The Domain Derivative of Time-Harmonic Electromagnetic Waves at Interfaces

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Abstract: We consider the scattering of time harmonic electromagnetic waves by a penetrable obstacle. In view of shape optimization or inverse reconstruction problems the domain derivative of the scattering problem is investigated. Existence of the derivative in the sense of a Fréchet derivative and a characterization by a transmission boundary value problem are shown.

Keywords: Maxwell equations, transmission problem, shape derivative

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1 Introduction

The domain derivative of solutions of boundary value problems and corresponding functionals is used in shape optimization and inverse identification problems [18, 20, 21]. Besides the existence of such a derivative with respect to a given shape, a representation by corresponding boundary value problems is of vital importance for further analytic and numerical investigations. The problem of determining a derivative of a solution of a partial differential equation with respect to variations of a boundary with given boundary condition has a long history. For instance early investigations on elastic problems are due to Hadamard [8].

Several approaches have been established to investigate the dependence of a solution of a partial differential equation or a corresponding functional on perturbations of the underlying geometry. For many functionals characterizations of these derivatives for higher order are also given (e.g. [2, 6, 11]). But, in case of Maxwell's equations, only a few results based on integral equation methods are known [5, 15, 19].

In this paper we will present a general variational approach to show the existence and a representation of the domain derivative in case of electromagnetic scattering from a penetrable obstacle. The method is quite general, since it can be applied also to other interface and boundary value problems for Maxwell's equations, which for instance confirms the representation formula presented in [15]. The weak approach to show existence of the domain derivative for scattering problems was first applied in [12] to the exterior Dirichlet problem for the Helmholtz equation. The method can be extended to any elliptic partial differential equation, [9, 10]. However, in the electromagnetic case a regularity problem occurs, which causes basic problems in considering shape derivatives for Maxwell's equations. We will show that a curl conserving transformation solves this problem. An existence prove of the domain derivative of scattered electromagnetic waves by a penetrable obstacle given from different electric properties compared to its surrounding medium is presented. A characterization of the derivative with respect to variations of the scattering obstacle in terms of Maxwell equations is also presented. The use of such a characterization can be seen for instance from recent results applying iterative regularization schemes to the corresponding inverse scattering problem, [7].

For convenience of the reader, we start with a short discussion of the underlying weak formulation of the scattering problem in the Sobolev space $H(\mathrm{curl}\,;\Omega)$. The third section will lead us to the existence of the so-called material derivative, by using a curl preserving transformation of the fields scattered by a perturbed obstacle. Then, in the last section, under some smoothness assumptions on the boundary ∂D, we conclude a representation of the essential part of the material derivative, the domain derivative, as a radiating electromagnetic field satisfying certain transmission boundary conditions at the interface.

2 The scattering problem

Let $D \subseteq \mathbb{R}^3$ be an open bounded domain. We consider the scattering of time-harmonic electromagnetic plane waves from an obstacle described by D with electric conductivity $\sigma_D \geq 0$, an electric permittivity $\epsilon_D > 0$ and a magnetic permeability $\mu_D > 0$. The obstacle is included in another isotropic and homogenous medium for instance vacuum with constant electric parameter $\epsilon_0, \mu_0 > 0$ and $\sigma_0 \geq 0$. For simplicity we assume in this work that D consists of an isotropic, homogeneous medium. However, it will become obvious, how to extend the results to anisotropic, inhomogenous situations.

Assuming absence of additional charges the spatially dependent part of an electromagnetic field is described by a solution of the reduced Maxwell equations

$$\mathrm{curl}\, E - ik\mu_r H = 0, \quad \mathrm{curl}\, H + ik\epsilon_r E = 0 \quad \text{in } D \cup \mathbb{R}^3 \backslash \overline{D}\,, \tag{2.1}$$

where $k = \omega\sqrt{\epsilon_0\mu_0}$ denotes the wave number with frequency $\omega > 0$ and

$$\mu_r(x) = \begin{cases} \dfrac{\mu_D}{\mu_0}\,, & x \in D\,, \\ 1\,, & x \in \mathbb{R}\backslash\overline{D}\,, \end{cases}$$

and

$$\epsilon_r(x) = \begin{cases} \dfrac{1}{\epsilon_0}\left(\epsilon_D + \dfrac{i\sigma_D}{\omega}\right)\,, & x \in D\,, \\ 1\,, & x \in \mathbb{R}\backslash\overline{D}\,, \end{cases}$$

are the relative permeability and the relative permittivity respectively. Thus, the interface ∂D is characterized by discontinuities of the permittivity and/or the permeability.

The scattering object is illuminated by an incident plane wave given by

$$E^i(x) = pe^{ikd^\top x} \quad \text{and} \quad H^i(x) = (d \times p)e^{ikd^\top x} = \frac{1}{ik}\mathrm{curl}E^i(x), \quad x \in \mathbb{R}^3\,,$$

with direction $d \in S^2$ and polarization $p \in \mathbb{R}^3$. We have $d^\top p = 0$, which ensures that E^i and H^i satisfy the reduced Maxwell equations in $\mathbb{R}^3\backslash\overline{D}$. Furthermore, we assume the exterior $\mathbb{R}^3\backslash\overline{D}$ of the obstacle to be simply connected and the decompositions $E = E^i + E^s$ and $H = H^i + H^s$ in $\Omega\backslash\overline{D}$ hold with scattered fields E^s, H^s, which are radiating solutions of the reduced Maxwell equations, i.e. E^s and H^s satisfy the Silver-Müller radiation condition

$$\frac{x}{|x|} \times H^s(x) + E^s(x) = o\left(\frac{1}{|x|}\right)\,, \qquad |x| \to \infty \tag{2.2}$$

uniformly in $x/|x| \in S^2$.

