Threshold Corrections in Grand Unified Theories

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1. Introduction

What are the fundamental laws of nature? Is there a unified mathematical framework that describes all observed (and not yet observed) phenomena consistently? These are the questions that drive theoretical physics and particle physics in particular. Starting from the beginning of the previous century, remarkable progress in finding possible answers to these questions has been made. With the early formulation of Quantum Electrodynamics (QED) by Paul Dirac [1] in 1927 the importance of Quantum Field Theories (QFT) in the description of fundamental interactions became evident. The quest for a valid unification of Quantum mechanics and Special Relativity then lead to the discovery of the Dirac equation and this in turn to the prediction of antimatter as an interpretation of "negative energy solutions".

Certainly, another milestone on the long way to the ultimate goal was the theory of electroweak interactions by Glashow, Salam and Weinberg [2–4]. This theory, as QED, is formulated as a gauge theory, but this time based on the Lie group SU(2)×U(1) and spontaneously broken in order to account for the masses of fermions and weak gauge bosons. Together with Quantum Chromodynamics (QCD) [5], the theory of strong interactions, this structure is known as the Standard Model (SM) of particle physics today. The proof of renormalizability of anomaly-free gauge theories [6] by 't Hooft and Veltman in 1972 paved the way for the success of this model. Since those days the SM has continuously been confirmed by a tremendous amount of precision measurements [7] establishing it as a starting point for any further investigations in particle theory.

Despite the enormous insights about nature the SM and General Relativity has bestowed us, these two theories at the same time have revealed that there is unequally much more that we do not understand. The mere fact that we need two theories to describe all fundamental interactions of nature is unsatisfactory. Further inherent weaknesses of the SM are the mystery of charge quantization, the unexplained origin of neutrino masses and the question of how the pattern of masses and mixings in the SM arises, just to name a few. This has opened the way for innumerable new ideas and theories with a varying degree of speculation and persuasiveness. Without a doubt, the concept of Grand Unification that was first introduced by Georgi and Glashow [8] pertains to the most promising candidates of physics beyond the SM. In its simplest form a Grand Unified Theory (GUT) can be constructed on the basis of the simple gauge group SU(5) unifying electromagnetic, weak and strong interactions. It combines all SM fermions into its two smallest representations and can be broken spontaneously by an adjoint Higgs to the SM gauge group. Even more alluring are SO(10) GUTs that allow for the unification of all SM fermion fields of one generation together with a right-handed neutrino in a single 16-dimensional multiplet. Here, however, the symmetry-breaking chain becomes typically more involved and therefore increasingly less appealing than in SU(5) GUTs. Among the most intriguing predictions of GUTs are the instability of nucleons and the unification of gauge couplings at some high scale M_{GUT}, the latter being the concern of this thesis.

At around the same period of time when first GUT models have been formulated, it has been realized that a consistent quantum field theory need not be constructed on the basis of a direct product of the Poincaré group and internal (gauge) symmetries [9]. The yet unnoticed possibility, called Supersymmetry (SUSY), involves anticommuting generators realizing a symmetry between fermions and bosons. In the same way as Dirac predicted antimatter as the "negative energy solutions" of his equation, exact SUSY predicts the existence of superpartners that differ from the original particle only by its spin. These findings constituted an a posteriori justification for the previously discovered Wess-Zumino model [10], the simplest supersymmetric, renormalizable quantum field theory. Softly broken SUSY with TeV scale superpartners, and in particular the Minimal Supersymmetric Standard Model (MSSM) [11], has numerous attractive features. Among them are a means of stabilizing the electroweak scale, the existence of a dark matter candidate particle, a natural mechanism to provide electroweak symmetry breaking and the possibility to embed gravity. Most notably, however, in the context of this work is the fact that at the one-loop level the three gauge couplings in the MSSM apparently unify at a high scale of about 10^{16} GeV which is to be contrasted to the situation in the SM [12]. This means that the concept of Grand Unification can be particularly easily realized in conjunction with SUSY.

In order to test the unification hypothesis of gauge couplings at the two-loop level, a concrete SUSY GUT model has to be specified. This is because a consistent n-loop Renormalization Group (RG) analysis must include n-loop running and $(n-1)$-loop threshold corrections[1]. These corrections arise because the MSSM is viewed as an effective theory that emerges from a SUSY GUT by integrating out the super-heavy particles at some unphysical scale $\mu_{\mathrm{GUT}} = \mathcal{O}(10^{16}\mathrm{GeV})$. Since the threshold corrections depend on the masses of those particles, one can constrain the super-heavy mass spectrum by requiring the gauge couplings to unify. Of special interest in this procedure is the prediction for the colored triplet Higgs mass $M_{\mathrm{H_c}}$. This is because this parameter can also be constrained from the experimental bound on the proton decay rate in SUSY GUTs via dimension-five operators. Both constraints together can then be checked for compatibility providing a means to test the model.

As experimental precision on the input values α_s, α_{em} and $\sin^2\Theta$ is increasing, also higher order corrections in the unification analysis become significant. The work in hand is part of a project that aims at establishing a consistent three-loop gauge coupling unification analysis for phenomenologically interesting GUT models, in particular SUSY GUTs. Up to now only consistent two-loop analyses are available [13–20]. They use two-loop Renormalization Group Equations (RGEs) for the MSSM and the SUSY GUT and one-loop decoupling at the SUSY scale μ_{SUSY} and the GUT scale μ_{GUT}. However, most of those analyses fix the SUSY decoupling scale to $\mu_{\mathrm{SUSY}} = M_Z$ in order to start with MSSM RGEs at the electroweak scale right away. In order to enhance this by one loop, we need three-loop RGEs for the SM, the MSSM and the SUSY GUT under consideration and two-loop matching corrections at the SUSY and GUT thresholds. Though a considerable part of the required input is already available [16, 21–33], the following ingredients are still missing: Three-loop electroweak corrections to the SM running, two-loop electroweak corrections to the decoupling at μ_{SUSY} and, most notably, two-loop matching corrections at μ_{GUT}.

[1] We use the terms threshold correction, matching correction and decoupling more ore less interchangeably in this thesis.

The aim of this thesis is twofold: In a first part we are going to analyze the unification of gauge couplings at (almost) three loops using the above mentioned state-of-the-art input and compare with previous two-loop analyses. This will provide a motivation for the second and main part of this thesis: The calculation of the yet missing two-loop threshold corrections at the GUT scale. Since there is a considerable number of well motivated GUT models, the aim is to set up the calculation in a general framework in order to not to be restricted to only one of them. The idea is to have a general formula depending on group theory invariants and mass matrices of the theory. Specifying to a certain GUT model means to assign certain values to these quantities and obtaining a formula that only depends on the mass spectrum of that particular GUT. To realize this, we will be facing a number of challenges: First of all, we will need to find an efficient method to treat all the group theory factors that appear in the diagrams. Since the theory is spontaneously broken and furthermore contains unspecified invariant tensors from scalar self-couplings and Yukawa couplings, we must reduce them to a basic set of primitive invariants making only use of gauge invariance and nothing else. Moreover, we will need to carry out a one-loop renormalization program for the model. The subtleties as regards gauge fixing, ghost interactions and tadpole terms that also appear in the renormalization of the likewise spontaneously broken electroweak model [34] are enhanced by the nontrivial group structure in our case.

The remainder of this thesis is organized as follows: In chapter 2 we will review the SM, SUSY and GUTs, also introducing several GUT models that will be used hereafter. This chapter will also treat a few important concepts that will be useful later, as RG methods, basics about proton decay and Schur's Lemma. Chapter 3 is then devoted to the analysis to gauge coupling unification in SUSY GUTs using state-of-the-art input and based on ref. [35]. The theoretical framework for the calculation of two-loop matching coefficients at the GUT scale is introduced in chapters 4 and 5. While the first focuses on field-theoretical aspects like gauge-fixing, tadpole terms and renormalization, the latter describes the treatment and reduction of group theory factors. As an application of the calculation we present numerical results for the case of the Georgi-Glashow SU(5) Model in chapter 6[2]. Finally, we summarize our findings and discuss possibilities for the future of this project in chapter 7. In the appendix the reader can find derivations of group theory reduction identities that appear in the main text and further useful supplementary material.

[2] A substantial part of chapters 4-6 has been published in ref. [36]

2. Supersymmetric Grand Unified Theories

The Standard Model (SM) of particle physics has been extremely successful in describing low-energy phenomena. However, there are good reasons to believe that it is only an effective theory of some extended model that is also valid at higher energies. In this chapter we briefly review the SM and discuss possibilities of physics beyond it, in particular Grand Unification and SUSY. Besides introducing particular GUT models, we will also cover some basic concepts that will be needed in the course of this work. Among them are RG methods, proton decay and Schur's Lemma.

2.1. The Standard Model and its Limitations

The SM is a gauge theory based on the gauge group $\mathbf{G}_{\mathrm{SM}} \equiv \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ with chiral fermions sitting in the following representations:

Field	Q_I	u_I^c	d_I^c	L_I	e_I^c
Representation	$(\mathbf{3}, \mathbf{2}, \frac{1}{6})$	$(\overline{\mathbf{3}}, \mathbf{1}, -\frac{2}{3})$	$(\overline{\mathbf{3}}, \mathbf{1}, \frac{1}{3})$	$(\mathbf{1}, \mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, \mathbf{1}, 1)$

where $I = 1, 2, 3$ is a generation index. The notation $(\mathbf{3}, \mathbf{2}, \frac{1}{6})$ for example denotes that the respective field transforms as a triplet of $\mathrm{SU}(3)_C$, a doublet of $\mathrm{SU}(2)_L$ and carries hypercharge $\frac{1}{6}$. Postulating local gauge invariance leads to the introduction of gauge fields that live in the adjoint representation of $\mathbf{G}_{\mathrm{SM}}$:

Field	G_μ^a	W_μ^a	B_μ
Representation	$(\mathbf{8}, \mathbf{1}, 0)$	$(\mathbf{1}, \mathbf{3}, 0)$	$(\mathbf{1}, \mathbf{1}, 0)$

Each of the three gauge fields interacts with the fermions via gauge-kinetic terms. The respective coupling strength is denoted by the gauge couplings g_3, g_2 and g_1. For convenience, we also define

$$\alpha_i = \frac{g_i^2}{4\pi}\,, \qquad i = 1, 2, 3\,. \tag{2.1}$$

Since all SM fermions sit in complex representations of the gauge group, fermion mass terms are not gauge invariant and are therefore forbidden. In order to obtain massive fermions and weak gauge bosons, a scalar Higgs field in the representation $(\mathbf{1}, \mathbf{2}, \frac{1}{2})$ with a vacuum expectation value (vev) that breaks the gauge symmetry to $\mathrm{SU}(3)_C \times \mathrm{U}(1)_{em}$ is introduced. Then the weak gauge bosons obtain masses through the gauge-kinetic terms of the Higgs field and the fermions via their Yukawa interactions to the Higgs field. Moreover, exploiting the freedom to perform unitary rotations in generation space, all flavor and CP violation in the SM can be encoded in the unitary so-called CKM matrix V_{CKM}.

Despite the great success of the model described above, there are a number of open questions it poses: The quantization of charge remains a mystery because hypercharge is described by an abelian group $\mathrm{U}(1)_Y$, the generators of which can be arbitrarily rescaled. Moreover, it may appear unaesthetic that $\mathbf{G}_{\mathrm{SM}}$ is not a simple group and that the choice of representations of the fermions seem arbitrary. This is even more mysterious as one observes that this particular choice of representations and hypercharges leads to the exact cancellation of anomalies in the SM, which is required in order to guarantee renormalizability. Other puzzles include neutrino masses, hierarchies of fermion masses and the smallness of flavor-violating parameters in the CKM matrix. On the other hand, after adding right-handed neutrinos to the SM in order to account for neutrino masses, via the seesaw mechanism one obtains an important hint to the scale of Grand Unification. Besides the large number of 19 parameters in the SM and the so-called strong CP problem, there is another issue that is worth mentioning here: In contrast to fermion masses in the SM, the mass of the Higgs field is not protected by some symmetry against large radiative corrections. At the one-loop level, these corrections are proportional to $\mathcal{O}(\Lambda^2) + \mathcal{O}(m_f^2)$, where Λ is some cutoff scale, e.g. M_{Pl}, and m_f the mass of any heavy fermion that can propagate inside the loop. In order to obtain a physical Higgs mass of $\mathcal{O}(M_Z)$, an extreme fine-tuning of about 30 orders of magnitude is needed, which is considered as unnatural. This so-called hierarchy problem [37–40] can be solved by SUSY whereas some of the aforementioned problems are addressed by GUTs, which will be discussed in the next sections.

2.2. The Georgi-Glashow SU(5) Model

As soon as shortly after the establishment of the SM in the late sixties of the last century, first proposals of gauge theories based on a simple gauge group, containing $\mathbf{G}_{\mathrm{SM}}$ as a subgroup, came up. These theories are attractive not only from the aesthetic point of view as we will see. The requirements for the choice of such a group $\mathbf{G}$ are:

- The rank of $\mathbf{G}$ must be at least four in order to contain all Cartan generators[1] of $\mathbf{G}_{\mathrm{SM}}$.
- Since SM fermions live in complex representations of $\mathbf{G}_{\mathrm{SM}}$, also $\mathbf{G}$ must possess complex representations. Otherwise a doubling of the particle content would be required.

The first GUT proposed by Georgi and Glashow [8] in 1974 was based on the gauge group SU(5), the simplest group that fulfills the above requirements, and therefore may be considered as the “prototype GUT”. All SM fermion representations of one generation can be unified in the representations $\overline{\mathbf{5}}$ and $\mathbf{10}$ of SU(5) as can be seen from the following decompositions[2]:

$$\overline{\mathbf{5}} \;\to\; (\overline{\mathbf{3}}, \mathbf{1}, \tfrac{1}{3}) \oplus (\mathbf{1}, \overline{\mathbf{2}}, -\tfrac{1}{2})\,, \tag{2.2}$$

$$\mathbf{10} = [\mathbf{5} \times \mathbf{5}]_a \;\to\; (\overline{\mathbf{3}}, \mathbf{1}, -\tfrac{2}{3}) \oplus (\mathbf{3}, \mathbf{2}, \tfrac{1}{6})_a \oplus (\mathbf{1}, \mathbf{1}, 1)\,. \tag{2.3}$$

[1] The rank of the group is equal to the number of simultaneously diagonalizable generators. These (diagonal) generators are called Cartan generators.

[2] The index a indicates that the antisymmetric part of the representation has to be taken.

More explicitly, the embedding of SM fermions into these representations can be written as

$$\overline{\mathbf{5}} = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ \hline e \\ -\nu_e \end{pmatrix}_L , \qquad \mathbf{10} = \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} 0 & u_3^c & -u_2^c & -u_1 & -d_1 \\ -u_3^c & 0 & u_1^c & -u_2 & -d_2 \\ u_2^c & -u_1^c & 0 & -u_3 & -d_3 \\ \hline u_1 & u_2 & u_3 & 0 & -e^c \\ d_1 & d_2 & d_3 & e^c & 0 \end{array}\right)_L . \tag{2.4}$$

All fields have been chosen to be left-handed and the indices $1, 2, 3$ denote the color of the field.

The gauge bosons live in the adjoint $\mathbf{24}$ representation:

$$\mathbf{24} \to (\mathbf{8}, \mathbf{1}, 0) \oplus (\mathbf{1}, \mathbf{3}, 0) \oplus (\mathbf{1}, \mathbf{1}, 0) \oplus (\mathbf{3}, \overline{\mathbf{2}}, -\tfrac{5}{6}) \oplus (\overline{\mathbf{3}}, \mathbf{2}, \tfrac{5}{6}) \tag{2.5}$$

that obviously contains gluons, weak gauge bosons and B bosons in its decomposition. Moreover, the last two summands of the decomposition constitute additional gauge bosons that are not present in the SM and obtain a super-heavy mass in the course of GUT symmetry breaking. Again, these gauge bosons can be written more explicitly as an hermitian 5×5 matrix:

$$\begin{aligned} A_\mu &= \sum_{\alpha=1}^{24} A_\mu^\alpha T^\alpha \qquad (2.6) \\ &= \frac{1}{\sqrt{2}} \left(\begin{array}{ccc|cc} & \frac{1}{\sqrt{2}} \sum_{a=1}^{8} G_\mu^a \lambda^a & & X_{1\mu}^\star & Y_{1\mu}^\star \\ & & & X_{2\mu}^\star & Y_{2\mu}^\star \\ & & & X_{3\mu}^\star & Y_{3\mu}^\star \\ \hline X_{1\mu} & X_{2\mu} & X_{3\mu} & \frac{1}{\sqrt{2}} W_\mu^3 & W_\mu^+ \\ Y_{1\mu} & Y_{2\mu} & Y_{3\mu} & W_\mu^- & -\frac{1}{\sqrt{2}} W_\mu^3 \end{array}\right) + \frac{B_\mu}{2\sqrt{15}} \left(\begin{array}{ccc|cc} -2 & & & & \\ & -2 & & & \\ & & -2 & & \\ \hline & & & 3 & \\ & & & & 3 \end{array}\right) \end{aligned}$$

where the SU(5) generators in the fundamental representation have been denoted by T^α and the Gell-Mann matrices by λ^a. From the above equation it is also visible that the normalization of the hypercharge generator (the one belonging to the B boson) is fixed, which automatically leads to the quantization of electric charge. In particular, the ratios of hypercharges of the various SM multiplets are fixed to their right values solely by group theory.

In order to break the SU(5) symmetry to $\mathbf{G}_{\text{SM}}$, the simplest possibility is to introduce a scalar $\mathbf{24}_H$ in the 24-dimensional representation which obtains a super-heavy vev in the direction of the SM singlet (cf. eq. (2.5)). Furthermore, to break the electroweak symmetry, one usually introduces another scalar $\mathbf{5}_H$ in the 5-dimensional representation with an uncharged vev of $\mathcal{O}(M_Z)$. The field $\mathbf{5}_H$ also contains the usual SM Higgs doublet, as can be seen from eq. (2.2). Then the Yukawa Lagrangian of the Georgi-Glashow model can be written as [8, 41]

$$\begin{aligned} \mathcal{L}_{\text{Y}}^{\text{SU(5)}} &= \frac{1}{4} \mathbf{10}_I Y_{IJ}^U \mathbf{10}_J \mathbf{5}_H + \sqrt{2}\; \mathbf{10}_I Y_{IJ}^D \overline{\mathbf{5}}_J\, \mathbf{5}_H^\star + h.c. \\ &\equiv \frac{1}{4} Y_{IJ}^U \epsilon_{ijklm} \mathbf{10}_{ijI} C \mathbf{10}_{klJ} \mathbf{5}_{H_m} + \sqrt{2}\; Y_{IJ}^D \mathbf{10}_{ijI} C \overline{\mathbf{5}}_{iJ} \mathbf{5}_{H_j}^\star + h.c.\,. \end{aligned} \tag{2.7}$$

In the first line we also have introduced a shorthand notation that is common in the literature, where fundamental SU(5) indices $i, j \ldots$ and the Dirac charge conjugation matrix $C = i\gamma_2\gamma_0$

are suppressed. Moreover, ϵ_{ijklm} is the totally antisymmetric symbol with $\epsilon_{12345} = 1$ and the generation indices have been denoted by I, J. An important prediction of this model becomes apparent, if one expresses the second term in the above expression through SM fields:

$$\mathcal{L}_{\rm Y}^{\rm SU(5)} \supset -Y^D_{ij} \left(Q^T_i C d^c_j + e^{cT}_i C L_j\right) \epsilon\, h^\star + h.c. \tag{2.8}$$

where the transposition is only with respect to Dirac space and h is the SM Higgs doublet. Comparing with the SM Yukawa Lagrangian yields:

$$Y^E = \left(Y^D\right)^T . \tag{2.9}$$

Which predicts the unification of lepton and down-quark Yukawa couplings at the Grand Unification scale $M_{\rm GUT}$. Though phenomenologically this condition is in quite good agreement with experiment for the third generation, there are considerable deviations for the first and second generation [41, p. 271]. This problem can be solved by additional scalar multiplets [42] or higher dimensional operators [43–45]. In the gauge coupling unification analyses performed later in this work, we will implicitly assume the second solution. Then we can safely ignore this issue because higher-dimensional operators will only affect our analysis in an easily predictable manner.

Finally, the scalar potential for the Georgi-Glashow model with the discrete symmetry $\mathbf{5}_H \to -\mathbf{5}_H$ and $\mathbf{24}_H \to -\mathbf{24}_H$ reads [41, 46]

$$\begin{aligned} V(\mathbf{5}_H, \mathbf{24}_H) = &\\ & -\mu^2_{\mathbf{5}}\, \mathbf{5}^\star_{H_i} \mathbf{5}_{H_i} + \frac{b}{6} (\mathbf{5}^\star_{H_i} \mathbf{5}_{H_i})^2 \\ & -\frac{1}{2}\mu^2_{\mathbf{24}}\, \mathbf{24}_{H_\alpha} \mathbf{24}_{H_\alpha} + \frac{A}{4!}\, \mathrm{Tr}(T^\alpha T^\beta T^\gamma T^\delta) \mathbf{24}_{H_\alpha} \mathbf{24}_{H_\beta} \mathbf{24}_{H_\gamma} \mathbf{24}_{H_\delta} + \frac{B}{4!} \left(\mathbf{24}_{H_\alpha} \mathbf{24}_{H_\alpha}\right)^2 \\ & + c\, (T^\alpha T^\beta)_{ij} \mathbf{24}_{H_\alpha} \mathbf{24}_{H_\beta} \mathbf{5}^\star_{H_i} \mathbf{5}_{H_j} . \end{aligned} \tag{2.10}$$

Note that we have written the GUT-breaking scalar $\mathbf{24}_H$ as a 24-dimensional vector multiplet and not as an hermitian 5×5 matrix as usually done. This is in order to be consistent with our notation later in chapter 4. After the GUT-breaking scalar has developed a super-heavy vev $\langle \mathbf{24}_H \rangle = v$ in the direction of the Hypercharge generator (i.e. $v_\alpha = v_0\, \delta_{\alpha,24}$), the physical mass spectrum of the theory can be calculated in terms of the parameters of the Lagrangian:

$$M^2_\Sigma = \tfrac{1}{144}\, A\, v_0^2 , \quad M^2_{24} = \tfrac{1}{3}(\tfrac{7}{120} A + B)\, v_0^2 , \quad M^2_{\rm H_c} = \tfrac{1}{12}\, c\, v_0^2 . \tag{2.11}$$

The color octet, the isotriplet and the singlet in $\mathbf{24}_H$ obtain masses M_Σ, $2\,M_\Sigma$ and M_{24}, respectively (cf. eq. (2.5)). The remaining terms in eq. (2.5) constitute the Goldstone bosons. After imposing the fine-tuning condition $\mu^2_{\mathbf{5}} = \frac{3}{20} c v_0^2$ in order to obtain massless Higgs doublets, the color triplet in $\mathbf{5}_H$ obtains the mass $M_{\rm H_c}$. The super-heavy gauge bosons in the representation $(\mathbf{3}, \overline{\mathbf{2}}, -\frac{5}{6}) \oplus (\overline{\mathbf{3}}, \mathbf{2}, \frac{5}{6})$ have a common mass

$$M^2_{\rm X} = \tfrac{5}{12}\, g^2\, v_0^2 . \tag{2.12}$$

where g is the unique gauge coupling of SU(5). We neglect the effects of electroweak symmetry breaking here because the strong hierarchy $\mathcal{O}(M_W) \ll \mathcal{O}(M_{\rm GUT})$ renders them completely negligible for the calculation of matching effects at the GUT scale later in this thesis.

2.3. Supersymmetry (SUSY)

The most popular solution to the aforementioned hierarchy problem is SUSY. To supersymmetrize a theory all fields are promoted to superfields containing both fermionic and bosonic components. This leads to a cancellation of quadratic divergences in the Higgs self-energy. Furthermore, the so-called non-renormalization theorem holds, which states that there is only (logarithmically divergent) wave function renormalization in supersymmetric theories [47].

Theoretically, there is another intriguing motivation for SUSY: In 1967 Coleman and Mandula proved on the basis of some very reasonable assumptions that all possible symmetries of the S matrix must be described by a direct sum of the Poincaré algebra and the algebra describing internal symmetries [48]. However, this theorem assumes that all generators of the algebra fulfill commutation relations. Therefore, it was realized soon how to evade it, namely by allowing for generators fulfilling anticommutation relations also [49]. Haag, Łopuszánski, Sohnius then proved that, roughly speaking, this possibility, called SUSY, is the only possible extension of the Poincaré algebra [9].

For our purposes it is not necessary to introduce the elegant superfield formalism where spacetime is enhanced by four anticommuting Grassmann dimensions. In a supersymmetric theory all interactions are fixed by the choice of the superpotential. For a renormalizable theory it would contain up to trilinear terms [50–52]:

$$\mathcal{W} \quad = \quad L^i \phi_i + \frac{1}{2} M^{ij} \phi_i \phi_j + \frac{1}{6} y^{ijk} \phi_i \phi_j \phi_k \tag{2.13}$$

where the ϕ_i can be interpreted as the complex scalar components of the respective superfields. No complex conjugated fields are allowed in the superpotential (holomorphy condition). Then the Lagrangian of the theory can simply be computed by the following formula:

$$\mathcal{L}_{\text{chiral}} = \partial^\mu \phi_i^\star \partial_\mu \phi_i + i\bar{\psi}_i \overline{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2}\left(\frac{\partial^2 \mathcal{W}}{\partial \phi_i \partial \phi_j} \psi_i \psi_j + \frac{\partial^2 \mathcal{W}^\star}{\partial \phi_i^\star \partial \phi_j^\star} \bar{\psi}_i \bar{\psi}_j\right) - \frac{\partial \mathcal{W}}{\partial \phi_i} \frac{\partial \mathcal{W}^\star}{\partial \phi_i^\star} \tag{2.14}$$

where ψ_i are the (Weyl) fermionic partners of ϕ_i. For a supersymmetric non-abelian gauge theory, as it will be considered here, additional gauge interactions appear and the Lagrangian can be written in the general form [52]:

$$\begin{aligned}
\mathcal{L} \quad = \quad & \phi_i^\star (\delta_{ij} \partial_\mu + igT_{ij}^\alpha A_\mu^\alpha)(\delta_{jk} \partial_\mu - igT_{jk}^\alpha A_\mu^\alpha) \phi_k + i\bar{\psi}_i \overline{\sigma}^\mu (\delta_{ij} \partial_\mu - igT_{ij}^\alpha A_\mu^\alpha) \psi_j \\
& -\frac{1}{4} F_{\mu\nu}^\alpha F^{\mu\nu\alpha} + i\bar{\lambda}_\alpha \overline{\sigma}^\mu (\delta_{\alpha\beta} \partial_\mu - ig(T_{\mathcal{A}}^\gamma)_{\alpha\beta} A_\mu^\gamma) \lambda_\beta \\
& -\sqrt{2} g (\bar{\lambda}_\alpha \bar{\psi}_i T_{ij}^\alpha \phi_j + \phi_i^\star T_{ij}^\alpha \psi_j \lambda_\alpha) - \frac{1}{2}\left(\frac{\partial^2 \mathcal{W}}{\partial \phi_i \partial \phi_j} \psi_i \psi_j + \frac{\partial^2 \mathcal{W}^\star}{\partial \phi_i^\star \partial \phi_j^\star} \bar{\psi}_i \bar{\psi}_j\right) \\
& -V(\phi_i, \phi_i^\star) \,,
\end{aligned} \tag{2.15}$$

$$\begin{aligned}
V(\phi_i, \phi_i^\star) \quad = \quad & F_i F_i^\star + \frac{1}{2} D^\alpha D^\alpha \,, \\
& F_i = -\frac{\partial \mathcal{W}^\star}{\partial \phi_i^\star} \,, \quad D^\alpha = -g \phi_i^\star T_{ij}^\alpha \phi_j \,,
\end{aligned} \tag{2.16}$$

$$F_\alpha^{\mu\nu} = \partial^\mu A_\alpha^\nu - \partial^\nu A_\alpha^\mu + g f_{\alpha\beta\gamma} A_\beta^\mu A_\gamma^\nu \,. \tag{2.17}$$

Here again ψ_i and ϕ_i are the fermionic and complex bosonic components of the chiral superfield under consideration. The index i can run over multiple irreducible representations of the gauge group generated by T^α_{ij} with structure constants $f^{\alpha\beta\gamma}$. The superpartners of the gauge field A^α_μ are denoted by λ^α both of which live in the adjoint representation of the gauge group generated by $(T^\alpha_{\mathcal{A}})_{\beta\gamma} \equiv -if^{\alpha\beta\gamma}$.

The minimal supersymmetric extension of the SM is obtained by working out eq. (2.15) for the gauge group $\mathbf{G}_{\text{SM}}$ and supermultiplets with the quantum numbers of the SM (cf. section 2.1). Since the superpotential does not contain complex conjugated fields and in order to guarantee the cancellation of anomalies, one further needs a second Higgs multiplet in the representation $(\mathbf{1}, \overline{\mathbf{2}}, -\frac{1}{2})$. Furthermore, since exact SUSY implies that the fields in one supermultiplet have a common mass and none of the superpartners has been observed yet, SUSY must be broken. Since finding a suitable breaking mechanism is not easy, one usually just parametrizes the breaking by adding so-called soft-breaking terms [53] to the Lagrangian. This makes the superpartners heavy and can lead to electroweak symmetry breaking. The resulting theory is called the Minimal Supersymmetric Standard Model (MSSM). It is convenient to define the ratio of the two vacuum expectation values as

$$\tan\beta \equiv \frac{v_u}{v_d} \tag{2.18}$$

where v_u gives masses to up-type fermions and v_d to down-type fermions. Specifying a particular SUSY breaking mechanism usually leads to a particular form of the soft-breaking terms. Popular breaking scenarios as, anomaly-mediated SUSY breaking (AMSB) [54], gauge-mediated SUSY breaking (GMSB) [55] and Minimal Supergravity (mSUGRA) [56] are implemented in so-called spectrum generators [57–60], which produce the low-energy values for the soft-breaking terms for a particular scenario via RG running. This defines all the necessary masses and mixing of the MSSM in terms of a few specific parameters.

2.4. Supersymmetric Grand Unification

Though SUSY and GUTs are good candidates for physics beyond the SM each by themselves, there is even more striking evidence for the combination of the two: Since GUTs are based on a simple gauge group, one of their most important predictions is the unification of gauge couplings at some high scale M_{GUT}. As became clear by early LEP data, this cannot be achieved in the context of the SM without SUSY [12] (at least for minimal GUT scenarios). On the other hand, unification of gauge couplings works perfect at the one-loop [12] and two-loop [13–20] level in the context of the MSSM (cf. fig. 2.1). One of the important goals of this thesis is to test this statement at the three-loop level. Moreover, we will focus on the calculation of the two-loop matching corrections at the GUT scale that are needed for a consistent three-loop RG analysis, as we will see later. This section is devoted to the introduction of two SUSY GUT models that will be used later in our analysis.

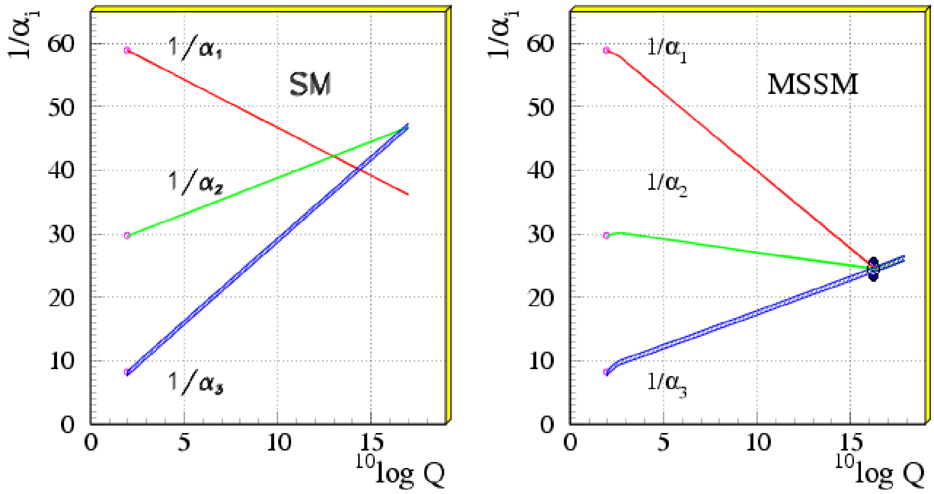


Figure 2.1.: *The running of gauge couplings at one loop in the SM (left) and the MSSM (right) [12]. Q is the renormalization scale.*

2.4.1. Minimal Supersymmetric SU(5)

The simplest supersymmetric GUT model is obtained by introducing superpartners for the fields of the Georgi-Glashow model. Due to the holomorphy condition of the superpotential and the request for anomaly freedom, also an additional Higgs superfield $\overline{\mathbf{5}}_H$ in the 5-dimensional representation is needed. Furthermore the scalar $\mathbf{24}_H$ that used to be real in the Georgi-Glashow model becomes complex (though it still transforms as the real adjoint representation) in order to maintain an equal number of fermionic and bosonic degrees of freedom within one supermultiplet. The superpotential of Minimal SUSY SU(5) [11] is then given by

$$\begin{aligned} \mathcal{W} \;=\;& M_1 \mathrm{Tr}(T^\alpha T^\beta)\mathbf{24}_{H_\alpha}\mathbf{24}_{H_\beta} + \lambda_1 \mathrm{Tr}(T^\alpha T^\beta T^\gamma)\mathbf{24}_{H_\alpha}\mathbf{24}_{H_\beta}\mathbf{24}_{H_\gamma} \\ & + \lambda_2 \overline{\mathbf{5}}_H T^\alpha \mathbf{24}_{H_\alpha}\mathbf{5}_H + M_2 \overline{\mathbf{5}}_H \mathbf{5}_H \\ & + \frac{1}{4}\mathbf{10}_I Y^U_{IJ}\mathbf{10}_J \mathbf{5}_H + \sqrt{2}\,\mathbf{10}_I Y^D_{IJ}\,\overline{\mathbf{5}}_J\,\overline{\mathbf{5}}_H \end{aligned} \tag{2.19}$$

where $I = 1,2,3$ is a generation index and the fields are understood to be the bosonic parts of the supermultiplet as described in the previous section. Again, SU(5) is broken to $\mathbf{G}_{\mathrm{SM}}$ by the adjoint Higgs boson obtaining a vev[3] $\langle \mathbf{24}_H \rangle = v/\sqrt{2}$ with $v_\alpha = v_0\,\delta_{\alpha,24}$ and $v_0 = -4\sqrt{30}M_1/(3\lambda_1)$ at tree-level. Choosing $\langle \overline{\mathbf{5}}_H \rangle$, $\langle \mathbf{5}_H \rangle \ll v_0$ and in addition imposing the (tree-level-) fine-tuning condition $M_2 = -\sqrt{3}\lambda_2 v_0/\sqrt{40} + \mathcal{O}(M_Z)$, the isodoublets in $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$ obtain masses of order M_Z. The super-heavy mass spectrum reads:

$$M^2_{\mathrm{X}} = \frac{5}{12} g^2 v_0^2\,, \quad M^2_{\mathrm{H_c}} = \frac{5}{24}\lambda_2^2 v_0^2\,, \quad M^2_\Sigma \equiv M^2_{(8,1)} = M^2_{(1,3)} = 25\,M^2_{(1,1)} = \frac{15}{32}\lambda_1^2 v_0^2\,, \tag{2.20}$$

[3]The factor $1/\sqrt{2}$ in the definition of the vev compared to the Georgi-Glashow model comes from the fact that in the supersymmetric version only the real part of the field (which comes with this factor) acquires a vev.

where the indices in parentheses refer to the SU(3) and SU(2) quantum numbers. Here M_Σ denotes the mass of the color octet part of the adjoint Higgs boson $\mathbf{24}_H$ and $M_{\mathrm{H_c}}$ stands for the mass of the color triplets of $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$, M_{X} is the mass of the gauge bosons and g the gauge coupling. The equality $M^2_{(8,1)} = M^2_{(1,3)}$ only holds if one neglects operators that are suppressed by $1/M_{\mathrm{Pl}}$ as we do for the moment [45, 61, 62]. Note that we have one physical mass parameter less than in the non-supersymmetric version of Minimal SU(5).

2.4.2. Missing Doublet Model

As already indicated previously, minimal SU(5) models suffer from a fine-tuning problem: In order to obtain Higgs doublets with masses at the electroweak scale and at the same time heavy color Higgs triplets, that suppress the proton decay rate to a reasonable level (cf. section 2.6), one has to fine-tune the parameters of the Lagrangian or the superpotential. Though the problem is, in some sense, less severe in supersymmetric models because due to the non-renormalization theorem the fine-tuning does not occur at each order of perturbation theory separately, it is still considered as unnatural. The Missing Doublet Model [63, 64] is designed to avoid this so-called doublet-triplet splitting problem [11]. This is achieved at the cost of introducing additional Higgs fields in the large SU(5) representations $\mathbf{50}_H$ and $\overline{\mathbf{50}}_H$ that do not contain any isodoublets and thus only couple to the color triplets in $\mathbf{5}$ and $\overline{\mathbf{5}}_H$. In order to break SU(5), another Higgs field in the 75-dimensional representation is used instead of the $\mathbf{24}_H$ in the minimal model. The superpotential reads [63, 64]

$$\begin{aligned}\mathcal{W} &= M_1 \mathrm{Tr}(\mathbf{75}_H^2) + \lambda_1 \mathrm{Tr}(\mathbf{75}_{\mathrm{H}}^3) + \lambda_2 \mathbf{5}_{\mathrm{H}} \mathbf{75}_{\mathrm{H}} \mathbf{50}_{\mathrm{H}} + \bar{\lambda}_2 \overline{\mathbf{5}}_{\mathrm{H}} \mathbf{75}_{\mathrm{H}} \overline{\mathbf{50}}_{\mathrm{H}} + \mathrm{M}_2 \overline{\mathbf{50}}_{\mathrm{H}} \mathbf{50}_{\mathrm{H}} \\ &\quad + \frac{1}{4} \mathbf{10}_I Y^U_{IJ} \mathbf{10}_J \mathbf{5}_H + \sqrt{2}\; \mathbf{10}_I Y^D_{IJ} \overline{\mathbf{5}}_J \overline{\mathbf{5}}_H \end{aligned} \tag{2.21}$$

After $\mathbf{75}_H$ develops a vev, the spectrum of the theory can be parametrized by the five mass parameters [18] $M_{\mathrm{X}}, M_{\mathrm{H_c}}, M_{\mathrm{H_{c'}}}, M_\Sigma$ and M_2. The last is usually assumed to be of $\mathcal{O}(M_{\mathrm{Pl}})$ such that the fields $\mathbf{50}_H$ and $\overline{\mathbf{50}}_H$ do not contribute to the running above the unification scale and below M_{Pl}. Otherwise, due to large group theory factors of these representations, perturbativity cannot be guaranteed. In chapter 3 we will show, that in GUTs that need large representations, as the Missing Doublet Model, the prediction of the gauge couplings at the electroweak scale have huge uncertainties coming from the insufficiently precise matching at the GUT scale. With the current theoretical input it is therefore not possible to reliably exclude such a model by an RG analysis. In order to facilitate a sufficiently precise unification analysis, it is indispensable to compute the two-loop matching corrections at the GUT scale. The foundations and the first step for such a calculation will be the subject of chapters 4 through 6.

2.5. Running and Decoupling

Since gauge coupling unification analyses are based on the RG, we shortly review its main properties here. We also consider the decoupling of heavy particles that plays a central role in this thesis. A more thorough introduction and treatment of these subjects can be found e.g. in refs. [65–67].

The main idea here is to test the prediction that there is only one (bare) gauge coupling in GUTs. For mass-independent renormalization schemes this prediction implies that the (renormalized) gauge couplings of the SM/MSSM should unify at a very high scale. The basic procedure for such an analysis is to take experimental measurements for α_i, $i = 1, 2, 3$ at the electroweak scale, run up to the SUSY scale, decouple the SUSY particles there, then run up to the GUT scale and decouple the heavy GUT particles there. If at the end the previously distinct couplings have a common value, the prediction is consistent with experiment. The details for the two ingredients, running and decoupling, will be described in the following two subsections.

2.5.1. Renormalization Group Equations

When perturbatively calculating Green's functions from a Lagrangian in Quantum Field Theory, one encounters divergences which are most commonly regulated using Dimensional Regularization (DREG) [6, 68] or Dimensional Reduction (DRED) [69–72]. The latter is more suitable for supersymmetric theories. In order to relate these divergent Green's functions to finite observables, one has to renormalize the (bare) parameters and fields of the Lagrangian. In this process of renormalization there is an arbitrariness of how much of the finite piece is subtracted together with the divergent part. One way in which this arbitrariness is parametrized, is by the renormalization scale μ which is inevitably introduced in order to maintain the original mass dimensions of the (renormalized) parameters in the Lagrangian when going from 4 to $4 - 2\epsilon$ spacetime dimension. The set of transformations on the parameters of the Lagrangian that describe the transition from a particular renormalization scale μ to another scale μ' can be parametrized by an abelian Lie group, the RG [73–75]. The action of the RG on the renormalized parameters can be described by RGEs which are first order differential equations.

As an example we consider the gauge coupling g of a gauge theory based on a simple group $\mathbf{G}$. The bare coupling g^0 gets renormalized by the following relation:

$$g^0 = \mu^\epsilon Z_g \, g \,, \tag{2.22}$$

with the renormalization constant Z_g in the modified minimal subtraction scheme ($\overline{\text{MS}}$). Since $\frac{dg^0}{d\mu} = 0$, we can derive a differential equation that describes the μ dependence of the renormalized coupling g:

$$\mu \frac{dg}{d\mu} = \beta \,, \qquad \beta = -\epsilon g - \frac{\mu}{Z_g} \frac{dZ_g}{d\mu} g \,, \tag{2.23}$$

where the gauge β function has been defined. Since β can be obtained by perturbative expansion, we can define $\overline{\text{MS}}$ β function coefficients and rewrite the above RGE as follows:

$$\frac{1}{2} \frac{d}{dt} \frac{\alpha}{4\pi} \quad = \quad \sum_{k=0}^{N-1} \left(\frac{\alpha}{4\pi} \right)^{k+2} \beta_k \,, \qquad \alpha = \frac{g^2}{4\pi} \tag{2.24}$$

We have also introduced the more convenient parameter $t = \ln(\mu)$ and used α instead of g. In QCD the gauge β function in the $\overline{\text{MS}}$ scheme is known to four loops [21, 22], though a complete three-loop SM gauge β function is still missing. For the MSSM [23,24] and a general supersymmetric GUT [26] full three-loop RGEs are available in the so-called $\overline{\mathrm{DR}}$ scheme which is a minimal subtraction scheme for DRED. The same is true for the most general single gauge coupling theory in $\overline{\mathrm{MS}}$ [25].

2.5.2. Decoupling of Heavy Particles

RGE analyses are most conveniently done using mass-independent renormalization schemes, such as $\overline{\mathrm{MS}}$ or $\overline{\mathrm{DR}}$. In these "unphysical" schemes the computation of β functions is simplified significantly, which makes them most suitable for this application. As GUTs predict a unique gauge coupling and the β function in $\overline{\mathrm{MS}}$ and $\overline{\mathrm{DR}}$ is mass-independent (and therefore independent of GUT-breaking effects) one would naively expect the gauge couping to stay unique up to low energy scales. Though in principle nothing is wrong with such an approach, we will see in a moment that it does not work practically. Mass-independent schemes have the well known drawback that the decoupling theorem [76] does not hold in its naive form. As a consequence, in the approach described above, observables of low-energy processes will depend logarithmically[4] on all the heavy particle masses of the GUT. This is unacceptable, since it would spoil perturbation theory by the presence of large logarithms $\ln(M_{\mathrm{GUT}}/\mu)$, where μ is the typical energy scale of the process and M_{GUT} a typical super-heavy particle mass. The way out of this dilemma is to use an effective theory [32, 77–85], where the heavy particles are integrated out at the GUT scale. This means that the heavy fields are removed from the original Lagrangian, which manifestly leads to power-suppressed contributions of $\mathcal{O}(1/M_{\mathrm{GUT}})$ in the effective Lagrangian. Moreover, the effects of the heavy particles are encoded in finite shifts of all the masses, couplings and fields of the theory. For the case of the gauge coupling this so-called decoupling relation reads:

$$\alpha_i(\mu_{\mathrm{GUT}}) \quad = \quad \zeta_{\alpha_i}(\mu_{\mathrm{GUT}}, \alpha(\mu_{\mathrm{GUT}}), M_{\mathrm{h}})\; \alpha(\mu_{\mathrm{GUT}})\,, \quad i = 1, 2, 3\,. \tag{2.25}$$

Here α_i and α stands for the $\overline{\mathrm{MS}}$ gauge coupling[5] in the effective theory (the SM or the MSSM) and full theory (GUT), respectively with the index $i = 1, 2, 3$ denoting the U(1), SU(2) and SU(3) coupling, respectively. μ_{GUT} is the unphysical scale, at which the decoupling is performed. At sufficiently high loop order predictions of physical observables must not depend on μ_{GUT} anymore. The remaining dependence on this scale gives us an estimation of the theory uncertainty of the prediction. ζ_{α_i} is the so-called matching coefficient that depends on all the mass parameters of the particles that have been integrated out. They are abbreviated by M_{h} in eq. (2.25). After performing the decoupling, eq. (2.25), at a scale $\mu_{\mathrm{GUT}} = \mathcal{O}(M_{\mathrm{GUT}})$ one uses the RGEs of the effective theory to compute the evolution of the three distinct gauge couplings down to the electroweak scale.

[4] At the two-loop order also more complicated functions of the heavy masses can appear.

[5] For simplicity we will only speak about $\overline{\mathrm{MS}}$ parameters from now on. If a SUSY GUT is considered, all $\overline{\mathrm{MS}}$ parameters will be replaced by $\overline{\mathrm{DR}}$ parameters.

