introduced in section 3.1 of this chapter. V denotes the set $\{k \in B : \lambda_s(k) \in I \text{ for some } s\}$ and $r := \max\{|s| : s \in \mathcal{J},\ \lambda_s(B) \cap I \neq \emptyset\}$. Let*

$$u_\delta(\lambda, x) := \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_B \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k) - (\lambda + i\delta)}\, dk \qquad (3.15)$$

be the unique $L^2(\mathbb{R}^d, \mathbb{C}^M)$-function solving

$$(P_\delta) \qquad (\mathcal{L} - \lambda - i\delta)u = f.$$

Define

$$\begin{aligned}
u^\pm(\lambda, x) \ :=\ & \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k) - \lambda}\, dk \qquad (3.16) \\
& + \sum_{s\in\mathcal{J};\ |s|\leq r} -\pi Hh_s(\lambda, x) \pm i\pi Th_s(\lambda, x),
\end{aligned}$$

where H is the Hilbert transform from section 5.6, T is the trace operator on the diagonal from section 5.5 and h_s is the level set integral

$$h_s(\lambda, \tau, x) = \frac{1}{\sqrt{|B|}} \int_{B^s_\tau} \frac{\langle (Uf)(\lambda,\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{|\nabla \lambda_s(k)|}\, d\mathcal{H}^{d-1}(k) \qquad (3.17)$$

discussed in section 3.5. Then

$$u_\delta(\lambda, x) \overset{\delta\to 0^\pm}{\longrightarrow} u^\pm(\lambda, x)$$

in $L^p(I_0, \mathcal{B}; d\lambda)$ *for any* $p \in (1, \infty)$ *with a weight function* w *as in theorem 3.19 and* $u^\pm$ *is a distributional solution of* (P_0)*, where the space of test functions is* $C_0^\infty(I_0 \times \mathbb{R}^d, \mathbb{C}^M)$.

Remark 3.32. *We recognize that we can improve the representation theorem in section 3.4 for* λ *in an interval* I_0 *of regular frequencies, since the distributional solution* $u^\pm$ *of* $(\mathcal{L}-\lambda)u = f$ *satisfies*

$$u^\pm[\varphi] = \int_{\mathbb{R}} \int_{\mathbb{R}^d} u^\pm(\lambda, x)\varphi(\lambda, x)\, dx\, d\lambda$$

for $\varphi \in C_0^\infty(I_0 \times \mathbb{R}^d, \mathbb{C}^M)$ *with* $u^\pm(\lambda, x)$ *from* (3.16).

Proof of theorem 3.31. The convergence result follows from lemma 3.28 and proposition 3.29. The solution property of $u^\pm$ follows from the convergence result, since for $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned}
& \left|u_\delta[(\mathcal{L} - \lambda - i\delta)\varphi] - u^\pm[(\mathcal{L} - \lambda)\varphi]\right| \\
= \; & \left|u_\delta[(\mathcal{L} - \lambda - i\delta)\varphi] - u^\pm[(\mathcal{L} - \lambda - i\delta)\varphi] - i\delta u^\pm[\varphi]\right| \\
\leq \; & \int_{I_0} \int_{\mathbb{R}^d} \left|(u_\delta(\lambda, x) - u^\pm(\lambda, x))\right| \left|(\mathcal{L} - \lambda - i\delta)\varphi(\lambda, x)\right| \, dx\, d\lambda + \left|i\delta u^\pm[\varphi]\right| \\
\leq \; & \int_{I_0} \left\|(u_\delta(\lambda, \cdot) - u^\pm(\lambda, \cdot))\sqrt{w}\right\|_{L^2(\mathbb{R}^d)} \left\|(\mathcal{L} - \lambda - i\delta)\varphi(\lambda, \cdot)\frac{1}{\sqrt{w}}\right\|_{L^2(\mathbb{R}^d)} d\lambda \\
& + \left|i\delta u^\pm[\varphi]\right| \\
= \; & \int_{I_0} \left\|u_\delta(\lambda, \cdot) - u^\pm(\lambda, \cdot)\right\|_{\mathcal{B}} \left\|(\mathcal{L} - \lambda - i\delta)\varphi(\lambda, \cdot)\frac{1}{\sqrt{w}}\right\|_{L^2(\mathbb{R}^d)} d\lambda \\
& + \left|\delta u^\pm[\varphi]\right| \\
\leq \; & \left\|u_\delta - u^\pm\right\|_{L^p(I_0, \mathcal{B})} \left\|(\mathcal{L} - \lambda - i\delta)\varphi\frac{1}{\sqrt{w}}\right\|_{L^q(I_0, L^2(\mathbb{R}^d))} + \left|\delta u^\pm[\varphi]\right| \\
\overset{\delta\to 0^\pm}{\longrightarrow} \; & 0 \cdot \left\|(\mathcal{L} - \lambda)\varphi\frac{1}{\sqrt{w}}\right\|_{L^q(I_0, L^2(\mathbb{R}^d))} + 0 = 0.
\end{aligned}$$

So

$$f[\varphi] = u_\delta[(\mathcal{L} - \lambda - i\delta)\varphi] \overset{\delta\to 0^\pm}{\longrightarrow} u^\pm[(\mathcal{L} - \lambda)\varphi]$$

giving

$$u^\pm[(\mathcal{L} - \lambda)\varphi] = f[\varphi] \quad (\forall\, \varphi \in C_0^\infty(I_0 \times \mathbb{R}^d, \mathbb{C}^M)).$$

□

3.7 Comments

Remark 3.33. *The "evanescent" part*[4] *of* $u^{\pm}$ *given by*

$$\frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_{B\setminus V}\frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{\lambda_s(k)-\lambda}\,dk$$

in fact is in $L^2(\mathbb{R}^d)$ *with respect to the variable* x*, which can be seen from the proof of lemma 3.28.*

The "point evaluation" part

$$\sum_{s\in\mathcal{J};\,|s|\leq r}\pm i\pi Th_s(\lambda,x)$$

of $u^{\pm}$*, in informal notation*

$$\begin{aligned} Th_s(\lambda,x) &= h_s(\lambda,\lambda,x) \\ &= \int_{\{\lambda_s(k)=\lambda\}}\frac{\langle (Uf)(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega)}\psi_s(x,k)}{|\nabla\lambda_s(k)|}\,d\mathcal{H}^{d-1}(k), \end{aligned} \tag{3.18}$$

describes the propagating part of $u^{\pm}$*, that is carried by Bloch modes, which correspond directly to the frequency* λ *at which the system is excited, because the integration in* (3.18) *accesses only Bloch waves* $\psi_s(\cdot,k)$ *with* s *and* k *such that* $\lambda_s(k)=\lambda$ *holds. To call this part "propagating" is justified by the fact that it is in a weighted* L^2*-space with respect to the variable* x *and by the properties of the weight function* w*, it decays (in general, depending on the right hand side* f*) slower than a* L^2*-function when* $|x|\to\infty$*, in contrast to other parts of* $u^{\pm}$ *that are truly in* $L^2(\mathbb{R}^d)$ *with respect to the variable* x*.*

It would be satisfying to show that

$$\sum_{s\in\mathcal{J};\,|s|\leq r}-\pi Hh_s(\lambda,x)$$

is also evanescent, i.e. in L^2 *with respect to the variable* x*. Note that this is true for* $H^{(\varepsilon)}h_s$ *for any* $\varepsilon>0$ *by lemma 3.24. But the convergence of* $H^{(\varepsilon)}h_s$ *as* $\varepsilon\to 0$ *is in the space* $L^p(I_0,\mathcal{B})$*. Since the properties of the weight function allow only the estimate*

$$\|\cdot\|_{L^2(\mathbb{R}^d,w(x)dx)}\leq C\,\|\cdot\|_{L^2(\mathbb{R}^d)}$$

[4] By this we mean a part of the solution with the fastest decay behavior compared with other parts of $u^{\pm}$. Usually an exponential decay is assumed, but for our purpose it is sufficient to distinguish parts in $L^2(\mathbb{R}^d)$ and parts in $L^2(\mathbb{R}^d;w(x)dx)$.

but not the converse, the evanescence of the Hilbert transform part of $u^\pm$ is not seen in such a direct way (but compare the quasi 1-D case in [28], where the only propagating part of the solution arises from "point evaluations"). This work leaves the question open.

Remark 3.34. *If f does not depend on λ, we can drop the compact support of f and demand only $f \in L^2(\mathbb{R}^d)$. Also the trace operator T is not needed and the Hilbert transform H is the standard Hilbert transform on L^2. Then the notation*

$$\begin{aligned} u^\pm(\lambda, x) \quad := \quad & \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{\langle (Uf)(\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{\lambda_s(k)-\lambda}\, dk \\ & + \sum_{s\in\mathcal{J};\, |s|\le r} -\pi H h_s(\lambda, x) \pm i\pi h_s(\lambda, x) \in L^2(I, \mathcal{B}; d\lambda) \end{aligned}$$

is rigorous.

Chapter 4

The limit amplitude principle for regular frequencies

In this chapter we prove the limit amplitude principle for the class of operators as in section 1.4. So the powerful tool of Floquet-Bloch transform with all its implications including the limiting absorption principles from chapter 2 and 3 will be available. To apply known theory on the asymptotics of solutions of the wave equation, we will need the additional assumption (6) in section 1.4, i.e. the spectrum of $\mathcal{L}$ is absolutely continuous and λ_s strictly positive for all $s \in \mathcal{J}$. From the continuity properties of λ_s follows that $\lambda_s(k) \geq q > 0$ for some $q \in \mathbb{R}$ and all $s \in \mathcal{J}, k \in B$.

An important tool to show the limit amplitude principle is the well-known representation formula

$$\begin{aligned} u(t) &= u_{\text{trans.}}(t) + u_{\text{as.}}(t) \\ &= \cos(\mathcal{L}^{1/2}t)u^0 + \mathcal{L}^{-1/2}\sin(\mathcal{L}^{1/2}t)u^1 \\ &\quad + \int_0^t \mathcal{L}^{-1/2}\sin\left(\mathcal{L}^{1/2}(t-\sigma)\right) e^{i\omega\sigma} g \, d\sigma, \end{aligned}$$

for the solution of the wave equation with initial conditions u^0, u^1 obtained by a diagonalization method (see [30], Chapter XV). ”trans.” stands for ”transitory term” which corresponds to the homogeneous case ($e^{i\omega t}g = 0$) and is a free oscillation term. ”as.” stands for ”asymptotic term” and corresponds to forced oscillations due to the source $e^{i\omega t}g$. From this formula we will show, using the Floquet-Bloch decomposition, a certain representation of $u_{\text{as.}}$ involving the Hilbert transform of the level set integrals h_s from the last chapter. This representation will be crucial in the study of the asymptotic behavior of $u_{\text{as.}}$ as $t \to \infty$, where we use the Fourier transform as the main tool.

The proof of the equivalence of the principle of limit amplitude and limit absorption consists basically of the fact that the function v satisfying

$$u(t) \sim e^{i\omega t} v \text{ as } t \to \infty$$

exists and is uniquely determined.

4.1 Functions of the operator $\mathcal{L}$

We will need to define operators $\rho(\mathcal{L}, t)$ where ρ is a continuous function depending in addition on the time t. Defining functions of self adjoint operators is a standard construction (see [23], chapter VII for continuous functional calculus and chapter VIII for bounded and unbounded Borel functional calculus for self adjoint operators). By virtue of the Floquet-Bloch transform U we can define directly $\rho(\mathcal{L}, t)$ for continuous ρ. To do this, neither ρ nor $\mathcal{L}$ needs to be bounded. This subsection therefore is used to introduce the operator $\rho(\mathcal{L}, t)$ and its domain and to collect some simple statements for later reference.

For abbreviation define

$$P_{s,k}[h](x) = \langle Uh(\cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega,\mathbb{C}^M)} \psi_s(x, k) \tag{4.1}$$

the projection on span$\{\psi_s(\cdot, k)\}$. To simplify the notation we will suppress the target space $\mathbb{C}^M$ in the proofs from now on.

Definition 4.1. *Let $\rho : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then for $t \in \mathbb{R}$ fixed we define the operator*

$$\begin{aligned} \rho(\mathcal{L}, t) : \quad & D(\rho(\mathcal{L}, t)) \subseteq L^2(\mathbb{R}^d, \mathbb{C}^M) \to L^2(\mathbb{R}^d, \mathbb{C}^M) \\ & \rho(\mathcal{L}, t)u(x) = \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_B \rho(\lambda_s(k), t) P_{s,k}[u](x)\, dk, \end{aligned} \tag{4.2}$$

with domain

$$D(\rho(\mathcal{L}, t)) = \{u \in L^2(\mathbb{R}^d, \mathbb{C}^M) : \sum_{s\in\mathcal{J}} \int_B \rho(\lambda_s(k), t) P_{s,k}[u]\, dk \text{ converges in } L^2(\mathbb{R}^d, \mathbb{C}^M)\}.$$

Note that due to the convergence in $L^2(\mathbb{R}^d, \mathbb{C}^M)$ one can interchange the sum over s and the integral over B.

Definition 4.2. *For any fixed $t \geq 0$ define operators $\mathcal{L}^{1/2}$, $\cos \mathcal{L}^{1/2}t$, $\sin \mathcal{L}^{1/2}t$ and $\mathcal{L}^{-1/2}$ according to definition 4.1. For example $\mathcal{L}^{-1/2} = \rho_1(\mathcal{L}, t)$ with $\rho_1(x,t) = x^{-1/2}$ and $\sin \mathcal{L}^{1/2}t = \rho_2(\mathcal{L}, t)$ with $\rho_2(x,t) = \sin(\sqrt{x}t)$. Hence in particular, with $\mathcal{L}$ also $\mathcal{L}^{1/2}$ is in the class of Floquet decomposable operators from section 1.4 (satisfying the additional requirement (6)) with strictly positive real spectrum.*

Lemma 4.3. *Let $t \in \mathbb{R}$ be fixed. If $\rho(\cdot, t)$ is bounded then $D(\rho(\mathcal{L}, t)) = L^2(\mathbb{R}^d, \mathbb{C}^M)$ and $\rho(\mathcal{L}, t)$ is bounded. For a continuous and bounded $\rho(\cdot, t)$ and any continuous $\widetilde{\rho}(\cdot, t)$, $\rho(\mathcal{L}, t)$ and $\widetilde{\rho}(\mathcal{L}, t)$ commute in the sense that*

$$D(\rho(\mathcal{L}, t)\widetilde{\rho}(\mathcal{L}, t)) = D(\widetilde{\rho}(\mathcal{L}, t)\rho(\mathcal{L}, t)) = D(\widetilde{\rho}(\mathcal{L}, t))$$

and for all $u \in D(\widetilde{\rho}(\mathcal{L}, t))$

$$\rho(\mathcal{L}, t)\widetilde{\rho}(\mathcal{L}, t)u = \widetilde{\rho}(\mathcal{L}, t)\rho(\mathcal{L}, t)u$$

holds.

