On the Equation of an Origami of Genus two with two Cusps

Diploma thesis

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Preface

Abstract

One of the guiding problems in the history of mathematics has been Riemann's moduli problem. It deals with the question "How many different complex structures can be put on a compact topological surface of genus g?". The moduli space M_g, the space of isomorphism classes of compact Riemann surfaces of genus g, is the corresponding classification space for this problem.

Interestingly, M_g itself carries the structure of an algebraic variety, but a description of its geometry is rather difficult. One approach to understand the geometry of M_g is to examine curves in M_g.

Origamis are combinatorial objects that provide a possibility to construct such curves. An origami is obtained by gluing finitely many unit squares of $\mathbb{C}$ in a certain way. Formally it can be described as a finite covering $p : X \to E$ of a topological torus E, ramified over a single point $\bar{P} \in E$. This results in a compact, square-tiled surface X of genus $g \geq 1$. One can put a translation structure on $X^* = X \backslash p^{-1}(\{\bar{P}\})$ with the help of the tiling and then vary this translation structure by shearing the squares into parallelograms. This yields a Teichmüller embedding $\iota : \mathbb{H} \hookrightarrow \Delta$ into the corresponding Teichmüller space $T_{g,n}$, where $n = |X \setminus X^*|$. The affine group $\mathrm{Aff}^+(X^*)$, the group of orientation preserving affine diffeomorphisms, acts on Δ as a subgroup of the mapping class group, and in fact is the stabilizer of Δ. This action can also be interpreted as an action of the Veech group $\Gamma(X^*) \subset \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ via Moebius transformations, where $\Gamma(X^*) = \mathrm{der}(\mathrm{Aff}^+(X^*))$ is the image of the derived map.

For an origami, the Veech group is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index. Therefore, its quotient $\mathbb{H}/\Gamma(X^*) \cong \Delta/\mathrm{Aff}^+(X^*)$ is an algebraic curve, and it is birationally equivalent to the image of the projection of Δ to M_g, which is thus an algebraic curve $\mathcal{C}$ in M_g.

In this thesis, we examine a particular origami of genus 2. We determine the equation of an affine curve that is birationally equivalent to the curve $\mathcal{C}$ in M_2, given by the origami.

The main result of this thesis is presented in Theorem IV.3.7. The remainder of the thesis is organized as follows.

In Chapter I, we give a first definition of an origami and we present the example origami S that we wish to study (cf. Figure I.2). We point out some results that have already been achieved for other origamis.

Chapter II contains known results about Riemann surfaces that are used later on; in particular, we present some properties of elliptic curves and hyperelliptic surfaces. Furthermore, we introduce the moduli space of curves and the corresponding Teichmüller space.

Chapter III again deals with origamis. First, we investigate translation surfaces, their affine groups and their Veech groups. Then we give another definition of an origami and show how one can construct a bunch of translation surfaces out of an origami. We define the affine group, the Veech group and the group of automorphisms of an origami and state some important properties of these groups (Proposition III.3.11 and Proposition III.3.14). Finally, we establish the relation to the Teichmüller space and to the moduli space and explain how an origami leads to a curve in the moduli space. As an example, we investigate some properties of the curve $\mathcal{C}$ associated to our origami S (cf. Example III.4.10).

In Chapter IV, we study the group of automorphisms of our origami S (cf. Proposition IV.1.5). We show that every translation surface coming from the origami S is birational to an affine plane curve depending on two complex parameters λ, μ (cf. Theorem IV.2.6). Finally, we establish a relation between λ and μ, which leads to our main result about the curve $\mathcal{C}$ associated to the origami S.

Acknowledgements

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Chapter I

Introduction

I.1 A First Approach to Origamis

Here, we introduce a first description of origamis. We present the orgami S that we will study in this thesis and reveal a few of its properties. Finally, we take a look at other origamis that one might be interested in.

An origami is best depicted by giving the following recipe.

I.1.1 Definition. Let us take a finite number of copies of the Euclidean unit square. We glue the squares together at their edges observing the following rules

- Each upper edge of a square is identified with the lower edge of a square by a translation.
- Each left edge of a square is identified with a right edge of a square by a translation.
- The topological space X obtained thereby is connected.

An *origami* is then the result of the gluing process together with the tiling of the topological space into squares.

The name origami goes back to Pierre Lochak [Loc05], where it refers to a more general object. In this thesis, we stick to the case of what one might call oriented origamis as they are presented by Schmithüsen in [Sch04].

There are some other ways in which one can describe origamis (see e.g. [Sch04]); one of these will be presented later in III.3.

I.1.2 Example. The basic example of an origami is made of only one unit square. In this case, there is only one possibility to glue the edges and the result is a compact surface of genus one, i.e. a torus (cf. Figure I.1). We denote E the *basic origami*. Note that there is one distinguished point $\bar{P}$ on E, where the edges meet.

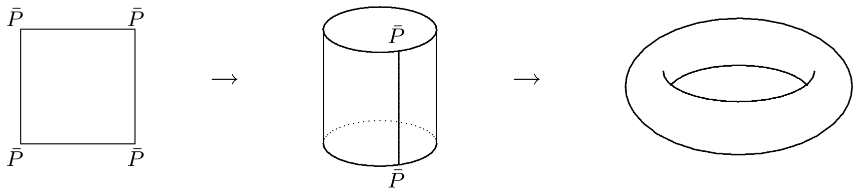


Figure I.1: *Gluing one unit square yields a torus*

A first observation is that an origami always defines a compact topological surface of genus $g \geq 1$.

I.1.3 Example. Throughout the whole thesis, we study a particular origami, which we name S. It consists of six unit squares that are glued as indicated in Figure I.2. Here, edges with the same letters are identified. The symbols □, ■, ○ and ● mark the four points of S where the edges meet.