For the computation of ζ_{α_i} we follow refs. [66, 67], where the construction of the effective Lagrangian is described for the case of QCD. We start with a GUT Lagrangian $\mathcal{L}^{\mathrm{GUT}}$ (full theory). It contains fields with the mass of order M_{GUT} and massless fields. (We neglect electroweak symmetry breaking here.)

$$\mathcal{L}^{\mathrm{GUT}} = \mathcal{L}^{\mathrm{SM}} + \mathcal{L}^{\mathrm{heavy}} \,. \tag{2.26}$$

First we define the bare decoupling coefficients for the light gauge fields, the light ghost fields and the gauge coupling. Renormalization will be performed afterwards.

$$A_\mu^{(0)\prime,a_i} = \sqrt{\zeta_{3_i}^{(0)}} A_\mu^{(0),a_i} \,, \qquad c^{(0)\prime,a_i} = \sqrt{\tilde{\zeta}_{3_i}^{(0)}} c^{(0),a_i} \,, \qquad g_i^{(0)} = \sqrt{\zeta_{g_i}^{(0)}} g^{(0)} \quad i = 1,2,3 \,. \tag{2.27}$$

The bare fields on the left-hand side belong to the effective theory, whereas the ones on the right-hand side are the fields that appear in $\mathcal{L}^{\mathrm{GUT}}$. Of course there are similar relations for all the other fields and couplings of the theory, but for our purposes these definitions are sufficient. The effective Lagrangian can then be defined by the following equation

$$\begin{aligned} \mathcal{L}^{\mathrm{eff}}(\ldots \sqrt{\zeta_{3_i}^{(0)}} A_\mu^{(0),a_i}, \sqrt{\tilde{\zeta}_{3_i}^{(0)}} c^{(0),a_i}, \sqrt{\zeta_{g_i}^{(0)}} g^{(0)}, \ldots) = \\ \mathcal{L}^{\mathrm{SM}}(\ldots, A_\mu^{(0)\prime,a_i}, c^{(0)\prime,a_i}, g_i^{(0)}, \ldots) + \mathcal{O}(\frac{1}{M_{\mathrm{GUT}}}) \end{aligned} \tag{2.28}$$

Green's functions of the light fields computed from $\mathcal{L}^{\mathrm{SM}}$ agree with those from $\mathcal{L}^{\mathrm{eff}}$ up to terms of order $1/M_{\mathrm{GUT}}$. We exploit this fact in order to compute the decoupling constant of the light gauge field. The following equalities hold up to terms of order $1/M_{\mathrm{GUT}}$:

$$\begin{aligned} \frac{\delta^{a_i b_i}\left(-g_{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right)}{-p^2(1+\Pi^0_{A,i}(p^2))} &= i \int d^4x e^{ipx} \langle T A_\mu^{(0),a_i}(x) A_\nu^{(0),b_i}(0)\rangle \\ &= \frac{i}{\zeta_{3_i}^{(0)}} \int d^4x e^{ipx} \langle T A_\mu^{(0)\prime,a_i}(x) A_\nu^{(0)\prime,b_i}(0)\rangle \\ &= \frac{1}{\zeta_{3_i}^{(0)}} \frac{\delta^{a_i b_i}\left(-g_{\mu\nu} + \frac{p^\mu p^\nu}{p^2}\right)}{-p^2(1+\Pi^{0\prime}_{A,i}(p^2))} \end{aligned} \tag{2.29}$$

where we have used eq. (2.27). $\Pi^0_{A,i}(p^2)$ and $\Pi^{0\prime}_{A,i}(p^2)$ are the transverse parts of the 1-particle irreducible gauge boson two-point function in the full and the effective theory. This yields an expression for the bare decoupling coefficient

$$\zeta_{3_i}^{(0)} = \frac{1+\Pi^0_{A,i}(p^2)}{1+\Pi^{0\prime}_{A,i}(p^2)} \,. \tag{2.30}$$

We expand the two-point functions for small external momentum p and keep only the leading term. Since $\Pi^{0\prime}_{A,i}(0)$ contains only light degrees of freedom, all the integrals that appear are scaleless and can be set to zero in the framework of DREG. Finally, we obtain a simple expression for $\zeta_{3_i}^{(0)}$:

$$\zeta_{3_i}^{(0)} = 1 + \Pi^{0,h}_{A,i}(0) \,. \tag{2.31}$$

The superscript h denotes the "hard part" of the Green's function, i.e. the bare decoupling coefficient is calculated from diagrams that contain at least one heavy line. Similar considerations for the two-point function with external light ghost fields and the light ghost-gauge-boson three-point function yields similar formulae for the bare decoupling of the ghost field and the vertex:

$$\begin{aligned}\tilde{\zeta}_{3i}^{(0)} &= 1 + \Pi_{c,i}^{0,h}(0)\,, \\ \tilde{\zeta}_{1i}^{(0)} &= 1 + \Gamma_{Ac^\dagger c,i}^{0,h}(0,0)\,.\end{aligned} \tag{2.32}$$

Finally, applying Slavnov-Taylor identities to the ghost-gauge-boson vertex, analogously as in the case of the gauge coupling renormalization constant, the bare decoupling coefficient for the gauge coupling can be derived,

$$\zeta_{g_i}^{(0)} = \frac{\tilde{\zeta}_{1i}^{(0)}}{\tilde{\zeta}_{3i}^{(0)}\sqrt{\zeta_{3i}^{(0)}}} \tag{2.33}$$

In order to obtain the decoupling coefficient for the renormalized gauge couplings, which is of more interest for us, we just have to take into account the relevant renormalization constants and arrive at our final formula:

$$\zeta_{\alpha_i} = \left(\frac{Z_g}{Z_{g_i}} \frac{\tilde{\zeta}_{1_i}^{(0)}}{\tilde{\zeta}_{3_i}^{(0)}\sqrt{\zeta_{3_i}^{(0)}}} \right)^2 , \quad i = 1, 2, 3\,. \tag{2.34}$$

The $\overline{\text{MS}}$ renormalization constants for the gauge coupling in the full and effective theory are denoted by Z_g and Z_{g_i}, respectively. Recall that the index i takes care of the fact that the SM gauge group is not simple and labels whether an external gauge bosons and ghosts belonging to U(1), SU(2) or SU(3) have to be taken.

At this point a remark concerning the μ_{GUT} dependence in different renormalization schemes might be helpful. In mass-dependent renormalization schemes the threshold effect around the GUT scale is obtained correctly by including the effect of the heavy GUT particles in the β function. As there is no explicit matching performed in such schemes, there is no dependence on the unphysical scale μ_{GUT} at all. Therefore, the question might arise how the μ_{GUT} dependence of ζ_{α_i} in mass-independent renormalization schemes, as in eq. (2.34), comes about. The answer is that by naively changing the renormalization scheme from mass-dependent to mass-independent, low-energy observables obtain a logarithmic dependence on an unphysical renormalization scale μ that has not been present before. These are the large logarithms $\ln(\mu/M_{\text{GUT}})$ that would spoil perturbation theory in the naive mass-independent renormalization scheme. In our (effective theory) approach, however, we absorb these logarithms in a redefinition of the gauge coupling and choose the unphysical scale $\mu = \mu_{\text{GUT}}$ in the vicinity of the heavy GUT masses, which introduces an explicit dependence of ζ_{α_i} on μ_{GUT}.

2.5.3. One-Loop Decoupling Coefficients for Various GUT Models

In this subsection the known one-loop decoupling coefficients are listed for the three GUT models that have been described earlier in this chapter. As a starting point we take the general formula from refs. [31, 33]. We also repeated their calculation and found agreement:

$$\begin{aligned}
\zeta_{\alpha_i}^{\overline{\mathrm{MS}}}(\mu_{\mathrm{GUT}}) &= 1+\frac{\alpha^{\overline{\mathrm{MS}}}(\mu_{\mathrm{GUT}})}{4\pi}\Bigg[-\frac{1}{12}\,\mathrm{Tr}(T^{a_i}T^{a_i}\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_H^2}\right)) \\
&\quad -\frac{1}{12}\,\mathrm{Tr}(T^{a_i}T^{a_i}\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_S^2}\right))+\frac{7}{4}\,\mathrm{Tr}(T_{\mathcal{A}}^{a_i}T_{\mathcal{A}}^{a_i}\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\mathrm{X}}^2}\right)) \\
&\quad -\frac{2}{3}\,\mathrm{Tr}(T^{a_i}T^{a_i}\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_D^2}\right))-\frac{1}{3}\,\mathrm{Tr}(T^{a_i}T^{a_i}\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_M^2}\right)) \\
&\quad +\frac{1}{6}\,\mathrm{Tr}(T_{\mathcal{A}}^{a_i}T_{\mathcal{A}}^{a_i})\Bigg], \qquad (\text{no sum over } a_i)\,. \qquad (2.35)
\end{aligned}$$

Here M_S, M_H M_{X} M_D and M_M are the mass matrices of GUT-breaking scalars, all other scalars, gauge bosons, Dirac fermions and Majorana fermions with the generators T^{a_i} taken in the appropriate representation. Note that there is no sum performed over the adjoint index a_i belonging to the group factor $\mathbf{G}_i$ in $\mathbf{G}_{\mathrm{SM}}=\prod_i \mathbf{G}_i$. Also due to the presence of the mass matrices in the logarithms, the trace is only performed over the subspace of super-massive particles. In $\overline{\mathrm{DR}}$ the constant term $\frac{1}{6}\,\mathrm{Tr}(T_{\mathcal{A}}^{a_i}T_{\mathcal{A}}^{a_i})$ with the adjoint generators $T_{\mathcal{A}}^{a_i}$ in eq. (2.35). is not present.

Computing the theory dependent group theory factors and mass matrices, our results for Minimal SUSY SU(5) and the Missing Doublet Model agree with the findings of e.g. refs. [13, 18]. For Minimal SUSY SU(5) the one-loop decoupling coefficients in $\overline{\mathrm{DR}}$ read

$$\begin{aligned}
\zeta_{\alpha_1}(\mu_{\mathrm{GUT}}) &= 1+\frac{\alpha(\mu_{\mathrm{GUT}})}{4\pi}\left[-\frac{2}{5}\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\mathrm{H_c}}^2}\right)+10\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\mathrm{X}}^2}\right)\right], \\
\zeta_{\alpha_2}(\mu_{\mathrm{GUT}}) &= 1+\frac{\alpha(\mu_{\mathrm{GUT}})}{4\pi}\left[-2\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\Sigma}^2}\right)+6\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\mathrm{X}}^2}\right)\right], \\
\zeta_{\alpha_3}(\mu_{\mathrm{GUT}}) &= 1+\frac{\alpha(\mu_{\mathrm{GUT}})}{4\pi}\left[-\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\mathrm{H_c}}^2}\right)-3\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\Sigma}^2}\right)+4\ln\left(\frac{\mu_{\mathrm{GUT}}^2}{M_{\mathrm{X}}^2}\right)\right], \quad (2.36)
\end{aligned}$$

with the mass parameters defined in subsection 2.4.1. For simplicity we keep from the list of arguments of the coefficients ζ_{α_i} only the decoupling scale. In the case of the Missing Doublet

Model the decoupling constants read [6]

$$
\begin{aligned}
\zeta_{\alpha_1}(\mu_{\rm GUT}) &= 1+\frac{\alpha(\mu_{\rm GUT})}{4\pi}\Bigg[-\frac{2}{5}\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm H_c}^2}\right)-\frac{2}{5}\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm H_{c'}}^2}\right)+10\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm X}^2}\right)\\
&\qquad -20\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\Sigma}^2}\right)+10\ln\left(\frac{64}{625}\right)\Bigg],\\
\zeta_{\alpha_2}(\mu_{\rm GUT}) &= 1+\frac{\alpha(\mu_{\rm GUT})}{4\pi}\Bigg[-22\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\Sigma}^2}\right)+6\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm X}^2}\right)+6\ln\left(\frac{4}{25}\right)\Bigg],\\
\zeta_{\alpha_3}(\mu_{\rm GUT}) &= 1+\frac{\alpha(\mu_{\rm GUT})}{4\pi}\Bigg[-\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm H_c}^2}\right)-\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm H_{c'}}^2}\right)-23\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\Sigma}^2}\right)\\
&\qquad +4\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm X}^2}\right)+4\ln\left(\frac{64}{78125}\right)\Bigg]. \qquad (2.37)
\end{aligned}
$$

As explained in subsection 2.4.2, the mass M_2 does not contribute here since it is usually assumed to be at or above the Planck scale in order to maintain perturbativity. Finally, the formulae for the Georgi-Glashow model are found to be:

$$
\begin{aligned}
\zeta_{\alpha_1}(\mu_{\rm GUT}) &= 1+\frac{\alpha(\mu_{\rm GUT})}{4\pi}\Bigg[-\frac{1}{15}\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm H_c}^2}\right)+\frac{35}{2}\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm X}^2}\right)+\frac{5}{3}\Bigg], \qquad (2.38)\\
\zeta_{\alpha_2}(\mu_{\rm GUT}) &= 1+\frac{\alpha(\mu_{\rm GUT})}{4\pi}\Bigg[-\frac{1}{3}\ln\left(\frac{\mu_{\rm GUT}^2}{4M_{\Sigma}^2}\right)+\frac{21}{2}\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm X}^2}\right)+1\Bigg],\\
\zeta_{\alpha_3}(\mu_{\rm GUT}) &= 1+\frac{\alpha(\mu_{\rm GUT})}{4\pi}\Bigg[-\frac{1}{6}\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm H_c}^2}\right)-\frac{1}{2}\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\Sigma}^2}\right)+7\ln\left(\frac{\mu_{\rm GUT}^2}{M_{\rm X}^2}\right)+\frac{2}{3}\Bigg].
\end{aligned}
$$

Here the result is given in $\overline{\rm MS}$ as can be seen from the presence of the constant terms. The mass parameters have been defined in subsection 2.2. The ultimate goal of our project is to extend these formulae to the two-loop level in the spirit of eq. (2.35).

2.5.4. One-Loop Decoupling Coefficients for the Matching of the MSSM to the SM

In the RG analysis that will be presented in chapter 3 we will also need to perform a matching of the MSSM to the SM. Therefore, for convenience we list all relevant gauge coupling one-loop decoupling coefficients here. For the decoupling of α_s actually the two-loop contributions are known [29, 30]. However, because the expression is huge, we only give the one-loop formula here [86]:

$$
\zeta_{\alpha_s}(\mu_{\rm SUSY}) = 1+\frac{\alpha_s(\mu_{\rm SUSY})}{4\pi}\left[-2\ln\left(\frac{\mu_{\rm SUSY}^2}{m_{\tilde{g}}^2}\right)-\frac{1}{3}\sum_{\tilde{q}}\ln\left(\frac{\mu_{\rm SUSY}^2}{m_{\tilde{q}_1}m_{\tilde{q}_2}}\right)\right]. \qquad (2.39)
$$

[6] The occurrence of the last term in the parentheses of each equation is due to the use of relations between the super-heavy masses.

The summation is performed over all six squark flavors $\tilde{q}$ and $m_{\tilde{g}}$ denotes the gluino mass. $\mu_{\rm SUSY}$ is the unphysical scale at which the matching is performed. No change of renormalization scheme is included in eq. (2.39), i.e. the formula can be used either to relate $\alpha_s^{\overline{\rm MS},{\rm MSSM}}$ to $\alpha_s^{\overline{\rm MS},{\rm SM}}$ or $\alpha_s^{\overline{\rm DR},{\rm MSSM}}$ to $\alpha_s^{\overline{\rm DR},{\rm SM}}$. Note also that the top-quark is not integrated out here since the effective theory is the full SM.

For the decoupling of α_1 and α_2 at $\mu_{\rm SUSY}$ one needs to take into account electroweak breaking effects which makes the calculation less straightforward. This is because the effective theory cannot be considered SU(2)×U(1) gauge invariant if the decoupling scale $\mu_{\rm SUSY}$ is relatively low, i.e. $\mathcal{O}(100{\rm GeV})$. In ref. [16] the treatment of these effects is described in detail and leads to the following formulae for the decoupling of the electromagnetic coupling α_{em} and for α_2:

$$\begin{aligned}
\zeta_{\alpha_{em}}(\mu_{\rm SUSY}) &= 1+\frac{\alpha_{em}(\mu_{\rm SUSY})}{4\pi}\Bigg[-\frac{8}{9}\sum_{\tilde{u}}\ln\left(\frac{\mu^2_{\rm SUSY}}{m_{\tilde{u}_1}m_{\tilde{u}_2}}\right)-\frac{2}{9}\sum_{\tilde{d}}\ln\left(\frac{\mu^2_{\rm SUSY}}{m_{\tilde{d}_1}m_{\tilde{d}_2}}\right)\\
&\quad -\frac{8}{3}\ln\left(\frac{\mu^2_{\rm SUSY}}{m_{\tilde{\chi}_1^\pm}m_{\tilde{\chi}_2^\pm}}\right)-\frac{2}{3}\sum_{\tilde{l}}\ln\left(\frac{\mu^2_{\rm SUSY}}{m_{\tilde{l}_1}m_{\tilde{l}_2}}\right)-\frac{1}{3}\ln\left(\frac{\mu^2_{\rm SUSY}}{m^2_{H^\pm}}\right)\Bigg]\\
\zeta_{\alpha_2}(\mu_{\rm SUSY}) &= 1+\frac{\alpha_2(\mu_{\rm SUSY})}{4\pi}\Bigg[-\frac{4}{3}\sum_{\tilde{u}}\left[\ln\left(\frac{\mu^2_{\rm SUSY}}{m^2_{\tilde{u}_1}}\right)+\sin^2\theta_{\tilde{u}}\ln\left(\frac{m^2_{\tilde{u}_2}}{m^2_{\tilde{u}_1}}\right)\right]\\
&\quad -\frac{2}{3}\sum_{\tilde{d}}\left[\ln\left(\frac{\mu^2_{\rm SUSY}}{m^2_{\tilde{d}_1}}\right)+\sin^2\theta_{\tilde{d}}\ln\left(\frac{m^2_{\tilde{d}_2}}{m^2_{\tilde{d}_1}}\right)\right]\\
&\quad -\frac{4}{3}\ln\left(\frac{\mu^2_{\rm SUSY}}{m^2_{\tilde{\chi}_1^\pm}}\right)-\frac{2}{3}\ln\left(\frac{\mu^2_{\rm SUSY}}{m^2_{\tilde{\chi}_2^\pm}}\right)-\frac{1}{3}\left[|(Z_-)_{12}|^2+|(Z_+)_{12}|^2\right]\ln\left(\frac{m^2_{\tilde{\chi}_1^\pm}}{m^2_{\tilde{\chi}_2^\pm}}\right)\\
&\quad -\frac{1}{6}\sum_{\tilde{l}}\ln\left(\frac{\mu^2_{\rm SUSY}}{m^2_{\tilde{l}_1}}\right)-\frac{1}{6}\ln\left(\frac{\mu^2_{\rm SUSY}}{m^2_{H^\pm}}\right)\Bigg], \qquad (2.40)
\end{aligned}$$

where again no change of renormalization scheme is included. The masses of up-squarks, down-squarks, charginos, charged sleptons and charged Higgses are denoted by $m_{\tilde{u}_i}$, $m_{\tilde{d}_i}$, $m_{\tilde{\chi}_i^\pm}$, $m_{\tilde{l}_i}$ and $m_{H^\pm}$, respectively. Furthermore Z_- and Z_+ are chargino mixing matrices defined by [87]

$$(Z_-)^T\begin{pmatrix} m_{\tilde{g}_2} & \frac{ev_u}{\sqrt{2}\sin\Theta}\\ \frac{ev_d}{\sqrt{2}\sin\Theta} & \mu\end{pmatrix}Z_+=\begin{pmatrix} m_{\tilde{\chi}_1^\pm} & 0\\ 0 & m_{\tilde{\chi}_2^\pm}\end{pmatrix} \qquad (2.41)$$

where the soft-breaking mass of the SU(2) gaugino $m_{\tilde{g}_2}$, the Weinberg angle Θ and the Higgs mass parameter μ from the MSSM superpotential appear. The squark field mixing angles are defined as

$$\begin{pmatrix}\tilde{q}_1\\ \tilde{q}_2\end{pmatrix}=\begin{pmatrix}\cos\theta_{\tilde{q}} & \sin\theta_{\tilde{q}}\\ -\sin\theta_{\tilde{q}} & \cos\theta_{\tilde{q}}\end{pmatrix}\begin{pmatrix}\tilde{q}_L\\ \tilde{q}_R\end{pmatrix}. \qquad (2.42)$$

The appearance of these mixing parameters in eq. (2.40) is a manifestation of the SU(2)×U(1)-breaking effects. In order to properly match the MSSM to the SM, one also has to take into account the change of renormalization scheme from $\overline{\mathrm{DR}}$ to $\overline{\mathrm{MS}}$. This is provided by the following general two-loop formula which is valid for a simple gauge group with only fermions in the theory [88, 89]:

$$\begin{aligned}\alpha^{\overline{\mathrm{DR}}}(\mu_{\mathrm{SUSY}}) &= \alpha_s^{\overline{\mathrm{MS}}}(\mu_{\mathrm{SUSY}})\Bigg[1+\frac{\alpha^{\overline{\mathrm{MS}}}(\mu_{\mathrm{SUSY}})}{\pi}\frac{I_2(\Pi^{\mathcal{A}})}{12}+\left(\frac{\alpha^{\overline{\mathrm{MS}}}(\mu_{\mathrm{SUSY}})}{\pi}\right)^2\frac{11}{72}I_2(\Pi^{\mathcal{A}})^2\\ &\quad -\frac{\alpha^{\overline{\mathrm{MS}}}(\mu_{\mathrm{SUSY}})}{\pi}\frac{\alpha_e(\mu_{\mathrm{SUSY}})}{\pi}\frac{1}{8}C_2(\Pi^{\mathcal{F}})I_2(\Pi^{\mathcal{F}})\Bigg]. \end{aligned}\tag{2.43}$$

For the bottom-quark mass the relation reads [88, 89]

$$\begin{aligned}m_b^{\overline{\mathrm{DR}}}(\mu_{\mathrm{SUSY}}) &= m_b^{\overline{\mathrm{MS}}}(\mu_{\mathrm{SUSY}})\Bigg[1-\frac{\alpha_e(\mu_{\mathrm{SUSY}})}{\pi}\frac{1}{4}C_2(\Pi^{\mathcal{F}})\\ &+\left(\frac{\alpha^{\overline{\mathrm{MS}}}(\mu_{\mathrm{SUSY}})}{\pi}\right)^2\frac{11}{192}I_2(\Pi^{\mathcal{A}})C_2(\Pi^{\mathcal{F}})\\ &-\frac{\alpha^{\overline{\mathrm{MS}}}(\mu_{\mathrm{SUSY}})}{\pi}\frac{\alpha_e(\mu_{\mathrm{SUSY}})}{\pi}\left(\frac{1}{4}C_2(\Pi^{\mathcal{F}})^2+\frac{3}{32}I_2(\Pi^{\mathcal{A}})C_2(\Pi^{\mathcal{F}})\right)\\ &+\left(\frac{\alpha_e(\mu_{\mathrm{SUSY}})}{\pi}\right)^2\left(\frac{3}{32}C_2(\Pi^{\mathcal{F}})^2+\frac{1}{32}C_2(\Pi^{\mathcal{F}})I_2(\Pi^{\mathcal{F}})\right)\Bigg]. \end{aligned}\tag{2.44}$$

Here we have used the notation of chapter 5 for the Casimir invariants and Dynkin indices. For the case of 6-flavor QCD they have the values

$$I_2(\Pi^{\mathcal{A}})=3\,,\qquad C_2(\Pi^{\mathcal{F}})=\frac{4}{3}\,,\qquad I_2(\Pi^{\mathcal{F}})=6\cdot\frac{1}{2}\,.\tag{2.45}$$

We cite the two-loop formula here in order to obtain sufficient precision also for the decoupling of α_s, where ζ_{α_s} is known up to two loops. For α_2 the one-loop part of eq. (2.43) still can be used by inserting the appropriate value for the quadratic Casimir invariant $I_2(\Pi^{\mathcal{A}})=2$. For α_1 this invariant vanishes and therefore there is no difference between $\alpha_1^{\overline{\mathrm{MS}}}$ and $\alpha_1^{\overline{\mathrm{DR}}}$ at the one-loop level. The evanescent coupling α_e that differs from α due to different renormalizations in DREG is computed for a degenerate squark mass $m_{\tilde q}$ by the following formula [90]

$$\begin{aligned}\alpha_e(\mu_{\mathrm{SUSY}}) &= \alpha^{\overline{\mathrm{DR}}}(\mu_{\mathrm{SUSY}})\Bigg\{1+\frac{\alpha^{\overline{\mathrm{DR}}}(\mu_{\mathrm{SUSY}})}{\pi}\Bigg[I_2(\Pi^{\mathcal{A}})\Bigg(-\frac{1}{4}\ln\left(\frac{\mu^2_{\mathrm{SUSY}}}{m^2_{\tilde g}}\right)\\ &+\frac{\left(m^2_{\tilde g}(1+\ln\left(\frac{\mu^2_{\mathrm{SUSY}}}{m^2_{\tilde g}}\right))-m^2_{\tilde q}(1+\ln\left(\frac{\mu^2_{\mathrm{SUSY}}}{m^2_{\tilde q}}\right))\right)}{2(m^2_{\tilde g}-m^2_{\tilde q})}\Bigg)\\ &+C_2(\Pi^{\mathcal{F}})\left(\frac{m^2_{\tilde q}-3m^2_{\tilde g}}{4(m^2_{\tilde g}-m^2_{\tilde q})}+\frac{m^2_{\tilde q}(2m^2_{\tilde g}-m^2_{\tilde q})\ln\left(\frac{\mu^2_{\mathrm{SUSY}}}{m^2_{\tilde q}}\right)-m^4_{\tilde g}\ln\left(\frac{\mu^2_{\mathrm{SUSY}}}{m^2_{\tilde g}}\right)}{2(m^2_{\tilde g}-m^2_{\tilde q})^2}\right)\Bigg]\Bigg\}. \end{aligned}\tag{2.46}$$

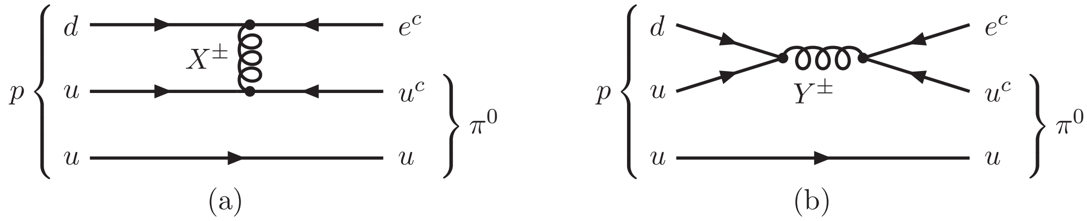


Figure 2.2.: *Dimension-six proton decay via heavy gauge bosons in SUSY and non-SUSY GUTs.*

2.6. Proton Decay

Besides gauge coupling unification, the instability of nucleons is one of the most remarkable predictions of GUTs (for a review see e.g. ref. [91] and references therein). Since baryons and leptons share gauge symmetry multiplets in such theories (cf. eq. (2.4)), gauge interactions inevitably lead to baryon and lepton number violating processes. These are mediated by the heavy X (electric charge $\pm\frac{4}{3}$) and Y bosons (electric charge $\pm\frac{1}{3}$) [92, 93]. Example diagrams that contribute to the process $p \to e^+\pi^0$ are depicted[7] in fig. 2.2. Note that these diagrams lead to operators in the effective theory that have dimension six and thus induce decay rates that are suppressed by $\frac{1}{M_{\mathrm{X}}^4}$. In SUSY GUTs the decay rates induced by these operators are generally too low to be in conflict with experiment [61] due to the higher unification scale as compared to non-SUSY GUTs.

However, SUSY allows for additional decay channels that are only suppressed by $\frac{1}{M_{\mathrm{H_c}}^2}$, where $M_{\mathrm{H_c}}$ is the mass of the colored Higgs triplet, that resides e.g. inside the $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$ of SU(5). This so-called dimension-five decay, with the dominant decay channel $p \to K^+\overline{\nu}$ (cf. fig. 2.3), can lead to much faster decay rates [95–97]. Since $M_{\mathrm{H_c}}$ can be predicted from the gauge coupling unification analysis, as we will see in chapter 3, one can check whether this prediction is consistent with the current bound on proton decay by calculating the dimension-five decay rate in the respective model. This kind of analysis lead Murayama and Pierce to claim the exclusion of the Minimal SUSY SU(5) model [19]. However, later careful analyses showed that this claim was a bit premature [45, 61]: First of all one should point out that the renormalizable version of minimal SUSY SU(5), as it has been defined in subsection 2.4.1, is not consistent with experiment anyway because the first and second generation lepton and quark Yukawa couplings fail to unify, as is predicted by this model. One way to reconcile this, is to introduce higher-dimensional operators into the superpotential, i.e. terms that are suppressed by $\frac{1}{M_{\mathrm{Pl}}}$. [43–45]. The inclusion of these higher-dimensional operators has two further consequences which both can serve to decrease the proton decay rate, attenuating the tension:

[7] This figure and all other figures that depict Feynman diagrams in this thesis have been created with help of the LaTeX package `AXODRAW` [94]

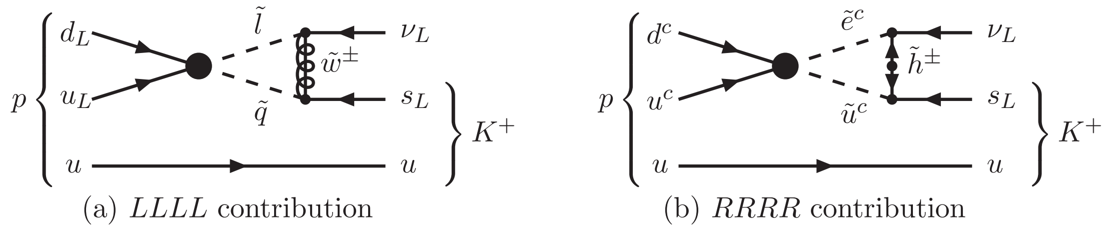


Figure 2.3.: *Dimension-five proton decay contributing to the dominant channel $p \to K^+\overline{\nu}$. The large blob indicates the dimension-five insertion that arises from integrating out the triplet Higgs fields. The contribution from four right-handed fields (b) has not been considered for a long time but turned out to be significant [97].*

1. The masses $M_{(8,1)}$ and $M_{(1,3)}$ in eq. (2.20) are not equal anymore. This also implies that one cannot determine $M_{\mathrm{H_c}}$ directly from the gauge coupling unification analysis but only the combination $\left(\frac{M_{(8,1)}}{M_{(1,3)}}\right)^{\frac{5}{2}} M_{\mathrm{H_c}}$. By choosing a lower ratio of these masses one can increase $M_{\mathrm{H_c}}$ and thus decrease the proton decay rate.

2. The proton decay rate depends on the first generation of down-quark-lepton Yukawa couplings Y^D in the renormalizable model (cf. eq. (2.19)). Since, however, Y^E and Y^D do not completely unify in the MSSM, it is not clear whether to choose the (larger) down-quark or (smaller) electron Yukawa coupling for the calculation. Most analyses before the publication of ref. [61], in particular ref. [19], have adopted the former choice, encountering larger proton decay rates. Note, however, that the other choice is equally justified and produces much lower decay rates. In the non-renormalizable model, where this inconsistency does not prevail, any value in between these extremes can effectively occur. It is shown in ref. [61] that due to this effect the proton decay rate can be pushed below the experimental limit and that the non-renormalizable Minimal SU(5) model is still perfectly viable.

2.7. Schur's Lemma

This section is somewhat out of context in this chapter. We review an important lemma from group theory that will be used throughout this thesis for the development of an appropriate apparatus for dealing with the group theory factors of spontaneously broken gauge theories as they appear in the calculation of the two-loop matching corrections at the GUT scale. In fact we will only need the first part of this famous Lemma by Issai Schur that states [98]:

> *Let $D(\mathbf{G})$ be an irreducible representation of a group $\mathbf{G}$ on the vector space V, and A be an arbitrary linear operator on V. If A commutes with all the operators $D(g)$, i.e. $AD(g) - D(g)A = 0, \forall g \in \mathbf{G}$, then A must be a multiple of the identity operator $\mathbb{1}$, i.e. $A = C\mathbb{1}$ where C is a number.*

(Simple) proofs of this statement can be found in virtually any textbook on group theory, e.g. in refs. [99, 100]. For the case of Lie groups the operators $D(g)$ are usually parametrized by the exponential map $D(g(\theta)) = \exp(-i\theta^\alpha T^\alpha)$ with the continuous parameters θ^α and the generator matrices T^α. Then the statement is: If T^α generate an irreducible representation and

$$\text{if } [A, T^\alpha] = 0, \quad \forall\alpha \quad \text{then } A = C\,\mathbb{1}\,. \tag{2.47}$$

If, on the other hand, the T^α generate a reducible representation, the lemma is still useful. Throughout this thesis, if not stated otherwise, we will assume that a reducible representation has block diagonal form. In such a case, a projector ϱ_i on the irreducible subspace i will have the form

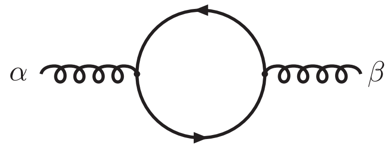
$$\varrho_i = \mathrm{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0) \tag{2.48}$$

i.e. it is a unit matrix on the particular irreducible subspace. Now we can consider generators T^α of a reducible representation. Then $\varrho_i\, T^\alpha$ will again generate an irreducible representation and we can apply Schur's Lemma:

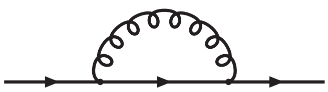
$$\text{if } [A, T^\alpha] = 0, \quad \forall\alpha \quad \text{then } \varrho_i\, A = C_i\, \varrho_i \tag{2.49}$$

with no sum over i.

An immediate application of this Lemma is known from QCD. where it allows us to define Dynkin indices and quadratic Casimir invariants:



$\sim \mathrm{Tr}(T^\alpha T^\beta) \equiv I_2\, \delta^{\alpha\beta}$



$\sim T^\alpha T^\alpha \equiv C_2\, \mathbb{1}$

We can define the invariants I_2 and C_2 because the matrices $A_{\alpha\beta} \equiv \mathrm{Tr}(T^\alpha T^\beta)$ and $A' \equiv T^\alpha T^\alpha$ commute with all the (irreducible) generators $T^\alpha_{\mathcal{A}}$ and T^α. Contracting α and β in the first line and taking the trace in the second line yields the relation

$$I_2\, \delta^{\alpha\alpha} \quad = \quad C_2\, \mathrm{Tr}(\mathbb{1})\,. \tag{2.50}$$

The rigorous treatment of group theory factors for higher loop orders in unbroken gauge theories with general Lie groups can be found e.g. in ref. [101]. In this thesis our focus will be on spontaneously broken gauge theories (GUTs). In chapter 5 we will therefore develop an appropriate framework of defining and reducing the group theory factors that appear in loop calculations in GUTs up to the two-loop order. This will involve distinguishing between broken and unbroken (reducible) generators and making use of the projectors ϱ_i defined in eq. (2.48).

3. Supersymmetric GUTs and Gauge Coupling Unification at Three Loops

In the previous chapter we introduced two of the simplest supersymmetric SU(5) models, the Minimal SUSY SU(5) Model and the Missing Doublet Model. Furthermore, we reviewed some historical highlights of the former, with particular emphasis on proton decay. There we mentioned that in 2002 this simplest SUSY GUT model has been claimed to be ruled out due to the combined constraint from proton decay and gauge coupling unification. However, later analyses have shown that the theory is still viable in its non-renormalizable version. Since in the meantime new experimental data and substantial progress on the theory side became available, we are encouraged to reanalyze the situation in the Minimal SUSY SU(5) Model with a focus on gauge coupling unification. In particular, our aim is to update the prediction for the triplet Higgs mass $M_{\mathrm{H_c}}$ by requiring the gauge couplings to unify. This is achieved by exploiting the $M_{\mathrm{H_c}}$ dependence of the GUT threshold corrections, eq. (2.36). The prediction gained in this way can then be compared with the lower bound coming from proton decay, using ref. [61] and the current experimental decay rates. For simplicity we stick to the renormalizable version of minimal SUSY SU(5) for the moment although it is not consistent with the measured values of first and second generation Yukawa couplings. This inconsistency does not play a role in our analysis. Furthermore, the predictions of the triplet Higgs mass can be translated to the non-renormalizable version simply by employing the rescaling relation

$$M_{\mathrm{H_c}}^{\mathrm{non-r}} = M_{\mathrm{H_c}} \left(\frac{M_{(1,3)}}{M_{(8,1)}} \right)^{\frac{5}{2}} , \tag{3.1}$$

where $M_{\mathrm{H_c}}^{\mathrm{non-r}}$ is the triplet Higgs mass in the non-renormalizable model and $M_{(1,3)}$ and $M_{(8,1)}$ are two independent mass parameters. Therefore, there is no loss of generality in this approach. We start by describing the procedure of running and decoupling that is used to determine $M_{\mathrm{H_c}}$. Then we will present our results and discuss the consequences for the viability of the model. There we will also mention results for the Missing Doublet Model which will provide a main motivation to perform the calculation that will be described in chapters 4 through 6. The present chapter is based on a paper that was published together with L. Mihaila, J. Salomon and M. Steinhauser [35].

3.1. Running and Decoupling

A consistent n-loop RGE analysis requires n-loop running and $(n-1)$-loop decoupling at each threshold [31]. Since we are aiming at three-loop precision, the required ingredients for

	run $M_Z \to \mu_{\rm SUSY}$	decouple $\alpha_i \to \alpha_i^{\rm (MSSM)}$	run $\mu_{\rm SUSY} \to \mu_{\rm GUT}$	decouple $\alpha_i^{\rm (MSSM)} \to \alpha^{\rm (GUT)}$	run $\mu_{\rm GUT} \to M_{\rm pl}$
EW	2(3)	1(2)	3	1(2)	$3^\star$
QCD	3	2	3		

Table 3.1.: *The loop precision available (and used) for the individual steps of the running-decoupling procedure. The number in parentheses indicate the loop-order needed for a consistent analysis.*
($\star$ The running of the gauge and Yukawa couplings is performed to three- and one-loop accuracy, respectively.)

the analysis are: Three-loop gauge β functions for the SM, the MSSM and the GUT under consideration, as well as two-loop decoupling coefficients at the SUSY and the GUT thresholds. Since the RGEs for the gauge couplings are coupled with those of the Yukawa couplings, we also need the latter at sufficiently high loop order. Unfortunately, not all of this theoretical input is available yet. Still, as will be argued later, we believe that the numerically most significant building blocks for the case of Minimal SUSY SU(5) are already known. The employed loop-orders for the individual steps are summarized in table 3.1.

In the following we will describe the theoretical and experimental input that is used for the individual steps:

1. **Experimental input at $\mathbf{M_Z}$**
   The three gauge couplings at the electroweak scale in the $\overline{\rm MS}$ scheme constitute crucial experimental input for our analysis. They are obtained from the weak mixing angle in the $\overline{\rm MS}$ scheme [7], the QED coupling constant at zero momentum transfer and its hadronic contribution [102] in order to obtain its value at the Z-boson scale. Furthermore, we use the strong coupling constant [103][1] at the scale M_Z. Considering all uncertainties on these parameters, our input reads:

$$\begin{aligned} \sin^2\Theta^{\overline{\rm MS}} &= 0.23119 \pm 0.00014\,, \\ \alpha_{em} &= 1/137.036\,, \\ \Delta\alpha^{(5)}_{\rm had} &= 0.02761 \pm 0.00015\,, \\ \alpha_s(M_Z) &= 0.1184 \pm 0.0020\,. \end{aligned} \tag{3.2}$$

   Whereas $\sin^2\Theta^{\overline{\rm MS}}$ and $\alpha_s(M_Z)$ are already defined in the $\overline{\rm MS}$ scheme, $\Delta\alpha^{(5)}_{\rm had}$ constitutes corrections to the on-shell value of α_{em}. In order to obtain the corresponding $\overline{\rm MS}$ result we add the leptonic [104] and top-quark [105] contribution, $\Delta\alpha^{(5)}_{\rm lep} = 314.97686\cdot 10^{-4}$ and $\Delta\alpha^{(5)}_{\rm top} = (-0.70\pm 0.05)\cdot 10^{-4}$, and apply the transition formula to the $\overline{\rm MS}$ scheme [7]

$$\Delta\alpha^{(5),\overline{\rm MS}} - \Delta\alpha^{(5),\rm OS} = \frac{\alpha_{em}}{\pi}\left(\frac{100}{27} - \frac{1}{6} - \frac{7}{4}\ln\frac{M_Z^2}{M_W^2}\right) \approx 0.0072\,. \tag{3.3}$$

[1] We adopt the central value from ref. [103], however, use as our default choice for the uncertainty 0.0020 instead of 0.0007.

This leads to

$$\alpha_{em}^{\overline{\rm MS}}(M_Z) \;=\; \frac{\alpha_{em}}{1-\Delta\alpha_{\rm lep}^{(5)}-\Delta\alpha_{\rm had}^{(5)}-\Delta\alpha_{\rm top}^{(5)}-0.0072} \;=\; \frac{1}{127.960\pm 0.021}\,. \quad (3.4)$$

In the quantities $\sin^2\Theta^{\overline{\rm MS}}$, $\alpha_{em}^{\overline{\rm MS}}(M_Z)$ and $\alpha_s(M_Z)$ the top-quark is still (partly) decoupled. Thus, in a next step we compute the six-flavor SM quantities using the relations [7,106]

$$\begin{aligned}
\alpha_{em}^{(6),\overline{\rm MS}} &= \alpha_{em}^{\overline{\rm MS}}\left\{1+\frac{4}{9}\frac{\alpha_{em}^{\overline{\rm MS}}}{\pi}\left[\ln\frac{M_Z^2}{M_t^2}\left(1+\frac{\alpha_s}{\pi}+\frac{\alpha_{em}^{\overline{\rm MS}}}{3\pi}\right)+\frac{15}{4}\left(\frac{\alpha_s}{\pi}+\frac{\alpha_{em}^{\overline{\rm MS}}}{3\pi}\right)\right]\right\}, \\
\sin^2\Theta^{(6),\overline{\rm MS}} &= \sin^2\Theta^{\overline{\rm MS}}\left\{1+\frac{1}{6}\frac{\alpha_{em}^{\overline{\rm MS}}}{\pi}\left(\frac{1}{\sin^2\Theta^{\overline{\rm MS}}}-\frac{8}{3}\right)\left[\left(1+\frac{\alpha_s}{\pi}\right)\ln\frac{M_t^2}{M_Z^2}-\frac{15}{4}\frac{\alpha_s}{\pi}\right]\right\},
\end{aligned} \quad (3.5)$$

where all couplings are evaluated at the scale $\mu = M_Z$. We obtain[2]

$$\begin{aligned}
\alpha_{em}^{(6),\overline{\rm MS}}(M_Z) &= 1/(128.129\pm 0.021)\,, \\
\sin^2\Theta^{(6),\overline{\rm MS}}(M_Z) &= 0.23138\pm 0.00014\,, \\
\alpha_s^{(6)}(M_Z) &= 0.1173\pm 0.0020\,.
\end{aligned} \quad (3.6)$$

These quantities are related to the three gauge couplings via the equations

$$\begin{aligned}
\alpha_1 &= \frac{5}{3}\frac{\alpha_{em}^{(6),\overline{\rm MS}}}{\cos^2\Theta^{(6),\overline{\rm MS}}}\,, \\
\alpha_2 &= \frac{\alpha_{em}^{(6),\overline{\rm MS}}}{\sin^2\Theta^{(6),\overline{\rm MS}}}\,, \\
\alpha_3 &= \alpha_s^{(6)}\,,
\end{aligned} \quad (3.7)$$

which holds for any renormalization scale μ.

Since the RGEs for gauge couplings are coupled with those of the Yukawa couplings, we have to evolve them simultaneously up to the GUT scale. Therefore, we also need starting values for the third-generation Yukawa couplings which are computed from the W- and Z-boson pole masses M_W and M_Z, the top-quark and tau-lepton pole masses M_t and M_τ and the running bottom-quark mass $m_b^{\overline{\rm MS}}$ [7,108,109].

$$\begin{aligned}
M_W &= 80.398 \text{ GeV}\,, \\
M_Z &= 91.1876 \text{ GeV}\,, \\
M_t &= 173.3 \text{ GeV}\,, \\
M_\tau &= 1.77684 \text{ GeV}\,, \\
m_b^{\overline{\rm MS}}(m_b^{\overline{\rm MS}}) &= 4.163 \text{ GeV}\,.
\end{aligned} \quad (3.8)$$

[2] Since we aim for gauge couplings at the electroweak scale with highest possible precision we use four-loop running and three-loop decoupling as implemented in `RunDec` [107] in order to obtain $\alpha_s^{(6)}$ from $\alpha_s(M_Z) \equiv \alpha_s^{(5)}(M_Z)$. At such high order in perturbation theory there is practically no dependence on the decoupling scale.

The corresponding uncertainties are not important for our analysis. The quark masses are converted to their $\overline{\mathrm{MS}}$ values in 6-flavor theory at M_Z using `RunDec` [107]. The difference between the on-shell and the $\overline{\mathrm{MS}}$ value of M_τ is neglected since it is only needed to obtain the starting value for $y_\tau(M_Z)$ which has a small impact on the running of the gauge couplings. To obtain the electroweak vev $v^{(6),\overline{\mathrm{MS}}}(M_Z)$ in $\overline{\mathrm{MS}}$ at M_Z in 6-flavor theory from the on-shell value of M_Z and the $\overline{\mathrm{MS}}$ values of the gauge couplings, we implement the one-loop SM contributions of the corrections given e.g. in ref. [86]:

$$v^{(6),\overline{\mathrm{MS}}}(M_Z) = 2\sqrt{M_Z^2 + \mathrm{Re}(\Pi_{ZZ}^T(M_Z^2))}\sqrt{\frac{\cos^2\Theta^{(6),\overline{\mathrm{MS}}}(M_Z)\sin^2\Theta^{(6),\overline{\mathrm{MS}}}(M_Z)}{4\pi\,\alpha_{em}^{(6),\overline{\mathrm{MS}}}(M_Z)}} \tag{3.9}$$

where Π_{ZZ}^T is the transverse Z-boson $\overline{\mathrm{MS}}$ self-energy in the SM

$$\begin{aligned}\Pi_{ZZ}^T(p^2) \;=\; & \frac{\alpha_{em}}{4\pi\cos^2\Theta\sin^2\Theta}\Bigg[-2\cos^2\Theta\Big(2p^2 + M_W^2 - M_Z^2\frac{\sin^4\Theta}{\cos^2\Theta}\Big)B_0(p,M_W,M_W)\\ & -\Big(8\cos^4\Theta + (2\cos^2\Theta - 1)^2\Big)\tilde{B}_{22}(p,M_W,M_W)\\ & +\sum_f N_c^f\Big\{(g_{f_L}^2 + g_{f_R}^2)\,H(p,m_f,m_f) - 4\,g_{f_L}\,g_{f_R}\,m_f^2\,B_0(p,m_f,m_f)\Big\}\Bigg].\end{aligned}$$

The electromagnetic coupling and the Weinberg angle have been taken in 6-flavor theory in $\overline{\mathrm{MS}}$ at the scale M_Z. The sum $\sum_f$ is over all quarks and leptons and N_c^f is 3 for quarks and 1 for leptons. Moreover, the quantities g_{f_L}, g_{f_R} as well as the loop functions are defined in the appendix of ref. [86]. The initial conditions for the Yukawa couplings are then given by

$$y_i(M_Z) \;=\; \frac{\sqrt{2}\,m_i^{(6),\overline{\mathrm{MS}}}(M_Z)}{v^{(6),\overline{\mathrm{MS}}}(M_Z)}\,, \qquad i = t, b, \tau\,. \tag{3.10}$$

2. **SUSY contributions to $\sin^2\Theta^{(6),\overline{\mathrm{MS}}}(M_Z)$**
The determination of the experimental values in eq. (3.6) implicitly assumes that the SM is valid up to high energy scales. Since in our case, however, the SM is only an effective theory of the MSSM, we have to worry about contributions to these values that are suppressed by $\frac{1}{M_{\mathrm{SUSY}}^2}$ in order to achieve the precision we are aiming at. Due to the presence of the weak gauge bosons in the loop corrections, the weak mixing angle receives the numerically largest contributions whereas the influence of supersymmetric particles on the electromagnetic and strong coupling can be neglected.

