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Stability of a conditional Cauchy equation implying the stability of the Jensen-Hosszú equation

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Abstract. For functions f taking values in a Banach space, the stability of the functional equation

$$f(x+y-xy)+f(xy)=f(x+y) \quad (x,y\in \mathbb{R})$$

will be shown. As a simple consequence we get the stability of the Jensen-Hosszú equation

$$f(x+y-xy)+f(xy)=2f\left(\frac{x+y}{2}\right) \quad (x,y\in \mathbb{R}),$$

which is already known from Kominek [1]; the constant 9/2 occurring there can be diminished to 2.

In this paper E denotes a (real) Banach space. For $\xi,\eta\in E$ the notation $\xi \overset{\varepsilon}{\sim} \eta$ means that $\|\xi-\eta\|\le\varepsilon$ (cf. Przebieracz [4]). In the sequel we use the following result from [5]:

Lemma. *Suppose $\varepsilon,\alpha\ge 0$ and let $g:\mathbb{R}\to E$ satisfy*

$$g(x)+g(y)\overset{\varepsilon}{\sim} g(x+y) \quad (x+y\ge\alpha).$$

Then we have $g(x)\overset{\varepsilon}{\sim} a(x)$ $(x\in\mathbb{R})$, where $a:\mathbb{R}\to E$ is additive.

Now our main result will be the stability of the conditional Cauchy equation

$$f(x+y-xy)+f(xy)=f(x+y) \quad (x,y\in \mathbb{R}).$$

Theorem. *Suppose $\varepsilon\ge 0$ and let $g:\mathbb{R}\to E$ satisfy*

$$g(x+y-xy)+g(xy)\overset{\varepsilon}{\sim} g(x+y) \quad (x,y\in \mathbb{R}). \tag{1}$$

Then we have $g(x)\overset{\varepsilon}{\sim} a(x)$ $(x\in\mathbb{R})$, where $a:\mathbb{R}\to E$ is additive.

Proof. It is sufficient to show that

$$g(u)+g(v)\overset{\varepsilon}{\sim} g(u+v) \quad (u+v\ge 4), \tag{2}$$

then the case $\alpha=4$ of the Lemma can be applied. For $u+v\ge 4$ we have $u\ge 2$ or $v\ge 2$, without loss of generality let us assume

$$u\ge 2.$$

Then

$$\delta := (u+v)^2 - 4v > 0 \tag{3}$$

holds. This is clear for $v < 0$; for $v \geq 0$ we get

$$\delta = u^2 + 2uv + v^2 - 4v > 2uv - 4v = 2v(u-2) \geq 0.$$

Now (3) implies the existence of

$$x = \frac{u+v}{2} + \sqrt{\frac{(u+v)^2}{4} - v}\ (> 0).$$

Then

$$x^2 - (u+v)x + v = 0,$$
$$x + v\frac{1}{x} - v = u.$$

When setting $y = v/x$, hence $v = xy$, we get

$$x + y - xy = u,\ xy = v,\ x + y = u + v,$$

thus (2) is a consequence of (1).

Corollary. *Suppose $\varepsilon \geq 0$ and let $g : \mathbb{R} \to E$ satisfy*

$$g(x+y-xy) + g(xy) \overset{\varepsilon}{\sim} 2g\left(\frac{x+y}{2}\right) \quad (x, y \in \mathbb{R}). \tag{4}$$

Then we have

$$g(x) \overset{2\varepsilon}{\sim} a(x) + b \quad (x \in \mathbb{R}), \tag{5}$$

where $a : \mathbb{R} \to E$ is additive and $b \in E$.

Proof. For $h(x) = g(x) - g(0)$ $(x \in \mathbb{R})$ we have $h(0) = \theta$ (the zero-element of E), and (4) leads to

$$h(x+y-xy) + h(xy) \overset{\varepsilon}{\sim} 2h\left(\frac{x+y}{2}\right) \quad (x, y \in \mathbb{R}). \tag{6}$$

Now $y = 0$ gives

$$h(x) \overset{\varepsilon}{\sim} 2h\left(\frac{x}{2}\right) \quad (x \in \mathbb{R}).$$

Replacing here x by $x + y$ and taking (6) into account, we get

$$h(x+y-xy) + h(xy) \overset{2\varepsilon}{\sim} h(x+y) \quad (x, y \in \mathbb{R}).$$

According to the Theorem we have $h(x) \overset{2\varepsilon}{\sim} a(x)$ $(x \in \mathbb{R})$, i.e., $g(x) - g(0) \overset{2\varepsilon}{\sim} a(x)$ $(x \in \mathbb{R})$, hence $g(x) \overset{2\varepsilon}{\sim} a(x) + b$ $(x \in \mathbb{R})$, where $a : \mathbb{R} \to E$ is additive and $b = g(0) \in E$.

Remarks. When taking $\varepsilon = 0$ in the Corollary, we get the solutions $f : I\!R \to E$ of the Jensen-Hosszú equation

$$f(x+y-xy) + f(xy) = 2f\left(\frac{x+y}{2}\right) \quad (x, y \in I\!R); \tag{7}$$

they are given by $f(x) = a(x) + b$ $(x \in I\!R)$, where $a : I\!R \to E$ is additive and $b \in E$.

The solutions of (7) are already known from Kominek [1]. From [1] also the stability of (7) is known; more precisely, Kominek obtains the result of the Corollary with $\frac{9}{2}\varepsilon$ instead of 2ε in (5). In a similar way the constant 20ε occurring in the proof of Losonczi [3] for the stability of the Hosszú equation could be diminished in [5] to 4ε. The question of best constants in these stability results is not settled by this.

Finally let us mention that Kominek and Sikorska [2] prove stability of the equation in (7) for functions $f : [0,1] \to E$ and $f :]0,1[\to E$, respectively. It would be of interest to have some concrete estimates of the constants occurring there.

References

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