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Orthogeodesic Point-Set Embedding of Trees*

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Abstract. Let S be a set of N grid points in the plane, and let G a graph with n vertices ($n \leq N$). An *orthogeodesic point-set embedding* of G on S is a drawing of G such that each vertex is drawn as a point of S and each edge is a chain of horizontal and vertical segments with bends on grid points whose length is equal to the Manhattan distance of its end vertices. We study the following problem. Given a family of trees $\mathcal{F}$ what is the minimum value $f(n)$ such that every n-vertex tree in $\mathcal{F}$ admits an orthogeodesic point-set embedding on every grid-point set of size $f(n)$? We provide polynomial upper bounds on $f(n)$ for both planar and non-planar orthogeodesic point-set embeddings as well as for the case when edges are required to be L-shaped chains. This report is an extended version of a paper by the same authors that is to appear in [6].

1 Introduction

Let S be a set of N points in the plane, and let G be an graph with n vertices such that $n \leq N$. A *point-set embedding* of G on S is a drawing of G such that each vertex of G is drawn as a point of S. If, in addition, the drawing of G is crossing-free, that is, edges are not allowed to intersect in their interior, then the point-set embedding is called *planar*. Point-set embeddings are a classical subject of investigation in graph drawing from both an algorithmic and a combinatorial point of view. From the algorithmic point of view we are typically interested in deciding whether a given graph has point-set embedding on a given set of points. From the combinatorial perspective, however, we typically wish to characterize point sets that admit point-set embeddings for a whole class of graphs, such as trees or planar graphs. Different types of point-set embeddings have been defined depending on the desired type of drawing, that is, depending on how the edges are mapped to the plane. Point-set embeddings have been considered for various classes of graphs, such trees, planar graphs and outerplanar graphs as well as for various types of drawings, such as straight-line drawings and polyline drawings.

Several algorithmic results are known for point-set embeddings in which edges are required to be straight-line segments. Deciding whether a planar graph admits a straight-line planar point-set embedding on a given point set is an NP-complete problem [5], while straight-line planar point-set embeddings of trees [3] and outerplanar graphs [2] can be computed efficiently. From the combinatorial perspective, Gritzmann et al. [12] prove that every planar graph with n vertices admits a straight-line planar point-set embedding on every set of n points in general position if and only if it is outerplanar. Kaufmann

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Table 1: Summary of the results in the paper. Each row corresponds to a family of trees $\mathcal{F}$ and each column corresponds to a type of drawing $\mathcal{D}$.The value in each entry is an upper bound to the minimum value $f(n)$ such that every n-vertex tree in $\mathcal{F}$ admits a point-set embedding of type $\mathcal{D}$ on every point set of size $f(n)$.

	L-Shaped		*Orthogeodesic*	
	Planar	*Non-Planar*	*Planar*	*Planar 2-spaced*
Caterpillars $\Delta = 3$	n [Th. 8]	n [Th. 8]	n [Th. 8]	n [Th. 1]
Trees $\Delta = 3$	n^2-2n+2 [Th. 6]	n [Th. 10][1]	n [Th. 3]	n [Th. 1]
Caterpillars $\Delta = 4$	$3n-2$ [Th. 7]	$n+1$ [Th. 11]	$\lfloor 1.5n \rfloor$ [Th. 4]	n [Th. 1]
Trees $\Delta = 4$	n^2-2n+2 [Th. 6]	$4n-3$ [Th. 9]	$4n$ [Th. 2]	n [Th. 1]

and Wiese show that every n-vertex planar graph admits a polyline planar point-set embedding on every set of n points with at most 2 bends per edge [14]. Colored versions of planar polyline point-set embeddings in which the points are colored and adjacent vertices must be mapped to points with different color have also been investigated [1, 8]. Special research efforts have been devoted to study *universal point sets* for planar graphs. A point set S is *universal* for a family $\mathcal{F}$ of graphs and for a type $\mathcal{D}$ of drawing if every graph in $\mathcal{F}$ admits a point-set embedding of type $\mathcal{D}$ on S. Every universal point set for straight-line planar drawings of planar graphs has size at least $1.235 \cdot n$ [15] while there exist universal point sets of size $\frac{8}{9}n^2$ [4]. For polyline point-set embeddings of planar graphs, on the other hand, there exist universal point sets of size n [10].

In this paper we study *orthogeodesic point-set embeddings* on the grid. Orthogeodesic point-set embeddings were introduced by Katz et al. [13] and require edges to be represented by *orthogeodesic chains*, i.e. by polygonal chains consisting of horizontal and vertical straight-line segments whose total length is equal to the L_1-metric, also called Manhattan metric, between the endpoints. Since orthogeodesic chains correspond to shortest orthogonal connections in the L_1 metric, they can be considered as the counter part of straight lines in the L_2 metric.

Katz et al. [13] considered orthogeodesic point-set embeddings from the algorithmic point of view and proved that it is NP-complete to decide whether a planar graph with n vertices and maximum degree 4 admits an orthogeodesic point-set embedding on n points, while the problem can be solved efficiently for cycles. Katz et al. [13] also show that, if the mapping between vertices and points is given and the bends are required to be at grid points, then the problem is NP-complete even for matchings, while the problem is polynomial-time solvable if bends need not be at grid points. A 2-colored version of the planar orthogeodesic point-set embedding has been studied by Di Giacomo et al. [7].

In this paper we consider orthogeodesic point-set embeddings *on the grid* from the combinatorial point of view. Let P be a set of *grid points* in the plane, i.e., $p = (i, j)$ with $i, j \in \mathbb{Z}$ for all $p \in P$. We write $x(p) := i$ and and $y(p) := j$. A set P of grid points with $x(p) \neq x(q)$ and $y(p) \neq y(q)$ for all $p, q \in P$ with $p \neq q$ is called *general*. For different classes of trees $\mathcal{F}$ and different drawing styles $\mathcal{D}$ we study the value $f(n)$ such that *every* general pointset is universal for orthogeodesic point-set embeddings of *all* trees in $\mathcal{F}$ using $\mathcal{D}$. The restriction to general point sets is necessary since there are arbitrarily large point sets that are not universal for orthogeodesic point-set embeddings of trees, e.g., a set of collinear points. That is, without this restriction $f(n)$ would not be well-defined for graphs other than paths. We consider both planar and non-planar orthogeodesic point-set embeddings as well as the case when edges can be arbitrary orthogeodesic chains or when edges are required to be L-shaped chains. An *L-shaped chain* is an orthogonal chain with only one bend, thus, it is an orthogeodesic chain with the minimum number of bends for general point sets. Table 1 summarizes our results.

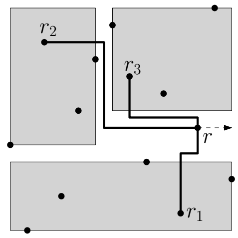

Fig. 1: Planar orthogeodesic point-set embedding of a tree on a general point set with bends allowed to have half-integer coordinates.

The rest of the paper is organized as follows. In Section 2 we study planar orthogeodesic point-set embeddings of trees without any further restriction. In Section 3 we consider the case when edges are required to be L-shaped, that is, if they are allowed to have only one bend. In Section 4 we study L-shaped orthogeodesic point-set embeddings without the planarity restriction. Finally, Section 5 concludes and lists some open problems.