The procedure to incorporate the supersymmetric effects on the numerical value of $\sin^2\Theta^{(6),\overline{\mathrm{MS}}}(M_Z)$ is as follows (cf. fig. 3.1): In a first step we transfer $\sin^2\Theta^{(6),\overline{\mathrm{MS}}}(M_Z)$ from eq. (3.6) to the $\overline{\mathrm{DR}}$ scheme [110] and apply afterwards the supersymmetric one-loop corrections evaluated in ref. [111] relating the weak mixing angle in the SM to the one in MSSM. In a next step we decouple the supersymmetric particles [16] and finally go back to the $\overline{\mathrm{MS}}$ scheme. As a result we obtain $\sin^2\Theta^{(6),\overline{\mathrm{MS}}}(M_Z)$ at the scale $\mu = M_Z$

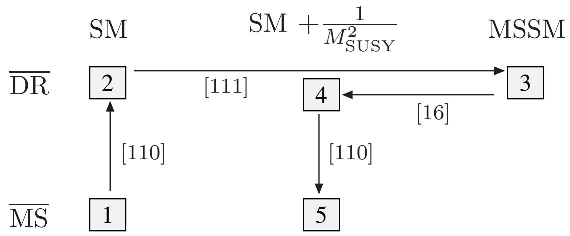


Figure 3.1.: *Illustration of the individual steps of how the* $\frac{1}{M_{\text{SUSY}}^2}$ *contributions to* $\sin^2\Theta^{(6),\overline{\text{MS}}}(M_Z)$ *are incorporated including the references used. The starting point is the numerical value given in eq. (3.6)*

including virtual MSSM contributions. The described procedure amounts to applying the following one-loop formula to the mixing angle

$$\sin^2\Theta^{(6),\overline{\text{MS}},\text{corr}}(M_Z) \quad = \quad \left(1 - \Delta\hat{k}_f + \zeta_{\alpha_{em}} - \zeta_{\alpha_2}\right)\sin^2\Theta^{(6),\overline{\text{MS}}}(M_Z) \qquad (3.11)$$

with $\zeta_{\alpha_{em}}$ and ζ_{α_2} given in eq. (2.40) and

$$\Delta\hat{k}_f = \frac{\cos\Theta^{(6),\overline{\text{MS}}}}{\sin\Theta^{(6),\overline{\text{MS}}}}\frac{\Pi_{Z\gamma}(M_Z^2) - \Pi_{Z\gamma}(0)}{M_Z^2}\,, \qquad (3.12)$$

where the SUSY contributions to the $Z-\gamma$ self-energy can be computed by [86]

$$\begin{aligned}\Pi_{Z\gamma}(p^2) = \sqrt{\frac{3}{5}\frac{\alpha_1\alpha_2}{4\pi}}\Bigg[& -2\cos 2\Theta\,\tilde{B}_{22}(m_{H^\pm}, m_{H^\pm}) \\ +\frac{1}{2}\sum_{i=1}^{2}&\left(|(Z_+)_{1,i}|^2 + |(Z_-)_{1,i}|^2 + 2\cos 2\Theta\right)\left(4\tilde{B}_{22}(m_{\chi_i}, m_{\chi_i}) + p^2 B_0(m_{\chi_i}, m_{\chi_i})\right) \\ -4\sum_f & N_c^f e_f\Big[(g_{f_L}\cos^2\theta_{\tilde{f}} - g_{f_R}\sin^2\theta_{\tilde{f}})\tilde{B}_{22}(m_{\tilde{f}_1}, m_{\tilde{f}_1}) \\ & \qquad +(g_{f_L}\sin^2\theta_{\tilde{f}} - g_{f_R}\cos^2\theta_{\tilde{f}})\tilde{B}_{22}(m_{\tilde{f}_2}, m_{\tilde{f}_2})\Big]\Bigg]\,. \end{aligned} \qquad (3.13)$$

Again, the gauge couplings and the Weinberg angle have been taken in 6-flavor theory in $\overline{\text{MS}}$ at the scale M_Z. The sum $\sum_f$ is over all quarks and leptons and N_c^f is 3 for quarks and 1 for leptons. For the definition of the quantities g_{f_L}, g_{f_R} and e_f as well as the loop functions, please refer to the appendix of ref. [86]. Z_+ and Z_- are the chargino mixing matrices from eq. (2.41) and the sfermion mixing angles have been given in eq. (2.42). We have taken care to write the various contributions in eq. (3.11) in such a way that the large logarithms $\ln(M_{\text{SUSY}}/M_Z)$ cancel completely.

Note that by construction these corrections to $\sin^2\Theta^{(6),\overline{\text{MS}}}(M_Z)$ are suppressed by the square of the supersymmetric mass scale. We anticipate that for typical supersymmetric benchmark scenarios the influence of supersymmetric corrections to $\sin^2\Theta^{(6),\overline{\text{MS}}}(M_Z)$ can lead to shifts in M_{H_c} which are of the order of 10%. For the mSUGRA parameters in eq. (3.18) the shift of $\sin^2\Theta^{(6),\overline{\text{MS}}}(M_Z)$ amounts to about $1.4\cdot 10^{-5}$.

3. **Running within the SM from $\mu = M_Z$ to the SUSY scale $\mu_{\rm SUSY}$**
Starting from eqs. (3.6), (3.7), (3.8) and (3.10) (and adding supersymmetric effects to the weak mixing angle as discussed above), we use the three-loop β function of QCD [112,113] and the two-loop RGEs in the electroweak sector [27,28,114,115] in order to obtain the values of the gauge couplings at $\mu_{\rm SUSY} \approx 1$ TeV. We take into account the tau, bottom and top Yukawa couplings and thus solve numerically a coupled system of six differential equations. The relevant RGEs are listed in appendix A.4 of this thesis. Since the quartic SM Higgs coupling λ enters the equations of the Yukawa couplings starting from two-loop order only, we neglect its contribution. The unphysical scale $\mu_{\rm SUSY}$ is not fixed but kept as a free parameter in our setup.

4. **Decoupling and conversion to $\overline{\rm DR}$ at $\mu_{\rm SUSY}$**
For energies of about $\mu_{\rm SUSY} \approx 1$ TeV the SUSY particles become active and the proper matching between the SM and the MSSM has to be performed. We decouple all heavy non-SM particles simultaneously at the scale $\mu_{\rm SUSY}$ using the one-loop relations for α_1 and α_2 [16] that have been listed in subsection 2.5.4. These contributions are numerically important since they help to flatten the $\mu_{\rm SUSY}$ dependence of $M_{\rm H_c}$ significantly as will be seen later in this chapter. For the Yukawa couplings we apply the relations

$$\begin{aligned}
y_t^{\rm SM}(\mu_{\rm SUSY}) &= \zeta_t^{\rm reg}\,\zeta_{m_t}(\mu_{\rm SUSY})\,\zeta_v^{-1}(\mu_{\rm SUSY})\,y_t^{\rm MSSM}(\mu_{\rm SUSY})\sin\beta\,,\\
y_b^{\rm SM}(\mu_{\rm SUSY}) &= \zeta_b^{\rm reg}\,\zeta_{m_b}(\mu_{\rm SUSY})\,\zeta_v^{-1}(\mu_{\rm SUSY})\,y_b^{\rm MSSM}(\mu_{\rm SUSY})\cos\beta\,,\\
y_\tau^{\rm SM}(\mu_{\rm SUSY}) &= \zeta_\tau^{\rm reg}\,\zeta_{m_\tau}(\mu_{\rm SUSY})\,\zeta_v^{-1}(\mu_{\rm SUSY})\,y_\tau^{\rm MSSM}(\mu_{\rm SUSY})\cos\beta\,.
\end{aligned}\qquad(3.14)$$

If not otherwise stated, a superscript SM denotes an $\overline{\rm MS}$ quantity and a superscript MSSM a $\overline{\rm DR}$ quantity. The various ζ coefficients take care of the change of regularization scheme, the decoupling of the mass and the decoupling of the vev, respectively. At the one-loop level the formulae read

$$\begin{aligned}
\zeta_{t,b}^{\rm reg} &= 1+\frac{\alpha_3}{4\pi}\frac{4}{3}\,, \qquad \zeta_\tau^{\rm reg} = 1\,,\\
\zeta_{m_i}(\mu_{\rm SUSY}) &= 1-\frac{\Sigma_i(m_i^2)}{m_i}\,, \qquad i = t,b,\tau\,,\\
\zeta_v^{-1}(\mu_{\rm SUSY}) &= \sqrt{\frac{\frac{3}{5}\alpha_1^{\rm SM}(\mu_{\rm SUSY})+\alpha_2^{\rm SM}(\mu_{\rm SUSY})}{\frac{3}{5}\alpha_1^{\rm MSSM}(\mu_{\rm SUSY})+\alpha_2^{\rm MSSM}(\mu_{\rm SUSY})}\left(1-\frac{{\rm Re}\,\Pi_{ZZ}^T(M_Z^2)}{M_Z^2}\right)}
\end{aligned}\qquad(3.15)$$

The self-energies $\Sigma_i(m_i^2)$ and $\Pi_{ZZ}^T(M_Z^2)$ can be extracted from eqs. (D.4) and (D.18) of ref [86]. However, we need to take into account only the contributions from the superpartners in these equations since we are decoupling to the full SM here. The SUSY QCD decoupling effects for α_3 and m_b are known to two-loop order [29, 30] and are also implemented to this precision in our setup. In particular the two-loop SUSY QCD corrections for α_3 are of utmost numerical importance.

As pointed out before, a fully consistent approach would require two-loop threshold corrections not only in the strong but also in the electroweak sector. They are not yet available, however, we also expect that their numerical impact is relatively small[3].

[3]The smallness of the two-loop contributions to $\zeta_{\alpha_{1,2}}(\mu_{\rm SUSY})$ at $\mu_{\rm SUSY}$ cannot be concluded solely from the

Furthermore, consistency with the RG running would require mixed QCD-Yukawa corrections for α_3. They have been calculated in the course of the Ph.D. thesis of Jens Salomon [116] and turned out to be negligible. This was anticipated since the effect of these kind of corrections on the (three-loop) running is numerically small.

In our numerical analysis we generate the SUSY mass spectrum with the help of the program `SOFTSUSY` [57] and also study the various SPS (Snowmass Points and Slopes) scenarios [117, 118].

At this stage also the change of renormalization scheme from $\overline{\text{MS}}$ to $\overline{\text{DR}}$ has to be taken into account. We employed the one-loop conversion relations [90, 110] for all parameters except α_3 and m_b where two-loop relations [90,119] have been used in order to be consistent with the decoupling at the SUSY scale. For convenience the relevant relations are listed in eqs (2.43), (2.44) and (3.15).

5. **Running within the MSSM from μ_{SUSY} to the high-energy scale μ_{GUT}**
We use the complete three-loop RGEs of the MSSM [23, 24] (cf. also appendix A.5) to evolve the gauge and Yukawa couplings from μ_{SUSY} to some very high scale of the order of 10^{16} GeV, that we denote by μ_{GUT}, where we expect that SUSY GUT particles become active.

6. **Decoupling at μ_{GUT} and determination of M_{H_c}**
At the scale μ_{GUT} we need to perform the decoupling of the super-heavy GUT particles. A consistent treatment would require two-loop decoupling relations. At the moment, however, only one-loop results are available [31–33]. As we will see later, for the Minimal SUSY SU(5) these missing two-loop effects are expected to be much less significant numerically than the SUSY QCD two-loop decoupling effects, which we included. For the Missing Doublet Model, however, we will see that this is not a good approximation and that at the moment no firm conclusions can be drawn here because the theoretical uncertainty due to the scale variation of μ_{GUT} is huge.

In order to determine the mass parameter M_{H_c}, one could take the values of the three gauge couplings that we computed in the previous step, apply eq. (2.25) together with eq. (2.36) and determine the values of the heavy GUT masses that yield a unique gauge coupling $\alpha(\mu_{\text{GUT}})$ by scanning over the parameter space. In fact, at this order of perturbation theory we can algebraically solve for those heavy masses by taking suitable linear combinations of eq. (2.36). For Minimal SUSY SU(5) we can therefore determine the colored Higgs triplet mass M_{H_c} and an additional parameter via

$$\begin{aligned}
-\frac{4\pi}{\alpha_1(\mu_{\text{GUT}})} + 3\,\frac{4\pi}{\alpha_2(\mu_{\text{GUT}})} - 2\,\frac{4\pi}{\alpha_3(\mu_{\text{GUT}})} &= -\frac{12}{5}\ln\left(\frac{\mu_{\text{GUT}}^2}{M_{\text{H}_\text{c}}^2}\right), \qquad (3.16)\\
5\,\frac{4\pi}{\alpha_1(\mu_{\text{GUT}})} - 3\,\frac{4\pi}{\alpha_2(\mu_{\text{GUT}})} - 2\,\frac{4\pi}{\alpha_3(\mu_{\text{GUT}})} &= -24\left[\ln\left(\frac{\mu_{\text{GUT}}^2}{M_{\text{X}}^2}\right) + \frac{1}{2}\ln\left(\frac{\mu_{\text{GUT}}^2}{M_{\Sigma}^2}\right)\right].
\end{aligned}$$

numerical smallness of of $\alpha_{1,2}(\mu_{\text{SUSY}})$ since their effect at μ_{GUT} can in principle still be of the same order of magnitude as those of $\zeta_{\alpha_3}(\mu_{\text{SUSY}})$. However, subsection 3.2.1 will provide an a posteriori motivation for the above statement: The flatness of the μ_{SUSY} dependence of M_{H_c} suggests that in the absence of large constant two-loop terms in the $\zeta_{\alpha_{1,2}}(\mu_{\text{SUSY}})$ and without miraculous cancellations between $\zeta_{\alpha_1}(\mu_{\text{SUSY}})$ and $\zeta_{\alpha_2}(\mu_{\text{SUSY}})$ their two-loop contributions should not be as important as the ones of $\zeta_{\alpha_3}(\mu_{\text{SUSY}})$.

It is common to define a new mass parameter $M_\mathrm{G} \equiv \sqrt[3]{M_\mathrm{X}^2 M_\Sigma}$, which is the second parameter, besides $M_{\mathrm{H_c}}$ that can be directly determined from the above equations. For the case of the Missing Doublet Model the same linear combinations, which are obtained from eq. (2.37), read

$$\begin{aligned}
-\frac{4\pi}{\alpha_1(\mu_\mathrm{GUT})} + 3\,\frac{4\pi}{\alpha_2(\mu_\mathrm{GUT})} - 2\,\frac{4\pi}{\alpha_3(\mu_\mathrm{GUT})} &= -\frac{12}{5}\left[\ln\left(\frac{\mu_\mathrm{GUT}^2}{M_{\mathrm{H_c}}^2}\right) + \ln\left(\frac{\mu_\mathrm{GUT}^2}{M_{\mathrm{H}_{c'}}^2}\right)\right] \\
&\quad +12\ln\left(\frac{64}{3125}\right), \\
5\,\frac{4\pi}{\alpha_1(\mu_\mathrm{GUT})} - 3\,\frac{4\pi}{\alpha_2(\mu_\mathrm{GUT})} - 2\,\frac{4\pi}{\alpha_3(\mu_\mathrm{GUT})} &= -24\left[\ln\left(\frac{\mu_\mathrm{GUT}^2}{M_\mathrm{X}^2}\right) + \frac{1}{2}\ln\left(\frac{\mu_\mathrm{GUT}^2}{M_\Sigma^2}\right)\right] \\
&\quad -12\ln\left(\frac{262144}{1953125}\right). \qquad (3.17)
\end{aligned}$$

7. **Running within the GUT from μ_GUT to the Planck scale $\mathrm{M_{Pl}}$**
   The last sequence of our approach consists in the running within the SUSY SU(5) model. We implemented the three-loop RGEs for the gauge [26], and the one-loop formulae for the Yukawa and Higgs self couplings [14, 18]. In the appendix of this thesis, section A.7, we derive the gauge β functions from the general formulae given in ref. [26]. The only purpose of this step is to check whether all parameters remain perturbative up to the Planck scale. Since eqs. (3.16) and (3.17) only predict the combination $M_\mathrm{G} \equiv \sqrt[3]{M_\mathrm{X}^2 M_\Sigma}$ (and $M_{\mathrm{H_c}} M_{H_{c'}}$ in the case of the Missing Doublet Model), we have some freedom in choosing the parameters that go into $\zeta_{\alpha_i}(\mu_\mathrm{GUT})$ which has an impact on the perturbativity constraint. We regard a parameter point as fulfilling the perturbativity constraint if e.g. for a given M_G a M_X can be found such that all the parameters remain perturbative.

The simultaneous decoupling, that has been adopted here, might be problematic in case there is a huge splitting among the SUSY or GUT masses. In that case a step-by-step decoupling would be preferable (see, e.g., ref. [120]); however, two-loop calculations in that framework are still missing. Furthermore, the mass splitting between the SUSY particles in almost all benchmark scenarios currently discussed in the literature is rather small. The same is true for the GUT masses that will appear in our analysis.

The described procedure has been implemented into a `Mathematica` package that provides several useful functions suitable for the numerical analysis that will be presented in the following section.

3.2. Predictions for $M_{\mathrm{H_c}}$ from Gauge Coupling Unification

Using the procedure described in the previous section, we now study the prediction of the two SUSY GUT masses $M_{\mathrm{H_c}}$ and $M_\mathrm{G} \equiv \sqrt[3]{M_\mathrm{X}^2 M_\Sigma}$ with a focus on the former as it is the relevant parameter for proton decay. In most cases we adopt the mSUGRA scenario for the generation of the SUSY mass spectrum using SOFTSUSY. For illustration the running of the gauge couplings at three-loop order is depicted in fig. 3.2 for $\mu_{\mathrm{SUSY}} = 1000\,\mathrm{GeV}$ and $\mu_{\mathrm{GUT}} = 10^{16}\,\mathrm{GeV}$. Additionally, we have chosen $M_\Sigma = 1 \cdot 10^{15}\mathrm{GeV}$, which, by requiring the gauge couplings to unify, implies $M_{\mathrm{H_c}} = 1.7 \cdot 10^{15}\mathrm{GeV}$ and $M_{\mathrm{X}} = 4.6 \cdot 10^{16}\mathrm{GeV}$. For the SUSY spectrum we have chosen the parameter point that will be introduced in eq. (3.18). Panel (b) illustrates how two different choices of the decoupling scale μ_{GUT} can lead to a similar value of the gauge coupling above μ_{GUT} which is the behavior we expect from a realiable analysis.

3.2.1. Dependence on the Decoupling Scales μ_{SUSY} and μ_{GUT}

We start by examining the dependence of $M_{\mathrm{H_c}}$ and M_{G} on the decoupling scales since this gives us an estimation for the improvement of the theoretical uncertainty due to the three-loop analysis compared to the two-loop analysis. An exact calculation would yield a flat curve, however, due to the use of perturbation theory, there is a remnant dependence which is gradually eliminated by successively including higher orders. Fig. 3.3 shows the dependence of $M_{\mathrm{H_c}}$ and M_{G} on μ_{SUSY} (a) and μ_{GUT} (b) for the mSUGRA parameters

$$\begin{aligned} m_0 &= m_{1/2} = -A_0 = 1000\,\mathrm{GeV}\,, \\ \tan\beta &= 3\,, \\ \mu &> 0\,. \end{aligned} \tag{3.18}$$

This parameter point results in squark masses of about 2 TeV which is at the upper range of what can be measured at the LHC. Lower values for the masses of the superpartners will generally result in lower values for $M_{\mathrm{H_c}}$ as will be shown later. Turning again to Fig. 3.3 (a), two things are noteworthy:

- The three-loop effects make the curve almost flat. (The remaining variation of $\log_{10}(M_{\mathrm{H_c}}/\mathrm{GeV})$ for the considered range of μ_{SUSY} is about 0.15.) We interpret this fact as a hint that despite the missing three-loop electroweak running in the SM and the two-loop electroweak decoupling from the SM to the MSSM, the numerically most decisive corrections have been taken into account.

- At the scale $\mu_{\mathrm{SUSY}} = M_Z$ (the left edge of the plot) the increase of $M_{\mathrm{H_c}}$ due to three-loop effects amounts to about one order of magnitude. This is important since many previous unification analyses used $\mu_{\mathrm{SUSY}} = M_Z$ as the default decoupling scale [13–17, 20]. Therefore, presumed tension between their predictions for $M_{\mathrm{H_c}}$ and bounds on $M_{\mathrm{H_c}}$ from proton decay is significantly attenuated by the three-loop effects.

The dependence of $M_{\mathrm{H_c}}$ on μ_{GUT} is shown in Fig. 3.3 (b). Recall that here only one-loop decoupling at μ_{GUT} is employed. Still the variation of $\log_{10}(M_{\mathrm{H_c}}/\mathrm{GeV})$ for the considered range of μ_{GUT} amounts to only 0.3 which is of the same order of magnitude as remaining variation due to the μ_{SUSY} dependence described above. Again, we consider this as a hint that the missing two-loop decoupling relations at the GUT scale will not change our result too much in the Minimal SUSY SU(5) Model[4]. However, as always, this is to be taken with

[4]This statement is not true for the Missing Doublet Model, as will be shown later.

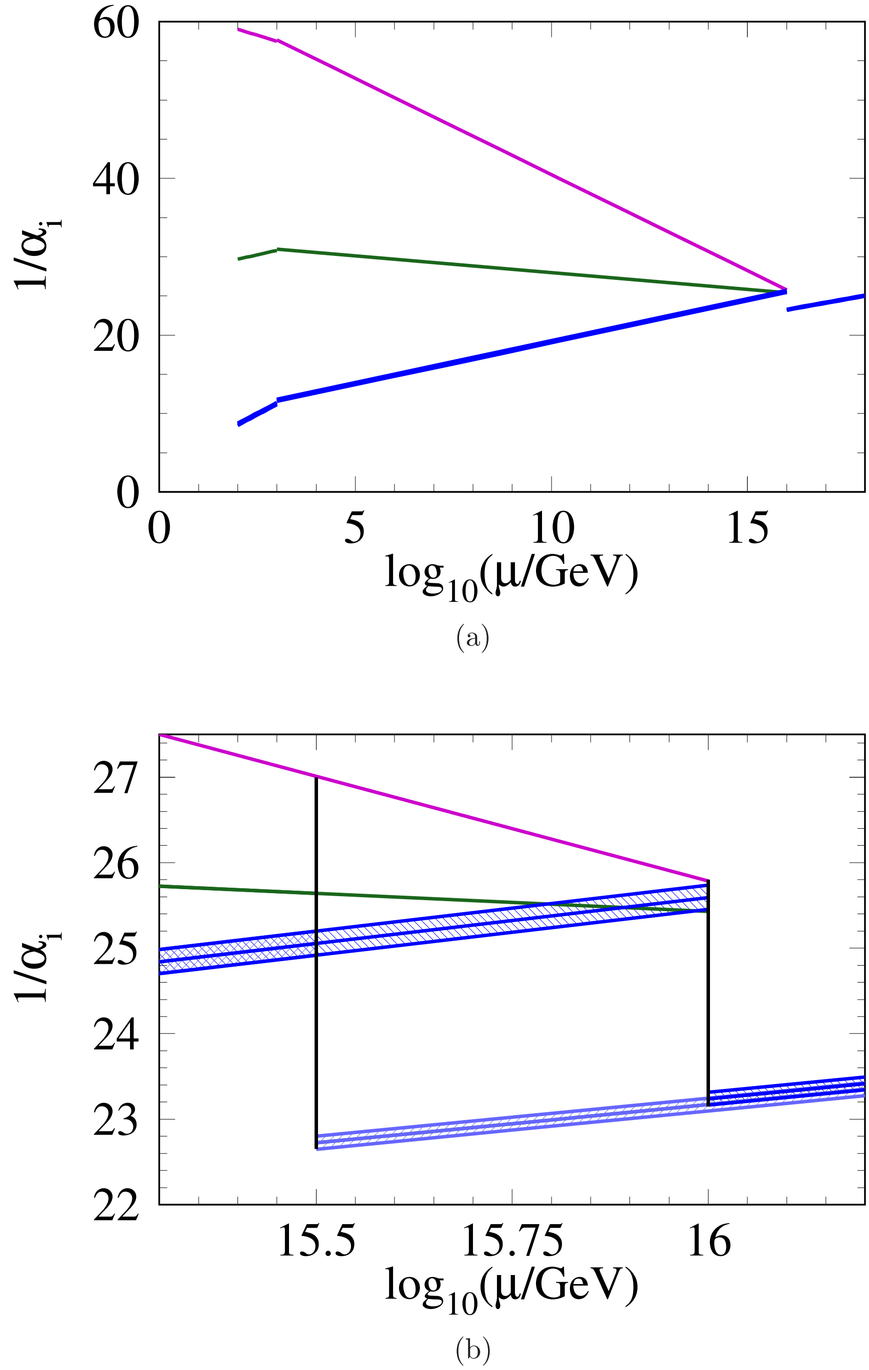
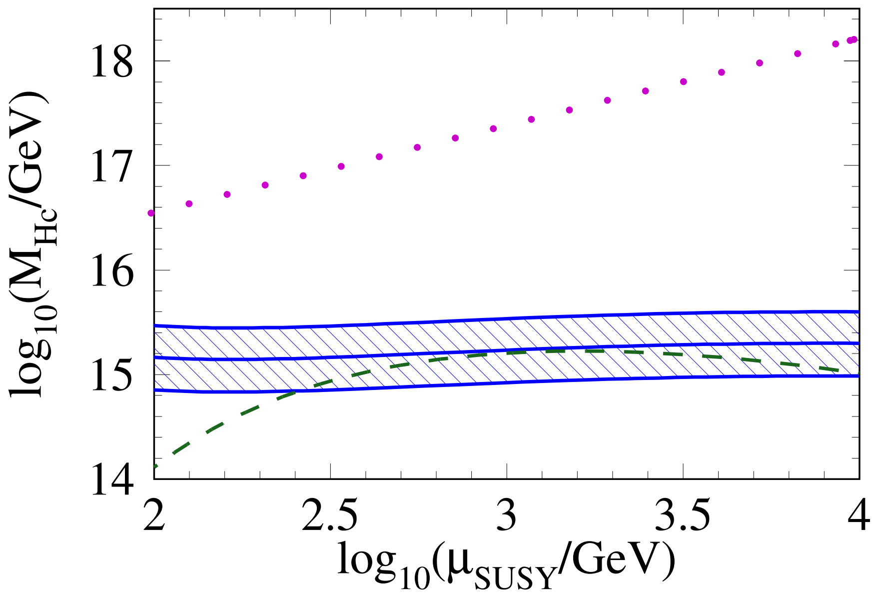
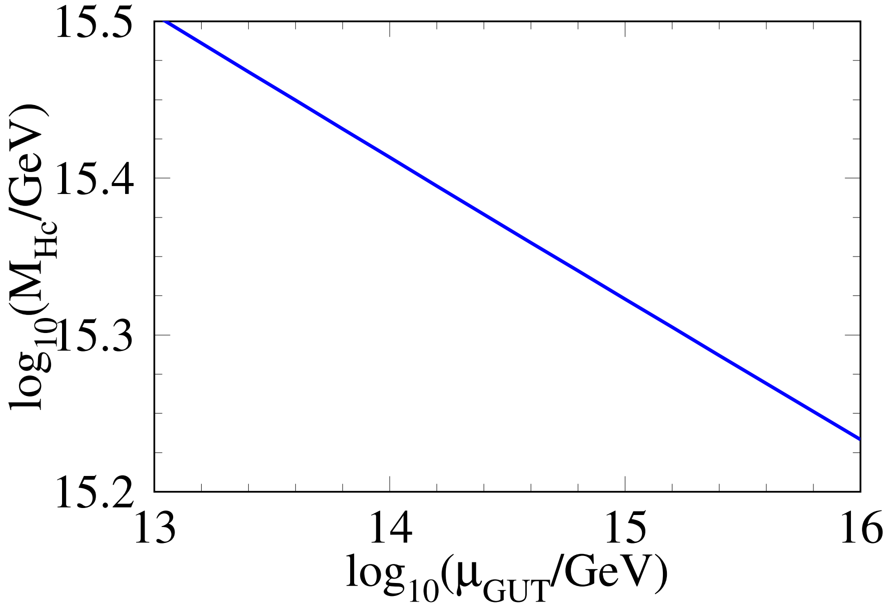


Figure 3.2.: *Illustration of the running and decoupling procedure for a specific parameter point. The discontinuities are due to matching corrections at the SUSY and GUT thresholds. In panel (b) the region around μ_{GUT} is enlarged and the behavior of the curve is shown if the decoupling scale μ_{GUT} is lowered from 10^{16} GeV to $10^{15.5}$ GeV $\approx 3.2 \cdot 10^{15}$ GeV. The shaded band indicates the impact of the uncertainty on $\alpha_3(M_Z)$.*



(a)



(b)

Figure 3.3.: *The dependence of $M_{\mathrm{H_c}}$ on μ_{SUSY} (a) and μ_{GUT} (b). The dotted, dashed and solid lines represent the one- two- and three-loop analysis. The shaded band indicates the uncertainty on the determination of $\alpha_s(M_Z)$ with the value $\delta\alpha_s = 0.0020$.*

care since the variation of physical parameters due to the dependence on a decoupling scale only provides a lower bound on the theoretical uncertainty.

3.2.2. Dependence on the SUSY Spectrum

Of course it is important to inspect how the result of the previous subsection changes depending on the SUSY spectrum. Here we restrict ourselves to the three-loop case. In the mSUGRA scenario only the parameter $m_{1/2}$ causes a significant variation of $M_{\mathrm{H_c}}$ and M_{G}. This dependence is depicted in fig. 3.4 (a). Still, varying $m_{1/2}$ up to 4 TeV does cause less variation in $M_{\mathrm{H_c}}$ than the one order of magnitude jump due to three loop effects that was discussed in the previous subsection. For further illustration the predictions of $M_{\mathrm{H_c}}$ and M_{G} for various SPS scenarios [117, 118] are shown in Fig. 3.4 (b). Again the variation of $M_{\mathrm{H_c}}$ is within one order of magnitude. Note that SPS9 (anomaly-mediated SUSY breaking) leads to the smallest values of $M_{\mathrm{H_c}}$.

3.2.3. Dependence on the Uncertainty on the Input Parameters

Up to now we have focused on using the central values of the gauge couplings and the weak mixing angle at the electroweak scale, eq. (3.6) as input for our analysis. The uncertainty in their determination, however, has a major impact on the prediction of $M_{\mathrm{H_c}}$ and M_{G} and translates into an uncertainty in these two parameters. Since $M_{\mathrm{H_c}}$ and M_{G} are complicated functions of the experimental input variables, their uncertainties will, in general, be correlated. In order to treat this in a proper way, we need to find a $\Delta\chi^2$ function that depends on $\log_{10}(M_{\mathrm{H_c}}/\mathrm{GeV})$ and $\log_{10}(M_{\mathrm{G}}/\mathrm{GeV})$. All values in the $\log_{10}(M_{\mathrm{H_c}}/\mathrm{GeV}) - \log_{10}(M_{\mathrm{G}}/\mathrm{GeV})$ plane that yield a $\Delta\chi^2$ below a certain fixed value, belong to a certain confidence level as will be explained in more detail in a moment. In the following we describe shortly the mathematical tools that are employed in such a situation (cf. e.g. ref. [121, 122]).

We start with some random variables x_i, $(i = 1, \ldots, n)$ with expectation values X_i. We assume that their uncertainties are not correlated, i.e. the covariance matrix is diagonal

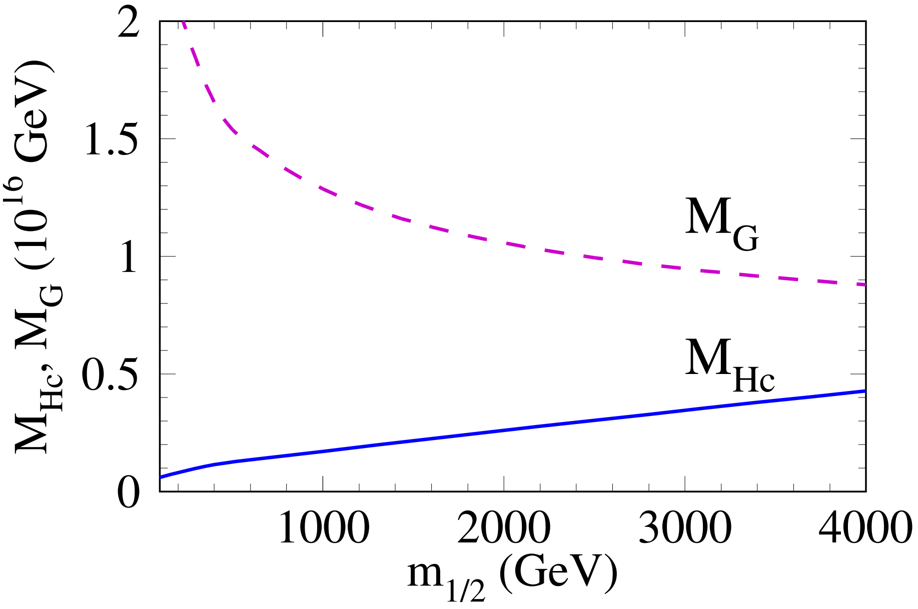
$$V^{(x)} = \mathrm{diag}(\sigma^2_{x_1}, \ldots, \sigma^2_{x_n}) \tag{3.19}$$

containing all squared standard deviations. Now we are interested in computing the covariance matrix $V^{(f)}$ of the quantities $f_j(\vec{x})$, $(j = 1, \ldots, m)$ with expectation values $F_j \equiv f_j(\vec{X})$. Assuming that the f_j can be approximated by a first order Taylor expansion in the region of about one standard deviation around X_i, we can compute the impact of the variation around $\vec{X}$ on the f_j:

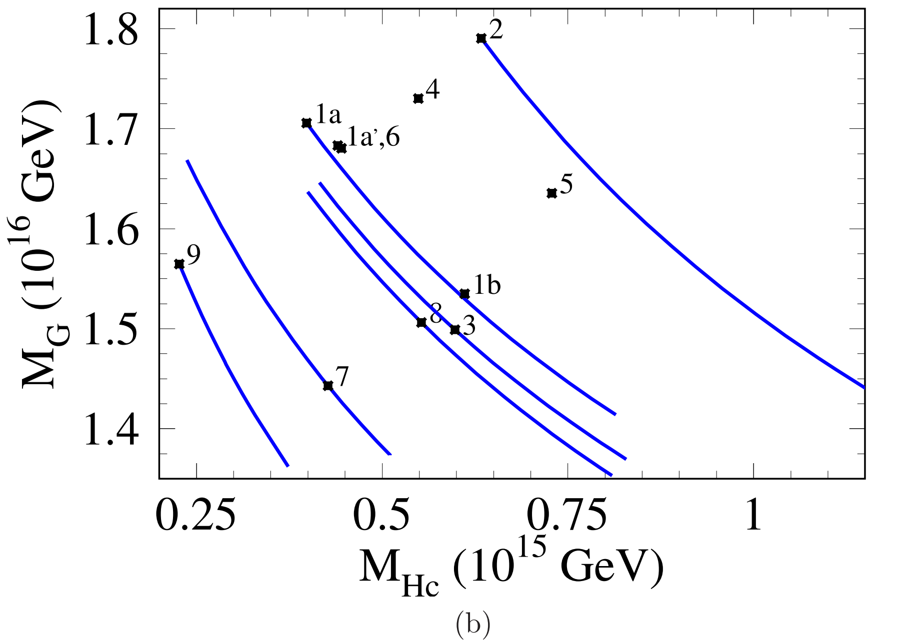
$$\delta f_j = f_j(\vec{X} + \delta\vec{x}) - F_j \approx \frac{\partial f_j(\vec{X})}{\partial X_i}\, \delta x_i \equiv J_{ji}\, \delta x_i\,, \tag{3.20}$$

where J is the $m \times n$ Jacobian matrix. Then the desired $m \times m$ covariance matrix for the quantities f_j is given by

$$V^{(f)} = J\, V^{(x)} J^T\,. \tag{3.21}$$



(a)



(b)

Figure 3.4.: *The dependence of* $M_{\mathrm{H_c}}$ *(solid in (a)) and* M_{G} *(dashed in (a)) on the SUSY spectrum. The labels and lines in panes (b) denote the various SPS points and slopes. Both plots have been generated with* $\mu_{\mathrm{SUSY}} = 1000\,\mathrm{GeV}$, $\mu_{\mathrm{GUT}} = 10^{16}\,\mathrm{GeV}$ *and three-loop running. The dependence on the mSUGRA parameters* m_0, $\tan\beta$ *and* A_0 *is rather mild.*

C.L. (%)	m=1	m=2	m=3
68.27	1.00	2.30	3.53
90.00	2.71	4.61	6.25
95.00	3.84	5.99	7.82
95.45	4.00	6.18	8.03
99.00	6.63	9.21	11.34
99.73	9.00	11.83	14.16

Table 3.2.: *Depending on the degrees of freedom m in $\vec{f}$, different values of $\Delta\chi^2$ correspond to different confidence levels (C.L.). The given values assume Gaussian distribution of the random variables f_j.*

This can now be used to calculate the $\Delta\chi^2$ function:

$$\Delta\chi^2(\vec{f}) = (\vec{f} - \vec{F})^T \, (V^{(f)})^{-1} \, (\vec{f} - \vec{F}) = (\vec{f} - \vec{F})^T \, (J \, V^{(x)} J^T)^{-1} \, (\vec{f} - \vec{F}) \tag{3.22}$$

where we have used eq. (3.21). If the standard deviations on x_i and their expectation values X_i together with the functions $f_j(\vec{x})$ are known, eq. (3.22) contains only known quantities. (The Jacobian matrix J can be computed numerically if the functions f_j are too complicated.) Computing $\Delta\chi^2$ for different values of $\vec{f}$ in eq. (3.22) yields a measure for the probability that this particular $\vec{f}$ is realized. A table that assigns confidence levels (C.L.) to particular values of $\Delta\chi^2$ assuming that the f_j are Gaussian distributed can be found e.g. in ref. [7] and is reproduced in table 3.2 for convenience.

For our particular case we have three relevant input values (i.e. $n = 3$) with non-negligible uncertainties and take

$$\begin{aligned} x_1 &= \Delta\alpha^{(5)}_{\text{had}} \,, \\ x_2 &= \sin^2\Theta^{\overline{\text{MS}}} \,, \\ x_3 &= \alpha_s(M_Z) \,, \end{aligned} \tag{3.23}$$

with mean values and standard deviations given in eq. (3.2). The functions $f_j(\vec{x})$ are defined by the running and decoupling procedure described in section 3.1 and

$$\begin{aligned} f_1 &= \log_{10}(M_{\text{H}_\text{c}}/\text{GeV}) \,, \\ f_2 &= \log_{10}(M_{\text{G}}/\text{GeV}) \,, \end{aligned} \tag{3.24}$$

i.e. m = 2. The Jacobian J is determined numerically by calculating difference quotients with sufficiently small step sizes such that a variation of the step size yields a stable result. Fig. 3.5 shows the result in the $M_{\text{H}_\text{c}} - M_{\text{G}}$ plane. We have adopted the choice $\mu_{\text{SUSY}} = M_Z$ here in order to be able to compare with the results in the literature. For the mSUGRA parameters we again adopt our default choice, eq. (3.18). Again, one can see that the three-loop effects are of comparable size as the uncertainty on the input parameters and cause a considerable shift of about one order of magnitude in M_{H_c}. Note, however, if we had chosen μ_{SUSY} in the vicinity of the average SUSY masses, the two- and three-loop ellipses would be almost on top of each other (cf. also fig. 3.3 (a)) and coinciding with the three-loop ellipses in fig. 3.5. Therefore, we can conclude from this analysis that in this case a two-loop running is sufficient if the decoupling scale is chosen close to the SUSY mass scales. Let us also mention that we can reproduce the results of ref. [19] after adopting their parameters and restricting ourselves to the perturbative input used in that publication.

Moreover, it is noteworthy that the error on the strong coupling has the largest impact on the uncertainty of $M_{\mathrm{H_c}}$ and essentially determines the semi-major axis of the ellipse. Fig. 3.5 (b) illustrates how the outcome would be changed if $\delta\alpha_s$ is halved.

Recently there have been a few extractions of α_s based on higher order perturbative corrections with uncertainties slightly above 1%, which, however, obtain central values for α_s close to 0.113 (see, e.g., ref. [123]). Since these results are significantly lower than the value given in eq. (3.2) it is interesting to show in Fig. 3.5 (a) also the corresponding 68% and 90% confidence level ellipses (for the two- and three-loop analyses) as dotted lines adopting $\alpha_s(M_Z) = 0.1135 \pm 0.0014$ [124]. One observes a big shift in the GUT masses, in the case of $M_{\mathrm{H_c}}$ the central value is about one order of magnitude lower than for the α_s value of eq. (3.2).

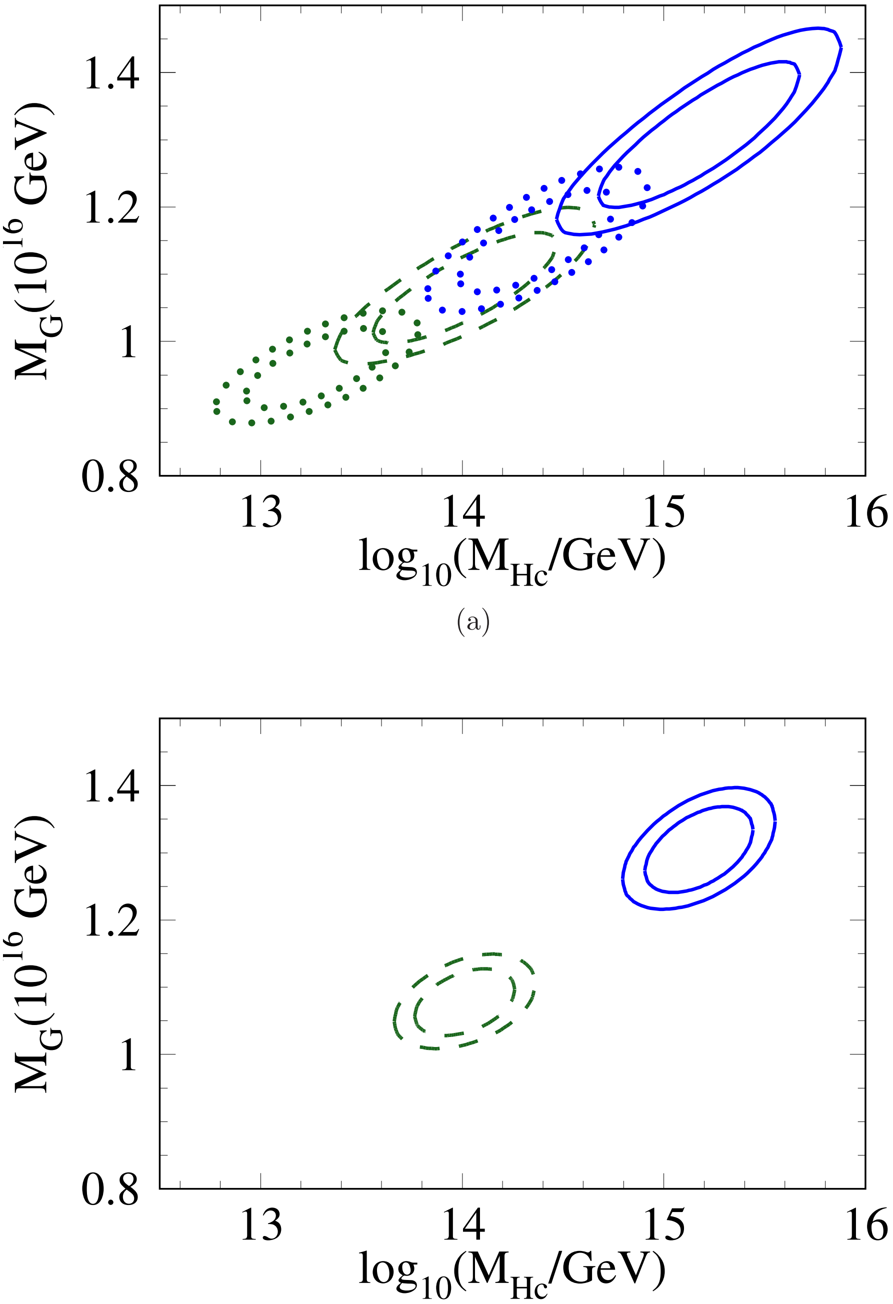
3.2.4. Top-Down Approach and the Missing Doublet Model

Finally, in this section we discuss the phenomenological results using the "top-down" approach, i.e. specifying the unique gauge coupling at some scale above μ_{GUT} and then predicting the three distinct gauge couplings at M_Z. Then we can again vary the unphysical scale μ_{GUT} and examine the impact on $\alpha_i(M_Z)$ in order to compare them to the experimental values. The reason that this approach gives additional information compared to the approach of subsection 3.2.1 is the following: Up to now the only way the threshold corrections at μ_{GUT} enter our analysis was via eqs. (3.16) and (3.17). However, these equations do not contain any information about the running above μ_{GUT} as will be argued in a moment, which has the consequence that the theoretical uncertainty due to the variation of μ_{GUT} may be underestimated :

In the general one-loop formula for the decoupling coefficient at μ_{GUT}, eq. (2.35), the μ_{GUT} dependent terms can be rewritten in the form (cf. section A.8 in the appendix)

$$\zeta_{\alpha_i}(\mu_{\mathrm{GUT}}) = 1 + \frac{\alpha(\mu_{\mathrm{GUT}})}{\pi}\Big[\tfrac{1}{2}(\beta_0^i - \beta_0)\,\ln(\mu_{\mathrm{GUT}}) - C_0(M_{\mathrm{h}})\Big] \tag{3.25}$$

where β_0^i and β_0 are the one-loop gauge β function coefficients for the effective theory (MSSM) and the GUT, respectively. $C_0(M_{\mathrm{h}})$ parametrizes the terms that depend on the super-heavy masses. Here, due to the presence of β_0 the decoupling coefficient depends on the running above μ_{GUT}. Therefore, for models that contain large representation, as the Missing Doublet Model, $\zeta_{\alpha_i}(\mu_{\mathrm{GUT}})$ will receive large contributions due to large numerical values of the Casimir invariants. In particular in the top-down approach, the impact on $\alpha_i(M_Z)$ due to the variation of μ_{GUT} will be huge in such a case. In forming the linear combinations, eqs. (3.16) and (3.17), it is easy to see that β_0 drops out. This is reflected by the moderate numerical coefficients in in front of the logs in eq. (3.17) as compared to eq. (2.37). Therefore, the large variation of physical parameters that has been observed before is not present in the approach of just using the one-loop formulae, eqs. (3.16) and (3.17), in order to determine $M_{\mathrm{H_c}}$ and M_{G}. Though this is no problem per se, it is not clear at all whether the same will happen if one



(a)

(b)

Figure 3.5.: *The impact of the uncertainties on the gauge couplings at the electroweak scale on* $M_{\mathrm{H_c}}$ *and* M_{G}*. The inner ellipses mark the 68% and the outer ones the 90% confidence level regions. Panel (a) shows the correlation for* $\delta\alpha_s = 0.0020$ *which is our default choice. The dashed ellipses are obtained from the two-loop analysis and the solid ellipses from the three-loop analysis. The dotted ellipses show how the result changes if one employs* $\alpha_s(M_Z) = 0.1135 \pm 0.0014$ *instead. In panel (b) we have adopted the more optimistic choice* $\delta\alpha_s = 0.0010$*. Again the dashed and solid curves correspond to the two- and three-loop analyses, respectively.*

moves to two-loop decoupling at μ_{GUT}, because there interference terms of various β function coefficients will appear in the μ_{GUT} dependence of $\zeta_{\alpha_i}(\mu_{\mathrm{GUT}})$. For this reason the estimation of theory uncertainty due to the variation of μ_{GUT} has to be handled with care when using eqs. (3.16) and (3.17). In order to estimate the full uncertainty, we employ the top-down approach in this last subsection of our analysis[5].

