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We present a residual-based a posteriori error estimator for Maxwell's equations in the electric field formulation. The error estimator is formulated in terms of the residual of the considered problem and we prove its *hp*-efficiency. Thus the error indicator bounds the energy error of the computed solution from above and below.

Keywords: adaptive mesh refinement, a posteriori error estimates, *hp* version of the finite element method, Maxwell's equations

65N15, 65N30, 65N50

1. Introduction

In recent years big interest in analyzing the behaviour of electromagnetic fields in nano-scaled environments has developed. In many situations these fields can – after exploiting some basic material properties – be described by Maxwell's equations in the electric field formulation

$$\begin{aligned} \frac{d^2}{dt^2}(\sigma E) + \nabla \times (\alpha \nabla \times E) &= -\frac{dJ}{dt} && \text{in } \Omega \\ \operatorname{div}(\sigma E) &= 0 && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $E : \Omega \times [0,T] \to \mathbb{R}^3$ denotes the electric field, $\sigma : \Omega \to \mathbb{R}^{3,3}$ the conductivity, $J : \Omega \times [0,T] \to \mathbb{R}^3$ the current, $\Omega \subseteq \mathbb{R}^3$ is a connected domain and $\alpha : \Omega \to \mathbb{R}^{3,3}$ denotes the inverse of the magnetic permeability. In many applications, c.f. Refs. 6, 21, Ω has the form $\mathbb{R}^3 \setminus B$ for some polyhedral domain $B \subset \mathbb{R}^3$. To be able to solve system (1.1) numerically one possibly has to restrict the domain Ω to some finite computational domain. On the newly created outer boundary one introduces artifical boundary conditions, c.f. Refs. 7, 10, 15, 22. In the following we restrict ourselves to possibly nonhomogeneous Dirichlet boundary conditions $n \times E = n \times g$ on $\partial\Omega$, where $g : \partial\Omega \times [0,T] \to \mathbb{R}^3$ and $\Omega \subset \mathbb{R}^3$ now denotes the restricted finite computational domain. Other types of boundary conditions can be analyzed as well. In realistic applications there usually is a sharp distinction between regions, where σ can be bounded away from zero, called the *conductor*, and regions, where $\sigma = 0$ holds (cf. Figure 1). Only outside the conductor we need the second equation of system (1.1), because, if we drop the condition $\operatorname{div}(\sigma E) = 0$ here, the solution E is

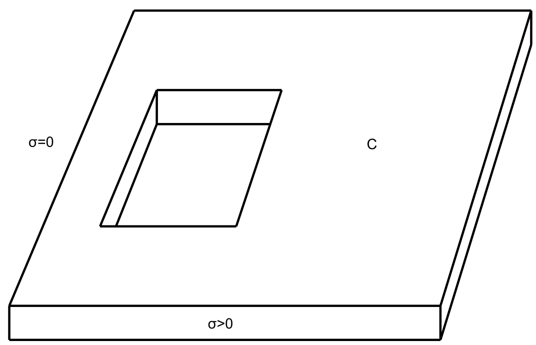

Fig. 1. Realistic conductor

not unique anymore. However, the quantity $\nabla \times E$, which actually is the interesting one in the outer space, is still uniquely determined. Inside the conductor the first equation of system (1.1) is sufficient to determine the electric field E uniquely and $\mathrm{div}(\sigma E) = 0$ follows from the fact that $\mathrm{div}(J) = 0$ always holds for physical reasons. Thus we may switch from problem (1.1) to the ungauged formulation

$$\frac{d^2}{dt^2}(\sigma E) + \nabla \times (\alpha \nabla \times E) = -\frac{dJ}{dt} \quad \text{in } \Omega$$
$$n \times E = n \times g \quad \text{on } \partial\Omega.$$

By applying some time-stepping scheme we obtain the problem to find $u : \Omega \to \mathbb{R}^3$ such that

$$\begin{aligned} \nabla \times (\alpha \nabla \times u) + \beta u &= f && \text{in } \Omega \\ n \times u &= n \times g && \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $f : [0, T] \times \Omega \to \mathbb{R}^3$ is some right-hand side function dependening on J and the values of u from the previous time-step(s) and β is given by σ scaled with the length of the current time-step. Since $\mathrm{div}(\nabla \times u) = 0$ for all $u : \Omega \to \mathbb{R}^3$ and it holds $\mathrm{div}(f) = 0$, too, this implies

$$\mathrm{div}(\beta u) = 0 \quad \text{in } \Omega. \tag{1.3}$$

Now equation (1.2) can be discretized with the finite element method and we get a numerical approximation of the analytic solution u. To solve this problem efficiently it is required to create problem-adapted approximation spaces. This can be obtained either by mesh refinement (h-refinement) or the use of higher order ansatz spaces (p-refinement). A combination of both (hp-FEM) can lead to exponentially fast convergence of the approximated towards the analytical solution[24]. For problem (1.2) h-adaptive mesh creation is discussed in e.g. Refs. 4, 8, 12. For the p- and especially the hp-FEM several results for adaptive mesh creation can be found in e.g. Refs. 12, 18, 26, 27. Since usually one does not know much about the exact solution of the problem one wants to solve, the only way to decide how good the computed approximation is and where it is favourable to refine the mesh any further is by means of the computed solution itself. Therefore one can use error indicators, which give an upper bound for the approximation error. For h-adaptive mesh creation for Maxwell's equations there have been proposed several different error estimators, e.g. Refs. 4, 8, 9, 12. However, for the p- and the hp-adaptive FEM the situation is a bit more demanding. There one also has to deal with varying polynomial degrees of the approximation space on different cells of the triangulation. Therefore an h-adaptive error indicator cannot lead to satisfying results, because one has to take into account the possible change of the polynomial degrees from cell to cell as well. In Ref. 26 a p-hierarchical a posteriori error estimator for Maxwell's equations in the electric field formulation was proposed.

In this paper we will introduce a residual-based a posteriori error estimator and prove its hp-efficiency. The estimator is quite similar to the FEM-part of the a posteriori error estimator derived in Ref. 19, but to the best of our knowledge there has not been any discussion about its hp-capabilities up to now. Thus we will derive a similiar residual-based error estimator, which is based on a pure finite element discretization, and prove its hp-efficiency, i.e. we derive upper and lower bounds for the estimator in terms of the exact error. To conclude this paper we give some numerical examples to illustrate the performance of the error estimator.

The paper is organized as follows: In section 2 we introduce some basic definitions and general assumptions, which we will use throughout the paper. In section 3 we present three interpolation operators and state some polynomial inverse estimates we require in the proofs of the following section. The main results are derived in section 4, where we introduce the residual-based a posteriori error estimator and prove upper and lower bounds for it in terms of the exact error of the approximated solution. Finally, section 5 gives some numerical examples, where the performance of the error estimator from the previous section is shown in various different types of problems.

2. Preliminaries

In this section we introduce the basic notations and state some general assumptions, which we require throughout the paper. Further we derive the weak formulation of problem (1.2).

Remark 1. Although a generalization of the results in this paper into the complex space $\mathbb{C}^3$ is straightforward under certain conditions, we restrict ourselves to real-valued functions u and coefficients α and β for simplicity. We do not want to make the statements artifically involved by having to take into account all the notational details required for complex-valued functions.

2.1. *Notations and General Assumptions*

Let $\Omega \subset \mathbb{R}^3$ be an open domain with Lipschitz continuous boundary. By $L^2(\Omega)$ we denote the Lebesgue space of all square-integrable functions in Ω and by $\gamma \in \mathbb{N}_0^3$ some multi-index. Then we define for $r \geq 0$ the Sobolev spaces H^r and $H^r(\mathrm{curl})$ by

$$H^r(\Omega) := \{u \in L^2(\Omega) : \partial^\gamma u \in L^2(\Omega) \text{ for all } \|\gamma\|_1 \leq r\}$$

and

$$H^r(\mathrm{curl}, \Omega) := \{u \in H^r(\Omega)^3 : \nabla \times u \in H^r(\Omega)^3\},$$

respectively. If $r = 0$, we simply write $H(\mathrm{curl}, \Omega) := H^0(\mathrm{curl}, \Omega)$. By $H_0^r(\mathrm{curl}, \Omega)$ we denote the functions $u \in H^r(\mathrm{curl}, \Omega)$, which additionally satisfy the homogeneous Dirichlet boundary conditions

$$n \times u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

For the space $H_0(\mathrm{curl}, \Omega)$ one can obtain the following decomposition.

Theorem 1 (Helmholtz decomposition). *For each $v \in H_0(\mathrm{curl}, \Omega)$ there exist some $z \in H_0(\mathrm{curl}, \Omega) \cap \{u \in L^2(\Omega)^3 : \mathrm{div}(u) \in L^2(\Omega)\}$ satisfying $\mathrm{div}(z) = 0$ on Ω and $q \in H^1(\Omega)$ such that*

$$v = z + \nabla q.$$

If $\partial\Omega$ is connected, we have $q \in H_0^1(\Omega)$. This splitting is orthogonal with respect to the $L^2(\Omega)$- and the $H(\mathrm{curl}, \Omega)$-inner product. Further there exists some constant $C_H > 0$ such that

$$\|z\|_{H(\mathrm{curl},\Omega)} + \|\nabla q\|_{L^2(\Omega)} \leq C_H \|v\|_{H(\mathrm{curl},\Omega)}.$$

This decomposition of v into z and ∇q is called Helmholtz decomposition.

Proof. See Theorem 1.2.3 in Ref. 14. □

By $\mathcal{K}$ we denote a triangulation of Ω. To avoid strong mesh size changes we assume that the shapes of the cells do not deteriorate too much. Therefore let $\mathcal{K}$ satisfies the following regularity property[24,25].

