Interpolatory Weighted-$\mathcal{H}_2$ Model Reduction

B. Anić

C. A. Beattie

S. Gugercin

A. C. Antoulas

Preprint Nr. 12/01

INSTITUT FÜR WISSENSCHAFTLICHES RECHNEN
UND MATHEMATISCHE MODELLBILDUNG

Anschriften der Verfasser:

M. Sc. Branimir Anić
Institut für Angewandte und Numerische Mathematik
Karlsruher Institut für Technologie (KIT)
D-76128 Karlsruhe

Prof. Dr. Christopher A. Beattie
Department of Mathematics
Virginia Tech
522 McBryde Hall
Blacksburg, VA 24061-0123
USA

Prof. Dr. Serkan Gugercin
Department of Mathematics
Virginia Tech
442 McBryde Hall
Blacksburg, VA 24061-0123
USA

Prof. Dr. Athanasios C. Antoulas
Department of Electrical and Computer Engineering
Rice University, MS 380
6100 South Main Street
Houston, Texas 77005-1892
USA

Interpolatory Weighted-$\mathcal{H}_2$ Model Reduction

Branimir Anić[*], Christopher A. Beattie[†], Serkan Gugercin[†‡], and Athanasios C. Antoulas[§]

Preprint submitted to Automatica, 12 January 2012

Abstract

This paper introduces an interpolation framework for the weighted-$\mathcal{H}_2$ model reduction problem. We obtain a new representation of the weighted-$\mathcal{H}_2$ norm of SISO systems that provides new interpolatory first order necessary conditions for an optimal reduced-order model. The $\mathcal{H}_2$ norm representation also provides an error expression that motivates a new weighted-$\mathcal{H}_2$ model reduction algorithm. Several numerical examples illustrate the effectiveness of the proposed approach.

1 Introduction

Consider a single input/single output (SISO) linear dynamical system with a realization

$$\mathbf{E}\,\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \quad y(t) = \mathbf{c}^T\mathbf{x}(t) \quad \Longleftrightarrow \quad G(s) = \mathbf{c}^T(s\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}, \tag{1}$$

for $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. $\mathbf{x}(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, are respectively the *state*, *input*, and *output* of the system. $G(s)$ is the *transfer function.* Following common usage, the underlying system will also be denoted by G. For many examples of scientific and industrial value, the state-space dimension n is quite large, leading to untenable demands on computational resources. *Model reduction* attempts to address this by finding a reduced-order system of the form,

$$\mathbf{E}_r\,\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t) + \mathbf{b}_r u(t), \quad y_r(t) = \mathbf{c}_r^T\mathbf{x}_r(t) \quad \Longleftrightarrow \quad G_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r-\mathbf{A}_r)^{-1}\mathbf{b}_r \tag{2}$$

for $\mathbf{E}_r$, $\mathbf{A}_r \in \mathbb{R}^{r\times r}$ and $\mathbf{b}_r$, $\mathbf{c}_r \in \mathbb{R}^r$ with $r \ll n$ such that $y_r(t) \approx y(t)$ over a large class of inputs $u(t)$. G_r is a low order, yet high fidelity, approximation to G. We construct G_r via state-space projection: two matrices $\mathbf{V}_r,\ \mathbf{W}_r \in \mathbb{R}^{n\times r}$ are chosen ("reduction bases") to produce

$$\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r, \qquad \mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r, \qquad \mathbf{b}_r = \mathbf{W}_r^T\mathbf{b}, \qquad \text{and} \qquad \mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}_r \tag{3}$$

See [2, 3] for more information on model reduction of linear dynamical systems.

1.1 Model Reduction by Interpolation

The reduction bases, $\mathbf{V}_r$ and $\mathbf{W}_r$, used in (3) will be chosen to force interpolation: the reduced-order transfer function, $G_r(s)$, will interpolate $G(s)$ (possibly together with higher order derivatives) at selected interpolation points. This approach to rational interpolation has been considered in [21, 22, 6, 9, 8, 3] and depends on the following result. (A set $\Sigma \subset \mathbb{C}$ is *closed under conjugation* if $\sigma \in \Sigma$ implies $\bar{\sigma} \in \Sigma$.)

[*]Department of Mathematics, Karlsruhe Institute of Technology, Germany, e-mail: anic@kit.edu
[†]Department of Mathematics, Virginia Tech, e-mail: {beattie,gugercin}@vt.edu
[‡]Corresponding Author. The work of S. Gugercin has been supported in part by NSF Grant DMS-0645347.
[§]Electrical and Computer Engineering Dept., Rice University, e-mail: aca@rice.edu

Theorem 1. *Given two sets of interpolation points* $\{\sigma_k\}_{k=1}^r$ *and* $\{\zeta_k\}_{k=1}^r$*, that are each closed under conjugation, and a dynamical system* G *as in (1), consider matrices* $\mathbf{V}_r$ *and* $\mathbf{W}_r$ *such that*

$$\begin{aligned} Ran(\mathbf{V}_r) &= span\left\{(\sigma_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}, \quad \cdots, \quad (\sigma_r\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}\right\} \quad and \\ Ran(\mathbf{W}_r) &= span\left\{(\zeta_1\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c}, \quad \cdots, \quad (\zeta_r\mathbf{E}^T-\mathbf{A}^T)^{-1}\mathbf{c}\right\}. \end{aligned} \tag{4}$$

Then, $\mathbf{V}_r$ *and* $\mathbf{W}_r$ *can be chosen to be real;* $G_r(s) = \mathbf{c}_r^T(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{b}_r$ *defined by (2)-(3) is a real dynamical system that satisfies* $G(\sigma_k) = G_r(\sigma_k)$ *and* $G(\zeta_k) = G_r(\zeta_k)$ *for* $k = 1, \ldots, r$*; and, if* $\sigma_j = \zeta_j$ *for some* j*, then* $G'(\sigma_j) = G_r'(\sigma_j)$*, as well.*

Theorem 1 can be generalized to higher-order derivative interpolation as well, see [21, 22, 6, 9, 8, 3]. The subspaces of Theorem 1 are *rational Krylov subspaces* and so, interpolatory model reduction methods for SISO systems are sometimes referred to as *rational Krylov methods*.

1.2 Weighted Model Reduction

The $\mathcal{H}_\infty$ norm of a stable linear system associated with a transfer function, $G(s)$, is defined as $\|G\|_{\mathcal{H}_\infty} = \max_{\omega\in\mathbb{R}} |G(\imath\omega)|$. The $\mathcal{H}_2$ norm of G is defined as $\|G\|_{\mathcal{H}_2} := \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\imath\omega)|^2\right)^{1/2}$. The vector spaces of meromorphic functions that are analytic in the right halfplane, having either bounded $\mathcal{H}_\infty$ norm or bounded $\mathcal{H}_2$ norm will be denoted simply as $\mathcal{H}_\infty$ or $\mathcal{H}_2$, respectively. Let $W \in \mathcal{H}_\infty$ be given. The (W-)*weighted* $\mathcal{H}_2$ *norm* is defined as $\|G\|_{\mathcal{H}_2(W)} = \|G \cdot W\|_{\mathcal{H}_2}$.

