The Galois action on Origami curves and a special class of Origamis

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Preface

The absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$, the group of field automorphisms of an algebraic closure of the field of rational numbers, has been a central object of interest in many areas of mathematics for a long time. While the calculation of absolute Galois groups of many other interesting fields can be explained to any moderately advanced mathematics student (for example we have $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$ for any prime p), the group $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ seems to be rather successful in eluding the efforts of researchers to gain insight into its structure. On the other hand, a look at the consequences that some deeper understanding of this group would have gives rise to the assumption that any such progress must be hard work. For example, the mere knowledge of the isomorphism types of groups that appear as finite quotients of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ would once and for all settle the field of inverse Galois theory.

One important line of progress in this study was the following: In 1979, Belyi published the article **[Bel79]**, containing what is now known as Belyi's theorem, which gives (depending on the point of view) a purely combinatorial or analytical condition for a complex curve to be defined over a number field: A curve X can be defined over $\overline{\mathbb{Q}}$ iff there exists a nonconstant morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ that is ramified over at most three points. Particularly, any *Belyi pair* (X, β) can be represented by a topological surface of genus $g(X)$ with an embedded bipartite graph. The simplicity and explicit nature of this construction made a deep impression on Grothendieck, who by that time had rather gained a reputation of proving deep results by working in the most abstract and general settings possible. As doing number theory seemed to be as easy as drawing stick-figures now, he coined the term *dessins d'enfants* for these embedded graphs. So, there is an action of the absolute Galois group on a set of combinatorially defined objects, which can be shown to be faithful in a stunningly explicit way. Based on these ideas, in 1984 Grothendieck formulated a programme to reveal the absolute Galois group as the group of automorphisms of a certain system of mapping class groups which he called the *Teichmüller tower*. This text, which he titled "Esquisse d'un Programme", was later published in **[Gro97]**. In Drinfel'd's celebrated article **[Dri90]**, $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ is realised as a subgroup of the *Grothendieck-Teichmüller group* $\widehat{\mathrm{GT}}$, a group that can be defined in a purely combinatorial way. Up to the present, it is unknown whether or not this inclusion is proper.

The idea behind the theory of dessins d'enfants can be stated as follows: Take objects that can be defined in an explicit combinatorial way, more precisely coverings, and study the action of the absolute Galois group on them. Taking a step forward, one could, instead of considering coverings of $\mathbb{P}^1$, take coverings of a genus 1 curve E. Here, one ramification point suffices to get an interesting theory, for whose objects of interest Lochak coined the name *Origamis*. However, the situation is a bit different here: While there is, up to isomorphism, only one complex curve of genus 0 with three punctures, and consequently the *Hurwitz spaces* of coverings of this curve with some prescribed genus and degree are finite sets of points, the moduli space of genus 1 curves is a curve itself. Though, by the rather deep result that Hurwitz spaces of coverings of curves in characteristic 0 are, under weak conditions, defined over $\mathbb{Q}$, the connected components of Hurwitz spaces parametrising Origami coverings are curves defined over $\overline{\mathbb{Q}}$, so one can study the Galois action on them. It is natural to ask if this action is faithful. In [**Möl05**], Möller gave a positive answer, making use of the faithfulness of the Galois action on dessins. In order to do so, he defined a way of constructing certain Origamis (which we will call *M-Origamis* in this work) from dessins. Using mainly algebro-geometric methods, he reveals just enough information on these Origamis to prove the faithfulness result.

The goal of this work is to shed a light on the deeply combinatorial nature of this construction. In doing so, we can not only reprove and enhance Möller's results by using mainly topological methods, but we are also able to explicitly deduce the properties of M-Origamis—like their genus, their number of punctures, their Veech group and their decomposition into maximal cylinders—given the defining permutations of the dessins they are associated to. Also, we will give some existence results and bounds for M-Origamis with certain properties. Finally we will be able to give explicit examples of Origami curves on which the absolute Galois group acts non-trivially. To the knowledge of the author, such examples have not been given in the literature before.*

A downside of Möller's construction is the following: Despite the faithfulness of the Galois action on the embedded Origami curves appearing in his construction, they are, as abstract curves, all defined over $\mathbb{Q}$. In Ellenberg's and McReynolds' more recent article [**EM09**], they prove, using different methods, that any subgroup of $\Gamma(2)$ of finite index can be realised as the Veech group of an Origami. Thus, by Belyi's theorem, every smooth projective curve over $\overline{\mathbb{Q}}$ appears as the isomorphism type of the normalisation of an Origami curve, showing that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ is faithful even on abstract Origami curves, i.e. after forgetting about their embedding in moduli space. However, it is not known to the author whether their construction allows giving examples of reasonable sizes.

The structure of this work is as follows: In Chapter 1, we present the theory of topological coverings in the way we will need it later on. In particular, we calculate the monodromy of pullbacks, fibre products and compositions of coverings, given the monodromy of the individual coverings. Since these results are essential for our subsequent calculations, we will prove them in detail.

In Chapter 2, we give an introduction to the theory of dessins d'enfants, putting an emphasis on the various equivalent definitions of a dessin. After defining the various notions of moduli fields and fields of definition, we sketch parts of the proof of Belyi's theorem. We close the chapter by explaining why the action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ on dessins is faithful.

Chapter 3 is dedicated to the theory of Origamis. Our way of defining them shall exhibit the similarities to the construction of dessins. After that, we put Origamis into a more general context, namely the theory of *translation surfaces*. These surfaces have been subject to study in the field of dynamical systems. We follow the classical construction of Teichmüller discs and Teichmüller curves,

*In the recent article [**BM10**], Bouw and Möller give a series of primitive Teichmüller curves in M_2 (so in particular not Origami curves) whose moduli fields are real quadratic extensions of $\mathbb{Q}$.

which is purely analytical. Then, we reveal the arithmetic nature of these curves. This enables us to prove parts of a conjecture that Kremer stated in [**Kre10a**], and to prove a relationship between the index of the Veech group and the degree of a certain field extension connected to an Origami.

Chapter 4 is the central part of this work. We begin by explaining Möller's algebro-geometric construction of Origamis from dessins, or, in slightly greater generality (which we will not need in this work), *pillow case coverings*. After that we justify, using results from classical algebraic geometry, that we can replace Möller's construction by a purely topological one, i.e. we can forget about the structures as algebraic curves and work with unramified coverings of punctured topological surfaces. Having done so, we calculate the monodromy of an M-Origami, given the monodromy of a dessin, using the results from Chapter 1. As its monodromy fully describes an M-Origami, we can go on from there to calculate its interesting properties, like the genus, the number of punctures, its Veech group, the Origamis in its affine equivalence class (which will turn out to be also M-Origamis) and the cylinder decomposition in its Strebel directions. This will finally allow us to reprove Möller's results using topological methods. Due to our detailed examination of the properties of M-Origamis, we are able to give finer conditions for such an Origami not to be defined over $\mathbb{Q}$. Also, we can show in a constructive way that the Galois action on Origamis is non-trivial in every genus $g \geq 4$.

In the last chapter, we exploit the explicit nature of our construction to present some examples. First, we pick some Galois orbits of trees from the catalogue [**BZ92**] and construct the associated M-Origamis and their Teichmüller curves. Besides actually illustrating the Galois action, these examples also show that different Veech groups can occur for M-Origamis, and that our Theorem 2 at the end of Chapter 3 is actually a non-trivial statement. Next, we construct a series of trees that we show not to be defined over $\mathbb{Q}$, which we use to prove the non-triviality result that appears at the end of Chapter 4. We also give an example of Origami curves originating from an orbit of dessins in genus 1. Finally, shifting our view away from the Galois action, we use our knowledge about the Veech groups of M-Origamis to construct an infinite series of Origamis of growing genus with Veech group $\mathrm{SL}_2(\mathbb{Z})$ that are not characteristic. It seems like not many examples of such Origamis are known so far.

Throughout this work, we use the following naming convention: Theorems that we cite from other authors (or which we believe to be folklore statements) are identified by capital letters, while for original theorems we use Arabic numerals as usual. For minor statements, such as propositions, lemmas and remarks we will not make that distinction. However, we make clear whether or not they are original work by the author. We try to stick to the convention of marking a term in italics when it appears for the first time.

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literature frequently (e.g. in [**Bra01**], [**Ham05**] and [**Sch05**], the first two of which mention a possible exceptional case) and that immediatly generalizes to Gabi's office door in a natural way. Also, their combination of patience with my pace of work and subtle indication that I might come to an end soon was just right.

Next, I wish to thank the past and present members of our workgroup, the *Kaffeerunde*. I could not have wished for a more friendly working atmosphere. Particularly I wish to thank: *Horst Hammer*, who encouraged me to study mathematics in the first place. *Karsten Kremer* for sharing his insights on Origamis with me, and for his beautiful thesis [**Kre10a**], which is still path-wise connected. *Stefan Kühnlein*, who in his mathematical omniscience (that he would never hesitate to deny) has helped me with many questions arising during my research. My office-mates *Myriam Finster* and *André Kappes* for many fruitful discussions on Origamis and the merry atmosphere in our office.

The *Karlsruhe Origami Library* [**WSKF$^+$11**] proved to be a great tool for testing hypotheses, verifying results and constructing examples. My thank goes to its various authors, in particular *Gabriela Weitze-Schmithüsen*, *Karsten Kremer*, *Myriam Finster* and *Joachim Breitner*.

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Karlsruhe, June 2011 Florian Nisbach

Contents

CHAPTER 1

Topological Preliminaries

The two classes of objects we will study closely in the following chapters, namely dessins d'enfants and Origamis, both have very deep and interesting geometric and arithmetic properties. However, they can be completely described in a topological way: They are topological coverings of 2-dimensional manifolds. A central idea of this work is to use topological reasoning as often as possible, which will help us to make many situations more explicit than it would be possible using more abstract, and seemingly more elegant techniques, such as algebraic geometry.

The goal of this chapter is to give an introduction to the classical topic of topological coverings, adapted to our needs in the forthcoming chapters. All of the results presented here should be well-known, but nevertheless we will prove them in detail, as we formulate them differently compared to classical textbooks on topology.

1. Reformulating the "Main Theorem for Coverings"

The central result in the theory of topological coverings is usually stated as follows: Given a sufficiently nice path-wise connected topological space X with fundamental group $G = \pi_1(X, x_0)$, the equivalence classes of path-wise connected spaces with a covering map to X are in bijection to the conjugacy classes of subgroups of G.

This view has two downsides: Firstly, this theorem is not helpful when we want to work with spaces covering X that are not necessarily path-wise connected. Secondly, the groups which describe the coverings, i.e. subgroups of G, are not very handy objects when we want to do calculations.

Our solution to these two problems will be to work with monodromy instead: For a covering $p: Y \to X$, we study the natural action of G on the fibre $p^{-1}(x_0)$. This is possible even if Y is not path-wise connected. Indeed, the only difference is that the action need not be transitive in this case. If $d := |p^{-1}(x_0)| < \infty$, we can describe the monodromy action by a homomorphism $G \to S_d$. In the cases we will consider later, the fundamental group G will be free of rank n, so after fixing generators, giving a homomorphism $G \to S_d$ simply amounts to giving an element of the set $(S_d)^n$. We get the equivalence classes of coverings of X of degree d as the orbits of $(S_d)^n$ under the diagonal action of S_d by conjugation.

Let us begin by making the notion of a covering that need not be path-wise connected:

Definition 1.1. Let X be a path-wise connected, locally path-wise connected, semi-locally simply connected topological space. (From now on, call such a topological space *-space.) A topological space Y together with a continuous surjective map $p : Y \to X$ is called *not necessarily path-wise connected covering (NPC)* if every point $x \in X$ has an open neighbourhood $U_x \ni x$ with the following property: $p^{-1}(U_x)$

is a disjoint union $\coprod_{i\in I} U_i$ such that for all $i \in I$ the restriction $p_{|U_i} : U_i \to U_x$ is a homeomorphism. U_x is then called *admissible neighbourhood* of x with respect to p. Moreover, as it is well known for ordinary coverings, $\deg p := |p^{-1}(x)|$ is well defined and is called *degree* of the NPC.

Note that, if in the above situation Y is path-wise connected, p is just a covering in the usual sense—see for example [**Sch71**, III 6.2].

We make the following agreement: If $\alpha, \beta \in \pi_1(X, x_0)$ are two elements of the fundamental group of a topological space then $\beta\alpha$ shall denote the homotopy class one gets by first passing through a representative of α and then one of β.

Definition and Remark 1.2. Let $p : Y \to X$ be a NPC of degree $d < \infty$.

a) If $g : [0, 1] \to X$ is a path, and if $y_1 \in p^{-1}(g(0))$ is a preimage of its starting point, then there is a unique lift $h := L^p_{y_1}(g) : [0, 1] \to Y$ with starting point $h(0) = y_1$. Its end point shall be denoted by $e^p_{y_1}(g)$. If g' is homotopic to g then $L^p_{y_1}(g')$ is homotopic to h and $e^p_{y_1}(g') = e^p_{y_1}(g)$.

b) Let $x_0 \in X$, $p^{-1}(x_0) = \{y_1, \ldots, y_d\}$. The map
$$m_p : \pi_1(X, x_0) \to S_d, \gamma \mapsto (i \mapsto j, \text{ if } y_j = e^p_{y_i}(\gamma))$$
is a group homomorphism and is called the *monodromy* of p.* It is uniquely determined by p up to conjugation in S_d.

c) p is a covering (i.e. Y is path-wise connected) iff the action induced by m_p is transitive.

Proof. Parts a) and b) are easily reduced to the case of ordinary coverings by restricting to the path-wise connected components of Y. In that case they are well-known, see for example [**Sch71**, III 6.3]. Let us prove part c).

"$\Leftarrow$": Let $y, y' \in Y$, w.l.o.g. $y = y_1$. Choose a path γ from $p(y')$ to x_0. Let $\tilde\gamma = L^p_{y'}(\gamma)$. There is an $i \in \{1, \ldots, d\}$ with $\tilde\gamma(1) = y_i$. m_p is transitive, so choose $\beta \in \pi_1(X, x_0)$ with $m_p(\beta)(1) = i$. Let $\tilde\beta = L^p_{y_1}(\beta)$. Then $\tilde\gamma^{-1}\tilde\beta$ is a path from y to y'.

"$\Rightarrow$": If m_p is not transitive, i.e. $\exists\, i, j \nexists\, \gamma : m_p(\gamma)(i) = j$, then there is no path from y_i to y_j. □

Theorem A. Let X be a *-space. Then there are the following bijections:

a)
$$\{X'/X \text{ Covering of degree } d\}_{/\text{Fibre preserving homeomorphisms}}$$
$$\overset{\text{i)}}{\leftrightarrow} \{U \subseteq \pi_1(X) \text{ subgroup of index } d\}_{/\text{conjugation}}$$
$$\overset{\text{ii)}}{\leftrightarrow} \{m : \pi_1(X) \to S_d \text{ transitive permutation representation}\}_{/\text{conjugation in } S_d}$$

b)
$$\{X'/X \text{ NPC of degree } d\}_{/\text{Fibre preserving homeomorphisms}}$$
$$\leftrightarrow \{m : \pi_1(X) \to S_d \text{ permutation representation }\}_{/\text{conjugation in } S_d}$$

In particular, a NPC is uniquely determined up to fibre preserving homeomorphism by its monodromy.

*This is actually the reason for the above convention to read words in the fundamental group from the right. One wants to consider elements of S_d as maps with group composition "$\circ$".

PROOF. a) The equivalence i) is the well known main theorem of the theory of coverings. For the second equivalence, we apply Lemma 1.3, which we state below. To be able to use it, note that S_d is generated by the permutations that fix 1, and $\{(1\,i),\ i \in 2, \dots, d\}$.

b) Decompose $p : X' \to X$ into the path connected components $X_1, \dots, X_k$ of X'. These are coverings of degree $d_1, \dots, d_k, \sum d_i = d$. Respectively, decompose a permutation representation into its orbits. With the following Remark 1.4, this reduces the claim to the situation of a). $\square$

LEMMA 1.3. Let G be a group. Then there is the following bijection:

$$\{U \subseteq G \text{ subgroup of index } d\} \leftrightarrow \{m : G \to S_d \text{ transitive}\}_{/\text{conjugations that fix 1}}$$

$$U \overset{\phi}{\mapsto} \text{action of } G \text{ on } u_1U, \dots, u_dU, u_1 = 1^\dagger$$

$$m^{-1}(\mathrm{Stab}(1)) \overset{\psi}{\leftarrow\!\shortmid} m$$

PROOF. $\psi \circ \phi = \mathrm{id}$ is obvious! To prove $\phi \circ \psi = \mathrm{id}$, let $m : G \to S_d$ be like in the above situation, $m' := (\phi \circ \psi)(m)$.

We claim that m and m' are conjugate to each other. Indeed, let $U = m^{-1}(\mathrm{Stab}(1))$, and let $u_1 = 1, \dots, u_d$ be coset representatives of U. W.l.o.g. we have $m(u_i)(1) = i$. (This is achieved by a conjugation fixing 1.)

Let $g \in G$. Then we have:

$$m(g)(i) = k \Leftrightarrow gu_iU = u_kU \Leftrightarrow m'(g)(i) = k$$

$\square$

For the results of the next section we need the following elementary

REMARK 1.4. a) Let $f : Y \to X$ be a covering of degree d, and let $g : Z \to Y$ be a NPC of degree e. Then, $f \circ g : Z \to X$ is a NPC of degree $d \cdot e$.

b) Let $f : Y \to X$ be a NPC of degree d. Then, for each path-wise connected component Y_i of Y, the restriction $f_{|Y_i} : Y_i \to X$ is a covering of some degree d_i such that we have $\sum d_i = d$.

c) In the situation of b), denote for each path-wise connected component the degree of the restriction $f_{|Y_i}$ by d_i, and let $g_i : Z_i \to Y_i$ be a NPC of degree e_i. Let $Z := \coprod Z_i$ and $g := \coprod g_i : Z \to Y$. Then, $f \circ g : Z \to X$ is a NPC of degree $\sum d_i \cdot e_i$.

We omit the straightforward proof here.

2. Fibre products of NPCs

Let $f : A \to X, g : B \to X$ be continuous maps of topological spaces. Then, the *fibre product* $A \times_X B$ is defined as

$$A \times_X B := \{(a, b) \in A \times B : f(a) = g(b)\}$$

endowed with the subspace topology of the product. Consequently, the projections $p_A : A \times_X B \to A, (a, b) \mapsto a$ and $p_B : A \times_X B \to B, (a, b) \mapsto b$ are continuous.

If now f and g are NPCs, then p_A, p_B and $f \circ p_A = g \circ p_B$ are also NPCs, possibly after restricting them such that their image is path-wise connected. We study the situation more closely in the following

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†Here, $u_1 = 1, \dots, u_d$ is a system of representatives of the right cosets, and the action is multiplication from the left.

Theorem B. Let X be a *-space, $f : A \to X$, $g : B \to X$ NPCs of degree d and d', respectively, with given monodromy maps m_f resp. m_g. Then, we have for the fibre product $A \times_X B$:

a) For each path-wise connected component $A_i \subseteq A$, the restriction
$$p_{A|p_A^{-1}(A_i)} : p_A^{-1}(A_i) = A_i \times_X B \to A_i$$
is a NPC of degree d' with monodromy
$$m_g \circ (f_{|A_i})_*.$$

b) The map $f \circ p_A = g \circ p_B : A \times_X B \to X$ is a NPC of degree dd' with monodromy
$$m_f \times m_g : \pi_1(X, x_0) \to S_d \times S_{d'} \subseteq S_{dd'}, \gamma \mapsto \Big((k, l) \mapsto (m_f(k), m_g(l))\Big),$$
where $(k, l) \in \{1, \ldots, d\} \times \{1, \ldots, d'\}$.

Proof. For part a), let A w.l.o.g. be path-wise connected, i.e. $A_i = A$, and let $x \in A$. Then we have:
$$p_A^{-1}(x) = \{(a, b) \in A \times B : a = x \wedge f(a) = g(b)\} = \{x\} \times g^{-1}(\{f(x)\}).$$
This shows the surjectivity of p_A.

Now choose admissible neighbourhoods U_A, U_B of $f(x)$ with respect to f and g, respectively. Then, $U := U_A \cap U_B$ is an admissible neighbourhood of $f(x)$ with respect to both f and g. Let $f^{-1}(U) = \coprod_{i=1}^{d} U_i$, and w.l.o.g. let $x \in U_1$.

We claim now that U_1 is an admissible neighbourhood of x with respect to p_A. To see this, let $g^{-1}(U) = \coprod_{j=1}^{d'} V_j$. Then we have:
$$\begin{aligned} p_A^{-1}(U_1) &= \{(a, b) \in A \times B, a \in U_1 : f(a) = g(b)\} \\ &= \left\{(a, b) \in U_1 \times \coprod_{i=1}^{d'} V_i : f(a) = g(b)\right\} \\ &= \coprod_{i=1}^{d'} \{(a, b) \in U_1 \times V_i : f(a) = g(b)\} = \coprod_{i=1}^{d'} U_1 \times_U V_i \\ &\cong \coprod_{i=1}^{d'} U_1 \end{aligned}$$
We have the rightmost homeomorphism because U, U_1 and V_i are homeomorphic. This shows that p_A is a NPC of degree d'.

Let us now calculate the monodromy of p_A. So, choose base points $x_0 \in X$ and $a_0 \in f^{-1}(x_0)$. Let $g^{-1}(x_0) = \{b_1, \ldots, b'_d\}$ be the fibre over x_0, and $p_A^{-1}(a_0) = \{c_1, \ldots, c'_d\}$ be the fibre over a_0, the numbering on the latter chosen such that $p_B(c_i) = b_i$.

Now, take a closed path $\gamma : [0, 1] \to A$ with $\gamma(0) = \gamma(1) = a_0$. Let $i \in \{1, \ldots, d'\}$, and let $\tilde{\gamma} = L_{c_i}^{p_A}(\gamma)$ be the lift of γ starting in c_i. Assume that $\tilde{\gamma}(1) = c_j$.

Consider now the path $\delta = f \circ \gamma$. It is a closed loop starting in x_0. Let $\tilde{\delta} = L_{b_i}^{g}(\delta)$ be its lift starting in b_i, then we have, because of the uniqueness of the lift and the commutativity of the diagram: $\tilde{\delta} = p_B \circ \tilde{\gamma}$, so particularly, as we asserted $p_B(c_i) = b_i$ for all i, we have $\tilde{\delta}(1) = b_j$.

So indeed, we have shown $m_{p_A}(i) = (m_g \circ f_*)(i)$.

For part b), we first note that as a consequence of Remark 1.4 b), $f \circ p_A$ is a NPC of degree $d \cdot d'$.

Let $\gamma \in \pi_1(X, x_0)$, $c_{ij} \in (f \circ p_A)^{-1}(x_0)$. Further let $e^{f\circ p_A}_{c_{ij}}(\gamma) = c_{kl}$, using the same notation as in Definition and Remark 1.2 a).

Then we have $e^f_{a_i}(\gamma) = p_A(c_{kl}) = a_k$ and $e^g_{b_j}(\gamma) = p_B(c_{kl}) = b_l$. This completes the proof. $\square$

The situation of the theorem is actually the main reason why we use NPCs instead of ordinary coverings: Even if we consider two ordinary coverings, their fibre product need not be connected. Indeed, take a covering $p: Y \to X$ of degree ≥ 2, then by part b) it is easy to see that $Y \times_X Y$ cannot be path-wise connected since the monodromy action of the projection to X is not transitive.

3. Compositions of NPCs

Now, we will discuss the monodromy of the composition of NPCs which are given in terms of their monodromy maps. Of course, the "lower" NPC has to be path-wise connected, i.e. a covering. The considered situation is the following:

Let X be a *-space, $f : Y \to X$ a cover of degree d, and $g : Z \to Y$ a NPC of degree d'. Furthermore, let $x_0 \in X$, $f^{-1}(x_0) = \{y_1, \ldots, y_d\}$, $g^{-1}(y_i) = \{z_{i1}, \ldots, z_{id'}\}$.

The fundamental group of X is denoted by $\Gamma := \pi_1(X, x_0)$, the given monodromy map by $m_f : \Gamma \to S_d$. Fix the notation

$$\Gamma_1 := m_f^{-1}(\mathrm{Stab}(1)) = \{\gamma \in \Gamma : m_f(\gamma)(1) = 1\}.$$

So, if we choose y_1 as a base point of Y and set $\Gamma' := \pi_1(Y, y_1)$, then we have $f_*(\Gamma') = \Gamma_1$. Denote, as usual, the monodromy map of the NPC g by $m_g : \Gamma' \to S_{d'}$.

Theorem C. In the situation described above, let γ_i, $i = 1, \ldots, d$, be right coset representatives of Γ_1 in Γ, with $\gamma_1 = 1$, such that $e^f_{y_i}(\gamma_i) = y_1$. So, we have $\Gamma = \dot{\bigcup} \Gamma_1 \cdot \gamma_i$.

Then, we have:

$$m_{f\circ g}(\gamma)(i, j) = \Big(m_f(\gamma)(i), m_g(c_i(\gamma))(j)\Big)$$

Here, we denote $c_i(\gamma) := (f_*)^{-1}(\gamma_k \gamma \gamma_i^{-1})$, $k := m_f(\gamma)(i)$

Proof. Let $\gamma \in \pi_1(X, x_0)$, $\alpha := L^{f\circ g}_{z_{ij}}(\gamma)$, and further let $\alpha(1) = e^{f\circ g}_{z_{ij}}(\gamma) =: z_{i'j'}$. We have to determine (i', j').

The path $\beta := L^f_{y_i}(\gamma) = g \circ \alpha$ has endpoint $\beta(1) = y_{m_f(\gamma)(i)}$. In particular, we have $i' = m_f(\gamma)(i)$.

Now, let us determine j'. So let $\beta_\nu := L^f_{y_\nu}(\gamma_\nu)$ be liftings (for $\nu = 1, \ldots, d$). Remember that by our choice of the numbering of the γ_ν, we have $\beta_\nu(1) = e^f_{y_\nu}(\gamma_\nu) = y_1$. Using the notation $k := m_f(\gamma)(i)$, we can write $\beta = \beta_k^{-1}\tilde{\beta}\beta_i$ with unique $\tilde{\beta} \in \pi_1(Y, y_1)$. Indeed, we have: $\tilde{\beta} = \beta_k \beta \beta_i^{-1} = L^f_{y_1}(\gamma_k \gamma \gamma_i^{-1})$.

W.l.o.g. we have that the lifting $\alpha_{ij} := L^g_{z_{ij}}(\beta_i)$ has endpoint z_{1j}, as we have chosen $\gamma_1 = 1$, and for $i \neq 1$ we can renumber the z_{ij}, $j = 1, \ldots, d'$.

Denote $\tilde{\alpha} := L^{g}_{z_{1j}}(\tilde{\beta})$ and $l := m_g(\tilde{\beta}(j))$, then we have $\alpha = \alpha_{kl}^{-1}\tilde{\alpha}\alpha_{ij}$, and because of $\tilde{\alpha}(1) = z_{1\,m_g(\tilde{\beta}(j))}$ we get:

$$z_{i'j'} = e^{f\circ g}_{z_{ij}}(\gamma) = \alpha(1) = \left(\alpha_{kl}^{-1}\tilde{\alpha}\alpha_{ij}\right)(1) = z_{kl}$$

So finally, $j' = l = m_g(\tilde{\beta})(j) = m_g(\beta_k\beta\beta_i^{-1})(j) = m_g(c_i(\gamma))(j)$. □

4. A lemma about normal coverings

The following topological lemma, which the author has learnt from Stefan Kühnlein, diverges a bit from the previous sections of this chapter, as it does not use the monodromy action. However, it will be very useful in the proof of one of the main results, where it replaces an argument that uses the group structure of elliptic curves. Thus, it also fits in our programme to use as much topological reasoning as possible.