Definition 1 (Shape regularity). Let $K \in \mathcal{K}$ be the image of reference cell $\widehat{K}$ under some map $F_K : \widehat{K} \to K$ and set $h_K := \mathrm{diam}(K)$. Then $\mathcal{K}$ is γ_1-*shape regular*, if and only if there exists some constant $\gamma_1 > 0$ such that

$$\frac{\|\nabla F_K\|_{L^\infty(\widehat{K})}}{h_K} + h_K \|(\nabla F_K)^{-1} \circ F_K\|_{L^\infty(\widehat{K})} \leq \gamma_1 \quad \forall K \in \mathcal{K}. \tag{2.2}$$

The polynomial degree vector on mesh $\mathcal{K}$ is denoted by $p := (p_K)_{K \in \mathcal{K}}$, $p_K \in \mathbb{N}_0$. To get reliable results also the polynomial degrees present on two neighbouring cells should not differ too much. Therefore we assume there exists some constant $\gamma_2 > 0$ (possibly different from γ_1) such that

$$\frac{p_{K_1}+1}{\gamma_2} \leq p_{K_2}+1 \leq \gamma_2(p_{K_1}+1) \tag{2.3}$$

for all $K_1, K_2 \in \mathcal{K}$ with $K_1 \cap K_2 \neq \emptyset$. Let $K \in \mathcal{K}$ be arbitrary. Then we define

$$\omega_K := K \cup \{L \in \mathcal{K} : K \cap L \neq \emptyset\}.$$

Let $\widehat{Q} := [0,1]^3$ be the reference cube and

$$\widehat{T} := \{x \in \mathbb{R}^3 : 0 \leq x_1, x_2, x_3, x_1+x_2+x_3 \leq 1\}$$

the reference tetrahedron. The finite dimensional approximation space of piecewise vector-valued polynomials is given by

$$V^p(\mathcal{K},\Omega) := \begin{cases} \left\{u \in H_0(\mathrm{curl},\Omega) : \left((\nabla F_K)^{-T} u\big|_K\right) \circ F_K \in Q_{p_K}\left(\widehat{K}\right) \quad \forall K \in \mathcal{K}\right\}, & \text{if } \widehat{K} = \widehat{Q} \\ \left\{u \in H_0(\mathrm{curl},\Omega) : \left((\nabla F_K)^{-T} u\big|_K\right) \circ F_K \in T_{p_K+1}\left(\widehat{K}\right)^3 \quad \forall K \in \mathcal{K}\right\}, & \text{if } \widehat{K} = \widehat{T} \end{cases},$$

where the polynomial space Q_{p_K} is given by

$$Q_{p_K} = Q_{p_K, p_K+1, p_K+1} \times Q_{p_K+1, p_K, p_K+1} \times Q_{p_K+1, p_K+1, p_K}$$

with

$$Q_{p,q,r}\left(\widehat{K}\right) = \mathrm{span}\left\{x_1^i x_2^j x_3^k : x \in \widehat{K}, i \in \{0,\dots,p\}, j \in \{0,\dots,q\}, k \in \{0,\dots,r\}\right\}$$

for $p, q, r \in \mathbb{N}_0$ and the polynomial space T_{p_K} is given by

$$T_{p_K}\left(\widehat{K}\right) = \mathrm{span}\left\{x_1^i x_2^j x_3^k : x \in \widehat{K}, 0 \leq i+j+k \leq p_K\right\}.$$

We denote the finite dimensional approximation space of piecewise scalar polynomials by

$$W^p(\mathcal{K},\Omega) := \begin{cases} \left\{u \in H^1(\Omega) : u\big|_K \circ F_K \in Q_{p_K+1,p_K+1,p_K+1}\left(\widehat{K}\right) \quad \forall K \in \mathcal{K}\right\}, & \text{if } \widehat{K} = \widehat{Q} \\ \left\{u \in H^1(\Omega) : u\big|_K \circ F_K \in T_{p_K+1}\left(\widehat{K}\right) \quad \forall K \in \mathcal{K}\right\}, & \text{if } \widehat{K} = \widehat{T} \end{cases}.$$

We assume that the matrix-valued coefficients $\alpha, \beta : \Omega \to \mathbb{R}^{3,3}$ are piecewise polynomials with polynomial degree vectors $p_\alpha = (p_{\alpha,K})_{K\in\mathcal{K}}$, $p_{\alpha,K} \in \mathbb{N}_0$, and $p_\beta = (p_{\beta,K})_{K\in\mathcal{K}}$, $p_{\beta,K} \in \mathbb{N}_0$, respectively. Further let α and β be uniformly positive definite, i.e. there exist constants $\alpha_{max} \geq \alpha_{min} > 0$ and $\beta_{max} \geq \beta_{min} > 0$ such that for all $u \in L^2(\Omega)^3$ it holds

$$\alpha_{min}\|u\|_{L^2(\Omega)} \leq |u^T \alpha u| \leq \alpha_{max}\|u\|_{L^2(\Omega)}$$

and

$$\beta_{min}\|u\|_{L^2(\Omega)} \leq |u^T \beta u| \leq \beta_{max}\|u\|_{L^2(\Omega)} \tag{2.4}$$

a.e. in Ω, respectively.

2.2. *Weak formulation*

To derive the weak formulation of problem (1.2) we assume that the boundary function $g : \partial\Omega \to \mathbb{R}^3$ is smooth enough such that there exists some lifting function $u_g \in H(\mathrm{curl}, \Omega)$ satisfying

$$\mathrm{div}(\beta u_g) = 0 \quad \text{on } \Omega$$

and $u_g = g$ on $\partial\Omega$. Then it suffices to consider the homogeneous version of problem (1.2) to find $u : \Omega \to \mathbb{R}^3$ such that

$$\begin{aligned} \nabla \times (\alpha \nabla \times u) + \beta u &= f \quad \text{in } \Omega \\ n \times u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.5}$$

By multiplying the first equation with some test function $\phi \in H_0(\mathrm{curl}, \Omega)$ and integration by parts we obtain the weak formulation

$$\int_\Omega ((\nabla \times \phi)^T \alpha \nabla \times u + \phi^T \beta u) = \int_\Omega \phi^T f \quad \forall \phi \in H_0(\mathrm{curl}, \Omega). \tag{2.6}$$

Analogously we obtain the discrete problem to find $u_{\mathrm{FE}} \in V^p(\mathcal{K}, \Omega)$ such that

$$\int_\Omega ((\nabla \times \phi)^T \alpha \nabla \times u_{\mathrm{FE}} + \phi^T \beta u_{\mathrm{FE}}) = \int_\Omega \phi^T f \quad \forall \phi \in V^p(\mathcal{K}, \Omega). \tag{2.7}$$

For $u, v \in H_0(\mathrm{curl}, \Omega)$ let the bilinear form $a : H_0(\mathrm{curl}, \Omega)^2 \to \mathbb{R}$ be given by

$$a(u, v) := \int_\Omega ((\nabla \times u)^T \alpha \nabla \times v + u^T \beta v)$$

and define the energy norm $\| \cdot \|_\Omega : H_0(\mathrm{curl}, \Omega) \to \mathbb{R}$ by

$$\|u\|_\Omega^2 := a(u, u).$$

The bilinear form a is elliptic, i.e. for some constant $C_{ell} > 0$ it holds

$$a(u, u) \geq C_{ell} \|u\|^2_{H(\mathrm{curl}, \Omega)} \quad \forall u \in H_0(\mathrm{curl}, \Omega), \tag{2.8}$$

and continuous, i.e. for some constant $C_c > 0$ it holds

$$|a(u, v)| \leq C_c \|u\|_{H(\mathrm{curl}, \Omega)} \|v\|_{H(\mathrm{curl}, \Omega)} \quad \forall u, v \in H_0(\mathrm{curl}, \Omega) \tag{2.9}$$

(for proofs see for example Ref. 21). Then the Lax-Milgram Theorem states that there exists a unique solution $u \in H_0(\mathrm{curl}, \Omega)$ satisfying (2.6) and a unique solution $u_{\mathrm{FE}} \in V^p(\mathcal{K}, \Omega)$ satisfying (2.7) for $f \in L^2(\Omega)^3$. If β is only semi-positive definite on some set of positive measure, then we still get uniqueness of the solutions in the quotient spaces $H_0(\mathrm{curl}, \Omega)/\operatorname{Ker}(\nabla\times)$ and $V^p(\mathcal{K}, \Omega)/\operatorname{Ker}(\nabla_{\mathrm{FE}}\times)$, where ∇_{FE} denotes the discrete gradient, respectively.

3. Interpolation Operators and Polynomial Inverse Estimates

In this section we introduce interpolation operators for the spaces $H_0^1(\Omega)$ and $H_0(\mathrm{curl},\Omega)$ and give some polynomial inverse estimates, which we require in the next section to prove the *hp*-efficiency of the residual-based error estimator.
We begin with the canonical interpolation operator $\Pi^{\mathrm{grad}} : H_0^1(\Omega) \to W^p(\mathcal{K},\Omega)$, which interpolates functions from the space $H_0^1(\Omega)$ into the scalar finite element approximation space $W^p(\mathcal{K},\Omega)$. For this operator the following estimate was proven in Ref. 24.

Theorem 2 (H^1-conforming interpolation). *Let $u \in H_0^1(\Omega)$. Then for all $K \in \mathcal{K}$ and all faces $\widetilde{f} \subset \partial K$ of K there exists some constant $C_{\mathrm{grad}} > 0$ depending only on γ_1 and γ_2 in $\mathbb{R}_+$ such that*

$$\|\Pi^{\mathrm{grad}}u - u\|_{L^2(K)} + \sqrt{\frac{h_{\widetilde{f}}}{p_{\widetilde{f}}+1}}\|\Pi^{\mathrm{grad}}u - u\|_{L^2(\widetilde{f})} \le C_{\mathrm{grad}}\frac{h_K}{p_K+1}\|\nabla u\|_{L^2(\omega_K)},$$

where $h_{\widetilde{f}} := \mathrm{diam}\left(\widetilde{f}\right)$ and $p_{\widetilde{f}}$ is the maximal polynomial degree essentially present at face $\widetilde{f}$.

Proof. From Ref. 24 and regularity assumptions (2.2) and (2.3) it follows

$$\|\Pi^{\mathrm{grad}}u - u\|_{L^2(K)} \le C_1\frac{h_K}{p_K+1}\|\nabla u\|_{L^2(\omega_K)} \tag{3.1}$$

for some constant $C_1 > 0$ independent of h_K and p_K. Futher, by using regularity assumptions (2.2) and (2.3) we obtain easily

$$\sqrt{\frac{h_{\widetilde{f}}}{p_{\widetilde{f}}+1}}\|\Pi^{\mathrm{grad}}u - u\|_{L^2(\widetilde{f})} \le C_2\|\Pi^{\mathrm{grad}}u - u\|_{L^2(\omega_K)}$$

for some constant $C_2 > 0$ depending solely on γ_1 and γ_2. Then the second estimate

$$\sqrt{\frac{h_{\widetilde{f}}}{p_{\widetilde{f}}+1}}\|\Pi^{\mathrm{grad}}u - u\|_{L^2(\widetilde{f})} \le C_1C_2\frac{h_K}{p_K+1}\|\nabla u\|_{L^2(\omega_K)}$$

follows immediately from inequality (3.1). Finally, setting $C_{\mathrm{grad}} := C_1(C_2+1)$ gives the desired result. □

The interpolation operator presented next maps functions from the space $H_0^r(\mathrm{curl},\Omega)$, $r > \frac{1}{2}$, to the vector-valued finite element approximation space $V^p(\mathcal{K},\Omega)$. In Ref. 11 Demkowicz and Buffa introduced a local $H(\mathrm{curl})$-conforming projection-based interpolation scheme $\Pi_K^{\mathrm{curl}} : H_0^r(\mathrm{curl},K) \to V^p\left(\mathcal{K}\big|_K, K\right)$ for $r > \frac{1}{2}$. Therefore we simply define the global $H(\mathrm{curl})$-conforming interpolation operator $\Pi^{\mathrm{curl}} : H_0^r(\mathrm{curl},\Omega) \to V^p(\mathcal{K},\Omega)$ by $\Pi^{\mathrm{curl}}u\big|_K := \Pi_K^{\mathrm{curl}}u$. Then one can prove the following estimate for the interpolation error.

Theorem 3 ($H(\mathrm{curl})$-conforming interpolation). *Let $K \in \mathcal{K}$, $\widetilde{f} \subset \partial K$, $\varepsilon > 0$, $r > \frac{1}{2} + \varepsilon$ and $u \in H^r(\mathrm{curl}, K)$. Then there exists some constant $C_{\mathrm{curl}} > 0$, $C_{\mathrm{curl}} \in O\left(\frac{1}{\varepsilon}\right)$ for $\varepsilon \to 0$, independent of h_K and p_K, such that*

$$\|\Pi^{\mathrm{curl}} u - u\|_{L^2(K)} + \sqrt{\frac{h_{\widetilde{f}}}{p_{\widetilde{f}} + 1}} \|\Pi^{\mathrm{curl}} u - u\|_{L^2(\widetilde{f})} \leq C_{\mathrm{curl}} \frac{h_K^k}{(p_K + 1)^{r-\varepsilon}} \|u\|_{H^r(\omega_K)},$$

where $k = \min\{r, p_K + 2\}$.

