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THE APPROXIMATE INVERSE IN ACTION IV: SEMI-DISCRETE EQUATIONS IN A BANACH SPACE SETTING

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Abstract. This article concernes the method of approximate inverse to solve semi-discrete, linear operator equations in Banach spaces. Semi-discrete means that we search a solution in an infinite dimensional Banach space having only a finite number of data available. In this sense the situation is applicalble to a large variety of applications where a measurement process delivers a discretization of an infinte dimensional data space. The method of approximate inverse computes scalar products of the data with pre-computed reconstruction kernels which are associated with mollifiers and the dual of the model operator. The convergence, approximation power and regularization property of this method when applied to semi-discrete operator equations in Hilbert spaces has been investigated in three prequels of that article. Here we extend these results to a Banach space setting. We show convergence and stability and reproduce the results for the integration operator acting on the space of continuous functions.

Key words. approximate inverse, mollifier property, reconstruction kernel, duality mapping

AMS subject classifications. 45Q05, 41A15, 65R32

1. Introduction.

The method of approximate inverse represents a regularization scheme for stably solving ill-posed equations

$$Af = g \tag{1.1}$$

where $A\colon X \to Y$ is a linear, bounded map between topolgical spaces. It was first introduced by Louis [6] and meanwhile bore efficient solvers for problems as computerized tomography [8], vector field tomography [22, 23, 28], X-ray diffractometry [25], sonar [12, 26], thermoacoustic computerized tomography [9], inverse scattering [1] and even feature reconstruction [7]. A concise monograph about this method is [24]. All these applications use a setting where X and Y being Hilbert spaces. Schuster and Schöpfer [27] extended the method to Banach spaces X and Y consisting of real valued functions with domain $\Omega \subset \mathbb{R}^q$. This setting includes e.g. L^p-spaces and continuous functions on compact sets. We briefly summarize the concept of approximate inverse as it was stated in [27]. We choose a family of mappings $\{e_\gamma\colon \Omega \to X^*\}_{\gamma>0}$ such that for any function $f \in X$ the convergence

$$f_\gamma(x) := \langle f, e_\gamma(x)\rangle_{X\times X^*} \to f(x), \qquad x \in \Omega,$$

holds in X as $\gamma \to 0$. Such a family $\{e_\gamma\}$ is called a *mollifier* and can be thought of as an approximation to Dirac's delta distribution. Supposed that there is a second family of mappings $\{v_\gamma\colon \Omega \to Y^*\}_{\gamma>0}$ satisfying

$$A^*[v_\gamma(x)] = e_\gamma(x), \qquad x \in \Omega,$$

then obviously $f_\gamma(x) = \langle v_\gamma, g\rangle_{Y^*\times Y}$. Hence computing the approximate inverse f_γ consists of the evaluation of dual pairings of the given data $g = Af$ with $v_\gamma(x)$. That is why we call the family $\{v_\gamma\}$ *reconstruction kernel*. In [27] the authors prove convergence and stability with

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respect to noisy data g^δ with rates and demonstrate their results in case that $X = L^p(\Omega)$, $1 \le p < \infty$, and $X = \mathcal{C}(K)$, K compact.

In this article we go now one step further considering semi-discrete operator equations

$$A_n f_n = g_n, \qquad f_n \in X, \tag{1.2}$$

where $A_n = \Psi_n A\colon X \to \mathbb{R}^n$, $g_n = \Psi_n g \in \mathbb{R}^n$ arise from A, g, respectively, by applying the so called *observation operator* $\Psi_n\colon Y \to \mathbb{R}^n$ to them. Ψ_n can be seen as the mathematical model of the measurement process that boils down the infinite dimensional data space Y to $\mathbb{R}^n$. The setting (1.2) is of large practical relevance, since any measurement device tears a finite number of observations from the data space Y. This is essential for applied inverse problems: We want to compute a quantity f in an infinite dimensional object space X observing another quantity $g \in Y$ which is linked to f via (1.1), but by using any measurement device only a finite number of observations $g_n = \Psi_n g$ of g are really accessible. Bertero et al [2, 3] investigated semi-discrete operator equations in Hilbert spaces and found solutions by means of the singular value decomposition (SVD) of A. They used the SVD to formulate regularization methods and proved convergence and stability with respect to noise. Krebs [5] considered the stable solution of semi-discrete equations in reproducing kernel Hilbert spaces by support vector regression.

The extension of the method of approximate inverse to this setting in a general Banach space framework needs other concepts than those for solving (1.1) as it was done in [27]. Moreover we pursue ideas that have been outlined in Rieder and Schuster [15, 16] for the Hilbert space setting. There the authors developed the concept of a *mollifier operator* $E_d\colon X \to X$ and extended the approximate inverse for equations as (1.2). They showed convergence with rates for exact and noisy data in a general framework as well as in applications as computerized tomography and Doppler tomography. The key idea is to say that a mollifier for arbitrary X consists of a sequence $\{e_{d,i}\}_{i=1}^d \subset X^*$ of elements in the dual space X^* with whom a Riesz system $\{b_{d,i}\}_{i=1}^d \subset X$ is associated such that the *mollifier operator*

$$E_d f := \sum_{i=1}^d \langle f, e_{d,i}\rangle_{X\times X^*} b_{d,i}, \qquad d \in \mathbb{N},$$

fulfills the convergence $E_d f \to f$ in X as $d \to \infty$. Provided that we have reconstruction kernels $v_i^n \in \mathbb{R}^n$ satisfying $A_n^* v_i^n = e_{d,i}$ for all $1 \le i \le d$ it makes perfectly sense to define the approximate inverse $\widetilde{A}_{n,d}\colon \mathbb{R}^n \to X$ as

$$\widetilde{A}_{n,d} g_n := \sum_{i=1}^d \langle v_i^n, g_n\rangle_{\mathbb{R}^n} b_{d,i}, \qquad d \in \mathbb{N},$$

since then the mollifier property of E_d guarantees the convergence

$$\lim_{d\to\infty} \widetilde{A}_{n,d} A_n f_n = f_n.$$

We give a short outline of the article's contents. First we introduce in Section 2 all necessary mappings and definitions to formulate the method of approximate inverse for the semi-discrete setting (1.2). We propose two different ways to obtain the reconstruction kernels v_i^n: one approximates v_i^n using a Landweber method (Section 3), the other one computes a replacement for v_i^n by means of a reconstruction kernel v_i for A, that is, for the underlying continuous problem (Section 4). In the latter case we not only give a complete convergence theory but we also prove the regularization property in Section 5 where the noise is modeled as a perturbation of the observation operator Ψ_n.

Throughout the paper all abstract concepts are made specific by applying them to a concrete example where A is the integration operator acting on the Banach space $\mathcal{C}(0,1)$. Further, we highlight the difference of a Hilbert space and a Banach space framework for the integration operator in the appendix.

2. Semi-discrete setting in Banach spaces. Consider equation (1.1) for a linear, continuous and injective operator $A\colon X \to Y$ where, if not indicated otherwise, X and Y are arbitrary, real Banach spaces equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$. By X^* and Y^* we denote the dual spaces of X and Y, respectively, consisting of all linear, continuous mappings from $X, Y \to \mathbb{R}$. The dual pairings are denoted by

$$\langle f^*, f\rangle_{X^*\times X} := f^*(f), \qquad \langle g^*, g\rangle_{Y^*\times Y} := g^*(g)$$

for $f^* \in X^*$, $f \in X$, $g^* \in Y^*$, $g \in Y$. Finally X^{**}, Y^{**} are the biduals of X, Y, respectively. E.g. X^{**} consists of all mappings $\varphi\colon X^* \to \mathbb{R}$ that are linear and continuous. Note that $X \subset X^{**}$ by

$$f(f^*) := f^*(f) \qquad f^* \in X^*,\ f \in X\,.$$

In practical situations we have only finitely many measurements at hand. The data acquisition is modeled by the observation operator $\Psi_n : Y \to \mathbb{R}^n$, and means that the measurement process tears n observations out from the infinite dimensional data space Y and this process is represented by Ψ_n. The map Ψ_n is assumed to be generated by n linear and continuous functionals $\psi_{n,k} \in Y^*$, that is

$$(\Psi_n v)_k = \langle \psi_{n,k}, v\rangle_{Y^*\times Y} = \psi_{n,k}(v)\,, \quad v \in Y, \quad k = 1,\dots,n, \tag{2.1}$$

and thus is linear and continuous, too. Hence we have to investigate the semi-discrete equation

$$A_n f_n = g_n \tag{2.2}$$

with $A_n = \Psi_n A$, $g_n = \Psi_n g$ rather than (1.1). Equation (2.2) in general does not have a solution for arbitrary $g_n \in \mathbb{R}^n$. This is why we rather consider the equation

$$A_n f_n = P_{\mathcal{R}(A_n)} g_n, \qquad g_n \in \mathbb{R}^n. \tag{2.3}$$

which is solvable but highly underdetermined. Let X be uniformly convex. Then we even can define the minimum norm solution $f_n^\dagger$ of (2.3), that is the unique $f_n^\dagger \in X$ with

$$\|f_n^\dagger\| = \min\{\|f_n\|_X : f_n \in X \text{ with (2.3) }\}\,.$$

Note that equation (2.3) is equivalent to

$$A_n^* A_n f_n = A_n^* g_n, \qquad g_n \in \mathbb{R}^n,$$

and solvable for any $g_n \in \mathbb{R}^n$ because of $\dim(\mathcal{R}(A_n)) < \infty$. Any solution of (2.3) minimizes the defect $\|A_n f_n - g_n\|_2$. Our aim is to extend the concept of approximate inverse to the given situation following the lines in Rieder and Schuster [15, 16] and thus to present a concept to approximate $f_n^\dagger$ in a stable way (see Theorem 4.8 below).