Proof. If $\rho(\cdot, t)$ is bounded, then there exists a constant $c > 0$ such that $|\rho(\lambda_s(k), t)|^2 \leq c$ for all $s \in \mathcal{J}$ and $k \in B$. For $u \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ the following calculation holds using the isometry property of U and the orthonormality of $\psi_s(\cdot, k)$ in $L^2(\Omega)$.

$$\begin{aligned}
\|\rho(\mathcal{L}, t)u\|^2_{L^2(\mathbb{R}^d)} &= \left\| \frac{1}{\sqrt{|B|}} \int_B \sum_{s \in \mathcal{J}} \rho(\lambda_s(k), t) \langle Uu, \psi_s \rangle_{L^2(\Omega)} \psi_s(\cdot, k)\, dk \right\|^2_{L^2(\mathbb{R}^d)} \\
&= \left\| \sum_{s \in \mathcal{J}} \rho(\lambda_s(k), t) \langle Uu, \psi_s \rangle_{L^2(\Omega)} \psi_s(\cdot, k) \right\|^2_{L^2(\Omega \times B)} \\
&= \int_B \left\| \sum_{s \in \mathcal{J}} \rho(\lambda_s(k), t) \langle Uu, \psi_s \rangle_{L^2(\Omega)} \psi_s(\cdot, k) \right\|^2_{L^2(\Omega)} dk \\
&= \int_B \sum_{s \in \mathcal{J}} |\rho(\lambda_s(k), t)|^2 |\langle Uu, \psi_s \rangle_{L^2(\Omega)}|^2 \, \|\psi_s(\cdot, k)\|^2_{L^2(\Omega)}\, dk \\
&\leq c \int_B \sum_{s \in \mathcal{J}} |\langle Uu, \psi_s \rangle_{L^2(\Omega)}|^2\, dk \\
&= c\, \|Uu\|^2_{L^2(\Omega \times B)} = c\, \|u\|^2_{L^2(\mathbb{R}^d)} .
\end{aligned}$$

For $u \in D(\widetilde{\rho}(\mathcal{L}, t))$, $\|\rho(\mathcal{L}, t)\widetilde{\rho}(\mathcal{L}, t)u\|^2_{L^2(\mathbb{R}^d)}$ can be estimated in an analogous way, from which the remaining statements immediately follow. □

Lemma 4.4. *For the domain of $\mathcal{L}^{1/2}$ the following holds*

(i) $D(\mathcal{L}) \subseteq D(\mathcal{L}^{1/2})$

(ii) $\mathcal{L}^{1/2} D(\mathcal{L}) \subseteq D(\mathcal{L}^{1/2})$

Proof. To show (i) we use the same technique as in the proof of lemma 4.3, noting that $\frac{1}{\sqrt{\lambda_s(k)}} \leq c$ for some $c > 0$.

$$\begin{aligned}
&\left\| \int_B \sum_{s\in\mathcal{J}} \sqrt{\lambda_s(k)} \langle Uu, \psi_s \rangle_{L^2(\Omega)} \psi_s(\cdot, k)\, dk \right\|^2_{L^2(\mathbb{R}^d)} \\
= &\left\| \int_B \sum_{s\in\mathcal{J}} \frac{1}{\sqrt{\lambda_s(k)}} \lambda_s(k) \langle Uu, \psi_s \rangle_{L^2(\Omega)} \psi_s(\cdot, k) \right\|^2_{L^2(\mathbb{R}^d)} dk \\
= &\int_B \sum_{s\in\mathcal{J}} \left| \frac{1}{\sqrt{\lambda_s(k)}} \right|^2 \left| \lambda_s(k) \langle Uu, \psi_s \rangle_{L^2(\Omega)} \right|^2 dk \\
\leq \ & c^2 \left\| \mathcal{L}u \right\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

and hence $u \in D(\mathcal{L}^{1/2})$ if $u \in D(\mathcal{L})$. For (ii) let $u \in \mathcal{L}^{1/2} D(\mathcal{L})$, i.e. $u = \mathcal{L}^{1/2} v$ with some $v \in D(\mathcal{L}^{1/2})$. Since $D(\mathcal{L}) \subseteq D(\mathcal{L}^{1/2})$ we obtain

$$\left\| \mathcal{L}^{1/2} u \right\|_{L^2(\mathbb{R}^d)} = \left\| \mathcal{L}^{1/2} \mathcal{L}^{1/2} v \right\|_{L^2(\mathbb{R}^d)} = \left\| \mathcal{L}v \right\|_{L^2(\mathbb{R}^d)} < \infty.$$

□

Lemma 4.5. *Assume that $\rho : (0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighborhood of $\sigma(\mathcal{L}) \times \mathbb{R}$ and $\left| \left(\frac{\partial}{\partial t} \rho \right)(y, \xi) \right| \leq c(\xi)$, where $\xi \mapsto c(\xi)$ is locally bounded. Then for all $u \in \bigcap_{\tau \in (t-\varepsilon, t+\varepsilon)} D(\rho(\mathcal{L}, \tau)) \cap D(\left(\frac{\partial}{\partial t} \rho \right)(\mathcal{L}, t))$ for some small ε the function $t \mapsto \rho(\mathcal{L}, t)u$ is classically differentiable as a function with values in $L^2(\mathbb{R}^d, \mathbb{C}^M)$ and*

$$\frac{\partial}{\partial t} \left(\rho(\mathcal{L}, t)u \right) = \left(\frac{\partial}{\partial t} \rho \right)(\mathcal{L}, t)u$$

holds.

Proof. For small $h \neq 0$, we have for the difference quotient

$$\frac{\rho(\mathcal{L}, t+h)u - \rho(\mathcal{L}, t)u}{h} - \left(\frac{\partial}{\partial t} \rho \right)(\mathcal{L}, t)u$$

$$= \frac{1}{\sqrt{|B|}} \int_B \sum_s \left(\frac{\rho(\lambda_s(k), t+h) - \rho(\lambda_s(k), t)}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\lambda_s(k), t) \right) P_{s,k}[u]\, dk.$$

Using the isometry property of U and Parseval's identity we perform the following calculation

$$\begin{aligned}
&\lim_{h\to 0} \left\| \frac{\rho(\mathcal{L}, t+h)u - \rho(\mathcal{L}, t)u}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\mathcal{L}, t)u \right\|^2_{L^2(\mathbb{R}^d)} \\
= &\lim_{h\to 0} \left\| \frac{1}{\sqrt{|B|}} \int_B \sum_s \left(\frac{\rho(\lambda_s(k), t+h) - \rho(\lambda_s(k), t)}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\lambda_s(k), t) \right) P_{s,k}[u]\, dk \right\|^2_{L^2(\mathbb{R}^d)} \\
= &\lim_{h\to 0} \left\| \sum_s \left(\frac{\rho(\lambda_s(k), t+h) - \rho(\lambda_s(k), t)}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\lambda_s(k), t) \right) P_{s,k}[u] \right\|^2_{L^2(\Omega\times B)} \\
= &\lim_{h\to 0} \int_B \sum_s \left| \frac{\rho(\lambda_s(k), t+h) - \rho(\lambda_s(k), t)}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\lambda_s(k), t) \right|^2 |\langle Uu, \psi_s \rangle_{L^2(\Omega)}|^2\, dk.
\end{aligned}$$

Since $\left|\left(\frac{\partial}{\partial t}\rho\right)(y,\xi)\right| \leq c(\xi)$, where $\xi \mapsto c(\xi)$ is locally bounded, for all $|h| \leq h_0$ there exists, by the mean value theorem of calculus, a constant $c > 0$ independent of h, s and k such that

$$\left| \frac{\rho(\lambda_s(k), t+h) - \rho(\lambda_s(k), t)}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\lambda_s(k), t) \right|^2 \leq c.$$

Since for the rest

$$\frac{1}{\sqrt{|B|}} \int_B \sum_s |\langle Uu, \psi_s \rangle_{L^2(\Omega)}|^2\, dk = \|Uu\|^2_{L^2(\Omega\times B)} = \|u\|^2_{L^2(\mathbb{R}^d)} < \infty$$

holds, Lebesgue's dominated convergence is applicable to move the limit $h \to 0$ inside the integral and the sum. So we arrive at

$$\begin{aligned}
&\sqrt{|B|} \lim_{h\to 0} \left\| \frac{\rho(\mathcal{L}, t+h)u - \rho(\mathcal{L}, t)u}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\mathcal{L}, t)u \right\|^2_{L^2(\mathbb{R}^d)} \\
= &\int_B \sum_s \lim_{h\to 0} \left| \frac{\rho(\lambda_s(k), t+h) - \rho(\lambda_s(k), t)}{h} - \left(\frac{\partial}{\partial t}\rho\right)(\lambda_s(k), t) \right|^2 |\langle Uu, \psi_s \rangle_{L^2(\Omega)}|^2\, dk \\
= &\ 0.
\end{aligned}$$

□

4.2 The solution of the wave equation for $t < \infty$

For an operator $\mathcal{L}$ as introduced in section 1.4 including assumption (6) we consider a wave type problem for $\mathcal{L}$ as in [30] p.420

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \mathcal{L}u = e^{i\omega t} g, \\ u(0) = u^0, \\ \frac{\partial u}{\partial t}(0) = u^1, \end{cases} \tag{4.3}$$

with given data $u^0 \in D(\mathcal{L})$, $u^1 \in D(\mathcal{L}^{1/2})$, $g \in D(\mathcal{L}^{1/2})$ and $\omega \in (0, \infty)$. We consider strong solutions of (4.3), i.e. $u \in C^2([0,\infty), L^2(\mathbb{R}^d, \mathbb{C}^M))$ such that for all $t \geq 0$, $u(t) \in D(\mathcal{L})$, $\frac{\partial u}{\partial t}(t) \in D(\mathcal{L}^{1/2})$, $\frac{\partial^2 u}{\partial t^2}(t) \in L^2(\mathbb{R}^d, \mathbb{C}^M)$.

Proposition 4.6. *The strong solution of* (4.3) *as described above is unique.*

Proof. If there is another solution $\widetilde{u}$ to (4.3) then $w := u - \widetilde{u}$ satisfies (4.3) with right hand side and initial conditions equal to 0. Then we deduce $w = 0$ adapting the proof of theorem 5, section 2.4.3 in [42]. The unboundedness of $\mathbb{R}^d$ is no obstruction, since the energy

$$\begin{aligned} e(t) &= \frac{1}{2} \int_{\mathbb{R}^d} \left| \frac{\partial w}{\partial t}(x,t) \right|^2_{\mathbb{C}^M} + |\mathcal{L}^{1/2} w(x,t)|^2_{\mathbb{C}^M} \, dx \\ &= \frac{1}{2} \left(\langle \frac{\partial w}{\partial t}(\cdot,t), \frac{\partial w}{\partial t}(\cdot,t) \rangle_{L^2(\mathbb{R}^d)} + \langle \mathcal{L}^{1/2} w(\cdot,t), \mathcal{L}^{1/2} w(\cdot,t) \rangle_{L^2(\mathbb{R}^d)} \right) \end{aligned}$$

is finite for each $t \geq 0$ by the hypotheses made on the strong solution. Differentiating e with respect to t and using that $\mathcal{L}^{1/2}$ is symmetric yields (skipping the subscripts $L^2(\mathbb{R}^d)$ of the inner products)

$$\begin{aligned} \frac{\partial e}{\partial t}(t) &= \frac{1}{2} \Big(\langle \frac{\partial^2 w}{\partial t^2}(\cdot,t), \frac{\partial w}{\partial t}(\cdot,t) \rangle + \langle \frac{\partial w}{\partial t}(\cdot,t), \frac{\partial^2 w}{\partial t^2}(\cdot,t) \rangle \\ &\quad + \langle \mathcal{L} w(\cdot,t), \frac{\partial w}{\partial t}(\cdot,t) \rangle + \langle \frac{\partial w}{\partial t}(\cdot,t), \mathcal{L} w(\cdot,t) \rangle \Big) = 0. \end{aligned}$$

Since $e(0) = 0$ it follows that $\mathcal{L}^{1/2} w = 0$, and by the assumptions on $\mathcal{L}^{1/2}$ (property (6) in section 1.4), $w = 0$. $\square$

The solution of the problem (4.3) can be written in the form (see [30])

$$u = u_{\text{trans.}} + u_{\text{as.}}, \tag{4.4}$$

with

$$u_{\text{trans.}}(t) = \cos(\mathcal{L}^{1/2}t)u^0 + \mathcal{L}^{-1/2}\sin(\mathcal{L}^{1/2}t)u^1, \tag{4.5}$$

$$u_{\text{as.}}(t) = \int_0^t \mathcal{L}^{-1/2}\sin\left(\mathcal{L}^{1/2}(t-\sigma)\right)e^{i\omega\sigma}g\,d\sigma, \tag{4.6}$$

which can be seen by directly differentiating u. Note that $u_{\text{trans.}}$ depends only on the initial conditions u^0, u^1 and not on $e^{i\omega t}g$, while $u_{\text{as.}}$ depends only on the source $e^{i\omega t}g$ and not on u^0, u^1.

From now on we restrict to an interval I of frequencies $\omega \in (0,\infty)$ that is compactly embedded in the set of regular frequencies with respect to the band functions $\sqrt{\lambda_s}$. In this chapter V denotes the preimage of I under all band functions $\sqrt{\lambda_s}$ of $\mathcal{L}^{1/2}$. Note that there is the parameter ω in the equation (4.3), and thus the actual strong solution of (4.3) depends on ω. We now start to regard the parameter ω and the variable x on an equal footing, meaning that the solution u of (4.3) is for any fixed t a function of ω and x, i.e. $u(t) = u(t)(\omega,x) = u(t,\omega,x)$. Then we have the following

Lemma 4.7. *For any fixed $t \geq 0$, we can regard $u_{trans.}(t)$ and $u_{as.}(t)$ as elements of $L^p(I, L^2(\mathbb{R}^d, \mathbb{C}^M; dx); d\omega)$, $p \in (1,\infty)$, and*

$$u_{as.}(t) = \frac{1}{\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \widehat{u}_{as.}(s,k,t,\omega)\,dk, \tag{4.7}$$

with

$$\widehat{u}_{as.}(s,k,t,\omega) = \int_0^t \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}}e^{i\omega\sigma}P_{s,k}[g]\,d\sigma. \tag{4.8}$$

Note that this means

$$u_{as.}(t) = \int_0^t \mathcal{L}^{-1/2}\sin\mathcal{L}^{1/2}(t-\sigma)e^{i\omega\sigma}g\,d\sigma = \rho_\omega(\mathcal{L},t)g \tag{4.9}$$

with

$$\rho_\omega(\lambda,t) = \int_0^t \frac{\sin(\sqrt{\lambda}(t-\sigma))}{\sqrt{\lambda}}e^{i\omega\sigma}\,d\sigma. \tag{4.10}$$

Proof. Note that

$$u_{\text{as.}}(t) = \frac{1}{\sqrt{|B|}}\int_0^t\sum_{s\in\mathcal{J}}\int_B \lambda_s(k)^{-1/2}\sin(\lambda_s(k)^{1/2}(t-\sigma))e^{i\omega\sigma}P_{s,k}[g]\,dk\,d\sigma$$

by (4.6) and definition 4.1. Thus, to show (4.7), we have to justify the following rearrangement of integrations and summation

$$\int_0^t \sum_{s\in\mathcal{J}} \int_B \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g]\, dk\, d\sigma$$

$$= \sum_{s\in\mathcal{J}} \int_B \int_0^t \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g]\, d\sigma\, dk. \qquad (4.11)$$

We denote the left hand side of (4.11) by u_1 and the right hand side by u_2. For any fixed $\omega \in (0,\infty)$ and $t \in [0,\infty)$, u_1 and u_2 are in $L^2(\mathbb{R}^d)$. For any test function $\varphi = \varphi(x) \in C_0^\infty(\mathbb{R}^d)$ we will show that

$$\int_{\mathbb{R}^d} u_1(x)\varphi(x)\, dx = u_1[\varphi] = u_2[\varphi],$$

so $u_1 = u_2$ in $L^2(\mathbb{R}^d; dx)$ for any fixed ω. Let $\varphi = \varphi(x) \in C_0^\infty(\mathbb{R}^d)$. For $t \geq 0, \omega \in (0,\infty)$ fixed, lemma 2.9 in chapter 2 implies that

$$\int_{\mathbb{R}^d} \sum_{s\in\mathcal{J}} \int_B \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dk\, dx$$

$$= \sum_{s\in\mathcal{J}} \int_B \int_{\mathbb{R}^d} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dx\, dk. \qquad (4.12)$$

Interchanging $\int_0^t$ and $\int_{\mathbb{R}^d}$ in

$$\int_0^t \int_{\mathbb{R}^d} \sum_{s\in\mathcal{J}} \int_B \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dk\, dx\, d\sigma$$

$$= \int_{\mathbb{R}^d} \int_0^t \sum_{s\in\mathcal{J}} \int_B \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dk\, d\sigma\, dx \qquad (4.13)$$

is justified by Fubini's theorem, since one can show (basically using Cauchy-Schwarz inequality and orthonormality of $\psi_s(\cdot,k)$ in $L^2(\Omega)$) that

$$\int_0^t \int_{\mathbb{R}^d} \left| \sum_{s\in\mathcal{J}} \int_B \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dk \right| dx\, d\sigma < \infty.$$

(4.12) and (4.13) imply

$$u_1[\varphi] = \int_{\mathbb{R}^d} \int_0^t \sum_{s\in\mathcal{J}} \int_B \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dk\, d\sigma\, dx$$

$$= \int_0^t \sum_{s\in\mathcal{J}} \int_B \int_{\mathbb{R}^d} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dx\, dk\, d\sigma. \tag{4.14}$$

By lemma 2.10 in chapter 2 we can change the order of the summation over s and the integrations $d\sigma$ over $(0,t)$ and dk over B in (4.14) arbitrarily while keeping the integration dx over $\mathbb{R}^d$ in the innermost position. Thus we arrive at

$$u_1[\varphi] = \sum_{s\in\mathcal{J}} \int_B \int_0^t \int_{\mathbb{R}^d} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, d\sigma\, dx\, dk.$$

For fixed s and k it is clear that one can interchange $\int_0^t$ and $\int_{\mathbb{R}^d}$. Moving the integration over $\mathbb{R}^d$ before the sum over s and the integral over B while keeping the integration over $(0,t)$ in the innermost position works similar to the first step (4.12). The result is then

$$\begin{aligned} u_1[\varphi] &= \int_0^t \sum_{s\in\mathcal{J}} \int_B \int_{\mathbb{R}^d} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, dx\, dk\, d\sigma \\ &= \int_{\mathbb{R}^d} \sum_{s\in\mathcal{J}} \int_B \int_0^t \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g](x)\varphi(x)\, d\sigma\, dk\, dx \\ &= u_2[\varphi]. \end{aligned}$$

This implies

$$u_{\text{as.}}(t) = \frac{1}{\sqrt{|B|}} u_1 = \frac{1}{\sqrt{|B|}} u_2 = \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_B \hat{u}_{\text{as.}}\, dk$$

with $\hat{u}_{\text{as.}}$ from (4.8).