Proof. The proof follows in the same fashion as the proof of Theorem 2. With Theorem 5 in Ref. 11, the Bramble-Hilbert Lemma and regularity assumptions (2.2) and (2.3) it follows

$$\|\Pi^{\mathrm{curl}} u - u\|_{L^2(K)} \leq C_1 \frac{h_K^k}{(p_K + 1)^{r-\varepsilon}} \|u\|_{H^r(\omega_K)}$$

for some constant $C_1 > 0$ independent of h_K and p_K. Then the second estimate can be derived analogously to the proof above. □

Remark 2.

(1) There have been proposed several approaches, e.g. in Refs. 9, 23, to overcome the strong regularity assumptions $u \in H^r(\mathrm{curl}, \Omega)$ for $r > \frac{1}{2} + \varepsilon$. Unfortunately all these solutions extend the domain of integration from edge e to some patch $\omega_e \supsetneq e$ embedding e. Thus one obtains a quasi-local Clément-type interpolation, which can be used for deriving p-estimates of the interpolation error, but an extension to the hp-context seems difficult, because those quasi-local interpolation operators preserve polynomials but not piecewise polynomials.

(2) Following the lines of the proof of Theorem 5 in Ref. 11 we observe that this extra regularity $u \in H^r(\mathrm{curl}, \Omega)$ for $r > \frac{1}{2} + \varepsilon$ instead of $u \in H(\mathrm{curl}, \Omega)$ is required only in the first interpolation step of the interpolation-projection scheme, where the function u is interpolated on the lowest order edge shape function ϕ_e by the Whitney interpolant

$$\Pi_e^{\mathrm{W}} u := \left(\int_e u^T t_e \right) \phi_e.$$

Here e denotes some edge of K and t_e is the unit tangential of e. In Ref. 1 Amrouche et al. showed that this regularity assumption could be weakened to $u \in H_\varepsilon(\Omega) := \{u \in L^{2+\varepsilon}(\Omega)^3 : \nabla \times u \in L^{2+\varepsilon}(\Omega)^3,\ n \times u \in L^{2+\varepsilon}(\partial\Omega)^3\}$.

Now let us state the following important result from Ref. 16 on the interplay of the two interpolation operators Π^{curl} and Π^{grad} defined above.

Theorem 4 (Commuting diagram property). *The interpolation operators $\Pi^{\mathrm{grad}} : H_0^1(\Omega) \to W^p(\mathcal{K}, \Omega)$ and $\Pi^{\mathrm{curl}} : H_0^r(\mathrm{curl}, \Omega) \to V^p(\mathcal{K}, \Omega)$ for $r > \frac{1}{2}$ make the* *following diagram commute:*

$$\begin{array}{ccc} H_0^{1+r}(\Omega) & \xrightarrow{\nabla} & H_0^r(\mathrm{curl},\Omega) \\ \Big\downarrow{\scriptstyle \Pi^{\mathrm{grad}}} & & \Big\downarrow{\scriptstyle \Pi^{\mathrm{curl}}} \\ W^p(\mathcal{K},\Omega) & \xrightarrow[\nabla]{} & V^p(\mathcal{K},\Omega) \end{array}$$

Proof. See Theorem 13 in Ref. 16. □

For completeness let us also define the L^2-interpolation $\Pi : L^2(\Omega)^3 \to X^p(\mathcal{K},\Omega)$, which maps functions from the space $L^2(\Omega)^3$ to the finite dimensional approximation space of piecewise vector-valued polynomials

$$X^p(\mathcal{K},\Omega) := \begin{cases} \left\{ f \in L^2(\Omega)^3 : f\big|_K \circ F_K \in Q_{p_K+1,p_K+1,p_K+1}\left(\widehat{K}\right)^3 \quad \forall K \in \mathcal{K} \right\}, & \text{if } \widehat{K} = \widehat{Q} \\ \left\{ f \in L^2(\Omega)^3 : f\big|_K \circ F_K \in T_{p_K+1}\left(\widehat{K}\right)^3 \quad \forall K \in \mathcal{K} \right\}, & \text{if } \widehat{K} = \widehat{T} \end{cases}$$

We need this interpolation operator to distinguish between the right-hand side function f and its implementation Πf.
Now we give some polynomial inverse estimates, which we require in the proofs of the next section. First we collect some inverse estimates on an arbitrary cell $K \in \mathcal{K}$.

Corollary 1 (Polynomial inverse estimates I). *Let $K \in \mathcal{K}$ be arbitrary and $u \in Q_{p_K+1,p_K+1,p_K+1}(K)$ or $u \in T_{p_K+1}(K)$ denote some polynomial.*

(1) Then there exists some constant $C_{pol,1} > 0$ independent of h_K and p_K such that

$$\|\partial^\gamma u\|_{L^2(K)} \le C_{pol,1} \frac{p_K + 1}{h_K} \|u\|_{L^2(K)}$$

for all multi-indices $\gamma \in \mathbb{N}_0^3$ satisfying $\|\gamma\|_1 = 1$.

(2) Let $a, b \in \mathbb{R}$ such that $b > a > -\frac{1}{2}$ and define the smoothing function $\phi_K : \mathbb{R}^3 \to \mathbb{R}_+$ by

$$\phi_K(x) := \frac{1}{h_K} \operatorname{dist}(x, \partial K).$$

Then there exists some constant $C_{pol,2} > 0$ independent of p_K such that

$$\|\phi_K^a u\|_{L^2(K)} \le C_{pol,2}(p_K + 1)^{b-a} \|\phi_K^b u\|_{L^2(K)}.$$

Proof. The proofs on the reference cell $\widehat{K}$ follow the lines of the derivation of the one-dimensional analogues Lemmata 4 and 5 in Ref. 5. Then one can apply a mapping from reference cell $\widehat{K}$ to the actual cell K to get the desired result. From the regularity assumptions (2.2) and (2.3) we know that the constants $C_{pol,1}, C_{pol,2} > 0$ are independent of h_K and p_K. □

Next we give some inverse estimates on a face $\widetilde{f} \subset \partial K$ of cell K.

Corollary 2 (Polynomial inverse estimates II). *Let $\widetilde{f} \subset \partial K$ be a face of some cell $K \in \mathcal{K}$ and define*

$$\omega_{\widetilde{f}} := \bigcup_{L \in \mathcal{K}} \{L : \widetilde{f} \text{ is a face of } L\}.$$

The smoothing function $\phi_{\omega_{\widetilde{f}}} : \mathbb{R}^3 \to \mathbb{R}_+$ given by

$$\phi_{\omega_{\widetilde{f}}}(x) := \frac{1}{\operatorname{diam}(\omega_{\widetilde{f}})} \operatorname{dist}(x, \partial\omega_{\widetilde{f}}).$$

Further let $a, b \in \mathbb{R}$ such that $b > a > -\frac{1}{2}$. Then, for every polynomial $u \in Q_{p_K+1,p_K+1,p_K+1}\left(K\big|_{\widetilde{f}}\right)$ or $u \in T_{p_K+1}\left(K\big|_{\widetilde{f}}\right)$ there exists some extension $v \in H_0^1(\omega_{\widetilde{f}})$ such that

(1) $v\big|_{\widetilde{f}} = \phi^a_{\omega_{\widetilde{f}}} u$

(2) There exist some constants $C_{pol,3}(a), C_{pol,4}(a) > 0$ depending solely on a such that

$$\|\partial^\gamma v\|_{L^2(\omega_{\widetilde{f}})} \leq C_{pol,3}(a) \frac{(p_{\widetilde{f}}+1)^{1-a}}{\sqrt{h_{\widetilde{f}}}} \left\| \phi^a_{\omega_{\widetilde{f}}} u \right\|_{L^2(\widetilde{f})}$$

for all multi-indices $\gamma \in \mathbb{N}_0^3$ with $\|\gamma\|_1 = 1$ and

$$\|v\|_{L^2(\omega_{\widetilde{f}})} \leq C_{pol,4}(a) \sqrt{h_{\widetilde{f}}} \|\phi^a_{\omega_{\widetilde{f}}} u\|_{L^2(\widetilde{f})}.$$

(3) There exists some constant $C_{pol,5} > 0$ independent of $p_{\widetilde{f}}$ such that

$$\|\phi^a_{\omega_{\widetilde{f}}} u\|_{L^2(\widetilde{f})} \leq C_{pol,5} (p_{\widetilde{f}}+1)^{b-a} \|\phi^b_{\omega_{\widetilde{f}}} u\|_{L^2(\widetilde{f})}.$$

Proof.

(1) $v \in H_0^1(\omega_{\widetilde{f}})$ can be constructed explicitly in the same way as in the proof of Lemma 2.6 in Ref. 20.

(2) We observe

$$0 \leq \phi_{\omega_{\widetilde{f}}} \leq \frac{1}{2} \tag{3.2}$$

and that for all $x \in \omega_{\widetilde{f}}$ it holds

$$\begin{aligned} \left|\nabla \phi_{\omega_{\widetilde{f}}}(x)\right| &\leq \frac{1}{\operatorname{diam}(\omega_{\widetilde{f}})} |\nabla \operatorname{dist}(x, \partial\omega_{\widetilde{f}})| \\ &\leq C_{\text{grad}} h^{-1}_{\widetilde{f}} \end{aligned} \tag{3.3}$$

for some constant $C_{\text{grad}} > 0$ independent of $h_{\widetilde{f}}$ by regularity assumption (2.2). Then the proof of the first inequality follows with Corollary 1, regularity assumptions (2.2) and (2.3) and the second inequality.
The second estimate follows by a direct calculation.

(3) The proof follows the lines of the proof of Theorem 2.5 in Ref. 20. □

4. Error estimator

In this section we define the residual-based a posteriori error estimator and show its *hp*-efficiency, i.e. we derive upper and lower bounds in terms of the exact error of the approximated solution for the error estimator.
We define the error estimator η as the sum of local error indicators η_K, $K \in \mathcal{K}$:

$$\eta^2 := \sum_{K \in \mathcal{K}} \eta_K^2. \tag{4.1}$$

For $K \in \mathcal{K}$ the local indicators η_K can be decomposed in the following way

$$\eta_K^2 := \eta_{R,K}^2 + \eta_{B,K}^2,$$

where $\eta_{R,K}$ denotes the residual-based term and $\eta_{B,K}$ the boundary term. The residual term $\eta_{R,K}$ is given by

$$\eta_{R,K}^2 := \frac{h_K^2}{(p_K+1)^2} \left(\| \operatorname{res} \|_{L^2(K)}^2 + \| \operatorname{div}(\beta u_{\mathrm{FE}}) \|_{L^2(K)}^2 \right), \tag{4.2}$$

where

$$\operatorname{res} := \Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}},$$

and the boundary term $\eta_{B,K}$ by

$$\eta_{B,K}^2 := \sum_{\widetilde{f} \subset \partial K \cap \Omega} \frac{h_{\widetilde{f}}}{2(p_{\widetilde{f}}+1)} \left(\| n_{\widetilde{f}} \times [\alpha \nabla \times u_{\mathrm{FE}}] \|_{L^2(\widetilde{f})}^2 + \| n_{\widetilde{f}}^T [\beta u_{\mathrm{FE}}] \|_{L^2(\widetilde{f})}^2 \right),$$

where $n_{\widetilde{f}}$ denotes the outward-pointing unit normal vector of cell K on face $\widetilde{f}$ and $[\cdot]$ denotes the jump over the face.
First we derive an upper bound for the energy error $\|u - u_{\mathrm{FE}}\|$ in terms of the error estimator η. This bound then serves as a lower bound for error estimator (4.1).