The key idea is to compute moments

$$\langle f_n^\dagger, e_{d,i}\rangle_{X\times X^*} = e_{d,i}(f_n^\dagger), \qquad i = 1,\dots,d, \tag{2.4}$$

of $f_n^\dagger$ with mollifiers $e_{d,i} \in X^*$, $i = 1,\dots,d$, and then approximate $f_n^\dagger$ by

$$E_d f_n^\dagger := \sum_{i=1}^{d} \langle f_n^\dagger, e_{d,i}\rangle_{X\times X^*}\, b_{d,i}\,.$$

Here, $\{b_{d,i}\}_{i=1}^d \subset X$ is a family of elements in X which are associated with the mollifiers $\{e_{d,i}\}_{i=1}^d$ and form a system in X which allows for an estimate like

$$\Big\| \sum_{i=1}^{d} \alpha_i\, b_{d,i} \Big\|_X \le \sigma(d) \max_{1\le i\le d} |\alpha_i|, \quad \alpha \in \mathbb{R}^d, \tag{2.5}$$

for a positive function $\sigma\colon \mathbb{N} \to \mathbb{R}_+$.

By now it is not clear what we understand by mollifiers $e_{d,i}$ in a general Banach space X. The sequence $\{e_{d,i}\}_{i=1}^d$ and thus the associated sequence $\{b_{d,i}\}_{i=1}^d$ have to be chosen such that E_d satisfies the *mollifier property*

$$\lim_{d\to\infty} \|E_d w - w\|_X = 0\,, \qquad w \in X\,, \tag{2.6}$$

which guarantees that $E_d w$ in fact approximates w for any $w \in X$.

EXAMPLE 2.1. *For $X = (\mathcal{C}([0,1]), \|\cdot\|_\infty)$ a family $\{b_{d,i}\}_{i=0}^d$ is given by linear B-splines. Let*

$$b(x) := \begin{cases} 1 - |x| : & |x| \le 1, \\ \quad 0 \quad : & \textit{otherwise.} \end{cases}$$

Then we define

$$b_{d,i}(x) := b(dx - i), \quad i = 1,\dots,d-1, \tag{2.7}$$

as well as[1]

$$b_{d,0}(x) := \chi_{[0,1/d]}(x)\, b(dx), \quad b_{d,d}(x) := \chi_{[1-1/d,1]}(x)\, b(dx-d). \tag{2.8}$$

Obviously $b_{d,i} \in X$, $i = 0,\dots,d$, and we have for $\alpha \in \mathbb{R}^{d+1}$

$$\Big\| \sum_{i=0}^{d} \alpha_i\, b_{d,i} \Big\|_\infty \le \max_{0\le i\le d} |\alpha_i| \tag{2.9}$$

Thus (2.5) *holds true with $\sigma(d) = 1$.*

Next we want to present mollifiers $\{e_{d,i}\}_{i=0}^d \subset X^*$ associated with $\{b_{d,i}\}_{i=0}^d$ such that (2.6) *holds true. To this end we introduce $NBV(0,1)$ which is the space of* normalized *functions of bounded variation over* $[0,1]$*. These functions vanish at* 0 *and are continuous from the right in* $(0,1)$*. Equipped with the total variation as norm $NBV(0,1)$ is a Banach space which can be identified with $\mathcal{C}(0,1)^*$, see, e.g., Taylor and Lay* [29, Sec. III.5]*. Any bounded*

[1] χ_A denotes the indicator function of the set A.

linear functional on $\mathcal{C}(0,1)$ *is uniquely induced by a* $\mu \in NBV(0,1)$ *via the Riemann-Stieltjes integral*

$$f \mapsto \langle f, \mu \rangle_{X \times X^*} := \int_0^1 f(x) \mathrm{d}\mu(x).$$

Now we define $\mathbf{E}_d \colon \mathcal{C}(0,1) \to \mathcal{C}(0,1)$ *by*

$$\mathbf{E}_d f = \sum_{i=0}^{d} \langle f, e_{d,i} \rangle_{X \times X^*} b_{d,i}, \quad e_{d,i} = \int_0^x E_{d,i}(t) \mathrm{d}t, \tag{2.10}$$

where

$$E_{d,0} = d\chi_{]0,x_{d,1}[} \quad \textit{as well as} \quad E_{d,d} = d\chi_{[x_{d,d-1},1[} \tag{2.11}$$

and

$$E_{d,1} = \frac{d}{2}\chi_{]0,x_{d,2}[}, \quad E_{d,i} = \frac{d}{2}\chi_{[x_{d,i-1},x_{d,i+1}[}, \quad i = 2, \ldots, d-1, \tag{2.12}$$

with $x_{d,k} = k/d$, $k = 0, \ldots, d$. *Since*

$$\langle f, e_{d,i} \rangle_{X \times X^*} = \int_0^1 f(x) \mathrm{d}e_{d,i}(x) = \int_0^1 f(x) e'_{d,i}(x) \mathrm{d}x = \int_0^1 f(x) E_{d,i}(x) \mathrm{d}x$$

the operator $\mathbf{E}_d$ *reproduces constant functions. Thus, we have the mollifier property*

$$\lim_{d \to \infty} \|f - \mathbf{E}_d f\|_\infty = 0 \quad \textit{for any } f \in \mathcal{C}(0,1) \tag{2.13}$$

as well as

$$\|f - \mathbf{E}_d f\|_\infty \le C_{\mathbf{E}}\, d^{-\alpha} \|f\|_{\mathcal{C}^\alpha(0,1)} \tag{2.14}$$

whenever f *is Hölder-continuous of order* $\alpha \in [0,1]$. *The constant* $C_{\mathbf{E}}$ *might depend on* α.

In modifying both boundary mollifiers $e_{d,0}$ *and* $e_{d,d}$ *we are able to achieve even higher convergence orders for smooth functions. For*

$$E^{\mathrm{b}} = 3\chi_{]0,0.5[} - \chi_{[0.5,1[} \tag{2.15}$$

we have

$$\int_0^1 E^{\mathrm{b}}(t) \mathrm{d}t = 1 \textit{ and } \int_0^1 t E^{\mathrm{b}}(t) \mathrm{d}t = 0.$$

Set

$$E_{d,0}(x) = dE^{\mathrm{b}}(dx) \;\textit{ and }\; E_{d,d}(x) = dE^{\mathrm{b}}(d - dx)$$ [2]

and keep $E_{d,i}$, $i = 1, \ldots, d-1$, *as in* (2.12)*. Then,* $\mathbf{E}_d$ *reproduces affin-linear functions since*

$$\langle p, e_{d,i} \rangle_{X \times X^*} = p(x_{d,i}), \; i = 0, \ldots, d, \; \textit{for all } p \in \Pi_1. \tag{2.16}$$

[2] $E_{d,d}$ needs yet to be normalized to be in $NBV(0,1)$. This is easily done by forcing continuity from the right at $1 - 1/(2d)$.

Hence, (2.14) *extends to*

$$\|f - \mathbf{E}_d f\|_\infty \le C_{\mathbf{E}}\, d^{-\alpha} \|f\|_{\mathcal{C}^\alpha(0,1)}, \quad 0 \le \alpha \le 2. \tag{2.17}$$

For the reader's convenience we prove (2.13)*,* (2.14)*, and* (2.17) *in Appendix* A.

To evaluate the approximation $E_d f_n^\dagger$ we need to calculate the moments (2.4) of $f_n^\dagger$ which is not accessible. Hence, we go one step further and search for solutions of the dual equations

$$A_n^* v_i^n = e_{d,i}\,, \qquad i = 1, \dots, d\,, \tag{2.18}$$

where the adjoint operator $A_n^* \colon \mathbb{R}^n \to X^*$ is given by

$$A_n^* \alpha = \sum_{k=1}^n \alpha_k\, A^*\, \psi_{n,k}\,, \qquad \alpha \in \mathbb{R}^n\,. \tag{2.19}$$

Assume for the moment that (2.18) has a solution. We deduce

$$\langle f_n^\dagger, e_{d,i} \rangle_{X \times X^*} = \langle A_n f_n^\dagger, v_i^n \rangle_2$$

and defining

$$\widetilde{A}_{n,d} \colon \mathbb{R}^n \to X\,, \quad \widetilde{A}_{n,d}\alpha := \sum_{i=1}^d \langle \alpha, v_i^n \rangle_2\, b_{d,i} \tag{2.20}$$

we obtain

$$\lim_{d\to\infty} \widetilde{A}_{n,d}\, A_n f_n^\dagger = \lim_{d\to\infty} E_d f_n^\dagger = f_n^\dagger\,.$$

This motivates to call $\widetilde{A}_{n,d}$ the (semi-discrete) *approximate inverse* of A_n, a solution v_i^n of (2.18) is again called reconstruction kernel.

