Paraproducts via H^∞-functional calculus and a T(1)-Theorem for non-integral operators

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"We must maintain the principle we laid down when dealing with astronomy, that our pupils must not leave their studies incomplete or stop short of the final objective. They can do this just as much in harmonics as they could in astronomy, by wasting their time on measuring audible concords and notes."
"Lord, yes, and pretty silly they look", he said. "They talk about 'intervals' of sound, and listen as carefully as if they were trying to hear a conversation next door. And some say they can distinguish a note between two others, which gives them a minimum unit of measurement, while others maintain that there's no difference between the notes in question. They are all using their ears instead of their minds."
"You mean those people who torment catgut, and try to wring the truth out of it by twisting it on pegs."

PLATO, Fourth Century BC.

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Contents

1 Introduction

In this thesis, we consider new types of paraproducts constructed via H^∞-functional calculus and develop a $T(1)$-Theorem for non-integral operators by combining methods used in the study of L^p theory for non-integral operators and the Kato problem with the recently developed theory of Hardy and BMO spaces associated to sectorial operators.

Let us first explain the setting and give a short overview of the main results of the thesis before coming to a more detailed discussion and an explanation of the background.

The underlying space (X, d, μ) is a **space of homogeneous type** as introduced by Coifman and Weiss in [CW71]. This is nowadays common practice in harmonic analysis and widens the scope of the theory for applications in comparison to the Euclidean space $\mathbb{R}^n$ at very little cost. In fact, in this thesis there is only one situation where it makes a relevant difference, namely in the context of Poincaré inequalities.

We consider a **sectorial operator** L of order $2m$ on $L^2(X)$ with the following properties:

- L has a bounded holomorphic functional calculus on $L^2(X)$;
- The semigroup e^{-tL} generated by L satisfies Davies-Gaffney estimates, also called L^2 off-diagonal estimates;
- The semigroup e^{-tL} satisfies an $L^p - L^2$ off-diagonal estimate for some $1 < p < 2$ and an $L^2 - L^q$ off-diagonal estimate for some $2 < q < \infty$.

Under the first two assumptions on L, there was recently developed a theory of Hardy spaces $H^p_L(X)$ and of a corresponding space $BMO_L(X)$ associated to the operator L. We give a unified presentation of the results, including several characterizations of the space $H^1_L(X)$ and the duality of the spaces $H^1_L(X)$ and $BMO_{L^*}(X)$. Under these assumptions on L, the results are to our knowledge nowhere stated before. Moreover, we generalize a Fefferman-Stein criterion, describing the connection of Carleson measures and elements of $BMO_L(X)$, and a Calderón reproducing formula for elements of $H^1_L(X)$ and $BMO_{L^*}(X)$. The basic tool for the whole theory is the bounded **holomorphic functional calculus** for L.

The connection of Carleson measures and elements of $BMO_L(X)$ sets the stage for a definition of **paraproducts** constructed via holomorphic functional calculus.
We show that, under the above three assumptions on L, for every $b \in BMO_L(X)$ the paraproduct operator

$$\Pi_b : f \mapsto \int_0^\infty \tilde{\psi}(t^{2m}L)[\psi(t^{2m}L)b \cdot A_t(e^{-t^{2m}L}f)] \, \frac{dt}{t} \tag{1.1}$$

is bounded on $L^2(X)$, where ψ, $\tilde{\psi}$ are taken from the set Ψ consisting of bounded holomorphic functions on a sector with decay at zero and infinity, and A_t denotes some averaging operator. The appearance of the operator A_t might seem to be surprising, but this is due to the fact that we do not impose any kernel estimates on the semigroup e^{-tL}.
Besides, we show that Π_b extends to a bounded operator from $L^p(X)$ to $H^p_L(X)$ for

$p \in (2, \infty)$ and from $L^\infty(X)$ to $BMO_L(X)$. A consideration of paraproducts as bilinear operators and the examination of differentiability properties of paraproducts complete the topic.

In a second part, we examine the L^2-boundedness of so-called **non-integral operators**. In our setting, they present as operators $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\text{loc}}(X)$ with $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\text{loc}}(X)$ such that for functions $\psi_1, \psi_2 \in \Psi$ with sufficient decay at zero the following **off-diagonal estimates** are valid:

$$\|T\psi_1(tL)f\|_{L^2(B_2)} \leq C \left(1 + \frac{\text{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)} \tag{1.2}$$

$$\|T^*\psi_2(tL^*)f\|_{L^2(B_2)} \leq C \left(1 + \frac{\text{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)} \tag{1.3}$$

for some $\gamma > 0$, for all $t > 0$, all balls B_1, B_2 with radius $r = t^{1/2m}$ and all $f \in L^2(X)$ supported in B_1.

On the Euclidean space $\mathbb{R}^n$ let us denote by G_L the Littlewood-Paley-Stein square function associated to L, i.e. let $G_L(f)(x) := \left(\int_0^\infty \left|t\nabla e^{-t^{2m}L}f(x)\right|^2 \frac{dt}{t}\right)^{1/2}$ for all $x \in \mathbb{R}^n$ and all $f \in L^2(\mathbb{R}^n)$. Then the main result of this thesis, a **$T(1)$-Theorem for non-integral operators** reads as follows:

Theorem *Let L be the sectorial operator of order $2m$ as specified above such that G_L and G_{L^*} are bounded on $L^2(\mathbb{R}^n)$. Let T be a non-integral operator satisfying* (1.2) *and* (1.3) *for sufficiently large $\gamma > 0$. Then T is bounded on $L^2(\mathbb{R}^n)$ if and only if*

$$T(1) \in BMO_L(\mathbb{R}^n) \qquad \textit{and} \qquad T^*(1) \in BMO_{L^*}(\mathbb{R}^n).$$

If the space $\mathbb{R}^n$ is replaced by some arbitrary space X of homogeneous type, we require in addition the validity of some Poincaré inequality and have to reformulate the boundedness of the Littlewood-Paley-Stein square functions.
The assumptions on the non-integral operator T are chosen in such a way that the boundedness on Hardy spaces $H^p_L(X)$ is an immediate consequence of the boundedness on $L^2(X)$.
With the same methods used in the proof of this $T(1)$-Theorem, we moreover show a second version of a $T(1)$-Theorem with weaker assumptions in the case that the conservation properties $e^{-tL}(1) = 1$ and $e^{-tL^*}(1) = 1$ hold.
Under the additional assumption that e^{-tL} is bounded on $L^\infty(X)$ uniformly in $t > 0$, we then apply this second version to prove the boundedness of the paraproduct operator $\tilde{\Pi}_f$ on $L^2(X)$, where $\tilde{\Pi}_f$ is defined by

$$\tilde{\Pi}_f(g) := \int_0^\infty \tilde{\psi}(t^{2m}L)[e^{-t^{2m}L}g \cdot e^{-t^{2m}L}f] \frac{dt}{t}$$

for $f \in L^\infty(X)$, $g \in L^2(X)$ and $\tilde{\psi} \in \Psi$ with sufficient decay at zero and infinity.
We end the thesis with an approach towards a $T(b)$-Theorem.

Let us go into a deeper discussion of the whole topic.

Non-integral operators To understand what a *non-integral operator* is, let us first clarify what we mean by an *integral operator*, or more precisely, a *singular integral operator*. The notion of the latter signals two properties of the operators we have in mind, namely that they are, at least formally, defined as integrals of the form

$$Tf(x) = \int k(x,y) f(y)\, dy,$$

and that the integral kernel k is in some sense singular. What we postulate in addition, is an estimate on the behaviour of the singularity of the kernel at $x = y$, i.e.

$$|k(x,y)| \leq C\, |x-y|^{-n},$$

and a Hölder-type estimate of the form

$$\left|k(x,y) - k(x,y')\right| + \left|k(y,x) - k(y',x)\right| \leq C \frac{|y-y'|^{\delta}}{|x-y|^{n+\delta}}$$

for all $x, y, y' \in \mathbb{R}^n$ with $x \neq y$ and $0 < |y - y'| \leq \frac{1}{2}\,|x-y|$. A weaker version of a Hölder-type estimate on the kernel is of the form

$$\int_{|x-y| \geq 2|y'-y|} \left|k(x,y) - k(x,y')\right|\, dx \leq C$$

and is called Hörmander-type estimate. The study of singular integral operators dates back to the beginning of the twentieth century, starting with the prototype of all singular integrals, the Hilbert transform. While the methods used there depended on techniques of complex analysis, Calderón and Zygmund systematically studied in the 1950's singular integral operators of convolution type, and later on also of non-convolution type, with the help of real variable methods. This led to an extensive study of this type of operators, in the literature subsequently also called *Calderón-Zygmund operators*. The theory of such operators has important applications to complex analysis, elliptic partial differential equations, pseudo-differential operators and many others.
One of the basic examples of Calderón-Zygmund operators are the Riesz transforms $R_j = -i\frac{\partial}{\partial x_j}(-\Delta)^{-1/2}$, $1 \leq j \leq n$, on $\mathbb{R}^n$, in generalization of the Hilbert transform on $\mathbb{R}$. They arise e.g. in the study of the Neumann problem on the upper half plane and their boundedness on $L^2(\mathbb{R}^n)$ can immediately be dealt with the Fourier transform. However, the boundedness of R_j on $L^p(\mathbb{R}^n)$ for $p \in (1,\infty)$ is not at all obvious. It was the motivating example for Calderón and Zygmund in their treatise of singular integral operators and reflects one of the most important properties of Calderón-Zygmund operators: If a Calderón-Zygmund operator is bounded on L^2, then it is also bounded on L^p for all $p \in (1,\infty)$ and satisfies a weak $(1,1)$ estimate.
For a precise definition of Calderón-Zygmund operators and an overview of Calderón-Zygmund theory we refer to standard textbooks of harmonic analysis such as [Ste93], [CM97] or [Gra04], cf. also [Chr90b]. Unlike the usual notion, we do not assume a Calderón-Zygmund operator to be bounded on L^2.

Even if in practice many operators fall under the scope of the Calderón-Zygmund theory, there are still numerous operators of interest that do not. In aim of a uniform treatment of some of these operators, Duong and McIntosh developed in [DM99] a theory

of so-called *singular integral operators with non-smooth kernels*, considering classes of operators, that do not satisfy the Hörmander condition, but are still of weak type $(1,1)$. Their result covers a kind of integral operators T such that suitable approximation operators $\{S_t\}_{t>0}$ satisfy upper Poisson bounds and the composite operator $T(I-S_t)$ satisfies a weakened Hörmander-type condition. Under the assumption that the operator T is L^2-bounded, they could, in generalization of Calderón-Zygmund theory, show a weak $(1,1)$ estimate for T. For concrete examples of such operators, we refer to [DM99] and the references given therein.
Blunck and Kunstmann went in [BK03] a large step further and generalized the result of Duong and McIntosh to *non-integral operators*. The authors replaced the weakened Hörmander estimate of [DM99] by a maximal estimate in terms of the Hardy-Littlewood p-maximal operator for some $p \in [1,2)$ and used instead of Poisson bounds for the approximation operators suitable weighted norm estimates. For such operators, the authors obtained a weak type (p,p) criterion under the assumption that they are bounded on L^2. A simplified version of this result, due to Auscher in [Aus07], reads as follows:

Let $p \in [1,2)$. Suppose that T is a sublinear operator of strong type $(2,2)$, and let A_r, $r>0$, be a family of linear operators acting on L^2. Assume that there exists some $\varepsilon > 0$ such that for $j \geq 2$

$$\|T(I - A_{r_B})f\|_{L^2(S_j(B))} \leq C2^{-j(n/2+\varepsilon)} |B|^{1/2-1/p} \|f\|_{L^p(B)} \tag{1.4}$$

and for $j \geq 1$

$$\|A_{r_B} f\|_{L^2(S_j(B))} \leq C2^{-j(n/2+\varepsilon)} |B|^{1/2-1/p} \|f\|_{L^p(B)}$$

for all balls B with radius r_B and all f supported in B. Then T is of weak type (p,p).

Besides, there exists a corresponding result for $p \geq 2$, due to Auscher, Coulhon, Duong and Hofmann in [ACDH04].
In the context of [BK03], and this will be the same how we understand it, the notion of *non-integral operators* indicates the following: Most obviously, it signals that the operators under consideration can no longer be represented by an integral operator with a Calderón-Zygmund kernel, sometimes even not with any other kernel in a suitable sense (besides the Schwartz kernel). At the same time, the operators lie *beyond* Calderón-Zygmund theory, still - or even more - being "singular" in some sense and generalizing the concept of Calderón-Zygmund operators. This includes that many ideas used in the treatise of such operators are generalizations of methods developed in Calderón-Zygmund theory. However, the ranges of p, where the operators are bounded on L^p, are often strictly smaller than the usual interval $(1,\infty)$. At last, the notion of non-integral operators, as we understand it, implicitly contains some regularity assumptions, in analogy to the notion of singular integral operators. In absence of pointwise kernel estimates, such a regularity assumption is given in terms of weigthed norm estimates, also called off-diagonal estimates.
One of the basic examples is again the Riesz transform, now in the context of more general elliptic operators. If L is a second order elliptic operators in divergence form, then for each $p < \frac{2n}{n+2}$ and for each $p > 2$ there exists some L as specified above such that $\nabla L^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$. The result for $p > 2$ is due to Kenig and is described in [AT98], the other one was recently shown by Hofmann, Mayboroda and McIntosh in

[HMM10].
However, Blunck and Kunstmann could show in [BK04] (cf. also [CD99] of Coulhon and Duong and [HM03] of Hofmann and Martell) by application of the above stated theorem that even in absence of pointwise Gaussian estimates the Riesz transform $\nabla^m L^{-1/2}$ of an elliptic operator of order $2m$ in divergence form is bounded on $L^p(\mathbb{R}^n)$ for all $p \in (\frac{2n}{n+2m} \vee 1, 2]$.

Off-diagonal estimates The main tool in the proof of the above result in [BK04] are weighted $L^p - L^q$ estimates of the form

$$\left\| \mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})} \right\|_{L^p \to L^q} \leq C_\gamma |B(x, t^{1/m})|^{\frac{1}{q}-\frac{1}{p}} \left(1 + \frac{d(x,y)}{t^{1/m}}\right)^{-\gamma},$$

where $1 \leq p \leq q \leq \infty$ and the estimate shall hold for all $\gamma > 0$. This type is also called *generalized Gaussian estimates*, indicating that, in view of a well-known theorem, the estimates are in the case of $(p,q) = (1,\infty)$ equivalent to pointwise Gaussian estimates. The idea to work with *weighted norm estimates*, or *off-diagonal estimates* as we will call them subsequently, has its origin in the paper [Sch94] of Schreieck and Voigt. They used an estimate of the form

$$\left\| e^{-\xi \cdot} T e^{\xi \cdot} \right\|_{L^p \to L^q} \leq C(\xi),$$

where $\xi \in \mathbb{R}^n$ and $1 \leq p \leq q \leq \infty$, as a substitute for pointwise Gaussian bounds on the semigroup in the context of L^p spectral independence of certain Schrödinger operators. This is what is often referred to as "box method" and in this case the notion *weighted estimates* becomes clearer, as one can rewrite the estimates in terms of norm bounds for the operator T in weighted spaces.
Estimates of the form

$$\|T_t(\mathbb{1}_E f)\|_{L^2(F)} \leq C e^{-\frac{\operatorname{dist}(E,F)^2}{ct}} \|f\|_{L^2(E)},$$

were first formulated by Davies in [Dav92], rewriting arguments of Gaffney in [Gaf59] in the context of heat equations on complete Riemannian manifolds. They are nowadays called *Davies-Gaffney estimates* and hold for most semigroups generated by elliptic operators, e.g. for elliptic higher order operators with bounded measurable coefficients and for Schrödinger operators with singular potentials.
In this thesis, off-diagonal estimates provide the main tool for the treatment of non-integral operators in absence of pointwise kernel estimates. First, we assume Davies-Gaffney estimates for the semigroup of the sectorial operator L. This is the basis for the development of the theory of Hardy and BMO spaces associated to operators. In the theory of paraproducts we need in addition an $L^p - L^2$ off-diagonal estimate for some $p < 2$. This extra assumption is correlated to the use of the Hardy-Littlewood maximal operator, as the 2-maximal operator is bounded on L^q for $q > 2$, but not on L^2.
For the non-integral operators T we have under consideration in the context of our $T(1)$-Theorem, we work with weaker off-diagonal estimates of the form (1.2) and (1.3). These estimates on approximations of T generalize the usual Hörmander condition of Calderón-Zygmund operators. Moreover, under the assumption that T is bounded on L^2, the estimates self-improve. That is, if e.g. (1.2) is satisfied for ψ_1, then the estimate is also satisfied for functions φ taken from a large class of bounded holomorphic functions.

This property in particular gives way to the application of L^p theory for non-integral operators.
For a detailed discussion of off-diagonal estimates, we refer the reader to the paper [BK05] of Blunck and Kunstmann and the series of papers [AM07a], [AM07b], [AM06], [AM08] of Auscher and Martell, in particular [AM07b].

$T(1)$**-Theorem** The fundamental question for Calderón-Zygmund operators is, whether they are bounded on L^2. For convolution operators, such as the Riesz transforms R_j on $\mathbb{R}^n$, this can immediately be shown by application of Fourier theory. But for most operators considered in applications, this is not at all obvious. Fefferman wrote in [Fef75] in 1975:

"When neither Plancherel's theorem nor Cotlar's lemma applies, L^2-boundedness of singular operators presents very hard problems, each of which must (so far) be dealt with on its own terms."

The question remained open until David and Journé presented in [DJ84] a characterization of Calderón-Zygmund operators to be bounded on $L^2(\mathbb{R}^n)$. This is what originally the term $T(1)$-*Theorem* denotes. In short, they prove that a Calderón-Zygmund operator T is bounded on $L^2(\mathbb{R}^n)$ if and only if it is weakly bounded (in some appropriate sense) and $T(1), T^*(1) \in BMO(\mathbb{R}^n)$. Thus, to check the boundedness of T on $L^2(\mathbb{R}^n)$, it is sufficient to check T on smooth, compactly supported test functions for weak boundedness and in addition, to check T and T^* on the constant function 1. There exist various types of weak boundedness properties, a common form is e.g. to assume that for all $x \in \mathbb{R}^n$ and all $t > 0$

$$\left|\langle T\varphi^{x,t}, \psi^{x,t}\rangle\right| \leq Ct^n,$$

where $\varphi^{x,t}(y) = \varphi((y-x)/t)$ and φ is a *normalized bump function*, i.e. $\varphi \in C^\infty(\mathbb{R}^n)$, supported in $B(0,1)$ and $\|\varphi\|_{C^N} \leq 1$ for some fixed N; and the same for ψ.
What is fascinating about this theorem is that it is both - a deep result of crucial importance and a theorem that can be formulated in only one sentence.
Many examples of operators, such as the Calderón commutators and pseudo-differential operators, can be covered by this result. But to one of the motivating examples for the development of the theory, the Cauchy integral operator along Lipschitz curves, the $T(1)$-Theorem is not directly applicable. This led to the development of the $T(b)$-Theorem of David, Journé and Semmes in [DJS85] (a first version in this direction is due to McIntosh and Meyer [MM85]), where the function 1 is replaced by a para-accretive function b.
There exist numerous variants and generalizations, among them generalizations to spaces of homogeneous type and non-homogeneous spaces, local $T(b)$-Theorems, quadratic $T(1)$-Theorems, and operator-valued versions. But in all cases one assumes kernel estimates for the operator T to be valid.
Let us again have a look at our leading example, the Riesz transform $\nabla L^{-1/2}$, now for an elliptic second order operator L in divergence form. The discussion before illustrates that the Riesz transform does in general not fall under the scope of Calderón-Zygmund operators and is what we call a non-integral operator. Thus, the $T(1)$-Theorem of David and Journé is not applicable for a proof of the boundedness of $\nabla L^{-1/2}$ on L^2. The question of L^2-boundedness for the Riesz transform is part of the Kato problem, which has been a long-standing conjecture. In [AHL$^+$02], Auscher, Hofmann, Lacey, McIntosh

and Tchamitchian solved this problem, that is, they showed that the domain of $\sqrt{L}$ is the Sobolev space $W^{1,2}(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \nabla f \in L^2(\mathbb{R}^n)\}$ with

$$\|\sqrt{L}f\|_{L^2(\mathbb{R}^n)} \approx \|\nabla f\|_{L^2(\mathbb{R}^n)} .$$

In particular, the result shows that the Riesz transform $\nabla L^{-1/2}$ is bounded on $L^2(\mathbb{R}^n)$. Let us at this point mention that the main technical tool in the proof of the Kato problem are off-diagonal estimates for the resolvent operator of L.

In view of the above, it therefore seems to be natural to reformulate what Fefferman said about singular integral operators:

In absence of Calderón-Zygmund theory, L^2-boundedness of non-integral operators presents very hard problems, each of which must (so far) be dealt with on its own terms.

This of course also imposes the following question, in analogy to what Auscher formulated in [Aus07] in the context of L^p theory:

Is there a general machinery to handle the L^2 theory of non-integral operators?

For the particular type of non-integral operators under consideration, this thesis gives a positive answer to the question. That is, for a sectorial operator L, satisfying the three assumptions specified at the beginning, with L^2-bounded Littlewood-Paley-Stein square functions G_L, G_{L^*} and an associated non-integral operator T satisfying (1.2) and (1.3), we obtain a characterization of L^2-boundedness of T. And, even more, in analogy to the $T(1)$-Theorem of David and Journé, the characterization can be formulated in terms of $T(1)$ and $T^*(1)$. This is what we call a $T(1)$-*Theorem for non-integral operators.*

Actually, many key elements used in the proof of the $T(1)$-Theorem of David and Journé stay applicable for the proof of our $T(1)$-Theorem, but now in a more general form. That is, we work with BMO spaces, Carleson measures, paraproducts and a Calderón reproducing formula that are constructed via functional calculus and are thus associated to a sectorial operator L.

The spaces H^p_L and BMO_L The main difference in our $T(1)$-*Theorem for non-integral operators* in comparison to the $T(1)$-Theorem for Calderón-Zygmund operators is the replacement of the space BMO by the spaces BMO_L and BMO_{L^*}, respectively. As is well-known, the space BMO, introduced by John and Nirenberg, consists of all functions f of *bounded mean oscillation* such that

$$\sup_B \frac{1}{|B|} \int_B |f(x) - \langle f \rangle_B| \, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$. In analogy to the space BMO that can also be characterized via the Laplacian, the space BMO_L is associated to a more general sectorial operator L. It consists of elements f, that need no longer be functions, such that

$$\sup_B \frac{1}{|B|} \int_B |(I - e^{-r_B^{2m} L})^M f(x)|^2 \, dx < \infty,$$

where again the supremum is taken over all balls B in $\mathbb{R}^n$, r_B denotes the radius of B and $M \in \mathbb{N}$ is chosen sufficiently large. Apparently, the main idea is to substitute the

averaging of f by a more general approximation associated to L. If one chooses L to be the Laplacian, then the spaces BMO and BMO_L coincide.
Most important for our applications is the fact that there exists an analogoue of the "Fefferman-Stein criterion" for the space BMO_L. This criterion, as stated in [FS72], describes the connection of Carleson measures and elements of BMO.
The spaces BMO_L were first introduced by Duong and Yan in [DY05b], where the semigroup of the operators L under consideration satisfied pointwise Poisson upper bounds. Hofmann and Mayboroda then gave in [HMa09] a generalization to second order elliptic operators in divergence form. For sectorial operators L of the form we have in mind, the theory is due to Duong and Li in [DL09].
What is closely related to - or, better to say, was the starting point for - the theory of BMO_L spaces, is the theory of Hardy spaces H^p_L associated to L. For the statement of the $T(1)$-Theorem for non-integral operators, these spaces only play a role in the background, as the space BMO_L is the dual of $H^1_{L^*}$. But let us indicate some facts that are correlated to the L^p theory of non-integral operators. Hofmann and Mayboroda gave in [HMa09] a sufficient condition for an operator to be bounded from H^1_L to L^1. In combination with an interpolation argument, this result, in a slightly more general form stated as Proposition 4.39 in the thesis, can be considered as a complement to the above stated theorem of Blunck and Kunstmann. In situations, where L^p-boundedness of non-integral operators fails, it is often possible to show that they are nevertheless bounded from some Hardy space H^p_L to L^p for $p \in [1, 2)$. The approximation operators A_r in (1.4) have obviously to be chosen in correlation to L. An example for such an operator is again the Riesz transform $\nabla L^{-1/2}$ of a second order elliptic operators in divergence form. Hofmann, Mayboroda and McIntosh show in [HMM10] that it is actually possible to characterize Hardy spaces associated to L via Riesz transforms. In particular, for all those p for which the Riesz transform is bounded on L^p itself, one obtains equivalence of the spaces L^p and H^p_L.
For a broader overview and a history of the theory of Hardy and BMO spaces associated to operators, we refer the reader to Section 4.1.

Paraproducts Paraproducts are a basic tool of harmonic analysis and play a crucial role in the proof of the $T(1)$-Theorem of David and Journé ([DJ84]). There, given a Calderón-Zygmund operator T, the authors first construct an operator $\tilde{T}$ as $\tilde{T} = T - L - M$, where L and M are paraproduct operators. The operators L and M are chosen such that they are $L^2(\mathbb{R}^n)$-bounded and such that $\tilde{T}(1) = 0$ and $\tilde{T}^*(1) = 0$. This reduces the original problem to the proof of the $L^2(\mathbb{R}^n)$-boundedness of $\tilde{T}$ which is handled via certain approximation operators and the use of the well-known Cotlar-Knapp-Stein lemma.
In the proof of our $T(1)$-Theorem for non-integral operators, the application of paraproducts persists to be very helpful, even if they do not reduce the operator T to an operator $\tilde{T}$ with $\tilde{T}(1) = \tilde{T}^*(1) = 0$. Nevertheless, we can decompose the operator T with their help into a main part and an error term.
For more details and a discussion of the role of the condition $\tilde{T}(1) = 0$, see Section 7.2.

To motivate our definition of paraproducts of the form (1.1), let us have a more detailed look at the paraproduct used in the proof of the $T(1)$-Theorem of David and Journé ([DJ84]). Given $b \in BMO(\mathbb{R}^n)$, they define an operator Π (in [DJ84] denoted by L) on

$L^2(\mathbb{R}^n)$ via

$$\Pi f = \int_0^\infty Q_t[(Q_t b)(P_t f)] \frac{dt}{t}, \qquad f \in L^2(\mathbb{R}^n),$$

where P_t and Q_t are convolution operators with $P_t(1) = 1$ and $Q_t(1) = 0$. Then they show that Π is a Calderón-Zygmund operator, bounded on $L^2(\mathbb{R}^n)$ and satisfying $\Pi(1) = b$ and $\Pi^*(1) = 0$.
In analogy to that, we define a paraproduct Π_b associated to the sectorial operator L. The convolution operator Q_t is replaced by $\psi(t^{2m}L)$ for some $\psi \in \Psi$, whereas the operator P_t is replaced by $A_t e^{-t^{2m}L}$. That we add the averaging operator A_t and do not only work with $e^{-t^{2m}L}$ itself, which would perhaps be more natural, is due to the fact that we do not have any kernel estimates of the operators. However, the averaging operator appears also to be quite useful for applications in the proof of our $T(1)$-Theorem for non-integral operators. We again refer to Section 7.2 for a discussion of the role of A_t.

But presenting paraproducts only as a tool in the context of $T(1)$-Theorems is too narrowly considered. Paraproducts emerged in the theory of paradifferential operators, see e.g. [CM78] and [Bon81], and are for themselves operators of interest. There is no canonical notion of paraproducts in the literature, but they are understood as bilinear operators of a similiar form to (1.1), representing "half" the product of two functions. For a short overview of the theory of paraproducts we refer to [BMN10].
In view of the recently developed theory of Hardy and BMO spaces associated to operators, it seems to be natural to consider also paraproducts associated to operators. In analogy to the fact that the paraproduct of David and Journé is a Calderón-Zygmund operator, we can show certain off-diagonal estimates for paraproducts associated to L, thus they are a prototype for a non-integral operator. This will then also enable us to extend the operators on certain $L^p(X)$ and $H^p_L(X)$ spaces. Moreover, via functional calculus we can show that there holds a Leibniz-type rule.
Let us finally mention that in some special cases, there also holds $\Pi_b(1) = b$ and $\Pi_b^*(1) = 0$, the latter at least formally.

Functional calculus Last, but not least, let us say a word about holomorphic functional calculus. It was introduced by McIntosh in [McI86], mainly motivated by the connection to the Kato problem. And indeed, the holomorphic functional calculus was one of the main tools in the solution of the Kato problem. The same is true for our setting. Where in the theory of Calderón-Zygmund operators Fourier analysis and later on Littlewood-Paley and wavelet theory is used, we work instead with approximation operators constructed via functional calculus. For example, the decomposition of the identity operator is done by application of a Calderón reproducing formula. If $\psi, \tilde{\psi} \in \Psi$ satisfy $\int_0^\infty \psi(t)\tilde{\psi}(t)\,\frac{dt}{t} = 1$, then the functional calculus yields that

$$\int_0^\infty \psi(tL)\tilde{\psi}(tL) f \,\frac{dt}{t} = f$$

for $f \in L^2$. In this way, we obtain approximation operators associated to L.

Comments on the $T(1)$-Theorem What seems to be astonishing while working in the general context of sectorial operators, is the assumption that the Littlewood-Paley-Stein square function G_L is bounded on L^2. This is an assumption which is more fitting for elliptic operators in divergence form. We do not know whether this is only for technical reasons or this is more intrinsic in the type of non-integral operators under consideration. We give a short comment on the topic in Section 7.2.
Another question that is only partly answered in the thesis is that of a weak boundedness property for non-integral operators. In the $T(1)$-Theorem for Calderón-Zygmund operators, one only postulates a very weak behaviour on the diagonal. In contrast to that, the assumptions (1.2) and (1.3) are rather strong, yielding an estimate not only "off-diagonal", but also "on-diagonal". One solution to this problem is given by a version with weaker off-diagonal estimates, as stated in Theorem 6.17, in the case that the conservation properties $e^{-tL}(1) = 1$ and $e^{-tL^*}(1) = 1$ are valid.

Comparison with a result of Bernicot While this thesis was under final preparation, we learned of the article [Ber10] of Bernicot, that also considers L^2-boundedness of non-integral operators. His result is a special case of our weak $T(1)$-Theorem, Theorem 6.17. The main difference in comparison to our results is the fact that he imposes pointwise kernel estimates on the semigroup e^{-tL}. This obviously restricts the operators L to a much smaller class than ours. Moreover, he only considers non-integral operators that satisfy off-diagonal estimates of the form (1.2) for some special ψ_1, namely $\psi_1(z) = z^M e^{-z}$ for some $M \in \mathbb{N}$. And, finally, his proof is completely different to ours. He takes at various places the pointwise estimates into account, e.g. in the proof of a Sobolev-type inequality. He himself says that "the pointwise bound seems to be very important" in his proof and then states as an open question:

"Can we expect a similar $T(1)$-Theorem under just off-diagonal decays for the heat kernel?"

This thesis gives a positive answer to the question. For a more detailed comparison, we refer to Section 7.1.

Structure of the thesis

This thesis is organized as follows: In **Chapter 2** we present fundamental notations and preliminaries. We recall the definition of spaces of homogeneous type and give the most basic facts of maximal functions, holomorphic functional calculus and tent spaces.
Chapter 3 is devoted to the study of off-diagonal estimates. We introduce three different notions of off-diagonal estimates and examine important properties of those. Moreover, we fix our assumptions on the operator L and show consequences of the assumed Davies-Gaffney estimates.
In **Chapter 4** we consider the theory of Hardy and BMO spaces associated to the operator L. We give two characterizations of Hardy spaces, one via molecules, the other one one via square functions, and then show the equivalence of both. In the second part of the chapter, we introduce the space BMO_L. We state a duality result for Hardy and BMO spaces and - what is important for the theory of paraproducts - the connection of Carleson measures and BMO functions.
Paraproducts are then the main topic of **Chapter 5**. We define paraproducts constructed via functional calculus and investigate their properties. Besides the most important property, the L^2-boundedness, we also consider the boundedness on L^p and H^p_L spaces and examine differentiability properties.
Chapter 6 presents the $T(1)$-Theorem for non-integral operators. We first fix our assumptions on the non-integral operator T, clarify how to define $T(1)$ and show necessary conditions for T to be bounded on L^2. We then give a concise introduction in Poincaré inequalities on metric spaces and fix the additional assumption on X and L for a Poincaré inequality to be vaild. The major part of the chapter is devoted to the statement and proof of the $T(1)$-Theorem for non-integral operators, followed by a second version with weaker assumptions. An application of the second $T(1)$-Theorem to paraproduct operators and an approach towards a $T(b)$-Theorem complete the chapter.
Finally, in **Chapter 7**, we give some concluding remarks, give a more detailed comparison with the result of Bernicot and comment on the role of constants.

2 Preliminaries

In this chapter, we give some fundamental notation and a definition of spaces of homogeneous type. We summarize the most basic facts about maximal operators, holomorphic functional calculus and tent spaces, that will be used in the sequel.

2.1 Notation

We introduce the following notation.
We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the natural, integer, real and complex numbers, respectively, and set in addition $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
For a set M, we denote by $\mathbb{1}_M$ its characteristic function, i.e. $\mathbb{1}_M(x) = 1$ for all $x \in M$ and $\mathbb{1}_M(x) = 0$ if $x \notin M$. For a finite set M, we denote by $\#M$ the cardinality of M.
We denote by $[\,.\,]$ the floor function, i.e. we define $[x] := \max\{k \,:\, k \in \mathbb{Z},\; k \leq x\}$ for every $x \in \mathbb{R}$.
We denote by (X, d, μ) a space of homogeneous type as introduced in Section 2.2. By $B(x,r) := \{y \in X \,:\, d(x,y) < r\}$, we denote the open ball in X with center $x \in X$ and radius $r > 0$. We moreover define $V(x,r) := \mu(B(x,r))$, and for any open set $\Omega \subseteq X$ we write $V(\Omega) := \mu(\Omega)$.
We fix some element $x_0 \in X$ that is henceforth denoted by 0. The ball $B_0 := B(0,1)$ is then referred to as *unit ball.*
For $p \in [1,\infty]$ and an open set $\Omega \subseteq X$, we denote by $L^p(\Omega)$ the usual Lebesgue space on the underlying measure space (Ω, μ). By $L^p_{\text{loc}}(X)$ we denote the space of all measurable functions f with $f \in L^p(B)$ for all balls $B \subseteq X$.
If Y, Y_1, Y_2 are normed spaces, we denote by $B(Y_1, Y_2)$ the space of continuous linear operators from Y_1 to Y_2 and set $B(Y) := B(Y,Y)$. We use the notation $Y' := B(Y, \mathbb{C})$ for the dual space.
We denote by $\mathcal{D}(S)$ the domain, by $\mathcal{R}(S)$ the range of an unbounded operator S, and by S^k the k-fold composition of S with itself, in the sense of unbounded operators.
Throughout the thesis, the letter "C" will denote (possibly different) positive constants that are independent of the essential variables. We will frequently write $a \lesssim b$, if there holds $a \leq Cb$ for non-negative quantities a, b.

2.2 Spaces of homogeneous type

In the following we will always assume X to be a space of homogeneous type. More precisely, we assume that (X, d) is a metric space and μ is a nonnegative Borel measure on X with $\mu(X) = \infty$ which satisfies the *doubling condition*:

There exists a constant $A_1 \geq 1$ such that for all $x \in X$ and all $r > 0$

$$V(x, 2r) \leq A_1 V(x,r) < \infty, \tag{2.1}$$

where we set $B(x,r) := \{y \in X \,:\, d(x,y) < r\}$ and $V(x,r) := \mu(B(x,r))$.

For example, the space $\mathbb{R}^n$, endowed with the Euclidean metric and the Lebesgue measure, or a graph of a Lipschitz function $F : \mathbb{R}^n \to \mathbb{R}$, with the induced Euclidean metric and with $\mu(F(E)) := |E|$, the Lebesgue measure of $E \subseteq \mathbb{R}^n$, are spaces of homogeneous type.

Note that the doubling property implies the following strong homogeneity property: There exists a constant $A_2 > 0$ and some $n > 0$ such that for all $\lambda \geq 1$, for all $x \in X$ and all $r > 0$ there holds

$$V(x, \lambda r) \leq A_2 \lambda^n V(x, r). \tag{2.2}$$

In an Euclidean space with the Lebesgue measure, the parameter n corresponds to the dimension of the space.
There also exist constants C and D, $0 \leq D \leq n$, so that

$$V(y, r) \leq C \left(1 + \frac{d(x,y)}{r}\right)^D V(x, r) \tag{2.3}$$

uniformly for all $x, y \in X$ and $r > 0$. For $D = n$, this is a direct consequence of (2.2) and the triangle inequality. If $X = \mathbb{R}^n$, then D can be chosen to be 0.

For a ball $B \subseteq X$ we denote by r_B the radius of B and set

$$S_0(B) := B \qquad \text{and} \qquad S_j(B) := 2^j B \setminus 2^{j-1} B \quad \text{for } j = 1, 2, \ldots, \tag{2.4}$$

where $2^j B$ is the ball with the same center as B and radius $2^j r_B$.

We recall the following construction of an analogue of a dyadic grid on Euclidean spaces for spaces of homogeneous type. The result is due to David [Da88] in slightly less generality and due to Christ [Chr90a] in the present formulation.

Lemma 2.1 *Let (X, d, μ) be a space of homogeneous type. Then there exists a collection $\mathcal{Q} := \{Q_\alpha^k \subseteq X \,:\, k \in \mathbb{Z},\, \alpha \in I_k\}$ of open subsets of X, where I_k is some index set, a constant $\delta \in (0,1)$ and constants $C_1, C_2 > 0$ such that*

(i) $\mu(X \setminus \bigcup_\alpha Q_\alpha^k) = 0$ *for each fixed k and* $Q_\alpha^k \cap Q_\beta^k = \emptyset$ *if* $\alpha \neq \beta$;

(ii) for any α, β, k, l with $l \geq k$, either $Q_\beta^l \subseteq Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;

(iii) for each $l < k$ there is a unique β such that $Q_\alpha^k \subseteq Q_\beta^l$;

(iv) $\operatorname{diam}(Q_\alpha^k) \leq C_1 \delta^k$;

(v) each Q_α^k contains some ball $B(z_\alpha^k, C_2 \delta^k)$, where $z_\alpha^k \in X$.

For a better understanding of the statement, one can think of $Q_\alpha^k \in \mathcal{Q}$ as being a dyadic cube with sidelength δ^k centered at z_α^k.
By abuse of notation we will sometimes call the elements of the collection $\mathcal{Q}$ "cubes".

We fix the following notation for further reference. It describes the covering of a dilated ball $2^j B$ with elements of $\mathcal{Q}$ whose diameters are related to the radius of the ball B.

Notation 2.2 Let $B = B(x_B, r_B)$ be an arbitrary ball in X. With the notation as in Lemma 2.1, we define $k_0 \in \mathbb{Z}$ to be the integer satisfying

$$C_1 \delta^{k_0} \leq r_B < C_1 \delta^{k_0 - 1} \tag{2.5}$$

and for each $j \in \mathbb{N}$ we define $k_j \in \mathbb{Z}$ to be the integer satisfying

$$\delta^{-k_j} \le 2^j < \delta^{-k_j - 1}. \tag{2.6}$$

We further define for each $j \in \mathbb{N}$ the index set M_j related to the ball $B = B(x_B, r_B)$ by

$$M_j := \{\beta \in I_{k_0} \, : \, Q_\beta^{k_0} \cap B(x_B, C_1 \delta^{k_0 - k_j - 2}) \neq \emptyset\}, \tag{2.7}$$

representing all "cubes" out of $\mathcal{Q}$ with "sidelength" approximately equal to r_B that have non-empty intersection with the dilated ball $2^j B$. More precisely, we observe that Lemma 2.1 yields - modulo null sets of μ - for every $j \in \mathbb{N}$ the following inclusions:

$$2^j B \subseteq B(x_B, C_1 \delta^{k_0 - k_j - 2}) \subseteq \bigcup_{\beta \in M_j} Q_\beta^{k_0} \subseteq B(x_B, 2C_1 \delta^{k_0 - k_j - 2}) \subseteq \delta^{-2} 2^{j+1} B. \tag{2.8}$$

The first and the fourth inclusions are simple consequences of the definition of k_0 and k_j, whereas the second one follows from Lemma 2.1 (i) and the third one uses Lemma 2.1 (iv). Further, Lemma 2.1 yields that the sets $Q_\beta^{k_0}$, $\beta \in M_j$, are disjoint and for each $\beta \in M_j$ there exists some $z_\beta^{k_0} \in X$ such that

$$B(z_\beta^{k_0}, c_1 r_B) \subseteq Q_\beta^{k_0} \subseteq B(z_\beta^{k_0}, r_B) \tag{2.9}$$

for some $c_1 \in (0,1)$ independent of j and β due to Lemma 2.1 (v) and (iv).

Remark 2.3 The cardinality of the set M_j defined in (2.7) is bounded from above by a constant times 2^{jn}. This fact is in analogy to the case of Euclidean spaces, saying that for an arbitrary ball $B = B(x_B, r_B)$ in X, one can cover the dilated ball $2^j B = B(x_B, 2^j r_B)$ by approximately 2^{jn} disjoint "cubes" out of $\mathcal{Q}$ of diameter approximately equal to r_B. The argument is a simple modification of the one given in [CW71], Chapitre III, comparing the constants A_1 and N, where N denotes the constant specified in Remark 2.4.

Let $B = B(x_B, r_B)$ be an arbitrary ball in X and let $j \in \mathbb{N}$. To get an estimate for $\# M_j$, observe that by definition of M_j and property (iv) of Lemma 2.1, for every $\beta \in M_j$ there holds the inclusion $B(x_B, C_1 \delta^{k_0 - k_j - 2}) \subseteq B(z_\beta^{k_0}, 3C_1 \delta^{k_0 - k_j - 2})$. Thus, the doubling condition (2.2) and property (v) of Lemma 2.1 yield

$$\begin{aligned}
\# M_j \cdot \mu(B(x_B, C_1 \delta^{k_0 - k_j - 2})) &\le \sum_{\beta \in M_j} \mu(B(z_\beta^{k_0}, 3C_1 \delta^{k_0 - k_j - 2})) \\
&\le A_2 (3C_1 C_2^{-1} \delta^{-k_j - 2})^n \sum_{\beta \in M_j} \mu(B(z_\beta^{k_0}, C_2 \delta^{k_0})) \\
&\le A_2 (3C_1 C_2^{-1} \delta^{-k_j - 2})^n \mu(\bigcup_{\beta \in M_j} Q_\beta^{k_0}),
\end{aligned}$$

using the disjointness of the sets $Q_\beta^{k_0}$ in the last step. With the help of (2.8) and the fact that $\delta^{-k_j} \le 2^j$ we further obtain that the above is bounded by

$$\begin{aligned}
&A_2 (3C_1 C_2^{-1} \delta^{-2})^n 2^{jn} \mu(B(x_B, 2C_1 \delta^{k_0 - k_j - 2})) \\
&\qquad \le A_2^2 (6C_1 C_2^{-1} \delta^{-2})^n 2^{jn} \mu(B(x_B, C_1 \delta^{k_0 - k_j - 2})),
\end{aligned}$$

again applying the doubling condition (2.2). Hence,

$$\# M_j \cdot \mu(B(x_B, C_1 \delta^{k_0 - k_j - 2})) \lesssim 2^{jn} \mu(B(x_B, C_1 \delta^{k_0 - k_j - 2}))$$

and therefore $\# M_j \lesssim 2^{jn}$.

Remark 2.4 Spaces of homogeneous type were first defined by Coifman and Weiss in [CW71], Chapitre III, in a slightly more general way. We shortly remark that the defining property for spaces of homogeneous type was originally the following, reflecting the covering property described in Remark 2.3.

"There exists some $N \in \mathbb{N}$ such that for every $x \in X$ and every $r > 0$ the ball $B(x, r)$ contains at most N points x_i with $d(x_i, x_j) > \frac{r}{2}$."

As can easily be seen, the doubling constant A_1 and the constant N depend on each other. For further details and examples of spaces of homogeneous type, we refer to [CW71], Chapitre III and [Chr90b], Chapter VI.

2.3 Averaging and maximal operators

Let $f \in L^1_{\text{loc}}(X)$. We denote the average of f over an open set $U \in X$ by

$$\langle f \rangle_U := \frac{1}{V(U)} \int_U f(x)\, d\mu(x).$$

Averaging operator With the notation as in Lemma 2.1 we define the following averaging operator on X. It substitutes the dyadic averaging operator on Euclidean spaces.
Let $t > 0$. We denote by $k_0 \in \mathbb{Z}$ the unique integer satisfying

$$C_1 \delta^{k_0} \leq t < C_1 \delta^{k_0 - 1}. \tag{2.10}$$

Then for almost every $x \in X$ there exists a unique $\alpha \in I_{k_0}$ such that $x \in Q^{k_0}_\alpha$. We will therefore define the uncentered averaging operator A_t by

$$A_t f(x) := \frac{1}{V(Q^{k_0}_\alpha)} \int_{Q^{k_0}_\alpha} f(y)\, d\mu(y), \qquad \text{for almost all } x \in X, \tag{2.11}$$

for every $f \in L^1_{\text{loc}}(X)$, where $Q^{k_0}_\alpha$ is the uniquely determined open set out of the collection $\{Q^{k_0}_\beta\}_{\beta \in I_{k_0}}$ with $x \in Q^{k_0}_\alpha$.
Observe that the operator A_t is constant on each open set $Q^{k_0}_\alpha$. Moreover, there holds

$$A_t f = \sum_{\alpha \in I_{k_0}} \langle f \rangle_{Q^{k_0}_\alpha} \mathbb{1}_{Q^{k_0}_\alpha},$$

where k_0 is determined by (2.10).
Let us also remark the following pointwise bound: There exists a constant $C > 0$ such that for almost every $x \in X$ and every $f \in L^1_{\text{loc}}(X)$

$$|A_t f(x)| \leq C \frac{1}{V(x, t)} \int_{B(x,t)} |f(y)|\, d\mu(y). \tag{2.12}$$

This follows immediately from Lemma 2.1, observing that whenever $x \in Q^{k_0}_\alpha$, then there holds $Q^{k_0}_\alpha \subseteq B(x, t) \subseteq B(z^{k_0}_\alpha, 2t)$ and, due to the doubling condition and (2.10), the inequality $V(y, t) \leq V(z^{k_0}_\alpha, 2t) \leq A_2 (2t)^n (C_2 \delta^{k_0})^{-n} V(z^{k_0}_\alpha, C_2 \delta^{k_0}) \lesssim V(Q^{k_0}_\alpha)$.

Maximal operators We denote by $\mathcal{M}$ the uncentered Hardy-Littlewood maximal operator, i.e. for a measurable function $f : X \to \mathbb{C}$ and a point $x \in X$ we set

$$\mathcal{M}f(x) = \sup_{\substack{r>0 \\ y \in B(x,r)}} \frac{1}{V(y,r)} \int_{B(y,r)} |f(z)| \, d\mu(z).$$

Further, for $p \in [1, \infty)$, we denote by $\mathcal{M}_p$ the *p-maximal operator*, i.e. for a measurable function $f : X \to \mathbb{C}$ we set

$$\mathcal{M}_p f = [\mathcal{M}(|f|^p)]^{1/p}.$$

For the sake of convenience, we state the well-known boundedness properties of Hardy-Littlewood maximal functions. For a proof of the theorem in the case of $p = 1$ (that is, for $\mathcal{M}_1 = \mathcal{M}$), we refer to [CW71], Chapitre III. The result for $p > 1$ is then an easy consequence.

Theorem 2.5 *Let $p \in [1, \infty)$. The sublinear operator $\mathcal{M}_p$ is bounded on $L^q(X)$ for every $q \in (p, \infty]$, but not on $L^p(X)$.*

2.4 Lebesgue differentiation theorem

Let us further recall the well-known Lebesgue differentiation theorem and the notion of Lebesgue points. Our presentation is taken from [HK00]. For a proof, we refer to any standard textbook of harmonic analysis, e.g. [Ste70], Chapter I.1.

For $c \geq 1$ and $x \in X$ we define $\mathcal{F}_c(x)$ as the family of all measurable sets $E \subseteq X$ such that $E \subseteq B(x,r)$ and $V(x,r) \leq cV(E)$ for some $r > 0$.
We say that a sequence of nonempty sets $\{E_i\}_{i=1}^{\infty}$ converges to x if there exists a sequence of radii $r_i > 0$ such that $E_i \subseteq B(x, r_i)$ and $r_i \to 0$ as $i \to \infty$.

Theorem 2.6 *Let $f \in L^1_{\mathrm{loc}}(X)$. For μ-almost every $x \in X$ there holds*

$$\lim_{r \to 0} \frac{1}{V(x,r)} \int_{B(x,r)} f(y) \, d\mu(y) = f(x). \tag{2.13}$$

Moreover, if we fix $c \geq 1$, then for μ-almost every $x \in X$ and every sequence of sets $\{E_i\}_{i=1}^{\infty} \subseteq \mathcal{F}_c(x)$ that converges to x we have

$$\lim_{i \to \infty} \frac{1}{V(E_i)} \int_{E_i} f(y) \, d\mu(y) = f(x). \tag{2.14}$$

Given $f \in L^1_{\mathrm{loc}}(X)$ it is often convenient to identify f with the representative given everywhere by the formula

$$f(x) := \limsup_{r \to 0} \frac{1}{V(x,r)} \int_{B(x,r)} f(y) \, d\mu(y). \tag{2.15}$$

Theorem 2.6 shows that in this way f is only modified on a set of measure zero.

Definition 2.7 *We say that $x \in X$ is a* Lebesgue point of f, *if*

$$\lim_{r \to 0} \frac{1}{V(x,r)} \int_{B(x,r)} |f(y) - f(x)| \, d\mu(y) = 0,$$

where $f(x)$ is given by (2.15).

It follows from Theorem 2.6 that almost all points of X are Lebesgue points of f. Observe that if $x \in X$ is a Lebesgue point of f, then both (2.13) and (2.14) are true.

2.5 Holomorphic functional calculus

One of the fundamental tools in the theory to be developed in the sequel is the holomorphic functional calculus introduced by McIntosh in [McI86]. In situations when in the standard Hardy space and Calderón-Zygmund theory convolution operators (e.g. convolution with the Poisson kernel) and Littlewood-Paley theory were used, we now work with approximation operators constructed by holomorphic functional calculus. These approximation operators are then associated to a general sectorial operator L instead of that they are associated to the Laplacian.
We only state the most important definitions and results. For more details on holomorphic functional calculi and proofs of the cited results below we refer to [McI86], [ADM96], [KW04] and [Haa06].

For $0 \leq \omega < \mu < \pi$ we define the closed and open sectors in the complex plane $\mathbb{C}$ by

$$\begin{aligned} S_{\omega+} &:= \{\zeta \in \mathbb{C} \setminus \{0\} : |\arg \zeta| \leq \omega\} \cup \{0\}, \\ \Sigma_\mu^0 &:= \{\zeta \in \mathbb{C} : \zeta \neq 0, |\arg \zeta| < \mu\}. \end{aligned}$$

We denote by $H(\Sigma_\mu^0)$ the space of all holomorphic functions on Σ_μ^0. We further define the space $H^\infty(\Sigma_\mu^0)$ consisting of all bounded holomorphic functions on Σ_μ^0 and subspaces $\Psi_{\sigma,\tau}(\Sigma_\mu^0)$ with specified decay at zero and infinity by

$$\begin{aligned} H^\infty(\Sigma_\mu^0) &:= \{\psi \in H(\Sigma_\mu^0) : \|\psi\|_{L^\infty(\Sigma_\mu^0)} < \infty\}, \\ \Psi_{\sigma,\tau}(\Sigma_\mu^0) &:= \{\psi \in H(\Sigma_\mu^0) : |\psi(\zeta)| \leq C\,|\zeta|^\sigma\,(1 + |\zeta|^{\sigma+\tau})^{-1} \text{ for every } \zeta \in \Sigma_\mu^0\} \end{aligned}$$

for every $\sigma, \tau > 0$. Alternatively, one can say that

$$\psi \in \Psi_{\sigma,\tau}(\Sigma_\mu^0) \;\Leftrightarrow\; \psi \in H^\infty(\Sigma_\mu^0) \text{ and } |\psi(\zeta)| \leq C \inf\{|\zeta|^\sigma, |\zeta|^{-\tau}\} \text{ for every } \zeta \in \Sigma_\mu^0.$$

Let $\Psi(\Sigma_\mu^0) := \bigcup_{\sigma,\tau>0} \Psi_{\sigma,\tau}(\Sigma_\mu^0)$.

Definition 2.8 *Let $\omega \in [0,\pi)$. A closed operator L on a Hilbert space H is said to be* sectorial of angle ω *if $\sigma(L) \subseteq S_{\omega+}$ and, for each $\mu > \omega$, there exists a constant $C_\mu > 0$ such that*

$$\left\|(\zeta I - L)^{-1}\right\| \leq C_\mu\,|\zeta|^{-1}, \qquad \zeta \notin S_{\mu+}.$$

Remark 2.9 Let $\omega \in [0,\pi)$ and let L be a sectorial operator of angle ω on a Hilbert space H. Then L has dense domain in H. If L is assumed to be injective, then L also has dense range in H. See e.g. [CDMY96], Theorem 2.3 and Theorem 3.8.

For a sectorial operator L, a functional calculus on $\Psi(\Sigma_\mu^0)$ can be defined as follows.

Definition and Theorem 2.10 *Let H be a Hilbert space and L be a sectorial operator of angle $\omega \in [0,\pi)$. For $\omega < \theta < \mu < \pi$ and $\psi \in \Psi(\Sigma_\mu^0)$ put*

$$\Phi_L(\psi) := \psi(L) := \frac{1}{2\pi i} \int_{\partial\Sigma_\theta^0} \psi(\lambda)(\lambda I - L)^{-1}\,d\lambda. \tag{2.16}$$

Then $\Phi_L : \Psi(\Sigma_\mu^0) \to B(H)$ defines a linear and multiplicative map with the following properties:

(i) Let $f_n, f \in H^\infty(\Sigma^0_\mu)$ be uniformly bounded and $f_n(\lambda) \to f(\lambda)$ for $\lambda \in \Sigma^0_\mu$. Then for all $\psi \in \Psi(\Sigma^0_\mu)$

$$\lim_{n\to\infty} \Phi_L(f_n \cdot \psi) = \Phi_L(f \cdot \psi) \qquad \text{in } B(H).$$

(ii) If $\psi(\lambda) = \frac{\lambda}{(\mu_1-\lambda)(\mu_2-\lambda)}$ with $\mu_1, \mu_2 \notin \overline{\Sigma^0_\mu}$, then

$$\psi(L) = L(\mu_1 I - L)^{-1}(\mu_2 I - L)^{-1}.$$

(iii) $\|\psi(L)\| \le \frac{c}{2\pi} \int_{\partial\Sigma^0_\theta} |\psi(\lambda)| \frac{d|\lambda|}{|\lambda|}$ for some positive constant c independent of ψ.

The integral in (2.16) is well-defined, since on $\partial\Sigma^0_\theta$ the estimate $\left\|(\lambda I - L)^{-1}\right\| \lesssim |\lambda|^{-1}$ holds. Moreover, an extension of Cauchy's theorem shows that the definition is independent of the choice of $\theta \in (\omega, \mu)$.
For a proof of the theorem, we refer to [KW04], Theorem 9.2.

With the help of the convergence property in Theorem 2.10 (i), one can extend the functional calculus on $\Psi(\Sigma^0_\mu)$ to functions from $H^\infty(\Sigma^0_\mu)$ in the following way.
Let L be an injective, sectorial operator of angle $\omega \in [0, \pi)$ and let $\mu \in (\omega, \pi)$. Let $f \in H^\infty(\Sigma^0_\mu)$ and $f_n \in \Psi(\Sigma^0_\mu)$ be uniformly bounded with $f_n \to f$ pointwise. We set $\psi(z) := z(1+z)^{-2}$. Theorem 2.10 then yields that

$$\lim_{n\to\infty} f_n(L)(\psi(L)x) = \lim_{n\to\infty} (f_n \cdot \psi)(L)x = (f \cdot \psi)(L)x$$

for every $x \in H$. Moreover, one can show that the operator $\psi(L) = L(1+L)^{-2}$ is injective and has dense range in H. Thus, one can define by

$$f(L) := [\psi(L)]^{-1}(f \cdot \psi)(L)$$

a closed operator on H, that satisfies the following properties.

Definition and Theorem 2.11 *Let H be a Hilbert space. If L is an injective, sectorial operator of angle $\omega \in [0, \pi)$ in H, and $\mu \in (\omega, \pi)$, then we say that L has a* bounded $H^\infty(\Sigma^0_\mu)$ functional calculus *if there exists a constant $c_\mu > 0$ such that for all $f \in H^\infty(\Sigma^0_\mu)$, there holds $f(L) \in B(H)$ and*

$$\|f(L)\| \le c_\mu \|f\|_{L^\infty(\Sigma^0_\mu)}.$$

For every $f \in H^\infty(\Sigma^0_\mu)$ we put $\bar{\Phi}_L(f) := f(L)$. If L has a bounded $H^\infty(\Sigma^0_\mu)$ functional calculus, then $\bar{\Phi}_L : H^\infty(\Sigma^0_\mu) \mapsto B(H)$ is an extension of Φ_L and defines a linear and multiplicative map.

Furthermore, the following convergence lemma is valid.

Lemma 2.12 *Let H be a Hilbert space, let L be an injective, sectorial operator of angle $\omega \in [0, \pi)$ in H, and let $\mu \in (\omega, \pi)$. If $f, f_n \in H^\infty(\Sigma^0_\mu)$ with $f_n(\lambda) \to f(\lambda)$ for $\lambda \in \Sigma^0_\mu$ and $\{f_n(L)\}_n$ is uniformly bounded in $B(H)$, then $f(L) \in B(H)$, $f_n(L)x \to f(L)x$ for all $x \in H$ and $\|f(L)\| \le \sup_n \|f_n(L)\|$.*

Let us in addition state a characterization for L to have a bounded holomorphic functional calculus. In particular the equivalence of (i) and (iv) will quite frequently be used in the sequel.

Theorem 2.13 *Let H be a Hilbert space and L be an injective sectorial operator of angle $\omega \in [0, \pi)$. Then the following statements are equivalent:*

(i) L has a bounded $H^\infty(\Sigma_\mu^0)$ functional calculus for all $\mu \in (\omega, \pi)$.

(ii) L has a bounded $H^\infty(\Sigma_\mu^0)$ functional calculus for some $\mu \in (\omega, \pi)$.

(iii) For some $\mu \in (\omega, \pi)$ there exists some $C > 0$ such that for all $\psi \in \Psi(\Sigma_\mu^0)$ there holds $\|\psi(L)\| \leq C \|\psi\|_{L^\infty(\Sigma_\mu^0)}$.

(iv) For some (all) $\mu \in (\omega, \pi)$ and some $\psi \in \Psi(\Sigma_\mu^0) \setminus \{0\}$ there exists some $C > 0$ such that for all $x \in H$

$$C^{-1} \|x\|^2 \leq \int_0^\infty \|\psi(tL)x\|^2 \frac{dt}{t} \leq C \|x\|^2 .$$

We end the section by giving a different representation of $\psi(L)$, as defined in (2.16), whenever $\psi \in \Psi(\Sigma_\mu^0)$ for some $\mu \in (\omega, \pi/2)$.

Remark 2.14 Let L be a sectorial operator of angle $\omega \in [0, \pi/2)$ and $\omega < \theta < \nu < \mu < \pi/2$. If $\psi \in \Psi(\Sigma_\mu^0)$, then $\psi(L)$ can alternatively be represented in terms of the semigroup instead of the resolvent as

$$\psi(L) = \int_{\Gamma_+} e^{-zL} \eta_+(z)\, dz + \int_{\Gamma_-} e^{-zL} \eta_-(z)\, dz, \tag{2.17}$$

where η_+ and η_- are defined by

$$\eta_\pm(z) = \frac{1}{2\pi i} \int_{\gamma_\pm} e^{\xi z} \psi(\xi)\, d\xi, \qquad z \in \Gamma_\pm, \tag{2.18}$$

and the paths of integration are given by $\Gamma_\pm = \mathbb{R}^+ e^{\pm i(\pi/2 - \theta)}$ and $\gamma_\pm = \mathbb{R}^+ e^{\pm i\nu}$.

2.6 Tent spaces and Carleson measures

Tent spaces on $\mathbb{R}^n$ were introduced by Coifman, Meyer and Stein in [CMS83]. Various ideas for tent spaces had been used before, but in [CMS83] they appear for the first time explicitly. Further development was then done by the same authors in [CMS85], where the most important results for the theory of tent spaces can be found. As was mentioned there already, tent spaces naturally arise in harmonic analysis and they provide the appropriate setting for the study of square functions, (non-tangential) maximal functions, Carleson measures and Hardy and BMO spaces and are deeply connected with the theory of singular integrals. And, as can be seen in the sequel, the same also stays true for the theory of Hardy and BMO spaces associated to operators and the $T(1)$-Theorem presented in Chapter 6.

We recall the most important definitions and properties of tent spaces and Carleson measures and functions. For more details we refer to [CMS85] and [Ste93], Chapter II. As mentioned in [Ste93], the proofs, given there in the case of the Euclidean space $\mathbb{R}^n$, take over to spaces of homogeneous type. For some of the results, we also give the corresponding reference to a proof in the setting of spaces of homogeneous type.

For any $x \in X$ and any $\alpha > 0$, we denote by $\Gamma^\alpha(x)$ the *cone* of aperture α with vertex x, namely

$$\Gamma^\alpha(x) := \{(y,t) \in X \times (0,\infty) \,:\, d(y,x) < \alpha t\}.$$

For any closed subset $F \subseteq X$ and any $\alpha > 0$ we denote by $\mathcal{R}^\alpha(F)$ the union of all cones with vertices in F (also called "saw-tooth region"), i.e.

$$\mathcal{R}^\alpha(F) := \bigcup_{x \in F} \Gamma^\alpha(x).$$

For simplicity we will write $\Gamma(x)$ instead of $\Gamma^1(x)$ and $\mathcal{R}(F)$ instead of $\mathcal{R}^1(F)$.
If O is an open subset of X, then the *tent* over O, denoted by $\hat{O}$, is defined as

$$\hat{O} := [\mathcal{R}(O^c)]^c = \{(x,t) \in X \times (0,\infty) \,:\, \mathrm{dist}(x, O^c) \geq t\}.$$

For balls $B = B(x_B, r_B)$ in X, one can instead of tents alternatively work with *cylindrical tents*, defined by

$$T(B) := \{(x,t) \in X \times (0,\infty) \,:\, x \in B,\ 0 < t \leq r_B\}.$$

Then there holds $\hat{B} \subseteq T(B) \subseteq \widehat{2B}$.

Definition 2.15 *For any measurable function F on $X \times (0,\infty)$, we define the conical square function $\mathscr{A}F$ by*

$$\mathscr{A}F(x) := \left(\iint_{\Gamma(x)} |F(y,t)|^2 \, \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \qquad x \in X,$$

and the Carleson function $\mathscr{C}F$ by

$$\mathscr{C}F(x) := \sup_{B\,:\, x \in B} \left(\frac{1}{V(B)} \iint_{\hat{B}} |F(y,t)|^2 \, \frac{d\mu(y) dt}{t} \right)^{1/2}, \qquad x \in X,$$

where the supremum is taken over all balls B in X that contain x.
For $0 < p < \infty$, the tent spaces on $X \times (0,\infty)$ are defined by

$$T^p(X) := \{F : X \times (0,\infty) \to \mathbb{C} \text{ measurable};\, \|F\|_{T^p(X)} := \|\mathscr{A}F\|_{L^p(X)} < \infty\}.$$

The tent space $T^\infty(X)$ is defined by

$$T^\infty(X) := \{F : X \times (0,\infty) \to \mathbb{C} \text{ measurable};\, \|F\|_{T^\infty(X)} := \|\mathscr{C}F\|_{L^\infty(X)} < \infty\}.$$

When $p \in [1,\infty]$, the space $(T^p(X), \|\,.\,\|_{T^p(X)})$ is a Banach space.

In [HLM+09], Lemma 4.7, the following density result for tent spaces in the case of spaces of homogeneous type was shown.

Lemma 2.16 *If $1 \leq p < \infty$, then $T^p(X) \cap T^2(X)$ is dense in $T^p(X)$.*

We recall the following duality results for tent spaces. For a proof, see [CMS85], Theorem 1 and 2.

Theorem 2.17 *(i) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There exists a constant $C > 0$ such that for all $F \in T^p(X)$ and all $G \in T^{p'}(X)$ there holds*

$$\iint_{X\times(0,\infty)} |F(x,t)G(x,t)| \, \frac{d\mu(x)dt}{t} \le C \int_X \mathscr{A}(F)(x)\mathscr{A}(G)(x)\, d\mu(x).$$

Further, there exists a constant $C > 0$ such that for all $F \in T^1(X)$ and all $G \in T^\infty(X)$ there holds

$$\iint_{X\times(0,\infty)} |F(x,t)G(x,t)| \, \frac{d\mu(x)dt}{t} \le C \int_X \mathscr{A}(F)(x)\mathscr{C}(G)(x)\, d\mu(x).$$

(ii) The pairing

$$\langle F, G\rangle \mapsto \iint_{X\times(0,\infty)} F(x,t)G(x,t)\, \frac{d\mu(x)dt}{t}$$

realizes $T^{p'}(X)$ as equivalent to the dual of $T^p(X)$ if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, and realizes $T^\infty(X)$ as equivalent to the dual of $T^1(X)$.

The relation between the functionals $\mathscr{A}$ and $\mathscr{C}$ are given as follows. For a proof, we again refer to [CMS85], Theorem 3.

Theorem 2.18 *(i) Let $0 < p < \infty$. There exists a constant $C > 0$ such that for all measurable functions F on $X \times (0,\infty)$*

$$\|\mathscr{A}(F)\|_{L^p(X)} \le C \, \|\mathscr{C}(F)\|_{L^p(X)}.$$

(ii) Let $2 < p \le \infty$. There exists a constant $C > 0$ such that for all measurable functions F on $X \times (0,\infty)$

$$\|\mathscr{C}(F)\|_{L^p(X)} \le C \, \|\mathscr{A}(F)\|_{L^p(X)}.$$

In particular, whenever $2 < p < \infty$ and $F \in T^p(X)$, then $\|\mathscr{C}(F)\|_{L^p(X)} \approx \|\mathscr{A}(F)\|_{L^p(X)}$.

We now come to the notion of atoms and the atomic decomposition of $T^1(X)$ on spaces of homogeneous type, as defined by Russ in [Rus07] in analogy to the notion of atoms on $\mathbb{R}^n$.

Definition 2.19 *A measurable function A on $X \times (0,\infty)$ is said to be a $T^1(X)$-atom, if there exists a ball $B \subseteq X$ such that A is supported in $\hat{B}$ and*

$$\iint_{X\times(0,\infty)} |A(x,t)|^2 \, \frac{d\mu(x)dt}{t} \le \frac{1}{V(B)}.$$

Note that a $T^1(X)$-atom belongs to $T^1(X)$ and its norm is controlled by a constant only depending on X. This can be seen as follows. If A is supported in $\hat{B}$, then $\mathscr{A}(A)$ is supported in B by definition of tent regions. Furthermore, the inequality (2.3), which is a consequence of the doubling property, implies that there exists a constant $C > 0$ such that for all $y \in X$

$$C^{-1} \le \int_{B(y,t)} V(x,t)^{-1}\, d\mu(x) \le C. \tag{2.19}$$

Hence, the Cauchy-Schwarz inequality, Fubini's theorem and the definition of atoms yield

$$\begin{aligned}
\|A\|_{T^1(X)} &= \int_X \left(\iint_{\Gamma(x)} |A(y,t)|^2 \, \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2} \, d\mu(x) \\
&= \int_B \left(\iint_{\Gamma(x)} |A(y,t)|^2 \, \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2} \, d\mu(x) \\
&\leq V(B)^{1/2} \left(\int_X \int_0^\infty \int_{B(x,t)} |A(y,t)|^2 \, \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} d\mu(x) \right)^{1/2} \\
&\leq V(B)^{1/2} \left(\int_X \int_0^\infty \left(\int_{B(y,t)} \frac{d\mu(x)}{V(x,t)} \right) |A(y,t)|^2 \, \frac{d\mu(y)dt}{t} \right)^{1/2} \leq C.
\end{aligned}$$

The proposition below shows that, conversely, any function in $T^1(X)$ has an atomic decomposition. The result is mainly taken from [Rus07], generalizing the analogous result of [CMS85] on $\mathbb{R}^n$. For the convergence in $T^2(X)$, we refer to [DL09], Proposition 3.6.