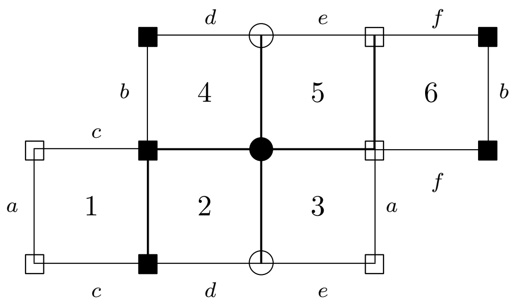


Figure I.2: *The origami S*

The genus g of S can be computed using Euler's formula. The surface is tiled into 6 squares with 12 edges, which meet in the four vertices □, ■, ○ and ● . Therefore,

$$4 - 12 + 6 = 2 - 2g,$$

and S is an origami of genus 2.

There are two points on S, where more than four squares abut, namely □ and ■.

We can use the fact that the squares are subsets of the Euclidean plane to define a *translation structure* (cf. III.1) on it. In particular, we can measure angles on S with respect to the translation structure. If we travel around one of the points $\square$ or $\blacksquare$ on a small closed loop, we see that the angle around it is equal to 4π, whereas it is 2π at every other point of S. These two points are singularities of the translation structure, and they are called *cusps* (cf. III.3.1).

It is a consequence of the Riemann-Roch Theorem that an origami of genus 2 has either one or two cusps (cf. Proposition III.3.8). Origamis of genus 2 with only one cusp are studied in [HL06].

The reason, why we chose to examine this special origami, is that it has a *translation*, i.e. there is a permutation of the squares that respects the gluings, namely $(1\,6)(2\,4)(3\,5)$. In addition to that the origami S has a *hyperelliptic involution*, since it is of genus 2 (cf. IV.1); these two *automorphisms* will be the ingredients to obtain our main result (cf. IV.3).

Having obtained a result for the origami S, one could ask whether this can be generalized to other origamis.

The origami S is in fact part of a whole family of origamis of genus 2 that all admit a translation. Let $n, k \in \mathbb{N}_{>0}$ and let $c = 2k$, $d = 2k + n$. Then the origami $S_{n,k}$ is defined by Figure I.3. Here, the gluings are made by identifying opposite sides. Then

$$(c+1\;\; d+1)(c+2\;\; d+2)\cdots(c+n\;\; d+n)\cdot(1\;\; k+1)(2\;\; k+2)\cdots(k\;\; 2k)$$

is a permutation of the squares of $S_{n,k}$ that respects the gluings.

Our example origami is the origami $S = S_{1,2}$.

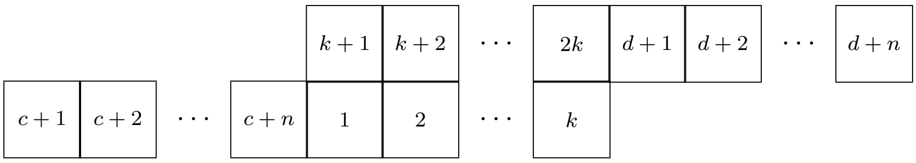


Figure I.3: *A family of origamis*

But these are far from being the only origamis in genus 2 that have a translation. In fact, any (normal) covering of degree 2 of a trivial origami (i.e. an origami of genus 1) that is ramified over precisely 2 points, leads to an origami of genus 2 with a translation.

I.2 Known Results

We want to give an overview of origamis of which one knows an equation. Möller [Möl05] studied two particular examples of origamis in genus 2 and was able to give an equation. One of them is the origami $S_{1,1}$ defined above, the other one is called $L(2,2)$ and is given by Figure I.4. (Again, the gluings are made by identifying opposite sides.)

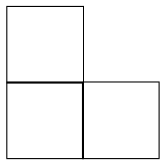


Figure I.4: *The origami* $L(2,2)$

Herrlich and Schmithüsen studied an extraordinary example of an origami in genus 3 in [HS05], whose curve in the moduli space M_3 intersects infinitely many other origami curves, and they determined its equation. In [Her06], we are given the equations of a whole infinite sequence of origami curves.

Chapter II

Fundamentals

II.1 Riemann Surfaces

We will see that origamis give rise to Riemann surfaces. Therefore, we recall some aspects of Riemann surfaces in this section. For a detailed introduction to this subject one may read for example [For81].

Recall that one defines a Riemann surface as a connected manifold of complex dimension 1. This implies that a Riemann surfaces can also be viewed as a smooth manifold of real dimension 2, since biholomorphic maps of $\mathbb{C}$ are also analytic maps of $\mathbb{R}^2$, and we will sometimes switch between these two points of view.

Moreover, a Riemann surface is an orientable manifold, because the determinant of the Jacobian of a biholomorphic map (which we consider as a map between open sets of $\mathbb{R}^2$) is positive. We always equip a Riemann surface with the orientation coming from its complex structure.

Speaking in the language of categories, the notion of a Riemann surface gives rise to a category $\mathfrak{Riem}$, whose objects are Riemann surfaces and whose morphisms are non-constant holomorphic maps between Riemann surfaces.

II.1.1 Notation. We recall some classical Riemann surfaces and their notations in this thesis. We denote the *complex plane* by $\mathbb{C}$ and the *Riemann sphere* by $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. The *upper half-plane* is denoted by

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\},$$

and the *unit disk* is denoted by

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

For a Riemann surface X, $\mathcal{O}(X)$ is the *ring of holomorphic functions* and $\mathcal{M}(X)$ denotes the *field of meromophic functions* on X. The *group of holomorphic automorphisms* of X is the group of biholomorphic maps $X \to X$, and we denote it $\mathrm{Aut}(X)$.

Riemann surfaces can be divided in two classes, according to whether they are compact or not. Among the non-compact Riemann surfaces, there are some that are "almost compact" in the following sense.

II.1.2 Definition. A Riemann surface X is said to be a *Riemann surface of finite type*, if there exists a compact Riemann surface $\bar{X}$, such that X can be embedded into $\bar{X}$ via a holomorphic, injective map $i : X \hookrightarrow \bar{X}$, such that $\bar{X} \setminus i(X)$ is finite. We define the genus of X to be the genus of $\bar{X}$. An element of $\bar{X} \setminus i(X)$ is called a puncture of X and the cardinality of $\bar{X} \setminus i(X)$ is called the number of punctures.