REMARK 2.2. *If X is a Hilbert space and thus uniformly convex, the minimum norm solution $f_n^\dagger$ of equation (2.3) exists and we have the interesting connection*

$$\langle f_n^\dagger, e_{d,i} \rangle_{X \times X^*} = \langle g_n, v_i^n \rangle_2, \qquad i = 1, \dots, d,$$

if only $g_n \in \mathcal{R}(A_n)$ or $v_i^n \in \mathcal{N}(A_n^*)^\perp$. And this identity holds true even in the case where v_i^n only solves the normal equation $A_n A_n^* v_i^n = A_n e_{d,i}$, see [16, Lemma 2.1]*. In Banach spaces this identity is valid only if $A_n^* v_i^n = e_{d,i}$ is solvable which cannot be expected.*

The image $\mathcal{R}(A_n^*)$ consists of the span of $\{A^*\psi_{n,k} : k = 1, \dots, n\}$ and we cannot expect $e_{d,i}$ to be an element of it. We outline two different ways to calculate reconstruction kernels v_i^n. The first one is an iterative method where the iterates converge to a minimizer of $\|A_n^* v_i^n - e_{d,i}\|_{X^*}$; the second one uses an approximate solution of $A^* v_i = e_{d,i}$ to construct v_i^n. The latter one is the strategy that was also pursued in [15, 16] and for which we give criteria to obtain convergence and stability with respect to noisy data g_n^δ.

3. Iterative calculation of reconstruction kernels. Let, for the moment, X be uniformly convex and smooth and thus reflexive. With $J^X \colon X \to X^*$, $J^{X^*} \colon X^* \to X$ we denote the single-valued duality mappings on X, X^*, respectively, that is the subdifferentials

$$J^X(f) = \partial\big\{\|f\|_X^2/2\big\}, \qquad J^{X^*}(f^*) = \partial\big\{\|f^*\|_{X^*}^2/2\big\}$$

of the norms $\|\cdot\|_X$, $\|\cdot\|_{X^*}$, respectively. E.g. $J^X(f) \in X^*$ is the unique element in the dual of X with

$$\frac{1}{2}\|g\|_X^2 - \frac{1}{2}\|f\|_X^2 \geq \langle J^X(f), g-f\rangle_{X^*\times X} \qquad \text{for all } f, g \in X.$$

Since X is uniformly convex and smooth, the duality mapping J^X is norm-to-weak continuous, J^{X^*} even continuous, and we have the interesting and important relations

$$f = J^{X^*}\big(J^X(f)\big), \qquad f^* = J^X\big(J^{X^*}(f^*)\big)\,, \qquad f \in X\,,\ f^* \in X^*.$$

Any minimizer of $\|A_n^* v_i^n - e_{d,i}\|_{X^*}$ is then characterized by the optimality condition

$$A_n J^{X^*}(A_n^* v_i^n - e_{d,i}) = \partial\{\|A_n^* v_i^n - e_{d,i}\|_{X^*}^2/2\} = 0 \tag{3.1}$$

and this equation has a solution since $\mathcal{R}(A_n^*)$ is closed and hence

$$\mathcal{R}(A_n^*) + J^X(\mathcal{N}(A_n)) = X^*\,. \tag{3.2}$$

Associated with the duality mappings are the concepts of Bregman distance and Bregman projection. The Bregman distance with respect to $\|\cdot\|_X^2/2$ is

$$D^X(f,g) = \frac{1}{2}\|f\|_X^2 - \frac{1}{2}\|g\|_X^2 - \langle J^X(g), f-g\rangle_{X^*\times X}\,, \qquad f, g \in X$$

which is not a metric but has some properties of one; e.g. $D^X(f,g) \geq 0$ and $D^X(f,g) = 0$, iff $f = g$. Analogously to the metric projection we can define the Bregman projection onto a nonempty, closed, convex set $C \subset X$ as the unique element $\Pi_C^X(f) \in C$ with

$$D^X(\Pi_C^X(f), f) = \min_{g\in C} D^X(g,f)$$

If especially $U \subset X$ is a subspace of X, then we have the interesting connection of the metric and Bregman projection, see [20, Lemma 3.9]

$$f = P_U(x) + J^{X^*}\Pi_{U^\perp}^{X^*} J^X(f)\,, \qquad f \in X, \tag{3.3}$$

where $U^\perp \subset X^*$ means the annihilator of U. We refer to the book of Cioranescu [4] for deeper insights of duality mappings and their properties.
One possibility to approximately compute v_i^n is to adopt the Landweber method from Schöpfer, Louis, and Schuster [18] to this situation. We consider the following iteration scheme, where we drop the indices of v_i^n and $e_{d,i}$ for a moment.

ALGORITHM 3.1.
(1) $v_0 = 0$,
(2) For $k = 0, 1, \ldots$ *iterate*

$$v_{k+1} = v_k - \mu_k A_n J^{X^*}(A_n^* v_k - e) \tag{3.4}$$

with appropriately chosen μ_k.

Using the results from [18] and [19, Prop. 1] we indeed can show convergence.

THEOREM 3.2. *Let X be uniformly convex and smooth and $e \in X^*$. Then there is a choice of μ_k such that the iterates $\{v_k\}_k \subset \mathbb{R}^n$ from Algorithm* 3.1 *converge strongly to the unique minimizer $v \in \mathbb{R}^n$ of $\|A_n^* v - e\|_{X^*}$ having minimum $\|\cdot\|_2$-norm.*

Proof. Applying A_n^* to the iteration (3.4) and subtracting e yield

$$A_n^* v_{k+1} - e = A_n^* v_k - e - \mu_k A_n^* A_n J^{X^*}(A_n^* v_k - e)$$

which corresponds to the Landweber iteration

$$x_{k+1}^* = x_k^* - \mu_k A_n^* A_n x_k \tag{3.5a}$$

$$x_{k+1} = J^{X^*}(x_{k+1}^*), \qquad k = 0, 1, \ldots \tag{3.5b}$$

with the settings $x_0^* := -e$, $x_0 = -J^{X^*}(e)$, $x_k^* := A_n^* v_k - e$, $x_k := J^{X^*}(A_n^* v_k - e)$. Proposition 1 in [19] says that the step sizes μ_k can be chosen such that x_k tends strongly to $-\Pi_{\mathcal{N}(A)} J^{X^*}(e)$ as $k \to \infty$. Note that here $Y = \mathbb{R}^n$ is finite-dimensional and that the Landweber method is included in the general framework of Algorithm SESOP as it was presented in [19, Sect. 3]. Since the duality mapping J^X is norm-to-weak continuous we have

$$x_k^* = J^X(x_k) \rightharpoonup -J^X \Pi_{\mathcal{N}(A)} J^{X^*}(e) =: x^* \qquad \text{as } k \to \infty$$

with respect to the weak topology. Hence $x_k^* + e \rightharpoonup x^* + e$ weakly, too. But $x_k^* + e \in \mathcal{R}(A_n^*)$ and since $\dim(\mathcal{R}(A_n^*)) < \infty$ this yields strong convergence of $x_k^* \to x^*$ as $k \to \infty$. Because of $x_k^* = A_n^* v_k - e$ we finally obtain

$$A_n^* v_k \to e - J^X \Pi_{\mathcal{N}(A)} J^{X^*}(e) = P_{\mathcal{R}(A_n^*)}(e) \qquad \text{as } k \to \infty\,. \tag{3.6}$$

Here we used the fact that

$$e = P_{\mathcal{R}(A_n^*)}(e) + J^X \Pi_{\mathcal{N}(A)} J^{X^*}(e)$$

which is a consequence of (3.2) and (3.3). Furthermore by (3.4) all iterates v_k are in $\mathcal{R}(A_n) = \big(\mathcal{N}(A_n^*)\big)^\perp$. Since the restriction of A_n^* to $A_n^* \colon \big(\mathcal{N}(A_n^*)\big)^\perp \to \mathcal{R}(A_n^*)$ is bijective between finite-dimensional spaces, we have convergence of the sequence $\{v_k\}$ to some $v \in \big(\mathcal{N}(A_n^*)\big)^\perp$. But from (3.6) it is clear that

$$A_n^* v = P_{\mathcal{R}(A_n^*)}(e)\,.$$

Hence v minimizes $\|A_n^* v - e\|_{X^*}$. Finally we show that v has minimal $\|\cdot\|_2$-norm among all minimizers. Let $z \in \mathbb{R}^n$ be an arbitrary minimizer of $\|A_n^* z - e\|_{X^*}$. Since the metric projection $P_{\mathcal{R}(A_n^*)}(e)$ is unique, we must have $A_n^* z = A_n^* v$ and thus $z = v + u$ with some $u \in \mathcal{N}(A_n^*)$. Hence we have $\langle v, u\rangle_2 = 0$ and get

$$\|z\|_2^2 = \|v\|_2^2 + 2\,\langle v, u\rangle_2 + \|u\|_2^2 = \|v\|_2^2 + \|u\|_2^2 \geq \|v\|_2^2\,,$$

where the last inequality is strict for $u \neq 0$. □

REMARK 3.3. *a) The proof of Theorem* 3.2 *shows that iteration* (3.4) *simultaneously approximates a reconstruction kernel v that minimizes $\|A_n^* v - e\|_{X^*}$ as well as the metric projection $A_n^* v = P_{\mathcal{R}(A_n^*)}(e) = \lim_{k\to\infty} x_k^*$. Note that this proof strongly relies on the facts that X is uniformly convex and smooth and that the range of A_n^* is of finite dimension.*
b) The iteration can be made more efficient by using the sequential subspace methods developed in [19]. *These achieve acceleration by using more search directions $A_n^* w_l$ than just $A_n^* A_n x_k$ in iteration* (3.5). *The assertion of Theorem* 3.2 *still holds if one assures that $w_l \in \mathcal{R}(A_n)$, which is always fulfilled for the canonical search directions suggested in* [19].