Theorem 5. *Let $u_{FE} \in V^p(\mathcal{K}, \Omega)$ be the solution of discrete problem (2.7) and $u \in H_0^r(\operatorname{curl}, \Omega)$ be the solution of weak problem (2.6) for some $\varepsilon > 0$ and $r > \frac{1}{2} + \varepsilon$. Further we assume that the triangulation $\mathcal{K}$ of Ω satisfies regularity assumptions (2.2) and (2.3). Then there exists some constant $C_1 > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|u - u_{FE}\|_\Omega^2 \le C_1 \sum_{K \in \mathcal{K}} (p_K+1)^{2\varepsilon} \left(\eta_K^2 + \frac{h_K^2}{(p_K+1)^2} \|\Pi f - f\|_{L^2(K)}^2 \right).$$

Proof. By definition we have

$$\|u - u_{\mathrm{FE}}\|_\Omega^2 = a(u - u_{\mathrm{FE}}, u - u_{\mathrm{FE}})$$

and, since $\Pi^{\mathrm{curl}}(u - u_{\mathrm{FE}}) \in V^p(\mathcal{K}, \Omega)$, using the Galerkin orthogonality yields

$$\|u - u_{\mathrm{FE}}\|_\Omega^2 = a((I - \Pi^{\mathrm{curl}})(u - u_{\mathrm{FE}}), u - u_{\mathrm{FE}}), \tag{4.3}$$

where I denotes the identity mapping. Set $e := u - u_{\mathrm{FE}}$. Since $e \in H_0(\mathrm{curl}, \Omega)$, we know from Theorem 1 that there exists some $z \in H_0(\mathrm{curl}, \Omega)$ and $q \in H^1(\Omega)$ such that $e = z + \nabla q$ and (4.3) reads

$$\begin{aligned}\|u - u_{\mathrm{FE}}\|_\Omega^2 &= a((I - \Pi^{\mathrm{curl}})e, u - u_{\mathrm{FE}}) \\ &= \sum_{K \in \mathcal{K}} \Bigg(\int_K \left(\nabla \times (I - \Pi^{\mathrm{curl}})(z + \nabla q)\right)^T \alpha \nabla \times (u - u_{\mathrm{FE}}) \\ &\qquad + \int_K (I - \Pi^{\mathrm{curl}})(z + \nabla q)^T \beta (u - u_{\mathrm{FE}}) \Bigg).\end{aligned}$$

By Theorem 4 we obtain

$$\begin{aligned}\|u - u_{\mathrm{FE}}\|_\Omega^2 = \sum_{K \in \mathcal{K}} \Bigg(&\int_K (\nabla \times ((I - \Pi^{\mathrm{curl}})z + \nabla (I - \Pi^{\mathrm{grad}})q))^T \alpha \nabla \times (u - u_{\mathrm{FE}}) \\ &+ \int_K ((I - \Pi^{\mathrm{curl}})z + \nabla (I - \Pi^{\mathrm{grad}})q)^T \beta (u - u_{\mathrm{FE}}) \Bigg)\end{aligned}$$

and, since $\nabla \times \nabla \widetilde{q} = 0$ for all $\widetilde{q} \in H^1(\Omega)$, this implies

$$\begin{aligned}\|u - u_{\mathrm{FE}}\|_\Omega^2 = \sum_{K \in \mathcal{K}} \Bigg(&\int_K \left(\nabla \times (I - \Pi^{\mathrm{curl}})z\right)^T \alpha \nabla \times (u - u_{\mathrm{FE}}) \\ &+ \int_K \left((I - \Pi^{\mathrm{curl}})z + \nabla (I - \Pi^{\mathrm{grad}})q\right)^T \beta (u - u_{\mathrm{FE}}) \Bigg).\end{aligned}$$

Then integration by parts yields

$$\begin{aligned}\|u - u_{\mathrm{FE}}\|_\Omega^2 = \sum_{K \in \mathcal{K}} \Bigg(&\int_K (I - \Pi^{\mathrm{curl}})z^T \left(\nabla \times (\alpha \nabla \times (u - u_{\mathrm{FE}})) + \beta (u - u_{\mathrm{FE}})\right) \\ &+ \int_K (\Pi^{\mathrm{grad}} - I) q \,\mathrm{div}(\beta (u - u_{\mathrm{FE}})) \\ &+ \int_{\partial K} (n \times (I - \Pi^{\mathrm{curl}})z)^T (n \times (\alpha \nabla \times (u - u_{\mathrm{FE}}))) \times n \\ &+ \int_{\partial K} (I - \Pi^{\mathrm{grad}}) q n^T \beta (u - u_{\mathrm{FE}}) \Bigg),\end{aligned}$$

where n denotes the outward-pointing unit normal vector to cell K. Using the strong formulation (2.5) in the first term and the divergence condition (1.3) in the second term yields

$$\begin{aligned}\|u-u_{\mathrm{FE}}\|_{\Omega}^2 = \sum_{K\in\mathcal{K}} \Bigg(&\int_K (I-\Pi^{\mathrm{curl}})z^T(f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}) \\ &+ \int_K (I-\Pi^{\mathrm{grad}})q\,\mathrm{div}(\beta u_{\mathrm{FE}}) \\ &+ \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2}\Bigg(\int_{\widetilde{f}} (n_{\widetilde{f}}\times(I-\Pi^{\mathrm{curl}})z)^T(n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}])\times n_{\widetilde{f}} \\ &\qquad + \int_{\widetilde{f}} (I-\Pi^{\mathrm{grad}})q n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\Bigg)\Bigg)\end{aligned}$$

and with the Cauchy-Schwarz inequality we have

$$\begin{aligned}\|u-u_{\mathrm{FE}}\|_{\Omega}^2 \le \sum_{K\in\mathcal{K}} \Bigg(&\|(I-\Pi^{\mathrm{curl}})z\|_{L^2(K)}\|f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}\|_{L^2(K)} \\ &+ \|(I-\Pi^{\mathrm{grad}})q\|_{L^2(K)}\|\,\mathrm{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} \\ &+ \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2}\Big(\|(I-\Pi^{\mathrm{curl}})z\|_{L^2(\widetilde{f})}\|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} \\ &\qquad + \|(I-\Pi^{\mathrm{grad}})q\|_{L^2(\widetilde{f})}\|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})}\Big)\Bigg).\end{aligned}$$

With Theorems 2 and 3 we get

$$\begin{aligned}\|u-u_{\mathrm{FE}}\|_{\Omega}^2 \le \sum_{K\in\mathcal{K}} \Bigg(&C_{\mathrm{curl}}\frac{h_K}{(p_K+1)^{1-\varepsilon}}\|z\|_{H^1(\omega_K)}\|f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}\|_{L^2(K)} \\ &+ C_{\mathrm{grad}}\frac{h_K}{p_K+1}\|\nabla q\|_{L^2(\omega_K)}\|\,\mathrm{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} \\ &+ \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2}\Bigg(\widetilde{C}_{\mathrm{curl}}\frac{\sqrt{h_K}}{(p_K+1)^{\frac{1}{2}-\varepsilon}}\|z\|_{H^1(\omega_K)}\|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} \\ &\qquad + \widetilde{C}_{\mathrm{grad}}\sqrt{\frac{h_K}{p_K+1}}\|\nabla q\|_{L^2(\omega_K)}\|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})}\Bigg)\Bigg)\end{aligned}$$

for some constants $\widetilde{C}_{\mathrm{curl}}, \widetilde{C}_{\mathrm{grad}} > 0$, which are independent of h_K and p_K by regularity assumptions (2.2) and (2.3). According to Ref. 1, Theorem 2.17, $H(\mathrm{curl}, K)$

is continuously embedded in $H^1(K)^3$ and, thus, it follows

$$\|u-u_{\mathrm{FE}}\|_\Omega^2 \le \sum_{K\in\mathcal{K}} \Bigg(C_{\mathrm{curl}} \frac{h_K}{(p_K+1)^{1-\varepsilon}} \|z\|_{H(\mathrm{curl},\omega_K)} \|f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}\|_{L^2(K)} + C_{\mathrm{grad}} \frac{h_K}{p_K+1} \|\nabla q\|_{L^2(\omega_K)} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} + \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2} \Bigg(\widetilde{C}_{\mathrm{curl}} \frac{\sqrt{h_K}}{(p_K+1)^{\frac{1}{2}-\varepsilon}} \|z\|_{H(\mathrm{curl},\omega_K)} \|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} + \widetilde{C}_{\mathrm{grad}} \sqrt{\frac{h_K}{p_K+1}} \|\nabla q\|_{L^2(\omega_K)} \|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} \Bigg) \Bigg)$$

and with Theorem 1 we obtain

$$\|u-u_{\mathrm{FE}}\|_\Omega^2 \le C_H \sum_{K\in\mathcal{K}} \Bigg(C_{\mathrm{curl}} \frac{h_K}{(p_K+1)^{1-\varepsilon}} \|f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}\|_{L^2(K)} + C_{\mathrm{grad}} \frac{h_K}{p_K+1} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} + \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2} \Bigg(\widetilde{C}_{\mathrm{curl}} \frac{\sqrt{h_K}}{(p_K+1)^{\frac{1}{2}-\varepsilon}} \|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} + \widetilde{C}_{\mathrm{grad}} \sqrt{\frac{h_K}{p_K+1}} \|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} \Bigg) \Bigg) \|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},\omega_K)}.$$

Then the Cauchy-Schwarz inequality yields

$$\|u-u_{\mathrm{FE}}\|_\Omega^2 \le C_H \sqrt{C} \Bigg(\sum_{K\in\mathcal{K}} \Bigg(C_{\mathrm{curl}} \frac{h_K}{(p_K+1)^{1-\varepsilon}} \|f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}\|_{L^2(K)} + C_{\mathrm{grad}} \frac{h_K}{p_K+1} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} + \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2} \Bigg(\widetilde{C}_{\mathrm{curl}} \frac{\sqrt{h_K}}{(p_K+1)^{\frac{1}{2}-\varepsilon}} \|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} + \widetilde{C}_{\mathrm{grad}} \sqrt{\frac{h_K}{p_K+1}} \|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} \Bigg) \Bigg)^2 \Bigg)^{\frac{1}{2}} \|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)}$$

for some constant $C>0$ independent of h_K and p_K and with the ellipticity of the bilinear form a (2.8) we obtain

$$\|u-u_{\mathrm{FE}}\|_\Omega^2 \le C_H^2 \frac{C}{C_{ell}} \sum_{K\in\mathcal{K}} \Bigg(C_{\mathrm{curl}} \frac{h_K}{(p_K+1)^{1-\varepsilon}} \|f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}\|_{L^2(K)} + C_{\mathrm{grad}} \frac{h_K}{p_K+1} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} + \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2} \Bigg(\widetilde{C}_{\mathrm{curl}} \frac{\sqrt{h_K}}{(p_K+1)^{\frac{1}{2}-\varepsilon}} \|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} + \widetilde{C}_{\mathrm{grad}} \sqrt{\frac{h_K}{p_K+1}} \|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} \Bigg)\Bigg)^2.$$