2 Planar Orthogeodesic Pointset Embeddings

We start by considering planar orthogeodesic point-set embeddings of trees. First, we show that every tree with maximum degree 4 can be embedded on every general point set with n points using at most two bends per edge if we require that the horizontal and vertical distance of any two points is at least two. This implies that we can embed every tree with n vertices on every general point set P with n points whose points are not horizontally or vertically aligned, if neither vertices nor bends are required to be grid points.

Theorem 1. *Every tree with n vertices and with maximum degree 4 admits a planar orthogeodesic point-set embedding on every general point set P with n points such that $\min\{|x(p) - x(q)|, |y(p) - y(q)|\} \geq 2$ for all $p, q \in P$ with $p \neq q$.*

Proof: Let T be any tree with n vertices and maximum degree 4. We root T at any vertex r of degree at most 3. Inductively, we prove that T admits a planar orthogeodesic point-set embedding on every general point set P with n points in which (i) each edge has two bends and (ii) no edge intersects a half-line h arbitrarily chosen among the two horizontal and two vertical half-lines starting at r.

The statement is trivially true for $n = 1$. We inductively prove that T admits the required embedding for the case that no edge may intersect the horizontal half-line starting at r and directed rightward. The other constructions are analogous. Let $n_1 \geq 0$, $n_2 \geq 0$, and $n_3 \geq 0$ denote the number of vertices in the subtrees T_1, T_2, and T_3 rooted at children r_1, r_2, and r_3 of the root r of T, respectively. Let P_1 denote the set of the n_1 bottommost points of P. Let P_2 denote the set of the n_2 leftmost points of $P \setminus P_1$. Let p be the bottommost point of $P \setminus (P_1 \cup P_2)$. Let $P_3 = P \setminus (P_1 \cup P_2 \cup \{p\})$ as illustrated in Fig. 1. We embed r on p and we inductively embed T_i on P_i $(i = 1, 2, 3)$ such that no edge intersects the vertical half-line h_1 starting at r_1 in the upward direction, the horizontal half-line h_2 starting at r_2 in the rightward direction and the vertical half-line h_3 starting in r_3 in the downward direction. We connect r with r_1 by an orthogeodesic edge vertically attached to r and to r_1, respectively, and connect the two vertical segments by an intermediate segment s on the horizontal line one unit above the

[1] Fink et al. [11] have independently obtained this result.

top side of the bounding box of P_1. Further, we connect r with r_2 and r_3 analogously as illustrated in Fig. 1.

To see why the induction hypothesis holds, first, note that the embeddings of $T_1, \dots, T_3$ are crossing-free by induction-hypothesis. Clearly, not edge intersects the horizontal half-line h starting in r in the rightward direction by construction. Hence, it suffices to show that the resulting drawing is crossing free, that is none of the edges connecting $r_1, \dots, r_3$ to r are involved in any crossings. Clearly, these edge can cross each other by choice of $P_1, \dots, P_3$ and the construction of the edges. Further, the straight-line segments incident to the vertices $r_1, \dots, r_3$ corresponding to the edges directed towards r, are mapped to the half-lines $h_1, \dots, h_3$ that are not crossed by any other edge by induction hypothesis. That is, there is no crossing in the bounding boxes of $P_1, \dots, P_3$, respectively. Next, consider the edge (r_1, r). The intermediate segment s is located on a horizontal grid line one unit above the highest point in P_1. Hence, this line does not contain any other point since we required $\min\{|x(p) - x(q)|, |y(p) - y(q)|\} \geq 2$ for all $p, q \in P$. Therefore, we can embed the edge as required by (i) and (ii). The remaining edges are analogous, which concludes the induction step. □

As an immediate consequence of Theorem 1 we obtain the following corollary for arbitrary point sets.

Corollary 1. *Let $P \subseteq \mathbb{R}^2$ be a set of points in the plane such $x(p) \neq x(q)$ and $y(p) \neq y(q)$ for all $p, q \in P$ such that $p \neq q$. Then every tree with maximum degree 4 has an orthogeodesic point-set embedding on P with at most two bends per edge.*

To see why Corollary 1 holds, we can consider a subdivision of the grid induced by the points in P. Let $x_1, \dots, x_n$ be the sorted sequence of the x-coordinates of the points in P and let $y_1, \dots, y_n$ be the sorted sequence of y-coordinates of the points in P. Let $\mathcal{G}$ be the grid induced by the horizontal and vertical lines through the points in P as well as by the horizontal lines $y = \frac{x_i + x_{i+1}}{2}$ and the vertical lines $x = \frac{y_i + y_{i+1}}{2}$ for $i = 1, \dots, n-1$. Then clearly, each point in $p \in P$ can be assigned a pair of integer coordinates (i_p, j_p) by numbering the horizontal grid-lines from bottom to top and the vertical grid lines from left to right such that $\min\{|i_p - i_q|, |j_p - j_q|\} \geq 2$. Then the corollary immediately follows from Theorem 1.

As another consequence of Theorem 1 we obtain the following theorem for general point sets on the grid without the restriction on the horizontal and vertical distance of the points.

Theorem 2. *Every tree with n vertices and with maximum degree 4 admits a planar orthogeodesic point-set embedding on every general point set with $4n$ points.*

Proof: We prove that any set P of $4n$ points contains a subset of n points such that no two points have a horizontal or vertical distance of less than two. The theorem then directly follows from Theorem 1. Let the points in P be $p_1, \dots, p_{4n}$ sorted from left to right. Let P_2 consist of the points p_{2i} $(1 \leq i \leq 2n)$ and let $P_1 = P \setminus P_2$. Clearly, the points in P_1 and P_2 have the desired horizontal spacing and one of the sets, say P_1 must contain at least $2n$ points. Repeating the argument for P_1 in the vertical direction yields the claim. □

For trees with maximum degree 3, however, we can improve this result by showing that every such tree has a planar orthogeodesic point-set embedding on every general point set with n points using at most two bends per edge. Hence, every general point set with n points is universal for planar orthogeodesic point-set embeddings of trees with maximum degree 3.

Theorem 3. *Every tree with n vertices and with maximum degree 3 admits a planar orthogeodesic point-set embedding on every general point set with n points.*

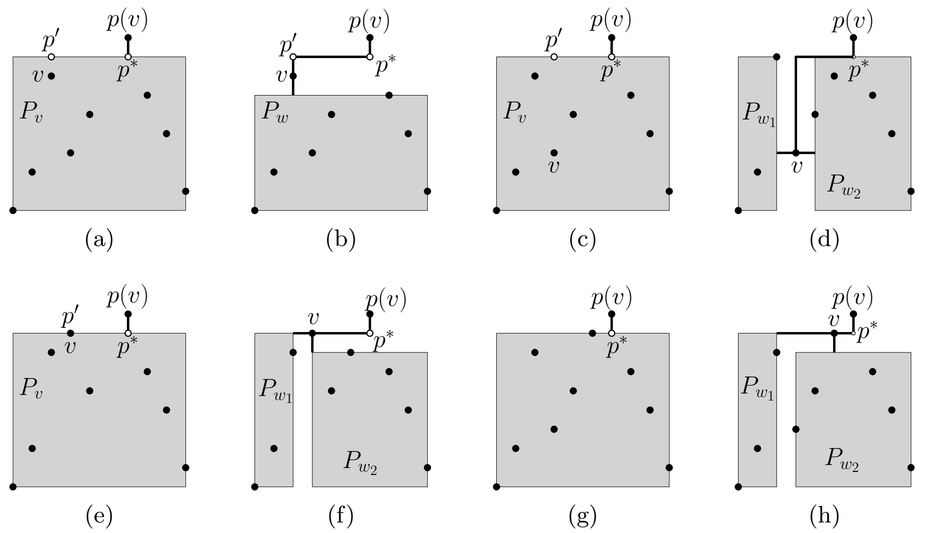

Fig. 2: Embedding a tree with maximum degree 3 on a set of n points. (a) Embedding r. (b)–(c) Embedding s with exactly one child. (d)–(g) Embedding s with two children.