At the interface ∂D the electric and the magnetic field satisfy the boundary conditions

$$\left.\begin{aligned} \big[\nu \times E\big]_\pm &= 0 \\ \big[\nu \times H\big]_\pm &= 0 \end{aligned}\right\} \text{ on } \partial D\,. \tag{2.3}$$

where ν denotes the outward directed unit normal to ∂D and by $[.]_\pm$ we denote the jump

$$[V]_\pm = \lim_{\substack{a\to 0 \\ x+a\in\Omega\backslash\overline{D}}} V(x+a) - \lim_{\substack{a\to 0 \\ x+a\in D}} V(x+a) \quad \text{for } x \in \partial D$$

of the continuous extension of a function V to the boundary from the exterior and the interior of D respectively. Assuming a negligible surface charge, the normal component of the electric and of the magnetic field satisfy

$$\left.\begin{array}{rcl}\left[\varepsilon_r\,\nu^\top E\right]_\pm &=& 0\\ \left[\mu_r\,\nu^\top H\right]_\pm &=& 0\end{array}\right\}\quad \text{on } \partial D\,, \tag{2.4}$$

Collecting (2.1)–(2.4) we obtain a complete description of the boundary value problem under consideration. For more details on electromagnetic scattering we refer to [3, 16, 17]. An existence proof of a unique solution of the scattering problem is given in [17] in the classical sense. A weak formulation in the Sobolev spaces $H^1(D) \times H^1_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{D})$ is presented in [1].

For our purpose a different weak formulation of the scattering problem is used as it is described in [16]. To begin with, we consider the vector valued version of Green's formula

$$(\operatorname{curl} U, V)_G - (U, \operatorname{curl} V)_G \;=\; \langle \nu \times U, V\rangle_{\partial G}$$

for sufficiently smooth functions U, V and domain G, where $(U,V) = \int_G U^\top \overline{V}\,dx$ denotes the $L^2(G)$ inner product and $\langle U, V\rangle_{\partial G} = \int_{\partial G} U^\top \overline{V}\,ds$ the inner product on the boundary ∂G.

Let Ω be a ball of radius $R > 0$ such that Ω contains the scattering object, $\overline{D} \subseteq \Omega$. Applying Green's formulae in D and in $\Omega\backslash\overline{D}$ and using Maxwell's equations together with the boundary condition 2.2 we obtain

$$\left(\frac{1}{\mu_r}\operatorname{curl} E, \operatorname{curl} V\right)_\Omega - k^2\,(\epsilon_r E, V)_\Omega = -\,\langle \nu \times \operatorname{curl} E, V\rangle_{\partial\Omega}$$

for any vector valued test function V with continuous tangential component $[\nu \times (V \times \nu)]_\pm = 0$ across the interface. Note that an analogous equation is satisfied by H if we exchange ϵ_r and μ_r.

We incorporate the radiation condition as a nonlocal boundary condition into this equation by the so-called Calderon operator Λ. It is defined by $\Lambda : \nu \times W \mapsto \nu \times \mathcal{H}^s$, where $\mathcal{E}^s, \mathcal{H}^s$ are the radiating electromagnetic fields satisfying the uniquely solvable exterior Maxwell problem

$$\operatorname{curl}\mathcal{E}^s - ik\mathcal{H}^s = 0, \quad \operatorname{curl}\mathcal{H}^s + ik\mathcal{E}^s = 0 \quad \text{in } \mathbb{R}^3\backslash\overline{\Omega}\,,$$

with boundary condition

$$\nu \times \mathcal{E}^s = \nu \times W \quad \text{on } \partial\Omega$$

and the Silver-Müller radiation condition (2.2). Thus the solution of the interface problem satisfies

$$\begin{aligned}\left(\frac{1}{\mu_r}\operatorname{curl} E, \operatorname{curl} V\right)_\Omega &- k^2\,(\epsilon_r E, V)_\Omega + ik\,\langle \Lambda(\nu \times E), V\rangle_{\partial\Omega} \\ &= \left\langle \left(ik\Lambda(\nu \times E^i) - \nu \times \operatorname{curl} E^i\right), V\right\rangle_{\partial\Omega}\end{aligned} \tag{2.5}$$

for test functions V.

On the other hand we observe equivalence of (2.5) and the transmission probelm (2.1)-(2.4) in the sense that a solution E for all $V \in C^\infty(\overline{\Omega})$ with regularity $E|_D \in C^2(D) \cap C^1(\overline{D})$ and $E_{\Omega\backslash\overline{D}} \in C^2(\Omega\backslash\overline{D}) \cap C^1(\overline{\Omega\backslash\overline{D}})$ satisfying $[\nu \times E]_\pm = 0$ at ∂D, can be extended by E and $H = \frac{1}{ik\mu_r}\operatorname{curl} E$ to a solution of the scattering problem. This is seen from appropriate test functions and the definition of the Calderon operator.

