Spectral multiplier theorems of Hörmander type via generalized Gaussian estimates

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

Matthias Uhl

aus Haslach im Kinzigtal

Tag der mündlichen Prüfung: 29. Juni 2011

Referent: Priv.-Doz. Dr. Peer Christian Kunstmann
Korreferent: Prof. Dr. Lutz Weis

Contents

1 Introduction **1**

2 Preliminaries **9**
- 2.1 Notations 9
- 2.2 Spaces of homogeneous type 9
- 2.3 Generalized Gaussian estimates and Davies-Gaffney estimates 12
- 2.4 The Hörmander condition 19

3 Spectral multipliers on Lebesgue spaces **25**
- 3.1 A Hörmander type multiplier theorem for operators satisfying generalized Gaussian estimates 25
- 3.2 Variation for operators with non-empty point spectrum 37

4 Hardy spaces **43**
- 4.1 Hardy spaces via square functions 45
- 4.2 Tent spaces 46
- 4.3 Hardy spaces via molecules 47
- 4.4 Relationship between Hardy spaces and Lebesgue spaces 62

5 Spectral multipliers on the Hardy space $H^1_L(X)$ **77**
- 5.1 A criterion for boundedness of spectral multipliers on $H^1_L(X)$ 77
- 5.2 A Hörmander type multiplier theorem on $H^1_L(X)$ 88

6 Boundedness of spectral multipliers on $H^p_L(X)$ and $L^p(X)$ **95**

7 Applications **101**
- 7.1 Maxwell operator 102
- 7.2 Stokes operator 115
- 7.3 Lamé operator 120

Bibliography **123**

1 Introduction

In this thesis we investigate spectral multiplier theorems on Lebesgue and Hardy spaces, where our focus is on the required regularity order.

Consider a non-negative, self-adjoint operator L on the Hilbert space $L^2(X)$, where X is an arbitrary measure space. If E_L denotes the resolution of the identity associated to L, then L can be represented in the form

$$L = \int_0^\infty \lambda \, dE_L(\lambda)$$

(see e.g. [Rud73, Theorem 13.33]). The spectral theorem asserts that the operator

$$F(L) := \int_0^\infty F(\lambda) \, dE_L(\lambda)$$

is well defined and acts as a bounded linear operator on $L^2(X)$ whenever $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function. Spectral multiplier theorems provide regularity assumptions on F which ensure that the operator $F(L)$ extends from $L^p(X) \cap L^2(X)$ to a bounded linear operator on $L^p(X)$ for all p ranging in some symmetric interval containing 2.

In 1960, L. Hörmander addressed this question for the Laplacian $L = -\Delta$ on $\mathbb{R}^D$ during his studies on the boundedness of Fourier multipliers on $\mathbb{R}^D$. His famous Fourier multiplier theorem ([Hör60, Theorem 2.5]) reads as follows:

If $\gamma \in \mathbb{N}$ with $\gamma > D/2$ and $F\colon [0,\infty) \to \mathbb{C}$ is a bounded function such that F is γ-times continuously differentiable on $(0,\infty)$ and the inequality

$$\sum_{k=0}^{\gamma} \sup_{R>0} \left(\frac{1}{R} \int_{R/2}^{2R} \left| R^k F^{(k)}(\xi) \right|^2 d\xi \right)^{1/2} < \infty \tag{1.1}$$

holds, then $F(-\Delta)$ is a bounded linear operator on $L^p(\mathbb{R}^D)$ for every $p \in (1,\infty)$.

Note that $F(-\Delta)$ corresponds to a Fourier multiplier operator and that the spectrum of $-\Delta$ is the set $[0,\infty)$ on which the function F is defined. This observation justifies the terminology “spectral multiplier” which will also be used for other operators than the Laplacian.

In honor of L. Hörmander, the condition (1.1) is called *(classical) Hörmander condition*. Nowadays, it is common practice to replace it by an upgraded version. For introducing the latter, fix a non-negative function $\omega \in C_c^\infty(0,\infty)$ with

$$\operatorname{supp}\omega \subseteq (1/4,1) \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}} \omega(2^{-n}\lambda) = 1 \quad \text{for all } \lambda > 0\,.$$

The substitute of (1.1), which will subsequently be referred to as *Hörmander condition*, reads as follows:

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty \tag{1.2}$$

for bounded Borel functions $F\colon [0,\infty) \to \mathbb{C}$, where $q \geq 2$, $s > 1/q$ and H_q^s denotes the Bessel potential space on $\mathbb{R}$. It is important to notice that (1.2) does not depend on the choice of ω (cf. Lemma 2.18). In contrast to the classical Hörmander condition (1.1), its substitute (1.2) also allows fractional derivation which improves the possibility to measure the required regularity of the considered function F. Further, it is not only formulated for the case $q = 2$. We refer to Lemma 2.19 for a detailed discussion of the relation between (1.1) and (1.2). Now Hörmander's Fourier multiplier theorem takes the following form:

If $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_2^s} < \infty$ for some $s > D/2$, then $F(-\Delta)$ is a bounded linear operator on $L^p(\mathbb{R}^D)$ for all $p \in (1,\infty)$.

By considering imaginary powers $(-\Delta)^{i\tau}$ for $\tau \in \mathbb{R}$, M. Christ ([Chr91, p. 73]) observed that the regularity order in Hörmander's statement cannot be improved beyond $D/2$. This means that for any $s < D/2$ there exists a bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ such that the Hörmander condition $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_2^s} < \infty$ holds, but $F(-\Delta)$ does not act as a bounded operator on $L^p(\mathbb{R}^D)$ for the whole range $p \in (1,\infty)$.

Another important example are the Bochner-Riesz means $\sigma_\alpha(-\Delta)$ for $\alpha > 0$, where

$$\sigma_\alpha(\lambda) := \begin{cases} (1-\lambda)^\alpha & \text{for } \lambda \leq 1, \\ 0 & \text{for } \lambda > 1. \end{cases}$$

Since the Hörmander condition $\sup_{n\in\mathbb{Z}} \|\omega \sigma_\alpha(2^n\cdot)\|_{H_2^s} < \infty$ holds if and only if $\alpha > s - 1/2$ (see e.g. [DOS02, p. 446]), Hörmander's result gives boundedness of Bochner-Riesz means $\sigma_\alpha(-\Delta)$ on $L^p(\mathbb{R}^D)$ for all $p \in (1,\infty)$ whenever $\alpha > (D-1)/2$.

Hörmander's multiplier theorem was generalized, on the one hand, to other spaces than $\mathbb{R}^D$ and, on the other hand, to more general operators than the Laplacian.

The development began in the early 1990's. G. Mauceri and S. Meda ([MM90]) and M. Christ ([Chr91]) extended the result to homogeneous Laplacians on stratified nilpotent Lie groups. Further generalizations were obtained by G. Alexopoulos ([Ale94]) who showed in the setting of connected Lie groups of polynomial volume growth a corresponding statement for the left invariant sub-Laplacian which was in turn extended by W. Hebisch ([Heb95]) to integral operators with kernels decaying polynomially away from the diagonal. More historical remarks about spectral multiplier theorems, including an overview about boundedness of Bochner-Riesz means, can be found in [DOS02] and the references therein.

The results in [DOS02] due to X.T. Duong, E.M. Ouhabaz, and A. Sikora mark an important step toward the study of more general operators. In the abstract framework of (subsets of) spaces of homogeneous type (X, d, μ) with dimension $D > 0$ in the sense of R.R. Coifman and G. Weiss ([CW71], see also Section 2.2) they investigated non-negative, self-adjoint operators L on $L^2(X)$ which satisfy *(classical) Gaussian estimates*, i.e. the semigroup $(e^{-tL})_{t>0}$ generated by $-L$ can be represented as integral operators

$$e^{-tL} f(x) = \int_X p_t(x, y) f(y) \, d\mu(y)$$

for all $f \in L^2(X)$, $t > 0$, μ-a.e. $x \in X$ and the kernels $p_t \colon X \times X \to \mathbb{C}$ enjoy the following pointwise upper bound

$$|p_t(x, y)| \leq C \, \mu(B(x, t^{1/m}))^{-1} \exp\left(-b \left(\frac{d(x, y)}{t^{1/m}} \right)^{\frac{m}{m-1}} \right) \tag{1.3}$$

for all $t > 0$ and all $x, y \in X$, where $b, C > 0$ and $m \geq 2$ are constants independent of t, x, y and $B(x, t^{1/m}) := \{y \in X : d(x, y) < t^{1/m}\}$ denotes the open ball in X with center x and radius $t^{1/m}$. Under these hypotheses X.T. Duong, E.M. Ouhabaz, and A. Sikora obtained that the operator $F(L)$ is of weak type $(1, 1)$ and thus bounded on $L^p(X)$ for all $p \in (1, \infty)$ whenever $F \colon [0, \infty) \to \mathbb{C}$ is a bounded Borel function with $\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H_2^s} < \infty$ for some $s > (D + 1)/2$.

However, the price for the generality lies in the requirement of an additional $1/2$ in the regularity order of the Hörmander condition. Unfortunately, sharp results as for the Laplacian are unknown at present time. In the general situation one can only say that the regularity assumption $s > D/2 + 1/6$ cannot be weakened as an example in [Tha89] by S. Thangavelu shows (see also [DOS02, Theorem 1.3]).

In order to get better multiplier results in the general situation as well, X.T. Duong, E.M. Ouhabaz, and A. Sikora introduced the so-called *Plancherel condition* ([DOS02, (3.1)]) which states the following: There exist $C > 0$ and $q \in [2, \infty)$ such that for all $R > 0$, $y \in X$, and all bounded Borel functions $F \colon [0, \infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [0, R]$

$$\int_X \left| K_{F(\sqrt[m]{L})}(x, y) \right|^2 d\mu(x) \leq C \mu(B(y, 1/R))^{-1} \|F(R \cdot)\|_q^2 \,, \tag{1.4}$$

where $K_{F(\sqrt[m]{L})} \colon X \times X \to \mathbb{C}$ denotes the kernel of the integral operator $F(\sqrt[m]{L})$. The result of X.T. Duong, E.M. Ouhabaz, and A. Sikora reads as follows ([DOS02, Theorem 3.1]):

Let (X, d, μ) be a space of homogeneous type with dimension D and L be a non-negative, self-adjoint operator on $L^2(X)$ which satisfies Gaussian estimates. Suppose that the Plancherel condition holds for some $q \in [2, \infty)$ and that $F\colon [0, \infty) \to \mathbb{C}$ is a bounded Borel function with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} < \infty$ for some $s > D/2$. Then the operator $F(L)$ is of weak type $(1, 1)$ and thus bounded on $L^p(X)$ for all $p \in (1, \infty)$.

According to [DOS02, Lemma 2.2], the inequality (1.4) always holds for $q = \infty$. In this case a corresponding statement is also valid when the Hörmander condition is formulated with respect to the norm in the Hölder space C^s.

Sometimes it is not clear whether, or even not true that, a non-negative, self-adjoint operator on $L^2(X)$ admits Gaussian estimates and thus the above result cannot be applied. This occurs, for example, for Schrödinger operators with bad potentials ([SV94]) or elliptic operators of higher order with bounded measurable coefficients ([Dav97]). Further examples are described in Chapter 7. Nevertheless, it is often possible to show a weakened version of Gaussian estimates, so-called generalized Gaussian estimates.

Let $p_0 \in [1, 2)$ and $m \geq 2$. A non-negative, self-adjoint operator L on $L^2(X)$ is said to fulfill *generalized Gaussian estimates* ($\mathrm{GGE}_{p_0,m}$) if there exist constants $b, C > 0$ such that

$$\left\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\right\|_{p_0 \to p_0'} \leq C\, \mu(B(x, t^{1/m}))^{-(\frac{1}{p_0} - \frac{1}{p_0'})} \exp\left(-b\left(\frac{d(x, y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right) \tag{1.5}$$

for all $t > 0$ and all $x, y \in X$. Here, $\mathbb{1}_E$ denotes the characteristic function of the set $E \subseteq X$ and $\|\mathbb{1}_{E_1} T \mathbb{1}_{E_2}\|_{p\to q}$ is defined by $\sup_{\|f\|_p \leq 1} \|\mathbb{1}_{E_1} \cdot T(\mathbb{1}_{E_2} f)\|_q$ for a bounded linear operator T on $L^2(X)$, $p, q \in [1, \infty]$, and measurable sets $E_1, E_2 \subseteq X$. Moreover, p_0' denotes the conjugate exponent of p_0, i.e. $1/p_0 + 1/p_0' = 1$ with the usual convention $1/\infty := 0$.

This definition covers classical Gaussian estimates for $p_0 = 1$ ([BK02, Proposition 2.9]). It is known that for the class of operators L with ($\mathrm{GGE}_{p_0,m}$) the interval $[p_0, p_0']$ is, in general, optimal for the existence of the semigroup $(e^{-tL})_{t>0}$ on $L^p(X)$ for each $p \in [p_0, p_0']$ in the sense that, if p does not lie in this range, one finds an operator L satisfying ($\mathrm{GGE}_{p_0,m}$), but e^{-tL} cannot be extended from $L^p(X) \cap L^2(X)$ to a bounded linear operator on $L^p(X)$ for any $t > 0$ (see e.g. [Dav97, Theorem 10]). In **Chapter 2** we take a closer look at the properties of generalized Gaussian estimates and the Hörmander condition.

In 2003, S. Blunck showed the following spectral multiplier theorem for operators satisfying generalized Gaussian estimates ([Blu03, Theorem 1.1]):

Let (X, d, μ) be a space of homogeneous type with dimension D and L be a non-negative, self-adjoint operator on $L^2(X)$ which satisfies generalized Gaussian estimates ($GGE_{p_0,m}$) for some $p_0 \in [1, 2)$ and $m \geq 2$. If $F\colon [0, \infty) \to \mathbb{C}$ is a bounded Borel function with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_2} < \infty$ for some $s > (D + 1)/2$, then the operator $F(L)$ is of weak type (p_0, p_0) and thus bounded on $L^p(X)$ for all $p \in (p_0, p_0')$.

In **Chapter 3** we present an improved spectral multiplier result with an adequate substitute of the Plancherel condition (1.4) which also works for operators L satisfying generalized Gaussian estimates. In order to motivate our replacement of (1.4), we rewrite (1.4) with the help of the identity $\sup_{y\in X} \|K_{F(\sqrt[m]{L})}(\cdot,y)\|_2 = \|F(\sqrt[m]{L})\|_{1\to 2}$ as a norm estimate for the operator $F(\sqrt[m]{L})$ itself

$$\left\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\right\|_{1\to 2} \le C\,|B(y,1/R)|^{-\frac{1}{2}}\|F(R\cdot)\|_q$$

for all $R>0$, $y\in X$, and all bounded Borel functions $F\colon [0,\infty)\to\mathbb{C}$ with $\operatorname{supp} F\subseteq[0,R]$, where the constants $C>0$ and $q\in[2,\infty)$ are independent of R,y,F. Inspired by this observation, we introduce our substitute of the Plancherel condition for operators L which fulfill generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$) for some $p_0\in[1,2)$ and $m\ge 2$ as follows:

$$\left\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\right\|_{p_0\to 2} \le C\,|B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\|F(R\cdot)\|_q \tag{1.6}$$

for all $R>0$, $y\in X$, and all bounded Borel functions $F\colon [0,\infty)\to\mathbb{C}$ with $\operatorname{supp} F\subseteq[0,R]$, where the constants $C>0$ and $q\in[2,\infty)$ are independent of R,y,F. Having this replacement of (1.4) at hand, we are able to show the following result (cf. Theorem 3.1) that provides a generalization of the statement [DOS02, Theorem 3.1] to operators which fulfill generalized Gaussian estimates.

Let (X,d,μ) be a space of homogeneous type with dimension D and L be a non-negative, self-adjoint operator on $L^2(X)$ such that generalized Gaussian estimates ($GGE_{p_0,m}$) for some $p_0\in[1,2)$ and $m\ge 2$ as well as the Plancherel condition (1.6) for some $q\in[2,\infty)$ hold. If $F\colon[0,\infty)\to\mathbb{C}$ is a bounded Borel function with $\sup_{n\in\mathbb{Z}}\|\omega F(2^n\cdot)\|_{H^s_q}<\infty$ for some $s>D/2$, then the operator $F(L)$ is of weak type (p_0,p_0) and thus bounded on $L^p(X)$ for all $p\in(p_0,p_0')$.

Note that in the case $p_0=1$ our statement matches with the one in [DOS02] which is sharp in the sense that it includes the same regularity assumptions as needed for the Laplacian in Hörmander's multiplier theorem which are sharp.

As the validity of the Plancherel condition (1.4) entails that the point spectrum of the considered operator L is empty, X.T. Duong, E.M. Ouhabaz, and A. Sikora presented a version of the Plancherel condition that applies for operators with non-empty point spectrum as well ([DOS02, Theorem 3.2]). We shall upgrade their result to operators which satisfy only generalized Gaussian estimates (cf. Theorem 3.8).

All the described results for operators L satisfying ($\mathrm{GGE}_{p_0,m}$) for some $p_0\in[1,2)$ have in common that the required regularity order in the Hörmander condition for getting a weak type (p_0,p_0)-bound is the same as needed for the weak type $(1,1)$-bound in the corresponding statements for operators enjoying classical Gaussian estimates. Their proofs rely on the weak type (p_0,p_0) criterion due to S. Blunck and P.C. Kunstmann ([BK03, Theorem 1.1]) and it seems to be impossible to soften the regularity assumptions with this approach. However, since for boundedness of $F(L)$ on $L^2(X)$ no regularity of F is needed, one expects, motivated by interpolation, $s > (D+1)(1/p_0 - 1/2)$ or $s > D(1/p_0 - 1/2)$ instead of $s > (D+1)/2$ or $s > D/2$, respectively, as sufficient regularity assumptions in the Hörmander condition when one asks about boundedness of $F(L)$ in $L^p(X)$ for all $p \in (p_0, 2]$. Since in the case of $(\mathrm{GGE}_{p_0,m})$ a weak type $(1,1)$-bound for $F(L)$ is, in general, not possible, we cannot hope to conclude such weakened regularity orders by interpolating the regularity orders $(D+1)/2$ or $D/2$ for the weak $(1,1)$-bound and 0 on $L^2(X)$.

In order to master the challenge, we consider Hardy spaces which serve as a substitute of Lebesgue spaces. For our purposes we shall consider specific Hardy spaces being associated to the operator L. They were introduced and revised during the past ten years. For a short survey about their development we refer to the beginning of **Chapter 4**. Those Hardy spaces possess nice properties, for example, they form an interpolation scale, coincide under the assumption of $(\mathrm{GGE}_{p_0,m})$ with $L^p(X)$ for all $p \in (p_0, 2]$, and allow spectral multiplier theorems even for all $p \in [1, p_0]$.

Such spaces can be defined for injective, non-negative, self-adjoint operators L on $L^2(X)$ which satisfy the $L^2(X)$-version of (1.5) with $p_0 = 2$, so-called *Davies-Gaffney estimates*

$$\left\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\right\|_{2\to 2} \le C \exp\left(-b\left(\frac{d(x,y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right)$$

for all $t > 0$ and all $x, y \in X$, where $m \ge 2$ and $b, C > 0$ are constants independent of t, x, y. Sometimes we refer to m as the order of L. For introducing Hardy spaces, define $\psi_0(z) := ze^{-z}$ und consider the conical square function

$$S_{\psi_0} f(x) := \left(\int_0^\infty \int_{B(x,t)} |\psi_0(t^m L) f(y)|^2 \, \frac{d\mu(y)}{|B(x,t)|} \, \frac{dt}{t}\right)^{1/2} \qquad (f \in L^2(X), x \in X).$$

For $p \in [1,2]$, the *Hardy space $H^p_{L,S_{\psi_0}}(X)$ associated to L via square functions* is said to be the completion of $\{f \in L^2(X) : S_{\psi_0} f \in L^p(X)\}$ with respect to the norm

$$\|f\|_{H^p_{L,S_{\psi_0}}} := \|S_{\psi_0} f\|_{L^p(X)} .$$

By the spectral theorem, it is plain to see that $H^2_{L,S_{\psi_0}}(X) = L^2(X)$. Under the assumption of $(\mathrm{GGE}_{p_0,m})$ for some $p_0 \in [1,2)$ we prove that $H^p_{L,S_{\psi_0}}(X)$ and $L^p(X)$ coincide for all $p \in (p_0, 2]$ (cf. Theorem 4.19).

There is an equivalent characterization of the space $H^1_{L,S_{\psi_0}}(X)$ in terms of a molecular decomposition (cf. Theorem 4.10). As we shall see, in order to verify boundedness of an operator on the Hardy space $H^1_{L,S_{\psi_0}}(X)$, the study of the operator on the whole Hardy space is not needed, one has just to understand the action of the operator on an individual molecule. Such an idea is classical in the more comfortable situation of the presence of an atomic decomposition and was used by various authors for obtaining boundedness of spectral multipliers on the Hardy space $H^1_{L,S_{\psi_0}}(X)$. For example, J. Dziubański ([Dzi99]) showed a spectral multiplier theorem for Schrödinger operators and, later, J. Dziubański and M. Preisner ([DP09]) established a generalization to arbitrary operators satisfying classical Gaussian estimates of order 2. Recently, X.T. Duong and L.X. Yan ([DY11]) obtained boundedness of spectral multipliers on the Hardy space $H^1_{L,S_{\psi_0}}(X)$ for operators L satisfying Davies-Gaffney estimates of order 2.

All the authors confined their studies to operators of order 2 because in this case the Davies-Gaffney estimates are equivalent to the finite speed propagation property for the corresponding wave equation (see e.g. [CS08, Theorem 3.4]) which contains information on the support of the integral kernel of $\cos(t\sqrt{L})$ and this in turn entails information on the kernel of $F(\sqrt{L})$ since they are related via the Fourier transform provided that F is an odd function. However, in general, such a relation fails (see e.g. [DOS02, Section 8.2]). In order to develop spectral multiplier results for operators satisfying Davies-Gaffney estimates of arbitrary order $m \geq 2$, we thus have to employ a different reasoning. In **Chapter 5** we shall establish a spectral multiplier theorem on the Hardy space $H^1_{L,S_{\psi_0}}(X)$ for such operators L by reducing the proof of the boundedness of $F(L)$ in $H^1_{L,S_{\psi_0}}(X)$ to the uniform boundedness of $F(L)a$ in $H^1_{L,S_{\psi_0}}(X)$ for every molecule a. That will be achieved by using similar tools as those prepared in Chapter 3. It turns out that the regularity assumptions in this statement are the same as needed for obtaining a weak type (p_0, p_0)-bound result in the presence of generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$).

In **Chapter 6** we state the main result of our work, namely a spectral multiplier result for operators satisfying generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$) for some $p_0 \in [1, 2)$ and $m \geq 2$. In a first step we combine our multiplier result on the Hardy space $H^1_{L,S_{\psi_0}}(X)$ with the interpolation procedure [Kri09, Corollary 4.84] which yields a multiplier result on $H^p_{L,S_{\psi_0}}(X)$ for all $p \in [1, 2]$ (cf. Theorem 6.3). But, according to Theorem 4.19, the spaces $H^p_{L,S_{\psi_0}}(X)$ and $L^p(X)$ coincide for each $p \in (p_0, 2]$, so that we obtain a spectral multiplier theorem on Lebesgue spaces, presented in Theorem 6.4 a), which reads as follows:

Let (X, d, μ) be a space of homogeneous type with dimension D and L be a non-negative, self-adjoint operator on $L^2(X)$ such that generalized Gaussian estimates ($GGE_{p_0,m}$) hold for some $p_0 \in [1, 2)$ and $m \geq 2$. For fixed $p \in (p_0, p_0')$, suppose that $s > (D + 1)\,|1/p - 1/2|$ and $1/q < |1/p - 1/2|$. Then, for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty$, the operator $F(L)$ is bounded on the Lebesgue space $L^p(X)$. More precisely, there exists a constant $C > 0$ such that

$$\|F(L)\|_{p\to p} \leq C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} + |F(0)|\Big)\,.$$

This statement improves the results [Blu03, Theorem 1.1] of S. Blunck and [Kri09, Theorem 4.95] of C. Kriegler concerning the required regularity order. We emphasize that in the presence of classical Gaussian estimates the multiplier theorem due to X.T. Duong, E.M. Ouhabaz, and A. Sikora in combination with interpolation would need the same regularity of F as our main result for ensuring the boundedness of $F(L)$ on $L^p(X)$ for $p \in (p_0, p_0')$.

We also quote a corresponding version of our main result with an installed version of the Plancherel condition that leads to weakened regularity assumptions (cf. Theorem 6.4 b)).

The final **Chapter 7** is devoted to the study of applications. We investigate the second order Maxwell operator with measurable coefficient matrices on bounded Lipschitz domains in $\mathbb{R}^3$ and show that the above spectral multiplier theorem is applicable to this operator (cf. Theorem 7.10). Our reasoning consists in using Davies' perturbation method (see e.g. [Dav95]) together with arguments similar to those in [MM09]. Moreover, we discuss the Stokes operator with Hodge boundary conditions. Finally, we verify, based on recent results in [MM10] due to M. Mitrea and S. Monniaux, the validity of generalized Gaussian estimates for the Lamé operator. From this it follows not only that our spectral multiplier result applies to this important operator, but also that its spectrum is p-independent. We mention that, in general, classical Gaussian estimates for the above operators fail.

Acknowledgement

In the first place, I would like to express my sincere gratitude to my supervisor Priv.-Doz. Dr. Peer Christian Kunstmann. I owe him my thanks for his great support, many fruitful discussions and lots of valuable advices which contributed to the success of this thesis.

Also, I am thankful to Prof. Dr. Lutz Weis for his kind readiness to co-examine this thesis. He filled me with enthusiasm for functional analysis during my studies in mathematics.

Many thanks also go to my colleagues and former colleagues in the Institute for Analysis for the brilliant working atmosphere.

Finally, I am grateful to my parents for supporting me throughout my whole life.

2 Preliminaries

In this chapter we introduce the basic notations, definitions, and background material needed for reading this thesis.

2.1 Notations

Throughout the whole thesis we assume that (X, d, μ) is a space of homogeneous type with dimension D, as defined in Section 2.2. To avoid repetition, we skip this assumption in all the subsequent statements.

We make use of the notation $B(x,r) := \{y \in X : d(y,x) < r\}$ for the open ball in X with center $x \in X$ and radius $r \geq 0$. We shall write $\lambda B(x,r)$ for the λ-dilated ball $B(x, \lambda r)$ and $A(x,r,k)$ for the annular region $B(x,(k+1)r) \setminus B(x,kr)$, where $k \in \mathbb{N}_0$, $\lambda > 0$, $r > 0$, and $x \in X$. The volume of a Borel set $\Omega \subseteq X$ will be denoted by $|\Omega| := \mu(\Omega)$.

For $p \in [1, \infty]$ the Lebesgue space $L^p(X, \mu)$ will be written in short $L^p(X)$. Additionally, we abbreviate the operator norm $\| \cdot \|_{p \to q} := \| \cdot \|_{L^p(X) \to L^q(X)}$ for $1 \leq p \leq q \leq \infty$.

The symbol $\mathbb{1}_E$ stands for the characteristic function of a Borel set $E \subseteq X$, whereas $\|\mathbb{1}_{E_1} T \mathbb{1}_{E_2}\|_{p \to q}$ is defined via $\sup_{\|f\|_p \leq 1} \|\mathbb{1}_{E_1} \cdot T(\mathbb{1}_{E_2} f)\|_q$ for a bounded linear operator T on $L^2(X)$, Borel sets $E_1, E_2 \subseteq X$, and $1 \leq p \leq q \leq \infty$.

For $p \in [1, \infty]$ the conjugate exponent p' is defined by $1/p + 1/p' = 1$ with the usual convention $1/\infty := 0$.

In the proofs, the letters b, C denote generic positive constants that are independent of the relevant parameters involved in the estimates and may take different values at different occurrences. We will often use the notation $a \lesssim b$ if there exists a constant $C > 0$ such that $a \leq Cb$ for two non-negative expressions a, b; $a \cong b$ stands for the validity of $a \lesssim b \lesssim a$.

2.2 Spaces of homogeneous type

Before we introduce spaces of homogeneous type, we briefly recall some basic definitions in measure theory. Fix a non-empty metric space (X, d). The *Borel σ-algebra* denotes the smallest σ-algebra of X that contains all the open subsets of X, its elements are also referred to as *Borel sets*. A measure μ defined on the Borel σ-algebra is called *Borel measure*. It is said to be *σ-finite* if X can be written as the union of a finite or countable family of Borel sets with finite measure. A Borel measure μ is called *regular* if it holds for any Borel set B

$$\mu(B) = \sup\{\mu(C) : C \subseteq B,\, C \text{ compact}\} = \inf\{\mu(O) : O \supseteq B,\, O \text{ open}\}.$$

We will present our results in the general framework of *spaces of homogeneous type* (X, d, μ), i.e. (X, d) is a non-empty metric space endowed with a σ-finite regular Borel measure μ with $\mu(X) > 0$ which satisfies the so-called *doubling condition*, that is, there exists a constant $C > 0$ such that for all $x \in X$ and all $r > 0$

$$|B(x, 2r)| \leq C\,|B(x, r)|. \tag{2.1}$$

This inequality entails immediately the *strong homogeneity property*, i.e. the existence of constants $C, D > 0$ such that for all $x \in X$, all $r > 0$, and all $\lambda \geq 1$

$$|B(x, \lambda r)| \leq C\lambda^D |B(x, r)|. \tag{2.2}$$

Indeed, for arbitrary $\lambda \geq 1$ write $\lambda = 2^m\rho$ with $m \in \mathbb{N}_0$ and $\rho \in [1, 2)$ and put $D := \log_2 C$, where C is the uniform constant in the doubling condition (2.1). Then we obtain for any $x \in X$ and $r > 0$ by using the doubling condition and the monotonicity of the measure μ

$$|B(x, \lambda r)| \leq C^m |B(x, 2r)| \leq C(\lambda/2^m)^D C^m |B(x, r)| = C\lambda^D |B(x, r)|.$$

In the sequel the value D always refers to the constant in (2.2) which will be also called *dimension of* (X, d, μ). Of course, D is not uniquely determined and for any $D' \geq D$ the inequality (2.2) is surely valid. However, the smaller D is, the stronger will be the multiplier theorems we are able to obtain. Therefore, we are interested in taking D as small as possible.

There is a multitude of examples of spaces of homogeneous type. The simplest one is the Euclidean space $\mathbb{R}^D$, $D \in \mathbb{N}$, equipped with the Euclidean metric and the Lebesgue measure. Bounded open subsets of $\mathbb{R}^D$ with Lipschitz boundary endowed with the Euclidean metric and the Lebesgue measure form also spaces of homogeneous type. For further examples we refer to [Rus07, p. 126].

More general definitions of spaces of homogeneous type can be found in [CW71, Chapitre III.1], where even a quasi-distance d was allowed, or in [Ste93, Section I.1.2], where the doubling condition is replaced by a weaker assumption in which the ball $B(x, 2r)$ on the left-hand side of (2.1) is changed to the union of balls of radius r meeting the ball $B(x, r)$.

We give a short review about well-known results concerning spaces of homogeneous type and start with a simple but useful observation which is an immediate consequence of the doubling condition (2.1) (see e.g. [BK05, Lemma 3.2]).

Fact 2.1. *Let (X, d, μ) be a space of homogeneous type. Then there exists a constant $C > 0$ such that for all $r > 0$, $x \in X$, and $y \in B(x, r)$*

$$C^{-1}|B(y,r)| \le |B(x,r)| \le C\,|B(y,r)|\,.$$

Consequently, it holds for any $r > 0$ and any $x \in X$

$$C^{-1} \le \int_{B(x,r)} \frac{1}{|B(y,r)|}\,d\mu(y) \le C\,. \tag{2.3}$$

An essential feature of spaces of homogeneous type is the validity of covering results which mean that, as in the Euclidean setting, one can cover an arbitrary ball of radius r by balls of radius s and their number is bounded from above by a term only involving the ratio r/s and the constants in (2.2) whenever $r \ge s > 0$.

Lemma 2.2. *Let (X, d, μ) be a space of homogeneous type with dimension D. Then for each $r \ge s > 0$ and $y \in X$ there exist finitely many points $y_1, \ldots, y_K$ in $B(y, r)$ such that*

i) $d(y_j, y_k) > s/2$ *for all* $j, k \in \{1, \ldots, K\}$ *with* $j \ne k$;

ii) $B(y, r) \subseteq \bigcup_{k=1}^{K} B(y_k, s)$;

iii) $K \lesssim (r/s)^D$;

iv) each $x \in B(y, r)$ is contained in at most M balls $B(y_k, s)$, where M depends only on the constants in (2.2) and is independent of r, s, x, y.

Proof. Let $r > 0$ and $y \in X$. The existence of points $y_1, \ldots, y_K \in B(y, r)$ with the properties i) and ii) is well-known (see e.g. [AM07b, Lemmas 6.1, 6.2] or [CW71, pp. 68 ff.], where i) was originally used to define spaces of homogeneous type). Let $y_1, \ldots, y_K$ be such a family in $B(y, r)$. We will show that then iii) and iv) are valid.

Concerning iii), observe that for $j \ne k$ the balls $B(y_j, s/4)$ and $B(y_k, s/4)$ are disjoint. Therefore, it holds

$$|B(y,2r)| \ge \sum_{k=1}^{K} |B(y_k, s/4)|\,.$$

For each $k \in \{1, \ldots, K\}$ one obtains by Fact 2.1 and the doubling property that $|B(y, 2r)| \cong |B(y_k, 2r)| \lesssim (8r/s)^D |B(y_k, s/4)|$. This yields

$$|B(y,2r)| \ge \sum_{k=1}^{K} |B(y_k, s/4)| \gtrsim \sum_{k=1}^{K} \frac{1}{(8r/s)^D}\,|B(y,2r)| = \frac{K}{(8r/s)^D}\,|B(y,2r)|$$

which leads to

$$K \lesssim (r/s)^D\,.$$

In order to show iv), fix $x \in B(y,r)$ and denote by $I_s(x)$ the set of all $k \in \{1,2,\dots,K\}$ for which $x \in B(y_k,s)$ holds. As the balls $B(y_j,s/4)$ and $B(y_k,s/4)$ are disjoint for $j \neq k$, one obtains

$$\begin{aligned} |B(x,2s)| &\geq \sum_{k\in I_s(x)} |B(y_k,s/4)| \gtrsim \sum_{k\in I_s(x)} 8^{-D}|B(y_k,2s)| \\ &\cong 8^{-D} \sum_{k\in I_s(x)} |B(x,2s)| = 8^{-D}|B(x,2s)|\,\#I_s(x)\,, \end{aligned} \tag{2.4}$$

where the second estimate follows from the doubling property $|B(y_k,2s)| \lesssim 8^D|B(y_k,s/4)|$ and the third estimate from Fact 2.1 because $x \in B(y_k,s) \subseteq B(y_k,2s)$ for any $k \in I_s(x)$. Now (2.4) gives the desired conclusion. Here, $\#I_s(x)$ denotes the cardinality of $I_s(x)$. ☐

2.3 Generalized Gaussian estimates and Davies-Gaffney estimates

In this section we introduce and describe the main hypotheses on the considered operator. We also discuss some properties of two-ball estimates which are frequently used in our later studies.

Definition 2.3. Let $m \geq 2$ and $1 \leq p \leq 2 \leq q \leq \infty$ with $p < q$. A non-negative, self-adjoint operator L on $L^2(X)$ satisfies *generalized Gaussian* (p,q)*-estimates (of order* m*)* if there exist constants $b, C > 0$ such that for all $t > 0$ and all $x, y \in X$

$$\left\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\right\|_{p\to q} \leq C\,|B(x,t^{1/m})|^{-(\frac{1}{p}-\frac{1}{q})} \exp\left(-b\left(\frac{d(x,y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right). \tag{2.5}$$

If $p_0 \in [1,2)$ and L fulfills generalized Gaussian (p_0,p_0')-estimates of order m, we also use the shorthand notation ($\mathrm{GGE}_{p_0,m}$).

For obvious reasons, we call bounds of the type (2.5) *two-ball estimates*. They were first formulated by G. Schreieck and J. Voigt ([SV94]) in the form of weighted norm estimates

$$\left\|e^{-\xi(\cdot)} T e^{\xi(\cdot)}\right\|_{p\to q} \leq C(\xi) \qquad (\xi \in \mathbb{R}^D)$$

as a substitute for classical Gaussian estimates in their approach to derive L^p-spectral independence of certain Schrödinger operators in $\mathbb{R}^D$.

There are a number of operators which satisfy generalized Gaussian estimates and among them there exist many for which classical Gaussian estimates fail. In Chapter 7 we will discuss generalized Gaussian estimates for the Maxwell, Stokes, and Lamé operator. More examples like higher order operators with complex coefficients or Schrödinger operators with singular potentials on $\mathbb{R}^D$ can be found e.g. in [BK03] or [Blu03].

An estimate of the form (2.5) is essentially of off-diagonal nature. The terminology “off-diagonal” is justified by the observation that the exponential factor on the right-hand side of (2.5) only becomes relevant when the distance $d(x,y)$ is large compared to $t^{1/m}$ or, roughly speaking, when x and y are away from the diagonal.

To make the notations shorter, we sometimes set $r_t := t^{1/m}$ and $g(\lambda) := C\exp(-b\lambda^{\frac{m}{m-1}})$, where b, C are the constants appearing in (2.5). We shall formulate the condition in the definition of generalized Gaussian estimates in an alternative manner for getting more familiar with bounds of this kind. To do so, we first note that the volumes of balls with the same radius but different centers are comparable. Indeed, for each $t > 0$ and $x, y \in X$ we have $B(x, r_t) \subseteq B(y, r_t + d(x,y))$ and thus we deduce from (2.2) that

$$|B(x,r_t)| \lesssim \left(1 + \frac{d(x,y)}{r_t}\right)^D |B(y,r_t)|\,. \tag{2.6}$$

If e.g. $X = \mathbb{R}^D$ is equipped with the Lebesgue measure and the Euclidean distance, this bound can be improved to $|B(x,r_t)| = |B(y,r_t)|$, of course, but that shall play no rôle here. By using (2.6), we can replace $|B(x,r_t)|^{-(1/p-1/q)}$ by $|B(x,r_t)|^{-1/p}|B(y,r_t)|^{1/q}$ in (2.5) provided that we change the constants b, C. In other words, (2.5) is equivalent to an estimate of the form

$$\left\|\mathbb{1}_{B(x,r_t)} e^{-tL} \mathbb{1}_{B(y,r_t)}\right\|_{p\to q} \le |B(y,r_t)|^{-1/p} |B(x,r_t)|^{1/q} g\left(\frac{d(x,y)}{r_t}\right)$$

which can be rewritten explicitly as

$$\left(\frac{1}{|B(x,r_t)|}\int_{B(x,r_t)} |e^{-tL}f|^q \, d\mu\right)^{1/q} \le g\left(\frac{d(x,y)}{r_t}\right)\left(\frac{1}{|B(y,r_t)|}\int_{B(y,r_t)} |f|^p \, d\mu\right)^{1/p}$$

for all $t > 0$, $x, y \in X$, and $f \in L^p(X)$ with $\operatorname{supp} f \subseteq B(y, r_t)$.

We collect some rather technical properties of two-ball estimates in the next result which is proved in [BK05, Proposition 2.1].

Fact 2.4. *Let $1 \le p \le q \le \infty$, $\omega > 1$, and $g(\lambda) := Ce^{-b\lambda^\omega}$ for some constants $b, C > 0$. Suppose that T is a bounded linear operator on $L^2(X)$ and $r > 0$. Then the following statements are equivalent:*

a) *For all $x, y \in X$, it holds*

$$\left\|\mathbb{1}_{B(x,r)} T \mathbb{1}_{B(y,r)}\right\|_{p\to q} \le |B(x,r)|^{-(\frac{1}{p}-\frac{1}{q})} g\left(\frac{d(x,y)}{r}\right).$$

b) *For all $x, y \in X$ and all $u, v \in [p,q]$ with $u \le v$, it holds*

$$\left\|\mathbb{1}_{B(x,r)} T \mathbb{1}_{B(y,r)}\right\|_{u\to v} \le |B(x,r)|^{-(\frac{1}{u}-\frac{1}{v})} g\left(\frac{d(x,y)}{r}\right).$$

c) *For all $x \in X$ and all $k \in \mathbb{N}$, it holds*

$$\left\|\mathbb{1}_{B(x,r)} T \mathbb{1}_{A(x,r,k)}\right\|_{p\to q} \le |B(x,r)|^{-(\frac{1}{p}-\frac{1}{q})} g(k)\,.$$

d) *For all balls $B_1, B_2 \subseteq X$ and all $\alpha, \beta \ge 0$ with $\alpha+\beta = \frac{1}{p}-\frac{1}{q}$, it holds*

$$\left\|\mathbb{1}_{B_1} v_r^{\alpha} T v_r^{\beta} \mathbb{1}_{B_2}\right\|_{p\to q} \le g\left(\frac{\operatorname{dist}(B_1,B_2)}{r}\right),$$

where $\operatorname{dist}(B_1,B_2) := \inf\{d(x,y) : x \in B_1, y \in B_2\}$ *and* $v_r(x) := |B(x,r)|$ *for* $x \in X$.

This statement is written modulo identification of g and $\widetilde{g}$, where $\widetilde{g}(\lambda) = ag(c\lambda)$ for some constants $a, c > 0$ independent of r, ω, T.

Since the estimate stated in c) involves an annular set $A(x,r,k)$, we call bounds of this kind *estimates of annular type.*

Another feature of generalized Gaussian estimates lies in the fact that they can be extended from real times $t > 0$ to complex times $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. The following result is taken from [Blu07, Theorem 2.1] whose proof relies on the Phragmén-Lindelöf theorem.

Fact 2.5. *Let $m \ge 2$, $1 \le p \le 2 \le q \le \infty$, and L be a non-negative, self-adjoint operator on $L^2(X)$. Assume that there are constants $b, C > 0$ such that for any $t > 0$ and $x, y \in X$*

$$\left\|\mathbb{1}_{B(x,t^{1/m})} e^{-tL} \mathbb{1}_{B(y,t^{1/m})}\right\|_{p\to q} \le C\,|B(x,t^{1/m})|^{-(\frac{1}{p}-\frac{1}{q})} \exp\left(-b\left(\frac{d(x,y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right).$$

Then there exist constants $b', C' > 0$ such that for all $x, y \in X$ and all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

$$\left\|\mathbb{1}_{B(x,r_z)} e^{-zL} \mathbb{1}_{B(y,r_z)}\right\|_{p\to q} \le C'\,|B(x,r_z)|^{-(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \exp\left(-b'\left(\frac{d(x,y)}{r_z}\right)^{\frac{m}{m-1}}\right),$$

where $r_z := (\operatorname{Re} z)^{1/m-1}|z|$.

Sometimes it is cumbersome that the radius of the balls in the above two-ball estimate for e^{-zL} depends on the value of z. The next lemma provides two-ball estimates with balls of arbitrary radius $r > 0$ by the cost of an additional factor involving the ratio of r and r_z as well as the dimension of the underlying space of homogeneous type. Also a corresponding version for estimates of annular type is given.

Lemma 2.6. *Suppose that the assumptions of Fact 2.5 are fulfilled and, as before, define $r_z := (\operatorname{Re} z)^{1/m-1}|z|$ for each $z \in \mathbb{C}$ with* $\operatorname{Re} z > 0$.

a) *There exist constants* $b', C' > 0$ *such that for all* $r > 0$, $x, y \in X$, *and* $z \in \mathbb{C}$ *with* $\operatorname{Re} z > 0$

$$\begin{aligned}&\left\|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)}\right\|_{p\to q}\\ &\quad\le C' \left|B(x,r)\right|^{-(\frac{1}{p}-\frac{1}{q})} \left(1+\frac{r}{r_z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \exp\left(-b'\left(\frac{d(x,y)}{r_z}\right)^{\frac{m}{m-1}}\right).\end{aligned}$$

b) *There exist constants* $b'', C'' > 0$ *such that for all* $k \in \mathbb{N}$, $r > 0$, $x \in X$, *and* $z \in \mathbb{C}$ *with* $\operatorname{Re} z > 0$

$$\begin{aligned}&\left\|\mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{A(x,r,k)}\right\|_{p\to q}\\ &\quad\le C'' \left|B(x,r)\right|^{-(\frac{1}{p}-\frac{1}{q})} \left(1+\frac{r}{r_z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} k^D \exp\left(-b''\left(\frac{r}{r_z}\,k\right)^{\frac{m}{m-1}}\right).\end{aligned}$$

Proof. **a)** In view of Fact 2.5, there are constants $b, C > 0$ such that

$$\left\|\mathbb{1}_{B(x,r_z)} e^{-zL} \mathbb{1}_{B(y,r_z)}\right\|_{p\to q} \le C \left|B(x,r_z)\right|^{-(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \exp\left(-b\left(\frac{d(x,y)}{r_z}\right)^{\frac{m}{m-1}}\right)$$

for all $x, y \in X$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. By Fact 2.4 (with $T := (|z|/\operatorname{Re} z)^{-D(1/p-1/q)} e^{-zL}$), one finds $b', C' > 0$ such that

$$\left\|\mathbb{1}_{B_1} v_{r_z}^{\frac{1}{p}-\frac{1}{q}} T \mathbb{1}_{B_2}\right\|_{p\to q} \le C' \exp\left(-b'\left(\frac{\operatorname{dist}(B_1,B_2)}{r_z}\right)^{\frac{m}{m-1}}\right) \tag{2.7}$$

for all balls $B_1, B_2 \subseteq X$ and all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Let $r > 0$ be fixed. The doubling property leads to

$$v_r(x) \lesssim \left(1+\frac{r}{r_z}\right)^D v_{r_z}(x)$$

for every $x \in X$ and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Now choose arbitrary $x, y \in X$ with $d(x,y) \ge 4r$ and consider the balls $B_1 := B(x,r)$ and $B_2 := B(y,r)$. Then it holds

$$\operatorname{dist}(B_1,B_2) = d(x,y) - 2r \ge \frac{1}{2}\, d(x,y)\,.$$

By inserting B_1, B_2 into (2.7) and collecting the estimates above together, one arrives at

$$\left\|\mathbb{1}_{B(x,r)} v_r^{\frac{1}{p}-\frac{1}{q}} e^{-zL} \mathbb{1}_{B(y,r)}\right\|_{p\to q} \lesssim \left(1+\frac{r}{r_z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \exp\left(-b'\left(\frac{d(x,y)}{2r_z}\right)^{\frac{m}{m-1}}\right).$$

Since $v_r(x) \cong v_r(z)$ for all $z \in B(x,r)$ (cf. Fact 2.1), one obtains the desired estimate

$$\begin{aligned}
&\left\| \mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)} \right\|_{p\to q} \\
&\quad \le C' \left| B(x,r) \right|^{-(\frac{1}{p}-\frac{1}{q})} \left(1+\frac{r}{r_z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \exp\left(-b'\left(\frac{d(x,y)}{2r_z}\right)^{\frac{m}{m-1}}\right)
\end{aligned}$$

for all $r>0$, $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, and $x, y \in X$ with $d(x,y) \ge 4r$. By the cost of changing the constants b', C', one is able to remove this restriction on $d(x,y)$.

b) Our approach mimics that of [BK05, (i)⇒(3), p. 359]. Observe that it suffices to prove the statement only for every $k \in \mathbb{N} \setminus \{1\}$. With the help of [BK05, Lemma 3.4], we can write for each $k \in \mathbb{N} \setminus \{1\}$, $r > 0$, $x \in X$, and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

$$\begin{aligned}
&\left\| \mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{A(x,r,k)} \right\|_{p\to q} \\
&\lesssim \int_X \left\| \mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)} \right\|_{p\to q} \left\| \mathbb{1}_{B(y,r)} \mathbb{1}_{A(x,r,k)} \right\|_{q\to q} v_r(y)^{-1} \, d\mu(y) \\
&= \int_{B(x,(k+2)r)\setminus B(x,(k-1)r)} \left\| \mathbb{1}_{B(x,r)} e^{-zL} \mathbb{1}_{B(y,r)} \right\|_{p\to q} v_r(y)^{-1} \, d\mu(y) \,.
\end{aligned}$$

By exploiting the bound from part a), we continue our estimation

$$\begin{aligned}
&\lesssim \left| B(x,r) \right|^{-(\frac{1}{p}-\frac{1}{q})} \left(1+\frac{r}{r_z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \times \\
&\quad \times \int_{B(x,(k+2)r)\setminus B(x,(k-1)r)} \exp\left(-b'\left(\frac{d(x,y)}{r_z}\right)^{\frac{m}{m-1}}\right) v_r(y)^{-1} \, d\mu(y) \,.
\end{aligned}$$

Using $d(x,y) \ge (k-1)r \ge \frac{1}{2}\,kr$ as well as $v_r(y)^{-1} \lesssim (k+2)^D v_{(k+2)r}(y)^{-1}$ leads to

$$\begin{aligned}
&\lesssim \left| B(x,r) \right|^{-(\frac{1}{p}-\frac{1}{q})} \left(1+\frac{r}{r_z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \times \\
&\quad \times \int_{B(x,(k+2)r)\setminus B(x,(k-1)r)} \exp\left(-2^{-\frac{m}{m-1}} b'\left(\frac{kr}{r_z}\right)^{\frac{m}{m-1}}\right) (k+2)^D v_{(k+2)r}(y)^{-1} \, d\mu(y) \\
&\lesssim \left| B(x,r) \right|^{-(\frac{1}{p}-\frac{1}{q})} \left(1+\frac{r}{r_z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \left(\frac{|z|}{\operatorname{Re} z}\right)^{D(\frac{1}{p}-\frac{1}{q})} \times \\
&\quad \times \; (k+2)^D \exp\left(-2^{-\frac{m}{m-1}} b'\left(\frac{kr}{r_z}\right)^{\frac{m}{m-1}}\right),
\end{aligned}$$

where the last inequality is thanks to (2.3). This proves the statement. □

In Chapter 4 we will introduce specific Hardy spaces associated to an operator L. As we shall see later, for defining and working with these spaces one does not need to assume that L enjoys generalized Gaussian estimates. It turns out that it will be enough to require a special form of two-ball estimates on $L^2(X)$ for the semigroup $(e^{-tL})_{t>0}$ generated by $-L$, so-called Davies-Gaffney estimates.

Definition 2.7. Let $m \geq 2$. We say that a family $\{S_t : t > 0\}$ of bounded linear operators acting on $L^2(X)$ satisfies *Davies-Gaffney estimates (of order m)* if there exist constants $b, C > 0$ such that for all $t > 0$ and all $x, y \in X$

$$\left\| \mathbb{1}_{B(x,t^{1/m})} S_t \mathbb{1}_{B(y,t^{1/m})} \right\|_{2\to 2} \leq C \exp\left(-b \left(\frac{d(x,y)}{t^{1/m}} \right)^{\frac{m}{m-1}} \right). \tag{2.8}$$

In order to indicate the validity of Davies-Gaffney estimates of order m, we later use the abbreviation (DG$_m$). If $\{S_t : t > 0\} = (e^{-tL})_{t>0}$ is a semigroup on $L^2(X)$ generated by $-L$, we shall also say that L satisfies Davies-Gaffney estimates when the semigroup $(e^{-tL})_{t>0}$ enjoys this property.

Note that, after writing out, the norm estimate (2.8) takes the following form

$$\left| (S_t f, g)_{L^2(X)} \right| \leq C \exp\left(-b \left(\frac{d(x,y)}{t^{1/m}} \right)^{\frac{m}{m-1}} \right) \|f\|_2 \|g\|_2$$

for any $t > 0$, $x, y \in X$, and $f, g \in L^2(X)$ with $\operatorname{supp} f \subseteq B(y, t^{1/m})$ and $\operatorname{supp} g \subseteq B(x, t^{1/m})$.

Estimates of the type (2.8) were first introduced by E.B. Davies ([Dav92]) inspired by ideas of M.P. Gaffney ([Gaf59]). They hold for a wide class of operators, including essentially all self-adjoint, elliptic second-order differential operators or Schrödinger operators with real-valued potentials (cf. e.g. [CS08]).

Davies-Gaffney estimates were extensively studied in the recent series of papers [AM06], [AM07a], [AM07b], [AM08] due to P. Auscher and J.M. Martell (see also [CS08], [DL10], [HLMMY08]). We mention that in the literature one usually finds a slightly different definition of Davies-Gaffney estimates in which the validity of (2.8) is required for all open subsets of X. Since we assume (2.8) only for each ball in X and, furthermore, its radius must be linked to the scale of the considered operator family, we need to take much care about our arguments. For this reason, we sometimes argue with coverings and thus our reasonings get more involved than usual.

We collect some helpful properties of Davies-Gaffney estimates. At first, we state an immediate consequence of Fact 2.4 which provides the connection between Davies-Gaffney estimates and generalized Gaussian estimates.

Corollary 2.8. *Let $m \geq 2$. If a non-negative, self-adjoint operator L satisfies (GGE$_{p_0,m}$) for some $p_0 \in [1, 2)$, then L fulfills (DG$_m$). Actually, (DG$_m$) corresponds to (GGE$_{2,m}$).*

Next, we quote a statement originally given in [HLMMY08, Proposition 3.1] for operators satisfying (DG_2). However, with some minor modifications the proof can be adapted to include Davies-Gaffney estimates of arbitrary order $m \geq 2$ as well ([Fre11, Proposition 3.13]). Since the above results have in common that the Davies-Gaffney condition (2.8) was assumed to hold for all the open subsets of X, we work out an adequate proof which applies in our more general situation. It is essentially based on the fact that Davies-Gaffney estimates for semigroups generated by non-negative, self-adjoint operators on $L^2(X)$ can be extended from real to complex times (cf. Fact 2.5).