Let $p : Y \to Z$ be a covering (i.e. in particular Y is path-wise connected). A *deck transformation* of this covering is a homeomorphism $\varphi : Y \to Y$ such that $p \circ \varphi = p$. We denote the group of deck transformations of p by $\mathrm{Deck}(p)$, or, when there is no ambiguity about p, by $\mathrm{Deck}(Y/Z)$. For any $z_0 \in Z$, the group $\mathrm{Deck}(p)$ acts on the fibre $p^{-1}(z_0)$. If this action is transitive, the covering is called *normal* and the deck transformation group is in this case sometimes denoted by $\mathrm{Gal}(Y/Z)$. It is well known that we have in this situation, for $y_0 \in p^{-1}(z_0)$:

$$\mathrm{Gal}(Y/Z) \cong \pi_1(Z, z_0)/p_*\left(\pi_1(Y, y_0)\right).$$

The lemma we are going to prove says the following: Given a normal covering $p : Y \to Z$ and two coverings of Y such that their compositions with p are equivalent coverings of Z, then any fibre preserving homeomorphism exhibiting this equivalence descends to an element of $\mathrm{Gal}(Y/Z)$.

Lemma 1.5. Let $f : X \to Y$, $f' : X' \to Y$ be coverings and $g : Y \to Z$ be a normal covering, where Z shall be a Hausdorff space. If $g \circ f \cong g \circ f'$, i.e. there is a homeomorphism $\varphi : X \to X'$ with $g \circ f = g \circ f' \circ \varphi$, then there is a deck transformation $\psi \in \mathrm{Deck}(g)$ such that $\psi \circ f = f' \circ \varphi$.

$$\begin{tikzcd}
X \arrow[rr, "\varphi"] \arrow[d, "f"'] & & X' \arrow[d, "f'"] \\
Y \arrow[rr, dotted, "\psi"] \arrow[dr, "g"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

Proof. Choose $z \in Z$, $y \in g^{-1}(z)$, $x \in f^{-1}(y)$. If we denote $x' := \varphi(x), y' := f'(x')$, then by hypothesis $y' \in g^{-1}(z)$. So by normality of g, there is a deck transformation $\psi \in \mathrm{Deck}(g)$ such that $\psi(y) = y'$ (which is even unique). Of course, $\psi \circ f(x) = f' \circ \varphi(x)$, and we claim now that we have $\psi \circ f = f' \circ \varphi$ globally.

Consider the set $A := \{a \in X \mid \psi \circ f(a) = f' \circ \varphi(a)\}$. Clearly $A \neq \emptyset$ because $x \in A$. Also, it is closed in X because all the spaces are Hausdorff. We want to show now that A is also open. Because X is connected, this implies $A = X$ and finishes the proof.

So let $a \in A$, and let $g(f(a)) \in U \subseteq Z$ be an admissible neighbourhood for both $g \circ f$ and $g \circ f'$ (and so particularly for g). Furthermore let $V \subseteq g^{-1}(U)$ be the connected component containing $f(a)$, and $W \subseteq f^{-1}(V)$ the one containing a. Denote $V' := \psi(V)$, and by W' denote the connected component of $f'^{-1}(V')$ containing $\varphi(a)$. As it is not clear by hypothesis that $W' = W'' := \varphi(W)$, set $\tilde{W}' := W' \cap W''$, which is still an open neighbourhood of $\varphi(a)$, and adjust the other neighbourhoods in the following way:

$$\tilde{W} := \varphi^{-1}(\tilde{W}') \subseteq W,\ \tilde{V} := f(\tilde{W}),\ \tilde{V}' := f'(\tilde{W}').$$

Note that we still have $a \in \tilde{W}$, that all these sets are still open, that $\tilde{U} := g(\tilde{V}) = g(\tilde{V}')$, and that the latter is still an admissible neighbourhood for $g \circ f$ and $g \circ f'$. By construction, by restricting all the maps to these neighbourhoods we get a commutative pentagon of homeomorphisms, so in particular $(\psi \circ f)_{|\tilde{W}} = (f' \circ \varphi)_{|\tilde{W}}$, which finishes the proof. $\square$

CHAPTER 2

Dessins d'enfants and Belyi's theorem

In this chapter, we will first introduce dessins d'enfants in the sense of Grothendieck and explain the equivalence of the topological, the complex analytic and the algebro-geometric view on a dessin.

We will go on by discussing the notions of moduli fields and fields of definition of a dessin. In order to do so, our point of view will be a scheme-theoretic one. This might seem unnecessarily abstract, as in terms of algebraic geometry dessins are just nonconstant functions on complex projective curves. However, when we restrict to subfields of $\mathbb{C}$ that are not algebraically closed, reasoning by classical algebraic geometry becomes a bit cumbersome. Furthermore, describing the action of $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ on dessins is much more natural in the language of $\overline{\mathbb{Q}}$-schemes, i.e. schemes together with a *structure morphism* to $\mathrm{Spec}(\overline{\mathbb{Q}})$, in contrast to working with explicit equations.

After that we are ready to state Belyi's famous theorem, which allows another point of view on dessins. We will sketch the proof, and finally state the faithfulness result about the action of $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ on dessins.

For the definitions and results stated in this chapter, we follow mainly **[Wol01]**, **[Sch94]** and **[Köc04]**.

1. Several definitions of a dessin

DEFINITION 2.1.
a) A *dessin d'enfant* (or *Grothendieck dessin*, or *children's drawing*) of degree d is a tuple (B, W, G, S), consisting of:
- A compact oriented connected real 2-dimensional manifold S,
- two finite disjoint subsets $B, W \subset S$ (called the *black* and *white vertices*),
- an embedded graph $G \subset S$ with vertex set $V(G) = B \dot{\cup} W$ which is bipartite with respect to that partition of $V(G)$, such that $S \setminus G$ is homeomorphic to a finite disjoint union of open discs (called the *cells* of the dessin, and such that $|\pi_0(G \setminus (B \cup W))| = d$.

b) An isomorphism between two dessins $D := (B, W, G, S)$ and $D' := (B', W', G', S')$ is an orientation preserving homeomorphism $f : S \to S'$, such that
$$f(B) = B', \; f(W) = W', \text{ and } f(G) = G'.$$

c) By $\mathrm{Aut}(D)$ we denote the group of automorphisms of D, i.e. the group of isomorphisms between D and itself.

So, from a naïve point of view, a dessin is given by drawing several black and white dots on a surface and connecting them in such a manner by edges that the cells which are bounded by these edges are simply connected. It is not obvious that a dessin d'enfant can be defined equally well in several other manners, revealing the rich structure of a dessin that is hidden in our first definition. We give an overview in the following

PROPOSITION 2.2. Giving a dessin in the above sense up to isomorphism is equivalent to giving each of the following data:

a) A finite topological covering $\beta : X^* \to \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ of degree d up to equivalence of coverings.
b) A conjugacy class of a subgroup $G \leq \pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})$ of index d.
c) A pair of permutations $(p_x, p_y) \in S_d^2$, such that $\langle p_x, p_y \rangle \leq S_d$ is a transitive subgroup, up to simultaneous conjugation in S_d.
d) A non-constant holomorphic map $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ of degree d, where X is a compact Riemann surface and β is ramified at most over the set $\{0, 1, \infty\}$, up to the following equivalence:

$$(\beta : X \to \mathbb{P}^1_{\mathbb{C}}) \cong (\beta' : X' \to \mathbb{P}^1_{\mathbb{C}}) :\Leftrightarrow \exists \text{ biholomorphic } \varphi : X \to X' : \beta = \beta' \circ \varphi.$$

e) A non-constant morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ of degree d, where X is a nonsingular connected projective curve over $\mathbb{C}$ and β is ramified at most over the set $\{0, 1, \infty\}$, up to the following equivalence:

$$(\beta : X \to \mathbb{P}^1_{\mathbb{C}}) \cong (\beta' : X' \to \mathbb{P}^1_{\mathbb{C}}) :\Leftrightarrow \exists \text{ isomorphism } \varphi : X \to X' : \beta = \beta' \circ \varphi.$$

PROOF. The equivalence between an isomorphism class of dessins in the sense of the definition, and a conjugacy class of a pair of permutations as in c) is shown in [**JS78**, § 3].

The equivalence of a), b) and c) is discussed in Theorem A, noting that the fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})$ is freely generated by two paths x and y, so giving a transitive homomorphism to S_d just amounts to picking their images p_x and p_y in such a way that they generate a transitive subgroup.

The equivalence between a) and d) is a well-known fact from complex analysis: According to [**For77**, Satz 1.8], we can lift the usual complex structure on $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ to X^* in exactly one way such that β becomes holomorphic. Then we can complete X^* to a compact Riemann surface X and continue β to a holomorphic map $X \to \mathbb{P}^1_{\mathbb{C}}$. In the other direction, we just remove the set $\{0, 1, \infty\}$ from $\mathbb{P}^1_{\mathbb{C}}$ and its finitely many preimages from X, so we end up with an unramified covering. Note that the equivalence between Definition 2.1 and d) is proven by Jones and Singerman in [**JS78**], and the direction from d) to our original definition can be understood in the following explicit way: As S, we take of course the Riemann surface X, forgetting its complex structure, as B and W we take the preimages of 0 and 1, respectively, and for the edges of G we take the preimages of the open interval $(0, 1) \subset \mathbb{P}^1_{\mathbb{C}}$. Then, $S \setminus G$ is the preimage of the set $\mathbb{P}^1_{\mathbb{C}} \setminus [0, 1]$, which is open and simply connected. So the connected components of $S \setminus G$ are open and simply connected proper subsets of a compact surface and thus homeomorphic to an open disc.

Finally, the equivalence between d) and e) follows from the well-known equivalence between nonsingular algebraic curves and Riemann surfaces, which we state below. □

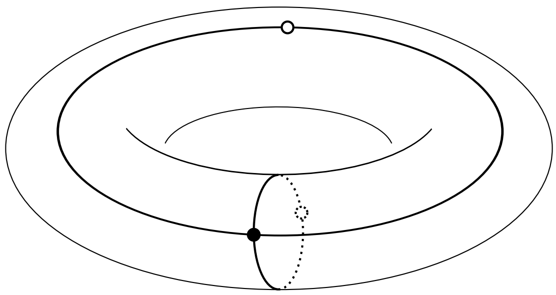
THEOREM D. There is an equivalence of categories between compact Riemann surfaces together with nonconstant morphisms and connected nonsingular projective curves over $\mathbb{C}$ together with nonconstant morphisms. This equivalence takes a Riemann surface to a complex curve with isomorphic function field.

Riemann himself already knew parts of that result, but the usual reference is [**Ser56**] whose title coined the name "GAGA-principle" for its main results, which generalise the above theorem.

For reasons that will become apparent soon, we will call the morphisms β appearing in the above proposition, specifically in part e), *Belyi morphisms*. Also, this proposition encourages us to slightly abuse the notation by using Belyi morphism and dessin d'enfant as synonyms.

It might now be nice to see a first simple example of a dessin d'enfant:

Example 2.3. We draw the following graph on a topological surface E of genus 1:



The graph, which has 4 edges, is obviously bipartite, and if we remove it from the surface, we get one connected component which is simply connected. So, what we have drawn is indeed a dessin D of degree 4 with one black vertex, two white vertices and one cell.

We want to understand at least the equivalent forms c) and e) from Proposition 2.2 to write down this dessin. For c), we first need to fix generators x and y of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\})$. We fix $\frac{1}{2}$ as a base point and define

$$x: [0,1] \to \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}, t \mapsto \frac{e^{2\pi i t}}{2}$$
$$y: [0,1] \to \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}, t \mapsto 1 - \frac{e^{2\pi i t}}{2}.$$

So x and y are simple closed loops around 0 and 1, respectively. Let us fix this choice for the rest of this work. Now we need to mark a point on each of the dessin's edges as the preimage of our base point and study the monodromy action of x and y, i.e. lift the paths, beginning in each of these preimages. This amounts, after choosing a numbering on the edges, to enumerating the edges ending in each black vertex in counterclockwise direction (this gives the cycles of p_x), and repeating the same for the white vertices, defining p_y. In the case of this example, we find (with the right numbering of edges):

$$p_x = (1\,2\,3\,4),\ p_y = (1\,3)(2\,4)$$

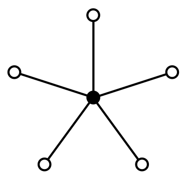
According to part e) of the above proposition, D also defines a structure as a complex projective curve on E and a morphism $\beta_D : E \to \mathbb{P}^1_{\mathbb{C}}$. In general, this can be hard to calculate, see [**Sch94**, §3] for an approach using Gröbner bases. Fortunately, the situation is relatively easy here, and we find (affine) equations in

[**AAD$^+$07**]:

$$E : y^2 = x^3 - x, \;\; \beta_D : (x, y) \mapsto x^2.$$

In the class of dessins d'enfants of genus 0, i.e. dessins drawn on the Riemann sphere $\mathbb{P}^1_{\mathbb{C}}$, one is often interested in the ones whose Belyi functions are polynomials. A polynomial $p \in \mathbb{C}[X]$ does not have poles, so as a function $p : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ it takes the value ∞ exactly once, i.e. it is totally ramified over ∞. If p has at most two more (finite) critical values, it can be rescaled to be a Belyi function by postcomposing a linear function. As we will see in Definition and Remark 2.9, Belyi polynomials are precisely the Belyi functions that correspond to trees. We will give two series of Belyi polynomials in the following example:

Example 2.4. First, consider for $n \geq 1$ the polynomial $P_n(z) = z^n$. It is totally ramified over 0 (and ∞), and unramified elsewhere. So it is a Belyi polynomial, and the corresponding dessin has n edges, one black vertex and n white vertices. This information already defines the dessin. Let us draw the dessin corresponding to $P_5(z) = z^5$. The picture shall be thought of to be a small part of the Riemann sphere the dessin is drawn on.

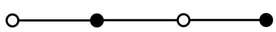


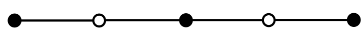
As a second series of examples, we consider the Chebyshev polynomials $(T_n)_{n\in\mathbb{N}}$. Remember that T_n is defined to be the function solving the equation $T_n(\cos\varphi) = \cos(n\varphi)$, and that the Chebyshev polynomials satisfy the recurrence relation

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z),$$

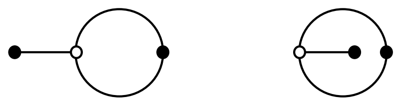
which also recursively defines them after setting $T_0(z) = 1$, $T_1(z) = z$. It is an easy exercise that $T_n'(z) = 0 \Rightarrow T_n(z) \in \{-1, 1\}$, i.e. the finite critical values are (at most) ± 1. So $\tilde{T}_n := \frac{1}{2}(T_n + 1)$ is a Belyi polynomial, and a closer inspection shows that if n is even, then $\tilde{T}_n$ has $\frac{n}{2} - 1$ double points and two unramified points lying over 0 and $\frac{n}{2}$ double points lying over 1. If n is odd, we have $\frac{n-1}{2}$ double points and one unramified point lying over 0 and 1, respectively. This is again enough information to draw the corresponding dessins. We draw the dessins corresponding to the Belyi polynomials

$$\tilde{T}_3(z) = 2z^3 - \tfrac{3}{2}z + \tfrac{1}{2} \text{ and } \tilde{T}_4(z) = 4z^4 - 4z^2 + 1 :$$





Before we continue, we want to mention a small warning about the intuition in the case of genus 0 dessins, i.e. dessins where the surface S is a sphere. In the above example, we have drawn them in the plane. This is possible in general: If such a dessin is given, then its graph G surely misses one point $P \in S$, so we can as well draw G in a plane, and since G is compact, we can draw it into a bounded domain. One usually makes use of this, because it makes it easy to draw pictures of genus 0 dessins. We only have to keep in mind that the choice of P puts a marking on one of the cells of the dessin (namely the unbounded cell, or the outside in the picture), that is not necessarily fixed by an isomorphism of dessins. Let us draw an example of two isomorphic dessins with different markings of the unbounded cell:



Let us note an easy corollary of Proposition 2.2 that may seem harmless but will almost prove the "obvious" part of Belyi's theorem later.

Corollary 2.5. For any $d \in \mathbb{N}$, there are only finitely many dessins d'enfants of degree d up to equivalence.

Proof. By Proposition 2.2, a dessin can be characterised by a pair of permutations $(p_x, p_y) \in (S_d)^2$. So, $(d!)^2$ is an upper bound for the number of isomorphism classes of dessins of degree d. $\square$

Next, we will establish the notion of a weak isomorphism between dessins.

Definition 2.6. We call two Belyi morphisms $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ and $\beta' : X' \to \mathbb{P}^1_{\mathbb{C}}$ *weakly isomorphic* if there are biholomorphic $\varphi : X \to X'$ and $\psi : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ such that the following square commutes:

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\beta"'] & X' \arrow[d, "\beta'"] \\ \mathbb{P}^1_{\mathbb{C}} \arrow[r, "\psi"] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd}$$

Note that in the above definition, if β and β' are ramified exactly over $\{0, 1, \infty\}$, then ψ has to be a Möbius transformation fixing this set. This subgroup $W \leq \mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}})$ is clearly isomorphic to S_3 and generated by

$$s : z \mapsto 1 - z \quad \text{and} \quad t : z \mapsto z^{-1}.$$

In the case of two branch points, ψ can of course still be taken from that group. The cases of exactly one and no branch point are trivial (the first one because it does not exist). So for a dessin β, we get up to isomorphism all weakly isomorphic dessins by postcomposing with all elements of W. Let us reformulate this on a more abstract level:

Definition and Remark 2.7. a) The group W acts on the set* of dessins from the left by $w \cdot \beta := w \circ \beta$. The orbits under that action are precisely the weak isomorphism classes of dessins.

*Actually we should be talking about isomorphism classes in order to get a set.

b) For a dessin β we call $W_\beta := \mathrm{Stab}(\beta)$, its stabiliser in W, the *group of nontrivial weak automorphisms*.[†]

c) If a dessin β is given by a pair of permutation (p_x, p_y), then its images under the action of W are described by the following table (where $p_z := p_x^{-1} p_y^{-1}$):

β	$s \cdot \beta$	$t \cdot \beta$	$(s \circ t) \cdot \beta$	$(t \circ s) \cdot \beta$	$(t \circ s \circ t) \cdot \beta$
(p_x, p_y)	(p_y, p_x)	(p_z, p_y)	(p_y, p_z)	(p_z, p_x)	(p_x, p_z)

Proof. Part a) was already discussed above. Proving c) amounts to checking what s and t do, and then composing, which has been done in [**Sij06**, 2.5]. □

It would be interesting to know all dessins that are fixed points under the action of W, which seems to be a rather strong condition. To the knowledge of the author, there is no complete classification of these dessins, but at least in Example 5.6 we give an infinite series of such dessins.

In the theory of dessins d'enfants, one often makes use of the notion of clean and pre-clean dessins. We define them here and introduce a class of "particularly un-clean" dessins that will be helpful in later considerations.

Definition and Remark 2.8. Let β be a dessin defined by a pair of permutations (p_x, p_y).

a) β is called *pre-clean* if $p_y^2 = 1$, i.e. if all white vertices are either of valence 1 or 2.

b) β is called *clean* if all preimages of 1 are ramification points of order precisely 2, i.e. if all white vertices are of valence 2.

c) If $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is a Belyi morphism of degree d, then if we define $a(z) := 4z(1-z) \in \mathbb{Q}[z]$ we find that $a \circ \beta$ is a clean dessin of degree $2d$.

d) We will call β *filthy* if it is not weakly isomorphic to a pre-clean dessin, i.e. $1 \notin \{p_x^2, p_y^2, p_z^2\}$.

Another common class of dessins consists of the unicellular ones. We briefly discuss them here.

Definition and Remark 2.9. a) A dessin d'enfant D is said to be *unicellular* if it consists of exactly one open cell.

b) If D is represented by a pair of permutations (p_x, p_y), it is unicellular iff $p_z = p_x^{-1} p_y^{-1}$ consists of exactly one cycle.

c) If D is represented by a Belyi morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$, it is unicellular iff β has exactly one pole.

d) If D is a dessin in genus 0, it is unicellular iff its graph is a tree.

Proof. For b) and c), note that every cell of a dessin D contains exactly one preimage of ∞. These are of course the poles of the associated Belyi morphism β, and they are in bijection to the cycles of p_z, which describes the monodromy of β around ∞.

For d), note that a graph is contractible iff it is a tree. Because of the requirement for a cell in a dessin to be simply connected, this is equivalent to D being unicellular. □

[†]A maybe more precise notion of a weak automorphism would be, for a dessin $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$, a pair (φ, ψ), where $\varphi \in \mathrm{Aut}(X)$, $\psi \in W$ and $\psi \circ \beta = \beta \circ \varphi$. This would exhibit W_β as the image of the group of such pairs under the projection $(\varphi, \psi) \mapsto \psi$.

2. Fields of definition and moduli fields

Before we can state Belyi's famous theorem, we have to introduce the notion of a field of definition of a variety and of a morphism between varieties, and the closely related notion of the moduli field. The naïve idea behind a field of definition is the following: Let V be a variety over some field K defined by some equations such that all the coefficients of these equations lie in some smaller field $k \subseteq K$. Then it appears reasonable to say that V can be defined over k, or to call k a field of definition of V. The same notion can be stated for morphisms between K-varieties. We could indeed formalise these ideas for, say, projective varieties $V \subseteq \mathbb{P}^n(K)$ defined by a set of equations, but the theory appears much clearer if we use the language of K-schemes, even if the schemes we will talk about are actually quasi-projective varieties in the sense of classical algebraic geometry. We follow more or less the presentation of the material in **[Köc04]** here. Let us start with a definition. Here, we denote as usual by

$$\mathrm{Spec}: \underline{\text{commutative rings}} \to \underline{\text{affine schemes}}$$

the contravariant functor which assigns to a ring its spectrum.

Definition 2.10. Let K be a field.

a) A *K-scheme* is a pair (S, p), where S is a scheme, and p a morphism $p: S \to \mathrm{Spec}(K)$, called the *structure morphism*.
b) A *K-morphism* $\varphi : (S, p) \to (S', p')$ is understood to be a scheme morphism $\varphi : S \to S'$ such that $p = p' \circ \varphi$.
c) A K-scheme (S, p) is called *K-variety* if S is a reduced scheme that is separated and of finite type over $\mathrm{Spec}(K)$.
d) We denote the category of K-schemes by Sch/K and the category of K-varieties by Var/K.

Now, we can study the action of $\mathrm{Aut}(K)$, the group of field automorphisms of K, on K-schemes. We introduce two versions of this action, and learn that they are actually equivalent.

Definition and Remark 2.11. Let (S, p) be a K-scheme, and $\sigma \in \mathrm{Aut}(K)$.

a) Define $(S, p)^\sigma := (S, \mathrm{Spec}(\sigma) \circ p)$ and $(S, p).\sigma := (S', p')$, where (S', p') shall be defined by the following Cartesian diagram:

$$\begin{tikzcd} S' \arrow[r, "\varphi"] \arrow[d, "p'"'] \arrow[dr, phantom, "\square"] & S \arrow[d, "p"] \\ \mathrm{Spec}(K) \arrow[r, "\mathrm{Spec}(\sigma^{-1})"] & \mathrm{Spec}(K) \end{tikzcd}$$

Then $(S, p)^\sigma \cong (S, p).\sigma$ in Sch/K.
b) Mapping $(S, p) \mapsto (S, p)^\sigma$ defines a right action of $\mathrm{Aut}(K)$ on Sch/K.

Proof. For part a) we first note that $\varphi : S' \to S$ in the Cartesian diagram is an isomorphism of schemes since $\mathrm{Spec}(\sigma^{-1}) \in \mathrm{Aut}(\mathrm{Spec}(K))$. Furthermore, by the commutativity of the diagram, we have $p' = \mathrm{Spec}(\sigma) \circ \mathrm{Spec}(\sigma^{-1}) \circ p' = \mathrm{Spec}(\sigma) \circ p \circ \varphi$, so φ is a K-isomorphism between $(S, \mathrm{Spec}(\sigma) \circ p)$ and (S', p').

Part b) is a direct consequence of the fact that Spec is a contravariant functor, so indeed $(S, p)^{(\sigma\tau)} = ((S, p)^\sigma)^\tau$.

For a reader more familiar with classical algebraic geometry, it might now be unintuitive that if we take (S, p) to be, say, a curve over $\mathbb{C}$, this action does anything visible at all, since it leaves the underlying scheme S completely unchanged and only changes the dubious structure morphism. So we hope to clarify the situation a bit by citing [**Köc04**, Remark 1.2], which translates our definitions back to the classical world of projective varieties:

Remark 2.12. Let K be a field, let $\sigma \in \mathrm{Aut}(K)$ and let $X = V(f_1, \dots, f_m) \subseteq \mathbb{P}^n(K)$ be a projective variety defined by homogeneous polynomials $f_1, \dots, f_m \in K[X_0, \dots, X_n]$. Then, X^σ is defined by the polynomials $\sigma^{-1}(f_1), \dots, \sigma^{-1}(f_m)$, where $\mathrm{Aut}(K)$ shall act on polynomials by acting on their coefficients.

We are now able to define the terms field of definition and moduli field.

Definition 2.13. a) A subfield $k \subseteq K$ is called *field of definition* of a K-scheme (resp. K-variety) (S, p) if there is a k-scheme (resp. k-variety) (S', p') such that there is a Cartesian diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & S' \arrow[d, "p'"] \\ \mathrm{Spec}(K) \arrow[r, "\mathrm{Spec}(\iota)"] & \mathrm{Spec}(k) \end{tikzcd}$$

where $\iota : k \to K$ is the inclusion. Alternatively, (S, p) is said to be *defined over k* then.

b) For a K-scheme (S, p), define the following subgroup $U(S, p) \leq \mathrm{Aut}(K)$:

$$U(S, p) := \{\sigma \in \mathrm{Aut}(K) \mid (S, p)^\sigma \cong (S, p)\}$$

The *moduli field* of (S, p) is then defined to be the fixed field under that group:

$$M(S, p) := K^{U(S,p)}$$

We will also need to understand the action of $\mathrm{Aut}(K)$ on morphisms. We will start the bad habit of omitting the structure morphisms here, which the reader should amend mentally.

Definition 2.14. Let $\beta : S \to T$ be a K-morphism (i.e. a morphism of K-schemes) and $\sigma \in \mathrm{Aut}(K)$.

a) The scheme morphism β is of course also a morphism between the K-schemes S^σ and T^σ. We denote this K-morphism by $\beta^\sigma : S^\sigma \to T^\sigma$.
b) Let $\beta' : S' \to T'$ be another K-morphism. Then we write $\beta \cong \beta'$ if there are K-isomorphisms $\varphi : S \to S'$ and $\psi : T \to T'$ such that $\psi \circ \beta = \beta' \circ \varphi$. Specifically we have $\beta \cong \beta^\sigma$ iff there are K-morphisms φ, ψ such that the following diagram (where we write down at least some structure morphisms) commutes:

$$\begin{tikzcd} S^\sigma \arrow[r, "\varphi"] \arrow[d, "\beta^\sigma"] & S \arrow[d, "\beta"] \\ T^\sigma \arrow[r, "\psi"] \arrow[d, "p"] & T \arrow[d, "p"] \\ \mathrm{Spec}(K) \arrow[r, "\mathrm{Spec}(\sigma)"] & \mathrm{Spec}(K) \end{tikzcd}$$

Let us now restate the definitions of field of definition and moduli field for morphisms. For that sake, note that taking the fibre product with $\mathrm{Spec}(K)$ via the inclusion $\iota : k \to K$ is a functor from Sch/k to Sch/K that we will denote by $\times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$: Indeed, it is just the base change to $\mathrm{Spec}(K)$.

DEFINITION 2.15.
a) A morphism $\beta : S \to T$ of K-schemes (or K-varieties, respectively) is said to be *defined over* a field $k \subseteq K$ if there is a morphism $\beta' : S' \to T'$ of k-schemes (k-varieties) such that $\beta = \beta' \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$.
b) For a morphism $\beta : S \to T$ of K-schemes, define the following subgroup $U(\beta) \leq \mathrm{Aut}(K)$:
$$U(\beta) := \{\sigma \in \mathrm{Aut}(K) \mid \beta^\sigma \cong \beta\}$$
The *moduli field* of β is then defined to be the fixed field under this group:
$$M(\beta) := K^{U(\beta)}$$

If, now, $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ is a Belyi morphism, the above notation gives already a version for a moduli field of β. But this is not the one we usually want, so we formulate a different version here:

DEFINITION 2.16. Let $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ be a Belyi morphism, and let $U_\beta \leq \mathrm{Aut}(\mathbb{C})$ be the subgroup of field automorphisms σ such that there exists a $\mathbb{C}$-isomorphism $f_\sigma : X^\sigma \to X$ such that the following diagram commutes:

$$\begin{tikzcd} X^\sigma \arrow[r, "f_\sigma"] \arrow[d, "\beta^\sigma"] & X \arrow[d, "\beta"] \\ (\mathbb{P}^1_{\mathbb{C}})^\sigma \arrow[r, "\mathrm{Proj}(\sigma)"] & \mathbb{P}^1_{\mathbb{C}} \end{tikzcd}$$

where $\mathrm{Proj}(\sigma)$ shall denote the scheme (not $\mathbb{C}$-scheme!) automorphism of $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[X_0, X_1])$ associated to the ring (not $\mathbb{C}$-algebra!) automorphism of $\mathbb{C}[X_0, X_1]$ which extends $\sigma \in \mathrm{Aut}(\mathbb{C})$ by acting trivially on X_0 and X_1.
Then, we call $M_\beta := \mathbb{C}^{U_\beta}$ the *moduli field* of the dessin corresponding to β.