Equation (2.5) implies an appropriate Sobolev space for a weak existence theory of the scattering problem, which is

$$H(\operatorname{curl}; \Omega) = \left\{ V \in (L^2(\Omega))^3 : \operatorname{curl} V \in L^2(\Omega) \right\} .$$

It is known that for a bounded Lipschitz domain G the tangential traces

$$\gamma_t V = \nu \times V \quad \text{and} \quad \gamma_T V = \nu \times (V \times \nu)$$

of $V \in H(\operatorname{curl}; G)$ on ∂G exist by the duality

$$(\operatorname{curl} V, W)_G - (V, \operatorname{curl} W)_G = \langle \gamma_t(V), \gamma_T(W) \rangle_{\partial G}$$

for $V, W \in H(\operatorname{curl}; G)$. It turns out that the trace operators

$$\gamma_t : H(\operatorname{curl}; G) \to H^{-1/2}(\mathrm{Div}; \partial G) , \quad \gamma_T : H(\operatorname{curl}; G) \to H^{-1/2}(\mathrm{Curl} : \partial G)$$

are surjective on image spaces defined with the help of the corresponding extension of the surface divergence and surface curl by

$$H^{-1/2}(\mathrm{Div}; \partial G) = \left\{ V \in H^{-1/2}(\partial G) : V_\nu = 0 \text{ a.e. and } \mathrm{Div}\, V \in H^{-1/2}(\partial G) \right\}$$

and

$$\begin{aligned} H^{-1/2}(\mathrm{Curl}; \partial G) &= (H^{-1/2}(\mathrm{Div}; \partial G))^* \\ &= \{ V \in H^{-1/2}(\partial G) : V_\nu = 0 \text{ a.e. and } \mathrm{Curl}\, V \in H^{-1/2}(\partial G) \} \end{aligned}$$

(see [4]). By the bounded extension of the Calderon operator $\Lambda : H^{-1/2}(\mathrm{Div}; \partial\Omega) \to H^{-1/2}(\mathrm{Div}; \partial\Omega)$ (see [13, 16]) the weak formulation (2.5) is well defined in $H(\operatorname{curl}; \Omega)$.

As an abbreviation we define the left hand side of (2.5) by the sesquilinear form

$$\mathcal{L}(E, V) = \left(\frac{1}{\mu_r} \operatorname{curl} E, \operatorname{curl} V \right)_\Omega - k^2 (\epsilon_r\, E, V)_\Omega + ik \, \langle \Lambda(\nu \times E), V \rangle_{\partial\Omega}$$

on $H(\operatorname{curl};\Omega)$. Then, the scattering problem is defined by seeking a solution $E \in H(\operatorname{curl};\Omega)$ of the variational equation

$$\mathcal{L}(E,V) = F(\overline{V}) \quad \text{for all } V \in H(\operatorname{curl};\Omega)\,, \tag{2.6}$$

where the functional $F \in (H(\operatorname{curl};\Omega))^*$ is given by

$$F\left(\overline{V}\right) = \left\langle \left(ik\Lambda(\nu \times E^i) - \nu \times \operatorname{curl} E^i\right), V \right\rangle_{\partial\Omega}.$$

There exists a unique solution $E \in H(\operatorname{curl};\Omega)$ of the interface problem (2.6) with

$$\|E\|_{H(\operatorname{curl};\Omega)} \leq c\|F\|_{H(\operatorname{curl},\Omega)^*}$$

(see Theorem 10.7 in [16]).

3 The domain derivative of the scattered field

We investigate the dependency of the electric field on variations of the boundary ∂D. A perturbation of the interface will be described by a vector field $h \in C_0^1(\Omega)$ with support in a neighborhood of the interface ∂D. If for instance $\|h\|_{C^1} < \frac{1}{2}$, h is a contraction and defines a diffeomorphism $\varphi : \Omega \to \Omega$ by $\varphi(x) = x + h(x)$ on Ω. A perturbed scattering obstacle is given by $D_h = \varphi(D)$ with boundary $\partial D_h = \{\varphi(x) : x \in \partial D\}$. Additionally we denote by μ_r^h and ε_r^h the relative parameter for perturbed objects, where the discontinuity occurs at ∂D_h.

From section 2 we know that the scattered field with respect to the perturbed object D_h is given by the unique solution $E_h \in H(\operatorname{curl};\Omega)$ of

$$\left(\frac{1}{\mu_r^h}\operatorname{curl} E_h, \operatorname{curl} V\right)_\Omega - k^2(\epsilon_r^h\, E_h, V)_\Omega + ik\left\langle \Lambda(\nu \times E_h), V\right\rangle_{\partial\Omega} = F(\overline{V}) \tag{3.1}$$

for all $V \in H(\operatorname{curl};\Omega)$, where F is defined as in equation (2.6).

We consider a change of variables $x = \varphi(\tilde{x})$, which maps the interface ∂D onto the interface ∂D_h. Let $\tilde{E}_h = E_h \circ \varphi$, then some elementary calculations show that

$$\operatorname{curl} \tilde{E}_h = \det(J_\varphi)\, J_\varphi^{-1}\left((J_\varphi^{-1})^\top \otimes (J_{E_h} \circ \varphi)\right)$$

where J_V denotes the Jacobian matrix of a vector field $V : \mathbb{R}^3 \to \mathbb{R}^3$ and $\otimes$ the tensor product

$$A \otimes B = \sum_{l=1}^{3} \begin{pmatrix} a_{3l}b_{l2} - a_{2l}b_{l3} \\ a_{1l}b_{l3} - a_{3l}b_{l1} \\ a_{2l}b_{l1} - a_{1l}b_{l2} \end{pmatrix} = \left(\sum_{j,k=1}^{3} \epsilon_{ijk} \sum_{l=1}^{3} a_{kl}b_{lj}\right)_{i=1,2,3}$$