4. Kernels from the underlying continuous setting and convergence. Although the calculation of the reconstruction kernels v_i^n can be done by Algorithm 3.1, this method has some drawbacks. Besides the conditions on X and A_n that have to be required in Theorem 3.2 to get convergence, an approximate reconstruction kernel might cause heavy artifacts. Furthermore A_n^* in general does not fulfill invariance properties as in Lemma 4.6 below and thus we had to perform the iteration for each mollifier $e_{d,i}$ which is very time consuming.

To cure this dilemma we seek a replacement for the kernel v_i^n that relies on a (maybe even exact) kernel v_i for A. Let again X and Y be arbitrary Banach spaces. We recall that A is injective and thus $\mathcal{R}(A^*)$ is weak*-dense in X^*. This implies that for given numbers $\varepsilon_i > 0$ and mollifiers $e_{d,i} \in X$ we find elements $v_i \in Y^*$ such that

$$|\langle A^* v_i - e_{d,i}, f\rangle_{X^*\times X}| = |(A^* v_i - e_{d,i})(f)| < \varepsilon_i \, \|f\|_X \,, \qquad i = 1,\dots,d \,. \tag{4.1}$$

Here, $f \in X$ denotes the (unique) solution of $Af = g$. This solution exists since A is injective and we assumed for the exact data $g \in \mathcal{R}(A)$. Note, that (4.1) does not mean that $\|A^* v_i - e_{d,i}\|_{X^*} < \varepsilon_i$ since each v_i in (4.1) depends on f and we have no uniform boundedness.

REMARK 4.1. *In case that X is reflexive, then $\mathcal{R}(A^*)$ is dense even with respect to the strong (norm-) topology of X and there exist v_i satisfying*

$$\|A^* v_i - e_{d,i}\|_{X^*} < \varepsilon_i$$

which is a stronger condition than (4.1)*.*

Having elements v_i satisfying (4.1) at our disposal we define

$$v_i^n = G_n \, \Psi_n' \, v_i, \qquad i = 1,\dots,d, \tag{4.2}$$

where $\Psi_n' \colon Y^* \to \mathbb{R}^n$ is linear and continuous and $G_n \in \mathbb{R}^{n\times n}$ is a suitable matrix. Both, Ψ_n' as well as G_n will be defined after the next example such that we gain pointwise convergence

$$\lim_{\substack{n\to\infty \\ d\to\infty}} \|\widetilde{A}_{n,d} \, A_n f - f\|_X = 0 \quad \text{for any } f \in X.$$

EXAMPLE 4.2. *In this example we compute explicitely reconstruction kernels satisfying* (4.1) *even with $\varepsilon_i = 0$. Let $X = (\mathcal{C}(0,1), \|\cdot\|_\infty)$ and consider the simple integration operator $A \colon X \to X$,*

$$Af(x) = \int_0^x f(t)\mathrm{d}t = \int_0^1 k(x,t) f(t)\mathrm{d}t, \quad k(x,t) = \begin{cases} 1 & : \ t \le x, \\ 0 & : \ \textit{otherwise.} \end{cases} \tag{4.3}$$

Obviously A is linear, bounded and injective. Note that X is neither uniformly convex nor smooth. Hence the computation of the kernels v_i^n by Algorithm 3.1 *fails and we have to take the replacements* (4.2)*.*

In a first step we determine the adjoint $A^* \colon X^* \to X^*$. Recall from Example 2.1 *that $X^* = NBV(0,1)$. Let $\mu \in NBV(0,1)$. Then,*

$$\begin{aligned} \langle \mu, Af\rangle_{X^*\times X} &= \int_0^1\int_0^1 k(x,t) f(t)\mathrm{d}t\mathrm{d}\mu(x) = \int_0^1 f(t) \int_0^1 k(x,t)\mathrm{d}\mu(x)\mathrm{d}t \\ &= \int_0^1 f(t) \int_t^1 \mathrm{d}\mu(x)\mathrm{d}t = \int_0^1 f(t)\big(\mu(1) - \mu(t)\big)\mathrm{d}t = \langle A^*\mu, f\rangle_{X^*\times X} \end{aligned}$$

where

$$A^*\mu(\xi) = \mu(1)\xi - \int_0^\xi \mu(x)\mathrm{d}x.$$

For $E \in NBV(0,1)$ *with* $\int_0^1 E(t)\mathrm{d}t = 1$ *and* $E(1) = 0$ *define* $e(x) := \int_0^x E(t)\mathrm{d}t = AE(x)$. *Then,* $v(x) = -E(x)$ *is a reconstruction kernel to the mollifier* e. *Indeed,*

$$A^*v(\xi) = v(1)\xi - \int_0^\xi v(x)\mathrm{d}x = -\underbrace{E(1)}_{=0}\xi + \int_0^\xi E(x)\mathrm{d}x = e(\xi). \tag{4.4}$$

We give two concrete examples. Let $E_\gamma = \chi_{[z-\gamma,z+\gamma[}/(2\gamma)$ *for a fixed* $z \in]0,1[$ *and let* $\gamma > 0$ *be so small that* $[z-\gamma, z+\gamma[\subset]0,1[$. *Then,*

$$\begin{aligned}\langle f, e_\gamma\rangle_{X\times X^*} &= \langle Af, v_\gamma\rangle_{X\times X^*} = \int_0^1 Af(t)\mathrm{d}v_\gamma(t) = -\int_0^1 Af(t)\mathrm{d}E_\gamma(t)\\ &= \frac{Af(z+\gamma) - Af(z-\gamma)}{2\gamma} = (Af)'(z) + \mathrm{O}(\gamma^2) = f(z) + \mathrm{O}(\gamma^2),\end{aligned}$$

that is, $\langle Af, v_\gamma\rangle$ *is the central difference of* Af *about* z *with step size* γ.

For the second example consider $E_\gamma(\cdot) = E^{\mathrm{b}}(\cdot/\gamma)/\gamma$ *where* E^{b} *is from* (2.15). *Here,*

$$\langle f, e_\gamma\rangle_{X\times X^*} = -\int_0^1 Af(t)\mathrm{d}E_\gamma(t) = \frac{4Af(\gamma/2) - Af(\gamma) - 3Af(0)}{\gamma} = f'(0) + \mathrm{O}(\gamma^2)$$

is a one-sided finite difference of Af *about* 0 *of second order. Similarly,* $v_\gamma(\cdot) = -E^{\mathrm{b}}((1-\cdot)/\gamma)/\gamma$ *gives rise to a second order one-sided finite difference of* Af *about* 1.

We still need to specify the matrix G_n and the mapping Ψ_n' appearing in (4.2). To this end we first introduce the family $\{\varphi_{n,k}\}_{k=1}^n \subset Y$ and the operator $\mathcal{I}_n \colon Y \to Y$ which are both connected to Ψ_n (2.1) via

$$\mathcal{I}_n y := \sum_{k=1}^n \langle \psi_{n,k}, y\rangle_{Y^*\times Y}\, \varphi_{n,k}, \qquad y \in Y. \tag{4.5}$$

The map $\mathcal{I}_n$ is supposed to fulfill the approximation condition

$$\lim_{n\to\infty} \|\mathcal{I}_n y - y\|_Y = 0 \quad \text{for any } y \in Y. \tag{4.6}$$

Besides we require the uniform boundedness

$$\|\mathcal{I}_n\|_{Y\to Y} \le C_{\mathrm{b}}, \tag{4.7}$$

i.e., $C_{\mathrm{b}} > 0$ is a constant. Note that (4.6) implicitly poses a condition on the space Y, too.

Now we define $\Psi_n' \colon Y^* \to \mathbb{R}^n$ by $(\Psi_n' v)_k := \langle v, \varphi_{n,k}\rangle_{Y^*\times Y}$, $k = 1, \dots, n$. Finally the matrix G_n is to be defined as

$$(G_n)_{j,k} := \langle \psi_{n,j}, \varphi_{n,k}\rangle_{Y^*\times Y}, \qquad j,k = 1,\dots,n,$$

yielding the important relation

$$\langle \Psi_n w, G_n\, \Psi_n' v\rangle_2 = \langle \mathcal{I}_n w, \mathcal{I}_n^* v\rangle_{Y\times Y^*}, \qquad w \in Y, \quad v \in Y^*. \tag{4.8}$$

$$\begin{tikzcd}
 & Y \arrow[d, "\mathcal{J}_n"] & & Y^* \arrow[d, "\mathcal{J}_n^*"] \\
X \arrow[r, "A"] \arrow[rr, bend right, "A_n"'] \arrow[d, "E_d"'] & Y \arrow[r, "\Psi_n"] & \mathbb{R}^n \arrow[dll, "\widetilde{A}_{n,d}"] & Y^* \arrow[l, "\Psi_n'"'] \\
\operatorname{span}\{b_{d,1},\dots,b_{d,d}\} \subset X & & &
\end{tikzcd}$$

FIG. 4.1. *A diagram of all operators involved in the convergence result for the approximate inverse $\widetilde{A}_{n,d}$ as stated in Theorem 4.3.*

We have all ingredients together (see Figure 4.1) to formulate an estimate of the approximation error which comes from applying the approximate inverse $\widetilde{A}_{n,d}$.