Fig. 3.6 (a) shows the prediction of $\alpha_i(M_Z)$ depending on μ_{GUT} for the following set of parameters:

$$\begin{aligned}
\alpha(10^{17}\text{ GeV}) &= 0.03986\,, \\
M_{\mathrm{H_c}} &= 3.67\cdot 10^{14}\text{ GeV}\,, \\
M_{\Sigma} &= 2\cdot 10^{16}\text{ GeV}\,, \\
M_{\mathrm{X}} &= 1.58\cdot 10^{16}\text{ GeV}\,, \\
\mu_{\mathrm{SUSY}} &= 500\text{GeV}\,,
\end{aligned} \tag{3.26}$$

and use the SPS1a benchmark scenario in order to be able to compare with the results for the Missing Doublet Model[6]. Moreover, the chosen set of GUT mass parameters guarantees the agreement of $\alpha^{(6),\overline{\mathrm{MS}}}(M_Z)$, $\sin^2\Theta^{(6),\overline{\mathrm{MS}}}(M_Z)$ and $\alpha_s^{(6),\overline{\mathrm{MS}}}(M_Z)$ with the values in eq. (3.2) for $\mu_{\mathrm{GUT}} = 10^{16}$GeV. It is noticeable that the one-loop curve for $\alpha_3(M_Z)$ is completely flat. This is due to the fact that the one-loop QCD β function coincides with the one of minimal SUSY SU(5). The two-loop and three-loop curve (both with one-loop decoupling) exhibit a variation over the considered range of μ_{GUT} that is comparable to the experimental uncertainty. This is in slight contrast to what has been observed in the previous sections where the variation of $M_{\mathrm{H_c}}$ depending on μ_{GUT} was very small. This teaches us that, despite all arguments given in the previous subsections, to be finally sure that the two-loop decoupling effects at μ_{GUT} are small for the Minimal SUSY SU(5), one has to compute them.

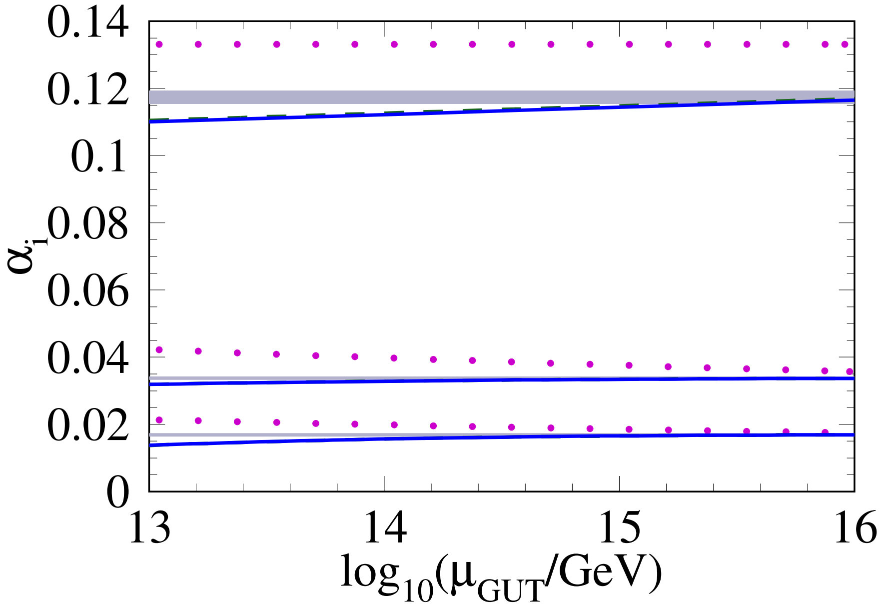
When inspecting the μ_{GUT} dependence in the same fashion for the Missing Doublet Model, things change dramatically as can be seen in fig. 3.6 (b). Here we have used as input parameters

$$\begin{aligned}
\alpha(10^{17}\text{ GeV}) &= 0.1504\,, \\
M_{\mathrm{H_c}} &= 6\cdot 10^{18}\text{ GeV}\,, \\
M_{\mathrm{H_{c'}}} &= 1\cdot 10^{16}\text{ GeV}\,, \\
M_{\Sigma} &= 2\cdot 10^{15}\text{ GeV}\,, \\
M_{\mathrm{X}} &= 3\cdot 10^{16}\text{ GeV}\,, \\
\mu_{\mathrm{SUSY}} &= 500\text{GeV}\,,
\end{aligned} \tag{3.27}$$

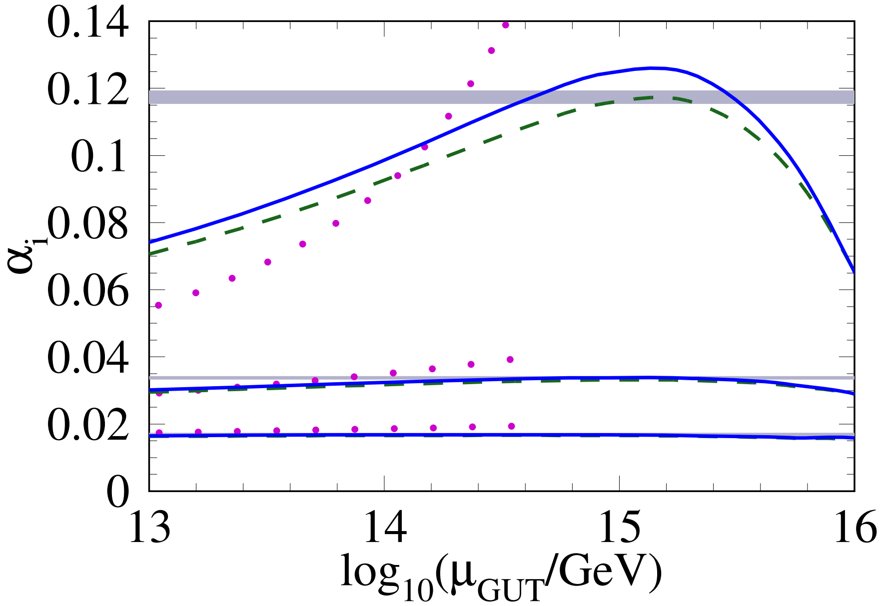
and again the SPS1a benchmark scenario. As can be seen, the dependence of $\alpha_i(M_Z)$ due to the variation of μ_{GUT} is huge here. The theoretical uncertainty due to the scale variation must be regarded as about one order of magnitude larger than the uncertainty on the experimental value of $\alpha_s(M_Z)$. Therefore, no conclusion on whether this particular parameter point is

[5] Of course, one could as well use a bottom-up approach and analyze the μ_{GUT} dependence of $\alpha(M_{\mathrm{Pl}})$. But then we would not have the direct comparison with experimental values.

[6] Since in the Missing Doublet Model the gauge coupling easily becomes non-perturbative above μ_{GUT} due to the large gauge β function, one has to tune the input parameters carefully.



(a)



(b)

Figure 3.6.: *Gauge couplings at the $\mu = M_Z$ obtained by a top-down approach within (a) Minimal SUSY SU(5) and (b) the Missing Doublet Model as a function of μ_{GUT}. The dotted, dashed and solid lines represent the one- two- and three-loop results, respectively. Furthermore, the experimental values for $\alpha_i(M_Z)$ including uncertainties are denoted by gray bands. In panel (b) the one-loop curves are depicted only up to $\mu_{\mathrm{GUT}} \approx 4 \cdot 10^{14}$GeV since beyond that scale α_3 becomes non-perturbative.*

excluded or not can be drawn here. This observation provides an even stronger motivation to compute the missing two-loop matching corrections at μ_{GUT} in order to allow for an sufficiently reliable analysis of models with large representations, as the Missing Doublet Model. These corrections are expected to reduce the scale dependence to a reasonable level.

As a further example where such large uncertainties are expected to occur, let us mention GUT models based on the group E_6 [125, 126]. Since the smallest representations in this group are 27-, 78- (adjoint) and 351-dimensional, very easily large contributions to the gauge β functions are produced. Due to eq. (3.25), this then induces a strong dependence on the decoupling scale in ζ_{α_i} and thus leads to the same effect as demonstrated for the Missing Doublet Model. As an example consider a SUSY E_6 model that employs a $\mathbf{351}_H$ for the symmetry breaking[7] and the three fermion generations are embedded in three copies of the $\mathbf{27}$. Then the minimal contribution to the one-loop gauge β function will be (cf. eq. (A.75))

$$\begin{aligned}\beta_0^{E_6} &= -3\, I_2(\Pi^{\mathcal{A}}) + \sum_x I_2(\Pi^x) \\ &= -3\, I_2(\Pi^{\mathbf{78}}) + 3\, I_2(\Pi^{\mathbf{27}}) + I_2(\Pi^{\mathbf{351}}) + \cdots \\ &= -3 \cdot 12 + 3 \cdot 3 + 84 + \cdots \\ &= 57 + \cdots \end{aligned} \tag{3.28}$$

where the ellipsis denotes further positive contributions. The numerical values for the Dynkin indices $I_2(\cdots)$ have been taken from ref. [128]. Compared to the Missing Doublet Model where $\beta_0 = 17$, we therefore expect an even greater uncertainty due to the variation of μ_{GUT} from such an E_6 model. Let alone when a $\mathbf{650}_H$ with a Dynkin index of

$$I_2(\Pi^{\mathbf{650}}) = 150 \tag{3.29}$$

is involved in the breaking. To the author's knowledge many gauge coupling unification analyses for E_6 models (e.g. ref. [129]) don't even include one-loop threshold effects at the GUT scale and rely purely on the apparent meeting of the gauge couplings due to RG running. From the above argument, care is needed with such an approach since the (one- and two-loop) threshold-corrections can change the picture in such a situation dramatically. These considerations show that a large GUT gauge β function not only implies a tendency for non-perturbative behavior, but also huge theoretical uncertainties.

3.2.5. Comparison with Proton Decay Constraints

Having derived all the phenomenological constraints on $M_{\mathrm{H_c}}$ from gauge coupling unification, it is in order to compare with the constraints that come from proton decay in the Minimal SUSY SU(5) Model. The latest upper bound on the proton decay rate for the dominant channel $p \to K^+\bar{\nu}$ [130] is $\Gamma_{\mathrm{exp}} = 4.35 \cdot 10^{-34}/\mathrm{y}$. In order to translate it into a lower bound for the the Higgs triplet mass in the minimal *renormalizable* SUSY SU(5) model, one needs an additional assumption about the Yukawa couplings that enter the expression of the decay rate $\Gamma(p \to K^+\bar{\nu})$. As pointed out in ref. [61] and at the end of section 2.6, one could either use

[7] Note that the presence of the $\mathbf{351}_H$ is not always mandatory. There are also consistent SUSY E_6 models where the Higgs content $\mathbf{78}_H \oplus 2 \times (\mathbf{27}_H \oplus \overline{\mathbf{27}}_H)$ is sufficient [127].

(i) the (larger) down-quark Yukawa coupling or

(ii) the (smaller) electron Yukawa coupling

for the calculation of the decay rate. Both cases are equally justified but yield strongly deviating results for the decay rate[8] For the case (i) and sparticle masses around 1 TeV the lower bound for the Higgs triplet mass can be read off from Fig. 2 of ref. [61] and amounts to $M_{\mathrm{H_c}} \geq 1.05 \cdot 10^{17}$GeV whereas for the second choice it becomes $M_{\mathrm{H_c}} \geq 5.25 \cdot 10^{15}$GeV. From our phenomenological analysis presented above it turns out that within the minimal renormalizable SUSY SU(5) model the upper bound for $M_{\mathrm{H_c}}$ is of about 10^{16} GeV at 90% C.L. which is about one order of magnitude higher that the two-loop result with decoupling at M_Z. Though this is still in conflict with case (i), we observe that the three loop effects push $M_{\mathrm{H_c}}$ up to the region that is allowed in case (ii) and one need not rely on higher-dimensional operators in this case.

Including higher-dimensional operators, finally, to make the formulation of the theory consistent, the tension between gauge coupling unification and proton decay is further reduced by the mechanisms described in section 2.6.

3.3. Summary

In this last section, we briefly summarize the most important findings of the present chapter:

- We have performed an (almost) three-loop RG analysis for the gauge couplings and examined phenomenological consequences for Minimal SUSY SU(5) and the Missing Doublet Model with a focus on the former.

- We have found that $\mu_{\mathrm{SUSY}} = M_Z$ is not a good choice for the decoupling scale when doing a two-loop analysis, though commonly used in the literature. With this choice the mass of the colored triplet Higgs gets raised by one order of magnitude by the three loop effect attenuating the tension with proton decay constraints (cf. subsections 3.2.1 and 3.2.3).

- We confirm that Minimal SUSY SU(5) cannot be ruled out by proton decay experiments (cf. subsection 3.2.5).

- For the Missing Doublet Model the variation of $\alpha_3(M_Z)$ due to the dependence on the decoupling scale μ_{GUT} is one order of magnitude larger that the current uncertainty on the experimental value of $\alpha_3(M_Z)$. This behavior is expected in any GUT model with large representations and is due to the missing two-loop GUT matching corrections. This observation provides a strong motivation to calculate these corrections (cf. subsection 3.2.4).

[8] The author is aware that this treatment is not completely consistent due to the non-unification of the first generation Yukawa couplings in the renormalizable minimal SUSY SU(5). However, including higher-dimensional operators complicates the treatment and only further weakens the bound compared to the case where the electron Yukawa coupling is used for the calculation in the renormalizable model.

4. Field-Theoretical Framework for the Two-Loop Matching Calculation

In the previous chapter an (almost) three-loop RGE gauge coupling unification analysis was presented for the Minimal SUSY SU(5) and the Missing Doublet Model. One of the missing pieces that would complete the analysis are the two-loop GUT matching corrections. Though their numerical impact is expected to be moderate in the case of Minimal SUSY SU(5), for the Missing Doublet Model and any other GUT model with large representations they constitute a decisive building block. Without them a reasonably reliable exclusion of such models by gauge coupling unification is virtually impossible because the theoretical uncertainty due to the variation of the unphysical parameter μ_{GUT} is so huge (cf. subsection 3.2.4).

This chapter is devoted to the description of a theoretical framework that is suitable for the calculation of these two-loop matching corrections. Since there is a host of well motivated GUT models that we want to apply our formula to, we want our final result to be applicable to as many of them as possible. Therefore, it is desirable to carry out the calculation of the two-loop matching corrections at the GUT scale in a framework that makes as few assumptions on the underlying GUT model as possible. The idea is to have a general formula that depends on the Casimir invariants and the mass spectrum of the theory. Such a formula has been available at the one-loop level for a long time and was presented in eq. (2.35). Choosing a specific model specifies those Casimir invariants and gives an expression that depends only on the masses and couplings of the model.

The actual calculation that has been carried out in the course of this thesis is not yet done in full generality, but makes some additional assumptions about the model. These assumptions will be described in section 4.3. Nevertheless, we present the theoretical framework that is needed for the calculation of the relevant Green's functions (almost) as general as possible below in order to be armed for future improvements of the calculation.

4.1. The Lagrangian

We consider a general renormalizable quantum field theory defined by the following Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_\alpha^{\mu\nu} F_{\mu\nu}^\alpha + \bar{\psi}\,(i\bar{\sigma}^\mu D_\mu - \mu_F)\psi + \frac{1}{2}(D^\mu\Phi)^T D_\mu \Phi - V(\Phi) + \mathcal{L}_{\mathrm{Y}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{gh}}. \tag{4.1}$$

The Weyl fermion field ψ and the real scalar field Φ reside in (not necessarily irreducible) representations of the gauge group $\mathbf{G}$. The matrix μ_F can contribute to Dirac and Majorana mass terms of the fermions. The dynamics of the gauge field that transforms as the adjoint representation of $\mathbf{G}$, is described by the Yang Mills curl $F_\alpha^{\mu\nu} = \partial^\mu A_\alpha^\nu - \partial^\nu A_\alpha^\mu + g f_{\alpha\beta\gamma} A_\beta^\mu A_\gamma^\nu$. Moreover, $V(\Phi)$, $\mathcal{L}_{\mathrm{Y}}$, $\mathcal{L}_{\mathrm{gf}}$ and $\mathcal{L}_{\mathrm{gh}}$ are the scalar potential, the Yukawa interactions, the gauge fixing and the ghost parts of the Lagrangian, respectively. They will be described in detail later in this section. $V(\Phi)$ is chosen such that the scalar field $\Phi = v + \Phi'$ contains one $\mathbf{G}$-irreducible subspace that develops a vacuum expectation value (vev) v, that breaks $\mathbf{G}$ down to the SM gauge group $\mathbf{G}_{\mathrm{SM}} \equiv \prod_k \mathbf{G}_k =$SU(3)×SU(2)×U(1). Models that have more than one vev of $\mathcal{O}(M_{\mathrm{GUT}})$, where M_{GUT} is a typical mass of a super-heavy particle, are not covered by our framework yet. The indices $\alpha, \beta, \gamma \ldots$ belong to the adjoint representation and label the generators of $\mathbf{G}$ which fulfill the commutation relations

$$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma}\, T^\gamma \quad \text{and also:} \quad [\tilde{T}^\alpha, \tilde{T}^\beta] = i f^{\alpha\beta\gamma}\, \tilde{T}^\gamma\,, \tag{4.2}$$

with the structure constants $f^{\alpha\beta\gamma}$. We use the tilde to denote the generators of the real[1] scalar representation which fulfill $(\tilde{T}^\alpha)^T = -\tilde{T}^\alpha$. The generator that acts on the fermion field satisfies $(T^\alpha)^\dagger = T^\alpha$. Again, T^α and $\tilde{T}^\alpha$ need not necessarily be defined on irreducible representations of $\mathbf{G}$, but can also have block diagonal form. In order to distinguish between broken and unbroken generators, we introduce the notation:

$$\{\alpha\} = \sum_i \{A_i\} + \sum_i \{a_i\} = \{A\} + \{a\}\,, \tag{4.3}$$

where A_i label the broken generators of $\mathbf{G}$ belonging to the $\mathbf{G}_{\mathrm{SM}}$-irreducible subspace labeled by i. If there is only one $\mathbf{G}_{\mathrm{SM}}$-irreducible subspace in the adjoint representation of $\mathbf{G}$, as e.g. is practically the case in SU(5), we can omit the sub-index i. In contrast, a_i labels the unbroken generators belonging to the subgroup[2] $\mathbf{G}_i$:

$$\begin{aligned} \tilde{T}^{a_i} v &= 0\,, \\ \tilde{T}^{A_i} v &\neq 0\,. \end{aligned} \tag{4.4}$$

As an example for this index labeling consider the decomposition of the adjoint representation of SU(5)

$$\underbrace{\mathbf{24}}_{\alpha} \to \underbrace{(\mathbf{8},\mathbf{1},0)}_{a_3} \oplus \underbrace{(\mathbf{1},\mathbf{3},0)}_{a_2} \oplus \underbrace{(\mathbf{1},\mathbf{1},0)}_{a_1} \oplus \underbrace{(\mathbf{3},\overline{\mathbf{2}},-\tfrac{5}{6}) \oplus (\overline{\mathbf{3}},\mathbf{2},\tfrac{5}{6})}_{A}\,. \tag{4.5}$$

In this case no distinction of indices belonging to $(\mathbf{3},\overline{\mathbf{2}},-\frac{5}{6})$ and $(\overline{\mathbf{3}},\mathbf{2},\frac{5}{6})$ is necessary because these two representations are complex conjugates of each other. In contrast, for the case of SO(10), the adjoint $\mathbf{45}$ decomposes under $\mathbf{G}_{\mathrm{SM}}$ as

$$\begin{aligned} \underbrace{\mathbf{45}}_{\alpha} \to\ & \underbrace{(\mathbf{8},\mathbf{1},0)}_{a_3} \oplus \underbrace{(\mathbf{1},\mathbf{3},0)}_{a_2} \oplus \underbrace{(\mathbf{1},\mathbf{1},0)}_{a_1} \oplus \underbrace{(\mathbf{3},\overline{\mathbf{2}},-\tfrac{5}{6}) \oplus (\overline{\mathbf{3}},\mathbf{2},\tfrac{5}{6})}_{A_1} \oplus \underbrace{(\mathbf{3},\overline{\mathbf{1}},\tfrac{2}{3}) \oplus (\overline{\mathbf{3}},\mathbf{1},-\tfrac{2}{3})}_{A_2} \\ & \oplus \underbrace{(\mathbf{3},\mathbf{2},\tfrac{1}{6}) \oplus (\overline{\mathbf{3}},\overline{\mathbf{2}},-\tfrac{1}{6})}_{A_3} \oplus \underbrace{(\mathbf{1},\mathbf{1},1) \oplus (\mathbf{1},\mathbf{1},-1)}_{A_4} \oplus \underbrace{(\mathbf{1},\mathbf{1},0)}_{A_5} \end{aligned} \tag{4.6}$$

[1]This is no loss of generality since any complex scalar can be written as two real scalars. Cf. also section A.3 in the Appendix.

[2]Please note the different meanings of the sub-index i when attached to the capital adjoint index opposed to when attached to a lowercase adjoint index.

Here we explicitly need to take into account sub-indices for broken generators.

The Lagrangian in eq. (4.1) is invariant under local gauge transformations with the real parameter $\theta = \theta(x)$:

$$\begin{aligned} \psi &\rightarrow \psi - i\theta^\alpha T^\alpha \psi \,, \\ \Phi &\rightarrow \Phi - i\theta^\alpha \tilde{T}^\alpha \Phi \,, \\ A_\mu^\alpha &\rightarrow A_\mu^\alpha + f^{\alpha\beta\gamma}\theta^\beta A_\mu^\gamma - \frac{1}{g}\partial_\mu \theta^\alpha \,. \end{aligned} \tag{4.7}$$

The covariant derivatives are defined as:

$$\begin{aligned} D_\mu \psi &= (\partial_\mu - igT^\alpha A_\mu^\alpha)\psi \,, \\ D_\mu \Phi &= (\partial_\mu - ig\tilde{T}^\alpha A_\mu^\alpha)\Phi \,. \end{aligned} \tag{4.8}$$

Note that $f^{a_i b_j A_k} = 0$ because otherwise the commutator $[T^{a_i}, T^{b_j}]$ would contain terms proportional to T^{A_k}. Then $\mathbf{G}_{\text{SM}}$ would not be closed (and hence would not be a group) which is not the case. Using this property and eqs. (4.2) and (4.4), the gauge-kinetic term for the scalar field $\Phi = v + \Phi'$ can be written as:

$$\begin{aligned} \frac{1}{2}(D^\mu \Phi)^T D_\mu \Phi &= \frac{1}{2}(\partial^\mu \Phi')^T \partial_\mu \Phi' + \frac{1}{2} g^2 v \tilde{T}^A \tilde{T}^B v A_A^\mu A_{\mu B} \\ &\quad + igv\tilde{T}^A \partial_\mu \Phi A_A^\mu + ig^2 f_{ABa} A_B^\mu A_{\mu a} v \tilde{T}^A \Phi' \\ &\quad + ig\Phi' \tilde{T}^\alpha \partial_\mu \Phi' A_\alpha^\mu + \frac{1}{2} g^2 \Phi' \tilde{T}^\alpha \tilde{T}^\beta \Phi' A_\alpha^\mu A_{\mu\alpha} \\ &\quad + g^2 v \tilde{T}^A \tilde{T}^B \Phi' A_A^\mu A_{\mu B} \,, \end{aligned} \tag{4.9}$$

where we can identify the (diagonal) gauge boson mass matrix

$$(M_{\text{X}})_{A_i B_i} \equiv g^2 v \tilde{T}^{A_i} \tilde{T}^{B_i} v \,, \tag{4.10}$$

with eigenvalues denoted by M_{X_i}. Again, the sub-index i labels the $\mathbf{G}_{\text{SM}}$-irreducible subspace that is meant, because each $\mathbf{G}_{\text{SM}}$-irreducible subspace can be assigned to a definite gauge boson mass. Note that the position of the adjoint indices $\alpha, A_i, a_i \ldots$ is irrelevant. Furthermore, it is understood that a partial derivative acts only on the single field, which is next to it. The gauge-kinetic term for the scalars contains the undesired quadratic mixing $igv\tilde{T}^A \partial_\mu \Phi A_A^\mu$ between Goldstone bosons and heavy gauge bosons. As we will see in a moment, the gauge fixing Lagrangian $\mathcal{L}_{\text{gf}}$ can be chosen in such a way that this term is canceled, at least at tree-level.

4.1.1. Gauge Fixing and Ghost Interactions

The quantization of gauge fields involves some peculiarities that are best dealt with in the functional integral formalism where an integration over all possible field configurations of the exponentiated classical action is performed. When carrying out this integration for the gauge fields A_μ^α, there are configurations that are connected via a gauge transformation, eq. (4.7), and thus are equivalent. They give an infinite contribution and thus cause the functional integral to diverge. In order to avoid this, we need to integrate only over a particular representative of such a gauge orbit. As described e.g. in ref. [65], this can be achieved by defining the generating functional as

$$Z[J] \quad = \quad \int [dA]\,\mathrm{det}M_f \exp\{i \int d^4x(\mathcal{L}_{\mathrm{class}} + \mathcal{L}_{\mathrm{gf}} + A^{\alpha}_{\mu} J^{\alpha\mu})\} \tag{4.11}$$

where $J^{\alpha\mu} = J^{\alpha\mu}(x)$ is the source function and

$$\mathcal{L}_{\mathrm{gf}} \equiv -\frac{1}{2}\sum_{\alpha} f_{\alpha}^2\,, \qquad (M_f(x,y))^{\alpha\beta} \equiv \frac{\delta f_{\alpha}(\theta)}{\delta\theta^{\beta}} \tag{4.12}$$

are the gauge-fixing Lagrangian and the Faddeev-Popov matrix, respectively. They are defined in terms of the gauge fixing functional f_{α} which is chosen such that $f_{\alpha} = 0$ defines a unique representative $A^{(\theta)\alpha}_{\mu}$ for a particular gauge orbit[3]. The functional derivative with respect to the gauge transformation parameter θ in eq. (4.12) is to be understood as the rate of change of f_{α} when A^{α}_{μ} and all other fields are gauge-transformed. Since we are dealing with a spontaneously broken gauge theory, we choose the so-called R_{ξ} gauge fixing functional [132]

$$\begin{aligned} f_{A_i} \quad &= \quad \frac{1}{\sqrt{\xi_{1i}}}\partial_{\mu}A^{\mu}_{A_i} - ig\sqrt{\xi_{2i}}v\tilde{T}^{A_i}\Phi'\,, \\ f_{a_i} \quad &= \quad \frac{1}{\sqrt{\eta_i}}\partial_{\mu}A^{\mu}_{a_i}\,. \end{aligned} \tag{4.13}$$

Note that we have chosen two distinct gauge parameters ξ_1 and ξ_2 for each $\mathbf{G}_{\mathrm{SM}}$-irreducible subspace of the heavy gauge bosons. They renormalize differently and thus can only be equated with each other after renormalization. Otherwise not all the Green's functions can be made finite. In the same way each SM group factor receives its own gauge parameter η_i. This subtlety arises first at the two-loop level and is not relevant for computing one-loop matching coefficients. Although there are other possibilities for the choice of the gauge fixing functional [31, 133–135], we find this one most convenient for our purposes. In order to account for the presence of the Faddeev-Popov determinant $\mathrm{det}M_f$ in the functional integral, eq. (4.11), we exponentiate the determinant [65, 136]:

$$\mathrm{det}M_f \quad = \quad \int [dc][dc^{\dagger}] \exp\{-i\int d^4x\, d^4y\, c^{\dagger}_{\alpha}(x)(M_f(x,y))^{\alpha\beta}c_{\beta}(y)\} \tag{4.14}$$

which leads to new (unphysical) ghost fields c_{α} and $c^{\dagger}_{\alpha}$ in the action. We compute the Faddeev-Popov matrix, eq. (4.12), for our choice of the gauge fixing functional, eq. (4.13). The result is proportional to $\delta^{(4)}(x-y)$ allowing us to carry out the integration over y in eq. (4.14). We absorb the factor $1/(g^{1/2}\xi_{1i}^{1/4})$ into c_{A_i} and $c^{\dagger}_{A_i}$, as well as the factor $1/(g^{1/2}\eta_i^{1/4})$ into c_{a_i} and $c^{\dagger}_{a_i}$ in order to guarantee for canonically normalized kinetic ghost terms. Finally, the whole procedure amounts to adding the following ghost interactions to the Lagrangian:

[3]This requirement may not be always fulfilled for non-abelian gauge theories due to Gribov ambiguities [131]. But since we are only interested in perturbative aspects in this work, we can assume that this is always possible.

$$
\begin{aligned}
\mathcal{L}_{\text{gh}}^{R_\xi} &= \sum_{ij} \Big[\partial^\mu c_{A_i}^\dagger (\delta_{A_i B_j} \partial_\mu - g f_{A_i B_j \alpha} A_\mu^\alpha) c_{B_j} \\
&\quad - g^2 \sqrt{\xi_{1i}\xi_{2i}}\; v \tilde{T}^{A_i} \tilde{T}^{B_j} v\; c_{A_i}^\dagger c_{B_j} - g^2 \sqrt{\xi_{1i}\xi_{2i}}\; v \tilde{T}^{A_i} \tilde{T}^{B_j} \Phi' c_{A_i}^\dagger c_{B_j} \Big] \\
&\quad - \sum_{ij} \Big[g \left(\frac{\eta_j}{\xi_{1i}} \right)^{\frac{1}{4}} f_{A_i b_j B_i} \partial^\mu c_{A_i}^\dagger A_\mu^{B_i} c_{b_j} + i g^2 \left(\xi_{1i} \eta_j \xi_{2i}^2 \right)^{\frac{1}{4}} f_{A_i b_j B_i}\; v \tilde{T}^{B_i} \Phi' c_{A_i}^\dagger c_{b_j} \Big] \\
&\quad - \sum_{ij} g \left(\frac{\xi_{1j}}{\eta_i} \right)^{\frac{1}{4}} f_{a_i B_j A_j} \partial^\mu c_{a_i}^\dagger A_\mu^{A_j} c_{B_j} \\
&\quad + \sum_i \partial^\mu c_{a_i}^\dagger (\delta_{a_i b_i} \partial_\mu - g f_{a_i b_i c_i} A_\mu^{c_i}) c_{b_i} \,.
\end{aligned} \tag{4.15}
$$

Here c_{A_i} and c_{a_i} denote the ghost fields belonging to the heavy and light gauge bosons, respectively. Note again that for an SU(5) GUT we could do the replacement $A_i \to A$ for the capital adjoint indices and the notation would become less clumsy. Here, however, we keep the sub-index i in order to stay as general as possible.

The gauge-fixing Lagrangian is easier to compute and reads

$$
\begin{aligned}
\mathcal{L}_{\text{gf}}^{R_\xi} &= -\frac{1}{2} \sum_i f_{A_i}^2 - \frac{1}{2} \sum_i f_{a_i}^2 \\
&= \sum_i \left[-\frac{1}{2\xi_{1i}} (\partial_\mu A_{A_i}^\mu)^2 - \frac{1}{2} g^2 \xi_{2i}\, \Phi' \tilde{T}^{A_i} v v \tilde{T}^{A_i} \Phi' + i g \sqrt{\frac{\xi_{2i}}{\xi_{1i}}} v \tilde{T}^{A_i} \Phi' \partial^\mu A_\mu^{A_i} \right] \\
&\quad - \sum_i \frac{1}{2\eta_i} (\partial_\mu A_{a_i}^\mu)^2 \,.
\end{aligned} \tag{4.16}
$$

As mentioned before, after partial integration, the term $ig\sqrt{\xi_{2i}/\xi_{1i}}\; v\tilde{T}^{A_i}\Phi'\partial^\mu A_\mu^{A_i}$ in eq. (4.16) exactly cancels the corresponding term in eq. (4.9) at tree level, where $\xi_{1i} = \xi_{2_i}$ is a valid choice. However, when considering higher orders in perturbation theory, the bare gauge parameters ξ_{1i} and ξ_{2_i} are not equal to each other and the above two terms must be kept explicitly as a counterterm in our calculation (cf. also the mixed counterterms in section 4.2). The quadratic term in eq. (4.16) can be identified with the (unphysical) Goldstone boson mass matrix:

$$
M_{\text{Gold}}^2 \equiv g^2 \sum_i \xi_{2i}\, \tilde{T}^{A_i} v v \tilde{T}^{A_i} \tag{4.17}
$$

with the property $\text{Tr}(M_{\text{Gold}}^2) = \sum_i \xi_{2i}\, M_{\text{X}_i}^2\, D_i^{\mathcal{A}}$, where $D_i^{\mathcal{A}}$ is the dimension of the i-th $\mathbf{G}_{\text{SM}}$-irreducible representation of the heavy gauge bosons. From the Goldstone theorem it follows that the matrix $v\tilde{T}^{A_i}$ projects on the subspace of Goldstone bosons, i.e. on the subspace that obtains no mass term from $V(\Phi)$. Hence, the matrix M_{Gold}^2 has non-zero entries only on the subspace of Goldstone bosons.

4.1.2. The Scalar Potential

For $V(\Phi)$ we consider the most general renormalizable scalar potential:

$$V(\Phi) = -\frac{1}{2}\mu^2_{ij}\Phi_i\Phi_j + \frac{1}{3!}\ \kappa_{ijk}\ \Phi_i\Phi_j\Phi_k + \frac{1}{4!}\ \lambda_{ijkl}\ \Phi_i\Phi_j\Phi_k\Phi_l\,, \tag{4.18}$$

with totally symmetric tensors μ^2_{ij}, κ_{ijk} and λ_{ijkl}. We impose the requirements

$$\begin{aligned} 0 &= [\mu^2, \tilde{T}^\alpha]\,, \\ 0 &= \tilde{T}^\alpha_{im}\ \kappa_{mjk} + \tilde{T}^\alpha_{jm}\ \kappa_{imk} + \tilde{T}^\alpha_{km}\ \kappa_{ijm}\,, \\ 0 &= \tilde{T}^\alpha_{im}\ \lambda_{mjkl} + \tilde{T}^\alpha_{jm}\ \lambda_{imkl} + \tilde{T}^\alpha_{km}\ \lambda_{ijml} + \tilde{T}^\alpha_{lm}\ \lambda_{ijkm}\,, \end{aligned} \tag{4.19}$$

in order to make $V(\Phi)$ gauge invariant under $\mathbf{G}$. Due to Schur's Lemma (cf. sec. 2.7) the first line in eq. (4.19) implies that the matrix μ^2 is proportional to the unit matrix on each subspace irreducible under $\mathbf{G}$.

To break the GUT symmetry, there has to be one $\mathbf{G}$-irreducible Higgs representation contained in Φ_i that develops a vev. In order to treat the symmetry breaking appropriately, we define the diagonal projector $\Pi^{\mathcal{H}}$ on this particular $\mathbf{G}$-irreducible subspace (clearly, $[\tilde{T}^\alpha, \Pi^{\mathcal{H}}] = 0$). This subspace is further divided into the subspace of Goldstone bosons and the subspace of physical Higgs bosons with projectors $P^{\mathcal{G}}$ and $P^{\tilde{\mathcal{H}}}$, respectively ($[\tilde{T}^{a_i}, P^{\mathcal{G}}] = 0 = [\tilde{T}^{a_i}, P^{\tilde{\mathcal{H}}}]$ and $\Pi^{\mathcal{H}} = P^{\tilde{\mathcal{H}}} + P^{\mathcal{G}}$). The physical Higgs bosons receive masses of order M_{GUT} from $V(\Phi)$, the Goldstone bosons do not. Using the Goldstone boson mass matrix from eq. (4.17), the projector on the space of Goldstone bosons can be explicitly written down as [41]:

$$(P^{\mathcal{G}})_{ij} \equiv g^2 \tilde{T}^A_{ik} v_k \left(\frac{1}{M_X^2}\right)_{AB} v_l \tilde{T}^B_{lj} = (\Pi^{\mathcal{H}} - P^{\tilde{\mathcal{H}}})_{ij}\,. \tag{4.20}$$

This form of the projector also follows from the Goldstone theorem, which states that $M_H^2\,\tilde{T}^A v = 0$, where M_H^2 is the scalar mass matrix arising from $V(\Phi)$, and therefore the matrix $\tilde{T}^A_{ij} v_j$ defines the subspace of Goldstone bosons. Constructing a proper projector out of this matrix yields eq. (4.20).

Now we can parametrize the scalars in the following way:

$$\Phi_i = v_i + \Phi'_i = v_i + H_i + G_i + S_i \tag{4.21}$$

where v, H and G live only in the subspace defined by $\Pi^{\mathcal{H}}$. S parametrizes all the other scalars[4]:

$$\begin{aligned} (\mathbb{1} - \Pi^{\mathcal{H}})_{ij}\Phi'_j &= S_i\,, \\ (\Pi^{\mathcal{H}})_{ij}\Phi'_j &= H_i + G_i\,, \\ (P^{\tilde{\mathcal{H}}})_{ij}\Phi'_j &= H_i\,, \\ (P^{\mathcal{G}}_{ij})\Phi'_j &= G_i\,. \end{aligned} \tag{4.22}$$

[4] To see that the number of Goldstone bosons is equal to the number of broken generators, it is also possible to define the Goldstone field as $G^A \equiv g\left(\frac{i}{M_X}\right)_{AB} v_i \tilde{T}^B_{ij}\Phi'_j$

First let us focus on the subspace that $\Pi^{\mathcal{H}}$ projects on. In order to develop a vev on this subspace, the parameter $\mu^2_{\mathcal{H}}$, defined by $\Pi^{\mathcal{H}}\mu^2 \equiv \mu^2_{\mathcal{H}}\mathbb{1}$, has to be positive[5]. If this is the case, it is convenient to parametrize this part of the scalar potential in terms of physical parameters as the Higgs mass M_H^2, the heavy gauge boson mass M_{X}, the gauge coupling g and the tadpole t instead of the unphysical couplings $\mu^2_{\mathcal{H}}$ and

$$\begin{aligned}
\lambda^{\mathcal{H}}_{ijkl} &\equiv \lambda_{i'j'k'l'}\Pi^{\mathcal{H}}_{i'i}\Pi^{\mathcal{H}}_{j'j}\Pi^{\mathcal{H}}_{k'k}\Pi^{\mathcal{H}}_{l'l}\,, \\
\kappa^{\mathcal{H}}_{ijk} &\equiv \kappa_{i'j'k'}\Pi^{\mathcal{H}}_{i'i}\Pi^{\mathcal{H}}_{j'j}\Pi^{\mathcal{H}}_{k'k}\,.
\end{aligned} \tag{4.23}$$

In principle, this is analogous to what is usually done for the SM Higgs potential (cf. chapter 3 of ref. [137]). Here, however, it is more involved due to the appearance of the general invariant tensors $\lambda^{\mathcal{H}}_{ijkl}$ and $\kappa^{\mathcal{H}}_{ijkl}$. Using essentially eq. (4.19) and Schur's Lemma, it is possible to rewrite the up to quadratic terms of the potential in terms of new parameters M_H^2 (diagonal Higgs mass matrix) and t (tadpole)[6]. For the trilinear and quartic terms this does not seem to be possible at the level of the Lagrangian. Thus, for the moment, we leave those terms expressed by the old parameters $\lambda^{\mathcal{H}}_{ijkl}$ and $\kappa^{\mathcal{H}}_{ijk}$. They have to be eliminated in favor of M_H^2, g and M_{X}^2 at diagram level to make our choice of parameters consistent.

Now, including also the scalars on the subspace defined by $\mathbb{1} - \Pi^{\mathcal{H}}$, which is straightforward, the scalar potential can be parametrized as follows:

$$\begin{aligned}
V(\Phi) \;=\;& t\, v_i H_i + \frac{1}{2}(M_H^2)_{ij}H_iH_j + \frac{1}{2}t\, H_iH_i + \frac{1}{2}t\, G_iG_i + \frac{1}{2}(M_S^2)_{ij}S_iS_j \\
&+ \frac{1}{2}(v_i\lambda^{\mathcal{H}}_{ijkl} + \kappa^{\mathcal{H}}_{jkl})H_jG_kG_l + \frac{1}{2}(v_i\lambda^{\mathcal{H}}_{ijkl} + \kappa^{\mathcal{H}}_{jkl})H_jH_kG_l + \frac{1}{6}(v_i\lambda^{\mathcal{H}}_{ijkl} + \kappa^{\mathcal{H}}_{jkl})H_jH_kH_l \\
&+ \frac{1}{2}(v_i\lambda_{ijkl} + \kappa_{jkl})H_jH_kS_l + (v_i\lambda_{ijkl} + \kappa_{jkl})H_jG_kS_l + \frac{1}{2}(v_i\lambda_{ijkl} + \kappa_{jkl})G_jG_kS_l \\
&+ \frac{1}{2}(v_i\lambda_{ijkl} + \kappa_{jkl})H_jS_kS_l + \frac{1}{2}(v_i\lambda_{ijkl} + \kappa_{jkl})G_jS_kS_l + \frac{1}{6}(v_i\lambda_{ijkl} + \kappa_{jkl})S_jS_kS_l \\
&+ \frac{1}{24}\lambda^{\mathcal{H}}_{ijkl}G_iG_jG_kG_l + \frac{1}{6}\lambda^{\mathcal{H}}_{ijkl}G_iG_jG_kH_l + \frac{1}{6}\lambda^{\mathcal{H}}_{ijkl}G_iH_jH_kH_l \\
&+ \frac{1}{4}\lambda^{\mathcal{H}}_{ijkl}G_iG_jH_kH_l + \frac{1}{24}\lambda^{\mathcal{H}}_{ijkl}H_iH_jH_kH_l \\
&+ \frac{1}{6}\lambda_{ijkl}H_iH_jH_kS_l + \frac{1}{2}\lambda_{ijkl}H_iH_jG_kS_l + \frac{1}{2}\lambda_{ijkl}H_iG_jG_kS_l \\
&+ \frac{1}{6}\lambda_{ijkl}G_iG_jG_kS_l + \frac{1}{4}\lambda_{ijkl}H_iH_jS_kS_l + \frac{1}{2}\lambda_{ijkl}H_iG_jS_kS_l \\
&+ \frac{1}{4}\lambda_{ijkl}G_iG_jS_kS_l + \frac{1}{6}\lambda_{ijkl}H_iS_jS_kS_l + \frac{1}{6}\lambda_{ijkl}G_iS_jS_kS_l \\
&+ \frac{1}{24}\lambda_{ijkl}S_iS_jS_kS_l
\end{aligned} \tag{4.24}$$

[5] This is true for $\kappa_{ijk} = 0$. For $\kappa_{ijk} \neq 0$ the condition is more complicated. However, we keep $\kappa_{ijk} \neq 0$ in the following and just assume that the condition is fulfilled.

[6] For more details of how this reparametrization is done, please refer to appendix A.2.

where

$$
\begin{aligned}
t &= -\mu^2_{\mathcal{H}} + \frac{1}{2v^2}\,\kappa^{\mathcal{H}}_{ijk}v_iv_jv_k + \frac{1}{6v^2}\,\lambda^{\mathcal{H}}_{ijkl}v_iv_jv_kv_l\,, && (4.25)\\
(M^2_H)_{ij} &= \kappa^{\mathcal{H}}_{ijk}v_k + \frac{1}{2}\lambda^{\mathcal{H}}_{ijkl}v_kv_l - \Pi^{\mathcal{H}}_{ij}\Big[\frac{1}{2v^2}\kappa_{ijk}v_iv_jv_k + \frac{1}{6v^2}\,\lambda^{\mathcal{H}}_{klmn}v_kv_lv_mv_n\Big]\,, && (4.26)\\
(M^2_{\mathcal{S}})_{ij} &= \kappa^{\mathcal{S}}_{ijk}v_k + \frac{1}{2}\lambda^{\mathcal{S}}_{ijkl}v_kv_l - \big[\mu^2(\mathbb{1}-\Pi^{\mathcal{H}})\big]_{ij}\,. && (4.27)
\end{aligned}
$$

Due to gauge invariance under the $\mathbf{G}_{\mathrm{SM}}$ (eqs. (4.4) and (4.19)), both mass matrices are diagonal and proportional to the unit matrix on each $\mathbf{G}_{\mathrm{SM}}$-irreducible subspace. M^2_H has only non-zero entries on the subspace defined by $P^{\tilde{\mathcal{H}}}$ and M^2_S only on the subspace defined by $(\mathbb{1}-\Pi^{\mathcal{H}})$ since we have defined

$$
\begin{aligned}
\lambda^{\mathcal{S}}_{ijkl}v_kv_l &\equiv (\mathbb{1}-\Pi^{\mathcal{H}})_{ii'}(\mathbb{1}-\Pi^{\mathcal{H}})_{jj'}\lambda_{i'j'kl}v_kv_l\,,\\
\kappa^{\mathcal{S}}_{ijk}v_k &\equiv (\mathbb{1}-\Pi^{\mathcal{H}})_{ii'}(\mathbb{1}-\Pi^{\mathcal{H}})_{jj'}\kappa_{i'j'k}v_k\,. && (4.28)
\end{aligned}
$$

Note that the $G_iG_jG_k$ interaction vanishes. This can be shown by using eq. (4.19), Schur's Lemma and the antisymmetry of the generators $\tilde{T}^\alpha$.