Proposition 2.20 *There exists a constant $C > 0$ with the following property: For every $F \in T^1(X)$, there exists a numerical sequence $\{\lambda_j\}_{j=0}^\infty$ and a sequence of $T^1(X)$-atoms $\{A_j\}_{j=0}^\infty$ such that*

$$F = \sum_{j=0}^{\infty} \lambda_j A_j \qquad \text{in } T^1(X) \text{ and a.e. in } X \times (0,\infty), \tag{2.20}$$

and

$$\sum_{j=0}^{\infty} |\lambda_j| \leq C \, \|F\|_{T^1(X)} \, .$$

Moreover, if $F \in T^1(X) \cap T^2(X)$, then the decomposition (2.20) *also converges in $T^2(X)$.*

We finally state the definition of non-tangential maximal functions and Carleson measures and the corresponding duality result.

Definition 2.21 *For any measurable function F on $X \times (0,\infty)$, the* non-tangential maximal function *F^* is defined by*

$$F^*(x) := \sup_{(y,t)\in\Gamma(x)} |F(y,t)| \, , \qquad x \in X. \tag{2.21}$$

The space $\mathcal{N}$ is defined by

$$\mathcal{N} := \{F : X \times (0,\infty) \to \mathbb{C} \text{ measurable}; \, \|F\|_{\mathcal{N}} := \|F^*\|_{L^1(X)} < \infty\}.$$

A Carleson measure *is a Borel measure ν on $X \times (0,\infty)$ such that*

$$\|\nu\|_{\mathcal{C}} := \sup_B \frac{1}{V(B)} \iint_{\hat{B}} |d\nu| < \infty,$$

where the supremum is taken over all balls B in X. We define $\mathcal{C}$ to be the space of all Carleson measures.

The spaces $(\mathcal{N}, \| \,.\, \|_{\mathcal{N}})$ and $(\mathcal{C}, \| \,.\, \|_{\mathcal{C}})$ are Banach spaces.
Observe that for $F \in T^\infty(X)$ there holds

$$\|F\|^2_{T^\infty(X)} = \|\mathscr{C}F\|^2_{L^\infty(X)} = \left\| |F(y,t)|^2 \ \frac{d\mu(y)dt}{t} \right\|_{\mathcal{C}}. \tag{2.22}$$

The connection between $\mathcal{N}$ and $\mathcal{C}$ is given by the following theorem. It is implicit already contained in [FS72], a proof is stated in [CMS85], Proposition 3. The space $\mathcal{C}$ is not exactly the dual space of $\mathcal{N}$. For a precise duality result of non-tangential maximal functions and Carleson measures in the case of $\mathbb{R}^n$, we refer to [CMS85], Proposition 1.

Theorem 2.22 *If $F \in \mathcal{N}$ and $\nu \in \mathcal{C}$, then*

$$\iint_{X \times (0,\infty)} |F(x,t)| \ d\nu(x,t) \leq C \, \|F\|_{\mathcal{N}} \cdot \|\nu\|_{\mathcal{C}} \, .$$

For applications, we also need the following corollary. The result has its origin in [CMS85], Remark (b) on p. 320.

Corollary 2.23 *Let $2 < p < \infty$. Let F be a measurable function on $X \times (0,\infty)$ with $F^* \in L^p(X)$ and let $G \in T^\infty(X)$. Then there holds*

$$\|\mathscr{C}(F \cdot G)\|_{L^p(X)} \leq C \, \|F^*\|_{L^p(X)} \, \|\mathscr{C}G\|_{L^\infty(X)} \, ,$$

with a constant $C > 0$ independent of F and G.

Proof: Let B be an arbitrary ball in X. The assumption $G \in T^\infty(X)$ implies that $|G(y,t)|^2 \, \frac{d\mu(y)dt}{t}$ is a Carleson measure. Replacing $|F|$ by $|F|^2$, Theorem 2.22 and (2.22) then yield that

$$\begin{aligned} \iint_{\hat{B}} |F(y,t)|^2 \, |G(y,t)|^2 \ \frac{d\mu(y)dt}{t} &\lesssim \|\mathscr{C}G\|^2_{L^\infty(X)} \int_X ((F \cdot \mathbb{1}_{\hat{B}})^*(z))^2 \, d\mu(z) \\ &= \|\mathscr{C}G\|^2_{L^\infty(X)} \int_B (F^*(z))^2 \, d\mu(z), \end{aligned}$$

by definition of tent regions. Hence, we get for every $x \in X$

$$\begin{aligned} \mathscr{C}(F \cdot G)(x) &= \sup_{B : x \in B} \left(\frac{1}{V(B)} \iint_{\hat{B}} |F(y,t)|^2 \, |G(y,t)|^2 \ \frac{d\mu(y)dt}{t} \right)^{1/2} \\ &\lesssim \sup_{B : x \in B} \left(\frac{1}{V(B)} \int_B (F^*(z))^2 \, d\mu(z) \right)^{1/2} \|\mathscr{C}G\|_{L^\infty(X)} \\ &= \mathcal{M}_2(F^*)(x) \, \|\mathscr{C}G\|_{L^\infty(X)} \, . \end{aligned}$$

Since $\mathcal{M}_2$ is bounded on $L^p(X)$ for every $p > 2$, we obtain

$$\begin{aligned} \|\mathscr{C}(F \cdot G)\|_{L^p(X)} &\lesssim \|\mathcal{M}_2(F^*)\|_{L^p(X)} \, \|\mathscr{C}G\|_{L^\infty(X)} \\ &\lesssim \|F^*\|_{L^p(X)} \, \|\mathscr{C}G\|_{L^\infty(X)} \, . \end{aligned}$$

□

3 Off-diagonal estimates and assumptions on the operator

Off-diagonal estimates are the most important technical tool in absence of pointwise kernel estimates. We introduce in this chapter three different types of off-diagonal estimates and present various features of those. In a second part, we fix our assumptions on the operator L and show certain self-improving properties of the assumed Davies-Gaffney estimates.
In the following, $m > 1$ will be a fixed constant, representing the order of the sectorial operator L. Later on, the letter m will in addition be used for molecules, but to our opinion there will not be any chance of confusion.

3.1 Davies-Gaffney and other off-diagonal estimates

For a family of linear operators $\{S_t\}_{t>0}$ acting on $L^2(X)$, we describe the notion of Davies-Gaffney estimates, off-diagonal estimates of a certain order and weak off-diagonal estimates of a certain order on $L^2(X)$.

Davies-Gaffney estimates We say that the family of operators $\{S_t\}_{t>0}$ satisfies *Davies-Gaffney estimates (L^2 off-diagonal estimates)* if there exist constants $C, c, \tau > 0$ such that for arbitrary open sets $E, F \subseteq X$

$$\|S_t f\|_{L^2(F)} \leq C e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{\tau}{2m-1}}} \|f\|_{L^2(E)}, \tag{3.1}$$

for every $t > 0$ and every $f \in L^2(X)$ supported in E.
Similarly, we say that a family of operators $\{S_z\}_{z\in\Sigma^0_\mu}$, $\mu \in (0, \frac{\pi}{2})$, satisfies *Davies-Gaffney estimates (L^2 off-diagonal estimates) in $z \in \Sigma^0_\mu$* if the analogue of (3.1) holds with $|z|$ in place of t on the right-hand side.

Off-diagonal estimates We say that a family of operators $\{S_t\}_{t>0}$ satisfies *L^2 off-diagonal estimates of order γ*, $\gamma > 0$, if there exists a constant $C > 0$ such that for arbitrary open sets $E, F \subseteq X$

$$\|S_t f\|_{L^2(F)} \leq C \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(E)},$$

for every $t > 0$ and every $f \in L^2(X)$ supported in E.

Weak off-diagonal estimates We say that a family of linear operators $\{S_t\}_{t>0}$ satisfies *weak L^2 off-diagonal estimates of order γ*, $\gamma > 0$, if there exists a constant $C > 0$ such that for every $t > 0$, arbitrary balls $B_1, B_2 \in X$ with radius $r = t^{1/2m}$ and every $f \in L^2(X)$ supported in B_1

$$\|S_t f\|_{L^2(B_2)} \leq C \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)}. \tag{3.2}$$

Unless otherwise specified, we always mean by (weak) off-diagonal estimates the definition of (weak) L^2 off-diagonal estimates.

We collect some important properties of the different concepts of off-diagonal estimates. Obviously, Davies-Gaffney estimates imply off-diagonal estimates of any order $\gamma > 0$ and off-diagonal estimates of a certain order $\gamma > 0$ imply weak off-diagonal estimates of the same order γ.
Moreover, the next lemma shows that a family of operators that satisfies weak off-diagonal estimates of any order larger than $\frac{n}{2m}$ is uniformly bounded on $L^2(X)$. The uniform boundedness of operator families that satisfy off-diagonal or Davies-Gaffney estimates on $L^2(X)$ follows immediately from the definition by taking $E = F = X$.

Lemma 3.1 *Assume that the family of operators $\{S_t\}_{t>0}$ satisfies weak L^2 off-diagonal estimates of order $\gamma > \frac{n}{2m}$. Then S_t is bounded on $L^2(X)$ uniformly in $t > 0$, i.e. there exists a constant $C > 0$ such that for all $f \in L^2(X)$ and every $t > 0$*

$$\|S_t f\|_{L^2(X)} \leq C \|f\|_{L^2(X)} .$$

Proof: Let $t > 0$ and $f, g \in L^2(X)$. In order to apply the weak L^2 off-diagonal estimates for S_t, we will split X with the help of Lemma 2.1 into "cubes" out of $\mathcal{Q}$ with diameter approximately equal to $t^{1/2m}$ and then order them into annuli around one fixed "cube" to get an estimate for the distance of the "cubes". With the notation as in Lemma 2.1, let $k_0 \in \mathbb{Z}$ be the integer satisfying $C_1\delta^{k_0} \leq t^{1/2m} < C_1\delta^{k_0-1}$. In addition, for every $\alpha \in I_{k_0}$ we denote by B_α the ball $B(z_\alpha^{k_0}, t^{1/2m})$ and observe that Lemma 2.1 (iv) and (v) yield the inclusion $Q_\alpha^{k_0} \subseteq B_\alpha$. Then there holds by assumptions

$$\begin{aligned}
|\langle S_t f, g\rangle| &\leq \sum_{\alpha\in I_{k_0}} \sum_{\beta\in I_{k_0}} \left|\langle S_t \mathbb{1}_{Q_\alpha^{k_0}} f, \mathbb{1}_{Q_\beta^{k_0}} g\rangle\right| \\
&\lesssim \sum_{\alpha\in I_{k_0}} \sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(Q_\alpha^{k_0})} \|g\|_{L^2(Q_\beta^{k_0})} \\
&\leq \left(\sum_{\alpha\in I_{k_0}} \sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t}\right)^{-\gamma} \|f\|^2_{L^2(Q_\alpha^{k_0})}\right)^{1/2} \\
&\qquad \times \left(\sum_{\alpha\in I_{k_0}} \sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t}\right)^{-\gamma} \|g\|^2_{L^2(Q_\beta^{k_0})}\right)^{1/2} , \qquad (3.3)
\end{aligned}$$

using the Cauchy-Schwarz inequality in the last step.
Let $\alpha \in I_{k_0}$ be fixed and let $j \in \mathbb{N}$. As in Notation 2.2 we define the index set M_j related to the ball B_α by

$$M_j := \{\beta \in I_{k_0} \,:\, Q_\beta^{k_0} \cap B(z_\alpha^{k_0}, C_1\delta^{k_0-k_j-2}) \neq \emptyset\}.$$

The inclusions (2.8) from Notation 2.2 yield that if $z_\beta^{k_0} \in S_j(B_\alpha)$, then $\beta \in M_j$ and, by definition of the annulus, $\operatorname{dist}(B_\alpha, B_\beta) \gtrsim 2^j t^{1/2m}$ for every $j \geq 3$. We therefore get for fixed $\alpha \in I_{k_0}$

$$\sum_{\beta\in I_{k_0}} \left(1+\frac{\operatorname{dist}(B_\alpha,B_\beta)^{2m}}{t}\right)^{-\gamma} \le \sum_{j=0}^{\infty} \sum_{\substack{\beta\in I_{k_0}\\ z_\beta^{k_0}\in S_j(B_\alpha)}} \left(1+\frac{\operatorname{dist}(B_\alpha,B_\beta)^{2m}}{t}\right)^{-\gamma}$$

$$\lesssim \sum_{j=0}^{\infty}\sum_{\beta\in M_j} (1+2^j)^{-2m\gamma} \lesssim \sum_{j=0}^{\infty} 2^{-2m\gamma j}2^{nj}, \tag{3.4}$$

where we used the result of Remark 2.3 in the last step, saying that the cardinality of M_j is less than a constant times 2^{jn}.
On the other hand, the disjointness of the cubes $\{Q_\alpha^{k_0}\}_{\alpha\in I_{k_0}}$ implies that

$$\sum_{\alpha\in I_{k_0}} \|f\|^2_{L^2(Q_\alpha^{k_0})} \le \|f\|^2_{L^2(X)}.$$

Hence, the expression in the first bracket of (3.3) is bounded by a constant times $\|f\|^2_{L^2(X)}$. Repeating the same procedure for the second bracket with the roles of α and β interchanged and f replaced by g finally shows that $|\langle S_t f, g\rangle| \lesssim \|f\|_{L^2(X)}\|g\|_{L^2(X)}$. □

Under the same assumptions as in Lemma 3.1, we can define the action of the operator S_t on $L^\infty(X)$ in the $L^2_{\text{loc}}(X)$ sense via duality. This will be, for instance, helpful to define the action of the semigroup $\{e^{-tL}\}_{t>0}$ on $L^\infty(X)$ for sectorial operators L satisfying Davies-Gaffney estimates, or to give a meaning to the assumption $T(1)\in BMO_L(X)$ in Theorem 6.13.

Remark 3.2 (i) Let $\{S_t\}_{t>0}$ be a family of linear operators on $L^2(X)$ that satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$. Then, for every $t>0$ and every ball B in X, the operator S_t^* also acts from $L^2(B)$ to $L^1(X)$ and one can thus define S_t as an operator from $L^\infty(X)$ to $L^2_{\text{loc}}(X)$ via duality. This works as follows:
Let $f\in L^\infty(X)$ and $t>0$. Further, let $B=B(x_B,t^{1/2m})$ be some ball in X and $\varphi\in L^2(X)$ with $\operatorname{supp}\varphi\subseteq B$. Similar to the proof of Lemma 3.1, we split X into annuli around B on the basis of Lemma 2.1 and Notation 2.2. That is, we denote by k_0 the integer defined in (2.5) and for each $j\in\mathbb{N}$ by M_j the set of indices defined in (2.7), so that $2^jB\subseteq\bigcup_{\beta\in M_j} Q_\beta^{k_0}$, where each open set $Q_\beta^{k_0}$ is contained in a ball $B_\beta = B(z_\beta^{k_0},t^{1/2m})$. Due to the Cauchy-Schwarz inequality and the weak off-diagonal estimates for S_t^* we obtain

$$|\langle f, S_t^*\varphi\rangle| \le \sum_{\beta\in I_{k_0}} |\langle \mathbb{1}_{Q_\beta^{k_0}} f, S_t^*\varphi\rangle| \le \sum_{j=0}^{\infty} \sum_{\substack{\beta\in I_{k_0}\\ z_\beta^{k_0}\in S_j(B)}} \|f\|_{L^2(Q_\beta^{k_0})}\|S_t^*\varphi\|_{L^2(Q_\beta^{k_0})}$$

$$\le \sum_{j=0}^{\infty}\Bigg(\sum_{\substack{\beta\in I_{k_0}\\ z_\beta^{k_0}\in S_j(B)}} \|f\|^2_{L^2(Q_\beta^{k_0})}\Bigg)^{1/2}\Bigg(\sum_{\substack{\beta\in I_{k_0}\\ z_\beta^{k_0}\in S_j(B)}} \|S_t^*\varphi\|^2_{L^2(Q_\beta^{k_0})}\Bigg)^{1/2}$$

$$\lesssim \|f\|_{L^\infty(X)} \sum_{j=0}^{\infty}\Bigg(\sum_{\beta\in M_j} V(Q_\beta^{k_0})\Bigg)^{1/2}\Bigg(\sum_{\substack{\beta\in I_{k_0}\\ z_\beta^{k_0}\in S_j(B)}} \left(1+\frac{\operatorname{dist}(B,B_\beta)^{2m}}{t}\right)^{-2\gamma}\|\varphi\|^2_{L^2(B)}\Bigg)^{1/2}, \tag{3.5}$$

using that if $z_\beta^{k_0} \in S_j(B)$, then β is in M_j. In addition, the disjointness of the open sets $Q_\beta^{k_0}$, the inclusions in (2.8) and the doubling property (2.2) of μ yield

$$\left(\sum_{\beta\in M_j} V(Q_\beta^{k_0})\right)^{1/2} \lesssim V(2^j B)^{1/2} \lesssim 2^{jn/2} V(B)^{1/2}.$$

On the other hand, for every $j \geq 3$ and every $\beta \in I_{k_0}$ with $z_\beta^{k_0} \in S_j(B)$ there holds $\beta \in M_j$ and $\operatorname{dist}(B, B_\beta) \gtrsim 2^j t^{1/2m}$. Hence, we observe that the second factor in (3.5) is for every $j \geq 0$ bounded by a constant times $(\# M_j)^{1/2}(1+2^j)^{-2m\gamma} \|\varphi\|_{L^2(B)}$. Taking into account that $\# M_j \lesssim 2^{jn}$ due to Remark 2.3, we finally end up with

$$\begin{aligned} |\langle f, S_t^*\varphi\rangle| &\lesssim \|f\|_{L^\infty(X)} \|\varphi\|_{L^2(B)} V(B)^{1/2} \sum_{j=0}^{\infty} 2^{jn/2} 2^{jn/2} 2^{-2m\gamma j} \\ &\lesssim \|f\|_{L^\infty(X)} \|\varphi\|_{L^2(B)} V(B)^{1/2}, \end{aligned}$$

since we assumed $\gamma > \frac{n}{2m}$.
Thus, for every $t > 0$ we can define $S_t f$ for $f \in L^\infty(X)$ via duality as

$$\langle S_t f, \varphi\rangle := \langle f, S_t^* \varphi\rangle,$$

where $\varphi \in L^2(X)$ is supported in some ball in X.

(ii) The calculation above yields in particular the following estimate: There exists a constant $C > 0$ such that for all $t > 0$, for all balls $B = B(x_B, t^{1/2m})$ and for all $f \in L^\infty(X)$ there holds

$$\|S_t f\|_{L^2(B)} \leq C V(B)^{1/2} \|f\|_{L^\infty(X)}.$$

Hence, the average of $S_t f$ over a ball $B = B(x_B, t^{1/2m})$ is bounded by

$$|\langle S_t f\rangle_B| \leq \frac{1}{V(B)} \int_B |S_t f(x)| \, d\mu(x) \leq V(B)^{-1/2} \|S_t f\|_{L^2(B)} \lesssim \|f\|_{L^\infty(X)}.$$

We continue with another important observation concerning the previously defined off-diagonal estimates: All notions of off-diagonal estimates are stable under composition.

Lemma 3.3 *If two families of operators $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ satisfy Davies-Gaffney estimates* (3.1) *with parameter $\tau > 0$, then so does $\{S_t T_t\}_{t>0}$. Moreover, there exist constants $C, c > 0$ such that for arbitrary open sets $E, F \subseteq X$*

$$\|S_s T_t f\|_{L^2(F)} \leq C e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{c\max\{t,s\}}\right)^{\frac{\tau}{2m-1}}} \|f\|_{L^2(E)},$$

for all $t, s > 0$ and all $f \in L^2(X)$ supported in E.

Proof: The case for $m = 1$ and $\tau = 1$ is proven in [HM03], Lemma 2.3. The proof for arbitrary m and τ follows along the same lines. Since the proof of Lemma 3.4 uses the same ideas, we omit the details. □

We get the corresponding result also for families of operators that satisfy off-diagonal estimates of a certain order. The lemma is formulated in a slightly different form to make it available for the proof of Lemma 6.24.
For a function $b \in L^\infty(X)$, we define the multiplication operator M_b by $M_b f := b \cdot f$ for all measurable functions $f : X \to \mathbb{C}$.

Lemma 3.4 *Let $b \in L^\infty(X)$. Let $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ be two families of bounded linear operators on $L^2(X)$ that satisfy off-diagonal estimates of order γ and δ, respectively. Then there exists some constant $C > 0$ such that for arbitrary open sets $E, F \subseteq X$*

$$\|S_s M_b T_t f\|_{L^2(F)} \leq C \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{\max(s,t)}\right)^{-\min(\gamma,\delta)} \|b\|_{L^\infty(X)} \|f\|_{L^2(E)},$$

for all $s, t > 0$ and all $f \in L^2(X)$ supported in E.

Proof: We follow the proof of [HM03], Lemma 2.3. Let $b \in L^\infty(X)$ and $s, t > 0$. Further, let $E, F \subseteq X$ be arbitrary open sets and let $f \in L^2(X)$ with $\operatorname{supp} f \subseteq E$. If $\operatorname{dist}(E,F) = 0$, then the result follows from the uniform boundedness of the operators S_s and T_t in $L^2(X)$.
Otherwise, let us set $\rho := \operatorname{dist}(E,F)$ and let $G_1 := \{x \in X : \operatorname{dist}(x,F) < \frac{\rho}{2}\}$ and $G_2 := \{x \in X : \operatorname{dist}(x,F) < \frac{\rho}{4}\}$. By construction there holds that G_1, G_2 are open with $\operatorname{dist}(E, G_1) \geq \frac{\rho}{2}$ and $\operatorname{dist}(F, X \setminus \bar{G}_2) \geq \frac{\rho}{4}$.
We split the operator $S_s M_b T_t$ into

$$S_s M_b T_t = S_s \mathbb{1}_{\bar{G}_2} M_b T_t + S_s \mathbb{1}_{X \setminus \bar{G}_2} M_b T_t.$$

Since S_s is uniformly bounded in $L^2(X)$ and T_t satisfies off-diagonal estimates of order δ, there holds on the one hand

$$\begin{aligned} \left\|S_s \mathbb{1}_{\bar{G}_2} M_b T_t f\right\|_{L^2(F)} &\lesssim \|M_b T_t f\|_{L^2(\bar{G}_2)} \leq \|b\|_{L^\infty(X)} \|T_t f\|_{L^2(G_1)} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(E,G_1)^{2m}}{t}\right)^{-\delta} \|b\|_{L^\infty(X)} \|f\|_{L^2(E)} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{\max(s,t)}\right)^{-\min(\gamma,\delta)} \|b\|_{L^\infty(X)} \|f\|_{L^2(E)}. \end{aligned} \tag{3.6}$$

On the other hand, since T_t is uniformly bounded and S_s satisfies off-diagonal estimates of order γ, we obtain

$$\begin{aligned} \left\|S_s \mathbb{1}_{X \setminus \bar{G}_2} M_b T_t f\right\|_{L^2(F)} &\lesssim \left(1 + \frac{\operatorname{dist}(F, X \setminus \bar{G}_2)^{2m}}{s}\right)^{-\gamma} \|M_b T_t f\|_{L^2(X \setminus \bar{G}_2)} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{\max(s,t)}\right)^{-\min(\gamma,\delta)} \|b\|_{L^\infty(X)} \|f\|_{L^2(E)}. \end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7) finishes the proof. □

Before coming to the corresponding result for families of operators that satisfy weak off-diagonal estimates, we state some auxiliary results.

Remark 3.5 Let $s, t > 0$ with $t \leq s$ and let B be an arbitrary ball in X with radius t. As in Notation 2.2, let k_0 be the uniquely determined integer satisfying $C_1\delta^{k_0} \leq t < C_1\delta^{k_0-1}$ and for each $\beta \in I_{k_0}$ let $B_\beta := B(z_\beta^{k_0}, t)$, where $z_\beta^{k_0}$ is given by Lemma 2.1. Further, suppose that $\gamma > n$. Then there holds for every $\varepsilon > 0$ with $\gamma \geq n + \varepsilon$

$$\sum_{\beta \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B, B_\beta)}{s}\right)^{-\gamma} = \sum_{j=0}^{\infty} \sum_{\substack{\beta \in I_{k_0} \\ z_\beta^{k_0} \in S_j(B)}} \left(1 + \frac{\operatorname{dist}(B, B_\beta)}{s}\right)^{-\gamma}$$
$$\lesssim \sum_{j=0}^{\infty} \sum_{\beta \in M_j} \left(1 + \frac{2^j t}{s}\right)^{-(n+\varepsilon)},$$

using the fact that for every $j \geq 3$ and all β with $z_\beta^{k_0} \in S_j(B)$ there holds $\operatorname{dist}(B, B_\beta) \gtrsim 2^j t$ and $\beta \in M_j$, where M_j was defined in (2.7). Moreover, Remark 2.3 shows that $\# M_j \lesssim 2^{jn}$, therefore the above is bounded by a constant times

$$\sum_{j=0}^{\infty} 2^{jn} \left(1 + \frac{2^j t}{s}\right)^{-(n+\varepsilon)} \lesssim \left(\frac{s}{t}\right)^{n+\varepsilon} \sum_{j=0}^{\infty} 2^{jn} 2^{-j(n+\varepsilon)} \lesssim \left(\frac{s}{t}\right)^{n+\varepsilon},$$

since we assumed $t \leq s$.
Thus, we finally obtain the following: For every $\varepsilon > 0$ there exists a constant $C > 0$ such that for all $t \leq s$ and every $\gamma \geq n + \varepsilon$

$$\sum_{\beta \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B, B_\beta)}{s}\right)^{-\gamma} \leq C \left(\frac{s}{t}\right)^{n+\varepsilon}, \tag{3.8}$$

where B is an arbitrary ball in X with radius t and the balls $B_\beta = B(z_\beta^{k_0}, t)$ are specified above. In view of the assumption $t \leq s$, one obviously aims at an application of sufficiently small chosen $\varepsilon > 0$.

Fundamental for the proof of Proposition 3.7 is the following lemma. It can be considered as an analogue of certain estimates for compositions of integral operators, see e.g. [Gra04], Appendix K.1.

Lemma 3.6 *Let $s, t > 0$ with $t \leq s$ and let B_1, B_2 be two arbitrary balls in X with radius t. If $\gamma, \delta > n$, then for every $\varepsilon > 0$ there exists some constant $C > 0$ such that*

$$\sum_{\beta \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_1, B_\beta)}{s}\right)^{-\gamma} \left(1 + \frac{\operatorname{dist}(B_\beta, B_2)}{s}\right)^{-\delta}$$
$$\leq C \left(\frac{s}{t}\right)^{n+\varepsilon} \left(1 + \frac{\operatorname{dist}(B_1, B_2)}{s}\right)^{-\min(\gamma,\delta)}, \tag{3.9}$$

where $B_\beta = B(z_\beta^{k_0}, t)$, $k_0 \in \mathbb{Z}$ is uniquely determined by $C_1\delta^{k_0} \leq t < C_1\delta^{k_0-1}$ and the index set I_{k_0} and $z_\beta^{k_0}$ are given in Lemma 2.1.

Proof: Let $\varepsilon > 0$. We denote by Σ the left-hand side of (3.9). If $\frac{\operatorname{dist}(B_1,B_2)}{s} \leq 3$, then we get, according to (3.8),

$$\Sigma \leq \sum_{\beta \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_1, B_\beta)}{s}\right)^{-\gamma} \lesssim \left(\frac{s}{t}\right)^{n+\varepsilon} \lesssim \left(\frac{s}{t}\right)^{n+\varepsilon} \left(1 + \frac{\operatorname{dist}(B_1, B_2)}{s}\right)^{-\min(\gamma,\delta)}.$$

If otherwise $\frac{\operatorname{dist}(B_1,B_2)}{s} \geq 3$, we split the space X into two parts. For this purpose, we set $\rho := \operatorname{dist}(B_1, B_2)$ and define $G := \{x \in X : \operatorname{dist}(x, B_2) < \frac{\rho}{2}\}$. Then there holds for every $\beta \in I_{k_0}$ with $z_\beta^{k_0} \in G$ the estimate

$$\operatorname{dist}(B_1, B_\beta) \geq \operatorname{dist}(B_1, G) - t \geq \frac{1}{2} \operatorname{dist}(B_1, B_2) - \frac{1}{3} \operatorname{dist}(B_1, B_2) = \frac{1}{6} \operatorname{dist}(B_1, B_2).$$

Using (3.8), this yields

$$\begin{aligned} &\sum_{\substack{\beta \in I_{k_0} \\ z_\beta^{k_0} \in G}} \left(1 + \frac{\operatorname{dist}(B_1, B_\beta)}{s}\right)^{-\gamma} \left(1 + \frac{\operatorname{dist}(B_\beta, B_2)}{s}\right)^{-\delta} \\ &\qquad \lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)}{s}\right)^{-\gamma} \sum_{\substack{\beta \in I_{k_0} \\ z_\beta^{k_0} \in G}} \left(1 + \frac{\operatorname{dist}(B_\beta, B_2)}{s}\right)^{-\delta} \\ &\qquad \lesssim \left(\frac{s}{t}\right)^{n+\varepsilon} \left(1 + \frac{\operatorname{dist}(B_1, B_2)}{s}\right)^{-\min(\gamma,\delta)}. \end{aligned} \tag{3.10}$$

Similarly, if $\beta \in I_{k_0}$ with $z_\beta^{k_0} \in X \backslash G$, we obtain the estimate $\operatorname{dist}(B_2, B_\beta) \gtrsim \operatorname{dist}(B_1, B_2)$. Hence, we can argue as before and end up with the same bound as in (3.10) for the sum over all $\beta \in I_{k_0}$ with $z_\beta^{k_0} \in X \setminus G$. This finishes the proof. □

We are now ready to state and prove the desired result that compositions of operator families with weak off-diagonal estimates do again satisfy weak off-diagonal estimates of the same order.

Proposition 3.7 *Let $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ be two families of linear operators on $L^2(X)$ that satisfy weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$ and $\delta > \frac{n}{2m}$, respectively.*
Then there exists some constant $C > 0$ such that for every $t > 0$ and arbitrary balls $B_1, B_2 \in X$ with radius $r = t^{1/2m}$

$$\|S_t T_t f\|_{L^2(B_2)} \leq C \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t}\right)^{-\min(\gamma,\delta)} \|f\|_{L^2(B_1)},$$

for all $f \in L^2(X)$ supported in B_1.

Proof: Let $t > 0$ and let B_1, B_2 be two balls in X with radius $t^{1/2m}$. We use Lemma 2.1 to cover the space X with balls of radius $t^{1/2m}$.
Let $k_0 \in \mathbb{Z}$ be defined by (2.5), so that $C_1\delta^{k_0} \leq t^{1/2m} < C_1\delta^{k_0-1}$. Moreover, let I_{k_0} be the index set defined in Lemma 2.1 and denote for every $\beta \in I_{k_0}$ by B_β the ball $B(z_\beta^{k_0}, t^{1/2m})$. Lemma 2.1 then yields in particular that $X = \bigcup_{\beta \in I_{k_0}} B_\beta$ (the union is

not necessarily disjoint).
Since we assumed $\gamma, \delta > \frac{n}{2m}$, we can apply Lemma 3.6 (now with $t^{1/2m}$ instead of t) and get for every $f \in L^2(X)$ with $\operatorname{supp} f \subseteq B_1$ by assumption on the operators

$$\begin{aligned}\|S_tT_tf\|_{L^2(B_2)} &\leq \sum_{\beta\in I_{k_0}} \left\|S_t\mathbb{1}_{B_\beta}T_tf\right\|_{L^2(B_2)} \\ &\lesssim \sum_{\beta\in I_{k_0}} \left(1+\frac{\operatorname{dist}(B_2,B_\beta)^{2m}}{t}\right)^{-\gamma}\left(1+\frac{\operatorname{dist}(B_\beta,B_1)^{2m}}{t}\right)^{-\delta}\|f\|_{L^2(B_1)} \\ &\lesssim \left(1+\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t}\right)^{-\min(\gamma,\delta)}\|f\|_{L^2(B_1)}.\end{aligned}$$

□

In the next remark, we will also give a formulation of weak off-diagonal estimates in terms of annuli of X centered around a fixed ball.

Remark 3.8 Let $\{S_t\}_{t>0}$ be a family of linear operators on $L^2(X)$ that satisfies weak off-diagonal estimates of order $\gamma > 0$. Then there exists a constant $C > 0$ such that for an arbitrary ball $B \in X$ with radius $r_B = t^{1/2m}$, for all $j \in \mathbb{N}_0$ and all $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq B$ and $\operatorname{supp} g \subseteq S_j(B)$ there holds

$$|\langle S_tf,g\rangle| \lesssim 2^{jn/2}\left(1+\frac{\operatorname{dist}(B,S_j(B))^{2m}}{t}\right)^{-\gamma}\|f\|_{L^2(B)}\|g\|_{L^2(S_j(B))}. \tag{3.11}$$

The proof works with the same methods as the one of Lemma 3.1. If $j \leq 3$, the proof is obvious. Otherwise, we can split the annulus $S_j(B)$ with the help of Lemma 2.1 into "cubes" out of $\mathcal{Q}$ with diameter approximately equal to $r_B = t^{1/2m}$. That is, let k_0 be defined by (2.5), let M_j be the set defined in (2.7) and for every $\beta \in I_{k_0}$ denote by B_β the ball $B(z_\beta^{k_0}, t^{1/2m})$. Then there holds $\#M_j \lesssim 2^{jn}$ and $2^jB \subseteq \bigcup_{\beta\in M_j} Q_\beta^{k_0}$. Moreover, denote by $\tilde{M}_j$ the set of all $\beta \in M_j$ such that $Q_\beta^{k_0} \cap S_j(B) \neq \emptyset$. Then there holds $\operatorname{dist}(B, B_\beta) \gtrsim \operatorname{dist}(B, S_j(B))$ for all $\beta \in \tilde{M}_j$ and we thus obtain

$$\begin{aligned}|\langle S_tf,g\rangle| &\leq \sum_{\beta\in\tilde{M}_j}\left|\langle S_tf,\mathbb{1}_{Q_\beta^{k_0}}g\rangle\right| \\ &\lesssim \sum_{\beta\in\tilde{M}_j}\left(1+\frac{\operatorname{dist}(B,B_\beta)^{2m}}{t}\right)^{-\gamma}\|f\|_{L^2(B)}\|g\|_{L^2(Q_\beta^{k_0})} \\ &\lesssim \left(1+\frac{\operatorname{dist}(B,S_j(B))^{2m}}{t}\right)^{-\gamma}\|f\|_{L^2(B)}\sum_{\beta\in\tilde{M}_j}\|g\|_{L^2(Q_\beta^{k_0})}.\end{aligned}$$

The Cauchy-Schwarz inequality now yields that

$$\sum_{\beta\in\tilde{M}_j}\|g\|_{L^2(Q_\beta^{k_0})} \leq (\#M_j)^{1/2}\left(\sum_{\beta\in\tilde{M}_j}\|g\|_{L^2(Q_\beta^{k_0})}^2\right)^{1/2} \lesssim 2^{jn/2}\|g\|_{L^2(S_j(B))},$$

which gives the assertion.

Let us finally remark how one can apply weak off-diagonal estimates for balls with some radius distinct from the scale of the operator family.

Remark 3.9 Let $\{S_t\}_{t>0}$ be a family of linear operators on $L^2(X)$ that satisfies weak off-diagonal estimates of order $\gamma > 0$. Let $s, t > 0$ and let B_1, B_2 be two arbitrary balls in X with radius s. Then for every $f \in L^2(X)$ with $\operatorname{supp} f \subseteq B_1$ there holds

$$\|S_t f\|_{L^2(B_2)} \lesssim \max\left\{1, \left(\tfrac{s}{t^{1/2m}}\right)^n\right\} \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)}.$$

For the proof, let $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq B_1$ and $\operatorname{supp} g \subseteq B_2$. Without restriction, let $\operatorname{dist}(B_1, B_2) > 2t$.
If $s > t^{1/2m}$, we again use a splitting of X according to Lemma 2.1 into "cubes" out of $\mathcal{Q}$ with diameter approximately equal to $t^{1/2m}$. That is, let k_0 be defined by (2.5) and set $B_\beta := B(z_\beta^{k_0}, t^{1/2m})$ for all $\beta \in I_{k_0}$. Then there exist index sets $M_1, M_2 \subseteq I_{k_0}$ such that $B_1 \subseteq \bigcup_{\alpha \in M_1} Q_\alpha^{k_0}$ and $B_2 \subseteq \bigcup_{\beta \in M_2} Q_\beta^{k_0}$ and, in addition, $\# M_i \lesssim \left(\frac{s}{t^{1/2m}}\right)^n$ for $i = 1, 2$ and $\operatorname{dist}(B_\alpha, B_\beta) \gtrsim \operatorname{dist}(B_1, B_2)$ for $\alpha \in M_1, \beta \in M_2$. Thus, by assumption on S_t in the first and by application of the Cauchy-Schwarz inequality in the second step, we obtain

$$\begin{aligned} |\langle S_t f, g\rangle| &\lesssim \sum_{\alpha \in M_1} \sum_{\beta \in M_2} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(Q_\alpha^{k_0})} \|g\|_{L^2(Q_\beta^{k_0})} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} (\# M_1)^{1/2} \left(\sum_{\alpha \in M_1} \|f\|^2_{L^2(Q_\alpha^{k_0})}\right)^{1/2} \\ &\qquad \times (\# M_2)^{1/2} \left(\sum_{\beta \in M_2} \|g\|^2_{L^2(Q_\beta^{k_0})}\right)^{1/2} \\ &\lesssim \left(\frac{s}{t^{1/2m}}\right)^n \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)}. \end{aligned}$$

The case for $s < t^{1/2m}$ follows immediately from the definition.

3.2 Assumptions on the operator

We fix our assumptions on the operator L. Unless otherwise specified, we will assume the following.

(H1) The operator L is an injective, sectorial operator in $L^2(X)$ of angle ω, where $0 \leq \omega < \pi/2$. Further, L has a bounded $H^\infty(\Sigma_\mu^0)$ functional calculus for all $\omega < \mu < \pi$.

(H2) The operator L generates an analytic semigroup $\{e^{-tL}\}_{t>0}$ which satisfies the Davies-Gaffney condition with parameter $\tau = 1$. That is, there exist constants $C, c > 0$ such that for arbitrary open subsets $E, F \subseteq X$

$$\left\| e^{-tL} f \right\|_{L^2(F)} \leq C \exp\left[-\left(\frac{\operatorname{dist}(E, F)^{2m}}{ct}\right)^{\frac{1}{2m-1}} \right] \|f\|_{L^2(E)} \tag{3.12}$$

for every $t > 0$ and every $f \in L^2(X)$ with $\operatorname{supp} f \subseteq E$.

These two assumptions will be all what we assume on L while developing the theory of Hardy and BMO spaces associated to L. To show $L^2(X)$-boundedness of the paraproduct, we need one additional assumption. Henceforth, we will explicitly mention whenever we take into account the following assumption.

(H3) The semigroup $\{e^{-tL}\}_{t>0}$ satisfies an $L^{\tilde{p}} - L^2$ off-diagonal estimate for some $\tilde{p} \in (1,2)$ and an $L^2 - L^{\tilde{q}}$ off-diagonal estimate for some $\tilde{q} \in (2,\infty)$, i.e. there exists a constant $C > 0$ and some $\varepsilon > 0$ such that for every $t > 0$, every $j \in \mathbb{N}_0$ and for an arbitrary ball B in X with radius $r = t^{1/2m}$ there holds

$$\left\| e^{-tL} \mathbb{1}_{S_j(B)} f \right\|_{L^2(B)} \leq C 2^{-j(\frac{n}{\tilde{p}}+\varepsilon)} V(B)^{\frac{1}{2}-\frac{1}{\tilde{p}}} \|f\|_{L^{\tilde{p}}(S_j(B))} \tag{3.13}$$

and

$$\left\| e^{-tL} \mathbb{1}_B g \right\|_{L^{\tilde{q}}(S_j(B))} \leq C 2^{-j(\frac{n}{\tilde{q}'}+\varepsilon)} V(B)^{\frac{1}{\tilde{q}}-\frac{1}{2}} \|g\|_{L^2(B)} \tag{3.14}$$

for all $f \in L^{\tilde{p}}(X)$ and all $g \in L^2(X)$.
Here, $\tilde{q}'$ is the conjugate exponent of $\tilde{q}$ defined by $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$.

Observe that (3.14) is just the dual estimate of (3.13). That is, if L satisfies (3.14) with exponent $\tilde{q}$, then L^* satisfies (3.13) with exponent $\tilde{q}'$ and vice versa.

Remark 3.10 From assumption (H1) follows that L has dense domain and dense range in $L^2(X)$. See e.g. [CDMY96], Theorem 2.3.

Remark 3.11 (i) One can show the following self-improving property of Davies-Gaffney estimates to be valid:
Assume that (H1) *is satisfied. If condition* (3.12) *holds for all balls B_1, B_2 in X, then the assertion is also true for arbitrary open sets E, F of X (in general with different constants $C, c > 0$).*
The proof is similar to the proof of Lemma 3.1 (cf. also [AM07b], Proposition 3.2(b)). One splits X into "cubes" out of the collection $\mathcal{Q}$ defined in Lemma 2.1, with diameter approximately equal to $\operatorname{dist}(E,F)$. In the case $t \gtrsim \operatorname{dist}(E,F)^{2m}$, the proof is obvious. Otherwise, one has to replace estimate (3.4) in the proof of Lemma 3.1 by the following. For fixed $\alpha \in I_{k_0}$ there holds, with $r \approx \operatorname{dist}(E,F)$ and the value of the constant c being different in each step,

$$\begin{aligned} \sum_{\beta \in I_{k_0}} e^{-\left(\frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{ct}\right)^{\frac{1}{2m-1}}} &\lesssim \sum_{j=0}^{\infty} e^{-\left(\frac{(2^j r)^{2m}}{ct}\right)^{\frac{1}{2m-1}}} 2^{jn} \\ &\lesssim e^{-\left(\frac{r^{2m}}{ct}\right)^{\frac{1}{2m-1}}} \sum_{j=0}^{\infty} e^{-c 2^{\frac{2m}{2m-1} j}} 2^{jn} \lesssim e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}}. \end{aligned}$$

The rest of the proof then works analogously to the one of Lemma 3.1.
In the special case of non-negative self-adjoint operators L and $m = 1$, Coulhon and Sikora show in [CS08], Lemma 3.1, that this self-improving property is even true with the same constants $C, c > 0$. Their proof is based on a refined Phragmén-Lindelöf theorem. But it seems to be unclear if - and is more likely to be false that - the same holds true for general $m > 1$.

(ii) With the same Phragmén-Lindelöf theorem, Coulhon and Sikora further show in [CS08], again in this special case, that condition (3.12) is equivalent to the following: *There exist some constants $C, c > 0$ such that for arbitrary open sets E, F in X with $\mu(E) < \infty$ and $\mu(F) < \infty$ and all $t > 0$*

$$\left|\langle e^{-tL}\mathbb{1}_E, \mathbb{1}_F\rangle\right| \leq C \exp\left(-\frac{\operatorname{dist}(E,F)^2}{ct}\right) \mu(E)^{1/2}\mu(F)^{1/2}.$$

This is the form of Davies-Gaffney conditions as they were considered in [Dav92], for instance.

Remark 3.12 If there exists a constant $C > 0$ such that $V(x,r) \geq Cr^n$ for all $x \in X$ and all $r > 0$, then (H3) is a consequence of the following estimates:

Let $\tilde{p} \in (1,2)$ and $\tilde{q} \in (2,\infty)$. There exist constants $C, c > 0$ such that for arbitrary open sets $E, F \subseteq X$ there holds

$$\left\|e^{-tL}f\right\|_{L^2(F)} \leq Ct^{-\frac{n}{2m}(\frac{1}{\tilde{p}}-\frac{1}{2})} \exp\left[-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}\right] \|f\|_{L^{\tilde{p}}(E)} \tag{3.15}$$

and

$$\left\|e^{-tL}g\right\|_{L^{\tilde{q}}(F)} \leq Ct^{-\frac{n}{2m}(\frac{1}{2}-\frac{1}{\tilde{q}})} \exp\left[-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}\right] \|g\|_{L^2(E)} \tag{3.16}$$

for every $t > 0$ and every $f \in L^{\tilde{p}}(X)$ and $g \in L^2(X)$ supported in E.

The proof is obvious. If (3.15) is satisfied, then, in particular, $e^{-tL} : L^{\tilde{p}}(X) \to L^2(X)$ is bounded for every $t > 0$. Analogously, if (3.16) is satisfied, then $e^{-tL} : L^2(X) \to L^{\tilde{q}}(X)$ is bounded for every $t > 0$. For sufficient conditions for (3.15) to be valid in terms of off-diagonal estimates of annular type, we refer to [AM07b], Proposition 3.2.
We refer to [BK05] and [AM07b] in general for further comparison of these types of off-diagonal estimates.

3.3 Properties of operators satisfying Davies-Gaffney estimates

We collect some important consequences of the above assumptions.

Proposition 3.13 *Assume that the operator L satisfies* (H1) *and* (H2). *Then for every $K \in \mathbb{N}$, the family of operators*

$$\{(tL)^K e^{-tL}\}_{t>0}$$

satisfies the Davies-Gaffney condition (3.1) *with parameter $\tau = 1$.*

The proof of Proposition 3.13 can be found in [HLM$^+$09], Prop. 3.1, in the case of self-adjoint operators L. For sectorial operators L one has to make only minor modifications, using the following Phragmén-Lindelöf type lemma. It is stated as Lemma 6.18 in [Ouh05], similar results can also be found in [Dav95], Lemma 9 and Section 2 of [CS08].

Lemma 3.14 *Let $\mu \in (0, \pi/2]$ and assume that $F : \Sigma^0_\mu \to \mathbb{C}$ is a holomorphic function such that*

$$|F(re^{i\theta})| \le a(r\cos\theta)^{-\beta} \qquad \textit{for all } re^{i\theta} \in \Sigma^0_\mu,$$

and

$$|F(r)| \le ar^{-\beta}e^{-br^{-\alpha}} \qquad \textit{for all } r > 0,$$

where a, b are positive constants, $\beta \ge 0$, and $0 \le \alpha \le 1$. Then for every $r > 0$ and $\theta \in (-\mu, \mu)$

$$|F(re^{i\theta})| \le a2^\beta(r\cos\theta)^{-\beta}\exp[-\frac{b\alpha}{2}r^{-\alpha}\sin(\mu - |\theta|)].$$

Proof (of Proposition 3.13): Let $z \in \mathbb{C}$ with $|\arg z| < \frac{\pi}{2} - \omega$, where ω is the sectoriality angle of the operator L assumed in (H1). Then $\lambda \mapsto e^{-\lambda z}$ belongs to $H^\infty(\Sigma^0_\sigma)$ for all σ with $\omega < \sigma < \frac{\pi}{2} - |\arg z|$. Thus, for $\nu \in (\omega, \frac{\pi}{2})$ the bounded H^∞-functional calculus of L assumed in (H1) implies that $\left(e^{-zL}\right)_{z\in\Sigma^0_{\frac{\pi}{2}-\nu}}$ is analytic with

$$\left\|e^{-zL}\right\|_{L^2(X)\to L^2(X)} \le C_\nu, \qquad z \in \Sigma^0_{\frac{\pi}{2}-\nu}. \tag{3.17}$$

Let $E, F \subseteq X$ be arbitrary open sets and let $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq E$, $\operatorname{supp} g \subseteq F$. We then define for every $z \in \Sigma^0_{\frac{\pi}{2}-\nu}$

$$G(z) := \langle e^{-zL}f, g\rangle = \int_X e^{-zL}f(x)\cdot\overline{g(x)}\,d\mu(x).$$

Since $(e^{-zL})_z$ is analytic, G is also analytic on $\Sigma^0_{\frac{\pi}{2}-\nu}$. Moreover, the Davies-Gaffney estimates for the semigroup yield for every $t > 0$

$$|G(t)| \le C\exp\left[-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}\right]\|f\|_{L^2(E)}\|g\|_{L^2(F)}$$

and (3.17) yields for every $z \in \Sigma^0_{\frac{\pi}{2}-\nu}$

$$|G(z)| \le C_\nu\|f\|_{L^2(E)}\|g\|_{L^2(F)}.$$

We apply Lemma 3.14 with $\alpha = \frac{1}{2m-1}$ and $\beta = 0$. For $z = re^{i\theta}$ with $r > 0$ and $|\theta| < \frac{\pi}{2} - \nu$ we get

$$|G(z)| \le C\exp\left[-\frac{1}{2(2m-1)}\left(\frac{\operatorname{dist}(E,F)^{2m}}{cr}\right)^{\frac{1}{2m-1}}\sin(\frac{\pi}{2} - \nu - |\theta|)\right]\|f\|_{L^2(E)}\|g\|_{L^2(F)}. \tag{3.18}$$

Let us fix now some $t > 0$. Using the Cauchy formula, we can write for every $K \in \mathbb{N}$

$$(tL)^Ke^{-tL} = (-1)^KK!\frac{t^K}{2\pi i}\int_{|\zeta-t|=\eta t} e^{-\zeta L}\frac{d\zeta}{(\zeta - t)^{K+1}}, \tag{3.19}$$

where we choose $\eta > 0$ so small that $B_t := \{\zeta \in \mathbb{C} : |\zeta - t| \le \eta t\}$ is contained in $\Sigma^0_{\frac{\pi}{2}-\nu}$. This is for example satisfied for $\eta = \frac{1}{2}\sin\frac{1}{2}(\frac{\pi}{2} - \nu)$.

Observe that for this choice of η, we obtain for every $z = re^{i\theta} \in B_t$ the estimates $|\theta| \leq \frac{1}{2}(\frac{\pi}{2} - \nu)$ and $r \leq (1+\eta)t$. Hence, the estimate (3.18) above yields that

$$\sup_{z \in B_t} |G(z)| \leq C \exp\left[-\left(\frac{\operatorname{dist}(E,F)^{2m}}{c't} \right)^{\frac{1}{2m-1}} \right] \|f\|_{L^2(E)} \|g\|_{L^2(F)}, \tag{3.20}$$

where the constant $c' > 0$ only depends on m, ν and the constant c given in the assumptions. Combining (3.19) and (3.20), we finally end up with

$$\begin{aligned} \left|\langle (tL)^K e^{-tL} f, g \rangle\right| &\leq K! \frac{t^K}{2\pi} \int_{|\zeta - t| = \eta t} \frac{|G(\zeta)|}{\eta^{K+1} t^{K+1}} \, |d\zeta| \\ &\leq \|G\|_{\infty, B_t} \frac{K!}{\eta^{K+1} t} \frac{1}{2\pi} 2\pi\eta t \\ &\leq C \exp\left[-\left(\frac{\operatorname{dist}(E,F)^{2m}}{c't} \right)^{\frac{1}{2m-1}} \right] \|f\|_{L^2(E)} \|g\|_{L^2(F)}. \qquad \square \end{aligned}$$

Remark 3.15 (i) The estimate (3.18) in the proof above also shows that if $\mu \in (0, \frac{\pi}{2} - \omega)$, then the family $\{e^{-zL}\}_{z \in \Sigma^0_\mu}$ satisfies Davies-Gaffney estimates in z with $\tau = 1$.
(ii) Copying the first lines of the proof with $(zL)^K e^{-zL}$, $K \in \mathbb{N}$, instead of e^{-zL}, one can also observe that the family $\{(zL)^K e^{-zL}\}_{z \in \Sigma^0_\mu}$, where $\mu \in (0, \frac{\pi}{2} - \omega)$, satisfies Davies-Gaffney estimates in z. Since we do not need the result any further, we omit the details.

Proposition 3.16 *Assume that the operator L satisfies* (H1) *and* (H2). *Then the family of resolvent operators $\{(I + tL)^{-1}\}_{t>0}$ satisfies Davies-Gaffney estimates in t with parameter $\tau = \frac{2m-1}{2m}$. That is, there exist constants $C, c > 0$ such that for arbitrary open subsets $E, F \subseteq X$*

$$\left\|(I + tL)^{-1} f\right\|_{L^2(F)} \leq C \exp\left(-\frac{\operatorname{dist}(E,F)}{ct^{1/2m}} \right) \|f\|_{L^2(E)}$$

for every $t > 0$ and every $f \in L^2(X)$ with $\operatorname{supp} f \subseteq E$.

Proof: We can recover the resolvent from the semigroup $(e^{-tL})_{t>0}$ via the Laplace transform, i.e. for every $t > 0$ we write

$$(I + tL)^{-1} = \int_0^\infty e^{-u} e^{-utL} \, du.$$

The Davies-Gaffney estimate for the resolvent is then a direct consequence of the corresponding estimate for the semigroup.
Let $E, F \subseteq X$ be two arbitrary open sets and let $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq E$ and $\operatorname{supp} g \subseteq F$. For every $t > 0$ we get from the Davies-Gaffney estimates of the semigroup

$$\begin{aligned} \left|\langle (I + tL)^{-1} f, g \rangle\right| &= \left| \int_0^\infty e^{-u} \langle e^{-utL} f, g \rangle \, du \right| \\ &\leq C \|f\|_{L^2(E)} \|g\|_{L^2(F)} \int_0^\infty e^{-u} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{cut} \right)^{\frac{1}{2m-1}}} \, du. \end{aligned}$$

To handle the integral, we set $\rho := \left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}$ and estimate

$$\int_0^\infty e^{-u} e^{-\rho u^{-\frac{1}{2m-1}}}\,du \le \int_0^{\rho^{\frac{2m-1}{2m}}} e^{-\rho u^{-\frac{1}{2m-1}}}\,du + \int_{\rho^{\frac{2m-1}{2m}}}^\infty e^{-u}\,du.$$

The substitution of $s = \rho u^{-\frac{1}{2m-1}}$ in the first integral shows that the above is equal to

$$(2m-1)\int_{\rho^{\frac{2m-1}{2m}}}^\infty e^{-s}\frac{\rho^{2m-1}}{s^{2m}}\,ds + \int_{\rho^{\frac{2m-1}{2m}}}^\infty e^{-s}\,ds \le 2m\int_{\rho^{\frac{2m-1}{2m}}}^\infty e^{-s}\,ds = 2me^{-\rho^{\frac{2m-1}{2m}}},$$

which finishes the proof. □

For further reference, we state another family of operators that satisfies Davies-Gaffney estimates.

Remark 3.17 Let L satisfy (H1) and (H2). Then, for every $N \in \mathbb{N}$, the family of operators

$$\left(\int_0^{t^{1/2m}} \frac{s^{2mN-1}}{t^N} e^{-s^{2m}L}\,ds\right)_{t>0}$$

satisfies Davies-Gaffney estimates in $t > 0$ with parameter $\tau = 1$.

Proof: Let $t > 0$, let $E, F \subseteq X$ be arbitrary open sets and let $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq E$ and $\operatorname{supp} g \subseteq F$. Then there holds

$$\begin{aligned}
\left|\left\langle\left(\int_0^{t^{1/2m}} \frac{s^{2mN-1}}{t^N} e^{-s^{2m}L}\,ds\right) f, g\right\rangle\right| &\le \int_0^{t^{1/2m}} \frac{s^{2mN-1}}{t^N}\left|\langle e^{-s^{2m}L} f, g\rangle\right|\,ds \\
&\lesssim \int_0^{t^{1/2m}} \frac{s^{2mN-1}}{t^N} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{cs^{2m}}\right)^{\frac{1}{2m-1}}}\,ds \cdot \|f\|_{L^2(E)}\|g\|_{L^2(F)} \\
&\le \frac{1}{2mN} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}}\left[\frac{s^{2mN}}{t^N}\right]_0^{t^{1/2m}} \cdot \|f\|_{L^2(E)}\|g\|_{L^2(F)} \\
&= \frac{1}{2mN} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}} \cdot \|f\|_{L^2(E)}\|g\|_{L^2(F)}\,.
\end{aligned}$$

□

It will often be very useful not only to have Davies-Gaffney estimates for the semigroup $\{e^{-tL}\}_{t>0}$ and the resolvent, but also to have L^2 off-diagonal estimates of some order σ for the operator family $\{\psi(tL)\}_{t>0}$, where ψ is a function in $\Psi(\Sigma^0_\mu)$. The order σ here depends on the decay of ψ at 0.

Proposition 3.18 *Let L satisfy* (H1) *and* (H2). *Let $\mu \in (\omega, \pi/2)$, $\psi \in \Psi_{\sigma,\tau}(\Sigma^0_\mu)$ for some $\sigma, \tau > 0$ and $f \in H^\infty(\Sigma^0_\mu)$. Then the family of operators $\{\psi(tL)f(L)\}_{t>0}$ satisfies L^2 off-diagonal estimates of order σ, with the constant controlled by $\|f\|_{L^\infty(\Sigma^0_\mu)}$.*

The result has its origins in [HMM10], Lemma 2.28. There, the proof is given for second order elliptic operators, but it easily carries over to the case of operators satisfying (H1) and (H2). This is due to the fact that the two main ingredients, the representation formulas (2.17), (2.18) and the Davies-Gaffney estimates for the semigroup $\{e^{-zL}\}_{z\in\Sigma^0_{\frac{\pi}{2}-\omega}}$, do also hold in our setting. For convenience of the reader and as we will use the result quite regularly, we give the proof here.

Proof: Let $\psi \in \Psi_{\sigma,\tau}(\Sigma_\mu^0)$, $f \in H^\infty(\Sigma_\mu^0)$ and let $t > 0$. To get the Davies-Gaffney estimates for the semigroup into play, we will apply the representation formulas (2.17), (2.18) to the function $\psi(tL)f(L)$. First, we get for every $z \in \Gamma_\pm = \mathbb{R}^+ e^{\pm i(\pi/2-\theta)}$ the basic estimate

$$\begin{aligned}|\eta_\pm(z)| &= \left|\frac{1}{2\pi i}\int_{\gamma_\pm} e^{\xi z}\psi(t\xi)f(\xi)\,d\xi\right| \lesssim \frac{1}{t}\int_{\gamma_\pm} |\psi(t\xi)|\,|f(\xi)|\,|d(t\xi)| \\ &\lesssim \frac{1}{t}\,\|f\|_{L^\infty(\Sigma_\mu^0)}\int_{\gamma_\pm}\frac{|t\xi|^\sigma}{1+|t\xi|^{\sigma+\tau}}\,|d(t\xi)| \lesssim \frac{1}{t}\,\|f\|_{L^\infty(\Sigma_\mu^0)}\,, \end{aligned} \tag{3.21}$$

where $\gamma_\pm = \mathbb{R}^+ e^{\pm i\nu}$ and $\omega < \theta < \nu < \mu < \pi/2$.
This estimate will be sufficient for the case $|z| \le t$. If otherwise $|z| > t$, then we need a more refined estimate. Thus, we brake $\eta_\pm(z)$ into two integrals: one over $\{\xi \in \gamma_\pm : |\xi| \le 1/t\}$ (called J_1) and the second one over $\{\xi \in \gamma_\pm : |\xi| \ge 1/t\}$ (called J_2). Since $\psi \in \Psi_{\sigma,\tau}(\Sigma_\mu^0)$, we obtain

$$\begin{aligned}J_1 &\lesssim \|f\|_{L^\infty(\Sigma_\mu^0)}\int_{\xi\in\gamma_\pm:|\xi|\le 1/t} e^{-|z||\xi|}\,|t\xi|^\sigma\,|d\xi| \\ &\lesssim \frac{1}{|z|}\,\|f\|_{L^\infty(\Sigma_\mu^0)}\,\frac{t^\sigma}{|z|^\sigma}\int_0^\infty e^{-\rho}\rho^\sigma\,d\rho \lesssim \frac{1}{t}\,\|f\|_{L^\infty(\Sigma_\mu^0)}\left(\frac{t}{|z|}\right)^{\sigma+1}, \end{aligned} \tag{3.22}$$

where we used the substitution $\rho = z\xi$ in the second step. For the second part, we use the assumed decay of ψ at infinity, which yields

$$\begin{aligned}J_2 &\lesssim \|f\|_{L^\infty(\Sigma_\mu^0)}\int_{\xi\in\gamma_\pm:|\xi|\ge 1/t} |z\xi|^{-\sigma-1}\,|t\xi|^{-\tau}\,|d\xi| \\ &\lesssim \|f\|_{L^\infty(\Sigma_\mu^0)}\left(\frac{t}{|z|}\right)^{\sigma+1} t^{-\tau-\sigma-1}\int_{\xi\in\gamma_\pm:|\xi|\ge 1/t}|\xi|^{-\sigma-1-\tau}\,|d\xi| \\ &\lesssim \frac{1}{t}\,\|f\|_{L^\infty(\Sigma_\mu^0)}\left(\frac{t}{|z|}\right)^{\sigma+1}. \end{aligned} \tag{3.23}$$

Hence, the combination of (3.21) with (3.22) and (3.23) yields for all $z \in \Gamma_\pm$

$$|\eta_\pm(z)| \lesssim \frac{1}{t}\,\|f\|_{L^\infty(\Sigma_\mu^0)}\min\left\{1, \left(\frac{t}{|z|}\right)^{\sigma+1}\right\}. \tag{3.24}$$

To get the desired off-diagonal estimate for $\{\psi(tL)f(L)\}_{t>0}$, we will plug in the obtained bound for $\eta_\pm$ into the representation formula (2.17). Let E, F be two arbitrary closed sets in X and let $g \in L^2(X)$ with $\operatorname{supp} g \subseteq E$. Then

$$\|\psi(tL)f(L)g\|_{L^2(F)} \le \int_{\Gamma_+}\left\|e^{-zL}g\right\|_{L^2(F)}|\eta_+(z)|\,|dz| + \int_{\Gamma_-}\left\|e^{-zL}g\right\|_{L^2(F)}|\eta_-(z)|\,|dz|\,.$$

According to Remark 3.15, the semigroup $\{e^{-zL}\}_{z\in\Sigma_\mu^0}$ also satisfies Davies-Gaffney estimates in $z \in \Sigma_\mu^0$. Thus, we further get from (3.24)

$$\begin{aligned}\int_{\Gamma_\pm}\left\|e^{-zL}g\right\|_{L^2(F)}|\eta_\pm(z)|\,|dz| &\lesssim \|g\|_{L^2(E)}\int_{\Gamma_\pm} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{c|z|}\right)^{\frac{1}{2m-1}}}|\eta_\pm(z)|\,|dz| \\ &\lesssim \|f\|_{L^\infty(\Sigma_\mu^0)}\|g\|_{L^2(E)}\int_{\Gamma_\pm} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{c|z|}\right)^{\frac{1}{2m-1}}}\min\left\{1, \left(\frac{t}{|z|}\right)^{\sigma+1}\right\}\frac{1}{t}\,|dz|\,. \end{aligned}$$

To handle the last integral, we split it into two parts I_1 and I_2, one over all $z \in \Gamma_\pm$ with $|z| \le t$ and the other over all $z \in \Gamma_\pm$ with $|z| > t$. The first part is simply estimated by

$$I_1 = \int_{z\in\Gamma_\pm:|z|\le t} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{c|z|}\right)^{\frac{1}{2m-1}}} \frac{1}{t}\,|dz| \le e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{ct}\right)^{\frac{1}{2m-1}}}.$$

On the other hand, observe that the second part is equal to

$$I_2 = \int_{z\in\Gamma_\pm:|z|\ge t} e^{-\left(\frac{\operatorname{dist}(E,F)^{2m}}{c|z|}\right)^{\frac{1}{2m-1}}} \left(\frac{t}{|z|}\right)^{\sigma+1} \frac{1}{t}\,|dz|\,.$$

If $t \ge \operatorname{dist}(E,F)^{2m}$, we obtain the bound

$$I_2 \le \int_{z\in\Gamma_\pm:|z|\ge t} \left(\frac{t}{|z|}\right)^{\sigma+1} \frac{1}{t}\,|dz| \le C.$$

Otherwise, if $t \le \operatorname{dist}(E,F)^{2m}$, then we get for every $N > 0$

$$\begin{aligned} I_2 \lesssim & \int_{z\in\Gamma_\pm:t\le|z|\le\operatorname{dist}(E,F)^{2m}} \left(\frac{|z|}{\operatorname{dist}(E,F)^{2m}}\right)^N \left(\frac{t}{|z|}\right)^{\sigma+1} \frac{1}{t}\,|dz| \\ & + \int_{z\in\Gamma_\pm:|z|\ge\operatorname{dist}(E,F)^{2m}} \left(\frac{t}{|z|}\right)^{\sigma+1} \frac{1}{t}\,|dz|\,. \end{aligned}$$

We choose some $N > \sigma$ and integrate the first part over $0 \le |z| \le \operatorname{dist}(E,F)^{2m}$ to obtain

$$\begin{aligned} I_2 &\lesssim \left(\frac{1}{\operatorname{dist}(E,F)^{2m}}\right)^N t^\sigma \operatorname{dist}(E,F)^{2m(N-\sigma)} + \left(\frac{t}{\operatorname{dist}(E,F)^{2m}}\right)^\sigma \\ &\lesssim \left(\frac{t}{\operatorname{dist}(E,F)^{2m}}\right)^\sigma. \end{aligned}$$

The combination of both estimates for I_1 and I_2 finally finishes the proof. □

We conclude this section with an almost orthogonality lemma, which is a slightly more general version than Lemma 4.6 in [HMM10]. The applications of the result will be in the spirit of the well-known Cotlar-Knapp-Stein lemma.

Lemma 3.19 *Let $\mu \in (\omega, \frac{\pi}{2})$, $\sigma, \tau > 0$ and $\psi \in \Psi_{\sigma,\tau}(\Sigma^0_\mu)$. Let further $\delta > 0$ and $\varphi \in H^\infty(\Sigma^0_\mu)$ with $|\varphi(z)| \le c\,|z|^\delta$ for every $z \in \Sigma^0_\mu$ with $|z| \le 1$ and some constant $c > 0$ independent of z. Then for any $a \ge 0$ with $a \le \delta$ and $a < \tau$, there is a family of operators $\{T_{s,t}\}_{s,t>0}$ such that*

$$\varphi(tL)\psi(sL) = \left(\frac{t}{s}\right)^a T_{s,t}, \qquad s,t > 0,$$

where $\{T_{s,t}\}_{s,t>0}$ satisfies L^2 off-diagonal estimates in s of order $\sigma + a$ uniformly in $t > 0$.