Note that the difference to Definition 2.15 b) is that there, we allow composing $\mathrm{Proj}(\sigma)$ with automorphisms of $\mathbb{P}^1_{\mathbb{C}}$, potentially making the subgroup of $\mathrm{Aut}(\mathbb{C})$ bigger and therefore the moduli field smaller. Let us make that precise, and add some more facts about all these fields, by citing [**Wol01**, Proposition 6]:

PROPOSITION 2.17. Let K be a field, and $\beta : S \to T$ a morphism of K-schemes, or of K-varieties. Then:

a) $M(S)$ and $M(\beta)$ depend only on the K-isomorphism type of S resp. β.

b) If furthermore β is a Belyi morphism, then the same goes for M_β.
c) Every field of definition of S (resp. β) contains $M(S)$ (resp. $M(\beta)$).
d) We have $M(S) \subseteq M(\beta)$.
e) If β is a Belyi morphism, then we also have $M(\beta) \subseteq M_\beta$.
f) In this case M_β (and therefore also $M(S)$ and $M(\beta)$) is a number field, i.e. a finite extension of $\mathbb{Q}$.

Proof. Parts a) to e) are direct consequences of the above definitions, and for f) we note that surely if $\sigma \in \mathrm{Aut}(\mathbb{C})$ then $\deg(\beta) = \deg(\beta^\sigma)$. So by Corollary 2.5, we have $[\mathrm{Aut}(\mathbb{C}) : U_\beta] = |\beta \cdot \mathrm{Aut}(\mathbb{C})| < \infty$ and so $[M_\beta : \mathbb{Q}] < \infty$. □

3. Belyi's Theorem

In the last part of the above proposition, we have learnt that if a Riemann surface S admits a Belyi morphism, i.e. a meromorphic function that is at most ramified over 0, 1 and ∞, then its (or rather the associated complex curve's) moduli field $M(S)$ is a number field. This is a rather surprising connection between a complex analytic and an algebraic property of Riemann surfaces. One might hope now that if S admits a Belyi morphism, it is even definable over a number field. Knowing that in genus $g \geq 3$ a nonsingular curve is generically definable over its moduli field supports this hope. But not only is this claim true in general, but also the converse statement. This is the famous Theorem of Belyi, which we state here:

Theorem E. Let C be a smooth projective complex curve. C is definable over a number field if and only if it admits a Belyi morphism $\beta : C \to \mathbb{P}^1_{\mathbb{C}}$.

We should make the historical note here that for reasons that are far from being obvious and probably even nontrivial, the "if"-direction of the theorem is often called the "obvious part", and the "only if"-direction the "trivial part".

We will now sketch the proof. The gap between Proposition 2.17 f) and the "obvious" part of the theorem can be filled by the following theorem due to H. Hammer and F. Herrlich that can be found in **[HH03]**:

Theorem F. Let K be a field, and X be a curve over K. Then X can be defined over a finite extension of $M(X)$.

The proof to the "trivial" part of Belyi's theorem is surprisingly explicit, giving a method to actually construct a Belyi morphism, given a complex curve defined over a number field. There are several versions of this construction that is sometimes called *Belyi's algorithm*. In our sketch we follow the lines of the Lemmas 3.4 to 3.6 in **[Köc04]** which stay quite close to Belyi's original work **[Bel79]**.

Proof (Theorem E, "trivial part"). Let C be a smooth projective complex curve definable over some number field, so in particular over $\overline{\mathbb{Q}}$. We begin by noting that this guarantees the existence of a nonconstant morphism $f : C \to \mathbb{P}^1_{\mathbb{C}}$ such that we have for its set of critical values: $\mathrm{crit}(f) \subseteq \overline{\mathbb{Q}} \cup \{\infty\}$. This is **[Köc04**, Lemma 3.4]. Define the set S to be the closure of $\mathrm{crit}(f) \setminus \{\infty\}$ under the action of $\mathrm{Aut}(\mathbb{C})$. Because $\mathrm{crit}(f)$ is finite and consists only of $\overline{\mathbb{Q}}$-rational points, S is also finite.

We claim now that there exists a nonconstant polynomial $p \in \mathbb{Q}[z]$ such that $\mathrm{crit}(p \circ f) = \mathrm{crit}(p) \cup p(\mathrm{crit}(f)) \subseteq \mathbb{Q} \cup \{\infty\}$. We may replace $\mathrm{crit}(f)$ by $S \cup \{\infty\}$ and note that if $n := |S| \leq 1$, then $\mathrm{crit}(f) \subseteq \mathbb{Q} \cup \{\infty\}$ and there is nothing to show, as f is itself, possibly up to a Möbius transformation, a Belyi morphism. So assume $n > 1$. Take p_1 to be the product of the minimal polynomials of all elements of S over $\mathbb{Q}$, then we have on the one hand $p_1(S) = \{0\}$, and on the other hand, as $\deg(p_1) = n$, we have $|\operatorname{crit}(p_1) \setminus \{\infty\}| \leq n - 1$, and $S_1 := \operatorname{crit}(p_1) \setminus \{\infty\}$ is closed again under the action of $\mathrm{Aut}(\mathbb{C})$. So we can conclude by induction on n.

Now that we found a morphism with a (potentially horribly large, but) finite set of critical values $T := \operatorname{crit}(p \circ f)$ that lies in $\mathbb{Q} \cup \{\infty\}$, we claim the existence of a polynomial $q \in \mathbb{Q}[z]$ such that $\operatorname{crit}(q \circ p \circ f) = \operatorname{crit}(q) \cup q(T) \subseteq \{0, 1, \infty\}$. Let $r = |T|$ and note that if $r \leq 3$ we can q just take to be a Möbius transformation. So let $r \geq 4$ and assume w.l.o.g. that we have $\{0, 1, \frac{m}{m+n}, \infty\} \subseteq T$ for some $\frac{m}{m+n} \in \mathbb{Q}$, which we can guarantee by composing with a Möbius transformation. Now we compose with the polynomial

$$q_1 : z \mapsto \frac{(m+n)^{m+n}}{m^m n^n} z^m (1-z)^n$$

We easily check that $q_1(0) = q_1(1) = 0$ and $q_1(\frac{m}{m+n}) = 1$, so $q_1(\{0, 1, \frac{m}{m+n}, \infty\}) \subseteq \{0, 1, \infty\}$. Furthermore we have $\operatorname{crit}(q_1) \subseteq \{0, 1, \infty\}$, and so in total we get $|\operatorname{crit}(q_1 \circ p \circ f)| \leq r - 1$. So again, we can conclude by induction on r, which finishes the proof. □

4. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ on dessins

Given a Belyi morphism $\beta : X \to \mathbb{P}^1_{\mathbb{C}}$ and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$, clearly $\beta^\sigma : X^\sigma \to \mathbb{P}^1_{\mathbb{C}}$ is again a Belyi morphism, as the number of branch points is an intrinsic property of the underlying scheme morphism β that is not changed by changing the structure morphisms. So, $\mathrm{Aut}(\mathbb{C})$ acts on the set of Belyi morphisms, or, due to Proposition 2.2, on the set of dessins d'enfants. As a consequence of Belyi's theorem, we even know that this action factors through $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$. So it is natural to ask if the action factors through an even smaller group. The answer is that indeed it already acts faithfully. From the standpoint that $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ is an intriguing group with a structure so rich that it does not allow much understanding, and dessins d'enfants are objects that can be described by purely combinatorial means, this is a rather astonishing fact. Let us state here a finer version of the faithfulness result here:

Theorem G. For every $g \in \mathbb{N}$, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ on the set of dessins d'enfants of genus g is faithful. This still holds for every g if we restrict to the clean, unicellular dessins in genus g.

Let us sketch the proof here:

Proof. First, fix a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$.

In genus 1, take an elliptic curve defined over $\overline{\mathbb{Q}}$ such that $j(E) \neq (j(E))^\sigma = j(E^\sigma)$. Now take any nonconstant morphism $f : E \to \mathbb{P}^1_{\mathbb{C}}$ and apply the Belyi-algorithm to find a Belyi morphism $\beta : E \to \mathbb{P}^1_{\mathbb{C}}$. Then by construction $\beta \not\cong \beta^\sigma$. If we want to restrict to unicellular dessins, we start with an f that has only one pole, whose existence is guaranteed by the Theorem of Riemann-Roch and note that in our version of Belyi's algorithm we only postcompose with polynomials, ensuring that the ramification above ∞ will stay total. In order to find a clean Belyi morphism, we postcompose with $z \mapsto 4z(1-z)$ as usual. It was noted by F. Armknecht in **[Arm01]** that this argument generalises to higher genera, by even restricting to hyperelliptic curves only.

In genus 0, this of course does not work, as $\mathbb{P}^1_{\mathbb{C}}$ is already defined over $\mathbb{Q}$. An elegant proof is written up in **[Sch94]**, where it is attributed to H. W. Lenstra, Jr.

The idea is to start with an algebraic number $\alpha \in \overline{\mathbb{Q}}$ such that $\sigma(\alpha) \neq \alpha$ and choose $f_\alpha \in \overline{\mathbb{Q}}[z]$ such that

$$f'_\alpha(z) = z^3(z-1)^2(z-\alpha).$$

Then by construction $\operatorname{crit}(f) = \{0, 1, \alpha, \infty\}$, and the ramification points are distinguished by their orders. Again, we apply the Belyi algorithm to find a Belyi polynomial $\beta \in \overline{\mathbb{Q}}[z]$ for which we check that $\beta^\sigma \not\cong \beta$. $\square$

CHAPTER 3

Origamis and Teichmüller curves

1. Origamis as coverings

We will first introduce Origamis in a way closely related to our way of defining dessins d'enfants, in order to point out the similarities between the two constructions.

The standard intuition for constructing an Origami of degree $d \in \mathbb{N}$ is the following: Take d copies of the unit square $[0, 1] \times [0, 1]$ and glue upper edges to lower edges and left to right edges, respecting the orientation, until there are no free edges left, in a way that we do not end up with more than one connected component. In this way, we get a compact topological surface X together with a tiling into d squares. Let us begin by noting two simple structures that such a tiling defines:

The first one is the following: If we remove all the vertices of squares from X, then we get in a natural way a topological covering of the unique Origami E of degree 1, i.e. a genus 1 surface with one point removed, by projecting the d squares of X to the one square of E. Note that if we are conversely given such a covering $p : X^* \to E^*$ (the asterisks denoting the punctured versions of the surfaces), we get a square tiling back on X by drawing a square on E (which amounts to choosing two nontrivial simple closed curves on E that are not homotopic and that intersect precisely in the puncture) and taking its preimage under p. This should remind of the construction of a dessin by taking the preimage of the segment $[0, 1]$ under some covering of $\mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$.

For the second one, note that we glued the right edge of each square to the left edge of exactly one square (otherwise we would not get a closed manifold). So if we choose a numbering $1, \ldots, d$ of the squares, this uniquely determines a permutation $p_A \in S_d$ telling us that the right edge of square i is glued to the left edge of square $p_A(i)$. In the same manner, the way of gluing upper to lower edges determines a permutation $p_B \in S_d$. If we choose another numbering on the squares, we get a pair of permutations $(p'_A, p'_B) \in S_d^2$ such that there is an element $c \in S_d$ with $p'_A = c^{-1} p_A c$, $p'_B = c^{-1} p_B c$. Conversely, given such a pair of permutations, we can reconstruct the surface X together with the tiling, as the orientation of the gluing of edges is determined already by the agreement that we want to respect the orientation of the squares.

Let us now turn the first of these two notions into a rigorous definition. We fix the standard identification $\mathbb{C} \cong \mathbb{R}^2$ by choosing $\{1, i\}$ as an $\mathbb{R}$-basis of $\mathbb{C}$, and furthermore we fix $\mathbb{C} / \mathbb{Z}^2$ as a model for the genus 1 surface E (which fixes a complex structure on E, and thus also a structure as a complex projective curve), and choose the point $0 + \mathbb{Z}^2$ as the puncture, that we will often denote by ∞. So, write $E^* := E \setminus \{\infty\}$.

DEFINITION 3.1. a) An *Origami* O of degree d is an unramified covering $O := (p : X^* \to E^*)$ of degree d, where X^* is a (noncompact) topological surface.

b) If $O' = (p' : X'^* \to E^*)$ is another Origami, then we say that O is *equivalent to* O' (which we denote by $O \cong O'$), if the defining coverings are isomorphic, i.e. if there is a homeomorphism $\varphi : X'^* \to X^*$ such that $p' = p \circ \varphi$.

c) $O = (p : X^* \to E^*)$ is called *normal* if p is a normal covering.

Like in the case of dessins, this is not the only possible way to define an Origami. We list several others here:

PROPOSITION 3.2. Giving an Origami in the above sense up to equivalence is equivalent to giving each of the following data:

a) A conjugacy class of a subgroup $G \leq \pi_1(E^*) \cong F_2$ of index d.

b) A pair of permutations $(p_A, p_B) \in S_d^2$, such that $\langle p_A, p_B \rangle \leq S_d$ is a transitive subgroup, up to simultaneous conjugation in S_d.

c) A non-constant holomorphic map $p : X \to E$ of degree d, where X is a compact Riemann surface and p is ramified at most over the set $\{\infty\}$, up to the following equivalence relation:

$$(p : X \to E) \cong (p' : X' \to E) :\Leftrightarrow \exists \text{ biholomorphic } \varphi : X \to X' : \ p = p' \circ \varphi.$$

d) A non-constant morphism $p : X \to E$ of degree d, where X is a nonsingular connected projective curve over $\mathbb{C}$ and p is ramified at most over the set $\{\infty\}$, up to the following equivalence:

$$(p : X \to E) \cong (p' : X' \to E) :\Leftrightarrow \exists \text{ isomorphism } \varphi : X \to X' : \ p = p' \circ \varphi.$$

Proving the above equivalences uses a subset of the arguments we used in the proof of Proposition 2.2. For a more detailed discussion that these equivalences respect equivalence, the reader may equivalently be referred to the proof of [**Kre10a**, Proposition 1.2].

2. Origamis as translation surfaces

We begin with the definition of a translation surface.

DEFINITION 3.3. Let X be a Riemann surface with some atlas $\mathfrak{X}$.

a) A *translation structure* μ on X is an atlas compatible with $\mathfrak{X}$ (as real analytic atlases, i.e. their union is an atlas of a real analytic surface), such that for any two charts $f, g \in \mu$, the transition map is locally a translation, i.e. a map

$$\varphi_{f,g} : U \subseteq \mathbb{C} \to U' \subseteq \mathbb{C}, \, x \mapsto x + t_{f,g}$$

for some $t_{f,g} \in \mathbb{C}$. We call the pair $X_\mu := (X, \mu)$ a translation surface.

b) A biholomorphic map $f : X_\mu \to Y_\nu$ between translation surfaces is called a *translation*, or an *isomorphism of translation surfaces*, if it is locally (i.e. on the level of charts) a translation. X_μ and Y_ν are then called isomorphic (as translation surfaces). If in this case $X = Y$, we say that the translation structures μ and ν are *equivalent*.

c) If μ is a translation structure on X, and $A \in \mathrm{SL}_2(\mathbb{R})$, then we define the translation structure

$$A \cdot \mu := \{A \cdot f \mid f \in \mu\}$$

where A shall act on $\mathbb{C}$ by identifying it with $\mathbb{R}^2$ as usual. Therefore, we get a left action of $\mathrm{SL}_2(\mathbb{R})$ on the set of translation structures on X.

Before we go on, a word of warning is advisable here. Let X be a Riemann surface with a complex atlas $\mathfrak{X}$ like in the definition, and let μ be a translation atlas on X. Because translations in the complex plane are clearly biholomorphic, μ is also a complex atlas. It should be kept in mind that generally, μ and $\mathfrak{X}$ will be *different* complex structures on the real analytic surface X! This will be the crucial point in defining Teichmüller discs later.

Next, let us generalise the notion of translations to define affine diffeomorphisms. We are particularly interested in affine diffeomorphisms from a translation surface to itself. When we speak of charts here, they should be understood to belong to the translation atlas, not necessarily to the complex atlas.

Definition and Remark 3.4. Let X_μ, Y_ν be translation surfaces.

a) An *affine diffeomorphism* $f: X_\mu \to Y_\nu$ is an orientation preserving diffeomorphism such that locally (i.e. when going down into the charts) it is a map of the form

$$x \mapsto A \cdot x + t,\ A \in \mathrm{GL}_2(\mathbb{R}),\ t \in \mathbb{C}\,.$$

We call X_μ and Y_ν *affinely equivalent* if there is such an affine diffeomorphism.

b) The matrix $A =: A_f$ in a) actually is a global datum of f, i.e. it is the same for every chart. We write $\mathrm{der}(f) := A_f$.
c) An affine diffeomorphism f is a translation iff $A_f = I$.
d) If $g : Y_\nu \to Z_\xi$ is another affine diffeomorphism, then $\mathrm{der}(g \circ f) = \mathrm{der}(g) \cdot \mathrm{der}(f)$.
e) We denote the group of all affine orientation preserving diffeomorphisms from X_μ to itself by $\mathrm{Aff}^+(X_\mu)$.
f) The map $\mathrm{der}: \mathrm{Aff}^+(X_\mu) \to \mathrm{GL}_2(\mathbb{R})$, is a group homomorphism.
g) $\mathrm{Trans}(X_\mu) := \ker(\mathrm{der})$ is called the *group of translations* of X_μ.
h) $\Gamma(X_\mu) := \mathrm{im}(\mathrm{der})$ is called the *Veech group* of X_μ. Its image under the projection map $\mathrm{GL}_2(\mathbb{R}) \to \mathrm{PGL}_2(\mathbb{R})$ is called the *projective Veech group* of X_μ. We denote it by $\mathrm{P}\Gamma(X_\mu)$.
i) If $A \in \mathrm{SL}_2(\mathbb{R})$, then we have $A \in \Gamma(X_\mu) \Leftrightarrow X_\mu \cong X_{A\cdot\mu}$ as translation surfaces.

Proof. Part b) is a consequence of the fact that we are using only translation atlases. Parts c) and d) are immediate consequences of the definitions, and f) is a consequence of d). Finally, i) is a consequence of c) and d) after checking the easy but totally counter-intuitive fact that for the identity $\mathrm{id}: X \to X$ on the Riemann surfaces, we have, if we study it as an affine diffeomorphism between X_μ and $X_{A\cdot\mu}$:

$$\mathrm{der}(\mathrm{id}) = A.$$

□

We further note that if X is a Riemann surface of finite volume then, as any $f \in \mathrm{Aff}^+(X_\mu)$ has to preserve the volume, the corresponding matrix $A_f \in \Gamma(X_\mu)$ has to have determinant ± 1. In fact, -1 is not possible since by our definition, affine diffeomorphisms are orientation preserving. So in this case we have $\Gamma(X_\mu) \subseteq \mathrm{SL}_2(\mathbb{R})$.

The Riemann surface $E = \mathbb{C} / \mathbb{Z}^2$ carries a natural translation structure: Indeed, the set of all local sections to the canonical projection $\pi : \mathbb{C} \to \mathbb{C} / \mathbb{Z}^2$ obviously form a translation atlas μ_0, as a transition map between two such sections is locally a translation by an element of $\mathbb{Z}^2$. Let us, as a first example, calculate the Veech group of the translation surface E_{μ_0}: First, note that every affine diffeomorphism of E can be lifted along π to an affine diffeomorphism of $\mathbb{C}$ with its natural translation structure given by the identity map as a translation atlas, in a unique way up to a translation. On the other hand, an affine diffeomorphism (i.e. an affine transformation in the ordinary sense $z \mapsto A \cdot z + t$) of $\mathbb{C}$ descends to E iff it respects the lattice $\mathbb{Z}^2$, i.e. $A \in \mathrm{SL}_2(\mathbb{Z})$. So, we get

$$\Gamma(E_{\mu_0}) = \mathrm{SL}_2(\mathbb{Z}).$$

It is well known that $\mathrm{SL}_2(\mathbb{Z})$ is generated by the following two matrices:

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

If we replace $\mathbb{Z}^2$ by another lattice $\Lambda_A := A \cdot \mathbb{Z}^2$ for some $A \in \mathrm{SL}_2(\mathbb{R})$, then we get for $E_{\Lambda_A} := \mathbb{C} / \Lambda_A$ a translation structure μ_A constructed in the same way as above. By the same arguments, we have for its Veech group: $\Gamma(E_{\Lambda_A}, \mu_A) = A\,\mathrm{SL}_2(\mathbb{Z})A^{-1}$. It is now easily seen that $(E,\, A \cdot \mu_0) \cong (E_{\Lambda_A},\, \mu_A)$ as translation surfaces. So for $B \in \mathrm{SL}_2(\mathbb{R})$, we have $\Gamma(E_{B \cdot \mu_0}) = B\,\mathrm{SL}_2(\mathbb{Z})B^{-1}$. We will state this in greater generality below.

Now consider an Origami $O = (p : X^* \to E^*)$. From our fixed translation structure μ_0 on E (or more precisely its restriction to E^*) we get a translation structure $\nu_0 := p^*\mu_0$ on X^* by defining

$$p^*\mu_0 := \left\{ f \circ p_{|U'} \mid f : U \to V \in \mu_0,\ U \text{ admissible for } p,\ U' \in \pi_0(p^{-1}(U)) \right\}.$$

It can now be shown that every two affine diffeomorphisms of $(X^*,\, \nu_0)$ that are lifts of the same affine diffeomorphism of E_{μ_0} differ only by a translation, and on the other hand any affine diffeomorphism of $(X^*,\, \nu_0)$ is a lift of an affine diffeomorphism of E_{μ_0}.

Let us now first define the Veech group of an Origami, and then make the above statements rigorous.

Definition 3.5. Let $O = (p : X^* \to E^*)$ be an Origami. Then we call

$$\Gamma(O) := \Gamma(X^*,\, p^*\mu_0)$$

the *Veech group* of O.

Proposition 3.6. If $O = (p : X^* \to E^*)$ is an Origami, and $\Gamma := \Gamma(O)$ its Veech group, then we have:

a) $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$.
b) $\Gamma(X^*, B \cdot p^*\mu_0) = B\Gamma B^{-1}$ for every $B \in \mathrm{SL}_2(\mathbb{R})$.

For a more detailed account than our short discussion above, the reader may refer to [**Sch05**, 1.3] which closes the gaps we left.

We call two Origamis $O = (p : X^* \to E^*)$ and $O' = (q : Y^* \to E^*)$ *affinely equivalent* if the corresponding translation surfaces $(X^*, p^*\mu_0)$ and $(Y^*, q^*\mu_0)$ are affinely equivalent. By Proposition 3.6 b), their Veech groups are then conjugate in $\mathrm{SL}_2(\mathbb{Z})$. The converse is not true: Indeed, we will construct an infinite series of Origamis of growing genus, all having Veech group $\mathrm{SL}_2(\mathbb{Z})$, in Example 5.6.

$\mathrm{SL}_2(\mathbb{Z})$ acts on the set of Origamis in the following way: For an Origami $O = (p : X^* \to E^*)$ and $A \in \mathrm{SL}_2(\mathbb{Z})$, we set $A \cdot O := (\varphi_A \circ p : X^* \to E^*)$, where $\varphi_A \colon E^* \to E^*$ is the map induced by $z \mapsto A \cdot z$ on $\mathbb{C}$. It is an instructional calculation to show that $A \cdot (p^*\mu_0) = (\varphi_A \circ p)^*\mu_0$, i.e. this action fits together with the action we defined earlier on translation structures. Let us now formulate the following

PROPOSITION 3.7. Let $O = (p : X^* \to E^*)$ be an Origami, then we have:

a) The isomorphism classes of Origamis that are affinely equivalent to O are in bijection with the left cosets of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$.
b) $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(O)] < \infty$.

PROOF. Let us first prove part a). Let $O' = (q : Y^* \to E^*)$ be an Origami that is affinely equivalent to O. So there has to exist, with respect to the pulled back translation structures, an affine diffeomorphism $\psi : X^* \to Y^*$. Let A be its derivative. By [**Sch05**, Proposition 3.3], ψ descends to an affine diffeomorphism $\varphi : E^* \to E^*$. As $\mathrm{der}(\varphi) = A$ (so in particular $A \in \mathrm{SL}_2(\mathbb{Z})$), and by the requirement that φ shall fix the marked point, we have that $\varphi = \varphi_A$. As $\psi : X^* \to Y^*$ is in particular a homeomorphism, $\varphi \circ p : X^* \to E^*$ is an Origami that coincides with $A \cdot O$, and by construction $\varphi_A \circ p = q \circ \psi$, we have shown that $O' \cong A \cdot O$.

Since O' was an arbitrary Origami from the affine equivalence class of O, we have shown that $\mathrm{SL}_2(\mathbb{Z})$ acts transitively from the left on this equivalence class. The stabiliser of O is precisely $\Gamma(O)$. We conclude by using [**Bos01**, Bemerkung 5].

To prove part b), we need, by part a), to give an argument why there are only finitely many Origamis that are affinely equivalent to O. We are going to use the argument stated above that an affine diffeomorphism has to respect the volume. On $\mathbb{C}$, we have the standard volume form $dx \wedge dy$, induced by the standard Lebesgue measure. So, for the induced volume form on $E = \mathbb{C} / \mathbb{Z}^2$, we get (for $I^2 := [0, 1] \times [0, 1]$)

$$\mathrm{vol}(E) = \int_{I^2} 1 \cdot dx \wedge dy = 1.$$

Removing a single point does not change the volume, and since X^* naturally carries the volume form pulled back from E^*, we find that p is locally measure preserving*, in the sense that for a measurable neighbourhood U of some point $x \in X^*$ that is small enough we have $\mathrm{vol}(U) = \mathrm{vol}(p(U))$. So, if O is an Origami of degree d, we have $\mathrm{vol}(X^*) = d$. If now $O' = (q : Y^* \to E^*)$ is an Origami that is affinely equivalent to O, then $\mathrm{vol}(Y^*) = d$, and by reversing the argument, O' is also an Origami of degree d. Using the same argument as in the proof of Corollary

*This may seem a bit arbitrary. A more precise way of arguing is that the volume form on E^* comes from the holomorphic differential dz that E^* carries naturally, and holomorphic differentials naturally correspond to translation structures. In that sense, the differential corresponding to $p^*\mu_0$ on X^* is $\omega := p^*dz$, and this again induces a natural volume form on X^*.

2.5, we get $(d!)^2$ as an upper bound for the number of Origamis of degree d. This closes the proof. □

Part b) also appears as Corollary 3.6 in [**Sch05**], where Schmithüsen uses different methods from the ones used here. Yet another proof of this fact with a more general scope can be found in [**GJ00**, Theorem 4.9]. Theorem 5.5 in the same article even provides a converse statement, in some sense: Given any translation surface such that its Veech group is a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, it admits a covering to some elliptic curve respecting the translation structure, i.e. it is an Origami.

3. Moduli and Teichmüller spaces of curves

We begin by giving a somewhat rough definition of different versions of the (coarse) moduli space of compact Riemann surfaces. A very detailed reference on this subject is provided in [**HM98**].