It is important to see that M^2_S must have only positive or zero entries. If there are negative entries, some of the S_i could develop a vev and our formalism would not apply. Strictly speaking, we have $(M^2_S)_{ij} < 0$ for the SM Higgs doublet that is contained in S_i, which would exclude it from our treatment. But since in that case the scales involved have the strong hierarchy $\mathcal{O}(M_W) \ll \mathcal{O}(M_{\mathrm{GUT}})$, we can safely set the entry to zero here. To do this, some of the $\lambda^{\mathcal{S}}_{ijkl}v_kv_l$ must be fine-tuned against the corresponding $(\mu^2(\mathbb{1}-\Pi^{\mathcal{H}}))_{ij}$ in eq. (4.27) which is known as the doublet triplet splitting problem, inherent to generic GUTs. Note that the classical minimum of the GUT-breaking Higgs potential is defined by the equation $t = 0$. However, if we compute higher-order corrections, the parameter $t \equiv 0 - \delta t$, where $\delta t = \mathcal{O}(\alpha)$ is a counterterm, has to be adjusted in such a way that the renormalized Higgs one point function is zero at all orders of perturbation theory.

4.1.3. Yukawa Interactions and Fermions

The last term in eq. (4.1) to be specified is $\mathcal{L}_{\mathrm{Y}}$. The most general Yukawa interaction of the Weyl fermion multiplet ψ with the real scalar multiplet Φ can be written as follows:

$$
\mathcal{L}_{\mathrm{Y}} = -\frac{1}{2}\left(Y^k_{ij}\psi_i\psi_j\Phi_k + Y^{k\star}_{ij}\bar{\psi}_i\bar{\psi}_j\Phi_k\right)\,. \tag{4.29}
$$

Here Y^k is a complex, symmetric matrix which, due to gauge invariance, satisfies the following relation:

$$
0 = Y^k_{mj}T^\alpha_{mi} + Y^k_{im}T^\alpha_{mj} + Y^m_{ij}\tilde{T}^\alpha_{mk}\,. \tag{4.30}
$$

In order to do calculations and derive Feynman rules with fermions, it is most convenient to use four-component spinors instead of Weyl fields. Generally, the matrix μ_F from eq. (4.1) and the matrix Y^kv_k from the Yukawa Lagrangian eq. (4.29) can give rise to both Dirac and Majorana mass terms. Some of the fields will stay massless. In order to combine the respective Weyl fermions in an appropriate way to four-component spinors, we split the reducible representation of ψ into several parts using projectors and accounting also for possible mass mixings and phases:

$$\psi = (\varrho^{\mathcal{D}_L} + \varrho^{\mathcal{D}_R} + \varrho^{\mathcal{M}} + p^{f_L})\psi = \mathcal{Z}_L\,\xi_+ + \mathcal{Z}_R\,\xi_- + \mathcal{Z}_M\,\lambda + \chi\,. \tag{4.31}$$

The Weyl fields ξ_+, ξ_-, λ and χ are defined only on the subspaces defined by the projectors $\varrho^{\mathcal{D}_L}$, $\varrho^{\mathcal{D}_R}$, $\varrho^{\mathcal{M}}$ and p^{f_L}, respectively. The unitary mixing matrices $\mathcal{Z}_L$, $\mathcal{Z}_R$ and $\mathcal{Z}_M$ are chosen such that the four component spinors

$$\Psi = \begin{pmatrix} \xi_+ \\ \bar{\xi}_-^T \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} \lambda \\ \bar{\lambda}^T \end{pmatrix} \tag{4.32}$$

are Dirac and Majorana mass eigenstates. Furthermore, there is also a massless (left-)chiral four-component spinor:

$$f = \begin{pmatrix} \chi \\ 0 \end{pmatrix}. \tag{4.33}$$

The real and diagonal Dirac and Majorana mass matrices are then given by

$$\begin{aligned} M_D &= \mathcal{Z}_R^T\,\varrho^{\mathcal{D}_R}\,(Y^k v_k + \mu_F)\,\varrho^{\mathcal{D}_L}\,\mathcal{Z}_L \\ &= \mathcal{Z}_L^\dagger\,\varrho^{\mathcal{D}_L}\,(Y^{k\star} v_k + \mu_F^\star)\,\varrho^{\mathcal{D}_R}\,\mathcal{Z}_R^\star \equiv \sum_n M_{D_n}\,\varrho_n^{\mathcal{D}}\,, \\ M_M &= \mathcal{Z}_M^T\,\varrho^{\mathcal{M}}\,(Y^k v_k + \mu_F)\,\varrho^{\mathcal{M}}\,\mathcal{Z}_M \\ &= \mathcal{Z}_M^\dagger\,\varrho^{\mathcal{M}}\,(Y^{k\star} v_k + \mu_F^\star)\,\varrho^{\mathcal{M}}\,\mathcal{Z}_M^\star \equiv \sum_n M_{M_n}\,\varrho_n^{\mathcal{M}}\,. \end{aligned} \tag{4.34}$$

where the eigenvalues are denoted by M_{D_n} and M_{M_n}, some of which may also be zero. Though explicitly stated here, we usually imply that the mixing matrices already contain the projector on the respective subspace, i.e.:

$$\varrho^{\mathcal{D}_R}\mathcal{Z}_R = \mathcal{Z}_R\,, \quad \varrho^{\mathcal{D}_L}\mathcal{Z}_L = \mathcal{Z}_L\,, \quad \varrho^{\mathcal{M}}\mathcal{Z}_M = \mathcal{Z}_M\,. \tag{4.35}$$

A few remarks are in order here:

- In principle, each of the four subspaces defined in eq. (4.31) can share a **G**-irreducible representation with any of the other subspaces, i.e. Majorana fermions can share **G**-irreducible subspace with Dirac fermions, Dirac fermions with chiral fermions etc.. The only restriction is that gauge invariance is maintained.

- Because ξ_+ and ξ_- form a Dirac spinor, we assume that they do not share a **G**-irreducible representation. Otherwise gauge interactions could cause a chirality flip which is not present in theories of interest. More specifically, if ξ_+ transforms as the $\mathbf{G}_{\mathrm{SM}}$-irreducible representation $\mathcal{R}_n^{\mathcal{D}_L}$, then, in order to maintain gauge invariance, ξ_+ must transform as the complex conjugate representation of $\mathcal{R}_n^{\mathcal{D}_L}$ which we denote by $\mathcal{R}_n^{\mathcal{D}_R}$. Therefore, if we introduce the generators t^α that transform the four-component Dirac fields, then

$$\varrho^{\mathcal{D}_L}\,t^\alpha = \varrho^{\mathcal{D}_L}\,T^\alpha\,, \qquad \varrho^{\mathcal{D}_R}\,t^\alpha = \varrho^{\mathcal{D}_R}\,(-T^{\alpha\star})\,. \tag{4.36}$$

- λ transforms as a real representation of $\mathbf{G}$ and $\mathbf{G}_{\mathrm{SM}}$.
- Care is needed, when using the invariance relation eq. (4.30) in connection with Dirac fermions. The generator T^α that appears there is the one that transforms the Weyl field. The Lagrangian, however, is now rewritten in terms of Dirac fields, that transform with the generators t^α. If we apply the relation to a coupling of the form $\xi_{-i} Y^k_{ij} \xi_{+j} = \overline{\Psi}_i Y^k_{ij} P_L \Psi_j$, then we have to replace T^α by t^α for the first index and by $-t^{\alpha\star}$ for the second index.

Taking into account these remarks, the gauge-kinetic Lagrangian in terms of four-component spinors takes the form:

$$\begin{aligned}
i\bar{\psi}\,\bar{\sigma}^\mu D_\mu \psi \;=\;& i\overline{\Psi}\,\gamma^\mu \partial_\mu \Psi + \tfrac{i}{2}\overline{\Lambda}\,\gamma^\mu \partial_\mu \Lambda + i\overline{f}\,\gamma^\mu \partial_\mu P_L f \\
& -g A^\alpha_\mu \Big\{ \overline{\Psi}\gamma^\mu \Big[(\mathcal{Z}_L^\dagger\, t^\alpha \mathcal{Z}_L) P_L + (\mathcal{Z}_R^T\, t^\alpha \mathcal{Z}_R^\star) P_R \Big] \Psi \\
& \qquad \overline{\Psi}\gamma^\mu \Big[(\mathcal{Z}_L^\dagger\, t^\alpha \mathcal{Z}_M) P_L + (\mathcal{Z}_R^T\, t^\alpha \mathcal{Z}_M^\star) P_R \Big] \Lambda \\
& \qquad \overline{\Lambda}\gamma^\mu \Big[(\mathcal{Z}_M^\dagger\, t^\alpha \mathcal{Z}_L) P_L + (\mathcal{Z}_M^T\, t^\alpha \mathcal{Z}_R^\star) P_R \Big] \Psi \\
& \qquad \overline{\Lambda}\gamma^\mu (\mathcal{Z}_M^\dagger\, t^\alpha \mathcal{Z}_M) P_L\, \Lambda \;+\; \overline{f}\gamma^\mu t^\alpha P_L\, f \\
& \qquad \overline{\Lambda}\gamma^\mu (\mathcal{Z}_M^\dagger\, t^\alpha) P_L\, f \;+\; \overline{f}\gamma^\mu (t^\alpha \mathcal{Z}_M) P_L\, \Lambda\,, \Big\}
\end{aligned} \tag{4.37}$$

where $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$ are the chirality projectors in Dirac space. Please note that in each term a different block of the reducible matrix t^α is projected out due to the appearance of different combinations of the mixing matrices. As stressed before, gauge interactions do not change the chirality of the fermion and therefore the number of terms in eq. (4.37) is limited. There is no such restriction for Yukawa terms. They can contain both chirality-preserving and chirality-flipping couplings. The latter can contribute to Dirac mass terms if the scalar field Φ develops a vev as can be seen in eq. (4.34). Therefore there is a larger number of possible couplings:

$$\begin{aligned}
\mathcal{L}_{\mathrm{Y}} \;=\;& -\tfrac{1}{2}\,\overline{\Psi^c}(\mathcal{Z}_L^T\, Y^k \mathcal{Z}_L) P_L \Psi\ \Phi_k \;-\; \tfrac{1}{2}\,\overline{\Psi}(\mathcal{Z}_L^\dagger\, Y^{k\star} \mathcal{Z}_L^\star) P_R \Psi^c\ \Phi_k \\
& -\tfrac{1}{2}\,\overline{\Psi}(\mathcal{Z}_R^T\, Y^k \mathcal{Z}_R) P_L \Psi^c\ \Phi_k \;-\; \tfrac{1}{2}\,\overline{\Psi^c}(\mathcal{Z}_R^\dagger\, Y^{k\star} \mathcal{Z}_R^\star) P_R \Psi\ \Phi_k \\
& -\overline{\Psi}(\mathcal{Z}_R^T\, Y^k \mathcal{Z}_L) P_L \Psi\ \Phi_k \;-\; \overline{\Psi}(\mathcal{Z}_L^\dagger\, Y^{k\star} \mathcal{Z}_R^\star) P_R \Psi\ \Phi_k \\
& -\tfrac{1}{2}\,\overline{\Lambda}(\mathcal{Z}_M^T\, Y^k \mathcal{Z}_M) P_L \Lambda\ \Phi_k \;-\; \tfrac{1}{2}\,\overline{\Lambda}(\mathcal{Z}_M^\dagger\, Y^{k\star} \mathcal{Z}_M^\star) P_R \Lambda\ \Phi_k \\
& -\tfrac{1}{2}\,\overline{f^c}\, Y^k P_L f\ \Phi_k \;-\; \tfrac{1}{2}\,\overline{f}\, Y^{k\star} P_R f^c\ \Phi_k \\[1em]
& -\overline{\Lambda}(\mathcal{Z}_M^T\, Y^k \mathcal{Z}_L) P_L \Psi\ \Phi_k \;-\; \overline{\Psi}(\mathcal{Z}_L^\dagger\, Y^{k\star} \mathcal{Z}_M^\star) P_R \Lambda\ \Phi_k \\
& -\overline{\Psi}(\mathcal{Z}_R^T\, Y^k \mathcal{Z}_M) P_L \Lambda\ \Phi_k \;-\; \overline{\Lambda}(\mathcal{Z}_M^\dagger\, Y^{k\star} \mathcal{Z}_R^\star) P_R \Psi\ \Phi_k \\[1em]
& -\overline{\Psi^c}(\mathcal{Z}_L^T\, Y^k) P_L f\ \Phi_k \;-\; \overline{\Psi}(\mathcal{Z}_L^\dagger\, Y^{k\star}) P_R f^c\ \Phi_k \\
& -\overline{\Psi}(\mathcal{Z}_R^T\, Y^k) P_L f\ \Phi_k \;-\; \overline{f}(Y^{k\star} \mathcal{Z}_R^\star) P_R \Psi\ \Phi_k \\
& -\overline{\Lambda}(\mathcal{Z}_M^T\, Y^k) P_L f\ \Phi_k \;-\; \overline{f}(Y^{k\star} \mathcal{Z}_M^\star) P_R \Lambda\ \Phi_k\,,
\end{aligned} \tag{4.38}$$

with the charge-conjugated Dirac spinor $\Psi^c \equiv i\gamma_2\gamma_0 \overline{\Psi}^T$. Again note that different combinations of mixing matrices project out different blocks of the matrix Y^k. The mass terms for

Dirac and Majorana fermions are given by

$$\mathcal{L}_{\text{fermionmass}} = -\overline{\Psi}\, M_D \Psi - \tfrac{1}{2}\overline{\Lambda}\, M_M \Lambda\,, \tag{4.39}$$

with M_D and M_M defined in eq. (4.34).

In the following we will need to distinguish between fields with the mass of $\mathcal{O}(M_{\text{GUT}})$ and massless fields. We will follow the convention of chapter 5 and use the projectors P_i^x for heavy fields and p_i^x for the light fields.

4.2. Renormalization

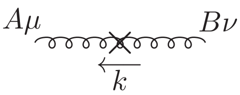
In order to do a two-loop calculation of the matching corrections, a one-loop renormalization program has to be carried out for the theory which, to the authors knowledge, has not been done for GUT models before. The counterterms are adjusted in such a way that all the one-loop Green's functions of the theory are finite. For convenience we use the on-shell scheme for the mass parameters of the theory and $\overline{\text{MS}}$ for the gauge couplings, the gauge parameters and the fields. The renormalized Lagrangian is obtained from eq. (4.1) by the following replacements:

$$\begin{aligned}
A_\mu^{a_i} &\to \sqrt{Z_{3i}}\; A_\mu^{a_i}\,, & A_\mu^{A_i} &\to \sqrt{Z_{3i}^X}\; A_\mu^{A_i}\,,\\
p_i^{\mathcal{D}}\Psi &\to \sqrt{Z_{\Psi i}}\; p_i^{\mathcal{D}}\Psi\,, & P_i^{\mathcal{D}}\Psi &\to \sqrt{Z_{\Psi i}^h}\; P_i^{\mathcal{D}}\Psi\,,\\
p_i^{\mathcal{M}}\Lambda &\to \sqrt{Z_{\Lambda i}}\; p_i^{\mathcal{M}}\Lambda\,, & P_i^{\mathcal{M}}\Lambda &\to \sqrt{Z_{\Lambda i}^h}\; P_i^{\mathcal{M}}\Lambda\,,\\
p_i^{f_L} f &\to \sqrt{Z_{fi}}\; p_i^{f_L} f\,, & &\\
c^{a_i} &\to \sqrt{\tilde{Z}_{3i}}\; c^{a_i}\,, & c^{A_i} &\to \sqrt{\tilde{Z}_{3i}^X}\; c^{A_i}\,,\\
P_i^{\tilde{\mathcal{H}}} H &\to \sqrt{Z_{Hi}}\; P_i^{\tilde{\mathcal{H}}} H\,, & P_i^{\mathcal{G}} G &\to \sqrt{Z_{Gi}}\; P_i^{\mathcal{G}} G\,,\\
p_i^{\mathcal{S}} S &\to \sqrt{Z_{Si}}\; p_i^{\mathcal{S}} S\,, & P_i^{\mathcal{S}} S &\to \sqrt{Z_{Si}^h}\; P_i^{\mathcal{S}} S\,,\\
M_{\text{X}_i}^2 &\to Z_{M_{\text{X}i}}^2\; M_{\text{X}_i}^2\,, & M_{H_i}^2 &\to Z_{M_{Hi}}^2\; M_{H_i}^2\,,\\
M_{D_i} &\to Z_{M_{Di}}\; M_{D_i}\,, & M_{M_i} &\to Z_{M_{Mi}}\; M_{M_i}\\
\xi_{1i} &\to Z_{\xi_{1i}}\; \xi\,, & \xi_{2i} &\to Z_{\xi_{2i}}\; \xi\,,\\
\eta_i &\to Z_{3i}\; \eta\,, & g &\to \mu^\epsilon Z_g\; g\,.
\end{aligned} \tag{4.40}$$

Again, we have used the sub-index i to take care of the fact that there might be several $\mathbf{G}_{\text{SM}}$-irreducible representations for a field that all renormalize differently. No summation is performed over that index. P_i^x and p_i^x are projectors on the various $\mathbf{G}_{\text{SM}}$-irreducible subspaces of heavy and light fields, respectively.

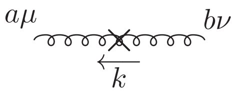
In the following we list the counterterm Feynman rules that are important for our calculation. They are obtained by inserting the renormalization prescriptions from eq. (4.40) into eq. (4.1) and considering the up to quadratic terms. For each counterterm we give an expression that is valid to arbitrary loop order in the first line and in the second line a more convenient expression that is valid only for one-loop renormalization. We use the notation $Z_i \equiv 1 - \delta Z_i$ and $t \equiv 0 - \delta t$ where δZ_i and δt are of order α. All the parameters that appear in the equations are renormalized ones.

Heavy gauge boson:



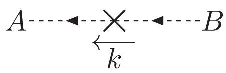
$$\begin{aligned} &= i\delta_{AB}\Big[(Z_3^X - \frac{Z_3^X}{\xi Z_{\xi_1}} + \frac{1}{\xi} - 1)k_\mu k_\nu - (Z_3^X - 1)k^2 g_{\mu\nu} + (Z_3^X Z_{M_{\mathrm{X}}}^2 - 1)M_{\mathrm{X}}^2 g_{\mu\nu}\Big] \\ &= i\delta_{AB}\Big[-\delta Z_{\xi_1} k_\mu k_\nu - (M_{\mathrm{X}}^2 - k^2)\delta Z_3^X g_{\mu\nu} - 2\delta Z_{M_{\mathrm{X}}} M_{\mathrm{X}}^2 g_{\mu\nu}\Big] \end{aligned}$$

Light gauge boson:



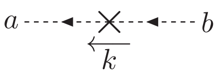
$$\begin{aligned} &= i\delta_{ab}(Z_3 - 1)\Big[k_\mu k_\nu - k^2 g_{\mu\nu}\Big] \\ &= -i\delta_{ab}\,\delta Z_3\Big[k_\mu k_\nu - k^2 g_{\mu\nu}\Big] \end{aligned}$$

Heavy ghost:



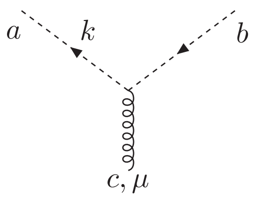
$$\begin{aligned} &= i\delta_{AB}\Big[(\tilde{Z}_3^X - 1)k^2 - (\tilde{Z}_3^X \sqrt{Z_{\xi_1}}\sqrt{Z_{\xi_2}} Z_{M_{\mathrm{X}}}^2 - 1)\xi M_{\mathrm{X}}^2\Big] \\ &= i\delta_{AB}\Big[\delta\tilde{Z}_3^X(\xi M_{\mathrm{X}}^2 - k^2) + (\frac{1}{2}\delta Z_{\xi_1} + \frac{1}{2}\delta Z_{\xi_2} + 2\delta Z_{M_{\mathrm{X}}})\xi M_{\mathrm{X}}^2\Big] \end{aligned}$$

Light ghost:



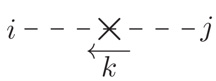
$$\begin{aligned} &= i\delta_{ab}(\tilde{Z}_3 - 1)k^2 \\ &= -i\delta_{ab}\,\delta\tilde{Z}_3\,k^2 \end{aligned}$$

Light ghost – gauge boson vertex:



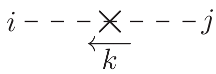
$$\begin{aligned} &= g\,(\tilde{Z}_3 - 1)\,f_{abc}\,k_\mu \\ &= -g\,\delta\tilde{Z}_3\,f_{abc}\,k_\mu \end{aligned}$$

Goldstone boson:



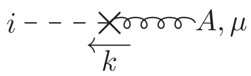
$$\begin{aligned} &= i(P^{\mathcal{G}})_{ij}\Big[(Z_G - 1)k^2 - (Z_G Z_{\xi_2} Z_{M_{\mathrm{X}}}^2 - 1)\xi M_{\mathrm{X}}^2 - t\Big] \\ &= i(P^{\mathcal{G}})_{ij}\Big[\delta Z_G(\xi M_{\mathrm{X}}^2 - k^2) + (\delta Z_{\xi_2} + 2\delta Z_{M_{\mathrm{X}}})\xi M_{\mathrm{X}}^2 + \delta t\Big] \end{aligned}$$

Physical Higgs boson:



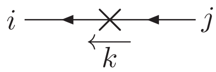
$$
\begin{aligned}
&= i\delta_{ij}\Big[(Z_H-1)k^2-(Z_H Z_{M_H}^2-1)M_H^2 - t\Big] \\
&= i\delta_{ij}\Big[\delta Z_H(M_H^2-k^2)+2\delta Z_{M_H}M_H^2 + \delta t\Big]
\end{aligned}
$$

Mixed counterterm:



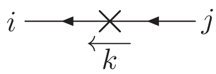
$$
\begin{aligned}
&= ig\Big[\sqrt{\frac{\xi_2}{\xi_1}} - 1\Big]v_k\tilde{T}^A_{ki}k_\mu \\
&= ig\ \frac{1}{2}(\delta Z_{\xi_1} - \delta Z_{\xi_2})\ v_k\tilde{T}^A_{ki}k_\mu
\end{aligned}
$$

Heavy Dirac fermion:



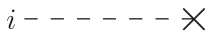
$$
\begin{aligned}
&= i\delta_{ij}\Big[(Z_\Psi^h-1)\not{k}-(Z_\Psi^h Z_{M_D}-1)M_D\Big]P_L \\
&= i\delta_{ij}\Big[\delta Z_\Psi^h(M_D-\not{k})+\delta Z_{M_D}M_D\Big]P_L
\end{aligned}
$$

Heavy Majorana fermion:



$$
\begin{aligned}
&= i\delta_{ij}\Big[(Z_\Lambda^h-1)\not{k}-(Z_\Lambda^h Z_{M_M}-1)M_M\Big]P_L \\
&= i\delta_{ij}\Big[\delta Z_\Lambda^h(M_M-\not{k})+\delta Z_{M_M}M_M\Big]P_L
\end{aligned}
$$

Higgs tadpole:

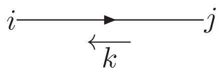


$$
= iv_i\delta t \tag{4.41}
$$

In order to avoid clutter with the notation, we have omitted the sub-index i here. From the context it is always unambiguous that the $\mathbf{G}_{\mathrm{SM}}$-irreducible representation of the field under consideration is meant. For our calculation we need the renormalization constants of all mass and gauge parameters at one-loop. As can be seen from the above Feynman rules, the set of equations that is used to determine δZ_{ξ_1}, δZ_{ξ_2} and δt is overconstrained (cf. also refs. [34, 138]). This provides a useful check for our calculation: we computed $\delta Z_{\xi_1} - \delta Z_{\xi_2}$ from the pole of a combination of the heavy gauge boson and heavy ghost propagator as well as from the mixed Goldstone boson – heavy gauge boson propagator. Both calculations lead to the same result. In the same way δt was computed from the pole of the physical Higgs tadpole as well as from the Goldstone propagator yielding the same result.

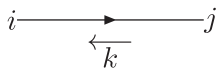
In the course of this section let us also specify the Feynman rules for the propagators that can be derived from the Lagrangian:

Heavy (Dirac or Majorana) fermion:



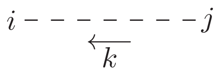
$$= \frac{i(P^{\mathcal{D},\mathcal{M}})_{ij}}{\not{k} - M_{D,M}}$$

Light fermion:



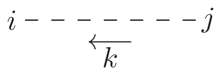
$$= \frac{i(p^{\mathcal{D},\mathcal{M},f_L})_{ij}}{\not{k}}$$

Physical Higgs:



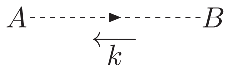
$$= \frac{i(P^{\tilde{\mathcal{H}}})_{ij}}{k^2 - M_H^2}$$

Goldstone boson:



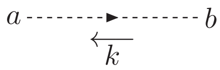
$$= \frac{i(P^{\mathcal{G}})_{ij}}{k^2 - \xi_2 M_{\mathrm{X}}^2}$$

Heavy ghost:



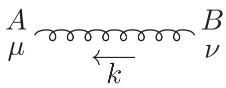
$$= \frac{i\delta_{AB}}{k^2 - \sqrt{\xi_1\xi_2} M_{\mathrm{X}}^2}$$

Light ghost:



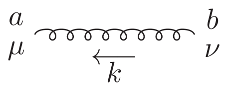
$$= \frac{i\delta_{ab}}{k^2}$$

Heavy gauge boson:



$$\begin{aligned} &= \frac{i\delta_{AB}}{k^2 - M_{\mathrm{X}}^2}\left[-g_{\mu\nu} + (1-\xi_1)\frac{k_\mu k_\nu}{k^2 - \xi_1 M_{\mathrm{X}}^2}\right] \\ &= i\delta_{AB}\left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{M_{\mathrm{X}}^2}\right)\frac{1}{k^2 - M_{\mathrm{X}}^2} - \frac{k_\mu k_\nu}{M_{\mathrm{X}}^2}\frac{1}{k^2 - \xi_1 M_{\mathrm{X}}^2}\right] \end{aligned}$$

Light gauge boson:



$$= \frac{i\delta_{ab}}{k^2}\left[-g_{\mu\nu} + (1-\eta)\frac{k_\mu k_\nu}{k^2}\right]$$

Again we have omitted the sub-index i for readability. Note furthermore that in the case of the heavy gauge boson the propagator has been split into a transverse and longitudinal part. Both parts are implemented as separate fields in our setup. We don't specify all the other Feynman rules since they can be easily read off the Lagrangian all parts of which have been given in the previous sections.

4.3. Calculation of the Two-Loop Matching Coefficients

In subsection 2.5.2 (with the final formula, eq. (2.34)) we have described how the matching coefficients $\zeta_{\alpha_i}(\mu_{\mathrm{GUT}})$ are calculated in principle. To do this in the theoretical framework that has been described in the present chapter, we need to calculate the following renormalized two Green's functions:

- The Green's function with two external light gauge bosons $A_\mu^{a_i}$.
- The Green's function with two external light ghosts c^{a_i}.

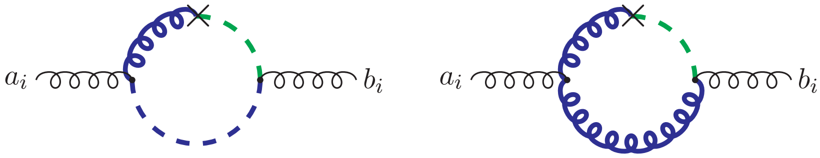


Figure 4.1.: *Sample diagrams where the mixed Goldstone boson gauge boson counterterm contributes to $\mathcal{O}(\alpha^2)$ terms. The insertion is proportional to $\frac{i}{2}g(\delta Z_{\xi_1} - \delta Z_{\xi_2})\ v_k \tilde{T}^A_{ki} k_\mu$ (cf. eq. (4.41)). These contributions are needed to subtract sub-divergences in corresponding two-loop diagrams. Colored (bold) lines represent fields with mass of $\mathcal{O}(M_{\mathrm{GUT}})$ and black (thin) lines massless fields. Goldstone bosons are marked green (light gray, short-dashed).*

- The vertex Green's function with two external light ghosts c^{a_i} and one light gauge bosons $A^{a_i}_\mu$.

To this end also the renormalization constants for the Higgs tadpole, GUT-breaking scalar masses, other scalar masses, fermion masses, gauge boson masses and gauge parameters ξ_{1i}, ξ_{2i} and η_i have to be computed at one loop order. Furthermore, also the two-loop gauge coupling renormalization, Z_g/Z_{g_i} is needed in the full and effective theory in $\overline{\mathrm{MS}}$. Though it is already known to the literature for a general renormalizable quantum field theory [25,27,28,115,139], we have repeated the calculation and found agreement. Actually, for our setup this calculation is more complicated than in the above references because we need a distinct field for each mass that appears in the theory, whereas for the calculation of above references it was sufficient to use generic scalars, generic fermions and so on. Therefore this provides a nontrivial check of our implementation.

The one-loop contributions to the above-mentioned Green's functions have to be carried out for (at least) two distinct (bare) gauge parameters ξ_{1_i} and ξ_{2_i}. Inserting the renormalization conditions, eq. (4.40), for the masses and gauge parameters, we obtain $\mathcal{O}(\alpha^2)$ terms that serve to subtract the sub-divergences of the two-loop contributions to the Green's functions. Afterwards we compute the limit $\xi_{1_i} \to \xi_{2_i} \equiv \xi$ for the renormalized gauge parameters using `Mathematica` in order to be compatible with the two-loop contributions that are computed with a single (renormalized) gauge parameter ξ. In order to further simplify the expressions, we have performed an expansion for small ξ up to $\mathcal{O}(\xi^3)$ and then checked in the final result that each order in ξ vanishes separately. Speaking about renormalization, let us also stress again that the mixed Goldstone boson – gauge boson counterterms contributes explicitly here via diagrams of the kind that are depicted in fig. 4.1. Moreover, the tadpole counterterm δt is needed for renormalizing gauge boson masses coming from Goldstone boson propagators (cf. also the Goldstone boson counterterm in eq. (4.41)). The $\mathcal{O}(\epsilon)$ parts of the on-shell counterterms are not needed in this calculation since the $1/\epsilon$ parts of the one-loop result do not depend on particle masses.

For the actual two-loop contributions to the three Green's functions we do not explicitly compute diagrams that contain a Higgs tadpole and compensate that by excluding the Higgs tadpoles from the on-shell mass renormalizations as well. Taking them into account explicitly would not provide a significantly powerful additional check and would only further increase the number of diagrams. The omission of tadpoles in the above described way is consistent since they do not depend on the kinematics of the process.

Given the large number of Feynman diagrams, an automated computation is indispensable. We have derived all the Feynman rules for the framework described in the present chapter and implemented them into model files, that are suitable for the programs used. Majorana fermions have been implemented using the methods developed in ref. [140]. The diagrams were generated with `QGRAF` [141] and further processed with `q2e` and `exp` [142, 143]. In the next step we used a `FORM` [144] implementation of the two-loop topologies of ref. [145] by the authors of ref. [30] and also the `FORM` packages `MINCER` [146] and `MATAD` [147]. Let us emphasize that no assumptions about the mass hierarchies of the heavy particles have been made. Therefore, the result is valid for arbitrary numerical values of the mass parameters as long as their mass splitting is not to large (empirically, the condition is $\frac{M_1}{M_2} \lesssim 100$ for any two masses M_1 and M_2) which would lead to power enhanced contributions and spoil perturbation theory. The group structure that appears in all vertices in this calculation has to be reduced to some primitive group theory invariants in order to allow for cancellations between the various contributions. This is a highly nontrivial task for these kind of calculations since the gauge symmetry is broken which complicates the situation compared to the unbroken case [25, 26, 139] significantly. Therefore, we have devoted a separate chapter to this topic and developed a framework that is suitable for our calculation. The reduction identities and algorithms that are described in chapter 5 have been implemented in a `FORM` routine and used in all steps of the calculation.

Let us now summarize all the checks that have been employed to ensure the correctness of our result for $\zeta_{\alpha_i}(\mu_{\rm GUT})$:

- $\zeta_{\alpha_i}(\mu_{\rm GUT})$ is UV finite after carrying out all the renormalization.
- The calculation has been carried out for arbitrary (renormalized) gauge parameters ξ and η . In the final result they all drop out.
- Some of the one-loop on-shell counterterms are overdetermined and yield a consistent result when calculating them from different Green's functions (cf. section 4.2).
- The $\mu_{\rm GUT}$ dependence of $\zeta_{\alpha_i}(\mu_{\rm GUT})$ is determined by the two-loop gauge beta functions of the full and effective theory. We can check analytically whether our result exhibits this correct dependence. (This will be explained in more detail in section 6.3.)

Though we have set up our calculation in the general framework that has been described in this chapter, we found it reasonable for the first step to impose some additional assumptions on the GUT theory under consideration. This is in order not to have to deal with too many difficulties simultaneously and to first obtain a result for a simple theory as a solid basis to check whether it exhibits the properties that we expect from it. Now it is in order to describe all these additional assumptions that have been made about the underlying GUT model:

- The trilinear coupling κ_{ijk} in $V(\Phi)$, eq. (4.18), has been set to zero.
- There is only one vev of $\mathcal{O}(M_{\rm GUT})$ in the theory.
- There are no heavy fermions in the theory.

- The heavy gauge bosons decompose in $\mathbf{G}_{\mathrm{SM}}$-irreducible representations with a common mass.
- The GUT-breaking Higgs decomposes into three $\mathbf{G}_{\mathrm{SM}}$-irreducible representations (+ Goldstone bosons) at most. They can all have different masses.
- The other scalars in the theory decompose in $\mathbf{G}_{\mathrm{SM}}$-irreducible representations that have a common mass (+ light scalars).
- The light particles in the theory can decompose in arbitrarily many $\mathbf{G}_{\mathrm{SM}}$-irreducible representations.

As can be seen, the main limitation comes from the number of heavy degrees of freedom in the theory. The above constraints are designed such that the resulting formula for ζ_{α_i} is applicable to the simplest GUT, the Georgi-Glashow model, yet keeping the calculation as simple as possible. The computational framework for our calculation is set up in such a way that it can be generalized to more heavy degrees of freedom, in order to apply it to SUSY GUTs in the future. In particular, the process of adding additional heavy degrees of freedom into the model files has been automated using `Perl` scripts. Therefore, it is certainly true to say that if we can establish a correct result for e.g. two-distinct masses of a certain field type (e.g. scalars), then the generalization to more distinct masses is only a matter of computer time. The most difficult obstacles among the above mentioned assumptions to overcome are probably the number of super-heavy vevs and the inclusion of heavy fermions. Though it is not clear whether it is necessary at all to overcome the first obstacle - since one could possibly treat theories with multiple super-heavy vevs as a sequence of effective theories with just one super-heavy vev - the second one certainly poses a challenge. In the course of this work we have already undertaken several steps towards including heavy fermions, as the calculation of the fermion mass on-shell counterterms, but at the two-loop level new group theory structures will appear which need to be dealt with. The inclusion of the trilinear coupling κ_{ijk} to the scalar potential is, though not trivial, still manageable, since most relevant reduction identities are already available in this thesis.

Although the present calculation is just general enough to be applied to the simplest GUT, the number of Feynman diagrams for the two-loop Green's functions described above already is considerable. For the light gauge boson two-point function it amounts to 6278, whereas for the ghost-gauge-boson vertex and the light ghost two-point function we have 4109 and 374 diagrams, respectively. Sample diagrams for all three processes are depicted in fig. 4.2.

The result for ζ_{α_i} is available in general form, i.e. with group theory factors and couplings not specified to a particular Lie group or model. In chapter 6 we will assign definite values to these quantities in order to show the application of the result exemplary. Note also that we always need three different sets of group theory factors for $i = 1, 2, 3$, respectively.

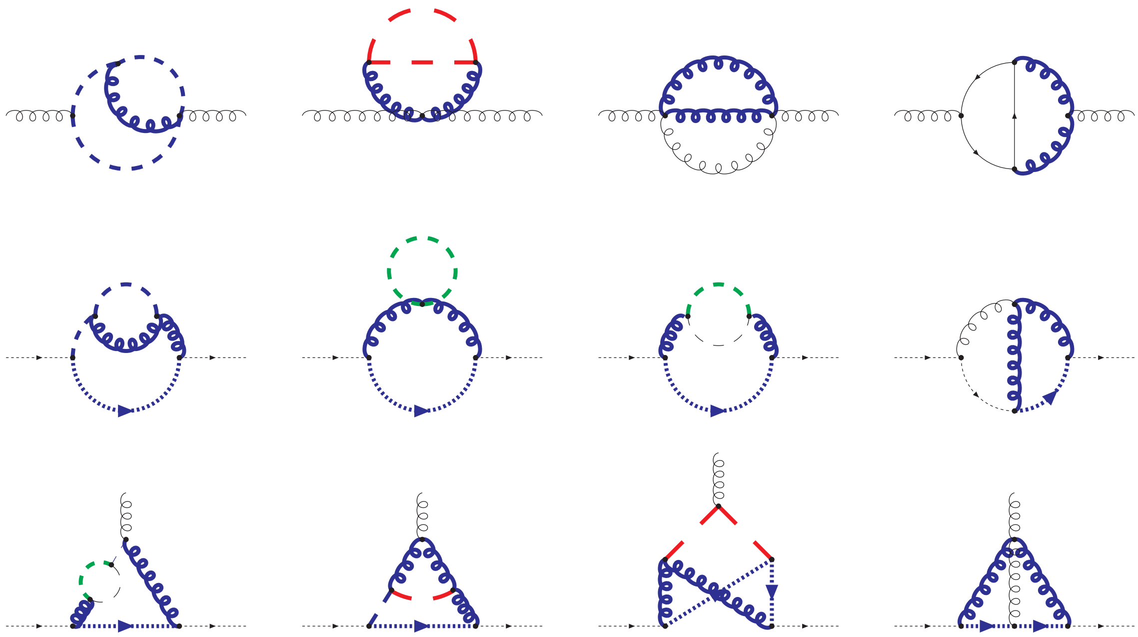


Figure 4.2.: *Sample two-loop diagrams that appear in the calculation of ζ_{α_i}. The first line shows the process $A_\mu^{a_i} \to A_\nu^{b_i}$ contributing to $\Pi_{A,i}^{0,h}(0)$. The second and third line depict $c^{a_i} \to c^{b_i}$ and $c^{a_i} \to c^{b_i} + A_\mu^{c_i}$ contributing to $\Pi_{c,i}^{0,h}(0)$ and $\Gamma_{Ac^\dagger c,i}^{0,h}(0,0)$, respectively. Colored (bold) lines represent fields with mass of $\mathcal{O}(M_{\mathrm{GUT}})$ and black (thin) lines massless fields. Furthermore, curly lines denote gauge bosons, dotted lines ghosts, dashed lines scalar fields and solid lines fermions. Goldstone bosons are marked green (light gray, short-dashed), physical Higgs bosons red (gray, long-dashed) and other heavy scalars blue (dark gray, short-dashed). Note also that two identical lines in one diagram need not have the same mass because of the non-degenerate mass spectrum.*

5. Group-Theoretical Framework for the Two-Loop Matching Calculation

In chapter 4 we have described the theoretical framework that is used for setting up the calculation of the two-loop matching corrections at the GUT scale. We have chosen, not to implement a particular model but to stay in that general framework in order to make extensions of the calculation to other GUT models as easy as possible. In the same spirit the reduction of group theory factors has been dealt with: The idea is, not to fix the representations where the fields live in, but write everything as depending on a set of group theory invariants. When specifying to a particular model, one chooses particular couplings and also calculates the numerical values of these invariants. Besides being easier expandable, this approach also has the advantage that the process of finding diagrams among which possible cancellations, of e.g. gauge parameter dependent terms, can occur is simplified. These are obviously all the diagrams that have the same group theory factor. Therefore, bugs in the implementation are found much easier in this way.

To realize the reduction of group theory factors in such a general manner, one first has to establish a notational framework of primitive group theory invariants. Then, of course, all reduction identities that are needed to reduce group theory factors to this primitive set of invariants have to be derived. Actually, we only need those reduction identities that appear in the processes under consideration up to the two-loop level.

In QCD and other unbroken gauge theories this idea has led to the common practice not to specify the calculation to SU(3) from the beginning but to give the result in terms of invariants as C_2, I_2 and C_A [21, 22, 25, 27, 28]. The state of the art of applying this to multi-loop calculations in unbroken gauge theories, as QCD, is described e.g. in refs. [101, 148]. Basically, here the number of loops is limited only by the available master integrals and not by the knowledge of how to the reduce group theory factors.

In broken gauge theories, as GUTs, the situation is quite different: To the author's knowledge, the reduction of group theory factors has only been dealt with up to the one-loop order in the literature [31, 149, 150] up to now. Certainly, one reason for this situation is that, compared to the unbroken case, the reduction is significantly more involved. The aim of this chapter is to provide a (more or less) self-contained treatment of how to reduce color factors[1] in spontaneously broken gauge theories up to the two-loop level. It constitutes one of the major achievements of this thesis. We have employed this framework for the two-loop calculation of $\zeta_{\alpha_i}(\mu_{\mathrm{GUT}})$ that has been described in the previous chapter. As a supplement to this chapter

[1] For convenience we will use the terms "color structure", "color factor" etc. in this chapter following QCD terminology. However, obviously our formalism is not restricted to $SU(3)$, but is meant to be applied to GUT groups.

the reader is advised to consult appendix A.1. There we have collected derivations for a number of nontrivial reduction identities that appear in the present chapter without proof.

5.1. General Definitions and Notations

In this section we start with some generalities and continue to establish the notation that will be used throughout this chapter. This involves many definitions but is inevitable in order to have a firm ground for further investigations.

We start with a simple GUT group $\mathbf{G}$ that is broken to a (in general not simple) gauge group $\prod_k \mathbf{G}_k$. In the following the $\mathbf{G}_k$ are called group factors. The generators of $\mathbf{G}$ in a general reducible representation $\mathcal{R}$ are denoted by T^α in this chapter[2]. They fulfill the commutation relation

$$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma}\, T^\gamma\,, \tag{5.1}$$

where $f^{\alpha\beta\gamma}$ are the structure constants of $\mathbf{G}$. In order to distinguish between broken and unbroken generators, we use the notation of section 4.1:

$$\{\alpha\} = \sum_i \{A_i\} + \sum_i \{a_i\} = \{A\} + \{a\} \tag{5.2}$$

where A_i label the broken generators of $\mathbf{G}$ belonging to the $\prod_k \mathbf{G}_k$-irreducible subspace labeled by i. If there is only one $\prod_k \mathbf{G}_k$-irreducible subspace in the adjoint representation of $\mathbf{G}$, we can omit the sub-index i in A_i. In contrast, a_i labels the unbroken generators belonging to the subgroup $\mathbf{G}_i$ (see also the examples given in eqs. (4.5) and (4.6)). As the $\mathbf{G}_i$ are regular subgroups of $\mathbf{G}$, also the following commutation relations hold:

$$[T^{a_i}, T^{b_j}] = i f^{a_i b_j c_k}\, T^{c_k} \tag{5.3}$$

where $f^{a_i b_j c_k} = 0$ unless $i = j = k$. Furthermore, because the subgroup $\mathbf{G}_i$ is closed, we have $f^{a_i b_j A_k} = 0$ for all i, j, k. Otherwise the commutator $[T^{a_i}, T^{b_j}]$ would contain terms proportional to the broken generators T^{A_k} and hence $\mathbf{G}_i$ would not be closed. Note that if not explicitly stated, the sub-indices $i, j \ldots$ of the indices $a_i, b_j \ldots, A_i, B_j \ldots$ are not summed. A repeated index $a, b \ldots$ or $A, B \ldots$ without sub-index, however, means that the sub-index has been summed over. For a particular GUT model, the representation $\mathcal{R}$ that T^α is defined on generally is reducible under $\mathbf{G}$. It decomposes into $\mathbf{G}$-irreducible representations where the gauge bosons, Weyl fermions and scalars of the theory live in:

$$\mathcal{R} \to \bigoplus_x \mathcal{R}^x = \mathcal{R}^{\mathcal{A}} \oplus \mathcal{R}^{\mathcal{H}} \oplus \mathcal{R}^{\mathcal{S}_{\mathrm{I}}} \oplus \mathcal{R}^{\mathcal{S}_{\mathrm{II}}} \oplus \ldots \oplus \mathcal{R}^{\mathcal{F}_{\mathrm{I}}} \oplus \mathcal{R}^{\mathcal{F}_{\mathrm{II}}} \oplus \ldots\,. \tag{5.4}$$

$\mathcal{R}^{\mathcal{A}}$ stands for the adjoint (gauge boson) representation and $\mathcal{R}^{\mathcal{H}}$ for the representation of the GUT-breaking scalar. The other symbols represent the irreducible representations for the scalars and fermions, respectively, numbered by roman numerals for convenience. We define projectors on these subspaces denoted by Π^x with $x = \mathcal{A}, \mathcal{H}, \mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}} \ldots, \mathcal{F}_{\mathrm{I}}, \mathcal{F}_{\mathrm{II}} \ldots$. Clearly,

[2] In chapter 4 we employed the symbol T^α only for the fermion representation. In this chapter, however, we will use the symbol for a generic generator of the gauge group.