It remains to show that $u_{\text{trans.}}(t)$ and $u_{\text{as.}}(t)$ are in $L^p(I, L^2(\mathbb{R}^d; dx); d\omega)$. For $t \geq 0$ fixed, $u_{\text{trans.}}(t)$ can be regarded as a function of ω and x that clearly lies in $L^p(I, L^2(\mathbb{R}^d; dx); d\omega)$, since it does not depend on ω and $u_{\text{trans.}}(t) \in L^2(\mathbb{R}^d; dx)$. Concerning $u_{\text{as.}}$, notice that for the function ρ_ω from (4.10)

$$|\rho_\omega(\lambda, t)| \leq 2t^2$$

holds, so $\rho_\omega(\mathcal{L}, t) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a bounded operator and since moreover $u_{\text{as.}}(t) = \rho_\omega(\mathcal{L}, t)g$ by (4.9)

$$\begin{aligned} \|u_{\text{as.}}(t)\|^p_{L^p(I,L^2(\mathbb{R}^d;dx);d\omega)} &= \int_I \|\rho_\omega(\mathcal{L}, t)g\|^p_{L^2(\mathbb{R}^d)}\, d\omega \\ &\leq c \int_I \|g\|^p_{L^2(\mathbb{R}^d)}\, d\omega = c|I|\, \|g\|^p_{L^2(\mathbb{R}^d)} < \infty. \end{aligned}$$

So we can regard $u_{\text{as.}}$ as an element of $L^p(I, L^2(\mathbb{R}^d; dx); d\omega)$. □

Lemma 4.8. *For* $\omega^2 \in \mathbb{C},\ \omega^2 \notin \sigma(\mathcal{L})$

$$\widehat{u}_{as.}(s,k,t,\omega) = \frac{P_{s,k}[g]}{\lambda_s(k)-\omega^2} G(t,\omega,\lambda_s(k)), \tag{4.15}$$

where

$$G(t,\omega,\eta) = e^{i\omega t} - \cos\left(\sqrt{\eta}t\right) - i\omega \frac{\sin\left(\sqrt{\eta}t\right)}{\sqrt{\eta}}. \tag{4.16}$$

Proof. We can literally transcribe a proof from [30], which uses integration by parts.

$$\begin{aligned}
\widehat{u}_{\text{as.}}(s,k,t) &= \int_0^t \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}} e^{i\omega\sigma} P_{s,k}[g]\, d\sigma \\
&= P_{s,k}[g] \int_0^t e^{i\omega\sigma} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}}\, d\sigma \\
&= P_{s,k}[g] \left(\left[e^{i\omega\sigma} \frac{\cos\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}^2} \right]_{\sigma=0}^{\sigma=t} \right. \\
&\quad \left. - \int_0^t i\omega e^{i\omega\sigma} \frac{\cos\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}^2}\, d\sigma \right) \\
&= P_{s,k}[g] \left(e^{i\omega t} \frac{1}{\sqrt{\lambda_s(k)}^2} - \frac{\cos\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}^2} \right. \\
&\quad \left. - \int_0^t i\omega e^{i\omega\sigma} \frac{\cos\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}^2}\, d\sigma \right) \\
&= P_{s,k}[g] \left(e^{i\omega t} \frac{1}{\sqrt{\lambda_s(k)}^2} - \frac{\cos\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}^2} \right. \\
&\quad - \left(\left[-i\omega e^{i\omega\sigma} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}^3} \right]_{\sigma=0}^{\sigma=t} \right. \\
&\quad \left. \left. - \int_0^t \omega^2 e^{i\omega\sigma} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}^3}\, d\sigma \right) \right) \\
&= P_{s,k}[g] \left(e^{i\omega t} \frac{1}{\sqrt{\lambda_s(k)}^2} - \frac{\cos\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}^2} - i\omega \frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}^3} \right. \\
&\quad \left. + \int_0^t \omega^2 e^{i\omega\sigma} \frac{\sin\left(\sqrt{\lambda_s(k)}(t-\sigma)\right)}{\sqrt{\lambda_s(k)}^3}\, d\sigma \right)
\end{aligned}$$

$$= \frac{P_{s,k}[g]}{\sqrt{\lambda_s(k)}^2} G(t,\omega,\lambda_s(k)) + \frac{\omega^2}{\sqrt{\lambda_s(k)}^2} \widehat{u}_{\text{as.}}(s,k,t),$$

with G from (4.16). Now for $\omega^2 \notin \sigma(\mathcal{L})$ we obtain the assertion. □

The next lemma shows that we can extend the formula (4.15) for $\widehat{u}_{\text{as.}}$ to $\omega^2 \in \sigma(\mathcal{L})$ such that (4.7) holds.

Lemma 4.9. *For any fixed* $\omega \in (0,\infty)$,

$$u_{as.}(t) = \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_B \frac{P_{s,k}[g]}{\lambda_s(k) - \omega^2} G(t,\omega,\lambda_s(k))\, dk,$$

where $G(t,\omega,\eta)$ *is defined in* (4.16)*. Note that* $\frac{G(t,\omega,\lambda_s(k))}{\lambda_s(k)-\omega^2}$ *is bounded in* k *for fixed* t,ω,s.

Proof. For $\omega^2 \notin \sigma(\mathcal{L})$ the assertion follows from lemma 4.7 and 4.8. So let $\omega^2 \in \sigma(\mathcal{L})$. We consider the equation (4.3) with ω replaced by $\omega + i\delta$ for some $\delta \neq 0$. Then (4.3) is solved by $u = u_{\text{trans.}} + u^\delta_{\text{as.}}$ with $u_{\text{trans.}}$ and $u^\delta_{\text{as.}}$ as in (4.6) with ω replaced by $\omega + i\delta$. Then we claim that the following convergence

$$\begin{aligned} u^\delta_{\text{as.}}(x,t) &= \int_0^t \mathcal{L}^{-1/2} \sin \mathcal{L}^{1/2}(t-\sigma) g e^{i(\omega+i\delta)\sigma} d\sigma \\ &\overset{\delta\to 0^\pm}{\longrightarrow} \int_0^t \mathcal{L}^{-1/2} \sin \mathcal{L}^{1/2}(t-\sigma) g e^{i\omega\sigma} d\sigma = u_{\text{as.}}(x,t) \end{aligned}$$

in $L^2(\mathbb{R}^d)$ holds for any fixed $t \geq 0$ and $\omega \in (0,\infty)$. To see this, first notice that as a consequence of lemma lemma 4.7

$$u^\delta_{\text{as.}}(t) - u_{\text{as.}}(t) = \zeta^\delta(\mathcal{L},t) g$$

with

$$\zeta^\delta(x,t) := \int_0^t \frac{\sin(\sqrt{x}(t-\sigma))}{\sqrt{x}} \left(e^{i(\omega+i\delta)\sigma} - e^{i\omega\sigma}\right)\, d\sigma$$

for all $|\delta| \leq \delta_0$. Clearly there exists a $c > 0$ independent of δ such that $|\zeta^\delta(x,t)| \leq c$. Furthermore

$$\lim_{\delta\to 0} \zeta^\delta(x,t) = 0,$$

so the convergence of $u^\delta_{\text{as.}}$ follows by Lebesgue's dominated convergence theorem. Since the spectrum of $\mathcal{L}$ is real, for $\omega^2 \in \sigma(\mathcal{L})$ and any $\delta \neq 0$ we have $(\omega + i\delta)^2 \notin \sigma(\mathcal{L})$ and so by lemma 4.8

$$\widehat{u}_{\text{as.}}(s,k,t,\omega+i\delta) = \frac{P_{s,k}[g]}{\lambda_s(k) - (\omega+i\delta)^2} G(t,\omega+i\delta,\lambda_s(k)). \tag{4.17}$$

By lemma 4.7

$$u^\delta_{\text{as.}}(x,t) = \frac{1}{\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_B \frac{P_{s,k}[g]}{\lambda_s(k) - (\omega+i\delta)^2} G(t,\omega+i\delta,\lambda_s(k))\, dk.$$

The limit $\delta \to 0$ of the right hand side of the above formula exists in $L^2(\mathbb{R}^d)$ and is equal to

$$\sum_{s\in\mathcal{J}} \int_B \frac{P_{s,k}[g]}{\lambda_s(k) - \omega^2} G(t,\omega,\lambda_s(k))\, dk$$

for any fixed $t \geq 0$ and $\omega \in (0,\infty)$ by Lebesgue's dominated convergence theorem since we can find a suitable bound as follows. Since only finitely many $\sqrt{\lambda_s}$ can intersect at a certain ω there is a maximal number r such that possibly $\lambda_s(k) = \omega^2$ for some k and s with $|s| \leq r$. For all s with $|s| > r$ the function

$$\frac{G(t,\omega+i\delta,\lambda_s(k))}{\lambda_s(k) - (\omega+i\delta)^2}$$

is clearly bounded by a constant c independent of $|\delta| \leq \delta_0$, $|s| > r$, $k \in B$. Consider now $|s| \leq r$. Using the Taylor series expansion for $G(t,\cdot,\lambda_s(k))$ at $\sqrt{\lambda_s(k)}$ yields ($R(\omega) = \sum_{n=2}^\infty \frac{1}{n!}(it)^n e^{i\sqrt{\lambda_s(k)}t} \left(\omega - \sqrt{\lambda_s(k)}\right)^n$ being the remainder term in the power series)

$$G(t,\omega+i\delta,\lambda_s(k)) = 0 + \left(ite^{i\sqrt{\lambda_s(k)}t} - i\frac{\sin\sqrt{\lambda_s(k)}t}{\sqrt{\lambda_s(k)}} \right) (\omega + i\delta - \sqrt{\lambda_s(k)}) + R(\omega+i\delta).$$

For the finite number of s with $|s| \leq r$ we show that for all small $|\delta| \leq \delta_0$

$$\left|\frac{G(t,\omega+i\delta,\lambda_s(k))}{\lambda_s(k) - (\omega+i\delta)^2}\right| = \left| \frac{e^{i(\omega+i\delta)t} - \cos\left(\sqrt{\lambda_s(k)}t\right) - i(\omega+i\delta)\frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}}}{\lambda_s(k) - (\omega+i\delta)^2} \right|$$

$$\leq \left| \frac{ite^{i\sqrt{\lambda_s(k)}t} - i\frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}}}{\sqrt{\lambda_s(k)} + \omega + i\delta} \right| + \left| \frac{R(\omega+i\delta)}{(\sqrt{\lambda_s(k)} + \omega + i\delta)(\sqrt{\lambda_s(k)} - (\omega+i\delta))} \right|$$

$$
\begin{aligned}
&\leq \frac{t+t}{\sqrt{\lambda_s(k)}+\omega} + \left| \frac{R(\omega+i\delta)}{(\sqrt{\lambda_s(k)}+\omega+i\delta)(\sqrt{\lambda_s(k)}-(\omega+i\delta))} \right| \\
&\leq \frac{t+t}{\sqrt{\lambda_s(k)}+\omega} + \frac{\exp(t(\omega+\delta_0+\sqrt{\lambda_s(k)}))}{(\sqrt{\lambda_s(k)}+\omega)(\omega+\delta_0+\sqrt{\lambda_s(k)})} \leq c
\end{aligned}
$$

holds for some $c > 0$ that is independent of $|\delta| \leq \delta_0$, $|s| \leq r$ and $k \in B$ (recall $\sqrt{\lambda_s(k)} \geq q > 0$). Since $u^{\delta}_{\text{as.}} \to u_{\text{as.}}$ for $\delta \to 0$ in $L^2(\mathbb{R}^d)$ the assertion follows. □

Now we are interested in the asymptotic behavior of $u_{\text{as.}}$ as $t \to \pm\infty$. To this end, it is favorable to rewrite $u_{\text{as.}}$ in terms of the standard Hilbert transform H after using partial fraction decomposition and the Coarea formula. Since the right hand side g of the corresponding Helmholtz equation of (4.3) is constant in ω this situation is simpler than in chapter 3 and we do not need the subinterval I_0 either. Thus, like in chapter 3 level set integrals will appear which lie in a weighted L^2-space.

Let w be a weight function as in theorem 3.19. Recall

$$
\mathcal{B} = L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx) \tag{4.18}
$$

with inner product denoted by $f \cdot g$ or $\langle f, g\rangle_{\mathcal{B}}$ given by

$$
\langle f, g\rangle_{\mathcal{B}} = \int_{\mathbb{R}^d} f\overline{g}\; w(x)dx.
$$

In the calculations we will prefer the notation $f \cdot g$.

Proposition 4.10. *Let I be an interval that is compactly embedded in the set of regular values of all $\sqrt{\lambda_s}$ and V its preimage under all $\sqrt{\lambda_s}$. Then with G from* (4.16)

$$
\begin{aligned}
u_{as.}(x,t) \;=\; & \frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{P_{s,k}[g]}{(\sqrt{\lambda_s(k)}-\omega)} G(t,\omega,\lambda_s(k))\; dk \\
& -\frac{1}{2\omega} \sum_{s\in\mathcal{J};|s|\leq r} \pi H\left(h_s(\tau,x)G(t,\omega,\tau)\right)(\omega) \\
& -\frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B} \frac{P_{s,k}[g]}{(\sqrt{\lambda_s(k)}+\omega)} G(t,\omega,\lambda_s(k))\; dk.
\end{aligned} \tag{4.19}
$$

with the level set integral h_s being

$$
h_s(\tau,x) = \frac{1}{\sqrt{|B|}} \int_{B^s_\tau} \frac{\langle Ug(\cdot,k), \psi_s(\cdot,k)\rangle_{L^2(\Omega)} \psi_s(x,k)}{|\nabla\sqrt{\lambda_s}(k)|}\; d\mathcal{H}^{d-1}(k), \tag{4.20}
$$

where we put $h_s(\tau, x) = 0$ if $\tau \notin I$. In contrast to chapter 3 the level sets are taken from the functions $\sqrt{\lambda_s}$, i.e. in this chapter

$$B^s_\tau = \{k \in B : \sqrt{\lambda_s}(k) = \tau\}.$$

Proof. Note that in the second line of (4.19) the Hilbert transform is not really a singular integral but rather

$$\begin{aligned}
& -\frac{1}{2\omega} H\left(h_s(\tau, x)G(t, \omega, \tau)\right)(\omega) \\
= \; & -\frac{1}{2\omega} \lim_{\varepsilon \to 0} \int_{|\tau-\omega| \geq \varepsilon} \frac{\chi_I(\tau) h_s(\tau, x) G(t, \omega, \tau)}{\omega - \tau}\, d\tau \\
= \; & -\frac{1}{2\omega} \int_I \frac{h_s(\tau, x)\big(e^{i\omega t} - \cos(\tau t) - i\omega \frac{\sin(\tau t)}{\tau}\big)}{\omega - \tau}\, d\tau \\
= \; & -\frac{1}{2\omega\sqrt{|B|}} \int_I \int_{B^s_\tau} \frac{\langle Ug(\cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)} \psi_s(x, k)}{|\nabla\sqrt{\lambda_s}(k)|(\omega - \sqrt{\lambda_s(k)})} G(t, \omega, \lambda_s(k))\, d\mathcal{H}^{d-1}(k)\, d\tau \\
= \; & \frac{1}{2\omega\sqrt{|B|}} \int_V \frac{\langle Ug(\cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)} \psi_s(x, k)}{\sqrt{\lambda_s(k)} - \omega} G(t, \omega, \lambda_s(k))\, dk,
\end{aligned}$$

unveiling how the Coarea formula comes into play. Undoing the partial fraction decomposition proves the statement (compare with lemma 4.9). □

Though the representation of $u_{\text{as.}}$ by (4.19) might seem to complicate things, it opens a way to regroup the terms on the right hand side using the linearity of H. Then the arising integrals will be really singular, but can be dealt with thanks to the properties of H. The partial fraction decomposition is for technical reasons which will become clear, when we will use Fourier transforms to obtain the asymptotical behavior as $t \to \pm\infty$ of parts of $u_{\text{as.}}$.

4.3 Asymptotic behavior of the solution of the wave equation for $t \to \pm\infty$

In this subsection we study the asymptotic behavior of $u_{\text{as.}}(t)$ and $u_{\text{trans.}}(t)$ for $t \to \pm\infty$. To keep the notation simpler, we will suppress the x variable in the level set integral (4.20) and regard

$$\tau \mapsto h_s(\tau)$$

as a L^2-function with values in $\mathcal{B} = L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx)$ and support in I (compare with section 3.5). Thus, integrating h_s over $\mathbb{R}$ means integrating in the sense of Bochner integral. Eventually we will see, that the asymptotic behavior of the solution u of the wave equation (4.3) is connected to the corresponding Helmholtz equation

$$(\mathcal{L} - \omega^2)v = g. \tag{4.21}$$

Since we will make excessive use of Fourier transforms with respect to several sets of variables, it will help to make the following

Definition 4.11. *We use this notation for Fourier transforms to emphasize what variables are involved*

$$\mathcal{F}_{ab}h(c) = \int_{\mathbb{R}} e^{-iab}h(b)db\bigg|_{a=c},$$

or even

$$\left(\mathcal{F}_{ab}h(b)\right)(c) = \int_{\mathbb{R}} e^{-iab}h(b)db\bigg|_{a=c}.$$

Fourier transforms on L^2 will also indicate which variables are involved if necessary, i.e. for a L^2 function h in the variable t, $\mathcal{F}_{\tau t}h$ is the Fourier transform of h in the variable τ.