with the antisymmetric tensor ϵ_{ijk}. We observe that $E_h \in H(\mathrm{curl}\,;\Omega)$ does not imply $\tilde{E}_h \in H(\mathrm{curl}\,;\Omega)$. But, it is known that the transformation

$$\hat{E}_h = J_\varphi^\top \tilde{E}_h\,. \tag{3.2}$$

conserves curl (see Lemma 3.58 in [16]), i.e. we have $\hat{E}_h \in H(\mathrm{curl}\,,\Omega)$ if and only if $E_h \in H(\mathrm{curl}_\sim;\Omega)$. If we indicate by $\mathrm{curl}_\sim$ the differential operator with respect to the transformed variables, it holds the identity

$$(\mathrm{curl}_\sim E_h)\circ\varphi = \frac{1}{\det(J_\varphi)} J_\varphi\,\mathrm{curl}\,\hat{E}_h\,.$$

A change of variables, $\tilde{x} = \varphi(x)$, in (3.1) leads to

$$\begin{aligned}
&\int_\Omega \frac{1}{\tilde{\mu}_r^h}(\mathrm{curl}\,\hat{E}_h)^\top \left(\frac{1}{\det(J_\varphi)} J_\varphi^\top J_\varphi\right)\mathrm{curl}\,\overline{V} - k^2 \tilde{\varepsilon}_r^h \hat{E}_h^\top J_\varphi^{-1} J_\varphi^{-\top}\overline{V}\det(J_\varphi)\,dx \\
&\qquad + ik\int_{\partial\Omega}\Big(\Lambda(\nu\times\hat{E}_h)\Big)^\top\overline{V}\,ds \\
&= \int_{\partial\Omega}\big(ik\Lambda(\nu\times E^i) - \nu\times\mathrm{curl}\,E^i\big)^\top\overline{V}\,ds
\end{aligned}\tag{3.3}$$

for all $V \in H(\mathrm{curl}\,;\Omega)$. Note that h is supposed to be compactly supported in a neighborhood of ∂D.

In view of the investigation of the dependency on h we compute the linearizations of the Jacobian of φ first.

Lemma 3.1 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open domain and $\varphi \in C_0^1(\Omega)$ denote a variation of ∂D defined as above. Then it holds that*

$$\frac{1}{\|h\|_{C^1}}\left\|\frac{J_\varphi^\top J_\varphi}{\det(J_\varphi)} - \Big(1 - \mathrm{div}(h)I + J_h + J_h^\top\Big)\right\|_\infty \to 0$$

and

$$\frac{1}{\|h\|_{C^1}}\left\|J_\varphi^{-1}J_\varphi^{-\top}\det(J_\varphi) - \Big(1 + \mathrm{div}(h)I - J_h - J_h^\top\Big)\right\|_\infty \to 0$$

for $\|h\|_{C^1} \to 0$.

Proof: The estimations can be computed from the Taylor expansions

$$\det(J_\varphi) = 1 + \mathrm{div}\,h + o(\|h\|_{C^1})\,, \qquad \frac{1}{\det(J_\varphi)} = 1 - \mathrm{div}\,h + o(\|h\|_{C^1})\,,$$

and

$$J_\varphi = I + J_h\,, \qquad J_\varphi^{-1} = I - J_h + o(\|h\|_{C^1})\,.$$

□

With the help of the lemma we show continuous dependency of the scattered field on variations of the domain D.

Theorem 3.2 *If $E \in H(\mathrm{curl}\,;\Omega)$ denote the solution of (2.6) and $E_h \in H(\mathrm{curl}\,;\Omega)$ of (3.1), then*

$$\lim_{\|h\|_{C^1}\to 0} \|\hat{E}_h - E\|_{H(\mathrm{curl}\,,\Omega)} = 0\,,$$

where $\hat{E}_h$ is defined by (3.2).

Proof We define the sesquilinear form $\mathcal{L}_h(\hat{E}_h, V)$ by the left hand side of equation (3.3). The Riesz representation theorem implies existence of bounded linear mappings $T, T_h : H(\mathrm{curl}\,;\Omega) \to H(\mathrm{curl}\,;\Omega)$ with $(TU, V)_{H(\mathrm{curl}\,;\Omega)} = \mathcal{L}(U,V)$ and $(T_hU, V)_{H(\mathrm{curl}\,;\Omega)} = \mathcal{L}_h(U,V)$.

By Lemma 3.1 we obtain

$$\begin{aligned}
\|(T_h - T)w\|^2_{H(\mathrm{curl}\,,\Omega)} &= \mathcal{L}_h(w, (T_h-T)w) - \mathcal{L}(w, (T_h-T)w) \\
&= \int_\Omega (\mathrm{curl}\,w)^\top \left(\frac{1}{\tilde{\mu}_r^h \det(J_\varphi)} J_\varphi^\top J_\varphi - \frac{1}{\mu_r} I \right) \mathrm{curl}\,(T_h - T)w \\
&\qquad - k^2 w^\top \left(\tilde{\varepsilon}_r^h J_\varphi^{-1} J_\varphi^{-\top} \det(J_\varphi) - \varepsilon_r I \right) (T_h - T)w\,dx \\
&\leq C\|h\|_{C^1} \|w\|_{H(\mathrm{curl}\,;\Omega)} \|(T_h - T)w\|_{H(\mathrm{curl}\,;\Omega)}
\end{aligned}$$

with a constant $C > 0$. Thus, we have $\|T_h - T\| \to 0$ if $\|h\|_{C_1} \to 0$.