We will see in Corollary II.3.7 that the genus g and the number of punctures n of a Riemann surface of finite type X are well-defined. The pair (g, n) is called the *type of* X.

II.2 Covering Maps

II.2.1 Definitions

Since our definition of a covering map is somewhat non-standard, we present it in this section. For a reference on this topic one may consult e.g. [For77].

II.2.1 Definition. Let X, Y be topological spaces.

a) A map $p : Y \to X$ is called *covering map*, if p is open, continuous and discrete (here, discrete means that the preimage $p^{-1}(\{x\})$ of every $x \in X$ is a discrete subset of Y).

b) Let $p : Y \to X$ be a covering map. A point $y \in Y$ is said to be a *ramification point* of p if there does not exist a neighborhood V of y such that $p|_V$ is injective. A point x is said to be a *branch point* of p, if it is the image of a ramification point.

c) A covering map with no ramification points is called an *unramified covering*, otherwise it is called a *ramified covering*.

The reason for studying covering maps is the following.

II.2.2 Remark. A holomorphic map $f : X \to Y$ between two Riemann surfaces is a covering map, provided that it is not constant. Clearly, f is open and continuous, and the identity theorem implies that it is discrete, since f is non-constant. A non-constant holomorphic map between Riemann surfaces is therefore also called a *holomorphic covering map*.

II.2.3 Definition. A *topological covering map* is a map $p : Y \to X$, such that every point $x \in X$ has an open neighborhood $U \subset X$, whose preimage has a decomposition

$$p^{-1}(U) = \bigcup_{i \in I} V_i$$

with open, disjoint sets $V_i \subset Y$, such that for all $i \in I$ the map $p|V_i \to U$ is a homeomorphism.

It is easy to see that a topological covering map is in particular an unramified covering.

II.2.2 Proper Covering Maps

When studying Riemann surfaces, it is of some importance to know about the properties of morphisms that preserve compactness. This leads to the notion of proper maps. The following discourse is taken partly from [For81].

II.2.4 Definition. Suppose that X, Y are locally compact topological spaces (i.e. X, Y are Hausdorff spaces and every point has a compact neighborhood). A covering map $p : Y \to X$ is called *proper*, if the preimage of any compact set in X under p is a compact set in Y.

Note that if Y is compact, then every covering map $p : Y \to X$ is proper. Let us sum up some properties of proper covering maps in the following proposition.

II.2.5 Proposition. *Let X, Y be locally compact topological spaces, and let $p : Y \to X$ be a covering map.*

a) If p is proper, then it is finite. Moreover, if X is connected and Y is non-empty, then p is surjective.

b) If p is a finite, topological covering map, then it is proper.

c) If p is a proper, unramified covering map, then it is a topological covering map.

d) If p is finite, surjective and unramified, then it is a topological covering map.

PROOF: *Part a).* Let p be proper and let $x \in X$. Since p is discrete, $p^{-1}(\{x\})$ is a discrete, compact subset of Y, and thus finite. The second statement follows from the fact, that a proper map is closed, i.e. the image of a closed subset is closed. The set $p(Y)$ is an open, closed, non-empty subset of X. Since X is connected, it follows that $X = p(Y)$.

Part b). Suppose that p is a finite, topological covering map and let $S \subset X$ be compact. Let $s \in S$. By our assumption, there exists an open neighborhood U_s of s such that $p^{-1}(U_s) = \bigcup_{j\in J} V_{s,j}$ where the sets $V_{s,j}$ are disjoint and $p|V_{s,j} \to U_s$ is a homeomorphism. Let $U'_s \subset U_s$ such that its closure satisfies $\overline{U}'_s \subset U_s$.

The set $R_s := S \cap \overline{U}'_s$ is compact. We have

$$p^{-1}(R_s) = \bigcup_{j\in J} (p|V_{s,j} \to U_s)^{-1}(R_s).$$

By our assumption, J is a finite set, and thus $p^{-1}(R_s)$ is compact as a finite union of compact sets.

Now we use the fact that S is compact. For $s \in S$ let U''_s be an open neighborhood of s contained in $\overline{U}'_s$. Then, $S \subset \bigcup_{s\in S} U''_s$, hence there exists a finite, open covering $S \subset \bigcup_{1\le k\le m} U''_{s_k}$. Thus, we also have $S \subset \bigcup_{1\le k\le m} \overline{U}'_{s_k}$, whereby

$$S = \bigcup_{1\le k\le m} (\overline{U}'_{s_k} \cap S) = \bigcup_{1\le k\le m} R_{s_k}.$$

Therefore,

$$p^{-1}(S) = p^{-1}(\bigcup_{1\le k\le m} R_{s_k}) = \bigcup_{1\le k\le m} p^{-1}(R_{s_k})$$

is a compact set, as it is a finite union of compact set.

For the proof Part c), we refer to [For81, Theorem 4.22].

Part d) Let $x \in X$. Since p is surjective, $p^{-1}(\{x\}) \neq \emptyset$. Suppose that $\{y_1, \dots, y_d\}$ are the (finitely many) preimages of x. Since Y is a Hausdorff space, there exist disjoint open sets $V_i \subset Y$, $1 \le i \le d$, such that V_i is a neighborhood of y_i. Since p is an unramified covering, we can assume that $p|V_i \to p(V_i) =: U_i$ is already a homeomorphism. The set

$$\tilde{U} = \bigcap_{i=1}^{d} U_i$$

is an open neighborhood of x. We set $\tilde{V}_i = (p|V_i \to U_i)^{-1}(\tilde{U})$. Then $\tilde{V}_i \cap \tilde{V}_j = \emptyset$ for $1 \le i, j \le d$, $i \neq j$, since V_i and V_j are disjoint. The map $p|\tilde{V}_i \to \tilde{U}$ is a homeomorphism and $p^{-1}(\tilde{U}) = \bigcup_{i=1}^{d} \tilde{V}_i$. This shows that p is a topological covering. □

II.2.3 Lifting Properties

Later on, we will be placed in the following situation. Given a covering map $p : Y \to X$ between topological spaces X,Y, when does an automorphism $f : X \to X$ induce an automorphism $\hat{f}$ of Y, such that $p \circ \hat{f} = f \circ p$? It is thus of some importance to know, under which conditions we can "lift" a map via a covering. We explain this now.