Proof: Let T be a tree with maximum degree 3 and let P be a general point set with n points. We root T in a leaf r. Let w be the unique vertex incident to r. For a vertex v in T we denote the tree rooted in v by T_v. Then we construct a point-set embedding of T on P as follows. First, we embed r on the topmost point p_t of P and assign the subtree T_w rooted in w to the point set $P_w := P \setminus \{p_t\}$ and an axis-parallel rectangle R_w whose opposite corners are the left-bottom corner of the bounding-box of P and the point one unit below the right-top corner of the bounding-box of P. We connect r with the top border of R_w by drawing a vertical segment from p_t to the point p^* one unit below p_t as illustrated in Figure 2a with $r = p(v)$.

Next, we traverse T in a top-down fashion. When considering the subtree T_v of T rooted in v we suppose that T_v has already been assigned to a pointset P_v and an axis-parallel rectangle R_v such that the following invariants hold.

(i) The sets $|T_v|$ and $|P_v|$ have equal size.
(ii) The set P_v lies inside the rectangle R_v.
(iii) The parent $p(v)$ of v lies outside of the rectangle R_v and is connected to R_v by a horizontal or vertical straight-line segment $\overline{p(v), p^*}$ such that p^* is located on the boundary of R_v.
(iv) Let T_u and T_v be two subtrees of T. If T_u is contained in T_v, then R_u is contained inside R_v. Similarly, if R_v is contained in R_u, then R_v is contained inside R_u. If neither T_u is contained in T_v nor T_v is contained in T_u, then $R_u \cap R_v = \emptyset$
(v) Let T be a tree containing the edges $e = (p(v), v)$ and consider the straight-line segment $\sigma = \overline{p(v), p^*}$. Then σ is contained in a rectangle R_v such that T_v has been assigned to R_v if and only if T_v contains e.

Clearly, these invariants are satisfied after we have handled r as described above. Let v be an internal vertex of T. Since T has maximum degree 3, the subtree rooted in v has at most two children. Suppose that p^* is on the top side of R_v; the cases in which p^* is on the bottom, left, or right side of R_v can be discussed analogously.

First, suppose that v has one child w and consider Figures 2a and 2b. We embed v on the topmost point p_t of P_v and assign T_w to the point set $P_w := P_v \setminus \{p_t\}$ and to the rectangle R_w whose opposite corners are the left-bottom corner of R_v and the point one unit below the right-top corner of R_v. Let p^* be the point on the boundary of R_w that is vertically below $p(v)$. and let p' be the point on the boundary of R_w that is vertically above v. We connect $p(v)$ to v extending the horizontal straight-line segment $\overline{p(v)p^*}$ that we have already drawn by the invariant by the horizontal segment $\overline{p^*p'}$ and the horizontal segment $\overline{p'v}$. Finally, we draw a vertical segment connecting v to the top side of R_w as illustrated in Figure 2b.

Next, suppose that v has two children w_1 and w_2. Let $P_{w_1} \subset P_v$ denote the point set composed of the leftmost $|T_{w_1}|$ points of P_v and let P_{w_2} denote the point set composed of the rightmost $|T_{w_2}|$ points in P_v. Further, we denote the single remaining point in $P_v \setminus (P_{w_1} \cup P_{w_2})$ by p. Let p^* be the point on the boundary of R_v vertically below $p(v)$ and let p' be the the point on the boundary of R_v that is vertically above p. Then we assign T_{w_1} the point set P_{w_1} and to the rectangle R_{w_1} whose opposite corners are the left-bottom corner of R_v and the intersection point between the top side of R_v and the vertical line one unit to the left of p. We consider two cases.

First, suppose that the straight-line segment $\overline{p^*p'}$ does not contain any point in P. This case is illustrated in Figures 2c and 2d. We embed v on p and we assign T_{w_2} the point set P_{w_2} and the rectangle R_{w_2} whose opposite corners are the right-bottom corner of R_v and the intersection point between the top side of R_v and the vertical line one unit to the right of p. Then we connect $p(v)$ to v with an edge by extending the straight-line segment $\overline{p(s), p^*}$ by a horizontal segment $\overline{p^*p'}$ and a vertical segment $\overline{p'p}$. Finally, we draw a horizontal segment connecting v with the right side of R_{w_1}. If and we draw a horizontal segment connecting v with the left side of R_{w_2} as illustrated in Figure 2d.

Second, suppose that the segment $\overline{p^*p'}$ contains a point $q \in P$. This case is illustrated in Figures 2e–2h. We consider two sub-cases. First, suppose that $q = p$. Then we embed v on p and we assign T_{w_2} the point set P_{w_2} and the rectangle R_{w_2} whose opposite corners are the right-bottom corner of R_v and the point one unit below the intersection point between the top side of R_v and the vertical line through p. We connect $p(v)$ to v with an edge by extending the straight-line segment $\overline{p(s), p^*}$ by a horizontal segment $\overline{p^*p}$ as illustrated in Figure 2f. Second, suppose that $q \neq p$ Then we embed v on q and we assign assign T_{w_2} to the point set $P_{w_2} \setminus \{q\} \cup \{p\}$ and to the rectangle R_{w_2} whose opposite corners are the right-bottom corner of R_v and the intersection point between the horizontal line one unit below the top side of R_v and the vertical line through p. We connect $p(v)$ to v by extending the vertical segment $\overline{p(s), p^*}$ by the horizontal segment $\overline{p^*q}$. Finally, in both cases, we draw a horizontal segment connecting v with the right side of R_{w_1} and we draw a vertical segment connecting v with the left side of R_{w_2} as illustrated in Figures 2f and 2h.

The case, when v is a leaf is handled similar to the case when v is an internal vertex with only one child.

Clearly, the invariants are maintained by the algorithm. The resulting drawing does not contain any crossings, since for each vertex v, the subtree T_v rooted in v is mapped to an axis-parallel rectangle that does not contain any vertex from $T - T_v$. Further, the constructed edges are orthogeodesic. Hence P admits and orthogeodesic point-set embedding of T □

A *caterpillar* is a tree such that by removing all leaves we are left with a path, called *spine*. In Theorem 2 we show that every tree with maximum degree 4 has a planar orthogeodesic point-set embedding on every general point set with $4n$ points. For caterpillars with maximum degree 4, however, this result is not tight.