We are interested in the problem of finding a reduced-order model G_r that minimizes a W-weighted $\mathcal{H}_2$ norm, i.e., that solves

$$\|G - G_r\|_{\mathcal{H}_2(W)} = \min_{\dim(\widetilde{G}_r) = r} \|G - \widetilde{G}_r\|_{\mathcal{H}_2(W)} \tag{5}$$

The introduction of the weight function, $W(s)$, allows one to penalize error in certain frequency ranges more heavily than in others.

An illustrative example: controller reduction Consider a linear dynamical system, P (the *plant*), with order n_P together with an associated stabilizing controller, G, having order n, that is connected to P in a feedback loop. Many control design methodologies, such as LQG and $\mathcal{H}_\infty$ methods, lead ultimately to controllers whose order is generically as high as the order of the plant, see [18, 23] and references therein. Thus, high-order plants will generally lead to high-order controllers. However, high-order controllers are usually undesirable in real-time applications because: (i) *Complex hardware:* A large-order controller typically requires complex hardware and a large investment in implementation; (ii) *Degraded accuracy*: Due to ill-conditioning in large-scale computations, it might not be possible to operate such a controller within the required accuracy margins; and, (iii) *Degraded computational speed:* The time needed to compute the output response for a complex controller might be too long, possibly longer than the system sampling time, yielding ineffective and potentially destructive feedback inputs. Thus, one may prefer to use a reduced order controller G_r with order $r \ll n$ to replace G.

Requiring G_r to be a good approximation to G is often not enough in terms of closed-loop performance; plant dynamics need to be taken into account during the reduction process. This may be achieved through frequency weighting: Given a stabilizing controller G, if G has the same number of unstable poles as G_r and if $\left\|[G - G_r]P[I + PG]^{-1}\right\|_{\mathcal{H}_\infty} < 1$, then G_r will also be a also stabilizing controller [1, 23]. Hence the controller reduction problem may be formulated as finding a reduced-order controller G_r that minimizes or reduces the weighted error $\|(G - G_r)W\|_{\mathcal{H}_\infty}$ with $W(s) := P(s)(I + P(s)G(s))^{-1}$; i.e., controller reduction becomes an application of weighted model reduction. This approach has been considered in

[18, 1, 15, 11, 7, 20, 13, 19, 17] and references therein, leading then to variants of frequency-weighted balanced truncation. Conversely, the methods in [12] and [16] are tailored instead towards minimizing a weighted-$\mathcal{H}_2$ error as in (5).

2 Weighted-$\mathcal{H}_2$ model reduction

The numerical methods proposed in [12] and [16] for approaching (5) require solving a sequence of large-scale Lyapunov or Riccati equations; they rapidly become computationally intractable as system order, n, increases. We will approach this problem within an interpolatory model reduction framework requiring only the solution of (generally sparse) linear systems and no need for dense matrix computations or solution of large-scale Lyapunov or Riccati equations. Interpolatory approaches can be effectively applied even when n reaches the tens of thousands.

2.1 A representation of the weighted-$\mathcal{H}_2$ norm

Given transfer functions G, $H \in \mathcal{H}_2$, and $W \in \mathcal{H}_\infty$, define the weighted $\mathcal{H}_2$ inner product as

$$\langle G,\ H\rangle_{\mathcal{H}_2(W)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G(\imath\omega)W(\imath\omega)}W(\imath\omega)H(\imath\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(-\imath\omega)W(-\imath\omega)W(\imath\omega)H(\imath\omega)\,d\omega,$$

so that $\|G\|_{\mathcal{H}_2(W)} = \Big(\langle G,\ G\rangle_{\mathcal{H}_2(W)}\Big)^{1/2}$. The following lemma gives a compact expression for the weighted $\mathcal{H}_2$ inner product based on the poles and residues of $G(s)$, $H(s)$ and $W(s)$.

Lemma 2. *Suppose G, $H \in \mathcal{H}_2$ have poles denoted respectively as $\{\lambda_1, \dots, \lambda_n\}$ and $\{\mu_1, \dots, \mu_m\}$, and suppose $W \in \mathcal{H}_\infty$ has poles denoted as $\{\gamma_1, \dots, \gamma_p\}$. Assume that $H(s)$ and $W(s)$ have no common poles, and the poles of $W(s)$ are simple. Then*

$$\langle G,\ H\rangle_{\mathcal{H}_2(W)} = \sum_{k=1}^{m} \mathsf{res}[G(-s)W(-s)W(s)H(s), \mu_k] + \sum_{i=1}^{p} G(-\gamma_i)W(-\gamma_i)H(\gamma_i)\cdot \mathsf{res}[W(s), \gamma_i].$$

- *If μ_k is a simple pole of $H(s)$ then*

$$\mathsf{res}[G(-s)W(-s)W(s)H(s), \mu_k] = G(-\mu_k)W(-\mu_k)W(\mu_k)\cdot \mathsf{res}[H(s), \mu_k].$$

- *If μ_k is a double pole of $H(s)$ then*

$$\begin{aligned}\mathsf{res}[G(-s)W(-s)W(s)H(s), \mu_k] =& G(-\mu_k)W(-\mu_k)W(\mu_k)\cdot \mathsf{res}[H(s), \mu_k] \\ &-h_{-2}(\mu_k)\cdot \frac{d}{ds}\,[G(s)W(s)W(-s)]\big|_{s=-\mu_k},\end{aligned}$$

  where $h_{-2}(\mu_k) = \lim_{s\to\mu_k}(s-\mu_k)^2 H(s)$.

Proof. $G(-s)W(-s)W(s)H(s)$ has poles at

$$\{-\lambda_1, \dots -\lambda_n\} \cup \{\pm\gamma_1, \dots, \pm\gamma_p\} \cup \{\mu_1, \dots, \mu_m\}.$$

For any $R > 0$, define a semicircular contour in the left halfplane:

$$\Gamma_R = \{z\,|\,z = \imath\omega \text{ with } \omega \in [-R, R]\} \cup \left\{z \,\middle|\, z = R\,e^{\imath\theta} \text{ with } \theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]\right\}.$$

For R large enough, the region bounded by Γ_R contains $\{\gamma_1,\dots,\gamma_p\}\cup\{\mu_1,\dots,\mu_m\}$, constituting all the poles of $W(s)H(s)$, and hence all the stable poles of $G(-s)W(-s)W(s)H(s)$. Then, the Residue Theorem yields

$$\begin{aligned}\langle G,\ H\rangle_{\mathcal{H}_2(W)} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(-\imath\omega)W(-\imath\omega)W(\imath\omega)H(\imath\omega)\,d\omega = \lim_{R\to\infty}\frac{1}{2\pi\imath}\int_{\Gamma_R} G(-s)W(-s)W(s)H(s)\ ds \\ &= \sum_{k=1}^{m}\mathsf{res}[G(-s)W(-s)W(s)H(s),\mu_k] + \sum_{i=1}^{p}\mathsf{res}[G(-s)W(-s)W(s)H(s),\gamma_i].\end{aligned}$$

This leads to the first assertion. Similarly, if μ_k is a simple pole for $H(s)$ then

$$\begin{aligned}\mathsf{res}[G(-s)W(-s)W(s)H(s),\ \mu_k] &= \lim_{s\to\mu_k}[(s-\mu_k)G(-s)W(-s)W(s)H(s)] \\ &= G(-\mu_k)W(-\mu_k)W(\mu_k)\lim_{s\to\mu_k}(s-\mu_k)H(s).\end{aligned}$$