Definition 3.8. a) Define the *coarse moduli space of Riemann surfaces of genus g with n distinguished marked points* as

$$M_{g,n} := \left\{(X, p_1, \dots, p_n) \mid X \text{ compact R. s. of genus } g,\ p_i \in X,\ p_i \neq p_j \text{ for } i \neq j\right\} /_\sim$$

where $(X, p_1, \dots, p_n) \sim (Y, q_1, \dots, q_n)$ if there is a biholomorphic map $\varphi :\ X \to Y$ with $\varphi(p_i) = q_i$, $i = 1, \dots, n$.

b) Define the *coarse moduli space of Riemann surfaces of genus g with n non-distinguished marked points* as

$$M_{g,[n]} := \left\{(X, p_1, \dots, p_n) \mid X \text{ compact R. s. of genus } g,\ p_i \in X,\ p_i \neq p_j \text{ for } i \neq j\right\} /_\sim$$

where $(X, p_1, \dots, p_n) \sim (Y, q_1, \dots, q_n)$ if there is a biholomorphic function $\varphi :\ X \to Y$ and a permutation $\pi \in S_n$, such that $\varphi(p_i) = q_{\pi(i)}$, $i = 1, \dots, n$.

c) Finally, define the *coarse moduli space of Riemann surfaces of genus g* as

$$M_g := M_{g,0} = M_{g,[0]}.$$

It is far from being obvious that $M_{g,n}$ and $M_{g,[n]}$, which we defined just as sets, can be turned into complex quasi-projective varieties, or complex analytic spaces, of dimension $3g - 3 + n$ (whenever this expression is positive—we have $\dim(M_{1,0}) = 1$, and $\dim(M_{0,n}) = 0$ for $n \leq 3$). There are natural projections

$$M_{g,n} \to M_{g,[n]} \to M_g$$

by forgetting the order of the marked points, and totally forgetting the marked points. Note that this widely generalises if we replace Riemann surfaces by algebraic curves. Indeed, there is a scheme $M^{\mathbb{Z}}_{g,n}$ such that we get back our moduli space as $M_{g,n} = M^{\mathbb{Z}}_{g,n} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{C}$, and for a suitable notion of projective curves with marked points, this even works if we replace $\operatorname{Spec}\mathbb{C}$ by any other base scheme S.

These spaces have a very rich and hard to understand geometry. In particular, they have complicated singular loci, typically coming from Riemann surfaces having "non-generic" nontrivial automorphisms (i.e. not the identity, or the (hyper-)elliptic involution on a surface of genus 1, 2 or 3) respecting the marked points. So in order to get simpler versions of classifying spaces for Riemann surfaces, it seems natural to introduce a kind of marking on Riemann surfaces that is surely not fixed by an automorphism. This leads to the definition of Teichmüller spaces.

Definition and Remark 3.9. Let S be a fixed compact Riemann surface of genus g with n marked points. (Let us write shortly that S is of type (g,n).)

a) If X is another surface of this type, a *marking* on X is an orientation preserving diffeomorphism $\varphi : S \to X$ which respects the marked points.
b) We define the *Teichmüller space* of the surface S as

$$\mathcal{T}(S) := \{(X, \varphi) \mid X \text{ R.s. of type } (g,n),\ \varphi : S \to X \text{ a marking}\}/_\sim$$

where $(X, \varphi) \sim (Y, \psi)$ if $\psi \circ \varphi^{-1} : X \to Y$ is homotopic to a biholomorphism respecting the marked points (where, of course, the homotopy shall fix the marked points).
c) If S' is another surface of type (g,n), then any choice of a marking on $\varphi : S \to S'$ yields a bijection $\mathcal{T}(S') \to \mathcal{T}(S)$ by precomposing all markings with φ, which gives us the right to just write $\mathcal{T}_{g,n}$.

Of course, in the same manner as above, we can also define versions $\mathcal{T}_{g,[n]}$ and $\mathcal{T}_g$ of Teichmüller spaces with unordered or no marked points. Again, they are complex analytic spaces of dimension $3g-3+n$ (again only if this expression is a positive number) in a natural way, and they are even nonsingular, i.e. they are actually complex manifolds. (As a reference, see Theorem 6.5.1 in **[Hub06]**.) On the other hand, the downside of this construction is that Teichmüller spaces are not algebraic varieties.

It appears to be natural to consider the projections $\mathcal{T}_{g,n} \to M_{g,n}$ that are given by forgetting the marking. Of course this is also possible for the two other versions of moduli and Teichmüller space. Let us exhibit this projection in another way: By Definition and Remark 3.9 we have a left action of $\mathrm{Diffeo}^+(S)$, the group of orientation preserving diffeomorphisms of S fixing each of the marked points, on $\mathcal{T}(S)$. By the definition of equivalence of marked surfaces, this action factors through quotienting by $\mathrm{Diffeo}^0(S)$, the group of diffeomorphisms of S homotopic to the identity. Clearly, this action is transitive on the possible markings for a fixed Riemann surface X, but it cannot change the isomorphism class of its complex structure. Indeed, we have the following

Proposition 3.10. The group $\Sigma(S) = \mathrm{Diffeo}^+(S)/\mathrm{Diffeo}^0(S)$ acts properly discontinuously on $\mathcal{T}(S)$, and the quotient space by this action is the moduli space $M_{g,n}$.

Before sketching the proof, let us make a note on the group $\Sigma(S)$. If we topologise $\mathrm{Diffeo}^+(S)$ with the compact open topology, we find $\Sigma(S) \cong \pi_0(\mathrm{Diffeo}^+(S))$ since S is locally compact and Hausdorff. Of course for another surface of type (g,n), we get an isomorphic group, so we can up to isomorphism study the abstract group $\Sigma_{g,n}$, called the *mapping class group* in genus g with n punctures. It has been subject to a great extent of research, a classical reference being **[Bir75]**, a more recent one **[FM11]**. Of course, it is possible to define other versions $\Sigma_{g,[n]}$ and Σ_g of the mapping class groups analogously. For them, we get analogous versions of the proposition.

Let us now make some remarks on the proof of the above proposition. The first part is Theorem 6.18 in **[IT92]**. However, mapping class groups are defined there a bit different from our version, so we should maybe also cite **[FM11**, Theorem 12.2]. One interesting aspect is why giving $\mathcal{T}_{g,n}$ a structure as a complex manifold endows $M_{g,n}$ with a structure as a complex analytic space. With the first part of the proposition, this amounts to checking that the stabilisers of points are finite. Indeed, if $\alpha \in \Sigma(S)$ stabilises $(X, \varphi) \in \mathcal{T}(S)$, then $\varphi\alpha\varphi^{-1}$ has to be homotopic to some biholomorphism of X. But due to Hurwitz's classical result **[Hur92]**, a compact Riemann surface of genus g can only have $84(g-1)$ automorphisms. Prescribing some marked points to be respected can only kill automorphisms.

Let us finish this section by making a remark on punctured Riemann surfaces, i.e. surfaces $X^* = X \setminus R$, where X is a compact Riemann surface and $R \subseteq X$ is a finite subset. These appeared naturally when defining dessins d'enfants and Origamis. We have seen earlier that such punctured surfaces can be uniquely completed to a compact Riemann surface, which is then of course isomorphic to X. Furthermore, clearly a diffeomorphism $f \in \mathrm{Diffeo}^+(X)$ restricts to an element of $\mathrm{Diffeo}^+(X^*)$ iff $f(R) = R$. So, it is natural to identify the moduli and Teichmüller spaces of punctured Riemann surfaces of genus g with n punctures with $M_{g,[n]}$ and $\mathcal{T}_{g,[n]}$, respectively.

4. Teichmüller discs and Teichmüller curves

We are now able to define the Teichmüller disc arising from a translation surface. Let (X, μ) be a translation surface of type $(g, [n])$, which by the considerations at the end of the last section we can imagine as a compact Riemann surface of genus g minus a finite set of points, endowed with a translation structure. For $B \in \mathrm{SL}_2(\mathbb{R})$, denote by X_B the Riemann surface that we get by endowing X with the complex structure induced by $B \cdot \mu$. Then the (set-theoretic) identity map $\mathrm{id} : X = X_I \to X_B$ is a marking in the sense of Definition and Remark 3.9 a). Note that this map is in general *not* holomorphic! So we get a map

$$\theta : \mathrm{SL}_2(\mathbb{R}) \to \mathcal{T}_{g,[n]},\ B \mapsto [(X_B, \mathrm{id} : X_I \to X_B)]\,.$$

Since for $B \in \mathrm{SL}_2(\mathbb{R})$ we have $z \mapsto B \cdot z$ is homotopic to a biholomorphism iff $B \in \mathrm{SO}(2)$ we find in particular that $\theta(A) = [(X_I, \mathrm{id}_{X_I})]$ iff $A \in \mathrm{SO}(2)$, i.e. θ factors through $\mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R}) \cong \mathbb{H}$. (For reasons we will explain below, we want to fix this well known bijection in the following way: Define $m : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{H}$, $A \mapsto \overline{-A^{-1}(i)}$, where $z \mapsto A(z)$ shall denote the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ by Möbius transformations. As the stabiliser of i is $\mathrm{SO}(2)$, we get the desired bijection by quotienting.) The factor map

$$\overline{\theta} : \mathbb{H} \to \mathcal{T}_{g,[n]}$$

is injective. It is in fact biholomorphic, and furthermore an isometry with respect to the standard hyperbolic metric on $\mathbb{H}$ and the Teichmüller metric on $\mathcal{T}_{g,[n]}$ as defined, for example, in **[Hub06**, 6.4]. See **[Nag88**, 2.6.5 and 2.6.6] for details. This leads to the following

Definition 3.11. Let $X_\mu := (X, \mu)$ be a translation surface of type $(g, [n])$. Then, the isometric image

$$\Delta_{X_\mu} := \overline{\theta}(\mathbb{H}) \subseteq \mathcal{T}_{g,[n]}$$

is called the *Teichmüller disc* associated with X_μ.

Using the Poincaré disc model for $\mathbb{H}$ justifies this name, as $\Delta_{X_\mu} \subseteq \mathcal{T}_{g,[n]}$ is actually isometric to an open unit disc, carrying the standard hyperbolic metric.

Now we are interested in the image of a Teichmüller disc Δ_{X_μ} under the projection map into moduli space. In general, it will not be an algebraic subvariety, but if the Veech group of the translation surface X_μ is a lattice in $\mathrm{SL}_2(\mathbb{R})$, i.e. if

$\mathrm{vol}(\mathbb{H}/\Gamma(X_\mu)) < \infty$, then in fact the image of Δ_{X_μ} in the moduli space is an algebraic curve. Indeed, we have the following

THEOREM H. Let X_μ be a translation surface of type $(g,[n])$, and Δ_{X_μ} its Teichmüller disc. Furthermore let $p : \mathcal{T}_{g,[n]} \to M_{g,[n]}$ be the projection. Then we have:

a) $p(\Delta_{X_\mu}) \subseteq M_{g,[n]}$ is an algebraic curve iff $\Gamma(X_\mu)$ is a lattice. It is then called the *Teichmüller curve* associated to X_μ.
b) In this case, the following diagram is commutative if we define $R := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$:

$$\begin{tikzcd}
\mathbb{H} \arrow[r, hook, "\overline{\theta}"] \arrow[d, "/R\Gamma(X_\mu)R^{-1}"'] & \Delta_{X_\mu} \arrow[r, hook] \arrow[d, "p_{|\Delta_{X_\mu}}"] & \mathcal{T}_{g,[n]} \arrow[d, "p"] \\
\mathbb{H}/R\Gamma(X_\mu)R^{-1} \arrow[r, "j"] & p(\Delta_{X_\mu}) \arrow[r, hook, "\iota"] & M_{g,[n]}
\end{tikzcd}$$

Furthermore, the map j is the normalisation map for the algebraic curve $p(\Delta_{X_\mu})$.

The statements of the above theorem have undergone quite some evolution until appearing in the form stated here. As a starting point, the famous paper [**Vee89**] should be named, where Veech proved that for the class of lattice surfaces, i.e. translation surfaces whose Veech group is a lattice, the so-called *Veech dichotomy* holds. At least part a) is often attributed to Smillie, who never seems to have published it in this completeness. Maybe the first published version is due to McMullen in [**McM03**]. A reader looking for a complete proof of the theorem as it is stated above may refer to [**HS07b**, 2.4], or [**Loc05**, Proposition 3.2].

Adapting the above theorem to our situation of Origamis, we write down the following simple

COROLLARY 3.12. Let $O = (f : X^* \to E^*)$ be an Origami, where X^* is of type $(g,[n])$. Then,

$$C(O) := p(\Delta_{X_{f^*\mu_0}}) \subseteq M_{g,[n]}$$

is an algebraic curve which we call the *Origami curve* defined by O.

PROOF. Let $k := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(O)]$. We have $k < \infty$ by Proposition 3.7. Take the standard fundamental domain $\Delta := \{z \in \mathbb{H} \mid -\frac{1}{2} \leq \mathrm{Im}\, z \leq \frac{1}{2}, |z| \geq 1\}$ of the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, then we have by Gauss-Bonnet

$$\mathrm{vol}(\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})) = \mathrm{vol}(\Delta) = \frac{\pi}{3},$$

and so, because the action of $\mathrm{SL}_2(\mathbb{Z})$ factors through $\mathrm{PSL}_2(\mathbb{Z})$, we have

$$\mathrm{vol}(\mathbb{H}/\Gamma(O)) = kl\frac{\pi}{3} < \infty,$$

where $l \in \{\frac{1}{2}, 1\}$, depending on whether or not $-I \in \Gamma(O)$. So $\Gamma(O)$ is a lattice and we can apply Theorem H a). $\square$

Before we go on, let us make a remark on Origami curves here. One might ask if the curve $C(O)$ uniquely characterises the Origami O. This is in general not true. Of course, in some sense, every point on $C(O)$ corresponds to an Origami, because its preimages on the Teichmüller disc define translation coverings of some elliptic curves. But in our stricter sense, i.e. an Origami covering has to cover the fixed elliptic curve $E = \mathbb{C} / \mathbb{Z}^2$, we have in general still more than one Origami on $C(O)$. As an answer to the above question, let us state the following

PROPOSITION 3.13. Let O, O' be Origamis. Then we have $C(O) = C(O')$ iff O and O' are affinely equivalent.

A proof can be found in [**HS07a**, Proposition 5 b)].

5. Cylinder decomposition

Let us give a short summary on Strebel directions and cylinders on a translation surface X_μ here. We will not give any proofs here, for details see for example Sections 3.2 and 3.3 in [**Kre10a**] and Section 4 in [**HS07b**].

Let X_μ be a translation surface of type (g, n), and $(f : U \subseteq X \to V \subseteq \mathbb{R}^2) \in \mu$ a map contained in its translation atlas. Then the standard metric on $\mathbb{R}^2$ (or rather its restriction to V) induces a metric on U. Since the transition maps of μ are all translations, under which the standard metric on $\mathbb{R}^2$ is invariant, these local metrics glue to a global one on X_μ, called the *flat metric* associated to μ. Due to its construction, a path $\gamma : (0,1) \to X_\mu$ is geodesic with respect to the flat metric iff it is locally of the form $t \mapsto t \cdot v + w$, where $0 \neq v, w \in \mathbb{R}^2$. Of course v is a global datum of γ. We call it (or more precisely its equivalence class in $\mathbb{P}^1(\mathbb{R})$) the *direction* of γ. We call a geodesic path *maximal* if its image is not properly contained in the image of another geodesic path.

DEFINITION 3.14. Let X_μ be a translation surface of type (g, n).

a) A direction $v \in \mathbb{P}^1(\mathbb{R})$ is called *Strebel* if every maximal geodesic path on X_μ with direction v is either closed, or a *saddle connection* (i.e. it connects two punctures of X_μ).
b) We call two Strebel directions $v, v' \in \mathbb{P}^1(\mathbb{R})$ *equivalent* if there is an $A \in \mathrm{P}\Gamma(X_\mu)$ such that $A \cdot v = v'$.

Now, if v is a Strebel direction for X_μ, a *cylinder* in X_μ is the image of a homeomorphism $c : (0, 1) \times S^1 \to U \subseteq X_\mu$, where U is an open subset of X_μ, with the condition that for every $s \in (0, 1)$, the restriction to $\{s\} \times S^1$ is a closed geodesic†. A cylinder is called *maximal* if it is not properly contained in another cylinder. We have the following fact:

REMARK 3.15. With the exception of the case $(g, n) = (1, 0)$, the maximal cylinders of X_μ in the Strebel direction v are the connected components of $X_\mu \setminus S$, where S is the union of the images of all saddle connections in direction v.

Let us restrict to Origamis now and summarise the situation in this case:

PROPOSITION 3.16. Let $O = (p : X^* \to E^*)$ be an Origami. Then we have:

a) There is a bijection between the following sets:
   - Equivalence classes of Strebel directions of O,
   - Conjugacy classes of maximal parabolic subgroups in $\mathrm{P}\Gamma(O)$,
   - Punctures (called *cusps*) of the normalisation of the Origami curve, $\mathbb{H} / R\Gamma(O)R^{-1}$.

†Note that this is a slightly different notion of a closed geodesic than above. A possibility to stay closer to that definition is to replace c by an infinite cyclic covering $\tilde{c} : (0, 1) \times (0, 1) \to U$.

b) The vector $\binom{1}{0}$ is a Strebel direction of O, called its *horizontal* Strebel direction.
c) Any maximal parabolic subgroup of $\mathrm{P}\Gamma(O)$ is generated by the equivalence class of a matrix of the form gT^wg^{-1}, for some $w \in \mathbb{N}$, $g \in \mathrm{SL}_2(\mathbb{Z})$.
d) The Strebel direction corresponding to a maximal parabolic subgroup $\langle \overline{gT^wg^{-1}} \rangle$ is $v_g := g \cdot \binom{1}{0}$. The maximal cylinders of O in this Strebel direction are the maximal horizontal cylinders of the Origami $g^{-1} \cdot O$.

Part a) of this proposition is discussed in Section 3.2 of **[Kre10a]**, parts b) and d) in Section 3.3. Specifically, part d) appears as Proposition 3.7 there. Part c) is just a consequence from the fact that any parabolic element of $\mathrm{SL}_2(\mathbb{Z})$ is conjugate to a power of $\pm T$.

6. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ on Origami curves

As the way we constructed Teichmüller curves is clearly of analytical nature, it may be surprising that they have interesting arithmetic properties. This kind of connection reminds of the Theorem of Belyi, which we can indeed use to prove a small part of the following

Proposition 3.17. Let $O = (p : X^* \to E^*)$ be an Origami and $C(O) \subseteq M_{g,[n]}$ its Teichmüller curve.

a) Then, the normalisation map $j : \mathbb{H}\,/R^{-1}\Gamma(O)R \to C(O)$ and the inclusion $\iota : C(O) \hookrightarrow M_{g,[n]}$ are defined over number fields.
b) Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ be a Galois automorphism, and $O^\sigma = (p^\sigma : (X^*)^\sigma \to E^*)$ the Galois conjugate Origami[‡]. Then we have[§]

$$\iota^\sigma((C(O))^\sigma) = \iota(C(O^\sigma)) \subseteq M_{g,[n]},$$

so in particular $C(O^\sigma) \cong (C(O))^\sigma$.

This result is proven by Möller in [**Möl05**, Proposition 3.2]. Let us first sketch the proof of part a): The main ingredient to the proof is the fact that $\mathcal{H}_{\mathcal{E}}$, the *Hurwitz stack* of coverings of elliptic curves ramified over one prescribed point of some prescribed genus and degree, is a smooth stack defined over $\mathbb{Q}$. This is a result of Wewers that can be found in **[Wew98]**. Möller identifies (an orbifold version of) $C(O)$ as a geometric connected component of $\mathcal{H}_{\mathcal{E}}$. The $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$-orbit of $C(O)$ consists precisely of the geometric connected components of the $\mathbb{Q}$-connected component containing $C(O)$. The definability of $C(O)$ over a finite extension of $\mathbb{Q}$ then follows from the fact that $\mathcal{H}_{\mathcal{E}}$ has only finitely many geometric connected components. Showing this amounts to showing that the number or Origamis of given degree is finite, which we did in the proof of Proposition 3.7.

As Möller's proof of part b) is quite short, we give a more detailed proof here. It is natural to use the language of algebraic stacks. We use the notation of **[Wew98]**, which is also used in [**Möl05**].

[‡]Note here that we fixed the choice of $E = \mathbb{C}\,/\,\mathbb{Z}^2$. We have $j(E) = 1728 \in \mathbb{Q}$, and thus E is defined over $\mathbb{Q}$.

[§]Strictly speaking, we should use a notation like $\iota_{C(O)}$ to distinguish the embeddings of different Origami curves. We suppress the index for reasons of simplicity and bear in mind that the following formula has two different morphisms called ι.

Proof. Let $\mathcal{E} \to \mathcal{M}_{1,1}$ be the universal family of elliptic curves, where we denote by $\mathcal{M}_{1,1} := \mathcal{M}_{1,1}(\overline{\mathbb{Q}})$ the *fine moduli stack* of elliptic curves with one puncture, over $\overline{\mathbb{Q}}$. We fix $d := \deg(p)$ and $g := g(X^*)$, and a divisor $\mathcal{D}$ on $\mathcal{E}$ encoding the ramification over each fibre. By [**Wew98**, Theorem 4.1.2] this data defines a smooth stack $\mathcal{H}_{\mathcal{E}}$ which is defined over $\mathbb{Q}$, the Hurwitz stack of coverings of elliptic curves of degree d and genus g, branched over $\mathcal{D}$. As $\mathcal{H}_{\mathcal{E}}$ is a fine moduli space for the Hurwitz problem given by the ramification data we also get a universal family $\mathcal{F} \to \mathcal{H}_{\mathcal{E}}$ such that there is a (2-)commutative diagram of the following form:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\mathfrak{p}"] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{H}_{\mathcal{E}} \arrow[r, "\mathfrak{q}"] & \mathcal{M}_{1,1}
\end{tikzcd}$$

This should be understood as follows: The closed points of $\mathcal{H}_{\mathcal{E}}$ parametrise the coverings $q : Y \to F$ of the prescribed ramification data (i.e. $g(Y) = g$, $g(F) = 1$, $\deg(q) = d$, and q is ramified precisely over the intersection of $\mathcal{D}$ and F, the latter as a fibre in $\mathcal{E}$). Given a closed point $P \in \mathcal{H}_{\mathcal{E}}$, we get the covering it parametrises by specialising to the fibre over $\{P\}$, i.e. it coincides with

$$\mathfrak{p} \times \mathrm{id}_{\{P\}} : \mathcal{F} \times_{\mathcal{H}_{\mathcal{E}}} \{P\} \to \mathcal{E} \times_{\mathcal{M}_{1,1}} \{\mathfrak{q}(P)\}.$$

As stated above, the (orbifold) Origami curve $\mathcal{C}(O)$ is a geometric connected component of $\mathcal{H}_{\mathcal{E}}$. Denote the corresponding part of the universal family by $\mathcal{F}_O$. Now we want to recover the Origami covering $p : X \to E$ (with the punctures filled in) from this Hurwitz stack. As explained above, we get it by taking the closed point $P \in \mathcal{H}_{\mathcal{E}}$ parametrizing p and its image in $\mathcal{M}_{1,1}$, and pulling the vertical arrows back to it by the inclusion map. Let us draw a diagram of the situation, where we also add one structure morphism $\varphi : \mathcal{M}_{1,1} \to \mathrm{Spec}(\overline{\mathbb{Q}})$. The requirement of all the arrows to be $\overline{\mathbb{Q}}$-morphisms defines the structure morphisms of the other stacks. Here, the left of the two inner squares and the outer, distorted one are Cartesian.

$$\begin{tikzcd}
E \arrow[rrr, hook, bend left] \arrow[dr] & X \arrow[l, "p"'] \arrow[r] \arrow[d] & \mathcal{F}_O \arrow[r, "\mathfrak{p}_O"'] \arrow[d] & \mathcal{E} \arrow[d] \\
& \{P\} \arrow[r, hook] \arrow[rr, hook, bend right] & \mathcal{C}(O) \arrow[r, "\mathfrak{q}_O"] & \mathcal{M}_{1,1} \arrow[d, "\varphi"] \\
& & & \mathrm{Spec}(\overline{\mathbb{Q}})
\end{tikzcd}$$

Now we change the structure morphism by $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$. As abstract stacks and morphisms, all the objects stay the same, but as $\overline{\mathbb{Q}}$-stacks (resp. -morphisms) they are changed, which we denote by decorating them with a σ. Let us redraw the diagram. Note that E and $\mathcal{M}_{1,1}$ are defined over $\mathbb{Q}$, so σ leaves them unchanged.

$$\begin{tikzcd}
E \arrow[drr] & X^\sigma \arrow[l, "p^\sigma"'] \arrow[r] \arrow[d] & \mathcal{F}_O^\sigma \arrow[r, "\mathfrak{p}_O^\sigma"'] \arrow[d] & \mathcal{E}^\sigma \arrow[d] \arrow[lll, bend right, leftarrow] \\
 & \{P\}^\sigma \arrow[r, hook] \arrow[rr, hook, bend right] & (\mathcal{C}(O))^\sigma \arrow[r, "\mathfrak{q}_O^\sigma"] & \mathcal{M}_{1,1} \arrow[d, "\mathrm{Spec}(\sigma) \circ \varphi"] \\
 & & & \mathrm{Spec}(\overline{\mathbb{Q}})
\end{tikzcd}$$

Here, the curve $(\mathcal{C}(O))^\sigma$ appears. We have to show that it is equal to the Origami curve $\mathcal{C}(O^\sigma)$. The former is of course also a geometric connected component of ${\mathcal{H}_\mathcal{E}}^\sigma = \mathcal{H}_\mathcal{E}$, so it is also an (orbifold) Origami curve (see [**Möl05**, Corollary 3.3]). So proving the equality amounts to checking that O^σ lies on the curve. This is easy: O^σ is defined by $p^\sigma : X^\sigma \to E$, and as the last diagram shows, this covering lies over the closed point $P^\sigma \in (\mathcal{C}(O))^\sigma$.

□

What is missing now is an argument that the "naïve", non-orbifold Origami curves $C(O^\sigma)$ and $(C(O))^\sigma$ coincide. To see that, we use that $\pi(\mathcal{C}(O)) = C(O)$, where $\pi : \mathcal{M}_{g,n} \to M_{g,n}$ is the canonical projection map from the fine moduli stack to the coarse moduli scheme.

If we wanted to stay outside the sinister world of stacks, we could also restate the above proof in a different setting: If we work over the punctured moduli space

$$M_{1,1}^* := M_{1,1} \setminus \{0, 1728\},$$

where we use the standard parametrisation of $M_{1,1}$ by the j-invariant (and thus remove exactly the two points corresponding to the isomorphism classes of elliptic curves possessing extra automorphisms), we have the existence of a universal family over $M_{1,1}^*$ and the corresponding punctured coarse version of the Hurwitz space (i.e. these schemes are fine moduli spaces for the corresponding "punctured" moduli problems). In this version of the proof, we obtain the equality of the punctured Origami curves $C^*(O^\sigma)$ and $(C^*(O))^\sigma$. So $C(O^\sigma)$ and $(C(O))^\sigma$ coincide on an open, dense subset and are therefore equal, as they are closed in $M_{g,[n]}$ and furthermore they, and $M_{g,[n]}$, are actually quasi-projective varieties over $\overline{\mathbb{Q}}$ and therefore very well-behaved (i.e. in particular separated over $\overline{\mathbb{Q}}$ and reduced). The downside of this construction is that the elliptic curve of our choice, $E = \mathbb{C} / \mathbb{Z}^2$, has j-invariant 1728, so in order to make the arguments work, we should restate our definition of an Origami, exchanging E by another elliptic curve E' with $j(E') \in \mathbb{Q} \setminus \{0, 1728\}$.[¶]

It is worth noting that a weaker version of part a) is accessible by using Belyi's theorem:

REMARK 3.18. Let $O = (p : X^* \to E^*)$ be an Origami and $C(O) \subseteq M_{g,[n]}$ its Teichmüller curve. Then its normalisation $\mathbb{H} / R^{-1}\Gamma(O)R$ is defined over a number field.

[¶]It seems like a particularly unlikely mishap has happened to us here, picking one of two bad curves from countably infinitely many choices. This seems to be quite a common phenomenon. For a recent account of this classical topic, see [**Spa03**].

