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On bounded perturbations of linear operators

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1. Introduction. Our starting-point is Theorem 1 of Donald H. Hyers [3]. We state it almost as in [3], but we do not require the space X to be a Banach space (because this is not necessary).

Hyers' Theorem. *Consider $f : X \to Y$, where X is a (real or complex) vector space and Y is a Banach space. Suppose $\beta > 0$. Then*

$$\text{I)} \qquad \|f(x) + f(y) - f(x+y)\| \leq \beta \qquad (x, y \in X)$$

implies the following:
II) *There exists an additive function $L : X \to Y$ such that*

$$\|f(x) - Lx\| \leq \beta \qquad (x \in X).$$

The function L is unique, and it is given by

$$(1) \qquad Lx = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \qquad (x \in X).$$

Observe that II) implies

$$\|f(x) + f(y) - f(x+y)\| \leq 3\beta \qquad (x, y \in X).$$

In the following example conditions I), II) are not equivalent: $X = Y = I\!R$, $f : I\!R \to I\!R$ given by $f(x) = \sin \frac{\pi}{12}(x^3 + 5x), L = 0$ (the zero operator). We have Range $f = [-1, 1]$, and from $2f(\pm 1) - f(\pm 2) = \pm 3$ we easily get the range of the function

$$f(x) + f(y) - f(x+y) \qquad (x, y \in I\!R)$$

to be the interval $[-3, 3]$. So we have II) with $\beta = 1$, but I) only can be satisfied by numbers $\beta \geq 3$.

In the next paragraph we shall use Hyers' Theorem to characterize functions $f = L + r$, where L is a linear and r is a bounded (non-linear) function. In paragraph 3 the case where the perturbation r has values in a compact set will be considered; for this we assume f to be continuous (being defined on a normed space X).

From all the numerous generalizations of Hyers' Theorem let us only refer to [7], which also had been used in the paper [1] by Roman Badora, Barbara Przebieracz, and the second author. László Székelyhidi [5] uses Hyers' Theorem, when characterizing linear operators.

2. Bounded perturbations. The following Remark will be used in the sequel.

Remark 1. Let $a : X \to Y$ be a bounded additive function, where X is a (real or complex) vector space and Y is a normed space. Then $a = 0$.

Indeed, if $\|a(x)\| \leq \gamma < \infty$ $(x \in X)$, then we get from $a(nx) = na(x)$ $(n \in I\!N, x \in X)$ that

$$\|a(x)\| = \frac{1}{n}\|a(nx)\| \leq \frac{1}{n}\gamma,$$

and $n \to \infty$ leads to $\|a(x)\| = 0$ $(x \in X)$.

Remark 1 also gives the uniqueness of L in Hyers' Theorem. Suppose $\|f(x) - Lx\| \leq \beta$ for additive $L = L_1$ and $L = L_2$. Then $a = L_1 - L_2$ is additive and $\|a(x)\| \leq 2\beta$ $(x \in X)$, hence $a = 0$, i.e., $L_1 = L_2$.

Theorem 1. *Consider $f : X \to Y$, where X is a vector space and Y is a Banach space, both spaces having the same scalar field Λ of real or complex numbers. Suppose $A \subseteq Y$, A being a bounded and closed set. Then the following two statements are equivalent:*
(P) *$f = L + r$, where $L : X \to Y$ is linear and $r(x) \in A$ $(x \in X)$.*
(Q) *There exist bounded sets $B, C \subseteq Y$ such that*

$$f(x) + f(y) - f(x + y) \in B \qquad (x, y \in X),$$

$$\lambda f(x) - f(\lambda x) \in \lambda A + C \qquad (\lambda \in \Lambda, x \in X).$$

Proof. (P) $\Rightarrow$ (Q): From (P) we get $f(x) + f(y) - f(x + y) = r(x) + r(y) - r(x + y) \in A + A - A$, hence we can take $B = A + A - A$.
Furthermore we get $\lambda f(x) - f(\lambda x) = \lambda r(x) - r(\lambda x) \in \lambda A - A$, hence we can take $C = -A$.

(Q) $\Rightarrow$ (P): The set B being bounded, we can apply Hyers' Theorem to get from (Q) the existence of an additive $L : X \to Y$ such that $r = f - L$ is bounded. Let us assume

$r(x) \in D$ $(x \in X)$, D being a bounded subset of Y.

(Q) implies for $\lambda = n \in I\!N$ that

$nr(x) - r(nx) \in nA + C \qquad (x \in X),$

hence

$$r(x) \in A + \frac{1}{n}C + \frac{1}{n}D \qquad (x \in X),$$

and $n \to \infty$ leads to $r(x) \in A$ $(x \in X)$.

It remains to show the homogeneity of L: We fix $\lambda \in \Lambda$, then (Q) implies

$$\lambda Lx + \lambda r(x) - L(\lambda x) - r(\lambda x) \in \lambda A + C \qquad (x \in X),$$

hence

$$\lambda Lx - L(\lambda x) \in \lambda A + C - \lambda A + A := R_\lambda \qquad (x \in X).$$

R_λ being a bounded set and $\lambda Lx - L(\lambda x)$ being additive with respect to x, we can apply Remark 1 to get $L(\lambda x) = \lambda Lx$ $(x \in X)$. Here $\lambda \in \Lambda$ is arbitrary, which gives the desired result.