With Minkowski's inequality this implies

$$\|u-u_{\mathrm{FE}}\|_\Omega^2 \le C_H^2 \frac{C}{C_{ell}} \sum_{K\in\mathcal{K}} \Bigg(C_{\mathrm{curl}} \frac{h_K}{(p_K+1)^{1-\varepsilon}} \left(\|\operatorname{res}\|_{L^2(K)} + \|\Pi f - f\|_{L^2(K)}\right) + C_{\mathrm{grad}} \frac{h_K}{p_K+1} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} + \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2} \Bigg(\widetilde{C}_{\mathrm{curl}} \frac{\sqrt{h_K}}{(p_K+1)^{\frac{1}{2}-\varepsilon}} \|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} + \widetilde{C}_{\mathrm{grad}} \sqrt{\frac{h_K}{p_K+1}} \|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})} \Bigg)\Bigg)^2$$

and it follows easily

$$\|u-u_{\mathrm{FE}}\|_\Omega^2 \le 2C_H^2 \frac{C}{C_{ell}} \sum_{K\in\mathcal{K}} \Bigg(2C_{\mathrm{curl}}^2 \frac{h_K^2}{(p_K+1)^{2(1-\varepsilon)}} \left(\|\operatorname{res}\|_{L^2(K)}^2 + \|\Pi f - f\|_{L^2(K)}^2\right) + C_{\mathrm{grad}}^2 \frac{h_K^2}{(p_K+1)^2} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}^2 + \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{1}{2} \Bigg(\widetilde{C}_{\mathrm{curl}}^2 \frac{h_K}{(p_K+1)^{1-2\varepsilon}} \|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})}^2 + \widetilde{C}_{\mathrm{grad}}^2 \frac{h_K}{p_K+1} \|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\|_{L^2(\widetilde{f})}^2 \Bigg)\Bigg).$$

Then there exists some constant $C_1 > 0$ independent of h and p such that

$$\|u-u_{\mathrm{FE}}\|_\Omega^2 \le C_1 \sum_{K\in\mathcal{K}} (p_K+1)^{2\varepsilon} \left(\eta_K^2 + \frac{h_K^2}{(p_K+1)^2} \|\Pi f - f\|_{L^2(K)}^2 \right)$$

and this concludes the proof. □

Next we derive an upper bound for the a posteriori error estimator η in terms of the energy error $\|u-u_{\mathrm{FE}}\|$. Therefore we first bound the local residual-based terms $\eta_{R,K}$ and the local boundary terms $\eta_{B,K}$ separately from above. Then we combine these results to obtain an upper bound for the resdiual-based a posteriori error estimator (4.1) in terms of the energy error.

Lemma 1. *Let $u_{FE} \in V^p(\mathcal{K},\Omega)$ be the solution of discrete problem (2.7) and $u \in H_0(\mathrm{curl},\Omega)$ be the solution of weak problem (2.6). Further we assume that the triangulation $\mathcal{K}$ of Ω satisfies regularity assumptions (2.2) and (2.3). Let $K \in \mathcal{K}$ and $\varepsilon > 0$ be arbitrary. Then there exists some constant $C_{R,K}(\varepsilon) > 0$ independent of h_K and p_K such that*

$$\eta_{R,K}^2 \leq C_{R,K}(\varepsilon)\left((p_K+1)^{1+\varepsilon}\|u-u_{FE}\|_K^2 + \frac{h_K^2}{(p_K+1)^{1-\varepsilon}}\|\Pi f - f\|_{L^2(K)}^2\right).$$

Proof. From Corollary 1 we know

$$\|\operatorname{res}\|_{L^2(K)} \leq C_{pol,2}(p_K+1)^{\frac{1+\varepsilon}{4}}\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{res}\right\|_{L^2(K)} \tag{4.4}$$

for some constant $C_{pol,2} > 0$, which depends only on $p_{\alpha,K}$ and $p_{\beta,K}$, and we observe

$$\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{res}\right\|_{L^2(K)}^2 = \int_K \phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}^T(f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}) + \int_K \phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}^T(\Pi f - f).$$

By setting

$$v_K := \begin{cases}\phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}, & \text{in } K\\ 0, & \text{in } \Omega\setminus K\end{cases}$$

we obtain

$$\begin{aligned}\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{res}\right\|_{L^2(K)}^2 &= \int_\Omega v_K^T(f-\nabla\times(\alpha\nabla\times u_{\mathrm{FE}})-\beta u_{\mathrm{FE}}) + \int_K \phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}^T(\Pi f - f)\\ &= \int_\Omega v_K^T(\nabla\times(\alpha\nabla\times(u-u_{\mathrm{FE}}))-\beta(u-u_{\mathrm{FE}})) + \int_K \phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}^T(\Pi f - f)\end{aligned}$$

with the strong formulation (2.5). Then integration by parts yields

$$\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{res}\right\|_{L^2(K)}^2 = a(v_K, u-u_{\mathrm{FE}}) + \int_K \phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}^T(\Pi f - f)$$

and with the definition of v_K, the continuity of the bilinear form a (2.9) in the first term and the Cauchy-Schwarz inequality in the second term we get

$$\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{res}\right\|_{L^2(K)}^2 \leq C_c\left\|\phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}\right\|_{H(\mathrm{curl},K)}\|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},K)} + \left\|\phi_K^{\frac{1+\varepsilon}{2}}\operatorname{res}\right\|_{L^2(K)}\|\Pi f - f\|_{L^2(K)}. \tag{4.5}$$

Now let us consider the norm $\left\| \phi_K^{\frac{1+\varepsilon}{2}} \operatorname{res} \right\|_{H(\operatorname{curl},K)}$ in more detail. By the definition of the $H(\operatorname{curl})$-norm we have

$$\begin{aligned}
&\left\| \phi_K^{\frac{1+\varepsilon}{2}} \operatorname{res} \right\|_{H(\operatorname{curl},K)}^2 \\
&\quad = \left\| \phi_K^{\frac{1+\varepsilon}{2}} \operatorname{res} \right\|_{L^2(K)}^2 + \left\| \nabla \times \left(\phi_K^{\frac{1+\varepsilon}{2}} \operatorname{res} \right) \right\|_{L^2(K)}^2 \\
&\quad \leq \left\| \phi_K^{\frac{1+\varepsilon}{2}} \operatorname{res} \right\|_{L^2(K)}^2 + \frac{(1+\varepsilon)^2}{2} \left\| \phi_K^{\frac{\varepsilon-1}{2}} \nabla \phi_K \times \operatorname{res} \right\|_{L^2(K)}^2 + 2 \left\| \phi_K^{\frac{1+\varepsilon}{2}} \nabla \times \operatorname{res} \right\|_{L^2(K)}^2
\end{aligned}$$

with Minkowski's inequality. We note

$$0 \leq \phi_K \leq \frac{1}{2} \tag{4.6}$$

and that for all $x \in K$ it holds

$$\begin{aligned}
|\nabla \phi_K(x)| &= \frac{1}{h_K} |\nabla \operatorname{dist}(x, \partial K)| \\
&\leq C_{\text{grad}} h_K^{-1}
\end{aligned} \tag{4.7}$$

for some constant $C_{\text{grad}} > 0$ independent of h_K. Then we obtain easily

$$\left\| \phi_K^{\frac{1+\varepsilon}{2}} \operatorname{res} \right\|_{H(\operatorname{curl},K)}^2 \leq \frac{1}{2^\varepsilon} \| \operatorname{res} \|_{H(\operatorname{curl},K)}^2 + \frac{(1+\varepsilon)^2 C_{\text{grad}}^2}{2} h_K^{-2} \left\| \phi_K^{\frac{\varepsilon-1}{2}} \operatorname{res} \right\|_{L^2(K)}^2$$

and with Corollary 1 it follows

$$\left\| \phi_K^{\frac{1+\varepsilon}{2}} \operatorname{res} \right\|_{H(\operatorname{curl},K)}^2 \leq \frac{(p_K+1)^{1-\varepsilon}}{h_K^2} \left(C_1(\varepsilon)(p_K+1)^{1+\varepsilon} + \frac{(1+\varepsilon)^2 C_{\text{grad}}^2 C_{pol,2}^2}{2} \right) \| \operatorname{res} \|_{L^2(K)}^2$$

for some constant $C_1(\varepsilon) > 0$ independent of h_K and p_K. Putting this into estimate (4.5) and using (4.6) yields

$$\begin{aligned}
&\left\| \phi_K^{\frac{1+\varepsilon}{4}} \operatorname{res} \right\|_{L^2(K)}^2 \\
&\quad \leq \left(C_c \frac{(p_K+1)^{\frac{1-\varepsilon}{2}}}{h_K} \sqrt{C_1(\varepsilon)(p_K+1)^{1+\varepsilon} + \frac{(1+\varepsilon)^2 C_{\text{grad}}^2 C_{pol,2}^2}{2}} \|u - u_{\text{FE}}\|_{H(\operatorname{curl},K)} \right. \\
&\qquad\qquad \left. + \frac{1}{2^{\frac{1+\varepsilon}{2}}} \|\Pi f - f\|_{L^2(K)} \right) \| \operatorname{res} \|_{L^2(K)}.
\end{aligned}$$

Then by putting this into inequality (4.4) we get

$$\begin{aligned}
&\| \operatorname{res} \|_{L^2(K)}^2 \\
&\quad \leq C_{pol,2}^2 \frac{p_K+1}{h_K} \left(C_c \sqrt{C_1(\varepsilon)(p_K+1)^{1+\varepsilon} + \frac{(1+\varepsilon)^2 C_{\text{grad}}^2 C_{pol,2}^2}{2}} \|u - u_{\text{FE}}\|_{H(\operatorname{curl},K)} \right. \\
&\qquad\qquad \left. + \frac{1}{2^{\frac{1+\varepsilon}{2}}} \frac{h_K}{(p_K+1)^{\frac{1-\varepsilon}{2}}} \|\Pi f - f\|_{L^2(K)} \right) \| \operatorname{res} \|_{L^2(K)}
\end{aligned}$$

and this implies

$$\frac{h_K}{p_K+1}\|\operatorname{res}\|_{L^2(K)} \le C_2(\varepsilon)\left((p_K+1)^{\frac{1+\varepsilon}{2}}\|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},K)} + \frac{h_K}{(p_K+1)^{\frac{1-\varepsilon}{2}}}\|\Pi f - f\|_{L^2(K)}\right) \tag{4.8}$$

for some constant $C_2(\varepsilon) > 0$, which is depending solely on $\varepsilon, p_{\alpha,K}$ and $p_{\beta,K}$.
Now we consider the second part $\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}$ of the residual-based term $\eta_{R,K}$.
From Corollary 1 we know

$$\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} \le C_{pol,2}(p_K+1)^{\frac{1+\varepsilon}{4}}\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)} \tag{4.9}$$

for some constant $C_{pol,2} > 0$, which depends only on $p_{\beta,K}$, and with the divergence condition (1.3) this implies