Theorem 4. *Every caterpillar with n vertices and with maximum degree 4 admits a planar orthogeodesic point-set embedding on every general point set with $\lfloor 1.5n \rfloor$ points.*

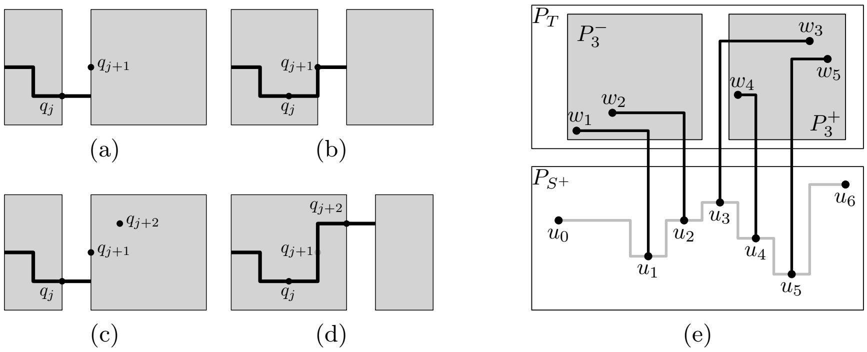

Fig. 3: Embedding a caterpillar on a set of $\lfloor 1.5n \rfloor$ points. (a)–(d) Embedding the spine S^+. (e) Embedding the leaves in T.

Proof: Let C be a caterpillar with n vertices and with maximum degree 4 and let n_i denote the number of vertices of C with degree $i = 1, \ldots, 4$. Let P^* be a general point set with $\lfloor 1.5n \rfloor$ points. From P^* we arbitrarily choose a point set P of size $N = n + n_3 + n_4$ points on which we embed C. First, we show that $N \leq 1.5n$, which implies $N \leq \lfloor 1.5n \rfloor$ since N is a natural number. Suppose for contradiction that $n_3 + n_4 > n/2$. Since each vertex with degree at least 3 is incident to a leaf this yields $n_1 \geq n_3 + n_4$. Summing up we have $n \geq n_1 + n_3 + n_4 \geq 2(n_3 + n_4) > n$, a contradiction.

Next, we show how to embed C on P. Each vertex $v \in V$ is mapped to a point $\pi(v) \in P$. Let $S = (u_1, \ldots, u_k)$ be the spine of C and let u_0 be a leaf incident to u_1 and let u_{k+1} be a leaf incident to u_k. By S^+ we denote the path $(u_0, u_1, \ldots, u_k, u_{k+1})$. Then we consider the vertices u_i for $i = 1, \ldots, k$. If u_i has two adjacent leaves not in S^+, label one of them "top" and one of them "bottom". If u_i has one adjacent leaf not in S^+, arbitrarily label it "top" or "bottom". Let B and T be the sets of leaves of C that have been labeled "bottom" and by "top", respectively.

Let P_T be the subset of the highest $|T|$ points of P and let P_B be the subset of the lowest $|B|$ points. Further, let $Q = P \setminus (P_T \cup P_B)$ be the remaining points. By construction Q contains $t = n_2 + 2(n_3 + n_4) + 2$ points. We embed C on P as follows.

(S1) The leaves in T will be embedded on P_T, the leaves in B will be embedded on P_B and the vertices in S^+ will be embedded on a subset $P_{S^+} \subseteq Q$.

(S2) The spine will be embedded as an x-monotone chain such that u_i is left of u_{i+1} for all $0 \leq i \leq k$.

(S3) Edge $\{u_i, u_{i+1}\}$ occupies the horizontal segment incident to u_i on the right for all $0 \leq i \leq k$. If, additionally, the degree of u_i is at least 3, then edge $\{u_{i-1}, u_i\}$ occupies the horizontal segment incident to u_i on the left for all $1 \leq i \leq k$.

Let $q_1, \ldots, q_t$ be the points in Q sorted from left to right. First, we map u_0 to the leftmost point q_1 in Q. Suppose, we have mapped $u_0, \ldots, u_i$ for some $i < k + 1$ and let $q_j = \pi(u_i)$. If u_{i+1} has degree 2, then we map u_{i+1} to q_{j+1} and we connect u_i and u_{i+1} by an L-shaped orthogeodesic chain composed of a horizontal segment incident to u_i and a vertical segment incident to u_{i+1} as illustrated in Figures 3a and 3b. If u_{i+1} has degree at least 3, then we map u_{i+1} to q_{j+2} skipping the point q_{j+1} in Q and we connect u_i by an orthogeodesic chain consisting of two horizontal segments incident to u_i and u_{i+1}, respectively, and a vertical segment in the column to the left of q_{j+2} as illustrated in Figures 3c and 3d. By construction, u_{k+1} is mapped to a point q_j such that $j \leq n_2 + 2(n_3 + n_4) + 2$ since we only skipped points for vertices with degree at least 3.

Now we describe how to embed the leaves in T on P_T. The leaves in B are embedded on P_B analogously. Let $w_1, \ldots, w_{|T|}$ be the vertices in T sorted such that their corresponding vertices on the spine are sorted from left to right and let T_i be the set of vertices in T that are incident to vertices u_j for $j < i$. For each i with $1 \leq i \leq k$ let P_i^- be the set of points in P_T to the left of $\pi(u_i)$ and let P_i^+ be the set of points in P_T to the right of $\pi(u_i)$, respectively, as illustrated in Figure 3e. Each leaf w_i is mapped to a point $\pi(w_i)$ and is attached to the spine by an L-shaped orthogeodesic chain. We maintain the following invariant.

(L1) If w_i is incident to u_j and $|P_j^-| > |T_j|$, then w_i is mapped to the lowest point $p \in P_i^- \setminus \bigcup_{l=1}^{i-1}\{\pi(w_l)\}$ by an L-shaped orthogeodesic chain consisting of the vertical segment incident to $\pi(u_j)$ and the horizontal segment incident to p. Otherwise, w_i is mapped to the highest unused point in $P_i^+ \setminus \bigcup_{l=1}^{i-1}\{\pi(w_l)\}$ as illustrated in Figure Figure 3e.

The resulting point-set embedding is orthogeodesic by construction. Planarity follows from the invariants as follows.

Due to invariants *(S1)* and *(S2)* the spine is mapped to an x-monotone chain such that the angle at vertices with degree at least 3 is 180 degrees. This implies that the spine does not cross itself and that the vertical segments incident to the vertices with degree at least 3 are unoccupied by the spine. Since, by invariant *(S1)*, we attached the leaves in T above the spine and the leaves in B below the spine, there cannot be a crossing between two edges incident to a leaf in T and a leaf in B, respectively. Suppose for contradiction that there is a crossing between two edges e_i and e_j incident to two leaves w_i and w_j in T, respectively. Without loss of generality we assume $i < j$. If $\pi(w_i) \in P_i^-$ and $\pi(w_j) \in P_j^+$ there cannot be a crossing by construction. If $\pi(w_i) \in P_i^- \subseteq P_j^-$ and $\pi(w_j) \in P_j^-$, then a crossing can only occur if $\pi(w_j) \in P_i^-$ and $\pi(w_j)$ is below $\pi(w_i)$, which contradicts invariant *(L1)*. Analogously, if $\pi(w_i) \in P_i^+$ and $\pi(w_j) \in P_j^+ \subseteq P_i^+$, then a crossing can only occur if $\pi(w_i) \in P_j^+$ and $\pi(w_i)$ is below $\pi(w_j)$, which contradicts invariant *(L1)*. Finally, if $\pi(w_j) \in P_i^-$ and $\pi(w_i) \in P_j^+ \subseteq P_i^+$, then this contradicts invariant *(L1)*, since w_i is only mapped to a point in P_i^+ if there is no unused point in P_i^-. Therefore, the embedding is crossing-free, which concludes the proof. □

3 Planar L-Shaped Orthogeodesic Pointset Embeddings

Next, we consider planar L-shaped orthogeodesic point-set embeddings of trees. First we prove that every tree with n vertices and with maximum degree 4 admits a planar L-shaped point-set embedding on every general point set with $n^2 - 2n + 2$ points. Every point set of this size contains a *diagonal* point set, which is universal for planar L-shaped point-set embeddings of trees with maximum degree 4. Let P be a point set and let $p_1, \ldots, p_n$ denote the points in P ordered by increasing x-coordinates. Then we refer to P as a *positive-diagonal point set* if $y(p_{i+1}) > y(p_i)$ for every $i = 1, \ldots, n-1$. Similarly, we refer to P as a *negative-diagonal point set* if $y(p_{i+1}) < y(p_i)$ for every $i = 1, \ldots, n-1$. If P is either a positive-diagonal point set or a negative-diagonal point set, then we call P a *diagonal point set.* First, we show that any diagonal point set is universal for L-shaped orthogeodesic point-set embeddings of trees with maximum degree 4.