Let X, Y, Z be topological spaces and let $p : Y \to X$ be a covering map. Let $f : Z \to X$ be a continuous map. A *lift* of f is a continuous map $\hat{f} : Z \to Y$ such that $f = p \circ \hat{f}$.

For a topological space X and a point $x \in X$, we denote by $\pi_1(X, x)$ the fundamental group of X with basepoint x. This is the group of homotopy classes of closed paths on X that start and end in the point x, and it encodes information on the topology of X. If Y is another topological space and if $f : X \to Y$ is a continuous map, then it induces a group homomorphism

$$f_* : \pi_1(X, x) \to \pi_1(Y, f(x))$$

between the fundamental groups. In particular, if $p : Y \to X$ is a topological covering map, then one can show that the corresponding group homomorphism p_* is injective (see [Hat02, Proposition 1.31].

The following proposition states under which conditions a lift exists.

II.2.6 Proposition. *Let X, Y be topological spaces and let $p : Y \to X$ be a topological covering. Let Z be a path-connected and locally path-connected topological space and let $f : Z \to X$ be continuous. Choose two points $z \in Z$, $y \in Y$, such that $f(z) = p(y)$. Then, there exists a lift $\hat{f} : Z \to Y$ of f, if and only if $f_*(\pi_1(Z, z)) \subset p_*(\pi(Y, y))$. Moreover, if two lifts $\hat{f}$, $\hat{f}'$ agree in one point $z_0 \in Z$, then $\hat{f} = \hat{f}'$.*

Proof: We refer to [Hat02, Proposition 1.33 and 1.34] for a proof of this statement. □

II.3 Holomorphic Covering Maps

In this section, we develop some properties of non-constant holomorphic maps that are used later on.

II.3.1 The Riemann-Hurwitz Formula

One of the main tools in the study of compact Riemann surfaces is the Riemann-Hurwitz formula, which relates the degree of the holomorphic map $f : X \to Y$ and its ramification behaviour to the genera of the compact Riemann surfaces X and Y.

II.3.1 Definition & Remark. Let X and Y be compact Riemann surfaces, and let $f : X \to Y$ be a non-constant holomorphic map.

a) We denote the *multiplicity of* f at the point $P \in X$ by $\mathrm{mult}_P(f)$, i.e. $\mathrm{mult}_P(f)$ is the unique integer $k \geq 1$ such that f can be expressed as
$$z \mapsto z^k$$
in local coordinates at the point $P \in X$. A point $P \in X$ with $\mathrm{mult}_P(X) > 1$ is a ramification point of f and the set of ramification points of f is a discrete, closed subset of X, and thus finite.

b) By Proposition II.2.5, f is a finite map. We denote the *degree of* f by $\deg(f)$, i.e. $\deg(f)$ is the well-defined integer $n \geq 1$ such that each point $y \in Y'$ has precisely n preimages in X. Here $Y' = Y \setminus B$, where B is the image of the set of ramification points of f.

c) The degree and the multiplicity are related by
$$\deg(f) = \sum_{P \in f^{-1}(\{Q\})} \mathrm{mult}_P(f),$$
for every $Q \in Y$.

PROOF: See e.g. [For81, p. 29]. □

II.3.2 Proposition. *(Riemann-Hurwitz formula)*
Let $f : X \to Y$ be a non-constant holomorphic map between compact Riemann surfaces X and Y of genus $g(X)$ and $g(Y)$. Then,
$$2g(X) - 2 = \deg(f) \cdot (2g(Y) - 2) + \sum_{P \in X} (\mathrm{mult}_P(f) - 1).$$

PROOF: See e.g. [Mir95, Chapter II, Theorem 4.16]. □

II.3.2 Proper Holomorphic Covering Maps

Next, we discuss the possibility of extending a holomorphic covering map in certain situations. Usually, the following propositions will be applied in the context of Riemann surfaces of finite type. Again, we follow mainly [For81].

II.3.3 Proposition. *Let X be a Riemann surface, let $A \subset X$ be a closed, discrete subset and let $X' = X \setminus A$. Suppose that Y' is a Riemann surface and that $p' : Y' \to X'$ is a proper, unramified, holomorphic covering. Then p' extends to a proper, holomorphic covering map p. More precisely, there exists a Riemann surface Y, a proper, holomorphic covering map $p : Y \to X$ and a biholomorphic map $\phi : Y' \to Y \setminus p^{-1}(A)$ such that the diagram*

$$\begin{tikzcd} Y' \arrow[r, "\phi"] \arrow[d, "p'"] & Y \setminus p^{-1}(A) \arrow[r, hook] & Y \arrow[d, "p"] \\ X' \arrow[rr, hook] & & X \end{tikzcd}$$

commutes.

Proof: See [For81, Theorem 8.4]. □

II.3.4 Proposition. *Let X, Y, Z be Riemann surfaces and let $p : Y \to X$, $q : Z \to X$ be proper, holomorphic covering maps. Let $A \subset X$ be closed and discrete and let $X' = X \setminus A$, $Y' = Y \setminus p^{-1}(A)$ and $Z' = Z \setminus q^{-1}(A)$. Then any biholomorphic map $f' : Y' \to Z'$, satisfying $p|_{Y'} = q|_{Z'} \circ f'$ extends to a biholomorphic map $f : Y \to Z$, such that $p = q \circ f$ and $f|_{Y'} = f'$.*

Proof: See [For81, Theorem 8.5]. □

II.3.5 Corollary. *If $p : Y \to X$ is the continuation of $p' : Y' \to X'$ as in Proposition II.3.3, then Y and p are unique up to biholomorphic maps.*