If μ_k is a double pole for $H(s)$, then it is also a double pole for $G(-s)W(-s)W(s)H(s)$ and

$$\begin{aligned}\mathsf{res}[G(-s)W(-s)W(s)H(s),\ \mu_k] &= \lim_{s\to\mu_k}\frac{d}{ds}[(s-\mu_k)^2G(-s)W(-s)W(s)H(s)] \\ &= \lim_{s\to\mu_k} G(-s)W(-s)W(s)\frac{d}{ds}\left[(s-\mu_k)^2H(s)\right] \\ &\quad + \lim_{s\to\mu_k}(s-\mu_k)^2H(s)\frac{d}{ds}\left[G(-s)W(-s)W(s)\right] \\ &= G(-\mu_k)W(-\mu_k)W(\mu_k)\ \cdot\ \mathsf{res}[H(s),\ \mu_k] \\ &\quad -h_{-2}(\mu_k)\ \cdot\ \frac{d}{ds}\left[G(s)W(s)W(-s)\right]\Big|_{s=-\mu_k}\end{aligned}$$

□

Corollary 3. *If $G(s)$ and $W(s)$ in Lemma 2 each have simple poles, then*

$$\|G\|^2_{\mathcal{H}_2(W)} = \sum_{k=1}^{n} G(-\lambda_k)W(-\lambda_k)W(\lambda_k)\cdot \mathit{res}[G(s),\lambda_k] + \sum_{k=1}^{p} G(-\gamma_k)W(-\gamma_k)G(\gamma_k)\cdot \mathit{res}[W(s),\gamma_k]. \tag{6}$$

This new formula (6) for the weighted-$\mathcal{H}_2$ norm contains as a special case (with $W(s)=1$), a similar expression for the (unweighted) $\mathcal{H}_2$ norm introduced in [10].

Suppose $W\in\mathcal{H}_\infty$ has simple poles at $\{\gamma_1,\dots,\gamma_p\}$ and define a linear mapping $\mathfrak{F}:\mathcal{H}_2\to\mathcal{H}_2$ by

$$\mathfrak{F}[G](s) = G(s)W(s)W(-s) + \sum_{k=1}^{p} G(-\gamma_k)W(-\gamma_k)\frac{\mathsf{res}[W(s),\gamma_k]}{s+\gamma_k} \tag{7}$$

Notice that $G(s)W(s)W(-s)$ has simple poles at $-\gamma_1,-\gamma_2,\dots,-\gamma_p$, and

$$\begin{aligned}\mathsf{res}[G(s)W(s)W(-s),-\gamma_k] &= \lim_{s\to-\gamma_k}(s+\gamma_k)G(s)W(s)W(-s) \\ &= G(-\gamma_k)W(-\gamma_k)\lim_{s\to-\gamma_k}(s+\gamma_k)W(-s) = -G(-\gamma_k)W(-\gamma_k)\lim_{s\to\gamma_k}(s-\gamma_k)W(s) \\ &= -G(-\gamma_k)W(-\gamma_k)\cdot\mathsf{res}[W(s),\gamma_k].\end{aligned}$$

Thus $\mathfrak{F}[G](s)$ has poles only in the left half plane and indeed $\mathfrak{F}:\mathcal{H}_2\to\mathcal{H}_2$.

Corollary 4. *Suppose G and W are stable with poles $\{\lambda_1,\dots,\lambda_n\}$ and $\{\gamma_1,\dots,\gamma_p\}$, respectively. Choose μ arbitrarily in the left half plane distinct from these points. Then for $F(s)=\mathfrak{F}[G](s)$,*

$$\left\langle G,\ \frac{1}{s-\mu}\right\rangle_{\mathcal{H}_2(W)} = F(-\mu) \quad \textit{and} \quad \left\langle G,\ \frac{1}{(s-\mu)^2}\right\rangle_{\mathcal{H}_2(W)} = -F'(-\mu)$$

Proof. By the theorem,

$$\left\langle G,\ \frac{1}{s-\mu}\right\rangle_{\mathcal{H}_2(W)} = G(-\mu)W(-\mu)W(\mu) + \sum_{k=1}^{p} G(-\gamma_k)W(-\gamma_k)\frac{\mathsf{res}[W(s),\gamma_k]}{\gamma_k-\mu} = F(-\mu)$$

and

$$\left\langle G,\ \frac{1}{(s-\mu)^2}\right\rangle_{\mathcal{H}_2(W)} = -\frac{d}{ds}\left[G(s)W(s)W(-s)\right]\big|_{s=-\mu} + \sum_{k=1}^{p} G(-\gamma_k)W(-\gamma_k)\frac{\mathsf{res}[W(s),\gamma_k]}{(\gamma_k-\mu)^2} = -F'(-\mu).$$

□

2.2 Weighted-$\mathcal{H}_2$ optimality conditions

Consider the problem of finding a reduced order system, G_r, that solves (5). The feasible set for (5) is nonconvex, so finding a true (global) minimizer is generally intractable. Nonetheless, we are able to obtain descriptive *necessary conditions* for G_r to satisfy (5).

Theorem 5. *If G_r has simple poles, $\{\hat\lambda_1, \ldots, \hat\lambda_r\}$, and solves (5), then G_r must satisfy:*

$$F_r(-\hat\lambda_k) = F(-\hat\lambda_k) \quad \textit{and} \quad F_r'(-\hat\lambda_k) = F'(-\hat\lambda_k) \quad \textit{for } k = 1, \ldots, r \tag{8}$$

where $F = \mathfrak{F}[G]$ and $F_r = \mathfrak{F}[G_r]$ is defined from (7).

Proof. Suppose by way of contradiction that, for some $\mu \in \{\hat\lambda_1, \ldots, \hat\lambda_r\}$,

$$\left\langle G - G_r,\ \frac{1}{s-\mu}\right\rangle_{\mathcal{H}_2(W)} = \alpha_0 \neq 0$$

By hypothesis, G_r can be represented as $G_r(s) = \sum_{i=1}^{r} \frac{\hat\varphi_i}{s-\hat\lambda_i}$ and for some index k, $\mu = \hat\lambda_k$. Define $\vartheta_0 = \arg(\alpha_0)$ and with $\varepsilon > 0$, define

$$\widetilde{G}_r^{(\varepsilon)}(s) = \frac{\hat\varphi_k + \varepsilon\, e^{-\imath\vartheta_0}}{s-\mu} + \sum_{i\neq k} \frac{\hat\varphi_i}{s-\hat\lambda_i}.$$

Then

$$\|G_r - \widetilde{G}_r^{(\varepsilon)}\|_{\mathcal{H}_2(W)} = \left\|\frac{-\varepsilon\, e^{-\imath\vartheta_0}}{s-\mu}\right\|_{\mathcal{H}_2(W)} \leq \|W\|_{\mathcal{H}_\infty}\frac{\varepsilon}{\sqrt{2|\mathsf{Re}(\mu)|}}$$

so that $\|G_r(s) - \widetilde{G}_r^{(\varepsilon)}(s)\|_{\mathcal{H}_2(W)} = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$. Since G_r solves (5),

$$\begin{aligned}\|G - G_r\|^2_{\mathcal{H}_2(W)} \leq& \|G - \widetilde{G}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)} \leq \|(G - G_r) + (G_r - \widetilde{G}_r^{(\varepsilon)})\|^2_{\mathcal{H}_2(W)} \\ \leq& \|G - G_r\|^2_{\mathcal{H}_2(W)} + 2\mathsf{Re}\left\langle G - G_r,\ G_r - \widetilde{G}_r^{(\varepsilon)}\right\rangle_{\mathcal{H}_2(W)} + \|G_r - \widetilde{G}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)}\end{aligned}$$

Thus, $0 \leq 2\ \mathsf{Re}\left\langle G - G_r,\ G_r - \widetilde{G}_r^{(\varepsilon)}\right\rangle_{\mathcal{H}_2(W)} + \|G_r - \widetilde{G}_r^{(\varepsilon)}\|^2_{\mathcal{H}_2(W)}$.