Subspace projected on	Projector	SU(5) example	Relation
$\mathbf{G}$-irreducible	Π^x	$\Pi^{\mathbf{5}_H}$	$\Pi^{\mathbf{5}_H} =$
Generic $\prod_k \mathbf{G}_k$-irreducible	ϱ_n^x	$\varrho^{\mathbf{5}_H}_{(\mathbf{3},\mathbf{1},-1/3)}, \varrho^{\mathbf{5}_H}_{(\mathbf{1},\mathbf{2},1/2)}$	$\varrho^{\mathbf{5}_H}_{(\mathbf{3},\mathbf{1},-1/3)} + \varrho^{\mathbf{5}_H}_{(\mathbf{1},\mathbf{2},1/2)} =$
Heavy $\prod_k \mathbf{G}_k$-irreducible	P_n^x	$P^{\mathbf{5}_H}_{(\mathbf{3},\mathbf{1},-1/3)}$	$P^{\mathbf{5}_H}_{(\mathbf{3},\mathbf{1},-1/3)} + p^{\mathbf{5}_H}_{(\mathbf{1},\mathbf{2},1/2)}$
Light $\prod_k \mathbf{G}_k$-irreducible	p_n^x	$p^{\mathbf{5}_H}_{(\mathbf{1},\mathbf{2},1/2)}$	

Table 5.1.: *Overview of all projectors that have been defined. The projector Π^x on a $\mathbf{G}$-irreducible subspace decomposes further into projectors on $\prod_k \mathbf{G}_k$-irreducible subspaces. These can either be denoted by the generic ϱ_n^x or by P_n^x and p_n^x which contain also information on whether the field living on the respective subspace is super-heavy of massless.*

$[\Pi^x, T^\alpha] = 0$ holds for all x and α. As an example, for the case of the Georgi-Glashow model eq. (5.4) would be

$$\mathcal{R} \to \mathbf{24} \oplus \mathbf{24}_H \oplus \mathbf{5}_H \oplus \overline{\mathbf{5}} \oplus \mathbf{10}\,, \tag{5.5}$$

and the projectors would be denoted as $\Pi^{\mathbf{24}}$ etc.. Each of the representations on the right-hand side decompose further under $\prod_k \mathbf{G}_k$:

$$\mathcal{R}^x \to \bigoplus_n \mathcal{R}^x_n = \mathcal{R}^x_1 \oplus \mathcal{R}^x_2 \oplus \mathcal{R}^x_3 \oplus \dots \tag{5.6}$$

with projectors P_n^x and p_n^x. We use a capital P to denote that the respective $\prod_k \mathbf{G}_k$-irreducible representation contains fields with a mass of $\mathcal{O}(M_{\mathrm{GUT}})$ and a lowercase p for projectors on a subspace with massless fields. Because the Lagrangian is still invariant under $\prod_k \mathbf{G}_k$ after symmetry breaking, each of those $\prod_k \mathbf{G}_k$-irreducible subspaces can be assigned to a definite mass of the respective field. The indices $n = 1, 2, 3\dots$ label the $\prod_k \mathbf{G}_k$-irreducible representation in $\mathcal{R}^x$. The projectors fulfill $[P_n^x, T^{a_i}] = 0 = [p_n^x, T^{a_i}]$ for all x, a, n and i. Picking out the $\mathbf{5}_H$ of SU(5) exemplary, eq. (5.6) would become

$$\mathbf{5}_H \to (\mathbf{3}, \mathbf{1}, -\tfrac{1}{3}) \oplus (\mathbf{1}, \mathbf{2}, \tfrac{1}{2})\,, \tag{5.7}$$

Furthermore, it generally holds that

$$\sum_n \varrho_n^x \equiv \sum_n p_n^x + \sum_n P_n^x \equiv p^x + P^x = \Pi^x \tag{5.8}$$

where ϱ can be P or p depending on n. Sticking to our SU(5) example, we would have $\varrho_n^{\mathbf{5}_H} = p_n^{\mathbf{5}_H}$ for $n = (\mathbf{1}, \mathbf{2}, \frac{1}{2})$ since this representation contains the light SM Higgs doublet, and $\varrho_n^{\mathbf{5}_H} = P_n^{\mathbf{5}_H}$ for $n = (\mathbf{3}, \mathbf{1}, -\frac{1}{3})$ which contains the super-heavy Higgs triplet. Table 5.1 summarizes all projector definitions that have been introduced so far.

Armed with these definitions, we can define some primitive Dynkin indices $I_2(\ldots)$ and Casimir invariants $C_2(\ldots)$ which have real numerical values for a given group.

$$\begin{aligned}
\mathrm{Tr}(\Pi^x T^\alpha T^\beta) &= I_2(\Pi^x)\ \delta^{\alpha\beta}\,,\\
\mathrm{Tr}(\varrho^x_n T^{a_i} T^{b_i}) &= I_2(\varrho^x_n)^i\ \delta^{a_i b_i}\,,\\
\mathrm{Tr}(\varrho^x_n T^{A_i} \varrho^x_m T^{B_i}) &= I_2(\varrho^x_n, \varrho^x_m)^i\ \delta^{A_i B_i}\,,\\
\Pi^x T^\alpha T^\alpha &= C_2(\Pi^x)\ \Pi^x\,,\\
\varrho^x_n T^{a_i} T^{a_i} &= C_2(\varrho^x_n)^i\ \varrho^x_n\,,\\
\varrho^x_n T^{A_i} \varrho^x_m T^{A_i} &= C_2(\varrho^x_n, \varrho^x_m)^i\ \varrho^x_n\,.
\end{aligned} \tag{5.9}$$

Again ϱ can stand for either P or p. The existence of these invariants follows from Schur's lemma (cf. section 2.7). Note that an invariant defined as e.g. $\mathrm{Tr}(\varrho^x_n T^{a_i} \varrho^x_m T^{b_i})$ is not primitive because $[T^{a_i}, \varrho_m] = 0$. Furthermore, we can define the dimensions of the various irreducible representations by

$$\begin{aligned}
\Delta^x &\equiv \mathrm{Tr}(\Pi^x)\,,\\
D^x_n &\equiv \mathrm{Tr}(P^x_n)\,,\\
d^x_n &\equiv \mathrm{Tr}(p^x_n)\,,
\end{aligned} \tag{5.10}$$

and specifically for the adjoint representation:

$$\begin{aligned}
\Delta^{\mathcal{A}} &\equiv \delta^{\alpha\alpha}\,,\\
D^{\mathcal{A}}_n &\equiv \delta^{A_n A_n}\,,\\
d^{\mathcal{A}}_n &\equiv \delta^{a_n a_n}\,.
\end{aligned} \tag{5.11}$$

which gives us the relations

$$\begin{aligned}
I_2(\Pi^x)\ \Delta^{\mathcal{A}} &= C_2(\Pi^x)\ \Delta^x\,,\\
I_2(P^x_n)^i\ d^{\mathcal{A}}_i &= C_2(P^x_n)^i\ D^x_n\,,\\
I_2(p^x_n)^i\ d^{\mathcal{A}}_i &= C_2(p^x_n)^i\ d^x_n\,,\\
I_2(P^x_n, \varrho^x_m)^i\ D^{\mathcal{A}}_i &= C_2(P^x_n, \varrho^x_m)^i\ D^x_n\,,\\
I_2(p^x_n, \varrho^x_m)^i\ D^{\mathcal{A}}_i &= C_2(p^x_n, \varrho^x_m)^i\ d^x_n\,.
\end{aligned} \tag{5.12}$$

Let us emphasize again that no summation over the sub-indices i that label the $\prod_k \mathbf{G}_k$-irreducible representation is implied here. However, we introduce the convention that omitting this sub-index implies summation:

$$\begin{aligned}
C_2(\ldots) &\equiv \sum_i C_2(\ldots)^i\,,\\
C_2(\varrho^x_n, \varrho^x)^i &\equiv \sum_m C_2(\varrho^x_n, \varrho^x_m)^i\,,\\
I_2(\ldots\varrho^x\ldots)^i &\equiv \sum_m I_2(\ldots\varrho^x_m\ldots)^i\,.
\end{aligned} \tag{5.13}$$

Note also that the invariants $I_2(\Pi^x)$ and $C_2(\Pi^x)$ have only been defined for convenience. Actually they can be decomposed into other more primitive invariants:

$$\begin{aligned}
I_2(\Pi^x) &= I_2(p^x)^i + I_2(P^x)^i = I_2(p^x, p^x)^i + I_2(P^x, P^x)^i + 2\, I_2(P^x, p^x)^i\,, \qquad (5.14)\\
C_2(\Pi^x) &= C_2(p^x_i) + C_2(p^x_i, p^x) + C_2(p^x_i, P^x) = C_2(P^x_i) + C_2(P^x_i, p^x) + C_2(P^x_i, P^x)\,,
\end{aligned}$$

where the summation convention introduced before has been used. The right hand side of eq. (5.14) does not depend on i anymore due to the summation of the projectors over the full representation space $\mathcal{R}^x$. Since in the calculation of matching coefficients one has to distinguish between heavy and light particles, it is more convenient to use the more primitive invariants on the right hand side.

Here we also give the first example of a nontrivial general reduction identity that can be applied to any representation at the two-loop level:

$$\begin{aligned} & f^{a_i B_j C_j}\ \mathrm{Tr}(\varrho_n^x T^{B_j} \varrho_m^x T^{C_j} T^{b_i}) = \\ & \frac{i}{2}\delta^{a_i b_i}\Big[I_2(P_j^{\mathcal{A}})^i I_2(\varrho_n^x, \varrho_m^x)^j + C_2(\varrho_n^x, \varrho_m^x)^j I_2(\varrho_n^x)^i - C_2(\varrho_m^x, \varrho_n^x)^j I_2(\varrho_m^x)^i\Big], \end{aligned} \tag{5.15}$$

where $x \in \{\mathcal{H}, \mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}}, \ldots, \mathcal{F}_{\mathrm{I}}, \mathcal{F}_{\mathrm{II}}\}$. The derivation of this identity can be found in appendix A.1.1.

5.2. The Adjoint Representation

The definitions that have been introduced so far are general enough to be applied to all color factors that appear in our calculations. However, some of the fields live in representations that deserve special attention. First let us focus on the adjoint representation. The generators here are defined by

$$(T_{\mathcal{A}}^\alpha)_{\beta\gamma} \equiv (\Pi^{\mathcal{A}}\, T^\alpha)_{\beta\gamma} = -i f^{\alpha\beta\gamma}\,. \tag{5.16}$$

Clearly, the operators $P_i^{\mathcal{A}}$ and $p_i^{\mathcal{A}}$ project on the subspaces with indices A_i and a_i, respectively. Because now $x = \mathcal{A}$ in eq. (5.12) and $f^{a_i b_j A_k} = 0$, things simplify for the adjoint representation since many Dynkin indices and Casimir invariants are directly related to each other. In fact, it is sufficient to define four invariants:

$$\begin{aligned} f^{\alpha\gamma\delta} f^{\beta\gamma\delta} &= I_2(\Pi^{\mathcal{A}})\ \delta^{\alpha\beta}\,, \\ f^{a_i c_i d_i} f^{b_i c_i d_i} &= I_2(p_i^{\mathcal{A}})^i\ \delta^{a_i b_i}\,, \\ f^{a_i C_j D_j} f^{b_i C_j D_j} &= I_2(P_j^{\mathcal{A}})^i\ \delta^{a_i b_i}\,, \\ f^{A_j c_i D_j} f^{B_j c_i D_j} &= I_2(P_j^{\mathcal{A}}, p_i^{\mathcal{A}})^j\ \delta^{A_j B_j} \end{aligned} \tag{5.17}$$

where we followed the notation of eq. (5.9). We keep the redundant sub-indices i and j in lines 2 and 4 of eq. (5.17) in order to be consistent with the notation of eq. (5.9). Note that $f^{a_i B_j C_k} = 0$ for $j \neq k$ because the indices j and k are used here to label the $\prod_k \mathbf{G}_k$-irreducible representation of the generator $T_{\mathcal{A}}^{a_i}$ which is assumed to have block-diagonal form. There is another quadratic Casimir invariant for the adjoint representation that can be expressed through these and one that vanishes:

$$\begin{aligned} f^{A_j CD} f^{B_j CD} &= \Big[I_2(\Pi^{\mathcal{A}})\ - 2\, I_2(P_j^{\mathcal{A}}, p^{\mathcal{A}})^j\Big]\ \delta^{A_j B_j}\,, \\ f^{A_i CD} f^{b_j CD} &= 0\,. \end{aligned} \tag{5.18}$$

Furthermore, there are relations among these invariants:

$$\begin{aligned} I_2(\Pi^{\mathcal{A}}) &= I_2(P^{\mathcal{A}})^i + I_2(p_i^{\mathcal{A}})^i\,, \\ I_2(P_j^{\mathcal{A}}, p_i^{\mathcal{A}})^j\ D_j^{\mathcal{A}} &= I_2(P_j^{\mathcal{A}})^i\ d_i^{\mathcal{A}}\,. \end{aligned} \tag{5.19}$$

At the two-loop level identities introduced so far are not sufficient to reduce all adjoint color factors because products of up to six structure constants with various contractions can appear in the diagrams. Using the Jacobi identity for $f^{\alpha\beta\gamma}$ (which is equivalent to the commutation relation for the adjoint generators)

$$\begin{aligned} [T^\alpha_\mathcal{A}, T^\beta_\mathcal{A}] - i f^{\alpha\beta\gamma} T^\gamma_\mathcal{A} &= 0\,, \quad \Leftrightarrow \\ f^{\beta\gamma\delta} f^{\alpha\delta\epsilon} + f^{\alpha\beta\delta} f^{\gamma\delta\epsilon} + f^{\gamma\alpha\delta} f^{\beta\delta\epsilon} &= 0 \end{aligned} \tag{5.20}$$

for different choices of α, β and γ, we can derive relations for products of three contracted structure constants that have three uncontracted indices:

$$\begin{aligned} f^{\alpha\delta\epsilon} f^{\beta\epsilon\phi} f^{\gamma\phi\delta} &= \frac{1}{2} I_2(\Pi^\mathcal{A})\, f^{\alpha\beta\gamma}\,, \\ f^{a_i d_i e_i} f^{b_i e_i f_i} f^{c_i f_i d_i} &= \frac{1}{2} I_2(p_i^\mathcal{A})^i\, f^{a_i b_i c_i}\,, \\ f^{a_i DE} f^{b_i EF} f^{c_i FD} &= \frac{1}{2} I_2(P^\mathcal{A})^i\, f^{a_i b_i c_i}\,, \\ f^{a_i DE} f^{b_i EF} f^{CFD} &= 0\,, \\ f^{a_i DE} f^{B_j EF} f^{C_j FD} &= \frac{1}{2}\Big[I_2(\Pi^\mathcal{A}) - 2\, I_2(P_j^\mathcal{A}, p^\mathcal{A})^j \Big]\, f^{a_i B_j C_j}\,, \\ f^{a_i d_i e_i} f^{B_j e_i F} f^{C_j F d_i} &= \frac{1}{2} I_2(p_i^\mathcal{A})^i\, f^{a_i B_j C_j}\,, \\ f^{a_i DE} f^{B_k E f_j} f^{C_k f_j D} &= \frac{1}{2}\Big[2\, I_2(P_k^\mathcal{A}, p_j^\mathcal{A})^k - \delta_{ij}\, I_2(p_i^\mathcal{A})^i \Big] f^{a_i B_k C_k}\,, \\ f^{A_j De} f^{B_j eF} f^{C_j FD} &= \frac{1}{2}\, I_2(P_j^\mathcal{A}, p^\mathcal{A})^j f^{A_j B_j C_j}\,, \\ f^{A_j DE} f^{B_j EF} f^{C_j FD} &= \frac{1}{2}\Big[I_2(\Pi^\mathcal{A}) - 3\, I_2(P_j^\mathcal{A}, p^\mathcal{A})^j \Big] f^{A_j B_j C_j}\,, \end{aligned} \tag{5.21}$$

where we have used the definitions in eq. (5.17). These relations are sufficient to do all the reduction for two-point and three-point Green's functions for the adjoint representation at the two-loop level. It may happen that one encounters contractions that are not listed in eq. (5.21). However, these contractions are then related to contractions that are listed in eq. (5.21) by commuting adjoint generators (i.e. applying the Jacobi identity, eq. (5.20)).

5.3. GUT-Breaking Scalars

Next, let's turn our attention to the representation $\mathcal{R}^\mathcal{H}$ where the GUT-breaking scalar field H and the Goldstone field G live in. Some peculiarities occur here due to the appearance of the vev v. v can be viewed as a $\Delta^\mathcal{H}$-dimensional vector with a single non-vanishing entry in the direction of the breaking. $\mathcal{R}^\mathcal{H}$ decomposes under $\prod_k \mathbf{G}_k$ into the part of physical Higgs fields and a part of Goldstone bosons:

$$\mathcal{R}^\mathcal{H} \to \mathcal{R}^{\tilde{\mathcal{H}}} \oplus \mathcal{R}^\mathcal{G} = \mathcal{R}_1^{\tilde{\mathcal{H}}} \oplus \mathcal{R}_2^{\tilde{\mathcal{H}}} \oplus \ldots \oplus \mathcal{R}_1^\mathcal{G} \oplus \mathcal{R}_2^\mathcal{G} \oplus \ldots \tag{5.22}$$

As already explained in section 4.1, the explicit form of the projector on the subspace $\mathcal{R}^\mathcal{G}$ is given by [41]:

$$P_i^\mathcal{G} = g^2 \tilde{T}^{A_i} v \left(\frac{1}{M_X^2}\right)_{A_i B_i} v \tilde{T}^{B_i}\,, \qquad P^\mathcal{G} = \sum_i P_i^\mathcal{G} \tag{5.23}$$

where the (diagonal) gauge boson mass matrix (cf. eq. (4.10))

$$(M_{\mathrm{X}}^2)_{A_iB_i} \equiv g^2 v\tilde{T}^{A_i}\tilde{T}^{B_i}v \equiv M_{\mathrm{X}_i}^2 \delta^{A_iB_i} \tag{5.24}$$

has been used. The antisymmetric generators in the real representation $\mathcal{R}^{\mathcal{H}}$ have been denoted by $\tilde{T}^\alpha$. Accordingly, the projectors on the subspace of physical Higgs bosons can be written as

$$\sum_i P_i^{\tilde{\mathcal{H}}} = P^{\tilde{\mathcal{H}}} \equiv \Pi^{\mathcal{H}} - P^{\mathcal{G}} = \Pi^{\mathcal{H}} - g^2\tilde{T}^A v \left(\frac{1}{M_X^2}\right)_{AB} v\tilde{T}^B . \tag{5.25}$$

where $P_i^{\tilde{\mathcal{H}}}$ projects on $\mathcal{R}_i^{\tilde{\mathcal{H}}}$. With these definitions and using eq. (5.3) as well as the antisymmetry of $\tilde{T}^\alpha$, we already can derive a useful reduction identity for an invariant tensor that appears frequently:

$$v\tilde{T}^{A_i}\tilde{T}^{B_j}\tilde{T}^{C_k}v = \frac{i}{2g^2}(M_{\mathrm{X}_i}^2 - M_{\mathrm{X}_j}^2 + M_{\mathrm{X}_k}^2)f^{A_iB_jC_k} . \tag{5.26}$$

One important property follows from $\tilde{T}^{a_i}v = 0$:

$$\tilde{T}^{a_i}\tilde{T}^{A_j}v = [\tilde{T}^{a_i}, \tilde{T}^{A_j}]\,v = -(T_{\mathcal{A}}^{a_i})_{A_jB_j}\tilde{T}^{B_j}v . \tag{5.27}$$

i.e. $\tilde{T}^{a_i}$ acts like the adjoint generator on the subspace of Goldstone bosons. This leads to various relations between invariants in $\mathcal{R}^{\mathcal{H}}$ and in $\mathcal{R}^{\mathcal{A}}$:

$$\begin{aligned}
D_j^{\mathcal{G}} &= D_j^{\mathcal{A}} , \\
I_2(P_j^{\mathcal{G}})^i &= I_2(P_j^{\mathcal{A}})^i , \\
C_2(P_j^{\mathcal{G}})^i &= I_2(P_j^{\mathcal{A}}, p_i^{\mathcal{A}})^j , \\
I_2(P^{\tilde{\mathcal{H}}})^i &= I_2(\Pi^{\mathcal{H}}) - I_2(P^{\mathcal{A}})^i , \\
I_2(P^{\tilde{\mathcal{H}}}, P^{\tilde{\mathcal{H}}})^i &= I_2(\Pi^{\mathcal{H}}) - 2\; C_2(\Pi^{\mathcal{H}}) + \frac{3}{2}\; I_2(P_i^{\mathcal{A}}, p^{\mathcal{A}})^i + \frac{1}{4}\; I_2(\Pi^{\mathcal{A}}) .
\end{aligned} \tag{5.28}$$

This again highlights the intimate relationship between the adjoint representation and the representation of Goldstone bosons. It is the manifestation of the fact that in spontaneously broken gauge theories Goldstone bosons become the longitudinal part of the (now) massive gauge bosons. Furthermore, from $v\tilde{T}^{A_i}\tilde{T}^{B_j}P_n^{\tilde{H}} = v\tilde{T}^{B_j}\tilde{T}^{A_i}P_n^{\tilde{H}}$ it follows that

$$\begin{aligned}
I_2(P_i^{\mathcal{G}}, P_n^{\tilde{\mathcal{H}}})^j D_j^{\mathcal{G}} &= I_2(P_j^{\mathcal{G}}, P_n^{\tilde{\mathcal{H}}})^i D_i^{\mathcal{G}} , \\
C_2(P_n^{\tilde{\mathcal{H}}}, P_i^{\mathcal{G}})^j &= C_2(P_n^{\tilde{\mathcal{H}}}, P_j^{\mathcal{G}})^i .
\end{aligned} \tag{5.29}$$

There is also a nontrivial important reduction identity that involves adjoint invariants as well as GUT-breaking scalar invariants:

$$\begin{aligned}
&\sum_{jk} f^{a_iB_jC_j} f^{b_iD_kE_k}\, v\tilde{T}^{B_j}\tilde{T}^{D_k}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{C_j}\tilde{T}^{E_k}v = \\
&\delta^{a_ib_i}\sum_j \frac{M_{\mathrm{X}j}^2}{g^2}\left[I_2(P_j^{\mathcal{A}})^i\, I_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j - \frac{1}{2}I_2(P_n^{\tilde{\mathcal{H}}})^i\, C_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j\right] .
\end{aligned} \tag{5.30}$$

The derivation of eq. (5.30) can be found in appendix A.1.2.

Furthermore, w.l.o.g. we now define $P_1^{\tilde{\mathcal{H}}}$ to be the operator that projects on the subspace where $v \neq 0$ (i.e. $(P_1^{\tilde{\mathcal{H}}})_{ij} = \frac{v_i v_j}{v^2}$ is a matrix with a single non-zero entry in the component (k,k) where $v_k \neq 0$). Then because of $\tilde{T}^{a_i} v = 0$ and $v\tilde{T}^{A_i} P_n^{\tilde{\mathcal{H}}} = v\tilde{T}^{A_i} P^{\mathcal{G}} P_n^{\tilde{\mathcal{H}}} = 0$, any invariant that contains $P_1^{\tilde{\mathcal{H}}}$ alone or together with some other $P_i^{\tilde{\mathcal{H}}}$ vanishes:

$$C_2(P_1^{\tilde{\mathcal{H}}}) = I_2(P_1^{\tilde{\mathcal{H}}}) = C_2(P_1^{\tilde{\mathcal{H}}}, P_i^{\tilde{\mathcal{H}}}) = I_2(P_1^{\tilde{\mathcal{H}}}, P_i^{\tilde{\mathcal{H}}}) = 0\,. \tag{5.31}$$

Since the Higgs mass matrix M_H^2 (cf. eq. (4.26)) commutes with all $\tilde{T}^{a_i}$, it is diagonal and proportional to the unit matrix on each $\prod_k \mathbf{G}_k$-irreducible subspace. Therefore, it can be written as:

$$M_H^2 = \sum_i P_i^{\tilde{\mathcal{H}}}\, M_{H_i}^2\,, \tag{5.32}$$

where $M_{H_i}^2$ are masses of the physical Higgs bosons and particularly $P^{\mathcal{G}}\, M_H^2 = 0$ due to the Goldstone theorem.

5.4. The Cubic and Quartic Scalar Couplings

Next, we will give some useful reduction identities that involve the cubic and quartic scalar couplings $\kappa_{ijk}^{\mathcal{H}} \equiv \kappa_{i'j'k'}\, P_{i'i}^{\mathcal{H}}\, P_{j'j}^{\mathcal{H}}\, P_{k'k}^{\mathcal{H}}$ and $\lambda_{ijkl}^{\mathcal{H}} \equiv \lambda_{i'j'k'l'}\, P_{i'i}^{\mathcal{H}}\, P_{j'j}^{\mathcal{H}}\, P_{k'k}^{\mathcal{H}}\, P_{l'l}^{\mathcal{H}}$, which are defined to be restricted to the space $\mathcal{R}^{\mathcal{H}}$. Firstly, we only consider this subspace because, again, due to the appearance of the vev on this subspace things are more involved. (In the case of the Georgi-Glashow model $\mathcal{R}^{\mathcal{H}}$ is the 24-dimensional representation.) $\kappa_{ijk}^{\mathcal{H}}$ and $\lambda_{ijkl}^{\mathcal{H}}$ are totally symmetric invariant tensors under $\mathbf{G}$ and appear in the scalar potential, eq. (4.18). These identities can be derived by using eq. (4.19) and Schur's Lemma as well as the definition of the Higgs mass matrix, eq. (4.26). Some of these identities can also be obtained by multiplying eq. (4.19) by $\Phi_i\Phi_j\Phi_k\Phi_l$ and performing derivatives w.r.t Φ_m. They are used to eliminate the coupling $\lambda_{ijkl}^{\mathcal{H}}$ from the result by expressing it through the physical Higgs masses and other physical parameters as the gauge boson mass matrix M_{X} and the gauge coupling g. Recall, that in order to make our choice of parameters consistent, $\kappa_{ijk}^{\mathcal{H}}$ and $\lambda_{ijkl}^{\mathcal{H}}$ should not be present in the final result of a loop calculation but expressed through physical quantities as masses and gauge couplings (cf. subsection 4.1.2). Using these identities in our calculation also guarantees manifest cancellation of the gauge parameter dependence in physical quantities as on-shell mass renormalizations or decoupling coefficients. The relation that are of relevance for our calculation are

$$\begin{aligned}
\lambda_{ijkl}^{\mathcal{H}} v_j v_k v_l &= \frac{v_i}{v^2}\big(3\, v M_H^2 v - \frac{3}{2}\kappa_{ijk} v_i v_j v_k\big)\,,\\
\lambda_{ijkl}^{\mathcal{H}} v_i v_j (P_s^{\mathcal{H}})_{kl} &= 2\,\mathrm{Tr}(M_{H_s}^2 P_s^{\mathcal{H}}) - 2\,\kappa_{jkl}^{\mathcal{H}} v_j (P_s^{\mathcal{H}})_{kl} + \frac{D_s^{\mathcal{H}}}{v^2}\big(\frac{1}{2}\kappa_{ijk}^{\mathcal{H}} v_i v_j v_k + v M_H^2 v\big)\,,\\
\lambda_{ijkn}^{\mathcal{H}}\; v_k (P_s^{\mathcal{G}})_{nl} &= -\kappa_{ijn}^{\mathcal{H}} (P_s^{\mathcal{G}})_{nl} + \big[(\tilde{T}^{A_s} M_H^2)_{ij} - (M_H^2 \tilde{T}^{A_s})_{ij}\big]\frac{g^2}{M_{\mathrm{X}_s}^2} v_n \tilde{T}_{nl}^{A_s}\,,
\end{aligned}$$

$$\begin{aligned}
\frac{M^2_{\mathrm{X}_m}}{g^2}\lambda^{\mathcal{H}}_{ijkl}(P^{\tilde{\mathcal{H}}}_n)_{ij}(P^{\mathcal{G}}_m)_{kl} &= C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^m\Big[2\,\mathrm{Tr}(M^2_H P^{\tilde{\mathcal{H}}}_n) - \kappa^{\mathcal{H}}_{jkl}v_j(P^{\tilde{\mathcal{H}}}_n)_{kl} \\
&\qquad + \frac{D^{\mathcal{H}}_s}{v^2}\big(\frac{1}{2}\kappa^{\mathcal{H}}_{ijk}v_iv_jv_k + vM^2_Hv\big)\Big] \\
&\quad +2\,\mathrm{Tr}(M^2_H\tilde{T}^{A_m}P^{\tilde{\mathcal{H}}}_n\tilde{T}^{A_m}) - 2\,\mathrm{Tr}(\tilde{T}^{A_m}M^2_HP^{\tilde{\mathcal{H}}}_n\tilde{T}^{A_m})\,, \\
\lambda^{\mathcal{H}}_{ijkl}(P^{\mathcal{G}}_n)_{ij}(P^{\mathcal{G}}_m)_{kl} &= \frac{g^4}{M^2_{\mathrm{X}_n}M^2_{\mathrm{X}_m}}\Big[v\tilde{T}^{A_n}\tilde{T}^{A_n}M^2_H\tilde{T}^{B_m}\tilde{T}^{B_m}v \\
&\qquad + 2\,v\tilde{T}^{A_n}\tilde{T}^{B_m}M^2_H\tilde{T}^{A_n}\tilde{T}^{B_m}v\Big]\,, \\
\lambda^{\mathcal{H}}_{ijkl}\lambda^{\mathcal{H}}_{mjkl}v_iv_m &= \frac{v^2}{\Delta^{\mathcal{H}}}\lambda^{\mathcal{H}}_{ijkl}\lambda^{\mathcal{H}}_{ijkl}\,.
\end{aligned} \tag{5.33}$$

For illustrative examples of how to derive these identities, the reader is advised to refer to subsection A.1.4 in the appendix. In the same way we obtain some relations for the coupling λ that is not restricted to the space $\mathcal{R}^{\mathcal{H}}$. Important examples are:

$$\begin{aligned}
\lambda_{ijkn}\,v_k(P^{\mathcal{G}}_s)_{nl} &= -\kappa_{ijn}(P^{\mathcal{G}}_s)_{nl} + \big[(\tilde{T}^{A_s}M^2_S)_{ij} - (M^2_S\tilde{T}^{A_s})_{ij}\big]\frac{g^2}{M^2_{\mathrm{X}_s}}v_n\tilde{T}^{A_s}_{nl}\,, \\
\frac{M^2_{\mathrm{X}_m}}{g^2}\lambda_{ijkl}(\varrho^{\mathcal{S}}_n)_{ij}(P^{\mathcal{G}}_m)_{kl} &= C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^m\,\lambda_{ijkl}v_iv_j(\varrho^{\mathcal{S}}_n)_{kl} \\
&\quad C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^m\,\kappa_{jkl}v_j(\varrho^{\mathcal{S}}_n)_{kl} \\
&\quad +2\,\mathrm{Tr}(M^2_S\tilde{T}^{A_m}\varrho^{\mathcal{S}}_n\tilde{T}^{A_m}) - 2\,\mathrm{Tr}(\tilde{T}^{A_m}M^2_S\varrho^{\mathcal{S}}_n\tilde{T}^{A_m})\,,
\end{aligned}$$

$$\begin{aligned}
\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4}\quad v_{i_1}v_{j_1}\quad & (\varrho^x_{n_1})_{i_2j_2}(\varrho^y_{n_2}T^{a_k})_{i_3j_3}(\varrho^z_{n_3}T^{a_k})_{i_4j_4} = \\
& \tfrac{1}{2}\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4}v_{i_1}v_{j_1}(\varrho^x_{n_1})_{i_2j_2}(\varrho^y_{n_2})_{i_3j_3}(\varrho^z_{n_3}T^{a_k}T^{a_k})_{i_4j_4} \\
& -\tfrac{1}{2}\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4}v_{i_1}v_{j_1}(\varrho^y_{n_2})_{i_2j_2}(\varrho^z_{n_3})_{i_3j_3}(\varrho^x_{n_1}T^{a_k}T^{a_k})_{i_4j_4} \\
& -\tfrac{1}{2}\lambda_{i_1i_2i_3i_4}\lambda_{j_1j_2j_3j_4}v_{i_1}v_{j_1}(\varrho^x_{n_1})_{i_2j_2}(\varrho^z_{n_3})_{i_3j_3}(\varrho^y_{n_2}T^{a_k}T^{a_k})_{i_4j_4}
\end{aligned} \tag{5.34}$$

with $x, y, z \in \{\mathcal{H}, \mathcal{S}_{\mathrm{I}}, \mathcal{S}_{\mathrm{II}}, \ldots\}$.

Up to now we have only given relations with the coupling λ_{ijkl} on the left-hand side. But there are also simplifications for the trilinear coupling κ_{ijk} that we need to consider in order to guarantee maximal mutual cancellations between different diagrams:

$$
\begin{aligned}
\kappa^{\mathcal{H}}_{ijk} v_i (P^{\mathcal{G}}_m)_{jk} &= -\frac{g^2}{2M^2_{\mathrm{X}_m}} \kappa_{ijk} v_i v_j v_k \, C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^m \,, \\
v_r \tilde{T}^{A_m}_{ri} \kappa^{\mathcal{H}}_{ijk} (P^{\tilde{\mathcal{H}}}_n \tilde{T}^{A_m} P^{\tilde{\mathcal{H}}}_l)_{jk} &= \kappa^{\mathcal{H}}_{ijk} v_i (P^{\tilde{\mathcal{H}}}_l)_{jk} C_2(P^{\tilde{\mathcal{H}}}_l, P^{\tilde{\mathcal{H}}}_n)^m \\
&\quad - \kappa^{\mathcal{H}}_{ijk} v_i (P^{\tilde{\mathcal{H}}}_n)_{jk} C_2(P^{\tilde{\mathcal{H}}}_n, P^{\tilde{\mathcal{H}}}_l)^m \,, \\
v_r \tilde{T}^{A_m}_{ri} \kappa^{\mathcal{H}}_{ijk} (P^{\tilde{\mathcal{H}}}_n \tilde{T}^{A_m} P^{\mathcal{G}}_l)_{jk} &= -\frac{g^2}{2M^2_{\mathrm{X}_l}} \kappa_{ijk} v_i v_j v_k \, C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^l \, C_2(P^{\mathcal{G}}_l, P^{\tilde{\mathcal{H}}}_n)^m \\
&\quad - \kappa^{\mathcal{H}}_{ijk} v_i (P^{\tilde{\mathcal{H}}}_n)_{jk} C_2(P^{\tilde{\mathcal{H}}}_n, P^{\mathcal{G}}_l)^m \,, \\
v_r \tilde{T}^{A_m}_{ri} \kappa^{\mathcal{H}}_{ijk} (P^{\mathcal{G}}_n \tilde{T}^{A_m} P^{\mathcal{G}}_l)_{jk} &= -\frac{g^2}{2M^2_{\mathrm{X}_l}} \kappa_{ijk} v_i v_j v_k \, C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^l \, C_2(P^{\mathcal{G}}_l, P^{\mathcal{G}}_n)^m \\
&\quad \frac{g^2}{2M^2_{\mathrm{X}_n}} \kappa_{ijk} v_i v_j v_k \, C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^n C_2(P^{\mathcal{G}}_n, P^{\mathcal{G}}_l)^m \,, \\
v_r \tilde{T}^{A_m}_{ri} \kappa_{ijk} (P^{\mathcal{S}}_n \tilde{T}^{A_m} P^{\mathcal{S}}_l)_{jk} &= \kappa_{ijk} v_i (P^{\mathcal{S}}_l)_{jk} C_2(P^{\mathcal{S}}_l, P^{\mathcal{S}}_n)^m \\
&\quad - \kappa_{ijk} v_i (P^{\mathcal{S}}_n)_{jk} C_2(P^{\mathcal{S}}_n, P^{\mathcal{S}}_l)^m \,, \\
v_r \tilde{T}^{A_m}_{ri} \kappa^{\mathcal{H}}_{ijk} (P^{\mathcal{G}}_n \tilde{T}^{A_m})_{jk} &= \frac{g^2}{4M^2_{\mathrm{X}_n}} C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^n C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^m \, \kappa_{ijk} v_i v_j v_k \,, \\
v_r \tilde{T}^{A_m}_{ri} \kappa_{ijk} (P^{x}_n \tilde{T}^{A_m})_{jk} &= -\frac{1}{2} C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^m \kappa_{ijk} v_i (P^{x}_n)_{jk} \,,
\end{aligned}
\tag{5.35}
$$

with $x \in \{\tilde{\mathcal{H}}, \mathcal{G}, \mathcal{S}_I, \mathcal{S}_{II}, \ldots\}$.

5.5. Fermion Representations and Yukawa Couplings

As explained in subsection 4.1.3, the splitting of representations for the fermions is more peculiar than for the real scalars because Weyl fields can combine either to Dirac or Majorana spinors or to chiral four-component spinors. These are the objects that we are actually interested in, and that appear in our calculation. Therefore, if we consider any **G**-irreducible representation $\mathcal{R}^{\mathcal{F}}$ of Weyl fermions that appears in our theory, the decomposition into $\prod_k \mathbf{G}_k$-irreducible representations à la eq. (5.6) looks like this:

$$
\mathcal{R}^{\mathcal{F}} \to \bigoplus_n \mathcal{R}^{\mathcal{F}}_n = \bigoplus_n \mathcal{R}^{\mathcal{D}_L}_n \oplus \bigoplus_n \mathcal{R}^{\mathcal{D}_R}_n \oplus \bigoplus_n \mathcal{R}^{\mathcal{M}}_n \oplus \bigoplus_n \mathcal{R}^{f_L}_n . \tag{5.36}
$$

where the symbols denote the left-handed Dirac component, the right-handed Dirac component, the Majorana field and the chiral massless field in this order. Not all terms on the right-hand side must be present for every representation $\mathcal{R}^{\mathcal{F}}$, because some **G**-irreducible representations may decompose just into Dirac spinor components for instance, as is the case for the **5** and $\overline{\mathbf{5}}$ Higgsino fields in the Minimal Supersymmetric SU(5) model. The projectors on these subspaces are denoted by $\varrho^{\mathcal{D}_L}_n$, $\varrho^{\mathcal{D}_R}_n$, $\varrho^{\mathcal{M}}_n$ and $p^{f_L}_n$, respectively.

As described in subsection 4.1.3, we are taking into account possible mixing matrices $\mathcal{Z}_L$, $\mathcal{Z}_R$ and $\mathcal{Z}_M$ for the heavy fermions. They are needed e.g. for describing the mixing between gauginos and Higgsinos in the 24-dimensional representation of the Minimal Supersymmetric SU(5) Model. We assume that mixing matrices commute with all unbroken generators because this is sufficient for the case of Minimal SUSY SU(5). Because of Schur's Lemma, this fact would imply that all mixing matrices are proportional to unity on each subspace irreducible under $\prod_k \mathbf{G}_k$, which would make them redundant. Therefore, in order to evade Schur's Lemma and have nontrivial mixings, the matrices must contain mixing of at least two $\prod_k \mathbf{G}_k$-irreducible subspaces, which is the case in the Minimal Supersymmetric SU(5) model. (There the mixing is between the $\mathbf{G}_{\text{SM}}$ representations $(\mathbf{3}, \overline{\mathbf{2}}, -\frac{5}{6})$ and $(\overline{\mathbf{3}}, \mathbf{2}, \frac{5}{6})$ in $\mathbf{24}$.) This again means that in case of a nontrivial mixing matrix on a subspace $\mathcal{R}_i^{\mathcal{D}}$ this particular subspace is reducible under $\prod_k \mathbf{G}_k$ and the respective projector $P_i^{\mathcal{D}}$ is only meaningfully defined on a reducible subspace. Therefore care is needed here when defining Casimir invariants and Dynkin indices for these subspaces: Using projectors on reducible subspaces and trying to define suitable invariants similar to eq. (5.9), we will observe that they will not necessarily be proportional to the unit matrix, as before, since they only commute with generators of a reducible representation. In such a case Schur's Lemma does not imply the existence of a Casimir invariant. Therefore, whenever mixing matrices occur in traces, we do not write them as an invariant, but leave the expression as it is.

In the following we list some important reduction identities which are derived using mainly the Yukawa invariance relation eq. (4.30).

$$
\begin{aligned}
\mathrm{Tr}(\varrho_j^x Y^n \varrho_k^y Y^{m\star})(\varrho_l^z T^{a_i} T^{a_i})_{nm} &= \mathrm{Tr}(\varrho_j^x Y^{n\star} \varrho_k^y Y^m T^{a_i} T^{a_i})(\varrho_l^z)_{nm} \\
&\quad + \mathrm{Tr}(\varrho_k^y Y^{n\star} \varrho_j^x Y^m T^{a_i} T^{a_i})(\varrho_l^z)_{nm} \\
&\quad + \mathrm{Tr}(\varrho_j^x Y^{n\star} \varrho_k^y T^{a_i\star} Y^m T^{a_i})(\varrho_l^z)_{nm} \\
&\quad + \mathrm{Tr}(\varrho_k^y Y^{n\star} \varrho_j^x T^{a_i\star} Y^m T^{a_i})(\varrho_l^z)_{nm}\,, \\
\mathrm{Tr}(\varrho_j^x T^{a_i} Y^{n\star} \varrho_k^y Y^m)(\varrho_l^z T^{a_i})_{nm} &= \mathrm{Tr}(\varrho_j^x Y^{n\star} \varrho_k^y Y^m T^{a_i} T^{a_i})(\varrho_l^z)_{nm} \\
&\quad + \mathrm{Tr}(\varrho_k^y Y^n \varrho_j^x T^{a_i} Y^{m\star} T^{a_i\star})(\varrho_l^z)_{nm}\,, \\
\mathrm{Tr}(\varrho_j^x T^{a_i} Y^n \varrho_k^y Y^{m\star})(\varrho_l^z T^{a_i})_{nm} &= -\mathrm{Tr}(\varrho_j^x Y^n \varrho_k^y Y^{m\star} T^{a_i} T^{a_i\star})(\varrho_l^z)_{nm} \\
&\quad - \mathrm{Tr}(\varrho_k^y Y^{n\star} \varrho_j^x T^{a_i} Y^m T^{a_i})(\varrho_l^z)_{nm}\,.
\end{aligned}
\tag{5.37}
$$

Here $x, y \in \{\mathcal{D}_L, \mathcal{D}_R, \mathcal{M}, f_L, \ldots\}$ and $z \in \{\mathcal{H}, \mathcal{S}_{\text{I}}, \mathcal{S}_{\text{II}}, \ldots\}$. The identities have been written in terms of the generators T^α that transform the Weyl components of the fermions. Depending on x and y, they will be either equal to t^α or $-t^{\alpha\star}$ (cf. subsection 4.1.3).

When computing the on-shell mass renormalization for Dirac and Majorana fermions, we want to check whether the gauge parameter dependence drops out explicitly. In order for this to happen, we use the Yukawa invariance relation, eq. (4.30), and the definition of the

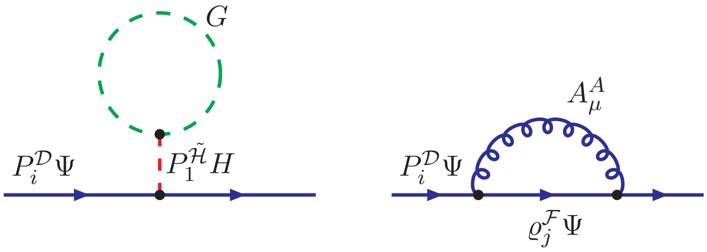


Figure 5.1.: *Relation eq. (5.38) guarantees the manifest cancellation of gauge parameter dependence in the divergent part of the on-shell mass renormalization between these two classes of diagrams. Goldstone bosons are marked green (light gray, short-dashed), and heavy gauge bosons and heavy Fermions blue (dark gray).*

fermion mass matrices, eq. (4.34), to derive the following nontrivial identity:

$$\begin{aligned}
C_2(P_1^{\tilde{\mathcal{H}}}, P^{\mathcal{G}}) &= \sum_j \Big[C_2(P_i^{\mathcal{D}_L}, \varrho_j^{\mathcal{F}}) + C_2(P_i^{\mathcal{D}_R}, \varrho_j^{\mathcal{F}})\Big] \\
&\quad -\frac{2}{M_{D_i}\mathrm{Tr}(P_i^{\mathcal{D}})} \sum_j \Big[\mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_L \mathcal{Z}_R^T t^A \varrho_j^{\mathcal{D}} \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger t^A) M_{D_j} \\
&\qquad\qquad + \mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_L \mathcal{Z}_R^T t^A \varrho_j^{\mathcal{M}} \mathcal{Z}_M^\star \mathcal{Z}_M^\dagger t^A) M_{M_j}\Big] \\
&= \sum_j \Big[C_2(P_i^{\mathcal{D}_L}, \varrho_j^{\mathcal{F}}) + C_2(P_i^{\mathcal{D}_R}, \varrho_j^{\mathcal{F}})\Big] \\
&\quad -\frac{2}{M_{D_i}\mathrm{Tr}(P_i^{\mathcal{D}})} \sum_j \Big[\mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger t^A \varrho_j^{\mathcal{D}} \mathcal{Z}_L \mathcal{Z}_R^T t^A) M_{D_j} \\
&\qquad\qquad + \mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger t^A \varrho_j^{\mathcal{M}} \mathcal{Z}_M \mathcal{Z}_M^T t^A) M_{M_j}\Big]
\end{aligned} \tag{5.38}$$

It relates the color factor of the tadpole with a Goldstone boson loop to the color factor of the self energy with a longitudinal heavy gauge boson (cf. fig. 5.1). The summation is over all $\prod_k \mathbf{G}_k$-irreducible representations that share a $\mathbf{G}$-irreducible representation with the Dirac fermion in $\mathcal{R}_i^D$. Note, however, that the right-hand side will be independent of i. The Casimir invariants that appear here are also defined in terms of the t^α. For a derivation of eq. (5.38), please refer to subsection A.1.3 in the appendix.

Another family of relations that is important for the renormalization of heavy fermion masses is illustrated here by two examples:

$$\begin{aligned}
\mathrm{Tr}(P_i^{\mathcal{D}_L} t^{a_l} t^{a_l} Y^{n\star} P_j^{\mathcal{D}_R} Y^m)(P_k^{\mathcal{G}})_{nm} &= \frac{g^2}{M_{\mathrm{X_k}}}\Big\{ M_{D_i}^2 \mathrm{Tr}(P_i^{\mathcal{D}_R}\, t^{a_l} t^{a_l} t^{A_k} P_j^{\mathcal{D}_R}\, t^{A_k}) \\
&+ M_{D_j}^2 \mathrm{Tr}(P_i^{\mathcal{D}_L}\, t^{a_l} t^{a_l} t^{A_k} P_j^{\mathcal{D}_L}\, t^{A_k}) \\
&- M_{D_i} M_{D_j} \mathrm{Tr}(P_i^{\mathcal{D}}\, t^{a_l} t^{a_l}\, \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger t^{A_k} P_j^{\mathcal{D}}\, \mathcal{Z}_L \mathcal{Z}_R^T t^{A_k}) \\
&- M_{D_i} M_{D_j} \mathrm{Tr}(P_i^{\mathcal{D}}\, t^{a_l} t^{a_l}\, \mathcal{Z}_L \mathcal{Z}_R^T t^{A_k} P_j^{\mathcal{D}}\, \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger t^{A_k})\Big\} \\
\mathrm{Tr}(P_i^{\mathcal{D}} t^{a_l} t^{a_l} \mathcal{Z}_R^T Y^n P_j^{\mathcal{D}} \mathcal{Z}_L \mathcal{Z}_R^T Y^m \mathcal{Z}_L)(P_k^{\mathcal{G}})_{nm} &= \frac{g^2}{M_{\mathrm{X_k}}}\Big\{ - M_{D_i} M_{D_j} \mathrm{Tr}(P_i^{\mathcal{D}_R}\, t^{a_l} t^{a_l} t^{A_k} P_j^{\mathcal{D}_R}\, t^{A_k}) \\
&- M_{D_i} M_{D_j} \mathrm{Tr}(P_i^{\mathcal{D}_L}\, t^{a_l} t^{a_l} t^{A_k} P_j^{\mathcal{D}_L}\, t^{A_k}) \\
&+ M_{D_i}^2 \mathrm{Tr}(P_i^{\mathcal{D}}\, t^{a_l} t^{a_l}\, \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger t^{A_k} P_j^{\mathcal{D}}\, \mathcal{Z}_L \mathcal{Z}_R^T t^{A_k}) \\
&+ M_{D_i}^2 \mathrm{Tr}(P_i^{\mathcal{D}}\, t^{a_l} t^{a_l}\, \mathcal{Z}_L \mathcal{Z}_R^T t^{A_k} P_j^{\mathcal{D}}\, \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger t^{A_k})\Big\}
\end{aligned} \tag{5.39}$$

Similar relations are obtained by exchanging $\mathcal{Z}_L$, $\mathcal{Z}_R$ and $\mathcal{Z}_M$ and building all possible combinations of Y^k and $Y^{k\star}$. Though our actual full calculation of $\zeta_{\alpha_i}(\mu_{\mathrm{GUT}})$ did not involve heavy fermions yet (cf. section 4.3), we have calculated the on-shell mass renormalization for them using the relations in this section and verified that the gauge parameter dependence drops out.