II.3.6 Corollary. *Let X, Y be compact Riemann surfaces and let $X' = X \setminus \{a_1, \dots, a_r\}$, $Y' = Y \setminus \{b_1, \dots, b_s\}$. Then, every isomorphism $f : X' \to Y'$ extends uniquely to an isomorphism $f : X \to Y$.*

Proof: We use meromorphic functions as covering maps in order to apply the propositions above. There exists a non-constant $h \in \mathcal{M}(Y)$. Let $A =$

$\{a_1, \ldots, a_r\}$ and let

$$V = \mathbb{P}^1 \setminus (h(A) \cup R),$$

where R are the branch points of h. Let $Y'' = Y \setminus h^{-1}(h(A) \cup R)$. Then $h|Y'' \to V$ is a proper, unramified, holomorphic covering map, and the same holds for $g'' := h|_{Y''} \circ f|_{X''}$, where $X'' := f^{-1}(Y'')$. Thus, we can extend g'' to a proper, holomorphic covering map $g : X \to \mathbb{P}^1$ by Proposition II.3.3. By Proposition II.3.4, we can now extend $f|X'' \to Y''$ to an isomorphism $f : X \to Y$. □

II.3.7 Corollary. *Let X be a Riemann surface of finite type. The compact Riemann surface $\bar{X}$, having the property that there exists an injective holomorphic map $i : X \to \bar{X}$, such that $\bar{X} \setminus i(X)$ is finite, is unique up to biholomorphic maps. In particular, the genus and the number of punctures of a Riemann surface of finite type are well-defined.*

II.3.3 Group Actions on a Riemann Surface

Another tool that we will make use of extensively later on, is to generate holomorphic covering maps from a Riemann surface X to another with the help of automorphisms in $\mathrm{Aut}(X)$. The next proposition states, in which cases this leads to meaningful results.

II.3.8 Proposition. *Let X be a Riemann surface, and let G be a finite group together with an effective group action $\rho : G \to \mathrm{Aut}(X)$ on X. Then the orbit space X/G can be given the structure of a Riemann surface such that the projection*

$$\pi : X \to X/G, \ P \mapsto \rho(G)(P)$$

is a holomorphic covering map of degree $\deg(\pi) = |G|$. Moreover, the multiplicity of π at $P \in X$ is equal to $\mathrm{mult}_P(\pi) = |G_P|$, where G_P is the stabilizer of P.

Proof: A detailed proof is given e.g. in [Mir95, Chapter III., Theorem 3.4]. □

II.4 Riemann Surfaces and Algebraic Curves

In this section, we establish the connection between compact Riemann surfaces and algebraic curves. It turns out that every compact Riemann surface is an algebraic curve, having certain additional properties and vice versa. In fact, we get an equivalence of categories.

An *affine, plane curve* $C \subset \mathbb{C}^2 = \mathbb{A}^2$ is the locus of zeros of an irreducible polynomial $f \in \mathbb{C}[X,Y]$. It is called nonsingular at a point $P \in C$, if either partial derivative $\partial f/\partial X$ or $\partial f/\partial Y$ is not zero at P. If this holds for every point, then C is a *regular* or *smooth* curve.

From the Implicit Function Theorem, we can deduce that a regular, affine, plane curve is always a Riemann surface, the local charts being projections on either the first or the second coordinate. However, it is not a compact Riemann surface, but only a Riemann surface of finite type. In order to make it compact, one could take its projective closure and obtain a *projective plane curve*, but possibly loses the regularity of the curve. The following formula can serve as a criterion to check, whether or not we can expect a regular, projective, plane curve.

II.4.1 Proposition. *(Plücker's formula)*
A regular, projective, plane curve of degree d (i.e. it is defined by a homogeneous polynomial of degree d) has genus $g = (d-1)(d-2)/2$.

PROOF: See e.g. [Mir95, p. 144, Proposition 2.15] □

To eliminate the singularities, one can "blow up" or desingularize the curve. This yields a curve in a higher dimensional space $\mathbb{P}^n$. Then, it is no longer described by a single polynomial, but it is the locus of zeros of a whole set of polynomials. More precisely, it is a local complete intersection (see e.g. [Mir95]). Such an object is called *projective, regular curve*, and one can show that it is a compact Riemann surface. In fact, one has more.

II.4.2 Theorem. *The category $\mathfrak{Proj}$ of projective, regular curves (together with non-constant morphisms) and the category $\mathfrak{Riem}_c$ of compact Riemann surfaces (together with non-constant holomorphic maps) are equivalent.*

PROOF: See e.g. [Rey89, Théorème 1.5] □

The true link between algebraic curves and compact Riemann surfaces is the function field. Let $\mathfrak{Funct}$ be the category of finitely generated field extensions of $\mathbb{C}$ with transcendence degree one, together with field homomorphisms. If C is an algebraic curve over $\mathbb{C}$, then its field of rational functions $k(C)$ belongs to $\mathfrak{Funct}$, and conversely C can be recovered from $k(C)$, i.e. there is an equivalence of categories in this case. For compact Riemann surfaces, one has the following.

II.4.3 Theorem. *To a compact Riemann surface X, we assign its field of meromorphic functions*

$$X \mapsto \mathcal{M}(X),$$

and to a non-constant holomorphic map $f : X \to Y$ between compact Riemann surfaces, we assign the field homomorphism

$$\mathcal{M}(f) = f^* : \mathcal{M}(Y) \to \mathcal{M}(X), \ \alpha \mapsto \alpha \circ f.$$

Then $\mathcal{M}$ defines a contravariant functor

$$\mathcal{M} : \mathfrak{Riem}_c \to \mathfrak{Funct}$$

that gives rise to an equivalence of categories.

Proof: See e.g. [Rey89, Théorème 7.2] □

An interesting property of the functor $\mathcal{M}$ is that it respects the degrees.

II.4.4 Proposition. *Let X and Y be compact Riemann surfaces and let $f : X \to Y$ be a holomorphic map of degree n. Then, the field extension*

$$f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$$

is a finite extension of degree n.