4.3.1 Asymptotic behavior of the parts of $u_{\text{as.}}$ involving the Hilbert transform H

First we take care of the the parts of $u_{\text{as.}}$ that contain the Hilbert transform H, i.e. the second line of (4.19). Inserting a zero in the form $-i\sin\tau t + i\sin\tau t$ we can split the second line of (4.19) in the following way:

$$\begin{aligned}
& -\pi H\left(h_s(\tau)\big(e^{i\omega t} - \cos(\tau t) - i\omega\frac{\sin(\tau t)}{\tau}\big)\right)(\omega) \\
= \; & -\pi H\left(h_s(\tau)\big(e^{i\omega t} - \cos(\tau t) - i\sin\tau t + i\sin\tau t - i\omega\frac{\sin(\tau t)}{\tau}\big)\right)(\omega) \\
= \; & -\pi H\left(h_s(\tau)\big(e^{i\omega t} - e^{i\tau t} + i\tau\frac{\sin\tau t}{\tau} - i\omega\frac{\sin(\tau t)}{\tau}\big)\right)(\omega) \\
= \; & -\pi H h_s(\omega)e^{i\omega t} + \pi H\left(h_s(\tau)e^{i\tau t}\right)(\omega) \\
& +\pi H\left(h_s(\tau)\big(i(\omega-\tau)\frac{\sin\tau t}{\tau}\big)\right)(\omega).
\end{aligned} \tag{4.22}$$

Now we will determine the asymptotic behavior for $t \to +\infty$ for each of the three resulting summands in (4.22), beginning with the last one.

Lemma 4.12. *$\pi H\left(i(\omega-\tau)\frac{\sin(\tau t)}{\tau}h_s(\tau)\right)(\omega)$ is constant in ω and tends to 0 in $\mathcal{B}$ as $t \to \pm\infty$.*

Proof. $\operatorname{supp}(h_s) \subseteq (0,\infty)$ and $h_s \in L^2(I,\mathcal{B})$ (see lemma 3.23) yields

$$\begin{aligned}\pi H\left(i(\omega-\tau)\frac{\sin(\tau t)}{\tau}h_s(\tau)\right)(\omega) &= \lim_{\varepsilon\to 0}\int\limits_{|\omega-\tau|\geq\varepsilon}\frac{1}{\omega-\tau}i\pi(\omega-\tau)\frac{\sin(\tau t)}{\tau}h_s(\tau)\,d\tau \\ &= i\pi\int_{\mathbb{R}}\sin(\tau t)\frac{h_s(\tau)}{\tau}\,d\tau \overset{t\to\pm\infty}{\longrightarrow} 0\end{aligned}$$

in $\mathcal{B}$ for any fixed ω according to the Riemann-Lebesgue lemma. □

To see the asymptotics of the second summand in (4.22) we will need the following two lemmata.

Lemma 4.13. *For $\mathcal{B}$*-valued *Schwarz functions φ and $f \in L^\infty(\mathbb{R},\mathbb{R})$ we have*

$$\mathcal{F}_{t\tau}\left(\int_{\mathbb{R}} f(\sigma)\varphi(\sigma)e^{-i\sigma\tau}d\sigma\right)(t) = f(-t)\varphi(-t).$$

Proof. Let $\psi = \psi(t)$ be a *$\mathcal{B}$-valued* Schwartz function on $\mathbb{R}$. Then the following calculation proves the statement, where $\langle\cdot,\cdot\rangle$ denotes the duality action of a distribution on the Schwartz space.

$$\begin{aligned} &\langle\mathcal{F}_{t\tau}\left(\int_{\mathbb{R}} f(\sigma)\varphi(\sigma)e^{-i\sigma\tau}d\sigma\right),\psi\rangle \\ =\ &\langle\int_{\mathbb{R}} f(\sigma)\varphi(\sigma)e^{-i\sigma\tau}d\sigma,\mathcal{F}_{\tau t}\psi\rangle \\ =\ &\int_{\mathbb{R}}\int_{\mathbb{R}} f(\sigma)\varphi(\sigma)e^{-i\sigma\tau}d\sigma\ \cdot\mathcal{F}_{\tau t}\psi(\tau)\,d\tau \\ =\ &\int_{\mathbb{R}} f(\sigma)\varphi(\sigma)\cdot\int_{\mathbb{R}} e^{-i\sigma\tau}\mathcal{F}_{\tau t}\psi(\tau)d\tau\,d\sigma \quad \text{by Fubini's theorem,} \\ =\ &\int_{\mathbb{R}} f(\sigma)\varphi(\sigma)\cdot\mathcal{F}_{\sigma\tau}\mathcal{F}_{\tau t}\psi(\sigma)\,d\sigma \\ =\ &\int_{\mathbb{R}} f(\sigma)\varphi(\sigma)\cdot\psi(-\sigma)\,d\sigma\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} f(-\sigma)\varphi(-\sigma) \cdot \psi(\sigma)\, d\sigma \\
&= \langle f(-\cdot)\varphi(-\cdot), \psi(\cdot)\rangle.
\end{aligned}
$$

Here we have used $\mathcal{F}_{\sigma\tau}\mathcal{F}_{\tau t}\psi(\sigma) = \psi(-\sigma)$. □

The following lemma is standard from harmonic analysis.

Lemma 4.14. *Let $f \in L^2(\mathbb{R}, \mathcal{B})$ have compact support and g be uniformly continuous on $\mathbb{R}$ and $\mathcal{F}g \in L^1(\mathbb{R}, \mathcal{B})$. Then $f \cdot g \in L^1(\mathbb{R}, \mathbb{C})$ and $\mathcal{F}(f \cdot g) = \mathcal{F}f * \mathcal{F}g$. (By $f \cdot g$ we mean the $\mathbb{C}$-valued function $x \mapsto f(x) \cdot g(x)$ and by the convolution $f * g$ of two $\mathcal{B}$-valued functions we mean the $\mathbb{C}$-valued function $x \mapsto (f * g)(x) = \int_{\mathbb{R}} f(x-y) \cdot g(y)\, dy$).*

Proof.

$$
\begin{aligned}
\mathcal{F}(f \cdot g)(\xi) &= \int_{\mathbb{R}} e^{-ix\xi} f(x) \cdot g(x)\, dx \\
&= \int_{\mathbb{R}} e^{-ix\xi} f(x) \cdot \int_{\mathbb{R}} e^{i\eta x} \mathcal{F}g(\eta)\, d\eta\, dx \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-ix\xi} e^{i\eta x} f(x) \cdot \mathcal{F}g(\eta)\, dx\, d\eta \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{-ix(\xi-\eta)} f(x)\, dx \right) \cdot \mathcal{F}g(\eta)\, d\eta \\
&= \int_{\mathbb{R}} \mathcal{F}f(\xi-\eta) \cdot \mathcal{F}g(\eta)\, d\eta \\
&= (\mathcal{F}f * \mathcal{F}g)(\xi).
\end{aligned}
$$

□

With this two lemmata we can prove the following, crucial

Proposition 4.15. *As $t \to \pm\infty$ we have the convergence*

$$
e^{-i\omega t} \pi H \left(h_s(\tau) e^{i\tau t} \right)(\omega) \to \mp i\pi h_s(\omega)
$$

in $L^2(I, \mathcal{B}; d\omega)$.

Proof. Now regard $T_t(\omega) := \pi H \left(h_s(\tau) e^{i\tau t} \right)(\omega)$ as a family of distributions on the Schwartz space in the variable ω with values in $\mathcal{B}$. It is helpful to compute the Fourier transform of T_t.

For a $\mathcal{B}$-valued Schwartz function φ we define $\widetilde{\varphi}(\omega) = \varphi(-\omega)$. Now by duality and using $\mathcal{F}\mathcal{F}\widetilde{\varphi} = \varphi$

$$\langle T_t, \varphi \rangle = \langle T_t, \mathcal{F}\mathcal{F}\widetilde{\varphi} \rangle = \langle \mathcal{F}T_t, \mathcal{F}\widetilde{\varphi} \rangle. \tag{4.23}$$

We compute the Fourier transform of T_t. Since $\tau \mapsto h_s(\tau)e^{i\tau t}$ is in $L^1(\mathbb{R}, \mathcal{B}) \cap L^2(\mathbb{R}, \mathcal{B})$ and $-i\text{sgn}(\cdot)$ is the Fourier multiplier of H in L^2 (see [41]) we see that

$$T_t(\omega) = -i\pi \mathcal{F}^{-1}_{\omega\sigma} \left(\text{sgn}(\sigma) \mathcal{F}_{\sigma\tau}(h_s(\tau)e^{i\tau t}) \right)(\omega)$$

and so

$$\mathcal{F}_{\sigma\omega} T_t(\sigma) = -i\pi\text{sgn}(\sigma)\mathcal{F}_{\sigma\tau}(h_s(\tau)e^{i\tau t}) = -i\pi\text{sgn}(\sigma)\mathcal{F}_{t\tau}(e^{i\sigma\tau}h_s(-\tau)), \tag{4.24}$$

where the last equality is seen as follows, using $h_s(\tau)e^{i\tau t} \in L^1(\mathbb{R}, \mathcal{B})$:

$$\begin{aligned} \mathcal{F}_{\sigma\tau}\left(h_s(\tau)e^{i\tau t}\right) &= \int_{\mathbb{R}} e^{-i\sigma\tau}\left(e^{i\tau t}h_s(\tau)\right) d\tau \\ &= \int_{\mathbb{R}} e^{-i\tau t}\left(e^{i\sigma\tau}h_s(-\tau)\right) d\tau = \mathcal{F}_{t\tau}(e^{i\sigma\tau}h_s(-\tau)). \end{aligned}$$

We use (4.24) to calculate $\langle \mathcal{F}T_t, \mathcal{F}\widetilde{\varphi} \rangle$:

$$\begin{aligned} \langle \mathcal{F}T_t, \mathcal{F}\widetilde{\varphi} \rangle &= \int_{\mathbb{R}} -i\pi\text{sgn}(\sigma)\mathcal{F}_{t\tau}(e^{i\sigma\tau}h_s(-\tau)) \cdot \mathcal{F}\widetilde{\varphi}(\sigma)\, d\sigma \\ &= -i\pi\mathcal{F}_{t\tau}\left(h_s(-\tau) \cdot \int_{\mathbb{R}} \text{sgn}(\sigma)\, e^{i\sigma\tau}\mathcal{F}\widetilde{\varphi}(\sigma)\, d\sigma\right) \quad \text{by Fubini's theorem} \\ &= -i\pi\mathcal{F}_{t\tau}\left(h_s(-\tau)\right) * \mathcal{F}_{t\tau}\left(\int_{\mathbb{R}} \text{sgn}(\sigma)\, e^{i\sigma\tau}\mathcal{F}\widetilde{\varphi}(\sigma)\, d\sigma\right). \end{aligned}$$

In the last step lemma 4.14 was used. Note that $h_s(-\tau)$ is a L^2-function of τ with compact support and $\tau \mapsto \int_{\mathbb{R}} \text{sgn}(\sigma)\, e^{i\sigma\tau}\mathcal{F}\widetilde{\varphi}(\sigma)\, d\sigma$ is uniformly continuous as the Fourier transform of the L^1-function $\text{sgn}(-\sigma)\mathcal{F}\widetilde{\varphi}(-\sigma)$. Further, using lemma 4.13, we see that

$$\begin{aligned} \mathcal{F}_{t\tau}\left(\int_{\mathbb{R}} \text{sgn}(\sigma)\, e^{i\sigma\tau}\mathcal{F}\widetilde{\varphi}(\sigma)\, d\sigma\right) &= \mathcal{F}_{t\tau}\left(\int_{\mathbb{R}} \text{sgn}(-\sigma)e^{-i\sigma\tau}\mathcal{F}\widetilde{\varphi}(-\sigma)\, d\sigma\right) \\ &= \text{sgn}(t)\, \mathcal{F}\widetilde{\varphi}(t), \end{aligned}$$

thus it is a L^1-function in t. So we continue the calculation with

$$\langle \mathcal{F}T_t, \mathcal{F}\widetilde{\varphi} \rangle = -i\pi\mathcal{F}_{t\tau}\left(h_s(-\tau)\right) * \text{sgn}(t)\, \mathcal{F}\widetilde{\varphi}(t)$$

$$
\begin{aligned}
&= -i\pi \int_{\mathbb{R}} \mathcal{F}_{\eta\tau}\left(h_s(-\tau)\right)(\eta)\, \mathrm{sgn}(t-\eta) \cdot \mathcal{F}\widetilde{\varphi}(t-\eta)\; d\eta \\
&= -i\pi \int_{\mathbb{R}} \mathcal{F}_{\eta\tau}\left(h_s(-\tau)\right)(\eta)\, \mathrm{sgn}(t-\eta) \cdot \int_{\mathbb{R}} e^{-i(t-\eta)\omega}\widetilde{\varphi}(\omega)\; d\omega\; d\eta
\end{aligned}
$$

Now we would like to change the order of the two integrations. We mark this step with $(\star)$ below and justify it after the following calculation at the end of the proof. Before continuing we define temporarily the family f_t of $L^2(\mathbb{R},\mathcal{B})$-functions

$$
f_t(\eta) := \mathcal{F}_{\eta\tau}\left(h_s(-\tau)\right)(\eta)\, \mathrm{sgn}(t-\eta).
$$

Then we continue the calculation

$$
\begin{aligned}
\langle \mathcal{F}T_t, \mathcal{F}\widetilde{\varphi}\rangle &= -i\pi \int_{\mathbb{R}} \mathcal{F}_{\eta\tau}\left(h_s(-\tau)\right)(\eta)\, \mathrm{sgn}(t-\eta) \cdot \int_{\mathbb{R}} e^{-i(t-\eta)\omega}\widetilde{\varphi}(\omega)\; d\omega\; d\eta \\
&\overset{(\star)}{=} -i\pi \int_{\mathbb{R}} e^{-it\omega} \left(\int_{\mathbb{R}} f_t(\eta) e^{i\eta\omega}\; d\eta \right) \cdot \widetilde{\varphi}(\omega)\; d\omega \\
&= -i\pi \int_{\mathbb{R}} e^{it\omega} \left(\int_{\mathbb{R}} f_t(\eta) e^{-i\eta\omega} d\eta \right) \cdot \varphi(\omega)\; d\omega \\
&= -i\pi \int_{\mathbb{R}} e^{it\omega} \mathcal{F}_{\omega\eta}(f_t(\eta))(\omega) \cdot \varphi(\omega)\; d\omega \\
&= \langle -i\pi e^{it\omega} \mathcal{F}_{\omega\eta}(f_t(\eta)), \varphi\rangle.
\end{aligned}
$$

Comparing this result with (4.23) we conclude that

$$
T_t = -i\pi e^{it\omega} \mathcal{F}_{\omega\eta}(f_t(\eta)) \in L^2(\mathbb{R},\mathcal{B}; d\omega).
$$

Since $\mathrm{sgn}(t-\eta) \to \pm 1$ pointwise as $t \to \pm\infty$, using Lebesgue dominated convergence we see that

$$
f_t \to \pm \mathcal{F}_{\eta\tau}\left(h_s(-\tau)\right)
$$

in $L^2(\mathbb{R},\mathcal{B}; d\eta)$ as $t \to \pm\infty$. Since the Fourier transform is a bounded operator in L^2 we conclude that

$$
\begin{aligned}
e^{-i\omega t} T_t &= -i\pi \mathcal{F}_{\omega\eta}(f_t(\eta)) \\
&\to -i\pi \mathcal{F}_{\omega\eta}(\pm \mathcal{F}_{\eta\tau}\left(h_s(-\tau)\right)) \\
&\quad = \mp i\pi \mathcal{F}_{\omega\eta}(\mathcal{F}^{-1}_{\eta\tau}\left(h_s(\tau)\right)) \\
&\quad = \mp i\pi h_s(\omega)
\end{aligned}
$$

in $L^2(\mathbb{R}, \mathcal{B}; d\omega)$ as $t \to \pm\infty$.