The list of reduction identities may not be exhaustive, but it contains the most important relations and shows how they can be derived in principle. In a similar fashion also other identities can be derived using the invariance relations eqs. (4.19) and (4.30) for the invariant tensors.

5.6. Automation of the Color Factor Reduction

In the following we briefly sketch the algorithm that is used for the reduction of all the color factors in the diagrams: we have written a `FORM` routine that treats the color factors for each individual diagram and reduces them to a basic set of invariants. In a first step, after all color factors have been brought into a suitable symbolic notation, all reduction identities that involve the quartic scalar coupling λ_{ijkl} and κ_{ijk} are applied to a given expression (eqs. (5.33), (5.34) and (5.35)). After that any expression will contain traces of strings of generators T^{a_i}, T^{A_i}, projectors ϱ_n^x, Yukawa matrices Y^n and mixing matrices $\mathcal{Z}_L, \mathcal{Z}_R, \mathcal{Z}_M$. The adjoint indices a_i and A_i are contracted either mutually or with some structure constants. First all the contracted adjoint indices are removed, i.e. traces of the form

$$\mathrm{Tr}(\ldots T^{a_i} \ldots T^{a_i} \ldots), \qquad \mathrm{Tr}(\ldots T^{A_i} \ldots T^{A_i} \ldots), \tag{5.40}$$

by applying the definitions of the quadratic Casimir invariants eq. (5.9). If two generators with mutually contracted indices are not next to each other, we commute them until they are and apply definitions of the quadratic Casimir invariants then. Eventually we arrive at traces that contain no more contracted indices. Next, expressions of the form

$$f^{c_i a_i b_i}\, \mathrm{Tr}(\ldots T^{a_i} \ldots T^{b_i} \ldots), \qquad f^{\alpha A_i B_i}\, \mathrm{Tr}(\ldots T^{A_i} \ldots T^{B_i} \ldots), \tag{5.41}$$

are reduced by using

$$f^{c_i a_i b_i} T^{a_i} T^{b_i} = \frac{1}{2} f^{c_i a_i b_i} \left[T^{a_i}, T^{b_i}\right] = \frac{i}{2} I_2(p_i^{\mathcal{A}})^i \, T^{c_i} \tag{5.42}$$

and eq. (5.15). Again generators that are not next to each other are commuted. Expression that contain the Goldstone projector $P_i^{\mathcal{G}}$ are treated separately. We insert the explicit form of $P_i^{\mathcal{G}}$ (eq. (5.23)) and write the traces in the form $v \ldots v$, where "…" stands for a string of generators T^{a_i}, T^{A_i}, and projectors ϱ_n^x. Here we additionally make use of the relations $T^{a_j} v = 0$ and $P_i^{\tilde{\mathcal{H}}} v = 0$ (for $i \neq 1$) in order to eliminate all generators inside the string that have a lowercase adjoint index. Afterwards, the same procedure as for traces is applied.

In the next step all color factors involving the Yukawa matrix Y^n are reduced to a basic set of invariants using eqs. (5.37), (5.38) and (5.39). Finally, we are left with various contractions of structure constants which are expressed by the respective invariants using eqs. (5.17) and (5.21). Lastly, some cosmetic manipulations are applied to the expression in order to make the notation of the final result as convenient as possible. The actual program is slightly more complicated than described above and one needs to introduce repetitive control structures because not all the reduction can be done by a single run. However, the basic procedure is along the lines described here. It is used at the final stage of the computation of each diagram.

6. Two-Loop Matching for the Georgi-Glashow SU(5) Model

The previous two chapters have described the details of our theoretical framework, both from the field theory perspective and from the group theory perspective. We also explained how the calculation of the two-loop matching corrections, $\zeta_{\alpha_i}(\mu_{\mathrm{GUT}})$, has been carried out in that general framework. In this chapter we want to apply our result to the simplest possible GUT, the Georgi-Glashow model [8] (cf. section 2.2). Although this theory is ruled out experimentally [12], we use it as a toy model to demonstrate some simple numerics and to check whether the result exhibits the expected properties. The goal for the future is, of course, to extend the calculation to supersymmetric SU(5) models and other realistic GUT models.

6.1. Specification to the Model

In order to specify our general calculation to this particular model, we must calculate the numerical values of all quadratic Casimir invariants and Dynkin indices that appear in the final result. This is done with `Mathematica` using explicit implementations of the group generators. They can be found e.g. in appendix B of ref. [62]. Furthermore, it is necessary to specify the quartic scalar coupling λ_{ijkl} (cf. eq. (4.18)) and the general Yukawa matrix Y^n_{ij} (cf. eq. (4.29)) such that the scalar potential and the Yukawa interactions of the model comes out. The contractions of these quantities that appear in the final result have to be calculated also and replaced in the general result. The other terms of the Lagrangian, eq. (4.1), are fixed solely by the particle content of the theory.

The explicit parametrization of the quartic scalar coupling λ_{ijkl} that leads to the scalar potential, eq. (2.10), of the Georgi-Glashow model is given in the following. It splits up into three parts. The first part is a quartic coupling of the $\mathbf{24}_H$ Higgs, the second one a quartic coupling of the $\mathbf{5}_H$ Higgs and the last one is a mixed $\mathbf{5}_H - \mathbf{24}_H$ coupling:

$$\begin{aligned}
\lambda^{24}_{\alpha\beta\gamma\delta} &= A\,\mathrm{sTr}(T^\alpha T^\beta T^\gamma T^\delta) + \frac{1}{3}\,B\,(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\,,\\
\lambda^{5}_{ijkl} &= \frac{1}{3}\,b\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,, \quad i,j,k,l = 1,\ldots,10\,,\\
\lambda^{5-24}_{\alpha\beta ij} &= c\,(\tau^\alpha\tau^\beta + \tau^\beta\tau^\alpha)_{ij}\,, \qquad \alpha,\beta,\gamma,\delta = 1,\ldots,24\,.
\end{aligned} \tag{6.1}$$

We have used the symmetrized trace

$$\mathrm{sTr}(T^{\alpha_1}\ldots T^{\alpha_n}) \equiv \frac{1}{n!}\sum_{\pi}\mathrm{Tr}(T^{\alpha_{\pi(1)}}\ldots T^{\alpha_{\pi(n)}}) \tag{6.2}$$

where the sum is over all the permutations of the indices. Furthermore, note that the indices i, j, k, l run from 1 to 10 although they belong to the fundamental $\mathbf{5}$ representation. This is because the corresponding scalar is complex and we have written it as twice as many real scalars that are transformed by the 10×10 generator matrices τ (cf. section A.3 in the appendix):

$$\tau^\alpha = \begin{pmatrix} i\,\mathrm{Im}(T^\alpha) & i\,\mathrm{Re}(T^\alpha) \\ -i\,\mathrm{Re}(T^\alpha) & i\,\mathrm{Im}(T^\alpha) \end{pmatrix}, \tag{6.3}$$

where in the present section T^α is the 5×5 generator matrix in the fundamental representation. Inserting eq. (6.1) into eqs. (4.26) and (4.27), we obtain the scalar mass matrices. Additionally, we need to impose the tree-level fine-tuning condition $\mu_{\mathbf{5}}^2 = \frac{3}{20} c v^2$, where $\mu_{\mathbf{5}}^2$ is the quadratic term of the $\mathbf{5}_H$ in eq. (4.18), in order to obtain massless Higgs doublets. Note that in principle one would need to calculate the one-loop fine-tuning condition in order to obtain massless Higgs doublets in a two-loop calculation. However, the light Higgs doublets show up the first time in the two-loop Green's functions in the matching calculation so that it is sufficient to use the tree-level fine-tuning condition. Using this parametrization, we can calculate the physical mass parameters, eq. (2.11).

The Yukawa interactions of the Georgi-Glashow model, eq. (2.7), are obtained by inserting the Yukawa matrix

$$Y^n_{sr} = \begin{pmatrix} -Y^U_{IJ}\epsilon_{ijklm}\, T^\alpha_{ij} T^\beta_{kl} S^\star_{mn} & 2i\, Y^D_{IJ}\,\mathrm{Im}(T^\alpha_{kl}) S_{ln} \\ 2i\, Y^D_{IJ}\,\mathrm{Im}(T^\alpha_{jl}) S_{ln} & 0 \end{pmatrix}_{sr} \tag{6.4}$$

into the general Yukawa Lagrangian, eq. (4.29). Here $s = (I, \tilde{s})$ and $r = (J, \tilde{r})$ are multi-indices, where I, J stand for the flavor indices of the SU(5) Yukawa matrices Y^U and Y^D. The indices $\tilde{s}, \tilde{r} = 1, \ldots, 29$ run over $\{\alpha, j\}$ and $\{\beta, k\}$, respectively. Note that we have written the fermions of the $\mathbf{10}$ representation as a 24-dimensional vector instead of an antisymmetric 5×5 matrix as usually. The Clebsch-Gordan coefficients for this transformation are given by the following equations:

$$\mathbf{10}^\alpha = \sqrt{2}\, T^\alpha_{ij}\, \mathbf{10}_{ij}\,, \quad \mathbf{10}_{ij} = -\sqrt{2}\, i\,\mathrm{Im}(T^\alpha_{ij})\, \mathbf{10}^\alpha\,, \tag{6.5}$$

where $\mathbf{10}_{ij}$ is the usual antisymmetric 5×5 matrix with the normalization as in eq. (2.4). The generator matrices that act on $\mathbf{10}^\alpha$ can then conveniently be rewritten as follows:

$$(T^\gamma_{\mathbf{10}})_{\alpha\beta} = \mathrm{Tr}(T^{\gamma\star} T^\alpha T^\beta) + \mathrm{Tr}(T^\gamma T^\beta T^\alpha) - \mathrm{Tr}(T^\gamma T^\beta T^{\alpha\star}) - \mathrm{Tr}(T^\gamma T^{\beta\star} T^\alpha)\,. \tag{6.6}$$

Furthermore, $S = \frac{1}{\sqrt{2}}(\mathbb{1}, i\mathbb{1})$ in eq. (6.4) is a 5×10 matrix and ϵ_{ijklm} is the totally antisymmetric tensor with $\epsilon_{12345} = 1$. As can be seen from eq. (6.4), the Weyl fermion multiplet ψ from section 4.1 is written as a $3\,(24 + 5) = 87$-dimensional vector for the case of Georgi-Glashow SU(5) model.

Using the definitions from this section, we computed the numerical values of all the group theory factors that appear in our general result. Furthermore, we set $V_{\mathrm{CKM}} = \mathbb{1}$ and kept only the third generation Yukawa couplings y_t and y_b. We obtained three two-loop formulae for ζ_{α_i} $(i = 1, 2, 3)$ that depend on the parameters

$$\alpha(\mu_{\mathrm{GUT}}),\ y_t(\mu_{\mathrm{GUT}}),\ y_b(\mu_{\mathrm{GUT}}),\ M_{\mathrm{X}},\ M_{\mathrm{H_c}},\ M_\Sigma,\ M_{24},\ \mu_{\mathrm{GUT}}\,. \tag{6.7}$$

where all the masses are renormalized on-shell and α, y_t and y_b in $\overline{\text{MS}}$. The resulting formulae are to long to be presented here, therefore the `Mathematica` package that contains the expressions can be downloaded from

`http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp10/ttp10-46/`

6.2. Running and Decoupling Setup

In order to examine the numerical impact of the two-loop matching corrections in this model, we have implemented a RGE analysis in `Mathematica`. Since in this model the gauge couplings do not unify, we just focus on examining the reduction of the decoupling scale dependence, as an illustration of our results. We start with the precise values of the three gauge couplings at the electroweak scale. In section 3.1 we have described how to obtain them from known experimental and theoretical input. For convenience we repeat our starting values here:

$$\begin{aligned}
\alpha_{em}^{(6),\overline{\text{MS}}}(M_Z) &= 1/(128.129 \pm 0.021)\,, \\
\sin^2\Theta^{(6),\overline{\text{MS}}}(M_Z) &= 0.23138 \pm 0.00014\,, \\
\alpha_s^{(6)}(M_Z) &= 0.1173 \pm 0.0020\,.
\end{aligned} \tag{6.8}$$

These quantities are related to the three gauge couplings via

$$\begin{aligned}
\alpha_1 &= \frac{5}{3}\frac{\alpha_{em}^{(6),\overline{\text{MS}}}}{\cos^2\Theta^{(6),\overline{\text{MS}}}}\,, \\
\alpha_2 &= \frac{\alpha_{em}^{(6),\overline{\text{MS}}}}{\sin^2\Theta^{(6),\overline{\text{MS}}}}\,, \\
\alpha_3 &= \alpha_s^{(6)}\,.
\end{aligned} \tag{6.9}$$

which holds for any renormalization scale μ. Furthermore, we also need the values of the Yukawa couplings at the electroweak scale. How to obtain them, has also been described in section 3.1.

The RG running in the SM was implemented at two loops [27, 28, 114, 139] for the electroweak sector and at three loops [112, 113] for QCD. We take into account the tau, bottom and top Yukawa couplings and thus solve the coupled system of six differential equations. Since the quartic SM Higgs coupling b enters the equations of the Yukawa couplings starting from two-loop order only, we neglect its contribution. After taking into account the two-loop decoupling relations, we compute the running from μ_{GUT} to the Planck scale using three-loop RGEs for the gauge coupling and one-loop RGEs for the Yukawa couplings. The RGEs are obtained by inserting the general expressions for the Yukawa and scalar couplings (eqs. (6.4) and (6.1)) as well as the numerical values for the group theory factors into the general formulae of refs. [25, 28] (see appendix A.6 for the details).

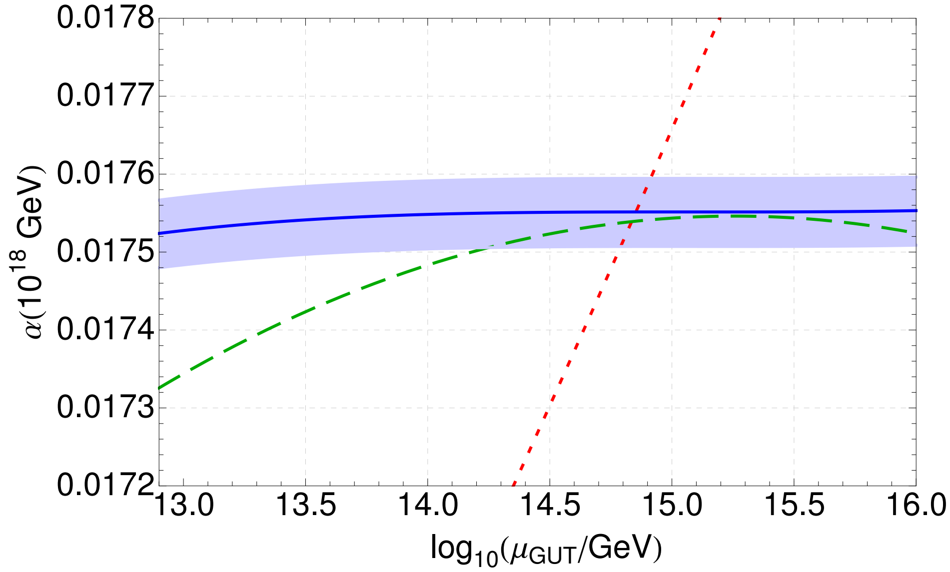


Figure 6.1.: *Dependence of $\alpha(10^{18}$ GeV$)$ on the decoupling scale μ_{GUT}. The red (dotted), green (dashed) and blue (solid) lines correspond to the one-, two- and three-loop analysis, respectively. For the three-loop curve also the impact of the uncertainty on $\alpha_3(M_Z)$ with $\delta\alpha_s = 0.0020$ has been indicated.*

6.3. Dependence on the Decoupling Scale

In fig. 6.1 the dependence on the decoupling scale of $\alpha(10^{18}\mathrm{GeV})$ in the bottom-up approach is shown. Since only for QCD the full three-loop β function could be implemented and there is no unification of gauge couplings anyway, we took $\alpha(\mu_{\mathrm{GUT}}) = \zeta_{\alpha_3}^{-1}(\mu_{\mathrm{GUT}})\ \alpha_3(\mu_{\mathrm{GUT}})$ as a starting value for the gauge coupling above the GUT scale. For illustration we use the following set of mass parameters:

$$
\begin{aligned}
M_{\mathrm{X}} &= 10^{15}\ \mathrm{GeV}\,, \\
M_{\mathrm{H_c}} &= 4\cdot 10^{13}\ \mathrm{GeV}\,, \\
M_{\Sigma} &= 10^{14}\ \mathrm{GeV}\,, \\
M_{24} &= 6\cdot 10^{13}\ \mathrm{GeV}
\end{aligned}
\tag{6.10}
$$

which are chosen to obey the restriction $M_{\mathrm{X}} \gtrsim M_i$ for $i = \mathrm{H_c}, \Sigma, 24$. Otherwise the scalar self-couplings easily become non-perturbative and blow up the gauge coupling above the GUT scale. The scale dependence is shown for n-loop running and $(n-1)$-loop decoupling with $n = 1, 2, 3$. We observe a dramatic improvement when going from $n = 1$ to $n = 2$, as well as when going from $n = 2$ to $n = 3$. In particular the three-loop corrections can be larger than the error band depending on μ_{GUT}. Note also that for $n = 2$ choosing μ_{GUT} naively as a mean value of the GUT masses, which would be of $\mathcal{O}(10^{14}$ GeV$)$ in our case, is not a good choice. The described qualitative behavior does not depend much on our choice of the GUT masses. Though the numerical effect of the two-loop matching is already significant in the Georgi-Glashow model, we emphasize that in certain models that contain large representations, as e.g. the Missing Doublet Model [63,64], we expect these corrections to be even larger [35] (cf.

also subsection 3.2.4). Our goal for the future, of course, is to generalize the formula for ζ_{α_i} to make it applicable to these models.

To provide a check of the result for the Georgi-Glashow model, we have verified analytically that the matching coefficients $\zeta_{\alpha_i}(\mu_{\rm GUT})$ exhibit the correct $\mu_{\rm GUT}$ dependence. This can be derived from the knowledge of the two-loop β functions of the SM and the SU(5) model by computing the derivative w.r.t. $t_{\rm GUT} \equiv \ln(\mu_{\rm GUT})$ of eq. (2.25). Solving the resulting differential equation order by order, we arrive at a general formula[1] for the $\mu_{\rm GUT}$ -dependent terms in $\zeta_{\alpha_i}(\mu_{\rm GUT})$:

$$\begin{aligned}\zeta_{\alpha_i}(\mu_{\rm GUT}) &= 1 + \frac{\alpha(\mu_{\rm GUT})}{\pi}\Big[\tfrac{1}{2}(\beta_0^i - \beta_0)\, t_{\rm GUT} - C_0(M_{\rm h})\Big] \qquad (6.11)\\ &+ \left(\frac{\alpha(\mu_{\rm GUT})}{\pi}\right)^2 \left[\tfrac{1}{4}(\beta_0^i - \beta_0)^2\, t_{\rm GUT}^2 + \Big[\tfrac{1}{8}(\Sigma_j \beta_1^{ij} - \beta_1) - C_0(M_{\rm h})\,(\beta_0^i - \beta_0)\Big]\, t_{\rm GUT} + C_1(M_{\rm h})\right].\end{aligned}$$

C_0 and C_1 are $\mu_{\rm GUT}$-independent terms that depend only on the heavy GUT masses. Recall that the β function coefficients are defined by:

$$\frac{1}{2}\frac{d}{dt}\frac{\alpha}{4\pi} \;=\; \sum_{k=0}^{N-1}\left(\frac{\alpha}{4\pi}\right)^{k+2}\beta_k\,, \qquad \alpha = \frac{g^2}{4\pi} \qquad (6.12)$$

and similarly for α_i. We find agreement in the $\mu_{\rm GUT}$ dependence of our explicit calculation with the form of eq. (6.11).

6.4. Summary

We have applied our result for the two-loop matching corrections to the simples GUT, the Georgi-Glashow model, and examined the numerical impact on $\alpha(10^{18}\text{GeV})$. Depending on the choice of the decoupling scale $\mu_{\rm GUT}$ this impact can be larger than the uncertainty on the input value $\alpha_s(M_Z)$. We have checked that the result exhibits the correct $\mu_{\rm GUT}$ dependence which is predicted by the two-loop gauge β functions of the full and effective theory. To the author's knowledge this is the first two-loop matching calculation at the GUT scale.

In order to apply our calculation to phenomenologically more interesting models, one needs to relax the assumptions named in section 4.3. For example, in order to extend the calculation to Minimal SUSY SU(5), the following changes compared to the current calculation have to be carried out:

- Consider the contributions from the trilinear term in the scalar potential, eq. (4.18).
- Add two massive Dirac and three massive Majorana fermion $\mathbf{G}_{\rm SM}$-irreducible representations as well as all possible kinds of Yukawa interactions for them. Mixing and phase matrices $\mathcal{Z}_L$, $\mathcal{Z}_R$ and $\mathcal{Z}_M$ have to be taken into account.

[1] See section A.8 for the detailed derivation. For simplicity we neglect the Yukawa corrections in this formula. However, the generalization is straightforward. Of course, in our analytical check we took care of them too.

- Increase the number of $\mathbf{G}_{\mathrm{SM}}$-irreducible representations within $\mathcal{R}^{\mathcal{S}}$ by four in order to account for the complexity of $\mathbf{24}_H$ as compared to non-SUSY SU(5).
- Convert the result to the $\overline{\mathrm{DR}}$ renormalization scheme.

However, let us stress, that the theoretical framework to carry out these generalizations has already been described in detail in chapters 4 and 5. In fact, we even have undertaken the first steps towards these generalization. For example we have calculated all the necessary one-loop renormalization constants and have confirmed that the on-shell mass renormalization constants are gauge parameter independent. Due to the additional fields, the number of two-loop diagrams blows up significantly and is expected to be of $\mathcal{O}(50000)$ in total.

It is also noteworthy that the specification of the Lagrangian to minimal SUSY SU(5) is expected to be much more involved than in the case of the Georgi-Glashow model. The scalar potential alone is much larger in Minimal SUSY SU(5) because of additional fields in the theory and scalar superpartners.

Summing up, by applying the the two-loop matching corrections at the GUT scale to the Georgi-Glashow model, we have laid a solid ground for extended calculations for more realistic models.

7. Conclusions and Outlook

The unification of gauge couplings is one of the crucial predictions of GUTs and therefore worth studying in detail. In this work we have undertaken first steps towards a consistent three-loop gauge coupling unification analysis with a focus on threshold corrections at the GUT scale. These threshold corrections can pose constraints on the colored triplet Higgs mass $M_{\mathrm{H_c}}$ that typically appears in GUTs. In combination with constraints on this mass coming from proton decay experiments, this can serve as a means to test SUSY GUT models.

In chapter 3 we have performed an (almost) three-loop gauge coupling unification analysis for the Minimal SUSY SU(5) and the Missing Doublet Model using state-of-the-art theoretical and experimental input. A consistent three-loop analysis would require three-loop running and two-loop decoupling at the SUSY and the GUT thresholds. In order to make our analysis fully consistent, the following additional theoretical input would have been needed: Three-loop electroweak corrections to the SM RGEs, two-loop threshold corrections to $\alpha_{1,2}(\mu_{\mathrm{SUSY}})$ and two-loop threshold corrections for $\alpha_{1,2,3}(\mu_{\mathrm{GUT}})$. For the case of Minimal SUSY SU(5), we argued that these missing pieces are not expected to change the outcome much. We have found that for the choice $\mu_{\mathrm{SUSY}} = M_Z$, which is often adopted in the literature, the prediction for $M_{\mathrm{H_c}}$ is raised by one order of magnitude by three-loop effects attenuating presumed tension with proton decay experiments. Therefore we have confirmed at the three-loop level that the non-renormalizable version of Minimal SUSY SU(5) cannot be ruled out with current experimental data.

In the case of the Missing Doublet Model, however, we have shown that the yet missing two-loop decoupling effects at the GUT scale are expected to be huge. This is because in the current setup the theoretical uncertainty of $\alpha_{1,2,3}(M_Z)$ due to the variation of μ_{GUT} exceeds the experimental uncertainty on these values by one order of magnitude. Since this behavior is typical of models with large representations, we are strongly motivated to compute these missing two-loop threshold effects at the GUT scale which was the subject of the rest of this thesis.

In chapters 4 and 5 we have introduced the field- and group-theoretical framework that has been used to carry out the two-loop matching calculation at the GUT scale. This framework is set up in a very general way and is therefore applicable to a wide range of GUT models. We have considered all parts of the Lagrangian and described how subtle tasks as gauge fixing, ghost interactions, tadpole terms and renormalization procedure are carried out in a proper way. Furthermore, chapter 5 introduces a notational framework that is suitable to treat the group theory factors that come up in the calculation. Moreover, it contains a wealth of nontrivial reduction identities in orderto allow for cancellations of e.g. gauge parameter dependent terms between different diagrams without inserting the actual numerical values for the group theory invariants. Therefore, these two chapters may be considered as a useful resource for future calculations of this kind.

Due to the complexity of the calculation, in a first step, we have imposed some additional assumptions about the underlying GUT model which are designed to match the simplest GUT, the Georgi-Glashow SU(5) Model. Though this model is ruled out experimentally, we have employed it as a useful toy model. The assumptions that have been made in the calculation have been described in section 4.3. As a first application, we have presented the numerical analysis using the two-loop matching corrections in the Georgi-Glashow Model in chapter 6. We have shown that in that model these matching corrections can be larger than the current uncertainty on $\alpha_s(M_Z)$. The checks that have been made to ensure the correctness of the result are also summarized in section 4.3. By this calculation we have laid a solid ground for future calculations of the two-loop matching effects in more complicated GUT models.

Looking into the future, the next logical step is to extend the calculation to Minimal SUSY SU(5). Several steps in this direction, as the formulation of the theoretical framework and the implementation of heavy fermions, have already been undertaken in the present work. A full list of steps to be considered for this project have been summarized in section 6.4. In this context it is certainly worthwhile also to explore whether supergraphs [47, 151–153] can be employed for a more efficient calculation of ζ_{α_i} in SUSY GUTs.

Looking even further into the future, most interesting results are expected for models with large representations, as e.g. the Missing Doublet model (cf. subsection 3.2.4). But also realistic non-supersymmetric SU(5) models as e.g. the model with an additional $\mathbf{15}_H$ Higgs field [154] might be of interest in this context. To extend the calculation to SO(10) scenarios, most probably additional changes in our theoretical framework have to be taken into account: Since it is not possible to break SO(10) to $\mathbf{G}_{\text{SM}}$ in a single step, we need to have two vevs of order M_{GUT} in our setup. It is not a priori obvious whether this will complicate our general treatment of the renormalization of the scalar potential (cf. section 4.1.2) significantly.

Of course, it would be desirable to finally extend the analysis to realistic SUSY GUT scenarios that have predictive power also in the flavor sector. As an example, let us mention the SUSY SO(10) model proposed by Chang, Masiero and Murayama [155] in 2002. Subsequent analyses of various flavor observables [156–159] confirmed the model as an interesting alternative to popular minimal flavor violation scenarios.

Summing up, this work has contributed to establish a consistent three-loop gauge coupling unification analysis for various SU(5) GUT models. In particular, we have performed a state-of-the-art (almost) three-loop RGE analysis for Minimal SUSY SU(5) and the Missing Doublet Model. Furthermore, in the main part of this thesis we have described a first calculation of the two-loop matching corrections for the gauge couplings at the GUT scale and applied our result to the Georgi-Glashow model.

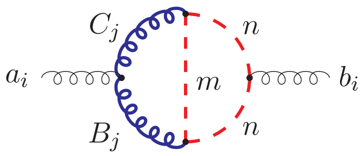
A. Appendix

A.1. Example Derivations of Group Theory Reduction Identities

This appendix is devoted to the illustrative derivation of a few reduction identities for color factors[1] that have been presented in chapter 5. It does by far not contain derivations for all identities that appear there, however, there is supposed to be an example for each class of identities such that the most important concepts in such derivations are still conveyed.

A.1.1. Derivation of Eq. (5.15)

As an example of the appearance of this particular color factor consider the following two-loop diagram, that is encountered in the calculation of $\zeta_{\alpha_1}(\mu_{\mathrm{GUT}})$:



$$\sim f^{a_iB_jC_j}\ \mathrm{Tr}(\varrho_n^x T^{B_j}\varrho_m^x T^{C_j}T^{b_i}) \equiv G^{a_ib_i}_{nm}\,.$$

The internal fields are heavy gauge bosons and scalars which reside not necessarily in the same irreducible representation. (In the above graph $x = \mathcal{S}$.) Note that in the absence of the projectors $\varrho^x_{n,m}$, we could easily compute the color factor by exploiting

$$f^{a_iB_jC_j}T^{B_j}T^{C_j} = \frac{1}{2}f^{a_iB_jC_j}[T^{B_j},T^{C_j}] = \frac{i}{2}f^{a_iB_jC_j}f^{\alpha B_jC_j}T^{\alpha} = \frac{i}{2}I_2(P_j^{\mathcal{A}})^i\,T^{a_i}\,, \tag{A.1}$$

where we have used the definition of the adjoint invariant, eq. (5.17). However, if we symmetrize the left-hand side of $G^{a_ib_i}_{nm}$ in n and m, something similar is possible. We arrive at this symmetrization by commuting T^{C_j} with T^{b_i} in $G^{a_ib_i}_{nm}$ and then taking the transpose of one of the traces that arises. This yields:

$$G^{a_ib_i}_{nm} - G^{a_ib_i\star}_{mn} = I_2(P_j^{\mathcal{A}})^i I_2(\varrho_n^x,\varrho_m)^j\,. \tag{A.2}$$

We have used $T^{\alpha T} = T^{\alpha\star}$, eqs. (5.9) and (5.17) as well as the fact that the structure constants are real. Moreover, the real part of $G^{a_ib_i}_{n,m}$ vanishes:

$$\begin{aligned}\mathrm{Re}(G^{a_ib_i}_{n,m}) \quad &= \quad \frac{1}{2}f^{a_iB_jC_j}\Big[\mathrm{Tr}(\varrho_n^x T^{B_j}\varrho_m^x T^{C_j}T^{b_i}) + \mathrm{Tr}(\varrho_n^x T^{B_j}\varrho_m^x T^{C_j}T^{b_i})^\star\Big] \\ &\quad\;\; \frac{1}{2}f^{a_iB_jC_j}\Big[\mathrm{Tr}(\varrho_n^x T^{B_j}\varrho_m^x T^{C_j}T^{b_i}) + \mathrm{Tr}(\varrho_n^x T^{C_j}\varrho_m^x T^{B_j}T^{b_i})\Big] = 0\end{aligned} \tag{A.3}$$

[1]For convenience we again adopt the terminology of chapter 5 and will speak of color factors instead of group theory factors etc..

We have again transposed the second trace and exploited the hermiticity of generators, as well as $[\varrho_n^x, T^{b_i}] = 0$. Eq. (A.3) gives zero because the structure constants are antisymmetric in B_j, C_j while the expression in parentheses is symmetric. Now that we know that $G_{n,m}^{a_i b_i}$ is purely imaginary, eq. (A.2) yields the symmetrization in n, m:

$$G_{nm}^{a_i b_i} + G_{mn}^{a_i b_i} = \delta^{a_i b_i} I_2(P_j^{\mathcal{A}})^i I_2(\varrho_n^x, \varrho_m)^j \,. \tag{A.4}$$

If we additionally can compute the antisymmetrized sum of $G_{nm}^{a_i b_i}$ and $G_{mn}^{a_i b_i}$, a proper linear combination of the two will give us an expression for $G_{nm}^{a_i b_i}$. This antisymmetrization is computed as follows: We start with the identity

$$\mathrm{Tr}(T^{a_i} T^{B_j} \varrho_n^x T^{b_i} T^{B_j} \varrho_m^x) \;=\; \mathrm{Tr}(T^{b_i} T^{B_j} \varrho_m^x T^{a_i} T^{B_j} \varrho_n^x) \tag{A.5}$$

which follows from the cyclicity of the trace. We now commute the left-most T^{a_i}, T^{b_i} with T^{B_j} on both sides:

$$\begin{aligned} \mathrm{Tr}(T^{B_j} T^{a_i} \varrho_n^x T^{b_i} T^{B_j} \varrho_m^x) + f^{a_i B_j C_j}\ \mathrm{Tr}(\varrho_n^x T^{B_j} \varrho_m^x T^{C_j} T^{b_i}) \;&=\; \\ \mathrm{Tr}(T^{B_j} T^{b_i} \varrho_m^x T^{a_i} T^{B_j} \varrho_n^x) + f^{b_i B_j C_j}\ \mathrm{Tr}(\varrho_n^x T^{B_j} \varrho_m^x T^{C_j} T^{a_i}) \,. \end{aligned} \tag{A.6}$$

Here we can identify $G_{nm}^{a_i b_i}$ and again apply the definitions of color invariants, eq. (5.9) for contracted generators that are side by side. Furthermore, we note that the matrix $A_{a_i b_i} \equiv G_{nm}^{a_i b_i}$ commutes with all unbroken generators $(T_{\mathcal{A}}^{C_i})_{a_i b_i}$ of the adjoint representation. Therefore, due to Schur's Lemma, it is diagonal and proportional to the unit matrix on each $\prod_i \mathbf{G}_i$-irreducible subspace. Particularly, $A_{a_i b_i}$ and thus $G_{nm}^{a_i b_i}$ is symmetric in a_i and b_i. Exploiting all this, eq. (A.6) can be rewritten as:

$$G_{nm}^{a_i b_i} - G_{mn}^{a_i b_i} \;=\; \delta^{a_i b_i}\Big[C_2(\varrho_n^x, \varrho_m^x)^j I_2(\varrho_n^x)^i - C_2(\varrho_m^x, \varrho_n^x)^j I_2(\varrho_m^x)^i\Big] \,. \tag{A.7}$$

This is the antisymmetrization of $G_{nm}^{a_i b_i}$ we have been looking for. Taking a half times the sum of eq. (A.4) and eq. (A.7), we obtain our final result

$$\begin{aligned} G_{nm}^{a_i b_i} \;&=\; f^{a_i B_j C_j}\ \mathrm{Tr}(\varrho_n^x T^{B_j} \varrho_m^x T^{C_j} T^{b_i}) = \\ &\frac{i}{2}\delta^{a_i b_i}\Big[I_2(P_j^{\mathcal{A}})^i I_2(\varrho_n^x, \varrho_m^x)^j + C_2(\varrho_n^x, \varrho_m^x)^j I_2(\varrho_n^x)^i - C_2(\varrho_m^x, \varrho_n^x)^j I_2(\varrho_m^x)^i\Big] \,, \end{aligned} \tag{A.8}$$

which agrees with eq. (5.15). Nothing has been assumed about the representation that is generated by T^α. Therefore, this identity is quite universal and applicable to a large class of diagrams.

A.1.2. Derivation of Eq. (5.30)

For this color factor there is no particular diagram that it is proportional to. It rather appears in the reduction process of a certain class of diagrams, examples of which are depicted in fig. A.1. Clearly, such a diagram must contain a Goldstone line that leads to the $v\tilde{T}^{B_j} \ldots \tilde{T}^{E_k} v$ structure and a GUT-breaking Higgs line that provides the projector $P_n^{\tilde{\mathcal{H}}}$. The structure we are interested in reads

$$\sum_{jk} f^{a_i B_j C_j} f^{b_i D_k E_k}\, v\tilde{T}^{B_j} \tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{C_j} \tilde{T}^{E_k} v \;\equiv\; \delta^{a_i b_i} \frac{G_n^i}{d_i^{\mathcal{A}}} \,. \tag{A.9}$$

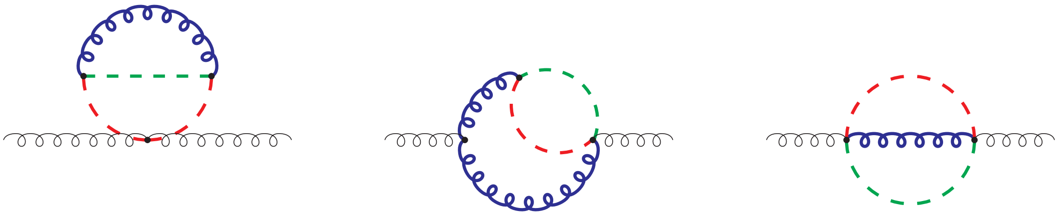


Figure A.1.: *Sample diagrams that lead to the color factor, eq. (5.30). All such diagrams have in common that they contain one Goldstone line and one GUT-breaking Higgs line. Colored (bold) lines represent fields with mass of $\mathcal{O}(M_{\mathrm{GUT}})$ and black (thin) lines massless fields. Goldstone bosons are marked green (light gray, short-dashed), GUT-breaking Higgs fields red (gray, long-dashed) and heavy gauge bosons blue (dark gray)*

The fact that the color factor is proportional to $\delta^{a_i b_i}$ again follows from Schur's Lemma and defines the quantity G_n^i that we want to compute in this subsection. The structure looks quite complicated since no unbroken generators T^{a_i} appear in between the two vevs such that we could exploit $T^{a_i}v = 0$ in order to simplify the expression. Moreover, no contracted adjoint indices appear in between the two vevs that could the eliminated using the definitions of Casimir invariants, eq. (5.9). Therefore, our strategy will be to rewrite the expression in several different ways by applying the relation

$$f^{a_i B_j C_j} T^{C_j} = -i[T^{a_i}, T^{B_j}]\,. \tag{A.10}$$

Using the resulting equations, we will finally be able to relate G_n^i to something we can apply the definitions of color invariants, eq. (5.9), to. There are three possible ways eq. (A.10) can be applied to G_n^i:

$$\begin{aligned} G_n^i &= \sum_{jk} f^{a_i B_j C_j} f^{a_i D_k E_k}\, v\tilde{T}^{B_j}\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{C_j}\tilde{T}^{E_k} v \\ &= \sum_{jk} v[\tilde{T}^{a_i}, \tilde{T}^{C_j}]\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{C_j}[\tilde{T}^{a_i}, \tilde{T}^{D_k}]v \\ &= -\sum_{jk} v\tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k} v\,, \end{aligned} \tag{A.11}$$

$$\begin{aligned} G_n^i &= \sum_{jk} v\tilde{T}^{B_j}[\tilde{T}^{a_i}, \tilde{T}^{E_j}] P_n^{\tilde{\mathcal{H}}} [\tilde{T}^{a_i}, \tilde{T}^{B_j}]\tilde{T}^{E_k} v \\ &= \sum_{jk} \Big[2\, v\tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{a_i}\tilde{T}^{C_j}\tilde{T}^{D_k} v - v\tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k} v \\ &\qquad - v\tilde{T}^{C_j}\tilde{T}^{D_k}\tilde{T}^{a_i} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{a_i}\tilde{T}^{C_j}\tilde{T}^{D_k} v\Big]\,, \end{aligned} \tag{A.12}$$

$$\begin{aligned} G_n^i &= \sum_{jk} v[\tilde{T}^{a_i}, \tilde{T}^{C_j}]\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} [\tilde{T}^{a_i}, \tilde{T}^{D_k}]\tilde{T}^{C_j} v \\ &= \sum_{jk} \Big[- v\tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{a_i}\tilde{T}^{D_k}\tilde{T}^{C_j} v + v\tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k} P_n^{\tilde{\mathcal{H}}} \tilde{T}^{D_k}\tilde{T}^{a_i}\tilde{T}^{C_j} v\Big]\,, \end{aligned} \tag{A.13}$$

where we have used $\tilde{T}^{a_i}v = 0$ and the antisymmetry of $\tilde{T}^{\alpha}$. Equating eqs. (A.11) and (A.12) yields

$$\begin{aligned} v\tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{a_i}\tilde{T}^{C_j}\tilde{T}^{D_k}v &= \frac{1}{2}v\tilde{T}^{C_j}\tilde{T}^{D_k}\tilde{T}^{a_i}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{a_i}\tilde{T}^{C_j}\tilde{T}^{D_k}v \\ &= \frac{1}{2}v\tilde{T}^{C_j}\tilde{T}^{D_k}\tilde{T}^{a_i}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{a_i}\tilde{T}^{D_k}\tilde{T}^{C_j}v\,. \end{aligned} \tag{A.14}$$

In the last line we have commuted $\tilde{T}^{D_k}$ and $\tilde{T}^{C_j}$. Then the term proportional to the structure constant vanishes because $P_n^{\tilde{\mathcal{H}}}\tilde{T}^{a_i}\tilde{T}^{F_j}v = \tilde{T}^{a_i}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{F_j}v = \tilde{T}^{a_i}P_n^{\tilde{\mathcal{H}}}P_j^{\mathcal{G}}\tilde{T}^{F_j}v = 0$. Inserting eq. (A.14) into eq. (A.13), we obtain

$$\begin{aligned} G_n^i &= \sum_{jk}\left[v\tilde{T}^{C_j}\tilde{T}^{a_i}\tilde{T}^{D_k}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{D_k}\tilde{T}^{a_i}\tilde{T}^{C_j}v - \frac{1}{2}v\tilde{T}^{C_j}\tilde{T}^{D_k}\tilde{T}^{a_i}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{a_i}\tilde{T}^{D_k}\tilde{T}^{C_j}v\right] \\ &= \sum_{jk}\frac{M_{\mathrm{X}_j}^2}{g^2}\left[\mathrm{Tr}(P_j^{\mathcal{G}}\tilde{T}^{a_i}\tilde{T}^{a_i}\tilde{T}^{D_k}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{D_k}) - \frac{1}{2}\mathrm{Tr}(P_j^{\mathcal{G}}\tilde{T}^{D_k}\tilde{T}^{a_i}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{a_i}\tilde{T}^{D_k})\right], \end{aligned} \tag{A.15}$$

where we have used eq. (5.23) and exploited $[\tilde{T}^{a_i}, \tilde{T}^{D_k}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{D_k}] = 0$. In this form the color factor looks much more manageable because there are contracted adjoint indices of generators that are next to each other. Applying the definitions of quadratic Casimir invariants and Dynkin indices, eq. (5.9), we finally obtain

$$\begin{aligned} G_n^i &= \sum_{jk}\frac{M_{\mathrm{X}_j}^2}{g^2}\left[C_2(P_j^{\mathcal{G}})^i I_2(P_j^{\mathcal{G}}, P_n^{\tilde{\mathcal{H}}})^k\, D_k^{\mathcal{A}} - \frac{1}{2}I_2(P_n^{\tilde{\mathcal{H}}})^i C_2(P_n^{\tilde{\mathcal{H}}}, P_j^{\mathcal{G}})^k\, d_i^{\mathcal{A}}\right] \\ & \sum_{j}\frac{M_{\mathrm{X}j}^2}{g^2}\left[I_2(P_j^{\mathcal{A}})^i\, I_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j - \frac{1}{2}I_2(P_n^{\tilde{\mathcal{H}}})^i\, C_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j\right]d_i^{\mathcal{A}}\,. \end{aligned} \tag{A.16}$$

In the final line we have used the third line of eq. (5.28), eqs. (5.29) and (5.19). This allowed us to carry out one of the two sums. By employing the definition of G_n^i, eq. (A.9), we are lead to the final result

$$\begin{aligned} &\sum_{jk} f^{a_iB_jC_j} f^{b_iD_kC_k}\, v\tilde{T}^{B_j}\tilde{T}^{D_k}P_n^{\tilde{\mathcal{H}}}\tilde{T}^{C_j}\tilde{T}^{E_k}v = \\ &\delta^{a_ib_i}\sum_{j}\frac{M_{\mathrm{X}j}^2}{g^2}\left[I_2(P_j^{\mathcal{A}})^i\, I_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j - \frac{1}{2}I_2(P_n^{\tilde{\mathcal{H}}})^i\, C_2(P_n^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})^j\right] \end{aligned} \tag{A.17}$$

in agreement with eq. (5.30).

A.1.3. Derivation of Eq. (5.38)

As explained in section 5.5, in order to check for gauge parameter independence of on-shell renormalization constants of heavy fermions, a manifest cancellation of contributions from diagrams of the type depicted in fig. 5.1 has to be guaranteed. The group theoretical identity that is needed for this, will be derived in the following. We start with the Yukawa invariance relation, eq. (4.30):

$$T^{\alpha\star} Y^k + Y^k T^\alpha + Y^m \tilde{T}^\alpha_{mk} = 0 \tag{A.18}$$

for $\alpha = A$. Since the color factors we want to relate do not contain explicit Yukawa couplings but only fermion masses, we are interested in the type of Yukawa couplings that couple right- to left-handed fermions. To this end we multiply eq. (A.18) by $P_i^{\mathcal{D}} \mathcal{Z}_R^T$ from the left and $\mathcal{Z}_L$ from the right. Then we take the trace of the equation and multiply by $(P^{\mathcal{G}} \tilde{T}^A P_1^{\tilde{\mathcal{H}}})_{kl}$ obtaining:

$$\begin{aligned} 0 \;&=\; \mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^T [T^{A\star} Y^k + Y^k T^A] \mathcal{Z}_L)\,(P^{\mathcal{G}} \tilde{T}^A P_1^{\tilde{\mathcal{H}}})_{kl} \\ &\quad + \mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^T Y^m \mathcal{Z}_L)(\tilde{T}^A P^{\mathcal{G}} \tilde{T}^A P_1^{\tilde{\mathcal{H}}})_{ml}\,. \end{aligned} \tag{A.19}$$

Using eqs. (4.36) and (5.9) as well as $(P_1^{\tilde{\mathcal{H}}})_{ml} = \frac{v_m v_l}{v^2}$ yields

$$0 = \mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^T [Y^k t^A - t^A Y^k] \mathcal{Z}_L)\, \tilde{T}^A_{kn} v_n + \mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^T Y^m \mathcal{Z}_L) v_m\, C_2(P_1^{\tilde{\mathcal{H}}}, P^{\mathcal{G}})\,. \tag{A.20}$$

In order to be able to introduce the Dirac mass matrix into this expression, we must first substitute

$$Y^k \tilde{T}^A_{kn} v_n = t^A Y^n v_n - Y^n v_n t^A\,, \tag{A.21}$$

which follows from eqs. (A.18) and (4.36) for the right-left type Yukawa coupling that appears in eq. (A.20). The resulting expression contains the Yukawa coupling only in the combination $Y^n v_n$, which is what we desired. We can now introduce the Dirac mass matrix, eq. (4.34), and exploit the unitarity of the mixing matrices $\mathcal{Z}_L$ and $\mathcal{Z}_R$. In a first step we assume $\mu_F = 0$ and then show later that the result does not change for $\mu_F \neq 0$. Introducing the Casimir invariants, eq. (5.9), this already leads us to the final result

$$\begin{aligned} C_2(P_1^{\tilde{\mathcal{H}}}, P^{\mathcal{G}}) \;&=\; \sum_j \Big[C_2(P_i^{\mathcal{D}_L}, \varrho_j^{\mathcal{F}}) + C_2(P_i^{\mathcal{D}_R}, \varrho_j^{\mathcal{F}}) \Big] \\ &\quad - \frac{2}{M_{D_i} \mathrm{Tr}(P_i^{\mathcal{D}})} \sum_j \Big[\mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_L \mathcal{Z}_R^T\, t^A \varrho_j^{\mathcal{D}} \mathcal{Z}_R^\star \mathcal{Z}_L^\dagger\, t^A) M_{D_j} \\ &\qquad\qquad + \mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_L \mathcal{Z}_R^T\, t^A \varrho_j^{\mathcal{M}} \mathcal{Z}_M^\star \mathcal{Z}_M^\dagger\, t^A) M_{M_j} \Big]\,, \end{aligned} \tag{A.22}$$

which is the first part of eq. (5.38). The second part is obtained by a similar derivation starting from the invariance relation for $Y^{k\star}$ and exchanging $\mathcal{Z}_L \leftrightarrow \mathcal{Z}_R$ in a proper way. It now remains to show that eq. (A.22) does not change if μ_F also contributes to the Dirac mass matrix. The right-hand side of eq. (A.22) is then changed by the amount

$$\Delta \equiv -2\,\mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^T\, \mu_F\, t^A t^A \mathcal{Z}_L) + 2\,\mathrm{Tr}(P_i^{\mathcal{D}} \mathcal{Z}_R^T t^A \mu_F\, t^A \mathcal{Z}_L)\,. \tag{A.23}$$

Since $[\mu_F, T^\alpha] = 0$ due to gauge invariance, Schur's Lemma implies that μ_F is diagonal and:

$$\mu_F = \sum_i \mu_{F_i}\, \Pi^{\mathcal{F}_i}\,. \tag{A.24}$$

We now assume that $\mathcal{Z}_L$ and $\mathcal{Z}_R$ diagonalize μ_F and $Y^k v_k$ in eq. (4.34) separately[2]. In such a case $\mathcal{Z}_L$ and $\mathcal{Z}_R$ can contain only phases, i.e.

$$P_i^{\mathcal{D}} \mathcal{Z}_{L,R} = P_i^{\mathcal{D}} \exp\left(i\,\phi^i_{L,R}\right). \tag{A.25}$$

Inserting this into eq. (A.23), yields

$$\begin{aligned}\Delta &= -2\,\exp\left[i(\phi^i_L + \phi^i_R)\right]\mu_{F_i}\,\mathrm{Tr}(P_i^{\mathcal{D}}\,t^A t^A) + 2\exp\left[i(\phi^i_L + \phi^i_R)\right]\mu_{F_i}\,\mathrm{Tr}(P_i^{\mathcal{D}} t^A t^A) \\ &= 0\,,\end{aligned} \tag{A.26}$$

i.e. eq. (A.22) remains valid for $\mu_F \neq 0$.