From section 2 we know that T has a bounded inverse. Thus with $T_h\hat{E}_h = F$ and $TE = F$ a perturbation argument shows $\|\hat{E}_h - E\|_{H(\mathrm{curl}\,,\Omega)} \to 0$ if $\|h\|_{C^1} \to 0$ (see for instance Theorem 10.1. in [14]). □

Considering the approximation of Lemma 3.1 more in detail leads to differentiability of E at $h = 0$ with respect to variations of the domain D.

Theorem 3.3 *Let $E \in H(\mathrm{curl}\,;\Omega)$ denote the solution of (2.6) and $E_h \in H(\mathrm{curl}\,;\Omega)$ of (3.1). Then there exists a bounded linear mapping $\mathcal{A} : C_0^1(\Omega) \to H(\mathrm{curl}\,;\Omega)$ such that*

$$\lim_{\|h\|_{C^1}\to 0} \frac{1}{\|h\|_{C^1}} \|\hat{E}_h - E - \mathcal{A}h\|_{H(\mathrm{curl}\,;\Omega)} = 0\,.$$

Proof: For $h \in C_0^1(\Omega)$ we define $W \in H(\mathrm{curl}\,;\Omega)$ by the unique solution of

$$\mathcal{L}(W,V) \;=\; \int_\Omega \Bigg[(\mathrm{curl}\,E)^\top \left(\frac{1}{\mu_r}\left(\mathrm{div}\,(h)I - J_h - J_h^\top\right)\right)\mathrm{curl}\,\overline{V} \\ - k^2 E^\top \left(\varepsilon_r\left(-\mathrm{div}\,(h)I + J_h + J_h^\top\right)\right)\overline{V}\Bigg]dx \qquad (3.4)$$

for all $V \in H(\mathrm{curl}\,;\Omega)$. By W we define the linear bounded operator $\mathcal{A} : C_0^1(\Omega) \to H(\mathrm{curl}\,;\Omega)$ with $\mathcal{A}h = W$ and from $\mathcal{L}(E,V) = F(\overline{V}) = \mathcal{L}_h(\hat{E}_h, V)$ with equation (3.3) we conclude

$$\begin{aligned}
\mathcal{L}(\hat{E}_h - E - \mathcal{A}h, V) \;&=\; \mathcal{L}(\hat{E}_h,V) - \mathcal{L}_h(\hat{E}_h,V) - \mathcal{L}(W,V) \\
= \int_\Omega (\mathrm{curl}\,\hat{E}_h)^\top &\Big[\frac{1}{\mu_r}I - \frac{J_\varphi^\top J_\varphi}{\tilde{\mu}_r^h \det(J_\varphi)} - \frac{1}{\mu_r}\left(\mathrm{div}\,(h)I - J_h - J_h^\top\right)\Big]\mathrm{curl}\,\overline{V}\,dx \\
- k^2 \int_\Omega \hat{E}_h^\top &\Big[\varepsilon_r I - \tilde{\varepsilon}_r^h J_\varphi^{-1} J_\varphi^{-\top}\det(J_\varphi) - \varepsilon_r(-\mathrm{div}\,(h)I + J_h + J_h^\top)\Big]\overline{V}\,dx \\
+ \int_\Omega (\mathrm{curl}\,(\hat{E}_h - E))^\top &\Big[\frac{1}{\mu_r}\left(\mathrm{div}\,(h)I - J_h - J_h^\top\right)\Big]\overline{V}\,dx \\
-k^2 \int_\Omega (\hat{E}_h - E))^\top &\Big[\varepsilon_r(-\mathrm{div}\,h + J_h + J_h^\top)\Big]\mathrm{curl}\,\overline{V}\,dx.
\end{aligned}$$

Thus by Lemma 3.1 and Theorem 3.3 we obtain

$$\begin{aligned}
&\frac{1}{\|h\|_{C^1}}\mathcal{L}(\hat{E}_h - E - \mathcal{A}h, V) \\
&\leq\; C\left(\|\hat{E}_h\|_{H(\mathrm{curl}\,,\Omega)}\, o(\|h\|_{C^1}) + \|\hat{E}_h - E\|_{H(\mathrm{curl}\,,\Omega)}\right)\|V\|_{H(\mathrm{curl}\,,\Omega)} \\
&\to\; 0 \quad \text{for } \|h\|_{C^1} \to 0
\end{aligned}$$

with a constant $C > 0$. The existence of a bounded inverse with respect to $\mathcal{L}$ and a perturbation argument as in the proof of Theorem 3.2 shows the assertion

$$\lim_{\|h\|_{C^1}\to 0} \frac{1}{\|h\|_{C^1}}\left\|\hat{E}_h - E - W\right\|_{H(\mathrm{curl}\,,\Omega)} \to 0 \quad \text{for } \|h\|_{C^1} \to 0\,.$$

□

We have shown differentiability of the solution of the scattering problem with respect to variations of D. Similar to the usual notation in shape optimization we call $\mathcal{A}h = W \in H(\mathrm{curl}\,;\Omega)$ defined by (3.4) the material derivative of the total field E in Ω with respect to h. Note, that W can be read as a Frechét derivative of E on appropriate spaces of admissible interfaces, different to the usual definition in the sense of a directional derivative [21].