Proof: Let ψ, φ as given in the assumptions and let $s, t > 0$. For every $a > 0$ with $a \le \delta$ and $a < \tau$ we write

$$\varphi(tL)\psi(sL) = \left(\frac{t}{s}\right)^a (tL)^{-a}\varphi(tL)(sL)^a\psi(sL) = \left(\frac{t}{s}\right)^a T_{s,t}$$

with $T_{s,t} := (tL)^{-a}\varphi(tL)(sL)^a\psi(sL)$.
Since we assumed $\delta \geq a$ and chose $\varphi \in H^\infty(\Sigma^0_\mu)$, there exists a constant $C > 0$ such that for every $z \in \Sigma^0_\mu$ with $|z| \leq 1$ there holds

$$\left|z^{-a}\varphi(z)\right| \leq c\,|z|^{-a}\,|z|^\delta \leq C$$

and, obviously, for every $z \in \Sigma^0_\mu$ with $|z| \geq 1$

$$\left|z^{-a}\varphi(z)\right| \leq C.$$

Hence, the function $z \mapsto z^{-a}\varphi(z)$, $z \in \Sigma^0_\mu$, belongs to $H^\infty(\Sigma^0_\mu)$ with

$$\sup_{t>0} \left\|(t\,\cdot\,)^{-a}\varphi(t\,\cdot\,)\right\|_{L^\infty(\Sigma^0_\mu)} \leq C.$$

As the function $z \mapsto z^a\psi(z)$ is in $\Psi_{\sigma+a,\tau-a}(\Sigma^0_\mu)$, Proposition 3.18 yields that $\{T_{s,t}\}_{s,t>0}$ satisfies L^2 off-diagonal estimates in s of order $\sigma + a$ uniformly in $t > 0$.
For $a = 0$ the claim follows directly from Proposition 3.18. □

3.4 Quadratic estimates

Remark 3.20 Let $\omega < \mu < \pi$ and $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$. Let us first recall the following fact, that is, according to Theorem 2.13, equivalent to the assumed bounded holomorphic functional calculus of L in (H1): For every $f \in L^2(X)$ there holds

$$\int_0^\infty \left\|\psi(t^{2m}L)f\right\|^2_{L^2(X)} \frac{dt}{t} \approx \|f\|^2_{L^2(X)}\,.$$

We will subsequently refer to this as *quadratic estimates*.
Moreover, the operator $Q_{\psi,L}$, defined for every $f \in L^2(X)$ by

$$(Q_{\psi,L}f)(x,t) := (\psi(t^{2m}L)f)(x), \qquad (x,t) \in X \times (0,\infty),$$

is bounded from $L^2(X)$ to $T^2(X)$. This follows from the fact that Fubini's theorem, (2.19) and Theorem 2.13 yield for every $f \in L^2(X)$

$$\begin{aligned}\|\mathscr{A}Q_{\psi,L}f\|_{T^2(X)} &= \left(\int_X\int_0^\infty\int_{B(x,t)} \left|\psi(t^{2m}L)f(y)\right|^2 \frac{d\mu(y)}{V(x,t)}\frac{dt}{t}\,d\mu(x)\right)^{1/2} \\ &= \left(\int_X\int_0^\infty\left(\int_{B(y,t)} V(x,t)^{-1}\,d\mu(x)\right)\left|\psi(t^{2m}L)f(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2} \\ &\approx \|f\|_{L^2(X)}\,.\end{aligned}$$

Let us further define the operator $\pi_{\psi,L}$ by

$$\pi_{\psi,L}F(x) := \int_0^\infty \psi(t^{2m}L)F(x,t)\,\frac{dt}{t}, \qquad x \in X,$$

for every $F \in T^2(X)$. The operator is well-defined for all $F \in T^2(X)$ and bounded from $T^2(X)$ to $L^2(X)$, as $\pi_{\psi,L}$ is the adjoint of the operator Q_{ψ,L^*}, and vice versa.

If $\tilde{\psi} \in \Psi(\Sigma_\mu^0)$ is chosen to satisfy $\int_0^\infty \psi(t)\tilde{\psi}(t)\,\frac{dt}{t} = 1$, then the functional calculus of L yields the following Calderón reproducing formula (see e.g. [KW04], Lemma 9.13): For every $f \in L^2(X)$ there holds

$$\int_0^\infty \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)f\,\frac{dt}{t} = f, \qquad \text{in } L^2(X),$$

or, equivalently,

$$\pi_{\psi,L} \circ Q_{\tilde{\psi},L} = \pi_{\tilde{\psi},L} \circ Q_{\psi,L} = I \qquad \text{in } L^2(X).$$

Such a function $\tilde{\psi} \in \Psi(\Sigma_\mu^0)$, that satisfies $\int_0^\infty \psi(t)\tilde{\psi}(t)\,\frac{dt}{t} = 1$, can e.g. be constructed by setting $\tilde{\psi}(z) := \{\int_0^\infty |\psi(t)|^2\,\frac{dt}{t}\}^{-1}\overline{\psi(\bar{z})}$ for $z \in \Sigma_\mu^0$. If one moreover requires a certain decay at zero or infinity, say $\tilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ for given $\alpha, \beta > 0$, then a possible construction is the following. We choose some $N \in \mathbb{N}$ with $N \geq \max(\alpha,\beta)$ and define $\rho_N \in \Psi_{N,N}(\Sigma_\mu^0)$ by $\rho_N(z) := \frac{z^N}{(1+z)^{2N}}$. Then there holds $\tilde{\psi} : z \mapsto C_N\overline{\psi(\bar{z})}\rho_N(z) \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ with $\int_0^\infty \psi(t)\tilde{\psi}(t)\,\frac{dt}{t} = 1$, where $C_N^{-1} := \int_0^\infty |\psi(t)|^2\,\rho_N(t)\,\frac{dt}{t}$.

The observations above also yield that for every $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$ the operator $Q_{\psi,L} \circ \pi_{\tilde{\psi},L}$ is bounded in $T^2(X)$. In [HMM10], Proposition 4.4, and [AMR08], Theorem 4.9, there was shown the following extension. The arguments used in the proof are similar to those of Section 4.3 and do also apply in our more general setting of a sectorial operator L.

Proposition 3.21 *Let $\mu \in (\omega, \pi/2)$ and let $\alpha > 0$, $\beta > \frac{n}{4m}$. Then for every $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ and every $\tilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, the operator $Q_{\psi,L} \circ \pi_{\tilde{\psi},L}$ originally defined on $T^2(X)$ extends by continuity to a bounded operator on $T^1(X)$.*

4 Hardy and BMO spaces associated to operators

In this chapter, we summarize the most important definitions and results of the recently developed theory of Hardy and BMO spaces associated to an operator L satisfying (H1) and (H2). We follow the outlines of [HMa09], [HMM10] and [DL09]. Since we work with more general assumptions on the operator L than it is done in these articles, we also give the proofs of most of the results.
Besides, we show a Calderón reproducing formula for functions from H^1_L and BMO_{L^*} and a relation between BMO_L functions and Carleson measures, that are new in this generality.

Throughout this chapter we assume that the operator L satisfies the assumptions (H1) and (H2).
Let us denote by $\mathcal{D}(S)$ the domain, by $\mathcal{R}(S)$ the range of an unbounded operator S, and by S^k the k-fold composition of S with itself, in the sense of unbounded operators.

4.1 Overview of the theory of Hardy and BMO spaces

Hardy and BMO spaces play an important role in harmonic analysis. They are deeply connected with the theory of singular integrals, give a substitute for the spaces L^1 and L^∞, which are in many contexts unsuitable, and naturally continue the scale of L^p spaces to the range of $p < 1$.
The theory of Hardy spaces dates back to the beginning of the last century. Hardy introduced in [Har15] in 1915 the space $H^p(\mathbb{D})$ to characterize boundary values of analytic functions on the unit disk $\mathbb{D}$. For every $0 < p < \infty$, he defined $H^p(\mathbb{D})$ as the space consisting of all analytic functions F on $\mathbb{D}$ such that $\sup_{0<r<1} \int_0^{2\pi} \left|F(re^{i\theta})\right|^p \, d\theta < \infty$. It is well known that this condition is sufficient to guarantee the existence of the boundary values $\lim_{r\to 1} F(re^{i\theta})$ almost everywhere. Moreover, if $1 < p < \infty$, the space $H^p(\mathbb{D})$ can be characterized as the space of all analytic functions on $\mathbb{D}$ whose real parts are Poisson integrals of a function in $L^p(0, 2\pi)$.
At its beginning, the theory of Hardy spaces was deeply connected with the theory of analytic functions (e.g. the existence of the boundary values was obtained by either of two methods, both of them reducing the problem from the case of analytic functions to the case of harmonic functions and then working with either Blaschke products or conjugate functions). The study of real variable Hardy spaces in $\mathbb{R}^n$ began in 1960 with the paper of Stein and Weiss [SW60]. Their basic idea was to adapt the notion of conjugate harmonic functions to real functions of n variables. A first characterization of Hardy spaces without the use of the notion of conjugacy was then given by Burkholder, Gundy and Silverstein [BGS71], working with non-tangential maximal functions instead. From then on, many real variable methods have been developed, for instance Fefferman and Stein gave in [FS72] new characterizations using maximal functions associated with a general approximate identity (replacing the Poisson kernel). These ideas led to characterizations of Hardy spaces via atomic or molecular decompositions, as it can be found for instance in [Coi74], [Lat79] and [TW80], which in turn enabled the extension from the definition of Hardy spaces on Euclidean spaces to the more general setting of spaces of homogeneous type introduced in [CW71].
But even if many techniques in the study of Hardy spaces were from then on based on

real variable methods and the role of the Poisson kernel was less important, the theory of Hardy spaces has still been intimately connected to properties of harmonic functions and the Poisson semigroup associated to the Laplacian. In recent years it turned out that there are situations in which the standard theory of Hardy spaces is not applicable to problems connected with more general elliptic operators instead of the Laplacian. For example, if one considers an elliptic second order operator L in divergence form, the corresponding Riesz transform $\nabla L^{-1/2}$ needs not to be bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. This led to the developement of a theory of Hardy spaces associated to certain sectorial operators. Many important properties of the standard Hardy space theory could be recovered in this setting, e.g. characterizations via square functions and non-tangential maximal functions and a molecular decomposition of Hardy spaces. Moreover, the Hardy spaces associated to the Laplacian coincide with the standard Hardy spaces introduced by Stein and Weiss.

In the context of Hardy spaces it seems to be natural also to consider the dual space of H^1. Fefferman and Stein proved in [FS72] that the space $BMO(\mathbb{R}^n)$, first introduced by John and Nirenberg in [JN61], is the dual space of $H^1(\mathbb{R}^n)$. In the recently developed theory of Hardy spaces associated to operators one could also show that there is an analogue of BMO which again is the dual of some Hardy space H^1.

An overview of the recent development can be found in [DL09]. At the beginning, in [ADM02], and [DY05b], [DY05a], Auscher, Duong, McIntosh and Duong and Yan considered Hardy spaces, and later on also BMO spaces, associated to an operator L, that has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$ and whose semigroup e^{-tL} has a kernel satisfying pointwise Poisson upper bounds. Then, Auscher, McIntosh and Russ in [AMR08] and Hofmann and Mayboroda in [HMa09] relaxed the assumptions on the operator L. In particular, they considered settings in which the pointwise bound on the heat kernel may fail and worked with Davies-Gaffney estimates instead. Further work was done by Hofmann, Lu, Mitrea, Mitrea and Yan in [HLM$^+$09] for non-negative, self-adjoint operators L on spaces of homogeneous type and by Hofmann, Mayboroda and McIntosh in [HMM10], considering also Hardy spaces H^p for $p \neq 1$. Finally, Duong and Li studied in [DL09] Hardy and BMO spaces associated to second order operators L satisfying the assumptions (H1) and (H2).

4.2 Hardy spaces associated to operators

Similar to the standard Hardy spaces of Stein and Weiss, there are different ways to define Hardy spaces associated to operators. We will present here two possibilities to define the Hardy space $H^1_L(X)$, namely one via molecules and the other one via square functions, as it is done in [HMa09], [HMM10], [HLM$^+$09] and [DL09]. Afterwards, we will show under which assumptions they are equivalent.

For other possible definitions of $H^1_L(X)$ we refer to the literature. A characterization of $H^1_L(X)$ in terms of non-tangential maximal functions is given in [HMa09] and [HLM$^+$09]. Moreover, in the case of non-negative, self-adjoint operators L, the authors of [HLM$^+$09] obtain an atomic decomposition of $H^1_L(X)$. The construction exploits the equivalence of Davies-Gaffney estimates for the semigroup e^{-tL} and the finite speed propagation for the corresponding wave equation for non-negative, self-adjoint operators. We refer to [HLM$^+$09] for details.

Hardy spaces via molecules

To motivate the definition of $H^1_L(X)$ via molecules, let us first recall the definition of the standard space $H^1(X)$ via molecules (see e.g. [CW77] for a definition of molecules on spaces of homogeneous type).

Let $\varepsilon > 0$ be fixed. A function $m \in L^1_{\text{loc}}(X)$ is called an *ε-molecule* associated to a ball B in X if

$$\int_X m(x)\, d\mu(x) = 0 \tag{4.1}$$

and for every $j \in \mathbb{N}_0$

$$\|m\|_{L^2(S_j(B))} \leq 2^{-j\varepsilon} V(2^j B)^{-1/2}. \tag{4.2}$$

Then a measurable function f belongs to $H^1(X)$ if there exists a decomposition

$$f = \sum_{j=0}^{\infty} \lambda_j m_j \qquad \mu\text{-a.e.},$$

where m_j are ε-molecules and λ_j are coefficients which satisfy $\sum_{j=0}^{\infty} |\lambda_j| < \infty$. It can be shown that the space $H^1(X)$ does not depend on $\varepsilon > 0$.

Given $M \in \mathbb{N}$ and $\varepsilon > 0$, we describe, in generalization of the above, the notion of a $(1, 2, M, \varepsilon)$*-molecule* associated to an operator L satisfying (H1) and (H2).

Definition 4.1 *Let $M \in \mathbb{N}$ and $\varepsilon > 0$. A function $m \in L^2(X)$ is called a $(1, 2, M, \varepsilon)$-molecule associated to L if there exists a function $b \in \mathcal{D}(L^M)$ and a ball B in X with radius $r_B > 0$ such that*

(i) $m = L^M b$;

(ii) For every $k = 0, 1, 2, \dots, M$ and all $j \in \mathbb{N}_0$ there holds

$$\left\| (r_B^{2m} L)^k b \right\|_{L^2(S_j(B))} \leq r_B^{2mM} 2^{-j\varepsilon} V(2^j B)^{-1/2}.$$

For $k = M$, assumption (ii) is the usual size condition estimate (4.2) for standard molecules of $H^1(X)$. For $k = 0, 1, \dots, M-1$ however, assumption (ii) describes the "L-cancellation" of molecules and gives a quantitive substitute of the vanishing moment condition (4.1), that is not applicable in many situations when working with a general sectorial operator L.

Remark 4.2 If $m = L^M b$ is a $(1, 2, M, \varepsilon)$-molecule associated to a ball B, then for every $k = 0, 1, 2, \dots, M$ the definition of molecules implies that

$$\begin{aligned}\left\| (r_B^{2m} L)^k b \right\|_{L^2(X)} &\leq \sum_{j=0}^{\infty} \left\| (r_B^{2m} L)^k b \right\|_{L^2(S_j(B))} \leq r_B^{2mM} \sum_{j=0}^{\infty} 2^{-j\varepsilon} V(2^j B)^{-1/2} \\ &\leq r_B^{2mM} V(B)^{-1/2} \sum_{j=0}^{\infty} 2^{-j\varepsilon} \lesssim r_B^{2mM} V(B)^{-1/2}.\end{aligned}$$

In particular, there holds $\|m\|_{L^2(X)} \lesssim V(B)^{-1/2}$. Moreover, the definition of molecules yields that

$$\|m\|_{L^1(X)} = \sum_{j=0}^{\infty} \|m\|_{L^1(S_j(B))} \leq \sum_{j=0}^{\infty} \|m\|_{L^2(S_j(B))} V(2^j B)^{1/2}$$
$$\leq \sum_{j=0}^{\infty} 2^{-j\varepsilon} V(2^j B)^{-1/2} V(2^j B)^{1/2} \lesssim 1.$$

Thus, $(1,2,M,\varepsilon)$-molecules are elements of $L^1(X)$, uniformly bounded by a constant only depending on $\varepsilon > 0$. This implies that the below defined space $H^1_L(X)$ is contained in $L^1(X)$.

The next lemma gives two simple, but essential examples of molecules - one constructed via the semigroup of L and one via the resolvent of L.
For the construction of more general examples of molecules, we refer to Lemma 4.14.

Lemma 4.3 *Assume that L satisfies* (H1), (H2) *and let $M \in \mathbb{N}$ and $\varepsilon > 0$. Let further B be an arbitrary ball in X with radius $r_B > 0$ and let $\varphi \in L^2(B)$ with $\|\varphi\|_{L^2(B)} = 1$. Then the functions*

$$\frac{1}{V(B)^{1/2}}(I - e^{-r_B^{2m}L})^M \varphi \qquad \text{and} \qquad \frac{1}{V(B)^{1/2}}(I - (I + r_B^{2m}L)^{-1})^M \varphi$$

are, up to a suitable normalizing constant, $(1,2,M,\varepsilon)$-molecules associated to B. The normalizing constant only depends on M, m, the doubling constant A_2 and the constants being implicit in the Davies-Gaffney estimates.

Proof: Let B be some ball in X and let $\varphi \in L^2(B)$ with $\|\varphi\|_{L^2(B)} = 1$. For $P_{r_B} := e^{-r_B^{2m}L}$ we write

$$\begin{aligned} I - P_{r_B} &= -\int_0^{r_B} \partial_\tau e^{-\tau^{2m}L}\, d\tau = 2m \int_0^{r_B} \tau^{2m-1} L e^{-\tau^{2m}L}\, d\tau \\ &= 2m \left(\int_0^{r_B} \frac{\tau^{2m-1}}{r_B^{2m}} e^{-\tau^{2m}L}\, d\tau \right) (r_B^{2m}L) = 2m(r_B^{2m}L) S_{r_B}, \end{aligned} \tag{4.3}$$

where we abbreviate

$$S_{r_B} := \int_0^{r_B} \frac{\tau^{2m-1}}{r_B^{2m}} e^{-\tau^{2m}L}\, d\tau$$

for convenience. In particular, we see that $(I - P_{r_B})^M \varphi \in \mathcal{R}(L^M)$.
Using (4.3), we get via the binomial formula for every $k = 0, \ldots, M-1$

$$\begin{aligned} (r_B^{2m}L)^{-(M-k)}(I - P_{r_B})^M &= (r_B^{2m}L)^{-(M-k)}(2m)^{M-k}(r_B^{2m}L)^{M-k} S_{r_B}^{M-k}(I - P_{r_B})^k \\ &= (2m)^{M-k} S_{r_B}^{M-k} \sum_{l=0}^{k} C_{k,l} P_{r_B}^l, \end{aligned} \tag{4.4}$$

where $C_{k,l}$ are appropriate binomial coefficients.
Observe that $P_{r_B} = e^{-r_B^{2m}L}$ and S_{r_B} satisfy Davies-Gaffney estimates in r_B^{2m} due to the assumption (H2) on L and Remark 3.17. Lemma 3.3 now yields that the composition

of powers of S_{r_B} and P_{r_B} and therefore the operator in (4.4) itself satisfies the same estimates. Thus, we get for every $k = 0, \dots, M-1$ and all $j \in \mathbb{N}_0$

$$\begin{aligned} V(B)^{-1/2} & \left\| (r_B^{2m} L)^{-(M-k)} (I - e^{-r_B^{2m} L})^M \varphi \right\|_{L^2(S_j(B))} \\ & \lesssim V(B)^{-1/2} \exp\left(- \left(\frac{\operatorname{dist}(S_j(B), B)^{2m}}{c r_B^{2m}} \right)^{\frac{1}{2m-1}} \right) \|\varphi\|_{L^2(B)} \\ & \lesssim 2^{-jN} V(B)^{-1/2} \lesssim 2^{-j(N-n/2)} V(2^j B)^{-1/2}, \end{aligned} \tag{4.5}$$

where $N \in \mathbb{N}$ can be chosen arbitrarily large and the last step is a consequence of the doubling property of μ.
Analogously, we get for $k = M$ with the help of the binomial formula and the Davies-Gaffney estimates

$$\begin{aligned} V(B)^{-1/2} & \left\| (I - e^{-r_B^{2m} L})^M \varphi \right\|_{L^2(S_j(B))} \lesssim V(B)^{-1/2} \sum_{k=0}^{M} \left\| e^{-k r_B^{2m} L} \varphi \right\|_{L^2(S_j(B))} \\ & \lesssim V(B)^{-1/2} \exp\left(- \left(\frac{\operatorname{dist}(S_j(B), B)^{2m}}{c r_B^{2m}} \right)^{\frac{1}{2m-1}} \right) \|\varphi\|_{L^2(B)} \\ & \lesssim 2^{-j(N-n/2)} V(2^j B)^{-1/2}, \end{aligned} \tag{4.6}$$

where again $N \in \mathbb{N}$ can be chosen arbitrarily large.
The estimates (4.5) and (4.6) now yield that, up to normalization by a constant only depending on M, m, the doubling constant A_2 and the constants being implicit in the Davies-Gaffney estimates, the function $\frac{1}{V(B)^{1/2}}(I - e^{-r_B^{2m} L})^M \varphi$ is a $(1, 2, M, \varepsilon)$-molecule.

In the same way, one can show that for $P_{r_B} = (I + r_B^{2m} L)^{-1}$ the function $\frac{1}{V(B)^{1/2}}(I - P_{r_B})^M \varphi$ is a $(1, 2, M, \varepsilon)$-molecule. To do so, let us write $I - P_{r_B}$ as

$$\begin{aligned} I - P_{r_B} &= I - (I + r_B^{2m} L)^{-1} \\ &= (r_B^{2m} L) \left[(r_B^{2m} L)^{-1} [(I + r_B^{2m} L) - I](I + r_B^{2m} L)^{-1} \right] = (r_B^{2m} L) P_{r_B}. \end{aligned}$$

On the one hand, we observe that again $(I - P_{r_B})^M \varphi \in \mathcal{R}(L^M)$. By Proposition 3.16, on the other hand, we can use the Davies-Gaffney estimates for the resolvent instead of those for the semigroup and proceed as before. □

We now come to the definition of molecular Hardy spaces associated to L.

Definition 4.4 *Given $M \in \mathbb{N}$, $\varepsilon > 0$ and $f \in L^1(X)$, we say that $f = \sum_j \lambda_j m_j$ is a* molecular *$(1, 2, M, \varepsilon)$-representation of f if $\sum_{j=0}^{\infty} |\lambda_j| < \infty$, each m_j is a $(1, 2, M, \varepsilon)$-molecule, and the sum converges in $L^2(X)$.*
Let $\varepsilon > 0$ be fixed. We set

$$\mathbb{H}^1_{L,mol,M}(X) := \{ f \in L^1(X) \, : \, f \textit{ has a } (1, 2, M, \varepsilon)\textit{-representation} \}$$

with the norm given by

$$\|f\|_{H^1_{L,mol,M}(X)} := \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j| \, : \, f = \sum_{j=0}^{\infty} \lambda_j m_j \textit{ is a } (1, 2, M, \varepsilon)\textit{-representation} \right\}.$$

The space $H^1_{L,mol,M}(X)$ *is defined to be the completion of* $\mathbb{H}^1_{L,mol,M}(X)$ *with respect to the norm* $\|f\|_{H^1_{L,mol,M}(X)}$ *above.*

Remark 4.5 The space $H^1_{L,mol,M}(X)$ is a Banach space. From the definition above follows that

$$H^1_{L,mol,M_2}(X) \subseteq H^1_{L,mol,M_1}(X)$$

whenever $M_1, M_2 \in \mathbb{N}$ with $1 \leq M_1 \leq M_2 < \infty$.

Hardy spaces via square functions

In analogy to the space $H^1(X)$, that can alternatively be defined via square functions associated to the Laplacian, we present a second possibility to define Hardy spaces associated to sectorial operators.

Suppose that L satisfies (H1) and (H2) and let $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$.
For $f \in L^2(X)$, we consider the square function $\mathscr{A}Q_{\psi,L}f$ associated to L, namely

$$\mathscr{A}Q_{\psi,L}(f)(x) = \left(\iint_{\Gamma(x)} \left|\psi(t^{2m}L)f(y)\right|^2 \, \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \quad x \in X.$$

Definition 4.6 *Let* $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$. *We define* $H^1_{\psi,L}(X)$ *to be the completion of the space*

$$\mathbb{H}^1_{\psi,L}(X) := \{f \in L^2(X) \, : \, \mathscr{A}Q_{\psi,L}f \in L^1(X)\}, \tag{4.7}$$

with respect to the norm

$$\|f\|_{H^1_{\psi,L}(X)} := \|\mathscr{A}Q_{\psi,L}f\|_{L^1(X)} = \|Q_{\psi,L}f\|_{T^1(X)}.$$

4.3 Characterizations of Hardy spaces

In this section we will show that - under certain assumptions on M and ψ - the above definitions of Hardy spaces via molecules and square functions are all equivalent. The exact result is stated in the theorem below, the rest of the section will then be designated to its proof.

Theorem 4.7 *Suppose that* $M \in \mathbb{N}$ *with* $M > \frac{n}{4m}$ *and* $\alpha > 0$, $\beta > \frac{n}{4m}$. *Assume either that* $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$ *and* $\{\psi(tL)\}_{t>0}$ *satisfies Davies-Gaffney estimates or that* $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu) \setminus \{0\}$. *Then*

$$H^1_{L,mol,M}(X) = H^1_{\psi,L}(X)$$

with

$$\|f\|_{H^1_{L,mol,M}(X)} \approx \|f\|_{H^1_{\psi,L}(X)}.$$

Consequently, the space $H^1_{L,mol,M}(X)$ is in fact independent of M, whenever $M > \frac{n}{4m}$. In addition, the space $H^1_{\psi,L}(X)$ does not depend on the special choice of the function $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$, whenever ψ satisfies the assumptions of Theorem 4.7. Hence, we can define the Hardy space $H^1_L(X)$ as follows.

Definition 4.8 *Let L be an operator satisfying* (H1) *and* (H2)*. The Hardy space $H^1_L(X)$ is the space*

$$H^1_L(X) := H^1_{L,mol,M}(X) = H^1_{\psi,L}(X),$$

where $M > \frac{n}{4m}$ and $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$ as in Theorem 4.7.

We split the proof of Theorem 4.7 into two steps: We first show in Proposition 4.10 and Corollary 4.13 that the inclusion $\mathbb{H}^1_{L,mol,M}(X) \subseteq \mathbb{H}^1_{\psi,L}(X)$ holds with

$$\|f\|_{H^1_{\psi,L}(X)} \lesssim \|f\|_{H^1_{L,mol,M}(X)}$$

for all $f \in \mathbb{H}^1_{L,mol,M}(X)$. Then, in a second step, we show in Proposition 4.17 that the reverse inclusion with the reverse inequality also holds. The assertion will then follow from the fact that $\mathbb{H}^1_{L,mol,M}(X)$ and $\mathbb{H}^1_{\psi,L}(X)$ are dense in $H^1_{L,mol,M}(X)$ and $H^1_{\psi,L}(X)$, respectively.

We begin with the following lemma, which is stated as Lemma 3.15 in [DL09]. It goes back to [HMa09], Lemma 3.2 and gives a criterion for an operator to be bounded from $H^1_{L,mol,M}(X)$ to $L^1(X)$. Basically, it says that it is enough to test the operator only on molecules and not on the whole space $H^1_{L,mol,M}(X)$.

Lemma 4.9 *Let $M \in \mathbb{N}$ and $\varepsilon > 0$. Assume that T is a linear operator or a non-negative sublinear operator, defined on $L^2(X)$ with values in the set of all measurable functions on X and satisfying a weak-type $(2,2)$ bound, i.e. assume that there exists some constant $C > 0$ such that for all $f \in L^2(X)$ and all $\eta > 0$*

$$\mu(\{x \in X \,:\, |Tf(x)| > \eta\}) \le C\eta^{-2} \|f\|^2_{L^2(X)} .$$

Assume further that there exists a constant $C_T > 0$ such that for every $(1,2,M,\varepsilon)$-molecule m, we have

$$\|Tm\|_{L^1(X)} \le C_T. \tag{4.8}$$

Then T is bounded from $\mathbb{H}^1_{L,mol,M}(X)$ to $L^1(X)$, and

$$\|Tf\|_{L^1(X)} \le C_T \|f\|_{H^1_{L,mol,M}(X)}$$

for all $f \in \mathbb{H}^1_{L,mol,M}(X)$. Consequently, by a standard density argument, T extends to a bounded operator from $H^1_{L,mol,M}(X)$ to $L^1(X)$.

Proof: Let $f \in \mathbb{H}^1_{L,mol,M}(X)$ and let $f = \sum_j \lambda_j m_j$ be a molecular $(1,2,M,\varepsilon)$-representation of f. For every $N \in \mathbb{N}$, we set $f^N := \sum_{j>N} \lambda_j m_j$.
The assumption that T is linear or non-negative sublinear implies that $|Tg - Th| \le |T(g-h)|$ for any $g, h \in L^2(X)$ and thus

$$|T(f)| - \sum_{j=0}^N |\lambda_j|\, |T(m_j)| \le |T(f)| - |T(\sum_{j=0}^N \lambda_j m_j)| \le |T(f - \sum_{j=0}^N \lambda_j m_j)| = \left|T(f^N)\right| . \tag{4.9}$$

As T is of weak type $(2,2)$, we get from (4.9) that for every $\eta > 0$

$$\mu(\{x \in X \,:\, \Big(|T(f)(x)| - \sum_{j=0}^{\infty} |\lambda_j|\,|T(m_j)(x)|\Big) > \eta\})$$
$$\leq \mu(\{x \in X \,:\, \left|T(f^N)(x)\right| > \eta\}) \leq C\eta^{-2}\left\|f^N\right\|^2_{L^2(X)}.$$

We further observe that $\limsup_{N\to\infty}\left\|f^N\right\|^2_{L^2(X)} = 0$, since the sum $\sum_j \lambda_j m_j$ converges in $L^2(X)$ by definition of the molecular representation. Therefore, we have that at almost every point $x \in X$

$$|T(f)(x)| \leq \sum_{j=0}^{\infty} |\lambda_j|\,|T(m_j)(x)|\,.$$

Together with (4.8), this shows that

$$\|Tf\|_{L^1(X)} \leq \sum_{j=0}^{\infty} |\lambda_j|\,\|Tm_j\|_{L^1(X)} \leq C_T \sum_{j=0}^{\infty} |\lambda_j|\,.$$

Recalling the definition of the norm on $H^1_{L,mol,M}(X)$ and taking the infimum over all $(1,2,M,\varepsilon)$-representations of f gives the desired conclusion. □

In view of Lemma 4.9, the inclusion $\mathbb{H}^1_{L,mol,M}(X) \subseteq \mathbb{H}^1_{\psi,L}(X)$ will be a consequence of the boundedness of the operator $\mathscr{A}Q_{\psi,L}$ on $L^2(X)$ (see further Corollary 4.13), as soon as we can show that $(1,2,M,\varepsilon)$-molecules are uniformly bounded in $H^1_{\psi,L}(X)$. This is what we will do in the next proposition, in analogy to Proposition 3.16 of [DL09].

Recall that for $f \in L^2(X)$ we defined in Remark 3.20

$$Q_{\psi,L}f(x,t) = \psi(t^{2m}L)f(x), \qquad (x,t) \in X \times (0,\infty),$$

and the norm on $H^1_{\psi,L}(X)$ was defined by

$$\|f\|_{H^1_{\psi,L}(X)} = \|Q_{\psi,L}f\|_{T^1(X)} = \left\|\left(\iint_{\Gamma(\cdot)} \left|\psi(t^{2m}L)f(y)\right|^2 \frac{d\mu(y)}{V(\,\cdot\,,t)}\frac{dt}{t}\right)^{1/2}\right\|_{L^1(X)}.$$

Proposition 4.10 *Suppose that L satisfies* (H1) *and* (H2). *Let $M \in \mathbb{N}$ with $M > \frac{n}{4m}$ and $\varepsilon > 0$. Further, let $\alpha > \frac{n}{4m}$, $\beta > M$ and let $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu) \setminus \{0\}$. Then there exists some constant $C > 0$ such that for all $(1,2,M,\varepsilon)$-molecules m, we have*

$$\|m\|_{H^1_{\psi,L}(X)} = \|Q_{\psi,L}m\|_{T^1(X)} \leq C.$$

Remark 4.11 A careful inspection of the proof below shows that Proposition 4.10 is also true for $\psi \in \Psi(\Sigma^0_\mu)$ (without the assumptions on the order of decay at zero and infinity), whenever $\{\psi(tL)\}_{t>0}$ satisfies Davies-Gaffney estimates. It was shown in Proposition 3.13 that in this case also $\{(tL)^M\psi(tL)\}_{t>0}$ satisfies Davies-Gaffney estimates.
For example, again due to Proposition 3.13, it is possible to take $z \mapsto \psi(z) = z^K e^{-z}$ for arbitrary $K \in \mathbb{N}$.

Remark 4.12 Let us mention that the arguments used in the proof of Proposition 4.10 are similar to those used in the proof of Proposition 3.21, whose main part is an estimate of the form

$$\left\| Q_{\psi,L} \circ \pi_{\tilde{\psi},L}(A) \right\|_{T^1(X)} \lesssim 1$$

uniformly for all $T^1(X)$-atoms A. See also Lemma 4.14, where it is shown that, under certain assumptions on $\tilde{\psi}$, $\pi_{\tilde{\psi},L}(A)$ is a molecule. The proof of Proposition 3.21, given in [HMM10], Proposition 4.4, and [AMR08], Theorem 4.9, is stated in a less general setting than ours, but it immediately takes over to our setting.

Proof: Let m be a $(1, 2, M, \varepsilon)$-molecule associated to a ball B of X with radius $r > 0$. In order to estimate $\|Q_{\psi,L}m\|_{T^1(X)}$, we use the following decomposition of $X \times (0,\infty)$. We first split $X \times (0,\infty)$ into annuli $S_k = [2^{k+1}B \backslash 2^k B \times (0, 2^{k+1}r)] \cup [2^k B \times (2^k r, 2^{k+1}r)]$, so that for all $k \in \mathbb{N}$ the annulus S_k is contained in the tent $\widehat{2^{k+2}B}$. Then, each annulus is again divided into three parts (represented by their characteristic functions η_k, η_k' and η_k'', see below). For η_k we can work with norm estimates on m and off-diagonal estimates coming from $\{\psi(tL)\}_{t>0}$. For η_k' and η_k'' we do not integrate over $(0,r)$, but over $(r, 2^{k+1}r)$ and $(2^k r, 2^{k+1}r)$. We therefore use the norm estimates of b and off-diagonal estimates of $\{(tL)^M\psi(tL)\}_{t>0}$ instead.
We define

$$\eta_0 = \mathbb{1}_{2B\times(0,2r)}$$

and for all $k \geq 1$

$$\eta_k = \mathbb{1}_{2^{k+1}B\backslash 2^k B\times(0,r)}, \qquad \eta_k' = \mathbb{1}_{2^{k+1}B\backslash 2^k B\times(r,2^{k+1}r)}, \qquad \eta_k'' = \mathbb{1}_{2^k B\times(2^k r,2^{k+1}r)}.$$

We can therefore decompose $F := Q_{\psi,L}m$ into

$$F = \eta_0 F + \sum_{k\geq 1}\eta_k F + \sum_{k\geq 1}\eta_k' F + \sum_{k\geq 1}\eta_k'' F.$$

If we can show that there exist constants $C > 0$ and $\sigma > 0$ such that

$$\begin{aligned}
&\text{(a)} \qquad \|\eta_k F\|_{T^1(X)} \leq C 2^{-k\sigma} \qquad \text{for each } k \geq 0,\\
&\text{(b)} \qquad \|\eta_k' F\|_{T^1(X)} \leq C 2^{-k\sigma} \qquad \text{for each } k \geq 1,\\
&\text{(c)} \qquad \|\eta_k'' F\|_{T^1(X)} \leq C 2^{-k\sigma} \qquad \text{for each } k \geq 1,
\end{aligned}$$

then the desired estimate $\|F\|_{T^1(X)} \lesssim 1$ will be an immediate consequence.
Since each $\eta_k F$, $\eta_k' F$ and $\eta_k'' F$ is supported in $\widehat{2^{k+2}B}$, the Cauchy-Schwarz inequality yields

$$\|\eta_k F\|_{T^1(X)} \leq \|\eta_k F\|_{T^2(X)}\, V(2^{k+2}B)^{1/2},$$

and the analogous estimate for $\eta_k' F$ and $\eta_k'' F$. Therefore, in view of the doubling property, it is enough to prove that the $T^2(X)$ norm of $\eta_k F$, $\eta_k' F$ and $\eta_k'' F$ is bounded by a constant times $2^{-k\sigma}V(2^k B)^{-1/2}$ for some $\sigma > 0$.
We first show (a). For $k = 0$ it is enough to estimate the $T^2(X)$ norm of F itself. To

do so, we observe that, according to (2.19), there exists a constant $C > 0$ such that for every $y \in X$

$$\int_{B(y,t)} V(x,t)^{-1}\, d\mu(x) \leq C. \tag{4.10}$$

Using Fubini's theorem, (4.10) and Remark 3.20 (which states that $\psi(tL)$ satisfies quadratic estimates as we assumed L to have a bounded functional calculus on $L^2(X)$), we then get

$$\begin{aligned}\|\eta_0 F\|^2_{T^2(X)} \leq \|F\|^2_{T^2(X)} &= \int_X \iint_{\Gamma(x)} \left|\psi(t^{2m}L)m(y)\right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} d\mu(x) \\ &\lesssim \int_0^\infty \int_X \left|\psi(t^{2m}L)m(y)\right|^2 \frac{d\mu(y)dt}{t} \\ &\lesssim \|m\|^2_{L^2(X)} \lesssim V(B)^{-1},\end{aligned}$$

where the last inequality is due to Remark 4.2, which is a direct consequence of the definition of molecules.

Fix now $k \geq 1$. Proposition 3.18 shows that $\{\psi(tL)\}_{t>0}$ satisfies off-diagonal estimates of order α. To apply these estimates, we use an annular decomposition of X, which yields

$$\begin{aligned}\|\eta_k F\|_{T^2(X)} &\leq \sum_{l=0}^\infty \left(\int_{2^{k+1}B\setminus 2^k B} \int_0^r \left|\psi(t^{2m}L)\mathbb{1}_{S_l(B)} m(y)\right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \\ &=: \sum_{l=0}^\infty I_l.\end{aligned}$$

Assume first that $0 \leq l \leq k-3$. Then, using the off-diagonal estimates, the definition of molecules and the fact that $\operatorname{dist}(S_l(B), 2^{k+1}B \setminus 2^k B) \geq (2^k - 2^l)r \gtrsim 2^k r$, we get

$$\begin{aligned}(I_l)^2 &\lesssim \int_0^r \left(1 + \frac{\operatorname{dist}(S_l(B), 2^{k+1}B\setminus 2^k B)^{2m}}{t^{2m}}\right)^{-2\alpha} \|m\|^2_{L^2(S_l(B))} \frac{dt}{t} \\ &\lesssim 2^{-2l\varepsilon} V(2^l B)^{-1} 2^{-4m\alpha k} \int_0^r \left(\frac{r}{t}\right)^{-4m\alpha} \frac{dt}{t} \\ &\lesssim 2^{-2l\varepsilon} V(2^l B)^{-1} 2^{-4m\alpha k}.\end{aligned} \tag{4.11}$$

Moreover, the doubling property (2.2) implies that $V(2^l B)^{-1} \lesssim 2^{kn} V(2^k B)^{-1}$, hence

$$\sum_{l=0}^{k-3} I_l \lesssim 2^{-k(2m\alpha - n/2)} V(2^k B)^{-1/2} \sum_{l=0}^{k-3} 2^{-l\varepsilon} \lesssim 2^{-k(2m\alpha-n/2)} V(2^k B)^{-1/2},$$

which gives the first part of the desired estimate since $\alpha > \frac{n}{4m}$.
Assume now that $k-2 \leq l \leq k+2$. Here, the off-diagonal estimates are of no use, as this case is the "on-diagonal" part. But since L has a bounded functional calculus on $L^2(X)$, we have by Remark 3.20

$$\begin{aligned}(I_l)^2 &\lesssim \int_0^\infty \int_X \left|\psi(t^{2m}L)\mathbb{1}_{S_l(B)} m(y)\right|^2 \frac{d\mu(y)dt}{t} \\ &\lesssim \|m\|^2_{L^2(S_l(B))} \lesssim 2^{-2k\varepsilon} V(2^k B)^{-1},\end{aligned}$$

again using the definition of molecules and the doubling property in the last line. Assume finally that $l \geq k+3$. With use of the off-diagonal estimates and taking into account that $\operatorname{dist}(S_l(B), 2^{k+1}B \setminus 2^k B) \gtrsim 2^l r$, we get in the same manner as in (4.11)

$$\begin{aligned}(I_l)^2 &\lesssim \int_0^r \left(1 + \frac{\operatorname{dist}(S_l(B), 2^{k+1}B \setminus 2^k B)^{2m}}{t^{2m}}\right)^{-2\alpha} \|m\|^2_{L^2(S_l(B))} \frac{dt}{t} \\ &\lesssim 2^{-2l\varepsilon} V(2^l B)^{-1} 2^{-4m\alpha l}.\end{aligned}$$

Since $l \geq k+3$, we obtain from the estimate above

$$\sum_{l=k+3}^{\infty} I_l \lesssim \sum_{l=k+3}^{\infty} 2^{-l\varepsilon} 2^{-2m\alpha l} V(2^l B)^{-1/2} \lesssim 2^{-2m\alpha k} V(2^k B)^{-1/2}.$$

This ends the proof of (a).

For the proof of (b), let us write $m = L^M b$ for some $b \in \mathcal{D}(L^M)$. In order to compensate for the integration over $(r, 2^{k+1}r)$ instead of $(0,r)$, we use the norm estimates of b instead of m. Recall that by definition for all $l \in \mathbb{N}_0$

$$\|b\|_{L^2(S_l(B))} \leq r^{2mM} 2^{-l\varepsilon} V(2^l B)^{-1/2}.$$

Inserting $m = L^M b$ into F and splitting X into annuli, we have

$$\begin{aligned}\|\eta_k' F\|_{T^2(X)} &= \left(\int_{2^{k+1}B\setminus 2^k B} \int_r^{2^{k+1}r} \left|(t^{2m}L)^M \psi(t^{2m}L) b(y)\right|^2 \frac{d\mu(y)dt}{t^{4mM+1}}\right)^{1/2} \\ &\leq \sum_{l=0}^{\infty} \left(\int_{2^{k+1}B\setminus 2^k B} \int_r^{2^{k+1}r} \left|(t^{2m}L)^M \psi(t^{2m}L) \mathbb{1}_{S_l(B)} b(y)\right|^2 \frac{d\mu(y)dt}{t^{4mM+1}}\right)^{1/2} \\ &=: \sum_{l=0}^{\infty} J_l.\end{aligned}$$

Observe that we assumed $\beta > M$, hence we have $z \mapsto z^M \psi(z) \in \Psi_{\alpha+M,\beta-M}(\Sigma_\mu^0)$ and $\{(tL)^M \psi(tL)\}_{t>0}$ satisfies off-diagonal estimates of order $\alpha + M$ due to Proposition 3.18. Assume first that $0 \leq l \leq k-3$. Then, similar to (4.11) we obtain

$$\begin{aligned}(J_l)^2 &\lesssim \int_r^{2^{k+1}r} \left(1 + \frac{\operatorname{dist}(S_l(B), 2^{k+1}B \setminus 2^k B)^{2m}}{t^{2m}}\right)^{-2(\alpha+M)} \|b\|^2_{L^2(S_l(B))} \frac{dt}{t^{4mM+1}} \\ &\lesssim r^{4mM} 2^{-2l\varepsilon} V(2^l B)^{-1} \int_r^{2^{k+1}r} \left(\frac{2^k r}{t}\right)^{-4m(\alpha+M)} \frac{dt}{t^{4mM+1}} \\ &= 2^{-2l\varepsilon} 2^{-4mMk} V(2^l B)^{-1} \int_r^{2^{k+1}r} \left(\frac{2^k r}{t}\right)^{-4m\alpha} \frac{dt}{t} \\ &\lesssim 2^{-2l\varepsilon} 2^{-k(4mM-n)} V(2^k B)^{-1},\end{aligned}$$

using the doubling property and a substitution $s = \frac{t}{2^k r}$ in the integral in the last step. Consequently, there holds

$$\sum_{l=0}^{k-3} J_l \lesssim 2^{-k(2mM-n/2)} V(2^k B)^{-1/2}.$$

Assume now $k-2 \leq l \leq k+2$. We first estimate t^{-4mM} in the integral against r^{-4mM}. Then, as for I_l, the functional calculus of L on $L^2(X)$, the norm estimate of b and the doubling property yield

$$\begin{aligned}
(J_l)^2 &\lesssim r^{-4mM} \int_0^\infty \int_X \left|(t^{2m}L)^M \psi(t^{2m}L) 1\!\!1_{S_l(B)} b(y)\right|^2 \frac{d\mu(y)dt}{t} \\
&\lesssim r^{-4mM} \|b\|^2_{L^2(S_l(B))} \lesssim 2^{-2k\varepsilon} V(2^k B)^{-1}.
\end{aligned}$$

Assume finally that $l \geq k+3$. As before, we get from the off-diagonal estimates, the fact that $\operatorname{dist}(S_l(B), 2^{k+1}B \setminus 2^k B) \gtrsim 2^l r$ and the definition of molecules

$$\begin{aligned}
(J_l)^2 &\lesssim \int_r^{2^{k+1}r} \left(\frac{2^l r}{t}\right)^{-4m(\alpha+M)} \|b\|^2_{L^2(S_l(B))} \frac{dt}{t^{4mM+1}} \\
&\lesssim r^{4mM} 2^{-2l\varepsilon} V(2^l B)^{-1} r^{-4mM} 2^{-4mMl} \int_r^{2^{k+1}r} \left(\frac{2^l r}{t}\right)^{-4m\alpha} \frac{dt}{t} \\
&\lesssim 2^{-2l\varepsilon} V(2^l B)^{-1} 2^{-4mMl}, \qquad (4.12)
\end{aligned}$$

where in the last inequality we estimated 2^{-l} against 2^{-k} in the integral and used the substitution $s = \frac{t}{2^k r}$.
Again using the assumption $l \geq k+3$, the above yields

$$\sum_{l=k+3}^{\infty} J_2 \lesssim 2^{-2mMk} V(2^k B)^{-1/2}.$$

For the proof of (c), we write in analogy to (b)

$$\begin{aligned}
\|\eta_k'' F\|_{T^2(X)} &= \left(\int_{2^k B} \int_{2^k r}^{2^{k+1}r} \left|(t^{2m}L)^M \psi(t^{2m}L) b(y)\right|^2 \frac{d\mu(y)dt}{t^{4mM+1}}\right)^{1/2} \\
&\leq \sum_{l=0}^{\infty} \left(\int_{2^k B} \int_{2^k r}^{2^{k+1}r} \left|(t^{2m}L)^M \psi(t^{2m}L) 1\!\!1_{S_l(B)} b(y)\right|^2 \frac{d\mu(y)dt}{t^{4mM+1}}\right)^{1/2} \\
&=: \sum_{l=0}^{\infty} K_l.
\end{aligned}$$

Let us first assume $0 \leq l \leq k+1$.
Since we now integrate over B_k instead of $2^{k+1}B \setminus 2^k B$ as in (b), we cannot use the off-diagonal estimates for small l. But on the other hand, we integrate over $(2^k r, 2^{k+1} r)$ instead of $(r, 2^{k+1}r)$, which enables us to estimate t against $2^k r$. The bounded functional calculus of L on $L^2(X)$ then yields

$$\begin{aligned}
(K_l)^2 &\lesssim (2^k r)^{-4mM} \int_0^\infty \int_X \left|(t^{2m}L)^M \psi(t^{2m}L) 1\!\!1_{S_l(B)} b(y)\right|^2 \frac{d\mu(y)dt}{t} \\
&\lesssim 2^{-4mMk} r^{-4mM} \|b\|^2_{L^2(S_l(B))} \\
&\lesssim 2^{-4mMk} 2^{-l\varepsilon} 2^{kn} V(2^k B)^{-1},
\end{aligned}$$

where we used the norm estimate of b and the doubling property in the last line.
Therefore

$$\sum_{l=0}^{k+1} K_l \lesssim 2^{-(2mM-n/2)k} V(2^k B)^{-1/2}.$$

Finally, assume that $l \geq k+2$. In this case, we have $\text{dist}(S_l(B), 2^k B) \gtrsim 2^l r$, hence we obtain by using the same arguments as in (4.12)

$$\begin{aligned}(K_l)^2 &\lesssim \int_{2^k r}^{2^{k+1} r} \left(\frac{2^l r}{t}\right)^{-4m(\alpha+M)} \|b\|_{L^2(S_l(B))}^2 \frac{dt}{t^{4mM+1}} \\ &\lesssim r^{4mM} 2^{-2l\varepsilon} V(2^l B)^{-1} r^{-4mM} 2^{-4mMl} \int_{2^k r}^{2^{k+1} r} \left(\frac{2^l r}{t}\right)^{-4m\alpha} \frac{dt}{t} \\ &\lesssim 2^{-2l\varepsilon} V(2^l B)^{-1} 2^{-4mMl}.\end{aligned}$$

This yields

$$\sum_{l=k+2}^{\infty} K_l \lesssim 2^{-2mMk} V(2^k B)^{-1/2}.$$

Combining all estimates above gives the desired conclusion. □

As mentioned before, the inclusion $\mathbb{H}^1_{L,mol,M}(X) \subseteq \mathbb{H}^1_{\psi,L}(X)$ follows immediately from Lemma 4.9, Proposition 4.10 and the boundedness of the operator $\mathscr{A} Q_{\psi,L}$ on $L^2(X)$.

Corollary 4.13 *Suppose that L satisfies* (H1) *and* (H2). *Let $M \in \mathbb{N}$ with $M > \frac{n}{4m}$ and $\varepsilon > 0$. Further, let $\alpha > \frac{n}{4m}$ and $\beta > M$. Assume either that $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$ and $\{\psi(tL)\}_{t>0}$ satisfies Davies-Gaffney estimates or that $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu) \setminus \{0\}$.*
Then $\mathbb{H}^1_{L,mol,M}(X) \subseteq \mathbb{H}^1_{\psi,L}(X)$ and there exists a constant $C > 0$ such that

$$\|f\|_{H^1_{\psi,L}(X)} \leq C \|f\|_{H^1_{L,mol,M}(X)}$$

for all $f \in \mathbb{H}^1_{L,mol,M}(X)$.

Proof: The inclusion $\mathbb{H}^1_{L,mol,M}(X) \subseteq L^2(X)$ holds by definition. For the inclusion $\mathbb{H}^1_{L,mol,M}(X) \subseteq H^1_{\psi,L}(X)$ we use Lemma 4.9. The operator $\mathscr{A} Q_{\psi,L}$ is bounded on $L^2(X)$, since for all $f \in L^2(X)$

$$\|\mathscr{A} Q_{\psi,L} f\|_{L^2(X)} = \|Q_{\psi,L} f\|_{T^2(X)} \lesssim \|f\|_{L^2(X)},$$

using that $Q_{\psi,L} : L^2(X) \to T^2(X)$ is bounded due to Remark 3.20 and the assumption that L has a bounded functional calculus on $L^2(X)$.
On the other hand, Proposition 4.10 shows that there exists a constant $C > 0$ such that for all $(1, 2, M, \varepsilon)$-molecules m we have

$$\|\mathscr{A} Q_{\psi,L} m\|_{L^1(X)} = \|m\|_{H^1_{\psi,L}(X)} \leq C.$$

Lemma 4.9 then yields that

$$\|f\|_{H^1_{\psi,L}(X)} = \|\mathscr{A} Q_{\psi,L} f\|_{L^1(X)} \leq C \|f\|_{H^1_{L,mol,M}(X)}$$

for all $f \in \mathbb{H}^1_{L,mol,M}(X)$. □

We now turn to the second step of the proof of Theorem 4.7, the inclusion $\mathbb{H}^1_{\psi,L}(X) \subseteq \mathbb{H}^1_{L,mol,M}(X)$. Thus, given a function $f \in \mathbb{H}^1_{\psi,L}(X)$, we have to find a molecular $(1,2,M,\varepsilon)$-representation of f (with an appropriate norm estimate). To obtain such a molecular decomposition, we will map f with the help of $Q_{\psi,L}$ into the tent space $T^1(X)$. Then, we will take into account the fact that there exists an atomic decomposition of $T^1(X)$, a result which was shown in [CMS85] for Euclidean spaces and in [Rus07] for spaces of homogeneous type. Having such an atomic decomposition of $T^1(X)$ at hand, we will map these atoms back into $L^2(X)$ with the help of $\pi_{\tilde{\psi},L}$, where $\pi_{\tilde{\psi},L}$ is the dual operator of $Q_{\tilde{\psi},L}$ defined in Remark 3.20 by

$$\pi_{\tilde{\psi},L}F(x) = \int_0^\infty \tilde{\psi}(t^{2m}L)(F(\cdot,t))(x)\,\frac{dt}{t}, \qquad x \in X,$$

for $F \in T^2(X)$ and some $\tilde{\psi} \in \Psi(\Sigma^0_\mu) \setminus \{0\}$.
The idea of such a reduction to tent spaces is taken from [HLM+09] and was also applied in Lemma 3.18 and Proposition 3.20 of [DL09].

To start with, we observe that $\pi_{\tilde{\psi},L}(X)$ maps $T^1(X)$ atoms into molecules of $H^1_{L,mol,M}(X)$. For convenience, we state the result with ψ instead of $\tilde{\psi}$, but remark that this function will in general be different from the one which defines the space $H^1_{\psi,L}(X)$.

Lemma 4.14 *Let $M \in \mathbb{N}$. Let $\alpha > \frac{n}{4m} + M$ and $\beta > 0$ and let $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$. Then there exist a constant $C > 0$ and an $\varepsilon > 0$ such that for every $T^1(X)$-atom A associated to some ball $B \subseteq X$ (or more precisely, to its tent $\hat{B}$), the function $C^{-1}\pi_{\psi,L}(A)$ is a $(1,2,M,\varepsilon)$-molecule associated to B.*

Remark 4.15 The result is also true for every $\psi \in \Psi(\Sigma^0_\mu)$ such that $z \mapsto z^{-M}\psi(z) \in \Psi(\Sigma^0_\mu)$ and $\{(tL)^{-M}\psi(tL)\}_{t>0}$ satisfies Davies-Gaffney estimates. This follows immediately from the proof below, observing that the two properties, that are used of the operator family $\{(tL)^{-M}\psi(tL)\}_{t>0}$, are quadratic estimates and off-diagonal estimates of some order larger than $\frac{n}{4m}$, and Proposition 3.13. In this case, one can also choose $\varepsilon > 0$ arbitrarily large.

Proof: Fix a ball B and let A be a $T^1(X)$-atom associated to $\hat{B}$. Thus,

$$\iint_{X\times(0,\infty)} |A(x,t)|^2 \,\frac{d\mu(x)dt}{t} \leq V(B)^{-1}. \tag{4.13}$$

Let us write $m := \pi_{\psi,L}(A) = L^M b$, where

$$b = \int_0^\infty L^{-M}\psi(t^{2m}L)A(\cdot,t)\,\frac{dt}{t}.$$

If we can show that there exist constants $C > 0$ and $\varepsilon > 0$ such that for all $k = 0,1,\ldots,M$ and all $l = 0,1,2,\ldots$ the estimate

$$\left\|(r_B^{2m}L)^{-(M-k)}m\right\|_{L^2(S_l(B))} \leq C2^{-l\varepsilon}V(2^lB)^{-1/2} \tag{4.14}$$

holds, then the conclusion follows.

Now for any $l \in \mathbb{N}_0$, consider $h_l \in L^2(S_l(B))$ such that $\|h_l\|_{L^2(S_l(B))} = 1$. For $0 \le k \le M$ the support condition on A, the Cauchy-Schwarz inequality and (4.13) yield

$$
\begin{aligned}
&\left| \int_X (r_B^{2m} L)^{-(M-k)} m(x) \overline{h_l(x)}\, d\mu(x) \right| \\
&\quad = \left| \int_X \left(\int_0^\infty (r_B^{2m} L)^{-(M-k)} \psi(t^{2m} L)(A(\cdot,t))(x) \frac{dt}{t} \right) \overline{h_l(x)}\, d\mu(x) \right| \\
&\quad \le \iint_{\hat{B}} |A(x,t)| \left| (r_B^{2m} L^*)^{-(M-k)} \psi(t^{2m} L^*) h_l(x) \right| \frac{d\mu(x)dt}{t} \\
&\quad \le V(B)^{-1/2} \times J, \qquad (4.15)
\end{aligned}
$$

where we set

$$
J := \left(\iint_{\hat{B}} \left| (r_B^{2m} L^*)^{-(M-k)} \psi(t^{2m} L^*) h_l(x) \right|^2 \frac{d\mu(x)dt}{t} \right)^{1/2}.
$$

Observe that by assumption $z \mapsto z^{-(M-k)}\psi(z) \in \Psi_{\alpha-(M-k),\beta}(\Sigma_\mu^0)$ for all $0 \le k \le M$. For $l \le 4$, we can use quadratic estimates due to Remark 3.20 and the assumption that L has a bounded functional calculus on $L^2(X)$. As $t < r_B$ inside the integral, this yields

$$
\begin{aligned}
J &\le \left(\int_0^{r_B} \int_B \left| (t^{2m} L^*)^{-(M-k)} \psi(t^{2m} L^*) h_l(x) \right|^2 \frac{d\mu(x)dt}{t} \right)^{1/2} \\
&\lesssim \|h_l\|_{L^2(S_l(B))} = 1. \qquad (4.16)
\end{aligned}
$$

For $l > 4$, we can take into account that $\{(tL)^{-(M-k)}\psi(tL)\}_{t>0}$ satisfies off-diagonal estimates of order $\alpha - (M-k)$ by Proposition 3.18. Using that $\operatorname{dist}(B, S_l(B)) \gtrsim 2^l r_B$, we obtain

$$
\begin{aligned}
J &\le \left(\int_0^{r_B} \int_B \left| (r_B^{2m} L^*)^{-(M-k)} \psi(t^{2m} L^*) h_l(x) \right|^2 \frac{d\mu(x)dt}{t} \right)^{1/2} \\
&\lesssim \left(\int_0^{r_B} \left(\frac{r_B^{2m}}{t^{2m}} \right)^{-2(M-k)} \left(1 + \frac{\operatorname{dist}(B, S_l(B))^{2m}}{t^{2m}} \right)^{-2(\alpha-(M-k))} \frac{dt}{t} \right)^{1/2} \|h_l\|_{L^2(S_l(B))} \\
&\lesssim 2^{-2m(\alpha-(M-k))l} \left(\int_0^{r_B} \left(\frac{r_B}{t} \right)^{-4m\alpha} \frac{dt}{t} \right)^{1/2} \lesssim 2^{-2m(\alpha-(M-k))l}. \qquad (4.17)
\end{aligned}
$$

From the combination of (4.15), (4.16) and (4.17), and the doubling property (2.2) of μ, we therefore get

$$
\begin{aligned}
&\left| \int_X (r_B^{2m} L)^{-(M-k)} m(x) \overline{h_l(x)}\, d\mu(x) \right| \\
&\quad \lesssim 2^{-2m(\alpha-(M-k))l} V(B)^{-1/2} \lesssim 2^{-(2m(\alpha-(M-k))-n/2)l} V(2^l B)^{-1/2}
\end{aligned}
$$

for every $h_l \in L^2(S_l(B))$ with $\|h_l\|_{L^2(S_l(B))} = 1$ and all $l \ge 0$.
If we take the supremum over all such h_l and choose for example $\varepsilon := 2m(\alpha - M) - \frac{n}{2}$ (which is larger than zero, since we assumed $\alpha > M + \frac{n}{4m}$), the desired estimate (4.14) follows. □

For convenience, we will state the following elementary fact. We will use this to show that the molecular decomposition of $f \in H^1_{\psi,L}(X)$, we are going to construct, converges in $L^2(X)$ and is therefore indeed a molecular representation of f.

Lemma 4.16 *Let B_1, B_2 be Banach spaces and let T be a bounded linear operator from B_1 to B_2. Suppose that the sum $\sum_j F_j$ converges in B_1. Then the sum $\sum_j f_j := \sum_j T(F_j)$ converges in B_2.*

We are now ready to finish the proof of the inclusion $\mathbb{H}^1_{\psi,L}(X) \subseteq \mathbb{H}^1_{L,mol,M}(X)$, following the outline described prior to Lemma 4.14. In contrast to the reverse inclusion, this inclusion holds for all $M \in \mathbb{N}$ and all $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$ without any further restrictions.

Proposition 4.17 *Let $M \in \mathbb{N}$ and let L satisfy* (H1) *and* (H2)*. Let $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$. If $f \in H^1_{\psi,L}(X) \cap L^2(X)$, then there exists some $\varepsilon > 0$, a family of $(1, 2, M, \varepsilon)$-molecules $\{m_j\}_{j=0}^\infty$ and a sequence of numbers $\{\lambda_j\}_{j=0}^\infty \in \ell^1$ such that f can be represented in the form $f = \sum_{j=0}^\infty \lambda_j m_j$, the sum converges in $L^2(X)$, and, for this choice of ε,*

$$\|f\|_{H^1_{L,mol,M}(X)} \leq C \sum_{j=0}^{\infty} |\lambda_j| \leq C' \|f\|_{H^1_{\psi,L}(X)},$$

where the constants $C, C' > 0$ are independent of f. In particular,

$$\mathbb{H}^1_{\psi,L}(X) \subseteq \mathbb{H}^1_{L,mol,M}(X).$$

Proof: Let $\psi \in \Psi(\Sigma^0_\mu) \setminus \{0\}$ and let $f \in \mathbb{H}^1_{\psi,L}(X) = H^1_{\psi,L}(X) \cap L^2(X)$. We set

$$F(x,t) := Q_{\psi,L} f(x,t) = \psi(t^{2m}L) f(x), \qquad (x,t) \in X \times (0,\infty).$$

The definiton of $H^1_{\psi,L}(X)$ and the fact that $Q_{\psi,L} : L^2(X) \to T^2(X)$ is bounded (due to Remark 3.20) immediately imply that $F \in T^1(X) \cap T^2(X)$. Further, Proposition 2.20 shows that every function in $T^1(X)$ has an atomic decomposition, hence there exist a numerical sequence $\{\lambda_j\}_{j=0}^\infty$ and a sequence of $T^1(X)$-atoms $\{A_j\}_{j=0}^\infty$ such that

$$F = \sum_{j=0}^{\infty} \lambda_j A_j, \tag{4.18}$$

and the sum converges both in $T^1(X)$ and $T^2(X)$. The proposition also yields the existence of a constant $C > 0$ with

$$\sum_{j=0}^{\infty} |\lambda_j| \leq C \|F\|_{T^1(X)} = C \|f\|_{H^1_{\psi,L}(X)}. \tag{4.19}$$

With the help of Lemma 4.14 we can now construct molecules m_j out of the $T^1(X)$-atoms A_j. For this, we choose some $\alpha > \frac{n}{4m} + M$, $\beta > 0$ and a function $\tilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma^0_\mu) \setminus \{0\}$ with $\int_0^\infty \tilde{\psi}(t)\psi(t)\, \frac{dt}{t} = 1$. Then, due to the functional calculus on $L^2(X)$, we have

$$\begin{aligned} f &= \int_0^\infty \tilde{\psi}(t^{2m}L)\psi(t^{2m}L) f \, \frac{dt}{t} = \pi_{\tilde{\psi},L}(F) \\ &= \sum_{j=0}^{\infty} \lambda_j \pi_{\tilde{\psi},L}(A_j), \end{aligned} \tag{4.20}$$

where by Lemma 4.16 the sum in the last line converges in $L^2(X)$, since $\pi_{\tilde{\psi},L} : T^2(X) \to L^2(X)$ is bounded due to Remark 3.20 and the sum in (4.18) converges in $T^2(X)$.

Moreover, by Lemma 4.14, there exist a constant $C > 0$ and some $\varepsilon > 0$ such that for every $j \in \mathbb{N}_0$ the function $m_j := C^{-1}\pi_{\tilde{\psi},L}(A_j)$ is a $(1, 2, M, \varepsilon)$-molecule. Consequently, the sum in (4.20) is a molecular $(1, 2, M, \varepsilon)$-representation, so that $f \in H^1_{L,mol,M}(X)$, and from (4.19) follows

$$\|f\|_{H^1_{L,mol,M}(X)} \leq C \sum_{j=0}^{\infty} |\lambda_j| \leq C' \|f\|_{H^1_{\psi,L}(X)} . \qquad \square$$

In principle, we are now done with the proof of Theorem 4.7. Combining Corollary 4.13 and Proposition 4.17, we get the equivalence of the spaces $H^1_{\psi,L}(X)$ and $H^1_{L,mol,M}(X)$, provided that $M > \frac{n}{4m}$ and $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu) \setminus \{0\}$ with $\alpha > \frac{n}{4m}$ and $\beta > \frac{n}{4m}$, and $\varepsilon > 0$ is chosen as in Lemma 4.14. Therefore, we can define $H^1_L(X)$ as one of these equivalent spaces, as it is done in Definition 4.8.
To relax the assumption on β and to show that it is indeed sufficient to assume $\beta > 0$ and, moreover, to get rid of the restriction on $\varepsilon > 0$, we use an argument given in [HMM10], Corollary 4.21. What is behind this argument, is the following observation. The assumption on β reflects the order of decay at 0 which is used in Proposition 3.18 to get the desired order of off-diagonal estimates for $\{\psi(tL)\}_{t>0}$. But if we think of a function $\psi_0 \in \Psi(\Sigma^0_\mu) \setminus \{0\}$ such that $\{\psi_0(tL)\}_{t>0}$ satisfies Davies-Gaffney estimates, Theorem 4.7 is also true, even if ψ_0 tends to 0 very slowly. With the help of a Calderón reproducing formula $\pi_{\tilde{\psi}_0,L} \circ Q_{\psi_0,L} = I$ we can now get an estimate

$$\|f\|_{H^1_{\psi,L}(X)} \lesssim \|f\|_{H^1_{\psi_0,L}(X)} ,$$

using the property that the operator $Q_{\psi,L} \circ \pi_{\tilde{\psi}_0,L}$ is defined and bounded on $T^1(X)$. The details of the argument are given below. So the order of decay at 0 turns out to be rather a technical assumption than a structural one.

Corollary 4.18 *Let $M \in \mathbb{N}$ with $M > \frac{n}{4m}$. Let further $\alpha > 0$, $\beta > \frac{n}{4m}$ and $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu) \setminus \{0\}$. Then*

$$H^1_{\psi,L}(X) = H^1_{L,mol,M}(X),$$

with equivalence of norms.

Proof: The inclusion $H^1_{\psi,L}(X) \subseteq H^1_{L,mol,M}(X)$ follows directly from Proposition 4.17. Concerning the reverse inclusion, we already know from Corollary 4.13 that $H^1_{L,mol,M}(X) \subseteq H^1_{\psi_0,L}(X)$, where $\psi_0 \in \Psi(\Sigma^0_\mu)$ is defined by $\psi_0(z) = ze^{-z}$. This follows from the fact that $\{(tL)e^{-tL}\}_{t>0}$ satisfies Davies-Gaffney estimates due to Proposition 3.13. Therefore, it remains to prove the inclusion $H^1_{\psi_0,L}(X) \subseteq H^1_{\psi,L}(X)$.
Let us now choose $\tilde{\psi}_0 \in \Psi(\Sigma^0_\mu)$ defined by $\tilde{\psi}_0(z) = C_M z^M e^{-z}$, where the constant C_M is chosen such that $\int_0^\infty \psi_0(t)\tilde{\psi}_0(t) \frac{dt}{t} = 1$. The Calderón reproducing formula then yields that $\pi_{\tilde{\psi}_0,L} \circ Q_{\psi_0,L} = I$ in $L^2(X)$.
Moreover, observe that $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu) \setminus \{0\}$ and $\tilde{\psi}_0 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$ for some $\alpha > 0$ and $\beta > \frac{n}{4m}$. Hence, according to Proposition 3.21, the operator $Q_\psi \circ \pi_{\tilde{\psi}_0,L}$, which was originally defined on $T^2(X)$, extends to a bounded operator on $T^1(X)$. This yields that

for every $f \in \mathbb{H}^1_{\psi_0,L}(X) = H^1_{\psi_0,L}(X) \cap L^2(X)$ there holds

$$\begin{aligned}\|f\|_{H^1_{\psi,L}(X)} = \|Q_{\psi,L}f\|_{T^1(X)} &= \left\|Q_{\psi,L} \circ \pi_{\tilde{\psi}_0,L} \circ Q_{\psi_0,L}f\right\|_{T^1(X)} \\ &\lesssim \|Q_{\psi_0,L}f\|_{T^1(X)} = \|f\|_{H^1_{\psi_0,L}(X)} .\end{aligned}$$

Since $\mathbb{H}^1_{\psi_0,L}(X)$ is dense in $H^1_{\psi_0,L}(X)$, the assertion follows. □

4.4 BMO spaces associated to operators

To motivate the definition of BMO spaces associated to operators, let us shortly recall the definition of the space $BMO(\mathbb{R}^n)$ introduced by John and Nirenberg in [JN61]. A function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is said to be in $BMO(\mathbb{R}^n)$, the space of functions of bounded mean oscillation, if and only if

$$\sup_B \frac{1}{|B|} \int_B |f(x) - \langle f \rangle_B| \, dx < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$.
The idea is now to replace the averaging of f over balls B by a more general approximation operator associated to L, namely the semigroup operator $e^{-r_B^{2m}L}$. More precisely, for every $M \in \mathbb{N}$ one defines a space $BMO_{L,M}(X)$ consisting of all elements $f \in \mathcal{E}_M(L)$, where $\mathcal{E}_M(L)$ will be defined below, that satisfy

$$\sup_B \frac{1}{V(B)} \int_B \left|(I - e^{-r_B^{2m}L})^M f(x)\right|^2 \, d\mu(x) < \infty, \tag{4.21}$$

where again the supremum is taken over all balls B in X.
If one chooses L to be the Laplacian, then one regains the standard space $BMO(X)$ (see [DY05b], Corollary 2.16).