Lemma 2.9. *Let $m \geq 2$ and L be a non-negative, self-adjoint operator on $L^2(X)$. If L fulfills Davies-Gaffney estimates (DG_m), then for each $K \in \mathbb{N}$ the family of operators*

$$\left\{(tL)^K e^{-tL} : t > 0\right\}$$

satisfies also Davies-Gaffney estimates (DG_m) with constants depending only on K and the constants in the Davies-Gaffney condition (2.8) for the semigroup $(e^{-tL})_{t>0}$ and in the doubling condition (2.2).

Proof. Let $K \in \mathbb{N}$ and $t > 0$ be arbitrary. The Cauchy formula gives the representation

$$(tL)^K e^{-tL} = t^K \frac{(-1)^K K!}{2\pi i} \int_{|z-t|=\eta t} e^{-zL} \frac{dz}{(z-t)^{K+1}} ,$$

where $\eta := 1/2 \sin(\theta/2)$ for some $\theta \in (0, \pi/2)$. Note that the choice of η ensures that the ball $\{z \in \mathbb{C} : |z - t| \leq \eta t\}$ is contained in the sector $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$. According to Lemma 2.6, it holds for every $x, y \in X$

$$\begin{aligned} &\left\|\mathbb{1}_{B(x,t^{1/m})}(tL)^K e^{-tL}\mathbb{1}_{B(y,t^{1/m})}\right\|_{2\to 2} \\ &\leq t^K \frac{K!}{2\pi} \int_{|z-t|=\eta t} \left\|\mathbb{1}_{B(x,t^{1/m})} e^{-zL}\mathbb{1}_{B(y,t^{1/m})}\right\|_{2\to 2} \frac{|dz|}{|z-t|^{K+1}} \\ &\lesssim t^K \frac{K!}{2\pi} \int_{|z-t|=\eta t} \exp\left(-b'\left(\frac{d(x,y)}{r_z}\right)^{\frac{m}{m-1}}\right) \frac{|dz|}{(\eta t)^{K+1}} , \end{aligned}$$

where $r_z := (\operatorname{Re} z)^{1/m}|z|/\operatorname{Re} z$. Due to $\operatorname{Re} z \in [(1-\eta)t, (1+\eta)t]$ and $1 \leq |z|/\operatorname{Re} z \leq 1/\cos\theta$ for all z belonging to the integration path, we have $r_z \cong t^{1/m}$ with implicit constants depending only on θ or m. Thus, we can finish our estimation as follows

$$\begin{aligned} &\lesssim t^K \frac{K!}{2\pi} 2\pi\eta t \exp\left(-b'\left(\frac{d(x,y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right) \frac{1}{(\eta t)^{K+1}} \\ &= \frac{K!}{\eta^K} \exp\left(-b'\left(\frac{d(x,y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right) . \end{aligned}$$

□

2.4 The Hörmander condition

In order to formulate the Hörmander condition, we recall the definitions and some properties of Bessel potential spaces and Hölder spaces.

Definition 2.10. For $q \in (1,\infty)$ and $s \geq 0$, the *Bessel potential space* H_q^s is defined via

$$H_q^s := \left\{ f \in L^q(\mathbb{R}) : \|f\|_{H_q^s} := \left\| \mathscr{F}^{-1}\big((1+|\cdot|^2)^{s/2}\mathscr{F}f\big)\right\|_q < \infty \right\}.$$

Here, $\mathscr{F}$ denotes the Fourier transform on the space of the tempered distributions $\mathscr{S}'(\mathbb{R})$. The expression $\mathscr{F}^{-1}((1+|\cdot|^2)^{s/2}\mathscr{F}f)$ is initially defined for $f \in \mathscr{S}'(\mathbb{R})$ and lies in $L^q(\mathbb{R})$ when it equals a regular distribution generated by an element of $L^q(\mathbb{R})$. Because of $s \geq 0$, this is always the case. More on this topic can be found e.g. in [Gra09, Section 6.2.1].

It is well-known that the Bessel potential space H_q^s coincides with the usual Sobolev space W_q^s and their norms are equivalent whenever s is an integer. Bessel potential spaces arise naturally as interpolation spaces that are obtained from Lebesgue and Sobolev spaces by means of the complex interpolation method (see e.g. [Cal64] or [BL76, Chapter 4]).

Fact 2.11. *Let $q \in (1,\infty)$, $0 \leq s_0 < s_1 < \infty$, and $\theta \in (0,1)$. Put $s := (1-\theta)s_0 + \theta s_1$, then*

$$\left[H_q^{s_0}, H_q^{s_1}\right]_\theta = H_q^s.$$

The proof of this statement and further details about Bessel potential spaces, including their appearance in the more general concept of Triebel-Lizorkin spaces, can be found e.g. in [Tri78, Section 2.4.2]. Next, we record the algebra and boundedness property of Bessel potential spaces (see e.g. [RS96, p. 32 and p. 222]).

Fact 2.12. *Let $q \in (1,\infty)$ and $s > 0$. The following assertions are equivalent:*

a) *The Bessel potential space H_q^s is closed under pointwise multiplication.*

b) *It holds $H_q^s \hookrightarrow L^\infty(\mathbb{R})$.*

c) *It holds $H_q^s \hookrightarrow C^0$, where C^0 denotes the space of all the bounded and continuous functions $\mathbb{R} \to \mathbb{C}$ endowed with the supremum norm.*

(The embedding $H_q^s \hookrightarrow C^0$ means that every $f \in H_q^s$ can be changed on a set of measure zero such that the modified function lies in C^0.)

d) *The inequality $s > 1/q$ holds.*

Additionally, one has the following embedding result (see e.g. [Tri78, Remark 2, p. 206]).

Fact 2.13. *If $1 < q \leq p < \infty$ and $t - 1/p \leq s - 1/q$, then it holds*

$$H_q^s \hookrightarrow H_p^t.$$

Regarding the definition of Hölder spaces, we recall the space C^γ, $\gamma \in \mathbb{N}$, of the γ-times continuously differentiable functions on $\mathbb{R}$ which is given by

$$C^\gamma := \left\{ f \in C^0 : \, f^{(k)} \in C^0 \text{ for all } k \in \{1, 2, \dots, \gamma\} \right\}$$

equipped with the norm

$$\|f\|_{C^\gamma} := \sum_{k=0}^{\gamma} \|f^{(k)}\|_\infty \,.$$

Definition 2.14. For $s \geq 0$ write $s = \gamma + \alpha$ for some $\gamma \in \mathbb{N}_0$ and $\alpha \in [0,1)$. The *Hölder space* C^s is said to be

$$C^s := \left\{ f \in C^\gamma : \, \|f\|_{C^s} < \infty \right\},$$

where

$$\|f\|_{C^s} := \begin{cases} \|f\|_{C^\gamma} & \text{for } s = \gamma \,, \\ \|f\|_{C^\gamma} + \sup\limits_{x \neq y} \dfrac{|f^{(\gamma)}(x) - f^{(\gamma)}(y)|}{|x-y|^\alpha} & \text{for } s \neq \gamma \,. \end{cases}$$

Hölder spaces are not as well behaved with respect to complex interpolation as Bessel potential spaces, but at least one has an embedding result (cf. [Ull10, (3.6.6)]) which will be enough for our purposes.

Fact 2.15. *Let $0 \leq s_0 < s_1$ and $\theta \in (0,1)$. Then for all $\varepsilon \in (0, \theta)$ and $\sigma > (1-\theta)s_0 + \theta s_1$ one has*

$$C^\sigma \hookrightarrow [C^{s_0}, C^{s_1}]_{\theta - \varepsilon} \,.$$

In particular, for every $\delta > 0$ there is a constant $C > 0$ such that for all $f \in C^{(1-\theta)s_0 + \theta s_1 + \delta}$

$$\|f\|_{[C^{s_0}, C^{s_1}]_\theta} \leq C \, \|f\|_{C^{(1-\theta)s_0 + \theta s_1 + \delta}} \,.$$

In order to avoid case distinctions, we shall also denote, with some abuse of notation, the Hölder space C^s by H^s_∞ for each $s \geq 0$. We emphasize that this convention will be used throughout the whole thesis.

For the following multiplication property we refer to [Tri83, Corollary (ii), p. 143] if $q < \infty$ and to [RS96, (12), p. 230] if $q = \infty$.

Fact 2.16. *Let $q \in [2, \infty]$ and $s \geq 0$. If $\gamma \in \mathbb{N}$ with $\gamma > s$, then there exists a constant $C > 0$ such that for all $\psi \in C^\gamma$ and $f \in H^s_q$*

$$\|\psi f\|_{H^s_q} \leq C \, \|\psi\|_{C^\gamma} \|f\|_{H^s_q} \,.$$

Next, we choose a non-negative function $\omega \in C_c^\infty(0,\infty)$ such that

$$\operatorname{supp}\omega \subseteq (1/4,1) \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}} \omega(2^{-n}\lambda) = 1 \quad \text{for all } \lambda > 0\,. \tag{2.9}$$

Such a function ω actually exists. Indeed, take a non-negative function $\psi \in C_c^\infty(0,\infty)$ with $\operatorname{supp}\psi \subseteq (1/4,1)$ and $\psi(\lambda) > 0$ for each $\lambda \in [1/3, 2/3]$. For all $\lambda > 0$ define

$$\omega(\lambda) := \psi(\lambda)\left(\sum_{n\in\mathbb{Z}} \psi(2^{-n}\lambda)\right)^{-1}.$$

Note that for every $\lambda > 0$ the denominator does not vanish. Now it is easy to see that ω has the desired properties. This reasoning is well-known and can be found e.g. in [Ouh05, p. 215] (see also [Hör60, Lemma 2.3] for a similar construction).

Armed with the function ω, we are able to formulate the Hörmander condition.

Definition 2.17. Let $q \in [2,\infty]$ and $s > 1/q$ (with the usual convention $1/\infty := 0$). A bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ is said to satisfy the *Hörmander condition (of regularity order s in L^q)* if

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty\,. \tag{2.10}$$

Let us take a more detailed look at the requirement $s > 1/q$. It ensures that the function $F\colon (0,\infty) \to \mathbb{C}$ is bounded whenever $F \in L^q_{loc}(0,\infty)$ fulfills the Hörmander condition (2.10). Indeed, due to the properties of ω and Fact 2.12, we get for every $F \in L^q_{loc}(0,\infty)$ with (2.10)

$$\begin{aligned}\|F\|_\infty &= \sup_{n\in\mathbb{Z}} \|\mathbb{1}_{[2^{n-1},2^n]}F\|_\infty \le \sup_{n\in\mathbb{Z}} \big\|\big(\omega(2^{-n-1}\cdot) + \omega(2^{-n}\cdot)\big)F\big\|_\infty \\ &\le 2\sup_{n\in\mathbb{Z}} \|\omega(2^{-n}\cdot)F\|_\infty = 2\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_\infty \lesssim \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty\,. \end{aligned} \tag{2.11}$$

Note that the above consideration does not take into account the value of $F(0)$.

We make the following important observation.

Lemma 2.18. *The Hörmander condition (2.10) is independent of the choice of ω.*

Proof. Let ψ be another non-negative C_c^∞-function such that

$$\operatorname{supp}\psi \subseteq (1/4,1) \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}} \psi(2^{-n}\lambda) = 1 \quad \text{for all } \lambda > 0\,.$$

Thanks to $\sum_{k=-1}^{1}\omega(2^{-k}\cdot) = 1$ on $(1/4,1)$, one then has $\psi = \sum_{k=-1}^{1}\omega(2^{-k}\cdot)\psi$ on $\mathbb{R}$. By Fact 2.16, one obtains for some $\gamma \in \mathbb{N}$ with $\gamma > s$ and for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty$

$$\begin{aligned}\sup_{n\in\mathbb{Z}} \|\psi F(2^n\cdot)\|_{H_q^s} &= \sup_{n\in\mathbb{Z}} \Big\|\sum_{k=-1}^{1} \omega(2^{-k}\cdot)\psi F(2^n\cdot)\Big\|_{H_q^s} \lesssim \sup_{n\in\mathbb{Z}} \|\psi\|_{C^\gamma} \Big\|\sum_{k=-1}^{1} \omega(2^{-k}\cdot)F(2^n\cdot)\Big\|_{H_q^s} \\ &\lesssim \|\psi\|_{C^\gamma} \sup_{n\in\mathbb{Z}} \sum_{k=-1}^{1} \|\omega(2^{-k}\cdot)F(2^n\cdot)\|_{H_q^s} \lesssim \|\psi\|_{C^\gamma} \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty\,.\end{aligned}$$

Interchanging the roles of ψ and ω in the above argument yields the reverse inequality. □

An equivalent formulation of the Hörmander condition (2.10) can be achieved if one replaces the dilations by 2^n, $n \in \mathbb{Z}$, with the dilations by t, $t > 0$, that is

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty \qquad \Longleftrightarrow \qquad \sup_{t>0} \|\omega F(t\cdot)\|_{H_q^s} < \infty\,.$$

The latter was used, for example, in [DOS02].

The relation between the Hörmander condition (2.10) and its classical version (1.1) reads as follows (cf. [Kri09, Proposition 4.11]).

Lemma 2.19. **a)** *Let $\gamma \in \mathbb{N}$ and $F\colon [0,\infty) \to \mathbb{C}$ be a C^γ-function such that the classical Hörmander condition (1.1) holds. Then F satisfies (2.10) with $q = 2$ for all $s \in [0,\gamma]$.*

b) *If $F\colon [0,\infty) \to \mathbb{C}$ is a continuous function which fulfills (2.10) with $q = 2$ for some $s > 0$, then F satisfies (1.1) for every $\gamma \in \mathbb{N}_0$ with $\gamma \le s$.*

Proof. We only present the proof of part a) since that of b) is of similar nature.

Let $\gamma \in \mathbb{N}$ and $F\colon [0,\infty) \to \mathbb{C}$ be a C^γ-function such that (1.1) is satisfied. With the help of the change of variables $\eta = \xi/2R$ in the integral of (1.1) and the chain rule, we obtain an equivalent formulation of (1.1)

$$\sum_{k=0}^{\gamma} 2^{1/2-k} \sup_{R>0} \left(\int_{1/4}^{1} |F(2R\cdot)^{(k)}(\eta)|^2 \, d\eta\right)^{1/2} < \infty\,. \tag{2.12}$$

Fix $k \in \{0,1,\dots,\gamma\}$. The Leibniz rule yields

$$[\omega F(R\cdot)]^{(k)}(\eta) = \sum_{j=0}^{k} \binom{k}{j} \omega^{(k-j)}(\eta)\, F(R\cdot)^{(j)}(\eta)$$

for all $R > 0$ and $\eta \in (1/4, 1)$. By recalling $\operatorname{supp}\omega \subseteq (1/4,1)$ and $\|\omega^{(k-j)}\|_\infty \le \|\omega\|_{C^\gamma}$ for each $j \in \{0,1,\dots,k\}$, we deduce

$$\sup_{n\in\mathbb{Z}} \big\| [\omega F(2^n\cdot)]^{(k)} \big\|_2 \le \sup_{n\in\mathbb{Z}} \sum_{j=0}^{k} \binom{k}{j} \|\omega\|_{C^\gamma} \big\| \mathbb{1}_{(1/4,1)} F(2^n\cdot)^{(j)} \big\|_2$$

which is finite due to (2.12) and its bound depends only on γ and ω. Therefore, it holds for any $\nu \in \{0,1,\dots,\gamma\}$

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_2^\nu} \cong \sup_{n\in\mathbb{Z}} \sum_{k=0}^{\nu} \big\| [\omega F(2^n\cdot)]^{(k)} \big\|_2 < \infty\,.$$

Let $s \in [0,\gamma]$. In the case $s = \nu \in \{1,2,\dots,\gamma\}$ we are already finished. Otherwise, write $s = \nu + \alpha$ for some $\nu \in \{1,2,\dots,\gamma-1\}$ and $\alpha \in (0,1)$. We have just shown that

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_2^\nu} < \infty \qquad \text{and} \qquad \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_2^{\nu+1}} < \infty\,.$$

Now (2.10) follows from the interpolation inequality (see e.g. [Tri78, (3), p. 59])

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_2^s} \le \sup_{n\in\mathbb{Z}} \Big(\|\omega F(2^n\cdot)\|_{H_2^\nu}^{1-\alpha} \, \|\omega F(2^n\cdot)\|_{H_2^{\nu+1}}^{\alpha} \Big) < \infty\,.$$

□

At the end of this section we give some examples of functions for which the Hörmander condition (2.10) holds. We start with a simple but useful criterion that guarantees the validity of (2.10). Its straightforward proof will be omitted.

Fact 2.20. *Let $\gamma \in \mathbb{N}$ and $F\colon [0,\infty) \to \mathbb{C}$ be a bounded function that is γ-times continuously differentiable on $(0,\infty)$. If F satisfies*

$$\sup_{\lambda>0} |\lambda^k F^{(k)}(\lambda)| < \infty \qquad \text{for all } k \in \{0,1,\dots,\gamma\}, \tag{2.13}$$

then F satisfies the classical Hörmander condition (1.1) of regularity order γ and thus, thanks to Lemma 2.19 a), also (2.10) with $q = 2$ for all $s \in [0,\gamma]$.

The assumption (2.13) was introduced by S.G. Michlin in [Mic56] and is nowadays known as *Michlin condition*. More information concerning this condition can be found e.g. in [Hyt04, Section 1] (see also [Kri09, Chapter 4]).

Next, we consider bounded, holomorphic functions on certain sectors in the complex plane and examine that their restrictions to the non-negative real line fulfill the Hörmander condition (2.10) of arbitrary regularity order. In particular, this property shows that our results developed in the subsequent chapters are applicable to a wider class of functions than those being allowed in the holomorphic functional calculus. The connection between the holomorphic functional calculus and spectral multiplier theorems is well-known ([CDMY96, Theorem 4.10], see also [DOS02, Section 8.1]).

Lemma 2.21. *Every bounded holomorphic function F defined on the sector*

$$\Sigma_\theta := \{ z \in \mathbb{C} \setminus \{0\} \,:\, |\arg z| < \theta \}$$

for some $\theta \in (0, \pi)$ fulfills the Hörmander condition (2.10) with $q = 2$ for any $s > 0$.

Proof. By Fact 2.20, it suffices to check that for each $k \in \mathbb{N}$ there is $C > 0$ such that

$$\sup_{\lambda>0} |\lambda^k F^{(k)}(\lambda)| \leq \frac{C}{\theta^k} \|F\|_{\infty,\Sigma_\theta} \,, \tag{2.14}$$

where $\|F\|_{\infty,\Sigma_\theta}$ denotes the supremum of F on Σ_θ and the constant C is independent of θ and F. Fix $\lambda > 0$ and $k \in \mathbb{N}$. According to the Cauchy formula, we can write

$$F^{(k)}(\lambda) = \frac{k!}{2\pi i} \int_{|z-\lambda|=\eta\lambda} \frac{F(z)}{(z-\lambda)^{k+1}} \, dz \,,$$

where $\eta := 1/2 \sin(\theta/2) > 0$. This choice of η guarantees that the ball $\{z \in \mathbb{C} \,:\, |z-\lambda| \leq \eta\lambda\}$ is contained in Σ_θ. The further proceeding is standard

$$\begin{aligned} |\lambda^k F^{(k)}(\lambda)| &\leq \frac{\lambda^k k!}{2\pi} \int_{|z-\lambda|=\eta\lambda} \frac{|F(z)|}{|z-\lambda|^{k+1}} \, |dz| \leq \frac{\lambda^k k!}{2\pi} \int_{|z-\lambda|=\eta\lambda} |dz| \, \frac{\|F\|_{\infty,\Sigma_\theta}}{(\eta\lambda)^{k+1}} \\ &= \frac{\lambda^k k!}{2\pi} \, 2\pi\eta\lambda \, \frac{\|F\|_{\infty,\Sigma_\theta}}{(\eta\lambda)^{k+1}} = \frac{k!}{\eta^k} \|F\|_{\infty,\Sigma_\theta} \,. \end{aligned}$$

Due to the elementary inequality $\sin(\theta/2) \geq \theta/\pi$, we get the desired estimate (2.14). □

3 Spectral multipliers on Lebesgue spaces

For the formulation of the Hörmander condition we fix once and for all a non-negative function $\omega \in C_c^\infty(0,\infty)$ such that (2.9) holds.

3.1 A Hörmander type multiplier theorem for operators satisfying generalized Gaussian estimates

We state a generalization of [DOS02, Theorem 3.1] which applies to operators without heat kernels as well if generalized Gaussian estimates hold. The main tools for our proof consist in weighted norm estimates that may be seen as a substitute for the missing bounds on kernels. Later in Chapter 5, we will observe that the auxiliary results prepared in this section are also applicable for developing spectral multiplier results on Hardy spaces.

Theorem 3.1. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying generalized Gaussian estimates ($GGE_{p_0,m}$) for some $p_0 \in [1,2)$ and $m \geq 2$. Suppose that there exist $C > 0$ and $q \in [2,\infty]$ such that for any $R > 0$, $y \in X$, and arbitrary bounded Borel functions $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [0,R]$*

$$\big\| F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)} \big\|_{p_0 \to 2} \leq C\, |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \|F(R\cdot)\|_q\,. \tag{3.1}$$

Take $s > \max\{D/2, 1/q\}$. Then, for any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty\,,$$

the operator $F(L)$ is of weak type (p_0,p_0) and there exists a constant $C > 0$ such that

$$\|F(L)\|_{L^{p_0}(X)\to L^{p_0,\infty}(X)} \leq C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} + \|F\|_\infty \Big)\,.$$

In particular, the operator $F(L)$ is bounded on $L^p(X)$ for each $p \in (p_0, p_0')$.

Recall that the operator $F(L)$ is called *of weak type* (p_0,p_0) if there exists $C > 0$ such that for all $\alpha > 0$ and all $f \in L^{p_0}(X)$

$$\mu\big(\big\{x \in X :\, |F(L)f(x)| > \alpha\big\}\big) \leq \left(\frac{C\,\|f\|_{p_0}}{\alpha}\right)^{p_0}.$$

In this case we denote

$$\|F(L)\|_{L^{p_0}(X)\to L^{p_0,\infty}(X)} := \sup \alpha\, \mu\big(\{x\in X : |F(L)f(x)|>\alpha\}\big)^{1/p_0},$$

where the supremum is taken over all $\alpha>0$ and $f\in L^{p_0}(X)$ with $\|f\|_{p_0}\le 1$.

Before we turn to the proof, we make some remarks.

For $p_0=1$ the above statement coincides with the one given in [DOS02, Theorem 3.1].

The assertion of the theorem remains even valid for open subsets Ω of X provided that the ball appearing on the right-hand side of (2.5) is the one in X. The reasoning is standard and relies on an observation quoted in [BK03, pp. 934-935] by adapting the argument given in [DM99, p. 245] (see also [Blu03, p. 452]). For this purpose, one has only to extend an operator $T\colon L^p(\Omega)\to L^q(\Omega)$ by zero to the operator $\widetilde{T}\colon L^p(X)\to L^q(X)$ defined via

$$\widetilde{T}u(x) := \begin{cases} T(\mathbb{1}_\Omega u)(x) & \text{for } x\in\Omega \\ 0 & \text{for } x\in X\setminus\Omega \end{cases} \qquad (u\in L^p(X),\ \mu\text{-a.e. } x\in X)$$

and observe that $\|\widetilde{T}\|_{L^p(X)\to L^q(X)} = \|T\|_{L^p(\Omega)\to L^q(\Omega)}$. The modified result allows to treat elliptic operators on irregular domains $\Omega\subseteq\mathbb{R}^D$ as well (cf. e.g. [Blu03, Section 2.1]).

In analogy to the terminology employed in [DOS02], we will refer to (3.1) as *Plancherel condition.* As already mentioned in the introduction, the statement of the theorem is false without the Plancherel condition (3.1). But this requirement is always fulfilled for $q=\infty$, as the next lemma shows (cp. [DOS02, Lemma 2.2]). In particular, the theorem then gives the weak type (p_0,p_0) boundedness of the operator $F(L)$ whenever $F\colon[0,\infty)\to\mathbb{C}$ is a bounded Borel function with $\sup_{n\in\mathbb{Z}}\|\omega F(2^n\cdot)\|_{C^s}<\infty$ for some $s>D/2$. In the special case $p_0=1$ the described result matches with [Ouh05, Theorem 7.23].

Lemma 3.2. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ which satisfies generalized Gaussian estimates ($GGE_{p_0,m}$) for some $p_0\in[1,2)$ and $m\ge 2$. Then there exists a constant $C>0$ such that for all $R>0$, $y\in X$, and bounded Borel functions $F\colon[0,\infty)\to\mathbb{C}$ with* $\operatorname{supp} F\subseteq[0,R]$

$$\big\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\big\|_{p_0\to 2} \le C\,|B(y,1/R)|^{-(\frac{1}{p_0}-\frac12)}\|F\|_\infty\,.$$

Proof. Consider $R>0$, $y\in X$, and a bounded Borel function $F\colon[0,\infty)\to\mathbb{C}$ whose support is contained in $[0,R]$. For any $\lambda\ge0$ define $G_1(\lambda):=F(\sqrt[m]{\lambda})\,e^{\lambda/R^m}$ and $G_2(\lambda):=e^{-\lambda/R^m}$, so that, by the spectral theorem for L on $L^2(X)$, one can write $F(\sqrt[m]{L})=G_1(L)G_2(L)$. Observe that $\operatorname{supp} G_1\subseteq[0,R^m]$ and thus $\|G_1(L)\|_{2\to2}\le\|G_1\|_\infty\le e\,\|F\|_\infty$. As L fulfills

($\mathrm{GGE}_{p_0,m}$), we deduce with the help of Fact 2.4 that

$$\begin{aligned}\left\|G_2(L)\,\mathbb{1}_{B(y,1/R)}\right\|_{p_0\to 2} &\le \sum_{k=0}^{\infty}\left\|\mathbb{1}_{A(y,1/R,k)}e^{-\frac{1}{R^m}L}\mathbb{1}_{B(y,1/R)}\right\|_{p_0\to 2}\\ &\lesssim \sum_{k=0}^{\infty}|B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}e^{-bk^{\frac{m}{m-1}}}\\ &\lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,.\end{aligned}$$

Combining the estimates gives

$$\begin{aligned}\left\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\right\|_{p_0\to 2} &\le \|G_1(L)\|_{2\to 2}\left\|G_2(L)\,\mathbb{1}_{B(y,1/R)}\right\|_{p_0\to 2}\\ &\lesssim \|F\|_\infty\,|B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,.\end{aligned}$$

□

We prepare the proof of Theorem 3.1 with the next three lemmas. The first one corresponds to [DOS02, Lemma 4.1] and gives an extension of generalized Gaussian estimates from real times to complex times in some weighted space. This is crucial for our proof of Lemma 3.4, where the operator $F(\sqrt[m]{L})$ will be represented in terms of the extended semigroup $(e^{-zL})_{\mathrm{Re}\,z>0}$ by a Fourier transform argument.

Lemma 3.3. *Let $s \ge 0$ and L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying ($GGE_{p_0,m}$) for some $p_0 \in [1,2)$ and $m \ge 2$. Then there exists a constant $C > 0$ such that for all $R > 0$, $\tau \in \mathbb{R}$, and $y \in X$*

$$\left\|e^{-(1+i\tau)R^{-m}L}\mathbb{1}_{B(y,1/R)}\right\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \le C\,|B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,(1+\tau^2)^{s/4}\,.$$

Proof. In a first step we verify

$$\left\|e^{-(1+i\tau)R^{-m}L}\mathbb{1}_{B(y,1/R)}\right\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \lesssim (1+\tau^2)^{s/4} \tag{3.2}$$

for any $R > 0$, $\tau \in \mathbb{R}$, and $y \in X$. By Fact 2.4, the assumption ($\mathrm{GGE}_{p_0,m}$) implies that there exist constants $b, C > 0$ such that for all $x, y \in X$ and all $t > 0$

$$\left\|\mathbb{1}_{B(x,t^{1/m})}e^{-tL}\mathbb{1}_{B(y,t^{1/m})}\right\|_{2\to 2} \le C\exp\left(-b\left(\frac{d(x,y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right).$$

According to Fact 2.5, this can be extended from real times t to complex times z. Precisely, there exist $b, C > 0$ such that for all $x, y \in X$ and all $z \in \mathbb{C}$ with $\mathrm{Re}\,z > 0$

$$\left\|\mathbb{1}_{B(x,r_z)}e^{-zL}\mathbb{1}_{B(y,r_z)}\right\|_{2\to 2} \le C\exp\left(-b\left(\frac{d(x,y)}{r_z}\right)^{\frac{m}{m-1}}\right),$$

where $r_z := (\operatorname{Re} z)^{1/m-1}|z|$. By Fact 2.4, this two-ball estimate is equivalent to the assertion that there exist $b, C > 0$ such that for every $k \in \mathbb{N}_0$, $y \in X$, and $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$

$$\big\|\mathbb{1}_{B(y,r_z)} e^{-zL} \mathbb{1}_{A(y,r_z,k)}\big\|_{2\to 2} \le C \exp\big(-bk^{\frac{m}{m-1}}\big)$$

or equivalently

$$\big\|\mathbb{1}_{A(y,r_z,k)} e^{-\overline{z}L} \mathbb{1}_{B(y,r_z)}\big\|_{2\to 2} \le C \exp\big(-bk^{\frac{m}{m-1}}\big)\,,$$

where, as usual, $A(y, r_z, k)$ denotes the annular region $B(y, (k+1)r_z) \setminus B(y, kr_z)$.

Now let $R > 0$, $s \ge 0$, $\tau \in \mathbb{R}$, and $y \in X$ be fixed. For $z := (1 + i\tau)R^{-m}$ we calculate $r_z = (1 + \tau^2)^{1/2}/R \ge 1/R$ and obtain

$$\begin{aligned}
&\big\|e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \\
&\le \sum_{k=0}^{\infty} \big\|\mathbb{1}_{A(y,r_z,k)} e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \\
&\le \sum_{k=0}^{\infty} \big(1 + R(k+1)r_z\big)^{s/2} \big\|\mathbb{1}_{A(y,r_z,k)} e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{2\to 2} \\
&\le \sum_{k=0}^{\infty} \big(1 + (k+1)(1+\tau^2)^{1/2}\big)^{s/2} \big\|\mathbb{1}_{A(y,r_z,k)} e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,r_z)}\big\|_{2\to 2} \\
&\le C(1+\tau^2)^{s/4} \sum_{k=0}^{\infty} (k+2)^{s/2} \exp\big(-bk^{\frac{m}{m-1}}\big) \\
&\lesssim (1+\tau^2)^{s/4}\,,
\end{aligned}$$

i.e. (3.2) is proven. In order to deduce the assertion of the lemma, we apply a version of [BK05, Lemma 3.4] for weighted spaces

$$\begin{aligned}
&\big\|e^{-(1+i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \\
&\quad\le \big\|e^{-(\frac{1}{2}+i\tau)R^{-m}L} e^{-\frac{1}{2}R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \\
&\quad\lesssim \int_X \big\|e^{-(\frac{1}{2}+i\tau)R^{-m}L} \mathbb{1}_{B(x,2^{-1/m}R^{-1})}\big\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \times \\
&\qquad\qquad \times \big\|\mathbb{1}_{B(x,1/R)} e^{-\frac{1}{2}R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{p_0\to 2} \frac{d\mu(x)}{|B(x, 2^{-1/m}R^{-1})|}\,.
\end{aligned} \tag{3.3}$$

Thanks to (3.2) and

$$\big(1 + Rd(\cdot,y)\big)^s \le \big(1 + Rd(\cdot,x)\big)^s \big(1 + Rd(x,y)\big)^s$$

for all $x \in X$, the first factor of the integrand is bounded by a constant times

$$(1+\tau^2)^{s/4}\big(1 + Rd(x,y)\big)^{s/2}\,.$$

The second term can be treated with the help of ($\mathrm{GGE}_{p_0,m}$) (cp. Lemma 2.6)

$$\big\|\mathbb{1}_{B(x,1/R)} e^{-\frac{1}{2}R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{p_0\to 2} \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \exp\big(-b(Rd(x,y))^{\frac{m}{m-1}}\big)\,.$$

Gathering the estimates above, we obtain that (3.3) is bounded by a constant times

$$|B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}(1+\tau^2)^{s/4}\int_X \big(1+Rd(x,y)\big)^{s/2}\exp\big(-b(Rd(x,y))^{\frac{m}{m-1}}\big)\frac{d\mu(x)}{|B(x,2^{-1/m}R^{-1})|}\,.$$

Hence, it remains to check that the integral is finite with a bound independent of R, y. To this end, we write X as a disjoint union of annuli and estimate the integral over an annulus

$$\begin{aligned}
&\int_X \big(1+Rd(x,y)\big)^{s/2}\exp\big(-b(Rd(x,y))^{\frac{m}{m-1}}\big)\frac{d\mu(x)}{|B(x,2^{-1/m}R^{-1})|}\\
&\quad\lesssim \sum_{k=0}^{\infty}\int_{A(y,1/R,k)}(k+2)^{s/2}\exp(-bk^{\frac{m}{m-1}})(k+1)^D\frac{d\mu(x)}{|B(x,(k+1)R^{-1})|}\\
&\quad\lesssim \sum_{k=0}^{\infty}(k+2)^{s/2+D}\exp(-bk^{\frac{m}{m-1}})\,,
\end{aligned}$$

where the last step is due to (2.3). As the series converges, the proof is finished. □

The second preparatory statement, which is our replacement of [DOS02, Lemma 4.3 a)], transfers the regularity of a function F to the weight in some weighted norm estimate for $F(\sqrt[m]{L})$. The only difference between (3.4) and (3.5) lies in the norm of $F(R\cdot)$.

Lemma 3.4. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying ($\mathrm{GGE}_{p_0,m}$) for some $p_0 \in [1,2)$ and $m \geq 2$.*

a) *Then for any $s \geq 0$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\big\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)}\\
&\qquad\leq C\,|B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,\|F(R\cdot)\|_{H_2^{(s+1)/2+\varepsilon}}
\end{aligned}\tag{3.4}$$

for every $R > 0$, $y \in X$, and every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [R/4, R]$ and $F(R\cdot) \in H_2^{(s+1)/2+\varepsilon}$.

b) *Suppose additionally that L fulfills the Plancherel condition (3.1) for some $q \in [2,\infty]$. Then for any $s \geq 2/q$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\begin{aligned}
&\big\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)}\\
&\qquad\leq C\,|B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,\|F(R\cdot)\|_{H_q^{s/2+\varepsilon}}
\end{aligned}\tag{3.5}$$

for every $R > 0$, $y \in X$, and every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [R/4, R]$ and $F(R\cdot) \in H_q^{s/2+\varepsilon}$.

Proof. Let $R > 0$ and $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function with $\operatorname{supp} F \subseteq [R/4, R]$. For all $\lambda \geq 0$ define $G(\lambda) := F(R \sqrt[m]{\lambda})\, e^{\lambda}$. If $\widehat{G}$ denotes the Fourier transform of G, then it holds in the strong convergence sense in $L^2(X)$

$$F(\sqrt[m]{L}) = G(R^{-m}L)\, e^{-R^{-m}L} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{G}(\tau) e^{-(1-i\tau)R^{-m}L}\, d\tau\,.$$

Thus, Lemma 3.3 and the Cauchy-Schwarz inequality yield for any $y \in X$, $s \geq 0$, and $\varepsilon > 0$ whenever $F(R\cdot) \in H_2^{(s+1)/2+\varepsilon}$

$$\begin{aligned}
&\big\|F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \\
&\lesssim \int_{-\infty}^{\infty} |\widehat{G}(\tau)|\, \big\|e^{-(1-i\tau)R^{-m}L} \mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)}\, d\tau \\
&\lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \int_{-\infty}^{\infty} |\widehat{G}(\tau)|\, (1+\tau^2)^{s/4}\, d\tau \\
&\leq |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \left(\int_{-\infty}^{\infty} |\widehat{G}(\tau)|^2\, (1+\tau^2)^{\frac{s+1+\varepsilon}{2}}\, d\tau\right)^{1/2} \left(\int_{-\infty}^{\infty} (1+\tau^2)^{-\frac{1+\varepsilon}{2}}\, d\tau\right)^{1/2} \\
&\lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \|G\|_{H_2^{(s+1+\varepsilon)/2}}\,.
\end{aligned} \tag{3.6}$$

Due to $\operatorname{supp} F(R\cdot) \subseteq [1/4, 1]$, it follows

$$\|G\|_{H_2^{(s+1+\varepsilon)/2}} \lesssim \|F(R\cdot)\|_{H_2^{(s+1+\varepsilon)/2}} \lesssim \|F(R\cdot)\|_{H_q^{(s+1+\varepsilon)/2}} \tag{3.7}$$

for each $q \in [2,\infty]$. From (3.6) and (3.7) we obtain part a) of the lemma.

Inserting (3.7) in (3.4) leads to a statement in which the required order of differentiability of the function $F(R\cdot)$ is $1/2$ larger than that of part b). In order to get rid of this additional $1/2$, we make use of the interpolation procedure as described in [DOS02, p. 455] (see also [MM90]) based on the Plancherel condition (3.1). By a simple scaling argument, we first observe that the claimed bound (3.5) is equivalent to the following estimate

$$\big\|H(R^{-1}\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\, \|H\|_{H_q^{s/2+\varepsilon}} \tag{3.8}$$

for any $\varepsilon > 0$, $s \geq 2/q$, $R > 0$, $y \in X$, and any bounded Borel function $H\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} H \subseteq [1/4, 1]$ and $H \in H_q^{s/2+\varepsilon}$.

For fixed $R > 0$, $y \in X$, and $\varphi \in L^{p_0}(X)$ with $\operatorname{supp} \varphi \subseteq B(y,1/R)$ and $\|\varphi\|_{p_0} = 1$ define

$$K_{y,R,q}\colon E_q \to L^2(X)\,, \quad H \mapsto H(R^{-1}\sqrt[m]{L})\varphi\,,$$

where $E_q := L^\infty([1/4,1])$ if $q < \infty$ and $E_q := C^0([1/4,1])$ if $q = \infty$. According to the Plancherel condition (3.1), we see after rescaling that

$$\|K_{y,R,q}(H)\|_2 \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \|H\|_{L^q([1/4,1])}$$

for every $H \in E_q$. Next, for $\alpha \geq 0$ we denote by $H_q^\alpha([1/4,1])$ the set of all $H \in H_q^\alpha$ with $\operatorname{supp} H \subseteq [1/4,1]$. The inequalities (3.6) and (3.7) lead to

$$\|K_{y,R,q}(H)\|_{L^2(X,(1+Rd(\cdot,y))^s d\mu)} \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \|H\|_{H_q^{(s+1+\varepsilon)/2}([1/4,1])}$$

for any $s \geq 0$, $\varepsilon > 0$, and $H \in H_q^{(s+1+\varepsilon)/2}([1/4,1])$. Now complex interpolation (cf. Fact 2.11 if $q < \infty$ and Fact 2.15 if $q = \infty$) yields for every $\theta \in (0,1)$

$$\|K_{y,R,q}(H)\|_{L^2(X,(1+Rd(\cdot,y))^{\theta s} d\mu)} \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \|H\|_{H_q^{(s+1+\varepsilon)\theta/2+\delta}([1/4,1])} \tag{3.9}$$

for any $s \geq 0$, $\varepsilon > 0$, $H \in H_q^{(s+1+\varepsilon)\theta/2+\delta}([1/4,1])$, and $\delta > 0$.

Let $s' \geq 2/q$ and $\varepsilon' > 0$ be arbitrary. Take $\theta \in (0,1)$ and $\delta > 0$ with $(1+\varepsilon)\theta/2 + \delta = \varepsilon'$. Next, choose $s \geq 0$ with $s\theta = s'$. Then inequality (3.9) reads

$$\|K_{y,R}(H)\|_{L^2(X,(1+Rd(\cdot,y))^{s'} d\mu)} \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \|H\|_{H_q^{s'/2+\varepsilon'}([1/4,1])}$$

for any $H \in H_q^{s'/2+\varepsilon'}([1/4,1])$. Taking the supremum over all $\varphi \in L^{p_0}(X)$ such that $\operatorname{supp} \varphi \subseteq B(y,1/R)$ and $\|\varphi\|_{p_0} = 1$ yields

$$\big\|H(R^{-1}\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Rd(\cdot,y))^{s'} d\mu)} \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})} \|H\|_{H_q^{s'/2+\varepsilon'}([1/4,1])}$$

for any $H \in H_q^{s'/2+\varepsilon'}([1/4,1])$. This proves (3.8) and thus (3.5). □

A reinspection of the above proof shows that it also works in the case $p_0 = 2$, i.e. the assertion of Lemma 3.4 is even true for operators satisfying Davies-Gaffney estimates. This result will be an important ingredient for our proof of the spectral multiplier theorem on the Hardy space $H_L^1(X)$, given in Theorem 5.4, and we state it explicitly.

Corollary 3.5. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG$_m$) for some $m \geq 2$.*

a) *Then for any $s \geq 0$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\big\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\big\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \leq C\,\|F(R\cdot)\|_{H_2^{(s+1)/2+\varepsilon}}$$

for every $R > 0$, $y \in X$, and every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [R/4,R]$ and $F(R\cdot) \in H_2^{(s+1)/2+\varepsilon}$.

b) *Suppose additionally that L fulfills the Plancherel condition (3.1) for $p_0 = 2$ and some $q \in [2,\infty]$. Then for any $s \geq 2/q$ and $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\big\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\big\|_{L^2(X)\to L^2(X,(1+Rd(\cdot,y))^s d\mu)} \leq C\,\|F(R\cdot)\|_{H_q^{s/2+\varepsilon}}$$

for every $R > 0$, $y \in X$, and every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [R/4,R]$ and $F(R\cdot) \in H_q^{s/2+\varepsilon}$.

As a final preparation for the proof of Theorem 3.1, we show the following statement which is an immediate consequence of the doubling property of X. Our reasoning resembles that of [DOS02, Lemma 4.4].

Lemma 3.6. *Let $\delta > D$. Then there exists a constant $C > 0$ such that*

a) *for all $g \in L^2_{loc}(X)$, $r \geq 1/R > 0$, $y \in X$, and $y', z \in B(y, r/4)$*

$$\|\mathbb{1}_{B(y',r)^c} g\|_{L^2(X,(1+Rd(\cdot,y))^{-\delta} d\mu)} \leq C\, |B(y',r)|^{1/2}\, (rR)^{-\delta/2} (\mathcal{M}_2 g)(z)\,;$$

b) *for all $g \in L^2_{loc}(X)$, $r, R > 0$, $y \in X$, and $z \in B(y, r/4)$*

$$\|\mathbb{1}_{B(y,r)^c} g\|_{L^2(X,(1+Rd(\cdot,y))^{-\delta} d\mu)} \leq C\, |B(y,1/R)|^{1/2}\, (1+rR)^{-(\delta-D)/2} (\mathcal{M}_2 g)(z)\,.$$

Here, $(\mathcal{M}_2 g)(z)$ is defined by

$$(\mathcal{M}_2 g)(z) := \sup_{\rho>0} \left(\frac{1}{|B(z,\rho)|} \int_{B(z,\rho)} |g|^2 \, d\mu \right)^{1/2}.$$

Proof. Let $\delta > D$, $g \in L^2_{loc}(X)$, $r > 0$, $y \in X$, and $y', z \in B(y, r/4)$ be arbitrary.

First, assume that $r \geq 1/R > 0$. We split $B(y', r)^c$ into annuli and obtain with the help of Fact 2.1

$$\begin{aligned}
\|\mathbb{1}_{B(y',r)^c} g\|^2_{L^2(X,(1+Rd(\cdot,y))^{-\delta} d\mu)} &\leq \sum_{k=0}^{\infty} \int_{2^k r \leq d(x,y') < 2^{k+1} r} |g(x)|^2 (Rd(x,y))^{-\delta} \, d\mu(x) \\
&\lesssim \sum_{k=0}^{\infty} (2^k rR)^{-\delta} \frac{2^{kD} |B(z,r)|}{|B(z,2^{k+2}r)|} \int_{2^{k-1} r \leq d(x,z) < 2^{k+2} r} |g(x)|^2 \, d\mu(x) \\
&\lesssim |B(y',r)|\, (rR)^{-\delta} \sum_{k=0}^{\infty} 2^{-(\delta-D)k} (\mathcal{M}_2 g)(z)^2 \lesssim |B(y',r)|\, (rR)^{-\delta} (\mathcal{M}_2 g)(z)^2\,.
\end{aligned}$$

This shows assertion a). Due to the doubling property, one can bound $|B(y', r)|$ by a constant times $(rR)^D |B(y', 1/R)|$, so that part b) is proven in the case $r \geq 1/R$.

Suppose now that $r < 1/R$. We integrate over the full space X and decompose X into $B(z, 1/R)^c \cup B(z, 1/R)$. The integral over $B(z, 1/R)^c$ can be estimated with the result proved above (take $r = 1/R$), whereas for the integral over $B(z, 1/R)$ one estimates the weight factor $(1 + Rd(\cdot, y))^{-\delta}$ by 1 and uses Fact 2.1

$$\begin{aligned}
&\|\mathbb{1}_{B(y,r)^c} g\|^2_{L^2(X,(1+Rd(\cdot,y))^{-\delta} d\mu)} \\
&\quad\lesssim \|\mathbb{1}_{B(z,1/R)^c} g\|^2_{L^2(X,(1+Rd(\cdot,y))^{-\delta} d\mu)} + \frac{|B(y,1/R)|}{|B(z,1/R)|} \|\mathbb{1}_{B(z,1/R)} g\|^2_{L^2(X)} \\
&\quad\lesssim |B(y,1/R)|\, (\mathcal{M}_2 g)(z)^2\,.
\end{aligned}$$

□

Now we are ready to prove the main result of this chapter.

Proof of Theorem 3.1. Consider a non-negative, self-adjoint operator L on $L^2(X)$ satisfying generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$) for some $p_0 \in [1,2)$ and $m \geq 2$. Additionally, assume that L fulfills the Plancherel condition (3.1) for some $q \in [2,\infty]$. Fix $s > \max\{D/2, 1/q\}$ and $\delta \in \mathbb{R}$ such that $2s > \delta > D$. Let $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function with $\sup_{l\in\mathbb{Z}} \|\omega F(2^l \cdot)\|_{H^s_q} < \infty$.

In view of $\operatorname{supp}\omega \subseteq (1/4,1)$, the condition $\sup_{l\in\mathbb{Z}} \|\omega F(2^l \cdot)\|_{H^s_q} < \infty$ holds if and only if the function $G\colon [0,\infty) \to \mathbb{C}$, $\lambda \mapsto F(\sqrt[m]{\lambda})$ satisfies the property $\sup_{l\in\mathbb{Z}} \|\omega G(2^l \cdot)\|_{H^s_q} < \infty$. For this reason, we shall consider in the proof $F(\sqrt[m]{L})$ rather than $F(L)$.

We will establish that $F(\sqrt[m]{L})$ is of weak type (p_0,p_0). Since $F(\sqrt[m]{L})$ is bounded on $L^2(X)$, boundedness of $F(\sqrt[m]{L})$ on $L^p(X)$ for all $p \in (p_0, 2]$ then follows from the Marcinkiewicz interpolation theorem. Hence, a straightforward dualization argument gives boundedness of $F(\sqrt[m]{L})$ on $L^p(X)$ for any $p \in (2, p_0')$, too.

We write $F(\lambda) = F(\lambda) - F(0) + F(0)$ and consequently

$$F(\sqrt[m]{L}) = (F - F(0))(\sqrt[m]{L}) + F(0)I\,.$$

Therefore, by replacing F by $F - F(0)$, we may assume in the sequel that $F(0) = 0$. By recalling the property (2.9) of ω, we then have for all $\lambda \geq 0$

$$F(\lambda) = \sum_{l\in\mathbb{Z}} \omega(2^{-l}\lambda) F(\lambda)\,. \tag{3.10}$$

Our main tool for the proof of the weak type (p_0,p_0) boundedness of $F(\sqrt[m]{L})$ is the following criterion [BK03, Theorem 1.1] due to S. Blunck and P.C. Kunstmann that generalizes Hörmander's weak type $(1,1)$ condition for integral operators ([Hör60], see also [DM99]).

Fact 3.7. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ for which ($GGE_{p_0,m}$) holds for some $p_0 \in [1,2)$ and $m \geq 2$. Suppose that T is a bounded linear operator on $L^2(X)$ such that there exist $C_T > 0$ and $n \in \mathbb{N}$ with*

$$N_{p_0', r_t/2}\big([TD^n e^{-tL}]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y) \leq C_T\,(\mathcal{M}_2 g)(z)$$

for all $t > 0$, $g \in L^{p_0'}(X)$, $z \in X$, and $y \in B(z, r_t/2)$. Then T is of weak type (p_0,p_0). More precisely, there exists a constant $C > 0$ such that

$$\|T\|_{L^{p_0}(X)\to L^{p_0,\infty}(X)} \leq C\,\big(\|T\|_{2\to 2} + C_T\big)\,.$$

Here, we used the notations

$$r_t := t^{1/m}\,, \qquad N_{q,\rho} g(y) := |B(y,\rho)|^{-1/q} \|g\|_{L^q(B(y,\rho))}\,,$$

$$\mathcal{M}_q g(y) := \sup_{\rho>0} N_{q,\rho} g(y)\,, \qquad D^n h(t) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k h(kt)\,.$$

In order to apply this criterion for the operator $T = F(\sqrt[m]{L})$, we have to check that there are constants $C > 0$ and $n \in \mathbb{N}$ with

$$N_{p_0',r_t/2}\big([F(\sqrt[m]{L})D^n e^{-tL}]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y) \le C \sup_{l\in\mathbb{Z}} \|\omega F(2^l\cdot)\|_{H^s_q} (\mathcal{M}_2 g)(z)$$

for all $t > 0$, $g \in L^{p_0'}(X)$, $z \in X$, and $y \in B(z, r_t/2)$.

To this end, let $t > 0$, $g \in L^{p_0'}(X)$, $z \in X$, and $y \in B(z, r_t/2)$ be arbitrary. Fix $n \in \mathbb{N}$ with $n > s$ and define the function $E_t\colon [0,\infty) \to \mathbb{R}$ via $E_t(\lambda) := E(t\lambda) := (1 - e^{-(t\lambda)^m})^n$, $\lambda \ge 0$. Then we have $D^n e^{-tL} = E_{r_t}(\sqrt[m]{L})$ and, thanks to (3.10),

$$\begin{aligned} N_{p_0',r_t/2}\big([F(\sqrt[m]{L})D^n e^{-tL}]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y) &= N_{p_0',r_t/2}\big([(FE_{r_t})(\sqrt[m]{L})]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y) \\ &\le \sum_{l\in\mathbb{Z}} N_{p_0',r_t/2}\big([(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y)\,. \end{aligned} \tag{3.11}$$

Now we analyze each summand separately. Fix $l \in \mathbb{Z}$ and $f \in L^{p_0}(X)$ such that $\operatorname{supp} f \subseteq B(y, r_t/2)$ and $\|f\|_{p_0} = 1$.