It remains to justify the step $(\star)$. For any fixed t we have $f_t \cdot \mathcal{F}\widetilde{\varphi}(\cdot - t) \in L^1(\mathbb{R}, \mathbb{C})$ by Cauchy-Schwarz inequality and

$$\int_{-R}^{R} f_t(\eta) \cdot \mathcal{F}\widetilde{\varphi}(\eta - t)\, d\eta \overset{R\to\infty}{\longrightarrow} \int_{\mathbb{R}} f_t(\eta) \cdot \mathcal{F}\widetilde{\varphi}(\eta - t)\, d\eta$$

in $\mathbb{C}$. On the other hand by Fubini's theorem

$$\begin{aligned} \int_{-R}^{R} f_t(\eta) \cdot \mathcal{F}\widetilde{\varphi}(\eta - t)\, d\eta &= \int_{-R}^{R} f_t(\eta) \cdot \int_{-\infty}^{\infty} e^{i\eta\omega} e^{-it\omega} \varphi(\omega)\, d\omega\, d\eta \\ &= \int_{-\infty}^{\infty} e^{-it\omega} \varphi(\omega) \cdot \underbrace{\int_{-R}^{R} f_t(\eta) e^{i\eta\omega}\, d\eta}_{=: g_{R,t}(\omega)}\, d\omega. \end{aligned}$$

Since $\chi_{(-R,R)} f_t \in L^1(\mathbb{R}, \mathcal{B}) \cap L^2(\mathbb{R}, \mathcal{B})$, we have

$$g_{R,t} = \mathcal{F}_1^{-1}(\chi_{(-R,R)} f_t) \overset{\text{a.e.}}{=} \mathcal{F}_2^{-1}(\chi_{(-R,R)} f_t) \in L^2(\mathbb{R}, \mathcal{B}),$$

where $\mathcal{F}_1$ is the Fourier transform in L^1 and $\mathcal{F}_2$ the Fourier transform in L^2. Since $\mathcal{F}_2$ is an isometry and $\chi_{(-R,R)} f_t \to f_t$ in $L^2(\mathbb{R}, \mathcal{B})$ as $R \to \infty$ we obtain $g_{R,t} \to \mathcal{F}_2^{-1} f_t$ in $L^2(\mathbb{R}, \mathcal{B})$ as $R \to \infty$. By the following calculation the step $(\star)$ will be justified. Note that for a $\mathcal{B}$-valued Schwartz function φ the function

$$\varphi(\omega) \cdot \mathcal{F}_2^{-1} f_t(\omega)$$

is in $L^1(\mathbb{R}, \mathbb{C}; d\omega)$.

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} e^{-i\omega t} \varphi(\omega) \cdot \int_{-R}^{R} e^{i\eta\omega} f_t(\eta)\, d\eta\, d\omega - \int_{-\infty}^{\infty} e^{-i\omega t} \varphi(\omega) \cdot \mathcal{F}_2^{-1} f_t(\omega)\, d\omega \right| \\ = & \left| \int_{-\infty}^{\infty} e^{-i\omega t} \varphi(\omega) \cdot \left(g_{R,t}(\omega) - \mathcal{F}_2^{-1} f_t(\omega) \right)\, d\omega \right| \\ \leq & \int_{-\infty}^{\infty} \|\varphi(\omega)\|_{\mathcal{B}} \left\| g_{R,t}(\omega) - \mathcal{F}_2^{-1} f_t(\omega) \right\|_{\mathcal{B}}\, d\omega \\ \leq & \|\varphi\|_{L^2(\mathbb{R},\mathcal{B})} \left\| g_{R,t} - \mathcal{F}_2^{-1} f_t \right\|_{L^2(\mathbb{R},\mathcal{B})} \overset{R\to\infty}{\longrightarrow} 0. \end{aligned}$$

□

4.3.2 Asymptotic behavior as $t \to +\infty$ of the nonsingular parts of $u_{\textbf{as.}}$ and of $u_{\textbf{trans.}}$

The remaining non-time-harmonic parts of $u_{\text{as.}}$, i.e the first and third line of (4.19) - except for the summands with $e^{i\omega t}$ - and $u_{\text{trans.}}$ converge weakly to 0, as it was known for the case $\omega^2 \notin \sigma(\mathcal{L})$ (see [30], Chapter XVII B, §4.). This fact is the issue of the following two lemmata.

Lemma 4.16. *As $t \to \pm\infty$, for any fixed $\omega \in I$, $g \in D(\mathcal{L}^{1/2})$*

$$\frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{P_{s,k}[g]}{(\sqrt{\lambda_s(k)} - \omega)} \left(\cos\left(\sqrt{\lambda_s(k)}t\right) + i\omega \frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}} \right) dk \to 0$$

weakly in $L^2(\mathbb{R}^d, \mathbb{C}^M; dx)$.

Proof. We have $\chi_{B\setminus V} P_{s,k}[g] = \chi_{B\setminus V}\langle Ug, \psi_s\rangle\psi_s = \langle \chi_{B\setminus V} Ug, \psi_s\rangle\psi_s$, since $\chi_{B\setminus V}$ depends only on k. Since $U : L^2(\mathbb{R}^d) \to L^2(\Omega \times B)$ is an isomorphism, we can find a $\hat{g} \in L^2(\mathbb{R}^d)$ with $U\hat{g} = \chi_{B\setminus V} Ug$. Note that although $\omega \in I$, $(\mathcal{L}^{1/2} - \omega)^{-1}\hat{g} \in L^2(\mathbb{R}^d)$ exists due to the special definition of $\hat{g}$. So

$$\begin{aligned}
\hat{u} &:= \frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{P_{s,k}[g]}{(\sqrt{\lambda_s(k)} - \omega)} \left(\cos\left(\sqrt{\lambda_s(k)}t\right) + i\omega \frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}} \right) dk \\
&= \frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B} \frac{P_{s,k}[\hat{g}]}{(\sqrt{\lambda_s(k)} - \omega)} \left(\cos\left(\sqrt{\lambda_s(k)}t\right) + i\omega \frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}} \right) dk \\
&= \frac{1}{2\omega} \left(\cos\left(\mathcal{L}^{1/2}t\right) (\mathcal{L}^{1/2} - \omega)^{-1}\, \hat{g} + i\omega \sin\left(\mathcal{L}^{1/2}t\right) \mathcal{L}^{-1/2} (\mathcal{L}^{1/2} - \omega)^{-1}\, \hat{g} \right).
\end{aligned}$$

Now since $\mathcal{L}^{1/2}$ is selfadjoint and the generator of the unitary group $e^{i\mathcal{L}^{1/2}t}$, moreover $(\mathcal{L}^{1/2} - \omega)^{-1}\hat{g},\ \mathcal{L}^{-1/2}(\mathcal{L}^{1/2} - \omega)^{-1}\hat{g} \in L^2(\mathbb{R}^d)$ and finally the spectrum of $\mathcal{L}^{1/2}$ is absolutely continuous, by a theorem found in [30] (Proposition 1, p.426) we have for any $\varphi \in L^2(\mathbb{R}^d)$:

$$\langle \hat{u}, \varphi \rangle_{L^2(\mathbb{R}^d)} \stackrel{t\to\pm\infty}{\longrightarrow} 0.$$

□

The same arguments lead to

Lemma 4.17. *As $t \to \pm\infty$, for any fixed $\omega \in I$, $g \in D(\mathcal{L}^{1/2})$, $u^0 \in D(\mathcal{L})$, $u^1 \in D(\mathcal{L}^{1/2})$*

$$\frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_B \frac{P_{s,k}[g]}{\left(\sqrt{\lambda_s(k)}+\omega\right)} \left(\cos\left(\sqrt{\lambda_s(k)}t\right) + i\omega\frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}}\right) dk \quad \to \quad 0,$$

$$u_{\textit{trans.}}(t) = \cos(\mathcal{L}^{1/2}t)u^0 + \mathcal{L}^{-1/2}\sin(\mathcal{L}^{1/2}t)u^1 \quad \to \quad 0,$$

weakly in $L^2(\mathbb{R}^d, \mathbb{C}^M; dx)$.

4.4 Formulation of the limit amplitude principle for regular frequencies

Recall $\mathcal{B} = L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx)$ with a weight function as in theorem 3.19 in chapter 3.

Theorem 4.18 (Limit amplitude principle for regular frequencies)**.** *Let $I \subset\subset \mathcal{R} \subseteq (0,\infty)$ be an interval that is compactly embedded in the set of regular (in the sense of 3.6) frequencies of the band functions $\sqrt{\lambda_s}$ of the operator $\mathcal{L}^{1/2}$. Then the strong solution $u = u(t,\omega,x)$[1] of the wave equation*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + \mathcal{L}u = e^{i\omega t}g, \\ u(0) = u^0, \\ \frac{\partial u}{\partial t}(0) = u^1, \end{cases} \tag{4.25}$$

with $u^0 \in D(\mathcal{L})$, $g, u^1 \in D(\mathcal{L}^{1/2})$, which is given by $u = u_{\textit{trans.}} + u_{\textit{as.}}$ from (4.5) *and* (4.6)*, the following asymptotics holds*

$$u(t,\omega,x) \sim e^{i\omega t}v^-(\omega,x) \text{ as } t \to +\infty, \tag{4.26}$$

in the following sense

$$\begin{aligned} & u \sim \widetilde{v} \text{ as } t \to +\infty \\ :\Longleftrightarrow\quad & \|\langle u(t,\omega,\cdot) - \widetilde{v}(t,\omega,\cdot), \phi(\cdot)\rangle_{\mathcal{B}}\|_{L^2(I,\mathbb{R};d\omega)} \\ & \overset{t\to+\infty}{\longrightarrow} 0 \quad \forall\, \phi \in L^2(\mathbb{R}^d, \mathbb{C}^M) \end{aligned} \tag{4.27}$$

[1] i.e. $u(t,\omega,\cdot) \in D(\mathcal{L})$, $\frac{\partial u}{\partial t}(t,\omega,\cdot) \in D(\mathcal{L}^{1/2})$, $\frac{\partial^2 u}{\partial t^2}(t,\omega,\cdot) \in L^2(\mathbb{R}^d, \mathbb{C}^M)$, $t \mapsto u(t,\omega,\cdot) \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ twice continuously differentiable.

with $v^- \in L^2(I, \mathcal{B}; d\omega)$ being the limiting absorption solution (in distributional sense) of the first kind of the corresponding Helmholtz equation

$$(\mathcal{L} - \omega^2)v = g, \tag{4.28}$$

i.e.

$$v^-(\omega, x) = \lim_{\delta\to 0^-} v_\delta(\omega, x) = \lim_{\delta\to 0^-} (\mathcal{L} - (\omega + i\delta)^2)^{-1} g,$$

the limit existing in $L^2(I, \mathcal{B}; d\omega)$.

Proof of theorem 4.18. Let V be the preimage of I under all band functions $\sqrt{\lambda_s}$. The theory from chapter 3 works also for the absorptive operators $\mathcal{L} - (\omega + i\delta)^2$ as for the absorptive operators $\mathcal{L} - \omega^2 - i\delta$. One has only to perform partial fraction decomposition after decomposing the resolvent by the Floquet-Bloch transform. Then adapting theorem 3.31 from chapter 3 have the following representation of the limiting absorption solution of (4.28)

$$\begin{aligned} v^\pm(\omega, x) \;=\; & \frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{P_{s,k}[g]}{\sqrt{\lambda_s}(k) - \omega}\, dk \\ & + \frac{1}{2\omega} \sum_{s\in\mathcal{J};|s|\le r} -\pi H h_s(\omega, x) \pm i\pi h_s(\omega, x) \\ & - \frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B} \frac{P_{s,k}[g]}{\sqrt{\lambda_s}(k) + \omega}\, dk \end{aligned} \tag{4.29}$$

with $P_{s,k}[g] = \langle Ug(\cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)} \psi_s(x, k)$,

$$h_s(\tau, x) = \int_{B^s_\tau} \frac{\langle Ug(\cdot, k), \psi_s(\cdot, k)\rangle_{L^2(\Omega)} \psi_s(x, k)}{|\nabla\sqrt{\lambda_s}(k)|}\, d\mathcal{H}^{d-1}(k)$$

and

$$B^s_\tau = \{k \in B : \sqrt{\lambda_s}(k) = \tau\}.$$

We can do without the trace operator T, since g depends not on ω. Again we skip the variable x. Comparing $v^\mp$ from (4.29) with the representation of $u_{\text{as.}}$ in equation (4.19) we find the following for the difference

$$\begin{aligned} & u_{\text{as.}} - e^{i\omega t} v^\mp \\ = \; & \frac{1}{2\omega\sqrt{|B|}} \sum_{s\in\mathcal{J}} \int_{B\setminus V} \frac{P_{s,k}[g]}{(\sqrt{\lambda_s(k)} - \omega)} \left(-\cos\left(\sqrt{\lambda_s(k)}t\right) - i\omega \frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}} \right) dk \end{aligned} \tag{4.30}$$

$$+\frac{1}{2\omega}\sum_{s\in\mathcal{J};\ |s|\le r} -\pi H\left(h_s(\tau)\big(-\cos(\tau t) - i\omega\frac{\sin(\tau t)}{\tau}\big)\right)(\omega) \pm i\pi e^{i\omega t} h_s(\omega)$$
$$-\frac{1}{2\omega\sqrt{|B|}}\sum_{s\in\mathcal{J}}\int_B \frac{P_{s,k}[g]}{(\sqrt{\lambda_s(k)}+\omega)}\left(-\cos\left(\sqrt{\lambda_s(k)}t\right) - i\omega\frac{\sin\left(\sqrt{\lambda_s(k)}t\right)}{\sqrt{\lambda_s(k)}}\right)\, dk.$$

Now the first and the third line converge weakly in $L^2(\mathbb{R}^d, \mathbb{C}^M; dx)$ to 0 for any fixed ω as $t \to \pm\infty$ as seen in lemma 4.16 and lemma 4.17. Since $\omega \in I$, one can show by Lebesgue dominated convergence that they converge to 0 in the sense of (4.27). An integrable majorant can be found noting that

$$\rho^{\pm}(\lambda_s(k), t) = \frac{\cos(\sqrt{\lambda_s(k)}t) + i\omega\frac{\sin(\sqrt{\lambda_s(k)}t)}{\sqrt{\lambda_s(k)}}}{\sqrt{\lambda_s(k)} \pm \omega}$$

can be bounded for all $\omega \in I$, $t \geq 0$, $s \in \mathcal{J}$ and $k \in B$, $k \in B \setminus V$ respectively, by a constant that is independent of all of these. The same applies to the term $u_{\text{trans.}}$. Let $\widetilde{u}$ (compare with (4.22)) denote the second line of the right hand side of (4.30). Then $\widetilde{u}$ converges to 0 as $t \to \pm\infty$ in $L^2(I, \mathcal{B}; d\omega)$ as discussed in lemma 4.12 and proposition 4.15. For $\phi \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ we have

$$\begin{aligned}
\|\langle\widetilde{u}(t,\omega,\cdot), \phi(\cdot)\rangle_{\mathcal{B}}\|^2_{L^2(I,\mathbb{R};d\omega)} &= \int_I |\langle\widetilde{u}(t,\omega,\cdot), \phi(\cdot)\rangle_{\mathcal{B}}|^2\, d\omega \\
&\leq \int_I \|\widetilde{u}(t,\omega,\cdot)\|^2_{\mathcal{B}}\, \|\phi\|^2_{\mathcal{B}}\ d\omega \\
&= \|\phi\|^2_{\mathcal{B}}\, \|\widetilde{u}(t,\omega,\cdot)\|^2_{L^2(I,\mathcal{B};d\omega)}
\end{aligned}$$

and so the second line the right hand side of (4.30) tends to 0 in the sense of (4.27). □

4.5 Equivalence of the presented principles of limit absorption and limit amplitude

By equivalence we mean that both principles select the same (distributional) solution to the Helmholtz equation.