Proof. The finite index inclusion $R^{-1}\Gamma(O)R \subseteq R^{-1}\operatorname{SL}_2(\mathbb{Z})R$ gives rise to a morphism of finite degree of Riemann surfaces:

$$\pi:\ C^* := \mathbb{H}\,/R^{-1}\Gamma(O)R \to R^{-1}\operatorname{SL}_2(\mathbb{Z})R \cong \mathbb{C}\,.$$

if we metrically complete C^* to C, we get a morphism $\pi : C \to \mathbb{P}^1_{\mathbb{C}}$ that is ramified over at most three points, as $\mathbb{H}\,/\operatorname{SL}_2(\mathbb{Z})$ has one cusp and two orbifold points. Thus, π is a Belyi morphism and therefore by Theorem E defined over a number field. $\square$

We end this train of thoughts by restating Proposition 3.17 in a different way:

Corollary 3.19. Let $\mathfrak{O}$ be a set containing one Origami of each isomorphism type. Then there is a natural right action of $\operatorname{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ on the set

$$C(\mathfrak{O}) := \{C(O) \mid O \in \mathfrak{O}\},$$

where $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ sends $C(O)$ to $C(O^\sigma)$.

7. Galois invariants and moduli fields

Let us now return to the notion of moduli fields, which we defined in Chapter 2. We begin by defining the moduli field of an Origami, and of an Origami curve:

Definition 3.20. Let $O = (p:\ X^* \to E^*)$ be an Origami, and $C(O)$ its Origami curve.

a) Consider the following subgroup of $\operatorname{Aut}(\mathbb{C})$:

$$U(O) := \{\sigma \in \operatorname{Aut}(\mathbb{C}) \mid \exists\, \mathbb{C}\text{-isomorphism } \varphi : X^\sigma \to X:\ p^\sigma = p \circ \varphi\}$$

Then, $M(O) := \mathbb{C}^{U(O)}$ is called the *moduli field* of O.

b) Remember that $M_{g,[n]}$ is defined over $\mathbb{Q}$ and define

$$U(C(O)) := \{\sigma \in \operatorname{Aut}(\mathbb{C}) \mid\ C(O) = (C(O))^\sigma\}\,,$$

where as usual we consider $C(O)$ and $(C(O))^\sigma$ as subsets of $M_{g,[n]}$. Then, we call $M(C(O)) := \mathbb{C}^{U(C(O))}$ the *moduli field* of the Origami curve $C(O)$.

Let us first note some easy to prove properties of these moduli fields:

Remark 3.21. Let, again, $O = (p:\ X^* \to E^*)$ be an Origami, and $C(O)$ its Origami curve. Then we have:

a) $M(C(O)) \subseteq M(O)$.
b) $[M(O) : \mathbb{Q}] = |O \cdot \operatorname{Aut}(\mathbb{C})| < \infty$ and $[M(C(O)) : \mathbb{Q}] = |C(O) \cdot \operatorname{Aut}(\mathbb{C})| < \infty$.

Proof. Part a) is a consequence of Corollary 3.19: If $\sigma \in \operatorname{Aut}(\mathbb{C})$ fixes O, it particularly fixes $C(O)$.

For part b) we begin by noting that we have $|O \cdot \operatorname{Aut}(\mathbb{C})| < \infty$ because the degree of O is an invariant under the action of $\operatorname{Aut}(\mathbb{C})$, and, as we have seen before, we can bound the number of Origamis of degree d by $(d!)^2$. Furthermore, we have $[\operatorname{Aut}(\mathbb{C}) : U(O)] = |O \cdot \operatorname{Aut}(\mathbb{C})|$, as $U(O)$ is the stabiliser of O under the action of $\operatorname{Aut}(\mathbb{C})$. From **[Köc04**, Lemma 1.6**]** follows the equality $[M(O) : \mathbb{Q}] = [\operatorname{Aut}(\mathbb{C}) : U(O)]$, given that we can show that $U(O)$ is a closed subgroup of $\operatorname{Aut}(\mathbb{C})$. Remember that a subgroup $G \leq \operatorname{Aut}(\mathbb{C})$ is closed iff there is a subfield $F \subseteq \mathbb{C}$ with $G = \operatorname{Aut}(\mathbb{C}\,/F)$. Lemma 1.5 in the same article tells us that $U(O)$ is closed if there is a finite extension $D/M(O)$ such that $\operatorname{Aut}(\mathbb{C}\,/D) \leq U(O)$. Let us now give a reason for the existence of such an extension D: As E and the branch locus $\{\infty\}$ are defined over $\mathbb{Q}$, it follows from [**GD06**, Theorem 4.1] that $p : X \to E$ can be defined over a number field. Choose such a field of definition D, which is hence a finite extension of $M(O)$. Obviously, any element $\sigma \in \mathrm{Aut}(\mathbb{C})$ that fixes D lies in $U(O)$, so we can apply Köck's Lemma 1.5 and finally deduce the first half of b).

Now we restate these arguments for the second equality: We use a) to deduce $[M(C(O)) : \mathbb{Q}] < \infty$. Furthermore, from Proposition 3.17 a) follows that the embedded Origami curve $C(O)$ can be defined over a number field, so we can use the same chain of arguments as above. $\square$

In [**Kre10a**, Conjecture 5.8], Kremer conjectures that all of the (affine) invariants of Origamis he lists in Chapter 3 of his thesis are also invariants under the action of $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$. We will now give a partial answer here:

THEOREM 1. The following properties of an Origami $O = (p :\ X^* \to E^*)$ are Galois invariants:

a) The index of the Veech group $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(O)]$.
b) The index of the projective Veech group $[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{P}\Gamma(O)]$.
c) The property whether or not $-I \in \Gamma(O)$.
d) The isomorphism type of the group of translations, $\mathrm{Trans}(O)$.

PROOF. We have $\varepsilon \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(O)] = [\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{P}\Gamma(O)]$ with $\varepsilon = \frac{1}{2}$ if $-I \notin \Gamma(O)$, and else $\varepsilon = 1$. So a) is a consequence of b) and c).

For part b), note that the morphism $\mathfrak{q}_O$ from the proof of Proposition 3.17 fits into a commutative diagram

$$\begin{tikzcd} \mathbb{H} / \overline{R}\, \mathrm{P}\Gamma(O)\, \overline{R}^{-1} \arrow[r] \arrow[d, "/\overline{R}\ \mathrm{PSL}_2(\mathbb{Z})\, \overline{R}^{-1}"'] & \mathcal{C}(O) \arrow[d, "\mathfrak{q}_O"] \\ \mathbb{H} / \overline{R}\ \mathrm{PSL}_2(\mathbb{Z})\, \overline{R}^{-1} \arrow[r] & \mathcal{M}_{1,1} \end{tikzcd}$$

where we denote by $\overline{R}$ the image of $R = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{PSL}_2(\mathbb{R})$. Note that this is a (2-)commutative diagram of stacks, i.e. we have the orbifold Origami curve, and the fine moduli stack on the right side, then we have to take the quotients on the left side as orbifold quotients. So, the horizontal morphisms are just isomorphisms, as the orbifold Origami curve $\mathcal{C}(O)$ is smooth. It should also be possible to state the proof without using stacks, but we would have to puncture the curves again, like in the "coarse version" of the proof of Proposition 3.17 b). In any case the horizontal morphisms are generically one-to-one, which shows $[\mathrm{PSL}_2(\mathbb{Z}) : \mathrm{P}\Gamma(O)] = \deg(\mathfrak{q}_O)$, which is clearly a Galois invariant of $\mathcal{C}(O)$, and thus also of O.

For part c), we note that $-I \in \Gamma(O)$ iff there is an automorphism $\phi \in \mathrm{Aut}(X)$ making the following diagram commutative, where $i : E \to E$ denotes the elliptic involution:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ E \arrow[r, "i"] & E \end{tikzcd}$$

Assume now $-I \in \Gamma(O)$, and take $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$. Applying it to the above diagram yields

$$\begin{tikzcd} X^\sigma \arrow[r, "\phi^\sigma"] \arrow[d, "p^\sigma"'] & X^\sigma \arrow[d, "p^\sigma"] \\ E \arrow[r, "i"] & E \end{tikzcd}$$

as E and i are defined over $\mathbb{Q}$. So $-I \in \Gamma(O^\sigma)$. By symmetry we can conclude the converse.

The proof of part d) works almost identically. Giving a translation is the same as giving an automorphism $\phi \in \mathrm{Aut}(X)$ that descends to the identity on E (which is certainly defined over $\mathbb{Q}$), i.e. that makes the following diagram commutative:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "p"'] & X \arrow[d, "p"] \\ E \arrow[r, "\mathrm{id}"] & E \end{tikzcd}$$

So, by repeating the above arguments, we get a bijection $\mathrm{Trans}(O) \to \mathrm{Trans}(O^\sigma)$, and the fact that the action of $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ is functorial shows that this bijection is a group isomorphism. □

Let us now apply this result to study the field extension $M(O)/M(C(O))$:

THEOREM 2. Let $O = (p : X^* \to E^*)$ be an Origami. Then we have

$$[M(O) : M(C(O))] \leq [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(O)].$$

PROOF. Let $O \cdot \mathrm{Aut}(\mathbb{C}) = \{O_1, \ldots, O_k\}$ and $C(O) \cdot \mathrm{Aut}(\mathbb{C}) = \{C_1, \ldots, C_l\}$. Then we have, as we have shown in Remark 3.21 b):

$$k = [M(O) : \mathbb{Q}], \; l = [M(C(O)) : \mathbb{Q}].$$

By Theorem 1 the Veech groups of all O_i have the same index $m := [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(O)]$, and by Corollary 3.19 we also have

$$\{C_1, \ldots, C_l\} = \{C(O_1), \ldots, C(O_k)\}.$$

From Proposition 3.7 a) we know that each curve C_j can be the Origami curve of at most m of the O_i's, so we have

$$l \geq \frac{k}{m},$$

or equivalently $\frac{k}{l} \leq m$. The left hand side of the latter inequality is by the multiplicity of degrees of field extensions equal to $[M(O) : M(C(O))]$, and the right hand side is by definition the index of the Veech group of O. □

One would expect that the typical case is $[M(O) : M(C(O))] = 1$, as it seems that two non-equivalent Origamis that are both Galois conjugate and affinely equivalent are a rather strange exception. But indeed, in Example 5.3 we will construct an Origami O such that $[M(O) : M(C(O))] = 2$. It is not known to the author, on the other hand, if (except in the case of $\Gamma(O) = \mathrm{SL}_2(\mathbb{Z})$) the inequality can become an equality, i.e. if it is possible to construct an Origami O such that all Origamis affinely equivalent to O are also Galois conjugate to O.

CHAPTER 4

The Galois action on M-Origamis

In [**Möl05**], Möller proved that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ on Origami curves is faithful, using a special class of Origamis which he constructed in an algebro-geometric way from dessins d'enfants. We will recover the construction of these Origamis, which we will call M-Origamis here, in a topological way, i.e. we will calculate their monodromy, using the techniques we introduced in Chapter 1. This will enable us to reprove the faithfulness result in an almost entirely topological setting. Not only does this make the situation understandable with less prerequisites, but it also enables us to generalise some of the results (along with fixing some minor errors in Möller's proof), and to understand the class of M-Origamis better. In particular, we will calculate properties like the genus, the number of punctures, the Veech group, the cylinder decompositions etc. of an M-Origami using the monodromy of the dessin it is associated to. Furthermore, we will shed a light on the relationship between the weak isomorphism class of a dessin and the affine equivalence class of the associated M-Origami. Finally, our results in this chapter will enable us to give some examples for M-Origamis on which the absolute Galois group acts non-trivially in the last chapter.

1. Defining pillow case Origamis and M-Origamis

We will first explain a construction to obtain an Origami from a pillow case cover—such Origamis will be called pillow case Origamis here. This construction has appeared in the literature before, and we will specialise it to the one given in [**Möl05**].

First, fix an elliptic curve E over $\mathbb{C}$, and let $h : E \to \mathbb{P}^1_{\mathbb{C}}$ be the quotient map by its elliptic involution. h is a double cover three of whose ramification points can be taken to be $0, 1$ and ∞. This defines the fourth ramification point, call it $\lambda \in \mathbb{C} \setminus \{0,1\}$. Now let $\gamma : Y \to \mathbb{P}^1_{\mathbb{C}}$ be a pillow case cover, i.e. let Y be a nonsingular projective curve over $\mathbb{C}$, and γ a nonconstant morphism with $\operatorname{crit}\gamma \subseteq \{0, 1, \lambda, \infty\}$. Let $\tilde{\pi} : \tilde{X} := E \times_{\mathbb{P}^1_{\mathbb{C}}} Y \to E$ be its pullback by h, and $\delta : X \to \tilde{X}$ be the desingularisation of $\tilde{X}$ (as $\tilde{X}$ will have singularities in general—we will discuss them in Proposition 4.3 in this chapter). Finally, let $[2] : E \to E$ be the multiplication by 2, which is an unramified cover of degree 4 which maps the four Weierstraß points of E, which we will, for reasons of simplicity, also denote by $0, 1, \lambda$ and ∞, to ∞, the neutral element of the group structure on E. In order not to get lost in all these morphisms, we draw a commutative diagram of the situation:

$$
\begin{tikzcd}
X \arrow[r, "\delta"] \arrow[dr, "\pi"'] & \tilde{X} \arrow[r] \arrow[d, "\tilde{\pi}"] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\gamma"] \\
& E \arrow[r, "h"] \arrow[d, "{[2]}"] & \mathbb{P}^1_{\mathbb{C}} \\
& E &
\end{tikzcd}
$$

Now, if X is connected, then $[2] \circ \pi : X \to E$ is a nonconstant morphism of a connected non-singular projective curve onto an elliptic curve, ramified over at most one point, so by definition an Origami cover. This leads to the wanted

Definition 4.1. a) In the situation above, we call $O(\gamma) := ([2] \circ \pi : X \to E)$ the *pillow case Origami* associated to the pillow case cover $\gamma : Y \to \mathbb{P}^1_{\mathbb{C}}$.
b) If furthermore $\beta := \gamma$ is unramified over λ, i.e. it is a Belyi morphism, then we call $O(\beta)$ the *M-Origami* associated to γ.

Before we go on, it is a good time to state the following simple

Remark 4.2. If $\gamma : Y \to \mathbb{P}^1_{\mathbb{C}}$ is a genus 0 pillow case cover, i.e. $g(Y) = 0$, then the associated pillow case Origami is hyperelliptic.

Proof. Under the assumption that we get an Origami, X is connected, and the fibre product construction yields a morphism $X \to \mathbb{P}^1_{\mathbb{C}}$ which is of degree 2. So, X is hyperelliptic. $\square$

2. The topological viewpoint

Let $\beta : Y \to \mathbb{P}^1_{\mathbb{C}}$ be a Belyi morphism, and $O(\beta)$ the associated M-Origami. (It is not yet clear that the pulled back and desingularised curve X is always connected, but at the end of the next section, we will give an argument why this is always true in the case of dessins, and so the construction of Definition 4.1 b) never fails.) All of the curves occurring in the construction, except $\tilde{X}$, are nonsingular, so they have natural structures as Riemann surfaces.

What we want to do now is remove all the critical and ramification loci and study the restrictions of the mappings as unramified coverings with respect to the complex topology. This is possible since the restriction of the desingularisation δ to the complement of the ramification locus of π is an isomorphism, as singularities in $\tilde{X}$ occur precisely over points of $\mathbb{P}^1_{\mathbb{C}}$ over which β and h are both ramified. This is shown by the following

Proposition 4.3. Let C, D be nonsingular projective curves over k, where k is an algebraically closed field, and $\Phi : C \to {\mathbb{P}^1}_k$, $\Psi : D \to {\mathbb{P}^1}_k$ nonconstant rational morphisms (i.e. ramified covers). Then we have for the singular locus of $C \times_{{\mathbb{P}^1}_k} D$:

$$\mathrm{Sing}(C \times_{{\mathbb{P}^1}_k} D) = \{(P, Q) \in C \times_{{\mathbb{P}^1}_k} D \mid \Phi \text{ is ramified at } P \wedge \Psi \text{ is ramified at } Q\}.$$

Proof. Since the property of a point of a variety to be singular can be decided locally, we can first pass on to an affine situation and then conclude by a calculation using the Jacobi criterion.

So let $(P,Q) \in F := C \times_{\mathbb{P}^1{}_k} D$, and let $U \subset \mathbb{P}^1{}_k$ be an affine neighbourhood of $\Phi(P) = \Psi(Q)$. Further let $P \in U' \subset C$, $Q \in U'' \subset D$ be affine neighbourhoods such that $\Phi(U') \subseteq U \supseteq \Psi(U'')$. This is possible since the Zariski topology of any variety admits a basis consisting of affine subvarieties (see **[Har04]** I, Prop. 4.3). Now let $p_C : F \to C$, $p_D : F \to D$ be the canonical projections, then, by the proof of [**Har04**, II Thm. 3.3] we have $p_C^{-1}(U') \cong U' \times_{\mathbb{P}^1{}_k} D$, and repeating that argument on the second factor gives

$$p_C^{-1}(U') \cap p_D^{-1}(U'') = (p_{D|p_C^{-1}(U')})^{-1}(U'') \cong U' \times_{\mathbb{P}^1{}_k} U'' \cong U' \times_U U''.$$

The last isomorphism is due to the easy fact that in any category, a monomorphism $S \to T$ induces an isomorphism $A \times_S B \cong A \times_T B$, given that either of the two exists. So we are, as desired, in an affine situation, as the fibre product of affine varieties is affine.

Now, let $U = \mathbb{A}_k^1$, and let

$$I' = (f_1, \dots, f_k) \subseteq k[x_1, \dots, x_n] =: R_n,\ I'' = (g_1, \dots, g_l) \subseteq k[y_1, \dots, y_m] =: R_m$$

be ideals such that $U' = V(I') \subseteq \mathbb{A}_k^n$, $U'' = V(I'') \subseteq \mathbb{A}_k^m$. Furthermore let $\varphi \in R_n$, $\psi \in R_m$ be polynomials representing the morphisms Φ and Ψ on the affine parts U' and U''. We denote their images in the affine coordinate rings by $\overline{\varphi} \in k[U']$, $\overline{\psi} \in k[U'']$. So we get

$$F := U' \times_U U'' = V\left(f_1, \dots, f_k, g_1, \dots, g_l, \varphi(x_1, \dots, x_n) - \psi(y_1, \dots, y_m)\right) \subseteq \mathbb{A}^{m+n}.$$

The Jacobi matrix of F is given by

$$J_F = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & & & \\ \vdots & & \vdots & & 0 & \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} & & & \\ & & & \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_m} \\ & 0 & & \vdots & & \vdots \\ & & & \frac{\partial g_l}{\partial y_1} & \cdots & \frac{\partial g_l}{\partial y_m} \\ \frac{\partial \varphi}{\partial x_1} & \cdots & \frac{\partial \varphi}{\partial x_n} & -\frac{\partial \psi}{\partial y_1} & \cdots & -\frac{\partial \psi}{\partial y_m} \end{pmatrix}$$

and by the Jacobi criterion a point $(P,Q) \in F$ is singular iff $\operatorname{rk}(J_F(P,Q)) < m + n - 1$.

For $P = (p_1, \dots, p_n) \in \mathbb{A}_k^n$, $Q = (q_1, \dots, q_m) \in \mathbb{A}_k^m$, denote by

$$M_P := \big((x_1 - p_1), \dots, (x_n - p_n)\big) \subseteq R_n \text{ and}$$
$$M_Q := \big((y_1 - q_1), \dots, (y_m - q_m)\big) \subseteq R_m$$

the corresponding maximal ideals, and, if $P \in U'$ and $Q \in U''$, denote by $m_P \subseteq k[U']$ and $m_Q \subseteq k[U'']$ the corresponding maximal ideals in the affine coordinate rings. Before we continue, we note that for $h \in R_n$, we have:

$$h \in M_P^2 \Leftrightarrow h(P) = 0 \wedge \forall i = 0, \dots, n : \frac{\partial h}{\partial x_i}(P) = 0. \tag{1}$$

Of course the corresponding statement is true for $M_Q^2 \subseteq R_m$.

Now let $P = (p_1, \dots, p_n) \in U'$, $Q = (q_1, \dots, q_m) \in U''$ be ramification points of $\overline{\varphi}$ and $\overline{\psi}$, respectively. This is by definition equivalent to

$$\overline{\varphi} - \overline{\varphi}(P) \in m_P^2 \ \wedge\ \overline{\psi} - \overline{\psi}(Q) \in m_Q^2, \text{ or,}$$

$$\varphi - \varphi(P) \in M_P^2 + I' \wedge \psi - \psi(Q) \in M_Q^2 + I''. \tag{2}$$

So, there exist $a_0 \in M_P^2$, $a_1, \ldots, a_k \in R_n$, $b_0 \in M_Q^2$, $b_1, \ldots, b_l \in R_m$ such that

$$\varphi - \varphi(P) = a_0 + \sum_{i=1}^{k} a_i f_i, \;\; \psi - \psi(Q) = b_0 + \sum_{i=1}^{l} b_i g_i.$$

Writing $R_n \ni h = h(P) + (h - h(P))$, we have the decomposition $R_n = k \oplus M_P$ as k-modules, and analogously $R_m = k \oplus M_Q$. So write

$$a_i = \lambda_i + \tilde{a}_i \in k \oplus M_P,\ i = 1, \ldots, k \text{ and } b_i = \mu_i + \tilde{b}_i \in k \oplus M_Q,\ i = 1, \ldots, l.$$

Because $I' \subseteq M_P$ and $I'' \subseteq M_Q$, we have $c := \sum_{i=1}^{k} \tilde{a}_i f_i \in M_P^2$ and $d := \sum_{i=1}^{l} \tilde{b}_i g_i \in M_Q^2$. So if we set $a := a_0 + c \in M_P^2$ and $b := b_0 + d \in M_Q^2$, we get

$$\varphi - \varphi(P) = a + \sum_{i=1}^{k} \lambda_i f_i \text{ and } \psi - \psi(Q) = b + \sum_{i=1}^{l} \mu_i g_i$$

Deriving on both sides of the equations with respect to all the variables, we get, using (1):

$$\forall i \in \{1, \ldots, n\} : \frac{\partial \varphi}{\partial x_i}(P) = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial x_i}(P) \text{ and}$$

$$\forall i \in \{1, \ldots, m\} : \frac{\partial \psi}{\partial y_i}(Q) = \sum_{j=1}^{l} \mu_j \frac{\partial g_j}{\partial y_i}(Q).$$

So the last row of $J_F(P, Q)$ is a linear combination of the first $m + n$ ones. As the two big non-zero blocks of J_F are simply the Jacobi matrices $J_{U'}$ and $J_{U''}$ of the nonsingular curves U' and U'', they have ranks $n - 1$ and $m - 1$, evaluated at P and Q respectively, and we get

$$\operatorname{rk}(J_F(P, Q)) = \operatorname{rk}(J_{U'}(P)) + \operatorname{rk}(J_{U''}(Q)) = (n - 1) + (m - 1) < m + n - 1,$$

so $(P, Q) \in \operatorname{Sing}(U' \times_U U'')$.

Conversely, let $(P, Q) \in \operatorname{Sing}(F)$ be a singular point. Then, by the nonsingularity of U' and U'', we have $\operatorname{rk}(J_F(P, Q)) = m + n - 2$. More specifically, the last row of this matrix is a linear combination of the others:

$$\exists \lambda_1, \ldots, \lambda_k \in k \ \forall i \in \{1, \ldots, n\} : \frac{\partial \varphi}{\partial x_i}(P) = \sum_{j=1}^{k} \lambda_j \frac{\partial f_j}{\partial x_i}(P) \text{ and}$$

$$\exists \mu_1, \ldots, \mu_l \in k \ \forall i \in \{1, \ldots, m\} : \frac{\partial \psi}{\partial y_i}(Q) = \sum_{j=1}^{l} \mu_j \frac{\partial g_j}{\partial y_i}(Q).$$

Now set $\tilde{\varphi} := \varphi - \varphi(P) - \sum \lambda_j f_j$ and $\tilde{\psi} := \psi - \psi(Q) - \sum \mu_j g_j$. Then we have $\tilde{\varphi} \in M_P$ and $\tilde{\psi} \in M_Q$, and furthermore

$$\forall i \in \{1, \ldots, n\} : \frac{\partial \tilde{\varphi}}{\partial x_i}(P) = 0 \text{ and } \forall i \in \{1, \ldots, m\} : \frac{\partial \tilde{\psi}}{\partial y_i}(Q) = 0.$$

So we can apply (1) again to get $\tilde{\varphi} \in M_P^2$, $\tilde{\psi} \in M_Q^2$, or,

$$\varphi - \varphi(P) \in M_P^2 + I' \wedge \psi - \psi(Q) \in M_Q^2 + I''.$$

This is precisely (2), which we have already shown above to be equivalent to P and Q being ramification points of $\overline{\varphi}$ and $\overline{\psi}$, respectively. $\square$

So if we denote $\mathbb{P}^{1^*} := \mathbb{P}^1_{\mathbb{C}} \setminus \{0,\, 1,\, \lambda,\, \infty\}$ and by E^*, Y^*, X^* its preimages under h, β and $\pi \circ h$, respectively, (and, for the sake of a simpler notation, do not change the names for the restricted maps,) then we have in particular:

Remark 4.4. The following diagram is Cartesian in the category of topological spaces together with covering maps:

$$\begin{tikzcd} X^* \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & Y^* \arrow[d, "\beta"] \\ E^* \arrow[r, "h"] & \mathbb{P}^{1^*} \end{tikzcd}$$

3. Monodromy of M-Origamis

Now, we are ready to calculate the monodromy of an M-Origami $O(\beta)$, given the monodromy of a Belyi morphism β, using the methods from Chapter 1. So let β be a Belyi morphism of degree $d := \deg \beta$ that is, as explained earlier, uniquely described by two permutations $p_x,\, p_y \in S_d$, and $O(\beta)$ the corresponding M-Origami. Then we have:

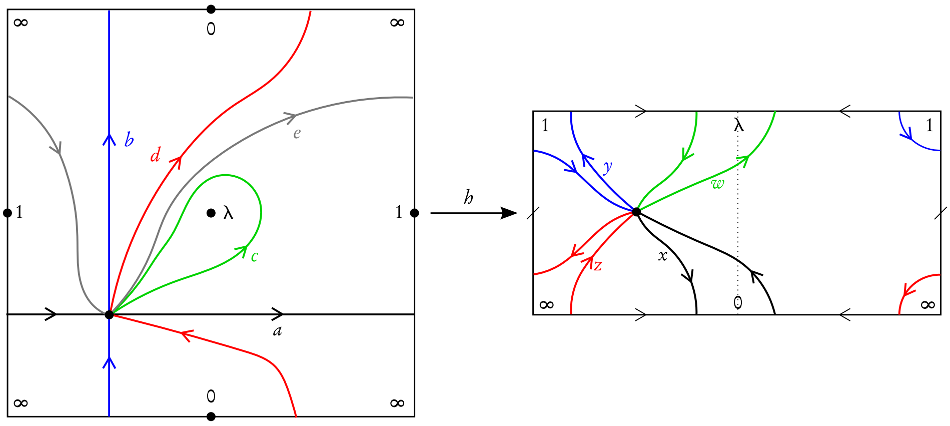
Theorem 3. Denote the standard generators of $\pi_1(E \setminus \{\infty\}) \cong F_2$ by A ("right") and B ("up"), and denote the standard generators of $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \cong F_2$ by x (counterclockwise path around 0) and y (counterclockwise path around 1). If the monodromy of β is given by $m_\beta(x) =: p_x \in S_d$, $m_\beta(y) =: p_y \in S_d$, then the monodromy of $[2] \circ \pi : \pi_1(E \setminus \{\infty\}) \to S_{4d}$ is given as follows:

$$m_{[2]\circ\pi} : \pi_1(E \setminus \{\infty\}) \to S_{4d}$$

$$m_{[2]\circ\pi}(A)(i,j) = \begin{cases} (2,j) & i = 1 \\ (1, p_y(j)) & i = 2 \\ (4,j) & i = 3 \\ (3, p_y^{-1}(j)) & i = 4 \end{cases} \text{ and } m_{[2]\circ\pi}(B)(i,j) = \begin{cases} (3,j) & i = 1 \\ (4,j) & i = 2 \\ (1, p_x^{-1}(j)) & i = 3 \\ (2, p_x(j)) & i = 4 \end{cases}$$

Note that we use the notation from Theorem C here, i.e. we let S_{4d} act on the set $\{1, \ldots, 4\} \times \{1, \ldots, d\}$.