In the case $r_t/2 \le 2^l$ we can apply Lemma 3.4 b) (with $\varepsilon := s - \delta/2 > 0$) directly and estimate as follows

$$\begin{aligned} &\big|\big\langle f, [(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})]^* \mathbb{1}_{B(y,4r_t)^c} g\big\rangle\big| \\ &\quad = \big|\big\langle (\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})f, \mathbb{1}_{B(y,4r_t)^c} g\big\rangle\big| \\ &\quad \le \big\|(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})f\big\|_{L^2(X,(1+2^{-l}d(\cdot,y))^\delta d\mu)} \|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y))^{-\delta} d\mu)} \\ &\quad \le \big\|(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})\mathbb{1}_{B(y,2^l)}\big\|_{L^{p_0}\to L^2(X,(1+2^{-l}d(\cdot,y))^\delta d\mu)} \|f\|_{p_0} \times \\ &\qquad \times \|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y))^{-\delta} d\mu)} \\ &\quad \lesssim |B(y,2^l)|^{-(\frac{1}{p_0}-\frac{1}{2})} \big\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\big\|_{H_q^{\delta/2+\varepsilon}} \|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y))^{-\delta} d\mu)}\,. \end{aligned}$$

Due to Lemma 3.6 b), the factor $\|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y))^{-\delta} d\mu)}$ is bounded by a constant times

$$|B(y,2^l)|^{1/2}(1 + 4r_t 2^{-l})^{-(\delta-D)/2}(\mathcal{M}_2 g)(z) \le |B(y,2^l)|^{1/2}(\mathcal{M}_2 g)(z)\,.$$

This, together with the doubling property, leads to

$$\begin{aligned} &\big|\big\langle f, [(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})]^* \mathbb{1}_{B(y,4r_t)^c} g\big\rangle\big| \\ &\quad \lesssim \left(\frac{2^l}{r_t}\right)^{\frac{D}{p_0'}} |B(y,r_t)|^{\frac{1}{p_0'}} \big\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\big\|_{H^s_q} (\mathcal{M}_2 g)(z)\,. \end{aligned}$$

Taking the supremum over all $f \in L^{p_0}(X)$ with $\operatorname{supp} f \subseteq B(y, r_t/2)$ and $\|f\|_{p_0} = 1$ yields

$$\begin{aligned} &N_{p_0',r_t/2}\big([(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y) \\ &\quad \lesssim \left(\frac{2^l}{r_t}\right)^{\frac{D}{p_0'}} \big\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\big\|_{H^s_q} (\mathcal{M}_2 g)(z)\,. \end{aligned} \tag{3.12}$$

Assume now $2^l < r_t/2$. In this case we cover $B(y, r_t/2)$ by balls of radius 2^l. Thanks to Lemma 2.2, one can construct $y_1, \ldots, y_K \in B(y, r_t/2)$ with $B(y, r_t/2) \subseteq \bigcup_{\nu=1}^K B(y_\nu, 2^l)$, $K \lesssim (r_t/2^{l+1})^D$, and each point of $B(y, r_t/2)$ is contained in at most M balls $B(y_\nu, 2^l)$, where M is independent of l and t. This yields

$$\begin{aligned}
&\left|\left\langle f, [(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})]^* \mathbb{1}_{B(y,4r_t)^c} g\right\rangle\right| \\
&\quad \le \sum_{\nu=1}^K \left|\left\langle(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})(f\mathbb{1}_{B(y_\nu,2^l)}), \mathbb{1}_{B(y,4r_t)^c} g\right\rangle\right| \\
&\quad \le \sum_{\nu=1}^K \left\|(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})(f\mathbb{1}_{B(y_\nu,2^l)})\right\|_{L^2(X,(1+2^{-l}d(\cdot,y_\nu))^\delta d\mu)} \times \\
&\qquad\quad \times \ \|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y_\nu))^{-\delta} d\mu)} \\
&\quad \le \sum_{\nu=1}^K \left\|(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})\,\mathbb{1}_{B(y_\nu,2^l)}\right\|_{L^{p_0}(X)\to L^2(X,(1+2^{-l}d(\cdot,y_\nu))^\delta d\mu)} \|f\mathbb{1}_{B(y_\nu,2^l)}\|_{p_0} \times \\
&\qquad\quad \times \ \|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y_\nu))^{-\delta} d\mu)} \\
&\quad \lesssim \sum_{\nu=1}^K |B(y_\nu,2^l)|^{-(\frac{1}{p_0}-\frac{1}{2})} \left\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\right\|_{H^s_q} \|f\mathbb{1}_{B(y_\nu,2^l)}\|_{p_0} \times \\
&\qquad\quad \times \ \|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y_\nu))^{-\delta} d\mu)}\,,
\end{aligned} \tag{3.13}$$

where the last step is due to Lemma 3.4 b) (with $\varepsilon := s - \delta/2 > 0$). Further, by Fact 2.1 and the doubling property, we have for every $\nu \in \{1, \ldots, K\}$

$$|B(y, r_t/2)| \cong |B(y_\nu, r_t/2)| \lesssim (r_t/2^{l+1})^D |B(y_\nu, 2^l)|$$

which gives

$$|B(y_\nu, 2^l)|^{-(\frac{1}{p_0}-\frac{1}{2})} \lesssim (r_t/2^{l+1})^{D(\frac{1}{p_0}-\frac{1}{2})} |B(y, r_t/2)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,. \tag{3.14}$$

From Lemma 3.6 a) we deduce that

$$\|\mathbb{1}_{B(y,4r_t)^c} g\|_{L^2(X,(1+2^{-l}d(\cdot,y_\nu))^{-\delta} d\mu)} \lesssim |B(y, r_t/2)|^{1/2} \left(\frac{r_t}{2^l}\right)^{-\delta/2} (\mathcal{M}_2 g)(z)\,. \tag{3.15}$$

It remains to estimate the sum $\sum_{\nu=1}^K \|f\mathbb{1}_{B(y_\nu,2^l)}\|_{p_0}$. Due to the properties of the covering and the function f, this is bounded by a constant times $(r_t/2^{l+1})^{D(1-1/p_0)}$. Indeed, Jensen's inequality in the version for concave functions (applied to $h(\lambda) := \lambda^{1/p_0}$, $\lambda \ge 0$) gives

$$\left(\sum_{\nu=1}^K \frac{1}{K} \int_{B(y_\nu,2^l)} |f|^{p_0}\, d\mu\right)^{1/p_0} \ge \sum_{\nu=1}^K \frac{1}{K} \|f\mathbb{1}_{B(y_\nu,2^l)}\|_{p_0}\,.$$

According to Lemma 2.2 iv), the left-hand side can be estimated by

$$\frac{1}{K^{1/p_0}}\left(M\int_{B(y,r_t/2)}|f|^{p_0}\,d\mu\right)^{1/p_0}=\left(\frac{M}{K}\right)^{1/p_0}$$

which leads to

$$\sum_{\nu=1}^{K}\|f\mathbb{1}_{B(y_\nu,2^l)}\|_{p_0}\le M^{1/p_0}K^{1-1/p_0}\,.$$

Since $K\lesssim(r_t/2^{l+1})^D$, the claimed estimate is shown. This, together with (3.13), (3.14), (3.15), gives in the case $r_t/2>2^l$

$$\begin{aligned}
&|B(y,r_t/2)|^{-\frac{1}{p_0'}}\left|\left\langle f,[(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})]^*\,\mathbb{1}_{B(y,4r_t)^c}g\right\rangle\right|\\
&\lesssim|B(y,r_t/2)|^{-\frac{1}{p_0'}}\sum_{\nu=1}^{K}(r_t/2^{l+1})^{D(\frac{1}{p_0}-\frac12)}|B(y,r_t/2)|^{-(\frac{1}{p_0}-\frac12)}\left\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\right\|_{H^s_q}\times\\
&\qquad\times\ \|f\mathbb{1}_{B(y_\nu,2^l)}\|_{p_0}\|\mathbb{1}_{B(y,4r_t)^c}g\|_{L^2(X,(1+2^{-l}d(\cdot,y_\nu))^{-\delta}d\mu)}\\
&\lesssim(r_t/2^{l+1})^{D(\frac{1}{p_0}-\frac12)}\left\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\right\|_{H^s_q}\times\\
&\qquad\times\ \sum_{\nu=1}^{K}\|f\mathbb{1}_{B(y_\nu,2^l)}\|_{p_0}|B(y,r_t/2)|^{-1/2}\|\mathbb{1}_{B(y,4r_t)^c}g\|_{L^2(X,(1+2^{-l}d(\cdot,y_\nu))^{-\delta}d\mu)}\\
&\lesssim(r_t/2^{l+1})^{D(\frac{1}{p_0}-\frac12)}\left\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\right\|_{H^s_q}(r_t/2^{l+1})^{D(1-\frac{1}{p_0})}\left(\frac{r_t}{2^l}\right)^{-\delta/2}(\mathcal{M}_2g)(z)\\
&\cong\left(\frac{r_t}{2^l}\right)^{-(\delta-D)/2}\left\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\right\|_{H^s_q}(\mathcal{M}_2g)(z)\,.
\end{aligned}$$

Hence, we arrive at

$$\begin{aligned}
&N_{p_0',r_t/2}\big([(\omega(2^l\cdot)FE_{r_t})(\sqrt[m]{L})]^*\,\mathbb{1}_{B(y,4r_t)^c}g\big)(y)\\
&\qquad\lesssim\left(\frac{r_t}{2^l}\right)^{-(\delta-D)/2}\left\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\right\|_{H^s_q}(\mathcal{M}_2g)(z)\,.
\end{aligned}\tag{3.16}$$

According to Fact 2.16, one gets

$$\left\|(\omega(2^l\cdot)FE_{r_t})(2^{-l}\cdot)\right\|_{H^s_q}\lesssim\left\|(\omega(2^l\cdot)F)(2^{-l}\cdot)\right\|_{H^s_q}\|E_{r_t}(2^{-l}\cdot)\|_{C^n([1/4,1])}\,.$$

Further, it follows from [Blu03, Lemma 3.5] for each $l\in\mathbb{Z}$ and $t>0$

$$\|E_{r_t}(2^{-l}\cdot)\|_{C^n([1/4,1])}\lesssim\min\big\{1,(2^{-l}r_t)^n\big\}\,.$$

By inserting (3.12) and (3.16) into (3.11) and by applying the two foregoing estimates, we obtain

$$\begin{aligned}
&N_{p_0',r_t/2}\big([F(\sqrt[m]{L})D^n e^{-tL}]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y)\\
&\lesssim \sum_{l\in\mathbb{Z}:\, 2^{l+1}<r_t} \left(\frac{r_t}{2^l}\right)^{-(\delta-D)/2} \big\|(\omega(2^l\cdot)F)(2^{-l}\cdot)\big\|_{H^s_q} \min\big\{1,(2^{-l}r_t)^n\big\}\,(\mathcal{M}_2 g)(z)\\
&\quad + \sum_{l\in\mathbb{Z}:\, 2^{l+1}\ge r_t} \left(\frac{2^l}{r_t}\right)^{\frac{D}{p_0'}} \big\|(\omega(2^l\cdot)F)(2^{-l}\cdot)\big\|_{H^s_q} \min\big\{1,(2^{-l}r_t)^n\big\}\,(\mathcal{M}_2 g)(z)\\
&\le \sup_{l\in\mathbb{Z}} \|\omega F(2^l\cdot)\|_{H^s_q}\,(\mathcal{M}_2 g)(z)\left(\sum_{l\in\mathbb{Z}:\, 2^{l+1}<r_t} \big(2^{-l}r_t\big)^{-(\delta-D)/2} + \sum_{l\in\mathbb{Z}:\, 2^{l+1}\ge r_t} \big(2^{-l}r_t\big)^{n-\frac{D}{p_0'}}\right)\\
&\lesssim \sup_{l\in\mathbb{Z}} \|\omega F(2^l\cdot)\|_{H^s_q}\,(\mathcal{M}_2 g)(z)\,.
\end{aligned}$$

In the last inequality we used that both series are convergent with upper bounds independent of t. To see this, write $2^{l_0}\le r_t<2^{l_0+1}$ for some $l_0\in\mathbb{Z}$, then

$$\begin{aligned}
&\sum_{l\in\mathbb{Z}:\, 2^{l+1}<r_t} \big(2^{-l}r_t\big)^{-(\delta-D)/2} + \sum_{l\in\mathbb{Z}:\, 2^{l+1}\ge r_t} \big(2^{-l}r_t\big)^{n-\frac{D}{p_0'}}\\
&\qquad\le \sum_{l=-\infty}^{l_0} 2^{-(l_0-l)(\delta-D)/2} + \sum_{l=l_0-1}^{\infty} 2^{(-l+l_0+1)(n-\frac{D}{p_0'})}\\
&\qquad\le \sum_{j=-\infty}^{0} 2^{j(\delta-D)/2} + \sum_{j=-1}^{\infty} 2^{(-j+1)(n-\frac{D}{p_0'})} < \infty
\end{aligned}$$

because $\delta-D>0$ and $n-\frac{D}{p_0'}>0$. □

Observe that, if one employs Lemma 3.4 a) instead of Lemma 3.4 b) in the above proof, the modified version of the proof then gives the weak type (p_0,p_0) boundedness of $F(L)$ whenever $F\colon[0,\infty)\to\mathbb{C}$ is a bounded Borel function with $\sup_{n\in\mathbb{Z}}\|\omega F(2^n\cdot)\|_{H^s_2}<\infty$ for some $s>(D+1)/2$ and $(\mathrm{GGE}_{p_0,m})$ holds for L. Since in this case the Plancherel condition (3.1) is not required, we have also worked out a proof of [Blu03, Theorem 1.1].

3.2 Variation for operators with non-empty point spectrum

In the last section we developed a spectral multiplier theorem for non-negative, self-adjoint operators L that fulfill generalized Gaussian estimates $(\mathrm{GGE}_{p_0,m})$ for some $p_0\in[1,2)$, $m\ge 2$ and the Plancherel condition (3.1) for some $q\in[2,\infty)$. Unfortunately, this result cannot be applied to operators whose point spectrum is non-empty because the validity of the Plancherel condition (3.1) for some $q\in[2,\infty)$ entails the emptiness of the point spectrum. Indeed, according to the Plancherel condition (3.1), one has for all $0 \leq a \leq R$ and $y \in X$

$$\left\|\mathbb{1}_{\{a\}}(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/R)}\right\|_{p_0\to 2} \lesssim |B(y,1/R)|^{-(\frac{1}{p_0}-\frac{1}{2})}\|\mathbb{1}_{\{a\}}(R\cdot)\|_q = 0$$

and therefore $\mathbb{1}_{\{a\}}(\sqrt[m]{L}) = 0$. Due to $\sigma(L) \subseteq [0,\infty)$, it follows that the point spectrum of L is empty. In order to treat operators with non-empty point spectrum as well, one may introduce some variation of the Plancherel condition (3.1). This approach originates in [CS01] and was also used in [DOS02]. For $N \in \mathbb{N}$, $q \in [1,\infty)$, and a bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [-1,2]$ define the norm $\|F\|_{N,q}$ via the formula

$$\|F\|_{N,q} := \left(\frac{1}{N}\sum_{k=1-N}^{2N} \sup_{\lambda\in[\frac{k-1}{N},\frac{k}{N})} |F(\lambda)|^q\right)^{1/q}.$$

It is easy to see that, for fixed F and N, $\|F\|_{N,q}$ increases in q.

With the help of this norm, we formulate a generalization of [DOS02, Theorem 3.2] that also applies to operators for which generalized Gaussian estimates are valid.

Theorem 3.8. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying generalized Gaussian estimates ($GGE_{p_0,m}$) for some $p_0 \in [1,2)$ and $m \geq 2$. Further, let $\kappa \in \mathbb{N}$ and $q \in [2,\infty)$. Suppose that there exists a constant $C > 0$ such that for any $N \in \mathbb{N}$, $y \in X$, and any bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with* $\operatorname{supp} F \subseteq [-1, N+1]$

$$\left\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/N)}\right\|_{p_0\to 2} \leq C\,|B(y,1/N)|^{-(\frac{1}{p_0}-\frac{1}{2})}\|F(N\cdot)\|_{N^\kappa,q}\,. \tag{3.17}$$

Additionally, assume that for every $\varepsilon > 0$ there exists a constant $C > 0$ such that for all $N \in \mathbb{N}$ and all bounded Borel functions $F\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [-1, N+1]$

$$\left\|F(\sqrt[m]{L})\right\|_{p_0\to p_0}^2 \leq CN^{\kappa D+\varepsilon}\|F(N\cdot)\|_{N^\kappa,q}^2\,. \tag{3.18}$$

Take $s > \max\{D/2, 1/q\}$. Then, for any bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with

$$\sup_{n\in\mathbb{N}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty\,, \tag{3.19}$$

the operator $F(L)$ is of weak type (p_0,p_0) and there exists a constant $C > 0$ such that

$$\|F(L)\|_{L^{p_0}(X)\to L^{p_0,\infty}(X)} \leq C\Big(\sup_{n\in\mathbb{N}} \|\omega F(2^n\cdot)\|_{H^s_q} + \|F\|_\infty\Big)\,. \tag{3.20}$$

In particular, $F(L)$ acts as a bounded linear operator on $L^p(X)$ for each $p \in (p_0,p_0')$.

By using the same reasoning as in the remark after Theorem 3.1, this statement can be extended to open subsets of X.

Inspired by [DOS02], we call the hypothesis (3.17) *Plancherel condition.* Note that (3.17) is weaker than (3.1) and so the secondary assumption (3.18) is needed in Theorem 3.8.

We prepare the proof of Theorem 3.8 with the next lemma which is of similar type as Lemma 3.4 b) and translates the derivation order in some sense to the weight in a weighted norm estimate. Our statement is based on [DOS02, Lemma 4.3 b)].

Lemma 3.9. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying ($GGE_{p_0,m}$) for some $p_0 \in [1,2)$ and $m \geq 2$. Fix $\kappa \in \mathbb{N}$. Suppose that L enjoys the Plancherel condition (3.17) for some $q \in [2,\infty)$. For $\xi \in C_c^\infty([-1,1])$ and $N \in \mathbb{N}$ define the function ξ_N via the formula $\xi_N(\lambda) := N\,\xi(N\lambda)$. Then for any $s \geq 2/q$, $\varepsilon > 0$, and any $\xi \in C_c^\infty([-1,1])$ there exists a constant $C > 0$ such that*

$$\begin{aligned}\big\|(F * \xi_{N^{\kappa-1}})(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/N)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Nd(\cdot,y))^s d\mu)} \\ \leq C\,|B(y,1/N)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,\|F(N\cdot)\|_{H_q^{s/2+\varepsilon}}\end{aligned} \tag{3.21}$$

for every $N \in \mathbb{N}$ with $N > 8$, every $y \in X$, and every bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [N/4, N]$ *and $F(N\cdot) \in H_q^{s/2+\varepsilon}$.*

Proof. The main idea of the proof resembles that of Lemma 3.4 b).

By a straightforward scaling argument, we see that it suffices to verify

$$\begin{aligned}\big\|[\xi_{N^{\kappa-1}} * (H(1/N\cdot))](\sqrt[m]{L})\,\mathbb{1}_{B(y,1/N)}\big\|_{L^{p_0}(X)\to L^2(X,(1+Nd(\cdot,y))^s d\mu)} \\ \lesssim |B(y,1/N)|^{-(\frac{1}{p_0}-\frac{1}{2})}\,\|H\|_{H_q^{s/2+\varepsilon}}\end{aligned} \tag{3.22}$$

for all $s \geq 2/q$, $y \in X$, $\varepsilon > 0$, $\xi \in C_c^\infty([-1,1])$, and all bounded Borel functions $H\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} H \subseteq [1/4,1]$ and $H \in H_q^{s/2+\varepsilon}$, where the implicit constant is independent of H, N, and y.

Let $N \in \mathbb{N}$ with $N > 8$ and $\xi \in C_c^\infty([-1,1])$ be fixed. In view of $\operatorname{supp} \xi_N \subseteq [-1/N, 1/N]$, we deduce for every bounded Borel function $H\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} H \subseteq [1/4,1]$ that $\operatorname{supp}(\xi_N * H) \subseteq [1/8, 9/8]$ and thus, by applying Hölder's inequality,

$$\begin{aligned}|(\xi_N * H)(\lambda)|^q &\leq \left(\int_{-1/N}^{1/N} |\xi_N(\tau)\,H(\lambda-\tau)|\,d\tau\right)^q \leq \|\xi_N\|_{q'}^q \int_{-1/N}^{1/N} |H(\lambda-\tau)|^q\,d\tau \\ &= \|\xi_N\|_{q'}^q \int_{\lambda-1/N}^{\lambda+1/N} |H(\tau)|^q\,d\tau\end{aligned}$$

which finally leads to the bound

$$\begin{aligned}
\|\xi_N * H\|_{N,q} &= \left(\frac{1}{N} \sum_{k=1-N}^{2N} \sup_{\lambda \in [\frac{k-1}{N}, \frac{k}{N})} |(\xi_N * H)(\lambda)|^q \right)^{1/q} \\
&\leq N^{-1/q} \|\xi_N\|_{q'} \left(\sum_{k=1-N}^{2N} \sup_{\lambda \in [\frac{k-1}{N}, \frac{k}{N})} \int_{\lambda - 1/N}^{\lambda + 1/N} |H(\tau)|^q \, d\tau \right)^{1/q} \\
&\leq N^{-1/q} N^{1-1/q'} \|\xi\|_{q'} \left(\sum_{k=1}^{N+1} \int_{(k-2)/N}^{(k+1)/N} |H(\tau)|^q \, d\tau \right)^{1/q} \\
&\leq \|\xi\|_{q'} \cdot 3 \, \|H\|_q \lesssim \|H\|_q \, .
\end{aligned}$$

This, together with (3.17), yields for any $y \in X$ and any bounded Borel function $H \colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} H \subseteq [1/4, 1]$

$$\begin{aligned}
&\big\| [\xi_{N^{\kappa-1}} * (H(1/N \cdot))](\sqrt[m]{L}) \, \mathbb{1}_{B(y,1/N)} \big\|_{p_0 \to 2} = \big\| [(\xi_{N^\kappa} * H)(1/N \cdot)](\sqrt[m]{L}) \, \mathbb{1}_{B(y,1/N)} \big\|_{p_0 \to 2} \\
&\quad \lesssim |B(y,1/N)|^{-(\frac{1}{p_0} - \frac{1}{2})} \|\xi_{N^\kappa} * H\|_{N^\kappa, q} \lesssim |B(y,1/N)|^{-(\frac{1}{p_0} - \frac{1}{2})} \|H\|_q \, . \qquad (3.23)
\end{aligned}$$

By inserting $F = \xi_{N^{\kappa-1}} * (H(1/N \cdot))$ in (3.6), we get for any $\varepsilon > 0$, $s \geq 0$, and any $y \in X$

$$\begin{aligned}
&\big\| [\xi_{N^{\kappa-1}} * (H(1/N \cdot))](\sqrt[m]{L}) \, \mathbb{1}_{B(y,1/N)} \big\|_{L^{p_0}(X) \to L^2(X,(1+Nd(\cdot,y))^s d\mu)} \\
&\quad \lesssim |B(y,1/N)|^{-(\frac{1}{p_0} - \frac{1}{2})} \|G\|_{H_2^{(s+1+\varepsilon)/2}} \, , \qquad (3.24)
\end{aligned}$$

where $G(\lambda) := (\xi_{N^{\kappa-1}} * H)(N \sqrt[m]{\lambda}) \, e^\lambda = (\xi_{N^\kappa} * H)(\sqrt[m]{\lambda}) \, e^\lambda$, $\lambda \geq 0$. Since $\xi_{N^\kappa} * H$ has support contained in $[1/8, 9/8]$, it follows for each $\varepsilon > 0$ and $s \geq 0$

$$\|G\|_{H_2^{(s+1+\varepsilon)/2}} \lesssim \|G\|_{H_q^{(s+1+\varepsilon)/2}} \lesssim \|\xi_{N^\kappa} * H\|_{H_q^{(s+1+\varepsilon)/2}} \lesssim \|H\|_{H_q^{(s+1+\varepsilon)/2}} \, . \qquad (3.25)$$

In dependence of $y \in X$ and N, let $K_{y,N}$ be defined as in the proof of Lemma 3.4 b) (case $q < \infty$), i.e. $K_{y,N} \colon L^\infty([1/4,1]) \to L^2(X)$, $K_{y,N}(H) := H(N^{-1} \sqrt[m]{L})\varphi$ for a fixed $\varphi \in L^{p_0}(X)$ with $\operatorname{supp} \varphi \subseteq B(y,1/N)$ and $\|\varphi\|_{p_0} = 1$. Introduce the operator

$$\begin{aligned}
\widetilde{K}_{y,N} \colon L^\infty([1/4,1]) \to L^2(X), \quad H \mapsto K_{y,N}(\xi_{N^\kappa} * H) &= (\xi_{N^\kappa} * H)(N^{-1} \sqrt[m]{L})\varphi \\
&= [\xi_{N^{\kappa-1}} * (H(1/N \cdot))](\sqrt[m]{L})\varphi \, .
\end{aligned}$$

Then it holds by (3.23)

$$\big\| \widetilde{K}_{y,N}(H) \big\|_2 \lesssim |B(y,1/N)|^{-(\frac{1}{p_0} - \frac{1}{2})} \|H\|_{L^q([1/4,1])}$$

and by (3.24), (3.25)

$$\big\| \widetilde{K}_{y,N}(H) \big\|_{L^2(X,(1+Nd(\cdot,y))^s d\mu)} \lesssim |B(y,1/N)|^{-(\frac{1}{p_0} - \frac{1}{2})} \|H\|_{H_q^{(s+1+\varepsilon)/2}([1/4,1])} \, .$$

In virtue of these bounds, the same interpolation argument as in the proof of Lemma 3.4 b) is possible and finally gives

$$\big\|\widetilde{K}_{y,N}(H)\big\|_{L^2(X,(1+Nd(\cdot,y))^{s'}d\mu)} \lesssim |B(y,1/N)|^{-(\frac{1}{p_0}-\frac{1}{2})}\|H\|_{H_q^{s'/2+\varepsilon'}([1/4,1])}$$

for all $s' \geq 2/q$ and $\varepsilon' > 0$. This proves (3.22) and thus (3.21). $\square$

A careful inspection of the above proof shows that a corresponding version of Lemma 3.9 is also valid in the case $p_0 = 2$. For later reference we record this observation as a corollary.

Corollary 3.10. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \geq 2$. Fix $\kappa \in \mathbb{N}$. Suppose that L enjoys the Plancherel condition (3.17) with $p_0 = 2$ for some $q \in [2,\infty)$. For $\xi \in C_c^\infty([-1,1])$ and $N \in \mathbb{N}$ define the function ξ_N via the formula $\xi_N(\lambda) := N\,\xi(N\lambda)$. Then for any $s \geq 2/q$, $\varepsilon > 0$, and any $\xi \in C_c^\infty([-1,1])$ there exists a constant $C > 0$ such that*

$$\big\|(F * \xi_{N^{\kappa-1}})(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/N)}\big\|_{L^2(X)\to L^2(X,(1+Nd(\cdot,y))^s d\mu)} \leq C\,\|F(N\cdot)\|_{H_q^{s/2+\varepsilon}} \tag{3.26}$$

for all $N \in \mathbb{N}$ with $N > 8$, all $y \in X$, and all bounded Borel functions $F\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [N/4, N]$ *and $F(N\cdot) \in H_q^{s/2+\varepsilon}$.*

Proof of Theorem 3.8. As in the proof of Theorem 3.1, it suffices to show the statement for $F(\sqrt[m]{L})$ if F is a bounded Borel function such that the Hörmander condition (3.19) holds.

Let $F\colon \mathbb{R} \to \mathbb{C}$ be a bounded Borel function with $\operatorname{supp} F \subseteq [0,2]$. Thanks to (3.18) with $\varepsilon = N = 1$, one can estimate

$$\big\|F(\sqrt[m]{L})\big\|_{p_0\to p_0} \lesssim \|F\|_{1,q} = \left(\sum_{k=0}^{2} \sup_{\lambda\in[k-1,k)} |F(\lambda)|^q\right)^{1/q} \lesssim \|F\|_\infty\,.$$

In particular, $F(\sqrt[m]{L})$ is of weak type (p_0,p_0) and the bound (3.20) is valid.

Therefore, it is enough to prove the statement for every bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ such that $\operatorname{supp} F \subseteq [1,\infty)$ and (3.19) hold. Due to the properties of ω, we can write

$$F = \sum_{l\in\mathbb{Z}} \omega(2^{-l}\cdot)F = \sum_{l=1}^{\infty} \omega_l F\,,$$

where $\omega_l := \omega(2^{-l}\cdot)$. Define the function

$$\widetilde{F} := \sum_{l=1}^{\infty} (\omega_l F) * \xi_{2^{l(\kappa-1)}}$$

and decompose

$$F(\sqrt[m]{L}) = \widetilde{F}(\sqrt[m]{L}) + (F - \widetilde{F})(\sqrt[m]{L})\,.$$

At first, we note that $\widetilde{F}(\sqrt[m]{L})$ is of weak type (p_0,p_0) with the desired bound. Indeed, by repeating the proof of Theorem 3.1 and using (3.21) in place of (3.5), we see that

$$N_{p_0',r_t/2}\big([\widetilde{F}(\sqrt[m]{L})D^n e^{-tL}]^* \mathbb{1}_{B(y,4r_t)^c} g\big)(y) \lesssim \sup_{l\in\mathbb{N}} \|\omega F(2^l\cdot)\|_{H^s_q} (\mathcal{M}_2 g)(z)$$

for every $t>0$, $g\in L^{p_0'}(X)$, $z\in X$, $y\in B(z,r_t/2)$ and for some $n\in\mathbb{N}$. Hence, Fact 3.7 applies, so that the weak type (p_0,p_0) boundedness of $\widetilde{F}(\sqrt[m]{L})$ is proven.

Next, we treat the operator $(F-\widetilde{F})(\sqrt[m]{L})$ and claim that it is bounded on $L^{p_0}(X)$ with

$$\big\|(F-\widetilde{F})(\sqrt[m]{L})\big\|_{p_0\to p_0} \lesssim \sup_{l\in\mathbb{N}} \|\omega F(2^l\cdot)\|_{H^s_q}\,.$$

For each $l\in\mathbb{N}$ define the function $H_l := \omega_l F - (\omega_l F)*\xi_{2^{l(\kappa-1)}}$. By observing that $\operatorname{supp}\omega_l F \subseteq (2^{l-2},2^l)$ and $\operatorname{supp}(\omega_l F * \xi_{2^{l(\kappa-1)}}) \subseteq (2^{l-2}-2^{l(1-\kappa)}, 2^l+2^{l(1-\kappa)})$, we conclude that the support of H_l is contained in $(2^{l-2}-2^{l(1-\kappa)}, 2^l+2^{l(1-\kappa)}) \subseteq [-1, 2^l+1]$. Put $\varepsilon := s - D/2 > 0$. According to (3.18), it then holds

$$\big\|H_l(\sqrt[m]{L})\big\|^2_{p_0\to p_0} \lesssim 2^{l(\kappa D+\varepsilon)} \|H_l(2^l\cdot)\|^2_{2^{l\kappa},q}\,. \tag{3.27}$$

In order to estimate the term $\|H_l(2^l\cdot)\|^2_{2^{l\kappa},q}$, we apply the following result from [CS01, (3.29)] (see also [DOS02, Proposition 4.6]) whose proof is based on Fourier analysis.

Fact 3.11. *Suppose that $\xi\in C_c^\infty(\mathbb{R})$ is a function that fulfills $\operatorname{supp}\xi\subseteq[-1,1]$, $\xi\geq 0$, $\widehat{\xi}(0)=1$, and $\widehat{\xi}^{(k)}(0)=1$ for all $k\in\{1,2,\dots,\lfloor s\rfloor+2\}$, where $\lfloor s\rfloor$ denotes the largest integer less than or equal to s. Then one finds a constant $C>0$ such that*

$$\|G - G*\xi_N\|_{N,q} \leq C N^{-s} \|G\|_{H^s_q}$$

for all $N\in\mathbb{N}$, $q\in[2,\infty)$, $s>1/q$, and all $G\in H^s_q$ with $\operatorname{supp} G\subseteq[0,1]$.

By recalling the definition of H_l and by using the inequality (3.27) as well as Fact 3.11, we deduce that

$$\begin{aligned}\big\|H_l(\sqrt[m]{L})\big\|^2_{p_0\to p_0} &\lesssim 2^{l(\kappa D+\varepsilon)} \big\|\omega_l F(2^l\cdot) - ((\omega_l F)(2^l\cdot)) * \xi_{2^{l\kappa}}\big\|^2_{2^{l\kappa},q} \\ &\lesssim 2^{l(\kappa D+\varepsilon)} 2^{-2l\kappa s} \|(\omega_l F)(2^l\cdot)\|^2_{H^s_q}\end{aligned}$$

and finally end up with

$$\begin{aligned}\big\|(F-\widetilde{F})(\sqrt[m]{L})\big\|_{p_0\to p_0} &\leq \sum_{l=1}^\infty \big\|H_l(\sqrt[m]{L})\big\|_{p_0\to p_0} \lesssim \sum_{l=1}^\infty 2^{l((D/2-s)\kappa+\varepsilon/2)} \|(\omega_l F)(2^l\cdot)\|_{H^s_q} \\ &\lesssim \sup_{l\in\mathbb{N}} \|\omega F(2^l\cdot)\|_{H^s_q}\,,\end{aligned}$$

as required.

□

4 Hardy spaces

Hardy spaces have a long history. Their origin lies in the complex analysis of one variable. In 1915, G.H. Hardy ([Har15]) investigated properties of analytic functions F on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $p \in (0, \infty)$, he studied the means

$$\mu_{p,F}(r) := \left(\int_{-\pi}^{\pi} |F(re^{i\theta})|^p \, d\theta \right)^{1/p}$$

as functions of $r > 0$ and proved that they behave similarly to the maximum modulus $\mu_{\infty,F}(r) := \sup\{|F(re^{i\theta})| : \theta \in [-\pi, \pi)\}$. In 1923, F. Riesz ([Rie23]) introduced for fixed $p \in (0, \infty]$ the class of holomorphic functions $F\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|F\|_{H^p(\mathbb{D})} := \sup_{0 \le r < 1} \mu_{p,F}(r)$$

is finite. In honor of G.H. Hardy, F. Riesz denoted this class by the symbol $H^p(\mathbb{D})$ and since then these spaces have become known as Hardy spaces. F. Riesz showed, among other things, that for every function $F \in H^p(\mathbb{D})$ the boundary values $\lim_{r \to 1-} F(re^{i\theta})$ exist for almost all $\theta \in [-\pi, \pi)$.

Analytic functions in the upper half plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ were also considered. This leads to the following classical definition of Hardy spaces $H^p(\mathbb{C}_+)$ for $p \in (0, \infty)$. A function F is said to belong to $H^p(\mathbb{C}_+)$ if F is holomorphic in $\mathbb{C}_+$ and

$$\|F\|_{H^p(\mathbb{C}_+)} := \sup_{y>0} \left(\int_{-\infty}^{\infty} |F(x+iy)|^p \, dx \right)^{1/p} < \infty \, .$$

Their theory was developed by V.I. Krylov ([Kry39]). For example, he showed that the boundary values $\lim_{y \to 0+} F(x+iy)$ exist for almost all $x \in \mathbb{R}$.

In 1960, E.M. Stein and G. Weiss extended the definition of Hardy spaces to higher dimensions ([SW60], see also [Ste70, Chapter VII]). They considered vector-valued functions which satisfy certain generalized Cauchy-Riemann equations in $\mathbb{R}^{D+1}_+ := \mathbb{R}^D \times (0, \infty)$. Such a function is said to belong to $H^p(\mathbb{R}^D)$ provided that

$$\|F\|_{H^p(\mathbb{R}^D)} := \sup_{y>0} \left(\int_{\mathbb{R}^D} |F(x,y)|^p \, dx \right)^{1/p} < \infty \, .$$

It is known that for any $p \in (1, \infty)$ the Hardy space $H^p(\mathbb{R}^D)$ is naturally equivalent to the Lebesgue space $L^p(\mathbb{R}^D)$ (cf. e.g. [Ste70, p. 220]).

In 1972, C. Fefferman and E.M. Stein ([FS72]) provided many characterizations of Hardy spaces on $\mathbb{R}^D$, in particular by means of square or maximal functions associated to the Poisson semigroup. For suitable functions f on $\mathbb{R}^D$ define the conical square function

$$(Sf)(x) := \left(\int_0^\infty \int_{B(x,t)} |t\nabla e^{-t\sqrt{-\Delta}}(f)(y)|^2 \, \frac{dy\, dt}{t^{D+1}} \right)^{1/2} \qquad (x \in \mathbb{R}^D).$$

Then f belongs to the Hardy space $H^1(\mathbb{R}^D)$ if and only if $Sf \in L^1(\mathbb{R}^D)$. Also a characterization in terms of the Riesz transforms was given.

R.R. Coifman and G. Weiss ([CW77]) extended the definition of Hardy spaces from the Euclidean setting to the more general framework of spaces of homogeneous type.

In the last years, a theory of Hardy spaces adapted to certain operators was introduced, similarly to the way that the Hardy spaces $H^p(\mathbb{R}^D)$ are adapted to the Laplacian. We refer to [DL10] for a survey of the recent development and only mention that their origin lies in the paper [ADM05] due to P. Auscher, X.T. Duong, and A. McIntosh, who defined the Hardy space $H^1_L(\mathbb{R}^D)$ associated to an operator L which has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^D)$ and whose kernels of the semigroup operators e^{-tL} have a pointwise Poisson upper bound. Afterwards, the assumptions on the associated operator were relaxed. S. Hofmann and S. Mayboroda ([HM09]) defined Hardy spaces associated to second order divergence form elliptic operators on $\mathbb{R}^D$ with complex coefficients. S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, and L.X. Yan ([HLMMY08]) made further progress toward the treatment of more general operators. They developed a theory of Hardy spaces adapted to non-negative, self-adjoint operators L on $L^2(X)$ which satisfy Davies-Gaffney estimates in the setting of spaces of homogeneous type. X.T. Duong and J. Li ([DL10]) considered even non-self-adjoint operators and introduced Hardy spaces associated to operators L which have a bounded holomorphic functional calculus on $L^2(X)$ and generate an analytic semigroup on $L^2(X)$ satisfying Davies-Gaffney estimates of order 2.

In this chapter we consider an injective, non-negative, self-adjoint operator L on $L^2(X)$ which satisfies Davies-Gaffney estimates (DG_m) for some $m \geq 2$. In Section 4.1 we introduce the Hardy spaces $H^p_{L,\psi}(X)$ associated to L defined in terms of square functions and classify them in the general setting of tent spaces (cf. Section 4.2). With the help of the atomic decomposition of tent spaces, we establish a characterization of $H^1_{L,\psi_0}(X)$ via molecules, where $\psi_0(z) := ze^{-z}$ (cf. Section 4.3). In the final Section 4.4 we verify that the Hardy spaces $H^p_{L,\psi}(X)$ and the Lebesgue spaces $L^p(X)$ coincide for all $p \in (p_0, 2]$ when L fulfills generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$) for some $p_0 \in [1, 2)$ and $m \geq 2$.

4.1 Hardy spaces via square functions

To start with, we introduce some notation. For $\theta \in (0,\pi)$ define the open sector

$$\Sigma_\theta := \{ z \in \mathbb{C} \setminus \{0\} \,:\, |\arg z| < \theta \}$$

and denote by $H_0^\infty(\Sigma_\theta)$ the space of all holomorphic functions $\psi\colon \Sigma_\theta \to \mathbb{C}$ such that there exist constants $C, \sigma > 0$ with $|\psi(z)| \le C \frac{|z|^\sigma}{1+|z|^{2\sigma}}$ for any $z \in \Sigma_\theta$. We will write shortly H_0^∞ when the angle θ is of no particular interest.

Note that $\int_0^\infty |\psi(t^m)|^2 \frac{dt}{t} < \infty$ for each $\psi \in H_0^\infty$. If L is a self-adjoint operator on $L^2(X)$, it follows from the spectral theorem for L that for all $\psi \in H_0^\infty \setminus \{0\}$, $f \in L^2(X)$, and $m \in \mathbb{N}$

$$\begin{aligned} \int_0^\infty \|\psi(t^m L) f\|_2^2 \,\frac{dt}{t} &= \int_0^\infty \big(\overline{\psi}(t^m L)\psi(t^m L) f, f\big)_{L^2(X)} \,\frac{dt}{t} \\ &= \Big(\int_0^\infty \overline{\psi}(t^m L)\psi(t^m L)\,\frac{dt}{t}\, f, f\Big)_{L^2(X)} = \int_0^\infty |\psi(t^m)|^2 \,\frac{dt}{t}\, \|f\|_2^2 \cong \|f\|_2^2 \end{aligned} \tag{4.1}$$

with implicit constants independent of f. Later, we refer to (4.1) as *quadratic estimate*.

Definition 4.1. Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \ge 2$. For a non-trivial $\psi \in H_0^\infty$ consider the conical square function

$$(S_\psi f)(x) := \Big(\int_0^\infty \int_{B(x,t)} |\psi(t^m L) f(y)|^2 \,\frac{d\mu(y)}{|B(x,t)|}\,\frac{dt}{t}\Big)^{1/2} \qquad (f \in L^2(X), x \in X).$$

For $p \in [1,2]$, the *Hardy space* $H^p_{L,S_\psi}(X)$ *associated to L via square functions* is said to be the completion of the space

$$\{ f \in L^2(X) \,:\, S_\psi f \in L^p(X) \}$$

with respect to the norm

$$\|f\|_{H^p_{L,S_\psi}} := \|S_\psi f\|_p \,.$$

If $\psi_0(z) := z e^{-z}$, we abbreviate $H^p_L(X) := H^p_{L,S_{\psi_0}}(X)$.

Note that in the special case of $X = \mathbb{R}^D$, $L = -\Delta$, and $\psi = \psi_0$ this definition gives the Hardy space $H^p(\mathbb{R}^D)$ as introduced by E.M. Stein and G. Weiss.

It can be easily verified that $H^2_{L,S_\psi}(X) = L^2(X)$ for every $\psi \in H_0^\infty \setminus \{0\}$. Indeed, due to Fubini's theorem, (2.3), and (4.1), one obtains for each $f \in L^2(X)$

$$\begin{aligned} \|S_\psi f\|_2^2 &= \int_0^\infty \int_X \int_{B(y,t)} |\psi(t^m L) f(y)|^2 \,\frac{d\mu(x)}{|B(x,t)|}\, d\mu(y)\,\frac{dt}{t} \\ &\cong \int_0^\infty \int_X |\psi(t^m L) f(y)|^2 \, d\mu(y)\,\frac{dt}{t} \cong \|f\|_2^2 \,. \end{aligned}$$

In particular, the space H^2_{L,S_ψ} is independent of the choice of ψ.

4.2 Tent spaces

We consider Hardy spaces via the abstract concept of tent spaces. In 1985, R.R. Coifman, Y. Meyer, and E.M. Stein ([CMS85]) introduced these spaces in the Euclidean setting $\mathbb{R}^D$. Among other things, they constructed a theory of atomic decomposition for tent spaces. Recently, E. Russ ([Rus07]) investigated tent spaces in the more general framework of spaces of homogeneous type and developed an atomic decomposition for these spaces as well. His result forms the basis for our proof that the Hardy space $H^1_{L,S_{\psi_0}}(X)$ admits a molecular decomposition, where $\psi_0(z) := ze^{-z}$ (cf. Theorem 4.10).

In this section we provide a short review on tent spaces. For any $x \in X$ let

$$\Gamma(x) := \big\{(y,t) \in X \times (0,\infty) \,:\, d(y,x) < t\big\}$$

denote the *cone of vertex* x.

Definition 4.2. For a measurable function $F\colon X \times (0,\infty) \to \mathbb{C}$, we define the *conical square function* $\mathcal{A}F$ via

$$\mathcal{A}F(x) := \left(\iint_{\Gamma(x)} |F(y,t)|^2 \,\frac{d\mu(y)\,dt}{t\,|B(x,t)|}\right)^{1/2} \qquad (x \in X).$$

Given $p \in [1,\infty)$, the *tent space* $T^p(X)$ is said to be the space of all measurable functions $F\colon X \times (0,\infty) \to \mathbb{C}$ such that $\mathcal{A}F \in L^p(X)$ holds. If $T^p(X)$ is equipped with the norm $\|F\|_{T^p(X)} := \|\mathcal{A}F\|_p$, then $T^p(X)$ becomes a Banach space.

Let us collect some well-known properties of tent spaces. First, we cite a density result and a characterization of the dual space of $T^p(X)$. We refer to [HLMMY08, Lemma 4.7] and to [HLMMY08, Proposition 4.8], respectively.

Fact 4.3. *For every* $p \in [1,\infty)$*, the space* $T^p(X) \cap T^2(X)$ *is dense in* $T^p(X)$*.*

Fact 4.4. *Let* $p \in (1,\infty)$ *and* $1/p + 1/p' = 1$*. The pairing*

$$\langle F, G\rangle \mapsto \int_0^\infty \int_X F(x,t)G(x,t)\,d\mu(x)\,\frac{dt}{t}$$

realizes $T^{p'}(X)$ *as equivalent to the dual of* $T^p(X)$*.*

Tent spaces behave very well with respect to the complex interpolation procedure, as the next statement shows (cf. e.g. [HLMMY08, Proposition 4.9]).

Fact 4.5. *Assume that* $1 \le p_0 < p < p_1 < \infty$ *with* $1/p = (1-\theta)/p_0 + \theta/p_1$ *for some* $\theta \in (0,1)$*. Then one has*

$$\big[T^{p_0}(X), T^{p_1}(X)\big]_\theta = T^p(X)\,.$$

Next, we recall the atomic decomposition for tent spaces.

If O is an open subset of X, then the *tent over* O, labeled by $\widehat{O}$, is defined as the set

$$\widehat{O} := \left\{(y,t) \in X \times (0,\infty) \,:\, d(x, O^c) \geq t\right\},$$

where O^c stands for the complement of O in X.

Definition 4.6. A measurable function $A\colon X \times (0,\infty) \to \mathbb{C}$ is called $T^1(X)$-*atom* if there exists a ball $B \subseteq X$ such that A is supported in the tent $\widehat{B}$ and it holds

$$\int_0^\infty \int_X |A(x,t)|^2 \, d\mu(x) \, \frac{dt}{t} \leq \frac{1}{\mu(B)} \, .$$

In this case we sometimes refer to A as a $T^1(X)$-atom associated with the ball B.

We remark that every $T^1(X)$-atom belongs to $T^1(X)$ and its norm is controlled by a constant depending only on the underlying space X, i.e. the constants in (2.2). This can be easily seen by using the Cauchy-Schwarz inequality, Fubini's theorem, and Fact 2.1. For details we refer to [Fre11, p. 22].

Conversely, any function in $T^1(X)$ admits an atomic decomposition. This was proved in [Rus07, Theorem 1.1]. For the final part of the statement below, concerning $T^2(X)$ convergence, we refer to [DL10, Proposition 3.6].

Fact 4.7. *There exists a constant $C > 0$ with the following property: For each $F \in T^1(X)$ there are a sequence $(\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1$ of complex numbers and a sequence $(A_j)_{j\in\mathbb{N}_0}$ of $T^1(X)$-atoms such that*

$$F = \sum_{j=0}^{\infty} \lambda_j A_j \, , \tag{4.2}$$

where the sum converges in $T^1(X)$ and almost everywhere in $X \times (0,\infty)$, and

$$\sum_{j=0}^{\infty} |\lambda_j| \leq C \, \|F\|_{T^1(X)} \, .$$

If, in addition, $F \in T^1(X) \cap T^2(X)$, then the decomposition (4.2) converges in $T^2(X)$, too.

4.3 Hardy spaces via molecules

A well-known feature of the classical Hardy space $H^1(\mathbb{R}^D)$ lies in the atomic decomposition which was originally developed by R.R. Coifman ([Coi74]) for $D = 1$ and by R.H. Latter ([Lat79]) for $D > 1$. One of the principal purposes of R.R. Coifman and G. Weiss ([CW77]) was to show that many of the properties of the Hardy space $H^1(\mathbb{R}^D)$ and operators acting on it can be obtained by focusing the attention on individual atoms. For example, the continuity of an operator T can often be proved by estimating $T(a)$ when a is an atom.

In [HLMMY08] the authors studied Hardy spaces associated to injective, non-negative, self-adjoint operators L satisfying Davies-Gaffney estimates of order 2. They showed that the Hardy space $H^1_{L,S_{\psi_0}}(X)$ possesses an atomic decomposition, where $\psi_0(z) := ze^{-z}$. To give a flavor of their results, we recall the definition of the atomic Hardy space (cf. e.g. [HLMMY08, Definitions 2.1 and 2.2]):

Let $M \in \mathbb{N}$. An element $a \in L^2(X)$ is called $(1,2,M)$*-atom associated to* L if there exist a function $b \in \mathcal{D}(L^M)$ and a ball $B \subseteq X$, whose radius is denoted by r, such that

i) $a = L^M b$;

ii) $\operatorname{supp} L^k b \subseteq B \quad$ for all $k \in \{0, 1, \ldots, M\}$;

iii) $\left\|(r^2 L)^k b\right\|_2 \leq r^{2M} \mu(B)^{-1/2} \qquad$ for all $k \in \{0, 1, \ldots, M\}$.

The *atomic Hardy space* $H^1_{L,at,M}(X)$ *associated to* L is defined as the completion of

$$\left\{ \sum_{j=0}^{\infty} \lambda_j a_j : \begin{array}{l} (\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1,\ a_j \text{ is a } (1,2,M)\text{-atom ass. to } L \text{ for any } j \in \mathbb{N}_0, \\ \text{and the series converges in } L^2(X) \end{array} \right\}$$

with respect to the norm given by

$$\|f\|_{H^1_{L,at,M}(X)} := \inf \left\{ \sum_{j=0}^{\infty} |\lambda_j| : \begin{array}{l} f = \sum_{j=0}^{\infty} \lambda_j a_j,\ (\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1,\ a_j \text{ is a } (1,2,M)\text{-atom ass.} \\ \text{to } L \text{ for any } j \in \mathbb{N}_0, \text{ and the series converges in } L^2(X) \end{array} \right\}.$$

According to [HLMMY08, Theorem 4.1], it holds $H^1_{L,at,M}(X) = H^1_{L,S_{\psi_0}}(X)$ if $M > D/4$. In order to show this result, S. Hofmann et al. established, among other things, the atomic decomposition of $H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$, i.e. for every $f \in H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$ there exist a sequence of complex numbers $(\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1$ and a family of $(1,2,M)$-atoms $(a_j)_{j\in\mathbb{N}_0}$ such that f can be represented in the form $f = \sum_{j=0}^{\infty} \lambda_j a_j$, where the series converges in $L^2(X)$ ([HLMMY08, Proposition 4.13]). Besides the atomic decomposition of the tent space $T^1(X)$ (cf. Fact 4.7), their proof relies heavily on the equivalence between the Davies-Gaffney estimates (DG_2) for L and the finite speed propagation property for the corresponding wave equation $Lu + u_{tt} = 0$ (cf. e.g. [CS08, Theorem 3.4]) which means that

$$\left(\cos(t\sqrt{L}) f_1, f_2\right)_{L^2(X)} = 0 \tag{4.3}$$

for all $0 < t < \operatorname{dist}(U_1, U_2)$, all $f_i \in L^2(U_i)$, and all disjoint open sets $U_i \subseteq X$, $i = 1, 2$.

Note that, if $\cos(t\sqrt{L})$ is an integral operator with kernel $K_t \in L^\infty(X \times X)$, then (4.3) simply says that $\operatorname{supp} K_t \subseteq \{(x,y) \in X \times X : d(x,y) \leq t\}$.

Unfortunately, it is not possible to deduce a result similar to the finite speed propagation property for operators L that fulfill (DG_m) for some $m > 2$. Due to this lack of information on the support, one is not able to develop an atomic decomposition of $H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$ for these operators L since one has no tools to show the support condition ii) in the above definition. Nevertheless, in the general situation one can decompose the Hardy space $H^1_{L,S_{\psi_0}}(X)$ by considering molecules instead of atoms. Molecules are building blocks similar to atoms, but the support property of the latter, i.e. ii) above, is relaxed.

Definition 4.8. Assume that L is an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \geq 2$. Let $\varepsilon > 0$ and $M \in \mathbb{N}$. A function $a \in L^2(X)$ is said to be a (M, ε, L)-*molecule* if there exist a function $b \in \mathcal{D}(L^M)$ and a ball $B \subseteq X$, whose radius is denoted by r, such that

i) $a = L^M b$;

ii) for every $k \in \{0, 1, \ldots, M\}$ and $j \in \mathbb{N}_0$, it holds

$$\left\|(r^m L)^k b\right\|_{L^2(U_j(B))} \leq r^{mM} 2^{-j\varepsilon} \mu(2^j B)^{-1/2}\,, \tag{4.4}$$

where the dyadic annuli $U_j(B)$ are defined by

$$U_0(B) := B \qquad \text{and} \qquad U_j(B) := 2^j B \setminus 2^{j-1} B \quad \text{for all } j \in \mathbb{N}\,. \tag{4.5}$$

As usual, ρB stands for the ball in X with the same center as B but radius ρr whenever $\rho > 0$ and B is a ball of radius r.

In this situation we sometimes refer to a as a (M, ε, L)-molecule associated with B.

In the literature (cf. e.g. [HLMMY08], [DL10]) the authors mostly study the case when $m = 2$ and typically use the terminology "$(1, 2, M, \varepsilon)$-molecule associated to L" instead of (M, ε, L)-molecule. To the best of our knowledge, a definition similar to ours for operators satisfying (DG_m) of arbitrary order $m \geq 2$ was first given in [Fre11, Definition 4.1].

Next, we introduce the molecular Hardy space $H^1_{L,mol,M,\varepsilon}(X)$. In the special case $m = 2$ our definition matches with the one given in [HLMMY08, Definition 2.4].

Definition 4.9. Fix $\varepsilon > 0$ and $M \in \mathbb{N}$. Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills Davies-Gaffney estimates (DG_m) for some $m \geq 2$.

We call $f = \sum_{j=0}^\infty \lambda_j m_j$ a *molecular* (M, ε, L)-*representation* of a given $f \in L^1(X)$ if $(\lambda_j)_{j \in \mathbb{N}_0}$ is a numerical sequence belonging to ℓ^1, m_j is a (M, ε, L)-molecule for any $j \in \mathbb{N}_0$, and the sum $\sum_{j=0}^\infty \lambda_j m_j$ converges in $L^2(X)$. Define

$$\mathbb{H}^1_{L,mol,M,\varepsilon}(X) := \left\{f \in L^1(X) \,:\, f \text{ has a molecular } (M, \varepsilon, L)\text{-representation}\right\}$$

with the norm given by

$$\|f\|_{H^1_{L,mol,M,\varepsilon}(X)} := \inf\left\{\sum_{j=0}^{\infty}|\lambda_j| \,:\, \sum_{j=0}^{\infty}\lambda_j m_j \text{ is a molecular } (M,\varepsilon,L)\text{-representation of } f\right\}.$$

The *Hardy space* $H^1_{L,mol,M,\varepsilon}(X)$ *associated to* L *via molecules* is said to be the completion of $\mathbb{H}^1_{L,mol,M,\varepsilon}(X)$ with respect to the norm $\|\cdot\|_{H^1_{L,mol,M,\varepsilon}(X)}$.

As a direct consequence of the definition, we note that $H^1_{L,mol,M_2,\varepsilon}(X) \subseteq H^1_{L,mol,M_1,\varepsilon}(X)$ for each $\varepsilon > 0$ and $M_1, M_2 \in \mathbb{N}$ with $M_1 \leq M_2$. In addition, the Hardy space $H^1_{L,mol,M,\varepsilon}(X)$ is contained in $L^1(X)$ because the $L^1(X)$-norm of (M,ε,L)-molecules is uniformly bounded by a constant depending only on ε and the constants in the doubling condition (cf. [Fre11, Remark 4.2]).