4 A characterization of the domain derivative

Considering first order terms with respect to h in $\widehat{E}_h$ lead to $W = J_h^\top E + J_E h + E'$ for the material derivative, where the function E' is usually called the domain derivative of E. Note that due to the assumptions on the variation h we have $W = E'$ on $\Omega\backslash\mathrm{supp}(h)$. Thus, the domain derivative E' is the common linearization considered in shape optimization and inverse obstacle problems. We are going to prove that under smoothness assumptions on ∂D the domain derivative E' is only dependent on the normal component $\nu^\top h$ of variations of the boundary ∂D and E' and $H' = \frac{1}{ik\mu_r}\mathrm{curl}\, E'$ satisfy a transmission boundary value problem for Maxwell's equations.

According to the following results we introduce the notations

$$V_\tau = \nu \times (V \times \nu) \quad \text{and} \quad V_\nu = \nu^\top V$$

for the tangential and the normal component of a vector on the boundary ∂D. Thus, on ∂D there holds $V = V_\tau + V_\nu \nu$. Additionally we use $\nabla_\tau : H^{1/2}(\partial D) \to H^{-1/2}(\partial D)$ for the surface gradient, which is given by

$$\nabla_\tau V = (\nabla V)_\tau = \nabla V - \frac{\partial V}{\partial \nu}\nu,$$

if V is a smooth function in a neighborhood of ∂D. With these notations we can formulate and prove the main result.

Theorem 4.1 *Assuming a regular boundary ∂D and $h \in C_0^1(\Omega)$. The domain derivative $E' = W - J_h^\top E - J_E h$ satisfies $E'|_D \in H(\mathrm{curl}\,, D)$ and $E'|_{\Omega\backslash\overline{D}} \in H(\mathrm{curl}\,, \Omega\backslash\overline{D})$ and can be uniquely extended to the radiating weak solution of the Maxwell's equations*

$$\mathrm{curl}\, E' - ik\mu_r H' = 0, \quad \mathrm{curl}\, H' + ik\epsilon_r E' = 0 \quad \text{in } D \cup \mathbb{R}^3\backslash\overline{D}\,, \tag{4.5}$$

defining H' by the first equation with transmission boundary conditions

$$\begin{aligned} [\nu \times E']_\pm &= -\Big[\nu \times \nabla_\tau(h_\nu E_\nu)\Big]_\pm - ik\Big[\mu_r\Big]_\pm h_\nu H_\tau \\ [\nu \times H']_\pm &= -\Big[\nu \times \nabla_\tau(h_\nu H_\nu)\Big]_\pm + ik\Big[\varepsilon_r\Big]_\pm h_\nu E_\tau \end{aligned}$$

on ∂D.

Proof: We define $E' = W - J_h^\top E - J_E h$ in D and in $\Omega\backslash\overline{D}$. By regularity of ∂D the solution of the scattering problem satisfy $E|_D, H|_D \in H^1(D)$ and $E|_{\Omega\backslash\overline{D}}, H|_{\Omega\backslash\overline{D}} \in H^1(\Omega\backslash\overline{D})$ (see [1]). Additionally by the polar vector, $\operatorname{curl} E$, of the skew symmetric matrix $J_E - J_E^\top$ we obtain

$$\begin{aligned}
\operatorname{curl}(J_h^\top E + J_E h) &= \operatorname{curl}\big((J_E - J_E^\top)h + \nabla(h^\top E)\big) \\
&= \operatorname{curl}\big(\operatorname{curl} E \times h\big) \\
&= \operatorname{div}(h)\operatorname{curl} E + J_{\operatorname{curl} E}h - J_h \operatorname{curl} E \qquad (4.6)
\end{aligned}$$

in D and in $\Omega\backslash\overline{D}$. Overall, together with $W \in H(\operatorname{curl};\Omega)$, we conclude that $E'|_D \in H(\operatorname{curl};D)$ and $E'|_{\Omega\backslash\overline{D}} \in H(\operatorname{curl};\Omega\backslash\overline{D})$.

Furthermore, from the definition (3.4) of the material derivative W we compute the identity

$$\begin{aligned}
\mathcal{L}(E', V) &= \mathcal{L}(W, V) - \mathcal{L}(J_h^\top E + J_E h, V) \\
&= \int_\Omega \Big(\frac{1}{\mu_r}(\operatorname{curl} E)^\top \big(\operatorname{div}(h)\, I - J_h - J_h^\top\big) \operatorname{curl} \overline{V} \\
&\qquad\qquad + k^2 \varepsilon_r E^\top (\operatorname{div}(h)) I - J_h - J_h^\top)\overline{V}\Big)\, dx \\
&\quad - \int_\Omega \frac{1}{\mu_r}\Big(\operatorname{curl}(J_h^\top E + J_E h)\Big)^\top \operatorname{curl}\overline{V} - k^2\varepsilon_r \Big(J_h^\top E + J_E h\Big)^\top \overline{V}\, dx\,.
\end{aligned}$$

Substituting the identity (4.6) into the previous equation leads to

$$\begin{aligned}
\mathcal{L}(E', V) &= -\int_\Omega \frac{1}{\mu_r}\big(J_{\operatorname{curl} E}h + J_h^\top \operatorname{curl} E\big)^\top \operatorname{curl}\overline{V}\, dx \\
&\quad + k^2 \int_\Omega \varepsilon_r \big(J_E h + \operatorname{div}(h) E - J_h E\big)^\top \overline{V}\, dx \\
&= -\int_\Omega \frac{1}{\mu_r}\big((J_{\operatorname{curl} E} - J_{\operatorname{curl} E}^\top)h + J_{\operatorname{curl} E}^\top h + J_h^\top \operatorname{curl} E\big)^\top \operatorname{curl}\overline{V}\, dx \\
&\quad + k^2 \int_\Omega \varepsilon_r \big(\operatorname{curl}(E \times h)\big)^\top \overline{V}\, dx
\end{aligned}$$