Theorem 4.19. *$I \subset\subset \mathcal{R}$ be an interval that is compactly embedded in the set of regular frequencies of the band functions $\sqrt{\lambda_s}$ of the operator $\mathcal{L}^{1/2}$. If $v \in L^2(I, \mathcal{B}; d\omega)$ satisfies* *the limit amplitude principle for the wave equation* (4.25)*, i.e. if for the solution u of* (4.25) $u \sim e^{i\omega t}v$ *for* $t \to +\infty$ *holds in the sense of* (4.27)*, then v is the limiting absorption solution of* (4.28) *for* $\delta \to 0^-$. *Conversely, if* $v \in L^2(I, \mathcal{B}; d\omega)$ *is the limiting absorption solution of* (4.28) *for* $\delta \to 0^-$*, then for the solution u of the wave equation* (4.25)*,* $u \sim e^{i\omega t}v$ *holds.*

Proof. Let u be the strong solution of (4.25) and $v_1, v_2 \in L^2(I, \mathcal{B}; d\omega)$ with $u \sim e^{i\omega t}v_1$ and $u \sim e^{i\omega t}v_2$ as $t \to +\infty$ in the sense of (4.27). Then $e^{i\omega t}v_1 \sim e^{i\omega t}v_2$ as $t \to +\infty$ because for $\phi \in L^2(\mathbb{R}^d, \mathbb{C}^M)$

$$
\begin{aligned}
& \left\|\langle e^{i\omega t}v_1 - e^{i\omega t}v_2, \phi\rangle_{\mathcal{B}}\right\|^2_{L^2(I,\mathbb{R};d\omega)} \\
\leq \ & \left\|\langle e^{i\omega t}v_1 - u(t,\omega,\cdot), \phi(\cdot)\rangle_{\mathcal{B}}\right\|^2_{L^2(I,\mathbb{R};d\omega)} + \left\|\langle u(t,\omega,\cdot) - e^{i\omega t}v_2, \phi(\cdot)\rangle_{\mathcal{B}}\right\|^2_{L^2(I,\mathbb{R};d\omega)} \\
\to \ & 0 \text{ as } t \to +\infty.
\end{aligned}
$$

But also

$$
\begin{aligned}
\left\|\langle e^{i\omega t}v_1 - e^{i\omega t}v_2, \phi\rangle_{\mathcal{B}}\right\|^2_{L^2(I,\mathbb{R};d\omega)} &= \int_I |\langle e^{i\omega t}v_1 - e^{i\omega t}v_2, \phi\rangle_{\mathcal{B}}|^2 \, d\omega \\
&= \int_I |e^{i\omega t} \int_{\mathbb{R}^d} (v_1 - v_2)\overline{\phi}\, w(x)dx|^2 \, d\omega \\
&= \int_I | \int_{\mathbb{R}^d} (v_1 - v_2)\overline{\phi}\, w(x)dx|^2 \, d\omega \\
&= \left\|\langle v_1 - v_2, \phi\rangle_{\mathcal{B}}\right\|^2_{L^2(I,\mathbb{R};d\omega)}.
\end{aligned}
$$

So we see that $\|\langle v_1 - v_2, \phi\rangle_{\mathcal{B}}\|^2_{L^2(I,\mathbb{R};d\omega)} = 0$ and hence $\langle v_1 - v_2, \phi\rangle_{\mathcal{B}} = 0$ for all $\phi \in L^2(\mathbb{R}^d, \mathbb{C}^M)$ for almost all $\omega \in I$. In particular, $\langle v_1 - v_2, \phi\rangle_{L^2(K,\mathbb{C}^M)} = 0$ for all $\phi \in C_0^\infty(K, \mathbb{C}^M)$, where K is an arbitrary ball in $\mathbb{R}^d$. So $v_1(\omega, x) = v_2(\omega, x)$ for almost all $x \in \mathbb{R}^d$, $\omega \in I$. From this we see, that the $v \in L^2(I, \mathcal{B}; d\omega)$ satisfying $u \sim e^{i\omega t}v$ as $t \to +\infty$ is unique. Theorem 4.18 states that the limiting absorption solution v^- of the Helmholtz equation (4.28) given by (4.29) satisfies $u \sim e^{i\omega t}v^-$ for $t \to +\infty$. □

Remark 4.20. *Theorem 4.19 shows the equivalence of the principles of limit amplitude and limit absorption of the first kind (compare chapter 1, section 1.1) provided I is an interval of regular frequencies. In an analogous way one can show the equivalence of the corresponding principles of the second kind for an interval I of regular frequencies, i.e. for the (strong) solution u of the wave equation* (4.25)*,* $u \sim e^{-i\omega t}v$ *as* $t \to +\infty$ *in the sense of* (4.27) *for some* $v \in L^2(I, \mathcal{B})$ *if and only if* $v = v^+ \in L^2(I, \mathcal{B})$ *is the limit absorption solution of* (4.28) *as* $\delta \to 0^+$*. Furthermore,* v^+ *is then given by* (4.29).

Chapter 5

Appendix

5.1 The Floquet-Bloch transform

The Floquet Bloch transform is one of the most important tools in this work. A general treatment can be found in [24]. $\Omega = [0,1]^d$ denotes the cell of periodicity and $B = [-\pi,\pi]^d$ the Brillouin zone.

Definition 5.1. *For a function f with compact support in $\mathbb{R}^d$ the Floquet Bloch transform of f is defined by*

$$Uf(x,k) := \frac{1}{\sqrt{|B|}} \sum_{n\in\mathbb{Z}^d} f(x-n)e^{ink} \quad (x \in \Omega,\ k \in B). \tag{5.1}$$

Theorem 5.2. *U extends to a bounded operator on $L^2(\mathbb{R}^d)$. Moreover,*

$$\begin{cases} U : L^2(\mathbb{R}^d) & \to & L^2(\Omega \times B) \\ f(x) & \mapsto & Uf(x,k) \end{cases} \tag{5.2}$$

is an isometric isomorphism with the inverse

$$\begin{cases} U^{-1} : L^2(\Omega \times B) & \to & L^2(\mathbb{R}^d) \\ g(x,k) & \mapsto & U^{-1}g(x) = \frac{1}{\sqrt{|B|}} \int_B g(x,k)\, dk, \end{cases} \tag{5.3}$$

where in the integral $g(x,k)$ is extended to all of $\mathbb{R}^d$ by the k-quasiperiodicity condition $g(x+n,k) = e^{ink}g(x,k)$ for $n \in \mathbb{Z}^d$.

A proof of this theorem can be found in [43].

For compactly supported functions in $L^2(\mathbb{R}^d)$ the following lemma holds by Cauchy-Schwaz inequality.

Lemma 5.3. *There exists a* $C = C(supp(f))$ *such that for all* $k \in B$

$$\|Uf(\cdot,k)\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{|B|}} \|f\|_{L^2(\mathbb{R}^d)} .$$

5.2 The Cauchy principal value integral

For reference see [23, 44].

Definition 5.4. *For* $h : \mathbb{R} \to \mathbb{C},\ p \in \mathbb{R}$ *the* Cauchy principal value *is defined as*

$$\begin{aligned} \mathcal{P}\!\int_{\mathbb{R}} h(x)\,dx &= \int\limits_{\mathbb{R}\setminus(p-\epsilon,p+\epsilon)} h(x)\,dx + \mathcal{P}\!\int\limits_{p-\epsilon}^{p+\epsilon} h(x)\,dx \\ &=: \int\limits_{\mathbb{R}\setminus(p-\epsilon,p+\epsilon)} h(x)\,dx + \lim_{\eta\to 0}\left[\int\limits_{p-\epsilon}^{p-\eta} h(x)\,dx + \int\limits_{p+\eta}^{p+\epsilon} h(x)\,dx\right]. \end{aligned} \tag{5.4}$$

In case of existence the Cauchy principal value is independent of the choice of ϵ.

Lemma 5.5. *Let* $h : \mathbb{R} \to \mathbb{C}$ *be locally Lipschitz continuous. Then*

$$\mathcal{P}\!\int_{p-\epsilon}^{p+\epsilon} \frac{h(\lambda)}{p-\lambda} d\lambda = \int_{p-\epsilon}^{p+\epsilon} \frac{h(\lambda)-h(p)}{p-\lambda} d\lambda.$$

Proof. We start with the definition of the $\mathcal{P}$ integral

$$\begin{aligned} &\mathcal{P}\!\int_{p-\epsilon}^{p+\epsilon} \frac{h(\lambda)}{p-\lambda} d\lambda \\ =\ &\lim_{\eta\to 0}\left[\int_{p-\epsilon}^{p-\eta} \frac{h(\lambda)}{p-\lambda} d\lambda + \int_{p+\eta}^{p+\epsilon} \frac{h(\lambda)}{p-\lambda} d\lambda\right] \\ =\ &\lim_{\eta\to 0}\Big[\underbrace{\int_{p-\epsilon}^{p-\eta} \frac{h(p)}{p-\lambda} d\lambda + \int_{p+\eta}^{p+\epsilon} \frac{h(p)}{p-\lambda} d\lambda}_{=0 \text{ because of symmetry}} \end{aligned}$$

$$
\begin{aligned}
&+\int_{p-\epsilon}^{p-\eta}\frac{h(\lambda)-h(p)}{p-\lambda}d\lambda+\int_{p+\eta}^{p+\epsilon}\frac{h(\lambda)-h(p)}{p-\lambda}d\lambda\Big] \\
=&\ \Big[\int_{p-\epsilon}^{p}\frac{h(\lambda)-h(p)}{p-\lambda}d\lambda+\int_{p}^{p+\epsilon}\frac{h(\lambda)-h(p)}{p-\lambda}d\lambda\Big] \\
=&\ \int_{p-\epsilon}^{p+\epsilon}\frac{h(\lambda)-h(p)}{p-\lambda}d\lambda,
\end{aligned}
$$

where in the third step we have used Lebesgue's dominated convergence theorem. □

We will need the following:

Theorem 5.6. *Let $h : \mathbb{R} \to \mathbb{C}$ be Lipschitz continuous in a neighborhood of p and integrable outside. Then*

$$\lim_{\delta\to 0^\pm}\int_{\mathbb{R}}\frac{h(\lambda)}{p-\lambda-i\delta}\,d\lambda=\mathcal{P}\!\int_{\mathbb{R}}\frac{h(\lambda)}{p-\lambda}\,d\lambda\pm i\pi h(p). \tag{5.5}$$

Proof. Write

$$\int_{\mathbb{R}}\frac{h(\lambda)}{p-\lambda-i\delta}\,d\lambda=\int_{\mathbb{R}}\frac{h(\lambda)-h(p)}{p-\lambda-i\delta}\,d\lambda+h(p)\int_{\mathbb{R}}\frac{1}{p-\lambda-i\delta}\,d\lambda$$

and use Lebesgue's dominated convergence theorem and lemma 5.5 on the first integral and Cauchy's integral formula on the second integral. □

5.3 The Hausdorff Measure

For reference see [31].

Definition 5.7.

(i) Let $A \subset \mathbb{R}^n,\ 0 \le s < \infty, 0 < \delta \le \infty$. Define

$$\mathcal{H}^s_\delta(A)\equiv\inf\left\{\sum_{j=1}^{\infty}\alpha(s)\left(\frac{diam\,C_j}{2}\right)^s \,|A\subset\bigcup_{j=1}^{\infty}C_j,\ diam\,C_j\le\delta\right\},$$

where

$$\alpha(s)\equiv\frac{\pi^{s/2}}{\Gamma(\frac{s}{2}+1)}.$$

Here $\Gamma(s)\equiv\int_0^\infty e^{-x}x^{s-1}\,dx$, $(0<s<\infty)$, is the usual gamma function.

(ii) For A and s as above, define

$$\mathcal{H}^s(A) \equiv \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

We call $\mathcal{H}^s$ s-dimensional Hausdorff measure *on $\mathbb{R}^n$.*

Theorem 5.8 (Properties of the Hausdorff measure)**.**

(a) $\mathcal{H}^s$ is a Borel regular measure $(0 \leq s < \infty)$.

(b) $\mathcal{H}^0$ is the counting measure.

(c) $\mathcal{H}^1 = \mathcal{L}^1$ (Lebesgue measure) on $\mathbb{R}^1$.

(d) $\mathcal{H}^s \equiv 0$ on $\mathbb{R}^n$ for all $s > n$.

(e) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for all $\lambda > 0$, $A \subset \mathbb{R}^n$.

(f) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ for each affine isometry $L : \mathbb{R}^n \to \mathbb{R}^n$, $A \subset \mathbb{R}^n$.

(g) $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.

Definition 5.9 (Hausdorff dimension)**.** *The* Hausdorff dimension *of a set $A \subset \mathbb{R}^n$ is defined to be*

$$\mathcal{H}_{dim}(A) \equiv \inf\{0 \leq s < \infty | \mathcal{H}^s(A) = 0\}.$$

5.4 The Area- and Coarea-Formula

For reference see [31], Theorem 2 in section 3.3.3 and proposition 3 in section 3.4.4.

Theorem 5.10 (Area formula)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each $\mathcal{L}^n$-integrable function $g : \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} g(x) Jf(x)\, dx = \int_{\mathbb{R}^m} \left[\sum_{x \in f^{-1}\{y\}} g(x) \right] d\mathcal{H}^n(y),$$

where Jf is the Jacobian of f in the sense of [31], section 3.2.2[1].

[1] i.e. $(Jf)^2 =$ sum of squares of the determinants of the $(n \times n)$-submatrices of Df, not to be confused with Clarke's generalized Jacobian ∂f of f or the Jacobian matrix Df of f.

Theorem 5.11 (Coarea formula)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz, with*

$$\operatorname{ess\,inf} |Df| > 0.$$

Suppose that $h : \mathbb{R}^d \to \mathbb{R}$ is integrable. Then

$$\int_{\{f>t\}} h \, dx = \int_t^\infty \left(\int_{\{f=s\}} \frac{h}{|Df|} \, d\mathcal{H}^{d-1} \right) \, ds.$$

5.5 A trace operator onto the diagonal for L^∞-valued Lipschitz functions

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be a Banach space. In our application in chapter 3 and 4, $\mathcal{B}$ will be the weighted space $L^2(\mathbb{R}^d, \mathbb{C}^M; w(x)dx)$. Let $I \subseteq \mathbb{R}$ a compact interval. We want to define a trace operator onto the diagonal for functions that are Lipschitz continuous with values in $L^\infty(I, \mathcal{B})$, i.e. for $\lambda \mapsto f(\lambda, \cdot) \in L^\infty(I, \mathcal{B})$ continuous on I we want a reasonable definition for the function $\lambda \mapsto f(\lambda, \lambda) \in \mathcal{B}$.
Define the set of step functions on I with values in $L^\infty(I, \mathcal{B})$ by

$$\begin{aligned} \mathrm{SF}(I, L^\infty(I, \mathcal{B})) \;\; := \;\; & \Big\{ g : I \to L^\infty(I, \mathcal{B}) : \exists\, N \in \mathbb{N}, \text{ disjoint intervals } J_j \\ & \text{and } g_j \in L^\infty(I, \mathcal{B}) \text{ such that} \\ & I = \bigcup_{j \in \mathbb{N}}^{N} J_j \text{ up to a set of measure 0} \\ & \text{and } g(\lambda, \cdot) = \sum_{j=1}^{N} \chi_{J_j}(\lambda) g_j(\cdot) \Big\}. \end{aligned}$$

$\mathrm{SF}(I, L^\infty(I, \mathcal{B}))$ is a normed space with the norm from $L^\infty(I, L^\infty(I, \mathcal{B}))$.

Definition 5.12. *For $p \in [1, \infty]$, $g \in SF(I, L^\infty(I, \mathcal{B}))$, $g(\lambda, \mu) = \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\mu)$ define the trace operator onto the diagonal by*

$$\begin{cases} T : SF(I, L^\infty(I, \mathcal{B})) & \to \;\; L^p(I, \mathcal{B}) \\ \qquad\qquad\qquad g & \mapsto \;\; Tg \\ \qquad\qquad\quad Tg(\lambda) & = \;\; \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\lambda). \end{cases}$$

Lemma 5.13. *$T : SF(I, L^\infty(I,\mathcal{B})) \to L^p(I,\mathcal{B})$ is well defined and bounded, and since the closure of $SF(I, L^\infty(I,\mathcal{B}))$ with respect to the $L^\infty(I, L^\infty(I,\mathcal{B}))$-norm contains all continuous functions on I with values in $L^\infty(I,\mathcal{B})$ we have $\|Tf\|_{L^p(I,\mathcal{B})} \le C_p \|f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}$ for all $f \in C(I, L^\infty(I,\mathcal{B}))$.*

Proof. Let $g(\lambda,\mu) = \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\mu) = \sum_{j=1}^M \chi_{K_j}(\lambda) h_j(\mu)$ and

$$T_1 g(\lambda) = \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\lambda), \qquad T_2 g(\lambda) = \sum_{j=1}^M \chi_{K_j}(\lambda) h_j(\lambda).$$

Consider the set $H_{ij} = \{J_i \cap K_j : 1 \le i \le N, 1 \le j \le M\}$. Then

$$\bigcup_{1\le i\le N, 1\le j\le M} H_{ij} = I \qquad \text{up to a set of measure 0}$$

and $T_1 g = T_2 g$ on each H_{ij}. Therefore T is well defined. Concerning the boundedness we first see that

$$\begin{aligned}\|g\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} &= \sup_{\lambda\in I} \|g(\lambda,\cdot)\|_{L^\infty(I,\mathcal{B})} = \sup_{\lambda\in I} \left\| \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\cdot) \right\|_{L^\infty(I,\mathcal{B})} \\ &= \max_{1\le j\le N} \|g_j\|_{L^\infty(I,\mathcal{B})}.\end{aligned}$$

For $p \in [1,\infty)$ we have

$$\begin{aligned}\|Tg\|^p_{L^p(I,\mathcal{B})} &= \int_I \|Tg(\lambda)\|^p_{\mathcal{B}}\, d\lambda = \int_I \left\| \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\lambda) \right\|^p_{\mathcal{B}} d\lambda \\ &= \sum_{j=1}^N \int_{J_j} \|g_j(\lambda)\|^p_{\mathcal{B}}\, d\lambda \\ &\le \sum_{j=1}^N \int_{J_j} \Big(\max_{1\le j\le N} \|g_j\|_{L^\infty(I,\mathcal{B})} \Big)^p d\lambda = |I|\, \|g\|^p_{L^\infty(I,L^\infty(I,\mathcal{B}))}.\end{aligned}$$

For $p = \infty$ we have

$$\begin{aligned}\|Tg\|_{L^\infty(I,\mathcal{B})} &= \sup_{\lambda\in I} \|Tg(\lambda)\|_{\mathcal{B}} = \sup_{\lambda\in I} \left\| \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\lambda) \right\|_{\mathcal{B}} \\ &\le \sup_{\lambda\in I} \left\| \sum_{j=1}^N \chi_{J_j}(\lambda) g_j \right\|_{L^\infty(I,\mathcal{B})} = \|g\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}.\end{aligned}$$

Putting $\frac{1}{\infty} = 0$ we conclude for all $p \in [1, \infty]$ and $g \in \mathrm{SF}(I, L^\infty(I, \mathcal{B}))$

$$\|Tg\|^p_{L^p(I,\mathcal{B})} \leq |I|^{1/p} \|g\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} . \tag{5.6}$$