THEOREM 4.3 (Noise-free case). *Let A, E_d, Ψ_n, Ψ_n', and $\mathcal{J}_n$ be as stated in this chapter. Further assume that the family $\{b_{d,i}\}_{i=1}^d \subset X$ satisfies (2.5) and that the triplets*

$$\{(e_{d,i}, v_i, b_{d,i})\}_{i=1}^d \subset X^* \times Y^* \times X$$

fulfill the conditions (2.6) *and* (4.1) *for $\varepsilon_i > 0$. Finally the discrete kernels in* (2.20) *are to be defined as in* (4.2). *Then,*

$$\begin{aligned}\|\widetilde{A}_{n,d}\,A_n f - f\|_X &\leq \|(E_d - I)f\|_X \\ &\quad + (C_\mathrm{b}+1)\sigma(d)\big(\|\mathcal{J}_n\,Af - Af\|_Y \max_{1\leq i\leq d}\|v_i\|_{Y^*} + \max_{1\leq i\leq d}\varepsilon_i\|f\|_X\big)\end{aligned} \tag{4.9}$$

for any $f \in X$. Choosing $d = d(n) = d(n,f)$ such that $d(n) \to \infty$ as $n \to \infty$ as well as

$$\sigma\big(d(n)\big)\|\mathcal{J}_n\,Af - Af\|_Y \max_{1\leq i\leq d(n)}\|v_i\|_{Y^*} \to 0 \qquad \text{as } n \to \infty \tag{4.10}$$

and

$$\sigma\big(d(n)\big) \max_{1\leq i\leq d(n)} \varepsilon_i \to 0 \qquad \text{as } n \to \infty \tag{4.11}$$

we have the convergence

$$\lim_{n\to\infty} \|\widetilde{A}_{n,d(n)}A_n f - f\|_X = 0.$$

Proof. An application of the triangle inequality gives

$$\|\widetilde{A}_{n,d}\,A_n f - f\|_X \leq \|(E_d - I)f\|_X + \|\widetilde{A}_{n,d}\,A_n f - E_d f\|_X.$$

and thus we need only to estimate the second part. Property (2.5) of the system $\{b_{d,i}\}_{i=1}^d$ yields

$$\|\widetilde{A}_{n,d}\,A_n f - E_d f\|_X \leq \sigma(d) \max_{1\leq i\leq d}\big|\langle A_n f, G_n\,\Psi_n' v_i\rangle_2 - \langle f, e_{d,i}\rangle_{X\times X^*}\big|.$$

Using (4.1) and the identity (4.8) we estimate

$$\begin{aligned}|\langle A_n f, G_n\,\Psi'_n v_i\rangle_2 - \langle f, e_{d,i}\rangle_{X\times X^*}| &\le |\langle \mathcal{I}_n\, Af, \mathcal{I}_n^* v_i\rangle_{Y\times Y^*} - \langle Af, v_i\rangle_{Y\times Y^*}| \\ &\qquad + |\langle f, A^* v_i - e_{d,i}\rangle_{X\times X^*}| \\ &\le \|\mathcal{I}_n^2\, Af - Af\|_Y\, \|v_i\|_{Y^*} + \varepsilon_i\, \|f\|_X.\end{aligned}$$

By (4.7) we obtain first

$$\begin{aligned}\|\mathcal{I}_n^2\, Af - Af\|_Y &\le \|\mathcal{I}_n(\mathcal{I}_n Af - Af)\|_Y + \|\mathcal{I}_n\, Af - Af\|_Y \\ &\le (C_b + 1)\|\mathcal{I}_n\, Af - Af\|_Y\end{aligned}$$

and then

$$|\langle A_n f, G_n\,\Psi'_n v_i\rangle_2 - \langle f, e_{d,i}\rangle_{X\times X^*}| \le (C_b+1)\|\mathcal{I}_n\, Af - Af\|_Y \|v_i\|_{Y^*} + \varepsilon_i \|f\|_{\mathcal{X}}$$

which finishes the proof. □

Typical operators of ill-posed problems have a kind of smoothing property which we abstractly state as the mapping property

$$A\colon \mathcal{X} \to \mathcal{Y} \quad \text{boundedly} \tag{4.12}$$

where the Banach spaces $\mathcal{X}$ and $\mathcal{Y}$ are boundedly embedded in X and Y, respectively. For instance, the integration operator (4.3) maps $\mathcal{C}^\alpha(0,1)$ continuously to $\mathcal{C}^{\alpha+1}(0,1)$ for any $\alpha \ge 0$.

In this context it is further natural to assume that the convergences (2.6) and (4.6) are uniform for smooth elements in X and Y, respectively: Let there exist non-negative sequences $\{\tau_d\}$ and $\{\rho_d\}$ converging to zero such that

$$\|E_d w - w\|_X \le \tau_d \|w\|_{\mathcal{X}} \quad \text{and} \quad \|\mathcal{I}_n y - y\|_Y \le \rho_n \|y\|_{\mathcal{Y}}. \tag{4.13}$$

COROLLARY 4.4. *Under* (4.12) *and* (4.13) *we have that*

$$\|\widetilde{A}_{n,d}\, A_n f - f\|_X \le C\Big(\tau_n + \sigma(d)\big(\rho_n \max_{1\le i\le d} \|v_i\|_{Y^*} + \max_{1\le i\le d} \varepsilon_i\big)\Big)\|f\|_{\mathcal{X}}$$

for a constant $C > 0$.

Proof. The stated estimate follows readily from (4.9) when taking into account that

$$\|\mathcal{I}_n\, Af - Af\|_Y \le \rho_n \|Af\|_{\mathcal{Y}} \le \rho_n \|A\|_{\mathcal{X}\to\mathcal{Y}} \|f\|_{\mathcal{X}}$$

and that $\|f\|_X \le C_{\mathrm{e}} \|f\|_{\mathcal{X}}$ which is the bounded embedding $\mathcal{X} \hookrightarrow X$. □

REMARK 4.5. *a) The coupling* (4.10) *of the regularization parameter d and the number of data n is a typical intertwining of regularization and discretization, cf. Plato and Vainikko* [11]. *We further remark that $\sigma(d)$ actually might be increasing. In a situation where $\sigma(d)$ increases polynomially we can choose $\varepsilon_i = \exp(-d)$ for $i = 0, \dots, d$ to insure* (4.11).

b) The crux is to find a $v_i \in Y^*$ satisfying (4.1) *since this condition depends on the minimum norm solution f which is not known. If there exists a linear mapping $B\colon Y \to X$ such that $I_X = BA$, then obviously $v_i = B^* e_{d,i}$ solves $A^* v_i = e_{d,i}$. Note that this does in general not guarantee that $v_i \in Y^*$. This depends on the specific setting for A, B, X and Y. An alternative would be an operator $B\colon Y \to X$ with the property that*

$\Lambda = BA$ is a pseudodifferential operator on X and X a Banach space of functions. Then $v_i = B^* e_{d,i}$ is a reconstruction kernel with $A^* v_i = \Lambda^* e_{d,i}$ which can be used to compute the moments $\langle \Lambda f, e_{d,i}\rangle_{X\times X^*} = \langle y, v_i\rangle_{Y\times Y^*}$. This technique e.g. is applied in local tomography, see Rieder, Dietz and Schuster [14]*, sonar, see Quinto, Rieder and Schuster* [12] *or feature reconstruction, see Louis* [7]*. Of course then $e_{d,i}$ has to be substituted by $\Lambda^* e_{d,i}$ in* (4.1)*. If this is not possible at least any a priori information such as e.g. $f \in U_r(f_*) = \{x \in X : \|x - f_*\|_X < r\}$ might be helpful.*

In general we have to compute d kernels v_i what might be time consuming. But there is a remedy. Provided that A^* obeys a certain invariance property, then it is possible to solve (4.1) only once. Lemma 4.6 can be seen as a generalization of Lemma 2.3 in [16] to Banach spaces.

LEMMA 4.6. *Assume that operators $T_i \in \mathcal{L}(X^*)$, $S_i \in \mathcal{L}(Y^*)$ are given with $T_i A^* = A^* S_i$, $i = 1,\dots,d$. Define $e_{d,i} = T_i e$, $i = 1,\dots,d$ for $e \in X^*$. Provided that*

$$|\langle A^* v - e, T_i^* f\rangle_{X^*\times X}| \le \varepsilon\, \|T_i^* f\|_X\,, \qquad i = 1,\dots,d \tag{4.14}$$

for $v \in Y^*$, then

$$|\langle A^* v_i - e_{d,i}, f\rangle_{X^*\times X}| \le \varepsilon_i\, \|f\|_X\,, \qquad i = 1,\dots,d\,, \tag{4.15}$$

where $v_i = S_i v$ and $\varepsilon_i = \varepsilon\, \|T_i\|_{X^*\to X^*}$.