Proof: See e.g. [For81, Theorem 8.3] □

II.4.1 The Riemann-Roch Theorem

Roughly speaking, the Riemann-Roch Theorem is a statement about the existence and number of meromorphic functions on a compact Riemann surface, whose zeros, respectively poles, lie in a given set. It is one of the main theorems for compact Riemann surfaces, respectively projective, regular curves and has very strong consequences.

First, we recall quickly some notations that are needed to formulate the Riemann-Roch Theorem and that will reappear later on in the discussion of hyperelliptic surfaces. A detailed treatment of this subject can be found in [FK80] or [For81].

In the following, let X always be a compact Riemann surface of genus $g \geq 0$. A *divisor* D on X is a finite formal sum

$$D = \sum_{P \in X} \mathrm{ord}_P(D) \cdot P$$

with $\mathrm{ord}_P(D) \in \mathbb{Z}$, i.e. D is an element of the free abelian group with basis X. Recall that the *degree of a divisor* is defined as $\deg(D) = \sum_{P \in X} \mathrm{ord}_P(D)$. A divisor is called effective, if $\mathrm{ord}_P(D) \geq 0$ for all $P \in X$.

A non-zero meromorphic function $f \in \mathcal{M}(X)$ defines a divisor (f), called *principal divisor*, by setting

$$(f) = \sum_{P \in X} \mathrm{ord}_P(f) \cdot P,$$

where $\mathrm{ord}_P(f)$ is the order of f at the point $P \in X$. Furthermore, we set

$$(f)_0 = \sum_{\substack{P \in X, \\ \mathrm{ord}_P(f) > 0}} \mathrm{ord}_P(f) \cdot P$$

the *divisor of zeros* of f and

$$(f)_\infty = \sum_{\substack{P \in X, \\ \mathrm{ord}_P(f) < 0}} -\mathrm{ord}_P(f) \cdot P$$

the *divisor of poles* of f. Hence, $(f) = (f)_0 - (f)_\infty$.

Two divisors D_1, D_2 on X are called *linearly equivalent*, if their difference is a principal divisor, i.e. there exists $f \in \mathcal{M}(X)$ such that

$$D_1 - D_2 = (f),$$

and we write $D_1 \sim D_2$, if this is the case. Note that $\sim$ is a congruence relation for the addition of divisors.

If ω is a non-zero meromorphic 1-form on X, we can associate the divisor (ω) to it in the following way: let $\omega = f dz$ on a complex chart (U, z) on X about a point P. Then $f \in \mathcal{M}(U) \setminus \{0\}$ and we set $\mathrm{ord}_P(\omega) = \mathrm{ord}_P(f)$. Now we define (ω) by

$$(\omega) = \sum_{P \in X} \mathrm{ord}_P(\omega) \cdot P.$$

A divisor of this form is called *canonical divisor*. Note that for any two meromorphic 1-forms $\omega_1, \omega_2 \neq 0$, there exists a meromorphic function $f \in$

$\mathcal{M}(X)\backslash\{0\}$, such that $\omega_1 = f\omega_2$. Therefore, all canonical divisors are linearly equivalent.

For a divisor D on X, we define

$$L(D) = \{f \in \mathcal{M}(X) \mid \mathrm{ord}_P(f) + \mathrm{ord}_P(D) \geq 0\} \cup \{0\}.$$

This is a finite dimensional $\mathbb{C}$-vector space and its dimension is denoted by $\ell(D)$.

Note that if D_1 and D_2 are linearly equivalent divisors on X, i.e. $D_1 - D_2 = (f)$ for an element $f \in \mathcal{M}(X)$, then

$$\begin{array}{rcl} L(D_1) & \longrightarrow & L(D_2) \\ g & \longmapsto & f \cdot g \end{array}$$

is an isomorphism of $\mathbb{C}$-vector spaces.

II.4.5 Theorem. *(Riemann-Roch Theorem)*
Let X be a compact Riemann surface of genus $g \geq 0$. Let D be a divisor on X and let W denote a canonical divisor on X. Then,

$$\ell(D) = \deg(D) - g + 1 + \ell(W - D)$$

Proof: A proof of the Riemann-Roch Theorem can for instance be found in [FK80, Section III.4]. Let us remark that $\ell(W - D)$ is well-defined, since any two canonical divisors are linearly equivalent, since linear equivalence is a congruence relation and since the dimension of $L(W-D)$ does not change, as long as we stay in the same equivalence class. $\square$

II.4.6 Corollary. *Let X be a compact Riemann surface.*

a) The canonical divisor satisfies $\ell(W) = g$ and $\deg(W) = 2g - 2$.

b) If the divisor D satisfies $\deg(D) > 2g - 2$, then $\ell(W - D) = 0$. In this case, the Riemann-Roch Theorem yields

$$\ell(D) = \deg(D) - g + 1.$$

Proof: *Part a)* If we apply the Riemann-Roch Theorem to the zero divisor 0, we get

$$\ell(0) = \deg(0) - g + 1 + \ell(W).$$

Since a holomorphic function on a compact Riemann surface is constant, $\ell(0) = 1$, and thus $\ell(W) = g$. Next, we apply the Riemann-Roch Theorem to the divisor W and obtain

$$\ell(W) = \deg(W) - g + 1 + \ell(0),$$

hence $\deg(W) = 2g - 2$.

Part b) If D is a divisor with $\deg(D) > 2g - 2$, then $W - D$ has degree $\deg(W - D) < 0$. But this implies $\ell(W - D) = 0$. □

II.4.7 Corollary. *Let X be a compact Riemann surface of genus $g \geq 0$. Then $\mathcal{M}(X)$ separates the points of X, i.e. given two distinct points $P_1, P_2 \in X$, there exists a meromorphic function $f \in \mathcal{M}(X)$, such that $f(P_1) \neq f(P_2)$.*

PROOF: Let $D = (g+1)P_1$. Then the Riemann-Roch Theorem yields

$$\ell(D) = \underbrace{\deg(D)}_{=(g+1)} - g + 1 + \underbrace{\ell(W - D)}_{\geq 0} \geq 2.$$

Thus there exists a non-constant meromorphic function $f \in L((g+1)P_1)$, and f has a pole (of order $\leq (g+1)$) at P_1, while it is holomorphic at P_2. Therefore, $f(P_1) \neq f(P_2)$. □

II.5 Elliptic Curves

Elliptic curves are not only involved in the definition of an origami, but they also play a role, when it comes to calculating the equation of our example origami, since we will make use of the group structure on a certain elliptic curve to find its equation. We present some basic facts about elliptic curves in this section. A reference for this subject is e.g. [Sil92].