Theorem 5. *Every tree with n vertices and with maximum degree 4 admits a planar L-shaped point-set embedding on every diagonal point set with n points.*

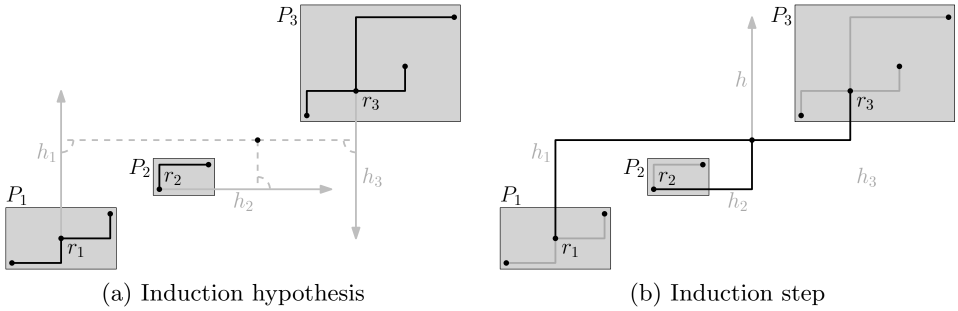

Fig. 4: Orthogeodesic point-set embedding of a tree with maximum degree 4 on a (positive-)diagonal point set.

Proof: Suppose that P is a positive-diagonal point set and let T be a tree with n vertices and with maximum degree 4. The case, when P is a negative-diagonal point set can be handled similarly. We root T in a vertex r with degree at most 3. By induction, we prove that T admits an orthogeodesic planar L-shaped point-set embedding on every diagonal point set with n points such that there is no edge overlapping or crossing a half-line h arbitrarily chosen among the two horizontal and two vertical half-lines starting at r.

In the base case $n = 1$ and the statement is trivially true. Suppose that the claim of the theorem is true for all $n' < n$. We show that T admits an orthogeodesic planar L-shaped point-set embedding on every diagonal point set P with n points such that no edge overlaps or crosses the vertical half-line h starting at r in the upward direction. The cases, when no edge overlaps the vertical half-line starting at r in the downward direction or the horizontal half-lines starting at r in the leftward or rightward direction, respectively, are handled analogously. Let $r_1, \dots r_k$ denote the children of r, that is, $k \leq 3$. Further, let n_i denote the number of vertices of the subtree T_i rooted in r_i for $i = 1, \dots, k$. If r has less than 3 children, we set $n_i = 0$ for $k < i \leq 3$. Let P_1, P_2, and P_3 be the point sets consisting of the bottommost n_1 points of P, the bottommost n_2 points of $P \setminus P_1$, and of the topmost n_3 points of P, respectively. Further, let p be the unique point in $P \setminus (P_1 \cup P_2 \cup P_3)$. By induction hypothesis, we can embed T_i on P_i for $1 \leq i \leq k$ as illustrated in Figure 4a such that no edge of T_1 intersects vertical half-line h_1 starting at r_1 in the upward direction, no edge of T_2 intersects the horizontal half-line h_2 starting at r_2 in the rightward direction and such that no edge of T_3 intersects the vertical half-line h_3 starting at r_3 in the downward direction. Since the bounding boxes of the point sets P_1, P_2 and P_3 are disjoint and since the geodesic chains corresponding to the edges of T_1, T_2 and T_3, respectively, are contained inside the bounding boxes of their respective point sets, the resulting embedding is crossing-free.

Then we embed r on p and we connect r to $r_1, \dots, r_k$ as illustrated in Figure 4b. That is, we connect r to r_i using the horizontal or vertical segment on h_i and the the straight-line segment incident to r that is orthogonal to h_i for all $1 \leq i \leq k$. By the choice of h_i for $1 \leq i \leq k$, the edges are mapped to no-intersecting orthogeodesic chains and we do not use or cross the vertical half-line h starting at r in the upward direction. This concludes the induction step. □

According to the Erdős-Szekeres theorem [9], every general point set with $n^2 - 2n + 2$ points contains either a positive-diagonal point set with n points or a negative-diagonal point set with n points. Hence, from Theorem 5 we immediately obtain the following theorem.

Theorem 6. *Every tree with n vertices and with maximum degree 4 admits a planar L-shaped point-set embedding on every general point set with $n^2 - 2n + 2$ points.*

For caterpillars with maximum degree 4 we can improve the bound of Theorem 6 as the following theorem shows.

Theorem 7. *Every caterpillar with n vertices and with maximum degree 4 admits a planar L-shaped point-set embedding on every general point set with $3n - 2$ points.*

Proof: Let C be a caterpillar with n vertices and with maximum degree 4 and let P be a general point set with $3n-2$ points. Let $(u_2, \ldots, u_{k-1})$ be the spine of C and let u_1 and u_k be two leaves of C adjacent to u_2 and to u_{k-1}, respectively. Let L denote the set of vertices of C containing all leaves of C, except u_1 and u_k. For $i = 1, \ldots, k-1$ we let C_i denote the subtree of C induced by the vertices $u_1, \ldots, u_i$ and by their adjacent leaves in $C - u_k$ and we let $C_k := C$. Observe that C_i is a caterpillar, for $i = 1, \ldots, k$. By induction on i we prove that C_i admits a planar L-shaped point-set embedding on every general point set with $3|C_i| - 2$ points for all $i = 1, \ldots, k$, such that the following invariant is satisfied.

(C1) The horizontal half-line starting at u_i directed rightward does not intersect any edge of the constructed drawing of C_i.

For $i = 1$ we have $|C_1| = 1$ and the induction hypothesis is trivially true. Suppose that the induction hypothesis is true for $i-1$ and consider an arbitrary point set P_i with $3|C_i| - 2$ points. By P_{i-1} we denote the point set consisting of the leftmost $3|C_{i-1}| - 2$ points of P_i. Using the induction hypothesis, we can construct an orthogeodesic planar L-shaped point-set embedding of C_{i-1} on P_{i-1} such that the horizontal half-line starting at u_{i-1} in the rightward direction is not intersected by any edge of the constructed embedding. We distinguish three cases.

First, assume that u_i is not adjacent to a leaf in L. Then, embed u_i on the rightmost point of P_i. Such a point exists since $|P_i \setminus P_{i-1}| = 3$. We connect u_i with u_{i-1} by an L-shaped edge using a horizontal straight-line segment incident to u_{i-1} that neither used nor crossed by any other edge by the induction hypothesis and a vertical straight-line segment attached to u_i.