Proof. Note, that the existence of a $v \in Y^*$ fulfilling (4.14) is guaranteed since the set $\{T_i^* f\}_{i=1}^d$ is finite and thus generates a neighborhood of zero in X^* with respect to the weak*-topology.
A simple calculation shows

$$|\langle A^* S_i v - T_i e, f\rangle_{X^*\times X}| = |\langle (A^* v - e), T_i^* f\rangle_{X^*\times X}| \le \varepsilon\, \|T_i^* f\|_X$$

which implies assertion (4.15) because of $\|T_i^*\|_{X\to X} = \|T_i\|_{X^*\to X^*}$, see, e.g., Rudin [17, Theorem 4.10]. □

EXAMPLE 4.7. *Our previous Examples 2.1 and 4.2 have been preparatory work to define and study an approximate inverse for the semi-discrete operator $A_n = \Psi_n A \colon X \to \mathbb{R}^{n+1}$, $X = (\mathcal{C}(0,1), \|\cdot\|_\infty)$, where $A \colon X \to X$ is the integration operator* (4.3) *and where $\Psi_n \colon X \to \mathbb{R}^{n+1}$ is evaluation at the n points $x_{n,k} = k/n$, $k = 0,\dots,n$. We emphasize that $Y = X$ and that $X^* = Y^* = NBV(0,1)$ in this example.*

Let $\psi_{n,0} = \chi_{]0,1]}$ and $\psi_{n,k} = \chi_{[x_{n,k},1]}$, $k = 1,\dots,n$. Then, $\psi_{n,k} \in X^*$ and

$$(\Psi_n g)_k = \langle \psi_{n,k}, g\rangle_{X^*\times X} = g(x_{n,k}). \tag{4.16}$$

With Ψ_n we associate the piecewise linear interpolation operator $\mathcal{I}_n \colon X \to X$,

$$\mathcal{I}_n g = \sum_{k=0}^{n} \langle \psi_{n,k}, g\rangle_{X^*\times X}\, b_{n,k} = \sum_{k=0}^{n} g(x_{n,k}) b_{n,k},$$

where the $b_{n,k}$'s are the linear B-splines from (2.7) *and* (2.8)*. Note that $(\mathcal{I}_n g)(x_{n,\ell}) = g(x_{n,\ell})$. Since $\mathcal{I}_n$ reproduces affine-linear functions we have*

$$\|\mathcal{I}_n g - g\|_\infty \le C_{\mathcal{I}}\, n^{-\beta} \|g\|_{\mathcal{C}^\beta(0,1)}, \quad 0 \le \beta \le 2, \tag{4.17}$$

by the arguments from Appendix A. Above estimate corresponds to right estimate in (4.13)*. For $\beta = 0$ we get the uniform boundedness* (4.7)*.*

In the present setting $\Psi_n'\colon X^* \to \mathbb{R}^{n+1}$ is given by

$$(\Psi_n' v)_k = \langle b_{n,k}, v\rangle_{X\times X^*}$$

and G_n is just the identity matrix of order $n+1$. In fact,

$$(G_n)_{k,\ell} = \langle \psi_{n,k}, b_{n,\ell}\rangle_{X^*\times X} = b_{n,\ell}(x_{n,k}) = \delta_{k,\ell}.$$

In view of Examples 2.1 and 4.2 we define the approximate inverse for A_n by

$$\widetilde{A}_{n,d} w = \sum_{i=0}^{d} \langle w, \Psi_n' v_{d,i}\rangle_2\, b_{d,i}, \quad v_{d,i} = -E_{d,i}, \tag{4.18}$$

with $E_{d,i}$ from (2.11) *and* (2.12)*. If $n = d$ then $\langle w, \Psi_n' v_{d,i}\rangle_2$ is easily evaluated to be*

$$\widetilde{A}_{n,n} w(x_{n,k}) = \langle w, \Psi_n' v_{n,k}\rangle_2 = n \begin{cases} w_1 - w_0 & : \ k = 0, \\ (w_{k+1} - w_{k-1})/2 & : \ k = 1, \dots, n-1, \\ w_n - w_{n-1} & : \ k = n. \end{cases}$$

All hypotheses of Theorem 4.3 are satisfied with $\sigma(d) = 1$, see (2.9)*, $\varepsilon_i = 0$, $i = 0, \dots, d$, cf.* (4.4)*. Further, $\|v_{d,i}\|_{X^*} \le 2\, d$ since the kernels $v_{d,i}$ are piecewise constant functions with two jumps of height d at most. Thus* (4.9) *reads in the present setting as*

$$\|f - \widetilde{A}_{n,d} A_n f\|_\infty \le \|f - \mathbf{E}_d f\|_\infty + 2(C_\mathrm{b} + 1)\, d\, \|\mathcal{I}_n A f - A f\|_\infty.$$

The smoothing property

$$A\colon \mathcal{C}^\alpha(0,1) \to \mathcal{C}^{1+\alpha}(0,1) \quad \textit{boundedly for any } \alpha \ge 0 \tag{4.19}$$

together with (4.17) *implies that*

$$\|f - \widetilde{A}_{n,d} A_n f\|_\infty \le \|f - \mathbf{E}_d f\|_\infty + 2(C_\mathrm{b} + 1)\, d\, n^{-1} \|A\|_{X \to \mathcal{C}^1(0,1)} \|f\|_\infty.$$

Setting $d(n) = n^{1-\gamma}$ for one $0 < \gamma < 1$ and recalling (2.13) *we have convergence:*

$$d(n) = n^{1-\gamma} \implies \lim_{n\to\infty} \|f - \widetilde{A}_{n,d(n)} A_n f\|_\infty = 0 \quad \textit{for any } f \in \mathcal{C}(0,1). \tag{4.20}$$

Consider (4.19) *for one $\alpha > 0$ and set $\mathcal{X} = \mathcal{C}^\alpha(0,1)$ and $\mathcal{Y} = \mathcal{C}^{1+\alpha}(0,1)$. In view of* (2.14) *and* (4.17) *the hypotheses of Corollary* 4.4 *are met. If $n = d$ and $f \in \mathcal{C}^\alpha(0,1)$ then* (4.9) *yields convergence with rates:*

$$\begin{aligned} \|f - \widetilde{A}_{n,n} A_n f\|_\infty &\le C_\mathbf{E}\, n^{-\min\{\alpha,1\}} \|f\|_{\mathcal{C}^\alpha(0,1)} + C\, n\, n^{-\min\{1+\alpha,2\}} \|f\|_{\mathcal{C}^\alpha(0,1)} \\ &\le \max\{C_\mathbf{E}, C\}\, n^{-\min\{\alpha,1\}} \|f\|_{\mathcal{C}^\alpha(0,1)}. \end{aligned} \tag{4.21}$$

At the end of this section we come back to our previous statement that the approximate inverse $\widetilde{A}_{n,d} g_n$ approximates the minimum norm solution $f_n^\dagger$ of (2.3). At first we realize that the assertions of Theorem 4.3 and Theorem 5.1 remain valid even if we set

$$v_i^n := P_{\mathcal{R}(A_n)} G_n \Psi_n' v_i, \qquad i = 1, \dots, d, \tag{4.22}$$

instead of (4.2), since in the corresponding proofs we take the inner product of the kernels v_i^n with $A_n f$ which is in $\mathcal{R}(A_n)$. Within this setting we are able to prove that the approximate inverse converges to $f_n^\dagger$ as $d \to \infty$ in case of a uniformly convex X.

THEOREM 4.8. *Let X be uniformly convex and let $f_n^\dagger \in X$ be the minimum norm solution of* (2.3) *with data $g_n \in \mathbb{R}^n$. Furthermore let v_i^n be defined in* (4.22) *where the $v_i \in Y^*$ fulfill $\|A^* v_i - e_{d,i}\|_{X^*} < \varepsilon_i$ for given real numbers $\varepsilon_i > 0$, $i = 1, \ldots, d$. Then,*

$$\lim_{d\to\infty} \|\widetilde{A}_{n,d} g_n - f_n^\dagger\|_X = 0 \tag{4.23}$$

provided that $\lim_{d\to\infty} \sigma(d) \max_{1\le i\le d} \varepsilon_i = 0$.

Proof. Using the triangle inequality yields

$$\|\widetilde{A}_{n,d} g_n - f_n^\dagger\|_X \le \|\widetilde{A}_{n,d} g_n - E_d f_n^\dagger\|_X + \|E_d f_n^\dagger - f_n^\dagger\|_X,$$

where the latter summand tends to zero as $d \to \infty$ due to the mollifier property (2.6). Since

$$g_n = P_{\mathcal{R}(A_n)} g_n + P_{\mathcal{R}(A_n)^\perp} g_n$$

and $A_n f_n^\dagger = P_{\mathcal{R}(A_n)} g_n$ we have

$$\langle v_i^n, g_n\rangle_2 = \langle A_n^* v_i^n, f_n^\dagger\rangle_{X^*\times X} + \langle v_i^n, P_{\mathcal{R}(A_n)^\perp} g_n\rangle_2 \overset{(4.22)}{=} \langle A_n^* v_i^n, f_n^\dagger\rangle_{X^*\times X}.$$

Taking this equality into account we may estimate

$$\begin{aligned}\|\widetilde{A}_{n,d} g_n - E_d f_n^\dagger\|_X &= \Big\| \sum_{i=1}^d \langle A_n^* v_i^n - e_{d,i}, f_n^\dagger\rangle_{X^*\times X} b_{d,i} \Big\|_X \\ &\overset{(2.5)}{\le} \sigma(d) \max_{1\le i\le d} \|A_n^* v_i^n - e_{d,i}\|_{X^*} \|f_n^\dagger\|_X \\ &\le \sigma(d) \max_{1\le i\le d} \varepsilon_i \|f_n^\dagger\|_X\end{aligned}$$

which finally proves (4.23). □

The orthogonal projection onto $\mathcal{R}(A_n)$ in (4.22) can be omitted if $g_n = A_n f$.