II.5.1 Definition. An *elliptic curve* (over $\mathbb{C}$) is a pair (X, N), where X is a compact Riemann surface of genus one and $N \in X$ is a point. A morphism between two elliptic curves (X, N), (X', N') is a non-constant holomorphic map $f : X \to X'$ such that $f(N) = N'$.

From the point of view of Riemann surfaces, an elliptic curve is described as follows.

A lattice Λ is a discrete subgroup of $(\mathbb{C}, +)$ of the form

$$\Lambda = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2,$$

with $\lambda_1, \lambda_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$. The quotient $\mathbb{C}/\Lambda$ is a Riemann surface and is called a *torus*.

II.5.2 Proposition. *For every lattice $\Lambda \subset \mathbb{C}$, the torus $\mathbb{C}/\Lambda$, together with the point $\bar{0} = 0 + \Lambda \in \mathbb{C}/\Lambda$, is an elliptic curve.*

Conversely, every elliptic curve (X, N) is isomorphic to a torus $(\mathbb{C}/\Lambda, \bar{0})$ for a lattice $\Lambda \subset \mathbb{C}$.

Proof: For the first statement, we refer to [For81, Corollary 17.13]. The second one can be found e.g. in [For81, Theorem 21.10]. □

On the other hand, we can describe elliptic curves in the language of algebraic geometry.

II.5.3 Proposition. *Let E be a regular, projective curve having the affine equation*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{II.1}$$

with $a_i \in \mathbb{C}$, $i = 1, 2, 3, 4, 6$, and let ∞ denote the point at infinity. Then (E, ∞) is an elliptic curve.

Conversely, for an elliptic curve (X, N), there exist parameters a_1, a_2, a_3, a_4, $a_6 \in \mathbb{C}$ such that $(X, N) \cong (E, \infty)$, where (E, ∞) is given by (II.1).

Proof: Given a regular, projective curve E corresponding to Equation (II.1), we can compute its genus with the help of Plücker's formula II.4.1. Since E belongs to a homogeneous polynomial of degree 3, its genus is given by

$$g = \frac{(d-1)(d-2)}{2} = 1.$$

Hence, E is a Riemann surface of genus one.

To show the converse, we look at the divisors jN, $2 \leq j \leq 6$. From Corollary II.4.6, we deduce that $\ell(jN) = j$ for all $j \geq 1$. Let $x \in L(2N)$ be a non-constant function. Since $\ell(3N) = 3$, there exists another non-constant function $y \in L(3N)$, such that $1, x, y$ are linearly independent. Then, $1, x, y, x^2, yx, x^3, y^2$ are seven functions in $L(6N)$, and as $\ell(6N) = 6$, there exist $c_1, \ldots, c_6 \in \mathbb{C}$, such that

$$y^2 + c_1yx + c_2y = c_3x^3 + c_4x^2 + c_5x + c_6.$$

The coefficient $c_3 \neq 0$, since the left hand side of the equation has a pole of order 6 and x^3 is the only function on the right hand side with a pole of order 6. Therefore, we can divide by c_3 and obtain a_1, a_2, a_3, a_4, $a_6 \in \mathbb{C}$ such that

$$y^2 + a_1yx + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

It remains to show that this also defines a regular curve in $\mathbb{P}^2$. But this is quite similar to what we will do later in the proof of Theorem IV.2.6, so we will not carry out this now. □

II.5.4 Corollary. *An elliptic curve (X, N) carries a natural structure of an abelian group with N as the zero element.*

There are several approaches to show that an elliptic curve (X, N) is also an abelian group. If we represent it as $\mathbb{C}/\Lambda$ with a lattice Λ, this is immediate. One can also show it with the help of the Riemann-Roch Theorem II.4.5 and prove that there is a bijection of X with the subgroup of its Picard group that consists of the divisors of degree zero (see e.g. [Har77, Example 1.3.7]). Finally, one has a concrete formula for the group law on an elliptic curve, represented by an equation of the form (II.1). This is the way, we will need it.

The Group Law. If we represent an elliptic curve (X, N) by an equation of the form (II.1), then the group law on (X, N) has the following form (see [Sil92]).

Let P_1, P_2, P_3 be three points on $X \setminus \{N\}$, $P_i = (x_i, y_i)$, $i = 1, 2, 3$, where (x_i, y_i) satisfy (II.1). We distinguish three cases:

a) If $x_1 = x_2$ and $y_1 + y_2 + a_1x_2 + a_3 = 0$, then

$$P_1 + P_2 = N.$$

b) If $x_1 \neq x_2$, let

$$\alpha = \frac{y_2 - y_1}{x_2 - x_1}, \qquad \text{and} \qquad \beta = \frac{y_1x_2 - y_2x_1}{x_2 - x_1}.$$

Otherwise, let

$$\begin{aligned} \alpha &= \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}, \\ \beta &= \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}. \end{aligned}$$

Then $P_3 = P_1 + P_2$ is given by

$$\begin{aligned} x_3 &= \alpha^2 + a_1\alpha - a_2 - x_1 - x_2 \\ y_3 &= -(\alpha + a_1)x_3 - \beta - a_3. \end{aligned}$$

II.6 Hyperelliptic Surfaces

As we will see, our origami S gives rise to Riemann surfaces of genus 2. In this section, we show that a Riemann surface of genus 2 is a hyperelliptic surface and shares some special properties, which will be presented here and on which we will rely later. This section is based upon [FK80].