We follow the approach in [HMa09] and [DL09], still assuming that L is an operator satisfying (H1) and (H2).

In order to define the space $BMO_{L,M}(X)$, let us first define the space $\mathcal{E}_M(L)$. The definition assures that $(I - e^{r_B^{2m}L})^M f \in L^2_{\text{loc}}(X)$, and therefore the expression in (4.21) is well-defined. Moreover, the definition is chosen such that one gets a theory that is consistent with the theory of $H^1_L(X)$. That is, one assures that functions from $BMO_{L,M}(X)$ interact well with molecules from $H^1_{L^*}(X)$ (see Proposition 4.30), to get a duality result for $H^1_{L^*}(X)$ and $BMO_{L,M}(X)$.

Let us fix some element $x_0 \in X$ that will henceforth be called 0. The ball $B_0 := B(0,1)$ will then be referred to as *unit ball.*

Definition 4.19 *Let $\varepsilon > 0$, $M \in \mathbb{N}$ and let $\phi \in \mathcal{R}(L^M) \subseteq L^2(X)$ with $\phi = L^M \nu$ for some $\nu \in \mathcal{D}(L^M)$. We introduce the norm*

$$\|\phi\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} := \sup_{j \geq 0} \left[2^{j\varepsilon} V(2^j B_0)^{1/2} \sum_{k=0}^{M} \left\|L^k \nu\right\|_{L^2(S_j(B_0))} \right],$$

where B_0 is the unit ball centered at 0 with radius 1, and we then set

$$\mathcal{M}_0^{1,2,M,\varepsilon}(L) := \{\phi \in \mathcal{R}(L^M) \,:\, \|\phi\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} < \infty\}.$$

We denote by $(\mathcal{M}_0^{1,2,M,\varepsilon}(L))'$ the dual of $\mathcal{M}_0^{1,2,M,\varepsilon}(L)$.
For any $M \in \mathbb{N}$, let $\mathcal{E}_M(L)$ be defined by

$$\mathcal{E}_M(L) := \bigcap_{\varepsilon>0} (\mathcal{M}_0^{1,2,M,\varepsilon}(L^*))'.$$

Remark 4.20 We note that if $\phi \in \mathcal{M}_0^{1,2,M,\varepsilon}(L)$ with norm 1, then ϕ is a $(1, 2, M, \varepsilon)$-molecule adapted to B_0. Conversely, if m is a $(1, 2, M, \varepsilon)$-molecule adapted to any ball B_1, then $m \in \mathcal{M}_0^{1,2,M,\varepsilon}(L)$. This follows from the observation that there exist integers K_0 and K_1, depending on r_{B_0} and r_{B_1} and $\operatorname{dist}(B_0, B_1)$, such that $2^{K_0}B_0 \supseteq B_1$ and $2^{K_1}B_1 \supseteq B_0$. One can therefore renormalize the molecule m such that it is a molecule adapted to the unit ball B_0.

Lemma 4.21 *Let L satisfy* (H1), (H2) *and let $M \in \mathbb{N}$ and $\varepsilon > 0$. For $t > 0$ let P_t denote either $e^{-t^{2m}L}$ or $(I + t^{2m}L)^{-1}$.*
Then for every $f \in (\mathcal{M}_0^{1,2,M,\varepsilon}(L^*))'$ and every $t > 0$, one can via duality define $(I-P_t)^M f$ as an element of $L^2_{\mathrm{loc}}(X)$. In the same way, $(t^{2m}L)^M e^{-t^{2m}L} f$ can be defined as an element of $L^2_{\mathrm{loc}}(X)$, too.

Proof: We obtain from Lemma 4.3, with L replaced by L^*, and Remark 4.20 the following: If $\varphi \in L^2(B)$ for some ball B of X with radius r_B, then $(I - P_t^*)^M \varphi \in \mathcal{M}_0^{1,2,M,\varepsilon}(L^*)$ with

$$\left\|(I - P_t^*)^M \varphi\right\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L^*)} \le C_1 \|\varphi\|_{L^2(B)}, \tag{4.22}$$

the constant $C_1 > 0$ being independent of φ (but depending on the constants t, r_B and $\operatorname{dist}(0, B)$ due to Remark 4.20 and depending on the normalizing constant described in Lemma 4.3).
The assertion of the lemma is now a simple consequence. Via duality we get

$$\begin{aligned}\left|\langle (I - P_t)^M f, \varphi\rangle\right| &= \left|\langle f, (I - P_t^*)^M \varphi\rangle\right| \\ &\le C_1 \|f\|_{(\mathcal{M}_0^{1,2,M,\varepsilon}(L^*))'} \|\varphi\|_{L^2(B)},\end{aligned}$$

where $C_1 > 0$ is the constant from (4.22). Since B and $\varphi \in L^2(B)$ were arbitrary, the claim follows.
The proof for $(t^{2m}L)^M e^{-t^{2m}L} f$ is similar to the above. □

Definition 4.22 *Let $M \in \mathbb{N}$ and let L be an operator satisfying* (H1) *and* (H2). *An element $f \in \mathcal{E}_M(L)$ is said to belong to $BMO_{L,M}(X)$ if*

$$\|f\|_{BMO_{L,M}(X)} := \sup_{B \subseteq X} \left(\frac{1}{V(B)} \int_B \left|(I - e^{-r_B^{2m}L})^M f(x)\right|^2 d\mu(x)\right)^{1/2} < \infty, \tag{4.23}$$

where the supremum is taken over all balls B in X.

The following proposition gives an equivalent characterization of $BMO_{L,M}(X)$ using the resolvent in place of the semigroup.

Proposition 4.23 *Let L be an operator satisfying* (H1), (H2) *and fix $M \in \mathbb{N}$. A functional $f \in \mathcal{E}_M(L)$ belongs to $BMO_{L,M}(X)$ if and only if*

$$\|f\|_{BMO_{L,M,res}(X)} := \sup_{B\subseteq X} \left(\frac{1}{V(B)} \int_B \left|(I-(I+r_B^{2m}L)^{-1})^M f(x)\right|^2 d\mu(x) \right)^{1/2} < \infty, \tag{4.24}$$

where the supremum is taken over all balls B in X. Further, there exists a constant $c>0$ such that for every $f \in BMO_{L,M}(X)$

$$c^{-1}\|f\|_{BMO_{L,M,res}(X)} \leq \|f\|_{BMO_{L,M}(X)} \leq c\|f\|_{BMO_{L,M,res}(X)}.$$

Before coming to the proof of Proposition 4.23, we add some auxiliary results.

Lemma 4.24 *Let $f \in L^2(X)$. Let $\{S_t\}_{t>0}$ and $\{T_t\}_{t>0}$ be two families of linear bounded operators on $L^2(X)$ such that $\{S_t\}_{t>0}$ satisfies Davies-Gaffney estimates and $\{T_t\}_{t>0}$ satisfies the estimate*

$$\sup_{B\subseteq X} V(B)^{-1/2} \left\|T_{r_B^{2m}} f\right\|_{L^2(B)} \leq C_f, \tag{4.25}$$

for some constant $C_f > 0$, where the supremum is taken over all balls B in X and $r_B > 0$ denotes the radius of B. Then, for every $\varepsilon > 0$ there exists some constant $C > 0$, independent of f, such that

$$\sup_{B\subseteq X} \sum_{j=0}^{\infty} 2^{-j\varepsilon} V(2^j B)^{-1/2} \left\|T_{r_B^{2m}} f\right\|_{L^2(S_j(B))} \leq CC_f \tag{4.26}$$

and there exists some constant $C' > 0$, independent of f, such that

$$\sup_{B\subseteq X} V(B)^{-1/2} \left\|S_{r_B^{2m}} T_{r_B^{2m}} f\right\|_{L^2(B)} \leq C'C_f,$$

where again the supremum is taken over all balls B in X.

Proof: Let $f \in L^2(X)$ and $B \subseteq X$ be an arbitrary ball with radius r_B. Moreover, let $\{S_t\}_{t>}$ and $\{T_t\}_{t>0}$ be as in the assumptions.
Let us denote the left-hand side of (4.26) by Σ. To estimate Σ against C_f, we will cover the annuli $S_j(B)$ with balls of radius r_B. To do so, we use Lemma 2.1, which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type. We use the notation given there and in Notation 2.2.
Given $\delta > 0$ and $C_1 > 0$ as in Lemma 2.1, we denote by k_0 the integer satisfying $C_1\delta^{k_0} \leq r_B < C_1\delta^{k_0-1}$ and, for each $j \in \mathbb{N}$, by k_j the integer satisfying $\delta^{-k_j} \leq 2^j < \delta^{-k_j-1}$. With x_B denoting the center of the ball B, we define for every $j \in \mathbb{N}$ the index set M_j related to $B = B(x_B, r_B)$ as in (2.7). Recall that M_j represents all "cubes" out of $\mathcal{Q}$ with "sidelength" approximately equal to r_B that have non-empty intersection with the dilated ball $2^j B$.
Due to Notation 2.2 there holds

$$2^j B \subseteq B(x_B, C_1\delta^{k_0-k_j-2}) \subseteq \bigcup_{\beta\in M_j} Q_\beta^{k_0} \subseteq B(x_B, 2C_1\delta^{k_0-k_j-2}) \subseteq \delta^{-2}2^{j+1}B. \tag{4.27}$$

Further, Lemma 2.1 yields that the sets $Q_\beta^{k_0}$, $\beta \in M_j$, are disjoint and for each $\beta \in M_j$ there exists some $z_\beta^{k_0} \in X$ such that

$$B(z_\beta^{k_0}, c_1 r_B) \subseteq Q_\beta^{k_0} \subseteq B(z_\beta^{k_0}, r_B) \tag{4.28}$$

for some $c_1 \in (0,1)$ independent of j.
Now, if we substitute (4.27) and (4.28) into (4.26), we obtain

$$\begin{aligned}
&\sum_{j=0}^\infty 2^{-j\varepsilon} V(2^j B)^{-1/2} \left\| T_{r_B^{2m}} f \right\|_{L^2(S_j(B))} \\
&\quad \le \sum_{j=0}^\infty 2^{-j\varepsilon} V(2^j B)^{-1/2} \left(\sum_{\beta \in M_j} \left\| T_{r_B^{2m}} f \right\|^2_{L^2(B(z_\beta^{k_0}, r_B))} \right)^{1/2} \\
&\quad \le C_f \sum_{j=0}^\infty 2^{-j\varepsilon} V(2^j B)^{-1/2} \left(\sum_{\beta \in M_j} V(z_\beta^{k_0}, r_B) \right)^{1/2},
\end{aligned} \tag{4.29}$$

where the last line follows from (4.25).
Further, the doubling property (2.2) of μ and (4.28) yield that

$$V(z_\beta^{k_0}, r_B) \le A_2 c_1^{-n} V(z_\beta^{k_0}, c_1 r_B).$$

We therefore get from (4.28), (4.27) and the disjointness of the sets $Q_\beta^{k_0}$, $\beta \in M_j$, that

$$\sum_{\beta \in M_j} V(z_\beta^{k_0}, r_B) \lesssim \sum_{\beta \in M_j} V(Q_\beta^{k_0}) \le V(x_B, 2C_1 \delta^{k_0 - k_j - 2}) \lesssim V(2^j B), \tag{4.30}$$

again using the doubling property (2.2) in the last step.
Combining (4.29) and (4.30), we can conclude that $\Sigma \lesssim C_f$, which finishes the proof of (4.26).

Let us now turn to the second assertion. By splitting X into annuli around B and applying the Davies-Gaffney estimates for $\{S_t\}_{t>0}$, we obtain

$$\begin{aligned}
\left\| S_{r_B^{2m}} T_{r_B^{2m}} f \right\|_{L^2(B)} &\le \sum_{j=0}^\infty \left\| S_{r_B^{2m}} \mathbb{1}_{S_j(B)} T_{r_B^{2m}} f \right\|_{L^2(B)} \\
&\lesssim \sum_{j=0}^\infty \exp\left[- \left(\frac{\operatorname{dist}(S_j(B), B)^{2m}}{c r_B^{2m}} \right)^{\frac{1}{2m-1}} \right] \left\| T_{r_B^{2m}} f \right\|_{L^2(S_j(B))} \\
&\lesssim \sum_{j=0}^\infty 2^{-jN} \left\| T_{r_B^{2m}} f \right\|_{L^2(S_j(B))}
\end{aligned}$$

for arbitrary $N \in \mathbb{N}$. The doubling property (2.2) and estimate (4.26) for $\varepsilon = N - n/2$ then yield

$$\begin{aligned}
V(B)^{-1/2} \left\| S_{r_B^{2m}} T_{r_B^{2m}} f \right\|_{L^2(B)} &\lesssim \sum_{j=0}^\infty 2^{-jN} 2^{jn/2} V(2^j B)^{-1/2} \left\| T_{r_B^{2m}} f \right\|_{L^2(S_j(B))} \\
&\lesssim C_f.
\end{aligned}$$

Since the ball B was arbitrary, this finishes the proof. □

The above lemma implies in particular that the following inclusion of BMO spaces is valid. In Theorem 4.28 we will moreover show that $BMO_{L,M}(X) = BMO_{L,N}(X)$ for every $M, N \in \mathbb{N}$ with $M, N > \frac{n}{4m}$.

Corollary 4.25 *Let $N, M \in \mathbb{N}$ with $M \leq N$. There exists a constant $C > 0$ such that for every $f \in BMO_{L,M}(X)$*

$$\|f\|_{BMO_{L,N}(X)} \leq C \|f\|_{BMO_{L,M}(X)}$$

and therefore $BMO_{L,M}(X) \subseteq BMO_{L,N}(X)$.

Proof (of Proposition 4.23): We follow the outline of the proof of [HMa09], Lemma 8.1 (using a slightly simpler decomposition).
Let $f \in \mathcal{E}_M(L)$ and $M \in \mathbb{N}$. We start with the proof of the inequality $\|f\|_{BMO_{L,M}(X)} \lesssim \|f\|_{BMO_{L,M,res}(X)}$. Observe first that for every ball B in X with radius $r_B > 0$ there holds

$$\begin{aligned}(r_B^{2m}L)^M &= (I + r_B^{2m}L)^M(I - (I + r_B^{2m}L)^{-1})^M \\ &= \left(\sum_{k=0}^{M} C_{k,M}(r_B^{2m}L)^{M-k}\right)(I - (I + r_B^{2m}L)^{-1})^M,\end{aligned}$$

where $C_{k,M}$ are the coefficients from the binomial formula.
By abbreviating $\tilde{f} := (I - (I + r_B^{2m}L)^{-1})^M f$, we therefore get

$$\begin{aligned}&V(B)^{-1/2}\left\|(I - e^{-r_B^{2m}L})^M f\right\|_{L^2(B)} \\ &= V(B)^{-1/2}\left\|(I - e^{-r_B^{2m}L})^M (r_B^{2m}L)^{-M}\left(\sum_{k=0}^{M} C_{k,M}(r_B^{2m}L)^{M-k}\right)\tilde{f}\right\|_{L^2(B)} \\ &\lesssim \sum_{k=0}^{M} V(B)^{-1/2}\left\|(I - e^{-r_B^{2m}L})^{M-k}\left(-\int_0^{r_B} \partial_\tau e^{-\tau^{2m}L}\, d\tau\right)^k (r_B^{2m}L)^{-k}\tilde{f}\right\|_{L^2(B)} \\ &\lesssim \sum_{k=0}^{M}\sum_{\nu=0}^{M-k} V(B)^{-1/2}\left\|e^{-\nu r_B^{2m}L}\left(-\int_0^{r_B} \partial_\tau e^{-\tau^{2m}L}\, d\tau\right)^k (r_B^{2m}L)^{-k}\tilde{f}\right\|_{L^2(B)}, \qquad (4.31)\end{aligned}$$

again using the binomial formula in the last step.
By assumption (H2), the semigroup $\{e^{-tL}\}_{t>0}$ satisfies Davies-Gaffney estimates. Hence, Remark 3.17 and Lemma 3.3 yield that the operator

$$\begin{aligned}&e^{-\nu r_B^{2m}L}\left(-\int_0^{r_B} \partial_\tau e^{-\tau^{2m}L}\, d\tau\right)^k (r_B^{2m}L)^{-k} \\ &= e^{-\nu r_B^{2m}L}\left(\int_0^{r_B} \frac{2m\tau^{2m-1}}{r_B^{2m}} e^{-\tau^{2m}L}\, d\tau\right)^k =: S_{r_B^{2m}}\end{aligned}$$

also satisfies Davies-Gaffney estimates in r_B^{2m} (except for $k = \nu = 0$). If $k = \nu = 0$, then the corresponding expression in (4.31) is equal to $\frac{1}{V(B)^{1/2}}\left\|(I - (I + r_B^{2m}L)^{-1})^M f\right\|_{L^2(B)}$ which obviously is bounded by $\|f\|_{BMO_{L,M,res}(X)}$.
In all other cases we apply Lemma 4.24 with $T_{r_B^{2m}} = (I - (I + r_B^{2m}L)^{-1})^M$ and constant

$C_f = \|f\|_{BMO_{L,M,res}(X)}$. The assumption (4.25) is satisfied by definition of the norm in $BMO_{L,M,res}(X)$, thus the lemma implies that

$$V(B)^{-1/2} \left\| (I - e^{-r_B^{2m}L})^M f \right\|_{L^2(B)} \lesssim \|f\|_{BMO_{L,M,res}(X)} .$$

Since $B \subseteq X$ was an arbitrary ball, we get the asserted estimate.

We now come to the inverse inequality $\|f\|_{BMO_{L,M,res}(X)} \lesssim \|f\|_{BMO_{L,M}(X)}$. In analogy to the above, we will write the operator $(I - (I + r_B^{2m}L)^{-1})^M$ as the combination of an operator satisfying Davies-Gaffney estimates and an operator (almost) of the form $(I - e^{-r_B^{2m}L})^M$. However, in this case, it needs a little more effort to get the desired representation.
Let us write the identity as

$$I = (2m)^M \left(r_B^{-2m} \int_{r_B}^{2^{1/2m}r_B} s^{2m-1} I \, ds \right)^M , \tag{4.32}$$

and, in addition, let us write the integral in (4.32) via the binomial formula as

$$\begin{aligned} &\int_{r_B}^{2^{1/2m}r_B} s^{2m-1} I \, ds \\ &\qquad = \int_{r_B}^{2^{1/2m}r_B} s^{2m-1} (I - e^{-s^{2m}L})^M \, ds + \sum_{k=1}^M C_{k,M} \int_{r_B}^{2^{1/2m}r_B} s^{2m-1} e^{-ks^{2m}L} \, ds, \end{aligned} \tag{4.33}$$

where $C_{k,M}$ are some constants depending on k and M only. To handle the second integral in (4.33), observe further that

$$\begin{aligned} 2mkL \int_{r_B}^{2^{1/2m}r_B} s^{2m-1} e^{-ks^{2m}L} \, ds &= - \int_{r_B}^{2^{1/2m}r_B} \partial_s e^{-ks^{2m}L} \, ds = e^{-kr_B^{2m}L} - e^{-2kr_B^{2m}L} \\ &= e^{-kr_B^{2m}L}(I - e^{-kr_B^{2m}L}) \\ &= e^{-kr_B^{2m}L}(I - e^{-r_B^{2m}L}) \sum_{j=0}^{k-1} e^{-jr_B^{2m}L}. \end{aligned} \tag{4.34}$$

For convenience, we set

$$\tilde{T}_{r_B^{2m}} := r_B^{-2m} \int_{r_B}^{2^{1/2m}r_B} s^{2m-1} (I - e^{-s^{2m}L})^M \, ds,$$

and for every $1 \leq k \leq M$

$$S_{r_B^{2m},k} := (r_B^{2m}L)^{-1} \sum_{j=0}^{k-1} e^{-(k+j)r_B^{2m}L}.$$

From (4.32), (4.33) and (4.34) we then get the following representation of the identity operator:

$$
\begin{aligned}
I &= (2m)^M \Bigg(r_B^{-2m} \int_{r_B}^{2^{1/2m} r_B} s^{2m-1} (I - e^{-s^{2m}L})^M \, ds \\
&\quad + \sum_{k=1}^{M} C_{k,M} r_B^{-2m} (2mkL)^{-1} e^{-kr_B^{2m}L} (I - e^{-r_B^{2m}L}) \sum_{j=0}^{k-1} e^{-jr_B^{2m}L} \Bigg)^M \\
&= \left(\tilde{T}_{r_B^{2m}} + (I - e^{-r_B^{2m}L}) \sum_{k=1}^{M} C_{k,M,m} S_{r_B^{2m},k} \right)^M \\
&= (I - e^{-r_B^{2m}L})^M \left(\sum_{k=1}^{M} C_{k,M,m} S_{r_B^{2m},k} \right)^M \\
&\quad + \tilde{T}_{r_B^{2m}} \sum_{\nu=1}^{M} (\tilde{T}_{r_B^{2m}})^{\nu-1} \left(\sum_{k=1}^{M} C_{\nu,k,M,m} (I - e^{-r_B^{2m}L}) S_{r_B^{2m},k} \right)^{M-\nu} \\
&=: P_{r_B^{2m}} + Q_{r_B^{2m}}. \qquad (4.35)
\end{aligned}
$$

With the help of these two operators we will estimate the $BMO_{L,M,res}(X)$ norm of f. We begin with $P_{r_B^{2m}}$. The definition of $P_{r_B^{2m}}$ yields

$$
\begin{aligned}
&V(B)^{-1/2} \left\| (I - (I + r_B^{2m}L)^{-1})^M P_{r_B^{2m}} f \right\|_{L^2(B)} \\
&\lesssim \sum_{k=1}^{M} V(B)^{-1/2} \left\| ((I - (I + r_B^{2m}L)^{-1}) S_{r_B^{2m},k})^M (I - e^{-r_B^{2m}L})^M f \right\|_{L^2(B)}. \qquad (4.36)
\end{aligned}
$$

By using the identity

$$
(I - (I + r_B^{2m}L)^{-1})(r_B^{2m}L)^{-1} = (I + r_B^{2m}L)^{-1},
$$

and the definition of $S_{r_B^{2m},k}$, we can write

$$
(I - (I + r_B^{2m}L)^{-1}) S_{r_B^{2m},k} = (I + r_B^{2m}L)^{-1} \sum_{j=0}^{k-1} e^{-(k+j)r_B^{2m}L}, \qquad (4.37)
$$

which is an operator that satisfies Davies-Gaffney estimates due to assumption (H2), Proposition 3.16 and Lemma 3.3. Thus, in view of (4.36) we can apply Lemma 4.24 with $T_{r_B^{2m}} = (I - e^{-r_B^{2m}L})^M$, which shows that

$$
V(B)^{-1/2} \left\| (I - (I + r_B^{2m}L)^{-1})^M P_{r_B^{2m}} f \right\|_{L^2(B)} \lesssim \|f\|_{BMO_{L,M}(X)}
$$

as desired.

We now turn to $Q_{r_B^{2m}}$. Let us abbreviate $\tilde{S}_{r_B^{2m}} := \sum_{k=1}^{M} (I - e^{-r_B^{2m}L}) S_{r_B^{2m},k}$. Then the definition of $Q_{r_B^{2m}}$ yields

$$
\begin{aligned}
&V(B)^{-1/2} \left\| (I - (I + r_B^{2m}L)^{-1})^M Q_{r_B^{2m}} f \right\|_{L^2(B)} \\
&\lesssim \sum_{\nu=1}^{M} V(B)^{-1/2} \left\| (I - (I + r_B^{2m}L)^{-1})^M (\tilde{T}_{r_B^{2m}})^{\nu-1} (\tilde{S}_{r_B^{2m}})^{M-\nu} \tilde{T}_{r_B^{2m}} f \right\|_{L^2(B)}. \qquad (4.38)
\end{aligned}
$$

We first observe the following norm estimate for the operator $\tilde{T}_{r_B^{2m}}$. If $\tilde{B}$ is an arbitrary ball of X with $r_{\tilde{B}} = r_B$, we get by changing the order of integration and using $s \approx r_{\tilde{B}}$ and the doubling condition (2.2)

$$\left\| \tilde{T}_{r_B^{2m}} f \right\|_{L^2(\tilde{B})} \le r_B^{-2m} \int_{r_B}^{2^{1/2m} r_B} s^{2m-1} \left\| (I - e^{-s^{2m} L})^M f \right\|_{L^2(\tilde{B})} ds$$
$$\lesssim V(\tilde{B})^{1/2} \left\| f \right\|_{BMO_{L,M}(X)} . \qquad (4.39)$$

On the other hand, with use of the binomial formula again, we can write $\tilde{T}_{r_B^{2m}}$ as

$$\tilde{T}_{r_B^{2m}} = r_B^{-2m} \int_{r_B}^{2^{1/2m} r_B} s^{2m-1} I \, ds + \sum_{k=1}^{M} C_{k,M} r_B^{-2m} \int_{r_B}^{2^{1/2m} r_B} s^{2m-1} e^{-k s^{2m} L} \, ds.$$

The first integral in the equation above is equal to $\frac{1}{2m}$, whereas the second one inside the sum can be rewritten with the help of (4.34). Hence, we obtain

$$\tilde{T}_{r_B^{2m}} = \frac{1}{2m} I + (I - e^{-r_B^{2m} L}) \sum_{k=1}^{M} C_{k,M,m} S_{r_B^{2m},k}.$$

Together with (4.37), this leads to the observation that $(I - (I + r_B^{2m} L)^{-1}) \tilde{T}_{r_B^{2m}}$ is, except for a multiplicative constant, the sum of the identity operator and an operator satisfying Davies-Gaffney estimates. In view of (4.37) again, the same is true for the operator $(I - (I + r_B^{2m} L)^{-1}) \tilde{S}_{r_B^{2m}}$ and thus also for the operator

$$(I - (I + r_B^{2m} L)^{-1})^M (\tilde{T}_{r_B^{2m}})^{\nu-1} (\tilde{S}_{r_B^{2m}})^{M-\nu}$$

occuring in (4.38). Together with (4.38) and (4.39), Lemma 4.24 then yields for $T_{r_B^{2m}} = \tilde{T}_{r_B^{2m}}$ the estimate

$$V(B)^{-1/2} \left\| (I - (I + r_B^{2m} L)^{-1})^M Q_{r_B^{2m}} f \right\|_{L^2(B)} \lesssim \left\| f \right\|_{BMO_{L,M}(X)} .$$

This finishes the proof. □

In analogy to [DY05b], Proposition 2.5 (see also [Mar04], Proposition 3.1), we can show that the classical $BMO(X)$ space is included in $BMO_{L,M}(X)$ if and only if $e^{-tL}(1) = 1$ holds for every $t > 0$. Recall that in view of Remark 3.2 we can define $e^{-tL}(1)$ as an element of $L^2_{\text{loc}}(X)$ for every fixed $t > 0$.

Proposition 4.26 *Assume that for every $t > 0$ there holds $e^{-tL}(1) = 1$ in $L^2_{\text{loc}}(X)$. Then we have $BMO(X) \subseteq BMO_{L,M}(X)$ for every $M \in \mathbb{N}$, and there exists a positive constant $C > 0$ such that for every $f \in BMO(X)$*

$$\left\| f \right\|_{BMO_{L,M}(X)} \le C \left\| f \right\|_{BMO(X)} . \qquad (4.40)$$

Conversely, if $BMO(X) \subseteq BMO_{L,1}(X)$, then there holds $e^{-tL}(1) = 1$ in $L^2_{\text{loc}}(X)$ for every $t > 0$.

Proof: In view of Corollary 4.25 it is enough to show the assertion for $M = 1$.
Let $B \subseteq X$ be an arbitrary ball. Due to the assumption $e^{-tL}(1) = 1$ in $L^2_{\text{loc}}(X)$, there holds for every $f \in BMO(X)$

$$\begin{aligned}
&\left(\frac{1}{V(B)}\int_B \left|(I-e^{-r_B^{2m}L})f(x)\right|^2\,d\mu(x)\right)^{1/2}\\
&\leq \left(\frac{1}{V(B)}\int_B |f(x)-\langle f\rangle_B|^2\,d\mu(x)\right)^{1/2} + \left(\frac{1}{V(B)}\int_B \left|e^{-r_B^{2m}L}(\langle f\rangle_B - f)(x)\right|^2\,d\mu(x)\right)^{1/2}\\
&=: I_1 + I_2.
\end{aligned}$$

Obviously, I_1 is bounded by $\|f\|_{BMO(X)}$, whereas for I_2 the Davies-Gaffney estimates for $\{e^{-tL}\}_{t>0}$ imply

$$\begin{aligned}
I_2 &\leq \sum_{j=0}^{\infty} V(B)^{-1/2}\left\|e^{-r_B^{2m}L}\mathbb{1}_{S_j(B)}(\langle f\rangle_B - f)\right\|_{L^2(B)}\\
&\lesssim \sum_{j=0}^{\infty} V(B)^{-1/2}\exp\left[-\left(\frac{\operatorname{dist}(B,S_j(B))^{2m}}{cr_B^{2m}}\right)^{\frac{1}{2m-1}}\right]\|f-\langle f\rangle_B\|_{L^2(S_j(B))}\\
&\lesssim V(B)^{-1/2}\left(\|f-\langle f\rangle_B\|_{L^2(B)} + \|f-\langle f\rangle_B\|_{L^2(2B)}\right)\\
&\qquad + \sum_{j=2}^{\infty} V(B)^{-1/2}2^{-jN}\|f-\langle f\rangle_B\|_{L^2(S_j(B))}
\end{aligned}$$

for arbitrary $N \in \mathbb{N}$.
Furthermore, for $j \geq 1$, one can due to the doubling property (2.2) easily compute that $|\langle f\rangle_{2^jB} - \langle f\rangle_B| \leq Cj\,\|f\|_{BMO(X)}$ for some constant $C > 0$ only depending on n and the doubling constant. This yields for every $j \geq 1$

$$\begin{aligned}
\|f-\langle f\rangle_B\|_{L^2(2^jB)} &\leq \|f-\langle f\rangle_{2^jB}\|_{L^2(2^jB)} + \|\langle f\rangle_{2^jB}-\langle f\rangle_B\|_{L^2(2^jB)}\\
&\leq V(2^jB)^{1/2}\|f\|_{BMO(X)} + CjV(2^jB)^{1/2}\|f\|_{BMO(X)}\,.
\end{aligned}$$

Putting all estimates together, we therefore obtain due to the doubling condition

$$I_2 \lesssim \|f\|_{BMO(X)}\sum_{j=0}^{\infty}(j+1)2^{-jN}2^{jn/2} \lesssim \|f\|_{BMO(X)}\,,$$

choosing $N > \frac{n}{2}$.

Finally, observe that the condition $e^{-tL}(1) = 1$ is necessary for the inclusion $BMO(X) \subseteq BMO_{L,1}(X)$. To see this, consider the constant function $f(x) = 1$. Since $\|1\|_{BMO(X)} = 0$, the inequality (4.40) implies that $\|1\|_{BMO_{L,1}} = 0$ and thus, for every $t > 0$, $e^{-tL}(1) = 1$ in $L^2_{\text{loc}}(X)$. □

For a further comparison of the standard $BMO(X)$ space with the spaces $BMO_{L,M}(X)$ associated to some operator L we refer to [DDSY08]. The authors consider operators L under the assumption that L is the generator of a semigroup satisfying Gaussian upper bounds and construct examples of operators for all possible containments of the two

spaces $BMO(X)$ and $BMO_{L,M}(X)$, that is, examples, where the two spaces are either equal, one contained in the other or both not contained in the other.
A sufficient criterion for the equivalence of $BMO(X)$ and $BMO_{L,M}(X)$ is given in [DY05b] in terms of Hölder-type estimates on the kernel of the semigroup.

4.5 A Carleson measure estimate

In consistence with the theory of BMO spaces, we can show that elements of $BMO_{L,M}(X)$ are intimately connected with Carleson measures (and tent spaces, respectively). That means, given a $BMO_{L,M}(X)$ function, one can construct a Carleson measure, where the Carleson norm depends in principle only on the norm of the $BMO_{L,M}(X)$ function. The exact result is stated in Proposition 4.27 below; it is a generalization of [HMa09], Lemma 8.3. The corresponding result for the space $BMO(\mathbb{R}^n)$, in the literature also called "Fefferman-Stein criterion", can for instance be found in [FS72].
For the converse result, i.e. the way how to get $BMO_{L,M}(X)$ functions back from special Carleson measures, we refer to Theorem 4.34. We postponed this to Section 4.7, since its proof is based on the duality of Hardy and BMO spaces associated to operators and the proof of this result in turn uses Proposition 4.27.
Besides the above mentioned duality result of Hardy and BMO spaces, our main application of Proposition 4.27 will be in the proof of $L^2(X)$-boundedness of paraproducts.

Proposition 4.27 *Let L satisfy* (H1), (H2) *and let $M \in \mathbb{N}$. Let further $\omega < \mu < \pi/2$ and let either $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$, where $\alpha > 0$ and $\beta > \frac{n}{4m} + M$, or let $\psi \in \Psi(\Sigma^0_\mu)$ be defined by $\psi(z) := z^M e^{-z}$, $z \in \Sigma^0_\mu$. Then the operator*

$$f \mapsto \psi(t^{2m}L)f$$

maps $BMO_{L,M}(X) \to T^\infty(X)$, i.e. for every $f \in BMO_{L,M}(X)$ is

$$\nu_{\psi,f} := \left|\psi(t^{2m}L)f(y)\right|^2 \frac{d\mu(y)\,dt}{t} \tag{4.41}$$

a Carleson measure and there exists a constant $C_\psi > 0$ such that for all $f \in BMO_{L,M}(X)$

$$\|\nu_{\psi,f}\|_{\mathcal{C}} \leq C_\psi \, \|f\|^2_{BMO_{L,M}(X)} \,.$$

Proof: Let $f \in BMO_{L,M}(X)$ and $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$. We aim to show that for every ball B in X the estimate

$$\left(\frac{1}{V(B)}\int_0^{r_B}\int_B \left|\psi(t^{2m}L)f(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2} \lesssim \|f\|_{BMO_{L,M}(X)}$$

is valid. To get an estimate against the $BMO_{L,M}(X)$ norm of f (to be more precise, against the norm $\|\,.\,\|_{BMO_{L,M,res}(X)}$, which is equivalent to $\|\,.\,\|_{BMO_{L,M}(X)}$ due to Proposition 4.23), we split f into

$$f = (I-(I+r_B^{2m}L)^{-1})^M f \;+\; [I-(I-(I+r_B^{2m}L)^{-1})^M]f$$

and write

$$
\begin{aligned}
&\left(\frac{1}{V(B)}\int_0^{r_B}\int_B \left|\psi(t^{2m}L)f(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}\\
&\quad\leq \left(\frac{1}{V(B)}\int_0^{r_B}\int_B \left|\psi(t^{2m}L)(I-(I+r_B^{2m}L)^{-1})^M f(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}\\
&\qquad+\left(\frac{1}{V(B)}\int_0^{r_B}\int_B \left|\psi(t^{2m}L)[I-(I-(I+r_B^{2m}L)^{-1})^M]f(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}\\
&\quad=: I_1 + I_2.
\end{aligned}
$$

For the sake of convenience, we set $\tilde{f} := (I-(I+r_B^{2m}L)^{-1})^M f$. We recall the estimate

$$
\sum_{j=0}^{\infty} 2^{-j\varepsilon} V(2^jB)^{-1/2}\|\tilde{f}\|_{L^2(S_j(B))} \lesssim \|f\|_{BMO_{L,M,res}(X)}\,, \tag{4.42}
$$

which was shown in Lemma 4.24. Obviously, there also holds

$$
V(B)^{-1/2}\|\tilde{f}\|_{L^2(2B)} \lesssim \|f\|_{BMO_{L,M,res}(X)} \tag{4.43}
$$

due to the doubling condition (2.2).

To handle I_1, we decompose X into annuli around B and use off-diagonal estimates together with quadratic estimates for the on-diagonal part. More precisely, we split I_1 into

$$
\begin{aligned}
I_1 &\leq V(B)^{-1/2}\left(\int_0^{r_B}\int_B \left|\psi(t^{2m}L)\mathbb{1}_{2B}\tilde{f}(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}\\
&\quad+\sum_{j=2}^{\infty} V(B)^{-1/2}\left(\int_0^{r_B}\int_B \left|\psi(t^{2m}L)\mathbb{1}_{S_j(B)}\tilde{f}(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}.
\end{aligned} \tag{4.44}
$$

Due to assumption (H1) and Remark 3.20, L satisfies quadratic estimates, which gives us a bound for the first term. In combination with (4.43), we get

$$
\begin{aligned}
V(B)^{-1/2}&\left(\int_0^{r_B}\int_B \left|\psi(t^{2m}L)\mathbb{1}_{2B}\tilde{f}(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}\\
&\lesssim V(B)^{-1/2}\|\tilde{f}\|_{L^2(2B)} \lesssim \|f\|_{BMO_{L,M,res}(X)}\,.
\end{aligned}
$$

Since we chose $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$ and assumed (H2), the family of operators $\{\psi(tL)\}_{t>0}$ satisfies off-diagonal estimates of order β according to Proposition 3.18. We therefore get for the second term in (4.44)

$$
\begin{aligned}
&\sum_{j=2}^{\infty} V(B)^{-1/2}\left(\int_0^{r_B}\int_B \left|\psi(t^{2m}L)\mathbb{1}_{S_j(B)}\tilde{f}(y)\right|^2 \frac{d\mu(y)dt}{t}\right)^{1/2}\\
&\quad\lesssim \sum_{j=2}^{\infty} V(B)^{-1/2}\left(\int_0^{r_B}\left(\frac{\operatorname{dist}(B,S_j(B))^{2m}}{t^{2m}}\right)^{-2\beta}\frac{dt}{t}\right)^{1/2}\|\tilde{f}\|_{L^2(S_j(B))}.
\end{aligned}
$$

Taking into account that $\operatorname{dist}(B, S_j(B)) \gtrsim 2^j r_B$, the substitution of $s = \frac{t}{r_B}$ and the doubling condition (2.2) yield that this again is bounded by a constant times

$$\sum_{j=2}^{\infty} V(B)^{-1/2} 2^{-2m\beta j} \left(\int_0^1 s^{4m\beta} \frac{ds}{s} \right)^{1/2} \|\tilde{f}\|_{L^2(S_j(B))}$$
$$\lesssim \sum_{j=2}^{\infty} 2^{-j(2m\beta - \frac{n}{2})} V(2^j B)^{-1/2} \|\tilde{f}\|_{L^2(S_j(B))} \lesssim \|f\|_{BMO_{L,M,res}(X)},$$

where we used the norm estimate (4.42) of $\tilde{f}$ together with the fact that $\beta > \frac{n}{4m}$ in the last step.
To estimate I_2 we first observe that

$$\begin{aligned}
&[I - (I - (I + r_B^{2m} L)^{-1})^M] \cdot (I - (I + r_B^{2m} L)^{-1})^{-M} \qquad (4.45)\\
&= (I - (I + r_B^{2m} L)^{-1})^{-M} - I = (I + (r_B^{2m} L)^{-1})^M - I = \sum_{k=1}^{M} \binom{M}{k} (r_B^{2m} L)^{-k}.
\end{aligned}$$

This allows us to handle I_2 in a similar way as I_1. If we insert (4.45) into the definition of I_2 and use the same annular decomposition of X as before, we find

$$\begin{aligned}
I_2 &\lesssim \sup_{1 \le k \le M} V(B)^{-1/2} \left(\int_0^{r_B} \int_B \left| \psi(t^{2m} L)(r_B^{2m} L)^{-k} (I - (I + r_B^{2m} L)^{-1})^M f(y) \right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \\
&= \sup_{1 \le k \le M} V(B)^{-1/2} \left(\int_0^{r_B} \left(\frac{t^{2m}}{r_B^{2m}} \right)^{2k} \int_B \left| \psi(t^{2m} L)(t^{2m} L)^{-k} \tilde{f}(y) \right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \\
&\lesssim \sup_{1 \le k \le M} V(B)^{-1/2} \left(\int_0^{r_B} \int_B \left| \psi(t^{2m} L)(t^{2m} L)^{-k} \mathbb{1}_{2B} \tilde{f}(y) \right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2} \\
&+ \sup_{1 \le k \le M} \sum_{j=2}^{\infty} V(B)^{-1/2} \left(\int_0^{r_B} \left(\frac{t^{2m}}{r_B^{2m}} \right)^{2k} \int_B \left| \psi(t^{2m} L)(t^{2m} L)^{-k} \mathbb{1}_{S_j(B)} \tilde{f}(y) \right|^2 \frac{d\mu(y)dt}{t} \right)^{1/2}.
\end{aligned} \qquad (4.46)$$

Since we assumed that $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$ with $\beta > \frac{n}{4m} + M$, we observe that $z \mapsto z^{-k} \psi(z) \in \Psi_{\beta-M,\alpha}(\Sigma_\mu^0)$ for every $1 \le k \le M$. Thus, we can again apply the quadratic estimates of $\{\psi(tL)(tL)^{-k}\}_{t>0}$ to the on-diagonal part in (4.46) and get the desired estimate as before.
Coming to the second part, we know from Proposition 3.18 that for every $1 \le k \le M$ the operator family $\{\psi(tL)(tL)^{-k}\}_{t>0}$ satisfies off-diagonal estimates of order $\beta - k$. Hence, the off-diagonal part in (4.46) is bounded by a constant times

$$\begin{aligned}
&\sup_{1 \le k \le M} \sum_{j=2}^{\infty} V(B)^{-1/2} \left(\int_0^{r_B} \left(\frac{t^{2m}}{r_B^{2m}} \right)^{2k} \left(\frac{\operatorname{dist}(B, S_j(B))^{2m}}{t^{2m}} \right)^{-2(\beta-k)} \frac{dt}{t} \right)^{1/2} \|\tilde{f}\|_{L^2(S_j(B))} \\
&\lesssim \sup_{1 \le k \le M} \sum_{j=2}^{\infty} V(B)^{-1/2} 2^{-2m(\beta-k)j} \left(\int_0^{r_B} \left(\frac{t^{2m}}{r_B^{2m}} \right)^{2k} \left(\frac{r_B^{2m}}{t^{2m}} \right)^{-2(\beta-k)} \frac{dt}{t} \right)^{1/2} \|\tilde{f}\|_{L^2(S_j(B))} \\
&\lesssim \sup_{1 \le k \le M} \sum_{j=2}^{\infty} V(2^j B)^{1/2} 2^{-j(2m(\beta-k) - \frac{n}{2})} \|\tilde{f}\|_{L^2(S_j(B))} \\
&\lesssim \|f\|_{BMO_{L,M,res}(X)},
\end{aligned}$$

where we used again the substitution $s = \frac{t}{r_B}$ in the last but one step and (4.42) together with the fact that $2m(\beta - k) - \frac{n}{2} > 0$ for every $1 \le k \le M$ in the last step.
In summary, we obtain $I_2 \lesssim \|f\|_{BMO_{L,M,res}(X)}$, which concludes the proof of the proposition for arbitrary $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$.
If $\psi \in \Psi(\Sigma_\mu^0)$ is defined by $\psi(z) = z^M e^{-z}$, $z \in \Sigma_\mu^0$, then the proof works analogously to the one above, taking into account that for every $0 \le k \le M$, the operator family $\{(tL)^k e^{-tL}\}_{t>0}$ satisfies Davies-Gaffney estimates due to Proposition 3.13. □

4.6 Duality of Hardy and BMO spaces

In this section we will determine the dual space of $H^1_L(X)$. The result is stated in Theorem 4.28 and the rest of the section will then be devoted to its proof.
We follow the lines of the proof of [HMM10], Theorem 3.52, which again is a modification of the proofs of [HMa09], Theorem 8.2 and Theorem 8.6.
In this context, we will give in Lemma 4.31 an extension of the Calderón reproducing formula, initially defined on $L^2(X)$, to functions from $H^1_L(X)$ and $BMO_{L^*,M}(X)$. This result is beside its application in Theorem 4.28 also interesting in itself and will further be used in Theorem 4.34 and Remark 5.9.

Theorem 4.28 *Suppose that L is an operator satisfying assumptions* (H1) *and* (H2)*. For any $M > \frac{n}{4m}$ and any $f \in BMO_{L^*,M}(X)$, the linear functional given by*

$$\ell(g) = \langle f, g \rangle, \tag{4.47}$$

initially defined on the dense subspace of $H^1_L(X)$, consisting of finite linear combinations of $(1, 2, M, \varepsilon)$-molecules, for some $\varepsilon > 0$, and where the pairing is that between $\mathcal{M}_0^{1,2,M,\varepsilon}(X)$ and its dual, has a unique bounded extension to $H^1_L(X)$ with

$$\|\ell\|_{(H^1_L(X))'} \le C \|f\|_{BMO_{L^*,M}(X)},$$

for some $C > 0$ independent of f.
Conversely, for every $M > \frac{n}{4m}$, every bounded linear functional ℓ on $H^1_L(X)$ can be realized as (4.47)*, i.e. there exists some $f \in BMO_{L^*,M}(X)$ such that* (4.47) *holds and*

$$\|f\|_{BMO_{L^*,M}(X)} \le C \|\ell\|_{(H^1_L(X))'},$$

for some $C > 0$ independent of ℓ.

In particular, the theorem yields that the definition of $BMO_{L,M}(X)$ is independent of the choice of $M > \frac{n}{4m}$. This leads to the following definition.

Definition 4.29 *Let L be an operator satisfying* (H1) *and* (H2)*. The space $BMO_L(X)$ is defined by*

$$BMO_L(X) := BMO_{L,M}(X),$$

where $M \in \mathbb{N}$ with $M > \frac{n}{4m}$.

We begin our proof of Theorem 4.28 with the observation that one can construct from every fixed $f \in BMO_{L^*,M}(X)$ a continuous linear functional on the space of all $(1, 2, M, \varepsilon)$-molecules for arbitrary $\varepsilon > 0$.
By Remark 4.20, every $(1, 2, M, \varepsilon)$-molecule m is in $\mathcal{M}_0^{1,2,M,\varepsilon}(X)$, hence the expression $\langle f, m \rangle$ is well-defined, where the pairing is that between $\mathcal{M}_0^{1,2,M,\varepsilon}(X)$ and its dual.

Proposition 4.30 *Let $M \in \mathbb{N}$ and $\varepsilon > 0$. For any $(1,2,M,\varepsilon)$-molecule m associated to a ball B of X, the mapping*

$$f \mapsto \langle f, m\rangle, \qquad f \in BMO_{L^*,M}(X),$$

is a bounded linear functional on $BMO_{L^*,M}(X)$.

Proof: Let m be an arbitrary $(1,2,M,\varepsilon)$-molecule associated to a ball B of X. We first observe that

$$\begin{aligned}(r_B^{2m}L)^M &= (I-(I+r_B^{2m}L)^{-1})^M(I+r_B^{2m}L)^M \\ &= (I-(I+r_B^{2m}L)^{-1})^M \sum_{k=0}^{M} C_{k,M}(r_B^{2m}L)^{M-k}, \end{aligned} \tag{4.48}$$

where $C_{k,M}$ are the coefficients from the binomial formula.
By definition of molecules, there exists some $b \in \mathcal{D}(L^M)$ with $m = L^M b$. Then, from (4.48), the Cauchy-Schwarz inequality and an annular decomposition of X, we obtain

$$\begin{aligned}|\langle f,m\rangle| &= r_B^{-2mM}\left|\langle f,(r_B^{2m}L)^M b\rangle\right| \\ &\lesssim r_B^{-2mM}\sum_{k=0}^{M}\left|\int_X (I-(I+r_B^{2m}L^*)^{-1})^M f(x)\overline{(r_B^{2m}L)^{M-k}b(x)}\,d\mu(x)\right| \\ &\le r_B^{-2mM}\sum_{k=0}^{M}\sum_{j=0}^{\infty}\left\|(I-(I+r_B^{2m}L^*)^{-1})^M f\right\|_{L^2(S_j(B))}\left\|(r_B^{2m}L)^{M-k}b\right\|_{L^2(S_j(B))}.\end{aligned}$$

The definition of molecules further yields that for every $j \in \mathbb{N}_0$ and every $0 \le k \le M$

$$\left\|(r_B^{2m}L)^{M-k}b\right\|_{L^2(S_j(B))} \le r_B^{2mM}2^{-j\varepsilon}V(2^jB)^{-1/2},$$

hence

$$|\langle f,m\rangle| \lesssim \sum_{j=0}^{\infty}2^{-j\varepsilon}V(2^jB)^{-1/2}\left\|(I-(I+r_B^{2m}L^*)^{-1})^M f\right\|_{L^2(S_j(B))}. \tag{4.49}$$

To estimate the remaining expression against the $BMO_{L^*,M}(X)$-norm of f, we use Lemma 4.24 with $C_f = \|f\|_{BMO_{L^*,M,res}(X)}$ and Proposition 4.23. We obtain

$$|\langle f,m\rangle| \lesssim \|f\|_{BMO_{L^*,M,res}(X)} \lesssim \|f\|_{BMO_{L^*,M}(X)},$$

which finishes the proof. □

Next, we will generalize the Calderón reproducing formula, originally given on $L^2(X)$ via functional calculus, to functions $f \in BMO_{L^*,M}(X)$ and $g \in H^1_L(X)$, that can be represented as a finite linear combination of molecules. The result is a generalization of Lemma 8.4 in [HMa09].

Lemma 4.31 *Let $M \in \mathbb{N}$ and suppose that $f \in \mathcal{E}_M(L^*)$ satisfies the "controlled growth estimate"*

$$\int_X \frac{\left|(I-(I+L^*)^{-1})^M f(x)\right|^2}{(1+d(x,0))^{\varepsilon_1}V(0,1+d(x,0))}\,d\mu(x) < \infty \tag{4.50}$$

for some $\varepsilon_1 > 0$. Let $\omega < \mu < \frac{\pi}{2}$. Let $\psi \in \Psi_{\beta_1,\alpha_1}(\Sigma^0_\mu) \setminus \{0\}$ and $\tilde{\psi} \in \Psi_{\beta_2,\alpha_2}(\Sigma^0_\mu) \setminus \{0\}$ for some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$, with $\beta_1 + \beta_2 > \frac{n+\varepsilon_1}{4m}$ and $\int_0^\infty \psi(t)\tilde{\psi}(t)\frac{dt}{t} = 1$. Then for every $g \in H^1_L(X)$ that can be represented as a finite linear combination of $(1, 2, M', \varepsilon)$-molecules, with $\varepsilon > \frac{\varepsilon_1}{2}$, $M' - M > \frac{n+\varepsilon_1}{4m}$ and $\alpha_1 + \alpha_2 > M'$, we have

$$\langle f, g\rangle = \lim_{\substack{\delta\to 0\\ R\to\infty}} \int_\delta^R \int_X \psi(t^{2m}L^*)f(x)\overline{\tilde{\psi}(t^{2m}L)g(x)}\, \frac{d\mu(x)dt}{t}.$$

Remark 4.32 If $f \in BMO_{L^*,M}(X)$, then condition (4.50) is fulfilled for every $\varepsilon_1 > 0$. This follows immediately from Lemma 4.24 and Proposition 4.23 by splitting the integral in (4.50) over X into annuli around B_0.

The proof of the lemma works in most parts analogously to the one of [HMa09]. We need one lemma in addition, which gives us a primitive of a function $\psi \in \Psi(\Sigma^0_\mu)$. The idea goes back to N. Kalton, a version of this is cited in [Ha04], Lemma 2.4.3.

Lemma 4.33 *Let $\mu \in (0, \pi)$, $\alpha, \beta > 0$ and $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$. Then for every $l \in \mathbb{N}$ with $l \geq \alpha$ there exists a function $\varphi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$ and some $\gamma \in \mathbb{C}$ such that*

$$\psi(z) = z\varphi'(z) + \gamma\frac{z}{(1+z)^{l+1}}, \qquad z \in \Sigma^0_\mu.$$

Proof: Let us define a function G on Σ^0_μ by setting

$$G(z) := \int_{\gamma_z} \frac{\psi(\zeta)}{\zeta}\, d\zeta, \qquad z \in \Sigma^0_\mu,$$

where $\gamma_z(t) := te^{i\arg z}$, $t \geq |z|$, is the parametrization of the half-ray with angle $\arg z$ starting at z. By assumption there holds $\frac{\psi(\zeta)}{\zeta} = \mathcal{O}(|\zeta|^{-\alpha-1})$ for $|\zeta| \to \infty$ and consequently, $G(z) = \mathcal{O}(|z|^{-\alpha})$ for $|z| \to \infty$. By definition of G, we further have

$$zG'(z) = \psi(z), \qquad z \in \Sigma^0_\mu.$$

To get the desired behaviour at 0, one has to do a little more work. We know by assumption that $\frac{\psi(z)}{z} = \mathcal{O}(|z|^{\beta-1})$ for $|z| \to 0$ and, since $\beta > 0$, the integral

$$\int_{\Gamma_\theta} \frac{\psi(\zeta)}{\zeta}\, d\zeta \tag{4.51}$$

converges for every $\theta \in (-\mu, \mu)$, where $\Gamma_\theta(t) := te^{i\theta}$, $0 < t < \infty$. Using the same arguments as in [KW04], Remark 9.3, one can show that due to Cauchy's theorem, the integral in (4.51) is independent of the angle $\theta \in (-\mu, \mu)$.
Therefore, let us set $c := \int_{\Gamma_\theta} \frac{\psi(\zeta)}{\zeta}\, d\zeta$ for any $\theta \in (-\mu, \mu)$. We then obtain

$$c - G(z) = \int_{\widetilde{\gamma}_z} \frac{\psi(\zeta)}{\zeta}\, d\zeta, \qquad z \in \Sigma^0_\mu,$$

where $\widetilde{\gamma}_z(t) := te^{i\arg z}$, $0 < t \leq |z|$, is the parametrization of the half-ray with angle $\arg z$ starting at 0 and ending at z.

From the assumption $\frac{\psi(\zeta)}{\zeta} = \mathcal{O}(|z|^{\beta-1})$ for $|z| \to 0$ we now get that $c - G(z) = \mathcal{O}(|z|^{\beta})$ for $|z| \to 0$. Therefore, by defining for a given $l \in \mathbb{N}$ with $l \geq \alpha$

$$\varphi(z) := G(z) - \frac{c}{(1+z)^l}, \qquad z \in \Sigma_\mu^0,$$

we obtain the following: By construction there holds $\varphi(z) = \mathcal{O}(|z|^{\beta})$ for $|z| \to 0$ and $\varphi(z) = \mathcal{O}(|z|^{-\alpha})$ for $|z| \to \infty$. In addition, a simple calculation shows that

$$\psi(z) = zG'(z) = z\varphi'(z) - \frac{lcz}{(1+z)^{l+1}},$$

which concludes the proof with $\gamma = -lc$. □

Proof (of Lemma 4.31): Without restriction we assume $\|g\|_{H^1_L(X)} \leq 1$. We will further assume that g can be represented as a finite linear combination of $(1, 2, M', \varepsilon)$-molecules, where all molecules are associated to the unit ball B_0 centered at 0 with radius 1. As described in Remark 4.20, this is possible due to the fact that for any ball B in X there exists some constant K_B such that $B \subseteq 2^{K_B} B_0$. Hence, it is possible to renormalize any molecule originally associated to B such that it is associated to B_0. A careful inspection of the limiting procedures below shows that we can omit these renormalization constants in the following.
For $0 < \delta < R$ let us consider

$$\begin{aligned}\int_X \int_\delta^R \psi(t^{2m}L^*)f(x)\overline{\tilde{\psi}(t^{2m}L)g(x)}\,\frac{dt d\mu(x)}{t} &= \left\langle f, \left(\int_\delta^R \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)g\,\frac{dt}{t}\right)\right\rangle \\ &= \langle f, g\rangle - \left\langle f, \left(g - \int_\delta^R \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)g\,\frac{dt}{t}\right)\right\rangle. \end{aligned} \tag{4.52}$$

On the one hand, we will write f in the following way. Using the binomial formula, we obtain

$$\begin{aligned} f &= (I - (I+L^*)^{-1} + (I+L^*)^{-1})^M f \\ &= \sum_{k=0}^{M} C_{k,M}(I - (I+L^*)^{-1})^{M-k}(I+L^*)^{-k} f \\ &= \sum_{k=0}^{M} C_{k,M}(L^*)^{-k}(I - (I+L^*)^{-1})^{M} f, \end{aligned} \tag{4.53}$$

where $C_{k,M}$ are the appropriate binomial coefficients.
On the other hand, since $g \in H^1_L(X)$ is a finite linear combination of $(1, 2, M', \varepsilon)$-molecules associated to B_0, we know that $L^{-k}g$ is in $L^2(X)$ with $\|L^{-k}g\|_{L^2(X)} \lesssim V(B_0)^{-1/2}$ for every $k = 0, \ldots, M'$. This allows us to use the Calderón reproducing formula in $L^2(X)$, and with the help of (4.53) we can write the second term of (4.52) as $\sum_{k=0}^{M} C_{k,M}$ times

$$\begin{aligned} &\left\langle (I - (I+L^*)^{-1})^M f, \left(L^{-k}g - \int_\delta^R \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)L^{-k}g\,\frac{dt}{t}\right)\right\rangle \\ &= \left\langle (I - (I+L^*)^{-1})^M f, \int_0^\delta \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)L^{-k}g\,\frac{dt}{t}\right\rangle \\ &\quad + \left\langle (I - (I+L^*)^{-1})^M f, \int_R^\infty \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)L^{-k}g\,\frac{dt}{t}\right\rangle =: I_1 + I_2. \end{aligned}$$

Let us first estimate integral I_2. For convenience, we denote by Υ the finite quantity in (4.50). The Cauchy-Schwarz inequality in the first step, an annular decomposition of X around B_0 in the second step and Minkowski's inequality in the third step imply

$$\begin{aligned}
|I_2| &\lesssim \left(\int_X \frac{\left|(I-(I+L^*)^{-1})^M f(x)\right|^2}{(1+d(x,0))^{\varepsilon_1} V(0,1+d(x,0))}\, d\mu(x) \right)^{1/2} \\
&\quad \times \sup_{0\le k\le M} \left(\int_X \left| \int_R^\infty \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)L^{-k}g(x)\,\frac{dt}{t} \right|^2 (1+d(x,0))^{\varepsilon_1} V(0,1+d(x,0))\, d\mu(x) \right)^{1/2} \\
&\lesssim \Upsilon \sup_{0\le k\le M} \sum_{j=0}^\infty 2^{j\varepsilon_1/2} V(2^jB_0)^{1/2} \left(\int_{S_j(B_0)} \left| \int_R^\infty \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)L^{-k}g(x)\,\frac{dt}{t} \right|^2 dx \right)^{1/2} \\
&\le \Upsilon \sup_{0\le k\le M} \sum_{j=0}^\infty 2^{j\varepsilon_1/2} V(2^jB_0)^{1/2} \\
&\qquad \times \int_R^\infty \left\| \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)(t^{2m}L)^{M'-k}L^{-M'}g \right\|_{L^2(S_j(B_0))} \frac{dt}{t^{2m(M'-k)+1}}.
\end{aligned}$$

In the last step we use that (following [HMa09]) molecules "absorb" negative powers (up to order M') of the operator L. Doing this increases the negative powers of t and delivers us with the needed decay as R goes to infinity.

Recall that we assumed $\psi \in \Psi_{\beta_1,\alpha_1}(\Sigma_\mu^0)$ and $\tilde{\psi} \in \Psi_{\beta_2,\alpha_2}(\Sigma_\mu^0)$ and, additionally, $M' < \alpha_1+\alpha_2$. Therefore one can easily see that for every $0 \le k \le M$

$$\psi_k : z \mapsto \psi(z)\tilde{\psi}(z)z^{M'-k} \;\in\; \Psi_{\beta,\alpha}(\Sigma_\mu^0),$$

with $\beta := \beta_1+\beta_2+M'-M$ and $\alpha := \alpha_1+\alpha_2-M'$.

On the one hand, this implies the uniform boundedness of the operator family $\{\psi_k(tL)\}_{t>0}$ in $L^2(X)$. From Proposition 3.18 follows, on the other hand, that $\{\psi_k(tL)\}_{t>0}$ satisfies L^2 off-diagonal estimates of order β.

Having this in mind, we can split $L^{-M'}g = \mathbb{1}_{X\setminus 2^{j-2}B_0}L^{-M'}g + \mathbb{1}_{2^{j-2}B_0}L^{-M'}g$ for each $j \ge 3$. For the first term we use the rapid decay of $L^{-M}g$ away from B_0, for the second we take advantage of the off-diagonal estimate of the operator. More precisely, we decompose the expression under the sup sign above and estimate it against a constant times

$$\begin{aligned}
&V(B_0)^{1/2} \int_R^\infty \left\| \psi_k(t^{2m}L)L^{-M'}g \right\|_{L^2(4B_0)} \frac{dt}{t^{2m(M'-k)+1}} \\
&\quad + \sum_{j=3}^\infty 2^{j\varepsilon_1/2} V(2^jB_0)^{1/2} \int_R^\infty \left\| \psi_k(t^{2m}L)[\mathbb{1}_{X\setminus 2^{j-2}B_0}L^{-M'}g] \right\|_{L^2(S_j(B_0))} \frac{dt}{t^{2m(M'-k)+1}} \\
&\quad + \sum_{j=3}^\infty 2^{j\varepsilon_1/2} V(2^jB_0)^{1/2} \int_R^\infty \left\| \psi_k(t^{2m}L)[\mathbb{1}_{2^{j-2}B_0}L^{-M'}g] \right\|_{L^2(S_j(B_0))} \frac{dt}{t^{2m(M'-k)+1}}
\end{aligned}$$

$$\lesssim \frac{1}{R^{2m(M'-k)}} V(B_0)^{1/2} \left\| L^{-M'} g \right\|_{L^2(X)}$$
$$+ \frac{1}{R^{2m(M'-k)}} \sum_{j=3}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \left\| L^{-M'} g \right\|_{L^2(X \setminus 2^{j-2} B_0)}$$
$$+ \sum_{j=3}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \int_R^\infty \left(1 + \frac{\operatorname{dist}(S_j(B_0), 2^{j-2}B_0)^{2m}}{t^{2m}} \right)^{-\beta} \frac{dt}{t^{2m(M'-k)+1}} \left\| L^{-M'} g \right\|_{L^2(X)}. \tag{4.54}$$

Let us first consider the third term in (4.54). Since $\operatorname{dist}(S_j(B_0), 2^{j-2}B_0) \approx 2^j$, we get by substituting $t = 2^j s$,

$$\int_R^\infty \left(1 + \frac{\operatorname{dist}(S_j(B_0), 2^{j-2}B_0)^{2m}}{t^{2m}} \right)^{-\beta} \frac{dt}{t^{2m(M'-k)+1}}$$
$$\lesssim \int_R^\infty \min\left(1, \frac{t}{2^j}\right)^{2m\beta} \frac{dt}{t^{2m(M'-k)+1}} = \frac{1}{2^{2m(M'-k)j}} \int_{R/2^j}^\infty \min(1,s)^{2m\beta} \frac{ds}{s^{2m(M'-k)+1}}. \tag{4.55}$$

Next, observe that by definition of β there holds $\beta > M'-M$ and $M'-M > 0$. Therefore, for every $\varepsilon_0 > 0$ with $\beta > M' - M - \varepsilon_0 > 0$ we can estimate (4.55) by

$$\frac{1}{2^{2m(M'-k)j}} \int_{R/2^j}^\infty \min(1,s)^{2m(M'-M-\varepsilon_0)} \frac{ds}{s^{2m(M'-k)+1}}$$
$$\leq \frac{1}{2^{2m(M'-k)j}} \int_{R/2^j}^\infty \frac{ds}{s^{2m(M-k+\varepsilon_0)+1}} \lesssim 2^{-2m(M'-M-\varepsilon_0)j} R^{-2m(M-k+\varepsilon_0)}.$$

Inserting this into (4.54) and using the doubling property for μ and the norm estimate $\left\| L^{-M'} g \right\|_{L^2(X)} \lesssim V(B_0)^{-1/2}$, one can see that the third term in (4.54) is bounded by a constant times

$$R^{-2m(M-k+\varepsilon_0)} \sum_{j=3}^{\infty} 2^{j(n+\varepsilon_1)/2} 2^{-2m(M'-M-\varepsilon_0)j}.$$

Choosing $\varepsilon_0 > 0$ small enough and taking into account that $M' - M > \frac{n+\varepsilon_1}{4m}$, this in turn is bounded by a constant times $R^{-\varepsilon_2}$ for some $\varepsilon_2 > 0$.