Eventually, we shall see that any choice of $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ and $\varepsilon \in (0, mM - D/2]$ leads to the same space $H^1_{L,mol,M,\varepsilon}(X)$. This follows from the more general fact that the Hardy space $H^1_{L,S_{\psi_0}}(X)$ defined via square functions and the Hardy space $H^1_{L,mol,M,\varepsilon}(X)$ defined via molecules coincide whenever $M \in \mathbb{N}$ with $M > \frac{D}{2m}$, $\varepsilon \in (0, mM - D/2]$, and $\psi_0(z) := ze^{-z}$.

Theorem 4.10. *Let* L *be an injective, non-negative, self-adjoint operator on* $L^2(X)$ *which satisfies Davies-Gaffney estimates* (DG_m) *for some* $m \geq 2$. *Assume that* $M \in \mathbb{N}$ *with* $M > \frac{D}{2m}$, $\varepsilon \in (0, mM - D/2]$, *and* $\psi_0(z) := ze^{-z}$. *Then*

$$H^1_{L,mol,M,\varepsilon}(X) = H^1_{L,S_{\psi_0}}(X)$$

with equivalent norms

$$\|f\|_{H^1_{L,mol,M,\varepsilon}(X)} \cong \|f\|_{H^1_{L,S_{\psi_0}}(X)},$$

where the implicit constants may depend only on ε, M *or the constants in the Davies-Gaffney and the doubling condition.*

Recently, X.T. Duong and J. Li investigated the case $m = 2$ for sectorial operators with bounded holomorphic functional calculus and showed the assertion of Theorem 4.10 in this situation (cf. [DL10, Theorem 3.12]). Their approach is based on the proof of [HMM10, Theorem 3.5] and extends with some modifications to the general case $m \geq 2$. This was already observed in [Fre11, Section 4.3], where $H^1_{L,mol,M,\varepsilon}(X) = H^1_{L,S_\psi}(X)$ was shown for a wide class of functions $\psi \in H^\infty_0$ under a certain decay assumption at ∞. As we only suppose the validity of the Davies-Gaffney condition (2.8) for all open balls in X and the molecular decomposition of the Hardy space $H^1_{L,S_{\psi_0}}(X)$ is for our further studies of great importance, we will present full details following the outline of X.T. Duong and J. Li in [DL10, Section 3.5].

During the proof, fix $\psi_0(z) := ze^{-z}$. Let $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ and $\varepsilon \in (0, mM - D/2]$. Recall that the spaces $H^1_{L,mol,M,\varepsilon}(X)$ and $H^1_{L,S_{\psi_0}}(X)$ are the completions of $\mathbb{H}^1_{L,mol,M,\varepsilon}(X)$ and $H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$ in the corresponding norms, respectively. We proceed in two steps.

Claim 1: $\mathbb{H}^1_{L,mol,M,\varepsilon}(X) \subseteq H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$ and for any $f \in \mathbb{H}^1_{L,mol,M,\varepsilon}(X)$

$$\|f\|_{H^1_{L,S_{\psi_0}}(X)} \lesssim \|f\|_{H^1_{L,mol,M,\varepsilon}(X)} .$$

Claim 2: $H^1_{L,S_{\psi_0}}(X) \cap L^2(X) \subseteq \mathbb{H}^1_{L,mol,M,\varepsilon}(X)$ and for any $f \in H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$

$$\|f\|_{H^1_{L,mol,M,\varepsilon}(X)} \lesssim \|f\|_{H^1_{L,S_{\psi_0}}(X)} .$$

As we will see in the proof, the assertion of Claim 1 is actually true for all $\varepsilon > 0$.

We prepare the proof of Claim 1 with the next statement which provides a generalization of [DL10, Lemma 3.15] to arbitrary $m \geq 2$. It says that an operator T is bounded from $H^1_{L,mol,M,\varepsilon}$ to $L^1(X)$ whenever $\|T(a)\|_1$ is uniformly bounded for any (M, ε, L)-molecule a. This is an important observation and, later in Section 5.1, we will establish a revised criterion that gives even the boundedness of operators from $H^1_{L,mol,M,\varepsilon}$ to $H^1_{L,mol,M,\varepsilon}$.

Lemma 4.11. *Consider an injective, non-negative, self-adjoint operator L on $L^2(X)$ for which Davies-Gaffney estimates (DG_m) hold for some $m \geq 2$. Let $\varepsilon > 0$ and $M \in \mathbb{N}$. Assume that T is a non-negative, sublinear operator such that T is of weak type $(2, 2)$ and there exists a constant $C_T > 0$ with*

$$\|T(a)\|_1 \leq C_T \tag{4.6}$$

for every (M, ε, L)-molecule a. Then T is bounded from $\mathbb{H}^1_{L,mol,M,\varepsilon}(X)$ to $L^1(X)$ and it holds for any $f \in \mathbb{H}^1_{L,mol,M,\varepsilon}(X)$

$$\|T(f)\|_1 \leq C_T \|f\|_{H^1_{L,mol,M,\varepsilon}(X)} .$$

Consequently, T extends to a bounded operator from $H^1_{L,mol,M,\varepsilon}(X)$ to $L^1(X)$.

Proof. Let $f \in \mathbb{H}^1_{L,mol,M,\varepsilon}(X)$ and $\delta > 0$ be fixed. By definition, we find a numerical sequence $(\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1$ and a family $(m_j)_{j\in\mathbb{N}_0}$ of (M, ε, L)-molecules such that

$$\|f\|_{H^1_{L,mol,M,\varepsilon}} \leq \sum_{j=0}^{\infty} |\lambda_j| \leq \|f\|_{H^1_{L,mol,M,\varepsilon}} + \delta$$

and $f = \sum_{j=0}^{\infty} \lambda_j m_j$ with respect to the $L^2(X)$-norm. Due to the sublinearity and non-negativity of T, we have for each $N \in \mathbb{N}$

$$T\Big(\sum_{j=N+1}^{\infty} \lambda_j m_j\Big) \geq T\Big(\sum_{j=0}^{\infty} \lambda_j m_j\Big) - T\Big(\sum_{j=0}^{N} \lambda_j m_j\Big) \geq T(f) - \sum_{j=0}^{N} |\lambda_j|\, T(m_j) \qquad \mu\text{-a.e.}$$

Hence, by exploiting Fatou's lemma and the weak type $(2,2)$ boundedness of T, it follows for any $\alpha > 0$

$$\begin{aligned}
&\mu\Big(\Big\{x \in X : T(f)(x) - \sum_{j=0}^{\infty} |\lambda_j|\, T(m_j)(x) > \alpha\Big\}\Big) \\
&\le \liminf_{N\to\infty} \mu\Big(\Big\{x \in X \,:\, T\Big(\sum_{j=N+1}^{\infty} \lambda_j m_j\Big)(x) > \alpha\Big\}\Big) \\
&\lesssim \liminf_{N\to\infty} \frac{1}{\alpha^2}\Big\|\sum_{j=N+1}^{\infty} \lambda_j m_j\Big\|_2^2 = 0
\end{aligned}$$

because the series $\sum_{j=0}^{\infty} \lambda_j m_j$ converges in $L^2(X)$. Thus, we have shown that

$$T(f) \le \sum_{j=0}^{\infty} |\lambda_j|\, T(m_j) \qquad \mu\text{-a.e.}$$

which, in combination with the hypothesis (4.6), leads to

$$\|T(f)\|_1 \le \sum_{j=0}^{\infty} |\lambda_j| \|T(m_j)\|_1 \le C_T \sum_{j=0}^{\infty} |\lambda_j| \le C_T\big(\|f\|_{H^1_{L,mol,M,\varepsilon}} + \delta\big)\,.$$

As $\delta > 0$ was arbitrary, the assertion of the lemma is proved. □

Since we only assume the validity of the Davies-Gaffney condition (2.8) for each ball, we present a corresponding version of (2.8) that works for dyadic annuli as well. It turns out that for our purposes polynomial decay is enough, so that we can estimate quite roughly.

Lemma 4.12. *Let $K \in \mathbb{N}$ and L be a non-negative, self-adjoint operator on $L^2(X)$ which fulfills Davies-Gaffney estimates (DG_m) for some $m \ge 2$. Define $\psi(z) := z^K e^{-z}$, $z \in \mathbb{C}$. Then there exists a constant $C > 0$ such that for all $\beta > 0$, $k \in \mathbb{N}$, $l \in \mathbb{N}_0$ with $0 \le l \le k-1$, $r > 0$, $t \in (0, 2^{k+1}r)$, and all $x \in X$*

$$\big\|\mathbb{1}_{2^{k+1}B\setminus 2^k B}\, \psi(t^m L)\, \mathbb{1}_{U_l(B)}\big\|_{2\to 2} \le C\left(\frac{t}{2^k r}\right)^{\beta},$$

where $B := B(x,r)$ and the dyadic annulus $U_l(B)$ is defined as in (4.5).

Proof. For arbitrary $r > 0$ and $x \in X$ write $B := B(x,r)$. Let $k \in \mathbb{N}$, $l \in \mathbb{N}_0$ with $0 \le l \le k-1$, and $t \in (0, 2^{k+1}r)$. By Lemma 2.2, one finds points $x_1, \ldots, x_{K_k} \in 2^{k+1}B\setminus 2^k B$ and $y_1, \ldots, y_{K_l} \in U_l(B)$ such that $2^{k+1}B \setminus 2^k B \subseteq \bigcup_{i=1}^{K_k} B(x_i, t)$ and $U_l(B) \subseteq \bigcup_{j=1}^{K_l} B(y_j, t)$, where $K_k \lesssim (2^k r/t)^D$ and $K_l \lesssim \max\{1, (2^l r/t)^D\} \le (2^k r/t)^D$. Observe that it holds

$d(x_i, y_j) \geq (2^k - 2^l)r \gtrsim 2^k r$ for all i, j. Due to Lemma 2.9, the family $\{\psi(tL) : t > 0\}$ satisfies Davies-Gaffney estimates (DG_m). This yields for any $\beta > 0$

$$\begin{aligned}
\big\|\mathbb{1}_{2^{k+1}B\setminus 2^k B}\,\psi(t^m L)\,\mathbb{1}_{U_l(B)}\big\|_{2\to 2} &\leq \sum_{i=1}^{K_k}\sum_{j=1}^{K_l}\big\|\mathbb{1}_{B(x_i,t)}\,\psi(t^m L)\,\mathbb{1}_{B(y_j,t)}\big\|_{2\to 2} \\
&\lesssim \sum_{i=1}^{K_k}\sum_{j=1}^{K_l}\exp\Big(-b\,\big(2^k r/t\big)^{\frac{m}{m-1}}\Big) \\
&\lesssim \big(2^k r/t\big)^{2D}\exp\big(-b\,2^k r/t\big) \lesssim \big(t/2^k r\big)^{\beta}\,.
\end{aligned}$$

□

Now we are ready to prove Claim 1.

Lemma 4.13. *Suppose that L satisfies the assumptions of Theorem 4.10. For any $\varepsilon > 0$ and $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ one has the embedding*

$$\mathbb{H}^1_{L,mol,M,\varepsilon}(X) \subseteq H^1_{L,S_{\psi_0}}(X) \cap L^2(X)\,.$$

More precisely, there exists a constant $C > 0$ depending only on ε, M and the constants in the Davies-Gaffney and the doubling condition such that for all $f \in \mathbb{H}^1_{L,mol,M,\varepsilon}(X)$

$$\|f\|_{H^1_{L,S_{\psi_0}}(X)} \leq C\,\|f\|_{H^1_{L,mol,M,\varepsilon}(X)}\,.$$

Proof. The inclusion $\mathbb{H}^1_{L,mol,M,\varepsilon}(X) \subseteq L^2(X)$ is valid by definition. Thus, by Lemma 4.11, it suffices to verify that there is a constant $C > 0$ such that for all (M, ε, L)-molecules a

$$\|S_{\psi_0}(a)\|_1 \leq C\,. \tag{4.7}$$

To this end, fix an arbitrary (M, ε, L)-molecule a and take a ball $B =: B(x, r)$ according to Definition 4.8. Thanks to the tent space theory discussed in Section 4.2, for the proof of (4.7) it is enough to show that the function $F\colon X \times (0,\infty) \to \mathbb{C}$ given by $F(y,t) := \psi_0(t^m L)a(y) = t^m L e^{-t^m L} a(y)$ satisfies

$$\|F\|_{T^1(X)} \leq C\,. \tag{4.8}$$

Motivated by the disjoint decomposition

$$\begin{aligned}
X \times (0,\infty) = \Big((2B) \times (0, 2r]\Big) &\cup \Big(\bigcup_{k=1}^{\infty} U_{k+1}(B) \times (0, r]\Big) \\
&\cup \Big(\bigcup_{k=1}^{\infty} U_{k+1}(B) \times (r, 2^{k+1}r]\Big) \cup \Big(\bigcup_{k=1}^{\infty} (2^k B) \times (2^k r, 2^{k+1} r]\Big)\,,
\end{aligned}$$

we define

$$\eta_0 := \mathbb{1}_{(2B)\times(0,2r]}$$

and for every $k \in \mathbb{N}$

$$\eta_k := \mathbb{1}_{2^{k+1}B\setminus 2^kB\times(0,r]}\,, \qquad \eta_k' := \mathbb{1}_{2^{k+1}B\setminus 2^kB\times(r,2^{k+1}r]}\,, \qquad \eta_k'' := \mathbb{1}_{(2^kB)\times(2^kr,2^{k+1}r]}\,.$$

In view of

$$F = \eta_0 F + \sum_{k=1}^{\infty} \eta_k F + \sum_{k=1}^{\infty} \eta_k' F + \sum_{k=1}^{\infty} \eta_k'' F\,,$$

the estimate (4.8) will be an immediate consequence of the following estimates:

(a) $\|\eta_k F\|_{T^1(X)} \leq C 2^{-k\sigma}$ for all $k \in \mathbb{N}_0$,

(b) $\|\eta_k' F\|_{T^1(X)} \leq C 2^{-k\sigma}$ for all $k \in \mathbb{N}$,

(c) $\|\eta_k'' F\|_{T^1(X)} \leq C 2^{-k\sigma}$ for all $k \in \mathbb{N}$,

where C, σ are some positive constants independent of a and k.

First, we show (a). Since, for every $k \in \mathbb{N}_0$, $\eta_k F$ is supported in $\widehat{2^{k+1}B}$, we just have to verify that its $T^2(X)$-norm is bounded by $C2^{-k\sigma}\mu(2^kB)^{-1/2}$ for some $C, \sigma > 0$. Indeed, after applying the Cauchy-Schwarz inequality, this means that $\frac{1}{C}2^{k\sigma}\eta_k F$ is a $T^1(X)$-atom and thus its $T^1(X)$-norm is controlled by a constant (cf. remark after Definition 4.6).

With the help of the doubling condition (2.3), the quadratic estimate (4.1), and the definition of the (M, ε, L)-molecule a, we obtain for $k = 0$

$$\begin{aligned}
\|\eta_0 F\|_{T^2(X)}^2 \leq \|F\|_{T^2(X)}^2 &\lesssim \int_0^\infty \int_X \left|\psi_0(t^m L)a(y)\right|^2 d\mu(y)\, \frac{dt}{t} \\
&\lesssim \|a\|_2^2 = r^{-2mM} \sum_{j=0}^{\infty} \|(r^m L)^M b\|_{L^2(U_j(B))}^2 \\
&\lesssim r^{-2mM} \sum_{j=0}^{\infty} r^{2mM} 2^{-2j\varepsilon} \mu(2^j B)^{-1} \lesssim \mu(B)^{-1}\,.
\end{aligned}$$

Now fix $k \in \mathbb{N}$. Then

$$\begin{aligned}
\|\eta_k F\|_{T^2(X)} &= \left(\int_{2^{k+1}B\setminus 2^kB} \int_0^r \left|\psi_0(t^m L)a(y)\right|^2 \frac{dt}{t}\, d\mu(y) \right)^{1/2} \\
&\leq \sum_{l=0}^{\infty} \left(\int_{2^{k+1}B\setminus 2^kB} \int_0^r \left|\psi_0(t^m L)(a\mathbb{1}_{U_l(B)})(y)\right|^2 \frac{dt}{t}\, d\mu(y) \right)^{1/2} =: \sum_{l=0}^{\infty} I_l\,.
\end{aligned}$$

We split the sum into three parts.

Assume that $0 \le l \le k-2$. Then, by Lemma 4.12 (with $\beta := mM$), the definition of the (M,ε,L)-molecule a, and the doubling property, one obtains

$$\begin{aligned}
I_l^2 &\lesssim \int_0^r \left(\frac{t}{2^k r}\right)^{2mM} \|a\mathbb{1}_{U_l(B)}\|_2^2 \,\frac{dt}{t} \\
&= 2^{-2kmM} \int_0^r \left(\frac{t}{r}\right)^{2mM} \frac{dt}{t} \; r^{-2mM} \big\|(r^m L)^M b\big\|_{L^2(U_l(B))}^2 \\
&\le 2^{-2kmM} r^{-2mM} \big(r^{mM} 2^{-l\varepsilon} \mu(2^l B)^{-1/2}\big)^2 \\
&\lesssim 2^{-2l\varepsilon} 2^{-2kmM} 2^{(k-l)D} \mu(2^k B)^{-1} \\
&\lesssim 2^{-2l\varepsilon} 2^{-k(2mM-D)} \mu(2^k B)^{-1} \,.
\end{aligned}$$

Consequently, it holds

$$\sum_{l=0}^{k-2} I_l \lesssim 2^{-k(mM-D/2)} \mu(2^k B)^{-1/2} \,.$$

Assume now that $k-1 \le l \le k+1$. We make use of the quadratic estimate (4.1) and get, due to the definition of the (M,ε,L)-molecule a and the doubling property,

$$\begin{aligned}
I_l^2 &\le \int_0^\infty \big\|\psi_0(t^m L)(a\mathbb{1}_{U_l(B)})\big\|_2^2 \,\frac{dt}{t} \lesssim \|a\mathbb{1}_{U_l(B)}\|_2^2 \\
&= r^{-2mM} \|(r^m L)^M b\|_{L^2(U_l(B))}^2 \lesssim 2^{-2k\varepsilon} \mu(2^k B)^{-1} \,.
\end{aligned}$$

Assume finally that $l \ge k+2$. Our argument resembles that in the case $0 \le l \le k-2$. Again, we employ Lemma 4.12 and the definition of the (M,ε,L)-molecule a

$$\begin{aligned}
I_l^2 &\lesssim \int_0^r \left(\frac{t}{2^l r}\right)^{2mM} \|a\mathbb{1}_{U_l(B)}\|_2^2 \,\frac{dt}{t} \\
&= 2^{-2lmM} \int_0^r \left(\frac{t}{r}\right)^{2mM} \frac{dt}{t} \; r^{-2mM} \big\|(r^m L)^M b\big\|_{L^2(U_l(B))}^2 \\
&\le 2^{-2lmM} 2^{-2l\varepsilon} \mu(2^l B)^{-1} \,.
\end{aligned}$$

It follows that

$$\sum_{l=k+2}^{\infty} I_l \lesssim \sum_{l=k+2}^{\infty} 2^{-lmM} 2^{-l\varepsilon} \mu(2^l B)^{-1/2} \lesssim 2^{-kmM} \mu(2^k B)^{-1/2} \,.$$

This ends the proof of (a).

Now we treat (b). Let $k \in \mathbb{N}$ be fixed. Similarly as in (a), it suffices to show that the $T^2(X)$-norm of $\eta_k' F$ is bounded by a constant times $2^{-k\sigma}\mu(2^k B)^{-1/2}$ for some $\sigma > 0$.

As a is a (M,ε,L)-molecule, there exists $b \in \mathcal{D}(L^M)$ with $a = L^M b$. Hence, we can write

$$\begin{aligned}
\|\eta_k' F\|_{T^2(X)} &= \left(\int_{2^{k+1}B\setminus 2^k B} \int_r^{2^{k+1}r} \left|(t^m L)^{M+1} e^{-t^m L} b(y)\right|^2 \frac{dt}{t^{2mM+1}}\, d\mu(y) \right)^{1/2} \\
&\le \sum_{l=0}^{\infty} \left(\int_{2^{k+1}B\setminus 2^k B} \int_r^{2^{k+1}r} \left|(t^m L)^{M+1} e^{-t^m L} (b \mathbb{1}_{U_l(B)})(y)\right|^2 \frac{dt}{t^{2mM+1}}\, d\mu(y) \right)^{1/2} \\
&=: \sum_{l=0}^{\infty} J_l \,.
\end{aligned}$$

As before, we distinguish three cases.

Assume that $0 \le l \le k-2$. Take $\widetilde{M} \in \mathbb{R}$ with $\widetilde{M} > 2mM+1$. Then Lemma 4.12 and the properties the (M,ε,L)-molecule a yield

$$\begin{aligned}
J_l^2 &\lesssim \int_r^{2^{k+1}r} \left(\frac{t}{2^k r}\right)^{\widetilde{M}} \|b\mathbb{1}_{U_l(B)}\|_2^2 \frac{dt}{t^{2mM+1}} \\
&\le (2^k r)^{-\widetilde{M}} \int_r^{2^{k+1}r} t^{\widetilde{M}-2mM-1}\, dt \;\left(r^{mM} 2^{-l\varepsilon} \mu(2^l B)^{-1/2}\right)^2 \\
&\le (2^k r)^{-\widetilde{M}} (2^{k+1} r)^{\widetilde{M}-2mM-1} \,(2^{k+1} r - r)\, r^{2mM} 2^{-2l\varepsilon} \mu(2^l B)^{-1} \\
&\lesssim 2^{-2kmM} 2^{-2l\varepsilon} \mu(2^l B)^{-1} \,.
\end{aligned}$$

In view of $\mu(2^l B)^{-1} \lesssim 2^{(k-l)D}\mu(2^k B)^{-1}$, it thus follows

$$\sum_{l=0}^{k-2} J_l \lesssim \sum_{l=0}^{k-2} 2^{-kmM} 2^{-l\varepsilon} 2^{kD/2} \mu(2^k B)^{-1/2} \lesssim 2^{-k(mM-D/2)} \mu(2^k B)^{-1/2} \,.$$

Assume now $k-1 \le l \le k+1$. As in the corresponding case of (a), we exploit the quadratic estimate (4.1) and deduce with the help of the definition of the (M,ε,L)-molecule a and the doubling condition

$$\begin{aligned}
J_l^2 &\le \int_0^\infty \int_X \left|(t^m L) e^{-t^m L} (L^M b \mathbb{1}_{U_l(B)})(y)\right|^2 d\mu(y)\, \frac{dt}{t} \lesssim \left\|L^M b \mathbb{1}_{U_l(B)}\right\|_2^2 \\
&\le r^{-2mM} \left(r^{mM} 2^{-l\varepsilon} \mu(2^l B)^{-1/2}\right)^2 \cong 2^{-2k\varepsilon} \mu(2^k B)^{-1} \,.
\end{aligned}$$

Assume finally that $l \geq k+2$. Then using Lemma 4.12 and the definition of the (M, ε, L)-molecule a lead to

$$\begin{aligned} J_l^2 &\lesssim \int_r^{2^{k+1}r} \left(\frac{t}{2^l r}\right)^{mM} \|b\mathbb{1}_{U_l(B)}\|_2^2 \frac{dt}{t^{2mM+1}} \\ &= (2^l r)^{-mM} \left(\int_r^{2^{k+1}r} t^{-mM-1} dt\right) \|b\|_{L^2(U_l(B))}^2 \\ &\lesssim (2^l r)^{-mM} r^{-mM} \left(r^{mM} 2^{-l\varepsilon} \mu(2^l B)^{-1/2}\right)^2 \\ &\leq 2^{-kmM} 2^{-2l\varepsilon} \mu(2^k B)^{-1} . \end{aligned}$$

Therefore, one has

$$\sum_{l=k+2}^{\infty} J_l \lesssim 2^{-kmM/2} \mu(2^k B)^{-1/2} .$$

This means that (b) is proved.

Now we examine (c). Similarly as before, we insert $a = L^M b$ for some $b \in \mathcal{D}(L^M)$

$$\begin{aligned} \|\eta_k'' F\|_{T^2(X)} &= \left(\int_{2^k B} \int_{2^k r}^{2^{k+1}r} \left|(t^m L)^{M+1} e^{-t^m L} b(y)\right|^2 \frac{dt}{t^{2mM+1}} \, d\mu(y)\right)^{1/2} \\ &\leq \sum_{l=0}^{\infty} \left(\int_{2^k B} \int_{2^k r}^{2^{k+1}r} \left|(t^m L)^{M+1} e^{-t^m L} (b\mathbb{1}_{U_l(B)})(y)\right|^2 \frac{dt}{t^{2mM+1}} \, d\mu(y)\right)^{1/2} \\ &=: \sum_{l=0}^{\infty} K_l . \end{aligned}$$

Assume first $0 \leq l \leq k$. Applying (4.1) with $\psi(z) := z^{M+1} e^{-z}$, $z \in \mathbb{C}$, and (4.4) gives

$$\begin{aligned} K_l^2 &\leq (2^k r)^{-2mM} \int_0^{\infty} \left\|(t^m L)^{M+1} e^{-t^m L} (b\mathbb{1}_{U_l(B)})\right\|_2^2 \frac{dt}{t} \\ &\lesssim 2^{-2kmM} r^{-2mM} \|b\mathbb{1}_{U_l(B)}\|_2^2 \leq 2^{-2kmM} 2^{-2l\varepsilon} \mu(2^l B)^{-1} . \end{aligned}$$

Due to the doubling condition, we have $\mu(2^l B)^{-1} \lesssim 2^{(k-l)D} \mu(2^k B)^{-1}$ and thus we obtain

$$\sum_{l=0}^{k} K_l \lesssim \sum_{l=0}^{k} 2^{-kmM} 2^{-l\varepsilon} 2^{(k-l)D/2} \mu(2^k B)^{-1/2} \lesssim 2^{-k(mM-D/2)} \mu(2^k B)^{-1/2} .$$

Observe that the initial assumption $M > \frac{D}{2m}$ ensures that $mM - D/2 > 0$.

At last assume that $l \geq k+1$. Then Lemma 4.12, in combination with the properties of the (M, ε, L)-molecule a, yields

$$\begin{aligned}
K_l^2 &\lesssim \int_{2^k r}^{2^{k+1} r} \left(\frac{t}{2^l r}\right)^{mM} \|b\mathbb{1}_{U_l(B)}\|_2^2 \, \frac{dt}{t^{2mM+1}} \\
&= (2^l r)^{-mM} \int_{2^k r}^{2^{k+1} r} t^{-mM-1} dt \;\; \|b\|_{L^2(U_l(B))}^2 \\
&\leq (2^k r)^{-mM} (2^k r)^{-mM} \big(r^{mM} 2^{-l\varepsilon} \mu(2^l B)^{-1/2}\big)^2 \\
&\leq 2^{-2kmM} 2^{-2l\varepsilon} \mu(2^k B)^{-1} \,.
\end{aligned}$$

Therefore, one ends up with

$$\sum_{l=k+1}^{\infty} K_l \lesssim 2^{-kmM} \mu(2^k B)^{-1/2} \,.$$

This proves (c).

In summary, we have shown the validity of (4.8), as desired. □

A detailed examination of the above proof shows that the $H^1_{L,S_{\psi_0}}(X)$-norm of a (M, ε, L)-molecule a depends only on ε, M and the constants in (2.2) and (2.8), but not on a itself. For future reference we record this below.

Corollary 4.14. *Suppose that an injective, non-negative, self-adjoint operator L fulfills Davies-Gaffney estimates (DG_m) for some $m \geq 2$. Let $\varepsilon > 0$ and $M \in \mathbb{N}$ with $M > \frac{D}{2m}$. Put $\psi_0(z) := ze^{-z}$. Then every (M, ε, L)-molecule a belongs to $H^1_{L,S_{\psi_0}}(X)$ and there is a constant $C > 0$ depending only on ε, M and the constants in the Davies-Gaffney and the doubling condition such that for all (M, ε, L)-molecules a*

$$\|a\|_{H^1_{L,S_{\psi_0}}(X)} \leq C \,.$$

This observation is of great significance for our further studies because our proof of the boundedness criterion for spectral multipliers on the Hardy space $H^1_{L,S_{\psi_0}}(X)$ is based on this fact, see Section 5.1 below.

Now we turn to the proof of Claim 2. In order to show that $H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$ is contained in $\mathbb{H}^1_{L,mol,M,\varepsilon}(X)$ for every $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ and every $\varepsilon \in (0, mM - D/2]$, we have to establish molecular (M, ε, L)-representations for functions belonging to the space $H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$. That will be achieved with the help of the atomic decomposition of the tent space $T^1(X)$ (cf. Fact 4.7).

To do so, we introduce the operators $Q_{L,M}$ and $\pi_{L,M}$ which map from $L^2(X)$ to $T^2(X)$ and vice versa, respectively. Thanks to the quadratic estimate (4.1), the operator given by

$$Q_{L,M}f(x,t) := \psi_0(t^m L)^M f(x) = \big(t^m L e^{-t^m L}\big)^M f(x) \qquad (x \in X, t > 0)$$

is bounded from $L^2(X)$ to $T^2(X)$. Define the operator $\pi_{L,M}$ via

$$\pi_{L,M}(F)(x) := \int_0^\infty \big(t^m L e^{-t^m L}\big)^M (F(\cdot,t))(x)\,\frac{dt}{t} \qquad (x \in X).$$

Then $\pi_{L,M}$ is well-defined for all $F \in T^2(X)$ and bounded from $T^2(X)$ to $L^2(X)$ since $\pi_{L,M}$ is the adjoint of the bounded operator $Q_{L,M}$.

As preparation for the proof of Claim 2, we state the following auxiliary lemma which says that $\pi_{L,M}$ maps $T^1(X)$-atoms into (M,ε,L)-molecules (cp. [DL10, Lemma 3.18]).

Lemma 4.15. *Consider an injective, non-negative, self-adjoint operator L on $L^2(X)$ which enjoys Davies-Gaffney estimates (DG_m) for some $m \geq 2$. Let A be a $T^1(X)$-atom associated with some ball $B \subseteq X$. Then for every $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ and every $\varepsilon \in (0, mM - D/2]$ there exists a constant C_M depending only on M such that $C_M \pi_{L,M}(A)$ is a (M,ε,L)-molecule associated with B.*

Proof. Fix a $T^1(X)$-atom A. By definition, there is a ball $B \subseteq X$ with $\operatorname{supp} A \subseteq \widehat{B}$ and

$$\int_0^\infty \int_X |A(x,t)|^2\,d\mu(x)\,\frac{dt}{t} \leq \mu(B)^{-1}.$$

Let $M \in \mathbb{N}$ with $M > \frac{D}{2m}$. We write

$$\pi_{L,M}(A) = L^M b,$$

where

$$b := \int_0^\infty t^{mM} (e^{-t^m L})^M A(\cdot,t)\,\frac{dt}{t}.$$

Now we choose an arbitrary $\varepsilon \in (0, mM - D/2]$ and check the condition ii) of the definition of a (M,ε,L)-molecule. Fix $k \in \{0,1,\ldots,M\}$ and let r denote the radius of the ball B. For every $j \in \mathbb{N}_0$ take $g_j \in L^2(X)$ such that $\operatorname{supp} g_j \subseteq U_j(B)$ and $\|g_j\|_2 = 1$. Then one has, due to the self-adjointness of L,

$$\begin{aligned}
\big|\big((r^m L)^k b, g_j\big)_{L^2(X)}\big| &= r^{mk} \left| \int_X \left(\int_0^\infty t^{mM} L^k (e^{-t^m L})^M (A(\cdot,t))(x)\,\frac{dt}{t}\right) \overline{g_j(x)}\,d\mu(x)\right| \\
&\leq r^{mk} \iint_{\widehat{B}} |A(x,t)|\,\big|\big(t^{mM} L^k e^{-Mt^m L}\,\overline{g_j}\big)(x)\big|\,\frac{d\mu(x)\,dt}{t} \\
&\leq r^{mk} \|A\|_{T^2(X)} \left(\iint_{\widehat{B}} \big|\big(t^{mM} L^k e^{-Mt^m L}\,\overline{g_j}\big)(x)\big|^2\,\frac{d\mu(x)\,dt}{t}\right)^{1/2} \\
&\leq \mu(B)^{-1/2}\, I_j, \qquad (4.9)
\end{aligned}$$

where

$$I_j := r^{mk}\Big(\iint_{\widehat{B}} t^{2mM}\big|\big(L^k e^{-Mt^m L}\,\overline{g_j}\big)(x)\big|^2\,\frac{d\mu(x)\,dt}{t}\Big)^{1/2}.$$

In view of the trivial fact that $\widehat{B} \subseteq B \times (0,r)$, the term I_j is bounded by

$$\begin{aligned} I_j &\le r^{mk}\Big(\int_0^r \int_B t^{2m(M-k)}\big|\big((t^m L)^k e^{-Mt^m L}\,\overline{g_j}\big)(x)\big|^2\,d\mu(x)\,\frac{dt}{t}\Big)^{1/2} \\ &\le r^{mM}\Big(\int_0^r \int_B \big|\big((t^m L)^k e^{-Mt^m L}\,\overline{g_j}\big)(x)\big|^2\,d\mu(x)\,\frac{dt}{t}\Big)^{1/2}. \end{aligned}$$

In the case $j \le \frac{1}{m}\log_2 M$ we have, due to the quadratic estimate (4.1),

$$I_j \lesssim r^{mM}\|g_j\|_2 = r^{mM}. \tag{4.10}$$

Next, let $j > \frac{1}{m}\log_2 M$. This guarantees that $M^{1/m}t < 2^j r$ for all $t \in (0,r)$ and so Lemma 4.12 is applicable which gives

$$\begin{aligned} I_j &\le r^{mM}\Big(\int_0^r \big\|\mathbb{1}_B\,((M^{1/m}t)^m L)^k e^{-(M^{1/m}t)^m L}\,\mathbb{1}_{U_j(B)}\big\|_{2\to 2}^2 \|g_j\|_{L^2(U_j(B))}^2\,\frac{dt}{t}\Big)^{1/2} \\ &\lesssim r^{mM}\Big(\int_0^r \Big(\frac{M^{1/m}t}{2^j r}\Big)^{2mM}\,\frac{dt}{t}\Big)^{1/2} \lesssim r^{mM}2^{-jmM}. \end{aligned} \tag{4.11}$$

By inserting the bounds (4.10) and (4.11) into (4.9) and by using the doubling property, we obtain for each $j \in \mathbb{N}_0$ and each $g_j \in L^2(X)$ with $\operatorname{supp} g_j \subseteq U_j(B)$ and $\|g_j\|_2 = 1$

$$\big|\big((r^m L)^k b, g_j\big)_{L^2(X)}\big| \lesssim r^{mM}2^{-j(mM-D/2)}\mu(2^j B)^{-1/2}.$$

Taking the supremum over all such g_j yields for every $j \in \mathbb{N}_0$

$$\|(r^m L)^k b\|_{L^2(U_j(B))} \lesssim r^{mM}2^{-j(mM-D/2)}\mu(2^j B)^{-1/2}.$$

Because of $\varepsilon \in (0, mM - D/2]$, we get for every $j \in \mathbb{N}_0$

$$\|(r^m L)^k b\|_{L^2(U_j(B))} \lesssim r^{mM}2^{-j\varepsilon}\mu(2^j B)^{-1/2}.$$

This shows that $\pi_{L,M}(A) = L^M b$ is, up to a multiplicative constant, a (M,ε,L)-molecule. □

After this preparatory lemma, we are ready for the proof of Claim 2, i.e. for establishing molecular decompositions of functions in the space $H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$. We mainly follow the outline of [DL10, Proposition 3.20].

Lemma 4.16. *Suppose that the operator L and the function ψ_0 are as in Theorem 4.10. Let $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ and $\varepsilon \in (0, mM - D/2]$. For every $f \in H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$ there exist a sequence of complex numbers $(\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1$ and a family of (M,ε,L)-molecules $(m_j)_{j\in\mathbb{N}_0}$ such that f can be decomposed in the form $f = \sum_{j=0}^{\infty} \lambda_j m_j$, with the sum converging in $L^2(X)$, and*

$$\|f\|_{H^1_{L,mol,M,\varepsilon}(X)} \le C_1 \sum_{j=0}^{\infty} |\lambda_j| \le C_2 \|f\|_{H^1_{L,S_{\psi_0}}(X)},$$

where the constants $C_1, C_2 > 0$ are independent of f. In particular, it holds

$$H^1_{L,S_{\psi_0}}(X) \cap L^2(X) \subseteq \mathbb{H}^1_{L,mol,M,\varepsilon}(X).$$

Proof. Let $f \in H^1_{L,S_{\psi_0}}(X) \cap L^2(X)$. Fix $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ and $\varepsilon \in (0, mM - D/2]$. Define for any $t > 0$, $x \in X$

$$F(x,t) := \psi_0(t^m L) f(x) = t^m L e^{-t^m L} f(x).$$

In view of the definition of $H^1_{L,S_{\psi_0}}(X)$ and the quadratic estimate (4.1), we deduce that $F \in T^1(X) \cap T^2(X)$. Hence, by Fact 4.7, there are a positive constant C, a sequence $(\lambda_j)_{j\in\mathbb{N}_0} \in \ell^1$ of complex numbers and a sequence $(A_j)_{j\in\mathbb{N}_0}$ of $T^1(X)$-atoms such that

$$F = \sum_{j=0}^{\infty} \lambda_j A_j, \tag{4.12}$$

where the sum converges in both $T^1(X)$ and $T^2(X)$, and

$$\sum_{j=0}^{\infty} |\lambda_j| \le C\, \|F\|_{T^1(X)} = C\, \|f\|_{H^1_{L,S_{\psi_0}}(X)}. \tag{4.13}$$

Further, by the spectral theorem for L, one can write for an appropriate constant C depending only on M

$$f = C \int_0^{\infty} (t^m L e^{-t^m L})^{M+1} f \, \frac{dt}{t} = C \pi_{L,M}(F) = C \sum_{j=0}^{\infty} \lambda_j \pi_{L,M}(A_j), \tag{4.14}$$

where the sum converges in $L^2(X)$ because $\pi_{L,M}$ acts as a bounded operator from $T^2(X)$ to $L^2(X)$ and the sum in (4.12) converges in $T^2(X)$.

Thanks to Lemma 4.15, there exists a constant $C_M > 0$ such that $C_M \pi_{L,M}(A_j)$ is a (M,ε,L)-molecule for every $j \in \mathbb{N}_0$. Consequently, the sum in (4.14) provides a molecular (M,ε,L)-representation of f, so that f belongs to $\mathbb{H}^1_{L,mol,M,\varepsilon}(X)$. Finally, by (4.13), we obtain

$$\|f\|_{H^1_{L,mol,M,\varepsilon}(X)} \lesssim \sum_{j=0}^{\infty} |\lambda_j| \lesssim \|f\|_{H^1_{L,S_{\psi_0}}(X)}.$$

□

In view of Theorem 4.10, the molecular Hardy space $H^1_{L,mol,M,\varepsilon}(X)$ does not depend on the choice of $M > \frac{D}{2m}$, nor on the choice of $\varepsilon \in (0, mM - D/2]$. Consequently, one may write $H^1_{L,mol}(X)$ in place of $H^1_{L,mol,M,\varepsilon}(X)$ whenever $M \in \mathbb{N}$ with $M > \frac{D}{2m}$ and $\varepsilon \in (0, mM - D/2]$. Additionally, the following definition makes sense.

Definition 4.17. Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \geq 2$. Suppose that $M \in \mathbb{N}$ with $M > \frac{D}{2m}$, $\varepsilon \in (0, mM - D/2]$, and $\psi_0(z) := ze^{-z}$. The *Hardy space* $H^1_L(X)$ *associated to* L is said to be the space

$$H^1_L(X) = H^1_{L,S_{\psi_0}}(X) = H^1_{L,mol}(X)\,.$$

4.4 Relationship between Hardy spaces and Lebesgue spaces

One of the most important features of the classical Hardy spaces lies in the fact that they form a complex interpolation scale. Hardy spaces associated to operators defined in terms of square functions also enjoy this property. This can be verified by viewing these spaces in the framework of tent spaces and then by using Fact 4.5 (cp. [HMM10, Lemma 4.24]).

Fact 4.18. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \geq 2$. Let $\psi \in H^\infty_0 \setminus \{0\}$. Suppose that $1 \leq p_0 < p < p_1 \leq 2$ with $1/p = (1-\theta)/p_0 + \theta/p_1$ for some $\theta \in (0,1)$. Then it holds*

$$\left[H^{p_0}_{L,S_\psi}(X), H^{p_1}_{L,S_\psi}(X)\right]_\theta = H^p_{L,S_\psi}(X)\,.$$

Put $\psi_0(z) := ze^{-z}$. Thanks to $H^1_{L,S_{\psi_0}}(X) \subseteq L^1(X)$ and $H^2_{L,S_{\psi_0}}(X) = L^2(X)$, Fact 4.18 yields that $H^p_{L,S_{\psi_0}}(X) \subseteq L^p(X)$ for each $p \in (1,2)$. In this section we study the question under which assumptions on L the reverse inclusion is valid.

This question is settled for the classical Hardy spaces $H^p(\mathbb{R}^D)$. It is well-known that they can be identified with the Lebesgue spaces $L^p(\mathbb{R}^D)$ for any $p \in (1,\infty)$ (see e.g. [Ste70, p. 220]).

However, if L is an injective, non-negative, self-adjoint operator on $L^2(\mathbb{R}^D)$ which satisfies Davies-Gaffney estimates (DG_m) for some $m \geq 2$, then $H^p_{L,S_{\psi_0}}(\mathbb{R}^D)$ may or may not coincide with $L^p(\mathbb{R}^D)$ for $p \in (1,2)$ (see e.g. [HMM10, Proposition 9.1 (v), (vi)], where Riesz transforms were studied). Recently, P. Auscher, X.T. Duong, and A. McIntosh showed in [ADM05, Theorem 6] that the presence of classical Gaussian estimates (1.3) ensures that $H^p_{L,S_\psi}(\mathbb{R}^D) = L^p(\mathbb{R}^D)$ for all $p \in (1,2]$ and all $\psi \in H^\infty_0 \setminus \{0\}$.

We will upgrade their result in two directions. On the one hand, we will only assume generalized Gaussian estimates and, on the other hand, we will study the situation in the more general framework of spaces of homogeneous type. Clearly, in this setting we cannot expect $H^p_{L,S_\psi}(X) = L^p(X)$ for all $p \in (1,2)$. Details are given in the next statement.

Theorem 4.19. *Assume that L is an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills generalized Gaussian estimates ($GGE_{p_0,m}$) for some $p_0 \in [1,2)$ and $m \geq 2$. Then, for each non-trivial $\psi \in H_0^\infty$ and each $p \in (p_0, 2]$, the Hardy space $H^p_{L,S_\psi}(X)$ and the Lebesgue space $L^p(X)$ coincide and their norms are equivalent.*

In particular, under the assumptions of Theorem 4.19, different choices of $\psi \in H_0^\infty \setminus \{0\}$ lead to the same Hardy space $H^p_{L,S_\psi}(X)$ for any $p \in (p_0, 2]$. Therefore, in the next chapters we omit the subscript S_ψ in the notation and write only $H^p_L(X)$ for the Hardy space associated to L defined via square functions.

The rest of this section is devoted to the proof of Theorem 4.19. Fix $\psi \in H_0^\infty \setminus \{0\}$. Since $H^p_{L,S_\psi}(X) \cap L^2(X)$ is dense in $H^p_{L,S_\psi}(X)$ and, of course, $L^p(X) \cap L^2(X)$ is dense in $L^p(X)$, it suffices to prove that $H^p_{L,S_\psi}(X) \cap L^2(X) = L^p(X) \cap L^2(X)$ with equivalent norms. To this end, we shall establish that for every $p \in (p_0, 2]$ there exist constants $C_1, C_2 > 0$ such that for any $f \in L^p(X) \cap L^2(X)$

$$C_1 \|f\|_p \leq \|S_\psi f\|_p \leq C_2 \|f\|_p \,. \tag{4.15}$$

We divide the proof of (4.15) into three steps. In a first step, and this will be the main work, we verify that $\|S_\psi f\|_p \lesssim \|f\|_p$ for all $f \in L^p(X) \cap L^2(X)$ and all $p \in (p_0, 2]$. In a second step we show that this estimate is actually valid for any $p \in (2, p_0')$. In the final step three we deduce the reverse inequality $\|f\|_p \lesssim \|S_\psi f\|_p$ for all $f \in L^p(X) \cap L^2(X)$ and all $p \in (p_0, 2]$ by a dualization argument based on the bound obtained in the second step.

Let us turn to the proof of step 1. Our idea consists in establishing a weak type (p_0, p_0)-estimate for S_ψ. This, in combination with the boundedness of S_ψ on $L^2(X)$, gives then the claimed estimates by applying the Marcinkiewicz interpolation theorem. Unfortunately, technical difficulties arise with the handling of the operator S_ψ which are caused to the definition of S_ψ via an area integral. Therefore, we make a detour and study the properties of what may be called *Littlewood-Paley-Stein $g^*_{\lambda,\psi}$-function adapted to L*

$$g^*_{\lambda,\psi}(f)(x) := \left(\int_0^\infty \int_X \left(\frac{s^{1/m}}{d(x,y) + s^{1/m}} \right)^{D\lambda} \left| \psi(sL) f(y) \right|^2 \frac{d\mu(y)}{|B(x, s^{1/m})|} \frac{ds}{s} \right)^{1/2}$$

for $\lambda > 0$, $x \in X$, and $f \in L^2(X)$. It turns out that $g^*_{\lambda,\psi}$ is better suited than S_ψ as far as Fubini arguments are concerned because it contains an integral over the full space. Nevertheless, thanks to the additional weight factor, the Littlewood-Paley-Stein $g^*_{\lambda,\psi}$-function behaves similar to the square function S_ψ. Due to $\frac{s^{1/m}}{d(x,y)+s^{1/m}} \in [1/2, 1]$ for all $y \in B(x, s^{1/m})$, $x \in X$, and $s > 0$, one sees after the substitution $s = t^{1/m}$ that $g^*_{\lambda,\psi}$ controls S_ψ for any $\lambda > 1$. Hence, the assertion of step 1 is an immediate consequence of the following statement.

Lemma 4.20. *Let L be as in Theorem 4.19 and $p \in (p_0, 2]$. For all $\lambda > 1$ and $\psi \in H_0^\infty \setminus \{0\}$, the Littlewood-Paley-Stein $g^*_{\lambda,\psi}$-function adapted to L is of strong type (p,p).*

Proof. Our argument mimics that of P. Auscher who sketched in [Aus07, Proposition 6.8] a proof for a corresponding assertion in the special case of second order divergence form operators on the Euclidean space $\mathbb{R}^D$.

Let $\psi \in H_0^\infty \setminus \{0\}$ be fixed, i.e. $\psi \in H_0^\infty(\Sigma_\theta) \setminus \{0\}$ for some $\theta \in (0, \pi/2)$. The keystone of our proof is the following identity that is obtained by applying Fubini's theorem

$$\int_F g^*_{\lambda,\psi}(f)(x)^2 \, dx = \int_0^\infty \int_X J_{\lambda,F}(y,s) \left|\psi(sL)f(y)\right|^2 d\mu(y) \, \frac{ds}{s} \qquad (f \in L^2(X))$$

with

$$J_{\lambda,F}(y,s) := \int_F \left(\frac{s^{1/m}}{d(x,y) + s^{1/m}} \right)^{D\lambda} \frac{1}{|B(x,s^{1/m})|} \, d\mu(x)$$

which holds for any $\lambda > 1$ and any closed set $F \subseteq X$. First, we observe that

$$J_{\lambda,F}(y,s) \le C_\lambda \qquad (y \in X, s > 0) \tag{4.16}$$

with a constant $C_\lambda > 0$ depending only on λ and the dimension D of the space X but not on F, y, s. In order to achieve this estimate, we split the integral over F into integrals over $B(y, s^{1/m})$ and over its complement $F \setminus B(y, s^{1/m})$ and ascertain that both integrals are finite with a bound independent of F, y, s. In fact, due to (2.3), it holds

$$\int_{B(y,s^{1/m})} \frac{1}{(d(x,y)s^{-1/m} + 1)^{D\lambda}} \frac{1}{|B(x,s^{1/m})|} \, d\mu(x) \le \int_{B(y,s^{1/m})} \frac{1}{|B(x,s^{1/m})|} \, d\mu(x) \lesssim 1\,.$$

By using $|B(x, 2^{k+1}s^{1/m})| \lesssim 2^{kD}|B(x, s^{1/m})|$ and (2.3) again, we obtain

$$\begin{aligned}
&\int_{F \setminus B(y,s^{1/m})} \frac{1}{(d(x,y)s^{-1/m} + 1)^{D\lambda}} \frac{1}{|B(x,s^{1/m})|} \, d\mu(x) \\
&\quad \le \sum_{k=0}^\infty \int_{B(y,2^{k+1}s^{1/m}) \setminus B(y,2^k s^{1/m})} \frac{1}{(d(x,y)s^{-1/m} + 1)^{D\lambda}} \frac{1}{|B(x,s^{1/m})|} \, d\mu(x) \\
&\quad \lesssim \sum_{k=0}^\infty \frac{1}{(2^k + 1)^{D\lambda}} \int_{B(y,2^{k+1}s^{1/m})} \frac{2^{kD}}{|B(x,2^{k+1}s^{1/m})|} \, d\mu(x) \lesssim \sum_{k=0}^\infty 2^{kD(1-\lambda)}
\end{aligned}$$

which is finite for any $\lambda > 1$.

In view of (4.16), for any $f \in L^2(X)$ the $L^2(X)$-norm of $g^*_{\lambda,\psi}(f)$ is majorized by a square function which can be estimated on $L^2(X)$ in the same manner as S_ψ (cf. proof of the boundedness of $S_\psi f$ on $L^2(X)$, p. 45)

$$\|g^*_{\lambda,\psi}(f)\|_2^2 \le C_\lambda \int_0^\infty \int_X \left|\psi(sL)f(y)\right|^2 d\mu(y) \, \frac{ds}{s} \lesssim \|f\|_2^2\,,$$

i.e. $g^*_{\lambda,\psi}$ is of strong type $(2,2)$. Thus, we only have to show that $g^*_{\lambda,\psi}$ is of weak type (p_0,p_0), the claimed strong type estimates can then be obtained with the Marcinkiewicz interpolation theorem.

Now let $\lambda > 1$. We shall prove that there exists a constant $C > 0$ such that for any $\alpha > 0$ and any $f \in L^{p_0}(X) \cap L^2(X)$

$$\mu\big(\{x \in X \,:\, g^*_{\lambda,\psi}(f)(x) > \alpha\}\big) \le \frac{C}{\alpha^{p_0}} \int_X |f|^{p_0}\, d\mu\,.$$

At the beginning, we recall the Calderón-Zygmund decomposition in $L^{p_0}(X)$ at height α. Loosely speaking, it says that an element of $L^{p_0}(X)$ can be written as the sum of "good" and "bad" functions (see e.g. [BK03, Theorem 3.1 and Remark 3.2]).

Fact 4.21. *There are constants $C > 0$ and $M \in \mathbb{N}$ depending only on p_0 and the dimension D of the space X such that for every $f \in L^{p_0}(X)$ and every $\alpha > 0$ we find a function g and a collection of balls (B_j) in X and functions (b_j) with disjoint supports such that*

$$f = g + \sum_j b_j$$

and the following properties hold

i) $\|g\|_\infty \le C\alpha\,;$

ii) $\operatorname{supp} b_j \subseteq B_j$ *and* $\int_{B_j} |b_j|^{p_0}\, d\mu \le C\alpha^{p_0}|B_j|\,;$

iii) $\sum_j |B_j| \le C\alpha^{-p_0} \int_X |f|^{p_0}\, d\mu\,;$

iv) $\sum_j \mathbb{1}_{B_j} \le M\,;$

v) $\|g\|_{p_0} \le C\,\|f\|_{p_0}\,.$

The items i) and v) immediately imply that $g \in L^2(X)$ with

$$\int_X |g|^2\, d\mu \le \|g\|_\infty^{2-p_0} \int_X |g|^{p_0}\, d\mu \lesssim \alpha^{2-p_0} \int_X |f|^{p_0}\, d\mu\,. \tag{4.17}$$

Now fix $\alpha > 0$ and $f \in L^{p_0}(X) \cap L^2(X)$. Choose a Calderón-Zygmund decomposition of f in $L^{p_0}(X)$ at height α according to Fact 4.21 and write $f = g + \sum_j b_j$.