This implies first that $0 \leq -\varepsilon|\alpha_0| + \mathcal{O}(\varepsilon^2)$, which then leads to a contradiction, $\alpha_0 = 0$.

To show the next assertion, suppose that for some $\mu \in \{\hat\lambda_1, \ldots, \hat\lambda_r\}$,

$$\left\langle G - G_r,\ \frac{1}{(s-\mu)^2}\right\rangle_{\mathcal{H}_2(W)} = \alpha_1 \neq 0.$$

Then for some k, $\mu = \hat{\lambda}_k$ and we define $\vartheta_1 = \arg(\hat{\varphi}_k \cdot \alpha_1)$. For $\varepsilon > 0$ sufficiently small, define

$$\widetilde{G}_r^{(\varepsilon)}(s) = \frac{\hat{\varphi}_k}{s - (\mu + \varepsilon\, e^{-\imath\vartheta_1})} + \sum_{i \neq k} \frac{\hat{\varphi}_i}{s - \hat{\lambda}_i}$$

As $\varepsilon \to 0$, we have

$$\|G_r - \widetilde{G}_r^{(\varepsilon)}\|_{\mathcal{H}_2(W)} = \left\| \frac{-\varepsilon\, \hat{\varphi}_k\, e^{-\imath\vartheta_1}}{(s-\mu)^2 - \varepsilon\, e^{-\imath\vartheta_1}} \right\|_{\mathcal{H}_2(W)} = \mathcal{O}(\varepsilon)$$

Following a similar argument as before, we find that $0 \leq -\varepsilon|\hat{\varphi}_k \cdot \alpha_1| + \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$, which leads to a contradiction, $\alpha_1 = 0$. $\square$

The interpolation conditions described in (8) give first order necessary conditions for G_r to solve the optimal weighted-$\mathcal{H}_2$ model reduction problem (5). Unfortunately, there does not appear to be a straightforward generalization of the corresponding computational approach as described in [10] for the optimal (unweighted) $\mathcal{H}_2$ model reduction problem. Instead, we consider a computational approach to this problem motivated by an expression for the weighted-$\mathcal{H}_2$ error.

2.3 A weighted-$\mathcal{H}_2$ error expression

The expression for the weighted-$\mathcal{H}_2$ norm in Corollary 3 leads immediately to an expression for the weighted-$\mathcal{H}_2$ error that forms the basis for our computational approach.

Corollary 6. *Suppose that G, G_r and W are stable with simple poles $\{\lambda_i\}_{i=1}^n$, $\left\{\hat{\lambda}_j\right\}_{j=1}^r$, and $\{\gamma_k\}_{k=1}^p$, respectively, and that there are no common poles. Define residues: $\phi_i := \mathsf{res}[G(s), \lambda_i]$; $\hat{\phi}_j := \mathsf{res}[G_r(s), \hat{\lambda}_j]$; and $\psi_k := \mathsf{res}[W(s), \gamma_k]$. The weighted-$\mathcal{H}_2$ error is given by*

$$\begin{aligned}
\|G - G_r\|_{\mathcal{H}_2(W)}^2 = &\sum_{i=1}^{n} (G(-\lambda_i) - G_r(-\lambda_i)) W(-\lambda_i) W(\lambda_i) \cdot \phi_i \\
&+ \sum_{j=1}^{r} (G_r(-\hat{\lambda}_j) - G(-\hat{\lambda}_j)) W(-\hat{\lambda}_j) W(\hat{\lambda}_j) \cdot \hat{\phi}_j \\
&+ \sum_{k=1}^{p} (G(-\gamma_k) - G_r(-\gamma_k)) W(-\gamma_k) (G(\gamma_k) - G_r(\gamma_k)) \cdot \psi_k
\end{aligned} \tag{9}$$

One may recover the (unweighted) $\mathcal{H}_2$ error expression of [10] as a special case by taking $W(s) = 1$. Notice that the weighted error depends on the mismatch of G and G_r at the reflected full system poles $\{-\lambda_i\}$, reflected reduced poles $\{-\hat{\lambda}_j\}$, and reflected weight poles $\{-\gamma_k\}$.

2.4 An algorithm for the weighted-$\mathcal{H}_2$ model reduction problem: W-IRKA

In order to reduce the weighted error, one may eliminate some terms in the error expression, by forcing *interpolation* at selected (mirrored) poles. Since r is required to be much smaller than n, there is not enough degrees of freedom to force interpolation at all the terms in the first and second components of the weighted-$\mathcal{H}_2$ error. However, the second-term, i.e. the mismatch at $\hat{\lambda}_j$ can be completely eliminated by enforcing $G(-\hat{\lambda}_j) = G_r(-\hat{\lambda}_j)$ for $j = 1, \ldots, r$. Hence, as in the unweighted $\mathcal{H}_2$ problem, the mirror images of the reduced-order poles play a crucial role. This motivates an algorithm with iterative rational Krylov steps to enforce the desired interpolation property as outlined in Algorithm 1 below. However, a crucial difference from the unweighted $\mathcal{H}_2$ problem is that we will not enforce interpolation of $G'(s)$

at these points; instead we will use the remaining r degrees of freedom to reflect the weight information $W(s)$ and also to eliminate terms from the first component of the error term. The error expression (6) shows that the interpolation errors are multiplied by the residues ϕ_i and ψ_k. Hence, we use the remaining r variables to eliminate the terms in the first and third components of the error expression corresponding to the dominant residues ϕ_k and ψ_k. Note that in several cases, such as in the controller reduction problem, the state-space dimension of the weight will be of the same as that of G; i.e. $\mathcal{O}(p) \approx \mathcal{O}(n)$. We will measure dominance in a relative sense; in other words, normalized by the largest (in amplitude) ϕ_k and ψ_k in every set. More details on this selection process can be found in Section 3 where several examples are used to illustrate these concepts. Note that one *never* needs to compute a full eigenvalue decomposition to obtain the residues of $G(s)$ and $W(s)$. Since only a small subset of poles is needed, one could use, for example, the *dominant pole algorithm* proposed by Rommes [14] which computes effectively eigenvalues corresponding to the dominant residues without requiring a full eigenvalue decomposition.