To handle the second term of (4.54), we recall that the definition of molecules yields the estimate

$$\left\| L^{-k} g \right\|_{L^2(S_\nu(B_0))} \lesssim 2^{-\nu\varepsilon} V(2^\nu B_0)^{-1/2}, \qquad \nu \in \mathbb{N}_0,\ 0 \leq k \leq M', \tag{4.56}$$

since g is a finite linear combination of $(1, 2, M', \varepsilon)$ molecules associated to B_0. Splitting $X \setminus 2^{j-2}B_0$ into annuli around B_0, this leads to

$$\sum_{j=3}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \left\| L^{-M'} g \right\|_{L^2(X \setminus 2^{j-2} B_0)}$$
$$\leq \sum_{j=3}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \sum_{\nu=j-2}^{\infty} \left\| L^{-M'} g \right\|_{L^2(S_\nu(B_0))}$$
$$\lesssim \sum_{j=3}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \sum_{\nu=j-2}^{\infty} 2^{-\nu\varepsilon} V(2^\nu B_0)^{-1/2} \lesssim \sum_{j=3}^{\infty} 2^{j\varepsilon_1/2} \sum_{\nu=j-2}^{\infty} 2^{-\nu\varepsilon} \lesssim 1$$

for every $\varepsilon > \frac{\varepsilon_1}{2}$. Thus, we observe that the second and, again using (4.56), also the first term of (4.54) are bounded by a constant times $R^{-2m(M'-k)}$.
In summary, we have the following: There exists some $\varepsilon' > 0$ such that

$$|I_2| \lesssim \Upsilon \sup_{0\le k\le M} \left[R^{-2m(M'-k)} + R^{-2m(M'-k)} + R^{-\varepsilon_2}\right] \lesssim \Upsilon R^{-\varepsilon'}.$$

We now come to deal with I_1. Denoting $\tilde{f} := (I-(I+L^*)^{-1})^M f$, this means we have to estimate

$$I_1 = \langle \tilde{f}, \int_0^\delta \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)L^{-k}g\,\frac{dt}{t}\rangle$$

for every $k = 0, \ldots, M$.
To get rid of the integral over t, we look for a primitive of $\psi\cdot\tilde{\psi}$. Set $l := [\alpha_1+\alpha_2]+1$. According to Lemma 4.33, there exists a function $\varphi \in \Psi_{\beta_1+\beta_2,\alpha_1+\alpha_2}(\Sigma_\mu^0)$ and a $\gamma \in \mathbb{C}$ such that

$$\psi(z)\tilde{\psi}(z) = z\varphi'(z) + \gamma\frac{z}{(1+z)^{l+1}}, \qquad z \in \Sigma_\mu^0.$$

This allows us to write

$$\begin{aligned}
&\int_0^\delta \psi(t^{2m}L)\tilde{\psi}(t^{2m}L)L^{-k}g\,\frac{dt}{t}\\
&\quad= \int_0^\delta (t^{2m}L)\varphi'(t^{2m}L)L^{-k}g\,\frac{dt}{t} + \gamma\int_0^\delta (t^{2m}L)(I+t^{2m}L)^{-(l+1)}L^{-k}g\,\frac{dt}{t}\\
&\quad= \frac{1}{2m}\int_0^\delta \partial_t(\varphi(t^{2m}L))L^{-k}g\,dt - \frac{1}{2ml}\gamma\int_0^\delta \partial_t((I+t^{2m}L)^{-l})L^{-k}g\,dt\\
&\quad= \frac{1}{2m}\varphi(\delta^{2m}L)L^{-k}g - \frac{1}{2ml}\gamma[(I+\delta^{2m}L)^{-l} - I]L^{-k}g.
\end{aligned}$$

We first handle the term with $\varphi(\delta^{2m}L)$. As in the treatment of I_2, the Cauchy-Schwarz inequality, assumption (4.50) and an annular decomposition of X around B_0 yield

$$\begin{aligned}
\left|\langle \tilde{f}, \varphi(\delta^{2m}L)L^{-k}g\rangle\right| &\lesssim \Upsilon\left(\int_X \left|\varphi(\delta^{2m}L)L^{-k}g(x)\right|^2 (1+d(x,0))^{\varepsilon_1}V(0,1+d(x,0))\,d\mu(x)\right)^{1/2}\\
&\lesssim \Upsilon\sum_{j=0}^\infty 2^{j\varepsilon_1/2}V(2^jB_0)^{1/2}\left\|\varphi(\delta^{2m}L)L^{-k}g\right\|_{L^2(S_j(B_0))}. \qquad (4.57)
\end{aligned}$$

Since adding extra negative powers of L as we did for I_2 does not help here, we need to split $L^{-k}g$ more carefully. We write $L^{-k}g = \mathbb{1}_{R_j}L^{-k}g + \mathbb{1}_{(R_j)^c}L^{-k}g$ with

$$\begin{aligned}
R_j &= 2^{j+2}B_0, \qquad \text{if } j = 0,1,2,\\
R_j &= 2^{j+2}B_0 \setminus 2^{j-2}B_0, \qquad \text{if } j = 3,4,\ldots.
\end{aligned}$$

Fix some $\eta > 0$. For the "on-diagonal" term $\mathbb{1}_{R_j}L^{-k}g$, the uniform boundedness of $\varphi(\delta^{2m}L)$ in $L^2(X)$ and (4.56) yield for arbitrary $N \in \mathbb{N}$ and for all $0 \le k \le M$

$$\begin{aligned}
&\sum_{j=N}^\infty 2^{j\varepsilon_1/2}V(2^jB_0)^{1/2}\left\|\varphi(\delta^{2m}L)\mathbb{1}_{R_j}L^{-k}g\right\|_{L^2(S_j(B_0))}\\
&\quad\lesssim \sum_{j=N}^\infty 2^{j\varepsilon_1/2}V(2^jB_0)^{1/2}\left\|L^{-k}g\right\|_{L^2(R_j)} \lesssim \sum_{j=N}^\infty 2^{j\varepsilon_1/2}V(2^jB_0)^{1/2}2^{-j\varepsilon}V(2^jB_0)^{-1/2}.
\end{aligned}$$

Due to the fact that we assumed $\tilde{\varepsilon} := \varepsilon - \varepsilon_1/2 > 0$, we can choose N depending on η such that the sum above is bounded by

$$\sum_{j=N}^{\infty} (2^{-\tilde{\varepsilon}})^j = 2^{-\tilde{\varepsilon}N} \frac{1}{1-2^{-\tilde{\varepsilon}}} \lesssim \eta.$$

The convergence of $\varphi(\delta^{2m}L) \to 0$ for $\delta \to 0$ in the strong operator topology and the fact that $L^{-k}g \in L^2(X)$ for every $0 \le k \le M$ allow us to estimate

$$\sum_{j=0}^{N} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \left\| \varphi(\delta^{2m}L) \mathbb{1}_{R_j} L^{-k} g \right\|_{L^2(S_j(B_0))} \lesssim \eta, \tag{4.58}$$

provided that $\delta > 0$ is small enough.

We now come to the "off-diagonal" term of (4.57) with $\mathbb{1}_{(R_j)^c} L^{-k} g$. We know that φ is in $\Psi_{\beta_1+\beta_2, \alpha_1+\alpha_2}(\Sigma^0_\mu)$, hence, according to Proposition 3.18, $\varphi(\delta^{2m}L)$ satisfies off-diagonal estimates of order $\beta_1 + \beta_2$. Therefore, using the off-diagonal estimates instead of the decay of molecules, we obtain for $\delta > 0$ as chosen in (4.58)

$$\begin{aligned} &\sum_{j=N}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \left\| \varphi(\delta^{2m}L) \mathbb{1}_{(R_j)^c} L^{-k} g \right\|_{L^2(S_j(B_0))} \\ &\lesssim \sum_{j=N}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \left(1 + \frac{\operatorname{dist}(S_j(B_0), (R_j)^c)^{2m}}{\delta^{2m}}\right)^{-(\beta_1+\beta_2)} \left\| L^{-k} g \right\|_{L^2(X)}. \end{aligned} \tag{4.59}$$

Observe that $\operatorname{dist}(S_j(B_0), (R_j)^c) \approx 2^j$ and that we assumed $\beta_1 + \beta_2 > \frac{n+\varepsilon_1}{4m}$. In view of the doubling property, we can choose some N depending on η such that the sum above is bounded by

$$\sum_{j=N}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} 2^{-2m(\beta_1+\beta_2)j} \delta^{2m(\beta_1+\beta_2)} V(B_0)^{-1/2} \lesssim \eta,$$

where we estimated $\delta > 0$ by some constant.
In the same way as in (4.58), the sum over $j = 0, \ldots, N$ can be bounded by a constant times η.

To conclude the proof we have to replace $\varphi(\delta^{2m}L)$ by $(I + \delta^{2m}L)^{-l} - I$ in (4.57) and do the same reasoning again. The operator $(I + \delta^{2m}L)^{-l} - I$ is also uniformly bounded in $L^2(X)$ with the convergence $(I + \delta^{2m}L)^{-l} - I = -L(\delta^{-2m} + L)^{-l} \to 0$ for $\delta \to 0$ in the strong operator topology. The only thing different is the estimate of the "off-diagonal" term. Instead of (4.59), we have

$$\begin{aligned} &\sum_{j=N}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \left\| [(I + \delta^{2m}L)^{-l} - I] \mathbb{1}_{(R_j)^c} L^{-k} g \right\|_{L^2(S_j(B_0))} \\ &\quad = \sum_{j=N}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \left\| (I + \delta^{2m}L)^{-l} \mathbb{1}_{(R_j)^c} L^{-k} g \right\|_{L^2(S_j(B_0))} \\ &\quad \lesssim \sum_{j=N}^{\infty} 2^{j\varepsilon_1/2} V(2^j B_0)^{1/2} \exp\left(-\frac{\operatorname{dist}(S_j(B_0), (R_j)^c)}{c\delta}\right) \left\| L^{-k} g \right\|_{L^2(X)}, \end{aligned}$$

due to the fact that $S_j(B_0) \cap (R_j)^c = \emptyset$ and the Davies-Gaffney estimate for the resolvent according to Proposition 3.16. The remaining works as above. $\square$

We now come to the proof of the main result of this section, Theorem 4.28.

Proof (of Theorem 4.28): Let $M \in \mathbb{N}$ with $M > \frac{n}{4m}$.
We begin with the inclusion $(H^1_L(X))' \subseteq BMO_{L^*,M}(X)$.
Assume that ℓ is a linear functional on $H^1_L(X)$. Then for every $g \in H^1_L(X)$

$$|\ell(g)| \leq \|\ell\|_{(H^1_L(X))'} \|g\|_{H^1_L(X)} .$$

Let $\varepsilon > 0$. Theorem 4.7 implies in particular, that there exists a constant $C > 0$ such that every $(1, 2, M, \varepsilon)$-molecule m belongs to $H^1_L(X)$ with $\|m\|_{H^1_L(X)} \leq C$. Hence,

$$|\ell(m)| \leq C \|\ell\|_{(H^1_L(X))'} . \tag{4.60}$$

However, due to Remark 4.20, we know that if $\phi \in \mathcal{M}_0^{1,2,M,\varepsilon}(L)$ with $\|\phi\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} = 1$, then ϕ is a $(1, 2, M, \varepsilon)$-molecule adapted to B_0. Therefore, by (4.60), ℓ defines a linear functional on $\mathcal{M}_0^{1,2,M,\varepsilon}(L)$. Since $\varepsilon > 0$ was arbitrary, this implies that $\ell \in \mathcal{E}_M(L^*)$.
Moreover, Lemma 4.3 yields that for every ball B in X and every $\varphi \in L^2(B)$ such that $\|\varphi\|_{L^2(B)} = 1$, the function

$$m_B := \frac{1}{V(B)^{1/2}}(I - e^{r_B^{2m}L})^M \varphi$$

is a $(1, 2, M, \varepsilon)$-molecule associated to B. Therefore, via duality, we can realize ℓ as an element of $BMO_{L^*,M}(X)$, i.e. there exists some $f \in BMO_{L^*,M}(X)$ such that

$$\frac{1}{V(B)^{1/2}} \langle \varphi, (I - e^{-r_B^{2m}L^*})^M f \rangle = \frac{1}{V(B)^{1/2}} \langle (I - e^{-r_B^{2m}L})^M \varphi, f \rangle = \ell(m_B),$$

and by taking the supremum over all balls B in X and all $\varphi \in L^2(B)$, we get from (4.60) the corresponding norm estimate

$$\|f\|_{BMO_{L^*,M}(X)} \lesssim \|\ell\|_{(H^1_L(X))'} .$$

Let us now turn to the converse inclusion $BMO_{L^*,M}(X) \subseteq (H^1_L(X))'$.
Let $f \in BMO_{L^*,M}(X)$. Via Proposition 4.30 we can initially define the mapping

$$\ell_f(g) := \langle f, g \rangle$$

on a dense subspace of $H^1_L(X)$, namely for all $g \in H^1_L(X)$ that are a finite linear combination of $(1, 2, M, \varepsilon)$-molecules, where $\varepsilon > 0$ is arbitrary. Hence, it is sufficient to show that there exists some constant $C > 0$ such that

$$|\ell_f(g)| \leq C \|f\|_{BMO_{L^*,M}(X)} \|g\|_{H^1_{L,mol,M}(X)} \tag{4.61}$$

for all $g \in H^1_L(X)$ that are a finite linear combination of $(1, 2, M, \varepsilon)$-molecules. Then ℓ_f extends by continuity to a continuous linear functional on $H^1_L(X)$.
First observe that, in view of Theorem 4.7, it is enough to prove (4.61) for all $g \in H^1_L(X)$ that are a finite linear combination of $(1, 2, M', \varepsilon)$-molecules, where $\varepsilon > 0$ is arbitrary

and $M' > M$ is fixed and chosen such that the assumptions of Lemma 4.31 are fulfilled. Recall that due to Remark 4.32 the growth estimate (4.50) is satisfied for every $\varepsilon_1 > 0$. As every $(1, 2, M', \varepsilon)$-molecule is also a $(1, 2, M, \varepsilon)$-molecule, $\ell_f(g)$ is for such a g still well-defined and Lemma 4.31 yields that the Calderón reproducing formula

$$\langle f, g\rangle = C_M \lim_{\substack{\delta\to 0\\ R\to\infty}} \int_\delta^R \int_X (t^{2m}L^*)^M e^{-t^{2m}L^*} f(x)\, \overline{t^{2m}Le^{-t^{2m}L}g(x)}\, \frac{d\mu(x)dt}{t} \tag{4.62}$$

is valid.
To estimate the above expression, we use on the one hand that due to Proposition 4.27 the function F, defined by

$$F(x,t) := (t^{2m}L^*)^M e^{-t^{2m}L^*} f(x), \qquad (x,t) \in X \times (0,\infty),$$

is in $T^\infty(X)$ with $\|F\|_{T^\infty(X)} \lesssim \|f\|_{BMO_{L^*,M}(X)}$. On the other hand, we have by Theorem 4.7 that the function G, defined by $G(x,t) := t^{2m}Le^{-t^{2m}L}g(x)$, $(x,t) \in X \times (0,\infty)$, is an element of $T^1(X)$. The atomic decomposition of $T^1(X)$ stated in Proposition 2.20 then yields that there exist a numerical sequence $\{\lambda_j\}_{j=0}^\infty$ and a sequence $\{A_j\}_{j=0}^\infty$ of $T^1(X)$ atoms supported in tents $\hat{B}_j$, where $B_j \subseteq X$ are balls, such that

$$G(\,.\,,t) = t^{2m}Le^{-t^{2m}L}g = \sum_{j=0}^\infty \lambda_j A_j$$

and there exists a constant $C > 0$ independent of G with

$$\sum_{j=0}^\infty |\lambda_j| \le C\, \|G\|_{T^1(X)} \approx \|g\|_{H^1_{L,mol,M}(X)}\,.$$

Hence, as each atom A_j is supported in the tent region $\hat{B}_j$, we can estimate (4.62) by

$$\begin{aligned}
|\langle f, g\rangle| &\lesssim \sum_{j=0}^\infty |\lambda_j| \int_0^\infty \int_X \left|(t^{2m}L^*)^M e^{-t^{2m}L^*} f(x)\right| |A_j(x,t)|\, \frac{d\mu(x)dt}{t}\\
&\lesssim \sum_{j=0}^\infty |\lambda_j| \int_X \mathscr{C}(F)(x)\mathscr{A}(A_j)(x)\, d\mu(x)\\
&\lesssim \sum_{j=0}^\infty |\lambda_j|\, \|F\|_{T^\infty(X)}\, \|A_j\|_{T^1(X)}\\
&\lesssim \|g\|_{H^1_{L,mol,M}(X)}\, \|f\|_{BMO_{L^*,M}(X)}\,,
\end{aligned}$$

where we have used Theorem 2.17 in the second step, Hölder's inequality in the third step and the norm estimates on F and G in the last step, taking into account that the norm of $T^1(X)$ atoms is controlled by a constant only depending on the space X.
This gives us the desired estimate (4.61), the proof is complete. □

4.7 Carleson measures revisited

In this section, we show the converse of Proposition 4.27, which gives the connection between functions from $BMO_L(X)$ and Carleson measures.
For a special choice of ψ, namely $\psi(z) = z^M e^{-z}$, the result is due to [HMa09], Theorem 9.1. In the generality as stated below, the result is new.

Theorem 4.34 *Let $M \in \mathbb{N}$, $M > \frac{n}{4m}$ and let $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$, where $\alpha > 0$ and $\beta > \frac{n}{4m}$. If $f \in \mathcal{E}_M(L)$ satisfies the controlled growth bound* (4.50) *(with L in place of L^*) for some $\varepsilon_1 > 0$, and if*

$$\nu_{\psi,f} := \left|\psi(t^{2m}L)f(y)\right|^2 \frac{d\mu(y)dt}{t} \tag{4.63}$$

is a Carleson measure, then $f \in BMO_L(X)$ and

$$\|f\|^2_{BMO_L(X)} \leq C \|\nu_{\psi,f}\|_{\mathcal{C}} .$$

For the proof, we essentially follow the lines of the proof [HMa09], Theorem 9.1. The main difference is that we use instead of [HMa09], Lemma 8.3, the more general Calderón reproducing formula shown in Lemma 4.31.

Proof: The result relies on the duality of the tent spaces $T^1(X)$ and $T^\infty(X)$ stated in Theorem 2.17. For $f \in \mathcal{E}_M(L)$ satisfying (4.50) and every $g \in H^1_{L^*}(X)$ that can be represented as a finite linear combination of $(1, 2, M', \varepsilon)$ molecules for some $M' > \frac{n}{4m}$ with $M' - M > \frac{n+\varepsilon_1}{4m}$ and $\varepsilon > \frac{\varepsilon_1}{2}$, we have by Lemma 4.31 that the duality pairing $\langle f, g\rangle$ can be represented by

$$\langle f, g\rangle = \iint_{X\times(0,\infty)} \psi(t^{2m}L)f(x)\, \overline{\tilde{\psi}(t^{2m}L^*)g(x)}\, \frac{d\mu(x)dt}{t},$$

where for some $\alpha_2 > M'$, $\beta_2 > 0$ the function $\tilde{\psi} \in \Psi_{\beta_2,\alpha_2}(\Sigma^0_\mu)$ is chosen such that $\int_0^\infty \psi(t)\tilde{\psi}(t)\, \frac{dt}{t} = 1$.
Moreover, according to Theorem 2.17, there holds

$$\begin{aligned}
&\iint_{X\times(0,\infty)} \left|\psi(t^{2m}L)f(x)\, \tilde{\psi}(t^{2m}L^*)g(x)\right| \frac{d\mu(x)dt}{t} \\
&\quad\lesssim \int_X \mathscr{C}(\psi(t^{2m}L)f)(x)\, \mathscr{A}(\tilde{\psi}(t^{2m}L^*)g)(x)\, d\mu(x) \\
&\quad\leq \left\|\mathscr{C}(\psi(t^{2m}L)f)\right\|_{L^\infty(X)} \left\|\mathscr{A}(\tilde{\psi}(t^{2m}L^*)g)\right\|_{L^1(X)}.
\end{aligned} \tag{4.64}$$

The first term in (4.64) is bounded by $\|\nu_{\psi,f}\|^{1/2}_{\mathcal{C}}$ by assumption, whereas the second is bounded by $\|g\|_{H^1_{L^*}(X)}$ due to Proposition 4.7. Therefore, we have

$$|\langle f, g\rangle| \lesssim \|\nu_{\psi,f}\|^{1/2}_{\mathcal{C}} \|g\|_{H^1_{L^*}(X)} ,$$

for every $g \in H^1_{L^*}(X)$ that can be represented as a finite linear combination of $(1, 2, M', \varepsilon)$ molecules. Since the space of all these functions is dense in $H^1_L(X)$, we obtain from Theorem 4.28, that $f \in BMO_L(X)$ with the desired norm estimate. □

4.8 The spaces $H^p_L(X)$ and interpolation

Hofmann, Mayboroda and McIntosh have shown in [HMM10] that there is a natural extension of the Hardy space H^1_L to Hardy spaces $H^p_L(X)$ for all $0 < p < \infty$. They give certain characterizations of $H^p_L(X)$ spaces, show duality and interpolation results and state the relation between $L^p(X)$ and $H^p_L(X)$ spaces. Additional results are presented by Duong and Li in [DL09] in the case of $0 < p \le 1$.

We present here the definition of $H^p_L(X)$ and the main results for the case $1 < p < \infty$. As the occasions, where we will apply the theory of $H^p_L(X)$ spaces, have a more supplementary character, we omit the proofs. The proofs can be found in [HMM10] in a slightly less general setting, but they take over with only minor changes.
In addition, we generalize a result of Hofmann and Mayboroda from [HMa09] that gives sufficient conditions for an operator to be bounded from $H^1_L(X)$ to $L^1(X)$.

Let us define $\psi_0 \in \Psi(\Sigma^0_\mu)$ by $\psi_0(z) := ze^{-z}$, $z \in \Sigma^0_\mu$, and consider for every $f \in L^2(X)$ the square function $\mathscr{A}Q_{\psi_0,L}f$ associated to L, namely

$$\mathscr{A}Q_{\psi_0,L}(f)(x) = \left(\iint_{\Gamma(x)} \left| t^{2m} L e^{-t^{2m}L} f(y) \right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2}, \quad x \in X.$$

Definition 4.35 *(i) Let $1 \le p \le 2$. We define $H^p_L(X)$ to be the completion of the space*

$$\mathbb{H}^p_L(X) := \{ f \in L^2(X) \, : \, \mathscr{A}Q_{\psi_0,L}f \in L^p(X) \},$$

with respect to the norm

$$\|f\|_{H^p_{\psi_0,L}(X)} := \|\mathscr{A}Q_{\psi_0,L}f\|_{L^p(X)} = \|Q_{\psi_0,L}f\|_{T^p(X)} .$$

(ii) Let $2 < p < \infty$. We define

$$H^p_L(X) := (H^{p'}_{L^*}(X))',$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and L^* is the adjoint operator of L.

Observe that due to Remark 3.20 there holds $H^2_L(X) = L^2(X)$.

In both cases, for $p \le 2$ and for $p > 2$, there is a characterization of $H^p_L(X)$ by general square functions $\psi \in \Psi(\Sigma^0_\mu)$ with a certain decay at infinity and at zero, respectively. This generalizes the result of $H^1_L(X)$ stated in Theorem 4.7. For a proof, we refer to Corollary 4.21 of [HMM10].

Theorem 4.36 *Let $\alpha > 0$ and $\beta > \frac{n}{4m}$. Further, let either $1 \le p \le 2$ and $\psi \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$ or $2 \le p < \infty$ and $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$. Define $H^p_{\psi,L}(X)$ to be the completion of the space*

$$\mathbb{H}^p_{\psi,L}(X) := \{ f \in L^2(X) \, : \, \mathscr{A}Q_{\psi,L}f \in L^p(X) \},$$

with respect to the norm

$$\|f\|_{H^p_{\psi,L}(X)} := \|\mathscr{A}Q_{\psi,L}f\|_{L^p(X)} = \|Q_{\psi,L}f\|_{T^p(X)} .$$

Then $H^p_L(X) = H^p_{\psi,L}(X)$, with equivalence of norms.

In analogy to the standard Hardy spaces $H^p(X)$ (which coincide with $L^p(X)$ for $1 < p < \infty$), the spaces $H^p_L(X)$ form a complex interpolation scale. For a proof, we refer to [HMM10], Lemma 4.24, where the authors reduce the problem to complex interpolation of tent spaces.

Proposition 4.37 *Let L be an operator satisfying* (H1) *and* (H2). *Let $1 \leq p_0 < p_1 < \infty$ and $0 < \theta < 1$. Then there holds*

$$[H_L^{p_0}(X), H_L^{p_1}(X)]_\theta = H_L^p(X) \qquad \text{where } 1/p = (1-\theta)/p_0 + \theta/p_1,$$

and

$$[H_L^{p_0}(X), BMO_L(X)]_\theta = H_L^p(X) \qquad \text{where } 1/p = (1-\theta)/p_0.$$

Moreover, the Hardy spaces $H_L^p(X)$ also satisfy a Marcinkiewicz-type interpolation theorem. For a proof, we refer to [DL09], Theorem 4.7.

Before stating the theorem, we introduce the following notion.
If T is defined on H_L^p for some $p \geq 1$, we say that it is of weak-type (H_L^p, p), provided that there exists a constant $C > 0$ such that

$$|\{x \in X : |T(f)(x)| > \alpha\}| \leq C\alpha^{-p} \|f\|_{H_L^p(X)}^p \tag{4.65}$$

for all $f \in H_L^p(X)$. The best constant C in (4.65) will be referred to as being the weak-type norm of T.

Theorem 4.38 *Let L be an operator satisfying* (H1) *and* (H2). *Suppose that $1 \leq p_1 < p_2 < \infty$, and let T be a sublinear operator from $H_L^{p_1}(X) + H_L^{p_2}(X)$ into the space of all measurable functions on X, which is of weak-type $(H_L^{p_1}, p_1)$ and $(H_L^{p_2}, p_2)$ with weak-type norms C_1 and C_2, respectively. If $p_1 < p < p_2$, then T is bounded from $H_L^p(X)$ into $L^p(X)$ and for all $f \in H_L^p(X)$ there holds*

$$\|Tf\|_{L^p(X)} \leq C \|f\|_{H_L^p(X)},$$

where C depends only on C_1, C_2, p_1, p_2 and p.

As usual, one of the endpoints in applications of Proposition 4.37 and Theorem 4.38 will often be the space $L^2(X) = H_L^2(X)$. To get an estimate on another endpoint, namely the space $H_L^1(X)$, the next proposition will be quite helpful. It is a generalization of [HMa09], Theorem 3.2.
To make it more apparent, where the required decay in the off-diagonal estimates (4.66) and (4.67), represented by γ and M, come into play, we state the proposition in terms of $H^1_{L,mol,M}(X)$ instead of $H_L^1(X)$ and recall that according to Theorem 4.7 there holds $H^1_{L,mol,M}(X) = H_L^1(X)$ whenever $M > \frac{n}{4m}$.

Proposition 4.39 *Let $M \in \mathbb{N}$. Assume that T is a linear or a non-negative sublinear operator defined on $L^2(X)$ such that*

$$T : L^2(X) \to L^2(X)$$

is bounded and T satisfies the following weak off-diagonal estimates:
There exists some $\gamma > \frac{n}{2m}$ and a constant $C > 0$ such that for every $t > 0$, arbitrary balls $B_1, B_2 \in X$ with radius $r = t^{1/2m}$ and every $f \in L^2(X)$ supported in B_1

$$\left\|T(I - e^{-tL})^M(f)\right\|_{L^2(B_2)} \leq C_T \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)}, \tag{4.66}$$

$$\left\|T(tLe^{-tL})^M(f)\right\|_{L^2(B_2)} \leq C_T \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)}. \tag{4.67}$$

Then

$$T : H^1_{L,mol,M}(X) \to L^1(X)$$

is bounded and there exists some $C > 0$*, independent of* C_T*, such that for all* $f \in H^1_{L,mol,M}(X)$ *there holds*

$$\|Tf\|_{L^1(X)} \leq CC_T \|f\|_{H^1_{L,mol,M}(X)} .$$

Remark 4.40 If one uses off-diagonal estimates instead of weak off-diagonal estimates, one only requires a decay of order $\gamma > \frac{n}{4m}$.

Similar to the Calderón-Zygmund theory, the application of Proposition 4.39 in our setting will be as follows. We will be able to show that an operator (e.g. the paraproduct in Chapter 5 or the operator T defined in Chapter 6) is bounded on $L^2(X) = H^2_L(X)$. To show that it is also bounded from $H^p_L(X)$ to $L^p(X)$ for any $1 \leq p < 2$, we first check the boundedness of the operator from $H^1_L(X)$ to $L^1(X)$ and then apply some interpolation result. Thus, the interpolation on the scale of $H^p_L(X)$ spaces replaces the interpolation on the scale of $L^p(X)$ spaces in the context of Calderón-Zygmund operators. In order to regain results on $L^p(X)$ spaces one can check that for some p the spaces $H^p_L(X)$ and $L^p(X)$ coincide. See Proposition 4.41 below.
In order to make Proposition 4.39 applicable to the kind of operators we are dealing with in Chapter 6 in the context of the $T(1)$-Theorem, we have generalized the result of Hofmann and Mayboroda to operators satisfying weak off-diagonal estimates instead of off-diagonal estimates.

Proof (of Proposition 4.39): We follow the proof of [HMa09]. Let $\varepsilon > 0$ and $M \in \mathbb{N}$. Since T is bounded on $L^2(X)$, it is according to Lemma 4.9 sufficient to show that there exists some constant $C > 0$ such that for every $(1, 2, M, \varepsilon)$-molecule m

$$\|Tm\|_{L^1(X)} \leq CC_T.$$

Then T extends to a bounded operator from $H^1_{L,mol,M}(X)$ to $L^1(X)$.
For convenience, we assume that T is linear. Let m be a $(1, 2, M, \varepsilon)$-molecule associated to some ball B in X with radius r_B. Recall that by definition of molecules for all $j \in \mathbb{N}_0$ there holds

$$\|m\|_{L^2(S_j(B))} \lesssim 2^{-j\varepsilon} V(2^j B)^{-1/2}. \tag{4.68}$$

To get the assumed weak off-diagonal estimates on T into play, we decompose Tm into

$$Tm = T(I - e^{-r_B^{2m} L})^M m + T[I - (I - e^{-r_B^{2m} L})^M]m.$$

We handle the two parts separately, using assumption (4.66) for the first and (4.67) for the second part.
Let us begin with the first part. We use two annular decompositions of X, one around B and the other, for fixed $j \in \mathbb{N}_0$, around $2^j B$. Using the Cauchy-Schwarz inequality,

this yields

$$\begin{aligned}
&\left\|T(I-e^{-r_B^{2m}L})^M m\right\|_{L^1(X)}\\
&\qquad\leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left\|T(I-e^{-r_B^{2m}L})^M(\mathbb{1}_{S_j(B)}m)\right\|_{L^1(S_k(2^jB))}\\
&\qquad\leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}V(2^{k+j}B)^{1/2}\left\|T(I-e^{-r_B^{2m}L})^M(\mathbb{1}_{S_j(B)}m)\right\|_{L^2(S_k(2^jB))}.
\end{aligned}$$

We will apply the weak off-diagonal estimates to the dilated balls 2^jB. Remark 3.9 yields that this is possible with an additional factor 2^{jn}. Moreover observe that, according to Remark 3.8, weak off-diagonal estimates imply annular estimates with an extra factor $2^{kn/2}$ as in (3.11). Hence, the above expression can be estimated by a constant times

$$\begin{aligned}
&C_T\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}V(2^{k+j}B)^{1/2}2^{jn}2^{kn/2}\left(1+\frac{\operatorname{dist}(2^jB,S_k(2^jB))^{2m}}{r_B^{2m}}\right)^{-\gamma}\|m\|_{L^2(S_j(B))}\\
&\quad\lesssim C_T\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}2^{jn}2^{kn}\left(\frac{2^{k+j}r_B}{r_B}\right)^{-2m\gamma}V(2^jB)^{1/2}\|m\|_{L^2(S_j(B))}\\
&\quad\lesssim C_T\sum_{j=0}^{\infty}2^{-j\varepsilon}2^{-(2m\gamma-n)j}\sum_{k=0}^{\infty}2^{-(2m\gamma-n)k}\lesssim 1,
\end{aligned}$$

using the doubling property (2.2) in the first inequality, (4.68) in the second and the assumption that $\gamma > \frac{n}{2m}$ in the third inequality.
For the second part, observe that the binomial formula yields

$$I-(I-e^{-r_B^{2m}L})^M=\sum_{k=1}^{M}C_{k,M}e^{-kr_B^{2m}L}$$

with constants $C_{k,M}$ only depending on k and M, and that

$$\sup_{1\leq k\leq M}\left\|Te^{-kr_B^{2m}L}m\right\|_{L^1(X)}\lesssim\sup_{1\leq k\leq M}\left\|T\left(\frac{k}{M}r_B^{2m}Le^{-\frac{k}{M}r_B^{2m}L}\right)^M(r_B^{2m}L)^{-M}m\right\|_{L^1(X)}$$

We now use the same arguments as before, applied to the operator $\left(\frac{k}{M}r_B^{2m}Le^{-\frac{k}{M}r_B^{2m}L}\right)^M$, together with the assumption

$$\left\|(r_B^{2m}L)^{-M}m\right\|_{L^2(S_j(B))}\lesssim 2^{-j\varepsilon}V(2^jB)^{-1/2}$$

for every $j\in\mathbb{N}_0$ instead of (4.68). Proceeding as before, we finally get the inequality $\left\|T[I-(I-e^{-r_B^{2m}L})^M]m\right\|_{L^1(X)}\lesssim C_T$, which finishes the proof. □

We conclude the chapter with a description of the containments of the spaces $H^p_L(X)$ and $L^p(X)$. Recall that for the standard Hardy spaces, there holds $H^p(X)=L^p(X)$ for all $1<p<\infty$.
In Remark 4.2, we have shown that $H^1_L(X)\subseteq L^1(X)$, and in Proposition 4.26 we have

characterized the containment of the spaces $BMO(X)$ and $BMO_{L,M}(X)$.

For further containments, we cite [HMM10], Proposition 9.1. There, $X = \mathbb{R}^n$ and L is assumed to be a (not necessarily injective) second order elliptic operator in divergence form with complex bounded measurable coefficients. We refer the reader to [HMM10], (1.1)-(1.3), for a precise definition of the operator L. We denote by $(p_-(L), p_+(L))$ the interior of the interval of L^p-boundedness of the heat semigroup e^{-tL} and recall that $p_-(L) < \frac{2n}{n+2}$ and $p_+(L) > \frac{2n}{n-2}$, if $n > 2$. For every $p \in [p_+(L), \infty)$, we define the null space

$$\mathcal{N}_p(L) := \{f \in L^p(\mathbb{R}^n) \cap W^{1,2}_{\text{loc}}(\mathbb{R}^n) \,:\, Lf = 0\}.$$

Proposition 4.41 *Let L be the operator defined in [HMM10], (1.1)-(1.3). We have the following containments and continuous embeddings:*

(i) $L^2(\mathbb{R}^n) \cap H^1_L(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$, *and*

$$\|f\|_{H^1(\mathbb{R}^n)} \lesssim \|f\|_{H^1_L(\mathbb{R}^n)}, \qquad f \in L^2(\mathbb{R}^n) \cap H^1_L(\mathbb{R}^n).$$

(ii) $L^2(\mathbb{R}^n) \cap H^p_L(\mathbb{R}^n) \subseteq L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \quad 1 < p \le p_-(L)$, *and*

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{H^p_L(\mathbb{R}^n)}, \qquad f \in L^2(\mathbb{R}^n) \cap H^p_L(\mathbb{R}^n).$$

(iii) $L^p(\mathbb{R}^n) \setminus \mathcal{N}_p(L) \hookrightarrow H^p_L(\mathbb{R}^n), \quad p_+(L) \le p < \infty$, *and*

$$\|f\|_{H^p_L(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \qquad f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n).$$

Moreover, there holds

(iv) $H^p_L(\mathbb{R}^n) = L^p(\mathbb{R}^n), \quad p_-(L) < p < p_+(L)$.

(v) $H^p_L(\mathbb{R}^n) \neq L^p(\mathbb{R}^n), \quad 1 < p \le p_-(L) \quad$ *or* $\quad p_+(L) \le p < \infty$.

For non-negative, self-adjoint operators L of order $2m$ on $L^2(X)$, in [Uhl11] of Uhl the following result in terms of generalized Gaussian estimates is shown. We refer the reader to a comparison with assumption (H3).

Proposition 4.42 *Let L be a non-negative, self-adjoint operator of order $2m$ on $L^2(X)$. If for some $p_0 \in [1, 2)$, there exist constants $C, c > 0$ such that for all $x, y \in X$ and all $t > 0$ there holds*

$$\left\| \mathbb{1}_{B(x,t^{1/2m})} e^{-tL} \mathbb{1}_{B(y,t^{1/2m})} \right\|_{L^{p_0}(X) \to L^{p_0'}(X)} \le C V(x, t^{1/2m})^{-(\frac{1}{p_0} - \frac{1}{p_0'})} \exp\left(- \left(\frac{d(x,y)^{2m}}{ct} \right)^{\frac{1}{2m-1}} \right),$$

then there holds

$$H^p_L(X) = L^p(X), \qquad p_0 < p \le 2.$$

5 Paraproducts via H^∞-functional calculus

In this chapter, we define paraproducts that are constructed via H^∞-functional calculus and present various properties of those. The most important property, and this is at the same time also the only one we need for the proof of our $T(1)$-Theorem for non-integral operators, Theorem 6.13, is its boundedness on $L^2(X)$. With similar methods, we can then show that the paraproducts under consideration extend to bounded operators from $L^p(X)$ to $H^p_L(X)$ for every $p \in (2,\infty)$ and from $L^\infty(X)$ to $BMO_L(X)$. With the help of certain off-diagonal estimates, we moreover obtain boundedness properties for paraproducts considered as bilinear operators.

5.1 Definition of paraproducts associated to operators

To motivate our definition of paraproducts, let us again recall the paraproduct used in the proof of the $T(1)$-Theorem of David and Journé ([DJ84]). Given $b \in BMO(\mathbb{R}^n)$, the authors define an operator Π on $L^2(\mathbb{R}^n)$ via

$$\Pi f = \int_0^\infty Q_t[(Q_t b)(P_t f)] \frac{dt}{t}, \qquad f \in L^2(\mathbb{R}^n), \tag{5.1}$$

where P_t and Q_t are certain convolution operators with $P_t(1) = 1$ and $Q_t(1) = 0$. They show that Π is an L^2-bounded Calderón-Zygmund operator with the additional properties $\Pi(1) = b$ and $\Pi^*(1) = 0$.

We generalize the above construction by replacing the convolution operators with approximation operators associated to L, that are constructed via functional calculus.

Definition 5.1 *Let L satisfy* (H1) *and let $M \in \mathbb{N}$. Let further $\omega < \mu < \frac{\pi}{2}$ and assume that $\psi, \tilde{\psi} \in \Psi(\Sigma^0_\mu)$. For $b \in BMO_{L,M}(X)$ and $f \in L^2(X)$ we define the paraproduct*

$$\Pi_b(f) := \int_0^\infty \tilde{\psi}(t^{2m}L)[\psi(t^{2m}L)b \cdot A_t(e^{-t^{2m}L} f)] \frac{dt}{t}, \tag{5.2}$$

where A_t is the averaging operator defined in (2.11).

For convenience, we do not index Π_b with the defining functions ψ and $\tilde{\psi}$. In the context, it will always become clear what the defining functions are.

5.2 Boundedness of paraproducts on $L^2(X)$

As for the paraproduct defined in [DJ84], the most important property of the paraproduct Π_b defined in (5.2) is clearly its boundedness on $L^2(X)$.

Theorem 5.2 *Assume that L satisfies* (H1), (H2) *and* (3.13) *of* (H3). *Let $M \in \mathbb{N}$, $\omega < \mu < \frac{\pi}{2}$ and $\alpha > 0$, $\beta > \frac{n}{4m} + M$ and assume that $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$ and $\tilde{\psi} \in \Psi(\Sigma^0_\mu)$. Then the operator Π_b, defined in* (5.2)*, is bounded in $L^2(X)$ for every $b \in BMO_{L,M}(X)$, i.e. there exists some constant $C > 0$ such that for every $f \in L^2(X)$ and every $b \in BMO_{L,M}(X)$*

$$\|\Pi_b(f)\|_{L^2(X)} \leq C \|b\|_{BMO_{L,M}(X)} \|f\|_{L^2(X)}.$$

Remark 5.3 If $b \in BMO_L(X)$, then it is due to Definition 4.29 sufficient to assume that $\beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$.

That means, we can show the boundedness of the operator Π_b on $L^2(X)$, if we assume a certain decay of ψ at 0 and, besides quadratic estimates and the Davies-Gaffney estimates of L, an $L^{\tilde{p}} - L^2$ off-diagonal estimate of L for some $\tilde{p} < 2$.
These assumptions also reflect the two main elements of the proof. On the one hand, one needs a Carleson measure estimate of $\left|\psi(t^{2m}L)b(y)\right|^2 \frac{d\mu(y)dt}{t}$ for $b \in BMO_{L,M}(X)$, which replaces the Carleson measure estimate of $|Q_t b(y)|^2 \frac{dydt}{t}$ for $b \in BMO(\mathbb{R}^n)$ in [DJ84]. This can be done by Proposition 4.27 for appropriate functions $\psi \in \Psi(\Sigma_\mu^0)$.
On the other hand, in analogy to [DJ84], we aim to use Theorem 2.22, which gives the well-known connection between Carleson measures and non-tangential maximal functions (see also [CM78], Lemma VI.3, where the result used in [DJ84] is stated, and [CMS85] for further details on non-tangential maximal functions). In absence of pointwise estimates, we need to replace the non-tangential maximal function, defined in (2.21), by the following modified version.

Definition 5.4 *Given an operator L satisfying* (H1) *and a function $f \in L^2(X)$ we define the non-tangential maximal operator $\mathcal{N}_{h,L}$ associated to the heat semigroup generated by L via*

$$\mathcal{N}_{h,L}f(x) := \sup_{(y,t)\in\Gamma(x)} \left(\frac{1}{V(y,t)} \int_{B(y,t)} \left| e^{-t^{2m}L} f(z) \right|^2 d\mu(z) \right)^{1/2}, \quad x \in X.$$

The additional averaging in the space variable is added (compared to the non-tangential maximal operator defined in (2.21)) in order to compensate for the lack of pointwise estimates on the heat semigroup. The idea has its origin in [KP93] and was e.g. recently applied in [HMa09] to give a characterization of $H^1_L(X)$ via non-tangential maximal functions.

For the proof of Theorem 5.2, we will first show that $\mathcal{N}_{h,L}$ is bounded on $L^2(X)$. This is done in Lemma 5.5 below, which is basically the analogue of a pointwise estimate of the non-tangential maximal function in (2.21) against the Hardy-Littlewood maximal function (see e.g. [Ste93], Proposition II.2). To get from the pointwise estimate to the boundedness of $\mathcal{N}_{h,L}$ on $L^2(X)$, we use the already mentioned $L^{\tilde{p}} - L^2$ off-diagonal estimate on L from (H3).
To make the result available for further application, we also state it in the more general setting of $L^p(X)$ spaces for $p > 2$.

Lemma 5.5 *(i) Assume that L satisfies* (H1) *and* (3.13) *of* (H3). *Then the operator $\mathcal{N}_{h,L}$ is bounded in $L^2(X)$, i.e. there exists a constant $C > 0$ such that for every $f \in L^2(X)$ there holds*

$$\|\mathcal{N}_{h,L}f\|_{L^2(X)} \leq C \|f\|_{L^2(X)}.$$

(ii) Assume that L satisfies (H1) *and* (H2). *Then the operator $\mathcal{N}_{h,L}$ is bounded in $L^p(X)$ for every $p \in (2, \infty]$.*

Proof: (i) We will show a pointwise estimate of $\mathcal{N}_{h,L}f$ against the uncentered maximal function $\mathcal{M}_{\tilde{p}}f$, where the index $\tilde{p} \in (1,2)$ comes from assumption (H3).
Let $f \in L^2(X)$ and $x \in X$. To apply the $L^{\tilde{p}} - L^2$ off-diagonal estimates for the semigroup,

we use an annular decomposition of f. This yields

$$\begin{aligned}\mathcal{N}_{h,L}f(x) &= \sup_{(y,t)\in\Gamma(x)} \left(\frac{1}{V(y,t)}\int_{B(y,t)} \left|e^{-t^{2m}L}f(z)\right|^2\,d\mu(z)\right)^{1/2}\\ &\leq \sup_{(y,t)\in\Gamma(x)} \sum_{j=0}^{\infty} V(y,t)^{-1/2}\left\|e^{-t^{2m}L}\mathbb{1}_{S_j(B(y,t))}f\right\|_{L^2(B(y,t))}\\ &\lesssim \sup_{(y,t)\in\Gamma(x)} \sum_{j=0}^{\infty} 2^{-j(\frac{n}{\tilde{p}}+\varepsilon)}V(y,t)^{-1/\tilde{p}}\left\|f\right\|_{L^{\tilde{p}}(S_j(B(y,t)))}.\end{aligned}$$

By application of the doubling condition (2.2), we further get that the above is bounded by a constant times

$$\begin{aligned}&\sup_{t>0}\sup_{y\in B(x,t)} \sum_{j=0}^{\infty} 2^{-j(\frac{n}{\tilde{p}}+\varepsilon)}\,2^{j\frac{n}{\tilde{p}}}V(y,2^jt)^{-1/\tilde{p}}\left\|f\right\|_{L^{\tilde{p}}(B(y,2^jt))}\\ &\lesssim \left[\mathcal{M}(|f|^{\tilde{p}})(x)\right]^{1/\tilde{p}} = \mathcal{M}_{\tilde{p}}f(x).\end{aligned}$$

As $\mathcal{M}_{\tilde{p}}$ is bounded on $L^p(X)$ for every $p\in(\tilde{p},\infty]$, the proof is finished.
(ii) First recall that due to Remark 3.2 the operator e^{-tL} can be defined via duality as an operator acting from $L^\infty(X)$ to $L^2_{\text{loc}}(X)$ for every $t>0$. With the same reasoning, one can also define for every $p\in(2,\infty)$ via duality e^{-tL} as an operator acting from $L^p(X)$ to $L^2_{\text{loc}}(X)$.
Let $p\in(2,\infty]$ and let $f\in L^p(X)$. Then, repeating the arguments in (i), but with the $L^{\tilde{p}}-L^2$ off-diagonal estimates replaced by the Davies-Gaffney estimates for the semigroup, we obtain for every $x\in X$

$$\begin{aligned}\mathcal{N}_{h,L}f(x) &\leq \sup_{(y,t)\in\Gamma(x)} \sum_{j=0}^{\infty} V(y,t)^{-1/2}\left\|e^{-t^{2m}L}\mathbb{1}_{S_j(B(y,t))}f\right\|_{L^2(B(y,t))}\\ &\lesssim \sup_{(y,t)\in\Gamma(x)} \sum_{j=0}^{\infty} V(y,t)^{-1/2}e^{-\left(\frac{\operatorname{dist}(S_j(B(y,t)),B(y,t))^{2m}}{ct^{2m}}\right)^{\frac{1}{2m-1}}}\left\|f\right\|_{L^2(B(y,2^jt))}\\ &\lesssim \sup_{t>0}\sup_{y\in B(x,t)} \sum_{j=0}^{\infty} 2^{-j(\frac{n}{2}+\varepsilon)}\,2^{j\frac{n}{2}}V(y,2^jt)^{-1/2}\left\|f\right\|_{L^2(B(y,2^jt))}\\ &\lesssim \mathcal{M}_2f(x).\end{aligned}$$

The claim follows from the fact that $\mathcal{M}_2$ is bounded on $L^p(X)$ for every $p\in(2,\infty]$. □

Remark 5.6 The boundedness of $\mathcal{N}_{h,L^*}$ in $L^2(X)$ immediately follows from Lemma 5.5 and the assumptions (H1) and (3.14) of (H3).
To show this, use the fact that L satisfies the L^2-L^q off-diagonal estimate (3.14) for some $q>2$ if and only if L^* satisfies the $L^{q'}-L^2$ off-diagonal estimate (3.13) (with $\tilde{p}$ replaced by q'), where q' is the conjugate exponent of q defined by $\frac{1}{q}+\frac{1}{q'}=1$. The claim follows from Lemma 5.5 with L replaced by L^*.

Together with the Carleson measure estimate in Proposition 4.27 and the quadratic estimates of L, the above lemma enables us to prove Theorem 5.2.

Proof (of Theorem 5.2): For $f,g \in L^2(X)$, the Cauchy-Schwarz inequality implies

$$\begin{aligned}|\langle \Pi_b(f), g\rangle| &= \left| \int_0^\infty \langle \psi(t^{2m}L)b \cdot A_t(e^{-t^{2m}L}f)\,,\, \tilde{\psi}(t^{2m}L^*)g\rangle \,\frac{dt}{t}\right| \\ &\le \left(\iint_{X\times(0,\infty)} \left|\psi(t^{2m}L)b(x)\cdot A_t(e^{-t^{2m}L}f)(x)\right|^2 \frac{d\mu(x)dt}{t}\right)^{1/2} \\ &\quad \times \left(\iint_{X\times(0,\infty)} \left|\tilde{\psi}(t^{2m}L^*)g(x)\right|^2 \frac{d\mu(x)dt}{t}\right)^{1/2}.\end{aligned}$$

The second factor is bounded by a constant times $\|g\|_{L^2(X)}$ according to Remark 3.20, since L has a bounded holomorphic functional calculus. Recalling the definition of $\nu_{\psi,b}$ in (4.41), we see that the first factor is equal to

$$\left(\iint_{X\times(0,\infty)} \left|A_t(e^{-t^{2m}L}f)(x)\right|^2 \, d\nu_{\psi,b}(x,t)\right)^{1/2}. \tag{5.3}$$

As we assumed $\beta > \frac{n}{4m} + M$, Proposition 4.27 yields that $\nu_{\psi,b}$ is a Carleson measure with $\|\nu_{\psi,b}\|_{\mathcal{C}}^{1/2} \lesssim \|b\|_{BMO_{L,M}(X)}$. On the other hand, observe that by definition of A_t and (2.12) we get for every $h \in L^2_{\text{loc}}(X)$ and every $y \in X$ the estimate $|A_t h(y)|^2 \lesssim \frac{1}{V(y,t)} \int_{B(y,t)} |h(z)|^2 \, d\mu(z)$. With the help of Theorem 2.22, which states the connection between Carleson measures and non-tangential maximal functions, we can therefore estimate (5.3) by a constant times

$$\begin{aligned}&\|\nu_{\psi,b}\|_{\mathcal{C}}^{1/2} \left(\int_X \sup_{(y,t)\in\Gamma(x)} \left|A_t(e^{-t^{2m}L}f)(y)\right|^2 \, d\mu(x)\right)^{1/2} \\ &\lesssim \|b\|_{BMO_{L,M}(X)} \left(\int_X \sup_{(y,t)\in\Gamma(x)} \frac{1}{V(y,t)} \int_{B(y,t)} \left|e^{-t^{2m}L}f(z)\right|^2 \, d\mu(z)\, d\mu(x)\right)^{1/2} \\ &= \|b\|_{BMO_{L,M}(X)} \|\mathcal{N}_{h,L}f\|_{L^2(X)} \lesssim \|b\|_{BMO_{L,M}(X)} \|f\|_{L^2(X)},\end{aligned}$$

using the boundedness of $\mathcal{N}_{h,L}$ on $L^2(X)$ in the last step. □

5.3 Boundedness of paraproducts on $L^p(X)$

Since the results of Section 5.2 do not involve any theory of Hardy spaces, we have decided to present them in terms of $BMO_{L,M}(X)$ instead of $BMO_L(X)$ (whose definition relies on the duality of $H^1_{L^*}(X)$ and $BMO_{L,M}(X)$). For simplicity, from now on we will restrict ourselves to the space $BMO_L(X)$ specified in Definition 4.29.

We will show that, in addition to its boundedness on $L^2(X)$, the paraproduct Π_b extends to a bounded operator from $L^p(X)$ to $H^p_L(X)$ for every $p \in (2,\infty)$ and from $L^\infty(X)$ to $BMO_L(X)$.
Let us first prove the latter, starting with the remark below, which enables us to define the action of Π_b on $L^\infty(X)$ and gives an appropriate estimate of $e^{-tL}f$ for $f \in L^\infty(X)$.

Remark 5.7 Let L satisfy (H1) and (H2). Let $p \in (2,\infty]$ and $f \in L^p(X)$. The proof of Lemma 5.5 (ii) in particular shows that for every $t>0$ and every $x \in X$

$$\left|A_t e^{-t^{2m}L} f(x)\right| \lesssim \frac{1}{V(x,t)} \int_{B(x,t)} \left|e^{-t^{2m}L} f(y)\right| \, d\mu(y) \lesssim \mathcal{M}_2 f(x).$$

The boundedness of $\mathcal{M}_2$ on $L^p(X)$ for every $p \in (2,\infty]$ then implies that

$$\left\|A_t e^{-t^{2m}L} f\right\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}$$

uniformly in $t>0$.

Via the duality result of Theorem 4.28 and with similar arguments as those used in the proof of Theorem 4.34 and in Section 8 of [HMa09], we now obtain the following.

Theorem 5.8 *Let L satisfy* (H1) *and* (H2). *Let $\omega < \mu < \frac{\pi}{2}$ and $\alpha > 0$, $\beta_1 > \frac{n}{2m}$, $\beta_2 > \frac{n}{4m}$. Assume that $\psi \in \Psi_{\beta_1,\alpha}(\Sigma^0_\mu)$ and $\tilde{\psi} \in \Psi_{\alpha,\beta_2}(\Sigma^0_\mu)$ and let further $b \in BMO_L(X)$. Then, the operator Π_b, initially defined on $L^2(X)$ in* (5.2) *extends to a bounded operator $\Pi_b : L^\infty(X) \to BMO_L(X)$, i.e. there exists some constant $C>0$ such that for every $b \in BMO_L(X)$ and every $f \in L^\infty(X)$ there holds*

$$\|\Pi_b(f)\|_{BMO_L(X)} \leq C \, \|b\|_{BMO_L(X)} \, \|f\|_{L^\infty(X)}.$$

Proof: Let $\psi, \tilde{\psi}$ as given in the assumptions and let $f \in L^\infty(X)$ and $b \in BMO_L(X)$. Moreover, let $\varepsilon > 0$ and $M \in \mathbb{N}$ with $M > \frac{n}{4m}$ and let $g \in \mathbb{H}^1_{L^*}(X)$, where $\mathbb{H}^1_{L^*}(X) = H^1_{L^*}(X) \cap L^2(X)$ as defined in (4.7). For every $R>0$ let us consider ℓ_R defined by

$$\ell_R(g) := \langle \int_{1/R}^{R} \tilde{\psi}(t^{2m}L) \mathbb{1}_{B_R}[\psi(t^{2m}L)b \cdot A_t e^{-t^{2m}L} f] \, \frac{dt}{t}, g \rangle, \tag{5.4}$$

where $B_R := B(0,R)$ and the pairing is that between $H^1_{L^*}(X)$ and its dual.
On the one hand, recall that we assumed $\tilde{\psi} \in \Psi_{\alpha,\beta_2}(\Sigma^0_\mu)$ with $\beta_2 > \frac{n}{4m}$. Hence, Theorem 4.7 yields that the function G, defined by

$$G(x,t) := \tilde{\psi}(t^{2m}L^*) g(x), \qquad (x,t) \in X \times (0,\infty), \tag{5.5}$$

is an element of $T^1(X)$ with

$$\|G\|_{T^1(X)} = \|\mathscr{A} G\|_{L^1(X)} \lesssim \|g\|_{H^1_{L^*}(x)}. \tag{5.6}$$

On the other hand, observe that $\nu_{\psi,b} := \left|\psi(t^{2m}L)b(y)\right|^2 \frac{d\mu(y)dt}{t}$ is a Carleson measure due to Proposition 4.27 and the assumption $\psi \in \Psi_{\beta_1,\alpha}(\Sigma^0_\mu)$ with $\beta_1 > \frac{n}{2m}$. We also obtain from Proposition 4.27 the estimate $\|\nu_{\psi,b}\|_{\mathcal{C}}^{1/2} \lesssim \|b\|_{BMO_L(X)}$. Thus, the function F, defined by

$$F(x,t) := \psi(t^{2m}L)b(x) \cdot A_t e^{-t^{2m}L} f(x), \qquad (x,t) \in X \times (0,\infty), \tag{5.7}$$

is an element of $T^\infty(X)$ with

$$\begin{aligned}
&\|F\|_{T^\infty(X)} = \|\mathscr{C} F\|_{L^\infty(X)} \\
&= \left\| x \mapsto \sup_{B: x \in B} \left(\frac{1}{V(B)} \int_0^{r_B} \int_B \left|\psi(t^{2m}L)b(y)\right|^2 \left|A_t e^{-t^{2m}L} f(y)\right|^2 \, \frac{d\mu(y)dt}{t} \right)^{1/2} \right\|_{L^\infty(X)} \\
&\lesssim \|f\|_{L^\infty(X)} \, \|\nu_{\psi,b}\|_{\mathcal{C}}^{1/2} \lesssim \|f\|_{L^\infty(X)} \, \|b\|_{BMO_L(X)},
\end{aligned} \tag{5.8}$$

where we used Remark 5.7 in the penultimate step.
This estimate also shows that $\ell_R \in L^2(X)$ for every $R > 0$, since Minkowski's inequality, the uniform boundedness of $\{\tilde{\psi}(tL)\}_{t>0}$ and the Cauchy-Schwarz inequality yield

$$\begin{aligned}\|\ell_R\|_{L^2(X)} &= \left\| \int_{1/R}^{R} \tilde{\psi}(t^{2m}L)\mathbb{1}_{B_R} F(\,.\,,t)\, \frac{dt}{t} \right\|_{L^2(X)} \lesssim \int_{1/R}^{R} \|F(\,.\,,t)\|_{L^2(B_R)}\, \frac{dt}{t} \\ &\leq C_R \left(\int_0^R \int_{B_R} |F(x,t)|^2\, \frac{d\mu(x)dt}{t} \right)^{1/2} \leq C_R V(B_R)^{1/2} \|F\|_{T^\infty(X)}\,.\end{aligned}$$

Therefore, according to Theorem 2.17, we obtain from (5.6) and (5.8)

$$\begin{aligned}|\ell_R(g)| &\leq \int_0^\infty \left| \langle \psi(t^{2m}L)b \cdot A_t e^{-t^{2m}L} f, \tilde{\psi}(t^{2m}L^*)g \rangle \right| \frac{dt}{t} \\ &\lesssim \int_X \mathscr{C}F(x)\, \mathscr{A}G(x)\, d\mu(x) \\ &\lesssim \|F\|_{T^\infty(X)} \|G\|_{T^1(X)} \lesssim \|f\|_{L^\infty(X)} \|b\|_{BMO_L(X)} \|g\|_{H^1_{L^*}(x)}\,.\end{aligned}$$

Since $\mathbb{H}^1_{L^*}(X)$ is dense in $H^1_{L^*}(X)$, the above implies that ℓ_R defines a continuous linear functional on $H^1_{L^*}(X)$ which can, due to Theorem 4.28, be identified as an element of $BMO_L(X)$ for every $R > 0$ with

$$\sup_{R>0} \|\ell_R\|_{BMO_L(X)} \lesssim \|f\|_{L^\infty(X)} \|b\|_{BMO_L(X)}\,. \tag{5.9}$$

Moreover, in view of the duality of $T^1(X)$ and $T^\infty(X)$ stated in Theorem 2.17, ℓ_R converges pointwise on $\mathbb{H}^1_{L^*}(X)$ for $R \to \infty$ with

$$\begin{aligned}\ell_R(g) &= \int_{1/R}^{R} \langle \mathbb{1}_{B_R} F(\,.\,,t), G(\,.\,,t) \rangle\, \frac{dt}{t} \\ &\to \int_0^\infty \langle F(\,.\,,t), G(\,.\,,t) \rangle\, \frac{dt}{t} \\ &= \int_0^\infty \langle \psi(t^{2m}L)b \cdot A_t e^{-t^{2m}L} f, \tilde{\psi}(t^{2m}L^*)g \rangle\, \frac{dt}{t}, \qquad R \to \infty.\end{aligned}$$

The principle of uniform boundedness then implies that in this sense we can define $\Pi_b(f)$ as an element of $BMO_L(X)$. The estimate (5.9) finally yields the desired norm estimate of the operator Π_b. □

Remark 5.9 Let us for a moment assume that the semigroup satisfies the conservation property

$$e^{-tL}(1) = 1 \qquad \text{in } L^2_{\mathrm{loc}}(X)$$

for every $t > 0$.
Let $\psi, \tilde{\psi} \in \Psi(\Sigma^0_\mu)$ and let $g \in H^1_{L^*}(X)$ be a finite linear combination of $(1, 2, M', \varepsilon)$-molecules for some $\varepsilon > 0$ and $M' \in \mathbb{N}$ such that the assumptions of Lemma 4.31 and Theorem 5.8 are satisfied.

If one chooses $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$ such that $\int_0^\infty \psi(t)\tilde{\psi}(t)\,\frac{dt}{t} = 1$, then Thereom 5.8 implies that $\Pi_b(1) \in BMO_L(X)$ with

$$\begin{aligned}\langle \Pi_b(1), g\rangle &= \int_0^\infty \langle \psi(t^{2m}L)b \cdot A_t e^{-t^{2m}L}1, \tilde{\psi}(t^{2m}L^*)g\rangle\,\frac{dt}{t}\\ &= \int_0^\infty \langle \psi(t^{2m}L)b, \tilde{\psi}(t^{2m}L^*)g\rangle\,\frac{dt}{t}\\ &= \langle b, g\rangle\end{aligned}$$

due to the reproducing formula of Lemma 4.31. Since g was arbitrarily chosen from a dense subset of $H^1_{L^*}(X)$, we thus obtain

$$\Pi_b(1) = b \qquad \text{in } BMO_L(X).$$

For the adjoint operator Π_b^* we do not know if it is defined on $L^\infty(X)$, but at least at a formal level we also obtain the equality

$$\Pi_b^*(1) = \int_0^\infty e^{-t^{2m}L^*} A_t[\overline{\psi(t^{2m}L)b} \cdot \tilde{\psi}(t^{2m}L^*)1]\,\frac{dt}{t} = 0,$$

whenever $\tilde{\psi}(tL^*)(1) = 0$.
The condition $\tilde{\psi}(tL^*)(1) = 0$ in $L^2_{\text{loc}}(X)$ is fulfilled in the case that $e^{-tL^*}(1) = 1$ in $L^2_{\text{loc}}(X)$ and $\tilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$ for some $\alpha > 0$ and $\beta > \frac{n}{4m}$. This can be seen as follows:
Let B be an arbitrary ball in X. With a similar estimate as in Remark 3.2, one can show that

$$\left\|e^{-tL}\right\|_{L^2(B)\to L^1(X)} \lesssim V(B)^{1/2} t^\gamma$$

for every $t > 0$ and every $\gamma > \frac{n}{4m}$. Hence, the operator

$$f \mapsto \int_0^\infty e^{-\lambda t} e^{-tL} f\,dt$$

converges for every $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ strongly as operator from $L^2(B)$ to $L^1(X)$ with the operator norm bounded by a constant times $V(B)^{1/2}\,|\lambda|^{-\gamma-1}$. This also implies that $\|\tilde{\psi}(\lambda)(\lambda+L)^{-1}\|_{L^2(B)\to L^1(X)} \lesssim V(B)^{1/2}|\tilde{\psi}(\lambda)|\,|\lambda|^{-\gamma-1}$ and the operator

$$\tilde{\psi}(L)f \mapsto \frac{1}{2\pi i}\int_\Gamma \tilde{\psi}(\lambda)(\lambda+L)^{-1} f\,d\lambda,$$

where Γ is an appropriately chosen path of integration in the right half-plane, converges strongly as operator from $L^2(B)$ to $L^1(X)$ whenever $\beta > \gamma$. The assumption $e^{-tL^*}(1) = 1$ then yields that for every $f \in L^2(B)$ there holds

$$\langle 1, (\lambda+L)^{-1}f\rangle = \langle 1, \int_0^\infty e^{-\lambda t} e^{-tL} f\,dt\rangle = \int_0^\infty e^{-\lambda t}\langle e^{-tL^*}(1), f\rangle\,dt = \frac{1}{\lambda}\langle 1, f\rangle.$$

We finally obtain for $\tilde{\psi}(L^*)(1)$ the equality

$$\begin{aligned}\langle \tilde{\psi}(L^*)(1), f\rangle &= \langle 1, \tilde{\psi}(L)f\rangle\\ &= \frac{1}{2\pi i}\int_\Gamma \tilde{\psi}(\lambda)\langle 1, (\lambda+L)^{-1}f\rangle\,d\lambda\\ &= \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{\psi}(\lambda)}{\lambda}\,d\lambda\,\langle 1, f\rangle = 0,\end{aligned}$$

where the last step is due to an extension of Cauchy's theorem and the assumption $\tilde{\psi} \in \Psi(\Sigma_\mu^0)$.

One possibility to show that Π_b also extends to a bounded operator from $L^p(X)$ to $H^p_L(X)$ is the use of the interpolation result for Hardy spaces stated in Proposition 4.37. We will present a more direct approach, that is similar to the proof of Theorem 5.8. The idea goes back to Hytönen and Weis, who showed in [HW10] the L^p-boundedness of (differently defined) paraproduct operators in a more general Banach space-valued setting.

Theorem 5.10 *Let $p \in (2,\infty)$. Let L satisfy* (H1) *and* (H2) *and let $\omega < \mu < \frac{\pi}{2}$ and $\alpha > 0$, $\beta > \frac{n}{4m}$. Assume that $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$ and $\tilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$ and let further $b \in BMO_L(X)$. Then, the operator Π_b, initially defined on $L^2(X)$ in* (5.2) *extends to a bounded operator $\Pi_b : L^p(X) \to H^p_L(X)$, i.e. there exists some constant $C > 0$ such that for every $b \in BMO_L(X)$ and every $f \in L^p(X)$ there holds*

$$\|\Pi_b(f)\|_{H^p_L(X)} \leq C\,\|b\|_{BMO_L(X)}\,\|f\|_{L^p(X)}\,.$$

Proof: Let $2 < p < \infty$ and p' the conjugate exponent of p defined by $\frac{1}{p}+\frac{1}{p'}=1$. Let $f \in L^p(X)$, $b \in BMO_L(X)$ and $g \in \mathbb{H}^{p'}_{L^*}(X)$.
For every $R > 0$, let ℓ_R be defined as in (5.4), where the pairing is now that between $H^p_L(X)$ and its dual. Further, let G and F be defined as in (5.5) and (5.7). Then, due to Theorem 4.36 and the assumption $\tilde{\psi} \in \Psi_{\alpha,\beta}(\Sigma^0_\mu)$ with $\beta > \frac{n}{4m}$, we obtain $G \in T^{p'}(X)$ with

$$\|G\|_{T^{p'}(X)} = \|\mathscr{A}G\|_{L^{p'}(X)} \lesssim \|g\|_{H^{p'}_{L^*}(X)}\,. \tag{5.10}$$

Let us now split F into $F = H \cdot F_0$ with $H(\,.\,,t) := \psi(t^{2m}L)b$ and $F_0(\,.\,,t) := A_t e^{-t^{2m}L}f$. On the one hand, Proposition 4.27 yields, as in the proof of Theorem 5.2, that there holds $H \in T^\infty(X)$ with $\|H\|_{T^\infty(X)} = \|\nu_{\psi,b}\|^{1/2}_{\mathscr{C}} \lesssim \|b\|_{BMO_L(X)}$. Observe that on the other hand $F_0^* = \mathcal{N}_{h,L}f$, thus we obtain from Lemma 5.5 that $F_0^* \in L^p(X)$ with $\|F_0^*\|_{L^p(X)} \lesssim \|f\|_{L^p(X)}$. Therefore, Corollary 2.23 implies that $F \in T^p(X)$ with

$$\begin{aligned}\|F\|_{T^p(X)} = \|\mathscr{C}(H\cdot F_0)\|_{L^p(X)} &\lesssim \|H\|_{T^\infty(X)}\,\|F_0^*\|_{L^p(X)}\\ &\lesssim \|b\|_{BMO_L(X)}\,\|f\|_{L^p(X)}\,.\end{aligned} \tag{5.11}$$

Hence, we get due to Theorem 2.17, Hölder's inequality and Theorem 2.18

$$\begin{aligned}|\ell_R(g)| &\leq \int_0^\infty \left|\langle\tilde{\psi}(t^{2m}L^*)g, \psi(t^{2m}L)b\cdot A_t e^{-t^{2m}L}f\rangle\right|\,\frac{dt}{t}\\ &\lesssim \int_X \mathscr{A}(F)(x)\mathscr{A}(G)(x)\,d\mu(x)\\ &\lesssim \|\mathscr{C}F\|_{L^p(X)}\,\|\mathscr{A}G\|_{L^{p'}(X)} \lesssim \|b\|_{BMO_L(X)}\,\|f\|_{L^p(X)}\,\|g\|_{H^{p'}_{L^*}(X)}\,,\end{aligned}$$

where the last step is a consequence of (5.10) and (5.11).
Since $\mathbb{H}^{p'}_{L^*}(X)$ is dense in $H^{p'}_{L^*}(X)$ and $H^p_L(X)$ was defined as the dual space of $H^{p'}_{L^*}(X)$, we can therefore identify ℓ_R with an element of $H^p_L(X)$. With the same reasoning as in the proof of Theorem 5.8 and in view of the duality of $T^p(X)$ and $T^{p'}(X)$, we can finally define $\Pi_b(f)$ as an element of $H^p_L(X)$ and Π_b as an operator acting from $L^p(X)$ to $H^p_L(X)$ with

$$\|\Pi_b(f)\|_{H^p_L(X)} \leq C\,\|b\|_{BMO_{L,M}(X)}\,\|f\|_{L^p(X)}\,. \qquad \square$$

5.4 Further properties of paraproducts

Throughout the section we will assume that L satisfies (H1), (H2) and also (H3). This is done to avoid technicalities, even if assumption (H3) will not always be necessary.

To obtain further boundedness properties of the paraproduct Π defined in (5.2), we will consider Π in this section as a bilinear operator, initially defined on $L^2(X) \times BMO_L(X)$ for $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$ by

$$\Pi(f,g) := \int_0^\infty \tilde{\psi}(t^{2m}L)[\psi(t^{2m}L)g \cdot A_t e^{-t^{2m}L} f] \, \frac{dt}{t} \tag{5.12}$$

for every $f \in L^2(X)$ and $g \in BMO_L(X)$.

In Section 5.2 and Section 5.3, we already showed that Π extends to a bounded bilinear operator

$$\begin{aligned}
&\Pi : L^2(X) \times BMO_L(X) \to L^2(X), \\
&\Pi : L^p(X) \times BMO_L(X) \to H_L^p(X), \qquad 2 < p < \infty, \\
&\Pi : L^\infty(X) \times BMO_L(X) \to BMO_L(X),
\end{aligned}$$

if the defining functions of the paraproduct, $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$, have enough decay at 0 and infinity, respectively.