As remarked before, $g^*_{\lambda,\psi}$ is bounded on $L^2(X)$. This, in combination with Chebyshev's inequality and (4.17), leads to

$$\mu\big(\{x \in X \,:\, g^*_{\lambda,\psi}(g)(x) > \alpha/3\}\big) \lesssim \frac{1}{\alpha^2} \int_X |g|^2\, d\mu \lesssim \frac{1}{\alpha^{p_0}} \int_X |f|^{p_0}\, d\mu\,.$$

In order to handle the remaining term, fix $N \in \mathbb{N}$ to be chosen later and introduce for $r \geq 0$ the regularization operator $A_r := I - (I - e^{-r^m L})^N$. Denote by r_j the radius of the ball B_j and by x_j its center, i.e. $B_j = B(x_j, r_j)$. Due to

$$g^*_{\lambda,\psi}\Big(\sum_j b_j\Big)(x) \leq g^*_{\lambda,\psi}\Big(\sum_j A_{r_j} b_j\Big)(x) + g^*_{\lambda,\psi}\Big(\sum_j (I - A_{r_j}) b_j\Big)(x)$$

for any $x \in X$, it is enough to estimate the volumes

$$\begin{aligned} A &:= \mu\Big(\Big\{x \in X \,:\, g^*_{\lambda,\psi}\Big(\sum_j A_{r_j} b_j\Big)(x) > \alpha/3\Big\}\Big)\,, \\ B &:= \mu\Big(\Big\{x \in X \,:\, g^*_{\lambda,\psi}\Big(\sum_j (I - A_{r_j}) b_j\Big)(x) > \alpha/3\Big\}\Big)\,. \end{aligned}$$

To establish a bound for A, we use again the $L^2(X)$-boundedness of $g^*_{\lambda,\psi}$ together with Chebyshev's inequality

$$A \lesssim \frac{1}{\alpha^2} \int_X \Big| g^*_{\lambda,\psi}\Big(\sum_j A_{r_j} b_j\Big)\Big|^2 \, d\mu \lesssim \frac{1}{\alpha^2} \int_X \Big|\sum_j A_{r_j} b_j\Big|^2 \, d\mu\,. \tag{4.18}$$

For estimating the $L^2(X)$-norm of $\sum_j A_{r_j} b_j$, we dualize against $\phi \in L^2(X)$ with $\|\phi\|_2 = 1$

$$\int_X \Big|\phi \sum_j A_{r_j} b_j\Big| \, d\mu \leq \int_X |\phi| \sum_j |A_{r_j} b_j| \, d\mu = \sum_{k=0}^{\infty} \sum_j C_{jk}\,,$$

where

$$C_{jk} := \int_{A(x_j, r_j, k)} |\phi| \, |A_{r_j} b_j| \, d\mu\,. \tag{4.19}$$

Recall that $A(x_j, r_j, k) = (k+1)B_j \setminus kB_j$. Due to the assumptions on L and the representation $A_r = I - (I - e^{-r^m L})^N = \sum_{\nu=1}^N c_{\nu,N} e^{-\nu r^m L}$ for some appropriate constants $c_{\nu,N}$, we have for any j and any $x, y \in X$

$$\begin{aligned} \big\|\mathbb{1}_{B(x,r_j)} A_{r_j} \mathbb{1}_{B(y,r_j)}\big\|_{p_0 \to p_0'} &\leq \sum_{\nu=1}^N |c_{\nu,N}| \, \big\|\mathbb{1}_{B(x,\nu^{1/m} r_j)} e^{-\nu r_j^m L} \mathbb{1}_{B(y,\nu^{1/m} r_j)}\big\|_{p_0 \to p_0'} \\ &\lesssim \sum_{\nu=1}^N |c_{\nu,N}| \, |B(x, \nu^{1/m} r_j)|^{-(\frac{1}{p_0} - \frac{1}{p_0'})} \exp\bigg(-b\Big(\frac{d(x,y)}{\nu^{1/m} r_j}\Big)^{\frac{m}{m-1}}\bigg) \\ &\leq \sum_{\nu=1}^N |c_{\nu,N}| \, |B(x, r_j)|^{-(\frac{1}{p_0} - \frac{1}{p_0'})} \exp\bigg(-b'\Big(\frac{d(x,y)}{r_j}\Big)^{\frac{m}{m-1}}\bigg) \\ &\lesssim |B(x, r_j)|^{-(\frac{1}{p_0} - \frac{1}{p_0'})} \exp\bigg(-b'\Big(\frac{d(x,y)}{r_j}\Big)^{\frac{m}{m-1}}\bigg) \end{aligned}$$

for some constant $b' > 0$ depending on N. Hence, Fact 2.4 gives for all j and all $k \in \mathbb{N}_0$

$$\begin{aligned}\left\|\mathbb{1}_{A(x_j,r_j,k)} A_{r_j} \mathbb{1}_{B(x_j,r_j)}\right\|_{p_0\to 2} &= \left\|\mathbb{1}_{B(x_j,r_j)} A_{r_j} \mathbb{1}_{A(x_j,r_j,k)}\right\|_{2\to p_0'} \\ &\lesssim |B(x_j,r_j)|^{-(\frac{1}{2}-\frac{1}{p_0'})} e^{-bk^{\frac{m}{m-1}}} = |B(x_j,r_j)|^{-(\frac{1}{p_0}-\frac{1}{2})} e^{-bk^{\frac{m}{m-1}}} . \end{aligned} \tag{4.20}$$

This, in combination with property ii) of the Calderón-Zygmund decomposition, yields

$$\begin{aligned}\|A_{r_j} b_j\|_{L^2(A(x_j,r_j,k))} &\le \left\|\mathbb{1}_{A(x_j,r_j,k)} A_{r_j} \mathbb{1}_{B(x_j,r_j)}\right\|_{p_0\to 2} \|b_j\|_{p_0} \\ &\lesssim |B(x_j,r_j)|^{-(\frac{1}{p_0}-\frac{1}{2})} e^{-bk^{\frac{m}{m-1}}} \alpha |B_j|^{1/p_0} = |B_j|^{1/2} e^{-bk^{\frac{m}{m-1}}} \alpha .\end{aligned}$$

Recall the definition of the *uncentered Hardy-Littlewood maximal operator* $\mathcal{M}$. For a locally integrable function $h\colon X \to \mathbb{C}$, $\mathcal{M}$ is defined by

$$\mathcal{M}h(x) := \sup_{B\ni x} \frac{1}{|B|} \int_B |h| \, d\mu \qquad (x \in X),$$

where the supremum is taken over all balls $B \subseteq X$ containing x. For any $y_j \in B_j$ and any $k \in \mathbb{N}_0$, we record

$$\int_{(k+1)B_j} |\phi|^2 \, d\mu \le \int_{2(k+1)B(y_j,r_j)} |\phi|^2 \, d\mu \le \left|2(k+1)B(y_j,r_j)\right| \mathcal{M}(|\phi|^2)(y_j)$$

and deduce with the help of Fact 2.1

$$\Big(\int_{A(x_j,r_j,k)} |\phi|^2 \, d\mu\Big)^{1/2} \le \Big(\int_{(k+1)B_j} |\phi|^2 \, d\mu\Big)^{1/2} \lesssim \left|(k+1)B_j\right|^{1/2} \big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2} .$$

Applying the Cauchy-Schwarz inequality in (4.19) and using the estimates above lead to

$$\begin{aligned} C_{jk} &\lesssim \left|(k+1)B_j\right|^{1/2} \big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2} |B_j|^{1/2} e^{-bk^{\frac{m}{m-1}}} \alpha \\ &\lesssim \alpha (k+1)^{D/2} e^{-bk^{\frac{m}{m-1}}} |B_j| \big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2} .\end{aligned}$$

Averaging over B_j yields

$$C_{jk} = \frac{1}{|B_j|} \int_{B_j} C_{jk} \, d\mu(y_j) \lesssim \alpha (k+1)^{D/2} e^{-bk^{\frac{m}{m-1}}} \int_{B_j} \big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2} \, d\mu(y_j) .$$

Keeping in mind the finite intersection property iv) of the Calderón-Zygmund decomposition, we sum over j and k

$$\begin{aligned} \int_X |\phi| \sum_j |A_{r_j} b_j| \, d\mu = \sum_{k=0}^{\infty} \sum_j C_{jk} &\lesssim \alpha \int_X \sum_j \mathbb{1}_{B_j}(y_j) \big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2} \, d\mu(y_j) \\ &\lesssim \alpha \int_{\bigcup_j B_j} \big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2} \, d\mu(y_j) .\end{aligned}$$

By exploiting a lemma due to A.N. Kolmogorov which reads

$$\int_E h^{1/2}\, d\mu \le 2|E|^{1/2}\Big(\sup_{t>0} t\,|\{x\in X\,:\, h(x)>t\}|\Big)^{1/2} \tag{4.21}$$

for each measurable function $h\colon E \to [0,\infty)$ and each Borel set $E \subseteq X$ (see e.g. [GR85, Lemma 2.8, p. 485]) and the weak type $(1,1)$ boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ (see e.g. [CW71, Théorème 2.1, pp. 71-72]), we obtain

$$\int_X |\phi| \sum_j |A_{r_j} b_j|\, d\mu \lesssim \alpha \Big|\bigcup_j B_j\Big|^{1/2} \||\phi|^2\|_1^{1/2} \lesssim \alpha\Big(\alpha^{-p_0}\int_X |f|^{p_0}\, d\mu\Big)^{1/2} \|\phi\|_2\,,$$

where the last inequality is due to iii) of Fact 4.21. Hence, together with (4.18), we conclude the desired bound for A

$$A \lesssim \frac{1}{\alpha^2}\int_X \Big|\sum_j A_{r_j} b_j\Big|^2 d\mu \lesssim \frac{1}{\alpha^{p_0}}\int_X |f|^{p_0}\, d\mu\,.$$

It remains to estimate B, i.e. the volume of

$$\Big\{x\in X\,:\, g^*_{\lambda,\psi}\Big(\sum_j (I-A_{r_j})b_j\Big)(x) > \alpha/3\Big\}\,.$$

Clearly, this set is contained in

$$\bigcup_j B_j \cup \Big\{x\in X\setminus \bigcup_j B_j\,:\, g^*_{\lambda,\psi}\Big(\sum_j (I-A_{r_j})b_j\Big)(x) > \alpha/3\Big\}\,.$$

The mass of $\bigcup_j B_j$ is under control with the bound we need. Indeed, by iii) of Fact 4.21

$$\Big|\bigcup_j B_j\Big| \le \sum_j |B_j| \lesssim \frac{1}{\alpha^{p_0}}\int_X |f|^{p_0}\, d\mu\,. \tag{4.22}$$

To handle the second term, we set $F := X\setminus\bigcup_j B_j$ and obtain, by Chebyshev's inequality,

$$\begin{aligned}\Big|\Big\{x\in F\,:\, g^*_{\lambda,\psi}\Big(\sum_j (I-A_{r_j})b_j\Big)(x) > \alpha/3\Big\}\Big| &\le \frac{9}{\alpha^2}\int_F g^*_{\lambda,\psi}\Big(\sum_j (I-A_{r_j})b_j\Big)(x)^2\, d\mu(x)\\ = \frac{9}{\alpha^2}\int_0^\infty\int_X J_{\lambda,F}(y,s)\,\Big|\sum_j \big(\psi(sL)(I-A_{r_j})b_j\big)(y)\Big|^2 d\mu(y)\,\frac{ds}{s}\,.\end{aligned}$$

In virtue of (4.16), one can bound $J_{\lambda,F}$ by a constant depending only on λ and D. We split the remaining integral into its local and non-local part

$$\begin{aligned}
&\int_0^\infty \int_X \Big|\sum_j \big(\psi(sL)(I-A_{r_j})b_j\big)(y)\Big|^2 \, d\mu(y)\, \frac{ds}{s} \\
&= \int_0^\infty \int_X \Big|\sum_j \big(\mathbb{1}_{B_j}(y) + (1-\mathbb{1}_{B_j}(y))\big)\, \big(\psi(sL)(I-A_{r_j})b_j\big)(y)\Big|^2 \, d\mu(y)\, \frac{ds}{s} \\
&\leq 2\int_0^\infty \int_X \Big|\sum_j \mathbb{1}_{B_j}(y)\, \big(\psi(sL)(I-A_{r_j})b_j\big)(y)\Big|^2 \, d\mu(y)\, \frac{ds}{s} \\
&\quad + 2\int_0^\infty \int_X \Big|\sum_j (1-\mathbb{1}_{B_j}(y))\, \big(\psi(sL)(I-A_{r_j})b_j\big)(y)\Big|^2 \, d\mu(y)\, \frac{ds}{s}\,. \qquad (4.23)
\end{aligned}$$

Now consider the term in the next to the last line. The finite intersection property iv) of Fact 4.21 ensures that for each $y \in X$ the sum over j has at most M non-zero terms. Therefore, the elementary inequality $(\sum_{\nu=1}^M a_\nu)^2 \leq M \sum_{\nu=1}^M a_\nu^2$, which is valid for arbitrary real numbers a_ν, leads to the upper bound

$$\begin{aligned}
&\int_0^\infty \int_X \Big|\sum_j \mathbb{1}_{B_j}(y)\, \big(\psi(sL)(I-A_{r_j})b_j\big)(y)\Big|^2 \, d\mu(y)\, \frac{ds}{s} \\
&\quad \leq M \sum_j \int_0^\infty \int_{B_j} \big|\big(\psi(sL)(I-A_{r_j})b_j\big)(y)\big|^2 \, d\mu(y)\, \frac{ds}{s}\,. \qquad (4.24)
\end{aligned}$$

We investigate each term separately. By integrating over the full space and by applying the quadratic estimate (4.1) in combination with $\operatorname{supp} b_j \subseteq B(x_j, r_j)$ and (4.20), we get for the j-th summand the bound

$$\begin{aligned}
&\int_0^\infty \int_X \big|\big(\psi(sL)(I-A_{r_j})b_j\big)(y)\big|^2 \, d\mu(y)\, \frac{ds}{s} \lesssim \big\|(I-A_{r_j})b_j\big\|_2^2 \\
&\leq \sum_{k=0}^\infty \big\|\mathbb{1}_{A(x_j,r_j,k)}(I-A_{r_j})\,\mathbb{1}_{B(x_j,r_j)}\big\|_{p_0\to 2}^2 \|b_j\|_{p_0}^2 \\
&\lesssim \sum_{k=0}^\infty \Big(|B(x_j,r_j)|^{-(\frac{1}{p_0}-\frac{1}{2})} e^{-bk^{\frac{m}{m-1}}}\Big)^2 \|b_j\|_{p_0}^2 \\
&\lesssim |B(x_j,r_j)|^{-(\frac{2}{p_0}-1)} \|b_j\|_{p_0}^2\,.
\end{aligned}$$

Due to the properties ii) and iii) of the Calderón-Zygmund decomposition, we then have, up to some multiplicative constant, the following bound of (4.24)

$$\sum_j |B(x_j,r_j)|^{-(\frac{2}{p_0}-1)} \|b_j\|_{p_0}^2 \lesssim \alpha^2 \sum_j |B(x_j,r_j)| \lesssim \alpha^{2-p_0} \int_X |f|^{p_0}\, d\mu\,. \qquad (4.25)$$

Now let us consider the non-local part of (4.23), i.e.

$$\int_0^\infty \int_X \Big| \sum_j \mathbb{1}_{B_j^c}(y) \big(\psi(sL)(I-A_{r_j})b_j\big)(y)\Big|^2 \, d\mu(y) \, \frac{ds}{s} = \int_0^\infty \Big\| \sum_j G_{j,s} b_j \Big\|_2^2 \frac{ds}{s} \,,$$

where

$$G_{j,s} := \mathbb{1}_{B_j^c} \cdot \psi(sL)(I-A_{r_j})\, \mathbb{1}_{B_j} \,.$$

In contrast to the local part, the change of integration and summation seems not to be fruitful for getting an appropriate bound. Nevertheless, we are able to argue similar as in [Yan02], where second order divergence form operators on $\mathbb{R}^D$ were studied. We shall establish

$$\int_0^\infty \Big\| \sum_j G_{j,s} b_j \Big\|_2^2 \frac{ds}{s} \lesssim \alpha^2 \Big| \bigcup_j B_j \Big| \,.$$

For estimating the $L^2(X)$-norm of the sum, we employ a dualization argument. Take $\phi \in L^2(X)$ and investigate

$$\Big| \Big\langle \phi, \sum_j G_{j,s} b_j \Big\rangle \Big| \leq \sum_j \int_{B_j^c} |\phi| \, |\psi(sL)(I-A_{r_j})b_j| \, d\mu \,.$$

Splitting B_j^c into annuli and applying the Cauchy-Schwarz inequality yield

$$\begin{aligned} &\sum_j \sum_{k=1}^\infty \int_{A(x_j,r_j,k)} |\phi| \, |\psi(sL)(I-A_{r_j})b_j| \, d\mu \\ &\leq \sum_j \sum_{k=1}^\infty \Big(\int_{A(x_j,r_j,k)} |\phi|^2 \, d\mu \Big)^{1/2} \Big(\int_{A(x_j,r_j,k)} |\psi(sL)(I-A_{r_j})b_j|^2 \, d\mu \Big)^{1/2} . \end{aligned}$$

The first factor can be treated as before

$$\Big(\int_{A(x_j,r_j,k)} |\phi|^2 \, d\mu \Big)^{1/2} \lesssim (k+1)^{D/2} |B(x_j,r_j)|^{1/2} \big(\mathcal{M}(|\phi|^2)(y_j) \big)^{1/2}$$

for any $y_j \in B(x_j,r_j)$ and any $k \in \mathbb{N}$. Thanks to $\operatorname{supp} b_j \subseteq B(x_j,r_j)$, the second factor can be estimated by

$$\big\| \mathbb{1}_{A(x_j,r_j,k)} \psi(sL)(I-A_{r_j})\, \mathbb{1}_{B(x_j,r_j)} \big\|_{p_0 \to 2} \|b_j\|_{p_0} \,.$$

Hence, we have the bound

$$\begin{aligned} \int_0^\infty \Big\| \sum_j G_{j,s} b_j \Big\|_2^2 \frac{ds}{s} &= \int_0^\infty \Big(\sup_{\|\phi\|_2 \leq 1} \Big| \Big\langle \phi, \sum_j G_{j,s} b_j \Big\rangle \Big| \Big)^2 \frac{ds}{s} \\ &\lesssim \int_0^\infty \Big(\sup_{\|\phi\|_2 \leq 1} \sum_j \sum_{k=1}^\infty (k+1)^{D/2} |B(x_j,r_j)|^{1/2} \big(\mathcal{M}(|\phi|^2)(y_j) \big)^{1/2} \times \\ &\quad \times \big\| \mathbb{1}_{A(x_j,r_j,k)} \psi(sL)(I-A_{r_j})\, \mathbb{1}_{B(x_j,r_j)} \big\|_{p_0 \to 2} \|b_j\|_{p_0} \Big)^2 \frac{ds}{s} \,. \end{aligned} \tag{4.26}$$

Now we shall estimate

$$\left\| \mathbb{1}_{A(x_j,r_j,k)} \psi(sL)(I-A_{r_j})\, \mathbb{1}_{B(x_j,r_j)} \right\|_{p_0\to 2} .$$

Clearly, the bound relies on the generalized Gaussian estimates for L. We will use similar arguments as in [Kun08, proof of Theorem 5.2]. Fix $j,k \in \mathbb{N}$, and $s>0$. At first, we provide a representation of the operator $\psi(sL)(I-A_{r_j})$ in terms of the semigroup $(e^{-zL})_{\operatorname{Re} z>0}$. We put $\varphi(z) := \sum_{\nu=0}^{N} \binom{N}{\nu}(-1)^{\nu} e^{-\nu z}$ for each $z\in\mathbb{C}$ with $\operatorname{Re} z>0$ and notice that $|\varphi(z)| = O(|z|^N)$ for $|z|\to 0$ in fixed sectors of half opening angle $<\pi/2$ and that $|\varphi(z)| = O(1)$ for $|z|\to\infty$. Therefore, it holds for every $\nu\in(0,\pi/2)$

$$|\varphi(z)| \lesssim \min\{|z|^N, 1\} \qquad (z\in\Sigma_\nu), \tag{4.27}$$

where the implicit constant depends only on ν. Take $\theta'\in(0,\theta)$. Due to $I-A_{r_j} = \varphi(r_j^m L)$, we can write with the help of the H_0^∞ calculus

$$\psi(sL)(I-A_{r_j})f = \frac{1}{2\pi i}\int_{\partial\Sigma_{\theta'}} \psi(s\zeta)\varphi(r_j^m\zeta)R(\zeta,L)f\, d\zeta \qquad (f\in L^2(X)),$$

where the unbounded contour $\partial\Sigma_{\theta'}$ is parameterized counterclockwise. We put $\sigma := \frac{\pi-\theta'}{2}$ and represent the resolvent $R(\zeta,L)$ as follows

$$R(\zeta,L)f = -\int_{\Gamma_{\pm\sigma}} e^{\zeta z} e^{-zL} f\, dz \qquad (\zeta\in\Gamma_{\pm\theta'}, f\in L^2(X)),$$

where Γ_ν denotes the half-ray $(0,\infty)e^{i\nu}$ for $\nu\in(-\pi,\pi)$. We thus have for any $f\in L^2(X)$

$$\psi(sL)(I-A_{r_j})f = \frac{1}{2\pi i}\left(\int_{\Gamma_\sigma}\int_{\Gamma_{\theta'}} + \int_{\Gamma_{-\sigma}}\int_{\Gamma_{-\theta'}}\right) \psi(s\zeta)\varphi(r_j^m\zeta)e^{\zeta z}\, d\zeta\; e^{-zL} f\, dz .$$

Therefore, we get

$$\begin{aligned} &\left\| \mathbb{1}_{A(x_j,r_j,k)} \psi(sL)(I-A_{r_j})\, \mathbb{1}_{B(x_j,r_j)} \right\|_{p_0\to 2} \\ &\le \left(\int_{\Gamma_\sigma}\int_{\Gamma_{\theta'}} + \int_{\Gamma_{-\sigma}}\int_{\Gamma_{-\theta'}}\right) |\psi(s\zeta)\varphi(r_j^m\zeta)e^{\zeta z}|\, |d\zeta| \left\| \mathbb{1}_{A(x_j,r_j,k)} e^{-zL} \mathbb{1}_{B(x_j,r_j)} \right\|_{p_0\to 2} |dz| . \end{aligned}$$

For all $\zeta\in\Gamma_{j\theta'}$ and $z\in\Gamma_{j\sigma}$, $j\in\{-1,1\}$, it holds

$$\operatorname{Re}(\zeta z) = |\zeta||z|\cos(j(\theta'+\sigma)) = -c|\zeta||z| ,$$

where $c := -\cos(\theta'+\sigma) > 0$ since $\theta'+\sigma\in(\pi/2,\pi)$. Due to $\psi\in H_0^\infty(\Sigma_\theta)$, we find constants $C,\beta>0$ such that $|\psi(s\zeta)| \le C\frac{|s\zeta|^\beta}{1+|s\zeta|^{2\beta}}$ for each $\zeta\in\Gamma_{\pm\theta'}$. This, in combination with (4.27), leads to

$$\begin{aligned} &\left\| \mathbb{1}_{A(x_j,r_j,k)} \psi(sL)(I-A_{r_j})\, \mathbb{1}_{B(x_j,r_j)} \right\|_{p_0\to 2} \\ &\lesssim \left(\int_{\Gamma_\sigma}\int_{\Gamma_{\theta'}} + \int_{\Gamma_{-\sigma}}\int_{\Gamma_{-\theta'}}\right) \frac{|s\zeta|^\beta}{1+|s\zeta|^{2\beta}} \min\{|r_j^m\zeta|^N, 1\}\, e^{-c|\zeta||z|}\, |d\zeta| \times \\ &\qquad \times \left\| \mathbb{1}_{A(x_j,r_j,k)} e^{-zL} \mathbb{1}_{B(x_j,r_j)} \right\|_{p_0\to 2} |dz| . \end{aligned}$$

To abbreviate, we set $g(\lambda) := \exp(-b\lambda^{m/(m-1)})$, $\lambda > 0$, with a constant $b > 0$ that may change from one appearance of the function g to the next without mentioning it. By applying Lemma 2.6, by recalling the definition of the contour integral and by writing $u = |\zeta|$ and $v = |z|$, we hence obtain

$$\begin{aligned}
&\left\| \mathbb{1}_{A(x_j,r_j,k)} \psi(sL)(I - A_{r_j}) \mathbb{1}_{B(x_j,r_j)} \right\|_{p_0 \to 2} \\
&\lesssim |B(x_j,r_j)|^{-(\frac{1}{p_0}-\frac{1}{2})} \int_0^\infty \int_0^\infty \frac{(su)^\beta}{1+(su)^{2\beta}} \min\{(r_j^m u)^N, 1\}\, e^{-cuv}\, du \times \\
&\quad \times\ (\cos\sigma)^{D(\frac{1}{p_0}-\frac{1}{2})} \left(1 + \frac{r_j}{v^{1/m}(\cos\sigma)^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})} k^D g\left(\frac{r_j}{v^{1/m}}\, k\right) dv\,.
\end{aligned}$$

After estimating $\cos\sigma$ by a constant depending only on θ', we insert this upper bound into (4.26) and utilize $\|b_j\|_{p_0} \lesssim \alpha |B(x_j,r_j)|^{1/p_0}$ (cf. Fact 4.21 ii)) for getting

$$\begin{aligned}
\int_0^\infty \Big\| \sum_j G_{j,s} b_j \Big\|_2^2 \frac{ds}{s} &\lesssim \int_0^\infty \Bigg(\sup_{\|\phi\|_2 \le 1} \sum_j \sum_{k=1}^\infty (k+1)^{D/2} |B(x_j,r_j)|^{1/2} \big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2} \times \\
&\times\ |B(x_j,r_j)|^{-(\frac{1}{p_0}-\frac{1}{2})} \int_0^\infty \int_0^\infty \frac{(su)^\beta}{1+(su)^{2\beta}} \min\{(r_j^m u)^N, 1\}\, e^{-cuv}\, du \times \\
&\times\ \left(1 + \frac{r_j}{v^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})} k^D g\left(\frac{r_j}{v^{1/m}}\, k\right) dv\ \ \alpha |B(x_j,r_j)|^{1/p_0} \Bigg)^2 \frac{ds}{s}\,.
\end{aligned}$$

In order to treat the integral with respect to s, we apply Minkowski's integral inequality and conclude an upper bound

$$\begin{aligned}
&\alpha^2 \int_0^\infty \Bigg(\sup_{\|\phi\|_2 \le 1} \sum_j \sum_{k=1}^\infty k^{3D/2} |B(x_j,r_j)| \left(\mathcal{M}(|\phi|^2)(y_j)\right)^{1/2} \int_0^\infty \int_0^\infty \frac{(su)^\beta}{1+(su)^{2\beta}} \times \\
&\quad \times\ \min\{(r_j^m u)^N, 1\}\, e^{-cuv}\, du\ \left(1 + \frac{r_j}{v^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})} g\left(\frac{r_j}{v^{1/m}}\, k\right) dv \Bigg)^2 \frac{ds}{s} \\
&\le \alpha^2 \Bigg(\int_0^\infty \int_0^\infty \Bigg(\int_0^\infty \Bigg(\sup_{\|\phi\|_2 \le 1} \sum_j \sum_{k=1}^\infty k^{3D/2} |B(x_j,r_j)| \left(\mathcal{M}(|\phi|^2)(y_j)\right)^{1/2} \frac{(su)^\beta}{1+(su)^{2\beta}} \times \\
&\quad \times\ \min\{(r_j^m u)^N, 1\}\, e^{-cuv} \left(1 + \frac{r_j}{v^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})} g\left(\frac{r_j}{v^{1/m}}\, k\right) \Bigg)^2 \frac{ds}{s} \Bigg)^{1/2} du\, dv \Bigg)^2.
\end{aligned}$$

By substituting $s = \tau/u$, one easily sees that $\int_0^\infty (\frac{(su)^\beta}{1+(su)^{2\beta}})^2 \frac{ds}{s} = \int_0^\infty (\frac{\tau^\beta}{1+\tau^{2\beta}})^2 \frac{d\tau}{\tau} \le \frac{1}{\beta}$.

Consequently, one has, up to a multiplicative constant, the bound

$$\alpha^2\Bigg(\sup_{\|\phi\|_2\le 1}\sum_j\sum_{k=1}^{\infty}k^{3D/2}|B(x_j,r_j)|\,\big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2}\times \\ \times\int_0^\infty\!\!\int_0^\infty \min\{(r_j^m u)^N,1\}\,e^{-cuv}\,du\,\left(1+\frac{r_j}{v^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})}g\left(\frac{r_j}{v^{1/m}}\,k\right)dv\Bigg)^2.$$

After the substitution $v=k^m r_j^m\eta$, the double integral in the last line is equal to

$$\int_0^\infty\!\!\int_0^\infty \min\{(r_j^m u)^N,1\}\,e^{-cuk^m r_j^m\eta}\,du\,\left(1+\frac{1}{k\eta^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})}g(1/\eta^{1/m})\,k^m r_j^m\,d\eta$$

and the substitution $u=\xi/(k^m r_j^m\eta)$ yields

$$\int_0^\infty\!\!\int_0^\infty \min\{(\xi/(k^m\eta))^N,1\}\,e^{-c\xi}\,\frac{d\xi}{k^m r_j^m\eta}\left(1+\frac{1}{k\eta^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})}g(1/\eta^{1/m})\,k^m r_j^m\,d\eta \\ \le\int_0^\infty\!\!\int_0^\infty \xi^N e^{-c\xi}\,d\xi\;\eta^{-1-N}\left(1+\frac{1}{\eta^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})}g(1/\eta^{1/m})\,d\eta\;\;k^{-mN} \\ =\int_0^\infty \xi^N e^{-c\xi}\,d\xi\;\int_0^\infty\eta^{-1-N}\left(1+\frac{1}{\eta^{1/m}}\right)^{D(\frac{1}{p_0}-\frac{1}{2})}g(1/\eta^{1/m})\,d\eta\;\;k^{-mN}\,.$$

Independent of the choice of $N\in\mathbb{N}$, both integrals are finite. If $N\in\mathbb{N}$ is taken such that $mN>3D/2+1$, then the series $\sum_{k=1}^\infty k^{3D/2}k^{-mN}$ converges. In summary, we have shown that

$$\int_0^\infty\Big\|\sum_j G_{j,s}b_j\Big\|_2^2\frac{ds}{s}\lesssim\alpha^2\Bigg(\sup_{\|\phi\|_2\le 1}\sum_j|B(x_j,r_j)|\,\big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2}\Bigg)^2. \tag{4.28}$$

The further arguments are the same as before. Averaging over $B(x_j,r_j)$ leads to

$$\sum_j|B(x_j,r_j)|\,\big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2}\le\sum_j\int_{B(x_j,r_j)}\big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2}\,d\mu(y_j) \\ \le M\int_{\bigcup_j B(x_j,r_j)}\big(\mathcal{M}(|\phi|^2)(y_j)\big)^{1/2}\,d\mu(y_j)\,,$$

by using the finite overlap property of the balls $B(x_j,r_j)$ (cf. Fact 4.21 iv))

$$\le M\Big|\bigcup_j B(x_j,r_j)\Big|^{1/2}\||\phi|^2\|_1^{1/2}\,,$$

by applying Kolmogorov's inequality (4.21) together with the weak type $(1,1)$ boundedness of the maximal operator $\mathcal{M}$. In view of (4.28) and Fact 4.21 iii), we finally arrive at

$$\int_0^\infty \Big\|\sum_j G_{j,s} b_j\Big\|_2^2 \frac{ds}{s} \lesssim \alpha^2 \Big|\bigcup_j B(x_j, r_j)\Big| \lesssim \alpha^{2-p_0} \int_X |f|^{p_0}\, d\mu\,.$$

This, combined with (4.22) and (4.25), gives the estimate

$$B \lesssim \frac{1}{\alpha^{p_0}} \int_X |f|^{p_0}\, d\mu\,,$$

as desired. □

Now let us discuss the second step of the proof of Theorem 4.19. For each $p \in (2, p_0')$, we have to show that $\|S_\psi f\|_p \lesssim \|f\|_p$ for all $f \in L^p(X) \cap L^2(X)$.

Fix $p \in (2, p_0')$. As L satisfies generalized Gaussian (p_0, p_0')-estimates, the result in [Blu07, Corollary 2.3] entails that L has a bounded H^∞ calculus on $L^p(X)$ (see e.g. [CDMY96] for the definition). Due to [CDMY96, Corollary 6.7], for any $\psi \in H_0^\infty$ there exists a constant $C > 0$ such that for all $f \in L^p(X)$

$$\|G_\psi f\|_p \le C\, \|f\|_p\,, \tag{4.29}$$

where the square function G_ψ is given by

$$G_\psi f(x) := \Big(\int_0^\infty |\psi(tL) f(x)|^2 \frac{dt}{t}\Big)^{1/2} \qquad (\mu\text{-a.e. } x \in X).$$

Due to (4.29), the assertion of step 2 is verified as soon as we establish $\|S_\psi f\|_p \lesssim \|G_\psi f\|_p$ for all $f \in L^p(X) \cap L^2(X)$. To do so, we borrow an idea from [ADM05, proof of Theorem 6]. Let $f \in L^p(X) \cap L^2(X)$. By definition and Fact 2.1, we observe that for all $x \in X$

$$(S_\psi f)(x)^2 \cong \int_0^\infty \int_{B(x,t)} |\psi(t^m L) f(y)|^2 \frac{d\mu(y)}{|B(y,t)|} \frac{dt}{t}\,.$$

For any $\phi \in L^{(p/2)'}(X)$ we thus have by Fubini's theorem

$$\begin{aligned}
\big|\big\langle (S_\psi f)^2, \phi\big\rangle\big| &\lesssim \int_X \int_0^\infty \int_{B(x,t)} |\psi(t^m L) f(y)|^2 \frac{d\mu(y)}{|B(y,t)|} \frac{dt}{t}\ |\phi(x)|\, d\mu(x) \\
&= \int_X \int_0^\infty \int_{B(y,t)} |\psi(t^m L) f(y)|^2 \frac{1}{|B(y,t)|} |\phi(x)|\, d\mu(x) \frac{dt}{t}\, d\mu(y) \\
&\le \int_X \int_0^\infty |\psi(t^m L) f(y)|^2 \frac{dt}{t}\ \mathcal{M}(|\phi|)(y)\, d\mu(y) \\
&\cong \big\langle (G_\psi f)^2, \mathcal{M}(|\phi|)\big\rangle\,.
\end{aligned}$$

It follows from the boundedness of the Hardy-Littlewood maximal operator $\mathcal{M}$ on $L^{(p/2)'}(X)$ that $\|(S_\psi f)^2\|_{p/2} \lesssim \|(G_\psi f)^2\|_{p/2}$ which gives $\|S_\psi f\|_p \lesssim \|G_\psi f\|_p$, as claimed.

Finally, we treat step three, i.e. the proof of the reverse inequality $\|f\|_p \lesssim \|S_\psi f\|_p$ for all $f \in L^p(X) \cap L^2(X)$ and all $p \in (p_0, 2]$, where the implicit constant may depend on p or ψ but not on f.

Define

$$\widetilde{\psi}(z) := \overline{\psi(\overline{z})} \left(\int_0^\infty |\psi(t^m)|^2 \, \frac{dt}{t} \right)^{-1} \qquad (z \in \Sigma_\theta).$$

Then $\widetilde{\psi}$ belongs to $H_0^\infty(\Sigma_\theta) \setminus \{0\}$ and it holds $\int_0^\infty \psi(t^m)\widetilde{\psi}(t^m) \, \frac{dt}{t} = 1$. From this we deduce that for any $z \in \Sigma_\theta$

$$\int_0^\infty \psi(t^m z)\widetilde{\psi}(t^m z) \, \frac{dt}{t} = 1 \, .$$

Indeed, this is obvious for positive real numbers z, and the general case follows by analytic continuation. Then the spectral theorem for L implies the following Calderón reproducing formula

$$\int_0^\infty \psi(t^m L)\widetilde{\psi}(t^m L) \, \frac{dt}{t} = I \, ,$$

where the integral converges strongly in $L^2(X)$. Let $p \in (p_0, 2]$, $f \in L^p(X) \cap L^2(X)$, and $g \in L^{p'}(X) \cap L^2(X)$ with $\|g\|_{p'} = 1$. By using the Calderón reproducing formula and Fubini's theorem, we obtain

$$\begin{aligned} \langle f, g \rangle &= \int_X \int_0^\infty \psi(t^m L) f(x) \, \overline{\widetilde{\psi}(t^m L) g(x)} \, \frac{dt}{t} \, d\mu(x) \\ &= \int_X \int_0^\infty \int_{B(y,t)} \psi(t^m L) f(x) \, \overline{\widetilde{\psi}(t^m L) g(x)} \, \frac{d\mu(x)}{|B(x,t)|} \, \frac{dt}{t} \, d\mu(y) \, . \end{aligned}$$

Applying Fact 2.1 and Hölder's inequality twice leads to

$$\begin{aligned} |\langle f, g \rangle| &\lesssim \int_X \int_0^\infty \int_{B(y,t)} |\psi(t^m L) f(x)| \, |\widetilde{\psi}(t^m L) g(x)| \, \frac{d\mu(x)}{|B(y,t)|} \, \frac{dt}{t} \, d\mu(y) \\ &\le \int_X (S_\psi f)(y) \, (S_{\widetilde{\psi}} g)(y) \, d\mu(y) \le \|S_\psi f\|_p \, \|S_{\widetilde{\psi}} g\|_{p'} \lesssim \|S_\psi f\|_p \, \|g\|_{p'} \, , \end{aligned}$$

where the last estimate is due to step two. By taking the supremum over all such g and by recalling the density of $L^{p'}(X) \cap L^2(X)$ in $L^{p'}(X)$, we deduce that

$$\|f\|_p \lesssim \|S_\psi f\|_p \, ,$$

as desired. This completes the proof of Theorem 4.19.

A careful examination of the proof above shows that the assumptions on the operator L can be relaxed. The assertion of Theorem 4.19 still remains true when L is an injective, sectorial operator on $L^2(X)$ of angle $\omega(L) \in [0, \pi/2)$ such that L has a bounded $H^\infty(\Sigma_\theta)$ calculus for all $\theta \in (\omega(L), \pi)$ and the analytic semigroup generated by $-L$ satisfies generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$) for some $p_0 \in [1, 2)$ and $m \ge 2$.

5 Spectral multipliers on the Hardy space $H^1_L(X)$

We consider an injective, non-negative, self-adjoint operator L on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \geq 2$ and provide a criterion for the boundedness of spectral multipliers on the Hardy space $H^1_L(X)$. Our result, presented in Theorem 5.1 below, generalizes the statement [DY11, Theorem 3.1] due to X.T. Duong and L.X. Yan which merely works for operators of order $m = 2$ under the more restrictive assumption that the Davies-Gaffney condition (2.8) is required for all the open subsets of X. As we will see, the fact that we suppose (2.8) only for each open ball in X may cause technical difficulties since the scale of the semigroup operators is related to the radii of the balls. In order to overcome these problems, we shall use various covering arguments.

In Section 5.2 we check that the assumption (5.1) of Theorem 5.1 holds whenever the involved function F satisfies the Hörmander condition of a certain regularity order depending on the dimension of the underlying space. This fact enables us to derive from Theorem 5.1 a Hörmander type multiplier theorem on $H^1_L(X)$, formulated in Theorem 5.4.

5.1 A criterion for boundedness of spectral multipliers on $H^1_L(X)$

Theorem 5.1. *Let L be a non-negative, self-adjoint operator on $L^2(X)$ which is injective on its domain and satisfies Davies-Gaffney estimates (DG_m) for some $m \geq 2$. Further, let $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function. Assume that there exist an integer $M > D/m$ and constants $C_F > 0$, $\delta > D/2$ such that*

$$\left\| \mathbb{1}_{U_j(B)} F(L)(I - e^{-r^m L})^M \mathbb{1}_B \right\|_{2\to 2} \leq C_F \, 2^{-j\delta} \tag{5.1}$$

for every $j \in \mathbb{N} \setminus \{1\}$ and every ball $B \subseteq X$, whose radius is denoted by r. As usual, $U_j(B)$ stands for the dyadic annular set as defined in (4.5). Then the operator $F(L)$ extends from $H^1_L(X) \cap L^2(X)$ to a bounded linear operator on $H^1_L(X)$. More precisely, there exists a constant $C > 0$ such that

$$\|F(L)f\|_{H^1_L(X)} \leq C C_F \|f\|_{H^1_L(X)}$$

for all $f \in H^1_L(X)$.

The proof strategy consists of reducing the statement to the uniform boundedness of $\|F(L)a\|_{H^1_L(X)}$ for every $(2M,\widetilde{\varepsilon},L)$-molecule a. Recall that a can be rewritten as $a = L^{2M}b$ for some $b \in \mathcal{D}(L^{2M})$. By lacking a support property of $L^k b$ for $k \in \{0,1,\dots,2M\}$, we cannot apply (5.1) directly. In order to master this challenge, we shall choose $\widetilde{\varepsilon}$ large enough and use an estimate of annular type furnished by the next lemma.

Lemma 5.2. *Suppose that the operator L and the function F have the same properties as in Theorem 5.1. Then there exists a constant $C>0$ such that*

$$\big\|\mathbb{1}_{U_j(B)}F(L)(I-e^{-r^mL})^M\mathbb{1}_{U_i(B)}\big\|_{2\to 2} \le CC_F\, 2^{iD}\, 2^{-|j-i|\delta}\,. \tag{5.2}$$

for every $i,j\in\mathbb{N}\setminus\{1\}$ and every ball $B\subseteq X$, whose radius is denoted by r.

Proof. It suffices to check (5.2) only for each $i,j\in\mathbb{N}\setminus\{1\}$ with $|j-i|>3$ since otherwise (5.2) is valid by the spectral theorem after choosing appropriate constants. Due to the self-adjointness of L, one can swap i and j in the term on the left-hand side of (5.2). Hence, it will be enough to show the assertion for every $i,j\in\mathbb{N}\setminus\{1\}$ with $j-i>3$. By applying [BK05, Lemma 3.4], (5.1), and the doubling property, we get for each $r>0$ and each $x\in X$

$$\begin{aligned}
&\big\|\mathbb{1}_{U_j(B(x,r))}F(L)(I-e^{-r^mL})^M\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2}\\
&\lesssim \int_X \big\|\mathbb{1}_{U_j(B(x,r))}F(L)(I-e^{-r^mL})^M\mathbb{1}_{B(z,r)}\big\|_{2\to 2}\big\|\mathbb{1}_{B(z,r)}\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2}\,\frac{d\mu(z)}{|B(z,r)|}\\
&\le \int_{B(x,2^{i+1}r)\setminus B(x,2^{i-2}r)} \sum_{\nu=j-i-3}^{j+i+1}\big\|\mathbb{1}_{U_\nu(B(z,r))}F(L)(I-e^{-r^mL})^M\mathbb{1}_{B(z,r)}\big\|_{2\to 2}\,\frac{d\mu(z)}{|B(z,r)|}\\
&\lesssim \int_{B(x,2^{i+1}r)} \sum_{\nu=j-i-3}^{j+i+1} C_F\,2^{-\nu\delta}\,2^{(i+1)D}\,\frac{d\mu(z)}{|B(z,2^{i+1}r)|}\,.
\end{aligned}$$

In the next to the last step we covered $U_j(B(x,r))$ by dyadic annuli around the point z. Here, we used, among other things, the elementary inequalities

$$|2^\alpha - 2^\beta| \ge 2^{|\alpha-\beta|-1} \qquad\text{and}\qquad 2^\alpha+2^\beta\le 2^{\alpha+\beta+1} \tag{5.3}$$

which are valid for each $\alpha,\beta\in\mathbb{N}_0$ with $\alpha\ne\beta$. With the help of

$$\sum_{\nu=j-i-3}^{j+i+1} 2^{-\nu\delta} = 2^{3\delta}\,2^{-(j-i)\delta}\sum_{\eta=0}^{2i+4}2^{-\eta\delta}\lesssim 2^{-(j-i)\delta}$$

and Fact 2.1, we finish our estimation as follows

$$\begin{aligned}
&\big\|\mathbb{1}_{U_j(B(x,r))}F(L)(I-e^{-r^mL})^M\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2}\\
&\quad\lesssim C_F 2^{-(j-i)\delta}\int_{B(x,2^{i+1}r)} 2^{(i+1)D}\frac{d\mu(z)}{|B(z,2^{i+1}r)|}\\
&\quad\lesssim C_F\,2^{iD}\,2^{-(j-i)\delta}\,.
\end{aligned}$$

□

Next, we provide the technical result that a integrated version of the regularization operator $(I - e^{-r^m L})^M$ satisfies $L^2(X)$-norm estimates of annular type if L fulfills (DG$_m$). This will be achieved with a similar reasoning as in the proof of the preceding statement.

Lemma 5.3. *Let $K \in \mathbb{N}$ and L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which fulfills Davies-Gaffney estimates (DG$_m$) for some $m \geq 2$. For $M \in \mathbb{N}$ and $r > 0$ define the operator*

$$P_{m,M,r}(L) := r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1}(I - e^{-s^m L})^M \, ds \,. \tag{5.4}$$

Then there exist $b, C > 0$ such that for any $i, j \in \mathbb{N}_0$ and arbitrary balls $B \subseteq X$ of radius r

$$\big\|\mathbb{1}_{U_j(B)} P_{m,M,r}(L)^K \mathbb{1}_{U_i(B)}\big\|_{2\to 2} \leq C \exp\big(-b\, 2^{|j-i|}\big) \,. \tag{5.5}$$

Here, the constants b, C depend exclusively on m, K, M and the constants appearing in the Davies-Gaffney and doubling condition.

Proof. Let $K, M \in \mathbb{N}$, $r > 0$, and $x \in X$. At the beginning, we note that the operator $P_{m,M,r}(L)$ is bounded on $L^2(X)$

$$\begin{aligned}\big\|P_{m,M,r}(L)\big\|_{2\to 2} &\leq r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} \big\|I - e^{-s^m L}\big\|^M_{2\to 2} \, ds \\ &\leq r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} 2^M \, ds = \frac{2^M}{m} \,.\end{aligned}$$

With analogous arguments as in the proof of Lemma 5.2, it is enough to verify (5.5) for each $i, j \in \mathbb{N}_0$ with $j - i > 6$. To this purpose, fix $k \in \{1, \dots, M\}$ and $s \in [r, \sqrt[m]{2}r]$ for a moment. We shall establish the estimate

$$\big\|\mathbb{1}_{U_j(B(x,r))} e^{-ks^m L} \mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2} \leq C \exp\big(-b(2^{j-1} - 2^{i+2})\big) \tag{5.6}$$

for some constants $b, C > 0$ depending only on m, M and the constants in the Davies-Gaffney or doubling condition, but not on the other parameters.

From the Davies-Gaffney estimates (DG$_m$) we obtain for each $y \in X$

$$\begin{aligned}\big\|\mathbb{1}_{B(x,r)} e^{-ks^m L} \mathbb{1}_{B(y,r)}\big\|_{2\to 2} &\leq \big\|\mathbb{1}_{B(x,k^{1/m}s)} e^{-ks^m L} \mathbb{1}_{B(y,k^{1/m}s)}\big\|_{2\to 2} \\ &\lesssim \exp\left(-b\left(\frac{d(x,y)}{k^{1/m}s}\right)^{\frac{m}{m-1}}\right) \leq \exp\left(-b(2M)^{-\frac{1}{m-1}}\left(\frac{d(x,y)}{r}\right)^{\frac{m}{m-1}}\right).\end{aligned}$$

Therefore, Fact 2.4 yields for any $\nu \in \mathbb{N}$

$$\big\|\mathbb{1}_{A(x,r,\nu)} e^{-ks^m L} \mathbb{1}_{B(x,r)}\big\|_{2\to 2} \lesssim \exp\big(-b\nu^{\frac{m}{m-1}}\big) \leq e^{-b\nu} \,.$$

By applying [BK05, Lemma 3.4] and the doubling property, we deduce

$$\begin{aligned}
&\big\|\mathbb{1}_{U_j(B(x,r))}e^{-ks^mL}\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2} \\
&\quad\lesssim \int_X \big\|\mathbb{1}_{U_j(B(x,r))}e^{-ks^mL}\mathbb{1}_{B(z,r)}\big\|_{2\to 2}\big\|\mathbb{1}_{B(z,r)}\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2}\,\frac{d\mu(z)}{|B(z,r)|} \\
&\quad\le \int_{B(x,2^{i+1}r)\setminus B(x,2^{i-2}r)} \sum_{\nu=2^{j-1}-2^{i+1}}^{2^j+2^{i+1}} \big\|\mathbb{1}_{A(z,r,\nu)}e^{-ks^mL}\mathbb{1}_{B(z,r)}\big\|_{2\to 2}\,\frac{d\mu(z)}{|B(z,r)|} \\
&\quad\lesssim \int_{B(x,2^{i+1}r)} \sum_{\nu=2^{j-1}-2^{i+1}}^{2^j+2^{i+1}} e^{-b\nu}\,2^{(i+1)D}\,\frac{d\mu(z)}{|B(z,2^{i+1}r)|}\,.
\end{aligned}$$

With the help of

$$\sum_{\nu=2^{j-1}-2^{i+1}}^{2^j+2^{i+1}} e^{-b\nu} \le \exp\big(-b(2^{j-1}-2^{i+1})\big)\sum_{\eta=0}^{\infty} e^{-b\eta} = \frac{1}{1-e^{-b}}\,\exp\big(-b(2^{j-1}-2^{i+1})\big)$$

and Fact 2.1, we finally arrive at the claimed estimate (5.6)

$$\big\|\mathbb{1}_{U_j(B(x,r))}e^{-ks^mL}\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2} \lesssim 2^{iD}\exp\big(-b(2^{j-1}-2^{i+1})\big) \lesssim \exp\big(-b(2^{j-1}-2^{i+2})\big)\,.$$

In view of the formula

$$(I-e^{-s^mL})^M = \sum_{k=0}^{M}\binom{M}{k}(-1)^k e^{-ks^mL}$$

and the disjointness of $U_i(B(x,r))$ and $U_j(B(x,r))$, we get from (5.6)

$$\begin{aligned}
&\big\|\mathbb{1}_{U_j(B(x,r))}P_{m,M,r}(L)\,\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2} \\
&\quad\le \sum_{k=0}^{M}\binom{M}{k}r^{-m}\int_r^{\sqrt[m]{2}r} s^{m-1}\big\|\mathbb{1}_{U_j(B(x,r))}e^{-ks^mL}\mathbb{1}_{U_i(B(x,r))}\big\|_{2\to 2}\,ds \\
&\quad\lesssim \sum_{k=1}^{M}\binom{M}{k}r^{-m}\int_r^{\sqrt[m]{2}r} s^{m-1}\,ds\,\exp\big(-b(2^{j-1}-2^{i+2})\big) \\
&\quad\lesssim \exp\big(-b(2^{j-1}-2^{i+2})\big)\,. \qquad (5.7)
\end{aligned}$$

Due to the inequality (5.3), the assertion (5.5) for $K=1$ is verified.

The general statement follows by induction, once (5.5) is checked for $K = 2$. That will be achieved by adapting the proof of [HM03, Lemma 2.3] to the present situation. For the rest of the proof we abbreviate $P := P_{m,M,r}(L)$. Let $f \in L^2(X)$ with $\operatorname{supp} f \subseteq U_i(B)$ and $\|f\|_2 = 1$ be fixed. We consider the set

$$\begin{aligned} G &:= \big\{y \in X \,:\, \operatorname{dist}(y, U_j(B)) < \tfrac{1}{2}\operatorname{dist}(U_i(B), U_j(B))\big\} \\ &= \big\{y \in X \,:\, (2^{j-2} + 2^{i-1})r < d(x,y) < (5 \cdot 2^{j-2} - 2^{i-1})r\big\} \end{aligned}$$

and analyze

$$\big\|\mathbb{1}_{U_j(B)} P^2 f\big\|_2 \le \big\|P(\mathbb{1}_G \cdot Pf)\big\|_{L^2(U_j(B))} + \big\|P(\mathbb{1}_{X\setminus G} \cdot Pf)\big\|_{L^2(U_j(B))}.$$

In order to estimate the first term on the right-hand side, we initially exploit the boundedness of P on $L^2(X)$ and then cover the set G by dyadic annuli in such a way as to enable us to apply (5.7)

$$\begin{aligned} \big\|P(\mathbb{1}_G \cdot Pf)\big\|_{L^2(U_j(B))} &\lesssim \|\mathbb{1}_G \cdot Pf\|_2 \le \sum_{k=\lfloor \log_2(2^{j-2}+2^{i-1})\rfloor}^{\lfloor \log_2(5\cdot 2^{j-2}-2^{i-1})\rfloor+1} \|\mathbb{1}_{U_k(B)} \cdot Pf\|_2 \\ &\lesssim \sum_{k=\lfloor \log_2(2^{j-2}+2^{i-1})\rfloor}^{\lfloor \log_2(5\cdot 2^{j-2}-2^{i-1})\rfloor+1} e^{-b(2^{k-1}-2^{i+2})}\|f\|_2 \\ &\le \big((\log_2(5\cdot 2^{j-2} - 2^{i-1}) + 3 - \log_2(2^{j-2} + 2^{i-1})\big) e^{-b((2^{j-2}+2^{i-1})/4-2^{i+2})} \\ &\lesssim e^{-b(2^{j-4}-2^{i+2})}. \end{aligned}$$

Thanks to (5.3), the latter is bounded by a constant times $\exp(-b\, 2^{j-i})$, as desired.

The second summand $\|P(\mathbb{1}_{X\setminus G} \cdot Pf)\|_{L^2(U_j(B))}$ can be treated in an analogous manner. One has only to interchange the sequence of the arguments. At first, one covers $X \setminus G$ by dyadic annuli, so that the off-diagonal estimates (5.7) are applicable, and then one utilizes the boundedness of P on $L^2(X)$ as well as (5.3). This gives a similar estimate as before and finishes the proof. □

With the two preceding lemmas at hand, we are prepared for the proof of the main result of this section.

Proof of Theorem 5.1. We imitate the proof of [DY11, Theorem 3.1]. Suppose that L is an injective, non-negative, self-adjoint operator on $L^2(X)$ that fulfills (DG_m) for some $m \ge 2$. Let $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function such that (5.1) holds for some constants $C_F > 0$, $\delta > D/2$, and $M \in \mathbb{N}$ with $M > D/m$.

First of all, we observe that one can define the operator $F(L)$ on the set $H^1_L(X) \cap L^2(X)$ which lies densely in $H^1_L(X)$. Once we have shown $H^1_L(X)$ boundedness of $F(L)$ on this dense set, the operator $F(L)$ can be extended to a bounded operator on $H^1_L(X)$.