Now let $f \in C(I, L^\infty(I, \mathcal{B}))$ and $\varepsilon > 0$ fixed. Since I is compact, f is uniformly continuous. Choose $N \in \mathbb{N}$ so big that $\|f(\lambda, \cdot) - f(\mu, \cdot)\|_{L^\infty(I,\mathcal{B})} \leq \varepsilon$ whenever $|\lambda - \mu| \leq \frac{1}{N}$. Let $g_k(\cdot) = f(\frac{k}{N}, \cdot)$ and $g_\varepsilon(\lambda, \cdot) := \sum_{k=1}^N \chi_{(\frac{k-1}{N}, \frac{k}{N})}(\lambda) g_k(\cdot)$. Then for almost all $\lambda \in I$ we have ($\lambda \in (\frac{k-1}{N}, \frac{k}{N})$ for some $k = k(\lambda)$)

$$\begin{aligned}
\|f(\lambda, \cdot) - g_\varepsilon(\lambda, \cdot)\|_{L^\infty(I,\mathcal{B})} &= \left\| f(\lambda, \cdot) - \sum_{k=1}^N \chi_{(\frac{k-1}{N}, \frac{k}{N})}(\lambda) g_k(\cdot) \right\|_{L^\infty(I,\mathcal{B})} \\
&= \left\| \sum_{k=1}^N \chi_{(\frac{k-1}{N}, \frac{k}{N})}(\lambda) \Big(f(\lambda, \cdot) - g_k(\cdot) \Big) \right\|_{L^\infty(I,\mathcal{B})} \\
&= \left\| \chi_{(\frac{k(\lambda)-1}{N}, \frac{k(\lambda)}{N})}(\lambda) \Big(f(\lambda, \cdot) - f(\frac{k(\lambda)}{N}, \cdot) \Big) \right\|_{L^\infty(I,\mathcal{B})} \\
&\leq \varepsilon
\end{aligned}$$

and thus we obtain $\|f - g_\varepsilon\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} \leq \varepsilon$ and so (5.6) also holds for $f \in C(I, L^\infty(I, \mathcal{B}))$. □

Definition 5.14. *For $I_0 \subset\subset I \backslash \partial I$, $p \in [1, \infty]$, $g \in SF(I, L^\infty(I, \mathcal{B}))$, $g(\lambda, \cdot) = \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\cdot)$ and all $h \geq 0$ so small that $I_0 - h \subseteq I$ define*

$$\begin{cases} T_h : SF(I, L^\infty(I, \mathcal{B})) & \rightarrow \quad L^p(I_0, \mathcal{B}) \\ \qquad\qquad T_h g(\lambda) & = \quad \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\lambda - h). \end{cases}$$

If h is so big that $I_0 - h \nsubseteq I$ then put $T_h g = 0$.

Lemma 5.15. *For fixed I_0, $h \geq 0$ small enough, T_h is well defined on $SF(I, L^\infty(I, \mathcal{B}))$ and bounded with norm independent of h. Moreover*

$$\|T_h f\|_{L^p(I_0,\mathcal{B})} \leq C_p \|f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}$$

and in the case $p \neq \infty$, $\|T_h f - T f\|_{L^p(I_0,\mathcal{B})} \overset{h\to 0}{\longrightarrow} 0$ for all $f \in C(I, L^\infty(I, \mathcal{B}))$.

Proof. If $p \in [1, \infty)$ we have

$$\|T_h g\|^p_{L^p(I_0,\mathcal{B})} = \int_{I_0} \|T_h g(\lambda)\|^p_{\mathcal{B}} \, d\lambda = \int_{I_0} \left\| \sum_{j=1}^N \chi_{J_j}(\lambda) g_j(\lambda - h) \right\|^p_{\mathcal{B}} d\lambda$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \int_{J_j \cap I_0} \|g_j(\lambda - h)\|_{\mathcal{B}}^p \, d\lambda \\
&\le \sum_{j=1}^{N} \int_{J_j \cap I_0} \Big(\max_{1 \le j \le N} \|g_j\|_{L^\infty(I,\mathcal{B})} \Big)^p d\lambda = |I_0| \, \|g\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}^p .
\end{aligned}
$$

For $p = \infty$ we have

$$
\begin{aligned}
\|T_h g\|_{L^\infty(I_0,\mathcal{B})} &= \sup_{\lambda \in I_0} \|T_h g(\lambda)\|_{\mathcal{B}} = \sup_{\lambda \in I_0} \left\| \sum_{j=1}^{N} \chi_{J_j \cap I_0}(\lambda) g_j(\lambda) \right\|_{\mathcal{B}} \\
&\le \sup_{\lambda \in I_0} \left\| \sum_{j=1}^{N} \chi_{J_j}(\lambda) g_j \right\|_{L^\infty(I,\mathcal{B})} \le \|g\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} .
\end{aligned}
$$

As in the preceding lemma we obtain for all $f \in C(I, L^\infty(I,\mathcal{B}))$

$$
\|T_h f\|_{L^p(I_0,\mathcal{B})}^p \le |I_0|^{1/p} \, \|f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} . \tag{5.7}
$$

For the last statement fix $\varepsilon > 0$ and choose a $g_\varepsilon \in \mathrm{SF}(I, L^\infty(I,\mathcal{B}))$,

$$
g(\lambda,\mu) = \sum_{j=1}^{N} \chi_{J_j}(\lambda) g_j^{(\varepsilon)}(\mu),
$$

$N = N(\varepsilon)$, $J_j = J_j(\varepsilon)$, such that $\|g_\varepsilon - f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} \le \varepsilon$. Then for fixed $\eta = \eta(\varepsilon) > 0$ choose a $h_{\varepsilon,\eta} \in \mathrm{SF}(I, C(I,\mathcal{B}))$,

$$
h_{\varepsilon,\eta}(\lambda,\mu) = \sum_{j=1}^{N} \chi_{J_j}(\lambda) h_j^{(\varepsilon,\eta)}(\mu),
$$

such that $\|g_\varepsilon - h_{\varepsilon,\eta}\|_{L^\infty(I,L^p(I,\mathcal{B}))} \le \eta$ (recall $p \ne \infty$). Then for almost all $\lambda \in I$ we have $\|h_{\varepsilon,\eta}(\lambda,\lambda - h) - h_{\varepsilon,\eta}(\lambda,\lambda)\|_{\mathcal{B}}^p \xrightarrow{h\to 0} 0$ and

$$
\|h_{\varepsilon,\eta}(\lambda,\lambda - h) - h_{\varepsilon,\eta}(\lambda,\lambda)\|_{\mathcal{B}}^p \le 2^p \, \|h_{\varepsilon,\eta}\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}^p
$$

and so by the Lebesgue dominated convergence $\|h_{\varepsilon,\eta}(\cdot,\cdot - h) - h_{\varepsilon,\eta}(\cdot,\cdot)\|_{L^p(I_0,\mathcal{B})} \xrightarrow{h\to 0} 0$. Moreover

$$
\|h_{\varepsilon,\eta} - g\|_{L^\infty(I,L^p(I))}^p = \sup_{\lambda \in I} \int_I \left\| \sum_{j=1}^{N} \chi_{J_j}(\lambda) \big(h_j^{(\varepsilon,\eta)}(\mu) - g_j^{(\varepsilon)}(\mu) \big) \right\|_{\mathcal{B}}^p d\mu
$$

$$
\begin{aligned}
&= \max_{1\le j\le N} \int_I \left\| h_j^{(\varepsilon,\eta)}(\mu) - g_j^{(\varepsilon)}(\mu)\right\|_{\mathcal{B}}^p d\mu, \\
\|Th_{\varepsilon,\eta} - Tg_\varepsilon\|_{L^p(I_0,\mathcal{B})}^p &= \sum_{j=1}^N \int_{J_j\cap I_0} \left\| h_j^{(\varepsilon,\eta)}(\mu) - g_j^{(\varepsilon)}(\mu)\right\|_{\mathcal{B}}^p d\mu \\
&\le N \max_{1\le j\le N} \int_I \left\| h_j^{(\varepsilon,\eta)}(\mu) - g_j^{(\varepsilon)}(\mu)\right\|_{\mathcal{B}}^p d\mu \\
&= N \|h_{\varepsilon,\eta} - g\|_{L^\infty(I,L^p(I))}^p, \\
\|T_h h_{\varepsilon,\eta} - T_h g_\varepsilon\|_{L^p(I_0,\mathcal{B})}^p &= \sum_{j=1}^N \int_{J_j\cap I_0} \left\| h_j^{(\varepsilon,\eta)}(\mu-h) - g_j^{(\varepsilon)}(\mu-h)\right\|_{\mathcal{B}}^p d\mu \\
&\le N \max_{1\le j\le N} \int_I \left\| h_j^{(\varepsilon,\eta)}(\mu) - g_j^{(\varepsilon)}(\mu)\right\|_{\mathcal{B}}^p d\mu \\
&= N \|h_{\varepsilon,\eta} - g\|_{L^\infty(I,L^p(I))}^p.
\end{aligned}
$$

Now we are prepared for the final estimation

$$
\begin{aligned}
\|T_h f - Tf\|_{L^p(I_0,\mathcal{B})} &\le \|T_h f - T_h g_\varepsilon\|_{L^p(I_0,\mathcal{B})} + \|T_h g_\varepsilon - Tg_\varepsilon\|_{L^p(I_0,\mathcal{B})} \\
&\quad + \|Tg_\varepsilon - Tf\|_{L^p(I_0,\mathcal{B})} \\
&\le 2|I_0|^{1/p} \|g_\varepsilon - f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} \\
&\quad + \|T_h g_\varepsilon - T_h h_{\varepsilon,\eta}\|_{L^p(I_0,\mathcal{B})} + \|T_h h_{\varepsilon,\eta} - Th_{\varepsilon,\eta}\|_{L^p(I_0,\mathcal{B})} \\
&\quad + \|Th_{\varepsilon,\eta} - Tg_\varepsilon\|_{L^p(I_0,\mathcal{B})} \\
&\le 2|I_0|^{1/p} \|g_\varepsilon - f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} + 2N \|h_{\varepsilon,\eta} - g_\varepsilon\|_{L^\infty(I,L^p(I,\mathcal{B}))} \\
&\quad + \|h_{\varepsilon,\eta}(\lambda,\lambda-h) - h_{\varepsilon,\eta}(\lambda,\lambda)\|_{(L^p(I_0,\mathcal{B}),d\lambda)} \\
&\le 2|I_0|^{1/p}\varepsilon + 2N\eta + \|h_{\varepsilon,\eta}(\lambda,\lambda-h) - h_{\varepsilon,\eta}(\lambda,\lambda)\|_{(L^p(I_0,\mathcal{B}),d\lambda)} \\
&\xrightarrow{h\to 0} 2|I_0|^{1/p}\varepsilon + 2N\eta,
\end{aligned}
$$

and so $\limsup_{h\to 0} \|T_h f - Tf\|_{L^p(I_0,\mathcal{B})} \le 2|I_0|^{1/p}\varepsilon + 2N\eta$ for any $\eta > 0$ and ε fixed.
It follows $\limsup_{h\to 0} \|T_h f - Tf\|_{L^p(I_0,\mathcal{B})} \le 2|I_0|^{1/p}\varepsilon$ for any ε and finally

$$
\lim_{h\to 0} \|T_h f - Tf\|_{L^p(I_0,\mathcal{B})} = 0.
$$

□

Let $I \subseteq \mathbb{R}$ be a compact interval, $I_0 \subset\subset I \setminus \partial I$ and $\lambda \mapsto f(\lambda,\cdot) \in L^\infty(I,\mathcal{B})$ continuous on I. For $p \in [1,\infty)$ we extend $f(\lambda,\cdot)$ to $L^p(\mathbb{R},\mathcal{B})$ by 0 outside of I. Moreover $f(\lambda,\cdot) = 0$ if

$\lambda \notin I$. By abuse of notation we write for $\lambda \in I$

$$(k_\varepsilon * f)(\lambda) = \int_{\mathbb{R}} k_\varepsilon(\lambda - \tau) f(\lambda, \tau) d\tau,$$

where k_ε is an approximate identity (see [41]).

Lemma 5.16. *For $f \in C(I, L^\infty(I, \mathcal{B}))$ and any $I_0 \subset\subset I \setminus \partial I$ we have*

$$k_\varepsilon * f - Tf \overset{\varepsilon \to 0}{\longrightarrow} 0$$

in $L^p(I_0, \mathcal{B})$.

Proof. Let $\varepsilon > 0$ fixed. First we prove that

$$\left\| \int_{\mathbb{R}} k_\varepsilon(\tau) \left(f(\lambda, \lambda - \tau) - T_\tau f(\lambda) \right) d\tau \right\|_{(L^p(I_0, \mathcal{B}), d\lambda)} \overset{\varepsilon \to 0}{\longrightarrow} 0.$$

Let $g_\eta \in \mathrm{SF}(I, L^\infty(I, \mathcal{B}))$ with $\|g_\eta - f\|_{L^\infty(I, L^\infty(I, \mathcal{B}))} \leq \eta$. Then there exists a set $M_1 \subseteq I_0$ with $|I_0 \setminus M_1| = 0$ such that $\lambda \in M_1 \Rightarrow \|g_\eta(\lambda, \cdot) - f(\lambda, \cdot)\|_{L^\infty(I, \mathcal{B})} \leq \eta$. Then for all $\lambda \in M_1$ there exists a set $M_2(\lambda) \subseteq I$ such that $|I \setminus M_2(\lambda)| = 0$ and $\mu \in M_2(\lambda) \Rightarrow \|g_\eta(\lambda, \mu) - f(\lambda, \mu)\|_{\mathcal{B}} \leq \eta$. Let $M_3(\lambda) = \lambda - M_2(\lambda)$. Then $M_3(\lambda) \subseteq \lambda - I$ and $|(\lambda - I) \setminus M_3(\lambda)| = 0$. For any $\mu \in M_2(\lambda)$ there exists a $\tau \in M_3(\lambda)$ such that $\mu = \lambda - \tau$ and we have $\lambda \in M_1, \tau \in M_3(\lambda) \Rightarrow \|g_\eta(\lambda, \lambda - \tau) - f(\lambda, \lambda - \tau)\|_{\mathcal{B}} \leq \eta$.

$$\begin{aligned}
& \left\| \int_{\mathbb{R}} k_\varepsilon(\tau) \left(f(\lambda, \lambda - \tau) - T_\tau f(\lambda) \right) d\tau \right\|_{(L^p(I_0, \mathcal{B}), d\lambda)} \\
= & \int_{M_1} \left\| \int_{\mathbb{R}} k_\varepsilon(\tau) \left(f(\lambda, \lambda - \tau) - T_\tau f(\lambda) \right) d\tau \right\|_{\mathcal{B}} d\lambda \\
= & \int_{M_1} \left\| \int_{M_3(\lambda)} k_\varepsilon(\tau) \left(f(\lambda, \lambda - \tau) - T_\tau f(\lambda) \right) d\tau \right\|_{\mathcal{B}} d\lambda \\
\leq & \int_{M_1} \int_{M_3(\lambda)} |k_\varepsilon(\tau)| \, \|f(\lambda, \lambda - \tau) - T_\tau f(\lambda)\|_{\mathcal{B}} \, d\tau d\lambda \\
\leq & \int_{M_1} \int_{M_3(\lambda)} |k_\varepsilon(\tau)| \, \|f(\lambda, \lambda - \tau) - g_\eta(\lambda, \lambda - \tau)\|_{\mathcal{B}} \\
& + |k_\varepsilon(\tau)| \, \|g_\eta(\lambda, \lambda - \tau) - T_\tau f(\lambda)\|_{\mathcal{B}} \, d\tau d\lambda \\
\leq & \int_{M_1} \int_{M_3(\lambda)} |k_\varepsilon(\tau)| \left(\eta + \|T_\tau(g_\eta - f)(\lambda)\|_{\mathcal{B}} \right) d\tau d\lambda
\end{aligned}$$

$$
\begin{aligned}
&\le \int_{M_1}\int_{M_3(\lambda)} |k_\varepsilon(\tau)| \left(\eta + \|T_\tau(g_\eta - f)\|_{L^\infty(I_0,\mathcal{B})}\right) d\tau d\lambda \\
&\le \int_{M_1}\int_{M_3(\lambda)} |k_\varepsilon(\tau)| \left(\eta + \|g_\eta - f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}\right) d\tau d\lambda \qquad \text{(by lemma 5.15)} \\
&\le 2\eta I_0 c,
\end{aligned}
$$

where $c > 0$ is a constant such that $\|k_\varepsilon\|_{L^1(\mathbb{R})} \le c$ for all $\varepsilon > 0$. The remainder of the proof is like in [41], theorem 1.2.19. Let $\delta > 0$. Since $\|T_\tau f - Tf\|_{L^p(I_0,\mathcal{B})} \overset{\tau\to 0}{\longrightarrow} 0$ by lemma 5.15 there exists a neighborhood V_δ of 0 such that for all $\tau \in V_\delta$

$$
\|T_\tau f - Tf\|_{L^p(I_0,\mathcal{B})} \le \frac{\delta}{2c}.
$$

Since k_ε has integral one we have

$$
\begin{aligned}
&k_\varepsilon(\lambda) * f - Tf(\lambda) \\
&= \int_{\mathbb{R}} k_\varepsilon(\lambda-\tau)\chi_I(\tau) f(\lambda,\tau) d\tau - \int_{\mathbb{R}} k_\varepsilon(\tau) Tf(\lambda) d\tau \\
&= \int_{\mathbb{R}} k_\varepsilon(\tau) f(\lambda,\lambda-\tau) d\tau - \int_{\mathbb{R}} k_\varepsilon(\tau) Tf(\lambda) d\tau \\
&= \int_{\mathbb{R}} k_\varepsilon(\tau)\big(f(\lambda,\lambda-\tau) - Tf(\lambda)\big) d\tau \\
&= \int_{V_\delta} k_\varepsilon(\tau)\big(T_\tau f(\lambda) - Tf(\lambda)\big) d\tau + \int_{\mathbb{R}\setminus V_\delta} k_\varepsilon(\tau)\big(T_\tau f(\lambda) - Tf(\lambda)\big) d\tau.
\end{aligned}
$$