In addition, we will now show that Π extends to a bounded bilinear operator

$$\begin{aligned}
&\Pi : L^\infty(X) \times H_L^p(X) \to L^p(X), \qquad 1 \leq p < 2, \\
&\Pi : L^\infty(X) \times L^2(X) \to L^2(X), \\
&\Pi : L^\infty(X) \times L^p(X) \to H_L^p(X), \qquad 2 < p < \infty.
\end{aligned}$$

We begin with the simplest case, namely the boundedness of $\Pi : L^\infty(X) \times L^2(X) \to L^2(X)$. This is an immediate consequence of quadratic estimates and Remark 5.7.

Lemma 5.11 *Let $\omega < \mu < \frac{\pi}{2}$ and let $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$. Then the operator Π defined in* (5.12) *extends to a bounded operator $\Pi : L^\infty(X) \times L^2(X) \to L^2(X)$. I.e. there exists a constant $C > 0$ such that for every $f \in L^\infty(X)$ and every $g \in L^2(X)$ there holds*

$$\|\Pi(f,g)\|_{L^2(X)} \leq C \|f\|_{L^\infty(X)} \|g\|_{L^2(X)} .$$

Proof: Let $f \in L^\infty(X)$ and $g, h \in L^2(X)$. The Cauchy-Schwarz inequality, Remark 5.7 and quadratic estimates for $\{\psi(tL)\}_{t>0}$ and $\{\tilde{\psi}(tL)\}_{t>0}$, which hold due to Remark 3.20, then yield

$$\begin{aligned}
&|\langle \Pi(f,g), h\rangle| \\
&\quad \leq \int_0^\infty \left| \langle \psi(t^{2m}L)g \cdot A_t e^{-t^{2m}L} f, \tilde{\psi}(t^{2m}L^*)h\rangle \right| \frac{dt}{t} \\
&\quad \leq \left(\int_0^\infty \left\| \psi(t^{2m}L)g \cdot A_t e^{-t^{2m}L} f \right\|_{L^2(X)}^2 \frac{dt}{t} \right)^{1/2} \left(\int_0^\infty \left\| \tilde{\psi}(t^{2m}L^*)h \right\|_{L^2(X)}^2 \frac{dt}{t} \right)^{1/2} \\
&\quad \lesssim \|f\|_{L^\infty(X)} \|g\|_{L^2(X)} \|h\|_{L^2(X)} .
\end{aligned}$$

This finishes the proof. □

Next, we will show that Π extends to a bounded operator $\Pi : L^\infty(X) \times H^1_L(X) \to L^1(X)$. To do so, we aim at an application of Proposition 4.39 and will therefore first check that the off-diagonal estimates (4.66) and (4.67) assumed in the proposition are satisfied.
We remark that one can relax the assumptions on the decay of ψ and $\tilde{\psi}$ at 0 and infinity, if one assumes Davies-Gaffney estimates on $\{\psi(tL)\}_{t>0}$ and $\{\tilde{\psi}(tL)\}_{t>0}$, respectively.

Lemma 5.12 *Let $\omega < \mu < \frac{\pi}{2}$, let $\alpha_1 > 0$ and $\alpha_2, \beta_1, \beta_2 > \frac{n}{2m}$ and let $\psi \in \Psi_{\beta_1,\alpha_1}(\Sigma^0_\mu)$ and $\tilde{\psi} \in \Psi_{\alpha_2,\beta_2}(\Sigma^0_\mu)$. Further, let $\delta > \frac{n}{2m}$ and $\varphi \in H^\infty(\Sigma^0_\mu)$ with $|\varphi(z)| \leq c\,|z|^\delta$ for every $z \in \Sigma^0_\mu$ with $|z| \leq 1$ and some constant $c > 0$ independent of z.*
Then, for every $\gamma > 0$ with $\gamma \leq \min(\beta_1, \alpha_2)$ and $\gamma < \min(\beta_2, \delta)$ there exists some constant $C > 0$ such that for every $f \in L^\infty(X)$, every $t > 0$, arbitrary balls $B_1, B_2 \in X$ with radius t and every $g \in L^2(X)$ supported in B_1 there holds

$$\left\|\varphi(t^{2m}L)\Pi(f,g)\right\|_{L^2(B_2)} \leq C\left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(B_1)}.$$

Proof: Let $t > 0$ and let $B_1, B_2 \subseteq X$ be two arbitrary balls with radius t. Let $f \in L^\infty(X)$ and let $g \in L^2(X)$ supported in B_1. Since we already know by Lemma 5.11 that $\Pi : L^\infty(X) \times L^2(X) \to L^2(X)$ is bounded, we can without restriction assume that $\operatorname{dist}(B_1, B_2) > t$.
First, via Minkowski's inequality, we obtain

$$\left\|\varphi(t^{2m}L)\Pi(f,g)\right\|_{L^2(B_2)} \leq \int_0^\infty \left\|\varphi(t^{2m}L)\tilde{\psi}(s^{2m}L)[\psi(s^{2m}L)g \cdot A_s e^{-s^{2m}L}f]\right\|_{L^2(B_2)} \frac{ds}{s}.$$

We split the integral over s into two parts, one over the interval $(0,t)$, called J_1, and one over the interval (t,∞), called J_2.
To handle J_1, let us make the following observations. Due to Proposition 3.18, the operator family $\{\psi(sL)\}_{s>0}$ satisfies off-diagonal estimates in s of order β_1. Moreover, since $\sup_{t>0}\|\varphi(t\,\cdot\,)\|_{L^\infty(\Sigma^0_\mu)} = \|\varphi\|_{L^\infty(\Sigma^0_\mu)} < \infty$, the same proposition also yields that $\{\varphi(tL)\tilde{\psi}(sL)\}_{s,t>0}$ satisfies off-diagonal estimates in s of order α_2. To apply these estimates, we cover X with the help of Lemma 2.1 by balls of radius $s > 0$, that is, we have $X = \bigcup_{\alpha \in I_{k_0}} B_\alpha$, where $k_0 \in \mathbb{Z}$ is determined by (2.5), $B_\alpha := B(z^{k_0}_\alpha, s)$ and $I_{k_0}, z^{k_0}_\alpha$ are as in Lemma 2.1 and Notation 2.2.
Also taking Remark 5.7 into account to estimate $A_s e^{-s^{2m}L}f$, we therefore obtain

$$\begin{aligned}
|J_1| &\leq \int_0^t \sum_{\alpha \in I_{k_0}} \left\|\varphi(t^{2m}L)\tilde{\psi}(s^{2m}L)\mathbb{1}_{B_\alpha}[\psi(s^{2m}L)g \cdot A_s e^{-s^{2m}L}f]\right\|_{L^2(B_2)} \frac{ds}{s} \\
&\lesssim \int_0^t \sum_{\alpha \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_2,B_\alpha)^{2m}}{s^{2m}}\right)^{-\alpha_2} \left\|\psi(s^{2m}L)g \cdot A_s e^{-s^{2m}L}f\right\|_{L^2(B_\alpha)} \frac{ds}{s} \\
&\lesssim \int_0^t \sum_{\alpha \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_2,B_\alpha)^{2m}}{s^{2m}}\right)^{-\alpha_2} \left(1 + \frac{\operatorname{dist}(B_\alpha,B_1)^{2m}}{s^{2m}}\right)^{-\beta_1} \frac{ds}{s} \\
&\qquad \times \|f\|_{L^\infty(X)} \|g\|_{L^2(B_1)}.
\end{aligned} \tag{5.13}$$

Moreover, since $\beta_1, \alpha_2 > \frac{n}{2m}$, Lemma 3.6 yields that

$$\sum_{\alpha \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_2, B_\alpha)^{2m}}{s^{2m}}\right)^{-\alpha_2} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_1)^{2m}}{s^{2m}}\right)^{-\beta_1} \lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{s^{2m}}\right)^{-\min(\beta_1,\alpha_2)}.$$

Since we assumed $\operatorname{dist}(B_1, B_2) > t$, we can therefore bound the last integral in (5.13) by a constant times

$$\begin{aligned}
&\int_0^t \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{s^{2m}}\right)^{-\min(\beta_1,\alpha_2)} \frac{ds}{s} \\
&\quad \lesssim \left(\frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\min(\beta_1,\alpha_2)} \int_0^t \left(\frac{s}{t}\right)^{2m\cdot\min(\beta_1,\alpha_2)} \frac{ds}{s} \\
&\quad \lesssim \left(\frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\min(\beta_1,\alpha_2)} \int_0^1 u^{2m\cdot\min(\beta_1,\alpha_2)} \frac{du}{u},
\end{aligned}$$

using the substitution $u = \frac{s}{t}$ in the last step. We end up with

$$|J_1| \lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\min(\beta_1,\alpha_2)} \|f\|_{L^\infty(X)} \|g\|_{L^2(B_1)}. \tag{5.14}$$

Let us now turn to J_2. We again use that $\{\psi(sL)\}_{s>0}$ satisfies off-diagonal estimates in s of order β_1. On the other hand, we get from Lemma 3.19 that for every $a > 0$ with $a \le \delta$ and $a < \beta_2$, there exists a family of operators $\{T_{s,t}\}_{s,t>0}$ such that

$$\varphi(tL)\tilde{\psi}(sL) = \left(\frac{t}{s}\right)^a T_{s,t},$$

where $\{T_{s,t}\}_{s,t>0}$ satisfies off-diagonal estimates in s of order $\alpha_2 + a$ (thus, in particular of order α_2) uniformly in $t > 0$. Hence, with the same covering of X by balls of radius s as before and following the same arguments as before, we get

$$\begin{aligned}
|J_2| &\le \int_t^\infty \sum_{\alpha \in I_{k_0}} \left\| \varphi(t^{2m}L)\tilde{\psi}(s^{2m}L)\mathbb{1}_{B_\alpha}[\psi(s^{2m}L)g \cdot A_s e^{-s^{2m}L} f] \right\|_{L^2(B_2)} \frac{ds}{s} \\
&\lesssim \int_t^\infty \sum_{\alpha \in I_{k_0}} \left(\frac{t}{s}\right)^{2ma} \left(1 + \frac{\operatorname{dist}(B_2, B_\alpha)^{2m}}{s^{2m}}\right)^{-\alpha_2} \left\| \psi(s^{2m}L)g \cdot A_s e^{-s^{2m}L} f \right\|_{L^2(B_\alpha)} \frac{ds}{s} \\
&\lesssim \int_t^\infty \left(\frac{t}{s}\right)^{2ma} \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{s^{2m}}\right)^{-\min(\beta_1,\alpha_2)} \frac{ds}{s} \|f\|_{L^\infty(X)} \|g\|_{L^2(B_1)}.
\end{aligned} \tag{5.15}$$

Recall that we assumed $\gamma \le \min(\beta_1, \alpha_2)$ and $\gamma < \min(\beta_2, \delta)$. Thus, we can fix some $a > \gamma$ with $a \le \delta$ and $a < \beta_2$. For such a choice of a we further get in view of the assumption $\operatorname{dist}(B_1, B_2) > t$, similar to the treatment of J_1,

$$\begin{aligned}
&\int_t^\infty \left(\frac{t}{s}\right)^{2ma} \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{s^{2m}}\right)^{-\min(\beta_1,\alpha_2)} \frac{ds}{s} \\
&\leq \int_t^\infty \left(\frac{t}{s}\right)^{2ma} \left(\frac{\operatorname{dist}(B_1, B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s} \\
&= \left(\frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\gamma} \int_t^\infty \left(\frac{t}{s}\right)^{2m(a-\gamma)} \frac{ds}{s} \\
&= \left(\frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\gamma} \int_1^\infty u^{-2m(a-\gamma)} \frac{du}{u} \lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\gamma}. \qquad (5.16)
\end{aligned}$$

Combining (5.14), (5.15) and (5.16) gives the desired estimate. □

Let us now apply these off-diagonal estimates in Proposition 4.39 to obtain the boundedness of $\Pi : L^\infty(X) \times H^1_L(X) \to L^1(X)$. Via interpolation and duality we then also get the remaining boundedness results for $p > 1$.

Theorem 5.13 *Let $\omega < \mu < \frac{\pi}{2}$, let $\alpha_1 > 0$ and $\alpha_2, \beta_1, \beta_2 > \frac{n}{2m}$.*
(i) Let $p \in [1, 2]$. If $\psi \in \Psi_{\beta_1,\alpha_1}(\Sigma^0_\mu)$ and $\tilde{\psi} \in \Psi_{\alpha_2,\beta_1}(\Sigma^0_\mu)$, then the operator Π defined in (5.12) extends to a bounded operator $\Pi : L^\infty(X) \times H^p_L(X) \to L^p(X)$. I.e. there exists a constant $C > 0$ such that for every $f \in L^\infty(X)$ and every $g \in H^p_L(X)$ there holds

$$\|\Pi(f, g)\|_{L^p(X)} \leq C \|f\|_{L^\infty(X)} \|g\|_{H^p_L(X)}.$$

(ii) Let $p \in [2, \infty)$. If $\psi \in \Psi_{\alpha_2,\beta_1}(\Sigma^0_\mu)$ and $\tilde{\psi} \in \Psi_{\beta_1,\alpha_1}(\Sigma^0_\mu)$, then the operator Π defined in (5.12) extends to a bounded operator $\Pi : L^\infty(X) \times L^p(X) \to H^p_L(X)$. I.e. there exists a constant $C > 0$ such that for every $f \in L^\infty(X)$ and every $g \in L^p(X)$ there holds

$$\|\Pi(f, g)\|_{H^p_L(X)} \leq C \|f\|_{L^\infty(X)} \|g\|_{L^p(X)}.$$

Proof: The assertion for $p = 2$ was proven in Lemma 5.11, since $H^2_L(X) = L^2(X)$.
Let $f \in L^\infty(X)$. Beginning with the assertion in (i), observe that Lemma 5.12 gives the needed off-diagonal estimates for Proposition 4.39. To see this, choose some $M \in \mathbb{N}$ with $M > \frac{n}{4m}$ and define $\varphi \in H^\infty(\Sigma^0_\mu)$ by either $\varphi(z) = (1 - e^{-z})^M$ or $\varphi(z) = (ze^{-z})^M$. In both cases, there holds $|\varphi(z)| \leq |z|^M$ for $z \in \Sigma^0_\mu$ with $|z| \leq 1$. Thus, we can choose some $\gamma > \frac{n}{2m}$ with $\gamma \leq \min(\beta, \alpha_2)$ and $\gamma < \min(\beta_2, M)$ and due to Lemma 5.12 the operator family $\{\varphi(t^{2m}L)\Pi(f, g)\}_{t>0}$ satisfies weak L^2 off-diagonal estimates of order γ with constant $C \|f\|_{L^\infty(X)}$ for some $C > 0$ independent of f. We therefore obtain from Proposition 4.39 that $\Pi(f, .)$ extends to a bounded operator from $H^1_L(X)$ to $L^1(X)$ with

$$\|\Pi(f, g)\|_{L^1(X)} \leq C \|f\|_{L^\infty(X)} \|g\|_{H^1_L(X)},$$

for all $g \in H^1_L(X)$ and some constant $C > 0$ independent of f and g.
Hence, Π extends to a bounded operator $\Pi : L^\infty(X) \times H^1_L(X) \to L^1(X)$. Via complex interpolation between $H^1_L(X)$ and $H^2_L(X)$, which holds due to Proposition 4.37, and interpolation between $L^1(X)$ and $L^2(X)$, we also obtain that Π extends to a bounded operator $\Pi : L^\infty(X) \times H^p_L(X) \to L^p(X)$ for every $p \in (1, 2)$.
The assertion (ii) is now obtained from (i) via duality. If p' denotes the conjugate

exponent of $p \in (2, \infty)$, then $H_L^p(X)$ was defined as the dual space of $H_{L^*}^{p'}(X)$. Observe that the dual operator of $\Pi(f, .)$ is the operator

$$h \mapsto \int_0^\infty \psi(t^{2m}L^*)[\tilde{\psi}(t^{2m}L^*)h \cdot \overline{A_t e^{-t^{2m}L} f}] \frac{dt}{t},$$

which is according to (i) bounded from $H_{L^*}^{p'}(X)$ to $L^{p'}(X)$ with its operator norm bounded by a constant times $\|f\|_{L^\infty(X)}$. Thus, $\Pi(f, .)$ is bounded from $L^p(X)$ to $H_L^p(X)$ with

$$\|\Pi(f, g)\|_{H_L^p(X)} \leq C \|f\|_{L^\infty(X)} \|g\|_{L^p(X)} .$$

□

5.5 Differentiability properties

Let us conclude the chapter with an observation on differentiability properties of paraproducts constructed via functional calculus. One of the fundamental properties of paraproducts, as they were e.g. considered in [Bon81] and [CM78] in the context of paradifferential operators, is that they satisfy a Leibniz-type rule and "preserve" Sobolev classes. We will show that there holds a corresponding result for the paraproduct Π defined in Section 5.4, according to the general philosophy, "differentiability" is not measured in terms of derivatives, but in terms of fractional powers of the operator L.

Let $\psi, \tilde{\psi} \in \Psi(\Sigma_\mu^0)$. Let us recall the paraproduct operator Π, now more precisely denoted by $\Pi_{\tilde{\psi},\psi}$, as defined in (5.12): For $f \in L^\infty(X)$ and $g \in L^2(X)$ we set

$$\Pi_{\tilde{\psi},\psi}(f, g) := \int_0^\infty \tilde{\psi}(t^{2m}L)[\psi(t^{2m}L)g \cdot A_t e^{-t^{2m}L} f] \frac{dt}{t}.$$

Then the following fractional Leibniz-type rule for paraproducts is valid.

Proposition 5.14 *Let $s > 0$, let $\tilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$ and $\psi \in \Psi_{\alpha,\beta}(\Sigma_\mu^0)$ for some $\alpha > \frac{s}{2m}$ and $\beta > 0$. For $f \in L^\infty(X)$ and $g \in \mathcal{D}(L^{s/2m})$ there holds*

$$L^{s/2m}\Pi_{\tilde{\psi},\psi}(f, g) = \Pi_{\tilde{\psi}_s,\psi_s}(f, L^{s/2m}g),$$

where $\tilde{\psi}_s, \psi_s$ are defined by $\tilde{\psi}_s(z) := z^{s/2m}\tilde{\psi}(z)$ and $\psi_s(z) := z^{-s/2m}\psi(z)$.
Moreover, there exists some constant $C > 0$ such that for all $f \in L^\infty(X)$ and all $g \in \mathcal{D}(L^{s/2m})$ there holds

$$\left\| L^{s/2m}\Pi(f, g) \right\|_{L^2(X)} \lesssim \|f\|_{L^\infty(X)} \left\| L^{s/2m} g \right\|_{L^2(X)} .$$

Proof: Due to functional calculus, the proposition is a consequence of the simple calculation

$$\begin{aligned} L^{s/2m}\Pi_{\tilde{\psi},\psi}(f, g) &= \int_0^\infty (t^{2m}L)^{s/2m}\tilde{\psi}(t^{2m}L)[(t^{2m}L)^{-s/2m}\psi(t^{2m}L)L^{s/2m}g \cdot A_t e^{-t^{2m}L} f] \frac{dt}{t} \\ &= \Pi_{\tilde{\psi}_s,\psi_s}(f, L^{s/2m}g), \end{aligned}$$

combined with Lemma 5.11. □

In view of Theorem 5.13, one can obviously obtain a similar result for the spaces $H^p_L(X)$ and $L^p(X)$, where $p \neq 2$. We refer the reader to Section 8.4 of [HMM10] for a discussion of Hardy-Sobolev spaces associated to a second order elliptic operator L in divergence form.
A corresponding result for paraproducts constructed via convolution operators is stated in [Chr90b], Proposition III.23.

To obtain a fractional Leibniz-type rule for products of functions, again in the sense that fractional derivatives are replaced by fractional powers of the operator L, let us now in addition assume that the operator $e^{-tL} : L^\infty(X) \to L^\infty(X)$ is bounded uniformly in $t > 0$.
In this case, we can omit the averaging operator A_t in the definition of paraproducts, which in turn enables us to represent the product of two functions with the help of paraproducts. That is, via functional calculus we can write (cf. below)

$$f \cdot g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_2(g,f), \tag{5.17}$$

where Π_1 and Π_2 are appropriately defined paraproduct operators.
With analogous arguments as in Proposition 5.14 we then obtain the following corollary. It can be understood as a generalization of an inequality of Kato and Ponce [KP88], where fractional derivatives are replaced by fractional powers of the operator L.

Corollary 5.15 *Let L satisfy* (H1), (H2) *and* (H3) *and assume in addition that $e^{-tL} : L^\infty(X) \to L^\infty(X)$ is bounded uniformly in $t > 0$. Let $s > 0$. Then there exists some constant $C > 0$ such that for all $f, g \in \mathcal{D}(L^s) \cap L^\infty(X)$*

$$\|L^s(fg)\|_{L^2(X)} \leq C \|L^s f\|_{L^2(X)} \|g\|_{L^\infty(X)} + C \|f\|_{L^\infty(X)} \|L^s g\|_{L^2(X)}.$$

Moreover, for $p \in [1,2) \cup (2,\infty)$, there exists some constant $C_p > 0$ such that for all $f, g \in \mathcal{D}((L_p)^s) \cap L^\infty(X)$

$$\|L^s(fg)\|_{L^p(X)} \leq C_p \|L^s f\|_{H^p_L(X)} \|g\|_{L^\infty(X)} + C_p \|f\|_{L^\infty(X)} \|L^s g\|_{H^p_L(X)}, \quad \text{if } 1 \leq p < 2,$$

$$\|L^s(fg)\|_{H^p_L(X)} \leq C_p \|L^s f\|_{L^p(X)} \|g\|_{L^\infty(X)} + C_p \|f\|_{L^\infty(X)} \|L^s g\|_{L^p(X)}, \quad \text{if } 2 < p < \infty,$$

where $-L_p$ denotes the generator of e^{-tL} in $H^p_L(X)$ for $1 \leq p < 2$ and in $L^p(X)$ for $2 < p < \infty$, respectively.

Remark 5.16 In the case that $H^p_L(X) = L^p(X)$, cf. Proposition 4.41, the above inequalities simplify to

$$\|L^s(fg)\|_{L^p(X)} \leq C_p \|L^s f\|_{L^p(X)} \|g\|_{L^\infty(X)} + C_p \|f\|_{L^\infty(X)} \|L^s g\|_{L^p(X)}.$$

Proof: The main part of the proof is to establish the decomposition (5.17) with appropriately chosen paraproducts Π_1 and Π_2. To do so, let $M \in \mathbb{N}$ to be chosen later and define functions φ, ψ and $\tilde{\varphi} \in H^\infty(\Sigma^0_\mu)$ by

$$\varphi : z \mapsto e^{-z}(1-e^{-z})^M, \qquad \psi : z \mapsto z\varphi'(z) \qquad \text{and} \qquad \tilde{\varphi} := 1 - \varphi.$$

Observe that in particular there holds $\varphi, \psi \in \Psi_{M,M}(\Sigma^0_\mu)$. Moreover, let $\tilde{\psi} \in \Psi_{M,M}(\Sigma^0_\mu)$ be chosen such that

$$\int_0^\infty \psi(t)\tilde{\psi}(t) \, \frac{dt}{t} = \frac{1}{2},$$

and define $\tilde{\psi}_1 \in \Psi_{M,M}(\Sigma_\mu^0)$ by $\tilde{\psi}_1(z) := z\tilde{\psi}'(z)$.
We can then represent the product $f \cdot g$ of two functions $f, g \in L^2(X)$ in the following way. Via functional calculus and using partial integration in the third step, we obtain

$$\begin{aligned} f \cdot g &= \int_0^\infty \tilde{\psi}(tL)\psi(tL)f \frac{dt}{t} \cdot g + \int_0^\infty \tilde{\psi}(tL)\psi(tL)g \frac{dt}{t} \cdot f \\ &= \int_0^\infty \tilde{\psi}(tL)\partial_t(\varphi(tL)f \cdot \varphi(tL)g)\, dt - \int_0^\infty \tilde{\psi}(tL)\partial_t[(I-\varphi(tL))f \cdot (I-\varphi(tL))g]\, dt \\ &= \int_0^\infty tL\tilde{\psi}'(tL)[\varphi(tL)f \cdot \varphi(tL)g] \frac{dt}{t} \\ &\quad + \int_0^\infty \tilde{\psi}(tL)[\psi(tL)f \cdot (I-\varphi(tL))g + (I-\varphi(tL))f \cdot \psi(tL)g] \frac{dt}{t}. \end{aligned}$$

By defining

$$\begin{aligned} \Pi_1(f,g) &:= \int_0^\infty \tilde{\psi}_1(tL)[\varphi(tL)f \cdot \varphi(tL)g] \frac{dt}{t}, \\ \Pi_2(f,g) &:= \int_0^\infty \tilde{\psi}(tL)[\psi(tL)f \cdot \tilde{\varphi}(tL)g] \frac{dt}{t}, \end{aligned}$$

we then observe that the product of f and g can be represented as the sum of three paraproducts, i.e.

$$f \cdot g = \Pi_1(f,g) + \Pi_2(f,g) + \Pi_2(g,f).$$

Similar to Proposition 5.14, there moreover holds for all $f, g \in \mathcal{D}((L_p)^s) \cap L^\infty(X)$

$$\begin{aligned} L^s\Pi_1(f,g) &= \int_0^\infty (tL)^s\tilde{\psi}_1(tL)[(tL)^{-s}\varphi(tL)L^sf \cdot \varphi(tL)g] \frac{dt}{t} \\ &=: \tilde{\Pi}_1(L^sf, g), \end{aligned}$$

and analogously

$$\begin{aligned} L^s\Pi_2(f,g) &= \int_0^\infty (tL)^s\tilde{\psi}(tL)[(tL)^{-s}\psi(tL)L^sf \cdot \tilde{\varphi}(tL)g] \frac{dt}{t} \\ &=: \tilde{\Pi}_2(L^sf, g). \end{aligned}$$

By choosing $M \in \mathbb{N}$ sufficiently large in comparison to $s > 0$, we can assure that both $\tilde{\Pi}_1$ and $\tilde{\Pi}_2$ are paraproduct operators which satisfy the assumptions of Theorem 5.13. Observe that omitting the averaging operator A_t, which appears in Theorem 5.13, is compensated by the additional assumption that e^{-tL} is bounded in $L^\infty(X)$ uniformly in $t > 0$. Hence, for $j = 1, 2$ the operator $\tilde{\Pi}_j(\,.\,, g)$ is bounded from $H^p_L(X)$ to $L^p(X)$, if $p \in [1, 2)$, and from $L^p(X)$ to $H^p_L(X)$, if $p \in (2, \infty)$, with the operator norm bounded by some constant times $\|g\|_{L^\infty(X)}$. We therefore obtain for $p \in [1, 2)$

$$\begin{aligned} \|L^s(fg)\|_{L^p(X)} &\le \left\|\tilde{\Pi}_1(L^sf, g)\right\|_{L^p(X)} + \left\|\tilde{\Pi}_2(L^sf, g)\right\|_{L^p(X)} + \left\|\tilde{\Pi}_2(L^sg, f)\right\|_{L^p(X)} \\ &\lesssim \|L^sf\|_{H^p_L(X)} \|g\|_{L^\infty(X)} + \|f\|_{L^\infty(X)} \|L^sg\|_{H^p_L(X)}, \end{aligned}$$

and the corresponding estimate for $p \in (2, \infty)$. The estimate for $p = 2$ is, in analogy to Lemma 5.11, an immediate consequence of quadratic estimates. □

6 A $T(1)$-Theorem for non-integral operators

This chapter is devoted to the statement and the proof of our main theorem, a $T(1)$-Theorem for non-integral operators. It characterizes the L^2-boundedness of operators T satisfying certain off-diagonal estimates associated to a sectorial operator L.
Before we come to the statement and the proof of our $T(1)$-Theorem, Theorem 6.13, we first fix our assumptions on the operator T, clarify how under these assumptions the expressions $T(1)$ and $T^*(1)$ can be defined and give necessary conditions for the boundedness of T on $L^2(X)$. Moreover, we discuss Poincaré estimates on metric spaces, that will be used in the proof of our main result.
After the statement and proof of Theorem 6.13, we also add a second version with weaker assumptions and apply this version to prove the boundedness of some paraproduct operator on $L^2(X)$. We finally present an approach towards a $T(b)$-Theorem and explain how to extend the theory to Hardy spaces $H^p_L(X)$ for $p \neq 2$.

Throughout the chapter, we will always assume L to be an operator satisfying (H1), (H2) and (H3).

6.1 Assumptions on the operator

Let us first fix our main assumptions on the operator T. These assumptions replace the kernel estimates of Calderón-Zygmund operators. Instead of a Hölder or Hörmander condition on the kernel, we assume weak L^2 off-diagonal estimates on the operator families $\{T\psi_1(tL)\}_{t>0}$ and $\{T^*\psi_2(tL^*)\}_{t>0}$, where ψ_1, ψ_2 are functions from $\Psi(\Sigma^0_\mu)$ with enough decay at 0.
Similar conditions were already used in Theorem 4.39 (which is essentially [HMa09], Theorem 3.2), to show the boundedness of operators $T : H^1_L(X) \to L^1(X)$ under the assumption that T is bounded on $L^2(X)$. The relation of the assumptions on T below stated and those used in Theorem 4.39 is given by Lemma 6.5 and Corollary 6.6.

Assumption Let $\mu \in (\omega, \frac{\pi}{2})$, and let $\alpha \geq 1$ and $\beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$.
Let $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\text{loc}}(X)$ be a linear operator with $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\text{loc}}(X)$, which satisfies the following off-diagonal estimates:

(OD1)$_\gamma$ There exists a function $\psi_1 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$, some $\gamma > 0$ and a constant $C > 0$ such that $\psi_1(L)$ is injective and for every $t > 0$, arbitrary balls $B_1, B_2 \in X$ with radius $r = t^{1/2m}$ and every $f \in L^2(X)$ supported in B_1 there holds

$$\|T\psi_1(tL)f\|_{L^2(B_2)} \leq C \left(1 + \frac{\text{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)}. \tag{6.1}$$

(OD2)$_\gamma$ There exists a function $\psi_2 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$, some $\gamma > 0$ and a constant $C > 0$ such that $\psi_2(L^*)$ is injective and for every $t > 0$, arbitrary balls $B_1, B_2 \in X$ with radius $r = t^{1/2m}$ and every $f \in L^2(X)$ supported in B_1 there holds

$$\|T^*\psi_2(tL^*)f\|_{L^2(B_2)} \leq C \left(1 + \frac{\text{dist}(B_1, B_2)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^2(B_1)}. \tag{6.2}$$

Whenever we say that a linear operator T satisfies $(\text{OD1})_\gamma$ or $(\text{OD2})_\gamma$, we mean that T satisfies (6.1) or (6.2), respectively, for $\mu, \alpha, \beta, \psi_1, \psi_2, C$ as specified above. The parameter $\gamma > 0$ will be specified in each situation separately.

The assumptions that $\psi_1(L)$ and $\psi_2(L^*)$ are injective are only used to define $T(1)$ and $T^*(1)$ in an appropriate way. If in applications this is clear, then the assumptions can be omitted.

The assumptions $(\text{OD1})_\gamma$ and $(\text{OD2})_\gamma$ will be essential for our $T(1)$-Theorem, Theorem 6.13. They are in some sense rather strong, since they are not only "off-diagonal" assumptions, but also include the "on-diagonal" case, postulating that (6.1) and (6.2) also hold for $\operatorname{dist}(B_1, B_2) = 0$.
In Section 6.6 we will show that it is possible to weaken them in some special cases, namely whenever the conservation property $e^{-tL}(1) = 1$, and the same for L^*, is valid.

6.2 Definition of $T(1)$ and $T^*(1)$

Before we can state our $T(1)$-Theorem, we first have to clarify how to understand the expressions $T(1)$ and $T^*(1)$ for a linear operator $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\text{loc}}(X)$ with $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\text{loc}}(X)$, that satisfies $(\text{OD2})_\gamma$ and $(\text{OD1})_\gamma$, respectively, for some $\gamma > \frac{n}{2m}$.
For convenience, we will only consider the definition of $T^*(1)$. How to define $T(1)$ will then be obvious.

The first observation is a simple consequence of Remark 3.2. If $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\text{loc}}(X)$ is a linear operator that satisfies $(\text{OD1})_\gamma$ for some $\gamma > \frac{n}{2m}$, then $\psi_1(tL^*)T^*(1)$ can be defined via duality as an element of $L^2_{\text{loc}}(X)$, i.e.

$$\langle \psi_1(tL^*)T^*(1), \varphi \rangle := \langle 1, T\psi_1(tL)\varphi \rangle$$

for all $\varphi \in L^2(X)$ that are supported in some ball $B \subseteq X$.

To motivate our definition of $T^*(1)$, let us first show how one can define $T^*(1)$ under slightly different assumptions. Instead of the assumption $(\text{OD1})_\gamma$ on T, let us assume for a moment that $\{T(I - (I + tL)^{-1})^M\}_{t>0}$ satisfies off-diagonal estimates of order $\gamma > \frac{n}{2m}$ and that T actually acts as a linear operator on $L^2(X)$. We can then show how to define $T^*(1)$ in $BMO_{L^*}(X)$.
We will later on also use the result to establish necessary conditions for non-integral operators to be bounded on $L^2(X)$.

Lemma 6.1 *Let $T : L^2(X) \to L^2(X)$ be a linear operator and $M \in \mathbb{N}$ with $M > \frac{n}{4m}$. If the operator family $\{T(I - (I + tL)^{-1})^M\}_{t>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$, then $T^*(1)$ can be defined as an element of $BMO_{L^*}(X)$ by setting*

$$\langle T^*(1), m \rangle := \lim_{R\to\infty} \langle (I - (I + L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)}), (I + L)^M b \rangle \tag{6.3}$$

for every $(1, 2, M, \varepsilon)$-molecule $m = L^M b$ associated to the unit ball B_0 and arbitrary $\varepsilon > 0$.

Observe that at a formal level, the right-hand side of (6.3) is equal to $\lim_{R\to\infty} \langle T^*(\mathbb{1}_{B(0,R)}), m\rangle$. Moreover, let us mention that (6.3) defines $T^*(1)$ as a linear functional on a dense subspace of $H^1_L(X)$, consisting of finite linear combinations of $(1,2,M,\varepsilon)$-molecules. This is due to the fact that any $(1,2,M,\varepsilon)$-molecule associated to an arbitrary ball B in X can be renormalized to a molecule associated to the unit ball B_0, see Remark 4.20.

Proof: Recall that according to Theorem 4.28 and Definition 4.29 the spaces $BMO_{L^*,M}(X)$ are equivalent whenever $M \in \mathbb{N}$ with $M > \frac{n}{4m}$ and that we defined $BMO_{L^*}(X)$ to be one of these equivalent spaces.
Let $\gamma > \frac{n}{2m}$ and $M \in \mathbb{N}$ with $M > \frac{n}{4m}$ as given in the assumptions. We will first show that there exists some constant $C_T > 0$ such that for every ball B in X with radius $r_B > 0$ and every $f \in L^\infty(X)$ there holds

$$V(B)^{-1/2} \left\|(I-(I+r_B^{2m}L^*)^{-1})^M T^*(f)\right\|_{L^2(B)} \leq C_T \left\|f\right\|_{L^\infty(X)}. \tag{6.4}$$

Observe that since $\{T(I-(I+tL)^{-1})^M\}_{t>0}$ satisfies weak off-diagonal estimates of some order larger than $\frac{n}{2m}$, the expression on the left hand side of (6.4) is well-defined via duality due to Remark 3.2. In addition, Remark 3.8 yields that the assumed weak off-diagonal estimates imply estimates over annuli of the form (3.11).
Let B be an arbitrary ball in X and let $g \in L^2(X)$ with $\operatorname{supp} g \subseteq B$. We obtain via the Cauchy-Schwarz inequality and the doubling property (2.2)

$$\begin{aligned}
&\left|\langle (I-(I+r_B^{2m}L^*)^{-1})^M T^*(f), g\rangle\right| = \left|\langle f, T(I-(I+r_B^{2m}L)^{-1})^M g\rangle\right| \\
&\quad\leq \sum_{j=0}^\infty \left|\langle \mathbb{1}_{S_j(B)} f, T(I-(I+r_B^{2m}L)^{-1})^M g\rangle\right| \\
&\quad\leq \|f\|_{L^\infty(X)} \sum_{j=0}^\infty V(2^jB)^{1/2} \left\|T(I-(I+r_B^{2m}L)^{-1})^M g\right\|_{L^2(S_j(B))} \\
&\quad\lesssim \|f\|_{L^\infty(X)} \sum_{j=0}^\infty V(2^jB)^{1/2} 2^{jn/2} \left(1+\frac{\operatorname{dist}(B,S_j(B))^{2m}}{r_B^{2m}}\right)^{-\gamma} \|g\|_{L^2(B)} \\
&\quad\lesssim \|f\|_{L^\infty(X)} V(B)^{1/2} \|g\|_{L^2(B)} \sum_{j=0}^\infty 2^{jn} 2^{-2m\gamma j} \lesssim \|f\|_{L^\infty(X)} V(B)^{1/2} \|g\|_{L^2(B)}, \quad (6.5)
\end{aligned}$$

again using that $\gamma > \frac{n}{2m}$ in the last step. As the ball B and $g \in L^2(B)$ were arbitrary, the above estimate implies (6.4).

Let us now show how $T^*(1)$ can be defined as an element of $\mathcal{E}_M(L^*) = \bigcap_{\varepsilon>0}(\mathcal{M}_0^{1,2,M,\varepsilon}(L))'$. Let $\varepsilon > 0$. Recall that $\mathcal{M}_0^{1,2,M,\varepsilon}(L)$ consists of all $m \in L^2(X)$ with $m = L^M b$ for some $b \in \mathcal{D}(L^M)$ and

$$\|m\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} = \sup_{j\geq 0}\left[2^{j\varepsilon} V(2^jB_0)^{1/2} \sum_{k=0}^M \left\|L^k b\right\|_{L^2(S_j(B_0))}\right] < \infty, \tag{6.6}$$

where B_0 denotes the unit ball in X centered at 0 as defined in Section 4.4. In addition, note that

$$L^M(I-(I+L)^{-1})^{-M} = (I+L)^M = \sum_{k=0}^M C_{k,M} L^{M-k},$$

i.e. $L^M = (I-(I+L)^{-1})^M (I+L)^M$, where $C_{k,M}$ are the coefficients from the binomial formula.
We define for every $R>0$ a linear functional ℓ_R on $\mathcal{M}_0^{1,2,M,\varepsilon}(L)$ by setting

$$\ell_R(m) := \langle (I-(I+L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)}), (I+L)^M b\rangle$$

for every $m = L^M b \in \mathcal{M}_0^{1,2,M,\varepsilon}(L)$. Then there holds

$$\begin{aligned}
|\ell_R(m)| &= \left|\langle (I-(I+L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)}), (I+L)^M b\rangle\right| \\
&\lesssim \sum_{k=0}^{M} \left|\langle (I-(I+L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)}), L^{M-k} b\rangle\right| \\
&\le \sum_{k=0}^{M}\sum_{j=0}^{\infty} \left\|(I-(I+L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)})\right\|_{L^2(S_j(B_0))} \left\|L^{M-k} b\right\|_{L^2(S_j(B_0))} \\
&\lesssim \sum_{j=0}^{\infty} 2^{-j\varepsilon} V(2^j B_0)^{-1/2} \left\|(I-(I+L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)})\right\|_{L^2(S_j(B_0))} \|m\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)}.
\end{aligned} \tag{6.7}$$

Using Lemma 4.24 and (6.4) with $f = \mathbb{1}_{B(0,R)}$, the above yields

$$\begin{aligned}
\sup_{R>0} |\ell_R(m)| &\lesssim \sup_{R>0}\sup_{B\subseteq X} V(B)^{-1/2} \left\|(I-(I+r_B^{2m}L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)})\right\|_{L^2(B)} \|m\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} \\
&\le C_T \|m\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)}.
\end{aligned}$$

It remains to show that ℓ_R converges pointwise for $R\to\infty$. Following the estimates in (6.7) and (6.5), one observes that $(\ell_R(m))_R$ is a Cauchy sequence for every $m \in \mathcal{M}_0^{1,2,M,\varepsilon}(L)$. Hence, $\lim_{R\to\infty} \ell_R(m)$ exists.
Since $\varepsilon > 0$ was arbitrary, we can thus define $T^*(1) \in \mathcal{E}_M(L^*)$ by

$$\langle T^*(1), m\rangle := \lim_{R\to\infty} \langle (I-(I+L^*)^{-1})^M T^*(\mathbb{1}_{B(0,R)}), (I+L)^M b\rangle$$

for every $\varepsilon > 0$ and every $m = L^M b \in \mathcal{M}_0^{1,2,M,\varepsilon}(L)$.

In a third step, we will show that the linear functional $T^*(1)$ of $\mathcal{E}_M(L^*)$ can also be considered as an element of $BMO_{L^*}(X)$. Since every $m \in \mathcal{M}_0^{1,2,M,\varepsilon}(L)$ is, up to normalization, a molecule of $H^1_L(X)$, we can define $T^*(1)$ as a linear functional on a dense subspace of $H^1_L(X)$. But similar to the proof of Theorem 4.28, it needs some extra work to show the continuity of $T^*(1)$.
We now denote ℓ_R by $T^*(\mathbb{1}_{B(0,R)})$. Then (6.4) also shows that $T^*(\mathbb{1}_{B(0,R)}) \in BMO_{L^*}(X)$ with

$$\sup_{R>0} \left\|T^*(\mathbb{1}_{B(0,R)})\right\|_{BMO_{L^*}(X)} \le C_T,$$

where for obvious reason we actually use the equivalent norm $\|\,.\,\|_{BMO_{L^*,M,res}(X)}$ defined via the resolvent operator; see Proposition 4.23. The duality of $H^1_L(X)$ and $BMO_{L^*}(X)$, proven in Theorem 4.28, now yields that $T^*(\mathbb{1}_{B(0,R)})$ is a continuous linear functional on $H^1_L(X)$ with

$$\sup_{R>0} \left\|T^*(\mathbb{1}_{B(0,R)})\right\|_{(H^1_L(X))'} \lesssim \sup_{R>0} \left\|T^*(\mathbb{1}_{B(0,R)})\right\|_{BMO_{L^*}(X)} \le C_T.$$

By uniform boundedness, we therefore obtain that $T^*(1) \in (H^1_L(X))'$ and thus $T^*(1) \in BMO_{L^*}(X)$ with $\|T^*(1)\|_{BMO_{L^*}(X)} \leq C_T$. This finishes the proof. □

For a similar definition of $T^*(1)$ in the case that $(\mathrm{OD1})_\gamma$ is assumed instead of off-diagonal estimates for $\{T(I-(I+tL)^{-1})^M\}_{t>0}$ and that T is defined as an operator $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\mathrm{loc}}(X)$, we will modify the above construction. To do so, let us first define spaces $Y^{\psi,\varepsilon}(L)$ that will replace the spaces $\mathcal{M}_0^{1,2,M,\varepsilon}(L)$.

Definition 6.2 *Let $\varepsilon > 0$ and let $\alpha \geq 1$ and $\beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$. Let $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$ such that $\psi(L)$ is injective. We define*

$$Y^{\psi,\varepsilon}(L) := \{m = \psi(L)b \,:\, b \in L^2(X),\ \lim_{j\to\infty} 2^{j\varepsilon} V(2^j B_0)^{1/2} \|b\|_{L^2(S_j(B_0))} = 0\},$$

with the norm given by

$$\|m\|_{Y^{\psi,\varepsilon}(L)} := \sup_{j \geq 0} \left[2^{j\varepsilon} V(2^j B_0)^{1/2} \|b\|_{L^2(S_j(B_0))} \right].$$

In addition, we define

$$Y_c^{\psi,\varepsilon}(L) := \{m = \psi(L)b \in Y^{\psi,\varepsilon}(L) \,:\, \operatorname{supp} b \subseteq B \text{ for some ball } B \subseteq X\}$$

and

$$\mathcal{E}_\psi(L) := \bigcap_{\varepsilon>0} (Y^{\psi,\varepsilon}(L^*))'.$$

Remark 6.3 For every $\psi \in \Psi(\Sigma^0_\mu)$ as specified in Definition 6.2 and every $\varepsilon > 0$, the space $Y^{\psi,\varepsilon}(L)$ is a Banach space and $Y_c^{\psi,\varepsilon}(L)$ is a dense subset of $Y^{\psi,\varepsilon}(L)$. Moreover, let us remark the following inclusion.
Let $M \in \mathbb{N}$ with $M > \frac{n}{4m}$. Let further $\alpha \geq 1$, $\beta > \frac{n}{4m} + M$ and $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$ such that $\psi(L)$ is injective. Then for every $\varepsilon > 0$ with $\frac{\varepsilon}{2m} \leq \beta - (M + \frac{n}{4m})$ there holds

$$Y^{\psi,\varepsilon}(L) \subseteq \mathcal{M}_0^{1,2,M,\varepsilon}(L).$$

The result is also true for functions $\psi \in \Psi(\Sigma^0_\mu)$ with $z \mapsto z^{-M}\psi(z) \in H^\infty(\Sigma^0_\mu)$ such that the family of operators $\{(tL)^{-M}\psi(tL)\}_{t>0}$ satisfies Davies-Gaffney estimates. In this case, the inclusion is valid for all $\varepsilon > 0$.

For the proof, let $m \in Y^{\psi,\varepsilon}(L)$, where $m = \psi(L)b$ for some $b \in L^2(X)$. Since $\beta > \frac{n}{4m} + M$, there obviously holds $m \in \mathcal{R}(L^M)$. In addition, we have to show that $\|m\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} < \infty$ (see (6.11) for a definition of the norm). First, observe that a similar calculation as in Remark 4.2 yields

$$\|b\|_{L^2(X)} \leq C_\varepsilon V(B_0)^{-1/2} \|m\|_{Y^{\psi,\varepsilon}(L)} \tag{6.8}$$

for some constant $C_\varepsilon > 0$ only depending on $\varepsilon > 0$. Moreover, observe that for every $k = 0, 1, \ldots, M$, the function $z \mapsto z^{-(M-k)}\psi(z)$ is an element of $\Psi_{\beta-M,\alpha}(\Sigma^0_\mu)$. Thus, Proposition 3.18 yields that the operator family $\{(tL)^{-(M-k)}\psi(tL)\}_{t>0}$ satisfies

off-diagonal estimates of order $\beta - M$.
Let us now write $b = \mathbb{1}_{R_j} b + \mathbb{1}_{(R_j)^c} b$ with

$$R_j = 2^{j+2} B_0, \qquad \text{if } j = 0, 1, 2,$$
$$R_j = 2^{j+2} B_0 \setminus 2^{j-2} B_0, \qquad \text{if } j = 3, 4, \ldots.$$

For every $k = 0, 1, \ldots, M$ and all $j \in \mathbb{N}_0$, we obtain due to the boundedness of $L^{-(M-k)}\psi(L)$ on $L^2(X)$

$$\left\| L^{-(M-k)}\psi(L)\mathbb{1}_{R_j} b \right\|_{L^2(S_j(B_0))} \lesssim \|b\|_{L^2(R_j)} \lesssim 2^{-j\varepsilon} V(2^j B_0)^{-1/2} \|m\|_{Y^{\psi,\varepsilon}(L)}, \tag{6.9}$$

where in the last step R_j is splitted into four annuli. On the other hand, the off-diagonal estimates for $\{(tL)^{-(M-k)}\psi(tL)\}_{t>0}$, (6.8) and the doubling property (2.2) yield

$$\begin{aligned}\left\| L^{-(M-k)}\psi(L)\mathbb{1}_{(R_j)^c} b \right\|_{L^2(S_j(B_0))} &\lesssim \left(1 + \operatorname{dist}(S_j(B_0), (R_j)^c)^{2m}\right)^{-(\beta-M)} \|b\|_{L^2(R_j)^c} \\ &\lesssim 2^{-2m(\beta-M)j} \|b\|_{L^2(X)} \\ &\lesssim 2^{-2m(\beta-M)j} 2^{jn/2} V(2^j B_0)^{-1/2} \|m\|_{Y^{\psi,\varepsilon}(L)}. \end{aligned} \tag{6.10}$$

We therefore obtain from (6.9), (6.10) and the assumption $\frac{\varepsilon}{2m} \leq \beta - (M + \frac{n}{4m})$

$$\begin{aligned}\|m\|_{\mathcal{M}_0^{1,2,M,\varepsilon}(L)} &= \sup_{j\geq 0} \left[2^{j\varepsilon} V(2^j B_0)^{1/2} \sum_{k=0}^{M} \left\| L^{-(M-k)}\psi(L) b \right\|_{L^2(S_j(B_0))} \right] \\ &\lesssim \sup_{j \geq 0} 2^{j\varepsilon} \left(2^{-j\varepsilon} + 2^{-2m(\beta - (M + \frac{n}{4m}))j} \right) \|m\|_{Y^{\psi,\varepsilon}(L)} \\ &\lesssim \|m\|_{Y^{\psi,\varepsilon}(L)}. \end{aligned} \tag{6.11}$$

Since Davies-Gaffney estimates imply off-diagonal estimates of any order, the second case is then obvious.

Let us now define $T^*(1)$ as an element of $\mathcal{E}_{\psi_1}(L^*)$ in the following way.

Lemma 6.4 *Let $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\text{loc}}(X)$ be a linear operator that satisfies* $(\text{OD1})_\gamma$ *for some $\gamma > \frac{n}{2m}$. Then $T^*(1)$ can be defined as an element of $\mathcal{E}_{\psi_1}(L^*)$ by*

$$\langle T^*(1), m \rangle := \lim_{R\to\infty} \langle \psi_1(L^*) T^*(\mathbb{1}_{B(0,R)}), b \rangle$$

for every $m \in Y_c^{\psi_1,\varepsilon}(L)$ with $m = \psi_1(L) b$ and every $\varepsilon > 0$.

Proof: Let $\gamma > \frac{n}{2m}$. Repeating the arguments used in (6.5) for the proof of (6.4), we can show that the assumption $(\text{OD1})_\gamma$ yields the following estimate: There exists some constant $C_T > 0$ such that for every ball B in X with radius $r_B > 0$ and every $f \in L^\infty(X)$ there holds

$$V(B)^{-1/2} \left\| \psi_1(r_B^{2m} L^*) T^*(f) \right\|_{L^2(B)} \leq C_T \|f\|_{L^\infty(X)}. \tag{6.12}$$

As mentioned before, the left hand side of (6.12) is well-defined via duality.
With the help of the above estimate, we can now define $T^*(1)$ as an element of $\mathcal{E}_{\psi_1}(L^*) =$

$\bigcap_{\varepsilon>0}(Y^{\psi_1,\varepsilon}(L))'$ as follows.
Let $\varepsilon > 0$. We define for every $R > 0$ a linear functional ℓ_R on $Y_c^{\psi_1,\varepsilon}(L)$ by setting

$$\ell_R(m) := \langle \psi_1(L^*)T^*(\mathbb{1}_{B(0,R)}), b\rangle$$

for every $m = \psi_1(L)b \in Y_c^{\psi_1,\varepsilon}(L)$. Observe that $\ell_R(m)$ is well-defined, since b is supported in some ball of X and $\psi_1(L^*)T^*(\mathbb{1}_{B(0,R)})$ is via duality defined as an element of $L^2_{\mathrm{loc}}(X)$. In analogy to the estimate in (6.7), we obtain from (6.12)

$$\begin{aligned}
|\ell_R(m)| &= \left|\langle \psi_1(L^*)T^*(\mathbb{1}_{B(0,R)}), b\rangle\right| \\
&\leq \sum_{j=0}^{\infty} \left\|\psi_1(L^*)T^*(\mathbb{1}_{B(0,R)})\right\|_{L^2(S_j(B_0))} \|b\|_{L^2(S_j(B_0))} \\
&\lesssim \sum_{j=0}^{\infty} 2^{-j\varepsilon} V(2^j B_0)^{-1/2} \left\|\psi_1(L^*)T^*(\mathbb{1}_{B(0,R)})\right\|_{L^2(S_j(B_0))} \|m\|_{Y^{\psi_1,\varepsilon}(L)} \\
&\lesssim C_T \|m\|_{Y^{\psi_1,\varepsilon}(L)},
\end{aligned} \tag{6.13}$$

where the implicit constants are independent of $R > 0$. Thus,

$$\sup_{R>0} |\ell_R(m)| \lesssim C_T \|m\|_{Y^{\psi_1,\varepsilon}(L)}.$$

With the same arguments used in the proof of Lemma 6.1, we can moreover show that $(\ell_R(m))_R$ is a Cauchy sequence for every $m \in Y_c^{\psi_1,\varepsilon}(L)$. Hence, $\lim_{R\to\infty} \ell_R(m)$ exists. Since $Y_c^{\psi_1,\varepsilon}(L)$ is dense in $Y^{\psi_1,\varepsilon}(L)$ and $\varepsilon > 0$ was arbitrary, we can now define $T^*(1) \in \mathcal{E}_{\psi_1}(L^*)$ by

$$\langle T^*(1), m\rangle := \lim_{R\to\infty} \langle \psi_1(L^*)T^*(\mathbb{1}_{B(0,R)}), b\rangle$$

for every $\varepsilon > 0$ and every $m \in Y_c^{\psi_1,\varepsilon}(L)$ with $m = \psi_1(L)b$. □

In the same way, one can then also define $T(1)$ as an element of $\mathcal{E}_{\psi_2}(L)$ under the assumption that $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\mathrm{loc}}(X)$ satisfies $(\mathrm{OD2})_\gamma$ for some $\gamma > \frac{n}{2m}$.

6.3 Necessary conditions

Let $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\mathrm{loc}}(X)$ be a linear operator that satisfies the off-diagonal estimate $(\mathrm{OD1})_\gamma$ for some $\gamma > \frac{n}{2m}$. We show that if T extends to a bounded operator on $L^2(X)$, then $T^*(1) \in BMO_{L^*}(X)$. Analogously, one can show that under the assumption $(\mathrm{OD2})_\gamma$, the condition $T(1) \in BMO_L(X)$ is necessary for T to be bounded on $L^2(X)$.

Let us recall what we have shown in Lemma 6.1 already: If for some $M \in \mathbb{N}$ with $M > \frac{n}{4m}$ the operator family $\{T(I-(I+tL)^{-1})^M\}_{t>0}$ satisfies off-diagonal estimates of order $\gamma > \frac{n}{2m}$, then $T^*(1)$ can be defined as an element of $BMO_{L^*}(X)$. To generalize this result to off-diagonal estimates on $\{T\psi(tL)\}_{t>0}$ for arbitrary $\psi \in \Psi(\Sigma_\mu^0) \setminus \{0\}$, we use the next lemma. It states a certain self-improving property of this kind of off-diagonal estimates and will later on also be applied in Corollary 6.22 to show the extension of T from $L^2(X)$ to spaces $H^p_L(X)$ for $p \neq 2$.

Lemma 6.5 *Let $\psi \in \Psi(\Sigma_\mu^0) \setminus \{0\}$ and let T be a linear operator on $L^2(X)$ such that $\{T\psi(tL)\}_{t>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$ on $L^2(X)$. Let $\delta > \gamma$ and let $\varphi \in H^\infty(\Sigma_\mu^0)$ with $|\varphi(z)| \lesssim |z|^\delta$ for $|z| \leq 1$. Moreover, assume that $\{T\varphi(tL)\}_{t>0}$ is uniformly bounded on $L^2(X)$. Then $\{T\varphi(tL)\}_{t>0}$ satisfies weak off-diagonal estimates of order γ on $L^2(X)$.*

The proof is similar to the one of Lemma 5.12, where off-diagonal estimates for paraproducts were shown. Again, the key idea in the proof is to represent $T\varphi(tL)$ with the help of a Calderón reproducing formula in the form (6.14), which enables us to apply the assumed weak off-diagonal estimates on $T\psi(tL)$.

Proof: Let $t > 0$ and let B_1, B_2 be arbitrary balls with radius t. Let $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq B_1$ and $\operatorname{supp} g \subseteq B_2$. Given $\psi \in \Psi(\Sigma_\mu^0) \setminus \{0\}$ from the assumptions, we choose some function $\tilde{\psi} \in \Psi_{\sigma,\tau}(\Sigma_\mu^0)$ with $\sigma, \tau > \gamma$ and $\int_0^\infty \psi(s)\tilde{\psi}(s)\,\frac{ds}{s} = 1$. The Calderón reproducing formula then yields

$$\langle T\varphi(t^{2m}L)f, g\rangle = \int_0^\infty \langle T\varphi(t^{2m}L)\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)f, g\rangle\,\frac{ds}{s}. \tag{6.14}$$

First, due to the Cauchy-Schwarz inequality and quadratic estimates for ψ and $\tilde{\psi}$, we obtain

$$\begin{aligned}
&\left|\langle T\varphi(t^{2m}L)f, g\rangle\right| \\
&\quad\leq \int_0^\infty \left|\langle T\varphi(t^{2m}L)\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)f, g\rangle\right|\,\frac{ds}{s} \\
&\quad\lesssim \left(\int_0^\infty \left\|\tilde{\psi}(s^{2m}L)f\right\|_{L^2(X)}^2 \frac{ds}{s}\right)^{1/2} \left(\int_0^\infty \left\|\psi(s^{2m}L^*)\varphi(t^{2m}L^*)T^*g\right\|_{L^2(X)}^2 \frac{ds}{s}\right)^{1/2} \\
&\quad\lesssim \|f\|_{L^2(B_1)} \left\|\varphi(t^{2m}L^*)T^*g\right\|_{L^2(X)} \lesssim \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)},
\end{aligned}$$

where we used the uniform boundedness of $\{T\varphi(tL)\}_{t>0}$ on $L^2(X)$ in the last step. If $\operatorname{dist}(B_1, B_2) \leq t$, this gives the desired estimate.
In the case of $\operatorname{dist}(B_1, B_2) > t$, we break the integral in (6.14) into two parts, one over $(0,t)$, which is called J_1, and one over (t,∞), which is called J_2.
We first turn to J_1. On the one hand, $\{T\psi(sL)\}_{s>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$. Proposition 3.18, on the other hand, yields that $\{\tilde{\psi}(sL)\varphi(tL)\}_{s,t>0}$ satisfies off-diagonal estimates in s of order σ, since $\sup_{t>0} \|\varphi(t\,\cdot)\|_{L^\infty(\Sigma_\mu^0)} = \|\varphi\|_{L^\infty(\Sigma_\mu^0)} < \infty$. Hence, the composition of the two operators $\{T\psi(sL)\tilde{\psi}(sL)\varphi(tL)\}_{s,t>0}$ satisfies weak off-diagonal estimates in s of order $\min(\gamma,\sigma) = \gamma > \frac{n}{2m}$ on $L^2(X)$ due to Proposition 3.7. Using Remark 3.9 (which provides us with weak off-diagonal estimates in s for balls of radius $t > s$), with the roles of s and t interchanged, we therefore get

$$\begin{aligned}
|J_1| &\leq \int_0^t \left|\langle T\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)\varphi(t^{2m}L)f, g\rangle\right|\,\frac{ds}{s} \\
&\lesssim \int_0^t \left(\frac{t}{s}\right)^n \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s}\,\|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)}.
\end{aligned}$$

Since we assumed $\gamma > \frac{n}{2m}$ and $\operatorname{dist}(B_1, B_2) > t$, there further holds

$$\int_0^t \left(\frac{t}{s}\right)^n \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s} \le \int_0^t \left(\frac{t}{s}\right)^{n-2m\gamma} \left(\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\gamma} \frac{ds}{s}$$
$$= \left(\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\gamma} \int_0^1 u^{2m\gamma-n} \frac{du}{u} \lesssim \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\gamma},$$

using the substitution $u = \frac{s}{t}$ in the penultimate step.
We now come to J_2. For the integral over (t, ∞), we again use that $\{T\psi(sL)\}_{s>0}$ satisfies weak off-diagonal estimates in s of order γ. The family of operators $\{\tilde{\psi}(sL)\varphi(tL)\}_{s,t>0}$, in contrast, is handled slightly different. Lemma 3.19 shows that these operators do not only satisfy off-diagonal estimates in s, but also provides us with an extra factor $\left(\frac{t}{s}\right)^a$. Precisely, Lemma 3.19 shows that for every $a > 0$ with $a \le \delta$ and $a < \tau$ (where δ describes the decay of φ at 0 and τ the decay of $\tilde{\psi}$ at infinity), there exists a family of operators $\{T_{s,t}\}_{s,t>0}$ such that

$$\tilde{\psi}(sL)\varphi(tL) = \left(\frac{t}{s}\right)^a T_{s,t},$$

where $\{T_{s,t}\}_{s,t>0}$ satisfies off-diagonal estimates in s of order $\sigma + a$ uniformly in $t > 0$. We again combine these operators and get, due to Proposition 3.7, that the family of operators $\{T\psi(sL)T_{s,t}\}_{s,t>0}$ satisfies weak off-diagonal estimates in s of order $\min(\gamma, \sigma + a) = \gamma$. Hence (recall that we can apply weak off-diagonal estimates to smaller balls without any change),

$$|J_2| \le \int_t^\infty \left|\langle T\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)\varphi(t^{2m}L)f, g\rangle\right| \frac{ds}{s}$$
$$\lesssim \int_t^\infty \left(\frac{t}{s}\right)^{2ma} \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s} \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)}.$$

Since we assumed $\delta > \gamma$ and $\tau > \gamma$, we can fix some $a > \gamma$ with $a \le \delta$ and $a < \tau$. For this choice of a we further get, similar to the treatment of J_1,

$$\int_t^\infty \left(\frac{t}{s}\right)^{2ma} \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s} \le \int_t^\infty \left(\frac{t}{s}\right)^{2m(a-\gamma)} \left(\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\gamma} \frac{ds}{s}$$
$$= \left(\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\gamma} \int_1^\infty u^{-2m(a-\gamma)} \frac{du}{u} \lesssim \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\gamma},$$

still assuming that $\operatorname{dist}(B_1, B_2) > t$. Combining the estimates of J_1 and J_2 finishes the proof. □

Let us now apply Lemma 6.5 to some special choices of $\varphi \in H^\infty(\Sigma_\mu^0)$.

Corollary 6.6 *Let $\psi \in \Psi(\Sigma_\mu^0) \setminus \{0\}$ and let T be a linear operator on $L^2(X)$ such that $\{T\psi(tL)\}_{t>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$. Let further $M \in \mathbb{N}$ with $M > \gamma$. If the families of operators $\{T(I - e^{-tL})^M\}_{t>0}$, $\{T(tLe^{-tL})^M\}_{t>0}$ and $\{T(I - (I + tL)^{-1})^M\}_{t>0}$ are uniformly bounded on $L^2(X)$, then they satisfy weak off-diagonal estimates of order γ in $L^2(X)$.*

Proof: This is an immediate consequence of Lemma 6.5. We simply observe that the functions $z \mapsto (1-e^{-z})^M$, $z \mapsto (ze^{-z})^M$ and $z \mapsto (1-(1+z)^{-1})^M = (z(1+z)^{-1})^M$ are in $H^\infty(\Sigma^0_\mu)$ with

$$\left|(1-e^{-z})^M\right| \leq |z|^M\,, \qquad \left|(ze^{-z})^M\right| \leq |z|^M \qquad \text{and} \qquad \left|(1-(1+z)^{-1})^M\right| \leq |z|^M\,.$$

for every $z \in \Sigma^0_\mu$ with $|z| \leq 1$ and apply Lemma 6.5 to these functions. □

The above results also imply the following necessary conditions for a non-integral operator to be bounded on $L^2(X)$.

Corollary 6.7 *Let $T : L^2(X) \to L^2(X)$ be a bounded linear operator. If T satisfies assumption* $(\mathrm{OD2})_\gamma$ *with $\gamma > \frac{n}{2m}$, then $T(1) \in BMO_L(X)$.*
Analogously, if T satisfies assumption $(\mathrm{OD1})_\gamma$ *with $\gamma > \frac{n}{2m}$, then $T^*(1) \in BMO_{L^*}(X)$.*

The proof is a simple combination of Lemma 6.1 and Corollary 6.6, observing that for every bounded operator T on $L^2(X)$ the operator families $\{T(I-(I+tL)^{-1})^M\}_{t>0}$ and $\{T^*(I-(I+tL^*)^{-1})^M\}_{t>0}$ are uniformly bounded in $L^2(X)$.
We remark that it is actually enough to assume uniform boundedness of $\{T^*(I-(I+tL^*)^{-1})^M\}_{t>0}$ and $\{T(I-(I+tL)^{-1})^M\}_{t>0}$ on $L^2(X)$, respectively, instead of the boundedness of T on $L^2(X)$, to show that $T(1) \in BMO_{L,M}(X)$ and $T^*(1) \in BMO_{L^*,M}(X)$.

We have shown in Proposition 4.27 and Theorem 4.34 that elements of $BMO_L(X)$ and $BMO_{L^*}(X)$ are intimately connected with Carleson measures. Due to this generalized Fefferman-Stein criterion, one could also formulate our $T(1)$-Theorem in terms of Carleson measures instead of elements from $BMO_L(X)$ and $BMO_{L^*}(X)$. Let us show how in this case a necessary condition for T to be bounded on $L^2(X)$ looks like. This formulation also avoids the discussion how to define $T(1)$ and $T^*(1)$, since one can simply define $\psi(tL)T(1)$ and $\psi(tL^*)T^*(1)$ via duality as elements of $L^2_{\mathrm{loc}}(X)$ for fixed $t > 0$.

Lemma 6.8 *Let T be a bounded linear operator on $L^2(X)$ and $\psi \in \Psi(\Sigma^0_\mu)$. If $\{T^*\psi(tL^*)\}_{t>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$, then*

$$\left|\psi(t^{2m}L)T(1)(x)\right|^2 \, \frac{d\mu(x)dt}{t}$$

is a Carleson measure.
Analogously, if $\{T\psi(tL)\}_{t>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$, then

$$\left|\psi(t^{2m}L^*)T^*(1)(x)\right|^2 \, \frac{d\mu(x)dt}{t}$$

is a Carleson measure.