REMARK 4.9. *As the proofs in this section show things get easier if X is uniformly smooth. But throughout this section Y is arbitrary. The only condition to Y is that it allows for an approximation as* (4.6).

5. Regularization property. Here we investigate the regularization property of the approximate inverse $\widetilde{A}_{n,d}$, that means the stability of the approximate solution with respect to noise in the given data. As in [15, 16] we interprete noise contaminated data as a perturbation of our observation operator Ψ_n. More explicitly, we define

$$(\Psi_n^\delta y)_k = (\Psi_n y)_k + \delta_k \|y\|_Y, \quad |\delta_k| \le \delta, \quad y \in Y, \tag{5.1}$$

for a positive number δ which represents the noise level. We can show that an appropriate coupling of the parameters n and d to the noise level δ gives convergence when δ goes to zero.

THEOREM 5.1 (Regularization property). *Adopt the hypotheses of Theorem* 4.3. *Assume further that the triplets $\{(e_{d,i}, v_i, b_{d,i})\}_{i=1}^d \subset X^* \times Y^* \times X$ allow a coupling of d with n such that $d = d(n) = d(n,f) \to \infty$ as $n \to \infty$ and* (4.10) *as well as* (4.11) *apply.*

If we couple $n = n_\delta$ with the noise level δ such that $n_\delta \to \infty$ when $\delta \to 0$ as well as

$$\delta\, \sigma\big(d(n_\delta)\big)\, \sqrt{n_\delta} \max_{1 \le i \le d(n_\delta)} \|G_{n_\delta}\, \Psi'_{n_\delta} v_i\|_2 \to 0 \qquad \text{as } \delta \to 0 \tag{5.2}$$

then

$$\lim_{\delta \to 0} \sup\{\|\widetilde{A}_{n_\delta, d(n_\delta)} \Psi^\delta_{n_\delta} Af - f\|_X : \Psi^\delta_{n_\delta} \text{ fulfills } (5.1)\} = 0.$$

Proof. We denote by $g_n = A_n f$ the exact data and by $g_n^\delta = \Psi_n^\delta Af$ the noise contaminated data. Using again property (2.5) of the family $\{b_{d,i}\}$ we find that

$$\begin{aligned} \|\widetilde{A}_{n,d}(g_n - g_n^\delta)\|_X &\le \sigma(d) \max_{1 \le i \le d} \Big|\langle (\Psi_n - \Psi_n^\delta)\, Af, G_n\, \Psi'_n v_i\rangle_2\Big| \\ &\le \sigma(d)\, \delta \sqrt{n}\, \|A\|_{X \to Y} \|f\|_X \max_{1 \le i \le d} \|G_n\, \Psi'_n v_i\|_2 \end{aligned} \tag{5.3}$$

yielding

$$\begin{aligned} \|\widetilde{A}_{n,d}\, g_n^\delta - f\|_X \le \|E_d f - f\|_X + C\,\sigma(d) \bigg(& \|\mathcal{I}_n Af - Af\|_Y \max_{1 \le i \le d} \|v_i\|_{Y^*} \\ & + \max_{1 \le i \le d} \varepsilon_i \|f\|_X + \delta \sqrt{n} \max_{1 \le i \le d} \|G_n\, \Psi'_n v_i\|_2 \|f\|_X \bigg) \end{aligned}$$

where $C > 0$ denotes a properly chosen constant. Replacing n by n_δ and d by $d(n_\delta)$ leads to the claimed convergence under the assumed coupling conditions. □

EXAMPLE 5.2. *We revisit Example* 4.7 *to apply Theorem* 5.1 *to the approximate inverse* (4.18)*. Let $\Psi_n^\delta : X \to \mathbb{R}^{n+1}$, $X = (\mathcal{C}(0,1), \|\cdot\|_\infty)$, be a perturbation of Ψ_n* (4.16)*:*

$$(\Psi_n^\delta g)_k = g(x_{n,k}) + \delta_k \|g\|_\infty, \quad |\delta_k| \le \delta. \tag{5.4}$$

As G_n is the identity matrix and the vector $\Psi'_n v_{d,i}$ has 4 non-zero entries at most we deduce that

$$\|G_n \Psi'_n v_{d,i}\|_2 = \|\Psi'_n v_{d,i}\|_2 \le 4 \max_{0 \le k \le n} |\langle b_{n,k}, v_{d,i}\rangle_{X \times X^*}| \le 8\, d. \tag{5.5}$$

Consider the case $f \in \mathcal{C}(0,1)$. Set $d(n) = n^{1-\gamma}$ for one positive $\gamma < 1$, see (4.20)*, and determine $n = n_\delta$ such that n_δ increases unboundedly as $\delta \to 0$ while $\lim_{\delta \to 0} n_\delta^{3/2-\gamma} \delta = 0$. Then,* (5.2) *holds true and we have the regularization property*

$$\lim_{\delta \to 0} \sup \big\{\|f - \widetilde{A}_{n_\delta, d(n_\delta)} w\|_\infty : w = \Psi^\delta_{n_\delta} Af,\ \Psi^\delta_{n_\delta} \text{ satisfies } (5.4)\big\} = 0.$$

An admissible choice for n_δ is $n_\delta \sim \delta^{\frac{\gamma-1}{3/2-\gamma}}$.

Under the smoothness assumption $f \in \mathcal{C}^\alpha(0,1)$, $\alpha > 0$, we even obtain a convergence order in δ. We derive from (4.21)*,* (5.3)*, and* (5.5) *that*

$$\begin{aligned} \|f - \widetilde{A}_{n,d} \Psi_n^\delta Af\|_\infty &\le \|f - \widetilde{A}_{n,d} \Psi_n Af\|_\infty + \|\widetilde{A}_{n,d}(\Psi_n - \Psi_n^\delta) Af\|_\infty \\ &\le C\big(n^{-\min\{1,\alpha\}} + \delta\, d \sqrt{n}\big) \|f\|_{\mathcal{C}^\alpha(0,1)}. \end{aligned}$$

Choosing $n_\delta \sim \delta^{\frac{-1}{\min\{5/2, 3/2+\alpha\}}}$ and $d(n_\delta) = n_\delta$ results in

$$\|f - \widetilde{A}_{n_\delta, n_\delta} \Psi^\delta_{n_\delta} Af\|_\infty = \mathrm{O}\big(\delta^{\frac{\min\{1,\alpha\}}{\min\{5/2, 3/2+\alpha\}}}\big) \quad \text{as } \delta \to 0.$$

The convergence order saturates at $2/5$.

A. Appendix: proofs of mollifier property (2.13) **and approximation properties** (2.14) **and** (2.17)**.** Let f be in $\mathcal{C}(0,1)$ and $\mathbf{E}_d$ be defined as in (2.10). We will prove that

$$\|f - \mathbf{E}_d f\|_\infty \le 2\,\omega(f;3h) \tag{A.1}$$

where $h = 1/d$ and ω is the modulus of continuity:

$$\omega(f;\tau) = \sup\{|f(x) - f(y)| \,:\, x,y \in [0,1],\, |x-y| \le \tau\}.$$

Note that (A.1) immediately yields (2.13) as well as (2.14).

We follow a standard procedure in approximation theory, see, e.g., Oswald [10, Sec. 2.2]. First we bound $\mathbf{E}_d\colon \mathcal{C}(I_{r,r+1}) \to \mathcal{C}(I_{r,r+1})$, $r = 0,\dots,d-1$, uniformly in d. Here, $I_{i,k}$ denotes the interval $[x_{d,i}, x_{d,k}]$ for $i < k$. We obtain

$$\begin{aligned}\|\mathbf{E}_d f\|_{\mathcal{C}(I_{r,r+1})} &= \sup_{x\in I_{r,r+1}} |\mathbf{E}_d f(x)| = \sup_{x\in I_{r,r+1}} |\langle f, e_{d,r}\rangle b_{d,r}(x) + \langle f, e_{d,r+1}\rangle b_{d,r+1}(x)|\\ &\le \max\{|\langle f, e_{d,r}\rangle|, |\langle f, e_{d,r+1}\rangle|\} \le \sup_{x\in I_{r-1,r+2}} |f(x)| = \|f\|_{\mathcal{C}(I_{r-1,r+2})},\end{aligned}$$

where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{X\times X^*}$ and where we set $x_{d,-1} = 0$ and $x_{d,d+1} = 1$. As $\mathbf{E}_d$ reproduces constant functions p we find

$$\begin{aligned}\|f - \mathbf{E}_d f\|_{\mathcal{C}(I_{r,r+1})} &= \|f - p\|_{\mathcal{C}(I_{r,r+1})} + \|\mathbf{E}_d(f-p)\|_{\mathcal{C}(I_{r,r+1})}\\ &\le 2\|f-p\|_{\mathcal{C}(I_{r-1,r+2})}.\end{aligned} \tag{A.2}$$