$$\begin{aligned}\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2 &= \int_K \operatorname{div}(\beta(u_{\mathrm{FE}}-u))\phi_K^{\frac{1+\varepsilon}{2}}\operatorname{div}(\beta u_{\mathrm{FE}})\\ &= \int_K (\beta(u-u_{\mathrm{FE}}))^T\nabla\left(\phi_K^{\frac{1+\varepsilon}{2}}\operatorname{div}(\beta u_{\mathrm{FE}})\right)\end{aligned}$$

by integration by parts. Then the Cauchy-Schwarz inequality gives

$$\begin{aligned}&\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2\\ &\quad\le \|\beta(u-u_{\mathrm{FE}})\|_{L^2(K)}\left\|\nabla\left(\phi_K^{\frac{1+\varepsilon}{2}}\operatorname{div}(\beta u_{\mathrm{FE}})\right)\right\|_{L^2(K)}\\ &\quad\le \|\beta(u-u_{\mathrm{FE}})\|_{L^2(K)}\left(\frac{1+\varepsilon}{2}\left\|\phi_K^{\frac{\varepsilon-1}{2}}\nabla\phi_K\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)} + \left\|\phi_K^{\frac{1+\varepsilon}{2}}\nabla\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}\right)\end{aligned}$$

with Minkowski's inequality. With inequalities (4.6) and (4.7) it follows

$$\begin{aligned}\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2 \le \|\beta(u-u_{\mathrm{FE}})\|_{L^2(K)}\Bigg(&\frac{(1+\varepsilon)C_{\mathrm{grad}}}{2}h_K^{-1}\left\|\phi_K^{\frac{\varepsilon-1}{2}}\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}\\ &+ \frac{1}{2^{\frac{1+\varepsilon}{2}}}\|\nabla\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}\Bigg)\end{aligned}$$

and with Corollary 1 we get

$$\begin{aligned}&\left\|\phi_K^{\frac{1+\varepsilon}{4}}\operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2\\ &\quad\le \frac{(p_K+1)^{\frac{1-\varepsilon}{2}}}{h_K}\Bigg(\frac{(1+\varepsilon)C_{\mathrm{grad}}C_{pol,2}}{2}\\ &\qquad\qquad + \frac{C_{pol,1}}{2^{\frac{1+\varepsilon}{2}}}(p_K+1)^{\frac{1+\varepsilon}{2}}\Bigg)\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}\|\beta(u-u_{\mathrm{FE}})\|_{L^2(K)}.\end{aligned}$$

for some constant $C_{pol,1} > 0$, which depends solely on $p_{\beta,K}$. Putting this into inequality (4.9) yields

$$\begin{aligned}\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}^2 \\ \leq C_{pol,2}^2 \frac{p_K+1}{h_K}\Bigg(&\frac{(1+\varepsilon)C_{\mathrm{grad}}C_{pol,2}}{2} \\ &+ \frac{C_{pol,1}}{2^{\frac{1+\varepsilon}{2}}}(p_K+1)^{\frac{1+\varepsilon}{2}}\Bigg)\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}\|\beta(u-u_{\mathrm{FE}})\|_{L^2(K)}\end{aligned}$$

Since β is uniformly positive definite (2.4) and the bilinear form a is elliptic (2.8), this implies

$$\frac{h_K}{p_K+1}\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} \leq C_3(\varepsilon)(p_K+1)^{\frac{1+\varepsilon}{2}}\|u-u_{\mathrm{FE}}\|_K \tag{4.10}$$

for some constant $C_3(\varepsilon) > 0$ independent of h_K and p_K. Combining this result with estimate (4.8) and using the ellipticity of the bilinear form a (2.8) implies

$$\begin{aligned}\eta_{R,K}^2 &= \frac{h_K^2}{(p_K+1)^2}\left(\|\operatorname{res}\|_{L^2(K)}^2 + \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}^2\right) \\ &\leq C_4(\varepsilon)(p_K+1)^{1+\varepsilon}\|u-u_{\mathrm{FE}}\|_K^2 + 2C_2(\varepsilon)^2\frac{h_K^2}{(p_K+1)^{1-\varepsilon}}\|\Pi f - f\|_{L^2(K)}^2\end{aligned}$$

for some constant $C_4(\varepsilon) > 0$ independent of h_K and p_K and setting

$$C_{R,K}(\varepsilon) := \max\left\{C_4(\varepsilon), 2C_2(\varepsilon)^2\right\}$$

concludes the proof. □

Now we consider the boundary term $\eta_{B,K}$.

Lemma 2. *Let $u_{FE} \in V^p(\mathcal{K},\Omega)$ be the solution of discrete problem (2.7) and $u \in H_0(\mathrm{curl},\Omega)$ be the solution of weak problem (2.6). Further we assume that the triangulation $\mathcal{K}$ of Ω satisfies regularity assumptions (2.2) and (2.3). Let $K \in \mathcal{K}$ and $\varepsilon > 0$ be arbitrary. Then there exists some constant $C_{B,K}(\varepsilon) > 0$ independent of h_K and p_K such that*

$$\eta_{B,K}^2 \leq C_{B,K}(\varepsilon)(p_K+1)^{1+\varepsilon}\left((p_K+1)^{2+\varepsilon}\|u-u_{FE}\|_{\omega_K}^2 + h_K^2\|\Pi f - f\|_{L^2(\omega_K)}^2\right).$$

Proof. Let $\widetilde{f} \subset \partial K \cap \Omega$ be a face of cell K. Then we know from Corollary 2 that there exists some polynomial extension $v \in H_0(\mathrm{curl},\omega_{\widetilde{f}})$ such that $v_{\widetilde{f}}\big|_{\widetilde{f}} = \phi_{\omega_{\widetilde{f}}}^{\frac{1+\varepsilon}{2}} n_{\widetilde{f}} \times [\alpha\nabla \times u_{\mathrm{FE}}]$ and

$$\left\|n_{\widetilde{f}} \times [\alpha\nabla \times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})} \leq C_{pol,5}(p_{\widetilde{f}}+1)^{\frac{1+\varepsilon}{4}}\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}. \tag{4.11}$$

for some constant $C_{pol,5} > 0$ depending solely on p_{α_K}. We observe

$$\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 = \int_{\widetilde{f}}(n_{\widetilde{f}} \times [\alpha\nabla \times u_{\mathrm{FE}}])^T v_{\widetilde{f}}$$

and w.l.o.g. we may assume $\omega_{\widetilde{f}} = K_l \cup K_r$ for some $K_l, K_r \in \mathcal{K}$. Then we get

$$\begin{aligned}\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 &= \int_{\widetilde{f}} \left(n_{\widetilde{f}} \times \left((\alpha\nabla \times (u - u_{\mathrm{FE}}))\,\big|_{K_l} - (\alpha\nabla \times (u - u_{\mathrm{FE}}))\,\big|_{K_r}\right)\right)^T v_{\widetilde{f}} \\ &= a(v_{\widetilde{f}}, u - u_{\mathrm{FE}}) \\ &\qquad - \int_{\omega_{\widetilde{f}}} v_{\widetilde{f}}^T (\nabla \times (\alpha\nabla \times (u - u_{\mathrm{FE}})) + \beta(u - u_{\mathrm{FE}}))\end{aligned}$$

with the integration by parts formula and using the strong formulation (2.5) yields

$$\begin{aligned}\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 &= a(v_{\widetilde{f}}, u - u_{\mathrm{FE}}) - \int_{\omega_{\widetilde{f}}} v_{\widetilde{f}}^T (f - \nabla \times (\alpha\nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}) \\ &= a(v_{\widetilde{f}}, u - u_{\mathrm{FE}}) - \int_{\omega_{\widetilde{f}}} v_{\widetilde{f}}^T \operatorname{res} + \int_{\omega_{\widetilde{f}}} v_{\widetilde{f}}^T (\Pi f - f) \\ &\le C_c \left\|v_{\widetilde{f}}\right\|_{H(\mathrm{curl},\omega_{\widetilde{f}})} \|u - u_{\mathrm{FE}}\|_{H(\mathrm{curl},\omega_{\widetilde{f}})} \\ &\qquad + \left\|v_{\widetilde{f}}\right\|_{L^2(\omega_{\widetilde{f}})} \left(\|\operatorname{res}\|_{L^2(\omega_{\widetilde{f}})} + \|\Pi f - f\|_{L^2(\omega_{\widetilde{f}})}\right)\end{aligned} \tag{4.12}$$

with the continuity of the bilinear form a (2.9) used in the first term and the Cauchy-Schwarz inequality used in the last two terms. Now let us consider the norm $\left\|v_{\widetilde{f}}\right\|_{H(\mathrm{curl},\omega_{\widetilde{f}})}$ in more detail. By the definition of the $H(\mathrm{curl})$-norm we have

$$\begin{aligned}\left\|v_{\widetilde{f}}\right\|_{H(\mathrm{curl},\omega_{\widetilde{f}})}^2 &= \left\|v_{\widetilde{f}}\right\|_{L^2(\omega_{\widetilde{f}})}^2 + \left\|\nabla \times v_{\widetilde{f}}\right\|_{L^2(\omega_{\widetilde{f}})}^2 \\ &\le \left(C_{pol,4}(\varepsilon)^2 h_{\widetilde{f}} + C_{pol,3}(\varepsilon)^2 \frac{(p_{\widetilde{f}}+1)^{1-\varepsilon}}{h_{\widetilde{f}}}\right) \left\|\phi_{\omega_{\widetilde{f}}}^{\frac{1+\varepsilon}{2}} n_{\widetilde{f}} \times [\alpha\nabla \times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}^2\end{aligned}$$

with Corollary 2 and it follows

$$\left\|v_{\widetilde{f}}\right\|_{H(\mathrm{curl},\omega_{\widetilde{f}})}^2 \le \frac{1}{2^{1+\varepsilon}} \left(C_{pol,4}(\varepsilon)^2 h_{\widetilde{f}} + C_{pol,3}(\varepsilon)^2 \frac{(p_{\widetilde{f}}+1)^{1-\varepsilon}}{h_{\widetilde{f}}}\right) \left\|n_{\widetilde{f}} \times [\alpha\nabla \times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}^2$$

from estimate (3.2). Putting this into inequality (4.12) yields

$$\begin{aligned}&\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 \\ &\quad \le \frac{C_c}{2^{\frac{1+\varepsilon}{2}}} \sqrt{C_{pol,4}(\varepsilon)^2 h_{\widetilde{f}} + C_{pol,3}(\varepsilon)^2 \frac{(p_{\widetilde{f}}+1)^{1-\varepsilon}}{h_{\widetilde{f}}}} \left\|n_{\widetilde{f}} \times [\alpha\nabla \times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})} \|u - u_{\mathrm{FE}}\|_{H(\mathrm{curl},\omega_{\widetilde{f}})} \\ &\qquad + \left\|v_{\widetilde{f}}\right\|_{L^2(\omega_{\widetilde{f}})} \left(\|\operatorname{res}\|_{L^2(\omega_{\widetilde{f}})} + \|\Pi f - f\|_{L^2(\omega_{\widetilde{f}})}\right)\end{aligned}$$

and with Corollary 2 we obtain

$$\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 \leq \left(\frac{C_c}{2^{\frac{1+\varepsilon}{2}}}\sqrt{C_{pol,4}(\varepsilon)^2 h_{\widetilde{f}} + C_{pol,3}(\varepsilon)^2\frac{(p_{\widetilde{f}}+1)^{1-\varepsilon}}{h_{\widetilde{f}}}}\|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},\omega_{\widetilde{f}})} + C_{pol,4}(\varepsilon)\sqrt{h_{\widetilde{f}}}\left(\|\operatorname{res}\|_{L^2(\omega_{\widetilde{f}})} + \|\Pi f - f\|_{L^2(\omega_{\widetilde{f}})}\right)\right)\left\|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}.$$