A.1.4. Example Derivations of Eq. (5.33)

We first derive the first two lines of eq. (5.33) starting with the definition of the GUT-breaking Higgs mass matrix, eq. (4.26):

$$(M_H^2)_{ij} = \kappa^{\mathcal{H}}_{ijk} v_k + \frac{1}{2}\lambda^{\mathcal{H}}_{ijkl} v_k v_l - \Pi^{\mathcal{H}}_{ij}\Big[\frac{1}{2v^2}\kappa_{ijk} v_i v_j v_k + \frac{1}{6v^2}\,\lambda^{\mathcal{H}}_{klmn} v_k v_l v_m v_n\Big]. \tag{A.27}$$

Multiplying by $v_i v_j$ and solving for $\lambda_{ijkl} v_i v_j v_k v_l$ already gives us the first line of eq. (5.33):

$$\lambda^{\mathcal{H}}_{ijkl} v_j v_k v_l = \frac{v_i}{v^2}\big(3\,v M_H^2 v - \frac{3}{2}\kappa_{ijk} v_i v_j v_k\big). \tag{A.28}$$

To obtain the second line, we multiply the Higgs mass matrix by $(P_s^{\tilde{\mathcal{H}}})_{ij}$:

$$\mathrm{Tr}(P_s^{\tilde{\mathcal{H}}} M_H^2) = \Big[\kappa^{\mathcal{H}}_{ijk} v_k + \frac{1}{2}\lambda^{\mathcal{H}}_{ijkl} v_k v_l\Big](P_s^{\tilde{\mathcal{H}}})_{ij} - \frac{\mathrm{Tr}(P^{\tilde{\mathcal{H}}})}{v^2}\Big[\frac{1}{2}\kappa_{ijk} v_i v_j v_k + \frac{1}{6}\,\lambda^{\mathcal{H}}_{klmn} v_k v_l v_m v_n\Big].$$

Substituting eq. (A.28) for $\lambda^{\mathcal{H}}_{klmn} v_k v_l v_m v_n$ and solving for $\lambda^{\mathcal{H}}_{ijkl} v_i v_j (P_s^{\mathcal{H}})_{kl}$ yields the second line of eq. (5.33):

$$\lambda^{\mathcal{H}}_{ijkl} v_i v_j (P_s^{\mathcal{H}})_{kl} = 2\,\mathrm{Tr}(M_{H_s}^2 P_s^{\mathcal{H}}) - 2\,\kappa^{\mathcal{H}}_{jkl} v_j (P_s^{\mathcal{H}})_{kl} + \frac{D_s^{\mathcal{H}}}{v^2}\big(\frac{1}{2}\kappa^{\mathcal{H}}_{ijk} v_i v_j v_k + v M_H^2 v\big)\,, \tag{A.29}$$

where we have used eq. (5.10). These identities are always used to eliminate λ_{ijkl} as far as possible from the result of a loop calculation and expressing it through physical quantities as the Higgs mass matrix.

Next, we want to reduce the expression $\lambda^{\mathcal{H}}_{ijkl}(P_n^{\mathcal{G}})_{ij}(P_m^{\mathcal{G}})_{kl}$ (fifth identity in eq. (5.33)) using the gauge invariance relation for λ_{ijkl}, eq. (4.19), for $\alpha = A_n$:

$$0 = \tilde{T}^{A_n}_{is}\,\lambda_{sjkl} + \tilde{T}^{A_n}_{js}\,\lambda_{iskl} + \tilde{T}^{A_n}_{ks}\,\lambda_{ijsl} + \tilde{T}^{A_n}_{ls}\,\lambda_{ijks}\,. \tag{A.30}$$

We multiply by $v_i(v\tilde{T}^{A_n})_j (P_m^{\mathcal{G}})_{kl}\frac{g^2}{M^2_{\mathrm{X}_n}}$, use the definition of $P_i^{\mathcal{G}}$, eq. (4.20), and obtain

$$\begin{aligned}\lambda^{\mathcal{H}}_{ijkl}(P_n^{\mathcal{G}})_{ij}(P_m^{\mathcal{G}})_{kl} &= \frac{g^4}{M^2_{\mathrm{X}_n} M^2_{\mathrm{X}_m}}\Big[(v\tilde{T}^{A_n}\tilde{T}^{A_n})_j \lambda_{ijkl}\, v_i (P_m^{\mathcal{G}})_{kl} \\ &\qquad + 2\,(v\tilde{T}^{A_n})_j \lambda_{ijkl}\, v_i (P_m^{\mathcal{G}}\tilde{T}^{A_n})_{kl}\Big].\end{aligned} \tag{A.31}$$

[2]This assumption is satisfied for the Higgsinos $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$ in Minimal SUSY SU(5). In that case $Y^k v_k$ is even zero.

Substituting both λs by the third line of eq. (4.19) and using the definition of the gauge boson mass matrix, eq. (5.24), and Casimir invariants, eq. (5.9), results in

$$\begin{aligned}\lambda^{\mathcal{H}}_{ijkl}(P^{\mathcal{G}}_n)_{ij}(P^{\mathcal{G}}_m)_{kl} &= \frac{g^4}{M^2_{\mathrm{X}_n}M^2_{\mathrm{X}_m}}\Big[-C_2(P^{\tilde{\mathcal{H}}}_1, P^{\mathcal{G}})^n\,\kappa_{ijk}(P^{\mathcal{G}}_m)_{ij}v_k - 2\,(v\tilde{T}^{A_n})_j\,\kappa_{jkl}(P^{\mathcal{G}}_m\tilde{T}^{A_n})_{kl} \\ &\qquad v\tilde{T}^{A_n}\tilde{T}^{A_n}M^2_H\tilde{T}^{B_m}\tilde{T}^{B_m}v + 2\,v\tilde{T}^{A_n}\tilde{T}^{B_m}M^2_H\tilde{T}^{A_n}\tilde{T}^{B_m}v\Big]\,. \qquad \text{(A.32)}\end{aligned}$$

Both κ terms can now be substituted by identities one and six from eq. (5.35), which results in their cancellation. Therefore, the final formula is

$$\begin{aligned}\lambda^{\mathcal{H}}_{ijkl}(P^{\mathcal{G}}_n)_{ij}(P^{\mathcal{G}}_m)_{kl} &= \frac{g^4}{M^2_{\mathrm{X}_n}M^2_{\mathrm{X}_m}}\Big[v\tilde{T}^{A_n}\tilde{T}^{A_n}M^2_H\tilde{T}^{B_m}\tilde{T}^{B_m}v \\ &\qquad + 2\,v\tilde{T}^{A_n}\tilde{T}^{B_m}M^2_H\tilde{T}^{A_n}\tilde{T}^{B_m}v\Big] \qquad \text{(A.33)}\end{aligned}$$

in agreement with the fifth identity in eq. (5.33).

A.2. Reparametrization of the Scalar Potential

In this appendix we give some details on how the parametrization of the up to quadratic terms in the scalar potential, eq. (4.24), arises. The presented parametrization is particularly useful for renormalization since it permits a proper treatment of tadpole terms at higher orders. For a more basic application of this idea, please refer to chapter 3 of ref. [137] where the reparametrization is done for the case of the SM. Here we develop it in full generality. We start with eq. (4.18) and insert the decomposition of the scalar field Φ:

$$\Phi_i = \underbrace{\underbrace{v_i + H_i}_{\mathcal{R}^{\tilde{\mathcal{H}}}} + \underbrace{G_i}_{\mathcal{R}^{\mathcal{G}}}}_{\mathcal{R}^{\mathcal{H}}} + \underbrace{S_i}_{\mathcal{R}^{\mathcal{S}}}\,, \qquad \text{(A.34)}$$

where v is the vev with a single non-zero component, H the physical Higgs field, G the Goldstone field and S a field, representing all the other scalars, present in the theory. For clarity we have also indicated the representations where the individual fields live in. Considering only the up to quadratic terms, we arrive at the following expression:

$$V(\Phi) = H_i\tilde{t}_i + \frac{1}{2}\tilde{M}^2_{ij}H_iH_j + \frac{1}{2}\tilde{M}^2_{ij}G_iG_j + \frac{1}{2}\tilde{M}^2_{ij}S_iS_j + \mathcal{O}(\Phi^3)\,, \qquad \text{(A.35)}$$

where we have defined the quantities

$$\begin{aligned}\tilde{t}_i &= -\mu^2_{ij}v_j + \frac{1}{2}\kappa_{ijk}\,v_jv_k + \frac{1}{6}\lambda_{ijkl}v_jv_kv_l\,, \\ \tilde{M}^2_{ij} &= -\mu^2_{ij} + \kappa_{ijk}\,v_k + \frac{1}{2}\lambda_{ijkl}v_kv_l\,. \qquad \text{(A.36)}\end{aligned}$$

Here we already have used the fact that, due to gauge invariance (eq. (4.19)), the matrix μ^2 is proportional to the unit matrix on each $\mathbf{G}$-irreducible subspace and therefore

$$\begin{aligned} \mu^2_{ij} v_i G_j &= 0\,, \qquad \mu^2_{ij} v_i S_j = 0\,, \qquad \mu^2_{ij} H_i G_j = 0\,, \\ \mu^2_{ij} H_i S_j &= 0\,, \qquad \mu^2_{ij} G_i S_j = 0\,. \end{aligned} \tag{A.37}$$

Also due to gauge invariance, eq. (4.19), we have $\lambda_{ijkl} v_j v_k v_l \sim v_i$ and $\kappa_{ijk} v_j v_k \sim v_i$, because the matrices K and K' defined by $K_{ij} = \lambda_{ijkl} v_k v_l$ and $K'_{ij} = \kappa_{ijk} v_k$ are diagonal and proportional to the unit matrix on each $\mathbf{G}_{\mathrm{SM}}$-irreducible subspace. Hence

$$\begin{aligned} \lambda_{ijkl} G_i v_j v_k v_l = \;& 0 \; = \lambda_{ijkl} S_i v_j v_k v_l\,, \\ \kappa_{ijk} G_i v_j v_k = \;& 0 \; = \kappa_{ijk} S_i v_j v_k\,, \end{aligned} \tag{A.38}$$

since $(P^{\mathcal{G}})_{ij} v_j = 0 = (P^{\mathcal{S}})_{ij} v_j$. The parametrization in eq. (A.35) has the disadvantage that the masslessness of Goldstone bosons is not manifest there due to the appearance of an explicit mass matrix for the Goldstone field. Note furthermore that on the subspace $\mathcal{R}^{\mathcal{H}}$ the parameter μ^2 is redundant, because it depends on the tadpole term $\tilde{t}$, which has been chosen as a physical parameter in the Lagrangian. Therefore, we want μ^2 also to disappear from the mass matrix of the physical Higgs bosons H by trading it for $\tilde{t}$. To solve these issues, we first rewrite the tadpole term by observing that $\tilde{t} \sim v_i$ due to arguments already given below eq. (A.37). Therefore we define

$$\begin{aligned} \tilde{t}_i \equiv t\, v_i\,, \qquad\qquad t \;&\equiv\; \frac{1}{v^2}\left(-\mu^2_{ij} v_i v_j + \tfrac{1}{2}\,\kappa_{ijk}\, v_i v_j v_k + \tfrac{1}{6}\lambda_{ijkl} v_i v_j v_k v_l\right) \\ &\equiv\; -\mu^2_{\mathcal{H}} + \frac{1}{2v^2}\,\kappa^{\mathcal{H}}_{ijk} v_i v_j v_k + \frac{1}{6v^2}\,\lambda^{\mathcal{H}}_{ijkl} v_i v_j v_k v_l\,. \end{aligned} \tag{A.39}$$

The classical minimum of the potential can be found by setting $t = 0$. In a quantum theory, however, we must keep t as a counterterm, as explained in sections 4.1.2 and 4.2, in order to guarantee that the vev v defines the minimum of the potential also at higher orders of perturbation theory. Next, we turn to the term $\frac{1}{2}\tilde{M}^2_{ij} G_i G_j$ in eq. (A.35). From the Goldstone theorem it follows that this mass term must vanish at the classical level. But since we are also interested in quantum corrections, care is needed here. We calculate the term by applying the explicit form of the Goldstone projector $P^{\mathcal{G}}$ which is given in eq. (4.20) to $\tilde{M}^2$. Using also the gauge invariance relations in eq. (4.19), we can prove the identities

$$\begin{aligned} (P^{\mathcal{G}})_{ij} \lambda_{jklm} v_l v_m \;&=\; \frac{1}{3v^2} (P^{\mathcal{G}})_{ik}\, \lambda_{jlmn} v_j v_l v_m v_n\,, \\ (P^{\mathcal{G}})_{ij} \kappa_{jkl} v_l \;&=\; \frac{1}{2v^2} (P^{\mathcal{G}})_{ik}\, \kappa_{jlm} v_j v_l v_m\,, \\ (P^{\mathcal{G}})_{ij} \mu^2_{jk} \;&=\; \frac{1}{v^2} (P^{\mathcal{G}})_{ik}\, \mu^2_{jl} v_j v_l = (P^{\mathcal{G}})_{ik}\, \mu^2_{\mathcal{H}}\,. \end{aligned} \tag{A.40}$$

From this it follows that:

$$(P^{\mathcal{G}})_{ij} \tilde{M}^2_{jk} = t\, (P^{\mathcal{G}})_{ik}\,. \tag{A.41}$$

As expected from the Goldstone theorem, the term vanishes at the classical level, but contributes as a counterterm to the two-point function of the Goldstone field via t. This is an important subtlety and crucial for a correct renormalization procedure (cf. also the counterterm Feynman rules given in section 4.2). Similarly, we calculate $\frac{1}{2}\tilde{M}^2_{ij}H_iH_j$ by applying the projector $P^{\tilde{\mathcal{H}}} = \Pi^{\mathcal{H}} - P^{\mathcal{G}}$ to $\tilde{M}^2$:

$$\begin{aligned}(P^{\tilde{\mathcal{H}}})_{ij}\tilde{M}^2_{jk} &= (\Pi^{\mathcal{H}})_{ij}\tilde{M}^2_{jk} - (P^{\mathcal{G}})_{ij}\tilde{M}^2_{jk} \\ &= t\,(P^{\tilde{\mathcal{H}}})_{ik} + \tilde{M}^2_{jk} - t\,(\Pi^{\mathcal{H}})_{ik} \\ &\equiv t\,(P^{\tilde{\mathcal{H}}})_{ik} + (M^2_H)_{jk}\,, \end{aligned} \tag{A.42}$$

where we have used eq. (A.41) and defined the physical Higgs mass matrix

$$(M^2_H)_{ij} = \kappa^{\mathcal{H}}_{ijk}\,v_k + \frac{1}{2}\lambda^{\mathcal{H}}_{ijkl}v_kv_l - \Pi^{\mathcal{H}}_{ij}\frac{1}{v^2}\Big[\frac{1}{2}\,\kappa^{\mathcal{H}}_{klm}v_kv_lv_m + \frac{1}{6}\,\lambda^{\mathcal{H}}_{klmn}v_kv_lv_mv_n\Big]\,. \tag{A.43}$$

Note that $P^{\mathcal{G}}M^2_H = 0$ and therefore the Higgs mass matrix, as we have defined it, can have non-zero entries only on the subspace $\mathcal{R}^{\tilde{\mathcal{H}}}$. Also the parameter $\mu^2_{\mathcal{H}}$ does not appear anymore in the definition of M^2_H, as desired. For the fields S_i no peculiarities occur, since they have no vev. Their mass matrix is essentially given by $\tilde{M}^2$:

$$(M^2_{\mathcal{S}})_{ij} = \kappa^{\mathcal{S}}_{ijk}v_k + \frac{1}{2}\lambda^{\mathcal{S}}_{ijkl}v_kv_l - (\mu^2(\mathbb{1} - \Pi^{\mathcal{H}}))_{ij}\,. \tag{A.44}$$

The up to quadratic terms of the scalar potential can now be written as

$$V(\Phi) = t\;v_iH_i + \frac{1}{2}(M^2_H)_{ij}H_iH_j + \frac{1}{2}t\;H_iH_i + \frac{1}{2}t\;G_iG_i + \frac{1}{2}(M^2_S)_{ij}S_iS_j + \mathcal{O}(\Phi^3)\,, \tag{A.45}$$

which is the appropriate parametrization for renormalizing the theory. The couplings λ_{ijkl} and κ_{ijk} still appear in the $\mathcal{O}(\Phi^3)$ terms of this equation. However, in the final result of a loop calculation we aim at expressing all cubic and quartic scalar couplings in terms of the physical parameters defined in this section (i.e. masses and tadpole terms).

A.3. Transition from Complex to Real Scalars

We show how any complex scalar multiplet can be rewritten as twice as many real scalars the generator matrices of which are antisymmetric. This is in order to demonstrate that the assumptions made in chapter 4 constitute no loss of generality and also to show how the transition is done practically, e.g. for the analysis in chapter 6. We start with a complex scalar multiplet φ transforming as a (not necessarily irreducible) representation of the group $\mathbf{G}$. The gauge-kinetic Lagrangian is then given by

$$\mathcal{L}_{\rm kin} = (D^\mu\varphi)^\dagger(D_\mu\varphi) \tag{A.46}$$

with

$$(D_\mu\varphi) = (\partial_\mu - ig\hat{T}^\alpha A^\alpha_\mu)\varphi\,, \tag{A.47}$$

where $\hat{T}^\alpha$ (note the hat) are hermitian generators. The gauge infinitesimal transformation is defined as

$$\varphi \;\rightarrow\; \varphi - ig\theta^\alpha\hat{T}^\alpha\varphi\,. \tag{A.48}$$

In order to make the transition to real scalars, we split the fields and the generators into real and imaginary part

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_R + i\varphi_I)\,, \qquad \hat{T}^\alpha = T_R^\alpha + iT_I^\alpha \tag{A.49}$$

with $(T_R^\alpha)^T = T_R^\alpha$ and $(T_I^\alpha)^T = -T_I^\alpha$. We want real scalar field to be defined as

$$\Phi = \begin{pmatrix} \varphi_R \\ \varphi_I \end{pmatrix}. \tag{A.50}$$

The question is now how to appropriately define antisymmetric generators $\tilde{T}^\alpha$ (note the tilde) such that the gauge transformation, eq. (A.48), can be rewritten as

$$\Phi \;\to\; \Phi - ig\theta^\alpha \tilde{T}^\alpha \Phi\,. \tag{A.51}$$

Inserting eq. (A.49) into eq. (A.48), we obtain

$$\varphi \to \varphi_R - ig\theta^\alpha\left[iT_I^\alpha\varphi_R + iT_R^\alpha\varphi_I\right] + i\left\{\varphi_I - ig\theta^\alpha\left[-iT_R^\alpha\varphi_R + iT_I^\alpha\varphi_I\right]\right\} \tag{A.52}$$

This can also be written as

$$\begin{pmatrix} \varphi_R \\ \varphi_I \end{pmatrix} \to \begin{pmatrix} \varphi_R \\ \varphi_I \end{pmatrix} - ig\theta^\alpha \begin{pmatrix} iT_I^\alpha\varphi_R + iT_R^\alpha\varphi_I \\ -iT_R^\alpha\varphi_R + iT_I^\alpha\varphi_I\,. \end{pmatrix} \tag{A.53}$$

Therefore, the appropriate definition of $\tilde{T}^\alpha$, in order to guarantee eq. (A.51), is

$$\tilde{T}^\alpha = \begin{pmatrix} iT_I^\alpha & iT_R^\alpha \\ -iT_R^\alpha & iT_I^\alpha \end{pmatrix}. \tag{A.54}$$

which manifestly fulfills $(\tilde{T}^\alpha)^T = -\tilde{T}^\alpha$. The gauge-kinetic term can then be written as

$$\mathcal{L}_{\text{kin}} = \frac{1}{2}(D^\mu\Phi)^T(D_\mu\Phi) \tag{A.55}$$

with

$$(D_\mu\Phi) = (\partial_\mu - ig\tilde{T}^\alpha A_\mu^\alpha)\Phi\,. \tag{A.56}$$

Moreover, it can can be checked that the generators $\tilde{T}^\alpha$ also fulfill the Lie algebra

$$[\tilde{T}^\alpha, \tilde{T}^\beta] = if^{\alpha\beta\gamma}\tilde{T}^\gamma \tag{A.57}$$

if it is fulfilled by $\hat{T}^\alpha$

$$[\hat{T}^\alpha, \hat{T}^\beta] = if^{\alpha\beta\gamma}\hat{T}^\gamma\,. \tag{A.58}$$

Of course, in case a complex scalar appears in the theory, we need to rewrite not only the gauge-kinetic term, but also all interaction terms that the field is involved in. As an example consider the $\mathbf{5}_H - \mathbf{24}_H$ mixing term in the Georgi-Glashow model that appears in eq. (2.10)

$$V(\mathbf{5}_H, \mathbf{24}_H) \supset c\,(\hat{T}^\alpha\hat{T}^\beta)_{ij}\mathbf{24}_{H_\alpha}\mathbf{24}_{H_\beta}\mathbf{5}^\star_{H_i}\mathbf{5}_{H_j}\,, \tag{A.59}$$

where $\hat{T}^\alpha$ are now the hermitian generators in the fundamental representation of SU(5). For convenience we write $\mathbf{5}_{H_j} \equiv \varphi_j$ henceforth. The task therefore is to rewrite

$$(T^\alpha T^\beta)_{ij}\varphi_i\varphi_j \tag{A.60}$$

in terms of Φ. Inserting eq. (A.49) into eq. (A.60), we obtain

$$\begin{aligned}(T^\alpha T^\beta)_{ij}\varphi_i\varphi_j &= \frac{c}{2}\left[\varphi_{R_i}\varphi_{R_j} + \varphi_{I_i}\varphi_{I_j} + i\varphi_{R_i}\varphi_{I_j} - i\varphi_{I_i}\varphi_{R_j}\right] \times \\ &\quad \left[(T_R^\alpha T_R^\beta)_{ij} - (T_I^\alpha T_I^\beta)_{ij} + i(T_R^\alpha T_I^\beta)_{ij} + i(T_I^\alpha T_R^\beta)_{ij}\right] \\ &= \frac{c}{2}\left[\varphi_{R_i}\varphi_{R_j} + \varphi_{I_i}\varphi_{I_j}\right]\left[(T_R^\alpha T_R^\beta)_{ij} - (T_I^\alpha T_I^\beta)_{ij}\right] \\ &\quad -\frac{c}{2}\left[\varphi_{R_i}\varphi_{I_j} - \varphi_{I_i}\varphi_{R_j}\right]\left[(T_R^\alpha T_I^\beta)_{ij} + (T_I^\alpha T_R^\beta)_{ij}\right],\end{aligned} \tag{A.61}$$

where symmetry properties in the indices i and j of the two terms in parentheses have been exploited. It is now possible to introduce the antisymmetric generator matrix $\tilde{T}^\alpha$ since

$$\begin{aligned}(T^\alpha T^\beta)_{ij}\varphi_i\varphi_j &= \frac{c}{2}(\varphi_{R_i}, \varphi_{I_i})\begin{pmatrix} (T_R^\alpha T_R^\beta)_{ij} - (T_I^\alpha T_I^\beta)_{ij} & -(T_R^\alpha T_I^\beta)_{ij} - (T_I^\alpha T_R^\beta)_{ij} \\ (T_R^\alpha T_I^\beta)_{ij} + (T_I^\alpha T_R^\beta)_{ij} & (T_R^\alpha T_R^\beta)_{ij} - (T_I^\alpha T_I^\beta)_{ij}\end{pmatrix}\begin{pmatrix}\varphi_{R_j} \\ \varphi_{I_j}\end{pmatrix} \\ &= \frac{c}{2}\Phi_i(\tilde{T}^\alpha\tilde{T}^\beta)_{ij}\Phi_j\end{aligned} \tag{A.62}$$

Therefore the mixing term is appropriately rewritten as

$$c\,(\hat{T}^\alpha\hat{T}^\beta)_{ij}\mathbf{24}_{H_\alpha}\mathbf{24}_{H_\beta}\mathbf{5}^\star_{H_i}\mathbf{5}_{H_j} = \frac{c}{4}\Phi_i\{\tilde{T}^\alpha, \tilde{T}^\beta\}_{ij}\Phi_j\mathbf{24}_{H_\alpha}\mathbf{24}_{H_\beta}, \tag{A.63}$$

where the anticommutator has been introduced in order to have a manifestly symmetric coupling in α, β and i, j. The result agrees with the form of $\lambda^{5-24}_{\alpha\beta ij}$ that has been used in eq. (6.1). Moreover, it is noteworthy that when going to Minimal SUSY SU(5), the undertaking of rewriting all complex scalars becomes unequally more complicated because of the large number of different scalar couplings in the potential.

A.4. β Functions for the SM

For the convenience of the reader, we list the RGEs for the gauge and Yukawa couplings in the SM here. We define the β function coefficients for the gauge couplings including Yukawa corrections as follows

$$\frac{1}{2}\frac{d\alpha_i}{dt} = \beta_0^i\left(\frac{\alpha_i}{4\pi}\right)^2 + \left(\frac{\alpha_i}{4\pi}\right)^2\left[\sum_{j=t,b,\tau}\beta^{ij}_{\text{Yuk}}\left(\frac{y_j}{4\pi}\right)^2 + \sum_{j=1}^{3}\beta_1^{ij}\frac{\alpha_j}{4\pi}\right] + \delta_{3i}\,\beta_2^{\text{QCD}}\left(\frac{\alpha_3}{4\pi}\right)^3, \tag{A.64}$$

where the three-loop contribution β_2 is only known for α_3. The coefficients have been determined in refs. [112, 113] (QCD) and [27, 28, 114] (electroweak sector)

$$\begin{aligned}\beta_0 &= \begin{pmatrix}\frac{1}{10} + \frac{4}{3}n_g \\ -\frac{43}{6} + \frac{4}{3}n_g \\ \frac{4}{3}n_g - 11\end{pmatrix}, \qquad \beta_1 = \begin{pmatrix}\frac{9}{50} + \frac{19}{15}n_g & \frac{9}{10} + \frac{3}{5}n_g & \frac{44}{15}n_g \\ \frac{3}{10} + \frac{1}{5}n_g & -\frac{259}{6} + \frac{49}{3}n_g & 4n_g \\ \frac{11}{30}n_g & \frac{3}{2}n_g & -102 + \frac{76}{3}n_g\end{pmatrix}, \\ \beta_2^{\text{QCD}} &= -\frac{2857}{2} + \frac{10066}{18}n_g - \frac{650}{27}n_g^2, \qquad \beta_{\text{Yuk}} = \begin{pmatrix}-\frac{17}{10} & -\frac{1}{2} & -\frac{3}{2} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{2} \\ -2 & -2 & 0\end{pmatrix}.\end{aligned} \tag{A.65}$$

Here $n_g = 3$ is the number of fermion generations in the SM. The RGEs for the Yukawa couplings are given by

$$\frac{dy_i}{dt} = y_i \left[\frac{1}{16\pi^2} \bar{\beta}_i^{(1)} + \left(\frac{1}{16\pi^2} \right)^2 \bar{\beta}_i^{(2)} \right] , \qquad i = t, b, \tau \tag{A.66}$$

where the functions $\bar{\beta}_i^{(1)}$ and $\bar{\beta}_i^{(2)}$ are defined as the $(3,3)$ components of the matrices $\bar{\boldsymbol{\beta}}^{\mathbf{(1)}}_{\mathbf{U,D,L}}$ and $\bar{\boldsymbol{\beta}}^{\mathbf{(2)}}_{\mathbf{U,D,L}}$ given in eqs. (B.4) to (B.10) of ref. [28]. There we need to insert the replacements

$$\mathbf{H} = \mathrm{diag}(0,0,y_t) , \qquad \mathbf{F_D} = \mathrm{diag}(0,0,y_b) , \qquad \mathbf{F_L} = \mathrm{diag}(0,0,y_\tau) . \tag{A.67}$$

We refrain from reproducing the full formulae for the Yukawa RGEs here because of their large size.

A.5. β Functions for the MSSM

For the case of the MSSM full three-loop RGEs are known [24]. We reproduce them partly for convenience and for the missing parts specify how to translate the result of ref. [24] into our notation. For the gauge β function we have

$$\frac{1}{2}\frac{d\alpha_i}{dt} = \beta_0^i \left(\frac{\alpha_i}{4\pi}\right)^2 + \left(\frac{\alpha_i}{4\pi}\right)^2 \left[\sum_{j=t,b,\tau} \beta^{ij}_{\mathrm{Yuk}} \left(\frac{y_j}{4\pi}\right)^2 + \sum_{j=1}^{3} \beta_1^{ij} \frac{\alpha_j}{4\pi} \right] + \beta_2^i \quad , \tag{A.68}$$

where

$$\beta_0 = \begin{pmatrix} \frac{3}{5} + 2n_g \\ -5 + 2n_g \\ 2n_g - 9 \end{pmatrix} , \qquad \beta_1 = \begin{pmatrix} \frac{9}{25} + \frac{38}{15}n_g & \frac{9}{5} + \frac{6}{5}n_g & \frac{88}{15}n_g \\ \frac{3}{5} + \frac{2}{5}n_g & -17 + 14n_g & 8n_g \\ \frac{11}{15}n_g & 3n_g & -54 + \frac{68}{3}n_g \end{pmatrix} ,$$
$$\beta_{\mathrm{Yuk}} = \begin{pmatrix} -\frac{26}{5} & -\frac{14}{5} & -\frac{18}{5} \\ -6 & -6 & -2 \\ -4 & -4 & 0 \end{pmatrix} , \tag{A.69}$$

where again $n_g = 3$ is the number of generations. Furthermore, the three-loop contribution β_2^i in eq. (A.68) is related to the quantities $\beta_{g_i}^{(3)}$ given in eqs. (6a) to (6c) in ref. [24] via

$$\beta_2^i = \frac{g_i}{(16\pi^2)^4} \beta_{g_i}^{(3)} \tag{A.70}$$

The formulae for the three-loop contributions to the running of the Yukawa couplings are huge. Therefore, we do not list them here. Taking into account only third generation Yukawa couplings and setting $V_{\mathrm{CKM}} = \mathbb{1}$, the RGEs read

$$\frac{dy_i}{dt} = (\beta_{Y_i})_{33} , \qquad i = t, b, \tau , \tag{A.71}$$

where the matrices β_{Y_i} are defined in eq. (11) of ref. [24].

A.6. Three-Loop Gauge β Function for the Georgi-Glashow SU(5) Model

For performing a consistent three-loop RGE analysis, apart from the two-loop GUT matching corrections also the three-loop gauge β function for the Georgi-Glashow model is needed. The authors of ref. [25] give a general formula for the gauge β function of a general single gauge coupling theory. Specifying the group theory factors that appear there to the Georgi-Glashow model and inserting the scalar self-couplings from eq. (6.1), as well as the Yukawa coupling from eq. (6.4) into their general result gives us the desired β function including scalar self-couplings and Yukawa corrections:

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\frac{\alpha}{4\pi} = & -\frac{40}{3}\left(\frac{\alpha}{4\pi}\right)^2 \\
& -\frac{1184}{15}\left(\frac{\alpha}{4\pi}\right)^3 + \left[-\frac{9}{2}\left(\frac{y_t}{4\pi}\right)^2 - 5\left(\frac{y_b}{4\pi}\right)^2\right]\left(\frac{\alpha}{4\pi}\right)^2 \\
& -\frac{1007357}{1080}\left(\frac{\alpha}{4\pi}\right)^4 \\
& +\Bigg[-\frac{1323}{4}\left(\frac{y_t}{4\pi}\right)^2 - \frac{3617}{10}\left(\frac{y_b}{4\pi}\right)^2 \\
& \quad +\frac{155}{96}\frac{A}{(4\pi)^2} + \frac{11}{20}\frac{b}{(4\pi)^2} + \frac{125}{12}\frac{B}{(4\pi)^2} + \frac{25}{4}\frac{c}{(4\pi)^2}\Bigg]\left(\frac{\alpha}{4\pi}\right)^3 \\
& +\Bigg[\frac{51}{4}\left(\frac{y_t}{4\pi}\right)^4 + \frac{47}{4}\left(\frac{y_b}{4\pi}\right)^4 + \frac{839}{8}\frac{y_t^2 y_b^2}{(4\pi)^4} \\
& \quad -\frac{493}{11520}\frac{A^2}{(4\pi)^4} - \frac{47}{144}\frac{AB}{(4\pi)^4} - \frac{1}{12}\frac{b^2}{(4\pi)^4} - \frac{65}{36}\frac{B^2}{(4\pi)^4} - \frac{851}{200}\frac{c^2}{(4\pi)^4}\Bigg]\left(\frac{\alpha}{4\pi}\right)^2 .
\end{aligned}
\tag{A.72}
$$

The first line of this equation represents the one-loop result, the second line the two-loop result and the rest corresponds to the three-loop corrections. Since the Yukawa couplings enter the gauge β function starting from two-loop level only, it is enough to employ the one-loop RGEs for the Yukawa couplings for the precision we are aiming at. These can be derived in a similar manner from the general formula in ref. [28]:

$$
\begin{aligned}
\frac{dy_t}{dt} &= y_t\left[-\frac{108}{5}\left(\frac{\alpha}{4\pi}\right) - 6\left(\frac{y_b}{4\pi}\right)^2 + 9\left(\frac{y_t}{4\pi}\right)^2\right], \\
\frac{dy_b}{dt} &= y_b\left[-18\left(\frac{\alpha}{4\pi}\right) + 11\left(\frac{y_b}{4\pi}\right)^2 - \frac{9}{2}\left(\frac{y_t}{4\pi}\right)^2\right].
\end{aligned}
\tag{A.73}
$$

The scalar self-couplings A, B and c that appear in eq. (A.72) only at the three-loop level are approximated as constants in our analysis by replacing them by their relations to the physical mass parameters $M_\Sigma, M_{24}, M_{\mathrm{H_c}}, M_{\mathrm{X}}$, and the gauge coupling α by using eqs. (2.11) and (2.12). The scalar self-coupling b that appears here can be approximated similarly by a constant using the SM Higgs mass $M^2_{\mathrm{H,SM}}$ and the mass of the W boson M_W:

$$b = \frac{3}{4} g^2 \frac{M^2_{\mathrm{H,SM}}}{M^2_W} . \tag{A.74}$$

A.7. Three-Loop Gauge β Function for Minimal SUSY SU(5) and the Missing Doublet Model

The analysis in chapter 3 required the knowledge of the three-loop gauge β function for Minimal SUSY SU(5) and the Missing Doublet Model. This can be derived from the general formula given in ref. [26] neglecting Yukawa contributions for simplicity. With our convention of β function coefficients, eq. (2.24), and the notation of eq. (5.9) their formulae read

$$\begin{aligned}
\beta_0 &= \sum_x I_2(\Pi^x) - 3 I_2(\Pi^{\mathcal{A}}) , \\
\beta_1 &= 2\, I_2(\Pi^{\mathcal{A}})\beta_0 + 4 \sum_x C_2(\Pi^x) I_2(\Pi^x) , \\
\beta_2 &= -8 \sum_x C_2(\Pi^x)^2 I_2(\Pi^x) + \left[8\, I_2(\Pi^{\mathcal{A}}) - 6\,\beta_0\right] \sum_x C_2(\Pi^x) I_2(\Pi^x) \\
&\quad + \beta_0 I_2(\Pi^{\mathcal{A}}) \left[4\, I_2(\Pi^{\mathcal{A}}) - \beta_0\right] ,
\end{aligned} \tag{A.75}$$

where the sums are performed over all supermultiplets that appear in the theory. For the case of Minimal SUSY SU(5) the superfield content is

$$\mathbf{24}_H \oplus \mathbf{5}_H \oplus \overline{\mathbf{5}}_H \oplus \overline{\mathbf{5}} \oplus \mathbf{10} . \tag{A.76}$$

An explicit calculation with `Mathematica` yield the following numerical values for the invariants

$$\begin{aligned}
I_2(\Pi^{\mathcal{A}}) &= I_2(\Pi^{\mathbf{24}}) = C_2(\Pi^{\mathbf{24}}) = 5 , \\
I_2(\Pi^{\mathbf{5}}) &= \frac{1}{2} , \quad C_2(\Pi^{\mathbf{5}}) = \frac{12}{5} , \\
I_2(\Pi^{\mathbf{10}}) &= \frac{3}{2} , \quad C_2(\Pi^{\mathbf{10}}) = \frac{18}{5} .
\end{aligned} \tag{A.77}$$

Inserting this into eq. (A.75) leads to the β function coefficients for Minimal SUSY SU(5)

$$\begin{aligned}
\beta_0 &= -3 , \\
\beta_1 &= \frac{794}{5} , \\
\beta_2 &= \frac{20271}{25} .
\end{aligned} \tag{A.78}$$

For the Missing Doublet Model a few additional invariants are needed

$$\begin{aligned}
I_2(\Pi^{\mathbf{75}}) &= \frac{50}{2} , \quad C_2(\Pi^{\mathbf{75}}) = 8 , \\
I_2(\Pi^{\mathbf{50}}) &= \frac{35}{2} , \quad C_2(\Pi^{\mathbf{50}}) = \frac{42}{5} .
\end{aligned} \tag{A.79}$$

However, as motivated in chapter 3, we do not take into account the contributions from $\mathbf{50}_H$ and $\overline{\mathbf{50}}_H$ in order to stay perturbative below the Planck scale. Then the β function coefficients for the Missing Doublet Model read

$$\begin{aligned} \beta_0 &= 17\,, \\ \beta_1 &= \frac{5294}{5}\,, \\ \beta_2 &= -\frac{672579}{25}\,. \end{aligned} \tag{A.80}$$

The one-loop RGEs for the Yukawa couplings in Minimal SUSY SU(5) read [18]

$$\begin{aligned} \frac{dy_t}{dt} &= y_t\left[\frac{12}{5}\left(\frac{\lambda_2}{4\pi}\right)^2+9\left(\frac{y_t}{4\pi}\right)^2+4\left(\frac{y_b}{4\pi}\right)^2-\frac{96}{5}\left(\frac{\alpha}{4\pi}\right)\right]\,, \\ \frac{dy_b}{dt} &= y_b\left[\frac{12}{5}\left(\frac{\lambda_2}{4\pi}\right)^2+3\left(\frac{y_t}{4\pi}\right)^2+10\left(\frac{y_b}{4\pi}\right)^2-\frac{84}{5}\left(\frac{\alpha}{4\pi}\right)\right]\,, \end{aligned} \tag{A.81}$$

where the Higgs self-coupling appear that have been defined in eq. (2.19). They evolve as follows

$$\begin{aligned} \frac{d\lambda_1}{dt} &= \lambda_1\left[\frac{567}{40}\left(\frac{\lambda_1}{4\pi}\right)^2+\frac{3}{2}\left(\frac{\lambda_2}{4\pi}\right)^2-30\left(\frac{\alpha}{4\pi}\right)\right]\,, \\ \frac{d\lambda_2}{dt} &= \lambda_2\left[\frac{189}{40}\left(\frac{\lambda_1}{4\pi}\right)^2+\frac{53}{10}\left(\frac{\lambda_2}{4\pi}\right)^2-\frac{98}{5}\left(\frac{\alpha}{4\pi}\right)+3\left(\frac{y_t}{4\pi}\right)^2+4\left(\frac{y_b}{4\pi}\right)^2\right]\,. \end{aligned} \tag{A.82}$$

For the case of the Missing Doublet Model the RGEs for the Yukawa couplings are obtained from the above system by setting $\lambda_2 = 0$ and λ_1 evolves as

$$\frac{d\lambda_1}{dt} = \lambda_1\left[448\left(\frac{\lambda_1}{4\pi}\right)^2-48\left(\frac{\alpha}{4\pi}\right)\right]\,. \tag{A.83}$$

These RGEs can be used in conjunction with e.g. eq. (2.20) to check whether the Higgs self-couplings remain perturbative up to the Planck scale.

A.8. Derivation of the $\mu_{\rm GUT}$-dependent Terms in ζ_{α_i} (eq. (6.11))

The $\mu_{\rm GUT}$-dependent terms of the n-loop decoupling coefficient ζ_{α_i} are fixed by n-loop gauge β functions as will be shown in this section. This fact has been exploited as a nontrivial check of our result in section 6.3. We use the shorthand notation $a_i \equiv \frac{\alpha_i}{\pi}$ and $a \equiv \frac{\alpha}{\pi}$ here and start with the decoupling relation

$$a_i(\mu_{\rm GUT}) = \zeta_{\alpha_i}(\mu_{\rm GUT})\,a(\mu_{\rm GUT})\,. \tag{A.84}$$

We take the derivative of eq. (A.84) with respect to $t_{\rm GUT} \equiv \ln(\mu_{\rm GUT})$

$$\frac{da_i}{d\,t_{\rm GUT}} = a\frac{d\zeta_{\alpha_i}}{d\,t_{\rm GUT}}+\zeta_{\alpha_i}\frac{da}{d\,t_{\rm GUT}} \tag{A.85}$$

and use the perturbative expansion of the gauge β functions and the decoupling coefficient up to the two-loop level

$$\begin{aligned} \frac{da}{d\,t_{\mathrm{GUT}}} &= \frac{1}{2}\beta_0 a^2 + \frac{1}{8}\beta_1 a^3\,, \\ \frac{da_i}{d\,t_{\mathrm{GUT}}} &= \frac{1}{2}\beta_0^i a_i^2 + \frac{1}{8}a_i^2 \sum_j \beta_1^{ij} a_j\,, \\ \zeta_{\alpha_i} &= 1 + aA_i + a^2 B_i\,, \end{aligned} \tag{A.86}$$

where the t_{GUT} dependence of a, a_i, A_i and B_i has been suppressed. Inserting this into eq.(A.85), expressing all a_i through a via eq. (A.84) and keeping only terms up to $\mathcal{O}(a^3)$ yields

$$\begin{aligned} 0 &= a^2 \left\{ \frac{dA_i}{d\,t_{\mathrm{GUT}}} - \frac{1}{2}\beta_0^i + \frac{1}{2}\beta_0 \right\} \\ &\quad + a^3 \left\{ \frac{dB_i}{d\,t_{\mathrm{GUT}}} - \beta_0^i A_i + \beta_0 A_i - \frac{1}{8}\sum_j \beta_1^{ij} + \frac{1}{8}\beta_1 \right\}\,. \end{aligned} \tag{A.87}$$

Each order in a must vanish separately, which defines two equations for A_i and B_i. Integrating the equation of $\mathcal{O}(a^2)$ leads to

$$A_i = \tfrac{1}{2}(\beta_0^i - \beta_0)t_{\mathrm{GUT}} - C_0 \tag{A.88}$$

with a t_{GUT}-independent constant C_0. Inserting this result into the $\mathcal{O}(a^3)$ equation and integrating yields

$$B_i = \tfrac{1}{4}(\beta_0^i - \beta_0)^2\, t_{\mathrm{GUT}}^2 + \left[\tfrac{1}{8}(\Sigma_j \beta_1^{ij} - \beta_1) - C_0\,(\beta_0^i - \beta_0)\right] t_{\mathrm{GUT}} + C_1 \tag{A.89}$$

with a t_{GUT}-independent constant C_1. C_0 and C_1 can, however, depend on the physical masses of the heavy particles that are integrated out, which we collectively denote by M_{h}. This dependence is actually what makes the full calculation of ζ_{α_i} nontrivial at the end. Inserting our results for A_i and B_i back into eq. (A.86), the final result is obtained:

$$\begin{aligned} \zeta_{\alpha_i}(\mu_{\mathrm{GUT}}) &= 1 + \frac{\alpha(\mu_{\mathrm{GUT}})}{\pi}\left[\tfrac{1}{2}(\beta_0^i - \beta_0)\, t_{\mathrm{GUT}} - C_0(M_{\mathrm{h}})\right] \\ &+ \left(\frac{\alpha(\mu_{\mathrm{GUT}})}{\pi}\right)^2 \left[\tfrac{1}{4}(\beta_0^i - \beta_0)^2\, t_{\mathrm{GUT}}^2 + \left[\tfrac{1}{8}(\Sigma_j \beta_1^{ij} - \beta_1) - C_0(M_{\mathrm{h}})\,(\beta_0^i - \beta_0)\right] t_{\mathrm{GUT}} + C_1(M_{\mathrm{h}})\right]. \end{aligned} \tag{A.90}$$

The inclusion of Yukawa corrections to this formula is straightforward.

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