Algorithm 1. ***Weighted Iterative Rational Krylov Algorithm (W-IRKA)***

Given realizations $G(s) = \mathbf{c}^T(s\mathbf{E} - A)^{-\mathbf{1}}\mathbf{b}$ *and* $W(s) = \mathbf{c}_w^T(s\mathbf{E}_w - \mathbf{A}_w)^{-1}\mathbf{b}_w$*, reduction order* $r = \nu + \varpi$ *with* $\nu, \varpi \geq 0$*, let* $\{\lambda_i\}_{i=1}^{\nu}$ *denote the* ν *dominant poles of* G *and* $\{\gamma_k\}_{k=1}^{\varpi}$ *the* ϖ *dominant poles of* W.

1. *Make an initial interpolation point selection:*
   $\zeta_i = -\lambda_i$ *for* $i = 1, \ldots, \nu$, $\quad \zeta_{j+\nu} = -\gamma_j$ *for* $j = 1, \ldots, \varpi$; $\quad \sigma_k = \zeta_k$ *for* $k = 1, \ldots, r$;
2. *Construct reduction bases,* $\mathbf{V}_r$ *and* $\mathbf{W}_r$*, that satisfy (4).*
3. *Repeat, while (relative change in* $\{\sigma_i\} >$ *tol)*
   (a) $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$ *and* $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$
   (b) *Assign* $\sigma_i \longleftarrow -\lambda_i(\mathbf{A}_r, \mathbf{E}_r)$ *for* $i = 1, \ldots, r$
   (c) *Update* $\mathbf{V}_r$ *so that* $Ran(\mathbf{V}_r) = span\left\{(\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{b},\ \cdots,\ (\sigma_r\mathbf{E} - \mathbf{A})^{-1}\mathbf{b}\right\}$.
4. $\mathbf{A}_r = \mathbf{W}_r^T\mathbf{A}\mathbf{V}_r$, $\mathbf{E}_r = \mathbf{W}_r^T\mathbf{E}\mathbf{V}_r$, $\mathbf{b}_r = \mathbf{W}_r^T\mathbf{b}$, $\mathbf{c}_r^T = \mathbf{c}^T\mathbf{V}_r$

Upon convergence of Algorithm 1, $\sigma_i = -\lambda_i(\mathbf{A}_r, \mathbf{E}_r)$; G_r interpolates G at these points, and the second sum in (9) is eliminated. $\mathbf{W}_r$ is unchanged throughout, so G_r interpolates G at r (aggregated) dominant poles of G and W, eliminating ν and ϖ terms from the first and third sums in (9), respectively. Examples in Section 3 illustrate the effectiveness of this approach.

3 Numerical examples

We illustrate the performance of Algorithm 1 with three examples related to controller reduction. $\Phi^{(N)}$ and $\Psi^{(N)}$ denote the set of normalized residues of $G(s)$ and $W(s)$, respectively.

3.1 A building model

The plant P, a model for the Los Angeles University Hospital, has order 48; see [4] for details. An LQG-based controller, G, of the same order, $n = 48$, is designed to dampen the oscillations in the impulse response. The ten highest normalized residues of $G(s)$ and of $W(s)$ are:

$$\Phi^{(N)} = [\ 1.0000\ \ 1.0000\ \ 0.0286\ \ 0.0286\ \ 0.0088\ \ 0.0088\ \ 0.0080\ \ 0.0080\ \ 0.0060\ \ 0.0060\ ]^T$$
$$\Psi^{(N)} = [\ 1.0000\ \ 1.0000\ \ 0.8416\ \ 0.8416\ \ 0.3935\ \ 0.3935\ \ 0.2646\ \ 0.2646\ \ 0.0951\ \ 0.0951]^T$$

There is a significant drop in $\Phi^{(N)}$ values after the second entry, so we take the first two residues of G as dominant. $\Psi^{(N)}$ remains at roughly the same order until the 9th entry. Thus, we choose $\nu = 2$; and $\varpi = r - \nu = r - 2$ for a given reduction order, r. To illustrate the effect of this dominant pole selection, we apply **W-IRKA**, varying ν from 0 to r. Tables 1 below lists the resulting weighted-$\mathcal{H}_2$ errors for three cases: $r = 12$, $r = 14$, and $r = 16$.

$r = 12$:

ν/ϖ	12/0	10/2	8/4	6/6	4/8	2/10	0/12
$\|G - G_r\|_{\mathcal{H}_2(W)}$	1.4021	1.1433	0.6548	0.6863	0.3576	0.2181	0.2853

$r = 14$:

ν/ϖ	14/0	12/2	10/4	8/6	6/8	4/10	2/12	0/14
$\|G - G_r\|_{\mathcal{H}_2(W)}$	1.4734	1.3436	0.6477	0.3019	0.1538	0.1425	0.1351	0.2224

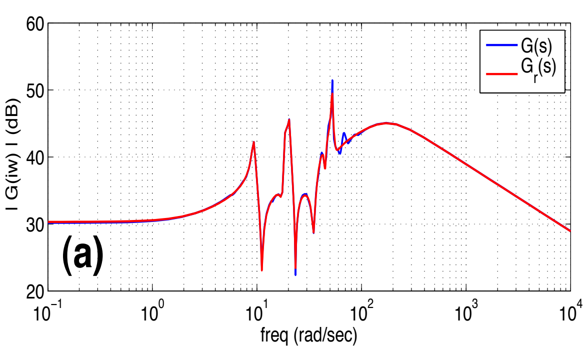
$r = 16$:

ν/ϖ	16/0	14/2	12/4	10/6	8/8	6/10	4/12	2/14	0/16
$\|G - G_r\|_{\mathcal{H}_2(W)}$	1.4206	1.1934	0.7258	0.2898	0.1917	0.1221	0.1154	0.1309	0.1388

Table 1: Weighted-$\mathcal{H}_2$ error as ν and ϖ vary

The weighted-$\mathcal{H}_2$ error decreases as we take more dominant poles of $W(s)$ over those of $G(s)$; suggesting the importance of the residues of $W(s)$ in the error expression 9. Choosing ν=2 is the best choice for most cases. Tables 1 illustrate that while the weighted error initially decreases as ν decreases, it starts increasing when $\nu < 2$, justifying the choice $\nu = 2$. For the case of $r = 16$, similar observations hold Although $\nu = 2$ is not the optimal choice when $r = 16$, the error for $\nu = 2$ is nearly smallest, making $\nu = 2$ still a very good candidate for **W-IRKA**. These numerical results support the idea of choosing ν and ϖ according to the decay of the normalized residues. Even though this choice seems to yield small weighted errors, there may be variations that are even better. The residues are multiplied by quantities such as $W(-\lambda_i)W(\lambda_i)$, so one might consider incorporating these multiplied quantities as well.

A satisfactory reduced-order controller should not only approximate the full-order controller, but also provide the same closed-loop behavior as the original controller. Let T and T_r denote the full-order and reduced-order closed-loop systems, respectively: T corresponds to the feedback connection of P with G; and T_r to the feedback connection of P with G_r. Figure 1-(a) depicts the amplitude Bode plots of G and G_r for $r = 14$ obtained with $\nu = 2$. G_r is an accurate match to G. Figure 1-(b) shows that the reduced-closed loop behavior T_r almost exactly replicates T.



