Stability of a conditional Cauchy equation implying the stability of the Jensen-Hosszú equation

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Abstract. For functions $f$ taking values in a Banach space, the stability of the functional equation

$$f(x + y - xy) + f(xy) = f(x) + f(y) \quad (x, y \in \mathbb{R})$$

will be shown. As a simple consequence we get the stability of the Jensen-Hosszú equation

$$f(x + y - xy) + f(xy) = 2f\left(\frac{x + y}{2}\right) \quad (x, y \in \mathbb{R}),$$

which is already known from Kominek [1]; the constant $9/2$ occurring there can be diminished to 2.

In this paper $E$ denotes a (real) Banach space. For $\xi, \eta \in E$ the notation $\xi \overset{\epsilon}{\sim} \eta$ means that $||\xi - \eta|| \leq \epsilon$ (cf. Przebieracz [4]). In the sequel we use the following result from [5]:

Lemma. Suppose $\epsilon, \alpha \geq 0$ and let $g : \mathbb{R} \to E$ satisfy

$$g(x) + g(y) \overset{\epsilon}{\sim} g(x + y) \quad (x + y \geq \alpha).$$

Then we have $g(x) \overset{\epsilon}{\sim} a(x) \quad (x \in \mathbb{R})$, where $a : \mathbb{R} \to E$ is additive.

Now our main result will be the stability of the conditional Cauchy equation

$$f(x + y - xy) + f(xy) = f(x) + f(y) \quad (x, y \in \mathbb{R}).$$

Theorem. Suppose $\epsilon \geq 0$ and let $g : \mathbb{R} \to E$ satisfy

(1) $g(x + y - xy) + g(xy) \overset{\epsilon}{\sim} g(x) + g(y) \quad (x, y \in \mathbb{R}).$

Then we have $g(x) \overset{\epsilon}{\sim} a(x) \quad (x \in \mathbb{R})$, where $a : \mathbb{R} \to E$ is additive.

Proof. It is sufficient to show that

(2) $g(u) + g(v) \overset{\epsilon}{\sim} g(u + v) \quad (u + v \geq 4),$

then the case $\alpha = 4$ of the Lemma can be applied. For $u + v \geq 4$ we have $u \geq 2$ or $v \geq 2$, without loss of generality let us assume $u \geq 2$.  

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Then
\[ \delta := (u + v)^2 - 4v > 0 \]
holds. This is clear for \( v < 0 \); for \( v \geq 0 \) we get
\[ \delta = u^2 + 2uv + v^2 - 4v > 2uv - 4v = 2v(u - 2) \geq 0. \]
Now (3) implies the existence of
\[ x = \frac{u + v}{2} + \sqrt{\frac{(u + v)^2}{4} - v} \quad (> 0). \]
Then
\[ x^2 - (u + v)x + v = 0, \]
\[ x + v \frac{1}{x} - v = u. \]
When setting \( y = v/x \), hence \( v = xy \), we get
\[ x + y - xy = u, \quad xy = v, \quad x + y = u + v, \]
thus (2) is a consequence of (1).

**Corollary.** Suppose \( \varepsilon \geq 0 \) and let \( g : \mathbb{R} \to E \) satisfy
\[ g(x + y - xy) + g(xy) \sim 2g \left( \frac{x + y}{2} \right) \quad (x, y \in \mathbb{R}). \]
Then we have
\[ g(x) \sim a(x) + b \quad (x \in \mathbb{R}), \]
where \( a : \mathbb{R} \to E \) is additive and \( b \in E \).

**Proof.** For \( h(x) = g(x) - g(0) \quad (x \in \mathbb{R}) \) we have \( h(0) = \theta \) (the zero-element of \( E \)), and (4) leads to
\[ h(x + y - xy) + h(xy) \sim 2h \left( \frac{x + y}{2} \right) \quad (x, y \in \mathbb{R}). \]
Now \( y = 0 \) gives
\[ h(x) \sim 2h \left( \frac{x}{2} \right) \quad (x \in \mathbb{R}). \]
Replacing here \( x \) by \( x + y \) and taking (6) into account, we get
\[ h(x + y - xy) + h(xy) \sim h(x + y) \quad (x, y \in \mathbb{R}). \]
According to the Theorem we have \( h(x) \sim a(x) \quad (x \in \mathbb{R}) \), i.e., \( g(x) - g(0) \sim a(x) \quad (x \in \mathbb{R}) \), hence \( g(x) \sim a(x) + b \quad (x \in \mathbb{R}) \), where \( a : \mathbb{R} \to E \) is additive and \( b = g(0) \in E \).
Remarks. When taking $\varepsilon = 0$ in the Corollary, we get the solutions $f : \mathbb{R} \to E$ of the Jensen-Hosszú equation

\[(7) \quad f(x+y-xy) + f(xy) = 2f\left(\frac{x+y}{2}\right) \quad (x, y \in \mathbb{R});\]

they are given by $f(x) = a(x) + b \ (x \in \mathbb{R})$, where $a : \mathbb{R} \to E$ is additive and $b \in E$.

The solutions of (7) are already known from Kominek [1]. From [1] also the stability of (7) is known; more precisely, Kominek obtains the result of the Corollary with $\frac{9}{2}\varepsilon$ instead of $2\varepsilon$ in (5). In a similar way the constant $20\varepsilon$ occurring in the proof of Losonczi [3] for the stability of the Hosszú equation could be diminished in [5] to $4\varepsilon$. The question of best constants in these stability results is not settled by this.

Finally let us mention that Kominek and Sikorska [2] prove stability of the equation in (7) for functions $f : [0,1] \to E$ and $f :]0,1[\to E$, respectively. It would be of interest to have some concrete estimates of the constants occurring there.

References


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