Proof: We only show the first claim, the other one will then be obvious. Thus, we assume that $\{T^*\psi(tL^*)\}_{t>0}$ satisfies weak off-diagonal estimates and aim to show that there exists a constant $C > 0$ such that

$$\sup_B \left(\frac{1}{V(B)} \int_0^{r_B} \int_B \left|\psi(t^{2m}L)T(1)(x)\right|^2 \, \frac{d\mu(x)dt}{t}\right)^{1/2} \leq C,$$

where the supremum is taken over all balls B in X and r_B denotes the radius of B.
We denote by B an arbitrary ball in X and split the expression on the left into an on- and an off-diagonal part by writing $\mathbb{1}_X = \mathbb{1}_{4B} + \mathbb{1}_{X\setminus 4B}$.
The on-diagonal part is handled with the help of quadratic estimates and the boundedness of T on $L^2(X)$. Due to Remark 3.20 and the doubling property (2.2), there holds

$$\left(\frac{1}{V(B)}\int_0^{r_B}\int_B \left|\psi(t^{2m}L)T(\mathbb{1}_{4B})(x)\right|^2 \frac{d\mu(x)dt}{t}\right)^{1/2} \lesssim V(B)^{-1/2}\left\|T(\mathbb{1}_{4B})\right\|_{L^2(X)} \lesssim 1. \tag{6.15}$$

For the off-diagonal part, on the other hand, we use the weak off-diagonal estimates from the assumption (and do no longer need the boundedness of T on $L^2(X)$). For this purpose, let $g \in L^2(X)$ with $\operatorname{supp} g \subseteq B$ and let $0 < t < r_B$. Splitting $\mathbb{1}_{X\setminus 4B}$ into annuli around B, we obtain

$$\begin{aligned}\left|\langle\psi(t^{2m}L)T(\mathbb{1}_{X\setminus 4B}), g\rangle\right| &= \left|\langle\mathbb{1}_{X\setminus 4B}, T^*\psi(t^{2m}L^*)g\rangle\right| \\ &\leq \sum_{j=2}^{\infty}\left|\langle\mathbb{1}_{S_j(B)}, T^*\psi(t^{2m}L^*)g\rangle\right| \\ &\leq \sum_{j=2}^{\infty} V(2^jB)^{1/2}\left\|T^*\psi(t^{2m}L^*)g\right\|_{L^2(S_j(B))}. \end{aligned}\tag{6.16}$$

Since we have assumed $0 < t < r_B$, Remark 3.9 shows that we get the appropriate weak off-diagonal estimates with an extra factor $\left(\frac{r_B}{t}\right)^n$. Moreover, due to Remark 3.8, we can apply weak off-diagonal estimates over annuli of the form (3.11). Hence, for every $j \geq 2$ and every $t < r_B$ there holds

$$\left\|T^*\psi(t^{2m}L^*)g\right\|_{L^2(S_j(B))} \lesssim 2^{jn/2}\left(\frac{r_B}{t}\right)^n\left(1+\frac{\operatorname{dist}(B,S_j(B))^{2m}}{t^{2m}}\right)^{-\gamma}\|g\|_{L^2(B)}$$

and therefore (6.16) is bounded by a constant times

$$\begin{aligned}&\sum_{j=2}^{\infty} V(2^jB)^{1/2}2^{jn/2}\left(\frac{r_B}{t}\right)^n\left(1+\frac{\operatorname{dist}(B,S_j(B))^{2m}}{t^{2m}}\right)^{-\gamma}\|g\|_{L^2(B)} \\ &\quad\lesssim V(B)^{1/2}\|g\|_{L^2(B)}\sum_{j=2}^{\infty}2^{jn}\left(\frac{r_B}{t}\right)^n\left(\frac{2^jr_B}{t}\right)^{-2m\gamma} \\ &\quad\lesssim V(B)^{1/2}\|g\|_{L^2(B)}\left(\frac{r_B}{t}\right)^{n-2m\gamma},\end{aligned}$$

where we used the doubling property (2.2) in the first and $\gamma > \frac{n}{2m}$ in the second inequality. This shows that

$$\left\|\psi(t^{2m}L)T(\mathbb{1}_{X\setminus 4B})\right\|_{L^2(B)} \lesssim V(B)^{1/2}\left(\frac{r_B}{t}\right)^{n-2m\gamma}.$$

Putting this estimate into the desired Carleson measure estimate, we finally get

$$\begin{aligned}&\left(\frac{1}{V(B)}\int_0^{r_B}\int_B\left|\psi(t^{2m}L)T(\mathbb{1}_{X\setminus 4B})(x)\right|^2\frac{d\mu(x)dt}{t}\right)^{1/2} \\ &\quad\lesssim\left(\int_0^{r_B}\left(\frac{t}{r_B}\right)^{2(2m\gamma-n)}\frac{dt}{t}\right)^{1/2} = \left(\int_0^1 s^{2(2m\gamma-n)}\frac{ds}{s}\right)^{1/2}\lesssim 1.\end{aligned}\tag{6.17}$$

Combining (6.15) and (6.17) shows that $\left|\psi(t^{2m}L)T(1)(x)\right|^2 \frac{d\mu(x)dt}{t}$ is a Carleson measure. $\square$

6.4 Poincaré inequalities

Let us add one last assumption we need for our $T(1)$-Theorem associated to sectorial operators.
For the proof of this $T(1)$-Theorem, we require some kind of Poincaré inequality. Since we have assumed X to be an arbitrary metric space and not only the Euclidean space $\mathbb{R}^n$ or a complete Riemannian manifold, the notion of partial derivatives is meaningless. Instead, we follow the approach of Hajłasz and Koskela in [HK95] and [HK00], who give generalizations of Poincaré inequalities and Sobolev spaces on metric spaces. Our basic tool will be the following definition, which is taken from Chapter 2 of [HK00].

Definition 6.9 *Assume that $u \in L^1_{\text{loc}}(X)$ and a measurable function $g \geq 0$ satisfy the inequality*

$$\frac{1}{V(B)} \int_B |u(x) - \langle u \rangle_B| \, d\mu(x) \leq C_P \, r_B \left(\frac{1}{V(\sigma B)} \int_{\sigma B} g(x)^p \, d\mu(x) \right)^{1/p}, \tag{6.18}$$

on each ball B in X, where r_B is the radius of B and $p > 0$, $\sigma \geq 1$, $C_P > 0$ are fixed constants. We then say that the pair (u, g) satisfies a p-Poincaré inquality.

Remark 6.10 If $u \in \text{Lip}(\mathbb{R}^n)$, $g = |\nabla u|$ and $p \geq 1$, then (6.18) is a corollary of the classical Poincaré inequality

$$\left(\frac{1}{V(B)} \int_B |u(x) - \langle u \rangle_B|^p \, dx \right)^{1/p} \leq C(n,p) \, r_B \left(\frac{1}{V(B)} \int_B |\nabla u(x)|^p \, dx \right)^{1/p}. \tag{6.19}$$

It is therefore natural to consider a pair (u, g) that satisfies a p-Poincaré inequality as a Sobolev function and its gradient. One approach to define Sobolev spaces on metric spaces is based on these p-Poincaré inequalities. We omit the details, since for us it will be enough to work with Definition 6.9. We refer to [HK00] for a survey on the topic, including a comparison of different definitions of Sobolev spaces on metric spaces, e.g. the one introduced by Hajłasz in [Haj96], also called Hajłasz-Sobolev space, and examples of pairs (u, g) on certain metric spaces that satisfy a p-Poincaré inequality.

Let us now formulate the needed assumption.

Assumption Let L satisfy (H1), (H2) and (H3).

(P) Assume that for every $f \in L^2(X)$ there exists a measurable function $g : X \times (0, \infty) \to \mathbb{C}$ such that

(i) for all $t > 0$ there holds $g_t := g(\,.\,, t) \geq 0$, and the pair $(e^{-t^{2m}L}f, g_t)$ satisfies a p-Poincaré inequality of the form (6.18) for some $p < 2$ and with constants $\sigma \geq 1$, $C_P > 0$ independent of t and f;

(ii) for all $t > 0$ there holds $g_t \in L^2(X)$, and there exists a constant $C > 0$ independent of f with

$$\int_0^\infty t^2 \, \|g_t\|^2_{L^2(X)} \, \frac{dt}{t} \leq C \, \|f\|^2_{L^2(X)}.$$

(P*) Assume that (P) holds with L replaced by L^*.

Remark 6.11 The assumptions (P) and (P*) are assumptions on the underlying space (X, d, μ) and the operator L, but not on the operator T itself.
If X is the Euclidean space $\mathbb{R}^n$, then the Poincaré inequality is automatically satisfied for the pairs $(e^{-t^{2m}L}f, |\nabla e^{-t^{2m}L}f|)$ and $(e^{-t^{2m}L^*}f, |\nabla e^{-t^{2m}L^*}f|)$, see e.g. [GT83], (7.45). In this case, (ii) is just the assumption that the Littlewood-Paley-Stein square function is bounded on $L^2(\mathbb{R}^n)$. For elliptic second order operators in divergence form, this can easily be shown with the help of the ellipticity condition, see e.g. [Aus07], Section 6.1.
In general, (ii) is fulfilled whenever the Riesz transforms

$$\begin{aligned} \nabla L^{-1/2m} :&\quad L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \\ \nabla (L^*)^{-1/2m} :&\quad L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \end{aligned}$$

are bounded, since then

$$\int_0^\infty \left\| t\nabla e^{-t^{2m}L}f \right\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} = \int_0^\infty \left\| \nabla L^{-1/2m}(t^{2m}L)^{1/2m} e^{-t^{2m}L} f \right\|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \lesssim \|f\|_{L^2(\mathbb{R}^n)}^2$$

and the analogous estimate for L^* hold due to quadratic estimates, see Remark 3.20.

Let us reformulate the assumptions (P) and (P*) also for another case. Let X be a complete Riemannian manifold, with the Riemannian measure μ on X satisfying the doubling property (2.2), and let ∇ denote the Riemannian gradient. To obtain (i) of (P), it is sufficient to assume that a 2-Poincaré inequality of the form (6.19) holds (with the Lebesgue measure replaced by μ). One can then again choose the pairs $(e^{-t^{2m}L}f, |\nabla e^{-t^{2m}L}f|)$ and $(e^{-t^{2m}L^*}f, |\nabla e^{-t^{2m}L^*}f|)$.
This is due to a certain self-improving property of Poincaré inequalities on Riemannian manifolds, stating that the interval of all p that satisfy a p-Poincaré inequality, is open. We refer to [KZ08] for details.
To get (ii), one can assume, as for the Euclidean space, that the mappings $f \mapsto |\nabla L^{-1/2m}f|$ and $f \mapsto |\nabla (L^*)^{-1/2m} f|$ are bounded on $L^2(X)$.

We conclude the section with the following theorem that describes a consequence of the p-Poincaré inequality (6.18). Essentially, it is [HK00], Theorem 3.2, but in a simplified form, which will be sufficient for the application in our situation. For a more general statement and the connection to Hajłasz-Sobolev spaces, we refer to Chapter 3 of [HK00].

Theorem 6.12 *Assume that the pair (u, g) satisfies a p-Poincaré inequality* (6.18) *for some $p > 0$. Then there exists some constant $C > 0$ such that*

$$|u(x) - u(y)| \leq C d(x, y) \left(\mathcal{M}_p g(x) + \mathcal{M}_p g(y) \right)$$

for almost every $x, y \in X$.

Proof: Let $x, y \in X$ be Lebesgue points of u. Recall that the Lebesgue differentiation theorem, Theorem 2.6, implies that this is true for almost all points.
For every $j \in \mathbb{N}_0$ we set $B_j(x) := B(x, r_j)$ and $B_j(y) := B(y, r_j)$ with $r_j := 2^{-j} d(x, y)$.

By Lebesgue's differentiation theorem there holds $\langle u\rangle_{B_j(x)} \to u(x)$ for $j \to \infty$. Thus, due to the fact that $B_{j+1}(x) \subseteq B_j(x)$ and the doubling property (2.2), we obtain

$$\begin{aligned}\left|u(x) - \langle u\rangle_{B_0(x)}\right| &\leq \sum_{j=0}^{\infty} \left|\langle u\rangle_{B_j(x)} - \langle u\rangle_{B_{j+1}(x)}\right| \\ &\leq \sum_{j=0}^{\infty} \frac{1}{V(B_{j+1})} \int_{B_{j+1}(x)} \left|u(z) - \langle u\rangle_{B_j(x)}\right| \, d\mu(z) \\ &\lesssim \sum_{j=0}^{\infty} \frac{1}{V(B_j)} \int_{B_j(x)} \left|u(z) - \langle u\rangle_{B_j(x)}\right| \, d\mu(z). \end{aligned} \tag{6.20}$$

The assumed p-Poincaré inequality for the pair (u, g) then implies that the above is bounded by a constant times

$$\begin{aligned}&\sum_{j=0}^{\infty} r_j \left(\frac{1}{V(\sigma B_j(x))} \int_{\sigma B_j(x)} g(z)^p \, d\mu(z)\right)^{1/p} \\ &\qquad \lesssim \sum_{j=0}^{\infty} 2^{-j} d(x,y) [\mathcal{M}(g^p)(x)]^{1/p} \lesssim d(x,y)\mathcal{M}_p(x). \end{aligned} \tag{6.21}$$

In analogy to the above, we also obtain

$$\left|u(y) - \langle u\rangle_{B_0(y)}\right| \lesssim d(x,y)\mathcal{M}_p(y). \tag{6.22}$$

It therefore remains to estimate the term

$$\left|\langle u\rangle_{B_0(x)} - \langle u\rangle_{B_0(y)}\right| \leq \left|\langle u\rangle_{B_0(x)} - \langle u\rangle_{2B_0(x)}\right| + \left|\langle u\rangle_{2B_0(x)} - \langle u\rangle_{B_0(y)}\right|. \tag{6.23}$$

By definition of r_0 there holds $B_0(y) \subseteq 2B_0(x)$, and due to the doubling property (2.3) and (2.2) we further get

$$V(2B_0(x)) = V(x, 2d(x,y)) \lesssim \left(1 + \frac{d(x,y)}{2d(x,y)}\right)^D V(y, 2d(x,y)) \lesssim V(B_0(y)).$$

Using that $B_0(x) \subseteq 2B_0(x)$ and $B_0(y) \subseteq 2B_0(x)$, we can therefore estimate (6.23) with the help of the p-Poincaré inequality for the pair (u, g) by

$$\begin{aligned}\left|\langle u\rangle_{B_0(x)} - \langle u\rangle_{B_0(y)}\right| &\lesssim \frac{2}{V(2B_0(x))} \int_{2B_0(x)} \left|u(z) - \langle u\rangle_{2B_0(x)}\right| \, d\mu(z) \\ &\lesssim 2r_0 \left(\frac{1}{V(2B_0(x))} \int_{2B_0(x)} g^p(z) \, d\mu(z)\right)^{1/p} \\ &\lesssim d(x,y)\mathcal{M}_p g(x). \end{aligned} \tag{6.24}$$

Combining (6.21), (6.22) and (6.24) yields the desired conclusion. □

6.5 Main theorem

We are now ready to state our main theorem.

For convenience, let us recall that assumptions (H1), (H2) and (H3) were defined in Section 3.2, assumptions $(\mathrm{OD1})_\gamma$ and $(\mathrm{OD2})_\gamma$ in Section 6.1 and assumptions (P) and (P*) in Section 6.4. Furthermore, (X, d, μ) was assumed to be the space of homogeneous type specified in Section 2.2 and the spaces $BMO_L(X)$ and $BMO_{L^*}(X)$ were defined in Sections 4.4 and 4.6.

Theorem 6.13 *Let L be an operator satisfying the assumptions* (H1), (H2) *and* (H3). *Additionally, let the assumptions* (P) *and* (P*) *be satisfied.*
Let $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\mathrm{loc}}(X)$ be a linear operator with $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\mathrm{loc}}(X)$, which satisfies the assumptions $(\mathrm{OD1})_\gamma$ and $(\mathrm{OD2})_\gamma$ for some $\gamma > \frac{n+D+2}{2m}$ and let $T(1) \in BMO_L(X)$, $T^*(1) \in BMO_{L^*}(X)$.
Then T is bounded in $L^2(X)$, i.e. there exists a constant $C > 0$ such that for all $f \in L^2(X)$ there holds

$$\|Tf\|_{L^2(X)} \leq C \|f\|_{L^2(X)} .$$

Let us sketch the two main ideas of the proof.
First, we decompose the operator T for each $t > 0$, at least formally, in the following way:

$$\begin{aligned} T &= T(I - e^{-t^{2m}L}) + Te^{-t^{2m}L} \\ &= T(I - e^{-t^{2m}L}) + [Te^{-t^{2m}L} - T(1) \cdot A_t e^{-t^{2m}L}] + T(1) \cdot A_t e^{-t^{2m}L}. \end{aligned} \tag{6.25}$$

This can be understood as a splitting of the operator into the "main term" or "principal part" $Te^{-t^{2m}L}$ and the "error term" $T(I-e^{-t^{2m}L})$. The main term is then further decomposed into the term in the squared brackets, which is handled via Poincaré inequalities and the term $T(1) \cdot A_t e^{-t^{2m}L}$, which can be estimated by application of the theory of paraproducts and use of the assumption $T(1) \in BMO_L(X)$.
The idea of such a decomposition is taken from articles of Axelsson, Keith and McIntosh, [AKM06], and Hytönen, McIntosh, Portal, [HMP08]; see e.g. (22) of [AKM06] or p.702, before Lemma 6.5, of [HMP08]. These articles treat perturbed Dirac operators in generalization of the Kato square root problem and are inspired by the proof of the Kato square root problem by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [AHL+02].
The use of paraproduct operators in this context is of course not new, they were already used in the proof of the $T(1)$-Theorem of David and Journé in [DJ84] to reduce the original problem to the boundedness of an operator satisfying $T(1) = T^*(1) = 0$. Even if this is not the case in our setting, the application of paraproducts persists to be very helpful.

Secondly, we approximate T by operators associated to L, namely, we write with the help of the Calderón reproducing formula for $f, g \in L^2(X)$

$$\langle Tf, g\rangle = \int_0^\infty \int_0^\infty \langle \psi_2(t^{2m}L) T \psi_1(s^{2m}L) \tilde{\psi}_1(s^{2m}L) f, \tilde{\psi}_2(t^{2m}L^*) g\rangle \, \frac{dt}{t} \frac{ds}{s}$$

and then estimate this expression with the help of decomposition (6.25). Here, ψ_1 and ψ_2 are the functions from assumptions $(\mathrm{OD1})_\gamma$ and $(\mathrm{OD2})_\gamma$. These approximation operators of the form $\psi(t^{2m}L)$ are associated to L and replace the convolution operators P_t and Q_t used in [DJ84], which are somehow associated to the Laplacian (one could e.g. take P_t as the convolution with the Poisson kernel).

We begin our proof of Theorem 6.13 with the estimate of the term in the squared brackets in (6.25). The idea of the proof of the proposition below is taken from [AKM06], Proposition 5.5, whose key elements are the following. On the one hand, one makes use of a Poincaré inequality. In [AKM06], the authors work with a weighted Poincaré inequality on $\mathbb{R}^n$, we will apply Theorem 6.12 instead. On the other hand, one takes into account the special form of the averaging operator A_t, which enables us to pull the function $e^{-t^{2m}L}f$ into $S_{t^{2m}}(1)$, see the proof below for details.
The proposition will be applied for $S_{t^{2m}} = \psi_2(t^{2m}L)T$ and $S_{t^{2m}} = T\psi_1(t^{2m}L)$ in the proof of Theorem 6.13.

Proposition 6.14 *Assume that* (P) *holds. Let* $\{S_t\}_{t>0}$ *be a family of linear operators on* $L^2(X)$ *that satisfies weak off-diagonal estimates of order* $\gamma > \frac{n+D+2}{2m}$*. Then there exists a constant* $C > 0$ *such that for all* $f \in L^2(X)$ *there holds*

$$\int_0^\infty \left\| S_{t^{2m}} e^{-t^{2m}L} f - S_{t^{2m}}(1) \cdot A_t e^{-t^{2m}L} f \right\|^2_{L^2(X)} \frac{dt}{t} \leq C \left\| f \right\|^2_{L^2(X)}.$$

Proof: Let $f \in L^2(X)$. The assumption (i) of (P) yields for every $t > 0$ the existence of some function $g_t \in L^2(X)$ such that the pair $(e^{-t^{2m}L}f, g_t)$ satisfies a p-Poincaré inequality for some $p < 2$.
If we can show that there exists some $C > 0$, independent of t and f, such that

$$\left\| S_{t^{2m}} e^{-t^{2m}L} f - S_{t^{2m}}(1) \cdot A_t e^{-t^{2m}L} f \right\|^2_{L^2(X)} \leq C t^2 \left\| g_t \right\|^2_{L^2(X)}, \tag{6.26}$$

then the assertion of the lemma is a consequence of assumption (ii) of (P).

Let $t > 0$ be fixed and abbreviate $u := e^{-t^{2m}L}f$. To apply the weak off-diagonal estimates on S_t, we decompose X with the help of Lemma 2.1 into "cubes" of "sidelength" approximately equal to t. That is, with the notation of Lemma 2.1, let $k_0 \in \mathbb{Z}$ be defined by $C_1\delta^{k_0} \leq t < C_1 \delta^{k_0 - 1}$ and write $X = \bigcup_{\alpha \in I_{k_0}} Q_\alpha^{k_0}$, where the equality holds modulo null sets of μ. By Lemma 2.1 we further know that for every $\alpha \in I_{k_0}$ there exists some $z_\alpha^{k_0} \in X$ such that

$$B(z_\alpha^{k_0}, c_1 t) \subseteq Q_\alpha^{k_0} \subseteq B(z_\alpha^{k_0}, t) \tag{6.27}$$

for some $c_1 \in (0,1)$ independent of t and α. Moreover, observe that the averaging operator A_t is, by definition, constant on each "cube" $Q_\alpha^{k_0}$. We therefore get

$$\begin{aligned} &\left\| S_{t^{2m}} u - S_{t^{2m}}(1) \cdot A_t u \right\|^2_{L^2(X)} \\ &\quad = \sum_{\alpha \in I_{k_0}} \left\| S_{t^{2m}} u - S_{t^{2m}}(1) \cdot A_t u \right\|^2_{L^2(Q_\alpha^{k_0})} \\ &\quad = \sum_{\alpha \in I_{k_0}} \left\| S_{t^{2m}} (u - \langle u \rangle_{Q_\alpha^{k_0}}) \right\|^2_{L^2(Q_\alpha^{k_0})} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\alpha\in I_{k_0}} \left(\sum_{\beta\in I_{k_0}} \left\| S_{t^{2m}} \mathbb{1}_{Q_\beta^{k_0}} (u - \langle u \rangle_{Q_\alpha^{k_0}}) \right\|_{L^2(Q_\alpha^{k_0})} \right)^2 \\
&\lesssim \sum_{\alpha\in I_{k_0}} \left(\sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t^{2m}} \right)^{-\gamma} \left\| u - \langle u \rangle_{Q_\alpha^{k_0}} \right\|_{L^2(Q_\beta^{k_0})} \right)^2 . \qquad (6.28)
\end{aligned}
$$

Observe that due to (3.8) there holds

$$
\sup_{\alpha\in I_{k_0}} \sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t^{2m}} \right)^{-\gamma} \lesssim 1, \qquad (6.29)
$$

since $\gamma > \frac{n}{2m}$. The Cauchy-Schwarz inequality then yields that the expression in (6.28) is bounded by

$$
\begin{aligned}
&\sum_{\alpha\in I_{k_0}} \left(\sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t^{2m}} \right)^{-\gamma} \right) \\
&\qquad \times \left(\sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t^{2m}} \right)^{-\gamma} \left\| u - \langle u \rangle_{Q_\alpha^{k_0}} \right\|_{L^2(Q_\beta^{k_0})}^2 \right) \\
&\qquad \lesssim \sum_{\alpha\in I_{k_0}} \sum_{\beta\in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)^{2m}}{t^{2m}} \right)^{-\gamma} \left\| u - \langle u \rangle_{Q_\alpha^{k_0}} \right\|_{L^2(Q_\beta^{k_0})}^2 . \qquad (6.30)
\end{aligned}
$$

The term $\left\| u - \langle u \rangle_{Q_\alpha^{k_0}} \right\|_{L^2(Q_\beta^{k_0})}^2$ is now handled via the assumed p-Poincaré inequality for the pair (u, g_t). Due to the Cauchy-Schwarz inequality and Theorem 6.12 we get

$$
\begin{aligned}
\left\| u - \langle u \rangle_{Q_\alpha^{k_0}} \right\|_{L^2(Q_\beta^{k_0})}^2 &= \int_{Q_\beta^{k_0}} \left| u(x) - \langle u \rangle_{Q_\alpha^{k_0}} \right|^2 \, d\mu(x) \\
&\leq \int_{Q_\beta^{k_0}} \left(\frac{1}{V(Q_\alpha^{k_0})} \int_{Q_\alpha^{k_0}} |u(x) - u(y)| \, d\mu(y) \right)^2 \, d\mu(x) \\
&\leq \frac{1}{V(Q_\alpha^{k_0})} \int_{Q_\beta^{k_0}} \int_{Q_\alpha^{k_0}} |u(x) - u(y)|^2 \, d\mu(y) \, d\mu(x) \\
&\lesssim \frac{1}{V(Q_\alpha^{k_0})} \int_{Q_\beta^{k_0}} \int_{Q_\alpha^{k_0}} d(x,y)^2 [\mathcal{M}_p g_t(x) + \mathcal{M}_p g_t(y)]^2 \, d\mu(y) \, d\mu(x). \qquad (6.31)
\end{aligned}
$$

Note that for $x \in Q_\beta^{k_0}$ and $y \in Q_\alpha^{k_0}$ there holds $d(x,y) \lesssim t\,(1 + \operatorname{dist}(B_\alpha, B_\beta)/t)$ due to (6.27). Moreover, the doubling property (2.3) and (6.27) yield that

$$
\frac{V(Q_\beta^{k_0})}{V(Q_\alpha^{k_0})} \lesssim \left(1 + \frac{\operatorname{dist}(B_\alpha, B_\beta)}{t} \right)^D .
$$

Taking these considerations into account and plugging (6.31) into (6.30), we end up with

$$\begin{aligned}
&\|S_{t^{2m}}u - S_{t^{2m}}(1)\cdot A_t u\|_{L^2(X)}^2\\
&\lesssim \sum_{\alpha\in I_{k_0}}\sum_{\beta\in I_{k_0}}\left(1+\frac{\mathrm{dist}(B_\alpha,B_\beta)^{2m}}{t^{2m}}\right)^{-\gamma}\left\|u-\langle u\rangle_{Q_\alpha^{k_0}}\right\|_{L^2(Q_\beta^{k_0})}^2\\
&\lesssim t^2\sum_{\alpha\in I_{k_0}}\sum_{\beta\in I_{k_0}}\left(1+\frac{\mathrm{dist}(B_\alpha,B_\beta)}{t}\right)^{-2m\gamma+2}\\
&\qquad\times\left[\int_{Q_\beta^{k_0}}[\mathcal{M}_p g_t(x)]^2\,d\mu(x)+\frac{V(Q_\beta^{k_0})}{V(Q_\alpha^{k_0})}\int_{Q_\alpha^{k_0}}[\mathcal{M}_p g_t(y)]^2\,d\mu(y)\right]\\
&\lesssim t^2\sum_{\alpha\in I_{k_0}}\sum_{\beta\in I_{k_0}}\left(1+\frac{\mathrm{dist}(B_\alpha,B_\beta)}{t}\right)^{-2m\gamma+D+2}\\
&\qquad\times\left[\int_{Q_\beta^{k_0}}[\mathcal{M}_p g_t(x)]^2\,d\mu(x)+\int_{Q_\alpha^{k_0}}[\mathcal{M}_p g_t(y)]^2\,d\mu(y)\right]\\
&\lesssim t^2\left[\sum_{\beta\in I_{k_0}}\int_{Q_\beta^{k_0}}[\mathcal{M}_p g_t(x)]^2\,d\mu(x)+\sum_{\alpha\in I_{k_0}}\int_{Q_\alpha^{k_0}}[\mathcal{M}_p g_t(y)]^2\,d\mu(y)\right]\\
&\lesssim t^2\,\|\mathcal{M}_p g_t\|_{L^2(X)}\lesssim t^2\,\|g_t\|_{L^2(X)}^2,
\end{aligned}$$

where we used (6.29) with the assumption $\gamma > \frac{n+D+2}{2m}$, the disjointness of the "cubes" and the boundedness of $\mathcal{M}_p$ on $L^2(X)$ for $p<2$ in the last three inequalities.
This shows (6.26), which again finishes the proof by assumption (ii) of (P). □

The next lemma gives a certain kind of almost orthogonality for operators constructed via H^∞-functional calculus and replaces the Cotlar-Knapp-Stein lemma used in [DJ84]. In particular, it enables us to estimate the "error term" $T(I-e^{-t^{2m}L})$ of the decomposition (6.25).
The first part of the lemma is a corollary of Lemma 3.19, whose idea has its origin in [HMM10], Lemma 4.6. A special case of this is due to Hofmann, Martell, see the proof of [HM03], Lemma 2.2.

Lemma 6.15 *Let $\alpha,\beta>0$ and let $\psi\in\Psi_{\beta,\alpha}(\Sigma_\mu^0)$. There exists a constant $C>0$ such that for every $s,t>0$ and every $f\in L^2(X)$ there holds*

$$\left\|(I-e^{-tL})\psi(sL)f\right\|_{L^2(X)}\le C\left(\frac{t}{s}\right)^{\min(\alpha,1)}\|f\|_{L^2(X)}$$

and

$$\left\|e^{-tL}\psi(sL)f\right\|_{L^2(X)}\le C\left(\frac{s}{t}\right)^{\beta}\|f\|_{L^2(X)}.$$

Proof: Let $s,t>0$. With $\delta:=\min(\alpha,1)$ we write

$$(I-e^{-tL})\psi(sL)=\left(\frac{t}{s}\right)^{\delta}(tL)^{-\delta}(I-e^{-tL})(sL)^\delta\psi(sL).$$

Observe that the function $z \mapsto z^{-\delta}(1-e^{-z})$ is in $H^\infty(\Sigma_\mu^0)$ with

$$\sup_{t>0} \left\| (tz)^{-\delta}(1-e^{-tz}) \right\|_{L^\infty(\Sigma_\mu^0)} = \left\| z^{-\delta}(1-e^{-z}) \right\|_{L^\infty(\Sigma_\mu^0)} < \infty.$$

Also the function $z \mapsto z^\delta \psi(z)$ belongs to $H^\infty(\Sigma_\mu^0)$ and $\sup_{s>0} \left\|(sz)^\delta \psi(sz)\right\|_{L^\infty(\Sigma_\mu^0)} < \infty$. From functional calculus we get the first desired estimate.
To show the second estimate, we write

$$e^{-tL}\psi(sL) = \left(\frac{s}{t}\right)^\beta (sL)^{-\beta}\psi(sL)(tL)^\beta e^{-tL}$$

and argue as before: the functions $z \mapsto z^{-\beta}\psi(z)$ and $z \mapsto z^\beta e^{-z}$ belong to $H^\infty(\Sigma_\mu^0)$ and therefore, $\left\|(sL)^{-\beta}\psi(sL)(tL)^\beta e^{-tL}\right\|_{L^2(X)\to L^2(X)}$ is, via functional calculus, bounded by a constant independent of $s > 0$ and $t > 0$. $\square$

Let us also recall that due to Lemma 3.1 the assumptions $(\mathrm{OD1})_\gamma$ and $(\mathrm{OD2})_\gamma$ immediately imply uniform boundedness of the operators $T\psi_1(t^{2m}L)$ and $T^*\psi_2(t^{2m}L^*)$, respectively, whenever $\gamma > \frac{n}{2m}$.

Corollary 6.16 *Let $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\mathrm{loc}}(X)$ be a linear operator with $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\mathrm{loc}}(X)$.*

(i) If T satisfies $(\mathrm{OD1})_\gamma$ for some $\gamma > \frac{n}{2m}$, then there exists some constant $C > 0$ such that

$$\left\|T\psi_1(t^{2m}L)f\right\|_{L^2(X)} \le C \left\|f\right\|_{L^2(X)}$$

for every $f \in L^2(X)$ and every $t > 0$.

(ii) If T satisfies $(\mathrm{OD2})_\gamma$ for some $\gamma > \frac{n}{2m}$, then there exists some constant $C > 0$ such that

$$\left\|T^*\psi_2(t^{2m}L^*)f\right\|_{L^2(X)} \le C \left\|f\right\|_{L^2(X)}$$

for every $f \in L^2(X)$ and every $t > 0$.

Now, we are ready to prove our main theorem.

Proof (of Theorem 6.13): Let $f, g \in L^2(X)$. Let $\alpha \ge 1$, $\beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$ and let $\psi_1, \psi_2 \in \Psi_{\beta,\alpha}(\Sigma_\mu^0) \setminus \{0\}$ as given in the assumption.
Corresponding to the functions ψ_1, ψ_2, we choose functions $\tilde{\psi}_1, \tilde{\psi}_2 \in \Psi(\Sigma_\mu^0)$ such that there holds $\int_0^\infty \psi_1(t)\tilde{\psi}_1(t)\,\frac{dt}{t} = 1$ and $\int_0^\infty \psi_2(t)\tilde{\psi}_2(t)\,\frac{dt}{t} = 1$ and decompose both f and g with the help of the Calderón reproducing formula. That is, we write

$$\langle Tf, g\rangle = \int_0^\infty \int_0^\infty \langle \psi_2(t^{2m}L)T\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \,\frac{dt}{t}\frac{ds}{s}$$

and show that the right-hand side is bounded by a constant times $\left\|f\right\|_{L^2(X)} \left\|g\right\|_{L^2(X)}$. In this way, T extends to a bounded operator on $L^2(X)$.
For the proof, we split the inner integral into two parts, one over $\{t \in (0,\infty) : 0 < t < s\}$,

called J_1, and the other one over $\{t \in (0,\infty) : s \leq t < \infty\}$, called J_2. We observe that for the second part J_2, Fubini's theorem yields

$$\begin{aligned} J_2 &= \int_0^\infty \int_s^\infty \langle \psi_2(t^{2m}L)T\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s} \\ &= \int_0^\infty \int_0^s \langle \tilde{\psi}_1(t^{2m}L)f, \psi_1(t^{2m}L^*)T^*\psi_2(s^{2m}L^*)\tilde{\psi}_2(s^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s}. \end{aligned}$$

The last line equals J_1 with T replaced by T^*, L by L^* and the roles of $\psi_1, \tilde{\psi}_1$ and $\psi_2, \tilde{\psi}_2$ interchanged. Note that all our assumptions are symmetric with respect to T, T^* and L, L^*. Moreover, instead of the weak off-diagonal estimates for $\{T\psi_1(t^{2m}L)\}_t$, assumed in $(\mathrm{OD1})_\gamma$, we can take into account the analogous estimates for $\{T^*\psi_2(t^{2m}L^*)\}_t$, assumed in $(\mathrm{OD2})_\gamma$. Thus, it will be sufficient only to treat J_1. Once we have proven this part, the estimate for J_2 will follow by duality.
In the following estimate for J_1, we will always assume that $0 < t < s$.
As described in (6.25), we decompose T into the two parts $Te^{-t^{2m}L}$ and $T(I - e^{-t^{2m}L})$ for every $t > 0$, which leads to

$$\begin{aligned} J_1 &= \int_0^\infty \int_0^s \langle \psi_2(t^{2m}L)T\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s} \\ &= \int_0^\infty \int_0^s \langle \psi_2(t^{2m}L)Te^{-t^{2m}L}\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s} \\ &\quad + \int_0^\infty \int_0^s \langle \psi_2(t^{2m}L)T(I - e^{-t^{2m}L})\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s} \\ &=: J_M + J_E. \end{aligned} \tag{6.32}$$

Let us first turn to the estimation of the error term J_E, the main term J_M will be treated below. In Lemma 6.15 there is shown that for every $s, t > 0$ and every $h \in L^2(X)$ there holds

$$\left\| (I - e^{-t^{2m}L})\psi_1(s^{2m}L)h \right\|_{L^2(X)} \lesssim \left(\frac{t^{2m}}{s^{2m}}\right)^{\min(\alpha,1)} \|h\|_{L^2(X)}. \tag{6.33}$$

In addition, due to assumption $(\mathrm{OD2})_\gamma$, with $\gamma > \frac{n}{2m}$, and Corollary 6.16 we have $\left\|\psi_2(t^{2m}L)T\right\|_{L^2(X)\to L^2(X)} \lesssim 1$ uniformly in $t > 0$. The combination of both estimates yields

$$\left\| \psi_2(t^{2m}L)T(I - e^{-t^{2m}L})\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f \right\|_{L^2(X)} \lesssim \left(\frac{t^{2m}}{s^{2m}}\right)^{\min(\alpha,1)} \left\| \tilde{\psi}_1(s^{2m}L)f \right\|_{L^2(X)}. \tag{6.34}$$

We can therefore estimate J_E with the help of the Cauchy-Schwarz inequality by

$$\begin{aligned}
|J_E| &\leq \int_0^\infty \int_0^s \left| \langle \psi_2(t^{2m}L) T (I - e^{-t^{2m}L}) \psi_1(s^{2m}L) \tilde{\psi}_1(s^{2m}L) f, \tilde{\psi}_2(t^{2m}L^*) g \rangle \right| \frac{dt}{t} \frac{ds}{s} \\
&\lesssim \int_0^\infty \int_0^s \left(\frac{t^{2m}}{s^{2m}} \right)^{\min(\alpha,1)} \left\| \tilde{\psi}_1(s^{2m}L) f \right\|_{L^2(X)} \left\| \tilde{\psi}_2(t^{2m}L^*) g \right\|_{L^2(X)} \frac{dt}{t} \frac{ds}{s} \\
&\leq \left(\int_0^\infty \int_0^s \left(\frac{t^{2m}}{s^{2m}} \right)^{\min(\alpha,1)} \left\| \tilde{\psi}_1(s^{2m}L) f \right\|_{L^2(X)}^2 \frac{dt}{t} \frac{ds}{s} \right)^{1/2} \\
&\qquad \times \left(\int_0^\infty \int_t^\infty \left(\frac{t^{2m}}{s^{2m}} \right)^{\min(\alpha,1)} \left\| \tilde{\psi}_2(t^{2m}L^*) g \right\|_{L^2(X)}^2 \frac{ds}{s} \frac{dt}{t} \right)^{1/2}, \qquad (6.35)
\end{aligned}$$

where we also used Fubini's theorem in the last step. By substitution of $u = \frac{t}{s}$, one easily observes that $\int_0^s \left(\frac{t}{s}\right)^\delta \frac{dt}{t} = \delta^{-1}$ for every $\delta > 0$. Since the operator family $\{\tilde{\psi}_1(sL)\}_{s>0}$ satisfies quadratic estimates due to Remark 3.20, the first factor in the last line of (6.35) can therefore be bounded by

$$\left(\int_0^\infty \int_0^s \left(\frac{t^{2m}}{s^{2m}} \right)^{\min(\alpha,1)} \left\| \tilde{\psi}_1(s^{2m}L) f \right\|_{L^2(X)}^2 \frac{dt}{t} \frac{ds}{s} \right)^{1/2} \lesssim \|f\|_{L^2(X)}. \qquad (6.36)$$

Changing the roles of s and t and using that $\int_t^\infty \left(\frac{t}{s}\right)^\delta \frac{ds}{s} = \delta^{-1}$ for every $\delta > 0$, we get the analogous estimate for the second factor in (6.35) and in summary

$$|J_E| \lesssim \|f\|_{L^2(X)} \|g\|_{L^2(X)}. \qquad (6.37)$$

To estimate the main term J_M, we use the extended decomposition in (6.25) of $Te^{-t^{2m}L}$ into the two parts $[Te^{-t^{2m}L} - T(1) \cdot A_t e^{-t^{2m}L}]$ and $T(1) \cdot A_t e^{-t^{2m}L}$. At the same time, we withdraw the decomposition of the function f by the Calderón reproducing formula at scale s. To do so, we do not consider J_M itself, but the same expression, now called J_M^0, with both paths of integration over the whole interval $(0, \infty)$. This leads to

$$\begin{aligned}
J_M^0 &= \int_0^\infty \int_0^\infty \langle \psi_2(t^{2m}L) T e^{-t^{2m}L} \psi_1(s^{2m}L) \tilde{\psi}_1(s^{2m}L) f, \tilde{\psi}_2(t^{2m}L^*) g \rangle \frac{ds}{s} \frac{dt}{t} \\
&= \int_0^\infty \langle \psi_2(t^{2m}L) T e^{-t^{2m}L} f, \tilde{\psi}_2(t^{2m}L^*) g \rangle \frac{dt}{t} \\
&= \int_0^\infty \langle \psi_2(t^{2m}L) T e^{-t^{2m}L} f - \psi_2(t^{2m}L) T(1) \cdot A_t e^{-t^{2m}L} f, \tilde{\psi}_2(t^{2m}L^*) g \rangle \frac{dt}{t} \\
&\qquad + \int_0^\infty \langle \psi_2(t^{2m}L) T(1) \cdot A_t e^{-t^{2m}L} f, \tilde{\psi}_2(t^{2m}L^*) g \rangle \frac{dt}{t} \\
&=: J_M^1 + J_M^2. \qquad (6.38)
\end{aligned}$$

It now becomes clear why we chose the decomposition of T like we did in (6.25). The term J_M^2 is exactly the paraproduct defined in (5.2) in Chapter 5, i.e.

$$J_M^2 = \langle \Pi_{T(1)}(f), g \rangle,$$

with the functions $\psi, \tilde{\psi}$ replaced by $\psi_2, \tilde{\psi}_2$. Recall that we assumed in $(\mathrm{OD2})_\gamma$ that $\psi_2 \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$ for some $\alpha > 0$ and $\beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$, and moreover assumed $T(1)$ to be an element of $BMO_L(X)$. Thus, $\Pi_{T(1)}$ is bounded on $L^2(X)$ due to Theorem 5.2 and we obtain the estimate

$$\left|J_M^2\right| \lesssim \|T(1)\|_{BMO_L(X)} \|f\|_{L^2(X)} \|g\|_{L^2(X)} . \tag{6.39}$$

It remains to find a bound for J_M^1. But the major part of this estimate was already done in Proposition 6.14 by application of the assumed Poincaré inequalities (P). Thus, if we set $S_{t^{2m}} := \psi_2(t^{2m}L)T$ and take into account the assumption $(\mathrm{OD2})_\gamma$ with $\gamma > \frac{n+D+2}{2m}$, then Proposition 6.14, in combination with the Cauchy-Schwarz inequality, yields

$$\begin{aligned} \left|J_M^1\right| &= \left| \int_0^\infty \langle \psi_2(t^{2m}L)Te^{-t^{2m}L}f - \psi_2(t^{2m}L)T(1)\cdot A_t e^{-t^{2m}L}f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \,\frac{dt}{t}\right| \\ &\leq \left(\int_0^\infty \left\|\psi_2(t^{2m}L)Te^{-t^{2m}L}f - \psi_2(t^{2m}L)T(1)\cdot A_t e^{-t^{2m}L}f\right\|_{L^2(X)}^2 \frac{dt}{t}\right)^{1/2} \\ &\qquad \times \left(\int_0^\infty \left\|\tilde{\psi}_2(t^{2m}L^*)g\right\|_{L^2(X)}^2 \frac{dt}{t}\right)^{1/2} \\ &\lesssim \|f\|_{L^2(X)} \|g\|_{L^2(X)} , \end{aligned} \tag{6.40}$$

where we also used quadratic estimates for the operator family $\{\tilde{\psi}_2(tL^*)\}_{t>0}$, due to Remark 3.20, in the last step.
Let us finally observe what we did wrong by considering J_M^0 instead of J_M. The combination of (6.39) and (6.40) provides us with the estimate

$$\left|J_M^0\right| \lesssim \left(\|T(1)\|_{BMO_L(X)} + 1\right) \|f\|_{L^2(X)} \|g\|_{L^2(X)} . \tag{6.41}$$

On the other hand, we have $J_M = J_M^0 - J_R$, where the remainder term J_R is defined by

$$J_R := \int_0^\infty \int_s^\infty \langle \psi_2(t^{2m}L)Te^{-t^{2m}L}\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \,\frac{dt}{t}\frac{ds}{s}.$$

This term can be handled in analogy to the treatment of J_E, replacing the estimate (6.33) by

$$\left\|e^{-t^{2m}L}\psi_1(s^{2m}L)h\right\|_{L^2(X)} \lesssim \left(\frac{s^{2m}}{t^{2m}}\right)^\beta \|h\|_{L^2(X)} ,$$

which again holds uniformly for all $s, t > 0$ and all $h \in L^2(X)$ according to Lemma 6.15. Together with the Cauchy-Schwarz inequality and Corollary 6.16, which states the uniform boundedness of $\left\|\psi_2(t^{2m}L)T\right\|_{L^2(X)\to L^2(X)}$, the above yields

$$\begin{aligned} |J_R| &\leq \int_0^\infty \int_s^\infty \left|\langle \psi_2(t^{2m}L)Te^{-t^{2m}L}\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle\right| \frac{dt}{t}\frac{ds}{s} \\ &\lesssim \int_0^\infty \int_s^\infty \left(\frac{s^{2m}}{t^{2m}}\right)^\beta \left\|\tilde{\psi}_1(s^{2m}L)f\right\|_{L^2(X)} \left\|\tilde{\psi}_2(t^{2m}L^*)g\right\|_{L^2(X)} \frac{dt}{t}\frac{ds}{s} \\ &\leq \left(\int_0^\infty \int_s^\infty \left(\frac{s^{2m}}{t^{2m}}\right)^\beta \left\|\tilde{\psi}_1(s^{2m}L)f\right\|_{L^2(X)}^2 \frac{dt}{t}\frac{ds}{s}\right)^{1/2} \\ &\qquad \times \left(\int_0^\infty \int_0^t \left(\frac{s^{2m}}{t^{2m}}\right)^\beta \left\|\tilde{\psi}_2(t^{2m}L^*)g\right\|_{L^2(X)}^2 \frac{ds}{s}\frac{dt}{t}\right)^{1/2} . \end{aligned} \tag{6.42}$$

If we now handle the last line of (6.42) with the same argument as used in (6.35) and (6.36), we end up with

$$|J_R| \lesssim \|f\|_{L^2(X)} \|g\|_{L^2(X)} . \tag{6.43}$$

By combining (6.37), (6.41) and (6.43), and repeating the same procedure for J_2 and recalling the splitting $\langle Tf, g\rangle = J_1 + J_2 = J_E + J_M^0 - J_R + J_2$, we finally obtain

$$|\langle Tf, g\rangle| \lesssim \Big(\|T(1)\|_{BMO_L(X)} + \|T^*\|_{BMO_{L^*}(X)} + 1\Big) \|f\|_{L^2(X)} \|g\|_{L^2(X)} .$$

This proves the theorem. □

6.6 A second version with weaker assumptions

As mentioned already, the assumptions $(\mathrm{OD1})_\gamma$ and $(\mathrm{OD2})_\gamma$ of Theorem 6.13 are rather strong, since they also contain an "on-diagonal" estimate on the operator families $\{T\psi_1(tL)\}_{t>0}$ and $\{T^*\psi_2(tL^*)\}_{t>0}$. We will give in this section a second version of Theorem 6.13 with weaker assumptions, that only requires (on- and off-diagonal) estimates on the operator families $\{\psi(tL)T\phi(tL)\}_{t>0}$ and $\{\psi(tL^*)T^*\phi(tL^*)\}_{t>0}$. To make the application of paraproducts available, we postulate in addition that the conservation properties $e^{-tL}(1) = 1$ and $e^{-tL^*}(1) = 1$ in $L^2_{\mathrm{loc}}(X)$ are valid.
The following result, Theorem 6.17, is in some sense nearer to the assumptions of the standard $T(1)$-Theorem of David and Journé for Calderón-Zygmund operators, where one only assumes some weak boundedness of T on the diagonal. However, we admit that their assumption is still much weaker than our new ones of Theorem 6.17 are.

In contrast to the assumption of Theorem 6.13, namely that T acts as a linear operator $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\mathrm{loc}}(X)$ with $T^* : \mathcal{D}(L^*) \cap \mathcal{R}(L^*) \to L^2_{\mathrm{loc}}(X)$, we postulate in the theorem below that T is a weakly continuous operator mapping from $L^2(X)$ to $L^2(X)$. This is a stronger assumption, but one thinks of an application to some kind of "truncations" T_ε of T with uniform L^2 bound. See e.g. [Ber10] of Bernicot for an example. This is also, where the basic idea of the construction is taken from. The proof, however, is completely different from [Ber10].

Theorem 6.17 *Let L be an operator satisfying the assumptions* (H1), (H2) *and* (H3). *Additionally, let the assumptions* (P) *and* (P*) *be satisfied.*
Let $\alpha > 0, \beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$ and $\psi, \tilde{\psi} \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$ with $\int_0^\infty \psi(t)\tilde{\psi}(t)\,\frac{dt}{t} = 1$ and define $\phi \in H^\infty(\Sigma^0_\mu)$ via

$$\phi(z) := \int_{\gamma_z} \psi(\zeta)\tilde{\psi}(\zeta)\,\frac{d\zeta}{\zeta}, \qquad z \in \Sigma^0_\mu,$$

where $\gamma_z(t) := te^{i\arg z}$, $t \in (|z|, \infty)$. Assume that the operator family $\{\phi(tL)\}_{t>0}$ satisfies off-diagonal estimates of order $\gamma > \frac{n+D+2}{2m}$ and moreover, assume that there holds

$$\phi(tL)(1) = \phi(tL^*)(1) = 1 \qquad \text{in } L^2_{\mathrm{loc}}(X) \tag{6.44}$$

for every $t > 0$.
Let $T : L^2(X) \to L^2(X)$ be a linear, weakly continuous operator such that $\{\psi(tL)T\phi(tL)\}_{t>0}$ and $\{\psi(tL^*)T^*\phi(tL^*)\}_{t>0}$ satisfy weak off-diagonal estimates of order $\gamma > \frac{n+D+2}{2m}$ and let $T(1) \in BMO_L(X)$ and $T^*(1) \in BMO_{L^*}(X)$.
Then $T : L^2(X) \to L^2(X)$ is bounded with a constant independent of the weak continuity parameters of T.

Note that one can get off-diagonal estimates for $\{\phi(tL)\}_{t>0}$ in the following way. By splitting $\phi(z) = (\phi(z) - e^{-z}) + e^{-z}$ for $z \in \Sigma_\mu^0$, one can on the one hand take into account Davies-Gaffney estimates for the semigroup $\{e^{-tL}\}_{t>0}$. On the other hand, it is clear by definition that $\phi(z) - e^{-z} \to 0$ for $|z| \to 0$ and for $|z| \to \infty$. Proposition 3.18 then yields the existence of off-diagonal estimates for $\{\phi(tL) - e^{-tL}\}_{t>0}$.
With a similar reasoning, one can show that the assumption (6.44) is a consequence of the property $e^{-tL}(1) = e^{-tL^*}(1) = 1$ in $L^2_{\text{loc}}(X)$. This is due to the fact that the latter implies $\psi(tL)(1) = \psi(tL^*)(1) = 0$ in $L^2_{\text{loc}}(X)$ for every $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$ with $\beta > \frac{n}{4m}$ and $\alpha > 0$, see Remark 5.9.

The proof of Theorem 6.17 is almost equal to the one of Theorem 6.13. The only difference is the replacement of the Calderón reproducing formula by the representation formula (6.45), which is a generalization of a construction in [Ber10].

Proof: Let $f, g \in L^2(X)$ and $\psi, \tilde{\psi}, \phi$ given as in the assumptions.
We first observe that by definition of ϕ there holds $\lim_{t\to 0} \phi(t) = 1$ and $\lim_{t\to\infty} \phi(t) = 0$. Since T is weakly continuous, we thus get by functional calculus

$$\begin{aligned} Tf &= \lim_{t\to 0} \phi^2(tL) T \phi^2(tL) f, \\ 0 &= \lim_{t\to\infty} \phi^2(tL) T \phi^2(tL) f, \end{aligned}$$

where the limit is interpreted in the weak sense in $L^2(X)$. Again by functional calculus, we obtain from the above as a special form of a Calderón reproducing formula that $\langle Tf, g\rangle$ can be represented as

$$\langle Tf, g\rangle = \langle \int_0^\infty \left(\left[t\frac{d}{dt}\phi^2(tL)\right] T\phi^2(tL)(f) + \phi^2(tL) T \left[t\frac{d}{dt}\phi^2(tL)\right] f \right) \frac{dt}{t}, g\rangle. \tag{6.45}$$

Once having handled the first summand in (6.45), in the following called J, the second one will work in the same way simply by duality. So let us have a more detailed look at the first part.
By definition of ϕ there holds $z\phi'(z) = \psi(z)\tilde{\psi}(z)$ for $z \in \Sigma_\mu^0$. This yields due to functional calculus,

$$\begin{aligned} &\int_0^\infty \left[t\frac{d}{dt}\phi^2(tL)\right] T\phi^2(tL)(f) \frac{dt}{t} \\ &\qquad = 2\int_0^\infty (tL)\phi'(tL)\phi(tL) T\phi^2(tL)(f) \frac{dt}{t} \\ &\qquad = 2\int_0^\infty \psi(tL)\psi_1(tL) T\phi^2(tL)(f) \frac{dt}{t}, \end{aligned} \tag{6.46}$$

where we set $\psi_1(z) := \tilde{\psi}(z)\phi(z)$. We further decompose f with the help of another Calderón reproducing formula as

$$f = \int_0^\infty \psi(sL)\tilde{\psi}(sL) f \frac{ds}{s}, \tag{6.47}$$

taking into account the assumption $\int_0^\infty \psi(t)\tilde{\psi}(t)\, \frac{dt}{t} = 1$. The combination of the two equations (6.46) and (6.47) then leads to

$$J = 2\int_0^\infty \int_0^\infty \langle \psi(t^{2m}L) T\phi^2(t^{2m}L)\psi(s^{2m}L)\tilde{\psi}(s^{2m}L) f, \psi_1(t^{2m}L^*) g\rangle \frac{dt}{t}\frac{ds}{s}. \tag{6.48}$$

Similar to the proof of Theorem 6.13, we split the inner integral into two parts, one over the interval $\{t \in (0,\infty) : 0 < t < s\}$, called J_1, and the other one over $\{t \in (0,\infty) : s \le t < \infty\}$, called J_2. In contrast to the proof of Theorem 6.13, for lack of symmetry in (6.48) we cannot handle J_2 simply by duality, but it can be dealt with similar to the remainder term J_R in Theorem 6.13.
Thus, let us first turn to J_2. The assumed weak off-diagonal estimates on the operator family $\{\psi(tL)T\phi(tL)\}_{t>0}$ yield due to Lemma 3.1

$$\left\|\psi(t^{2m}L)T\phi(t^{2m}L)\right\|_{L^2(X)\to L^2(X)} \lesssim 1$$

uniformly in $t > 0$. Moreover, observe that by assumption there holds $\frac{\psi(\zeta)\tilde{\psi}(\zeta)}{\zeta} = \mathcal{O}(|\zeta|^{-2\alpha-1})$ for $|\zeta| \to \infty$ and consequently, $\phi(z) = \mathcal{O}(|z|^{-2\alpha})$ for $|z| \to \infty$. Replacing e^{-z} by $\phi(z)$ in Lemma 6.15, it is therefore easy to check that there exists some $\delta > 0$ such that for all $h \in L^2(X)$

$$\left\|\phi(t^{2m}L)\psi(s^{2m}L)h\right\|_{L^2(X)} \lesssim \left(\frac{s^{2m}}{t^{2m}}\right)^{\delta} \|h\|_{L^2(X)},$$

uniformly in $s, t > 0$. With exactly the same arguments as in (6.42), we end up with

$$\begin{aligned}|J_2| &\le \int_0^\infty \int_s^\infty \left|\langle \psi(t^{2m}L)T\phi^2(t^{2m}L)\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)f, \psi_1(t^{2m}L^*)g\rangle\right| \frac{dt}{t}\frac{ds}{s} \\ &\lesssim \|f\|_{L^2(X)} \|g\|_{L^2(X)}.\end{aligned}$$

To handle J_1, we apply for every $t > 0$ the splitting

$$T\phi^2(t^{2m}L) = T\phi^2(t^{2m}L)e^{-t^{2m}L} + T\phi^2(t^{2m}L)(I - e^{-t^{2m}L}),$$

representing the splitting of J_1 into the main term J_M and the error term J_E just as in (6.32), i.e.

$$\begin{aligned}J_1 &= \int_0^\infty \int_0^s \langle \psi(t^{2m}L)T\phi^2(t^{2m}L)e^{-t^{2m}L}\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)f, \psi_1(t^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s} \\ &\quad + \int_0^\infty \int_0^s \langle \psi(t^{2m}L)T\phi^2(t^{2m}L)(I - e^{-t^{2m}L})\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)f, \psi_1(t^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s} \\ &=: J_M + J_E.\end{aligned}$$

The treatment of J_E works analogously to (6.35), using the weak off-diagonal estimates for $\{\psi(tL)T\phi(tL)\}_{t>0}$ instead of assumption $(\mathrm{OD2})_\gamma$ and the uniform boundedness of $\{\phi(tL)\}_{t>0}$ in $L^2(X)$.
To estimate the main term J_M, we also aim to apply a paraproduct estimate and therefore write $J_M = J_M^0 + J_R$ with a remainder J_R that can be handled with the same arguments as in (6.42), and

$$\begin{aligned}J_M^0 &= \int_0^\infty \int_0^\infty \langle \psi(t^{2m}L)T\phi^2(t^{2m}L)e^{-t^{2m}L}\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)f, \psi_1(t^{2m}L^*)g\rangle \frac{dt}{t}\frac{ds}{s} \\ &= \int_0^\infty \langle \psi(t^{2m}L)T\phi^2(t^{2m}L)e^{-t^{2m}L}f - \psi(t^{2m}L)T\phi^2(t^{2m}L)(1)\cdot A_t e^{-t^{2m}L}f, \psi_1(t^{2m}L^*)g\rangle \frac{dt}{t} \\ &\quad + \int_0^\infty \langle \psi(t^{2m}L)T\phi^2(t^{2m}L)(1)\cdot A_t e^{-t^{2m}L}f, \psi_1(t^{2m}L^*)g\rangle \frac{dt}{t} \\ &=: J_M^1 + J_M^2,\end{aligned}$$

in analogy to (6.38).
Observe that the operator family $\{\psi(tL)T\phi^2(tL)\}_{t>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n+D+2}{2m}$ due to the assumptions and Proposition 3.7. By taking assumption (P) into account, we can thus apply Proposition 6.14 with $S_t := \psi(tL)T\phi^2(tL)$, which yields the desired estimate for J_M^1 just as in (6.40).
We finally note that assumption (6.44) yields

$$J_M^2 = \int_0^\infty \langle \psi(t^{2m}L)T(1) \cdot A_t e^{-t^{2m}L} f, \psi_1(t^{2m}L^*)g\rangle \, \frac{dt}{t} = \langle \Pi_{T(1)} f, g\rangle,$$

and J_M^2 can therefore be treated by Theorem 5.2 and the assumption $T(1) \in BMO_L(X)$. This finishes the proof. □

6.7 Application to paraproducts

In this section, we will present an application of Theorem 6.17 to a different type of paraproduct operator.

We will do this under more restrictive assumptions on L. Let again L be an operator satisfying (H1), (H2) and (H3). Additionally, let us assume that the following is valid.

(H4) The operator $e^{-tL} : L^\infty(X) \to L^\infty(X)$ is bounded uniformly in $t > 0$.

(H5) For every $t > 0$ there holds $e^{-tL}(1) = 1$ in $L^\infty(X)$ and $e^{-tL^*}(1) = 1$ in $L^2_{\text{loc}}(X)$.

Let us remark that we do not assume $e^{-tL^*} : L^\infty(X) \to L^\infty(X)$ to be bounded.
The assumption (H5) in particular implies that there holds $\psi(tL^*)(1) = 0$ in $L^2_{\text{loc}}(X)$ for every $t > 0$ and every $\psi \in \Psi_{\beta,\alpha}(\Sigma_\mu^0)$, where $\beta > \frac{n}{4m}$ and $\alpha > 0$, see Remark 5.9.

Definition 6.18 *Let $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$. Assume that $\psi_1 \in \Psi_{\beta_1,\alpha_1}(\Sigma_\mu^0) \setminus \{0\}$ and $\psi_2 \in \Psi_{\beta_2,\alpha_2}(\Sigma_\mu^0) \setminus \{0\}$ and abbreviate $\tilde\psi := \psi_1 \cdot \psi_2$. For every $f \in L^\infty(X)$ and every $g \in L^2(X)$ we define the paraproduct*

$$\tilde\Pi_f(g) := \int_0^\infty \tilde\psi(t^{2m}L)[e^{-t^{2m}L} g \cdot e^{-t^{2m}L} f] \, \frac{dt}{t}. \tag{6.49}$$

We refer the reader to compare the operator $\tilde\Pi_f$ with the paraproduct operator $\Pi(f, .)$ defined in (5.12). The boundedness of $\Pi(f, .)$ on $L^2(X)$ is an immediate consequence of quadratic estimates for the operator families $\{\tilde\psi(tL)\}_{t>0}$ and $\{\psi(tL)\}_{t>0}$, see Lemma 5.11. In contrast to that, the boundedness of $\tilde\Pi_f$ is not obvious. To give a sufficient criterion for $\tilde\Pi_f$ to be bounded on $L^2(X)$, we apply Theorem 6.17 to approximations of the newly defined paraproduct. We then obtain the following result.

Theorem 6.19 *Let L satisfy* (H1)-(H5) *and let the assumptions* (P) *and* (P*) *be valid. For every $f \in L^\infty(X)$ let $\tilde\Pi_f$ be the operator defined in* (6.49) *with* $\min(\alpha_1, \beta_1, \alpha_2, \beta_2) > \max(\frac{n}{4m} + [\frac{n}{4m}] + 1, \frac{n+D+2}{2m})$. *Then there exists some constant $C > 0$ such that for every $f \in L^\infty(X)$ and every $g \in L^2(X)$ there holds*

$$\left\| \tilde\Pi_f(g) \right\|_{L^2(X)} \le C \, \|f\|_{L^\infty(X)} \, \|g\|_{L^2(X)} \, .$$

For the proof of Theorem 6.19, let us first define suitable approximations of the paraproduct operator.
Let $f \in L^\infty(X)$ be fixed. We define for every $R > 0$ the operator $T_R : L^2(X) \to L^2(X)$ by

$$T_R(g) := \int_{1/R}^{R} \psi_1(t^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(t^{2m}L)[e^{-t^{2m}L}g \cdot e^{-t^{2m}L}f]\,\frac{dt}{t} \tag{6.50}$$

for every $g \in L^2(X)$.

Remark 6.20 A careful inspection of the proof below shows that it is possible to replace $e^{-t^{2m}L}f$ in the definition of the paraproduct $\tilde{\Pi}_f$ in (6.49) by $S_t f$ for some different operator family $\{S_t\}_{t>0}$. One can then again obtain $L^2(X)$-boundedness of the corresponding paraproduct if one makes the following assumptions.
To assure the validity of the off-diagonal estimates in Lemma 6.21 below, one has to assume that $S_t : L^\infty(X) \to L^\infty(X)$ is bounded uniformly in $t > 0$, whereas the assumption (H4) is no longer needed.
Moreover, observe that one does not only have off-diagonal estimates for $\{\psi(tL)T_R\phi(tL)\}_{t>0}$, but even for $\{\psi(tL)T_R\}_{t>0}$ itself, see again Lemma 6.21 below. One can therefore apply a variant of Theorem 6.17, such that only the assumption $e^{-tL^*}(1) = 1$ and no longer the assumption $e^{-tL}(1) = 1$ is required.
Finally, one has to check that (6.51) is satisfied. The second condition in (6.51), i.e. the uniform boundedness of $T_R^*(1)$ in $BMO_{L^*}(X)$ is true due to the uniform boundedness of the operator family $\{S_t\}_{t>0}$ in $L^\infty(X)$ instead of (H4). To show the first condition, i.e. the uniform boundedness of $T_R(1)$ in $BMO_L(X)$, in the original proof one uses the assumptions (H4) and $e^{-tL}(1) = 1$. If one replaces $e^{-t^{2m}L}$ by S_t, one does not need any longer those two assumptions, but has to *suppose* in addition that $T_R(1) \in BMO_L(X)$ uniformly in $R > 0$.
In summary, one can omit the assumptions (H4) and $e^{tL}(1) = 1$ and replace $e^{-t^{2m}L}f$ by $S_t f$, whenver one can assure that $\{S_t\}_{t>0}$ is uniformly bounded in $L^\infty(X)$ and $T_R(1) \in BMO_L(X)$ uniformly in $R > 0$.
In comparison to Theorem 4.5 of [Ber10], the above result, Theorem 6.19, is thus applicable to a larger class of operators L than it is considered in [Ber10].

For convenience, let us set $\delta := \min(\alpha_1, \beta_1, \alpha_2, \beta_2)$. Let $\psi \in \Psi_{\delta,\delta}(\Sigma_\mu^0)$ and choose $\phi \in H^\infty(\Sigma_\mu^0)$ according to the assumptions of Theorem 6.17, such that $\{\phi(tL)\}_{t>0}$ satisfies off-diagonal estimates of order δ. Then the following off-diagonal estimates are valid.

Lemma 6.21 *Let $f \in L^\infty(X)$ and let $R > 0$. The operator families $\{\psi(tL)T_R\}_{t>0}$ and $\{\phi(tL)T_R\psi(tL)\}_{t>0}$ satisfy off-diagonal estimates of order γ for every $0 < \gamma < \delta$. More precisely, there exists some constant $C > 0$, independent of $R > 0$, such that for arbitrary open sets E, F in X, all $g \in L^2(X)$ with $\operatorname{supp} g \subseteq E$ and all $f \in L^\infty(X)$*

$$\|\psi(tL)T_R g\|_{L^2(F)} \leq C \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)}$$

and

$$\|\phi(tL)T_R\psi(tL)g\|_{L^2(F)} \leq C \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)}\,.$$

We postpone the proof of the lemma to the end of the section and first turn to the proof of Theorem 6.19.

Proof (of Theorem 6.19): Let $f \in L^\infty(X)$. We apply Theorem 6.17 to the approximation operators T_R defined in (6.50). First observe that due to the uniform boundedness of the operator families $\{\psi_1(tL)\}_{t>0}$, $\{\psi_2(tL)\}_{t>0}$, $\{e^{-tL}\}_{t>0}$ in $L^2(X)$ and of $\{e^{-tL}\}_{t>0}$ in $L^\infty(X)$, every operator T_R is bounded in $L^2(X)$ with the operator norm bounded by some constant depending on $R > 0$.
Using Lemma 3.4, we obtain from Lemma 6.21 the required off-diagonal estimates for the operator families $\{\psi(tL)T_R\phi(tL)\}_{t>0}$ and $\{\psi(tL^*)T_R^*\phi(tL^*)\}_{t>0}$ with constants independent of $R > 0$.
It remains to check that

$$\sup_{R>0} \|T_R(1)\|_{BMO_L(X)} < \infty \qquad \text{and} \qquad \sup_{R>0} \|T_R^*(1)\|_{BMO_{L^*}(X)} < \infty. \tag{6.51}$$

Starting with the first assertion, let us define for every $h \in H^1_{L^*}(X)$ a function H by

$$H(x,t) := \psi_1(t^{2m}L^*)h(x), \qquad (x,t) \in X \times (0,\infty).$$

Since $\psi_1 \in \Psi_{\beta_1,\alpha_1}(\Sigma^0_\mu)$ with $\alpha_1 > \frac{n}{4m}$, Theorem 4.7 yields that $H \in T^1(X)$ with $\|H\|_{T^1(X)} \approx \|h\|_{H^1_{L^*}(X)}$.
Using that $\psi_2 \in \Psi_{\beta_2,\alpha_2}(\Sigma^0_\mu)$ with $\beta_2 > \frac{n}{4m} + [\frac{n}{4m}] + 1$, there holds on the other hand that the function F, defined by

$$F(x,t) := \psi_2(t^{2m}L)e^{-t^{2m}L}f(x), \qquad (x,t) \in X \times (0,\infty),$$

is according to Proposition 4.27 an element of $T^\infty(X)$ with $\|F\|_{T^\infty(X)} \lesssim \|f\|_{BMO_L(X)}$.
Due to the assumption $e^{-tL}(1) = 1$ and Proposition 4.26, there actually holds $L^\infty(X) \subseteq BMO(X) \subseteq BMO_L(X)$ and therefore $\|F\|_{T^\infty(X)} \lesssim \|f\|_{L^\infty(X)}$.
Again taking into account the assumption $e^{-tL}(1) = 1$ in $L^\infty(X)$, we thus obtain

$$\begin{aligned}
\langle T_R(1), h\rangle &= \int_{1/R}^R \langle \mathbb{1}_{B(0,R)}\psi_2(t^{2m}L)[e^{-t^{2m}L}1 \cdot e^{-t^{2m}L}f], \psi_1(t^{2m}L^*)h\rangle \,\frac{dt}{t} \\
&= \int_{1/R}^R \langle \mathbb{1}_{B(0,R)}\psi_2(t^{2m}L)e^{-t^{2m}L}f, \psi_1(t^{2m}L^*)h\rangle \,\frac{dt}{t} \\
&= \int_{1/R}^R \langle \mathbb{1}_{B(0,R)}F(x,t), H(x,t)\rangle \,\frac{dt}{t}.
\end{aligned}$$

The duality of tent spaces, described in Theorem 2.17, then yields that

$$|\langle T_R(1), h\rangle| \lesssim \|F\|_{T^\infty(X)} \|H\|_{T^1(X)} \lesssim \|f\|_{L^\infty(X)} \|h\|_{H^1_{L^*}(X)},$$

where the implicit constants are independent of $R > 0$. Due to the duality of $H^1_{L^*}(X)$ and $BMO_L(X)$, see Theorem 4.28, we finally obtain that $T_R(1) \in BMO_L(X)$ with

$$\sup_{R>0} \|T_R(1)\|_{BMO_L(X)} \lesssim \|f\|_{L^\infty(X)}.$$

Coming to the second assertion in (6.51), observe that the adjoint operator of T_R is given by

$$T_R^*(g) := \int_{1/R}^{R} e^{-t^{2m}L^*}[\psi_2(t^{2m}L^*)\mathbb{1}_{B(0,r)}\psi_1(t^{2m}L^*)g \cdot \overline{e^{-t^{2m}L}f}]\, \frac{dt}{t}$$

for every $g \in L^2(X)$.
As already mentioned, there holds $\psi_1(tL^*)(1) = 0$ in $L^2_{\text{loc}}(X)$ due to the assumption $e^{-tL^*}(1) = 1$ in $L^2_{\text{loc}}(X)$. Thus,

$$T_R^*(1) = \int_{1/R}^{R} e^{-t^{2m}L^*}[\psi_2(t^{2m}L^*)\mathbb{1}_{B(0,r)}\psi_1(t^{2m}L^*)(1) \cdot \overline{e^{-t^{2m}L}f}]\, \frac{dt}{t} = 0$$

in $L^2(X)$ and therefore also in $BMO_{L^*}(X)$. □

Let us now prove that the approximation operators T_R of the paraproduct $\tilde{\Pi}_f$ satisfy the required off-diagonal estimates.