Let $\widetilde{\delta} \in (D/2, \min\{\delta, mM - D/2\})$ be fixed. Define $\varepsilon := \widetilde{\delta} - D/2 > 0$ and $\widetilde{\varepsilon} := D + \widetilde{\delta}$. In order to prove Theorem 5.1, we claim that, for every $(2M, \widetilde{\varepsilon}, L)$-molecule a, $F(L)a$ is, up to multiplication by a constant independent of a, a (M, ε, L)-molecule.

The conclusion of Theorem 5.1 is then an immediate consequence of Corollary 4.14. Indeed, by Lemma 4.16, every $f \in H^1_L(X) \cap L^2(X)$ admits a molecular $(2M, \widetilde{\varepsilon}, L)$-representation, i.e. there exist a scalar sequence $(\lambda_j)_{j\in\mathbb{N}_0}$ and a sequence $(m_j)_{j\in\mathbb{N}_0}$ of $(2M, \widetilde{\varepsilon}, L)$-molecules such that

$$f = \sum_{j=0}^{\infty} \lambda_j m_j$$

in $L^2(X)$ and

$$\|f\|_{H^1_L(X)} \cong \sum_{j=0}^{\infty} |\lambda_j|$$

with implicit constants independent of f. Therefore, it holds

$$\|F(L)f\|_{H^1_L(X)} \leq \sum_{j=0}^{\infty} |\lambda_j| \, \|F(L)m_j\|_{H^1_L(X)} .$$

But by the claim above, $F(L)m_j$ is a constant multiple of a (M, ε, L)-molecule. Hence, we conclude from Corollary 4.14 that $\|F(L)m_j\|_{H^1_L(X)}$ is bounded by a constant $C > 0$ and we emphasize that this constant is independent of j. Thus, once the above claim is proved, the boundedness of $F(L)$ on $H^1_L(X)$ is shown because for any $f \in H^1_L(X) \cap L^2(X)$ one has

$$\|F(L)f\|_{H^1_L(X)} \leq \sum_{j=0}^{\infty} |\lambda_j| \, \|F(L)m_j\|_{H^1_L(X)} \leq C \sum_{j=0}^{\infty} |\lambda_j| \cong \|f\|_{H^1_L(X)}$$

and $H^1_L(X) \cap L^2(X)$ is dense in the Hardy space $H^1_L(X)$.

Now we proceed with the proof of the claim stated above. Let a be a $(2M, \widetilde{\varepsilon}, L)$-molecule. According to Definition 4.8, we find a function $b \in \mathcal{D}(L^{2M})$ and a ball $B \subseteq X$ such that $a = L^{2M} b$ and (4.4) hold. By the spectral theorem for L, we may write

$$F(L)a = L^M \big(F(L) L^M b \big) .$$

In particular, $F(L)L^M b$ belongs to $\mathcal{D}(L^M)$. For the proof that $F(L)a$ is a constant multiple of a (M, ε, L)-molecule it remains to check ii) from Definition 4.8, i.e. the existence of a constant $C > 0$ such that for all $j \in \mathbb{N}_0$ and all $k \in \{0, 1, \ldots, M\}$

$$\big\| (r^m L)^k \big(F(L) L^M b \big) \big\|_{L^2(U_j(B))} \leq C C_F r^{mM} 2^{-j\varepsilon} \mu(2^j B)^{-1/2} , \tag{5.8}$$

where r denotes the radius of the ball B.

For $j \in \{0,1,2\}$, we employ the boundedness of $F(L)$ on $L^2(X)$ as well as the properties of the $(2M, \widetilde{\varepsilon}, L)$-molecule a. For any $k \in \{0,1,\ldots,M\}$, this leads to

$$\begin{aligned}\big\|(r^m L)^k \big(F(L)L^M b\big)\big\|_{L^2(U_j(B))} &\leq r^{mk}\|F(L)\|_{2\to 2}\, r^{-m(M+k)}\|(r^m L)^{M+k} b\|_2 \\ &= \|F\|_\infty\, r^{-mM}\sum_{i=0}^{\infty}\|(r^m L)^{M+k} b\|_{L^2(U_i(B))} \\ &\leq \|F\|_\infty\, r^{-mM}\sum_{i=0}^{\infty} r^{2mM}2^{-i\widetilde{\varepsilon}}\mu(2^i B)^{-1/2} \\ &\lesssim \|F\|_\infty\, r^{mM}\mu(B)^{-1/2}\,. \end{aligned} \tag{5.9}$$

By the doubling property, we have $\mu(2^j B) \lesssim 2^{jD}\mu(B)$ and thus $\mu(B)^{-1} \lesssim 2^{2D}\mu(2^j B)^{-1}$. This, together with (5.9), shows that for each $j \in \{0,1,2\}$ and $k \in \{0,1,\ldots,M\}$

$$\big\|(r^m L)^k \big(F(L)L^M b\big)\big\|_{L^2(U_j(B))} \lesssim \big(\|F\|_\infty 2^{D+2\varepsilon}\big)\, r^{mM}2^{-j\varepsilon}\mu(2^j B)^{-1/2}\,.$$

Now assume that $j \geq 3$. We start by representing the identity on $L^2(X)$ with the help of the operators $e^{-\nu r^m L}$ and $P_{m,M,r}(L)$, where the latter was defined in (5.4). After applying this to $(r^m L)^k(F(L)L^M b)$, the procedure produces a regularizing effect for the operator $F(L)$ and finally permits us to insert the assumption (5.1) in the version of Lemma 5.2 and the Davies-Gaffney estimates in the form of Lemma 5.3. Inspired from [HM09, (8.7), (8.8)], we use the elementary equations

$$1 = mr^{-m}\int_r^{\sqrt[m]{2}r} s^{m-1}\,ds$$

and

$$1 = (1-e^{-s^m\lambda})^M - \sum_{\nu=1}^{M}\binom{M}{\nu}(-1)^\nu e^{-\nu s^m \lambda} \qquad (\lambda \geq 0, s > 0)$$

to deduce, by applying the spectral theorem for L,

$$I = mr^{-m}\int_r^{\sqrt[m]{2}r} s^{m-1}(I-e^{-s^m L})^M\,ds + \sum_{\nu=1}^{M}\nu C_{\nu,M} m r^{-m}\int_r^{\sqrt[m]{2}r} s^{m-1}e^{-\nu s^m L}\,ds\,, \tag{5.10}$$

where $C_{\nu,M} := \frac{(-1)^{\nu+1}}{\nu}\binom{M}{\nu}$. Further, it holds $\partial_s e^{-\nu s^m L} = -\nu m s^{m-1} L e^{-\nu s^m L}$ and therefore

$$\begin{aligned}\nu m L\int_r^{\sqrt[m]{2}r} s^{m-1}e^{-\nu s^m L}\,ds &= e^{-\nu r^m L} - e^{-2\nu r^m L} = e^{-\nu r^m L}(I - e^{-\nu r^m L}) \\ &= e^{-\nu r^m L}(I-e^{-r^m L})\sum_{\eta=0}^{\nu-1} e^{-\eta r^m L}\,. \end{aligned} \tag{5.11}$$

By recalling the definition of $P_{m,M,r}(L)$ and by inserting the equation (5.11) into (5.10), we end up with the following formula for the identity on $L^2(X)$

$$I = mP_{m,M,r}(L) + \sum_{\nu=1}^{M} C_{\nu,M} r^{-m} L^{-1}(I - e^{-r^m L}) \sum_{\eta=\nu}^{2\nu-1} e^{-\eta r^m L} .$$

Expanding the identity I^M by means of the binomial formula leads to

$$\begin{aligned} I &= \big(mP_{m,M,r}(L)\big)^M \\ &\quad + \sum_{l=1}^{M} \binom{M}{l} \left(\sum_{\nu=1}^{M} C_{\nu,M} r^{-m} L^{-1}(I - e^{-r^m L}) \sum_{\eta=\nu}^{2\nu-1} e^{-\eta r^m L} \right)^l \big(mP_{m,M,r}(L)\big)^{M-l} \\ &= m^M P_{m,M,r}(L)^M + \sum_{l=1}^{M} r^{-ml} L^{-l}(I - e^{-r^m L})^l P_{m,M,r}(L)^{M-l} \sum_{\nu=1}^{(2M-1)l} C_{l,\nu,m,M} e^{-\nu r^m L} \end{aligned}$$

for appropriate constants $C_{l,\nu,m,M}$ depending on the subscripted parameters.

Now fix $k \in \{0, 1, \ldots, M\}$. The above identity allows us to represent $(r^m L)^k (F(L)L^M b)$ in the following way

$$\begin{aligned} (r^m L)^k \big(F(L)L^M b\big) &= m^M r^{mk} P_{m,M,r}(L)^M F(L)(L^{M+k} b) \\ &\quad + \sum_{l=1}^{M} r^{mk-ml} L^{-l}(I - e^{-r^m L})^l P_{m,M,r}(L)^{M-l} \sum_{\nu=1}^{(2M-1)l} C_{l,\nu,m,M} e^{-\nu r^m L} F(L)(L^{M+k} b) \\ &=: \sum_{l=0}^{M} G^{(k)}_{l,M,r} . \end{aligned}$$

We shall establish an adequate bound on $\|G^{(k)}_{l,M,r}\|_{L^2(U_j(B))}$ by distinguishing the three cases $l = 0$, $l \in \{1, \ldots, M-1\}$, and $l = M$.

Case 1: $l = 0$.
First, we write for μ-a.e. $x \in X$

$$\begin{aligned} \big|G^{(k)}_{0,M,r}(x)\big| &= m^M r^{mk} \big|P_{m,M,r}(L)\big(P_{m,M,r}(L)^{M-1} F(L)(L^{M+k} b)\big)(x)\big| \\ &= m^M r^{mk} r^{-m} \int_r^{\sqrt[m]{2}r} s^{m-1} \big|P_{m,M,r}(L)^{M-1}\big(F(L)(I - e^{-s^m L})^M (L^{M+k} b)\big)(x)\big| \, ds \\ &\le \sum_{i=0}^{\infty} m^M r^{mk} r^{-m} \times \\ &\qquad \times \int_r^{\sqrt[m]{2}r} s^{m-1} \Big|P_{m,M,r}(L)^{M-1}\Big(\mathbb{1}_{U_i(B)}\big(F(L)(I - e^{-s^m L})^M (L^{M+k} b)\big)\Big)(x)\Big| \, ds . \end{aligned}$$

As seen in Lemma 5.3, the operator $P_{m,M,r}(L)^{M-1}$ enjoys the off-diagonal estimate (5.5). This yields

$$\begin{aligned}
&\big\|G^{(k)}_{0,M,r}\big\|_{L^2(U_j(B))}\\
&\le m^M r^{mk}\sum_{i=0}^{\infty} r^{-m}\ \times\\
&\qquad\times\int_r^{\sqrt[m]{2}r} s^{m-1}\Big\|P_{m,M,r}(L)^{M-1}\Big(\mathbb{1}_{U_i(B)}\big(F(L)(I-e^{-s^mL})^M(L^{M+k}b)\big)\Big)\Big\|_{L^2(U_j(B))}\,ds\\
&\lesssim r^{mk}\sum_{i=0}^{\infty}\exp\big(-b\,2^{|j-i|}\big)\,r^{-m}\int_r^{\sqrt[m]{2}r} s^{m-1}\big\|F(L)(I-e^{-s^mL})^M(L^{M+k}b)\big\|_{L^2(U_i(B))}\,ds\,.
\end{aligned}$$

In order to apply Lemma 5.2, we first observe that for every $s\in[r,\ \sqrt[m]{2}r]$ the ball $U_0(B)$ is contained in $U_0(B(x_B,s))$ and the annulus $U_i(B)$ in $U_{i-1}(B(x_B,s))\cup U_i(B(x_B,s))$ for each $i\in\mathbb{N}$ if x_B denotes the center of B. These inclusions give for every $s\in[r,\ \sqrt[m]{2}r]$

$$\begin{aligned}
&\big\|F(L)(I-e^{-s^mL})^M(L^{M+k}b)\big\|_{L^2(U_i(B))}\\
&\quad\le\sum_{\nu=i-1}^{i}\big\|F(L)(I-e^{-s^mL})^M(L^{M+k}b)\big\|_{L^2(U_\nu(B(x_B,s)))}\\
&\quad\le\sum_{\nu=i-1}^{i}\Bigg(\big\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{B(x_B,s)}L^{M+k}b)\big\|_{L^2(U_\nu(B(x_B,s)))}\\
&\qquad\qquad+\sum_{\eta=1}^{\infty}\big\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{U_\eta(B(x_B,s))}L^{M+k}b)\big\|_{L^2(U_\nu(B(x_B,s)))}\Bigg)\,. \qquad (5.12)
\end{aligned}$$

Due to (5.1), the first summand in the bracket is bounded by

$$C_F 2^{-\nu\delta}\|L^{M+k}b\|_{L^2(B(x_B,s))}\le C_F 2^{-\nu\delta}\big(\|L^{M+k}b\|_{L^2(B)}+\|L^{M+k}b\|_{L^2(U_1(B))}\big)\,.$$

By recalling the properties of the $(2M,\widetilde{\varepsilon},L)$-molecule a, we obtain

$$\begin{aligned}
\|L^{M+k}b\|_{L^2(B)}&=r^{-m(M+k)}\|(r^mL)^{M+k}b\|_{L^2(B)}\\
&\le r^{-m(M+k)}r^{2mM}\mu(B)^{-1/2}=r^{mM-mk}\mu(B)^{-1/2}
\end{aligned}$$

as well as

$$\begin{aligned}
\|L^{M+k}b\|_{L^2(U_1(B))}&=r^{-m(M+k)}\|(r^mL)^{M+k}b\|_{L^2(U_1(B))}\\
&\le r^{-m(M+k)}r^{2mM}2^{-\widetilde{\varepsilon}}\mu(2B)^{-1/2}\le r^{mM-mk}\mu(B)^{-1/2}\,.
\end{aligned}$$

Hence, we have the bound

$$\big\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{B(x_B,s)}L^{M+k}b)\big\|_{L^2(U_\nu(B(x_B,s)))}\lesssim C_F r^{mM-mk}2^{-\nu\delta}\mu(B)^{-1/2}\,. \quad (5.13)$$

The series in the bracket of (5.12) can be estimated with the help of Lemma 5.2

$$\sum_{\eta=1}^{\infty}\big\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{U_\eta(B(x_B,s))}L^{M+k}b)\big\|_{L^2(U_\nu(B(x_B,s)))}$$
$$\lesssim \sum_{\eta=1}^{\infty} C_F 2^{\eta D}2^{-|\nu-\eta|\delta}\|L^{M+k}b\|_{L^2(U_\eta(B(x_B,s)))}\,.$$

Since a is a $(2M,\widetilde{\varepsilon},L)$-molecule, we obtain

$$\begin{aligned}
&\|L^{M+k}b\|_{L^2(U_\eta(B(x_B,s)))}\\
&\quad\le r^{-m(M+k)}\big(\|(r^mL)^{M+k}b\|_{L^2(U_\eta(B(x_B,r)))}+\|(r^mL)^{M+k}b\|_{L^2(U_{\eta+1}(B(x_B,r)))}\big)\\
&\quad\le r^{-m(M+k)}\big(r^{2mM}2^{-\eta\widetilde{\varepsilon}}\mu(2^\eta B(x_B,r))^{-1/2}+r^{2mM}2^{-(\eta+1)\widetilde{\varepsilon}}\mu(2^{\eta+1}B(x_B,r))^{-1/2}\big)\\
&\quad\lesssim r^{mM-mk}2^{-\eta\widetilde{\varepsilon}}\mu(B(x_B,r))^{-1/2}
\end{aligned}$$

and thus

$$\begin{aligned}
&\sum_{\eta=1}^{\infty}\big\|F(L)(I-e^{-s^mL})^M(\mathbb{1}_{U_\eta(B(x_B,s))}L^{M+k}b)\big\|_{L^2(U_\nu(B(x_B,s)))}\\
&\qquad\lesssim C_F r^{mM-mk}\mu(B(x_B,r))^{-1/2}\sum_{\eta=1}^{\infty}2^{-\eta(\widetilde{\varepsilon}-D)}2^{-|\nu-\eta|\delta}\\
&\qquad\lesssim C_F r^{mM-mk}2^{-\nu\widetilde{\delta}}\mu(B(x_B,r))^{-1/2}\,.
\end{aligned}\tag{5.14}$$

In the last step we used the fact that

$$\begin{aligned}
\sum_{\eta=1}^{\infty}2^{-\eta(\widetilde{\varepsilon}-D)}2^{-|\nu-\eta|\delta}&=2^{-\nu(\widetilde{\varepsilon}-D)}\left(\sum_{n=-\infty}^{0}2^{n(\widetilde{\varepsilon}-D)}2^{-|n|\delta}+\sum_{n=1}^{\nu-1}2^{n(\widetilde{\varepsilon}-D)}2^{-n\delta}\right)\\
&\le 2^{-\nu\widetilde{\delta}}\left(\sum_{n=-\infty}^{0}2^{-|n|\delta}+\sum_{n=1}^{\infty}2^{-n(D+\delta-\widetilde{\varepsilon})}\right)\lesssim 2^{-\nu\widetilde{\delta}}\,.
\end{aligned}$$

In view of the inequalities (5.13) and (5.14), we have the following estimate of (5.12)

$$\big\|F(L)(I-e^{-s^mL})^M(L^{M+k}b)\big\|_{L^2(U_i(B))}\lesssim C_F r^{mM-mk}2^{-i\widetilde{\delta}}\mu(B)^{-1/2}\,.$$

With the help of this bound and the doubling property, we continue

$$\begin{aligned}
&\big\|G^{(k)}_{0,M,r}\big\|_{L^2(U_j(B))}\\
&\quad\lesssim r^{mk}\sum_{i=0}^{\infty}\exp\big(-b\,2^{|j-i|}\big)\,r^{-m}\int_r^{\sqrt[m]{2}r}s^{m-1}\big\|F(L)(I-e^{-s^mL})^M(L^{M+k}b)\big\|_{L^2(U_i(B))}\,ds\\
&\quad\lesssim r^{mk}\sum_{i=0}^{\infty}\exp\big(-b\,2^{|j-i|}\big)\,r^{-m}\int_r^{\sqrt[m]{2}r}s^{m-1}\,ds\;C_F r^{mM-mk}2^{-i\widetilde{\delta}}\mu(B)^{-1/2}\\
&\quad\lesssim C_F r^{mM}2^{-j\widetilde{\delta}}\mu(B)^{-1/2}\lesssim C_F r^{mM}2^{-j(\widetilde{\delta}-D/2)}\mu(2^jB)^{-1/2}\,.
\end{aligned}\tag{5.15}$$

In the second to the last step we used, among other things, the following fact which is easily verified by an index shift

$$\begin{aligned}\sum_{i=0}^{\infty}\exp\big(-b\,2^{|j-i|}\big)\,2^{-i\widetilde{\delta}} &= \sum_{n=-\infty}^{0}\exp\big(-b\,2^{|n|}\big)\,2^{-(j-n)\widetilde{\delta}}+\sum_{n=1}^{j}\exp\big(-b\,2^{|n|}\big)\,2^{-(j-n)\widetilde{\delta}}\\ &\le 2^{-j\widetilde{\delta}}\sum_{n=-\infty}^{\infty}\exp\big(-b\,2^{|n|}\big)\,2^{n\widetilde{\delta}}\lesssim 2^{-j\widetilde{\delta}}\,. \end{aligned}\tag{5.16}$$

Case 2: $l\in\{1,2,\dots,M-1\}$.
We have for μ-a.e. $x\in X$

$$\begin{aligned}\big|G^{(k)}_{l,M,r}(x)\big| &\le r^{m(k-l)}\sum_{\nu=1}^{(2M-1)l}|C_{l,\nu,m,M}|\int_r^{\sqrt[m]{2}r}\Big(\frac{s}{r}\Big)^m\Big|L^{M-l}e^{-\nu r^mL}(I-e^{-r^mL})^l\circ\\ &\qquad\circ\ P_{m,M,r}(L)^{M-l-1}\big(F(L)(I-e^{-s^mL})^M(L^kb)\big)(x)\Big|\,\frac{ds}{s}\\ &\lesssim r^{m(k-M)}\sum_{\nu=1}^{(2M-1)l}\sum_{i=0}^{\infty}\int_r^{\sqrt[m]{2}r}\Big|(r^mL)^{M-l}e^{-\nu r^mL}(I-e^{-r^mL})^l\circ\\ &\qquad\circ\ P_{m,M,r}(L)^{M-l-1}\Big(\mathbb{1}_{U_i(B)}\big(F(L)(I-e^{-s^mL})^M(L^kb)\big)\Big)(x)\Big|\,\frac{ds}{s}\,.\end{aligned}$$

By Lemma 2.9, the operator family $\{(tL)^{M-l}e^{-\nu tL}:t>0\}$ satisfies Davies-Gaffney estimates of order m. After writing $(I-e^{-tL})^l$ with the help of the binomial formula, it is straightforward to prove that (DG_m) also holds for $\{(tL)^{M-l}e^{-\nu tL}(I-e^{-tL})^l:t>0\}$. Hence, one can show $L^2(X)$-norm estimates of annular type similar to that in (5.6) for operators of the form $(r^mL)^{M-l}e^{-\nu r^mL}(I-e^{-r^mL})^l$ whenever r denotes the radius of the considered ball. Thanks to Lemma 5.3, $P_{m,M,r}(L)^{M-l-1}$ fulfills (5.5). If one adapts the arguments given at the end of the proof of Lemma 5.3, one can verify that the composition of these operators enjoys the following version of (5.5)

$$\big\|\mathbb{1}_{U_j(B)}(r^mL)^{M-l}e^{-\nu r^mL}(I-e^{-r^mL})^lP_{m,M,r}(L)^{M-l-1}\mathbb{1}_{U_i(B)}\big\|_{2\to2}\le C\exp\big(-b\,2^{|j-i|}\big)$$

for some constants $b,C>0$ depending only on m,K,M and the constants in the Davies-Gaffney and doubling condition.

This estimate leads to

$$\begin{aligned}\big\|G^{(k)}_{l,M,r}\big\|_{L^2(U_j(B))}&\lesssim r^{m(k-M)}\sum_{\nu=1}^{(2M-1)l}\sum_{i=0}^{\infty}\exp\big(-b\,2^{|j-i|}\big)\times\\ &\qquad\times\int_r^{\sqrt[m]{2}r}\big\|F(L)(I-e^{-s^mL})^M(L^kb)\big\|_{L^2(U_i(B))}\frac{ds}{s}\,.\end{aligned}\tag{5.17}$$

By employing similar arguments as in Case 1 (just replace $L^{M+k}b$ by $L^k b$), we conclude that for any $i \in \mathbb{N}_0$ and $s \in [r, \sqrt[m]{2}r]$

$$\big\|F(L)(I - e^{-s^m L})^M (L^k b)\big\|_{L^2(U_i(B))} \lesssim C_F r^{2mM-mk} 2^{-i\widetilde{\delta}} \mu(B)^{-1/2}\,. \tag{5.18}$$

Inserting this estimate into (5.17) yields readily

$$\big\|G^{(k)}_{l,M,r}\big\|_{L^2(U_j(B))} \lesssim C_F r^{mM} 2^{-j(\widetilde{\delta}-D/2)} \mu(2^j B)^{-1/2}\,. \tag{5.19}$$

Case 3: $l = M$.
In this case we have

$$\begin{aligned} G^{(k)}_{M,M,r} &= r^{m(k-M)} \sum_{\nu=1}^{(2M-1)M} C_{M,\nu,m,M} e^{-\nu r^m L}\big(F(L)(I - e^{-r^m L})^M (L^k b)\big) \\ &= r^{m(k-M)} \sum_{\nu=1}^{(2M-1)M} C_{M,\nu,m,M} \sum_{i=0}^{\infty} e^{-\nu r^m L}\Big(\mathbb{1}_{U_i(B)}\big(F(L)(I - e^{-r^m L})^M (L^k b)\big)\Big)\,. \end{aligned}$$

With the help of (5.6), (5.3), (5.18), and (5.16), we obtain

$$\begin{aligned} \big\|G^{(k)}_{M,M,r}\big\|_{L^2(U_j(B))} &\lesssim r^{m(k-M)} \sum_{i=0}^{\infty} \exp\big(-b\, 2^{|j-i|}\big) \big\|F(L)(I - e^{-r^m L})^M (L^k b)\big\|_{L^2(U_i(B))} \\ &\lesssim C_F r^{mM} \sum_{i=0}^{\infty} \exp\big(-b\, 2^{|j-i|}\big)\, 2^{-i\widetilde{\delta}} \mu(B)^{-1/2} \\ &\lesssim C_F r^{mM} 2^{-j\widetilde{\delta}} \mu(B)^{-1/2} \\ &\lesssim C_F r^{mM} 2^{-j(\widetilde{\delta}-D/2)} \mu(2^j B)^{-1/2}\,. \end{aligned}$$

This, in combination with (5.9), (5.15), and (5.19), gives the desired estimate (5.8). □

5.2 A Hörmander type multiplier theorem on $H^1_L(X)$

After the preparations in the last section, we are able to formulate Hörmander type spectral multiplier results on the Hardy space $H^1_L(X)$. We will state two versions of it, namely a more classical one, given in Theorem 5.4, and one including the Plancherel condition which leads to weakened regularity assumptions on the involved function, given in Theorem 5.5.

In order to formulate the Hörmander condition, we fix for the rest of this section a non-negative function $\omega \in C^\infty_c(\mathbb{R})$ with

$$\operatorname{supp}\omega \subseteq (1/4, 1) \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}} \omega(2^{-n}\lambda) = 1 \quad \text{for all } \lambda > 0\,.$$

Theorem 5.4. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \geq 2$. If a bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ fulfills*

$$\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_2} < \infty \tag{5.20}$$

for some $s > (D+1)/2$, then $F(L)$ can be extended from $H^1_L(X) \cap L^2(X)$ to a bounded linear operator on $H^1_L(X)$. More precisely, there exists a constant $C > 0$ such that

$$\|F(L)f\|_{H^1_L(X)} \leq C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_2} + |F(0)|\Big)\|f\|_{H^1_L(X)}$$

for all $f \in H^1_L(X)$.

In the special case $m = 2$ the statement corresponds to [DY11, Theorem 1.1]. However, X.T. Duong and L.X. Yan formulated the Hörmander condition (5.20) with respect to the norm in the Hölder space C^s which leads to a stronger assumption than our formulation of the Hörmander condition with respect to the norm in the Bessel potential space H^s_2. For that reason, it was enough to require $s > D/2$ in [DY11, Theorem 1.1]. As we will see at the end of this section, the result due to X.T. Duong and L.X. Yan is contained in one of our statements (cf. Corollary 5.6).

The proof of [DY11, Theorem 1.1] relies essentially on the equivalence between Davies-Gaffney estimates (DG_2) and finite speed propagation. As for Davies-Gaffney estimates of arbitrary order $m \geq 2$ this relationship is no longer valid, we shall argue in a different way by employing the tools of weighted norm estimates developed in Chapter 3.

Proof of Theorem 5.4. Let $F\colon [0,\infty) \to \mathbb{C}$ be a bounded Borel function. Observe that F satisfies (5.20) if and only if the function $\lambda \mapsto F(\sqrt[m]{\lambda})$ satisfies (5.20). Hence, we can consider $F(\sqrt[m]{L})$ in lieu of $F(L)$ during the proof. First, we write

$$F(\sqrt[m]{L}) = (F - F(0))(\sqrt[m]{L}) + F(0)I$$

and notice, after replacing F by $F - F(0)$, that we may assume $F(0) = 0$ in the sequel. Due to the properties of ω, for every $\lambda \geq 0$ we then have the decomposition

$$F(\lambda) = \sum_{l=-\infty}^{\infty} \omega(2^{-l}\lambda)F(\lambda) = \sum_{l=-\infty}^{\infty} F_l(\lambda)\,,$$

where $F_l(\lambda) := \omega(2^{-l}\lambda)F(\lambda)$.

Fix $s > D/2$ and $M \in \mathbb{N}$ with $M > 2s/m$. Further, assume that F fulfills the Hörmander condition (5.20) of order $s+1/2$. For verifying the uniform boundedness of $\sum_{l=-N}^{N} F_l(\sqrt[m]{L})$ in $H^1_L(X)$, we apply Theorem 5.1. To this end, we only need to check that condition (5.1) holds for the operator $\sum_{l=-N}^{N} F_l(\sqrt[m]{L})$ with a constant C_F independent of $N \in \mathbb{N}$.

For each $l \in \mathbb{Z}$ and $r > 0$, we introduce the abbreviations

$$F_{r,M}(\lambda) := F(\lambda)(1 - e^{-(r\lambda)^m})^M ,$$
$$F^l_{r,M}(\lambda) := F_l(\lambda)(I - e^{-(r\lambda)^m})^M = \omega(2^{-l}\lambda)F(\lambda)(1 - e^{-(r\lambda)^m})^M ,$$

where $\lambda \geq 0$. In this notation, we may write

$$F(\sqrt[m]{L})(I - e^{-r^m L})^M = F_{r,M}(\sqrt[m]{L}) = \lim_{N\to\infty} \sum_{l=-N}^{N} F^l_{r,M}(\sqrt[m]{L}) . \tag{5.21}$$

We choose $s' \in (D/2, s)$ and claim that for all $j \in \mathbb{N} \setminus \{1\}$, all $l \in \mathbb{Z}$, and all balls $B \subseteq X$ of radius r

$$\big\|\mathbb{1}_{U_j(B)} F^l_{r,M}(\sqrt[m]{L})\, \mathbb{1}_B\big\|_{2\to 2} \lesssim C_{\omega,s} 2^{-js'} (2^l r)^{-s'} \min\big\{1, (2^l r)^{mM}\big\} \max\big\{1, (2^l r)^{D/2}\big\} , \tag{5.22}$$

where $C_{\omega,s} := \sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^{s+1/2}_2}$ and the implicit constant depends only on m, M, s and the constants in the Davies-Gaffney or doubling condition.

This, together with (5.21), shows that for any $j \in \mathbb{N} \setminus \{1\}$ and any ball $B \subseteq X$ of radius r

$$\begin{aligned}
&\big\|\mathbb{1}_{U_j(B)} F(\sqrt[m]{L})(I - e^{-r^m L})^M \mathbb{1}_B\big\|_{2\to 2} \\
&\lesssim C_{\omega,s} 2^{-js'} \lim_{N\to\infty} \sum_{l=-N}^{N} (2^l r)^{-s'} \min\big\{1, (2^l r)^{mM}\big\} \max\big\{1, (2^l r)^{D/2}\big\} \\
&\leq C_{\omega,s} 2^{-js'} \left(\sum_{l\in\mathbb{Z}:\ 2^l r > 1} (2^l r)^{D/2 - s'} + \sum_{l\in\mathbb{Z}:\ 2^l r \leq 1} (2^l r)^{mM - s'} \right) .
\end{aligned}$$

With a similar reasoning as in the proof of Theorem 3.1 (cf. p. 37) we see that both sums converge and have an upper bound independent of r. This means that (5.1) holds for the function $F(\sqrt[m]{\cdot})$, as desired.

It remains to prove our claim (5.22). Consider a ball $B \subseteq X$ with center $y \in X$ and radius $r > 0$. First, we observe that $\operatorname{supp} F^l_{r,M} \subseteq (2^{l-2}, 2^l)$. Corollary 3.5 a) then says that for any $l \in \mathbb{Z}$ and any $\varepsilon > 0$

$$\big\|F^l_{r,M}(\sqrt[m]{L})\, \mathbb{1}_{B(y,2^{-l})}\big\|_{L^2(X)\to L^2(X,(1+2^l d(\cdot,y))^{2s'} d\mu)} \lesssim \|F^l_{r,M}(2^l \cdot)\|_{H^{s'+1/2+\varepsilon}_2} . \tag{5.23}$$

Let $j \in \mathbb{N} \setminus \{1\}$. For each $x \in U_j(B)$ we obtain, due to $d(x,y) \geq 2^{j-1} r$, the estimate $(1 + 2^l d(x,y))^{s'} \geq 2^{(j-1)s'} (2^l r)^{s'}$. Hence, we get for $\varepsilon := s - s' > 0$

$$\begin{aligned}
&2^{-s'} 2^{js'} (2^l r)^{s'} \big\|\mathbb{1}_{U_j(B)} F^l_{r,M}(\sqrt[m]{L})\, \mathbb{1}_{B(y,2^{-l})}\big\|_{2\to 2} \\
&\quad \leq \big\|\mathbb{1}_{U_j(B)} F^l_{r,M}(\sqrt[m]{L})\, \mathbb{1}_{B(y,2^{-l})}\big\|_{L^2(X)\to L^2(X,(1+2^l d(\cdot,y))^{2s'} d\mu)} \\
&\quad \lesssim \|F^l_{r,M}(2^l \cdot)\|_{H^{s+1/2}_2}
\end{aligned}$$

or equivalently

$$\big\|\mathbb{1}_{U_j(B)} F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y,2^{-l})}\big\|_{2\to 2} \lesssim 2^{-js'}(2^l r)^{-s'}\,\|F^l_{r,M}(2^l\cdot)\|_{H_2^{s+1/2}}\,. \tag{5.24}$$

For $l\in\mathbb{Z}$ with $r\le 2^{-l}$ the left-hand side is an upper bound for $\|\mathbb{1}_{U_j(B)}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_B\|_{2\to 2}$.

In the case $l\in\mathbb{Z}$ with $r>2^{-l}$, we cover $B=B(y,r)$ by balls of radius 2^{-l}. This procedure eventually leads to an additional factor depending on the ratio of r and 2^{-l} and the dimension of the underlying space X. By Lemma 2.2, one can construct a family of points $y_1,\dots,y_K\in B(y,r)$ such that $B(y,r)\subseteq\bigcup_{\nu=1}^K B(y_\nu,2^{-l})$, $K\lesssim(2^l r)^D$, and every $x\in B(y,r)$ is contained in at most M balls $B(y_\nu,2^{-l})$, where M depends only on the constants in the doubling condition. Observe that

$$U_j(B(y,r))\subseteq\bigcup_{\eta=j-1}^{j+1}U_\eta(B(y_\nu,r))$$

for all $j\in\mathbb{N}\setminus\{1\}$ and $\nu\in\{1,2,\dots,K\}$. Therefore, by (5.24), one obtains

$$\begin{aligned}
\big\|\mathbb{1}_{U_j(B(y,r))}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y_\nu,2^{-l})}\big\|_{2\to 2} &\le \sum_{\eta=j-1}^{j+1}\big\|\mathbb{1}_{U_\eta(B(y_\nu,r))}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y_\nu,2^{-l})}\big\|_{2\to 2}\\
&\lesssim \sum_{\eta=j-1}^{j+1}2^{-\eta s'}(2^l r)^{-s'}\|F^l_{r,M}(2^l\cdot)\|_{H_2^{s+1/2}}\\
&\lesssim 2^{-js'}(2^l r)^{-s'}\|F^l_{r,M}(2^l\cdot)\|_{H_2^{s+1/2}}\,.
\end{aligned}$$

Consider $g,h\in L^2(X)$ with $\|g\|_2=1$ and $\|h\|_2=1$. Then we obtain for every $j\in\mathbb{N}\setminus\{1\}$ and every $l\in\mathbb{Z}$ with $r>2^{-l}$

$$\begin{aligned}
\big|\big\langle h,\mathbb{1}_{U_j(B(y,r))}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y,r)}g\big\rangle\big|^2 &= \big|\big\langle\mathbb{1}_{B(y,r)}F^l_{r,M}(\sqrt[m]{L})^*\,\mathbb{1}_{U_j(B(y,r))}h,g\big\rangle\big|^2\\
&\le \big\|\mathbb{1}_{B(y,r)}F^l_{r,M}(\sqrt[m]{L})^*\,\mathbb{1}_{U_j(B(y,r))}h\big\|_2^2\,\|g\|_2^2\\
&= \int_{B(y,r)}\big|F^l_{r,M}(\sqrt[m]{L})^*(\mathbb{1}_{U_j(B(y,r))}h)(x)\big|^2\,d\mu(x)\\
&\le \sum_{\nu=1}^K\int_{B(y_\nu,2^{-l})}\big|F^l_{r,M}(\sqrt[m]{L})^*(\mathbb{1}_{U_j(B(y,r))}h)(x)\big|^2\,d\mu(x)\\
&\le \sum_{\nu=1}^K\big\|\mathbb{1}_{B(y_\nu,2^{-l})}F^l_{r,M}(\sqrt[m]{L})^*\,\mathbb{1}_{U_j(B(y,r))}\big\|_{2\to 2}^2\|h\|_2^2\\
&= \sum_{\nu=1}^K\big\|\mathbb{1}_{U_j(B(y,r))}F^l_{r,M}(\sqrt[m]{L})\,\mathbb{1}_{B(y_\nu,2^{-l})}\big\|_{2\to 2}^2\\
&\lesssim \sum_{\nu=1}^K\big(2^{-js'}(2^l r)^{-s'}\|F^l_{r,M}(2^l\cdot)\|_{H_2^{s+1/2}}\big)^2\,.
\end{aligned}$$

Thus, by taking the supremum over all such g, h and by recalling $\sqrt{K} \lesssim (2^l r)^{D/2}$, we deduce

$$\big\| \mathbb{1}_{U_j(B(y,r))} F^l_{r,M}(\sqrt[m]{L})\, \mathbb{1}_{B(y,r)} \big\|_{2\to 2} \lesssim (2^l r)^{D/2}\, 2^{-js'} (2^l r)^{-s'} \| F^l_{r,M}(2^l \cdot) \|_{H_2^{s+1/2}} \,.$$

In summary, we have shown that

$$\big\| \mathbb{1}_{U_j(B)} F^l_{r,M}(\sqrt[m]{L})\, \mathbb{1}_{B} \big\|_{2\to 2} \lesssim \max\big\{1, (2^l r)^{D/2}\big\}\, 2^{-js'} (2^l r)^{-s'} \| F^l_{r,M}(2^l \cdot) \|_{H_2^{s+1/2}} \tag{5.25}$$

for any $j \in \mathbb{N} \setminus \{1\}$, $l \in \mathbb{Z}$, and any ball $B \subseteq X$ of radius r.

If γ is an integer larger than $s + 1/2$, then it holds

$$\begin{aligned} \| F^l_{r,M}(2^l \cdot) \|_{H_2^{s+1/2}} &= \big\| \lambda \mapsto \omega(\lambda) F(2^l \lambda)(1 - e^{-(2^l r \lambda)^m})^M \big\|_{H_2^{s+1/2}} \\ &\lesssim \| \omega F(2^l \cdot) \|_{H_2^{s+1/2}} \big\| \lambda \mapsto (1 - e^{-(2^l r \lambda)^m})^M \big\|_{C^\gamma([\frac{1}{4},1])} \\ &\lesssim \sup_{n\in\mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_2^{s+1/2}} \min\big\{1, (2^l r)^{mM}\big\} \,. \end{aligned} \tag{5.26}$$

The first inequality is due to Fact 2.16, whereas the second inequality follows from [Blu03, Lemma 3.5].

In view of (5.25) and (5.26), the claim (5.22) is confirmed. This completes the proof. □

Now we discuss the aforementioned spectral multiplier theorem on the Hardy space $H^1_L(X)$ in which an adequate $L^2(X)$-version of the Plancherel condition (3.1), see (5.27) below, is installed. On the one hand, this assumption guarantees that the class of functions for which the multiplier result applies is extended. However, on the other hand, the validity of (5.27) for some $q \in [2, \infty)$ entails the emptiness of the point spectrum of the considered operator (this can be seen in the same manner as for the original Plancherel condition (3.1), cf. p. 38). In order to treat operators with non-empty point spectrum as well, we present a corresponding statement which is valid under the assumptions formulated in part b) by adapting those of Theorem 3.8 to the situation on the Hardy space $H^1_L(X)$.

Theorem 5.5. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ for which Davies-Gaffney estimates of order $m \geq 2$ hold.*

a) *Suppose that there exist $C > 0$ and $q \in [2, \infty]$ such that for any $R > 0$, $y \in X$, and any bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [0, R]$*

$$\big\| F(\sqrt[m]{L})\, \mathbb{1}_{B(y,1/R)} \big\|_{2\to 2} \leq C\, \| F(R\cdot) \|_q \,. \tag{5.27}$$

If $s > \max\{D/2, 1/q\}$ and $F\colon [0,\infty) \to \mathbb{C}$ is a bounded Borel function with

$$\sup_{n\in\mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s} < \infty \,,$$

then there exists a constant $C > 0$ such that for all $f \in H^1_L(X)$

$$\| F(L) f \|_{H^1_L(X)} \leq C \Big(\sup_{n\in\mathbb{Z}} \| \omega F(2^n \cdot) \|_{H_q^s} + |F(0)| \Big) \| f \|_{H^1_L(X)} \,.$$

b) *Fix $\kappa \in \mathbb{N}$ and $q \in [2,\infty)$. Suppose that there is $C > 0$ such that for any $N \in \mathbb{N}$, $y \in X$, and any bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with* $\operatorname{supp} F \subseteq [-1, N+1]$

$$\big\|F(\sqrt[m]{L})\,\mathbb{1}_{B(y,1/N)}\big\|_{2\to 2} \leq C\,\|F(N\cdot)\|_{N^\kappa,q}\,.$$

In addition, assume that for every $\varepsilon > 0$ there is $C > 0$ such that for all $N \in \mathbb{N}$ and all bounded Borel functions $F\colon \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} F \subseteq [-1, N+1]$

$$\big\|F(\sqrt[m]{L})\big\|^2_{H^1_L(X)\to H^1_L(X)} \leq CN^{\kappa D+\varepsilon}\|F(N\cdot)\|^2_{N^\kappa,q}\,. \tag{5.28}$$

Let $s > \max\{D/2, 1/q\}$. Then, for any bounded Borel function $F\colon \mathbb{R} \to \mathbb{C}$ with

$$\sup_{n\in\mathbb{N}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty\,, \tag{5.29}$$

there exists a constant $C > 0$ such that for all $f \in H^1_L(X)$

$$\|F(L)f\|_{H^1_L(X)} \leq C\Big(\sup_{n\in\mathbb{N}} \|\omega F(2^n\cdot)\|_{H^s_q} + \|F\|_\infty\Big)\|f\|_{H^1_L(X)}\,. \tag{5.30}$$

As in the proof of Theorem 5.4, we shall use the tools of weighted norm estimates provided in Chapter 3. Since they were well prepared for the Plancherel condition, they can be adapted to the present situation with only minor changes.

Proof. The proof of the assertion a) follows along the same lines as that of Theorem 5.4 with one small modification. Instead of Corollary 3.5 a) one has to employ part b) of the same corollary to obtain the desired regularity order in the Hörmander condition.

In order to verify the statement b), we imitate our proof of Theorem 3.8. Due to the close resemblance, we will only briefly describe the main steps.

As usual, we are allowed to consider $F(\sqrt[m]{L})$ in place of $F(L)$.

If $F\colon \mathbb{R} \to \mathbb{C}$ is a bounded Borel function with $\operatorname{supp} F \subseteq [0,2]$ and (5.29), the estimate (5.28) with $\varepsilon = N = 1$ yields

$$\big\|F(\sqrt[m]{L})\big\|_{H^1_L(X)\to H^1_L(X)} \lesssim \|F\|_{1,q} \lesssim \|F\|_\infty\,.$$

Therefore, we can restrict ourselves to bounded Borel functions $F\colon \mathbb{R} \to \mathbb{C}$ such that $\operatorname{supp} F \subseteq [1,\infty)$ and (5.29) hold. As in the proof of Theorem 3.8, we put $\omega_l := \omega(2^{-l}\cdot)$ for each $l \in \mathbb{N}$ and write

$$\widetilde{F} := \sum_{l=1}^\infty (\omega_l F) * \xi_{2^{l(\kappa-1)}}\,, \tag{5.31}$$

where $\xi \in C_c^\infty([-1,1])$ and $\xi_N := N\xi(N\cdot)$ for $N \in \mathbb{N}$.

Next, we split

$$F(\sqrt[m]{L}) = \widetilde{F}(\sqrt[m]{L}) + (F - \widetilde{F})(\sqrt[m]{L})$$

and show that both operators on the right-hand side are bounded on $H^1_L(X)$ with the bound we need. The estimate

$$\big\| \widetilde{F}(\sqrt[m]{L}) \big\|_{H^1_L(X) \to H^1_L(X)} \lesssim \sup_{n \in \mathbb{N}} \| \omega F(2^n \cdot) \|_{H^s_q}$$

can be checked by repeating the proof of Theorem 5.4. In view of the modified decomposition (5.31), there arises a convolution term on the left-hand side of (5.22). In order to get rid of this and to obtain a corresponding version of (5.23), one just has to utilize Corollary 3.10 instead of Corollary 3.5 a). Observe that the bound (3.26) in Corollary 3.10 also delivers the desired regularity order in the Hörmander condition.

The second operator $(F - \widetilde{F})(\sqrt[m]{L})$ is bounded on $H^1_L(X)$ with

$$\big\| (F - \widetilde{F})(\sqrt[m]{L}) \big\|_{H^1_L(X) \to H^1_L(X)} \lesssim \sup_{n \in \mathbb{N}} \| \omega F(2^n \cdot) \|_{H^s_q} .$$

The reasoning is similar to that in the proof of Theorem 3.8. Again, one defines the function $H_l := \omega_l F - (\omega_l F) * \xi_{2^{l(\kappa-1)}}$ for each $l \in \mathbb{N}$. Then (5.28) and Fact 3.11 yield

$$\big\| H_l(\sqrt[m]{L}) \big\|^2_{H^1_L(X) \to H^1_L(X)} \lesssim 2^{l(\kappa D + \varepsilon)} \| H_l(2^l \cdot) \|^2_{2^{l\kappa}, q} \lesssim 2^{l(\kappa D + \varepsilon)} 2^{-2l\kappa s} \| (\omega_l F)(2^l \cdot) \|^2_{H^s_q} \tag{5.32}$$

for any $l \in \mathbb{N}$, where $\varepsilon := s - D/2 > 0$. The proof is completed by summing up (5.32). □

The assumption (5.28) seems to be very restrictive and it is not clear how to establish such an estimate. Maybe, one can prove a corresponding version of (5.28) with the operator norm in $H^1_L(X)$ replaced by that in $H^p_L(X)$ for some $p \in (1, 2)$. It turns out that our proof, after obvious modifications, then also works and gives the boundedness of the operator $F(L)$ on $H^p_L(X)$ whenever the function F fulfills the Hörmander condition (5.29) of regularity order $s > \max\{D/2, 1/q\}$. In this case one may think that the regularity assumption $s > \max\{D/2, 1/q\}$ can be weakened. Unfortunately, this is not possible with our technique since in the last step of the proof sketched above, more precisely in the corresponding version of (5.32), we need $s > D/2$ to ensure the summability of the terms on the right-hand side.

By a proof similar to that of Lemma 3.2, we can verify that the Plancherel condition (5.27) is always valid for $q = \infty$. Hence, we get from Theorem 5.5 a) the following result.

Corollary 5.6. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ which satisfies (DG_m) for some $m \geq 2$. If $s > D/2$ and $F\colon [0, \infty) \to \mathbb{C}$ is a bounded Borel function with $\sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{C^s} < \infty$, then $F(L)$ extends to a bounded linear operator on the Hardy space $H^1_L(X)$. To be more precise, there is a constant $C > 0$ such that*

$$\| F(L) \|_{H^1_L(X) \to H^1_L(X)} \leq C \Big(\sup_{n \in \mathbb{Z}} \| \omega F(2^n \cdot) \|_{C^s} + |F(0)| \Big) .$$

6 Boundedness of spectral multipliers on $H^p_L(X)$ and $L^p(X)$

In the last chapter we developed spectral multiplier theorems on the Hardy space $H^1_L(X)$ which ensure the boundedness of the operator $F(L)$ on $H^1_L(X)$, where F is a bounded Borel function satisfying the Hörmander condition of a certain regularity order and L is an injective, non-negative, self-adjoint operator on $L^2(X)$ for which Davies-Gaffney estimates hold. Clearly, by interpolation between the spaces $H^1_L(X)$ and $L^2(X)$, the operator $F(L)$ is then bounded on $H^p_L(X)$ for every $p \in (1,2)$. With this naive approach one deduces boundedness of $F(L)$ on $H^p_L(X)$ by requiring the same regularity order in the Hörmander condition as for the boundedness of $F(L)$ on $H^1_L(X)$.

Since self-adjoint operators admit the classical functional calculus on $L^2(X)$, allowing arbitrary bounded Borel functions $\mathbb{R} \to \mathbb{C}$ without any regularity hypothesis, one expects that the regularity assumptions on F can be weakened when one asks about boundedness of $F(L)$ on $H^p_L(X)$ for some $p \in (1,2)$. This is actually true, as the interpolation procedure described in [Kri09, Section 4.6.1] shows. In order to apply this method, we first introduce the setting of [Kri09].

As usual, take a non-negative function $\omega \in C_c^\infty(\mathbb{R})$ with

$$\operatorname{supp} \omega \subseteq (1/4, 1) \qquad \text{and} \qquad \sum_{n \in \mathbb{Z}} \omega(2^{-n}\lambda) = 1 \quad \text{for all } \lambda > 0 \,.$$

Let $q \geq 2$ and $s > 1/q$ be fixed. The *Hörmander class* $\mathcal{H}^s_{q,\omega}$ *(of regularity order s in L^q)* is said to be the space consisting of all $F \in L^q_{loc}(0,\infty)$ for which the Hörmander condition $\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} < \infty$ holds, i.e.

$$\mathcal{H}^s_{q,\omega} := \left\{ F \in L^q_{loc}(0,\infty) : \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} < \infty \right\} .$$

If $\mathcal{H}^s_{q,\omega}$ is equipped with the norm

$$\|F\|_{\mathcal{H}^s_{q,\omega}} := \sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} \,,$$

then the Hörmander class $\mathcal{H}^s_{q,\omega}$ becomes a Banach algebra, i.e. $\mathcal{H}^s_{q,\omega}$ is a Banach space and there exists a constant $C > 0$ such that $\|FG\|_{\mathcal{H}^s_{q,\omega}} \leq C \, \|F\|_{\mathcal{H}^s_{q,\omega}} \|G\|_{\mathcal{H}^s_{q,\omega}}$ for all $F, G \in \mathcal{H}^s_{q,\omega}$ (see [Kri09, Propositions 4.8 and 4.11]). According to Lemma 2.18, different choices of ω

lead to the same spaces $\mathcal{H}^s_{q,\omega}$ with equivalent norms. Therefore, it is justified to drop the subscript ω in the notation and simply write $\mathcal{H}^s_q$ in the sequel.

The embedding properties of the Bessel potential spaces, recalled in Fact 2.13, confer on the Hörmander classes (see [Kri09, Proposition 4.9]).

Fact 6.1. *If $2 \le q \le p < \infty$ and $0 < t - 1/p \le s - 1/q$, then it holds*

$$\mathcal{H}^s_q \hookrightarrow \mathcal{H}^t_p .$$

Suppose that $q \ge 2$ and $s > 1/q$. As already remarked after Definition 2.17, every $F \in \mathcal{H}^s_q$ is bounded and, conversely, it is clear that every bounded Borel function $(0,\infty) \to \mathbb{C}$ is contained in $L^q_{loc}(0,\infty)$. Hence, the Hörmander class $\mathcal{H}^s_q$ matches with the set of restrictions on $(0,\infty)$ of the bounded Borel functions $[0,\infty) \to \mathbb{C}$ which were considered in the spectral multiplier theorems presented in the foregoing chapter. However, since the Hörmander condition $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty$ contains no information concerning the value of $F(0)$, the value $F(0)$ is not regarded in the Hörmander class. But this causes no problems as long as one treats injective operators.

Let $p \in [1,2]$, $q \ge 2$, and $s > 1/q$. An injective, non-negative, self-adjoint operator L on $L^2(X)$ which fulfills Davies-Gaffney estimates is said to have a *$\mathcal{H}^s_q$ calculus on $H^p_L(X)$* if there exists a constant $C > 0$ such that for all $F \in \mathcal{H}^s_q$

$$\|F(L)\|_{H^p_L(X)\to H^p_L(X)} \le C \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} .$$

Note that, thanks to Theorem 5.4, L has a $\mathcal{H}^s_2$ calculus on $H^1_L(X)$ for any $s > (D+1)/2$ whenever L is injective, non-negative, and self-adjoint on $L^2(X)$ and satisfies (DG_m) for some $m \ge 2$.

The interpolation statement concerning the Hörmander functional calculus, adapted to the present situation, reads as follows (cf. [Kri09, Corollary 4.84]).

Fact 6.2. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ such that Davies-Gaffney estimates (DG_m) hold for some $m \ge 2$. Assume that L has a $\mathcal{H}^s_q$ calculus on the Hardy space $H^1_L(X)$ for some $q \ge 2$ and $s > 1/q$. Then, for any $\theta \in (0,1)$, the operator L has a $\mathcal{H}^{s_\theta}_{q_\theta}$ calculus on $[L^2(X), H^1_L(X)]_\theta$ whenever $s_\theta > \theta s$ and $q_\theta > q/\theta$.*

With the help of this interpolation result, we are able to prove a spectral multiplier theorem on the Hardy space $H^p_L(X)$ for each $p \in [1,2]$. We also state a version including the Plancherel condition which yields a weakened regularity order in the Hörmander condition.