Then, by estimate (4.8) and regularity assumptions (2.2) and (2.3) it follows

$$\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 \leq C_1(\varepsilon)(p_{\widetilde{f}}+1)^{\frac{1+\varepsilon}{2}}\left(\frac{p_{\widetilde{f}}+1}{\sqrt{h_{\widetilde{f}}}}\|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},\omega_{\widetilde{f}})} + \sqrt{h_{\widetilde{f}}}\|\Pi f - f\|_{L^2(\omega_{\widetilde{f}})}\right)\left\|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}$$

for some constant $C_1(\varepsilon) > 0$ independent of $h_{\widetilde{f}}$ and $p_{\widetilde{f}}$. Putting this into inequality (4.11) gives

$$\left\|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}^2 \leq C_{pol,5}^2 C_1(\varepsilon)\frac{(p_{\widetilde{f}}+1)^{1+\varepsilon}}{\sqrt{h_{\widetilde{f}}}}\left((p_{\widetilde{f}}+1)\|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},\omega_{\widetilde{f}})} + h_{\widetilde{f}}\|\Pi f - f\|_{L^2(\omega_{\widetilde{f}})}\right)\left\|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}$$

and this implies

$$\sqrt{\frac{h_{\widetilde{f}}}{p_{\widetilde{f}}+1}}\left\|n_{\widetilde{f}}\times[\alpha\nabla\times u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})} \leq C_{pol,5}^2 C_1(\varepsilon)(p_{\widetilde{f}}+1)^{\frac{1}{2}+\varepsilon}\left((p_{\widetilde{f}}+1)\|u-u_{\mathrm{FE}}\|_{H(\mathrm{curl},\omega_{\widetilde{f}})} + h_{\widetilde{f}}\|\Pi f - f\|_{L^2(\omega_{\widetilde{f}})}\right). \tag{4.13}$$

Now let us consider the second term $\|n_{\widetilde{f}}^T\beta[u_{FE}]\|_{L^2(\widetilde{f})}$ of $\eta_{B,K}$. From Corollary 2 we know that there exists some polynomial extension $v_{\widetilde{f}} \in H_0^1(\omega_{\widetilde{f}})$ such that $v_{\widetilde{f}}\big|_{\widetilde{f}} = n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]$ and

$$\left\|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})} \leq C_{pol,5}(p_{\widetilde{f}}+1)^{\frac{1+\varepsilon}{4}}\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}. \tag{4.14}$$

for some constant $C_{pol,5} > 0$ depending solely on $p_{\beta,K}$. We observe

$$\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 = \int_{\widetilde{f}} v_{\widetilde{f}} n_{\widetilde{f}}^T [\beta(u-u_{\mathrm{FE}})]$$

and w.l.o.g. we may assume $\omega_{\widetilde{f}} = K_l \cap K_r$ for some $K_l, K_r \in K$. This yields

$$\begin{aligned}\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 &= \int_{\widetilde{f}} v_{\widetilde{f}} n_{\widetilde{f}}^T \left((\beta(u-u_{\mathrm{FE}}))\,\big|_{K_l} - (\beta(u-u_{\mathrm{FE}}))\,\big|_{K_r} \right) \\ &= \int_\Omega (\nabla v_{\widetilde{f}})^T \beta(u-u_{\mathrm{FE}}) + \int_\Omega v_{\widetilde{f}} \operatorname{div}(\beta(u-u_{\mathrm{FE}}))\end{aligned}$$

with the integration by parts formula and by using the divergence condition (1.3) and the Cauchy-Schwarz inequality we get

$$\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 \le \left\|\nabla v_{\widetilde{f}}\right\|_{L^2(\omega_{\widetilde{f}})} \|\beta(u-u_{\mathrm{FE}})\|_{L^2(\omega_{\widetilde{f}})} + \left\|v_{\widetilde{f}}\right\|_{L^2(\omega_{\widetilde{f}})} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(\omega_{\widetilde{f}})}.$$

Then Corollary 2 implies

$$\begin{aligned}\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 \le \Bigg(& C_{pol,3}(\varepsilon)\sqrt{\frac{(p_{\widetilde{f}}+1)^{1-\varepsilon}}{h_{\widetilde{f}}}} \|\beta(u-u_{\mathrm{FE}})\|_{L^2(\omega_{\widetilde{f}})} \\ & + C_{pol,4}(\varepsilon)\sqrt{h_{\widetilde{f}}} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(\omega_{\widetilde{f}})} \Bigg) \left\|\phi_{\omega_{\widetilde{f}}}^{\frac{1+\varepsilon}{2}} n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}.\end{aligned}$$

It follows with estimates (3.2), (4.10) and regularity assumptions (2.2), (2.3)

$$\left\|\phi_{\omega_{\widetilde{f}}}^{-\frac{1+\varepsilon}{4}} v_{\widetilde{f}}\right\|_{L^2(\widetilde{f})}^2 \le \frac{C_2(\varepsilon)}{2^{\frac{1+\varepsilon}{2}}} \frac{(p_{\widetilde{f}}+1)^{\frac{3+\varepsilon}{2}}}{\sqrt{h_{\widetilde{f}}}} \|u-u_{\mathrm{FE}}\|_{\omega_{\widetilde{f}}} \left\|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}$$

for some constant $C_2(\varepsilon) > 0$ independent of $h_{\widetilde{f}}$ and $p_{\widetilde{f}}$, since β is uniformly positive definite (2.4) and the bilinear form a is elliptic (2.8). By putting this into inequality (4.14) we obtain

$$\left\|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}^2 \le \frac{C_2(\varepsilon) C_{pol,5}^2}{2^{\frac{1+\varepsilon}{2}}} \frac{(p_{\widetilde{f}}+1)^{2+\varepsilon}}{\sqrt{h_{\widetilde{f}}}} \|u-u_{\mathrm{FE}}\|_{\omega_{\widetilde{f}}} \left\|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})}$$

and this implies

$$\sqrt{\frac{h_{\widetilde{f}}}{p_{\widetilde{f}}+1}} \left\|n_{\widetilde{f}}^T[\beta u_{\mathrm{FE}}]\right\|_{L^2(\widetilde{f})} \le \frac{C_2(\varepsilon) C_{pol,5}^2}{2^{\frac{1+\varepsilon}{2}}} (p_{\widetilde{f}}+1)^{\frac{3}{2}+\varepsilon} \|u-u_{\mathrm{FE}}\|_{\omega_{\widetilde{f}}}.$$

Combining this result with (4.13) and summing over all faces $\widetilde{f} \subset \partial K \cap \Omega$ gives

$$\begin{aligned}\eta_{B,K}^2 &= \sum_{\widetilde{f}\subset\partial K\cap\Omega} \frac{h_{\widetilde{f}}}{2(p_{\widetilde{f}}+1)} \left(\|n_{\widetilde{f}} \times [\alpha \nabla \times u_{FE}]\|_{L^2(\widetilde{f})}^2 + \|n_{\widetilde{f}}^T[\beta u_{FE}]\|_{L^2(\widetilde{f})}^2 \right) \\ &\le \sum_{\widetilde{f}\subset\partial K\cap\Omega} (p_{\widetilde{f}}+1)^{1+\varepsilon} \left(C_3(\varepsilon)(p_{\widetilde{f}}+1)^{2+\varepsilon} \|u-u_{\mathrm{FE}}\|_{\omega_{\widetilde{f}}}^2 + 2C_{pol,5}^4 C_2(\varepsilon)^2 h_{\widetilde{f}}^2 \|\Pi f - f\|_{L^2(\omega_{\widetilde{f}})}^2 \right)\end{aligned}$$

for some constant $C_3(\varepsilon) > 0$ independent of $h_{\widetilde{f}}$ and $p_{\widetilde{f}}$ by the ellipticity of the bilinear form a (2.8) and with regularity assumptions (2.2) and (2.3) there exists some constant $C_{B,K}(\varepsilon) > 0$ independent of $h_{\widetilde{f}}$ and $p_{\widetilde{f}}$ such that

$$\eta_{B,K}^2 \le C_{B,K}(\varepsilon)(p_K+1)^{1+\varepsilon}\left((p_K+1)^{2+\varepsilon}\|u-u_{FE}\|_{\omega_K}^2 + h_K^2\|\Pi f - f\|_{L^2(\omega_K)}^2\right). \square$$

By combining the results from Lemmas 1 and 2 above we can derive an upper bound for the residual-based a posteriori error indicator η in terms of the energy error $\|u - u_{FE}\|$.

Theorem 6. *Let $u_{FE} \in V^p(\mathcal{K}, \Omega)$ be the solution of discrete problem (2.7) and $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of weak problem (2.6). Further we assume that the triangulation $\mathcal{K}$ of Ω satisfies regularity assumptions (2.2) and (2.3). Let $\varepsilon > 0$ be arbitrary. Then there exists some constant $C_2(\varepsilon) > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\eta^2 \le C_2(\varepsilon)\sum_{K\in\mathcal{K}}(p_K+1)^{3+2\varepsilon}\left(\|u-u_{FE}\|_{\omega_K}^2 + \frac{h_K^2}{(p_K+1)^{2+\varepsilon}}\|\Pi f - f\|_{L^2(\omega_K)}^2\right).$$

Proof. The result follows immediately by summing up the estimates from Lemmas 1 and 2 and using regularity assumptions (2.2) and (2.3). $\square$

Remark 3.

(1) The upper bound for error indicator η is dominated by the estimate for the boundary term $\eta_{B,K}$ derived in Lemma 2, because this term cannot be bounded uniformly in p. However for the residual-based term $\eta_{R,K}$ one can derive a bound, which is uniform in h and p, by inserting a smoothing function – similar to the one defined in Corollary 2 – into the error estimator η (c.f. Ref. 20, where this technique was used for the Poisson problem, and Ref. 17, where it was used in a discontinuous Galerkin framework). But, since the exact evaluation of such a smoothing function for nonaffine mappings $F_K : \widehat{K} \to K$ is not an easy task, we do not want to include this term into our estimator.

(2) The upper bound for the a posteriori error estimator (4.1) cannot be determined fully cell-wise local. This is due to the way we applied Lemmata 1 and 2. There we had to extend the cell boundary function defined on the face to a polynomial, which is defined on a patch including the neighboring cells of the face.

(3) Note that in Theorem 6 we did not require the extra regularity $u \in H^r(\mathrm{curl}, \Omega)$, $r > \varepsilon + \frac{1}{2}$ for $\varepsilon > 0$, instead of $u \in H(\mathrm{curl}, \Omega)$ for the derivation of the upper bound. This is due to the fact that we did not use the $H(\mathrm{curl})$-conforming interpolation operator Π^{curl} in its proof.

With Theorems 5 and 6 we have shown that there exist upper and lower bounds of the estimated error η in terms of the energy error $\|u - u_{FE}\|$. Thus the error indicator can be considered to be hp-efficient.