Second, assume that u_i is adjacent to exactly one leaf a_i and consider the three leftmost points of $P_i \setminus P_{i-1}$. These points exist since $|P_i \setminus P_{i-1}| = 6$. Then, either two of the three points are above the horizontal line h_{i-1} through u_{i-1} or two of the points are below h_{i-1}. Suppose that two of the points, say p_1 and p_2, are above h_{i-1}. The other case can be handled in a similar fashion. Without loss of generality we may assume that p_1 is to the left of p_2. Then, we embed u_i on the rightmost point p_2 and we embed a_i on the leftmost point p_1. Further, we connect u_i with u_{i-1} by an L-shaped edge horizontally attached to u_{i-1} and vertically attached to u_i and we connect u_i with a_i by an L-shaped edge horizontally attached to u_i and vertically attached to a_i.

Third, assume that u_i is adjacent to two leaves a_i and b_i and consider the nine leftmost points of $P_i \setminus P_{i-1}$. These points exist since $|P_i \setminus P_{i-1}| = 9$. Then, either five of such nine points are above the horizontal line h_{i-1} through u_{i-1} or five are below. Suppose that five points $p_1, \ldots, p_5$ are above h_{i-1}. The other case can be handled in a similar fashion. As a consequence of the Erdős-Szekeres-Theorem [9] every general point set with at least 5 points contains a diagonal point set with 3 points, hence the points $p_1, \ldots p_5$ contain a diagonal pointset Without loss of generality we may assume that $p_1, \ldots, p_3$ form a diagonal pointset and that $x(p_1) < x(p_2) < x(p_3)$. If $y(p_1) < y(p_2) < y(p_3)$ as illustrated in Figure 5a), that is, if the points $p_1, \ldots, p_3$ form a positive-diagonal point set, then we embed u_i on p_2, a_i on p_1 and b_i on p_3. Similarly, if $y(p_1) >$

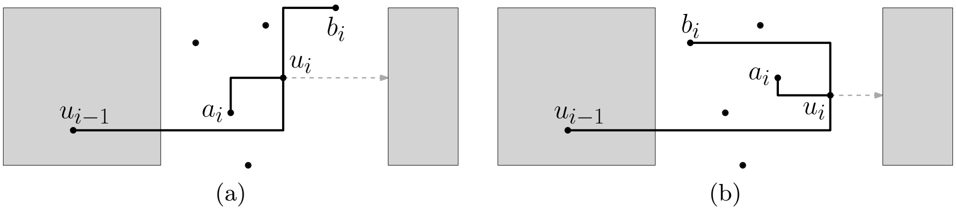

Fig. 5: Planar L-shaped point-set embedding of caterpillars on general point sets. (a) $y(p_1) < y(p_2) < y(p_3)$. (b) $y(p_1) > y(p_2) > y(p_3)$.

$y(p_2) > y(p_3)$ as illustrated in Figure 5b), that is if the points $p_1, \dots, p_3$ for a negative-diagonal point set, then we embed u_i on p_3, a_i on p_2 and b_i on p_1. In both cases, we connect u_i with u_{i-1} by an L-shaped edge that is horizontally attached to u_{i-1} and vertically attached to u_i and we connect u_i to a_i by an L-shaped edge horizontally attached to u_i and vertically attached to a_i, and we connect u_i to b_i by an L-shaped edge vertically attached to u_i and horizontally attached to b_i as illustrated in Figures 5a) and 5b), respectively. Note that we did not use or cross the horizontal half-line starting at u_i in the rightward direction. Hence, the invariant *(C1)* is maintained, which concludes the induction. □

For caterpillars with maximum degree 3 we can improve this bound even further by showing that every such a caterpillar can be embedded on every general point set with n points using L-shaped edges.

Theorem 8. *Every caterpillar with n vertices and with maximum degree 3 admits a planar L-shaped point-set embedding on every general point set with n points.*

Proof: Let C be a caterpillar with n vertices and let P be a general point set consisting of n points $p_1, \dots, p_n$. Assume that the points are sorted such that $x(p_i) < x(p_{i+1})$ for all $1 \leq i \leq n-1$ and let $P_i := \{p_1, \dots, p_i\}$. Let $u_2, \dots, u_{k-1}$ denote the spine of C and let u_1 and u_k be two vertices adjacent to u_2 and u_{k-1}, respectively. Let C_i be the sub-tree of $C - u_k$ induced by the vertices $u_1, \dots, u_i$ and the leaves incident to these vertices for $1 \leq i \leq k-1$. Further, let $C_k := C$.

By induction on i we prove that we can find a planar L-shaped point-set embedding of C_i on P_j such that u_i such that the following invariants are maintained.

- *(C1)* The size of $|P_j|$ equals the size of C_i, that is, $j = |C_i|$.
- *(C2)* The vertex u_i is mapped either to p_j or to p_{j-1}.
- *(C3)* Both the horizontal half-line h_i starting at p_j in the rightward direction as well at least one vertical half-line ℓ_i starting at p_j either in the upward or in the downward direction do not intersect the drawing of C_i.

The induction hypothesis is trivially true for $i = 1$. We map u_1 to p_1 and let h_1 denote the horizontal half-line starting at p_1 in the rightward direction. Further, we let ℓ_1 denote the vertical half-line starting at p_1 in the downward direction. Now, suppose that the induction hypothesis is true for $i-1$ and consider the caterpillar C_i. By the induction hypothesis, we can find a planar L-shaped point-set embedding of C_{i-1} on P_j such that $j = |C_{i-1}|$. Assume, without loss of generality, that ℓ_{i-1} denotes the vertical half-line starting at u_{i-1} in the downward direction as illustrated in Figure 6a and suppose that u_{i-1} is mapped to p_c such that $c \in \{|C_i|, |C_{i-1}|\}$.

First, suppose that u_i has degree at most two, that is it is adjacent to at most two vertices u_{i-1} and, possibly, u_{i+1} if it exists. Then we map u_i to p_{c+1} and connect u_i

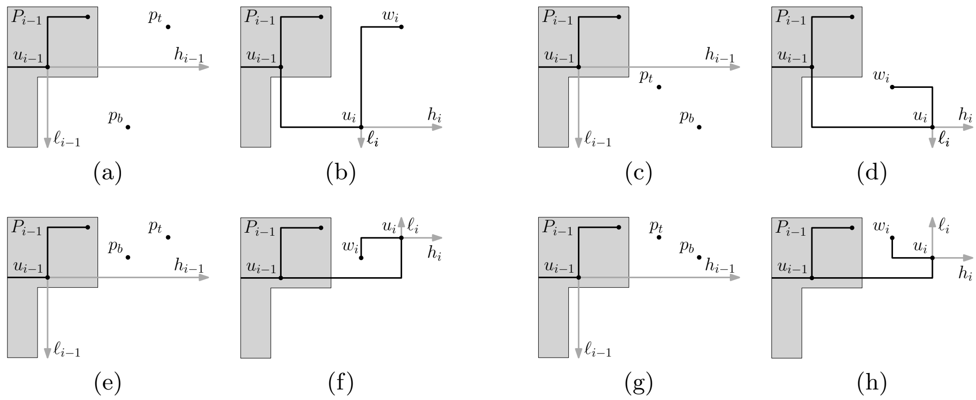

Fig. 6: Orthogeodesic planar L-shaped point-set embedding of a caterpillar with maximum degree 3.

to u_{i-1} by an L-shaped edge that is horizontally attached to u_{i-1} and vertically attached to u_i. Clearly the invariants are maintained.