Remark 2. If X in Theorem 1 is a normed space, then f is continuous if and only if L, r are continuous. Indeed, if f is continuous, then the linear operator $L = f - r$ is bounded in a neighborhood of the origin of X, hence L is continuous, and finally also $r = f - L$ is continuous.

Remark 3. For $\Lambda = I\!\!R$ and $A = \{x \mid x \in Y, \|x\| \leq \varepsilon\}$ (where $\varepsilon > 0$), Theorem 1 is known from the paper [2] by Roman Ger and the second author.

3. Compact perturbations. Under the assumptions of Hyers' Theorem, suppose $V \subseteq Y, V$ being bounded, closed, and convex. If

$$f(x) + f(y) - f(x+y) \in V \qquad (x, y \in X), \tag{2}$$

then

$$f(x) - Lx \in V \qquad (x \in X) \tag{3}$$

easily follows (cf., e.g., [6] by Jacek Tabor or [7]). Indeed, for $x \in X$ we have

$$f(x) - \frac{1}{2^n} f(2^n x) = \sum_{\nu=1}^{n} \frac{1}{2^\nu} \left[2f\left(2^{\nu-1}x\right) - f\left(2^\nu x\right)\right], \tag{4}$$

where $2f(2^{\nu-1}x) - f(2^\nu x) \in V$ (cf. (2)). Let us take (1) into account, then $n \to \infty$ in (4) leads to (3).

Theorem 2. *Let $f : X \to Y$ be continuous, where X is a real normed space and Y a real Banach space. Then the following two statements are equivalent:*
(R) *There is a continuous linear operator $L : X \to Y$ and a compact set $C \subseteq Y$ such that $f(x) - Lx \in C$ $(x \in X)$.*
(S) *There is a compact set $V \subseteq Y$ such that*

$$f(x) + f(y) - f(x+y) \in V \ (x, y \in X).$$

Proof. (R) $\Rightarrow$ (S): From (R) we get

$$f(x)+f(y)-f(x+y)\in C+C-C\ (x,y\in X),$$

and obviously $V:=C+C-C$ is compact.

(S) $\Rightarrow$ (R): We assume V to be compact and convex (otherwise we replace this compact set by its closed convex hull, which is compact by a result of Stanisław Mazur [4]). Then we get (3), $L:X\to Y$ being additive (and continuous according to Remark 2). Thus $L:X\to Y$ is a continuous linear operator and (R) holds for $C=V$.

Observe that the foregoing proof has a simple structure: From (R) we get (S) by taking $V=C+C-C$, and from (S) we arrive at (R) by choosing C to be the closed convex hull of V.

Remark 4. Theorem 2 also holds for complex spaces X,Y, if to (S) we add the condition

$$\sup_{x\in X}\|if(x)-f(ix)\|<\infty. \tag{5}$$

Indeed, (S) already gives (R), where $L:X\to Y$ is a continuous $\mathbb{R}$-linear operator. By the boundedness of $f-L$ and by (5) we have

$$\|f(x)-Lx\|\le\alpha,\ \|if(x)-f(ix)\|\le\gamma\ (x\in X)$$

for some $\alpha,\gamma\ge 0$. Then

$$\|iLx-L(ix)\|\le$$

$$\le\|iLx-if(x)\|+\|if(x)-f(ix)\|+\|f(ix)-L(ix)\|\le 2\alpha+\gamma$$

holds for all $x\in X$. Consequently, the additive function $a(x)=iLx-L(ix)$ $(x\in X)$ is bounded, and from Remark 1 we get $a=0$. This shows $L(ix)=iLx$ $(x\in X)$, hence the $\mathbb{R}$-linear operator L also is complex-linear.

References

1. Roman Badora, Barbara Przebieracz, Peter Volkmann, *Stability of the Pexider functional equation.* Ann. Math. Silesianae **24** (2010), 7-13 (2011).

2. Roman Ger, Peter Volkmann, *On sums of linear and bounded mappings.* Abh. Math. Sem. Univ. Hamburg **68**, 103-108 (1998).

3. Donald H. Hyers, *On the stability of the linear functional equation.* Proc. Nat. Acad. Sci. U.S.A. **27**, 222-224 (1941).

4. Stanisław Mazur, *Über die kleinste konvexe Menge, die eine gegebene kompakte Menge enthält.* Studia Math. **2**, 7-9 (1930).

5. László Székelyhidi, *On a characterization of linear operators.* Acta Math. Hungar. **71**, 293-295 (1996).

6. Jacek Tabor, *Ideally convex sets and Hyers theorem.* Funkcialaj Ekvac. **43**, 121-125 (2000).

7. Peter Volkmann, *O stabilności równań funkcyjnych o jednej zmiennej.* Sem. LV, No. 11, 6 pp. (2001), Errata ibid. No. 11bis, 1 p. (2003), http://www.math.us.edu.pl/smdk

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