Proof. The proof is a straightforward calculation, but we should get a clear idea of the involved fundamental groups here. As a topological model for $\mathbb{P}^{1^*}$ we use the *pillow case*, i.e. an euclidean rectangle R with width-to-height ratio 2 : 1, folded in half and stitched together in the obvious way, the four resulting cusps forming the punctures. The two to one covering h is then realised by taking two copies of R, rotating one of them by an angle of π, stitching them together and then identifying opposite edges in the usual way. The result is a genus 1 surface with 4 punctures. The following picture should give an idea of the whole situation.



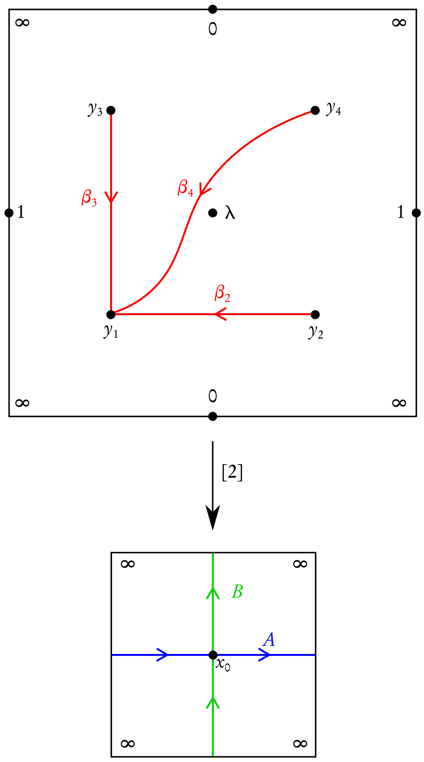
We have $\pi_1(\mathbb{P}^{1*}) \cong F_3$. As depicted, we can take the paths w, x, y, z as generators, subject to the relation $wxzy = 1$. (Remember that we use the convention to read words in the fundamental group from the right to the left.) Now, we have $\pi_1(E \setminus \{0, 1, \lambda, \infty\}) \cong F_5$, and as in the picture we choose a, b, c, d, e as free generators. We can verify easily that

$$h_* : \pi_1(E \setminus \{0, 1, \lambda, \infty\}) \to \pi_1(\mathbb{P}^{1*}), \begin{cases} a & \mapsto & yw \\ b & \mapsto & x^{-1}w^{-1} \\ c & \mapsto & w^2 \\ d & \mapsto & xw^{-1} \\ e & \mapsto & y^{-1}w^{-1} \end{cases}.$$

Now, let us calculate the monodromy of the covering $\pi : X \setminus \pi^{-1}(\{0, 1, \lambda, \infty\}) \to E \setminus \{0, 1, \lambda, \infty\}$. Using the notation of the theorem, we have (as β is unramified over λ) $m_\beta(w) = 1$ and $m_\beta(z) = p_x^{-1}p_y^{-1}$. Thus, by Theorem B a), we have

$$m_\pi : \pi_1(E \setminus \{0, 1, \lambda, \infty\}) \to S_d, \begin{cases} a & \mapsto & p_y \\ b & \mapsto & p_x^{-1} \\ c & \mapsto & 1 \\ d & \mapsto & p_x \\ e & \mapsto & p_y^{-1} \end{cases}.$$

Now, we will use Theorem C to calculate the monodromy of $[2] \circ \pi$. We need only one puncture in the "bottom" elliptic curve E, so its fundamental group is isomorphic to F_2. As sketched in the figure below, we choose a base point x_0 and paths A and B as generators of $\pi_1(E \setminus \{\infty\})$.



If we number the preimages of x_0 as in the picture, of course the monodromy map of [2] is given by

$$m_{[2]}(A) = (1\,2)(3\,4),\ m_{[2]}(B) = (1\,3)(2\,4).$$

The theorem requires us to choose right coset representatives of $m_{[2]}^{-1}(\mathrm{Stab}(1))$ in $\pi_1(E \setminus \{\infty\})$. We take

$$\gamma_1 := 1,\ \gamma_2 := A^{-1},\ \gamma_3 := B^{-1},\ \gamma_4 := B^{-1}A^{-1}$$

and check that they do what they should by seeing that they lift along [2] to paths β_i connecting y_i to y_1. The next step is to calculate the c_i's. We get

$$c_i(A) = [2]_*^{-1}(\gamma_k A \gamma_i^{-1}) = \begin{cases} [2]_*^{-1}(A^{-1}A \cdot 1) = 1, & i = 1 \\ [2]_*^{-1}(1 \cdot AA) = a, & i = 2 \\ [2]_*^{-1}(B^{-1}A^{-1}AB) = 1, & i = 3 \\ [2]_*^{-1}(B^{-1}AAB) = e, & i = 4 \end{cases} \quad \text{and}$$

$$c_i(B) = [2]_*^{-1}(\gamma_k B \gamma_i^{-1}) = \begin{cases} [2]_*^{-1}(B^{-1}B \cdot 1) = 1, & i = 1 \\ [2]_*^{-1}(B^{-1}A^{-1}BA) = c, & i = 2 \\ [2]_*^{-1}(1 \cdot BB) = b, & i = 3 \\ [2]_*^{-1}(A^{-1}BAB) = d, & i = 4 \end{cases}.$$

Putting all together, we get

$$m_{[2]\circ\pi}(A)(i,j) = \Big(m_{[2]}(A)(i),\, m_\pi(c_i(A))(j)\Big) = \begin{cases} (2,j) & i=1 \\ (1,p_y(j)) & i=2 \\ (4,j) & i=3 \\ (3,p_y^{-1}(j)) & i=4 \end{cases} \quad \text{and}$$

$$m_{[2]\circ\pi}(B)(i,j) = \Big(m_{[2]}(B)(i),\, m_\pi(c_i(B))(j)\Big) = \begin{cases} (3,j) & i=1 \\ (4,j) & i=2 \\ (1,p_x^{-1}(j)) & i=3 \\ (2,p_x(j)) & i=4 \end{cases},$$

which is, seemingly by a strange coincidence, exactly what we have claimed. □

Now we can easily fill in the gap left open in the beginning of the previous section:

Remark 4.5. If we start with a Belyi morphism β, the topological space X^* arising in the construction is always connected, so $O(\beta)$ is indeed an Origami.

Proof. What we have to show is that $m_A := m_{[2]\circ\pi}(A)$ and $m_B := m_{[2]\circ\pi}(B)$ generate a transitive subgroup of S_{4d}. So choose $i \in \{1,\ldots,4\}$, $j \in \{1,\ldots,d\}$, then it clearly suffices to construct a path $\gamma \in \pi_1(E \setminus \{\infty\})$ such that $m_{[2]\circ\pi}(\gamma)(1,1) = (i,j)$.
Now, as Y^* is connected, there is a path $\gamma' \in \pi_1(\mathbb{P}^{1^*})$ such that $m_\beta(\gamma')(1) = j$. Consider the homomorphism

$$\phi : \pi_1(\mathbb{P}^{1^*}) \to \pi_1(E \setminus \{\infty\}), \begin{cases} x \mapsto & B^{-2}, \\ y \mapsto & A^2 \end{cases}$$

By the above theorem, a lift of the path A^2 connects the square of $O(\beta)$ labelled with $(1,k)$ to the one labelled with$(1,p_y(k))$, and a lift of B^{-2} connects $(1,k)$ to $(1,p_x(k))$. So we get $m_{[2]\circ\pi}(\phi(\gamma'))(1,1) = (1,j)$. So to conclude, we set $\gamma := \epsilon\phi(\gamma')$, where $\epsilon := 1$, A, B or AB for $i = 1$, 2, 3 or 4, respectively, and get $m_{[2]\circ\pi}(\gamma)(1,1) = (i,j)$. □

For some calculations in the following sections, it will be useful to know the monodromy of the map π around the Weierstraß points 0, 1, λ, $\infty \in E$, respectively. Therefore, we do the following quick calculation here:

Lemma 4.6. If we choose the following simple loops around the Weierstraß points

$$x' := l_0 := db^{-1} \qquad y' := l_1 := ac^{-1}e^{-1}$$
$$z' := l_\infty := bed^{-1}a^{-1} \qquad w' := l_\lambda := c,$$

then we have

$$h_*(x') = x^2 \qquad h_*(y') = y^2$$
$$h_*(z') = z^2 \qquad h_*(w') = w^2.$$

PROOF. Indeed, by the calculations in the proof of Theorem 3, we have:

$$\begin{aligned}
h_*(x') &= xw^{-1}wx = x^2,\\
h_*(y') &= yww^{-2}wy = y^2,\\
h_*(z') &= x^{-1}w^{-1}y^{-1}w^{-1}wx^{-1}w^{-1}y^{-1} = (ywx)^{-2} = z^2,\\
h_*(w') &= w^2.
\end{aligned}$$

□

4. The genus of M-Origamis

We calculate the genus of the M-Origami associated to a Belyi morphism β and give lower and upper bounds depending only on $g(Y)$ and $\deg\beta$. We have the following

PROPOSITION 4.7. Let $\beta : Y \to \mathbb{P}^1_{\mathbb{C}}$ be a Belyi morphism and $d = \deg\beta$ its degree, $m_0 := p_x$, $m_1 := p_y$, $m_\infty := p_z = p_x^{-1}p_y^{-1} \in S_d$ the monodromy of the standard loops around 0, 1 and ∞. Further let g_0, g_1, g_∞ be the number of cycles of even length in a disjoint cycle decomposition of m_0, m_1 and m_∞, respectively. Then we have:

a)
$$g(X) = g(Y) + d - \frac{1}{2}\sum_{i=0}^{\infty} g_i.$$

b)
$$g(Y) + \left\lceil \frac{d}{4} \right\rceil \le g(X) \le g(Y) + d.$$

PROOF. The Riemann-Hurwitz formula for β says:

$$2g(Y) - 2 = d(2g(\mathbb{P}^1) - 2) + \sum_{p\in Y}(e_p - 1)$$

Let us now split up $\sum_{p\in Y}(e_p - 1) = v_0 + v_1 + v_\infty$, where $v_i := \sum_{p\in\beta^{-1}(i)}(e_p - 1)$. Of course, $v_i = \sum_{c \text{ cycle in } m_i}(\mathrm{len}(c) - 1)$, where $\mathrm{len}(c)$ is the length of a cycle c. Putting that and the fact that $g(\mathbb{P}^1) = 0$ in the last equation yields

$$2g(Y) - 2 = -2d + \sum_{i=0}^{\infty} v_i.$$

Now we write down the Riemann-Hurwitz formula for π. Note that π is at most ramified over the preimages of 0, 1 and ∞ under h, which we denote also by 0, 1 and ∞. β is unramified over the (image of the) fourth Weierstraß point, and so is π. So again the ramification term splits up into $\sum_{p\in X}(e_p - 1) = v'_0 + v'_1 + v'_\infty$ with $v'_i = \sum_{c \text{ cycle in } m'_i}(\mathrm{len}(c) - 1)$, where m'_0, m'_1, m'_∞ are permutations describing the monodromy of π going around the Weierstraß points $0, 1, \infty$ respectively. Of course $g(E) = 1$, and so we get

$$2g(X) - 2 = \sum_{i=0}^{\infty} v'_i.$$

Subtracting from that equation the one above and dividing by 2 yields

$$g(X) = g(Y) + d - \frac{1}{2}\sum_{i=0}^{\infty}(v_i - v'_i),$$

so we can conclude a) with the following

LEMMA 4.8. $v_i - v_i' = g_i$, where g_i is the number of cycles of even length in m_i.

The proof of this lemma is elementary. Write $m_0' := m_\pi(x')$, $m_1' := m_\pi(y')$, $m_\infty' := m_\pi(z')$ as in Lemma 4.6, then we get $m_i' = m_i^2$ by that lemma and Theorem B. Now, note that the square of a cycle of odd length yields a cycle of the same length, while the square of a cycle of even length is the product of two disjoint cycles of half length. So, if c is such an even length cycle and $c^2 = c_1 c_2$, then obviously $(\mathrm{len}(c) - 1) - ((\mathrm{len}(c_1) - 1) + (\mathrm{len}(c_2) - 1)) = 1$. Summing up proves the lemma. $\square$

In part b) of the proposition, the second inequality is obvious as all g_i are non-negative. For the first one, note that we have $g_i \leq \frac{d}{2}$, so $\sum g_i \leq \frac{3}{2}d$, and finally $d - \frac{1}{2}\sum g_i \geq \frac{d}{4}$. Of course $g(X)$ and $g(Y)$ are natural numbers, so we can take the ceiling function. $\square$

With the notations of the above proposition and the considerations in the proof, we can easily count the number of punctures of an M-Origami:

REMARK 4.9. Let $\beta : Y \to \mathbb{P}^1_{\mathbb{C}}$ be a Belyi morphism, $\pi : X \to E$ the normalisation of its pullback by the elliptic involution as in Definition 4.1, and $O_\beta = (p : \; X \to E)$ the associated M-Origami.

a) Let $W := \{0, 1, \lambda, \infty\} \subseteq E$ be the set of Weierstraß points of E, then we have:

$$n(P) := |\pi^{-1}(P)| = \#(\text{cycles in } m_P) + g_P, \;\; (P \in \{0, 1, \infty\})$$
$$n(\lambda) := |\pi^{-1}(\lambda)| = \deg \pi = \deg \beta.$$

b) For $O_\beta = (p : \; X \to E)$, we have:

$$|p^{-1}(\infty)| = n(0) + n(1) + n(\lambda) + n(\infty).$$

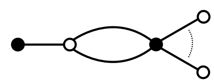
PROOF. Part b) is a direct consequence of a), as by definition $p = [2] \circ \pi$, and $[2] : \; E \to E$ is an unramified cover with $[2]^{-1}(\infty) = W$.

For a), the second equality is clear, since, as we noted above, π is unramified over λ. If $P \in \{0, 1\, \infty\}$, then $n(P)$ is the number of cycles in $m_P' = m_P^2$ (where we reuse the notation from the proof above). As a cycle in m_P corresponds to one cycle in m_P^2 if its length is odd, and splits up into two cycles of half length if its length is even, we get the desired statement. $\square$

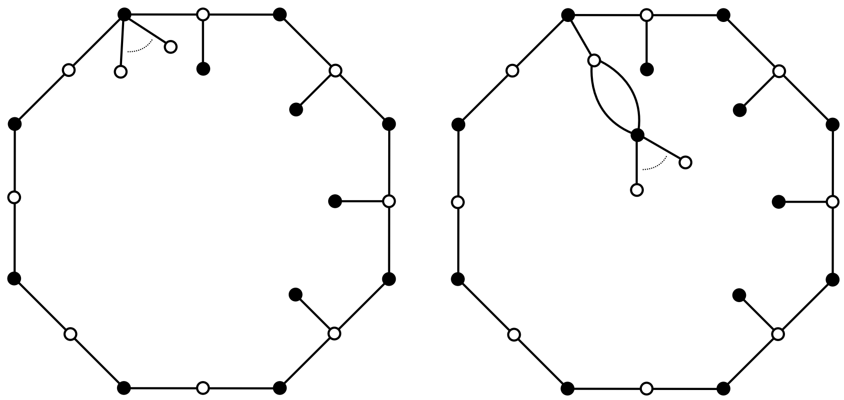
Let us now return to the genus of M-Origamis. We show that the upper bound in part b) of Proposition 4.7 is sharp in “almost all” cases in the following sense:

PROPOSITION 4.10. For $g \in \mathbb{N}$, $d \geq 6g + 1$, but $d \neq 6g + 2$, there is a dessin of genus g and degree d such that the resulting Origami has genus $d + g$.

PROOF. We have to construct dessins with $\sum g_i = 0$. For $g = 0$ and d odd, we can obviously just take the d-star. For $g = 0$ and $d \geq 4$ even, we can take the following dessin with $d - 3$ edges on the right:



Now let $g \geq 1$. We take a regular $4g$-gon where we mark the vertices with black dots and the midpoints of edges with white dots. If we glue opposite edges, we get a genus g dessin of degree $4g$. Now, add a "sting" at each of the $2g$ white vertices. For $d \geq 6g+1$ odd, add $d-6g$ "stings" at the black vertex. For $d \geq 6g+4$ even, concatenate the dessin shown above (with $d-6g$ edges) to the black vertex. To illustrate this, we draw this dessin for $g=2$. The left one corresponds to the odd case, the right one to the even.



□

If X is an M-Origami fulfilling $g(X) \leq G$ for some $G \in \mathbb{N}$, then by the above proposition we have $d \leq 4G$. There are only finitely many dessins up to a given degree, so as an interesting consequence we get the following corollary which might seem surprising at first:

COROLLARY 4.11. Given a natural number $G \in \mathbb{N}$, there are only finitely many M-Origamis with genus less or equal to G.

In the special case of unicellular dessins (i.e. m_∞ consists of one cycle of length $d = \deg(\beta)$), we can obviously strengthen the estimation of Proposition 4.7 b). In this case we have $g_\infty = 0$ or 1, depending on the parity of d, so we have the following

REMARK 4.12. In the situation of Proposition 4.7, for unicellular dessins we get

$$g(Y) + \left\lceil \frac{d-1}{2} \right\rceil \leq g(X) \leq g(Y) + d.$$

In particular, for trees we get

$$\left\lceil \frac{d-1}{2} \right\rceil \leq g(X) \leq d.$$

It is often convenient to work with clean dessins, i.e. all preimages of 1 are ramification points of order 2. As explained in Definition and Remark 2.8 c), it is possible to replace a Belyi morphism β by a clean one by postcomposing $\tau : z \mapsto 4z(1-z)$. We want to calculate how this process changes the genus of the obtained Origami. What happens to the dessin is that all vertices become black, and in the centre of each edge a new white vertex is inserted. So the numbers

g_0, g_1, g_∞ are replaced by

$$\tilde{g}_0 = g_0 + g_1, \quad \tilde{g}_1 = d \quad \tilde{g}_\infty = g_\infty + u_\infty$$

where u_∞ is the number of cycles of odd length in m_∞. Note also that $\tilde{d} := \deg(\tilde{\beta}) = 2d$, so altogether we get:

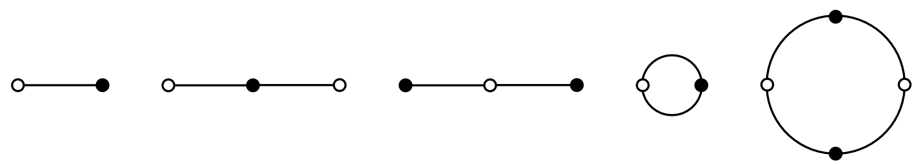
REMARK 4.13. Let X be the Origami obtained from some Belyi morphism β, and $\tilde{X}$ the one obtained from $\tilde{\beta} = 4\beta(1-\beta)$. Then we have

$$g(\tilde{X}) = g(X) + \frac{1}{2}(d - u_\infty).$$

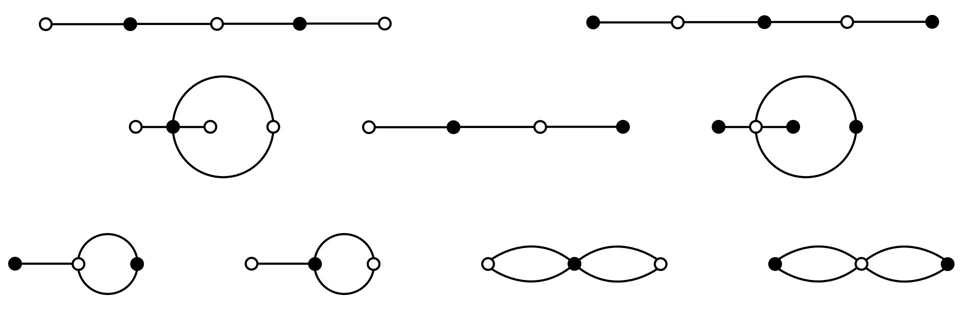
From Corollary 4.11 follows that we can (at least in theory) list all M-Origamis of a given genus. We do this for genera 1 and 2.

PROPOSITION 4.14. There are five dessins that yield genus 1 Origamis, and nine dessins that yield genus 2 Origamis.

The calculations were done with the Sage computer algebra system **[Sag10]** together with Karsten Kremer's Origami package **[Kre10b]**. Note that we do not mark the cell drawn as unbounded, but we distinguish between dessins with black and white vertices interchanged. The following five dessins correspond to the genus 1 Origamis:



The nine dessins below correspond to the genus 2 Origamis:



Of these dessins, the middle one from the second row and the two left ones from the third row yield genus 2 Origamis with one double root of the holomorphic differential associated to its translation structure, the other ones yield Origamis with two single roots.

Note that all the above dessins are determined by their cycle structure, thus are defined over $\mathbb{Q}$, and so are the resulting Origami curves.

5. The Veech group

To calculate the Veech group of an M-Origami, we use the charactarisation that Weitze-Schmithüsen gave in [**Sch05**]. According to Theorem 1 in that work, the Veech group of an Origami $O = (p : X^* \to E^*)$ is the image in $\mathrm{Out}^+(\pi_1(E^*)) \cong \mathrm{SL}_2(\mathbb{Z})$ of the stabiliser $\mathrm{Stab}(p_*\pi_1(X^*))$ under the action of $\mathrm{Aut}^+(\pi_1(E^*))$. More generally, if $\phi \in \mathrm{Aut}^+(\pi_1(E^*))$ is an "orientation preserving" automorphism, and $A \in \mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(\pi_1(E^*))$ is its image, then the monodromy of the Origami $A \cdot O$ is given by $m_p \circ \phi$, where $m_p : \pi_1(E^*) \to S_d$ is the monodromy map of O. This is shown in [**Sch05**, Proposition 3.5].

Proposition 4.15. Let β be the dessin of degree d given by a pair of permutations (p_x, p_y), and O_β the associated M-Origami. Then we have for the standard generators S, T, $-I$ of $\mathrm{SL}_2(\mathbb{Z})$:

- $S \cdot O_\beta$ is the M-Origami associated to the pair of permutations (p_y, p_x).
- $T \cdot O_\beta$ is the M-Origami associated to the pair of permutations (p_z, p_y), where as usual $p_z = p_x^{-1} p_y^{-1}$.
- $(-I) \cdot O_\beta \cong O_\beta$, i.e. $-I \in \Gamma(O_\beta)$.

Proof. We lift S, T, $-I \in \mathrm{SL}_2(\mathbb{Z}) \cong \mathrm{Out}^+(F_2)$ to the following automorphisms of $F_2 = \langle A, B \rangle$, respectively:

$$\phi_S : A \mapsto B,\ B \mapsto A^{-1}$$
$$\phi_T : A \mapsto A,\ B \mapsto AB$$
$$\phi_{-I} : A \mapsto A^{-1},\ B \mapsto B^{-1}$$

Let us, for brevity, write $m_w := m_{[2]\circ\pi}(w)$ for an element $w \in F_2$, so in particular O_β is given by the pair of permutations (m_A, m_B) as in Theorem 3.
First we calculate the monodromy of the Origami $S \cdot O_\beta$, which is given by $(m_{\phi_S^{-1}(A)}, m_{\phi_S^{-1}(B)})$: We get

$$m_{\phi_S^{-1}(A)}(i,j) = m_B^{-1}(i,j) = \begin{cases} (3, p_x(j)), & i = 1 \\ (4, p_x^{-1}(j)), & i = 2 \\ (1, j), & i = 3 \\ (2, j), & i = 4 \end{cases} \text{ and } m_{\phi_S^{-1}(B)} = m_A.$$

Now we conjugate this pair by the following permutation

$$c_1 : (1, j) \mapsto (2, j) \mapsto (4, j) \mapsto (3, j) \mapsto (1, j),$$

which does of course not change the Origami it defines, and indeed we get:

$$c_1 m_B^{-1} c_1^{-1}(i,j) = \begin{cases} (2, j), & i = 1 \\ (1, p_x(j)), & i = 2 \\ (4, j), & i = 3 \\ (3, p_x^{-1}(j)), & i = 4 \end{cases}, \ c_1 m_A c_1^{-1}(i,j) = \begin{cases} (3, j), & i = 1 \\ (4, j), & i = 2 \\ (1, p_y^{-1}(j)), & i = 3 \\ (2, p_y(j)), & i = 4 \end{cases},$$

which is clearly the M-Origami associated to the dessin given by the pair (p_y, p_x).

Next, we discuss the action of the element T in the same manner, and we get:

$$m_{\phi_T^{-1}(A)} = m_A \text{ and } m_{\phi_T^{-1}(B)}(i,j) = m_A^{-1} m_B(i,j) = \begin{cases} (4, p_y(j)), & i=1 \\ (3,j), & i=2 \\ (2, p_y^{-1} p_x^{-1}(j)), & i=3 \\ (1, p_x(j)), & i=4 \end{cases}.$$

The reader is invited to follow the author in not losing hope and verifying that with

$$c_2 : (i,j) \mapsto \begin{cases} (1, p_y(j)), & i=1 \\ (2, p_y(j)), & i=2 \\ (4, p_y(j)), & i=3 \\ (3,j), & i=4 \end{cases}$$

we have

$$c_2 m_A c_2^{-1}(i,j) = \begin{cases} (2,j), & i=1 \\ (1, p_y(j)), & i=2 \\ (4,j), & i=3 \\ (3, p_y^{-1}(j)), & i=4 \end{cases} \text{ and}$$

$$c_2 m_A^{-1} m_B c_2^{-1}(i,j) = \begin{cases} (3,j), & i=1 \\ (4,j), & i=2 \\ (1, p_y p_x(j)), & i=3 \\ (2, p_x^{-1} p_y^{-1}(j)), & i=4 \end{cases},$$

which is the monodromy of the M-Origami associated to (p_z, p_y).

For the element $-I \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$m_{\phi_{-I}^{-1}(A)}(i,j) = m_A^{-1}(i,j) = \begin{cases} (2, p_y^{-1}(j)), & i=1 \\ (1,j), & i=2 \\ (4, p_y(j)), & i=3 \\ (3,j), & i=4 \end{cases} \text{ and}$$

$$m_{\phi_{-I}^{-1}(B)}(i,j) = m_B^{-1}(i,j) = \begin{cases} (3, p_x(j)), & i=1 \\ (4, p_x^{-1}(j)), & i=2 \\ (1,j), & i=3 \\ (2,j), & i=4 \end{cases}.$$

In this case,

$$c_3 : \begin{cases} (1,j) \leftrightarrow (4,j) \\ (2,j) \leftrightarrow (3,j) \end{cases}$$

does the trick and we verify that $c_3 m_A c_3 = m_A^{-1}$, $c_3 m_B c_3 = m_B^{-1}$, so $-I \in \Gamma(O_\beta)$. □

It will turn out in the next theorem that the Veech group of M-Origamis often is $\Gamma(2)$. Therefore we list, as an easy corollary from the above proposition, the action of a set of coset representatives of $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{Z})$.

Corollary 4.16. For an M-Origami O_β associated to a dessin β given by (p_x, p_y), and

$$\alpha \in \{I, S, T, ST, TS, TST\},$$

$\alpha \cdot O_\beta$ is again an M-Origami, and it is associated to the dessin with the monodromy indicated in the following table:

I	S	T	ST	TS	TST
p_x	p_y	p_z	p_y	p_z	p_x
p_y	p_x	p_y	p_z	p_x	p_z

Proof. The first three columns of the above table are true by the above proposition. If we write $M(p_x, p_y)$ for the M-Origami associated to the dessin given by (p_x, p_y), then we calculate

$$
\begin{aligned}
ST \cdot M(p_x, p_y) &= S \cdot M(p_z, p_y) = M(p_y, p_z),\\
TS \cdot M(p_x, p_y) &= T \cdot M(p_y, p_x) = M(p_y^{-1}p_x^{-1}, p_x) \cong M(p_z, p_x),\\
TST \cdot M(p_x, p_y) &= T \cdot M(p_y, p_z) = M(p_y^{-1}p_z^{-1}, p_z) = M(p_x, p_z).
\end{aligned}
$$

□

Theorem 4. Let, again, β be a dessin given by the pair of permutations (p_x, p_y).

a) For the associated M-Origami O_β, we have $\Gamma(2) \subseteq \Gamma(O_\beta)$.
b) The orbit of O_β under $\mathrm{SL}_2(\mathbb{Z})$ precisely consists of the M-Origamis associated to the dessins weakly isomorphic to β.
c) If $\Gamma(2) = \Gamma(O_\beta)$ then β has no nontrivial weak automorphism, i.e. $W_\beta = \{\mathrm{id}\}$. In the case that β is filthy, the converse is also true.