Second, suppose that u_i has degree three, that is $i < k$ and u_i is adjacent to two vertices u_{i-1} and u_{i+1} as well as an additional leaf w_i. Note that $|C_i| = c + 2$, that is, $|C_i| - |C_{i-1}| = 2$. Let p_b and p_t denote the two vertices in $P_{|C_i|} \setminus P_{|C_{i-1}|}$ such that $y(p_b) < y(p_t)$, that is p_b is below p_t. We distinguish two sub-cases.

First, suppose that p_b is below the vertical half-line h_{i-1} as illustrated in Figures 6a and 6c. Then we map u_i to p_b and we map w_i to p_t. Further, we connect u_{i-1} to u_i by an L-shaped edge that is vertically attached to u_{i-1} and horizontally attached to u_i and we attach w_i to u_i by an L-shaped edge that is vertically attached to u_i and horizontally attached to w_i as illustrated in Figures 6b and 6d. This way, the horizontal half-line h_i starting at u_i in the rightward direction as well as the vertical half-line ℓ_i starting at u_i in the downward direction do not intersect the constructed embedding. Hence, the invariants are maintained.

Second, suppose that p_b is above the vertical half-line h_{i-1}, that is, p_t is also above h_{i-1} as illustrated in Figures 6e and 6g. We map u_i to the rightmost point of p_b and p_t and we map w_i to the leftmost point of p_b and p_t, respectively. Further, we connect u_i to u_{i-1} by an L-shaped edge that is horizontally attached to u_{i-1} and vertically attached to u_i and we connect w_i to u_i by an L-shaped edge that is horizontally attached to u_i and vertically attached to w_i as illustrated in Figures 6f and 6h. Note that the newly constructed edges do not intersect. Clearly, the constructed embedding does not intersect the horizontal half-line h_i starting at u_i in the rightward direction and the vertical half-line ℓ_i starting at u_i in the upward direction. Hence, the invariants are maintained. This concludes the induction.

□

4 Non-Planar *L*-Shaped Orthogeodesic Point-Set Embeddings

Next, we consider non-planar L-shaped orthogeodesic point-set embeddings. We start by showing that every tree with n vertices as a non-planar L-shaped orthogeodesic point-set embedding on every general point set with $4n-3$ points.

Theorem 9. *Every tree with* n *vertices and with maximum degree 4 admits a non-planar* L*-shaped point-set embedding on every general point set with* $4n-3$ *points.*

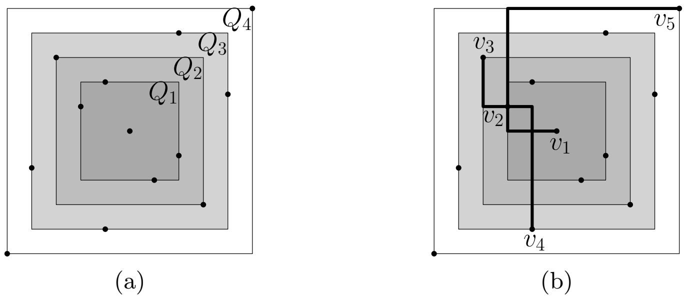

Fig. 7: Non-planar L-shaped point-set embedding of a tree with maximum degree 4.

Proof: Let $T = (V, E)$ be a tree with n vertices and let P be a general point set with $4n - 3$ points. Let T be rooted in a leaf $r \in V$ and let the vertices of T be labeled $r = v_1, \ldots, v_n$ according to a depth-first search in T. Let $Q_n = P$. For $n \geq i \geq 1$, let P_i consist of the points on the boundary of the bounding box of Q_i, and for $n \geq i \geq 2$ let $Q_{i-1} = Q_i \setminus P_i$ as illustrated in Figure 7a. Since the boundary of the bounding box of a general point set contains at least two and at most four points, and since P contains $4n - 3$ points, we have that each set P_i contains at least two and at most four vertices, except for P_1, which contains at least one vertex.

We embed T using L-shaped orthogeodesic chains such that vertex v_i is mapped to a point in P_i for all $1 \leq i \leq n$. We start by mapping the root v_1 to an arbitrary point $p^* \in P_1$. Suppose we have embedded all vertices $v_1, \ldots v_i$ for some $i \geq 1$ and we would like to embed v_{i+1}. Since the vertices are ordered according to a depth-first search, we have already embedded the parent v_j of v_{i+1}. Further, the vertices $v_1, \ldots, v_j$ have been embedded inside the bounding box of the point set Q_i which in the interior of the bounding box of the points in Q_{i+1}. Since v_j has degree at most 4 and since, we have not yet mapped v_{i+1}, at least one of the segments incident to v_j in the drawing is unoccupied by the drawing. Without loss of generality we may assume that the vertical segment above v_j is unoccupied (otherwise we can rotate the instance accordingly). By construction the points in P_{i+1} are on the bounding box of Q_{i+1}, which contains Q_i in its interior. Hence, P_{i+1} contains a point p_t on the top side of the bounding box of Q_{i+1}. Then we map v_{i+1} to p_t and we connect it to v_j by an L-shaped edge that is vertically attached to v_j and horizontally attached to p_t as illustrated in Figure 7b. □

Next, we improve on this by showing that a general point set of size n allows an L-shaped point-set embedding for the class of trees with n vertices and maximum degree 3.

Theorem 10. *Every tree with n vertices and with maximum degree 3 admits a non-planar L-shaped point-set embedding on every general point set with n points.*

Proof: Let T be a tree with n vertices and with maximum degree 3 and let P be a general point set with n points. Assume that T is rooted in a vertex r with degree at most 2. By induction on n we prove that we can find an L-shaped point-set embedding of T on P such that none of the edges occupies the vertical line through r.

If $n = 1$, we map the single vertex of T to the single point in P and we are done. Suppose that the induction hypothesis holds for all $n' < n$. Let $n_1 \geq 0$ and $n_2 \geq 0$ denote the number of vertices in the subtrees T_1 and T_2 rooted at the children r_1 and r_2 of r, respectively. Further, let P_1 and P_2 be the point sets consisting of the leftmost n_1 points and the rightmost n_2 points of P, respectively. Let p be the unique point of P not in P_1 and not in P_2. Then we embed r on p. By induction we can find an L-shaped point set embedding of T_1 on P_1 and of T_2 on P_2 such that the vertical line through r_1 and r_2

(a) induction hypothesis (b) induction step

Fig. 8: Non-planar L-shaped point-set embedding of a tree on a general point set.

is unoccupied by any edge of the resulting drawings, respectively. Then, we connect r to r_1 by an L-shaped edge that is horizontally attached to r and vertically attached to r_1. Similarly, we connect r to r_2 by an L-shaped edge that is horizontally attached to r and vertically attached to r_2 as illustrated in Figure 8. Since the constructed edges are attached to r horizontally and since the embeddings of T_1 and T_2 are contained in the bounding boxes of their respective point sets, which do not intersect the vertical line through r, the maintain the invariant as claimed, which concludes the induction step. □

For caterpillars with maximum degree 4 we can improve this by showing that every general point set with $n + 1$ points admits an orthogeodesic L-shaped point-set embedding of every caterpillar with n vertices and with maximum degree 4.