where we have used $\operatorname{div} E = 0$ in D and in $\Omega\backslash\overline{D}$ to get $\operatorname{curl}(E \times h) = \operatorname{div}(h)E + J_E h - J_h E$. With $J_h^\top \operatorname{curl} E + J_{\operatorname{curl} E}^\top h = \nabla(h^\top \operatorname{curl} E)$ and the polar vector of $(J_{\operatorname{curl} E} - J_{\operatorname{curl} E}^\top)$, which is given from Maxwell equations by $\operatorname{curl}(\operatorname{curl} E) = k^2 \mu_r \varepsilon_r E$ on D and $\Omega\backslash\overline{D}$, we conclude

$$\begin{aligned} \mathcal{L}(E', V) &= -\int_\Omega \frac{1}{\mu_r}\Big(\nabla(h^\top \operatorname{curl} E)\Big)^\top \operatorname{curl} \overline{V}\, dx \\ &\quad + k^2 \int_\Omega \varepsilon_r \operatorname{div}\left((E\times h)\times \overline{V}\right)\, dx\,, \end{aligned}$$

where we use the vector identity $\operatorname{div}((E\times h)\times\overline{V}) = (\operatorname{curl}(E\times h))^\top \overline{V} - (E\times h)^\top \operatorname{curl}\overline{V}$. Furthermore, by $\Big(\nabla(h^\top\operatorname{curl} E)\Big)^\top \operatorname{curl}\overline{V} = \operatorname{div}\Big((h^\top \operatorname{curl} E)\operatorname{curl}\overline{V}\Big)$ and the divergence theorem it follows

$$\begin{aligned} \mathcal{L}(E', V) &= -\int_\Omega \left[\operatorname{div}\left(\frac{1}{\mu_r}(h^\top \operatorname{curl} E)\operatorname{curl}\overline{V}\right) - k^2 \operatorname{div}\left(\varepsilon_r (E\times h)\times\overline{V}\right)\right] dx \\ &= \int_{\partial D} \left[\frac{1}{\mu_r}(h^\top \operatorname{curl} E)\,\nu^\top \operatorname{curl}\overline{V}\right]_\pm - k^2\Big[\left(\varepsilon_r(E\times h)\times\overline{V}\right)^\top \nu\Big]_\pm ds\,. \end{aligned}$$

Note that the variation h is supposed to be compactly supported in Ω and therefore no boundary term on $\partial\Omega$ occurs on the right hand side. By Stokes' theorem for the surface gradient we have

$$\int_{\partial D} \psi\, \nu^\top \operatorname{curl}\overline{V}\, ds = \int_{\partial D} (\nabla_\tau \psi)^\top \left(\nu\times\overline{V}\right) ds$$

for a function $\psi : \partial D \to \mathbb{C}$. Therefore we obtain

$$\mathcal{L}(E', V) = \int_{\partial D} \left[\frac{1}{\mu_r}\nabla_\tau(h^\top \operatorname{curl} E)\right]_\pm^\top \left(\nu\times\overline{V}\right) + k^2 \left[\varepsilon_r(E\times h)\right]_\pm^\top \Big(\nu\times\overline{V}\Big)\, ds$$

for all $V \in H(\operatorname{curl};\Omega)$, since the tangential trace of test functions $V \in H(\operatorname{curl};\Omega)$ is continuous at ∂D.

We substitute $\frac{1}{\mu_r}\operatorname{curl} E = ikH$ and obtain from the boundary condition $[\nu\times H]_\pm = 0$ the identity

$$\left[\frac{1}{\mu_r}\nabla_\tau(h^\top \operatorname{curl} E)\right]_\pm = ik\nabla_\tau(h_\nu H_\nu)$$

at ∂D. Furthermore, we find by the boundary condition $[\varepsilon_r E_\nu]_\pm = 0$ the equation

$$\begin{aligned} [\varepsilon_r(E\times h)]_\pm^\top(\nu\times\overline{V}) &= [\varepsilon_r(E_\tau\times h)_\tau]_\pm^\top(\nu\times\overline{V}) \\ &= -[\varepsilon_r \nu\times(E_\tau\times h)]_\pm^\top\,\overline{V}_\tau \;=\; -[\varepsilon_r h_\nu E_\tau]_\pm^\top\,\overline{V}_\tau\,. \end{aligned}$$

Finally we obtain

$$\mathcal{L}(E',V) = -\int_{\partial D}\left(\Big[ik\nu \times \nabla_\tau(h_\nu H_\nu)\Big]_\pm + k^2\Big[\varepsilon_r\Big]_\pm h_\nu E_\tau\right)^\top \overline{V}_\tau\, ds \tag{4.7}$$

for all $V \in H(\operatorname{curl};\Omega)$.

We define $H' = \frac{1}{ik\mu_r}\operatorname{curl} E'$ and obtain that $E'|_D, H'|_D \in H(\operatorname{curl};D)$ and $E'|_{\Omega\setminus\overline{D}}, H'|_{\Omega\setminus\overline{D}} \in H(\operatorname{curl};\Omega\setminus\overline{D})$ constitute weak solutions of Maxwell's equations in D and in $\Omega\setminus\overline{D}$ using test functions with compact support in D and in $\Omega\setminus\overline{D}$, respectively. Next using the Calderon operator we see that E' and H' can be extended to radiating solutions of the Maxwell equations in $\mathbb{R}^3\setminus\overline{D}$. Additionally applying Green's formula with $V \in H_0(\operatorname{curl};\Omega)$ we find by (4.7) the boundary condition

$$[\nu \times H']_\pm = [\nu \times \nabla_\tau(h_\nu H_\nu)]_\pm - ik[\varepsilon_r]_\pm h_\nu E_\tau$$

on ∂D.