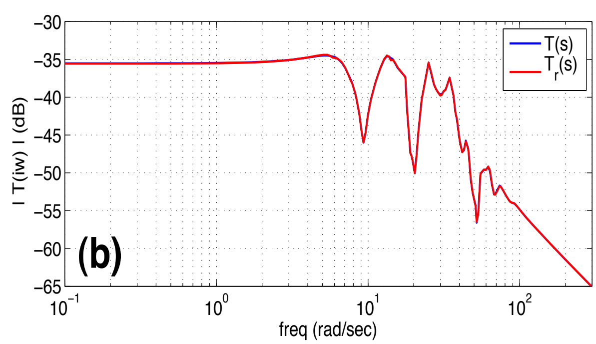


Figure 1: Bode Plots (a) Full and reduced controller (b) Full and reduced closed-loop system

We now compare **W-IRKA** with Frequency Weighted Balanced Truncation (**WFBT**). We vary the reduction order from $r = 10$ to $r = 20$ in increments of 2, and compute weighted $\mathcal{H}_\infty$ and $\mathcal{H}_2$ errors for each case. We use $\nu = 2$ for all cases even though it might not the best choice for **W-IRKA**. Results are listed in Table 2. Note that for every r value, **W-IRKA** outperforms **FWBT** with respect to the weighted-$\mathcal{H}_2$ norm. This might be anticipated since **W-IRKA** is designed to reduce the $\mathcal{H}_2$ error. But **W-IRKA** outperforms **FWBT** with respect to the weighted-$\mathcal{H}_\infty$ norm as well in all except the $r = 18$

case. This is significant since balanced truncation approaches generally yield small $\mathcal{H}_\infty$ norms. This behavior is similar to the behavior of **IRKA** for the (unweighted) $\mathcal{H}_2$ problem where one often observes that **IRKA** consistently yields satisfactory $\mathcal{H}_\infty$ approximants as well [10]. Note that for $r = 10$, the reduced-order controller due to **FWBT** fails to produce a stable closed-loop system.

r	10	12	14	16	18	20
FWBT	2.1080	1.1723	0.1415	0.1386	0.1214	0.1310
W-IRKA	0.6677	0.2180	0.1351	0.1309	0.1028	0.0956

$\|G - G_r\|_{\mathcal{H}_2(W)}$

r	10	12	14	16	18	20
FWBT	1.409	0.5286	0.0723	0.0811	0.0498	0.0830
W-IRKA	0.9175	0.1562	0.0723	0.0721	0.0722	0.0516

$\|G - G_r\|_{\mathcal{H}_\infty(W)}$

Table 2: Comparison of **W-IRKA** and **FWBT**

3.2 International Space Station 12A Module

The plant P is a model for the International Space Station 12A Module with dimension $n_P = 1412$. It is lightly damped and its impulse response exhibits long-lasting oscillations. A state-feedback, full-order, observer-based controller of order $n = n_P = 1412$ is designed to dampen these oscillations. Figures 2-(a) and 2-(a) illustrate the impulse responses P and T, respectively. Notice that the controller dampens oscillations significantly.

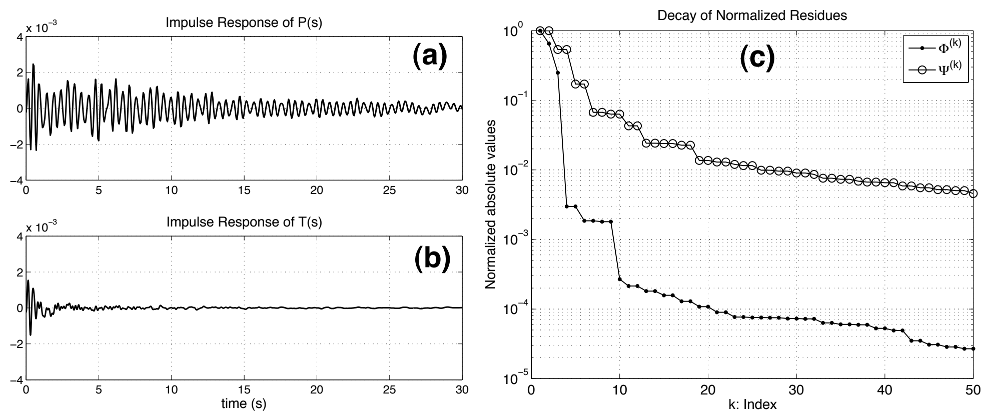


Figure 2: (a) Impulse response of P (b) Impulse response of T (c) Decay of the normalized residues

The decay rate of the first 50 normalized residues $\Phi^{(k)}$ and $\Psi^{(k)}$ are shown in Figure 2-(c). While there is almost a two order-of-magnitude drop in $\Phi^{(k)}$ between the third and fourth components, $\Psi^{(k)}$ continues to stay significant. Hence, we take $\nu = 3$ and reduce order from $n = 1412$ to $r = 60$ using **W-IRKA**. For comparison, we also apply **FBWT**. We denote the resulting reduced-order closed-loop systems due to **W-IRKA** and **FWBT** by T_r and T_{fwbt}, respectively. Note that T_{fwbt} was unstable for $r = 60$. Indeed, $r = 88$ is the *smallest* order **FWBT**-derived reduced controller that lead to a stable closed-system. All **FWBT**-derived G_r are stable; however for $r < 88$ when G_r is connected to P, the resulting T_{fwbt} is unstable. Hence, we compare below the $r = 60$ case for **W-IRKA** with the $r = 88$ case for **FWBT**.

In Figure 3-(a) we plot the impulse responses of T, T_r and T_{fwbt}. T_r almost exactly replicates T. In Figure 3-(b), we plot the absolute value of the errors in the impulse responses due to both methods. **W-IRKA** outperforms **FWBT** even with a lower-order controller. We also simulate both T and T_r for a sinusoidal input of $u(t) = \cos(2t)$. Results shown in Figures 3-(c) and 3-(d) illustrate the superior performance of **W-IRKA** even more clearly.

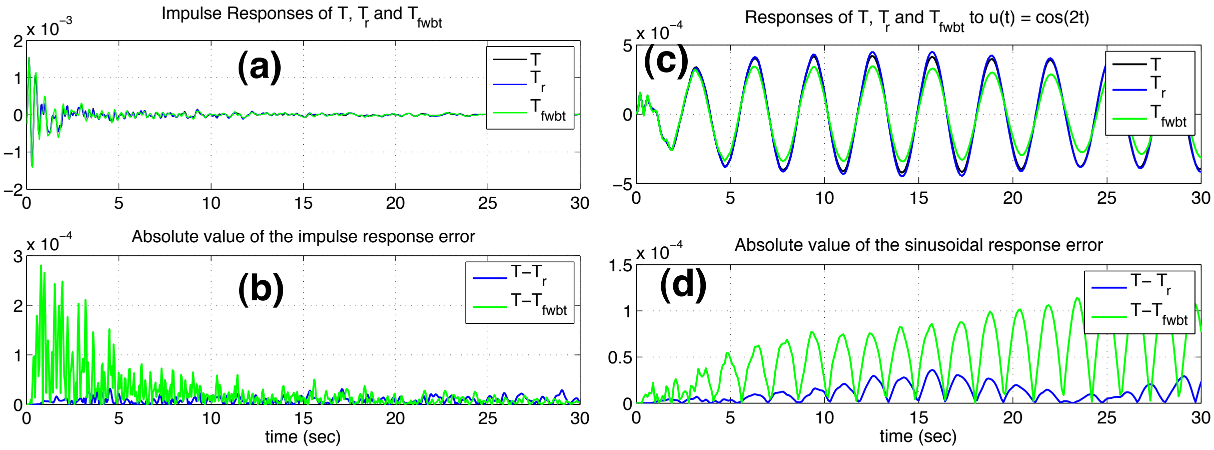


Figure 3: Comparison of **W-IRKA** and **FWBT** using closed-loop system responses