Theorem 11. *Every caterpillar with n vertices and with maximum degree 4 admits a non-planar L-shaped orthogeodesic point-set embedding on every general point set with $n + 1$ points.*

Proof: Let P be a general point set with $n + 1$ points. Let C be a caterpillar with maximum degree 4 and let $(u_1, \dots, u_k)$ denote the vertices of its spine. Further, let S^+ denote the path $(u_0, u_1, \dots, u_k, u_{k+1})$ where u_0 and u_{k+1} are two leaves incident to u_1 and u_k, respectively. We embed C on P using L-shaped orthogeodesic chains for the edges such that the following invariants are maintained.

(S1) The spine is embedded as a monotone chain starting in the leftmost point in P.
(S2) The spine leaves each vertex along the horizontal segment to its right and enters each vertex along a vertical segment either above or below it.
(S3) All but possibly one point to the left of u_i are occupied by the vertices u_j for $i < j$ and the leaves adjacent to these vertices.

By applying (S3) to u_{k+1} it is clear that $n + 1$ points are sufficient for the embedding.

First, we embed u_0 on the leftmost point in P. Suppose we have mapped all vertices $u_0, \dots, u_i$ for some $0 \leq i \leq k$. Let u_i be mapped to p_j and let P_i^+ be the remaining points to the right of u_i that are not yet occupied by a point.

In order to embed u_{i+1} as well as the leaves incident to it, we distinguish four cases:

Case 1: u_{i+1} has degree at most two. Let p be the leftmost point in P_i^+. We map u_{i+1} to p and connect it to u_i an L-shaped orthogeodesic chain as illustrated in Figures 9a and 9b.

Case 2: u_{i+1} has degree 3. Let w be a leaf incident to u_{i+1}. Let p_1 be the leftmost point in P_i^+ and let p_2 be the leftmost point in P_i^+ to the right of p_1. We map u_{i+1} to p_2 and connect it to u_i by an L-shaped orthogeodesic chain starting with a horizontal segment in u_{i+1}. Further, we map w to p_1 and connect it to u_{i+1} by the horizontal

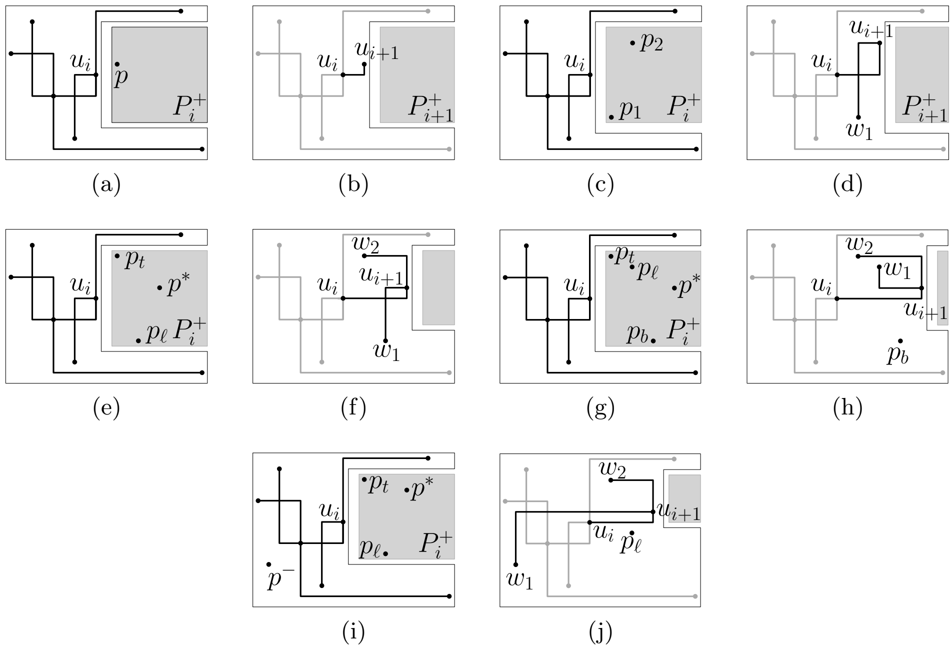

Fig. 9: Embedding a caterpillar on $n+1$ points using L-shaped edges.

segment incident to p_2 to the left and the vertical segment incident to p_1 as illustrated in Figures 9c and 9d.

Case 3: u_{i+1} has degree 4 and there is no unoccupied point to the left of u_i. Let w_1 and w_2 be two leaves incident to u_{i+1}. Recall that we assumed that the vertex u_i be mapped to a point p_j. Let p^* be the leftmost point in P_i^+ with the following property.

> Either *(i)* p^* is above p_j and p^* contains two distinct points p_ℓ and p_t to its left and above, respectively, as illustrated in Figures 9e and 9g or, similarly, *(ii)* p^* is below p_j and p^* contains two distinct points p_ℓ and p_b to its left and below, respectively.

Let Q denote the set of the leftmost 4 points in P_i^+. We claim that Q contains a point p^* with the desired property. Let p_t be the topmost point in Q and let p_b be the bottommost point in Q. Further, let p_ℓ be the leftmost point such that $p_\ell \neq p_t, p_b$ and let q be the remaining point. By construction, q has the desired properties. Hence, there can be at most three points to the left of p^*.

We assume that p^* is above p_j as illustrated in Figures 9e and 9g, respectively. The case when p^* is below p_j is analogous. First, we consider the case that there are only two points in P_i^+ to the left of p^*, namely a point p_t above p^* and a point p_ℓ left of p^* as illustrated in Figure 9e. We map u_{i+1} to p^* and connect it to u_i by an L-shaped orthogeodesic chain consisting of the horizontal segment incident to u_i and the vertical segment incident to u_{i+1}. Further, we map w_1 to p_ℓ and connect it to p^* by an L-shaped orthogeodesic chain consisting of the horizontal segment incident to p^* and the vertical segment incident to p_ℓ. Further, we map w_2 to p_t and connect it by the respective orthogeodesic chain as illustrated in Figure 9f.

Next, we consider the case that there are three points to the left of p^* as illustrated in Figure 9g. Let Q, p_t, p_b and p_ℓ be chosen as described above. We embed u_{i+1} on p^*, w_1 on p_ℓ and w_2 on p_t as in the above description and we leave the point p_b to the left of p^* unused as illustrated in Figure 9h.

Case 4: u_{i+1} has degree 4 and there is a single unoccupied point p^- to the left of u_i. This case is analogous to the Case 3, except that we do not require that p^* contains a point p_ℓ to its left in P_i^+, since p^- will substitute p_ℓ. Note that, as in Case 3, one single point to the left of p^* may remain unoccupied as illustrated in Figures 9i and 9j. □

5 Conclusions

In this paper we studied orthogeodesic point-set embeddings of trees on the grid. For various types of drawings $\mathcal{D}$ and various families of trees $\mathcal{F}$ we proved upper bounds on the minimum value $f(n)$ such that every n-vertex tree in $\mathcal{F}$ admits a point-set embedding of type $\mathcal{D}$ on every point set of size $f(n)$. Since n is a trivial lower bound for $f(n)$ in all considered variants of the problem and since the upper bounds we provided are larger than n for some of the considered variants, it is an interesting topic for future research to close the gap between n and $f(n)$. The gap is especially large for planar L-shaped point-set embeddings for which we only proved a quadratic upper bound. Hence it would be interesting to come up with a sub-quadratic upper bound or a non-trivial lower bound. Further, we restricted our attention to trees, but we may consider the same problem for different classes of graphs.

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