Now we take the $L^p(I_0,\mathcal{B})$-norm with respect to λ.

$$
\begin{aligned}
&\left\| \int_{V_\delta} k_\varepsilon(\tau)\big(T_\tau f(\lambda) - Tf(\lambda)\big) d\tau \right\|_{(L^p(I_0,\mathcal{B}),d\lambda)} \\
&\le \int_{V_\delta} |k_\varepsilon(\tau)| \, \|T_\tau f(\lambda) - Tf(\lambda)\|_{(L^p(I_0,\mathcal{B}),d\lambda)} d\tau \\
&\le \int_{V_\delta} |k_\varepsilon(\tau)| \frac{\delta}{2c} d\tau \le \frac{\delta}{2}
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\| \int_{\mathbb{R}\setminus V_\delta} k_\varepsilon(\tau)\big(T_\tau f(\lambda) - Tf(\lambda)\big) d\tau \right\|_{(L^p(I_0,\mathcal{B}),d\lambda)} \\
&\le \int_{\mathbb{R}\setminus V_\delta} |k_\varepsilon(\tau)| \, \|T_\tau f(\lambda) - Tf(\lambda)\|_{(L^p(I_0,\mathcal{B}),d\lambda)} d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}\setminus V_\delta} |k_\varepsilon(\tau)|(|I|^{1/p}+|I_0|^{1/p})\,\|f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}\,d\tau \\
&\le \frac{\delta}{2},
\end{aligned}
$$

provided that

$$
\int_{\mathbb{R}\setminus V_\delta} |k_\varepsilon(\tau)|d\tau \le \frac{\delta}{2(|I|^{1/p}+|I_0|^{1/p})\,\|f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}}.
$$

This can be achieved by choosing ε small enough, since k_ε is an approximate identity. So $\|k_\varepsilon * f - Tf\|_{L^p(I_0,\mathcal{B})} \le \delta$ for any $\varepsilon > 0$ small enough and

$$
\limsup_{\varepsilon\to 0} \|k_\varepsilon * f - Tf\|_{L^p(I_0,\mathcal{B})} \le \delta
$$

follows. Since $\delta > 0$ was arbitrary, the assertion follows. □

Suppose that $\lambda \mapsto f(\lambda,\cdot) \in L^\infty(I,\mathcal{B})$ is Lipschitz continuous on I and let

$$
\mathrm{Lip}(f)_I := \inf\left\{C > 0 : \|f(\lambda,\cdot) - f(\mu,\cdot)\|_{L^\infty(I,\mathcal{B})} \le C|\lambda-\mu|,\ \lambda,\mu \in I\right\}.
$$

Then we have the following

Lemma 5.17. *For all $\lambda \in I$ and almost all $\tau \in I$ we have*

$$
\|f(\lambda,\tau) - Tf(\tau)\|_{\mathcal{B}} \le \mathit{Lip}(f)_I|\lambda-\tau|.
$$

Proof. Let $\lambda \in I$ be fixed. Then we can choose a sequence $g_\varepsilon \in \mathrm{SF}(I, L^\infty(I,\mathcal{B}))$, $g_\varepsilon(\mu,\cdot) = \sum_{j=1}^{N(\varepsilon)} \chi_{J_j^{(\varepsilon)}}(\mu) g_j^{(\varepsilon)}$, such that for all $\varepsilon > 0$ the fixed λ is in the interior of one of the $J_j^{(\varepsilon)}$ and $\|g_\varepsilon - f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))} \le \varepsilon$. Then for almost all $\tau \in I_0$

$$
\begin{aligned}
&\|f(\lambda,\tau) - Tf(\tau)\|_{\mathcal{B}} \\
\le\ & \|f(\lambda,\tau) - g_\varepsilon(\tau,\tau)\|_{\mathcal{B}} + \|g_\varepsilon(\tau,\tau) - Tf(\tau)\|_{\mathcal{B}} \\
=\ & \left\|\sum_{j=1}^{N(\varepsilon)} \chi_{J_j^{(\varepsilon)}}(\tau)\left(f(\lambda,\tau) - g_j^{(\varepsilon)}(\tau)\right)\right\|_{\mathcal{B}} + \|T(g_\varepsilon - f)(\tau)\|_{\mathcal{B}} \\
\le\ & \sum_{j=1}^{N(\varepsilon)} \chi_{J_j^{(\varepsilon)}}(\tau)\sup_{\nu\in I}\left\|f(\lambda,\nu) - g_j^{(\varepsilon)}(\nu)\right\|_{\mathcal{B}} + \sup_{\nu\in I}\|T(g_\varepsilon - f)(\nu)\|_{\mathcal{B}} \\
=\ & \sup_{\nu\in I}\sum_{j=1}^{N(\varepsilon)} \chi_{J_j^{(\varepsilon)}}(\tau)\left\|f(\lambda,\nu) - g_j^{(\varepsilon)}(\nu)\right\|_{\mathcal{B}} + \|T(g_\varepsilon - f)\|_{L^\infty(I,\mathcal{B})}
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{\nu \in I} \left\| \sum_{j=1}^{N(\varepsilon)} \chi_{J_j^{(\varepsilon)}}(\tau) \left(f(\lambda,\nu) - g_j^{(\varepsilon)}(\nu) \right) \right\|_{\mathcal{B}} + \|T(g_\varepsilon - f)\|_{L^\infty(I,\mathcal{B})} \\
&= \|f(\lambda,\cdot) - g_\varepsilon(\tau,\cdot)\|_{L^\infty(I,\mathcal{B})} + \|T(g_\varepsilon - f)\|_{L^\infty(I,\mathcal{B})} \\
&\leq \underbrace{\|f(\lambda,\cdot) - g_\varepsilon(\tau,\cdot)\|_{L^\infty(I,\mathcal{B})}}_{\stackrel{\varepsilon\to 0}{\longrightarrow}\|f(\lambda,\cdot)-f(\tau,\cdot)\|_{L^\infty(I,\mathcal{B})}} + \underbrace{\|g_\varepsilon - f\|_{L^\infty(I,L^\infty(I,\mathcal{B}))}}_{\stackrel{\varepsilon\to 0}{\longrightarrow}0} \\
&\stackrel{\varepsilon\to 0}{\longrightarrow} \|f(\lambda,\cdot) - f(\tau,\cdot)\|_{L^\infty(I,\mathcal{B})}.
\end{aligned}
$$

Hence we conclude

$$
\begin{aligned}
\|f(\lambda,\tau) - Tf(\tau)\|_{\mathcal{B}} &\leq \|f(\lambda,\cdot) - f(\tau,\cdot)\|_{L^\infty(I,\mathcal{B})} \\
&\leq \mathrm{Lip}(f)_I |\lambda - \tau|
\end{aligned}
$$

for all $\lambda \in I$ and almost all $\tau \in I$. □

5.6 A variant of the Hilbert transform

Let $f \in \mathrm{Lip}(I, L^\infty(I,\mathcal{B}))$. Recall that we put $f(\lambda,\cdot) = 0$ for $\lambda \notin I$ and both $f(\lambda,\tau) = 0$ and $Tf(\tau) = 0$ for $\tau \notin I$. Define the truncated Hilbert transform by

$$
H^{(\varepsilon)}(f)(\lambda) = \frac{1}{\pi} \int\limits_{|\lambda-\tau|\geq\varepsilon} \frac{f(\lambda,\tau)}{\lambda-\tau} d\tau.
$$

We can write the truncated Hilbert transform of f on I the following way

$$
\begin{aligned}
H^{(\varepsilon)}f(\lambda) &= \frac{1}{\pi} \int\limits_{|\lambda-\tau|\geq\varepsilon} \frac{f(\lambda,\tau)}{\lambda-\tau} d\tau = \chi_I(\lambda) \frac{1}{\pi} \int\limits_{|\lambda-\tau|\geq\varepsilon} \frac{f(\lambda,\tau)}{\lambda-\tau} d\tau \\
&= \chi_I(\lambda) \frac{1}{\pi} \int\limits_{|\lambda-\tau|\geq\varepsilon} \frac{f(\lambda,\tau) - Tf(\tau)}{\lambda-\tau} d\tau + \chi_I(\lambda) \frac{1}{\pi} \int\limits_{|\lambda-\tau|\geq\varepsilon} \frac{Tf(\tau)}{\lambda-\tau} d\tau \\
&=: \chi_I(\lambda) G^{(\varepsilon)} f(\lambda) + \chi_I(\lambda) H^{(\varepsilon)}(Tf)(\lambda). \qquad (5.8)
\end{aligned}
$$

Now for all $p \in (1,\infty)$ we know from theorem 4.1.12 in [41] that $H^{(\varepsilon)}(Tf) \stackrel{\varepsilon\to 0}{\longrightarrow} H(Tf)$ in $L^p(\mathbb{R},\mathcal{B})$ and a.e. since $Tf \in L^p(\mathbb{R},\mathcal{B})$. Moreover form theorem 4.1.7 in [41][2] we know $\|H(Tf)\|_{L^p(\mathbb{R},\mathcal{B})} \leq C_p \|Tf\|_{L^p(\mathbb{R},\mathcal{B})}$.

[2]The cited theorem is proved for real valued functions. For $\mathcal{B}$-valued functions take section 4.6 in [41] into account.

Lemma 5.18. *For $p \in [1,\infty]$, $\chi_I G^{(\varepsilon)} f - Gf \overset{\varepsilon\to 0}{\longrightarrow} 0$ in $L^p(I,\mathcal{B})$ where*

$$Gf(\lambda) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\chi_I(\lambda)\,(f(\lambda,\tau) - Tf(\tau))}{\lambda-\tau} d\tau.$$

Moreover $\|Gf\|_{L^p(I,\mathcal{B})} \le C_p Lip(f)_I$.

Proof. Let $p \in [1,\infty)$

$$\begin{aligned}
\left\|\chi_I G^{(\varepsilon)} f - Gf\right\|^p_{L^p(I,\mathcal{B})} &= \int_I \left\|G^{(\varepsilon)} f(\lambda) - Gf(\lambda)\right\|^p_{\mathcal{B}} d\lambda \\
&= \int_I \left\| \frac{1}{\pi}\int_{|\lambda-\tau|\le\varepsilon} \frac{f(\lambda,\tau) - Tf(\tau)}{\lambda-\tau} d\tau \right\|^p_{\mathcal{B}} d\lambda \\
&\le \int_I \left(\frac{1}{\pi}\int_{|\lambda-\tau|\le\varepsilon} \frac{\|f(\lambda,\tau) - Tf(\tau)\|_{\mathcal{B}}}{|\lambda-\tau|} d\tau \right)^p d\lambda \\
&\le \frac{1}{\pi}\mathrm{Lip}(f)^p_I 2^p \varepsilon^p I \overset{\varepsilon\to 0}{\longrightarrow} 0
\end{aligned}$$

using lemma 5.17. For $p = \infty$ we have

$$\begin{aligned}
\left\|\chi_I G^{(\varepsilon)} f - Gf\right\|^p_{L^\infty(I,\mathcal{B})} &= \sup_{\lambda\in I} \left\|G^{(\varepsilon)} f(\lambda) - Gf(\lambda)\right\|_{\mathcal{B}} \\
&= \sup_{\lambda\in I} \left\| \frac{1}{\pi}\int_{|\lambda-\tau|\le\varepsilon} \frac{f(\lambda,\tau) - Tf(\tau)}{\lambda-\tau} d\tau \right\|_{\mathcal{B}} \\
&\le \frac{1}{\pi}\mathrm{Lip}(f)_I 2\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}$$

Moreover

$$\|Gf\|_{L^p(I,\mathcal{B})} \le \begin{cases} \frac{1}{\pi}\mathrm{Lip}(f)_I |I|^{(p+1)/p} & \text{if } p \ne \infty \\ \frac{1}{\pi}\mathrm{Lip}(f)_I |I| & \text{if } p = \infty. \end{cases}$$

□

Definition 5.19. *We define the Hilbert transform of $f \in \mathrm{Lip}(I, L^\infty(I,\mathcal{B}))$ on I by*

$$Hf(\lambda) := \chi_I(\lambda) Gf(\lambda) + \chi_I(\lambda) H(Tf)(\lambda).$$

5.7 List of Symbols

Ω	$\Omega = [0,1]^d$, d-dimensional cell of periodicity
B	$B = [-\pi,\pi]^d$, d-dimensional Brillouin zone
χ_A	characteristic function of the set A
x	space variable, $x \in \mathbb{R}^d$
k	wave vector variable, $k \in B$
λ, τ	frequency variables, $\lambda, \tau \in \mathbb{R}$
ω	frequency variable, $\omega \in \mathbb{R}$, $\omega^2 = \lambda$
$\mathcal{L}$	Floquet-Bloch decomposable operator
$\lambda_s(k)$	band function of $\mathcal{L}$
$\psi_s(x,k)$	Bloch wave of $\mathcal{L}$
U	Floquet-Bloch transform
$\Phi_{s,k}(\lambda)$	$\Phi_{s,k}(\lambda) = \sqrt{\lvert B\rvert}\,\overline{\langle U\overline{\varphi}(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle}_{L^2(\Omega,\mathbb{C}^M)}\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}$
$\mathrm{Lip}(f)_{\mathfrak{C}}$	local Lipschitz constant of f on $\mathfrak{C}$
$\lVert\varphi\rVert_\infty$	$\lVert\varphi\rVert_\infty = \sup_{\lambda\in\mathbb{R},\ x\in\mathbb{R}^d} \lvert\varphi(\lambda,x)\rvert$
B^s_τ	$B^s_\tau = \{k \in B:\ \lambda_s(k) = \tau\}$ in chapter 3, $B^s_\tau = \{k \in B:\ \sqrt{\lambda_s(k)} = \tau\}$ in chapter 4
I_0, I	intervals of regular frequencies
h_s	level set integral, $h_s(\lambda,\tau,x) = \int_{B^s_\tau} \frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle}{\lvert\nabla\lambda_s(k)\rvert}\, d\mathcal{H}^{d-1}(k)$ in chapter 3, $h_s(\lambda,\tau,x) = \int_{B^s_\tau} \frac{\langle Uf(\lambda,\cdot,k),\psi_s(\cdot,k)\rangle}{\lvert\nabla\sqrt{\lambda_s}(k)\rvert}\, d\mathcal{H}^{d-1}(k)$ in chapter 4
$U^{-1}_{B^s_\tau}$	inverse Floquet-Bloch transform restricted to B^s_τ
$\partial\lambda_s(k)$	generalized gradient of λ_s in k
$\mathcal{R}, \mathcal{R}_s$	regular values, regular values of λ_s
$w(x)$	weight function
$P_{s,k}$	$P_{s,k}[h](x) = \langle Uh(\cdot,k),\psi_s(\cdot,k)\rangle_{L^2(\Omega,\mathbb{C}^M)}\psi_s(x,k)$
$\mathcal{B}$	weighted L^2-space
H	Hilbert transform or variant of Hilbert transform
T	Trace operator on the diagonal/point evaluation operator
$\mathcal{F}$	Fourier transform

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Curriculum Vitae Auctoris

Maria Barbara Radosz, Dipl. Phys.
geboren am 10.10.1979 in Ratibor, Polen.

Eltern: Claudia Radosz, Christoph Radosz.

- 10.2010 - : Wissenschaftlicher Mitarbeiter am Institut für Analysis, Karlsruher Institut für Technologie
- 10.2006 - 7.2010: Doktorand am Karlsruher Institut für Technologie unter der Betreuung von Prof. Dr. Michael Plum im Rahmen des Graduiertenkollegs 1294 "Analysis, Simulation und Design nanotechnologischer Prozesse".
- 10.2000 - 9.2006: Studium der Physik an der Universität Karlsruhe.
  Diplomarbeit:*Untersuchung der Abbildungseigenschaften eines Auger-Fluoreszenz-teleskopes mit einer künstlichen UV-Lichtquelle*
- 10.2002 - 9.2006 Studium der Mathematik an der Universität Karlsruhe.
- 1991 - 2000: Besuch des Progymnasium Baiersbronn und Hebel-Gymnasium Pforzheim.