3.3 Control of an Unstable Plant

In the previous two examples, the plant was stable and the goal was to dampen the oscillations in the impulse response. In this example, we start with an unstable plant P of order $n_P = 2000$. Hence, the goal in this case is to stabilize the plant. An observer-based state-feedback controller of the same order, $n = 2000$, has been designed to stabilize P. We note that the full-order controller G has 4 unstable poles. Therefore, as stated in Section 1.2, we would like to obtain a reduced-order controller with the same number of unstable poles. The general approach in the frequency weighted balanced truncation setting would be to decompose the controller into stable and anti-stable parts, and to apply the reduction to the stable part. However, obtaining such a decomposition requires a full eigendecomposition of the original controller and and is numerically intractable in large-scale setting. To investigate the behavior of the proposed approach in the setting of an unstable controller, we have applied **W-IRKA** to G without the stable and anti-stable decomposition. The ten highest normalized residues are listed below:

$$\Phi^{(N)} = [\ 1.0000 \ \ 0.6705 \ \ 0.0275 \ \ 0.0275 \ \ 0.0133 \ \ 0.0111 \ \ 0.0111 \ \ 0.0091 \ \ 0.0091 \ \ 0.0089\ ]^T$$
$$\Psi^{(N)} = [\ 1.0000 \ \ 1.0000 \ \ 0.3767 \ \ 0.3767 \ \ 0.2570 \ \ 0.2570 \ \ 0.0154 \ \ 0.0154 \ \ 0.0004 \ \ 0.0004\ ]^T$$

For reduction order $r = 30$, we choose $\nu = 2$ in **W-IRKA** since there is an order of magnitude gap between the second and third highest residues in $\Phi^{(N)}$. The reduced-order controller G_r has exactly 4 unstable poles (like the full-order controller, G) and stabilizes P. Moreover, **W-IRKA** yields unstable poles in G_r that are accurate approximations to the unstable poles of G; see (10) below. $\boldsymbol{\lambda}_{\text{uns}}$ and $\widehat{\boldsymbol{\lambda}}_{\text{uns}}$ denote the unstable poles of $G(s)$ and $G_r(s)$, respectively:

$$\begin{aligned} \boldsymbol{\lambda}_{\text{uns}} &= [\ 1.2778 \times 10^{1} \quad 3.9599 \times 10^{-1} \quad 2.5213 \times 10^{-2} \pm 1.0482 \times 10^{-1}\,\imath\ ] \\ \widehat{\boldsymbol{\lambda}}_{\text{uns}}^{(r\,=\,30)} &= [\ 1.2778 \times 10^{1} \quad 3.9599 \times 10^{-1} \quad 2.5276 \times 10^{-2} \pm 1.0487 \times 10^{-1}\,\imath\ ] \end{aligned} \tag{10}$$

We do not claim that **W-IRKA** will always retain the same number of unstable poles, nonetheless we have observed this behavior in several examples where the unstable poles of the controller are farther away from the imaginary axis than a significant number of the stable poles. Intuitively, this behavior is similar to the one observed in eigenvalue computations using rational Krylov subspaces where outliers in the spectrum are captured more quickly and accurately [5]. In this example, the controller had 1996 stable poles and 4 unstable poles, the outliers in the spectrum. Typically, the rational Krylov subspaces $\mathbf{V}_r$ captured eigenvectors associated with these outlying eigenvalues very rapidly and produced reduced-order

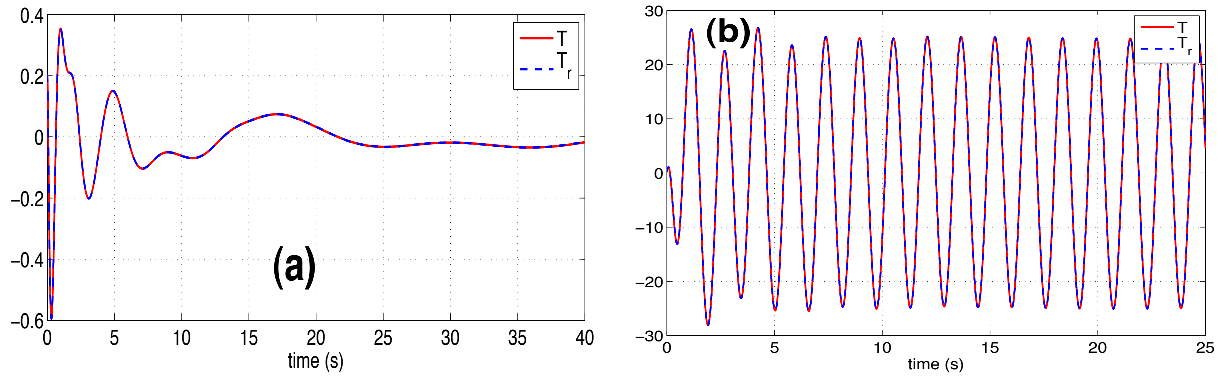


Figure 4: Comparison of T and T_r: (a) Impulse responses (b) Sinusoidal responses

controllers with quite accurate replication of the original unstable poles. To investigate this further, we decreased the reduction order to $r = 25$. **W-IRKA** again produced a stabilizing reduced-order controller with 4 unstable poles:

$$\widehat{\boldsymbol{\lambda}}_{\text{uns}}^{(r\,=\,25)} = [\ 1.2778 \times 10^1 \qquad 3.9399 \times 10^{-1} \qquad 2.7971 \times 10^{-2} \pm 1.0558 \times 10^{-1}\,\imath\ ]$$

Notice that the two unstable poles at 12.778 and 0.39399 (the two farthest from the imaginary axis) are still captured very accurately. The remaining two poles, situated closer to the imaginary axis, show a modest loss in accuracy. Note also that the original controller has two stable poles very close to the imaginary axis at $-7.0320 \times 10^{-2} \pm 1.4035 \times 10^{-1}\imath$.

In Figure 4, we show a comparison of the closed-loop responses of $T(s)$ and $T_r(s)$ to a unit pulse, $u(t) = \delta(t)$, and to a sinusoid, $u(t) = \cos(4t)$. The reduced closed-loop behavior almost exactly replicates the full-order behavior.

4 Conclusions

We presented new formulae for the weighted-$\mathcal{H}_2$ inner product and the weighted-$\mathcal{H}_2$ norm that explicitly reveals the contribution of the poles and residues of the full-order model and of the weight. One of the major consequences of this new representation is the interpolatory optimality conditions for the weighted-$\mathcal{H}_2$ approximation. Moreover, we introduced a heuristic method to produce high-fidelity weighted-$\mathcal{H}_2$ reduced models. The effectiveness of the proposed method has been illustrated via several examples.