Proof. a) We have $\Gamma(2) = \langle T^2, ST^{-2}S^{-1}, -I\rangle$, so we have to show that these three matrices are elements of $\Gamma(O_\beta)$. We already know by Proposition 4.15 that $-I$ acts trivially. Using the notation of the proof of the above corollary, we calculate:

$$
\begin{aligned}
T^2 \cdot M(p_x, p_y) &= T \cdot M(p_z, p_y) = M(p_z^{-1}p_y^{-1}, p_y) \cong M(p_x, p_y),\\
ST^{-2}S^{-1} \cdot M(p_x, p_y) &= ST^{-2} \cdot M(p_y, p_x) = S \cdot M(p_y, p_x) = M(p_x, p_y).
\end{aligned}
$$

b) By a), the orbit of O_β under $\mathrm{SL}_2(\mathbb{Z})$ is the set of translates of O_β under a set of coset representatives of $\Gamma(2)$ in $\mathrm{SL}_2(\mathbb{Z})$. We have calculated them in the above corollary, and indeed they are associated to the dessins weakly isomorphic to β.

c) "$\Rightarrow$": Let $\Gamma(2) = \Gamma(O_\beta)$, then by part b) we have

$$6 = |\,\mathrm{SL}_2(\mathbb{Z}) \cdot O_\beta| \leq |W \cdot \beta| \leq 6$$

so we have equality and indeed W_β is trivial.

"$\Leftarrow$": For now, fix an element $\mathrm{id} \neq w \in W$. By assumption, $\beta \not\cong \beta' := w \cdot \beta$. Let π, π' be their pullbacks by h as in Definition 4.1. By Lemma 4.19*, we have $\pi \not\cong \pi'$. Now assume $O_\beta \cong O_{\beta'}$, so by Lemma 4.20, there exists a deck transformation $\varphi \in \mathrm{Deck}([2])$ such that $\pi' \cong \varphi \circ \pi$. But since β (and so β') is filthy, φ has to fix $\lambda \in E$, so it is the identity. This is the desired contradiction to $\pi \not\cong \pi'$. By varying w we get $|\,\mathrm{SL}_2(\mathbb{Z}) \cdot O_\beta| = 6$, so in particular $\Gamma(O_\beta) = \Gamma(2)$. □

Part b) of the above theorem indicates a relationship between weakly isomorphic dessins and affinely equivalent M-Origamis. Let us understand this a bit more conceptually:

*Note that this and the following lemma logically depend only on the calculations in the proof of Theorem 3.

Proposition 4.17. The group W from Definition and Remark 2.7 acts on the set of Origamis whose Veech group contains $\Gamma(2)$ via the group isomorphism

$$\phi : W \to \mathrm{SL}_2(\mathbb{Z})/\Gamma(2), \begin{cases} s \mapsto \overline{S} \\ t \mapsto \overline{T} \end{cases} .$$

Furthermore, the map $M : \beta \mapsto O_\beta$, sending a dessin to the corresponding M-Origami, is W-equivariant.

Proof. By the proof of Proposition 3.7, the action of $\mathrm{SL}_2(\mathbb{Z})$ on the set of Origamis whose Veech group contains $\Gamma(2)$ factors through $\Gamma(2)$. So any group homomorphism $G \to \mathrm{SL}_2(\mathbb{Z})/\Gamma(2)$ defines an action of G on this set. To see that the map M is equivariant with respect to the actions of W on dessins and M-Origamis, respectively, amounts to comparing the tables in Definition and Remark 2.7 and Corollary 4.16. $\square$

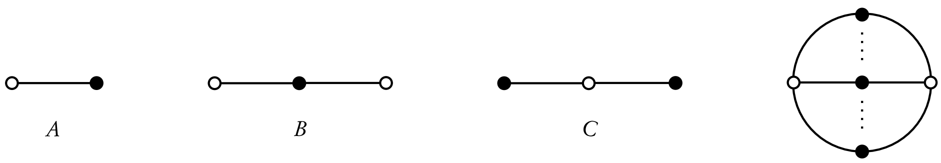
6. Cylinder decomposition

By the results of the above section, for an M-Origami O_β we find that $\mathbb{H}/\Gamma(2) \cong \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$ covers its Origami curve $C(O_\beta)$ which therefore has at most three cusps. By Proposition 3.16 a) this means that O_β has at most three non-equivalent Strebel directions, namely $\binom{1}{0}$, $\binom{0}{1}$ and $\binom{1}{1}$. For each of these, we will calculate the cylinder decomposition. Before, let us prove the following lemma which will help us assert a peculiar condition appearing in the calculation of the decomposition:

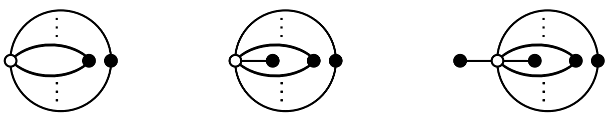
Lemma 4.18. Let β be a Belyi morphism, and let p_x, p_y, p_z be the monodromy around 0, 1, ∞ as usual. If, for one cycle of p_y that we denote w.l.o.g. by $(1 \ldots k)$, $k \leq \deg(\beta)$, we have

$$\forall i = 1, \ldots, k : \; p_x^2(i) = p_z^2(i) = i,$$

then the dessin representing β is in the following list (where D_n, E_n, F_n and G_n are dessins with n black vertices—so we can even write $A = F_1$, $B = D_1$, $C = G_2$):



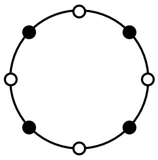
A B C D_n



E_n F_n G_n

In particular under these conditions β is defined over $\mathbb{Q}$.

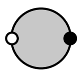
Note: If we impose the cycle condition on p_x instead of p_y, we get of course the dessins with black and white vertices interchanged, and if we impose it on p_z, it is obvious that the series D_n becomes the *daisy chain* series D'_n (the below image showing D'_4 as an example), and the series E_n, F_n and G_n become the dessins of the Chebyshev-polynomials (see Example 2.4). The fact that they are defined over $\mathbb{Q}$ does not change, of course.



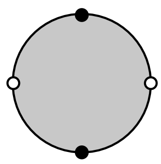
D'_4

Proof. The idea of the proof is to take cells of length 1 and 2 (this is the condition $p_z^2(i) = i$), bounded by edges, and glue them (preserving the orientation) around a white vertex w until the cycle around this vertex is finished, i.e. until there are no more un-glued edges ending in w. The building blocks for this procedure are shown as 1a and 2a in the figure below. Note that it is not a priori clear that we end up with a closed surface in this process, but the proof will show that this is inevitable.

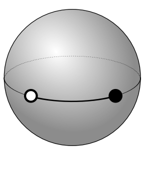
As a first step we consider in which ways we can identify edges or vertices within a cell of length 1 or 2, meeting the requirements that no black vertex shall have a valence > 2 (this is the condition $p_x^2(i) = i$) and that the gluing respects the colouring of the vertices. The reader is invited to check that the figure below lists all the possible ways of doing this.



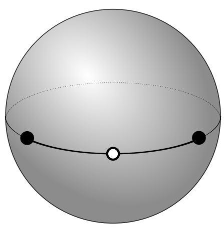
1a



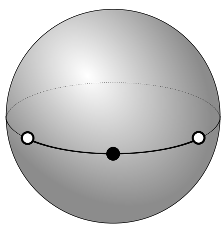
2a



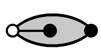
1b



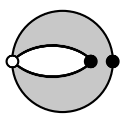
2b



2c



2d



2e

Now we want to see in which ways we can glue these seven building blocks around the white vertex w. In the cases 1b, 2b and 2c we already have a closed surface, so we cannot attach any more cells, and end up with the dessins A, B and C. If we start by attaching the cell 2a to w, we can only attach other cells of that type to w. Note that we cannot attach anything else to the other white vertex, as this would force us to attach a cell of length > 2 to w in order to finish the cycle. So, in this case, we get the dessins of series D. Now consider the cell 2e. It has 4 bounding edges. To each of its two sides, we can attach another copy of 2e, leaving the number of bounding edges of the resulting object invariant, or a copy of 1a or 2d, both choices diminishing the number of boundary edges by 2. Having a chain of 2e's, we cannot glue their two boundary components, as this would not yield a topological surface but rather a sphere with two points identified. So to close the cycle around the white vertex w, we have to attach on either side either 1a or 2d. This way we get the series E, F and G. If, on the other hand, we start with a cell of types 1a or 2d, we note that we can only attach cells of type 1a, 2d or 2e, so we get nothing new.

It might be surprising at first that constructing one cycle of p_y with the given properties already determines the whole dessin. Again, the reader is invited to list the possibilities we seem to have forgotten, and check that they are in fact already in our list.

Now, clearly all of the dessins of type $A, \dots, E$ are determined by their cycle structure and hence defined over $\mathbb{Q}$. $\square$

After this lemma, we are able to calculate the decomposition of an M-Origami into maximal cylinders now. For a maximal cylinder of width w and height h, we write that it is of type (w, h).

Theorem 5. Let O_β be an M-Origami associated to a dessin β which is given by a pair of permutations $(p_x,\ p_y)$. Then we have:

a) If $(p_x,\ p_y)$ does not define one of the dessins listed in Lemma 4.18, then, in the Strebel direction $\binom{1}{0}$, O_β has:
- for each fixed point of p_y one maximal cylinder of type $(2, 2)$,
- for each cycle of length 2 of p_y one maximal cylinder of type $(4, 2)$,
- for each cycle of length $l > 2$ of p_y two maximal cylinders of type $(2l, 1)$.

b) In particular, we have in this case:

$$\#\,\text{max. horizontal cylinders} = 2 \cdot \#\,\text{cycles in } p_y - \#\,\text{fixed points of } p_y^2.$$

c) We get the maximal cylinders of O_β in the Strebel directions $\binom{0}{1}$ and $\binom{1}{1}$ if we replace in a) the pair $(p_x,\ p_y)$ by $(p_y,\ p_x)$ and $(p_y,\ p_z)$, respectively.

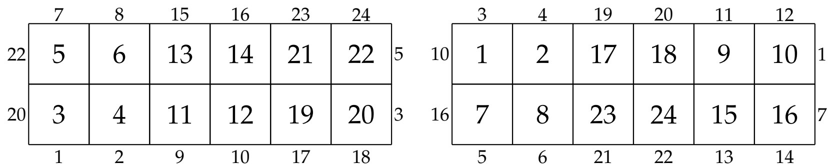
Before we write down the proof, we list, for the sake for completeness, the maximal horizontal cylinders of the M-Origamis associated to the dessins of Lemma 4.18:

- The M-Origami coming from D_n has two maximal horizontal cylinders of type $(2n, 2)$ for $n \geq 3$ (and else one of type $(2n, 4)$).
- The M-Origami coming from E_n has one maximal horizontal cylinder of type $(4n, 2)$.
- The M-Origami coming from F_n has one maximal horizontal cylinder of type $(4n - 2, 2)$.

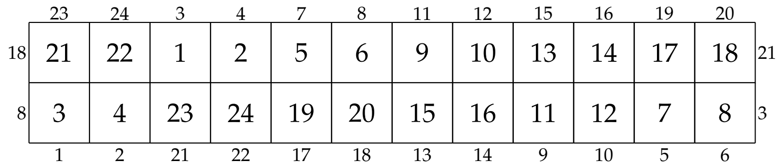
- The M-Origami coming from G_n has one maximal horizontal cylinder of type $(4n-4, 2)$.

Let us draw the M-Origamis associated to D_3, E_3, F_3 and G_3 here:

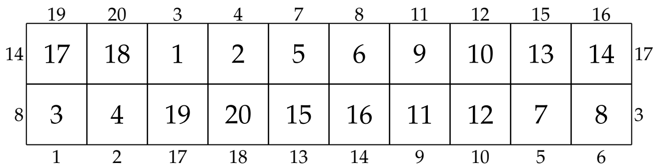
O_{D_3} :



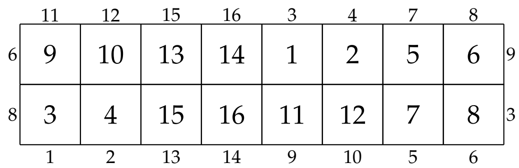
O_{E_3} :



O_{F_3} :



O_{G_3} :



Proof (of Theorem 5).

a) First we note that for each cycle c of p_y of length l, we get two horizontal cylinders of length $2l$, one consisting of squares labelled $(1, j)$ and $(2, j)$, and one consisting of squares labelled $(3, j)$ and $(4, j)$—the latter rather belonging to the corresponding inverse cycle in p_y^{-1}. They are maximal iff there lie ramification points on both their boundaries (except in the trivial case where $g(O_\beta) = 1$). Since the map π from the construction of an M-Origami is always unramified over λ (see Theorem 3), there are ramification points on the boundary in the middle of such a pair of cylinders iff the corresponding cycle is not self inverse, i.e. $l > 2$. On the other two boundary components, there are ramification points iff not for every entry j appearing in c, we have $p_x^2(j) = p_z^2(j) = 1$. This is due to Lemma 4.6, and it is exactly the condition in Lemma 4.18.

b) is a direct consequence of a).

c) By Proposition 3.16 d) we know that the cylinders in vertical and diagonal direction are the horizontal ones of $S^{-1} \cdot O_\beta$ and $(TS)^{-1} \cdot O_\beta$. As we know that $S \equiv S^{-1}$ and $(TS)^{-1} \equiv ST \pmod{\Gamma(2)}$ we read off the claim from the table in Corollary 4.16.

7. Möller's theorem and variations

We are now able to reprove Möller's main result in **[Möl05]** in an almost purely topological way. This will allow us to construct explicit examples of Origamis such that $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ acts non-trivially on the corresponding Origami curves, which was not obviously possible in the original setting.

Let us first state the following theorem, reformulating Theorem 5.4 from **[Möl05]**:

THEOREM I.
a) Let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ be an element of the absolute Galois group, and β be a Belyi morphism corresponding to a clean tree, i.e. a dessin of genus 0, totally ramified over ∞, such that all the preimages of 1 are ramification points of order precisely 2, and assume that β is not fixed by σ. Then we also have for the Origami curve $C(O_\beta)$ of the associated M-Origami: $C(O_\beta) \neq C(O_\beta)^\sigma$ (as subvarieties of $M_{g,[n]}$).
b) In particular, the action of $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ on the set of all Origami curves is faithful.

We will gather some lemmas which will enable us to reprove the above theorem within the scope of this work and to prove similar statements to part a) for other classes of dessins.

Note that when defining M-Origamis in Section 1 of this chapter, we did not impose any conditions on the elliptic curve E. We will now use the model $E = \mathbb{C} / \mathbb{Z}^2$ again, whose j-invariant is 1728. Hence, E is defined over $\mathbb{Q}$, so in particular $E^\sigma \cong E$ for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$. Let us now begin with the following simple

LEMMA 4.19. Let $\beta \ncong \beta'$ be two dessins, defined by (p_x, p_y) and (p'_x, p'_y), respectively. Then, for their pullbacks π, π' as in Definition 4.1 we also have $\pi \ncong \pi'$.

PROOF. Let $\deg(\beta) = \deg(\beta') = d$ (if their degrees differ, the statement is trivially true). Now assume $\pi \cong \pi'$. This would imply the existence of a permutation $\alpha \in S_d$ such that $c_\alpha \circ m_\pi = m_{\pi'}$, where c_α is the conjugation by α. By the proof of Theorem 3, we have $m_\pi(d) = p_x$, $m_\pi(a) = p_y$, so in particular $(c_\alpha(p_x), c_\alpha(p_y)) = (p'_x, p'_y)$, which contradicts the assertion that $\beta \ncong \beta'$. □

Next, let us check what happens after postcomposing [2], the multiplication by 2 on the elliptic curve. But we first need the following

LEMMA 4.20. Let $\pi : X \to E, \pi' : X' \to E$ be two coverings. If we have $[2] \circ \pi \cong [2] \circ \pi'$, then there is a deck transformation $\varphi \in \mathrm{Deck}([2])$ such that $\varphi \circ \pi \cong \pi'$.

PROOF. Clearly, E is Hausdorff, and [2] is a normal covering, so we can apply Lemma 1.5. □

PROPOSITION 4.21. Assume we have a dessin β given by (p_x, p_y), and a Galois automorphism $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ such that $\beta \ncong \beta^\sigma$. If furthermore the 4-tuple $(p_x^2, p_y^2, p_z^2, 1) \in (S_d)^4$ contains one permutation with cycle structure distinct from the others, then we have $O_\beta \ncong (O_\beta)^\sigma$.

PROOF. First of all, what is $(O_\beta)^\sigma$? We chose E to be defined over $\mathbb{Q}$, so $E \cong E^\sigma$, and [2] is also defined over $\mathbb{Q}$, so $[2] = [2]^\sigma$, and so $([2] \circ \pi)^\sigma = [2] \circ \pi^\sigma$ which means by definition that $(O_\beta)^\sigma = O_{\beta^\sigma}$.

Assume now $O_\beta \cong O_{\beta^\sigma}$. So by the above lemma, there is a deck transformation $\varphi \in \mathrm{Deck}([2])$ such that $\varphi \circ \pi \cong \pi^\sigma$. The deck transformation group here acts by translations, so in particular without fixed points. Note that by Lemma 4.6, the tuple $(p_x^2, p_y^2, p_z^2, 1)$ describes the ramification of π around the Weierstrass points, and the ramification behaviour of π^σ is the same. So, imposing the condition that one entry in this tuple shall have a cycle structure distinct from the others, it follows that $\varphi = \mathrm{id}$. So we have even $\pi \cong \pi^\sigma$, and so by Lemma 4.19 $\beta \cong \beta^\sigma$, which contradicts the assumption. □

Now, we have all the tools together to prove Theorem I.

Proof (of Theorem I). The second claim follows from the first, as the action of $\mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ is faithful on trees, and it stays faithful if we restrict to clean ones. We have seen that in Theorem G.

So, choose a nontrivial Galois automorphism σ and a clean tree β of degree d defined by (p_x, p_y) such that $\beta \not\cong \beta^\sigma$.

First we check the condition of Proposition 4.21 by showing that the cycle structure of p_z^2 is distinct from the others. p_z^2 consists of two cycles because purity implies even parity of d. As $\beta \not\cong \beta^\sigma$, surely $d > 2$, so $p_z^2 \neq 1$, and so it is distinct from $p_y^2 = 1$. Because of the purity, the dessin β has $\frac{d}{2}$ white vertices, and so $\frac{d}{2} + 1$ black vertices, which is a lower bound for the number of cycles in p_x^2. Again, from $d > 2$ we conclude that p_x^2 must consist of at least 3 cycles and therefore cannot be conjugate to p_z^2.

We claim now that O_β and O_{β^σ} are not affinely equivalent. By Theorem 4 and Corollary 4.16 this amounts to checking that β and β^σ are not weakly isomorphic. As we will see later in an example, this cannot be assumed in general, but in this case p_x, p_y, p_z consist of $\frac{d}{2} + 1$ cycles, $\frac{d}{2}$ cycles and 1 cycle, respectively, so any weak isomorphism would actually have to be an isomorphism, which we excluded.

So, by Proposition 3.13 , $C(O_\beta) \neq C(O_{\beta^\sigma})$ as embedded curves in the moduli space. But as we saw in the proof of Proposition 4.21, the latter is equal to $C((O_\beta)^\sigma)$, which is in turn equal to $(C(O_\beta))^\sigma$ by Proposition 3.17 b). Altogether we found, for an arbitrary $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$, an Origami O such that

$$C(O) \neq (C(O))^\sigma.$$

So indeed, the absolute Galois group acts faithfully on the set of Origami curves. □

Inspecting our results that we used to prove Möller's theorem more closely, we see that we can actually use them to give a bigger class of dessins β for which we know that from $\beta^\sigma \not\cong \beta$ follows $C(O_\beta) \neq (C(O_\beta))^\sigma$:

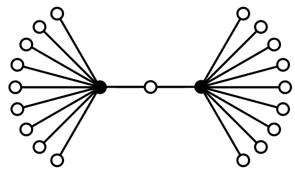
Theorem 6. Let β be a dessin such that $\beta^\sigma \not\cong \beta$ for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$.

a) If β is a tree or a filthy dessin, then $O_\beta \not\cong O_\beta^\sigma$.
b) If furthermore $M(\beta) = M_\beta$ (in the sense of Definitions 2.15 and 2.16), then we have $C(O_\beta) \neq (C(O_\beta))^\sigma$.

Proof. For part a), remember that a dessin with monodromy given by (p_x, p_y) is said to be filthy if it is not weakly isomorphic to a pre-clean one, i.e. $1 \notin \{p_x^2, p_y^2, p_z^2\}$. So the condition of Proposition 4.21 is satisfied for the permutation 1.

The case of β being a tree is a little bit more tricky. We want to show that the cycle structure of p_z^2 is unique. β is totally ramified over ∞, so p_z^2 has one cycle if $d = \deg(\beta)$ is odd and two if it is even. First assume it to be odd. If p_x^2 (and so p_x) also had only one cycle, then we had $\beta : z \mapsto z^d$ which contradicts the assumption $\beta \ncong \beta^\sigma$. We repeat the argument for p_y. Clearly $d > 1$, so $p_z^2 \neq 1$.

Assume now $d \in 2\,\mathbb{Z}$, so p_z^2 has two cycles of length $\frac{d}{2}$. Assume p_x^2 to be conjugate to it, then p_x has either one cycle of length d or two of length $\frac{d}{2}$. We already discussed the first case, and in the latter one, there is, for every d, only one tree with that property:



So in particular it is defined over $\mathbb{Q}$, which contradicts the hypothesis $\beta \ncong \beta^\sigma$. Again we repeat the argument for p_y, and as surely $d > 2$ we have $p_z^2 \neq 1$.

For part b), we have to check that O_β and O_β^σ are not affinely equivalent. Again by Theorem 4 b) this is equivalent to β and β^σ not being weakly isomorphic. But indeed, the conditon $M(\beta) = M_\beta$ implies that whenever β and β^σ are weakly isomorphic for some $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$, we have actually $\beta \cong \beta^\sigma$. □

Note that Example 5.3 shows that the condition $M(\beta) = M_\beta$ is actually necessary: the Galois orbit of dessins considered in that example contains a pair of filthy trees which are weakly isomorphic. By the above arguments, the corresponding M-Origamis are distinct but affinely equivalent. □

From Corollary 4.11 we know that in order to act faithfully on the Teichmüller curves of M-Origamis, the absolute Galois group must act non-trivially on the Teichmüller curves of M-Origamis of genus g for infinitely many g. The question is if our construction can provide examples for Teichmüller curves not defined over $\mathbb{Q}$ for many low genera g. The following proposition gives an answer:

Proposition 4.22. For every $g \geq 4$, there is an Origami of genus g whose Teichmüller curve is not fixed by the complex conjugation.

Proof. See Example 5.4. □

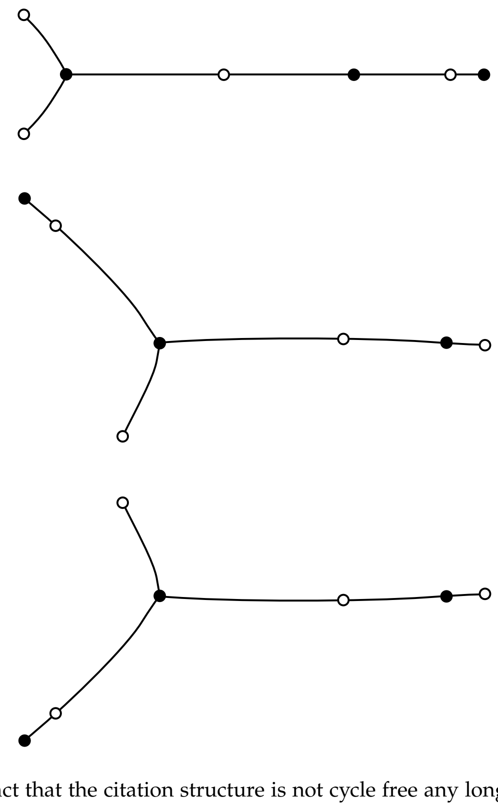
CHAPTER 5

Examples

1. M-Origamis from trees

We will now give some examples to enlighten the Galois action on Origami curves. In order to do this, we will, of course, use our construction of M-Origamis, so first of all, we need explicit examples of Galois orbits of dessins d'enfants. While the arguments for the existence of a dessin in genus 0 and 1 that is not fixed by a certain Galois automorphism σ given in [**Sch94**] are in principle constructive, they are not really appropriate for constructing examples as they make use of Belyi's algorithm, which produces ludicrously high degrees even in simple cases. A first source for examples is Bétréma's and Zvonkin's catalogue [**BZ92**] where they give equations for all trees up to 7 edges and classify the Galois orbits.

Example 5.1. In the catalogue, we find that all trees up to 5 edges are defined over $\mathbb{Q}$, so they do not lead to better understanding of the Galois action. There are two nontrivial Galois orbits of trees of degree 6, one of length 2, and one of length 3. The first one is studied in detail in [**Kre10a**, 5.4]*, so we discuss the second one here. Let us draw the dessins in that orbit:



*Be aware of the fact that the citation structure is not cycle free any longer!

As usual, the vertices depicted as filled dots are the preimages of 0, and the white dots are the preimages of 1. All three dessins have Belyi polynomials of the form

$$\beta(z) = z^3(z-1)^2(z-a),$$

where $a \in \overline{\mathbb{Q}}$ runs through the three roots of the following polynomial, which is irreducible over $\mathbb{Q}$:

$$25a^3 - 12a^2 - 24a - 16$$

For the sake of completeness, let us list them:

$$a_1 = \frac{1}{25}\left(4 + 18\sqrt[3]{2} + 6\sqrt[3]{4}\right),\ a_{2,3} = \frac{1}{25}\left(4 - 3\sqrt[3]{4}(1 \mp \sqrt{3}i) - 9\sqrt[3]{2}(1 \pm \sqrt{3}i)\right)$$

The real root corresponds to the first dessin which is mirror symmetric and hence defined over $\mathbb{R}$, and the two complex conjugate ones correspond to the lower two dessins which are mirror images of each other.

We choose a numbering on the edges and write down the monodromy of each of these dessins: $p_x = (1\,2\,3)(4\,5)$ for all three dessins, and from top to bottom $p_y^{(1)} = (3\,4)(5\,6), p_y^{(2)} = (1\,6)(3\,4), p_y^{(3)} = (2\,6)(3\,4)$.