Theorem 6.3. *Let L be an injective, non-negative, self-adjoint operator on $L^2(X)$ satisfying Davies-Gaffney estimates (DG_m) for some $m \ge 2$.*

a) *Fix* $p \in [1,2]$. *Let* $s > (D+1)(1/p - 1/2)$ *and* $1/q < 1/p - 1/2$. *Then* L *has a* $\mathcal{H}_q^s$ *calculus on* $H_L^p(X)$, *i.e. for every bounded Borel function* $F\colon (0,\infty) \to \mathbb{C}$ *with* $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty$, *there exists a constant* $C > 0$ *such that*

$$\|F(L)\|_{H_L^p(X)\to H_L^p(X)} \leq C \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s}\,.$$

b) *Assume further that* L *fulfills the Plancherel condition (5.27) for some* $q_0 \in [2,\infty)$. *Fix* $p \in [1,2]$. *Let* $s > \max\{D, 2/q_0\}\,(1/p - 1/2)$ *and* $1/q < 2/q_0\,(1/p - 1/2)$. *Then* L *has a* $\mathcal{H}_q^s$ *calculus on* $H_L^p(X)$, *i.e. for every bounded Borel function* $F\colon (0,\infty) \to \mathbb{C}$ *with* $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty$, *there exists a constant* $C > 0$ *such that*

$$\|F(L)\|_{H_L^p(X)\to H_L^p(X)} \leq C \sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s}\,.$$

Proof. Let $p \in [1,2]$. The assertion of part a) follows directly by combining Theorem 5.4 and Fact 6.2 with $\theta := 2(1/p - 1/2)$.

Suppose that L additionally satisfies the Plancherel condition (5.27) for some $q_0 \in [2,\infty)$. Then Theorem 5.5 a) applies, and it follows from (5.30) and (2.11) that L has a $\mathcal{H}_{q_0}^s$ calculus on $H_L^1(X)$ for each $s > \max\{D/2, 1/q_0\}$. Now Fact 6.2 with $\theta := 2(1/p-1/2)$ yields b). □

If the operator L actually enjoys generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$) for some $p_0 \in [1,2)$, then Theorem 4.19 ensures $H_L^p(X) = L^p(X)$ for every $p \in (p_0, 2]$. Therefore we deduce from Theorem 6.3 a spectral multiplier theorem on the Lebesgue space $L^p(X)$ as well. The regularity assumptions in our statement are weakened compared to that of [Blu03, Theorem 1.1]. As we will see at the end of this chapter, our result also ameliorates [Kri09, Theorem 4.95] concerning the regularity order.

Theorem 6.4. *Let* L *be a non-negative, self-adjoint operator on* $L^2(X)$ *such that generalized Gaussian estimates* ($GGE_{p_0,m}$) *hold for some* $p_0 \in [1,2)$ *and* $m \geq 2$.

a) *For fixed* $p \in (p_0, p_0')$ *suppose that* $s > (D+1)\,|1/p-1/2|$ *and* $1/q < |1/p-1/2|$. *Then, for every bounded Borel function* $F\colon [0,\infty) \to \mathbb{C}$ *with* $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty$, *the operator* $F(L)$ *is bounded on the Lebesgue space* $L^p(X)$. *More precisely, there exists a constant* $C > 0$ *such that*

$$\|F(L)\|_{p\to p} \leq C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} + |F(0)|\Big)\,.$$

b) *In addition, assume that* L *fulfills the Plancherel condition (5.27) for some* $q_0 \in [2,\infty)$. *Fix* $p \in (p_0, p_0')$. *Let* $s > \max\{D, 2/q_0\}\,|1/p - 1/2|$ *and* $1/q < 2/q_0\,|1/p - 1/2|$. *Then, for every bounded Borel function* $F\colon [0,\infty) \to \mathbb{C}$ *with* $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} < \infty$, *the operator* $F(L)$ *is bounded on the Lebesgue space* $L^p(X)$. *More precisely, there exists a constant* $C > 0$ *such that*

$$\|F(L)\|_{p\to p} \leq C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H_q^s} + |F(0)|\Big)\,.$$

Proof. Let $p \in (p_0, 2)$. We shall prove both assertions simultaneously. For the proof of part a) suppose that $s > (D+1)(1/p - 1/2)$ and $1/q < 1/p - 1/2$, whereas for the proof of part b) suppose that L fulfills the Plancherel condition (5.27) for some $q_0 \in [2, \infty)$ as well as $s > \max\{D, 2/q_0\}\,(1/p - 1/2)$ and $1/q < 2/q_0\,(1/p - 1/2)$.

Since injectivity of L is not assumed, Theorem 6.3 cannot be applied directly. In order to overcome this difficulty, one can use the concept of [KW04, Proposition 15.2] (see also [CDMY96, Theorem 3.8]) that provides a decomposition of the space $L^2(X)$ as the orthogonal sum of the closure of the range $\overline{R(L)}$ of L and the null space $N(L)$ of L. The operator L then takes the form

$$L = \begin{pmatrix} L_0 & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $L^2(X) = \overline{R(L)} \oplus N(L)$, where L_0 is the part of L in $\overline{R(L)}$, i.e. the restriction of L to $\mathcal{D}(L_0) := \{x \in \overline{R(L)} \cap \mathcal{D}(L) : Lx \in \overline{R(L)}\}$. But L_0 is injective on its domain, so that Theorem 6.3 applies to L_0. This approach was already made in [Kri09, Section 4.6.1] and, as remarked in [Kri09, Illustration 4.87], the decomposition and the interpolation result cited in Fact 6.2 can be combined. Hence, L_0 has a $\mathcal{H}^s_q$ calculus on $H^p_{L_0}(X)$. Consider a bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q} < \infty$. Then it holds

$$F(L) = \begin{pmatrix} (F|_{(0,\infty)})(L_0) & 0 \\ 0 & F(0)\, I_{N(L)} \end{pmatrix}$$

on $H^p_{L_0}(X) \cap L^2(X)$. Because of $F|_{(0,\infty)} \in \mathcal{H}^s_q$, one has moreover

$$\|(F|_{(0,\infty)})(L_0)\|_{H^p_{L_0}(X) \to H^p_{L_0}(X)} \lesssim \sup_{n\in\mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_q}$$

as well as

$$\|F(0)\, I_{N(L)}\|_{H^p_{L_0}(X) \to H^p_{L_0}(X)} \le |F(0)|\,.$$

Since, by Theorem 4.19, the spaces $H^p_{L_0}(X)$ and $L^p(X)$ coincide, the statements a) and b) are proven for any $p \in (p_0, 2)$.

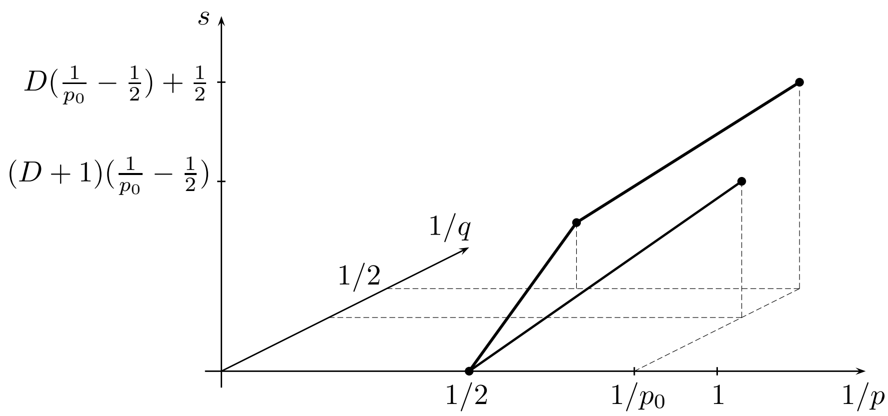
Let $p \in (2, p_0')$. Due to the self-adjointness of L on $L^2(X)$, boundedness of spectral multipliers on $L^p(X)$ follows by the case proved above and dualization. The claim for $p = 2$ is trivial. □

By standard methods (cp. remark after Theorem 3.1), Theorem 6.4 can be extended to the case when X is replaced by an open subset of a space of homogeneous type.

At the end of this chapter we remark that the assertion of Theorem 6.4 a) gives an improvement of the result in [Kri09, Theorem 4.95] concerning the regularity order in the Hörmander condition.

Let L be a non-negative, self-adjoint operator on $L^2(X)$ such that generalized Gaussian estimates ($\mathrm{GGE}_{p_0,m}$) hold for some $p_0 \in [1,2)$ and $m \geq 2$. Take an arbitrary $p \in (p_0, p_0')$. Then [Kri09, Theorem 4.95] ensures that L has a $\mathcal{H}_2^\alpha$ calculus on $L^p(X)$ provided that $\alpha > D\,|\frac{1}{p} - \frac{1}{2}| + \frac{1}{2}$, whereas the statement of Theorem 6.4 a) says that L has a $\mathcal{H}_q^s$ calculus on $L^p(X)$ whenever $s > (D+1)\,|1/p - 1/2|$ and $1/q < |1/p - 1/2|$.

It follows from Fact 6.1 that the spectral multiplier theorem given in Theorem 6.4 a) is applicable to a wider class of functions than that of [Kri09, Theorem 4.95]. The comparison is illustrated in the figure below.



7 Applications

In this chapter we give examples of operators to which our results of the preceding chapters apply. They have in common that, in general, classical Gaussian estimates are not satisfied.

In the first section we discuss the Maxwell operator. This operator is of great importance in the studies of electrodynamics. Following the outline in [CK98, Chapter 6], we explain briefly its appearance without demand for mathematical precision. The Maxwell equations

$$\operatorname{rot}\mathcal{E} + \partial_t\mathcal{H} = 0\,, \qquad \operatorname{rot}\mathcal{H} - \varepsilon(\cdot)\partial_t\mathcal{E} = 0\,, \qquad \operatorname{div}\mathcal{H} = 0 \qquad \text{in } \Omega$$

govern the propagation of electromagnetic waves in a region $\Omega \subseteq \mathbb{R}^3$. Here, $\mathcal{E}\colon \Omega \times \mathbb{R} \to \mathbb{R}^3$ and $\mathcal{H}\colon \Omega \times \mathbb{R} \to \mathbb{R}^3$ denote the electric and magnetic field, respectively, whereas the matrix-valued function $\varepsilon(\cdot)\colon \Omega \to \mathbb{R}^{3\times 3}$ describes the electric permittivity. The magnetic permeability was taken to be the identity matrix and the electric conductivity to be zero. We take perfect conductor boundary conditions

$$\nu \times \mathcal{E} = 0\,, \qquad \nu \cdot \mathcal{H} = 0 \qquad \text{on } \partial\Omega\,.$$

If the waves behave time periodically with respect to the same frequency $\omega > 0$, the ansatz $\mathcal{E}(x,t) = e^{-i\omega t}E(x)$ and $\mathcal{H}(x,t) = e^{-i\omega t}H(x)$ lead to the so-called time-harmonic Maxwell equations $\operatorname{rot} E - i\omega H = 0$ and $\operatorname{rot} H + i\omega\varepsilon(\cdot)E = 0$. Elimination of E finally yields

$$\begin{aligned} \operatorname{rot}\varepsilon(\cdot)^{-1}\operatorname{rot} H - \omega^2 H &= 0 \qquad \text{in } \Omega\,,\\ \operatorname{div} H &= 0 \qquad \text{in } \Omega\,,\\ \nu \cdot H &= 0 \qquad \text{in } \partial\Omega\,,\\ \nu \times \varepsilon(\cdot)^{-1}\operatorname{rot} H &= 0 \qquad \text{in } \partial\Omega\,. \end{aligned}$$

The operator $\operatorname{rot}\varepsilon(\cdot)^{-1}\operatorname{rot}$ then corresponds to the Maxwell operator.

Our studies of the Maxwell operator take place in the following framework. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $\varepsilon(\cdot) \in L^\infty(\Omega, \mathbb{C}^{3\times 3})$ a matrix-valued function such that $\varepsilon(\cdot)^{-1} \in L^\infty(\Omega, \mathbb{C}^{3\times 3})$ and $\varepsilon(x) \in \mathbb{C}^{3\times 3}$ is a positive definite, hermitian matrix for almost all $x \in \Omega$. We emphasize that no regularity assumptions on $\varepsilon(\cdot)$ are made. In a first step we introduce, inspired by the approach in [MM09], an operator A_2 acting on $L^2(\Omega, \mathbb{C}^3)$ via the form method in such a manner that, in the special case of $\varepsilon(x)$ being the identity matrix for any $x \in \Omega$, the operator A_2 can be characterized by $A_2 u = \operatorname{rot}\operatorname{rot} u - \nabla \operatorname{div} u$ for each $u \in L^2(\Omega, \mathbb{C}^3)$ such that $\operatorname{rot} u, \operatorname{rot}\operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3)$, $\operatorname{div} u \in H^1(\Omega, \mathbb{C})$, $\nu \cdot u|_{\partial\Omega} = 0$, and $\nu \times \operatorname{rot} u|_{\partial\Omega} = 0$. We then define the Maxwell operator M_2 as the restriction of A_2 on the space of divergence-free functions.

By using Davies' perturbation method, we will establish generalized Gaussian estimates for the operator A_2 (cf. Theorem 7.3). Since the Helmholtz projection and A_2 are commuting (cf. Lemma 7.7), many properties of A_2 transfer to the Maxwell operator M_2, including the validity of spectral multiplier theorems (cf. Theorem 7.10).

The subsequent two sections are devoted to the study of the Stokes operator and the Lamé operator. Our arguments are based on recently published results due to M. Mitrea and S. Monniaux ([MM09] and [MM10], respectively) in which certain two-ball estimates for the resolvents of these operators were verified. We shall prove that these kind of bounds entail generalized Gaussian estimates for the corresponding semigroup operators (cf. Lemma 7.12). Since the authors used them only with regard to obtaining analyticity of the associated semigroup on L^p and maximal regularity in L^p for p ranging in some interval containing 2, we close the thesis by presenting further consequences of generalized Gaussian estimates for the Lamé operator which were not mentioned so far. Besides the property of having a bounded holomorphic functional calculus, we state the L^p-independence of the spectrum.

Unfortunately, there does not seem to be any literature on the Plancherel condition for the Maxwell, Stokes or Lamé operator. Even for operators satisfying Gaussian estimates in the classical sense the situation is far from being understood (cf. [DOS02]).

7.1 Maxwell operator

First, we provide a short overview of the definitions and some basic properties of the natural function spaces needed for introducing the Maxwell operator. We start with the specification of the underlying domain.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$, i.e. a bounded, connected, open subset of $\mathbb{R}^3$ with a Lipschitz continuous boundary $\partial\Omega$. Roughly speaking, this means that $\partial\Omega$ can be locally represented by graphs of Lipschitz continuous functions in two variables and Ω lies locally on one side of the graph. As the regularity property of $\partial\Omega$ is crucial for the validity of Sobolev embedding results and our arguments will depend heavily on the latter, we cite the precise definition. Its formulation is taken from [ABDG98, Notation 2.1].

Definition 7.1. A bounded domain Ω in $\mathbb{R}^3$ is said to be *Lipschitz* if for any point x on the boundary $\partial\Omega$ there exist a system of orthogonal coordinates (y_1, y_2, y_3), a cube U_x containing x, $U_x =: (-a_1, a_1) \times (-a_2, a_2) \times (-a_3, a_3)$, and a Lipschitz continuous mapping Φ_x defined from $(-a_1, a_1) \times (-a_2, a_2)$ into $(-\frac{1}{2}a_3, \frac{1}{2}a_3)$ such that

$$\begin{aligned} \Omega \cap U_x &= \big\{(y_1, y_2, y_3) \in U_x : \, y_3 > \Phi_x(y_1, y_2)\big\} \, , \\ \partial\Omega \cap U_x &= \big\{(y_1, y_2, y_3) \in U_x : \, y_3 = \Phi_x(y_1, y_2)\big\} \, . \end{aligned}$$

Let us mention that this definition allows domains with corners, but cuts or cusps are excluded. Further, we remark that a Lipschitz continuous boundary $\partial\Omega$ enables the definition of a unit exterior normal field $\nu\colon \partial\Omega \to \mathbb{R}^3$ almost everywhere on the boundary. Additionally, the surface measure $d\sigma$ is well defined on $\partial\Omega$.

We consider the differential operators divergence div and rotation rot on $L^2(\Omega, \mathbb{C}^3)$ in the distributional sense and first recall their definitions. For $n \in \{1, 3\}$ denote by $\mathscr{D}(\Omega, \mathbb{C}^n)$ the space of test functions, whereas $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathscr{D}(\Omega, \mathbb{C}^n)$ and its dual $\mathscr{D}'(\Omega, \mathbb{C}^n)$. For a vector-valued function $u = (u_1, u_2, u_3) \in L^2(\Omega, \mathbb{C}^3)$ the *divergence* $\operatorname{div} u$ is given by

$$\langle \operatorname{div} u, \varphi \rangle := -\int_\Omega u \cdot \nabla\varphi \, dx = -\int_\Omega \big(u_1 \, \partial_1\varphi + u_2 \, \partial_2\varphi + u_3 \, \partial_3\varphi\big) \, dx$$

for any $\varphi \in \mathscr{D}(\Omega, \mathbb{C})$ and the *rotation* $\operatorname{rot} u$ by

$$\langle \operatorname{rot} u, \varphi \rangle := \int_\Omega u \cdot \operatorname{rot} \varphi \, dx = \int_\Omega u_1(\partial_2\varphi_3 - \partial_3\varphi_2) + u_2(\partial_3\varphi_1 - \partial_1\varphi_3) + u_3(\partial_1\varphi_2 - \partial_2\varphi_1) \, dx$$

for any $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathscr{D}(\Omega, \mathbb{C}^3)$. The domains of these operators in $L^2(\Omega, \mathbb{C}^3)$ are

$$\begin{aligned} H(\operatorname{div}, \Omega) &:= \big\{ u \in L^2(\Omega, \mathbb{C}^3) \, : \, \operatorname{div} u \in L^2(\Omega, \mathbb{C}) \big\} , \\ H(\operatorname{rot}, \Omega) &:= \big\{ u \in L^2(\Omega, \mathbb{C}^3) \, : \, \operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3) \big\} \end{aligned}$$

equipped with their natural graph norms

$$\begin{aligned} \|u\|_{H(\operatorname{div},\Omega)} &:= \|u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{div} u\|_{L^2(\Omega,\mathbb{C})} , \\ \|u\|_{H(\operatorname{rot},\Omega)} &:= \|u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{rot} u\|_{L^2(\Omega,\mathbb{C}^3)} , \end{aligned}$$

respectively.

As $L^2(\Omega, \mathbb{C}^3)$ consists of equivalence classes of functions that are defined on Ω almost everywhere and $\partial\Omega$ is a set of Lebesgue measure zero, we have to specify the meaning of the boundary values $\nu \cdot u|_{\partial\Omega}$ or $\nu \times u|_{\partial\Omega}$ for $u \in H(\operatorname{div}, \Omega)$ or $u \in H(\operatorname{rot}, \Omega)$. This will be achieved via the normal component trace mapping $\eta_n(u) := \nu \cdot u|_{\partial\Omega}$ and the tangential trace mapping $\eta_t(u) := \nu \times u|_{\partial\Omega}$ which are initially defined for $u \in \mathscr{D}(\overline{\Omega}, \mathbb{C}^3)$. By exploiting the density of $\mathscr{D}(\overline{\Omega}, \mathbb{C}^3)$ in $H(\operatorname{div}, \Omega)$ and $H(\operatorname{rot}, \Omega)$ (see e.g. [GR86, Theorems 2.4 and 2.10 in Chapter I]), one is able to show that the normal trace $\eta_n\colon \mathscr{D}(\overline{\Omega}, \mathbb{C}^3) \to L^\infty(\partial\Omega, \mathbb{C})$ and the tangential trace $\eta_t\colon \mathscr{D}(\overline{\Omega}, \mathbb{C}^3) \to L^\infty(\partial\Omega, \mathbb{C}^3)$ can be extended by continuity to bounded linear mappings $H(\operatorname{div}, \Omega) \to H^{-1/2}(\partial\Omega, \mathbb{C})$ and $H(\operatorname{rot}, \Omega) \to H^{-1/2}(\partial\Omega, \mathbb{C}^3)$, respectively (see e.g. [GR86, Theorems 2.5 and 2.11 in Chapter I]). Here, for $n \in \{1, 3\}$, $H^{-1/2}(\partial\Omega, \mathbb{C}^n)$ denotes the dual space of $H^{1/2}(\partial\Omega, \mathbb{C}^n)$. In the sequel we shall only write $H(\operatorname{div}, \Omega) \to H^{-1/2}(\partial\Omega, \mathbb{C})$, $u \mapsto \nu \cdot u|_{\partial\Omega}$ and $H(\operatorname{rot}, \Omega) \to H^{-1/2}(\partial\Omega, \mathbb{C}^3)$, $u \mapsto \nu \times u|_{\partial\Omega}$ for the extended mappings.

The theory sketched above leads to the following integration by parts formulae (see e.g. [GR86, (2.17) and (2.22) in Chapter I])

$$(\operatorname{div} u, v)_{L^2(\Omega,\mathbb{C})} + (u, \nabla v)_{L^2(\Omega,\mathbb{C}^3)} = \int_{\partial\Omega} (\nu \cdot u)\, \overline{v}\, d\sigma$$

for each $u \in H(\operatorname{div}, \Omega)$ and $v \in H^1(\Omega, \mathbb{C})$ as well as

$$(\operatorname{rot} u, v)_{L^2(\Omega,\mathbb{C}^3)} - (u, \operatorname{rot} v)_{L^2(\Omega,\mathbb{C}^3)} = \int_{\partial\Omega} (\nu \times u) \cdot \overline{v}\, d\sigma$$

for each $u \in H(\operatorname{rot}, \Omega)$ and $v \in H^1(\Omega, \mathbb{C}^3)$.

Now we are prepared for defining the function space

$$V(\Omega) := \big\{u \in L^2(\Omega, \mathbb{C}^3) \,:\, \operatorname{div} u \in L^2(\Omega, \mathbb{C}),\, \operatorname{rot} u \in L^2(\Omega, \mathbb{C}^3),\, \nu \cdot u|_{\partial\Omega} = 0\big\} \tag{7.1}$$

equipped with the inner product

$$(u, v)_{V(\Omega)} := (u, v)_{L^2(\Omega,\mathbb{C}^3)} + (\operatorname{div} u, \operatorname{div} v)_{L^2(\Omega,\mathbb{C})} + (\operatorname{rot} u, \operatorname{rot} v)_{L^2(\Omega,\mathbb{C}^3)}\,.$$

Then $V(\Omega)$ becomes a Hilbert space which is dense in $L^2(\Omega, \mathbb{C}^3)$ since it includes $\mathscr{D}(\Omega, \mathbb{C}^3)$. In general, $V(\Omega)$ is not contained in $H^1(\Omega, \mathbb{C}^3)$ (cf. e.g. [ABDG98, p. 832] for a counter-example consisting of a domain Ω with a "re-entrant edge"). However, under additional assumptions on the domain Ω the space $V(\Omega)$ is continuously embedded into $H^1(\Omega, \mathbb{C}^3)$. For example, this is the case when Ω has a $C^{1,1}$-boundary or Ω is convex (cf. e.g. [ABDG98, Theorems 2.9 and 2.17]).

Nevertheless, the following statement due to D. Mitrea, M. Mitrea, and M. Taylor ([MMT01, p. 87]) holds for arbitrary bounded Lipschitz domains Ω in $\mathbb{R}^3$ (see also [Cos90] for a simpler version of the result that is valid for bounded, simply connected domains in $\mathbb{R}^3$ with connected Lipschitz boundary).

Fact 7.2. *The space $V(\Omega)$ is continuously embedded into $H^{1/2}(\Omega, \mathbb{C}^3)$. More precisely, there exists a constant $C > 0$ depending only on the boundary $\partial\Omega$ and on the diameter diam(Ω) of Ω such that for every $u \in V(\Omega)$*

$$\|u\|_{H^{1/2}(\Omega,\mathbb{C}^3)} \le C\big(\|u\|_{L^2(\Omega,\mathbb{C}^3)} + \|\operatorname{div} u\|_{L^2(\Omega,\mathbb{C})} + \|\operatorname{rot} u\|_{L^2(\Omega,\mathbb{C}^3)}\big)\,.$$

Now we turn towards the definition of the Maxwell operator on $L^2(\Omega, \mathbb{C}^3)$ which will be given in a quite general framework without stating any regularity assumptions on the coefficient matrix.

In a first stage, we introduce a form $\mathfrak{a}$ with the form domain $V(\Omega)$ and establish generalized Gaussian estimates for its associated operator A_2 on $L^2(\Omega, \mathbb{C}^3)$ (cf. Theorem 7.3) by using Davies' perturbation method (see e.g. [Dav95], [LSV02, Section 2]). To the best of our knowledge, this procedure was never elaborated before in this context.

The Maxwell operator M_2 on $\mathbb{P}_2 L^2(\Omega,\mathbb{C}^3)$ is then defined as the restriction of A_2 on the subspace of the divergence-free functions. Since A_2 and the Helmholtz projection $\mathbb{P}_2$ are commuting, the Maxwell operator M_2 inherits many properties from A_2.

Fix, once and for all, a matrix-valued function $\varepsilon(\cdot) \in L^\infty(\Omega,\mathbb{C}^{3\times 3})$ taking values in the set of positive definite, hermitian matrices. Assume additionally that $\varepsilon(\cdot)^{-1} \in L^\infty(\Omega,\mathbb{C}^{3\times 3})$. As immediate consequences we have that, for almost every $x \in \Omega$, the matrix $\varepsilon(x)^{-1}$ is also hermitian and that $\varepsilon(\cdot)^{-1}$ fulfills the following *uniform ellipticity condition*

$$\varepsilon(x)^{-1}\xi \cdot \overline{\xi} \geq \varepsilon_0 |\xi|^2 \tag{7.2}$$

for all $\xi \in \mathbb{C}^3$ and almost all $x \in \Omega$, where the constant $\varepsilon_0 > 0$ is independent of ξ and x. We consider the densely defined, sesquilinear form

$$\mathfrak{a}(u,v) := \int_\Omega \varepsilon(\cdot)^{-1} \mathrm{rot}\, u \cdot \overline{\mathrm{rot}\, v}\, dx + \int_\Omega \mathrm{div}\, u\, \overline{\mathrm{div}\, v}\, dx \qquad (u,v \in \mathcal{D}(\mathfrak{a}))$$

with form domain $\mathcal{D}(\mathfrak{a}) := V(\Omega)$. Due to the properties of the coefficient matrix $\varepsilon(\cdot)^{-1}$, the form $\mathfrak{a}$ is continuous and coercive in the sense that there exist constants $C_1 \geq 0$, $C_2 > 0$ such that for all $u \in V(\Omega)$

$$\mathrm{Re}\, \mathfrak{a}(u,u) + C_1 \|u\|^2_{L^2(\Omega,\mathbb{C}^3)} \geq C_2 \|u\|^2_{V(\Omega)} \tag{7.3}$$

(in fact one can take $C_1 = C_2 = \min\{\varepsilon_0, 1\}$). If the *operator A_2 associated with the form $\mathfrak{a}$* is defined via

$$u \in \mathcal{D}(A_2),\ A_2 u = f \text{ if and only if } u \in V(\Omega) \text{ and } \mathfrak{a}(u,v) = (f,v)_{L^2(\Omega,\mathbb{C}^3)} \text{ for all } v \in V(\Omega),$$

then A_2 is sectorial and $-A_2$ generates an analytic semigroup acting on $L^2(\Omega,\mathbb{C}^3)$ (see e.g. [DL88, p. 450]) which is actually bounded because of $\mathfrak{a}(u,u) \geq 0$ for all $u \in V(\Omega)$. Since the form $\mathfrak{a}$ is symmetric, the operator A_2 is even self-adjoint.

In the special case of $\varepsilon(x)$ being the identity matrix for all $x \in \Omega$ the associated operator A_2 with the form $\mathfrak{a}$ is given by

$$\begin{aligned} \mathcal{D}(A_2) &= \big\{ u \in V(\Omega) :\ \mathrm{rot}\, u \in H(\mathrm{rot},\Omega),\ \mathrm{div}\, u \in H^1(\Omega,\mathbb{C}),\ \nu \times \mathrm{rot}\, u|_{\partial\Omega} = 0 \big\}, \\ A_2 u &= \mathrm{rot}\,\mathrm{rot}\, u - \nabla \mathrm{div}\, u = -\Delta u \qquad \text{for } u \in \mathcal{D}(A_2)\,. \end{aligned}$$

This characterization was proved by M. Mitrea and S. Monniaux ([MM09, (3.17) and (3.18)]). They called the boundary conditions of the set $\mathcal{D}(A_2)$, namely $\nu \cdot u|_{\partial\Omega} = 0$ and $\nu \times \mathrm{rot}\, u|_{\partial\Omega} = 0$, *Hodge boundary conditions*.

If $\varepsilon(\cdot)$ is smooth, one can show by adapting the arguments of M. Mitrea and S. Monniaux

$$\mathcal{D}(A_2) = \big\{u \in V(\Omega) : \operatorname{rot}\varepsilon(\cdot)^{-1}\operatorname{rot} u \in L^2(\Omega,\mathbb{C}^3),\, \operatorname{div} u \in H^1(\Omega,\mathbb{C}),\, \nu \times \varepsilon(\cdot)^{-1}\operatorname{rot} u|_{\partial\Omega} = 0\big\},$$
$$A_2 u = \operatorname{rot}\varepsilon(\cdot)^{-1}\operatorname{rot} u - \nabla\operatorname{div} u \qquad \text{for } u \in \mathcal{D}(A_2)\,.$$

Since in the present situation no regularity assumptions on $\varepsilon(\cdot)$ were made, it seems to be unlikely that a corresponding description of A_2 is possible.

It turns out that we do not need to know about the concrete domain of A_2 in order to establish generalized Gaussian estimates for A_2. This is caused by the fact that our approach is based on Davies' perturbation method for which only the form $\mathfrak{a}$ and the abstract definition of its associated operator A_2 are relevant.

Theorem 7.3. *The operator A_2 associated with the form $\mathfrak{a}$ enjoys generalized Gaussian $(3/2,3)$-estimates of order 2.*

Proof. We just have to show that A_2 fulfills generalized Gaussian $(2,3)$-estimates of order 2. Thanks to the self-adjointness of A_2, generalized Gaussian $(3/2,2)$-estimates then follows by dualization and the claimed generalized Gaussian $(3/2,3)$-estimates by composition and the semigroup law.

The proof is divided into several steps. The first three steps are devoted to establishing Davies-Gaffney estimates (DG_2) for the operator families $(e^{-tA_2})_{t>0}$, $\{t^{1/2}\operatorname{div} e^{-tA_2} : t > 0\}$, and $\{t^{1/2}\operatorname{rot} e^{-tA_2} : t > 0\}$. In order to derive these bounds, we will use Davies' perturbation method. It consists in studying "twisted" forms

$$\mathfrak{a}_{\varrho\phi}(u,v) := \mathfrak{a}(e^{\varrho\phi}u, e^{-\varrho\phi}v)$$

for $u, v \in V(\Omega)$, where $\varrho \in \mathbb{R}$ and $\phi \in C_c^\infty(\overline{\Omega})$ is a real-valued function satisfying $\|\partial_j\phi\|_\infty \le 1$ for all $j \in \{1,2,3\}$. The space of all such functions ϕ will be denoted by $\mathcal{E}$. Observe that the multiplication with a function of the form $e^{\varrho\phi}$ leaves the space $V(\Omega)$ invariant and hence the form $\mathfrak{a}_{\varrho\phi}$ is well-defined.

In the remaining two steps we deduce generalized Gaussian $(2,3)$-estimates for A_2 by combining the Davies-Gaffney estimates and the Sobolev embedding theorem.

Step 1: We claim that for each $\gamma \in (0,1)$ there exists a constant $\omega_0 \ge 0$ such that for all $u \in V(\Omega)$, $\varrho \in \mathbb{R}$, and $\phi \in \mathcal{E}$

$$\big|\mathfrak{a}_{\varrho\phi}(u,u) - \mathfrak{a}(u,u)\big| \le \gamma\mathfrak{a}(u,u) + \omega_0\varrho^2\|u\|_2^2\,. \tag{7.4}$$

For the proof of (7.4) we take a more detailed look at the form $\mathfrak{a}_{\varrho\phi}$. It holds for any $u \in V(\Omega)$, $\varrho \in \mathbb{R}$, and $\phi \in \mathcal{E}$

$$\operatorname{div}(e^{\varrho\phi}u) = e^{\varrho\phi}(\operatorname{div} u + \varrho\,\nabla\phi \cdot u)$$

and consequently

$$\operatorname{div}(e^{\varrho\phi}u)\,\overline{\operatorname{div}(e^{-\varrho\phi}u)} = \operatorname{div} u\,\overline{\operatorname{div} u} + \varrho(\nabla\phi\cdot u)\,\overline{\operatorname{div} u} - \varrho\operatorname{div} u\,(\nabla\phi\cdot\overline{u}) - \varrho^2(\nabla\phi\cdot u)(\nabla\phi\cdot\overline{u})\,.$$

In addition, one has

$$\operatorname{rot}(e^{\varrho\phi}u) = e^{\varrho\phi}(\operatorname{rot} u + \varrho\,\nabla\phi\times u)$$

and thus

$$\begin{aligned}\varepsilon(\cdot)^{-1}\operatorname{rot}(e^{\varrho\phi}u)\cdot\overline{\operatorname{rot}(e^{-\varrho\phi}u)} &= \varepsilon(\cdot)^{-1}\operatorname{rot} u\cdot\overline{\operatorname{rot} u} + \varrho\,\varepsilon(\cdot)^{-1}(\nabla\phi\times u)\cdot\overline{\operatorname{rot} u}\\ &\quad - \varrho\,\varepsilon(\cdot)^{-1}\operatorname{rot} u\cdot(\nabla\phi\times\overline{u}) - \varrho^2\varepsilon(\cdot)^{-1}(\nabla\phi\times u)\cdot(\nabla\phi\times\overline{u})\,.\end{aligned}$$

This leads to

$$\begin{aligned}\big|\mathfrak{a}_{\varrho\phi}(u,u) - \mathfrak{a}(u,u)\big| &\le |\varrho|\int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi\times u)\cdot\overline{\operatorname{rot} u}|\,dx + |\varrho|\int_\Omega |(\nabla\phi\cdot u)\,\overline{\operatorname{div} u}|\,dx\\ &\quad + |\varrho|\int_\Omega |\varepsilon(\cdot)^{-1}\operatorname{rot} u\cdot(\nabla\phi\times\overline{u})|\,dx + |\varrho|\int_\Omega |\operatorname{div} u\,(\nabla\phi\cdot\overline{u})|\,dx\\ &\quad + \varrho^2\int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi\times u)\cdot(\nabla\phi\times\overline{u})|\,dx + \varrho^2\int_\Omega |\nabla\phi\cdot u|^2\,dx\,.\end{aligned}$$

Next, let us analyze each of the summands on the right-hand side separately. Let $\delta > 0$ to be chosen later. By applying the Cauchy-Schwarz inequality, by using the elementary inequality $ab \le \delta a^2 + \frac{1}{4\delta}b^2$, which is valid for any real numbers a, b, and by recalling the properties of ϕ, we can estimate the first term in the following way

$$\begin{aligned}&|\varrho|\int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi\times u)\cdot\overline{\operatorname{rot} u}|\,dx \le |\varrho|\int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi\times u)|\,|\operatorname{rot} u|\,dx\\ &\le |\varrho|\int_\Omega \|\varepsilon(\cdot)^{-1}\|_\infty|\nabla\phi|\,|u|\,|\operatorname{rot} u|\,dx \le \|\varepsilon(\cdot)^{-1}\|_\infty\sqrt{3}\,\|\nabla\phi\|_\infty\int_\Omega |\operatorname{rot} u|\,|\varrho||u|\,dx\\ &\le \sqrt{3}\,\|\varepsilon(\cdot)^{-1}\|_\infty\int_\Omega \delta|\operatorname{rot} u|^2 + \frac{1}{4\delta}\,\varrho^2|u|^2\,dx = \sqrt{3}\,\|\varepsilon(\cdot)^{-1}\|_\infty\Big(\delta\|\operatorname{rot} u\|_2^2 + \frac{1}{4\delta}\,\varrho^2\|u\|_2^2\Big)\,.\end{aligned}$$

The second term is bounded by

$$\begin{aligned}|\varrho|\int_\Omega |(\nabla\phi\cdot u)\,\overline{\operatorname{div} u}|\,dx &\le |\varrho|\int_\Omega |\nabla\phi|\,|u|\,|\operatorname{div} u|\,dx\\ &\le \sqrt{3}\int_\Omega |\varrho||u|\,|\operatorname{div} u|\,dx \le \sqrt{3}\Big(\delta\|\operatorname{div} u\|_2^2 + \frac{1}{4\delta}\,\varrho^2\|u\|_2^2\Big)\,.\end{aligned}$$

The third term can be treated analogously to the first term

$$|\varrho|\int_\Omega |\varepsilon(\cdot)^{-1}\operatorname{rot} u\cdot(\nabla\phi\times\overline{u})|\,dx \le \sqrt{3}\,\|\varepsilon(\cdot)^{-1}\|_\infty\Big(\delta\|\operatorname{rot} u\|_2^2 + \frac{1}{4\delta}\,\varrho^2\|u\|_2^2\Big)\,.$$

The estimate for the fourth term is prepared in a similar manner as that for the second term

$$|\varrho| \int_\Omega |\operatorname{div} u\,(\nabla\phi \cdot \overline{u})|\,dx \le \sqrt{3}\Big(\delta\|\operatorname{div} u\|_2^2 + \frac{1}{4\delta}\,\varrho^2\|u\|_2^2\Big)\,.$$

The dealing with the fifth term consists in

$$\begin{aligned}\varrho^2 \int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi \times u)\cdot(\nabla\phi\times\overline{u})|\,dx &\le \varrho^2 \int_\Omega |\varepsilon(\cdot)^{-1}(\nabla\phi\times u)|\,|\nabla\phi\times\overline{u}|\,dx \\ &\le \varrho^2 \int_\Omega \sqrt{3}\,\|\varepsilon(\cdot)^{-1}\|_\infty\,|u|\,\sqrt{3}\,|u|\,dx = 3\|\varepsilon(\cdot)^{-1}\|_\infty\varrho^2\|u\|_2^2\,,\end{aligned}$$

whereas the sixth term is bounded by

$$\varrho^2 \int_\Omega |\nabla\phi\cdot u|^2\,dx \le \varrho^2\int_\Omega |\nabla\phi|^2|u|^2\,dx \le 3\varrho^2\|u\|_2^2\,.$$

By putting all these estimates together, we finally end up with

$$\begin{aligned}\big|\mathfrak{a}_{\varrho\phi}(u,u) - \mathfrak{a}(u,u)\big| &\le \big(2\sqrt{3}\,\|\varepsilon(\cdot)^{-1}\|_\infty + 2\sqrt{3}\big)\,\delta\,\big(\|\operatorname{rot} u\|_2^2 + \|\operatorname{div} u\|_2^2\big) \\ &\quad + \Big(\big(2\sqrt{3}\,\|\varepsilon(\cdot)^{-1}\|_\infty + 2\sqrt{3}\big)\frac{1}{4\delta} + 3\|\varepsilon(\cdot)^{-1}\|_\infty + 3\Big)\varrho^2\|u\|_2^2\,.\end{aligned}$$

The ellipticity property (7.2) of the coefficient matrix $\varepsilon(\cdot)^{-1}$ yields for each $u \in V(\Omega)$

$$\mathfrak{a}(u,u) \ge \min\{\varepsilon_0, 1\}\big(\|\operatorname{rot} u\|_2^2 + \|\operatorname{div} u\|_2^2\big)\,. \tag{7.5}$$

Now let $\gamma \in (0,1)$ be arbitrary. Take $\delta > 0$ with $\gamma = (2\sqrt{3}\,\|\varepsilon(\cdot)^{-1}\|_\infty + 2\sqrt{3})\,\delta/\min\{\varepsilon_0, 1\}$. Then we deduce for each $u \in V(\Omega)$, $\varrho \in \mathbb{R}$, and $\phi \in \mathcal{E}$

$$\big|\mathfrak{a}_{\varrho\phi}(u,u) - \mathfrak{a}(u,u)\big| \le \gamma\mathfrak{a}(u,u) + \omega_0\varrho^2\|u\|_2^2$$

with some constant $\omega_0 \ge 0$ depending exclusively on γ, ε_0, $\|\varepsilon(\cdot)^{-1}\|_\infty$. This shows (7.4).

Step 2: Due to (7.4), if $\omega > \omega_0$, we can write for any $u \in V(\Omega)$, $\varrho \in \mathbb{R}$, and $\phi \in \mathcal{E}$

$$\begin{aligned}\operatorname{Re}\mathfrak{a}_{\varrho\phi}(u,u) &= \mathfrak{a}(u,u) - \operatorname{Re}\big(\mathfrak{a}(u,u) - \mathfrak{a}_{\varrho\phi}(u,u)\big) \ge \mathfrak{a}(u,u) - \big|\mathfrak{a}(u,u) - \mathfrak{a}_{\varrho\phi}(u,u)\big| \\ &\ge \mathfrak{a}(u,u) - \big(\gamma\mathfrak{a}(u,u) + \omega\varrho^2\|u\|_2^2\big) = (1-\gamma)\mathfrak{a}(u,u) - \omega\varrho^2\|u\|_2^2\,.\end{aligned}$$

By recalling (7.5), we thus have shown that the form $\mathfrak{a}_{\varrho\phi\omega} := \mathfrak{a}_{\varrho\phi} + \omega\varrho^2$ is coercive in the sense of (7.3) with $C_1 = C_2 = (1-\gamma)\min\{\varepsilon_0, 1\}$. This entails that the operator $A_{\varrho\phi\omega}$ associated with the form $\mathfrak{a}_{\varrho\phi\omega}$ is sectorial of some angle $\theta_0 \in (0, \pi/2)$. Therefore, $-A_{\varrho\phi\omega}$ generates a bounded analytic semigroup $(e^{-tA_{\varrho\phi\omega}})_{t>0}$ on $L^2(\Omega, \mathbb{C}^3)$ and additionally

$$\big\|e^{-zA_{\varrho\phi\omega}}\big\|_{2\to 2} \le 1 \tag{7.6}$$

for all $z \in \mathbb{C} \setminus \{0\}$ with $|\arg z| \leq \theta_0$. In view of [LSV02, Lemma 3.2], this yields for any $\varrho \in \mathbb{R}$, $\phi \in \mathcal{E}$, and $z \in \mathbb{C} \setminus \{0\}$ with $|\arg z| \leq \theta_0$

$$\left\| e^{-\varrho\phi} e^{-zA_2} e^{\varrho\phi} \right\|_{2\to 2} \leq e^{\omega\varrho^2 \operatorname{Re} z} \tag{7.7}$$

and thus, by [Kun08, Lemma 3.4 and Remark 3.6], the operator A_2 satisfies Davies-Gaffney estimates of order 2. Additionally, we have for each $\varrho \in \mathbb{R}$, $\phi \in \mathcal{E}$, and $t > 0$

$$\left\| A_{\varrho\phi\omega} e^{-tA_{\varrho\phi\omega}} \right\|_{2\to 2} \leq \frac{1}{t \sin\theta_0}\,.$$

This estimate follows easily from Cauchy's formula and (7.6)

$$\begin{aligned}\left\| A_{\varrho\phi\omega} e^{-tA_{\varrho\phi\omega}} \right\|_{2\to 2} &= \Big\| \frac{1}{2\pi i} \int_{|z-t| = t\sin\theta_0} \frac{1}{(z-t)^2}\, e^{-zA_{\varrho\phi\omega}}\, dz \Big\|_{2\to 2} \\ &\leq \frac{1}{2\pi}\, 2\pi t \sin\theta_0\, \frac{1}{(t\sin\theta_0)^2} = \frac{1}{t\sin\theta_0}\,.\end{aligned}$$

The above argument is well-known and can be found e.g. in [Kun08, p. 2742].

Step 3: Our next task consists in verifying that Davies-Gaffney estimates of order 2 hold for the operator families $\{t^{1/2} \operatorname{div} e^{-tA_2} : t > 0\}$ and $\{t^{1/2} \operatorname{rot} e^{-tA_2} : t > 0\}$.

For arbitrary $f \in \mathscr{D}(\Omega, \mathbb{C}^3)$, $\varrho \in \mathbb{R}$, $\phi \in \mathcal{E}$, $\omega > \omega_0$, and $t > 0$ define $v(t) := e^{-tA_{\varrho\phi\omega}} f$. Then $v(t)$ belongs to $\mathcal{D}(A_{\varrho\phi\omega})$ and, due to (7.5) and the estimates in Step 2, we obtain

$$\begin{aligned}\left\| \operatorname{rot} v(t) \right\|_2^2 + \left\| \operatorname{div} v(t) \right\|_2^2 &\leq \frac{1}{\min\{\varepsilon_0, 1\}}\, \mathfrak{a}(v(t), v(t)) \\ &\leq \frac{1}{(1-\gamma)\min\{\varepsilon_0, 1\}}\, \operatorname{Re} \mathfrak{a}_{\varrho\phi\omega}(v(t), v(t)) \\ &\leq \frac{1}{(1-\gamma)\min\{\varepsilon_0, 1\}}\, \left| (A_{\varrho\phi\omega} v(t), v(t))_{L^2(\Omega,\mathbb{C}^3)} \right| \\ &\leq \frac{1}{(1-\gamma)\min\{\varepsilon_0, 1\}}\, \|A_{\varrho\phi\omega} v(t)\|_2 \|v(t)\|_2 \\ &\leq \frac{1}{(1-\gamma)\min\{\varepsilon_0, 1\}\sin\theta_0}\, t^{-1} \|f\|_2^2\,.\end{aligned}$$

As the space of test functions $\mathscr{D}(\Omega, \mathbb{C}^3)$ is dense in $L^2(\Omega, \mathbb{C}^3)$, we conclude that

$$\left\| \operatorname{div} e^{-tA_{\varrho\phi}} \right\|_{2\to 2} \leq \frac{1}{\sqrt{(1-\gamma)\min\{\varepsilon_0, 1\}\sin\theta_0}}\, t^{-1/2} e^{\omega\varrho^2 t}$$

and

$$\left\| \operatorname{rot} e^{-tA_{\varrho\phi}} \right\|_{2\to 2} \leq \frac{1}{\sqrt{(1-\gamma)\min\{\varepsilon_0, 1\}\sin\theta_0}}\, t^{-1/2} e^{\omega\varrho^2 t} \tag{7.8}$$

for all $\varrho \in \mathbb{R}$, $\phi \in \mathcal{E}$, $\omega > \omega_0$, and $t > 0$.

In order to obtain weighted norm estimates for $t^{1/2}\mathrm{rot}\, e^{-tA_2}$, we have to interchange rot and multiplication by $e^{-\varrho\phi}$ (see e.g. [Kun08, Corollary 3.2]). To this end, we represent $e^{-\varrho\phi}\mathrm{rot}\, h$ in terms of $\mathrm{rot}(e^{-\varrho\phi}h)$ and apply this representation to $h := e^{-tA_2}e^{\varrho\phi}f$. It holds

$$\mathrm{rot}(e^{-\varrho\phi}h) = e^{-\varrho\phi}\mathrm{rot}\, h - \varrho e^{-\varrho\phi}\nabla\phi \times h$$

and thus

$$e^{-\varrho\phi}\mathrm{rot}\, h = \mathrm{rot}(e^{-\varrho\phi}h) + \varrho\nabla\phi \times (e^{-\varrho\phi}h)\,.$$

The L^2-norm of the first term on the right-hand side can be estimated by (7.8), whereas for the second term we use $\|\nabla\phi\|_\infty \leq \sqrt{3}$, the elementary fact that $|\varrho| \leq C_\delta t^{-1/2}e^{\delta\varrho^2 t}$ for arbitrary $\delta > 0$ and some constant $C_\delta > 0$ depending only on δ, and (7.7)

$$\begin{aligned}\big\|e^{-\varrho\phi}\mathrm{rot}\, e^{-tA_2}e^{\varrho\phi}f\big\|_2 = \big\|e^{-\varrho\phi}\mathrm{rot}\, h\big\|_2 &\leq \big\|\mathrm{rot}(e^{-\varrho\phi}h)\big\|_2 + |\varrho|\,\|\nabla\phi\|_\infty\|e^{-\varrho\phi}h\|_2 \\ &\lesssim t^{-1/2}e^{(\omega+\delta)\varrho^2 t}\|f\|_2\end{aligned}$$

which yields

$$\big\|e^{-\varrho\phi}\mathrm{rot}\, e^{-tA_2}e^{\varrho\phi}\big\|_{2\to 2} \lesssim t^{-1/2}e^{(\omega+\delta)\varrho^2 t}\,.$$

According to [Kun08, Lemma 3.4 and Remark 3.6], there are constants $b, C > 0$ such that for all $t > 0$ and $x, y \in \Omega$

$$\big\|\mathbb{1}_{B(x,t^{1/2})}t^{1/2}\mathrm{rot}\, e^{-tA_2}\mathbb{1}_{B(y,t^{1/2})}\big\|_{2\to 2} \leq C\exp\left(-b\,\frac{|x-y|^2}{t}\right),$$

i.e. the family of operators $\{t^{1/2}\mathrm{rot}\, e^{-tA_2} : t > 0\}$ fulfills Davies-Gaffney estimates (DG_2). Similar arguments show that $\{t^{1/2}\mathrm{div}\, e^{-tA_2} : t > 0\}$ enjoys the same property.

Step 4: Let Ω_0 be a bounded Lipschitz domain in $\mathbb{R}^3$. In view of Fact 7.2 and the Sobolev embedding $H^{1/2}(\Omega_0, \mathbb{C}^3) \hookrightarrow L^{p^*}(\Omega_0, \mathbb{C}^3)$ for $p^* := \frac{3\cdot 2}{3-1} = 3$, we find a constant C depending only on $\partial\Omega_0$ and $\mathrm{diam}(\Omega_0)$ such that for every $u \in V(\Omega_0)$

$$\|u\|_{L^3(\Omega_0,\mathbb{C}^3)} \leq C\big(\|u\|_{L^2(\Omega_0,\mathbb{C}^3)} + \|\mathrm{div}\, u\|_{L^2(\Omega_0,\mathbb{C})} + \|\mathrm{rot}\, u\|_{L^2(\Omega_0,\mathbb{C}^3)}\big)\,. \tag{7.9}$$

With the help of the rescaling procedure used in [MM09, p. 3145], we get for all $w \in V(\Omega_0)$

$$\|w\|_{L^3(\Omega_0,\mathbb{C}^3)} \leq CR^{-1/2}\big(\|w\|_{L^2(\Omega_0,\mathbb{C}^3)} + R\,\|\mathrm{div}\, w\|_{L^2(\Omega_0,\mathbb{C})} + R\,\|\mathrm{rot}\, w\|_{L^2(\Omega_0,\mathbb{C}^3)}\big)\,, \tag{7.10}$$

where $R := \mathrm{diam}(\Omega_0)$ and the constant C depends exclusively on the Lipschitz character of the domain Ω_0.

The *Lipschitz character* of the bounded Lipschitz domain Ω_0 reflects the number of coordinate systems needed to cover $\partial\Omega_0$ by cubes such that inside each cube Ω_0 is the domain above the graph of a Lipschitz function, the side lengths of these cubes, and the supremum of the Lipschitz constants of the involved Lipschitz functions (see e.g. [AK04, p. 11]).

Step 5: The desired generalized Gaussian $(2,3)$-estimates for A_2 follow by combining the Davies-Gaffney estimates from Step 2 and 3 together with the inequality (7.10). A similar reasoning was applied in [MM09, Section 5].

Let $t > 0$, $x, y \in \Omega$, and $f \in \mathscr{D}(\Omega, \mathbb{C}^3)$ with $\operatorname{supp} f \subseteq B(y, t^{1/2})$ be arbitrary. We put $\Omega_0 := B(x, 2t^{1/2}) \subseteq \Omega$ and choose a cut-off function $\eta \in \mathscr{D}(\Omega_0, \mathbb{R})$ such that

$$0 \le \eta \le 1\,, \qquad \eta = 1 \quad \text{on } B(x, t^{1/2})\,, \qquad \text{and} \qquad \|\nabla \eta\|_\infty \le t^{-1/2}\,.$$

First, we remark that

$$\begin{aligned}\big\|\operatorname{div}(\eta e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C})} &\le \big\|\eta \operatorname{div}(e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C})} + \big\|\nabla\eta \cdot e^{-tA_2} f\big\|_{L^2(\Omega_0,\mathbb{C})} \\ &\lesssim \big\|\operatorname{div}(e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C})} + t^{-1/2}\big\|e^{-tA_2} f\big\|_{L^2(\Omega_0,\mathbb{C}^3)}\end{aligned}$$

and similarly

$$\big\|\operatorname{rot}(\eta e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C}^3)} \lesssim \big\|\operatorname{rot}(e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C}^3)} + t^{-1/2}\big\|e^{-tA_2} f\big\|_{L^2(\Omega_0,\mathbb{C}^3)}\,.$$

Since $\nu \cdot (\eta e^{-tA_2} f)|_{\partial\Omega_0} = 0$ and the Lipschitz character of Ω_0 is controlled by that of Ω, we may use (7.10) and arrive at

$$\begin{aligned}&\big\|e^{-tA_2} f\big\|_{L^3(B(x,t^{1/2}),\mathbb{C}^3)} \le \big\|\eta e^{-tA_2} f\big\|_{L^3(\Omega_0,\mathbb{C}^3)} \\ &\quad\lesssim t^{-1/4}\big(\big\|\eta e^{-tA_2} f\big\|_{L^2(\Omega_0,\mathbb{C}^3)} + t^{1/2}\big\|\operatorname{div}(\eta e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C})} + \\ &\qquad\quad + t^{1/2}\big\|\operatorname{rot}(\eta e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C}^3)}\big) \\ &\quad\lesssim t^{-1/4}\big(3\big\|e^{-tA_2} f\big\|_{L^2(\Omega_0,\mathbb{C}^3)} + t^{1/2}\big\|\operatorname{div}(e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C})} + \\ &\qquad\quad + t^{1/2}\big\|\operatorname{rot}(e^{-tA_2} f)\big\|_{L^2(\Omega_0,\mathbb{C}^3)}\big) \\ &\quad\lesssim t^{-1/4} \exp\Big(-b\,\frac{|x-y|^2}{t}\Big)\|f\|_2\,,\end{aligned}$$

where the implicit constants are independent of f, t, x, y. The last inequality is due to the Davies-Gaffney estimates for $(e^{-tA_2})_{t>0}$, $\{t^{1/2}\operatorname{div} e^{-tA_2} : t > 0\}$, and $\{t^{1/2}\operatorname{rot} e^{-tA_2} : t > 0\}$.