The other transmission condition can be computed from the definition $E' = W - J_E h - J_h^\top E$, the vanishing jump of the tangential trace of $W \in H(\operatorname{curl};\Omega)$, and again the transmission boundary conditions of E and H. To begin with we use the polar vector of $J_E - J_E^\top$ and obtain

$$\begin{aligned}[\nu \times E']_\pm &= -[\nu \times (J_E h + J_h^\top E)]_\pm \\ &= -[\nu \times (J_E - J_E^\top)h]_\pm - [\nu \times (J_h^\top E + J_E^\top h]_\pm \\ &= -[\nu \times (\operatorname{curl} E \times h)]_\pm - [\nu \times \nabla(h^\top E)]_\pm\,.\end{aligned}$$

For the first term we conclude by the jump condition $[\mu_r H_\nu]_\pm = 0$

$$\begin{aligned}[\nu \times (\operatorname{curl} E \times h)]_\pm &= ik[\mu_r H_\tau]_\pm h_\nu + ik[\mu_r \nu \times (H \times h_\tau)]_\pm \\ &= ik[\mu_r H_\tau]_\pm h_\nu - ik[\mu_r H_\nu h_\tau]_\pm \\ &= ik[\mu_r H_\tau]_\pm h_\nu\,.\end{aligned}$$

From $[\nu \times E]_\pm = 0$ the second term reduces to

$$\begin{aligned}\Big[\nu \times \nabla(h^\top E)\Big]_\pm &= \Big[\nu \times \nabla_\tau(h^\top E)\Big]_\pm \\ &= \Big[\nu \times \nabla_\tau(h_\nu E_\nu)\Big]_\pm\end{aligned}$$

Finally, the boundary condition

$$[\nu \times E']_\pm = -\Big[\nu \times \nabla_\tau(h_\nu E_\nu)\Big]_\pm - ik\Big[\mu_r\Big]_\pm h_\nu H_\tau$$

on ∂D follows. □

References

[1] T. Abboud and J. C. Nédélec, *Electromagnetic waves in an inhomogeneous medium*, J. Math. Anal. Appl. **164** (1992), 40–58.

[2] L. Afraites, M. Dambrine and D. Kateb. *On second order shape optimization methods for electrical impedance tomography*, SIAM Control and Optimization **47** (2008), 1556–1590.

[3] D.L. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, Berlin, Heidelberg, New York, etc. 1992.

[4] A. Buffa, M. Costabel and D. Sheen *On the traces of $H(\mathrm{curl}\,;\Omega)$ in Lipschitz domains* J. Math. Anal. Appl., **92** (2002), 679–710.

[5] M. Costabel, F. Le Louër, *Shape Derivatives of Boundary Integral Operators in Electromagnetic Scattering*, Preprint (2010).

[6] K. Eppler and H. Harbrecht, *Second-order shape optimization using wavelet BEM*, Optim. Methods Softw., **21** (2006), 135–153.

[7] M. Ganesh, S.C. Hawkins *A high-order algorithm for multiple electromagnetic scattering in three Dimensions* Numer. Algorithms **50** (2009), 469–510.

[8] J. Hadamard, *Lessons on the Calculus of Variation*, Gauthier-Villards, Paris 1910.

[9] F. Hettlich, *Fréchet Derivatives in Inverse Obstacle Scattering,* Inverse Problems **11** (1995), 371–382.

[10] F. Hettlich, *The Domain derivative in Inverse Obstacle Scattering*, Habilitation Thesis, Erlangen, 1999

[11] F. Hettlich and W. Rundell, *A Second Degree Method for Nonlinear Inverse Problems,* SIAM J. Numer. Anal., **37** (2000), 587–620.

[12] A. Kirsch, *The Domain Derivative and Two Applications in Inverse Scattering Theory* Inverse Problems **9** (1993), 81–96.

[13] A. Kirsch and P. Monk, *A Finite Element/Spectral Method for Approximating the Time-Harmonic Maxwell System in $\mathbb{R}^3$*, SIAM J. Appl. Math. **55** (1995), 1324–1344.

[14] R. Kress, *Linear Integral Equations*, Springer-Verlag, Berlin, 1989.

[15] R. Kress, *Electromagnetic waves scattering: Scattering by obstacles* In: Scattering (Pike, Sabatier, eds.), Academic Press, London (2001), 191–210.

[16] P. Monk *Finite element Methods for Maxwell's Equations*, Oxford University Press, 2003.

[17] C. Müller, *Foundations of the mathematical theory of electromagnetic waves,* Springer Verlag, Berlin, 1969.

[18] O. Pironneau, *Optimal Shape Design for Elliptic Systems,* Springer-Verlag, Berlin, Heidelberg, New York, etc. 1984.

[19] R. Potthast, *Domain Derivatives in Electromagnetic Scattering*, Math. Meth. Appl. Sc. **19** (1996), 1157–1175.

[20] J. Simon, *Differentiation with Respect to the Domain in Boundary Value Problems,* Numer. Funct. Anal. Opt., **2** (1980), 649–687.

[21] J. Sokolowski and J.-P. Zolesio, *Introduction to Shape Optimization*, Springer Verlag, Berlin, Heidelberg, New York, etc. 1992.