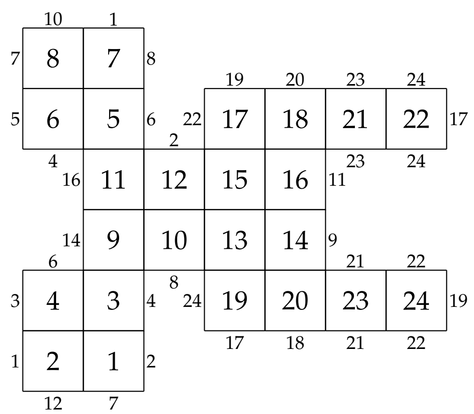
To calculate the monodromy of the corresponding M-Origamis, we use Theorem 3. We know that if the degree of a dessin β is d, the degree of the corresponding M-Origami O_β is $4d$. In the theorem, the permutations describing O_β are given as bijections on the set $\{1, 2, 3, 4\} \times \{1, \ldots, d\}$. We identify this set with $\{1, \ldots, 4d\}$ by the map $(i, j) \mapsto i + 4(j-1)$ in order to be able to write up the permutations in a usual manner. So, applying Theorem 3 yields

$$p_B = (1\,3\,9\,11\,5\,7)(2\,4\,6\,8\,10\,12)(13\,15\,17\,19)(14\,16\,18\,20)(21\,23)(22\,24)$$

for all three corresponding Origamis (note that p_B only depends on p_x, which can be chosen to be the same) and, again from top to bottom

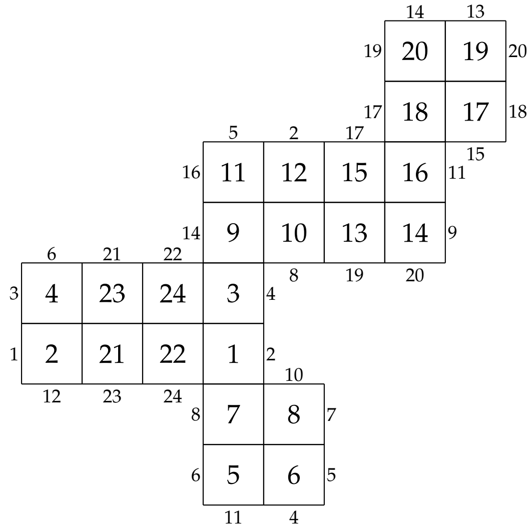
$$p_A^{(1)} = (1\,2)(3\,4)(5\,6)(7\,8)(9\,10\,13\,14)(11\,12\,15\,16)(17\,18\,21\,22)(19\,20\,23\,24)$$

which looks as follows:



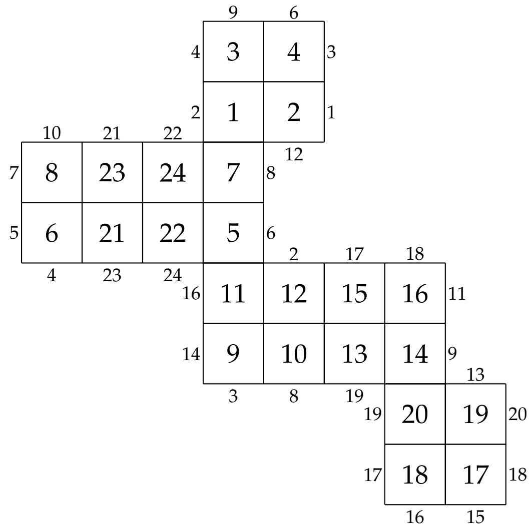
$$p_A^{(2)} = (1\,2\,21\,22)(3\,4\,23\,24)(5\,6)(7\,8)(9\,10\,13\,14)(11\,12\,15\,16)(17\,18)(19\,20)$$

which looks as follows:



$$p_A^{(3)} = (1\,2)(3\,4)(5\,6\,21\,22)(7\,8\,23\,24)(9\,10\,13\,14)(11\,12\,15\,16)(17\,18)(19\,20)$$

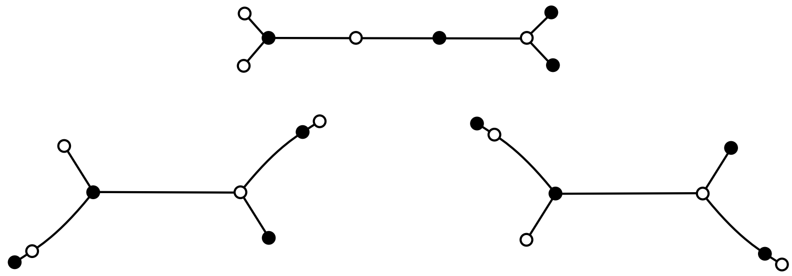
which looks as follows:



Of course also for the Origamis we have that the first one is mirror symmetric, and the other two are mirror images of each other. All three are hyperelliptic Origamis (due to Remark 4.2) of genus 4 with 18 punctures. It follows from Theorem 6 a) that they are pairwise distinct. Furthermore we check that no two of them are weakly isomorphic. So, by part b) of the mentioned theorem they lie on different Teichmüller curves. A direct calculation (e.g. by using [**WSKF$^+$11**]) shows that their veech groups are equal to $\Gamma(2)$.

Next, we look at trees for which both permutations p_x and p_y have cycle structures $(3\,2\,1\,1)$. So, trouble is brewing in the sense of Theorem 4, as some of the dessins could have weak automorphisms or be weakly isomorphic to each other. Let us check what is happening there. According to the catalogue, the nine trees with this cycle structure fall into two Galois orbits, one of length 3, the other one of length 6.

EXAMPLE 5.2. We first look at the orbit of length 3, consisting of the following dessins:



They are given by Belyi polynomials of the form

$$\beta(z) = z^3(z-a)^2\left(z^2 + \left(2a - \frac{7}{2}\right)z + \frac{8}{5}a^2 - \frac{28}{5}a + \frac{21}{5}\right),$$

where a runs through the three roots of the polynomial

$$24a^3 - 84a^2 + 98a - 35,$$

one of which is real and the other two of which are complex conjugate.
We number the edges in such a way that for all three we get $p_x = (1\,2\,3)(4\,5)$ and, in the ordering of the picture, we get

$$p_y^{(1)} = (3\,4)(5\,6\,7),\ p_y^{(2)} = (2\,7)(3\,6\,4),\ p_y^{(3)} = (1\,7)(3\,4\,6).$$

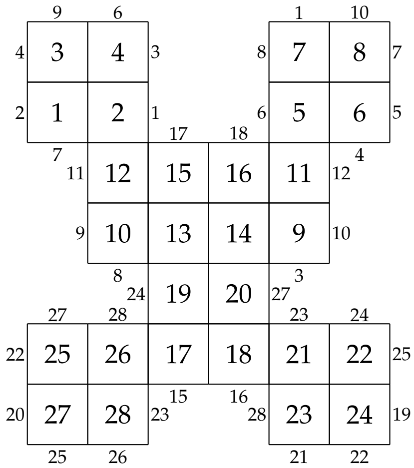
First, we write down the monodromy of the corresponding M-Origamis, using Theorem 3: For all of them, we can choose p_B to be

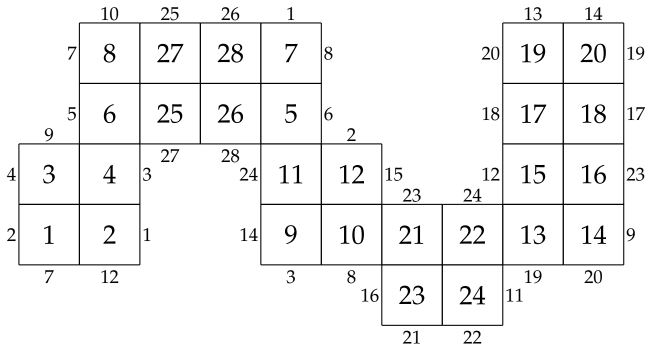
$$\begin{aligned} p_B = {} & (1\,3\,9\,11\,5\,7)(2\,4\,6\,8\,10\,12)(13\,15\,17\,19)(14\,16\,18\,20) \\ & (21\,23)(22\,24)(25\,27)(26\,28) \end{aligned}$$

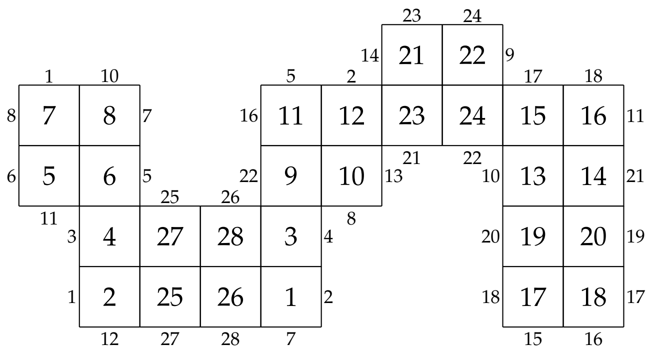
and further we calculate

$$\begin{aligned} p_A^{(1)} &= (1\,2)(3\,4)(5\,6)(7\,8)(9\,10\,13\,14)(11\,12\,15\,16) \\ &\quad (17\,18\,21\,22\,25\,26)(19\,20\,27\,28\,23\,24), \\ p_A^{(2)} &= (1\,2)(3\,4)(5\,6\,25\,26)(7\,8\,27\,28)(9\,10\,21\,22\,13\,14) \\ &\quad (11\,12\,15\,16\,23\,24)(17\,18)(19\,20), \\ p_A^{(3)} &= (1\,2\,25\,26)(3\,4\,27\,28)(5\,6)(7\,8)(9\,10\,13\,14\,21\,22) \\ &\quad (11\,12\,23\,24\,15\,16)(17\,18)(19\,20). \end{aligned}$$

Now let us draw the Origamis. Like in the previous example, we can do that in a way that exhibits the mirror symmetry of the first one (which shows that the Origami, and thus its curve, is defined over $\mathbb{R}$), and the fact that the other two are mirror images of each other, i.e. they are exchanged by the complex conjugation.

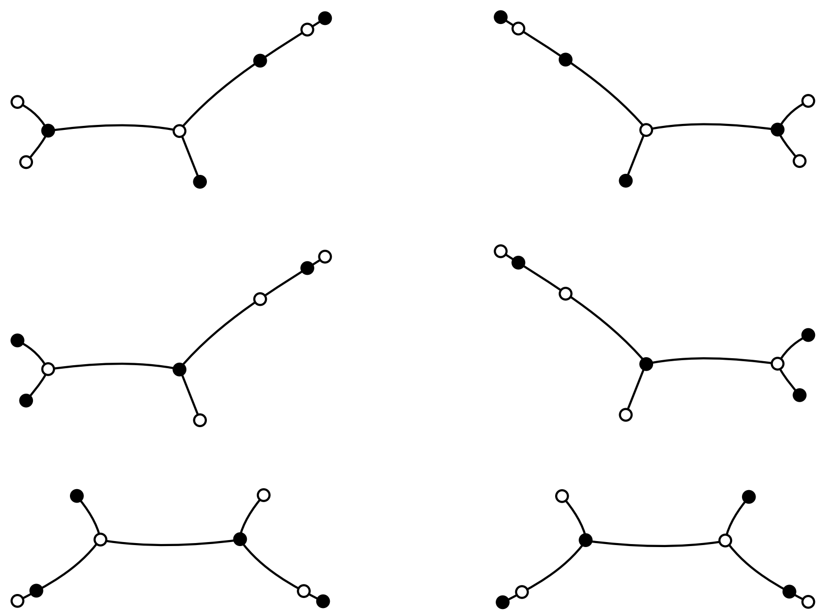






Using Proposition 4.7 and Remark 4.9, we see that all of these three M-Origamis of degree 28 have genus 6 and 18 punctures. The interesting fact is that the three dessins admit a weak automorphism lying over $z \mapsto 1 - z$, i.e. they stay the same after exchanging white and black vertices. By Corollary 4.16, this means that S is contained in each of their Veech groups. T is contained in neither of them, because by the same corollary this cannot happen for nontrivial trees, and so all three have the Veech group generated by $\Gamma(2)$ and S, so their Teichmüller curves have genus 0 with 2 cusps, and no two of these Teichmüller curves coincide.

Example 5.3. Now let us investigate the second Galois orbit with the same cycle structure. The six dessins are depicted below:



From the catalogue we learn that their Belyi polynomials are of the form

$$\beta(z) = z^3(z-a)^2\Bigg(z^2 + \Big(\frac{4}{3}a^5 - \frac{34}{15}a^4 - \frac{26}{15}a^3 + \frac{7}{5}a^2 + \frac{20}{3}a - \frac{28}{5}\Big)z - \frac{8}{15}a^5 - \frac{32}{75}a^4 + \frac{172}{75}a^3 + \frac{148}{75}a^2 - \frac{14}{5}a - \frac{287}{75}\Bigg),$$

where a runs through the six complex roots of the following polynomial:

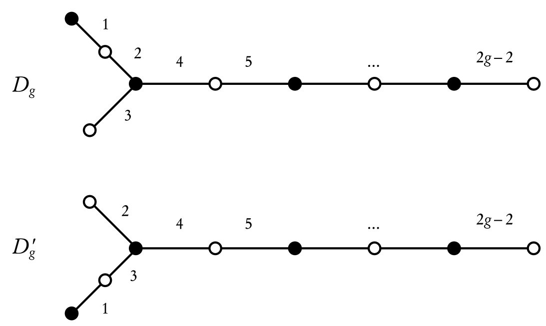
$$20a^6 - 84a^5 + 84a^4 + 56a^3 - 294a + 245$$

As they have the same cycle structure as the three in the Galois orbit discussed in the previous example, the resulting M-Origamis are of course combinatorially equivalent to them: They also are of degree 28 and genus 6, and they have 18 punctures. But something is different here: the weak automorphism group of each of these dessins is trivial. To see this, note that an element of the Group W stabilising a tree (that is not the dessin of $z \mapsto z^n$, or $z \mapsto (1-z)^n$) has to fix ∞, so it is either the identity or the element s which acts by exchanging the white and black vertices. Indeed, none of these six dessins keep fixed under s (but interestingly the whole orbit does). As these dessins are filthy, we conclude by Theorem 4 c) that all have Veech group $\Gamma(2)$. Furthermore, the two columns of the picture are complex conjugates to each other, and we get the first row from the second by applying s—which is also the case for the two dessins from the last row. So what is the situation here? We have three Teichmüller curves, the first two containing the Origamis associated to the two upper left and upper right dessins, respectively. They are interchanged by the complex conjugation. The third curve contains the two other Origamis, associated to the bottom row, so this Teichmüller curve is stabilised under the action of the complex conjugation, and hence defined over $\mathbb{R}$.

2. Curves for (almost) every genus

We construct a series of M-Origamis $(O_g)_{g\geq 4}$ here, such that O_g has genus g, and the corresponding Teichmüller curve is not fixed by the complex conjugation.

EXAMPLE 5.4. Consider, for $g \geq 4$, the following pair of dessins D_g and D'_g:



Clearly, they are interchanged by the complex conjugation. We want to show that it acts non-trivially, i.e. $D_g \ncong D'_g$. With the numbering of edges indicated in the picture, we get for both dessins the following permutation as the monodromy around 0:

$$p_x = (2\,3\,4)(5\,6)(7\,8)\cdots(2g-3\;2g-2)$$

For the monodromy around 1, we get

$$p_y = (1\,2)(4\,5)(6\,7)\cdots(2g-4\;2g-3) \text{ and } p'_y = (1\,3)(4\,5)(6\,7)\cdots(2g-4\;2g-3),$$

respectively. Let us assume $(p_x,\,p_y) \cong (p_x,\,p'_y)$, so we claim the existence of some $c \in S_{2g-2}$ that commutes with p_x, such that $c^{-1}p_y c = p'_y$. In order to commute with p_x, we must have $c(1) = 1$, and the cycle decomposition must contain a power of $(2\,3\,4)$. So from $c^{-1}p_y c(1) = p'_y(1) = 3$ we get $c(3) = 2$, so $(4\,3\,2)$ must be a cycle of c. As we must have $c^{-1}p_y c(2) = p'_y(2) = 2$ we get $c(2) = 5$, which is a contradiction. $\square$

Note that D_4 and D'_4 appear in Example 5.1, where we calculated the genus of the resulting M-Origamis to be 4. In increasing g by one, we increase the degree by 2 and (in the sense of Proposition 4.7) g_0 and g_1 by 1, respectively. So by that proposition, we get that the genus of the associated M-Origamis O_g and O'_g is g. By Proposition 4.21, they are distinct, and as p_x is distinguished by its 3-cycle we find that D_g and D'_g are not weakly isomorphic. We conclude that $C(O_g) \neq C(O'_g)$, and we remember that $C(O_g) = C(O'_g)^{\Xi}$, where $\Xi \in \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ is the complex conjugation.

3. An orbit of genus 1 dessins

Theorem 6 gives us conditions under which Galois orbits of dessins in genus greater than 0 result in Galois orbits of Origami curves of the same length. We give an example in genus 1.

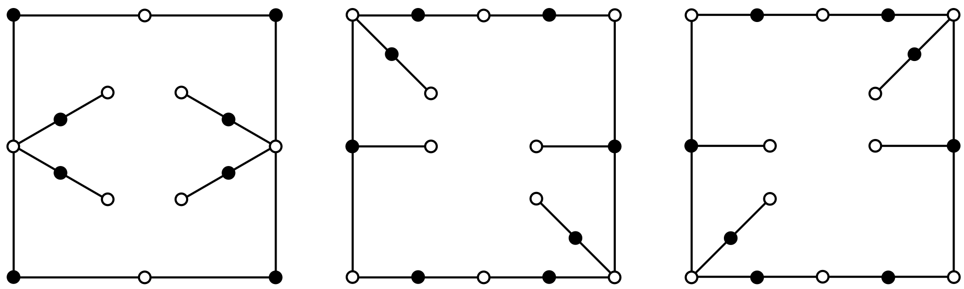
EXAMPLE 5.5. Consider, for $k \in \{0, 1, 2\}$, the elliptic curve X_k, given by the following affine equation:

$$y^2 = x(x-1)\left(x - \frac{e^{\frac{2k\pi i}{3}}}{\sqrt[3]{2}}\right).$$

On each of them, the morphism

$$\beta_k : (x, y) \mapsto 4x^3(1 - x^3)$$

is a Belyi morphism of degree 12. Let us draw the corresponding dessins, which appear as Beispiel 5 and Beispiel 5′ in **[Wol01]**:



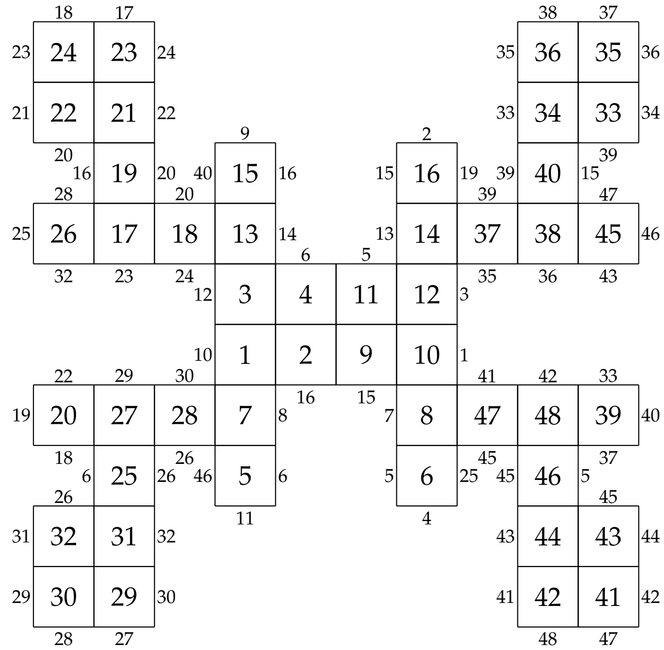
For all three of them, the top and bottom edges and the left and right edges shall be identified in an orientation preserving way, in order to obtain dessins on genus 1 surfaces.

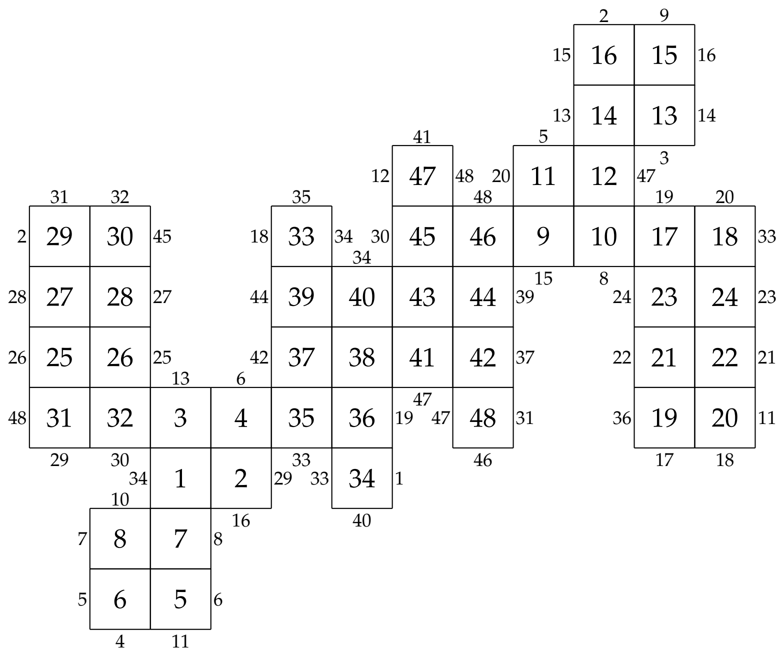
We choose a numbering on their edges in such a way that we find for their defining permutations $p_x^{(k)} = (1\,2\,3\,4)(5\,6)(7\,8)(9\,10)(11\,12)$ for $k = 0, \ldots, 2$ and

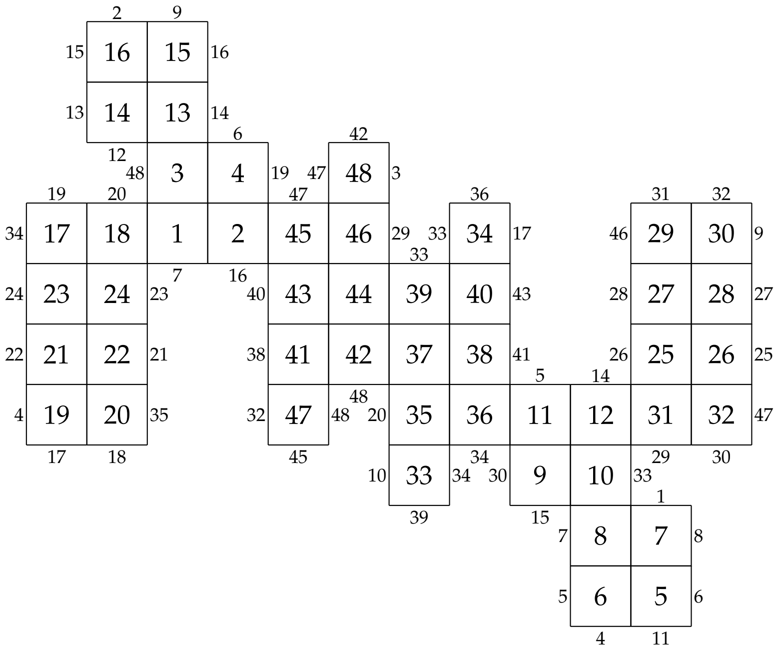
$$p_y^{(0)} = (1\,3)(2\,7\,5\,4\,10\,12),\ p_y^{(1)} = (1\,8\,12\,3\,5\,9)(10\,11),\ p_y^{(2)} = (1\,12\,8\,3\,9\,5)(10\,11).$$

Clearly these dessins are filthy, so by Theorem 6 a) we get three distinct Origamis O_0, O_1 and O_2. Also, for each of these dessins, the cycle structures of $p_x^{(k)}$, $p_y^{(k)}$ and $p_z^{(k)}$ are pairwise distinct, which has two consequences: First, for each of them, the group of weak automorphisms W_{β_k} is trivial, and since they are filthy, this implies that the Origamis have Veech group $\Gamma(2)$. Secondly, any weak isomorphism between them would have to be an isomorphism, so for each of them we have $M_{\beta_k} = M(\beta_k)$, so by part b) of the mentioned theorem we get three distinct curves. Let us omit writing down permutations defining the three Origamis, and rather draw pictures of them. They are each of degree 48, of genus 9, and they have 32 punctures.

Note that we draw them in a way that reveals that O_0 is defined over $\mathbb{R}$, O_1 and O_2 are exchanged by the complex conjugation, and that for each of them we have $-I \in \Gamma(O_k)$.







4. M-Origamis with Veech group $\mathrm{SL}_2(\mathbb{Z})$

The following series of examples arose in a discussion with Stefan Kühnlein.

Example 5.6. Here, we construct an infinite series of M-Origamis with Veech group $\mathrm{SL}_2(\mathbb{Z})$. It is quite noteworthy that only the two simplest Origamis in this series are characteristic. An Origami $O = (f : X^* \to E^*)$ is called *characteristic* if $f_*(\pi_1(X^*)) \leq \pi_1(E^*) \cong F_2$ is a characteristic subgroup. For a detailed account of these Origamis see **[Her06]**.

If a dessin β is a fixed point under the action of the group W of Möbius transformations fixing the set $\{0, 1, \infty\}$, i.e. $W_\beta = W$, then we know, according to Theorem 4 b), that the associated M-Origami has Veech group $\mathrm{SL}_2(\mathbb{Z})$. It might seem at first that there are no such dessins except $\beta = \mathrm{id}_{\mathbb{P}^1_{\mathbb{C}}}$, but indeed we will construct, for every $n \geq 1$, a dessin K_n of degree n^2 with $W_{K_n} = W$. First, there is no other choice than defining $K_1 = \mathrm{id}_{\mathbb{P}^1_{\mathbb{C}}}$.

Now let $n \geq 2$. Consider on the set $(\mathbb{Z}/n\,\mathbb{Z})^2$ the following two maps:

$$p_x : (k, l) \mapsto (k+1, l), \;\; p_y : (k, l) \mapsto (k, l+1).$$

They are clearly bijective, so we can regard them as elements of S_{n^2}. Note that they commute. Also, as $p_x^k p_y^l(0, 0) = (k, l)$, they generate a transitive subgroup of S_{n^2} and so (p_x, p_y) defines a dessin K_n of degree n^2. We calculate its monodromy around ∞ as $p_z = p_x^{-1} p_y^{-1} : (k, l) \mapsto (k-1, l-1)$ and define

$$c : (k,l) \mapsto (l, k), \;\; d : (k, l) \mapsto (-k, l-k).$$

As $c^2 = d^2 = \mathrm{id}$, both c and d are also bijective and thus elements of S_{n^2}. Furthermore we easily verify that

$$c^{-1}p_x c = p_y,\ c^{-1}p_y c = p_x,\ d^{-1}p_x d = p_z,\ d^{-1}p_y d = p_y,$$

so we find that $s \cdot K_n \cong t \cdot K_n \cong K_n$ and thus by Definition and Remark 2.7, $W_{K_n} = W$. Before we go on, we calculate the genus of K_n. The permutation p_x consists of n cycles (of length n), and so do p_y and p_z. As its degree is n^2, by the Euler formula we get $2 - 2g(K_n) = 2n - n^2 + n$ and thus

$$g(K_n) = \frac{n^2 - 3n + 2}{2}.$$

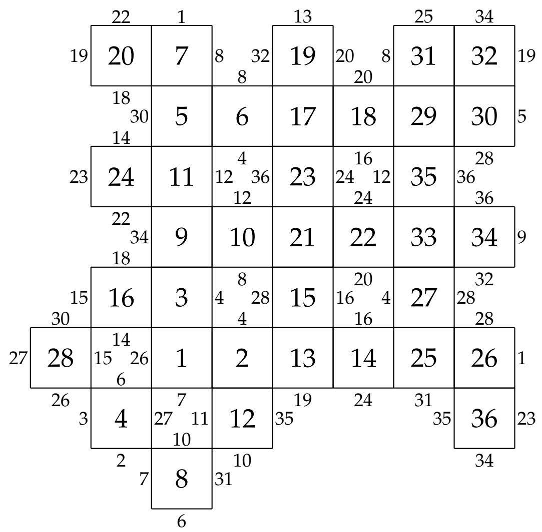
Now, look at the associated M-Origami O_{K_n} of degree $4n^2$. By the considerations above, we know that it has Veech group $\mathrm{SL}_2(\mathbb{Z})$. But since for $n \geq 3$, we have $1 \notin \{p_x^2, p_y^2, p_z^2\}$, the group of translations cannot act transitively on the squares of O_{K_n} which therefore is not a normal and specifically not a characteristic Origami. According to a remark in **[Her06]**, it seems as if not many examples are known for non-characteristic Origamis with full Veech group $\mathrm{SL}_2(\mathbb{Z})$. Let us close the example by calculating the genus of O_{K_n}. We use the formula from Proposition 4.7 and therefore we have to count the cycles of even length in p_x, p_y and p_z. We have

$$g_0 = g_1 = g_\infty = \begin{cases} n, & n \in 2\,\mathbb{Z} \\ 0, & n \in 2\,\mathbb{Z}+1 \end{cases}$$

and therefore

$$g(O_{K_n}) = g(K_n) + n^2 - \frac{1}{2}(g_0 + g_1 + g_\infty) = \begin{cases} \frac{3n^2-6n+2}{2}, & n \in 2\,\mathbb{Z} \\ \frac{3n^2-3n+2}{2}, & n \in 2\,\mathbb{Z}+1 \end{cases}.$$

The Origami O_{K_3} has degree 36 and genus 10. We draw a picture of it that almost reveals its mirror symmetry, i.e. the fact that it is defined over $\mathbb{R}$. It is not known to the author if this Origami can be drawn as a connected figure, or in a way revealing some more of its symmetries[†].



[†]We could actually make the picture mirror symmetric to a diagonal axis by cutting square 24 diagonally in half, and gluing one half to the bottom edge of square 14.

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