In other words, we have checked that there are constants $b, C > 0$ such that for all $t > 0$ and all $x, y \in \Omega$

$$\big\|\mathbb{1}_{B(x,t^{1/2})} e^{-tA_2} \mathbb{1}_{B(y,t^{1/2})}\big\|_{2\to 3} \le C t^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{3})} \exp\Big(-b\,\frac{|x-y|^2}{t}\Big)\,. \tag{7.11}$$

This ends the proof. □

As noted at the beginning of this section, $V(\Omega)$ enjoys better embedding properties when Ω is convex or its boundary is of class $C^{1,1}$. In these cases the space $V(\Omega)$ is continuously embedded into $H^1(\Omega,\mathbb{C}^3)$ which in turn is continuously embedded into $L^6(\Omega,\mathbb{C}^3)$. Hence, in this situation one can take the $L^6(\Omega,\mathbb{C}^3)$-norm on the left-hand side of (7.9). Observe that this automatically gives the desired exponent of t in (7.11) (cp. e.g. [Kun08, proof of Theorem 3.1]) and thus in Step 4 rescaling would not be needed. All in all, the following statement holds.

Corollary 7.4. *In the situation of Theorem 7.3 suppose additionally that the domain Ω is convex or has a $C^{1,1}$-boundary. Then the operator A_2 associated with the form $\mathfrak{a}$ satisfies generalized Gaussian $(6/5,6)$-estimates.*

Since A_2 satisfies generalized Gaussian (p_0,p_0')-estimates for some $p_0\in[1,3/2]$, the semigroup generated by $-A_2$ can be extended to a bounded analytic semigroup on $L^p(\Omega,\mathbb{C}^3)$ for every $p\in[p_0,p_0']$ with $p\neq\infty$ (this can be seen as in the scalar-valued case which is proven in [Blu07, Theorem 1.1]). For the rest of this section, we denote by $-A_p$ its generator.

In order to introduce the Maxwell operator, we first recall some basic facts concerning the Helmholtz decomposition in $L^p(\Omega,\mathbb{C}^3)$ which is loosely described as follows: Any vector field $v\in L^p(\Omega,\mathbb{C}^3)$ can be splitted into the sum of a divergence-free and rotation-free component. As we will see later, the range of p's for which such a decomposition holds is related to the regularity of the boundary $\partial\Omega$ of the bounded Lipschitz domain Ω.

For $p\in(1,\infty)$ we introduce the space of divergence-free functions

$$L^p_\sigma(\Omega):=\left\{v\in L^p(\Omega,\mathbb{C}^3):\,\operatorname{div}v=0,\,\nu\cdot v|_{\partial\Omega}=0\right\}$$

and the space of gradients

$$G^p(\Omega):=\left\{\nabla g:\,g\in W^1_p(\Omega,\mathbb{C})\right\}.$$

Then both are closed subspaces of $L^p(\Omega,\mathbb{C}^3)$. In the case $p=2$ the corresponding orthogonal projection from $L^2(\Omega,\mathbb{C}^3)$ onto $L^2_\sigma(\Omega)$ is called *Helmholtz projection*, denoted by $\mathbb{P}_2$. The properties of this mapping are very well-known. An overview on $\mathbb{P}_2$ can be found e.g. in [Soh01, Section II.2.5].

E. Fabes, O. Mendez, and M. Mitrea established a Helmholtz decomposition in $L^p(\Omega,\mathbb{C}^3)$ ([FMM98, Theorems 11.1 and 12.2]) which reads as follows.

Fact 7.5. *For every bounded Lipschitz domain Ω in $\mathbb{R}^3$ there exists $\varepsilon>0$ such that $\mathbb{P}_2$ extends to a bounded linear operator $\mathbb{P}_p$ from $L^p(\Omega,\mathbb{C}^3)$ onto $L^p_\sigma(\Omega)$ for all $p\in(3/2-\varepsilon,3+\varepsilon)$. In this range, one has an L^p-*Helmholtz decomposition

$$L^p(\Omega,\mathbb{C}^3)=L^p_\sigma(\Omega)\oplus G^p(\Omega)\tag{7.12}$$

*as a topological direct sum. The operator $\mathbb{P}_p$ is then called L^p-*Helmholtz projection.

In the class of bounded Lipschitz domains, this result is sharp in the sense that for any $p \notin [3/2, 3]$ *there exists a bounded Lipschitz domain* $\Omega \subseteq \mathbb{R}^3$ *for which the* L^p*-Helmholtz decomposition (7.12) fails.*

If, however, Ω *has a regular boundary* $\partial\Omega \in C^1$*, then the result is true for all* $p \in (1, \infty)$*.*

For convenience, we introduce the following abbreviation.

Notation 7.6. We denote by I_Ω the largest interval on the real line such that for each $p \in I_\Omega$ the semigroup $(e^{-tA_2})_{t>0}$ extends to a bounded analytic semigroup on $L^p(\Omega, \mathbb{C}^3)$ and that there exists an L^p-Helmholtz decomposition.

In view of the foregoing statements, the length of I_Ω is deeply connected to the regularity properties of the boundary $\partial\Omega$ and the interval $[3/2, 3]$ is always contained in I_Ω.

As we will immediately see, the operators A_2 and $\mathbb{P}_2$ are commuting. This relies on the fact that $\mathbb{P}_2$ leaves the domain $V(\Omega)$ of the form $\mathfrak{a}$ invariant which is essentially due to the boundary condition of $V(\Omega)$. We remark that this property stands in contrast to the situation of the presence of Dirichlet boundary conditions ($\nu \cdot v|_{\partial\Omega} = 0$ and $\nu \times v|_{\partial\Omega} = 0$). The latter is implicitly mentioned in [CF88, Chapter 4].

Lemma 7.7. *For any* $p \in I_\Omega$*, the operator* A_p *and the Helmholtz projection* $\mathbb{P}_p$ *are commuting, i.e.* $\mathbb{P}_p(\mathcal{D}(A_p))$ *is contained in* $\mathcal{D}(A_p)$ *and it holds for all* $u \in \mathcal{D}(A_p)$

$$\mathbb{P}_p A_p u = A_p \mathbb{P}_p u \,.$$

Proof. At first, we treat the case $p = 2$. The statement for arbitrary $p \in I_\Omega$ then follows by density and consistency.

We claim that $\mathbb{P}_2 \colon V(\Omega) \to V(\Omega)$. Indeed, let $u \in V(\Omega)$. By definition of $\mathbb{P}_2$, it is evident that $\operatorname{div}(\mathbb{P}_2 u) = 0$ as well as $\nu \cdot (\mathbb{P}_2 u)|_{\partial\Omega} = 0$. In order to check $\operatorname{rot}(\mathbb{P}_2 u) \in L^2(\Omega, \mathbb{C}^3)$, we write $\mathbb{P}_2 u = u - \nabla g$ for some $g \in W_2^1(\Omega, \mathbb{C})$ and note that it suffices to show $\operatorname{rot}(\nabla g) = 0$. This can be easily verified via the distributional definitions of rot and ∇ which transfer the assertion on test functions, but the validity of $\operatorname{div}(\operatorname{rot} \varphi) = 0$ for each $\varphi \in \mathscr{D}(\Omega, \mathbb{C}^3)$ is well-known. In particular, we have just computed $\operatorname{rot}(\mathbb{P}_2 u) = \operatorname{rot} u$ for every $u \in V(\Omega)$.

Now consider $u \in \mathcal{D}(A_2)$. We get for each $v \in V(\Omega)$

$$(\mathbb{P}_2 A_2 u, v)_{L^2(\Omega, \mathbb{C}^3)} = (A_2 u, \mathbb{P}_2 v)_{L^2(\Omega, \mathbb{C}^3)} = \mathfrak{a}(u, \mathbb{P}_2 v) = \mathfrak{a}(\mathbb{P}_2 u, v) \,,$$

where the last equality is obtained with the help of $\operatorname{rot}(\mathbb{P}_2 u) = \operatorname{rot} u$. This means that $\mathbb{P}_2 u \in \mathcal{D}(A_2)$ and $\mathbb{P}_2 A_2 u = A_2 \mathbb{P}_2 u$.

Let $p \in I_\Omega$. Observe that A_p and $\mathbb{P}_p$ are commuting if and only if resolvents of A_p commute with $\mathbb{P}_p$ on $L^p(\Omega, \mathbb{C}^3)$.

Hence, we have $\mathbb{P}_p(\lambda + A_p)^{-1} = (\lambda + A_p)^{-1}\mathbb{P}_p$ on $L^p(\Omega, \mathbb{C}^3) \cap L^2(\Omega, \mathbb{C}^3)$ for any $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Notice that $-A_2$ as well as $-A_p$ are generators of bounded analytic semigroups and thus their resolvent sets include the right-half complex plane and, moreover, their resolvents are consistent. By the density of $L^p(\Omega, \mathbb{C}^3) \cap L^2(\Omega, \mathbb{C}^3)$ in $L^p(\Omega, \mathbb{C}^3)$ and by the boundedness of resolvent operators, the equality $\mathbb{P}_p(\lambda + A_p)^{-1} = (\lambda + A_p)^{-1}\mathbb{P}_p$ extends to $L^p(\Omega, \mathbb{C}^3)$. This gives the lemma. □

Now we are prepared to introduce the Maxwell operator.

Definition 7.8. For $p \in I_\Omega$ we define the *Maxwell operator* M_p on $L^p_\sigma(\Omega)$ by setting

$$\begin{aligned} \mathcal{D}(M_p) &:= \mathbb{P}_p\mathcal{D}(A_p) = \mathcal{D}(A_p) \cap L^p_\sigma(\Omega)\,, \\ M_p u &:= A_p u \qquad \text{for } u \in \mathcal{D}(M_p)\,. \end{aligned}$$

Since A_p and $\mathbb{P}_p$ are commuting, one obtains, by representing e^{-tA_p} in terms of resolvents of A_p (cp. (7.19) below), that $\mathbb{P}_p e^{-tA_p} = e^{-tA_p}\mathbb{P}_p$ on $L^p(\Omega, \mathbb{C}^3)$ for every $t > 0$. Further, it is obvious that $\mathbb{P}_p A_p = M_p \mathbb{P}_p$ on $\mathcal{D}(A_p)$ which entails that $\mathbb{P}_p e^{-tA_p} = e^{-tM_p}\mathbb{P}_p$ on $L^p(\Omega, \mathbb{C}^3)$ (cp. [MM09, Theorem 7.4]).

Observe that the assertion of Theorem 6.4 remains valid in the current vector-valued situation. Indeed, its proof extends without problems to this case just by replacing the modulus by the Euclidean norm. We mention that in the vector-valued setting the Calderón-Zygmund decomposition also holds. Since A_2 satisfies generalized Gaussian $(3/2, 3)$-estimates (cf. Theorem 7.3), Theorem 6.4 a) yields the following result.

Theorem 7.9. *Let $p \in (3/2, 3)$. Suppose that $s > 4\,|1/p - 1/2|$ and $1/q < |1/p - 1/2|$. Then, for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty$, the operator $F(A_2)$ is bounded on $L^p(\Omega, \mathbb{C}^3)$ and there exists a constant $C > 0$ such that*

$$\|F(A_2)\|_{p\to p} \le C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} + |F(0)|\Big)\,.$$

As A_p and $\mathbb{P}_p$ are commuting, the functional calculus for A_2 on $L^p(\Omega, \mathbb{C}^3)$ and the Helmholtz projection $\mathbb{P}_p$ are commuting as well. Hence, we deduce a spectral multiplier theorem for the Maxwell operator by restricting $F(A_2)$ to the space of divergence-free functions.

Theorem 7.10. *Let $p \in (3/2, 3)$. Suppose that $s > 4\,|1/p - 1/2|$ and $1/q < |1/p - 1/2|$. Then, for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} < \infty$, the operator $F(M_2)$ is bounded on $L^p_\sigma(\Omega)$ and there exists a constant $C > 0$ such that*

$$\|F(M_2)\|_{L^p_\sigma(\Omega)\to L^p_\sigma(\Omega)} \le C\Big(\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_q} + |F(0)|\Big)\,.$$

Finally, we remark that the situation on the full space $\Omega = \mathbb{R}^3$ is more comfortable since no boundary terms occur. In particular, the form $\mathfrak{a}$ is better suited concerning partial integration. Note that the range of values $p \in [1,2)$ for which our method gives generalized Gaussian (p,p')-estimates then depends only on the regularity of the coefficient matrix $\varepsilon(\cdot)$. In the case of smooth coefficients one can even prove classical Gaussian estimates.

7.2 Stokes operator

In this section we show a spectral multiplier theorem for the Stokes operator with Hodge boundary conditions. Our argument is based on certain off-diagonal estimates for the resolvents of the Hodge-Laplacian which were recently established by M. Mitrea and S. Monniaux ([MM09]). At first, we recall the definition of the *Hodge-Laplacian* B which is the operator associated with the densely defined, sesquilinear, symmetric form

$$\mathfrak{b}(u,v) := \int_\Omega \operatorname{rot} u \cdot \overline{\operatorname{rot} v}\, dx + \int_\Omega \operatorname{div} u\, \overline{\operatorname{div} v}\, dx \qquad (u, v \in V(\Omega)),$$

where $\Omega \subseteq \mathbb{R}^3$ is a bounded Lipschitz domain and $V(\Omega)$ denotes the function space as introduced in (7.1). Then B is self-adjoint, invertible, and $-B$ generates an analytic semigroup on $L^2(\Omega,\mathbb{C}^3)$. According to [MM09, (3.17) and (3.18)], the Hodge-Laplacian B can be characterized by

$$\begin{aligned} \mathcal{D}(B) &= \big\{ u \in V(\Omega) \,:\, \operatorname{rot} u \in H(\operatorname{rot},\Omega),\, \operatorname{div} u \in H^1(\Omega,\mathbb{C}),\, \nu \times \operatorname{rot} u|_{\partial\Omega} = 0 \big\}\,, \\ Bu &= -\Delta u \qquad \text{for } u \in \mathcal{D}(B)\,. \end{aligned}$$

M. Mitrea and S. Monniaux ([MM09, Lemma 3.7]) observed that the Hodge-Laplacian B and the Helmholtz projection $\mathbb{P}_2$ are commuting.

Definition 7.11. The *Stokes operator A with Hodge boundary conditions* on $L^2_\sigma(\Omega)$ is defined via $A := \mathbb{P}_2 B$ with the domain $\mathcal{D}(A) := \mathbb{P}_2\mathcal{D}(B)$.

Starting from norm estimates of annular type on $L^p(\Omega,\mathbb{C}^3)$ with $p = 2$ for resolvents of the Hodge-Laplacian B, M. Mitrea and S. Monniaux developed an iterative bootstrap argument ([MM09, Lemma 5.1]) that allows to incrementally increase the value of p to $p^* := \frac{3}{2}\,p$, which is caused by Sobolev embedding, as long as $p < q_\Omega$, where q_Ω denotes the critical index for the well-posedness of the Poisson type problem for the Hodge-Laplacian ([MM09, (1.9)]). In the present situation of a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$, it is known that $q_\Omega > 3$ (cf. [Mit04]). M. Mitrea and S. Monniaux ([MM09, Section 6]) showed that for any $\theta \in (0,\pi)$ there exist $q \in (3,\infty]$ and constants $b, C > 0$ such that for all $j \in \mathbb{N}$, $x \in \Omega$, and $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| < \pi - \theta$

$$\big\|\mathbb{1}_{B(x,|\lambda|^{-1/2})}\lambda(\lambda+B)^{-1}\mathbb{1}_{B(x,2^{j+1}|\lambda|^{-1/2})\setminus B(x,2^{j-1}|\lambda|^{-1/2})}\big\|_{2\to q} \le C\,|\lambda|^{\frac{3}{2}(\frac{1}{2}-\frac{1}{q})}\, e^{-b\,2^j}\,. \quad (7.13)$$

As we shall see, in the Euclidean setting the validity of those estimates for the resolvents ensures generalized Gaussian estimates for the semigroup operators.

Since an analytic semigroup $(e^{-tL})_{t>0}$ and resolvents of its generator $-L$ are intimately related via integral representations, one obtains a nearly equivalent formulation of generalized Gaussian estimates if one replaces in the two-ball estimate (2.5) the semigroup operators with resolvent operators of the form $\lambda(\lambda+L)^{-1}$ for $\lambda \in \rho(-L)$. To be precise, the transfer from semigroup operators to resolvent operators and vice versa reads as follows.

Lemma 7.12. *Let $\Omega \subseteq \mathbb{R}^D$ be a Borel set and L be a non-negative, self-adjoint operator on $L^2(\Omega)$. Assume that $1 \le p \le 2 \le q \le \infty$ and $m \ge 2$ with $D/m(1/p - 1/q) < 1$.*

a) *Fix $\theta \in (0, \pi/2)$ and suppose that there exist constants $b, C > 0$ such that for all $x, y \in \Omega$ and all $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| < \pi - \theta$*

$$\left\|\mathbb{1}_{B(x,|\lambda|^{-1/m})}\lambda(\lambda+L)^{-1}\mathbb{1}_{B(y,|\lambda|^{-1/m})}\right\|_{p\to q} \le C\,|\lambda|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}e^{-b\,|\lambda|^{1/m}|x-y|}\,. \tag{7.14}$$

Then there are constants $b', C' > 0$ such that the semigroup operators satisfy

$$\left\|\mathbb{1}_{B(x,t^{1/m})}e^{-tL}\mathbb{1}_{B(y,t^{1/m})}\right\|_{p\to q} \le C'\,t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\exp\left(-b'\left(\frac{|x-y|}{t^{1/m}}\right)^{\frac{m}{m-1}}\right) \tag{7.15}$$

for any $t > 0$ and any $x, y \in \Omega$.

b) *Suppose that there exist constants $b, C > 0$ such that for all $t > 0$ and all $x, y \in \Omega$*

$$\left\|\mathbb{1}_{B(x,t^{1/m})}e^{-tL}\mathbb{1}_{B(y,t^{1/m})}\right\|_{p\to q} \le C\,t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\exp\left(-b\left(\frac{|x-y|}{t^{1/m}}\right)^{\frac{m}{m-1}}\right). \tag{7.16}$$

Then for any $\theta \in (0, \pi/2)$ there are constants $b', C' > 0$ such that for all $x, y \in \Omega$ and all $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| < \theta$

$$\left\|\mathbb{1}_{B(x,|\lambda|^{-1/m})}\lambda(\lambda+L)^{-1}\mathbb{1}_{B(y,|\lambda|^{-1/m})}\right\|_{p\to q} \le C'\,|\lambda|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}e^{-b'\,|\lambda|^{1/m}|x-y|}\,.$$

Proof. As noted in [BK03, pp. 934-935] (see also remark on p. 26), one can assume that $\Omega = \mathbb{R}^D$. Otherwise, instead of an operator $T\colon L^p(\Omega) \to L^q(\Omega)$, one considers the extended operator $\widetilde{T}\colon L^p(\mathbb{R}^D) \to L^q(\mathbb{R}^D)$ defined by

$$\widetilde{T}u(x) := \begin{cases} T(\mathbb{1}_\Omega u)(x) & \text{for } x \in \Omega \\ 0 & \text{for } x \notin \Omega \end{cases} \qquad (u \in L^p(\mathbb{R}^D), x \in \mathbb{R}^D).$$

Then it is straightforward to check that $\|\widetilde{T}\|_{L^p(\mathbb{R}^D)\to L^q(\mathbb{R}^D)} = \|T\|_{L^p(\Omega)\to L^q(\Omega)}$. In the following, we will shortly write $\|\cdot\|_{p\to q}$ for the norm $\|\cdot\|_{L^p(\mathbb{R}^D)\to L^q(\mathbb{R}^D)}$.

For the proof of part a), fix $t > 0$ and $x, y \in \mathbb{R}^D$. In order to verify (7.15), we use weighted norm estimates for the resolvent operators similar to those of Davies' perturbation method presented in the previous section and an integral representation for the semigroup operators based on the Cauchy formula.

Put $h(\tau) := \beta\tau$ for some constant $\beta > 0$. Then one gets for the Legendre transform $h^{\#} \colon \mathbb{R} \to [-h(0), \infty]$ of h

$$h^{\#}(\sigma) := \sup_{\tau \geq 0} \sigma\tau - h(\tau) = \sup_{\tau \geq 0} (\sigma - \beta)\tau = \begin{cases} 0 & \text{for } \sigma \leq \beta\,, \\ \infty & \text{for } \sigma > \beta\,. \end{cases} \tag{7.17}$$

As before, $\mathcal{E}$ denotes the space of all real-valued functions $\phi \in C_c^\infty(\mathbb{R}^D)$ with $\|\partial_j \phi\|_\infty \leq 1$ for any $j \in \{1, 2, \ldots, D\}$. Then $d_{\mathcal{E}}(x, y) := \sup\{\phi(x) - \phi(y) : \phi \in \mathcal{E}\}$ defines a metric on $\mathbb{R}^D$ which is actually equivalent to the Euclidean distance (see e.g. [Dav95, Lemma 4]). Therefore, [BK05, Theorem 1.2] is applicable and gives that (7.14) is equivalent to

$$\left\| e^{-\varrho\phi} v_{|\lambda|^{-1/m}}^{\frac{1}{p}-\frac{1}{q}} \lambda(\lambda + L)^{-1} e^{\varrho\phi} \right\|_{p\to q} \lesssim e^{h^{\#}(\varrho|\lambda|^{-1/m})}\,,$$

where $v_{|\lambda|^{-1/m}}(x) := |B(x, |\lambda|^{-1/m})| \cong |\lambda|^{-\frac{D}{m}}$, and consequently

$$\left\| e^{-\varrho\phi} \lambda(\lambda + L)^{-1} e^{\varrho\phi} \right\|_{p\to q} \lesssim |\lambda|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})} e^{h^{\#}(\varrho|\lambda|^{-1/m})}$$

for any $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg\lambda| < \pi - \theta$, $\varrho \geq 0$, and any $\phi \in \mathcal{E}$. By exploiting (7.17), we have for any $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg\lambda| < \pi - \theta$, $0 \leq \varrho \leq \beta|\lambda|^{1/m}$, and $\phi \in \mathcal{E}$

$$\left\| e^{-\varrho\phi} \lambda(\lambda + L)^{-1} e^{\varrho\phi} \right\|_{p\to q} \lesssim |\lambda|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\,. \tag{7.18}$$

Based on the Cauchy integral formula, one can represent the semigroup operator e^{-tL} in terms of resolvent operators (see e.g. [EN00, pp. 96 ff.])

$$e^{-tL} = \frac{1}{2\pi i} \int_\Gamma e^{t\lambda} (\lambda + L)^{-1}\, d\lambda\,, \tag{7.19}$$

where Γ is, as usual, a piecewise smooth curve in $\Sigma_{\pi-\theta}$ going from $\infty e^{-i(\pi-\theta')}$ to $\infty e^{i(\pi-\theta')}$ for some $\theta' \in (\theta, \pi/2)$. Put $\eta := \frac{1}{2}(\pi - \theta + \frac{\pi}{2}) = \frac{3}{4}\pi - \frac{\theta}{2}$ and $\omega_\varrho := |\sin\eta|^{-1}\, \beta^{-m} \varrho^m$ for $\varrho \geq 0$ with β being the constant in the definition of the function h. We consider shifted versions of e^{-tL} and shall establish a bound on $\|e^{-\varrho\phi} e^{-\omega_\varrho t} e^{-tL} e^{\varrho\phi}\|_{p\to q}$ for any $\varrho \geq 0$ and $\phi \in \mathcal{E}$ by using the above integral representation for e^{-tL} with the counterclockwise oriented integration path $\Gamma = \Gamma_{t^{-1},\eta} + \omega_\varrho$, where

$$\Gamma_{t^{-1},\eta} := -(-\infty, -t^{-1}] e^{-i\eta} \cup t^{-1} e^{i[-\eta,\eta]} \cup [t^{-1}, \infty) e^{i\eta}\,.$$

It holds for each $\varrho \geq 0$ and $\phi \in \mathcal{E}$

$$\begin{aligned}\left\|e^{-\varrho\phi}e^{-\omega_\varrho t}e^{-tL}e^{\varrho\phi}\right\|_{p\to q} &\leq \int_{\Gamma_{t^{-1},\eta}+\omega_\varrho} e^{t(\operatorname{Re}\lambda-\omega_\varrho)}\left\|e^{-\varrho\phi}(\lambda+L)^{-1}e^{\varrho\phi}\right\|_{p\to q}|d\lambda| \\ &= \int_{\Gamma_{t^{-1},\eta}} \frac{e^{t\operatorname{Re}\zeta}}{|\zeta+\omega_\varrho|}\left\|e^{-\varrho\phi}(\zeta+\omega_\varrho)(\zeta+\omega_\varrho+L)^{-1}e^{\varrho\phi}\right\|_{p\to q}|d\zeta|\,.\end{aligned}$$

For every $\zeta \in \Gamma_{t^{-1},\eta}$ we can bound the operator norm with the help of (7.18) when the condition $\varrho \leq \beta\,|\zeta+\omega_\varrho|^{1/m}$ is valid. A simple geometric argument gives that $|\zeta+\omega_\varrho| \geq |\sin\eta|\,\omega_\varrho$ and thus (7.18) surely applies for $\varrho \leq \beta\,|\sin\eta|^{1/m}\,\omega_\varrho^{1/m}$. But, due to the definition of ω_ϱ, this requirement imposes no restrictions on ϱ. Therefore, we can continue our estimation by applying (7.18) and the elementary fact $|\zeta+\omega_\varrho| \cong |\zeta|+\omega_\varrho$ (cf. e.g. [MM09, (5.2)])

$$\lesssim \int_{\Gamma_{t^{-1},\eta}} \frac{e^{t\operatorname{Re}\zeta}}{|\zeta|+\omega_\varrho}\,(|\zeta|+\omega_\varrho)^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\,|d\zeta| \leq \int_{\Gamma_{t^{-1},\eta}} e^{t\operatorname{Re}\zeta}\,|\zeta|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})-1}\,|d\zeta|\,.$$

Here, we made use of the condition $D/m(1/p-1/q) < 1$. Next, we estimate the integral on each of the three segments of the integration path $\Gamma_{t^{-1},\eta}$ separately. We begin with a bound for the integral on the half ray $[t^{-1},\infty)e^{i\eta}$

$$\begin{aligned}\int_{[t^{-1},\infty)e^{i\eta}} e^{t\operatorname{Re}\zeta}\,|\zeta|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})-1}\,|d\zeta| &= \int_{t^{-1}}^{\infty} e^{tu\cos\eta}\,u^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})-1}\,du \\ &= t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\int_1^\infty e^{v\cos\eta}\,v^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})-1}\,dv \lesssim t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\,,\end{aligned}$$

where the last step is due to $\cos\eta < 0$. The integral on the half ray $-(-\infty,-t^{-1}]e^{-i\eta}$ can be treated in the same manner. A bound for the remaining integral over the circular arc $t^{-1}e^{i[-\eta,\eta]}$ is obtained by using the canonical parametrization $\zeta(\alpha) = t^{-1}e^{i\alpha}$ for $\alpha \in [-\eta,\eta]$

$$\int_{t^{-1}e^{i[-\eta,\eta]}} e^{t\operatorname{Re}\zeta}\,|\zeta|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})-1}\,|d\zeta| = t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\int_{-\eta}^{\eta} e^{\cos\alpha}\,d\alpha \lesssim t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\,.$$

Putting things together, we have shown that for all $\varrho \geq 0$, $\phi \in \mathcal{E}$, and $t > 0$

$$\left\|e^{-\varrho\phi}e^{-\omega_\varrho t}e^{-tL}e^{\varrho\phi}\right\|_{p\to q} \lesssim t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}$$

and after recalling $\omega_\varrho = |\sin\eta|^{-1}\,\beta^{-m}\,\varrho^m$

$$\left\|e^{-\varrho\phi}e^{-tL}e^{\varrho\phi}\right\|_{p\to q} \lesssim t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}e^{|\sin\eta|^{-1}\,\beta^{-m}\varrho^m t}\,.$$

By [Kun08, Lemma 3.4 and Remark 3.6], this entails the desired two-ball estimate (7.15).

The proof of part b) is similar to that of part a). It will be achieved by showing the equivalent statement (cf. [BK05, Theorem 1.2])

$$\left\| e^{-\varrho\phi}\lambda(\lambda+L)^{-1}e^{\varrho\phi}\right\|_{p\to q} \lesssim |\lambda|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})} e^{h^{\#}(\varrho|\lambda|^{-1/m})}$$

for all $\varrho \geq 0$, $\phi \in \mathcal{E}$, and all $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| < \theta$, where $h^{\#}$ is given via (7.17) and $\theta \in (0, \pi/2)$ is a fixed number.

According to [BK05, Theorem 1.2], the assumption (7.16) can be equivalently written as

$$\left\| e^{-\varrho\phi}e^{-tL}e^{\varrho\phi}\right\|_{p\to q} \lesssim t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})} e^{g^{\#}(\varrho t^{1/m})}$$

for any $\varrho \geq 0$, $\phi \in \mathcal{E}$, and $t > 0$, where $g(\tau) := \gamma\tau^{m/(m-1)}$ for some constant $\gamma > 0$. The Legendre transform of g is given by $g^{\#}(\sigma) = \delta\sigma^m$ with $\delta := (m-1)^{m-1}/(\gamma^{m-1}m^m)$.

Let $\lambda \in \mathbb{C}\setminus\{0\}$ with $|\arg\lambda| < \theta$. Since the resolvent $(\lambda+L)^{-1}$ is the Laplace transform of the semigroup $(e^{-tL})_{t>0}$

$$(\lambda+L)^{-1} = \int_0^\infty e^{-\lambda t}e^{-tL}\,dt\,,$$

one gets for any $\varrho \geq 0$ and $\phi \in \mathcal{E}$

$$\begin{aligned}
\left\| e^{-\varrho\phi}\lambda(\lambda+L)^{-1}e^{\varrho\phi}\right\|_{p\to q} &\leq \int_0^\infty e^{-t\operatorname{Re}\lambda}\,|\lambda|\,\left\| e^{-\varrho\phi}e^{-tL}e^{\varrho\phi}\right\|_{p\to q} dt \\
&\lesssim \int_0^\infty e^{-t\operatorname{Re}\lambda}\,|\lambda|\,t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\,e^{\delta\varrho^m t}\,dt \\
&= |\lambda|^{\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\int_0^\infty s^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}e^{-(\frac{\operatorname{Re}\lambda}{|\lambda|}-\frac{\delta}{|\lambda|}\varrho^m)s}\,ds\,.
\end{aligned}$$

We split the integral at $s = 1$. Due to $D/m(1/p-1/q) < 1$, the integral from 0 to 1 is finite. The integral over the interval $(1,\infty)$ converges for $\varrho < (\frac{\operatorname{Re}\lambda}{|\lambda|\delta})^{1/m}|\lambda|^{1/m} \leq \delta^{-1/m}|\lambda|^{1/m}$, whereas it diverges for $\varrho \geq \delta^{-1/m}|\lambda|^{1/m}$. Hence, the integral is bounded by a constant times $\exp(h^{\#}(\varrho|\lambda|^{-1/m}))$. This completes the proof. □

In view of (7.13) and Fact 2.4, Lemma 7.12 ensures the validity of generalized Gaussian $(2,q)$-estimates for the Hodge-Laplacian for some $q \in (3,\infty]$. Similar as in the previous section, Theorem 6.4 a) entails the boundedness of spectral multipliers at first for the Hodge-Laplacian and then, by restriction, for the Stokes operator A with Hodge boundary conditions. This leads to the following statement.

Theorem 7.13. *Assume that (7.13) holds for some $q \in (3,\infty]$ and that there is an L^q-Helmholtz decomposition. Fix $p \in (q', q)$ and take $s > 4\,|1/p - 1/2|$ and $1/r < |1/p - 1/2|$. Then, for every bounded Borel function $F\colon [0,\infty) \to \mathbb{C}$ with $\sup_{n\in\mathbb{Z}} \|\omega F(2^n\cdot)\|_{H^s_r} < \infty$, the operator $F(A)$ is bounded on $L^p_\sigma(\Omega)$ and there exists a constant $C > 0$ such that*

$$\|F(A)\|_{L^p_\sigma(\Omega)\to L^p_\sigma(\Omega)} \leq C\Big(\sup_{n\in\mathbb{Z}}\|\omega F(2^n\cdot)\|_{H^s_r} + |F(0)|\Big)\,.$$

7.3 Lamé operator

Recently, M. Mitrea and S. Monniaux ([MM10]) studied the properties of the Lamé operator which appears in the linearization of the compressible Navier-Stokes equations. They showed analyticity of the semigroup generated by the Lamé operator and maximal regularity for the time-dependent Lamé system equipped with homogeneous Dirichlet boundary conditions. Their approach is essentially based on off-diagonal estimates for the resolvents of the Lamé operator and, according to Lemma 7.12, the latter are basically equivalent to generalized Gaussian estimates. Hence, the results given in the previous chapters are also applicable to the Lamé operator.

At first, we describe the setting of [MM10]. Although our results apply in the general framework of [MM10] as well, we will restrict ourselves to the three-dimensional case. This restriction serves only to introduce less notation. Furthermore, we consider complex-valued functions.

Let Ω be a bounded, open subset of $\mathbb{R}^3$ such that the *interior ball condition* holds, i.e. there exists a positive constant c such that for all $x \in \Omega$ and all $r \in (0, \frac{1}{2}\operatorname{diam}(\Omega))$

$$|B(x,r)| \geq cr^3 .$$

This condition ensures that Ω becomes a space of homogeneous type when Ω is equipped with the three-dimensional Lebesgue measure and the Euclidean distance. For example, any bounded Lipschitz domain in $\mathbb{R}^3$ or domains satisfying an interior corkscrew condition (see e.g. [JK82, p. 93]) enjoy the interior ball condition.

Fix $\eta, \eta' \in \mathbb{R}$ with $\eta > 0$ and $\eta + \eta' > 0$. We consider the sesquilinear form $\mathfrak{c}$ defined by

$$\mathfrak{c}(u,v) := \eta \int_\Omega \operatorname{rot} u \cdot \overline{\operatorname{rot} v}\, dx + (\eta + \eta') \int_\Omega \operatorname{div} u\, \overline{\operatorname{div} v}\, dx$$

for $u, v \in H_0^1(\Omega, \mathbb{C}^3)$, where $H_0^1(\Omega, \mathbb{C}^3)$ denotes the closure of the test function space $\mathscr{D}(\Omega, \mathbb{C}^3)$ with respect to the norm of the Sobolev space $H^1(\Omega, \mathbb{C}^3)$. Then it is easy to see that the form $\mathfrak{c}$ is closed, continuous, symmetric, and coercive. Therefore, the operator L associated with the form $\mathfrak{c}$ on $L^2(\Omega, \mathbb{C}^3)$ is self-adjoint and $-L$ generates a bounded analytic semigroup on $L^2(\Omega, \mathbb{C}^3)$. In [MM10, Section 1.1] it is checked that L is given by

$$\begin{aligned} \mathcal{D}(L) &= \left\{ u \in H_0^1(\Omega, \mathbb{C}^3) \,:\, \eta\Delta u + \eta'\nabla \operatorname{div} u \in L^2(\Omega, \mathbb{C}^3) \right\}, \\ Lu &= -\eta\Delta u - \eta'\nabla \operatorname{div} u \qquad \text{for } u \in \mathcal{D}(L)\,. \end{aligned}$$

The operator L is called *Lamé operator with Dirichlet boundary conditions.*

In [MM10, Section 2] M. Mitrea and S. Monniaux adapt their approach of [MM09] to the Lamé operator L and establish the following statement: For any fixed angle $\theta \in (0, \pi)$ one finds $q \in (2, \infty]$ and constants $b, C > 0$ such that for all $j \in \mathbb{N}$, $x \in \Omega$, and $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\arg \lambda| < \pi - \theta$

$$\left\| \mathbb{1}_{B(x,|\lambda|^{-1/2})} \lambda(\lambda + L)^{-1} \mathbb{1}_{B(x,2^{j+1}|\lambda|^{-1/2}) \setminus B(x,2^{j-1}|\lambda|^{-1/2})} \right\|_{2 \to q} \leq C \, |\lambda|^{\frac{3}{2}(\frac{1}{2} - \frac{1}{q})} \, e^{-b \, 2^j} . \quad (7.20)$$

Since this guarantees the validity of (7.14), Lemma 7.12 yields generalized Gaussian $(2, q)$-estimates for the Lamé operator L.

As remarked in [MM10, Remark 1.5], the estimate (7.20) is always valid for $q = 6$ which is caused by the Sobolev embedding $H^1(\Omega, \mathbb{C}^3) \hookrightarrow L^6(\Omega, \mathbb{C}^3)$. If the Poisson problem for the Lamé operator (cf. [MM10, (1.15)]) is well-posed in $L^6(\Omega, \mathbb{C}^3)$, then, according to [MM10, Lemma 2.2], (7.20) also holds for $q^* = \infty$. It turns out that the largest value $q_0 \in (2, \infty]$, for which the iterative method of M. Mitrea and S. Monniaux delivers (7.20) and thus generalized Gaussian estimates for L, depends on the well-posedness of the Poisson problem for the Lamé operator and this is deeply connected to the regularity properties of the boundary $\partial\Omega$. Only for certain domains Ω the exact characterization of q_0 is known. We refer to [MM10, Theorem 4.1] for a discussion of this topic and only mention that, if Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then one can even prove (7.20) for $q = \infty$ (cf. [MM10, Remark 1.6]), i.e. L actually satisfies classical Gaussian estimates. However, in general, (7.20) with $q = \infty$ does not hold.

All in all, we have seen that the Lamé operator L fulfills generalized Gaussian (q_0', q_0)-estimates for some $q_0 \in [6, \infty]$. Therefore, Theorem 6.4 a) applies for L and gives the following result.

Theorem 7.14. *Fix $p \in (q_0', q_0)$. Suppose that $s > 4\,|1/p - 1/2|$ and $1/r < |1/p - 1/2|$. Then, for every bounded Borel function $F \colon [0, \infty) \to \mathbb{C}$ with $\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_r} < \infty$, the operator $F(L)$ is bounded on $L^p(\Omega, \mathbb{C}^3)$ and there exists a constant $C > 0$ such that*

$$\|F(L)\|_{p \to p} \leq C \Big(\sup_{n \in \mathbb{Z}} \|\omega F(2^n \cdot)\|_{H^s_r} + |F(0)| \Big) .$$

In particular, the Lamé operator L admits a bounded holomorphic functional calculus on $L^p(\Omega, \mathbb{C}^3)$ for every $p \in (q_0', q_0)$. Even this result is new.

Additionally, for any $p \in [q_0', q_0]$ with $p \neq \infty$, the semigroup generated by (minus) the Lamé operator extends to a bounded analytic semigroup on $L^p(\Omega, \mathbb{C}^3)$. Due to [BK05, Corollary 1.5], the spectrum of its generator on $L^p(\Omega, \mathbb{C}^3)$ is independent of p.

Bibliography

[Ale94] G. Alexopoulos: *Spectral multipliers on Lie groups of polynomial growth.* Proc. Am. Math. Soc. 120, No. 3, 973-979, 1994.

[AK04] H. Ammari and H. Kang: *Reconstruction of small inhomogeneities from boundary measurements.* Lecture Notes in Mathematics 1846, Springer, 2004.

[ABDG98] C. Amrouche, C. Bernardi, M. Dauge, and V. Girault: *Vector potentials in three-dimensional non-smooth domains.* Math. Methods Appl. Sci. 21, No. 9, 823-864, 1998.

[Aus07] P. Auscher: *On necessary and sufficient conditions for L^p-estimates of Riesz transforms associated to elliptic operators on $\mathbb{R}^n$ and related estimates.* Mem. Am. Math. Soc. 871, 2007.

[ADM05] P. Auscher, X.T. Duong, and A. McIntosh: *Boundedness of Banach space valued singular integral operators and Hardy spaces.* Unpublished preprint, 2005.

[AM06] P. Auscher and J.M. Martell: *Weighted norm inequalities, off-diagonal estimates and elliptic operators. III: Harmonic analysis of elliptic operators.* J. Funct. Anal. 241, No. 2, 703-746, 2006.

[AM07a] P. Auscher and J.M. Martell: *Weighted norm inequalities, off-diagonal estimates and elliptic operators. I: General operator theory and weights.* Adv. Math. 212, No. 1, 225-276, 2007.

[AM07b] P. Auscher and J.M. Martell: *Weighted norm inequalities, off-diagonal estimates and elliptic operators. II: Off-diagonal estimates on spaces of homogeneous type.* J. Evol. Equ. 7, No. 2, 265-316, 2007.

[AM08] P. Auscher and J.M. Martell: *Weighted norm inequalities, off-diagonal estimates and elliptic operators. IV: Riesz transforms on manifolds and weights.* Math. Z. 260, No. 3, 527-539, 2008.

[BL76] J. Bergh and J. Löfström: *Interpolation spaces: an introduction.* Grundlehren der mathematischen Wissenschaften 223, Springer, 1976.

[Blu03] S. Blunck: *A Hörmander-type spectral multiplier theorem for operators without heat kernel.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 2, No. 3, 449-459, 2003.

[Blu07] S. Blunck: *Generalized Gaussian estimates and Riesz means of Schrödinger groups.* J. Aust. Math. Soc. 82, No. 2, 149-162, 2007.

[BK02] S. Blunck and P.C. Kunstmann: *Weighted norm estimates and maximal regularity.* Adv. Differ. Equ. 7, No. 12, 1513-1532, 2002.

[BK03] S. Blunck and P.C. Kunstmann: *Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus.* Rev. Mat. Iberoam. 19, No. 3, 919-942, 2003.

[BK05] S. Blunck and P.C. Kunstmann: *Generalized Gaussian estimates and the Legendre transform.* J. Oper. Theory 53 (2), 351-365, 2005.

[Cal64] A.P. Calderón: *Intermediate spaces and interpolation, the complex method.* Stud. Math. 24, 113-190, 1964.

[Chr91] M. Christ: *L^p bounds for spectral multipliers on nilpotent groups.* Trans. Am. Math. Soc. 328, No. 1, 73-81, 1991.

[Coi74] R.R. Coifman: *A real variable characterization of H^p.* Stud. Math. 51, 269-274, 1974.

[CMS85] R.R. Coifman, Y. Meyer, and E.M. Stein: *Some new functions and their applications to harmonic analysis.* J. Funct. Anal. 62, 304-335, 1985.

[CW71] R.R. Coifman and G. Weiss: *Analyse harmonique non-commutative sur certains espaces homogènes.* Lecture Notes in Mathematics 242, Springer, 1971.

[CW77] R.R. Coifman and G. Weiss: *Extensions of Hardy spaces and their use in analysis.* Bull. Am. Math. Soc. 83, 569-645, 1977.

[CK98] D. Colton and R. Kress: *Inverse acoustic and electromagnetic scattering theory.* 2nd edition, Applied Mathematical Sciences 93, Springer, 1998.

[CF88] P. Constantin and C. Foias: *Navier-Stokes equations.* Chicago Lectures in Mathematics, University of Chicago Press, 1988.

[Cos90] M. Costabel: *A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains.* Math. Methods Appl. Sci. 12, No. 4, 365-368, 1990.

[CS08] T. Coulhon and A. Sikora: *Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem.* Proc. Lond. Math. Soc. (3) 96, No. 2, 507-544, 2008.

[CDMY96] M. Cowling, I. Doust, A. McIntosh, and A. Yagi: *Banach space operators with a bounded H^∞ functional calculus.* J. Aust. Math. Soc., Ser. A 60, No. 1, 51-89, 1996.

[CS01] M. Cowling and A. Sikora: *A spectral multiplier theorem for a sublaplacian on $SU(2)$.* Math. Z. 238, No. 1, 1-36, 2001.

[DL88] R. Dautray and J.-L. Lions: *Analyse mathématique et calcul numérique pour les sciences et les techniques.* Vol. 8, Evolution: semi-groupe, variationnel, Masson, 1988.

[Dav92] E.B. Davies: *Heat kernel bounds, conservation of probability and the Feller property.* J. Anal. Math. 58, 99-119, 1992.

[Dav95] E.B. Davies: *Uniformly elliptic operators with measurable coefficients.* J. Funct. Anal. 132, 141-169, 1995.

[Dav97] E.B. Davies: *Limits on L^p regularity of self-adjoint elliptic operators.* J. Differ. Equations 135, No. 1, 83-102, 1997.

[DL10] X.T. Duong and J. Li: *Hardy spaces associated to operators satisfying Davies-Gaffney estimates and bounded holomorphic functional calculus.* Preprint, 2010.

[DM99] X.T. Duong and A. McIntosh: *Singular integral operators with non-smooth kernels on irregular domains.* Rev. Mat. Iberoam. 15, No. 2, 233-265, 1999.

[DOS02] X.T. Duong, E.M. Ouhabaz, and A. Sikora: *Plancherel-type estimates and sharp spectral multipliers.* J. Funct. Anal. 196, 443-485, 2002.

[DY11] X.T. Duong and L.X. Yan: *Spectral multipliers for Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates.* J. Math. Soc. Japan 63, No. 1, 295-319, 2011.

[Dzi99] J. Dziubański: *Spectral multiplier theorem for H^1 spaces associated with some Schrödinger operators.* Proc. Am. Math. Soc. 127, No. 12, 3605-3613, 1999.

[DP09] J. Dziubański and M. Preisner: *Remarks on spectral multiplier theorems on Hardy spaces associated with semigroups of operators.* Rev. Unión Mat. Argent. 50, No. 2, 201-215, 2009.

[EN00] K.-J. Engel and R. Nagel: *One-parameter semigroups for linear evolution equations.* Springer, 2000.

[FMM98] E. Fabes, O. Mendez, and M. Mitrea: *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains.* J. Funct. Anal. 159, No. 2, 323-368, 1998.

[FS72] C. Fefferman and E.M. Stein: *H^p spaces of several variables.* Acta Math. 129, 137-193, 1972.

[Fre11] D. Frey: *Paraproducts via H^∞-functional calculus and a $T(1)$-Theorem for non-integral operators.* Dissertation, Karlsruher Institut für Technologie (KIT), 2011. URL: `http://digbib.ubka.uni-karlsruhe.de/volltexte/1000022796`.

[Gaf59] M.P. Gaffney: *The conservation property of the heat equation on Riemannian manifolds.* Commun. Pure Appl. Math. 12, 1-11, 1959.

[GR85] J. García-Cuerva and J.L. Rubio de Francia: *Weighted norm inequalities and related topics.* North-Holland Mathematics Studies 116, 1985.

[GR86] V. Girault and P. Raviart: *Finite element methods for Navier-Stokes equations: theory and algorithms.* Springer, 1986.

[Gra09] L. Grafakos: *Modern Fourier analysis.* 2nd edition, Graduate Texts in Mathematics 250, Springer, 2009.

[Har15] G.H. Hardy: *The mean value of the modulus of an analytic function.* Lond. M. S. Proc. (2) 14, 269-277, 1915.

[Heb95] W. Hebisch: *Functional calculus for slowly decaying kernels.* Preprint, 1995.

[HLMMY08] S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea, and L.X. Yan: *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates.* Preprint, 2008.

[HM03] S. Hofmann and J.M. Martell: *L^p bounds for Riesz transforms and square roots associated to second order elliptic operators.* Publ. Mat. 47, No. 2, 497-515, 2003.

[HM09] S. Hofmann and S. Mayboroda: *Hardy and BMO spaces associated to divergence form elliptic operators.* Math. Ann. 344, No. 1, 37-116, 2009.

[HMM10] S. Hofmann, S. Mayboroda, and A. McIntosh: *Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces.* Preprint, 2010.

[Hör60] L. Hörmander: *Estimates for translation invariant operators in L^p spaces.* Acta Math. 104, 93-140, 1960.

[Hyt04] T. Hytönen: *Fourier embeddings and Mihlin-type multiplier theorems.* Math. Nachr. 274-275, 74-103, 2004.

[JK82] D.S. Jerison and C.E. Kenig: *Boundary behavior of harmonic functions in non-tangentially accessible domains.* Adv. Math. 46, 80-147, 1982.

[Kri09] C. Kriegler: *Spectral multipliers, R-bounded homomorphisms, and analytic diffusion semigroups.* Dissertation, Universität Karlsruhe (TH), 2009. URL: `http://digbib.ubka.uni-karlsruhe.de/volltexte/1000015866`.

[Kry39] V.I. Krylov: *On functions regular in a half-plane.* Amer. Math. Soc. Transl. 32, No. 2, 37-81, 1963; translation from Mat. Sb. 6 (48), 95-138, 1939.

[Kun08] P.C. Kunstmann: *On maximal regularity of type $L^p - L^q$ under minimal assumptions for elliptic non-divergence operators.* J. Funct. Anal. 255, No. 10, 2732-2759, 2008.

[KW04] P.C. Kunstmann and L. Weis: *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus.* Functional analytic methods for evolution equations, Lecture Notes in Mathematics 1855, Springer, 65-311, 2004.

[Lat79] R.H. Latter: *The atomic decomposition of Hardy spaces.* Harmonic analysis in Euclidean spaces, Part 1, Williamstown/Massachusetts 1978, Proc. Symp. Pure Math., vol. 35, 275-279, 1979.

[LSV02] V. Liskevich, Z. Sobol, and H. Vogt: *On the L_p-theory of C_0-semigroups associated with second-order elliptic operators II.* J. Funct. Anal. 193, No. 1, 55-76, 2002.

[MM90] G. Mauceri and S. Meda: *Vector-valued multipliers on stratified groups.* Rev. Mat. Iberoam. 6, No. 3-4, 141-154, 1990.

[Mic56] S.G. Michlin: *On the multipliers of Fourier integrals.* Dokl. Akad. Nauk SSSR 109, 701-703, 1956.

[Mit04] M. Mitrea: *Sharp Hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds.* Duke Math. J. 125, No. 3, 467-547, 2004.

[MMT01] D. Mitrea, M. Mitrea, and M. Taylor: *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds.* Mem. Am. Math. Soc. 713, 2001.

[MM09] M. Mitrea and S. Monniaux: *On the analyticity of the semigroup generated by the Stokes operator with Neumann-type boundary conditions on Lipschitz subdomains of Riemannian manifolds.* Trans. Am. Math. Soc. 361, No. 6, 3125-3157, 2009.

[MM10] M. Mitrea and S. Monniaux: *Maximal regularity for the Lamé system in certain classes of non-smooth domains.* J. Evol. Equ. 10, No. 4, 811-833, 2010.

[Ouh05] E.M. Ouhabaz: *Analysis of heat equations on domains.* London Mathematical Society Monographs Series 31, Princeton Univ. Press, 2005.

[Rie23] F. Riesz: *Über die Randwerte einer analytischen Funktion.* Math. Zs. 18, 87-95, 1923.

[Rud73] W. Rudin: *Functional analysis.* McGraw-Hill Series in Higher Mathematics, 1973.

[RS96] T. Runst and W. Sickel: *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations.* De Gruyter Series in Nonlinear Analysis and Applications 3, 1996.

[Rus07] E. Russ: *The atomic decomposition for tent spaces of homogeneous type.* In: CMA/AMSI research symposium "Asymptotic geometric analysis, harmonic analysis, and related topics". Proc. of the Centre for Math. and Appl., vol. 42, 125-135. Australian National University, Canberra, 2007.

[SV94] G. Schreieck and J. Voigt: *Stability of the L_p-spectrum of Schrödinger operators with form-small negative part of the potential.* "Functional analysis". Proceedings of the Essen conference, 1991. Marcel-Dekker, New York. Lect. Notes Pure Appl. Math. 150, 95-105, 1994.

[Soh01] H. Sohr: *The Navier-Stokes equations. An elementary functional analytic approach.* Birkhäuser Advanced Texts, 2001.

[Ste70] E.M. Stein: *Singular integrals and differentiability of functions.* Princeton Univ. Press, 1970.

[Ste93] E.M. Stein: *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals.* Princeton Univ. Press, 1993.

[SW60] E.M. Stein and G. Weiss: *On the theory of harmonic functions of several variables, I. The theory of H^p-spaces.* Acta Math. 103, 25-62, 1960.

[Tha89] S. Thangavelu: *Summability of Hermite expansions. I, II.* Trans. Am. Math. Soc. 314, No. 1, 119-142, 143-170, 1989.

[Tri78] H. Triebel: *Interpolation theory, function spaces, differential operators.* North-Holland, 1978.

[Tri83] H. Triebel: *Theory of function spaces.* Monographs in Mathematics, vol. 78, Birkhäuser, 1983.

[Ull10] A. Ullmann: *Maximal functions, functional calculus, and generalized Triebel-Lizorkin spaces for sectorial operators.* Dissertation, Karlsruher Institut für Technologie (KIT), 2010. URL: `http://digbib.ubka.uni-karlsruhe.de/volltexte/1000021376`.

[Yan02] L.X. Yan: *A remark on Littlewood-Paley g-function.* Bull. Aust. Math. Soc. 66, No. 1, 33-41, 2002.