Proof (of Lemma 6.21): Let E, F be two arbitrary open sets in X and let $f \in L^\infty(X)$ and $g \in L^2(X)$ with $\operatorname{supp} g \subseteq E$. We begin with the estimate

$$\|\psi(tL)T_R g\|_{L^2(F)} \leq C\left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)}.$$

Let $\delta = \min(\alpha_1, \beta_1, \alpha_2, \beta_2)$ as defined before and fix some $\gamma > 0$ with $\gamma < \delta$. Then for every $s, t > 0$ there holds

$$\|\psi(tL)\psi_1(sL)\|_{L^2(X)\to L^2(X)} \lesssim \min\left(\frac{s}{t}, \frac{t}{s}\right)^{\delta}, \tag{6.52}$$

using the same arguments as e.g. in Remark 6.27. Hence, due to Minkowski's inequality and the uniform boundedness of the operator families $\{\psi_2(sL)\}_{s>0}$, $\{e^{-sL}\}_{s>0}$ in $L^2(X)$ and $\{e^{-sL}\}_{s>0}$ in $L^\infty(X)$ we obtain

$$\begin{aligned}
&\left\|\psi(t^{2m}L)T_R(g)\right\|_{L^2(X)} \\
&\quad \leq \int_0^\infty \left\|\psi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)[e^{-s^{2m}L}g \cdot e^{-s^{2m}L}f]\right\|_{L^2(X)} \frac{ds}{s} \\
&\quad \lesssim \int_0^\infty \min\left(\frac{s}{t}, \frac{t}{s}\right)^{2m\delta} \frac{ds}{s}\, \|g\|_{L^2(X)} \|f\|_{L^\infty(X)} \lesssim \|g\|_{L^2(X)} \|f\|_{L^\infty(X)}.
\end{aligned}$$

If $\operatorname{dist}(E,F) \leq t$, the above estimate yields the desired conclusion. Otherwise, let $\rho := \operatorname{dist}(E,F) > t$, and define $G_1 := \{x \in X : \operatorname{dist}(x,F) < \frac{\rho}{2}\}$ and $G_2 := \{x \in X : \operatorname{dist}(x,F) < \frac{\rho}{4}\}$. Then there holds that G_1, G_2 are open with $\operatorname{dist}(E, G_1) \geq \frac{\rho}{2}$ and $\operatorname{dist}(F, X \setminus \bar{G}_2) \geq \frac{\rho}{4}$. We split X into $X = \bar{G}_2 \cup X \setminus \bar{G}_2$ and obtain

$$\begin{aligned}
&\left\|\psi(t^{2m}L)T_R(g)\right\|_{L^2(F)} \\
&\quad \leq \int_0^\infty \left\|\psi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{\bar{G}_2}[e^{-s^{2m}L}g \cdot e^{-s^{2m}L}f]\right\|_{L^2(F)} \frac{ds}{s} \\
&\qquad + \int_0^\infty \left\|\psi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{X\setminus\bar{G}_2}[e^{-s^{2m}L}g \cdot e^{-s^{2m}L}f]\right\|_{L^2(F)} \frac{ds}{s} \\
&\quad =: J_{\bar{G}_2} + J_{X\setminus\bar{G}_2}.
\end{aligned}$$

The estimate (6.52), the uniform boundedness of $\{\psi_2(sL)\}_{s>0}$ in $L^2(X)$ and $\bar{G}_2 \subseteq G_1$ yield

$$\begin{aligned} J_{\bar{G}_2} &\lesssim \int_0^\infty \min\left(\frac{s}{t}, \frac{t}{s}\right)^{2m\delta} \left\| \psi_2(s^{2m}L)\mathbb{1}_{\bar{G}_2}[e^{-s^{2m}L}g \cdot e^{-s^{2m}L}f] \right\|_{L^2(X)} \frac{ds}{s} \\ &\lesssim \int_0^\infty \min\left(\frac{s}{t}, \frac{t}{s}\right)^{2m\delta} \left\| e^{-s^{2m}L}g \cdot e^{-s^{2m}L}f \right\|_{L^2(G_1)} \frac{ds}{s}. \end{aligned}$$

Since on the one hand, $\{e^{-sL}\}_{s>0}$ satisfies Davies-Gaffney estimates and is on the other hand uniformly bounded in $L^\infty(X)$, we can estimate the above by a constant times $\|f\|_{L^\infty(X)} \|g\|_{L^2(E)}$ times

$$\begin{aligned} &\int_0^\infty \min\left(\frac{s}{t}, \frac{t}{s}\right)^{2m\delta} \left(1 + \frac{\operatorname{dist}(E, G_1)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s} \\ &\lesssim \int_0^t \left(\frac{s}{t}\right)^{2m\delta} \left(1 + \frac{\operatorname{dist}(E, F)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s} + \int_t^\infty \left(\frac{t}{s}\right)^{2m\delta} \left(1 + \frac{\operatorname{dist}(E, F)^{2m}}{s^{2m}}\right)^{-\gamma} \frac{ds}{s} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(E, F)^{2m}}{t^{2m}}\right)^{-\gamma} \left[\int_0^t \left(\frac{s}{t}\right)^{2m\delta} \frac{ds}{s} + \int_t^\infty \left(\frac{t}{s}\right)^{2m\delta} \left(\frac{t}{s}\right)^{-2m\gamma} \frac{ds}{s}\right] \\ &\lesssim \left(1 + \frac{\operatorname{dist}(E, F)^{2m}}{t^{2m}}\right)^{-\gamma}, \end{aligned}$$

since $\gamma < \delta$. Hence, there holds

$$J_{\bar{G}_2} \lesssim \left(1 + \frac{\operatorname{dist}(E, F)^{2m}}{t^{2m}}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)}. \tag{6.53}$$

For the analogous estimate of $J_{X \setminus \bar{G}_2}$, we instead use the off-diagonal estimates of the operator family $\{\psi(tL)\psi_1(sL)\}_{s,t>0}$. We split $J_{X \setminus \bar{G}_2}$ into two parts $J^1_{X \setminus \bar{G}_2}$ and $J^2_{X \setminus \bar{G}_2}$, representing the integration over $(0,t)$ and (t,∞), respectively. Considering $J^1_{X \setminus \bar{G}_2}$, we take into account that Lemma 3.19 yields off-diagonal estimates in t of order γ for the operator family $\{\psi(tL)\psi_1(sL)\}_{s,t>0}$ with an extra term $\left(\frac{s}{t}\right)^\gamma$. In addition, $\{\psi_2(sL)\}_{s>0}$ satisfies off-diagonal estimates in s of order γ due to Proposition 3.18. Lemma 3.4 then yields that

$$\begin{aligned} J^1_{X \setminus \bar{G}_2} &\leq \int_0^t \left\| \psi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{X \setminus \bar{G}_2}[e^{-s^{2m}L}g \cdot e^{-s^{2m}L}f] \right\|_{L^2(F)} \frac{ds}{s} \\ &\lesssim \int_0^t \left(\frac{s}{t}\right)^{2m\gamma} \left(1 + \frac{\operatorname{dist}(F, X \setminus \bar{G}_2)^{2m}}{t^{2m}}\right)^{-\gamma} \left\| e^{-s^{2m}L}g \cdot e^{-s^{2m}L}f \right\|_{L^2(X)} \frac{ds}{s} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(E, F)^{2m}}{t^{2m}}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)}. \end{aligned} \tag{6.54}$$

For the part $J^2_{X \setminus \bar{G}_2}$, we in turn use that $\{\psi(tL)\psi_1(sL)\}_{s,t>0}$ satisfies off-diagonal estimates in s of order γ_1 with an extra term $\left(\frac{t}{s}\right)^{\gamma_1}$, where $\gamma < \gamma_1 < \delta$. With similar arguments as in (6.54), we then obtain

$$\begin{aligned}
J^2_{X\setminus\bar G_2} &\le \int_t^\infty \left\| \psi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{X\setminus\bar G_2}[e^{-s^{2m}L}g\cdot e^{-s^{2m}L}f]\right\|_{L^2(F)}\frac{ds}{s}\\
&\lesssim \int_t^\infty \left(\frac{t}{s}\right)^{2m\gamma_1}\left(1+\frac{\operatorname{dist}(F,X\setminus\bar G_2)^{2m}}{s^{2m}}\right)^{-\gamma_1}\left\|e^{-s^{2m}L}g\cdot e^{-s^{2m}L}f\right\|_{L^2(X)}\frac{ds}{s}\\
&\lesssim \left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\gamma}\|f\|_{L^\infty(X)}\|g\|_{L^2(E)}\int_t^\infty\left(\frac{t}{s}\right)^{2m\gamma_1}\left(\frac{t}{s}\right)^{-2m\gamma}\frac{ds}{s}\\
&\lesssim \left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\gamma}\|f\|_{L^\infty(X)}\|g\|_{L^2(E)}\,. \qquad (6.55)
\end{aligned}$$

Hence, from the combination of (6.54) and (6.55) we get

$$J_{X\setminus\bar G_2}\lesssim\left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\gamma}\|f\|_{L^\infty(X)}\|g\|_{L^2(E)}\,, \qquad (6.56)$$

and the combination of (6.53) and (6.56) finally yields the desired conclusion. Observe that all implicit constants in the inequalities are independent of $R>0$.

We continue with the estimation of

$$\|\phi(tL)T_R\psi(tL)g\|_{L^2(F)}\le C\left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t}\right)^{-\gamma}\|f\|_{L^\infty(X)}\|g\|_{L^2(E)}\,.$$

By definition of ϕ there holds $|\phi(z)|=\mathcal{O}(|z|^\delta)$ for $|z|\to\infty$. Hence, using similar arguments as in Lemma 6.15, there holds

$$\|\phi(tL)\psi_1(sL)\|_{L^2(X)\to L^2(X)}\lesssim\left(\frac{s}{t}\right)^\delta \qquad (6.57)$$

and

$$\left\|e^{-sL}\psi(tL)\right\|_{L^2(X)\to L^2(X)}\lesssim\left(\frac{t}{s}\right)^\delta. \qquad (6.58)$$

We therefore obtain, again using the uniform boundedness of the occuring operator families in $L^2(X)$ and $L^\infty(X)$, respectively,

$$\begin{aligned}
&\left\|\phi(t^{2m}L)T_R\psi(t^{2m}L)g\right\|_{L^2(X)}\\
&\le\int_0^\infty\left\|\phi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)[e^{-s^{2m}L}\psi(t^{2m}L)g\cdot e^{-s^{2m}L}f]\right\|_{L^2(X)}\frac{ds}{s}\\
&\lesssim\int_0^t\left(\frac{s}{t}\right)^{2m\delta}\left\|\psi_2(s^{2m}L)[e^{-s^{2m}L}\psi(t^{2m}L)g\cdot e^{-s^{2m}L}f]\right\|_{L^2(X)}\frac{ds}{s}\\
&\quad+\int_t^\infty\left\|e^{-s^{2m}L}\psi(t^{2m}L)g\cdot e^{-s^{2m}L}f\right\|_{L^2(X)}\frac{ds}{s}\\
&\lesssim\|f\|_{L^\infty(X)}\|g\|_{L^2(X)}\int_0^\infty\min\left(\frac{s}{t},\frac{t}{s}\right)^{2m\delta}\frac{ds}{s}\lesssim\|f\|_{L^\infty(X)}\|g\|_{L^2(X)}\,.
\end{aligned}$$

If $\operatorname{dist}(E,F)\le t$, the above estimate yields the desired conclusion. Otherwise, with the notation as before, we split X into $X=\bar G_2\cup X\setminus\bar G_2$. Let us moreover split the integrals into two parts over $(0,t)$ and (t,∞). Taking into account the fact that $\{e^{-sL}\psi(tL)\}_{s,t>0}$ satisfies off-diagonal estimates in t of order γ and using $\bar{G}_2 \subseteq G_1$ and (6.57), we then obtain

$$\begin{aligned}
J^1_{\bar{G}_2} &\leq \int_0^t \left\| \phi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{\bar{G}_2}[e^{-s^{2m}L}\psi(t^{2m}L)g \cdot e^{-s^{2m}L}f] \right\|_{L^2(X)} \frac{ds}{s} \\
&\lesssim \int_0^t \left(\frac{s}{t}\right)^{2m\gamma} \left\| e^{-s^{2m}L}\psi(t^{2m}L)g \cdot e^{-s^{2m}L}f \right\|_{L^2(G_1)} \frac{ds}{s} \\
&\lesssim \left(1 + \frac{\operatorname{dist}(E,G_1)^{2m}}{t^{2m}}\right)^{-\gamma} \int_0^t \left(\frac{s}{t}\right)^{2m\gamma} \frac{ds}{s} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)} \\
&\lesssim \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)} \,.
\end{aligned} \tag{6.59}$$

Moreover, for $\varepsilon = \delta - \gamma > 0$, the operator family $\{e^{-sL}\psi(tL)\}_{s,t>0}$ satisfies off-diagonal estimates in t of order γ with an extra factor $\left(\frac{t}{s}\right)^{\varepsilon}$, using that

$$e^{-sL}\psi(tL) = \left(\frac{t}{s}\right)^{\varepsilon} (sL)^{\varepsilon} e^{-sL} (tL)^{-\varepsilon}\psi(tL)$$

together with Proposition 3.18. Thus,

$$\begin{aligned}
J^2_{\bar{G}_2} &\leq \int_t^\infty \left\| \phi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{\bar{G}_2}[e^{-s^{2m}L}\psi(t^{2m}L)g \cdot e^{-s^{2m}L}f] \right\|_{L^2(X)} \frac{ds}{s} \\
&\lesssim \int_t^\infty \left\| e^{-s^{2m}L}\psi(t^{2m}L)g \cdot e^{-s^{2m}L}f \right\|_{L^2(G_1)} \frac{ds}{s} \\
&\lesssim \left(1 + \frac{\operatorname{dist}(E,G_1)^{2m}}{t^{2m}}\right)^{-\gamma} \int_t^\infty \left(\frac{t}{s}\right)^{2m\varepsilon} \frac{ds}{s} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)} \\
&\lesssim \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)} \,.
\end{aligned} \tag{6.60}$$

Let us turn to the calculation of $J_{X\setminus\bar{G}_2}$. We now use that for $\varepsilon = \delta - \gamma > 0$

$$\phi(tL)\psi_1(sL) = \left(\frac{s}{t}\right)^{\varepsilon} (tL)^{\varepsilon}\phi(tL)(sL)^{-\varepsilon}\psi_1(sL),$$

therefore $\{\phi(tL)\psi_1(sL)\}_{s,t>0}$ satisfies off-diagonal estimates in t of order γ with an extra factor $\left(\frac{s}{t}\right)^{\varepsilon}$. Hence,

$$\begin{aligned}
J^1_{X\setminus\bar{G}_2} &\leq \int_0^t \left\| \phi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{X\setminus\bar{G}_2}[e^{-s^{2m}L}\psi(t^{2m}L)g \cdot e^{-s^{2m}L}f] \right\|_{L^2(X)} \frac{ds}{s} \\
&\lesssim \left(1 + \frac{\operatorname{dist}(F, X\setminus\bar{G}_2)^{2m}}{t^{2m}}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)} \int_0^t \left(\frac{s}{t}\right)^{2m\varepsilon} \frac{ds}{s} \\
&\lesssim \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\gamma} \|f\|_{L^\infty(X)} \|g\|_{L^2(E)} \,.
\end{aligned} \tag{6.61}$$

For the remaining part, we apply (6.58) and off-diagonal estimates of $\{\phi(tL)\psi_1(sL)\}_{s,t>0}$

in s of order γ, which yields

$$\begin{aligned}
J^2_{X\setminus\bar{G}_2} &\le \int_t^\infty \left\| \phi(t^{2m}L)\psi_1(s^{2m}L)\mathbb{1}_{B(0,R)}\psi_2(s^{2m}L)\mathbb{1}_{X\setminus\bar{G}_2}[e^{-s^{2m}L}\psi(t^{2m}L)g\cdot e^{-s^{2m}L}f]\right\|_{L^2(X)} \frac{ds}{s} \\
&\lesssim \int_t^\infty \left(1+\frac{\operatorname{dist}(F,X\setminus\bar{G}_2)^{2m}}{s^{2m}}\right)^{-\gamma}\left(\frac{t}{s}\right)^{2m\delta}\frac{ds}{s}\,\|f\|_{L^\infty(X)}\,\|g\|_{L^2(E)} \\
&\lesssim \left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\gamma}\|f\|_{L^\infty(X)}\,\|g\|_{L^2(E)}\,, \qquad (6.62)
\end{aligned}$$

since $\delta > \gamma$. Combining (6.59) and (6.60) with (6.61) and (6.62) finishes the proof. □

6.8 Extension to Hardy spaces $H^p_L(X)$ for $p \neq 2$

From now on, let again L be an operator satisfying (H1),(H2) and (H3), but not necessarily (H4) and (H5).

Once having settled the boundedness on $L^2(X)$ for an operator T satisfying off-diagonal estimates of the form $(\mathrm{OD1})_\gamma$ and $(\mathrm{OD2})_\gamma$, the extension to Hardy spaces $H^p_L(X)$ for $p \neq 2$ is almost immediate. Such a property is similar to the behaviour of Calderón-Zygmund operators, in respect of the fact that every Calderón-Zygmund operator, that is bounded on $L^2(X)$, is automatically also bounded on $L^p(X)$ for all $p \in (1,\infty)$.

Corollary 6.22 *Let L be an operator satisfying the assumptions* (H1) *and* (H2). *Let $T : L^2(X) \to L^2(X)$ be a bounded linear operator that satisfies* $(\mathrm{OD1})_\gamma$ *for some* $\gamma > \frac{n}{2m}$. *Then T extends to a bounded operator*

$$T : H^p_L(X) \to L^p(X), \qquad 1 \le p \le 2,$$

and T^* extends to a bounded operator

$$\begin{aligned}
T^* &: L^p(X) \to H^p_{L^*}(X), \qquad 2 \le p < \infty, \\
T^* &: L^\infty(X) \to BMO_{L^*}(X).
\end{aligned}$$

One can obviously obtain the corresponding results for T replaced by T^* and L by L^*, if one uses $(\mathrm{OD2})_\gamma$ instead of $(\mathrm{OD1})_\gamma$.

Proof: To show that T extends to a bounded operator $T : H^1_L(X) \to L^1(X)$, one combines Proposition 4.39 with Corollary 6.6, taking into account that the operator families $\{(I-e^{-tL})^M\}_{t>0}$, $\{(tLe^{-tL})^M\}_{t>0}$ and therefore also $\{T(I-e^{-tL})^M\}_{t>0}$, $\{T(tLe^{-tL})^M\}_{t>0}$ are uniformly bounded on $L^2(X)$.
One then uses the interpolation scales for the spaces $L^p(X)$ and $H^p_L(X)$, see Proposition 4.37, and obtains the boundedness of $T : H^p_L(X) \to L^p(X)$ for $1 \le p \le 2$.
Since Theorem 4.28 yields that $(H^1_L(X))' = BMO_{L^*}(X)$ and the space $H^p_{L^*}(X)$ was defined as the dual space of $H^{p'}_L(X)$ for $2 < p < \infty$ and $\frac{1}{p}+\frac{1}{p'}=1$ (see Definition 4.35), one finally gets the remaining assertions of the corollary via duality. □

6.9 Towards a $T(b)$-Theorem

Whenever one states some kind of $T(1)$-Theorem, there naturally arises the question if there exists a generalization to a $T(b)$-Theorem for some accretive function b (see

Definition 6.23 below for the definition of accretive functions). This is due to the fact that in many applications the assumption $T(1) \in BMO$ is not directly verifiable. For the $T(1)$-Theorem of David and Journé, the question was positively answered by David, Journé and Semmes in [DJS85], who were able to show that one can replace the condition $T(1) \in BMO$ by the condition $T(b) \in BMO$ for some para-accretive function b.
In this section, we will now give a criterion, under which a $T(b)$-Theorem in our setting holds. Since we are not sure if this criterion is the right one for applications, we only call this "an approach towards a $T(b)$-Theorem". For a short discussion of the topic, we refer to Remark 6.27.

Let us begin with the definition of accretive functions.

Definition 6.23 *A function $b \in L^\infty(X)$ is said to be* accretive *if there exists a constant $c_0 > 0$ such that $\operatorname{Re} b(x) \geq c_0$ for almost all $x \in X$.*

Before coming to the statement and proof of the the $T(b)$-Theorem, Theorem 6.28, we first state two auxiliary results. We make use of both results in the proof of Theorem 6.28, and they represent the major changes in comparision to the proof of the $T(1)$-Theorem, Theorem 6.13. Their proofs are shifted to the end of the section.

For every $b \in L^\infty(X)$, we denote by M_b the multiplication operator defined by $M_b f := b \cdot f$ for all measurable functions $f : X \to \mathbb{C}$.

Lemma 6.24 *Let L satisfy* (H1) *and* (H2)*. Let $\alpha, \beta \geq 1$, $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$ and let $T : \mathcal{D}(L) \cap \mathcal{R}(L) \to L^2_{\mathrm{loc}}(X)$ be a linear operator such that the operator family $\{T\psi(tL)\}_{t>0}$ satisfies weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$. Moreover, let $b \in L^\infty(X)$ and assume that there exist $\delta > 0$ and $\tilde{\psi} \in \Psi_{\beta_1,\alpha_1}(\Sigma^0_\mu)$ for some $\alpha_1 \geq \alpha$ and $\beta_1 \geq \beta$ such that there holds $\int_0^\infty \psi(t)\tilde{\psi}(t)\, \frac{dt}{t} = 1$ and such that there exists some constant $C > 0$ with*

$$\left\| \tilde{\psi}(sL) M_b \psi(tL) f \right\|_{L^2(X)} \leq C \min\left(\frac{s}{t}, \frac{t}{s}\right)^\delta \|f\|_{L^2(X)} \|b\|_{L^\infty(X)} \tag{6.63}$$

for all $s, t > 0$ and all $f \in L^2(X)$. Additionally, assume that there exists some $\varepsilon_0 \in (0,1)$ such that $\varepsilon_0 \beta > \gamma$ and $(1-\varepsilon_0)\delta > \frac{n}{2m} + \gamma$.
Then the operator family $\{TM_b\psi(tL)\}_{t>0}$, originally defined by (6.68)*, satisfies weak off-diagonal estimates of order γ.*

Remark 6.25 If one replaces the *weak* off-diagonal estimates by off-diagonal estimates in Lemma 6.24, one no longer needs the assumption $\gamma > \frac{n}{2m}$. Also the assumption $(1-\varepsilon_0)\delta > \frac{n}{2m} + \gamma$ reduces to $(1-\varepsilon_0)\delta > \gamma$.
The proof follows the same lines as the one of Lemma 6.24, replacing the splitting of X into balls of radius t by a splitting into two complementary sets, as it is done in the proof of Lemma 3.4 and Lemma 6.26 below.

Lemma 6.26 *Let $\alpha > 0$, $\beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$ and $\psi \in \Psi_{\beta,\alpha}(\Sigma^0_\mu)$. Let $b \in L^\infty(X)$ and assume that there exist $\delta > \beta$ and $\tilde{\psi} \in \Psi_{\beta_1,\alpha_1}(\Sigma^0_\mu)$ for some $\alpha_1 \geq \alpha$ and $\beta_1 \geq \beta$ such that there holds $\int_0^\infty \psi(t)\tilde{\psi}(t)\, \frac{dt}{t} = 1$ and such that* (6.63) *is satisfied with b replaced by $\overline{b}$. Additionally, assume that there exists some $\varepsilon_0 \in (0,1)$ with $\varepsilon_0 \beta > \frac{n}{4m}$ and $(1-\varepsilon_0)\delta >$*

$\varepsilon_0\beta + [\frac{n}{4m}] + 1$.
Then for every $f \in BMO_L(X)$ is

$$\nu_{\psi,f} := \left|\psi(t^{2m}L)M_b f(y)\right|^2 \frac{d\mu(y)dt}{t}$$

a Carleson measure and there exists a constant $C_\psi > 0$ such that for all $f \in BMO_L(X)$

$$\|\nu_{\psi,f}\|_{\mathcal{C}} \le C_\psi \|b\|^2_{L^\infty(X)} \|f\|^2_{BMO_L(X)}.$$

Remark 6.27 We admit that it remains unclear if condition (6.63) is the right one for applications. Let us shortly explain this. If one chooses $b = 1$, then condition (6.63) is a continuous version of similar conditions that are usually used in the Cotlar-Knapp-Stein lemma, and can be proven immediately. In analogy to Lemma 3.19 and Lemma 6.15, one can show that there holds

$$\left\|\tilde{\psi}(sL)\psi(tL)\right\|_{L^2(X)\to L^2(X)} \lesssim \min\left\{\left(\frac{s}{t}\right)^{\delta_1}, \left(\frac{t}{s}\right)^{\delta_2}\right\}$$

with $\delta_1 = \min(\alpha,\beta_1)$ and $\delta_2 = \min(\alpha_1,\beta)$. To see this e.g. in the case $s \le t$, one writes

$$\tilde{\psi}(sL)\psi(tL) = \left(\frac{s}{t}\right)(sL)^{-\delta_1}\tilde{\psi}(sL)(tL)^{\delta_1}\psi(tL)$$

and observes that $z \mapsto z^{-\delta_1}\tilde{\psi}(z) \in H^\infty(\Sigma^0_\mu)$ and $z \mapsto z^{\delta_1}\psi(z) \in H^\infty(\Sigma^0_\mu)$ due to the assumptions on ψ and $\tilde{\psi}$. Thus, (6.63) is satisfied with $\delta = \min(\alpha,\alpha_1,\beta,\beta_1)$.
If we now take $b \in L^\infty(X)$ arbitrary, such an estimate is no longer obvious. Similar conditions have already been used in generalized Cotlar-Knapp-Stein lemmata in a discrete setting in the context of Calderón-Zygmund operators, see e.g. the article of Han, Zhang, [HZ01]. But there the approximation operators, in [HZ01] e.g. called T_j, were adapted to the function b in such a way that there holds $T_j(b) = 0$. This is in contrast to our setting, where the operators $\psi(tL)$, that resemble the operators T_j, are in general not adapted to b.

We are now ready to state our $T(b)$-Theorem.

Theorem 6.28 *Let L be an operator satisfying the assumptions* (H1), (H2) *and* (H3). *Additionally, let the assumptions* (P) *and* (P*) *be satisfied.*
Let $T : \mathcal{D}(L)\cap\mathcal{R}(L) \to L^2_{\text{loc}}(X)$ be a linear operator with $T^* : \mathcal{D}(L^*)\cap\mathcal{R}(L^*) \to L^2_{\text{loc}}(X)$ such that the assumptions $(\text{OD1})_\gamma$ *and* $(\text{OD2})_\gamma$ *are satisfied for some $\gamma > \frac{n+D+2}{2m}$.*
Let $b_1, b_2 \in L^\infty(X)$ be two accretive functions such that the assumptions of Lemma 6.24 are satisfied for the operator families $\{T\psi_1(tL)\}_{t>0}$ with b_1 and for $\{T^*\psi_2(tL^*)\}_{t>0}$ with $\overline{b}_2$ and such that the assumptions of Lemma 6.26 are satisfied for the triples $\psi_1, \overline{b}_1, L^*$ and ψ_2, b_2, L.
Moreover, let $T(b_1) \in BMO_L(X)$ and $T^*(\overline{b}_2) \in BMO_{L^*}(X)$.
Then T is bounded in $L^2(X)$, i.e. there exists a constant $C > 0$ such that for all $f \in L^2(X)$ there holds

$$\|Tf\|_{L^2(X)} \le C\|f\|_{L^2(X)}.$$

Proof (of Theorem 6.28): The proof works analogously to the one of Theorem 6.13. We will not give the proof in all details, but only state the differences to the one of Theorem 6.13.
Let $f, g \in L^2(X)$. Let $b_1, b_2 \in L^\infty(X)$ be the two accretive functions given in the assumption with constants c_1 and c_2, respectively. Moreover, let $\alpha \geq 1$, $\beta > \frac{n}{4m} + [\frac{n}{4m}] + 1$ and let $\psi_1, \psi_2 \in \Psi_{\beta,\alpha}(\Sigma^0_\mu) \setminus \{0\}$ as given in the assumption. Denote by $\tilde{\psi}_1, \tilde{\psi}_2 \in \Psi(\Sigma^0_\mu)$ the functions given in the assumptions of Lemma 6.24 and Lemma 6.26 that satisfy $\int_0^\infty \psi_1(t)\tilde{\psi}_1(t)\,\frac{dt}{t} = 1$ and $\int_0^\infty \psi_2(t)\tilde{\psi}_2(t)\,\frac{dt}{t} = 1$.
Since b_1, b_2 are accretive functions, it will be sufficient to estimate $M_{b_2}TM_{b_1}$ instead of T. Once it is shown that $M_{b_2}TM_{b_1}$ is bounded on $L^2(X)$, one also obtains the boundedness of T itself on $L^2(X)$, since $\|b_1^{-1}\|_{L^\infty(X)} \leq c_1^{-1}$ and $\|b_2^{-1}\|_{L^\infty(X)} \leq c_2^{-1}$ and therefore

$$\begin{aligned}|\langle Tf, g\rangle| &= \left|\langle M_{b_2}TM_{b_1}M_{b_1^{-1}}f, M_{(\bar{b}_2)^{-1}}g\rangle\right| \\ &\lesssim \|b_1^{-1}\cdot f\|_{L^2(X)}\,\|b_2^{-1}\cdot g\|_{L^2(X)} \leq c_1^{-1}c_2^{-1}\,\|f\|_{L^2(X)}\,\|g\|_{L^2(X)}\,.\end{aligned}$$

In analogy to the proof of Theorem 6.13, we first decompose both f and g with the help of the Calderón reproducing formula, which yields

$$\langle M_{b_2}TM_{b_1}f, g\rangle = \int_0^\infty \int_0^\infty \langle \psi_2(t^{2m}L)M_{b_2}TM_{b_1}\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle\,\frac{dt}{t}\frac{ds}{s}. \tag{6.64}$$

The two main differences will be the following. Observe that due to Lemma 6.24 and the assumption $\gamma > \frac{n+D+2}{2m}$, the operator families

$$\{TM_{b_1}\psi_1(tL)\}_{t>0} \qquad \text{and} \qquad \{T^*M_{\bar{b}_2}\psi_2(tL^*)\}_{t>0} \tag{6.65}$$

satisfy weak off-diagonal estimates of order γ. Moreover, together with the assumptions $T(b_1) \in BMO_L(X)$ and $T^*(\bar{b}_2) \in BMO_{L^*}(X)$, Lemma 6.26 yields that

$$\left|\psi_2(t^{2m}L)M_{b_2}T(b_1)(y)\right|^2 \frac{d\mu(y)dt}{t} \qquad \text{and} \qquad \left|\psi_1(t^{2m}L^*)M_{\bar{b}_1}T^*(\bar{b}_2)(y)\right|^2 \frac{d\mu(y)dt}{t} \tag{6.66}$$

are Carleson measures.
As in the proof of Theorem 6.13, it is enough to consider the part J_1, where in the inner integral of (6.64) one only integrates over the interval $\{t \in (0,\infty) : 0 < t < s\}$. Then, one also uses the first line of the decomposition (6.25), but now applied for the operator $M_{b_2}TM_{b_1}$ instead of T. The error term J_E is then equal to

$$J_E = \int_0^\infty \int_0^s \langle \psi_2(t^{2m}L)M_{b_2}TM_{b_1}(I - e^{-t^{2m}L})\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle\,\frac{dt}{t}\frac{ds}{s}.$$

Due to the weak off-diagonal estimates for the operator family $\{T^*M_{\bar{b}_2}\psi_2(tL^*)\}_{t>0}$ and the fact that

$$\left\|M_{b_1}(I - e^{-t^{2m}L})\psi_1(s^{2m}L)h\right\|_{L^2(X)} \lesssim \left\|(I - e^{-t^{2m}L})\psi_1(s^{2m}L)h\right\|_{L^2(X)},$$

we can simply copy the estimates in (6.33), (6.34), (6.35) and (6.36) and obtain

$$|J_E| \lesssim \|f\|_{L^2(X)}\,\|g\|_{L^2(X)}\,.$$

To handle the main term J_M, we now split $M_{b_2}TM_{b_1}e^{-t^{2m}L}$ into

$$[M_{b_2}TM_{b_1}e^{-t^{2m}L} - M_{b_2}T(b_1)\cdot A_t e^{-t^{2m}L}] + M_{b_2}T(b_1)\cdot A_t e^{-t^{2m}L}.$$

Then, following the same procedure as in (6.38), we get $J_M^0 = J_M^1 + J_M^2$ with

$$J_M^1 = \int_0^\infty \langle \psi_2(t^{2m}L)M_{b_2}TM_{b_1}e^{-t^{2m}L}f - \psi_2(t^{2m}L)M_{b_2}T(b_1)\cdot A_t e^{-t^{2m}L}f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \,\frac{dt}{t}$$

and

$$J_M^2 = \int_0^\infty \langle \psi_2(t^{2m}L)M_{b_2}T(b_1)\cdot A_t e^{-t^{2m}L}f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \,\frac{dt}{t}.$$

The term J_M^1 can again be estimated by application of Proposition 6.14, with a slight modification. We set $S_{t^{2m}} := \psi_2(t^{2m}L)M_{b_2}T$ and observe that this operator satisfies weak off-diagonal estimates of order $\gamma > \frac{n+D+2}{2m}$ via (6.65). It remains to check that the constant function 1 in Proposition 6.14 can be replaced by some arbitrary function $b_1 \in L^\infty(X)$, i.e. that one can also obtain the estimate

$$\int_0^\infty \left\| S_{t^{2m}}M_{b_1}e^{-t^{2m}L}f - S_{t^{2m}}(b_1)\cdot A_t e^{-t^{2m}L}f \right\|_{L^2(X)}^2 \frac{dt}{t} \leq C\, \|b_1\|_{L^\infty(X)}^2 \|f\|_{L^2(X)}^2 .$$

This can easily be seen in the calculations of (6.28), where one can pull the function b_1 out of the L^2-norm in the last step.
The term J_M^2 is, up to the multiplication operator M_{b_2}, a paraproduct. To handle this term, let us have a short look at the proof of Theorem 5.2, which states the boundedness of paraproducts on $L^2(X)$. There, one only exploits the fact that $\left|\psi(t^{2m}L)b(y)\right|^2 \frac{d\mu(y)dt}{t}$ is a Carleson measure whenever $b \in BMO_L(X)$ and does not explicitly use that $b \in BMO_L(X)$. Since we have by assumptions that $\left|\psi_2(t^{2m}L)M_{b_2}T(b_1)(y)\right|^2 \frac{d\mu(y)dt}{t}$ is a Carleson measure, see (6.66), we also get the desired estimate for J_M^2.
The proof of the remainder term J_R, defined by

$$J_R := \int_0^\infty \int_s^\infty \langle \psi_2(t^{2m}L)M_{b_2}TM_{b_1}e^{-t^{2m}L}\psi_1(s^{2m}L)\tilde{\psi}_1(s^{2m}L)f, \tilde{\psi}_2(t^{2m}L^*)g\rangle \,\frac{dt}{t}\frac{ds}{s},$$

is again handled as in (6.42), with the same changes as those for the treatment of J_E. This finishes the proof. □

Let us finally give the proofs of Lemma 6.24 and 6.26. The proofs are similar to those of Lemma 5.12 and Lemma 6.5, again transferring off-diagonal estimates from one operator to another with the help of a Calderón reproducing formula.

Proof (of Lemma 6.24): Let $b \in L^\infty(X)$ and let $t > 0$. Further, let B_1, B_2 be two arbitrary ball in X with radius t and let $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq B_1$ and $\operatorname{supp} g \subseteq B_2$. We will show the following estimate:

$$\left|\langle TM_b\psi(t^{2m}L)f, g\rangle\right| \lesssim \left(1 + \frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)} \|b\|_{L^\infty(X)} \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)} . \tag{6.67}$$

To be able to apply the weak off-diagonal estimates for $\{T\psi(tL)\}_{t>0}$, we decompose the expression on the left with the help of the Calderón reproducing formula, which gives

$$\langle TM_b\psi(t^{2m}L)f,g\rangle = \int_0^\infty \langle\tilde{\psi}(s^{2m}L)M_b\psi(t^{2m}L)f,\psi(s^{2m}L^*)T^*g\rangle \,\frac{ds}{s}, \tag{6.68}$$

where $\tilde{\psi}$ is the function taken from the assumptions, satisfying $\int_0^\infty \psi(t)\tilde{\psi}(t)\,\frac{dt}{t} = 1$.
We deduce from Lemma 3.1 that due to the weak off-diagonal estimates of order $\gamma > \frac{n}{2m}$, the operator family $\{T\psi(tL)\}_{t>0}$ is uniformly bounded on $L^2(X)$. Together with assumption (6.63) and the Cauchy-Schwarz inequality, this yields

$$\begin{aligned}\left|\langle TM_b\psi(t^{2m}L)f,g\rangle\right| &\leq \int_0^\infty \left\|\tilde{\psi}(s^{2m}L)M_b\psi(t^{2m}L)f\right\|_{L^2(X)} \left\|\psi(s^{2m}L^*)T^*g\right\|_{L^2(X)} \,\frac{ds}{s}\\ &\lesssim \int_0^\infty \min\left(\frac{s}{t},\frac{t}{s}\right)^{2m\delta} \frac{ds}{s}\, \|b\|_{L^\infty(X)}\, \|f\|_{L^2(B_1)}\, \|g\|_{L^2(B_2)}\\ &\lesssim \|b\|_{L^\infty(X)}\, \|f\|_{L^2(B_1)}\, \|g\|_{L^2(B_2)}\,,\end{aligned}$$

where we used in the last step the fact that

$$\sup_{t>0}\int_0^\infty \min\left(\frac{s}{t},\frac{t}{s}\right)^{2m\delta} \frac{ds}{s} \leq C_{2m\delta}$$

for every $\delta > 0$. This shows (6.67) in the case of $\operatorname{dist}(B_1,B_2) \leq t$.
For $\operatorname{dist}(B_1,B_2) > t$, we split the integral in (6.68) into two parts, one over $(0,t)$, which is called J_1, and one over (t,∞), which is called J_2.
To handle J_1, we cover X with the help of Lemma 2.1 by balls of radius t. That is, we have $X = \bigcup_{\alpha\in I_{k_0}} B_\alpha$, where $k_0 \in \mathbb{Z}$ is determined by $C_1\delta^{k_0} \leq t < C_1\delta^{k_0-1}$, the balls are defined by $B_\alpha := B(z_\alpha^{k_0},t)$ and $I_{k_0}, z_\alpha^{k_0}$ are as in Lemma 2.1 and Notation 2.2.
Applying this decomposition of X and using the Cauchy-Schwarz inequality, we then get

$$\begin{aligned}|J_1| &\leq \int_0^t \left|\langle\tilde{\psi}(s^{2m}L)M_b\psi(t^{2m}L)f,\psi(s^{2m}L^*)T^*g\rangle\right| \,\frac{ds}{s}\\ &\leq \sum_{\alpha\in I_{k_0}} \int_0^t \left\|\tilde{\psi}(s^{2m}L)M_b\psi(t^{2m}L)f\right\|_{L^2(B_\alpha)} \left\|\psi(s^{2m}L^*)T^*g\right\|_{L^2(B_\alpha)} \,\frac{ds}{s}.\end{aligned}$$

Due to the weak off-diagonal estimates for $\{\psi(sL^*)T^*\}_{s>0}$ and Remark 3.9, we have for all $s < t$ and all $\alpha \in I_{k_0}$

$$\left\|\psi(s^{2m}L^*)T^*g\right\|_{L^2(B_\alpha)} \lesssim \left(\frac{t}{s}\right)^n \left(1+\frac{\operatorname{dist}(B_\alpha,B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \|g\|_{L^2(B_2)}\,. \tag{6.69}$$

On the other hand, as a result of Proposition 3.18, $\{\tilde{\psi}(tL)\}_{t>0}$ and $\{\psi(tL)\}_{t>0}$ satisfy off-diagonal estimates in t of order β_1 and β, respectively. Hence, Lemma 3.4 shows that $\{\tilde{\psi}(sL)M_b\psi(tL)\}_{s,t>0}$ satisfies off-diagonal estimates in $\max(s,t)$ of order $\beta = \min(\beta,\beta_1)$. Together with assumption (6.63), this yields for all $s < t$ and all $\alpha \in I_{k_0}$

$$\begin{aligned}&\left\|\tilde{\psi}(s^{2m}L)M_b\psi(t^{2m}L)f\right\|_{L^2(B_\alpha)}\\ &\quad\lesssim \min\left\{\left(1+\frac{\operatorname{dist}(B_1,B_\alpha)^{2m}}{t^{2m}}\right)^{-\beta}, \left(\frac{s}{t}\right)^{2m\delta}\right\} \|b\|_{L^\infty(X)}\, \|f\|_{L^2(B_1)}\\ &\quad\lesssim \left(1+\frac{\operatorname{dist}(B_1,B_\alpha)^{2m}}{t^{2m}}\right)^{-\varepsilon\beta} \left(\frac{s}{t}\right)^{(1-\varepsilon)2m\delta} \|b\|_{L^\infty(X)}\, \|f\|_{L^2(B_1)}\,,\end{aligned} \tag{6.70}$$

for every $\varepsilon \in (0,1)$.
Recall that we assumed the existence of some $\varepsilon_0 \in (0,1)$ such that $\varepsilon_0\beta > \frac{n}{2m}$ and $(1-\varepsilon_0)\delta > \frac{n}{2m} + \min(\varepsilon_0\beta, \gamma)$. Since we also assumed $\gamma > \frac{n}{2m}$, we therefore have $\min(\varepsilon_0\beta,\gamma) > \frac{n}{2m}$. This enables us to apply Lemma 3.6 to get

$$\begin{aligned}\sum_{\alpha \in I_{k_0}} \left(1 + \frac{\operatorname{dist}(B_1, B_\alpha)^{2m}}{t^{2m}}\right)^{-\varepsilon_0\beta} \left(1 + \frac{\operatorname{dist}(B_\alpha, B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \\ \lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)}, \qquad (6.71)\end{aligned}$$

where we estimated the occuring s in (6.71) simply by t.
Combining the estimates (6.69) and (6.70) with (6.71), we therefore obtain

$$\begin{aligned}|J_1| &\leq \sum_{\alpha \in I_{k_0}} \int_0^t \left\| \tilde{\psi}(s^{2m}L) M_b \psi(t^{2m}L) f \right\|_{L^2(B_\alpha)} \left\| \psi(s^{2m}L^*) T^* g \right\|_{L^2(B_\alpha)} \frac{ds}{s} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)} \int_0^t \left(\frac{s}{t}\right)^{(1-\varepsilon_0)2m\delta - n} \frac{ds}{s} \|b\|_{L^\infty(X)} \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)} \\ &\lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_2)^{2m}}{t^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)} \|b\|_{L^\infty(X)} \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)},\end{aligned}$$

where in the last step we used the fact that the integral is bounded by a constant independent of s and t due to the assumption $(1-\varepsilon_0)\delta > \frac{n}{2m}$. Therefore, the last line gives the desired estimate for J_1.

We now turn to the integral J_2. As before, we cover X with balls of radius t and use the Cauchy-Schwarz inequality to get

$$\begin{aligned}|J_2| &\leq \int_t^\infty \left| \langle \tilde{\psi}(s^{2m}L) M_b \psi(t^{2m}L) f, \psi(s^{2m}L^*) T^* g \rangle \right| \frac{ds}{s} \\ &\leq \sum_{\alpha \in I_{k_0}} \int_t^\infty \left\| \tilde{\psi}(s^{2m}L) M_b \psi(t^{2m}L) f \right\|_{L^2(B_\alpha)} \left\| \psi(s^{2m}L^*) T^* g \right\|_{L^2(B_\alpha)} \frac{ds}{s}.\end{aligned}$$

On the one hand, we again use the weak off-diagonal estimates for $\{\psi(sL^*)T^*\}_{s>0}$ and get for $s > t$ and $\alpha \in I_{k_0}$ (observe that this estimate also works for balls of radius t by embedding them into larger balls of radius s)

$$\left\| \psi(s^{2m}L^*) T^* g \right\|_{L^2(B_\alpha)} \lesssim \left(1 + \frac{\operatorname{dist}(B_\alpha, B_2)^{2m}}{s^{2m}}\right)^{-\gamma} \|g\|_{L^2(B_2)}. \qquad (6.72)$$

In analogy to (6.70), on the other hand, we obtain by application of Proposition 3.18, Lemma 3.4 and assumption (6.63) for every $s > t$ and every $\alpha \in I_{k_0}$

$$\begin{aligned}&\left\| \tilde{\psi}(s^{2m}L) M_b \psi(t^{2m}L) f \right\|_{L^2(B_\alpha)} \\ &\quad\lesssim \left(1 + \frac{\operatorname{dist}(B_1, B_\alpha)^{2m}}{s^{2m}}\right)^{-\varepsilon_0\beta} \left(\frac{t}{s}\right)^{(1-\varepsilon_0)2m\delta} \|b\|_{L^\infty(X)} \|f\|_{L^2(B_1)}. \qquad (6.73)\end{aligned}$$

Lemma 3.6 in turn yields that for all $s > t$ there holds

$$\sum_{\alpha\in I_{k_0}} \left(1+\frac{\operatorname{dist}(B_1,B_\alpha)^{2m}}{s^{2m}}\right)^{-\varepsilon_0\beta} \left(1+\frac{\operatorname{dist}(B_\alpha,B_2)^{2m}}{s^{2m}}\right)^{-\gamma}$$
$$\lesssim \left(\frac{s}{t}\right)^{(n+\varepsilon)} \left(1+\frac{\operatorname{dist}(B_1,B_2)^{2m}}{s^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)} \tag{6.74}$$

for arbitrary $\varepsilon > 0$. As above, the combination of (6.72), (6.73) and (6.74) provides us with

$$|J_2| \le \sum_{\alpha\in I_{k_0}} \int_t^\infty \left\|\tilde{\psi}(s^{2m}L)M_b\psi(t^{2m}L)f\right\|_{L^2(B_\alpha)} \left\|\psi(s^{2m}L^*)T^*g\right\|_{L^2(B_\alpha)} \frac{ds}{s}$$
$$\lesssim \int_t^\infty \left(1+\frac{\operatorname{dist}(B_1,B_2)^{2m}}{s^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)} \left(\frac{t}{s}\right)^{(1-\varepsilon_0)2m\delta-(n+\varepsilon)} \frac{ds}{s}$$
$$\times \|b\|_{L^\infty(X)} \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)}\,. \tag{6.75}$$

Finally observe that the integral in (6.75) can in view of the assumption $\operatorname{dist}(B_1,B_2) > t$ be bounded by a constant times

$$\left(\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)} \int_t^\infty \left(\frac{t}{s}\right)^{-2m\min(\varepsilon_0\beta,\gamma)} \left(\frac{t}{s}\right)^{(1-\varepsilon_0)2m\delta-(n+\varepsilon)} \frac{ds}{s}$$
$$\lesssim \left(1+\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)},$$

since we postulated $(1-\varepsilon_0)\delta - \min(\varepsilon_0\beta,\gamma) > \frac{n+\varepsilon}{2m}$ for sufficiently small $\varepsilon > 0$. Thus,

$$|J_2| \lesssim \left(1+\frac{\operatorname{dist}(B_1,B_2)^{2m}}{t^{2m}}\right)^{-\min(\varepsilon_0\beta,\gamma)} \|b\|_{L^\infty(X)} \|f\|_{L^2(B_1)} \|g\|_{L^2(B_2)}\,.$$

This finishes the proof. □

Proof (of Lemma 6.26): We set $M := [\frac{n}{4m}] + 1$. Then there holds $BMO_L(X) = BMO_{L,M}(X)$ according to Definition 4.29.
We follow the proof of Proposition 4.27, replacing the operator family $\{\psi(tL)\}_{t>0}$ by the operator family $\{\psi(tL)M_b\}_{t>0}$. The corresponding term I_1 can be handled with just the same methods, once one has checked that with $\{\psi(tL)\}_{t>0}$ also $\{\psi(tL)M_b\}_{t>0}$ satisfies quadratic estimates and off-diagonal estimates of the same order.
For the term I_2, it needs a more careful treatment. What is essential for this part is the fact that the operator family $\{\psi(tL)(tL)^{-k}\}_{t>0}$, now replaced by $\{\psi(tL)M_b(tL)^{-k}\}_{t>0}$, satisfies off-diagonal estimates of order $\beta - k > \frac{n}{4m}$ for every $1 \le k \le M$. If one can establish these estimates, the proof for the second part I_2 can be copied from the one of Proposition 4.27.
Thus, let us show, in analogy to Lemma 6.24, that $\{\psi(tL)M_b(tL)^{-k}\}_{t>0}$ satisfies off-diagonal estimates of some order larger than $\frac{n}{4m}$. Let E, F be two open sets in X and let $g \in \mathcal{D}(L^{-k})$ with $\operatorname{supp} g \subseteq E$, $h \in L^2(X)$ with $\operatorname{supp} h \subseteq F$. Via the Calderón reproducing formula, we write

$$\langle \psi(t^{2m}L)M_b(t^{2m}L)^{-k}g, h\rangle = \int_0^\infty \langle \psi(t^{2m}L)M_b\psi(s^{2m}L)\tilde{\psi}(s^{2m}L)(t^{2m}L)^{-k}g, h\rangle \frac{ds}{s}.$$

Due to the Cauchy-Schwarz inequality, the uniform boundedness of $\{\psi(sL)(sL)^{-k}\}_{s>0}$ and assumption (6.63) we then obtain

$$\begin{aligned}
&\left|\langle \psi(t^{2m}L)M_b(t^{2m}L)^{-k}g,h\rangle\right| \\
&\leq \int_0^\infty \left(\frac{s}{t}\right)^{2mk} \left\|\psi(s^{2m}L)(s^{2m}L)^{-k}g\right\|_{L^2(X)} \left\|\tilde{\psi}(s^{2m}L^*)M_{\bar{b}}\psi(t^{2m}L^*)h\right\|_{L^2(X)} \frac{ds}{s} \\
&\lesssim \int_0^\infty \min\left(\frac{s}{t},\frac{t}{s}\right)^{2m\delta} \left(\frac{s}{t}\right)^{2mk} \frac{ds}{s} \left\|b\right\|_{L^\infty(X)} \left\|g\right\|_{L^2(X)} \left\|h\right\|_{L^2(X)} \\
&\lesssim \left\|b\right\|_{L^\infty(X)} \left\|g\right\|_{L^2(E)} \left\|h\right\|_{L^2(F)},
\end{aligned}$$

where for the case $s>t$ we take into account that $\delta>M$ and therefore $\delta>k$ for all $1\leq k\leq M$. This yields the desired estimate for $\operatorname{dist}(E,F)\leq t$.
For the case $\rho:=\operatorname{dist}(E,F)>t$, we define the sets $G_1:=\{x\in X : \operatorname{dist}(x,F)<\frac{\rho}{2}\}$ and $G_2:=\{x\in X : \operatorname{dist}(x,F)<\frac{\rho}{4}\}$ and then split X into $X=\bar{G}_2\cup X\setminus\bar{G}_2$. By construction there holds that G_1,G_2 are open with $\operatorname{dist}(E,G_1)\geq\frac{\rho}{2}$ and $\operatorname{dist}(F,X\setminus\bar{G}_2)\geq\frac{\rho}{4}$. Using that $\bar{G}_2\subseteq G_1$, this leads to

$$\begin{aligned}
&\left|\langle \psi(t^{2m}L)M_b(t^{2m}L)^{-k}g,h\rangle\right| \\
&\leq \int_0^\infty \left(\frac{s}{t}\right)^{2mk} \left\|\psi(s^{2m}L)(s^{2m}L)^{-k}g\right\|_{L^2(G_1)} \left\|\tilde{\psi}(s^{2m}L^*)M_{\bar{b}}\psi(t^{2m}L^*)h\right\|_{L^2(G_1)} \frac{ds}{s} \\
&\quad + \int_0^\infty \left(\frac{s}{t}\right)^{2mk} \left\|\psi(s^{2m}L)(s^{2m}L)^{-k}g\right\|_{L^2(X\setminus\bar{G}_2)} \left\|\tilde{\psi}(s^{2m}L^*)M_{\bar{b}}\psi(t^{2m}L^*)h\right\|_{L^2(X\setminus\bar{G}_2)} \frac{ds}{s} \\
&=: J_{G_1}+J_{X\setminus\bar{G}_2}.
\end{aligned}$$

For the term $J_{X\setminus\bar{G}_2}$ we get via Lemma 3.4, applied to $\{\tilde{\psi}(sL^*)M_{\bar{b}}\psi(tL^*)\}_{s,t>0}$, assumption (6.63) and the uniform boundedness of $\{\psi(sL)(sL)^{-k}\}_{s>0}$

$$\begin{aligned}
J_{X\setminus\bar{G}_2} \lesssim \int_0^\infty \min\left(\left(1+\frac{\operatorname{dist}(F,X\setminus\bar{G}_2)^{2m}}{\max(s,t)^{2m}}\right)^{-\beta}, \min\left(\frac{s}{t},\frac{t}{s}\right)^{2m\delta}\right)\left(\frac{s}{t}\right)^{2mk}\frac{ds}{s} \\
\times \left\|b\right\|_{L^\infty(X)} \left\|g\right\|_{L^2(E)} \left\|h\right\|_{L^2(F)}. \qquad (6.76)
\end{aligned}$$

Since by construction there holds $\operatorname{dist}(F,X\setminus\bar{G}_2)\gtrsim\operatorname{dist}(E,F)>t$, we can bound the integral in (6.76) in a similar way as in the proof of Lemma 6.24 by a constant times

$$\begin{aligned}
&\int_0^t \min\left(\left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\beta}, \left(\frac{s}{t}\right)^{2m\delta}\right)\left(\frac{s}{t}\right)^{2mk}\frac{ds}{s} \\
&\qquad + \int_t^\infty \min\left(\left(1+\frac{\operatorname{dist}(E,F)^{2m}}{s^{2m}}\right)^{-\beta}, \left(\frac{t}{s}\right)^{2m\delta}\right)\left(\frac{s}{t}\right)^{2mk}\frac{ds}{s} \\
&\lesssim \left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\beta}\int_0^t \left(\frac{s}{t}\right)^{2mk}\frac{ds}{s} \\
&\qquad + \left(\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\varepsilon_0\beta}\int_t^\infty \left(\frac{t}{s}\right)^{-2m\varepsilon_0\beta}\left(\frac{t}{s}\right)^{2m(1-\varepsilon_0)\delta-2mk}\frac{ds}{s} \\
&\lesssim \left(1+\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-\varepsilon_0\beta}
\end{aligned}$$

for $\varepsilon_0 \in (0,1)$ as given in the assumptions with $(1-\varepsilon_0)\delta > \varepsilon_0\beta + k$ for all $1 \le k \le M$. It remains to estimate J_{G_1}. Observe that $\{\psi(sL)(sL)^{-k}\}_{s>0}$ satisfies off-diagonal estimates of order $\beta - k$ due to Proposition 3.18. With the help of assumption (6.63), we therefore obtain

$$J_{G_1} \lesssim \int_0^\infty \left(1 + \frac{\operatorname{dist}(E, G_1)^{2m}}{s^{2m}}\right)^{-(\beta-k)} \min\left(\frac{s}{t}, \frac{t}{s}\right)^{2m\delta} \left(\frac{s}{t}\right)^{2mk} \frac{ds}{s} \\ \times \|b\|_{L^\infty(X)} \|g\|_{L^2(E)} \|h\|_{L^2(F)} . \tag{6.77}$$

Using the fact that $\operatorname{dist}(E, G_1) \gtrsim \operatorname{dist}(E, F) > t$ and the assumption $\delta > \beta$, we can show that the integral in (6.77) is bounded by a constant times

$$\left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-(\beta-k)} \int_0^t \left(\frac{s}{t}\right)^{2m\delta} \left(\frac{s}{t}\right)^{2mk} \frac{ds}{s} \\ + \left(\frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-(\beta-k)} \int_t^\infty \left(\frac{t}{s}\right)^{-2m(\beta-k)} \left(\frac{t}{s}\right)^{2m\delta} \left(\frac{s}{t}\right)^{2mk} \frac{ds}{s} \\ \lesssim \left(1 + \frac{\operatorname{dist}(E,F)^{2m}}{t^{2m}}\right)^{-(\beta-k)} .$$

In summary, the above estimates yield that the operator family $\{\psi(tL)M_b(tL)^{-k}\}_{t>0}$ satisfies off-diagonal estimates of order $\min(\beta - k, \varepsilon_0\beta) > \frac{n}{4m}$ for every $1 \le k \le M$. This finishes the proof. □

7 Concluding remarks

7.1 Comparison with a $T(1)$-Theorem of Bernicot

While this thesis was under final preparation, we learned of an article [Ber10] of Bernicot with the title *"A T(1)-Theorem in relation to a semigroup of operators and applications to new paraproducts"*. Let us compare his results with those of us presented in this thesis.

We begin with a comparison of the underlying spaces. Bernicot assumes X to be a complete Riemannian manifold with a Riemannian measure satisfying the doubling condition, where we instead work in the more general setting of a space of homogeneous type. The major difference between these assumptions is, that we do not have the notion of a gradient at hand, as it is the case for Riemannian manifolds. We replace the notion of gradients by generalizations of Poincaré inequalities on metric spaces and reformulate the boundedness of Littlewood-Paley-Stein square functions on $L^2(X)$, cf. Section 6.4.

The most important difference between Bernicot's and our result is the assumption on the operator L. He assumes L to be a sectorial operator on $L^2(X)$ of order $2m$ with the properties

- L has a bounded holomorphic functional calculus on $L^2(X)$.
- The semigroup e^{-zL} is given by a kernel a_z that satisfies upper Poisson bounds of the form

$$|a_z(x,y)| \leq C \frac{1}{\mu(B(x,|z|^{1/2m}))} \left(1 + \frac{d(x,y)}{|z|^{1/2m}}\right)^{-\gamma}$$

  for sufficiently large $\gamma > 0$.
- There holds $L(1) = L^*(1) = 0$.

We also assume L to be a sectorial operator on $L^2(X)$ with a bounded holomorphic functional calculus. However, the second assumption on L of Bernicot is a fundamental constraint in comparison to our setting. The Poisson upper bounds in particular imply that the semigroup is uniformly bounded on L^p for all $p > 1$. We instead do only assume that e^{-tL} satisfies Davies-Gaffney estimates and some $L^p - L^2$ off-diagonal estimate and do not impose any kernel estimates on the semigroup. Our results can therefore be applied to a much larger class of operators, e.g to an elliptic operator whose heat semigroup fails pointwise bounds.

Then, of course, also the spaces $BMO_L(X)$ are different. Bernicot uses the definition of BMO spaces associated to operators as introduced by Duong and Yan in [DY05b], whereas we work with the enlarged class of spaces $BMO_L(X)$ due to Hofmann, Mayboroda, [HMa09], and, more generally, due to Duong and Li, [DL09], for operators L satisfying Davies-Gaffney estimates.

Under the above assumptions, Bernicot states a $T(1)$-Theorem associated to the operator L. His assumptions on the operator T are the following:

- $T : L^2(X) \to L^2(X)$ is linear and weakly continuous.
- The operator families $\{(tL)^M e^{-tL} T e^{-tL}\}_{t>0}$ and $\{(tL^*)^M e^{-tL^*} T^* e^{-tL^*}\}_{t>0}$ satisfy weak off-diagonal estimates.

- $T(1) \in BMO_L(X)$ and $T^*(1) \in BMO_{L^*}(X)$.

In comparison to that, in our $T(1)$-Theorem, Theorem 6.13, we only assume T to be a linear operator acting as $T : \mathcal{D}(L)\cap\mathcal{R}(L) \to L^2_{\text{loc}}(X)$ with $T^* : \mathcal{D}(L^*)\cap\mathcal{R}(L^*) \to L^2_{\text{loc}}(X)$ and do not impose any weak continuity on the operator. In some cases, such as for the application of the $T(1)$-Theorem to paraproducts which Bernicot presents, this might be less important, since one can work with suitable truncations of the operator. But it is in general not clear how to approximate or truncate the operator T in absence of kernel estimates. Moreover, Bernicot omits to define $T(1)$ in an appropriate way. As can be seen in Section 6.2, this task is not at all trivial when T is assumed to be a non-integral operator.
The type of off-diagonal estimates Bernicot assumes on the operator T, is weaker than ours. In Theorem 6.13, we assume weak off-diagonal estimates on $\{T\psi(tL)\}_{t>0}$ and $\{T^*\psi(tL^*)\}_{t>0}$ for some $\psi \in \Psi(\Sigma^0_\mu)$ and in the "on-diagonal" case, they might be too restrictive for some applications. This is the reason why we also stated Theorem 6.17, where the assumptions on T are similar to Bernicot's $T(1)$-Theorem. The formulation of this theorem was inspired by his result, however, in the formulation of the weak off-diagonal estimates, we do not restrict ourselves to the special functions $\psi(z) = z^M e^{-z}$ and $\phi(z) = e^{-z}$.
The statements of our $T(1)$-Theorems, Theorem 6.13 and Theorem 6.17, and Bernicot's $T(1)$-Theorem are at a formal level similar. Besides the above three assumptions on T, Bernicot also assumes a Poincaré inequality and the boundedness of the Littlewood-Paley-Stein square functions on $L^2(X)$ to be valid. In our setting, these conditions are resembled as assumptions (P) and (P*). But in view of the fact that the assumptions on the operator L are much more restrictive, the scope of Bernicot's theorem is much smaller.
Moreover, the proof Bernicot uses is completely different from ours. He follows the concept of proof due to Coifman and Meyer in [CM86], which is a simplified version of David and Journé's proof of the $T(1)$-Theorem for Calderón-Zygmund operators in [DJ84]. Bernicot does not work with paraproducts in the proof as we do, but directly applies some Carleson measure estimates. Most important, at various points he takes into account the existence of pointwise bounds for the semigroup e^{-tL}. He himself says in a comment in [Ber10], that in his proof, "the pointwise bound seems to be very important". In particular, Bernicot uses a kind of Sobolev inequality whose proof heavily relies on the existence of pointwise bounds for the semigroup.
He then also formulates as an open question:

"Can we expect a similar $T(1)$-Theorem under just off-diagonal decays for the heat kernel?"

This thesis gives a positive answer to his question.

Let us finally make some comments on paraproducts. Bernicot considers paraproduct operators of the form

$$\tilde{\Pi}(f) = \int_0^\infty \tilde{\psi}(tL)[\psi(tL)f\phi(tL)h]\,\frac{dt}{t},$$

where $h \in L^\infty(X)$, $\tilde{\psi}(z) = z^M e^{-z}(1-e^{-z})$, $\phi(z) = e^{-z}$ and ψ is either chosen as $\psi = \tilde{\psi}$ or $\psi = \phi$. In comparison to that, our definition of the paraproduct Π in Section 5.4 is

much more general, since we do not only consider operators $\tilde{\psi}(tL)$ that are constructed by the semigroup, but in general via functional calculus. Due to the fact that e^{-tL} is uniformly bounded on $L^\infty(X)$, Bernicot does not need any averaging operator as we do in the definition of Π.
If $\psi = \tilde{\psi}$, the proof of the boundedness of $\tilde{\Pi}$ on $L^2(X)$ is a simplified version of our Lemma 5.11. For the choice $\psi = \phi$, Bernicot applies his $T(1)$-Theorem to show that $\tilde{\Pi}$ is bounded on $L^2(X)$. We prove in Theorem 6.19 the same result in our more general setting, but with the additional assumption that e^{-tL} is uniformly bounded on $L^\infty(X)$. What we do not require, is the uniform boundedness of e^{-tL^*} on $L^\infty(X)$.
Bernicot then examines various extensions of $\tilde{\Pi}$ to L^p spaces, where $\tilde{\Pi}$ is considered as a bilinear operator. The results he obtains reach on the one hand further, since he considers the boundedness of paraproducts on a larger scale of L^p spaces and on weighted Lebesgue spaces, but these results are at the same time owed to the more restrictive assumptions on the operator L. In those cases, where the spaces $H^p_L(X)$ coincide with $L^p(X)$, cf. Proposition 4.41, one can obtain corresponding results for Π.

7.2 The role of constants

One of the challenges on the way to a $T(1)$-Theorem associated to sectorial operators was the question of the role of constants and cancellation conditions. If one examines the theory of Calderón-Zygmund operators, one observes that constants play an outstanding role. Let us illustrate their importance on the basis of two examples. First, if T is a singular integral operator with kernel k and f is a smooth function with compact support that satisfies the cancellation condition $\int f(x)\,dx = 0$, then one can write

$$Tf(x) = \int k(x,y)f(y)\,dy = \int [k(x,y) - k(x,y')]f(y)\,dy.$$

For the difference in the squared brackets, it is now possible to apply Hölder or Hörmander conditions for Calderón-Zygmund kernels.
Secondly, let us have a short look at the proof of the $T(1)$-Theorem of David and Journé in [DJ84]. There, the original operator T is via paraproducts reduced to an operator $\tilde{T}$ satisfying $\tilde{T}(1) = \tilde{T}^*(1) = 0$. Under this assumptions, one can construct appropriate approximation operators T_j with $T_j(1) = T_j^*(1) = 0$. The authors then aim to apply the Cotlar-Knapp-Stein lemma and thus have to check that the condition

$$\left\|T_j^* T_k\right\|_{L^2\to L^2} + \|T_j T_k^*\|_{L^2\to L^2} \le \omega(j-k)$$

with $\sum_k \sqrt{\omega(k)} < \infty$ is satisfied. In doing so, one considers the kernel $k_{j,k}$ of the composite operator $T_j^* T_k$, that can be written as

$$k_{j,k}(x,y) = \int k_j(z,x)k_k(z,y)\,dz = \int k_j(z,x)[k_k(z,y) - k_k(x,y)]\,dz$$

in view of the cancellation condition $T_j^*(1) = 0$. It is thus again possible to apply Hölder or Hörmander conditions for the kernel of T and within that, appropriate estimates for the kernel k_k of the approximation operator T_k.

If one now considers operators T that only satisfy certain off-diagonal estimates, e.g. the assumptions $(\mathrm{OD1})_\gamma$ and $(\mathrm{OD2})_\gamma$, instead of Calderón-Zygmund kernel estimates,

cancellation conditions are no longer applicable. In particular, it is in no way easier to consider operators T with the additional assumption $T(1) = T^*(1) = 0$. But the paraproduct again reduces T to a simplified operator $\tilde{T}$. For this operator $\tilde{T}$, we now work with a Poincaré inequality instead of Hölder or Hörmander estimates. See the proof of Theorem 6.13 and the splitting $J_M^0 = J_M^1 + J_M^2$ in (6.38) for details. The application of a Poincaré inequality involves the additional assumption (P), that resembles the boundedness of the Littlewood-Paley-Stein square function on $L^2(X)$.
In this context, also the averaging operator A_t comes into play. At first sight, the application of the averaging operator A_t in our context seems to be a reminiscent of the theory of standard BMO spaces, as this is just the operator that was replaced in the theory of the spaces $BMO_L(X)$ by an approximation operator associated to L. But it enables us to estimate paraproduct operators in lack of pointwise bounds for the heat semigroup and at the same time, appears to be very helpful for the use of the Poincaré inequality.
It would be interesting to know if one can get rid of the boundedness assumption for the Littlewood-Paley-Stein square function. One approach in this direction could be to substitute the usual Poincaré inequality by generalized Poincaré inequalities associated to L. Such generalized Poincaré inequalities were considered by Yan and Yang in [YY07], by Jiménez-del-Toro and Martell in [JM09] and by Badr, Jiménez-del-Toro and Martell in [BJM10]. The idea of the generalization is just the same as the one for generalized BMO spaces, namely to replace the averaging operator A_t by approximation operators associated to L. We leave this as an open question.
Let us finally mention that in the construction of Hardy spaces associated to operators, the lack of cancellation condition for molecules was compensated by quantitative analogues as described in Definition 4.1.

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