Choosing especially $p = \frac{1}{|I_{r-1,r+2}|}\int_{I_{r-1,r+2}} f(t)\mathrm{d}t$ we get

$$\|f-p\|_{\mathcal{C}(I_{r-1,r+2})} = \frac{1}{|I_{r-1,r+2}|}\sup_{x\in I_{r-1,r+2}} \Big|\int_{I_{r-1,r+2}} \big(f(x) - f(t)\big)\mathrm{d}t\Big| \le \omega(f;3h).$$

Thus

$$\|f - \mathbf{E}_d f\|_{\mathcal{C}(0,1)} = \max_{r\in\{0,\dots,d-1\}} \|f - \mathbf{E}_d f\|_{\mathcal{C}(I_{r,r+1})} \le 2\,\omega(f;3h),$$

which is (A.1). Finally, we validate (2.17). By (2.16) estimate (A.2) remains valid for $p \in \Pi_1$, that is,

$$\|f - \mathbf{E}_d f\|_{\mathcal{C}(I_{r,r+1})} \le 2\inf\{\|f-p\|_{\mathcal{C}(I_{r-1,r+2})} : p \in \Pi_1\}.$$

Applying Jackson's theorem, see, e.g., Schumaker [21, Theorem 3.12], yields (2.17).

B. Appendix: Integration operator in a Hilbert space setting. The integration operator A (4.3) can also be considered in a Hilbert space setting, that is, as a bounded operator $A\colon L^2(0,1) \to L^2(0,1)$. Our results from the previous sections apply here as well.

First we define the bounded observation operator (2.1) by

$$\Psi_n\colon L^2(0,1) \to \mathbb{R}^{n+1}, \quad (\Psi_n g)_k = \langle \psi_{n,k}, g\rangle_{L^2(0,1)},\ k = 0,\dots,n, \tag{B.1}$$

where the $\psi_{n,k}$'s are suitable L^2-functions to be specified later. To set up the approximate inverse for $A_n = \Psi_n A\colon L^2(0,1) \to \mathbb{R}^{n+1}$ we start with computing reconstruction kernels for $A\colon L^2(0,1) \to L^2(0,1)$ whose adjoint is

$$A^*\colon L^2(0,1) \to L^2(0,1), \quad g \mapsto A^* g(x) = \int_x^1 g(t)\mathrm{d}t.$$

Set $v = -e'$ for $e \in H_0^1(0,1)$ with $\int_0^1 e(t)\mathrm{d}t = 1$. Then, $A^*v = e$, i.e., v is a reconstruction kernel corresponding to the mollifier e. Indeed, if e is localized about x we have

$$f(x) \approx \langle f, e\rangle_{L^2} = \langle Af, v\rangle_{L^2} = -\langle Af, e'\rangle_{L^2} = \langle (Af)', e\rangle_{L^2}$$

using integration by parts for the last equality.

We define the mollification operator $\mathbf{E}_d\colon L^2(0,1) \to L^2(0,1)$, $d \geq 2$, by

$$\mathbf{E}_d f = \sum_{i=0}^{d} \langle f, e_{d,i}\rangle_{L^2} b_{d,i}$$

where the $b_{d,i}$'s are as in (2.7) and (2.8). Estimate (2.9) holds true even if we replace the sup-norm by the L^2-norm. Thus, (2.5) is valid with $\sigma(d) = 1$.

The mollifiers in $\mathbf{E}_d$ are generated in the following way:

$$e_{d,i}(x) = de(dx - i), \quad i = 1, \dots, d-1,$$

with $e \in H_0^1(-1,1)$, for instance, $e(x) = \frac{35}{32}(1-x^2)^3$, $|x| \leq 1$, and $e(x) = 0$, otherwise. Further,

$$e_{d,0} = 2de(2dx - 1), \quad e_{d,d} = 2de(2dx - (2d-1)).$$

Observe that $\operatorname{supp} e_{d,i} = [x_{d,i-1}, x_{d,i+1}]$, $i = 1, \dots, d-1$, and $\operatorname{supp} e_{d,0} = [0, x_{d,1}]$, $\operatorname{supp} e_{d,d} = [x_{d,d-1}, 1]$. The reconstruction kernels are $v_{d,i} = -(e_{d,i})'$.

Since $\mathbf{E}_d$ reproduces constant functions we have

$$\lim_{d\to\infty} \|f - \mathbf{E}_d f\|_{L^2} = 0 \quad \text{and} \quad \|f - \mathbf{E}_d f\|_{L^2} \leq C_{\mathbf{E}}\, d^{-\min\{1,\alpha\}} \|f\|_{H^\alpha}$$

by the arguments of Appendix A.

The last ingredient of the approximate inverse is the operator $\mathcal{I}_n\colon L^2(0,1) \to L^2(0,1)$, cf. (4.5), which we define to be

$$\mathcal{I}_n g = \sum_{k=0}^{n} \langle \psi_{n,k}, g\rangle_{L^2} b_{n,k}.$$

There are several choices for the $\psi_{n,k}$'s leading to[3]

$$\|\mathcal{I}_n g - g\|_{L^2} \leq C_{\mathcal{I}}\, n^{-\beta} \|g\|_{H^\beta}, \quad 0 \leq \beta \leq 2.$$

Thus, we have (4.6), (4.7), and (4.13).

Now we can assemble the approximate inverse $\widetilde{A}_{n,d}\colon \mathbb{R}^{n+1} \to L^2(0,1)$ as

$$\widetilde{A}_{n,d} w = \sum_{i=0}^{d} \langle w, G_n \Psi_n' v_{d,i}\rangle_2\, b_{d,i}, \quad v_{d,i} = -(e_{d,i})',$$

where $\Psi_n'\colon L^2(0,1) \to \mathbb{R}^{n+1}$, $(\Psi_n' u)_k = \langle b_{n,k}, u\rangle_{L^2}$ and G_n is the matrix of order $n+1$ with entries $(G_n)_{k,\ell} = \langle \psi_{n,k}, b_{n,\ell}\rangle_{L^2}$. Typically, the $\psi_{n,k}$'s will have a small support and thus, G_n will be a banded matrix with a band-width being independent of n.

[3]One choice can be found in [15, Example 3.1].

Similar to the Banach space setting the integration operator smooths in terms of Sobolev scales,

$$A\colon H^\alpha(0,1) \to H^{\alpha+1}(0,1) \quad \text{boundedly for any } \alpha \geq 0,$$

cf. (4.12). Hence, the hypotheses of Theorem 4.3 and Corollary 4.4 are fulfilled. Since $\|v_{d,i}\|_{L^2} = \|(e_{d,i})'\|_{L^2} = d^{3/2}\|e'\|_{L^2}$ the error estimate (4.9) reads

$$\|f - \widetilde{A}_{n,d}A_n f\|_{L^2} \leq \|f - \mathbf{E}_d f\|_\infty + C\, d^{3/2}\, n^{-1}\, \|f\|_{L^2}$$

and leads to convergence for $d(n) = n^{2/3-\gamma}$ where $0 < \gamma < 2/3$:

$$d(n) = n^{2/3-\gamma} \implies \lim_{n\to\infty} \|f - \widetilde{A}_{n,d(n)}A_n f\|_{L^2} = 0 \quad \text{for any } f \in L^2(0,1).$$

Assuming smoothness of f, i.e., $f \in H^\alpha(0,1)$ for an $\alpha > 0$, and choosing $d(n) = n^{2/3}$ the error bound of Corollary 4.4 yields

$$\|f - \widetilde{A}_{n,d(n)}A_n f\|_{L^2} \leq C\, n^{-\min\{\alpha,1\}}\, \|f\|_{H^\alpha}.$$

Let us compare the above results with the situation of Example 4.7: If convergence and convergence with rates should hold then we can recover about n moments of the searched for solution from n measurements in the Banach space framework, see (4.20) and (4.21). Whereas the Hilbert space approach allows only for a reliable reconstruction of about $n^{2/3}$ moments from the same amount of data. This seemingly unfavorable result for the L^2-scenery reflects the proper inclusion $\mathcal{C}^\alpha(0,1) \subset H^\alpha(0,1)$, $\alpha \geq 0$, that is, the Hilbert space contains irregular elements being not in its Banach space equivalent.

REMARK B.1. *Note that our above Banach and Hilbert space approximate inverses of the integration operator are directly comparable since all their ingredients match their respective counterparts of the other setting. For instance, since point evaluation cannot be defined boundedly on L^2 we replaced the observation operator* (4.16) *by the operator* (B.1) *which takes averages. However, point evaluation can be defined boundedly on the range of $A\colon L^2(0,1) \to L^2(0,1)$ and we may replace the observation operator* (B.1) *by*

$$\Psi_n\colon H^1(0,1) \to \mathbb{R}^{n+1}, \quad (\Psi_n g)_k = g(x_{n,k}),\ k = 0,\ldots,n,$$

compare (4.16)*. Here we can apply the approximate inverse according to our theory developed in* [15, 16]*, see also* [13] *and* [24]*. The relation of number of measurements and number of reconstructible moments we obtain gets worse: To yield convergence for an L^2-function we can only recover about $n^{1/2}$ moments.*

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