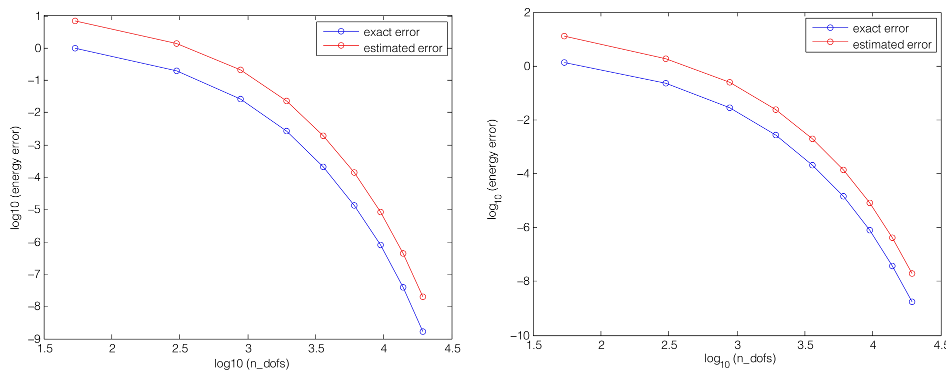

Fig. 2. Example 1: Error reduction of the exact and the estimated error. Left: $\beta = 10^{-2}$. Right: $\beta = 10^2$.

5. Numerical Examples

In this section we will apply the residual-based a posteriori error estimator from Section 4 to some numerical examples. Due to the lack of examples for matrix-valued coefficients α and β with known analytic solutions we consider only scalar-valued coefficients here. First we consider some academic problems with smooth solutions to investigate, whether the error indicator is robust with respect to changes in the coefficients α and β. Then we go ahead to a more realistic example, where β is discontinuous. In the fourth example we consider the special case of a problem, which violates assumption (1.3). To conclude this section we consider a problem admitting a singular solution and thus a real hp-adaptive grid should pay off. All computations are performed with the finite element library deal.II[2,3].

5.1. *Example 1*

In our first experiment we consider a rather simple case, where the coefficients α and β are constant. We choose $\alpha := 1$ and $\beta \in \{10^{-2}, 10^2\}$. The domain is set to $\Omega := (0,1)^3$ and the analytic solution is given by

$$u(x) := \begin{pmatrix} 0 \\ 0 \\ \sin(\pi x_1) \end{pmatrix}.$$

Then the right-hand side reads $f = (\pi^2 + \beta)u$. We start with a coarse grid of 8 hexahedrals of equal size and polynomial degree $p_K = 0$, $K \in \mathcal{K}$, on all cells. Since the solution is smooth and does not possess any local features to detect we perform global p-refinements only. In Figure 2 we show the resulting behaviour of the true energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale.

We do not observe much difference in the behaviour of the error estimator for different values of β and thus can expect some robustness with respect to β in

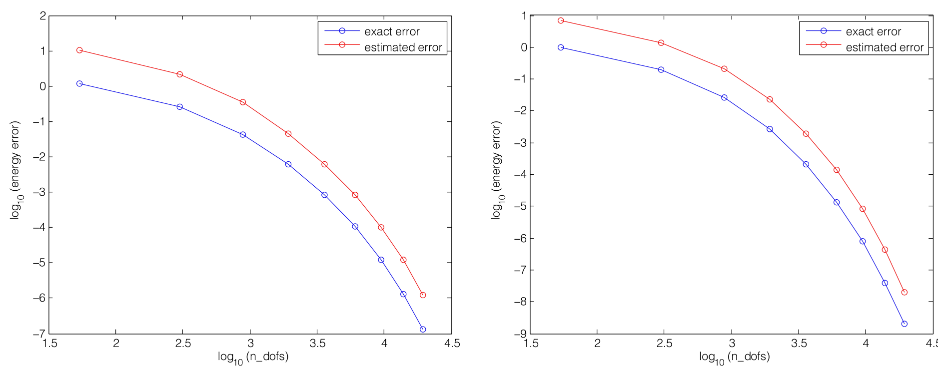

Fig. 3. Error reduction of the exact and the estimated error. Left: Example 2. Right: Example 3.

the case $f \sim \beta$. This is an important feature of an error indicator for Maxwell's equations, because in time-dependent problems β gets scaled by the length of the time-step, and this should not effect the performance of the estimator too much.

5.2. *Example 2*

In this example we choose

$$\alpha(x) := \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) + 1.5$$

and $\beta := 1$ is kept constant. The domain Ω and the analytic solution u are the same as in Example 1. As above we only perform global p-refinement. The results are shown in Figure 3 on the left-hand side.

Also in this situation the error estimator seems to perform well. The estimated error approaches the exact error as the polynomial degree increases.

5.3. *Example 3*

In this experiment we consider a more realistic example than the previous ones. We set $\alpha := 1$ and choose

$$\beta(x) := \begin{cases} 1 & , \text{ if } \max\{|x_1 - 0.5|, |x_2 - 0.5|, |x_3 - 0.5|\} \leq 0.25 \\ 0 & , \text{ else} \end{cases}$$

to be discontinuous. As already mentioned in the introduction this is a common situation in realistic applications, where we have a conduction region ($\beta = 1$) and an outer space ($\beta = 0$). The domain Ω and the smooth analytic solution u are carried over from Example 1 again. Due to the discontinuity of β inside the cells of our rather coarse grid consisting of only 8 hexahedrals we have to use high-order quadrature rules to approximate the integrals sufficiently accurate. However we are still able to benefit from the smoothness of the analytical solution and can peform global p-refinement only. The error curves are shown in Figure 3 on the right-hand side. Again the error estimator shows a satisfying performance.

5.4. *Example 4*

Again we choose $\alpha := 1$, but this time

$$\beta(x) := \sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) + 1.5.$$

Again, the domain Ω and the analytic solution u are the same as in Example 1. We observe that this special choice of β and u does not satisfy assumption (1.3), since

$$\operatorname{div}(\beta u) = 2\pi \sin(2\pi x_1)\sin(\pi x_1)\sin(2\pi x_2)\cos(2\pi x_3) \neq 0$$

and thus this example actually is out of the scope of this paper. Therefore we cannot hope for a good performance of the originally presented a posteriori error estimator (4.1). This can also be seen in Ref. 4, Table 5, where the ratio

$$\varepsilon := \frac{\eta}{\|u - u_{FE}\|}$$

for a similar h-adaptive a posteriori error estimator does not approach a constant value contrary to the statement of the authors, but increases dramatically. However a simple modification of the indicator gives quite satisfactory numerical results also for this example. Therefore we change the second part of the residual-based term $\eta_{R,K}$ to

$$\|\operatorname{div}(\beta(u - u_{FE}))\|_{L^2(K)}.$$

This modification takes into account the non-vanishing divergence $\operatorname{div}(\beta u)$ and, hence, also this term should converge to zero, which would not be the case for the one proposed in (4.2). As in the previous examples we carry out global p-refinement only. The numerical results are plotted in Figure 4 on the left-hand side and show a satisfactory performance of the modified a posteriori error estimator.

5.5. *Example 5*

In the last example we consider a problem with a singular solution. Let $\Omega := (-1,1)^3 \setminus ([0,1) \times (-1,0] \times (-1,1))$, $\alpha := \beta := 1$ and

$$u(r,\phi,x_3) := \frac{2}{3} r^{-\frac{1}{3}} \begin{pmatrix} -\sin\left(\frac{\phi}{3}\right) \\ \cos\left(\frac{\phi}{3}\right) \\ 0 \end{pmatrix},$$

where $(r,\phi,z) \in \mathbb{R}_+ \times [0,2\pi) \times \mathbb{R}$ denote the cylindrical coordinates. Thus u has an edge singularity along the reentrant edge at the axis $x_3 = 0$. The right-hand side function f equals u. Since u is constant along the x_3-axis we construct a problem-adapted mesh by the following refinement strategy: All cells, which are close to the singularity are bisected, whereas on all other cells the polynomial degree is

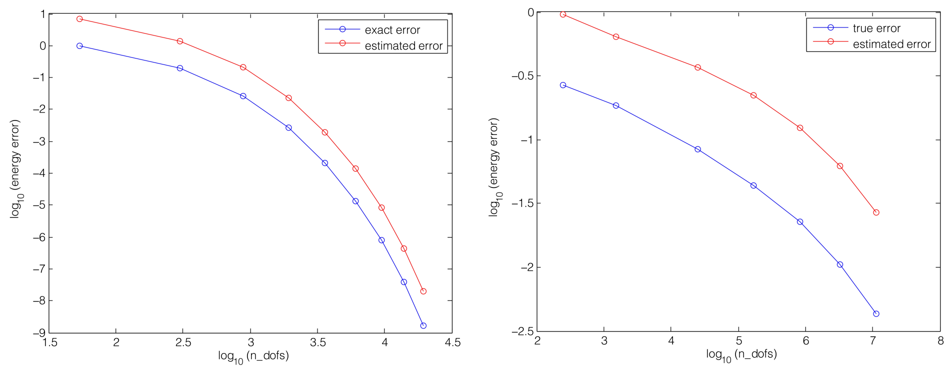

Fig. 4. Left: Example 4: Error reduction of the exact and the estimated error. Right: Example 5: Error reduction of the exact and the estimated error.

increased. To obtain an adaptively refined grid we select a subset of cells to be refined in every refinement step by the following marking strategy from Ref. 13 with $\theta = 0.3$. Find a minimal set $\mathcal{A} \subset \mathcal{K}$ such that

$$\sum_{K \in \mathcal{A}} \eta_K^2 \geq \theta^2 \sum_{K \in \mathcal{K}} \eta_K^2$$

and refine all cells contained in $\mathcal{A}$ according to the foregoing rule. We start our computations on a mesh consisting of 48 equally-sized hexahedrals and polynomial degree vector $p = 0$. The final grid is shown in Figure 5. The performance of the error estimator can be seen in Figure 4 on the right hand side. We observe that the varying mesh sizes h_K and different polynomial degrees p_K, $K \in \mathcal{K}$, over the triangulation do not have a big influence on the performance of the error indicator. Again the behaviour of the exact energy error can be seen clearly in the curve of the estimated error.

6. Conclusion

We have derived a residual-based a posteriori error estimator for the finite element solution of Maxwell's equations in the electric field formulation. Moreover we have proven its *hp*-efficiency and presented a set of testing examples to investigate the behaviour of the error indicator in a broad range of applications.

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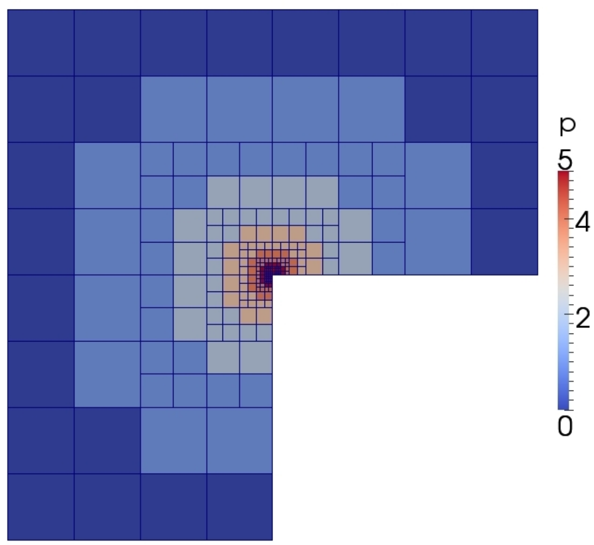

Fig. 5. Example 5: Final grid.

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