References

[1] B.D.O. Anderson and Y. Liu. Controller reduction: concepts and approaches. *IEEE Trans. Automat. Control,*, 34(8):802–812, 2002.

[2] A.C. Antoulas. *Approximation of Large-Scale Dynamical Systems.* SIAM, 2005.

[3] A.C. Antoulas, C.A. Beattie, and S. Gugercin. Interpolatory model reduction of large-scale dynamical systems. In J. Mohammadpour and K. Grigoriadis, editors, *Efficient Modeling and Control of Large-Scale Systems.* Springer, 2010.

[4] A.C. Antoulas, D.C. Sorensen, and S. Gugercin. A survey of model reduction methods for large-scale systems. *Contemporary Mathematics*, 280:193–219, 2001.

[5] C. Beattie, M. Embree, and J. Rossi. Convergence of restarted Krylov subspaces to invariant subspaces. *SIAM Journal on Matrix Analysis and Applications*, 25(4):1074–1109, 2004.

[6] C. De Villemagne and R.E. Skelton. Model reductions using a projection formulation. *International Journal of Control*, 46(6):2141–2169, 1987.

[7] D.F. Enns. Model reduction with balanced realizations: An error bound and a frequency weighted generalization. In *23rd IEEE Conf. on Decision and Control 1984*, volume 23, pages 127–132, 1984.

[8] P. Feldmann and R.W. Freund. Efficient linear circuit analysis by Padé approximation via the Lanczos process. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 14(5):639–649, 1995.

[9] E. Grimme. *Krylov Projection Methods for Model Reduction.* PhD thesis, Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1997.

[10] S. Gugercin, A. C. Antoulas, and C. A. Beattie. $\mathcal{H}_2$ model reduction for large-scale linear dynamical systems. *SIAM J. Matrix Anal. Appl.*, 30(2):609–638, 2008.

[11] S. Gugercin and A.C. Antoulas. A survey of model reduction by balanced truncation and some new results. *International Journal of Control*, 77(8):748–766, 2004.

[12] Y. Halevi. Frequency weighted model reduction via optimal projection. *IEEE Trans. Automat. Control*, 37(10):1537–1542, 1992.

[13] C.-A. Lin and T.-Y. Chiu. Model reduction via frequency weighted balanced realization. *Control Theory and Advanced Technol.*, 8:341–351, 1992.

[14] J. Rommes and M. Martins. Efficient computation of multivariable transfer function dominant poles using subspace acceleration. *IEEE Trans. on Power Systems*, 21:1471–1483, 2006.

[15] G. Schelfhout and B. De Moor. A note on closed-loop balanced truncation. *IEEE Transactions on Automatic Control*, 41(10):1498–1500, 2002.

[16] J. T. Spanos, M. H. Milman, and D. L. Mingori. A new algorithm for L^2 optimal model reduction. *Automatica (Journal of IFAC)*, 28(5):897–909, 1992.

[17] V. Sreeram and A. Ghafoor. Frequency weighted model reduction technique with error bounds. In *American Control Conference, 2005.*, volume 4, pages 2584 – 2589, 2005.

[18] A. Varga and B.D.O. Anderson. Accuracy-enhancing methods for balancing-related frequency-weighted model and controller reduction. *Automatica*, 39(5):919–927, 2003.

[19] G. Wang, V. Sreeram, and WQ Liu. A new frequency-weighted balanced truncation method and an error bound. *IEEE Transactions on Automatic Control*, 44(9):1734–1737, 1999.

[20] G. Wang, V. Sreeram, and WQ Liu. Balanced performance preserving controller reduction. *Systems & Control Letters*, 46(2):99–110, 2002.

[21] A. Yousouff and RE Skelton. Covariance equivalent realizations with applications to model reduction of large-scale systems. *Control and Dynamic Systems*, 22:273–348, 1985.

[22] A. Yousuff, D.A. Wagie, and R.E. Skelton. Linear system approximation via covariance equivalent realizations. *Journal of mathematical analysis and applications*, 106(1):91–115, 1985.

[23] K. Zhou, J. Doyle, and K. Glover. *Robust and Optimal Control.* Prentice-Hall, 1996.

IWRMM-Preprints seit 2009

Nr. 09/01 Armin Lechleiter, Andreas Rieder: Towards A General Convergence Theory For Inexact Newton Regularizations
Nr. 09/02 Christian Wieners: A geometric data structure for parallel finite elements and the application to multigrid methods with block smoothing
Nr. 09/03 Arne Schneck: Constrained Hardy Space Approximation
Nr. 09/04 Arne Schneck: Constrained Hardy Space Approximation II: Numerics
Nr. 10/01 Ulrich Kulisch, Van Snyder : The Exact Dot Product As Basic Tool For Long Interval Arithmetic
Nr. 10/02 Tobias Jahnke : An Adaptive Wavelet Method for The Chemical Master Equation
Nr. 10/03 Christof Schütte, Tobias Jahnke : Towards Effective Dynamics in Complex Systems by Markov Kernel Approximation
Nr. 10/04 Tobias Jahnke, Tudor Udrescu : Solving chemical master equations by adaptive wavelet compression
Nr. 10/05 Christian Wieners, Barbara Wohlmuth : A Primal-Dual Finite Element Approximation For A Nonlocal Model in Plasticity
Nr. 10/06 Markus Bürg, Willy Dörfler: Convergence of an adaptive hp finite element strategy in higher space-dimensions
Nr. 10/07 Eric Todd Quinto, Andreas Rieder, Thomas Schuster: Local Inversion of the Sonar Transform Regularized by the Approximate Inverse
Nr. 10/08 Marlis Hochbruck, Alexander Ostermann: Exponential integrators
Nr. 11/01 Tobias Jahnke, Derya Altintan : Efficient simulation of discret stochastic reaction systems with a splitting method
Nr. 11/02 Tobias Jahnke : On Reduced Models for the Chemical Master Equation
Nr. 11/03 Martin Sauter, Christian Wieners : On the superconvergence in computational elastoplasticity
Nr. 11/04 B.D. Reddy, Christian Wieners, Barbara Wohlmuth : Finite Element Analysis and Algorithms for Single-Crystal Strain-Gradient Plasticity
Nr. 11/05 Markus Bürg: An hp-Efficient Residual-Based A Posteriori Error Estimator for Maxwell's Equations
Nr. 12/01 Branimir Anic, Christopher A. Beattie, Serkan Gugercin, Athanasios C. Antoulas: Interpolatory Weighted-H2 Model Reduction

Eine aktuelle Liste aller IWRMM-Preprints finden Sie auf:

www.math.kit.edu/iwrmm/seite/preprints

Kontakt

Karlsruher Institut für Technologie (KIT)
Institut für Wissenschaftliches Rechnen
und Mathematische Modellbildung

Prof. Dr. Christian Wieners
Geschäftsführender Direktor

Campus Süd
Engesserstr. 6
76131 Karlsruhe

E-Mail:Bettina.Haindl@kit.edu

www.math.kit.edu/iwrmm/

Herausgeber

Karlsruher Institut für Technologie (KIT)
Kaiserstraße 12 | 76131 Karlsruhe

Februar 2012

www.kit.edu