II.6.1 Weierstraß points

Let X be a compact Riemann surface of genus $g \geq 1$ and let $P \in X$. We start by taking a look at the divisors jP, where j is a positive integer. For most of the points of X, there does not exist a meromorphic function, which is holomorphic everywhere except at P and has a pole of order $\leq g$ at P. The points of X that admit such a function are called Weierstraß points. Such points exist, if the genus satisfies $g \geq 2$, and one can show that there are always finitely many of them. The consideration of the Weierstraß points of X will be very useful in our discussion of hyperelliptic surfaces later.

II.6.1 Lemma. *Let $P \in X$ be a point of a compact Riemann surface X. Then,*

$$\ell(jP) - \ell((j-1)P) \in \{0, 1\}$$

for all $j \in \mathbb{N}$.

PROOF: Let $f \in L(jP)$ be a meromorphic function. Expanding f in a Laurent series in local coordinates (U, z) about P yields

$$f = \sum_{k=-j}^{\infty} a_k (z - z(P))^k.$$

We define a $\mathbb{C}$-linear map

$$\Phi : L(jP) \to \mathbb{C}, f \mapsto a_{-j},$$

which sends f to its highest coefficient. Then $\ker \Phi = L((j-1)P)$ and Φ is either surjective or the zero map. Therefore, $\ell(jP) - \ell((j-1)P) = \dim_{\mathbb{C}}(\Phi(L(jP)) \in \{0, 1\}$. □

Let X be a compact Riemann surface of genus $g \geq 1$ and let $P \in X$. We consider the numbers $\ell(jP)$, where $j \in \{0, \ldots, 2g\}$. Since $g \geq 1$, we know that there are only constant functions in $L(P)$, i.e. $\ell(P) = 1$. On the other hand, since $\deg(2gP) > 2g - 2$, Corollary II.4.6 implies that

$$\ell(2gP) = \deg(2gP) - g + 1 = g + 1.$$

Thus one has an increasing sequence

$$1 = \ell(0) = \ell(P) \leq \ell(2P) \leq \ldots \leq \ell(2gP) = g + 1.$$

Since $\ell(jP) - \ell((j-1)P) \in \{0, 1\}$ by the preceding lemma, there must be g numbers in $\{1, \ldots, 2g\}$, where the sequence increases by 1 and g numbers in $\{1, \ldots, 2g\}$, where the sequence remains constant. Note that it is a consequence of Corollary II.4.6 that $\ell(jP) - \ell((j-1)P) = 1$, if $j \geq 2g$, so only the case $j \leq 2g$ leads to interesting results.

II.6.2 Definition. Let $j \geq 1$. We will call j a *gap* for P, provided that $\ell(jP) - \ell((j-1)P) = 0$, i.e. there is no meromorphic function $f \in \mathcal{M}(X)$, holomorphic in $X \setminus \{P\}$ with a pole of order j at P. Otherwise j will be called a *non-gap*.

Let $(n_1, \ldots, n_g)$ with $1 = n_1 < n_2 < \ldots < n_g \leq 2g$ be the *gap sequence at* P, i.e. $n_1, \ldots, n_g$ are precisely the gaps at P. Likewise, its complement $(a_1, \ldots, a_g)$ in the set $\{1, \ldots, 2g\}$ will be called the *sequence of non-gaps.*

II.6.3 Remark. Let $P \in X$. If $r, s \in \{1, \ldots, 2g\}$ are non-gaps, then $r + s$ is a non-gap.

PROOF: Let r be a non-gap at P. Thus, there exists a function $f \in \mathcal{M}(X)$, holomorphic in $X \setminus \{P\}$ with a pole of order r at P. Similarly, there exists a function $g \in \mathcal{M}(X)$, holomorphic in $X \setminus \{P\}$ with a pole of order s at P. Then $f \cdot g$ is holomorphic in $X \setminus \{P\}$ and has a pole of order $r + s$ at P. Thus $r + s$ is a non-gap. $\square$

II.6.4 Definition. Let $P \in X$ and let $(n_1, \ldots, n_g)$ be the gap sequence at P. The *weight of* P is defined as

$$\tau(P) := \sum_{i=1}^{g} (n_i - i).$$

A point P is called a *Weierstraß point*, if $\tau(P) > 0$, i.e. if $(n_1, \ldots, n_g) \neq (1, 2, \ldots, g)$.

An important statement with regard to the Weierstraß points on a compact Riemann surface X is the following.

II.6.5 Theorem. *Let X be a compact Riemann surface of genus g. Then,*

$$\sum_{P\in X}\tau(P) = (g-1)g(g+1)$$

PROOF: See [FK80], p.84, III.5.10. for a proof of this statement. □

Thus for every compact Riemann surface X of genus $g \geq 2$, there exist Weierstraß points on X. Moreover, there are only finitely many of them.

II.6.2 Hyperelliptic Surfaces

II.6.6 Definition. Let X be a compact Riemann surface of genus $g \geq 2$. X is called *hyperelliptic*, if it admits a non-constant meromorphic function $f \in \mathcal{M}(X)$ such that $f : X \to \mathbb{P}^1$ is a two-sheeted covering map.

II.6.7 Remark. A compact Riemann surface X of genus $g \geq 2$ is hyperelliptic, if there exists an effective divisor D on X satisfying

$$\deg(D) = 2 \quad \text{and} \quad \ell(D) = 2,$$

for the existence of such a divisor D ensures that there is a non-constant meromorphic function $f \in \mathcal{M}(X)$ with exactly 2 poles (counted with multiplicities), which makes f a two-sheeted covering map.

II.6.8 Proposition. *If X is a compact Riemann surface of genus 2, then X is hyperelliptic.*

PROOF: We consider a Weierstraß point $P \in X$, whose existence is provided by Theorem II.6.5. Then, 2 cannot be a gap at P, and $\ell(2P) = 2$. Thus there exists a non-constant meromorphic function $f \in L(2P)$, and X is hyperelliptic. □

II.6.9 Proposition. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$ and let $f \in \mathcal{M}(X)$ be a two-sheeted covering map. Then f is ramified over exactly $2g+2$ points, and the ramification points of f are precisely the Weierstraß points of X.*

Proof: Applying the Riemann-Hurwitz formula II.3.2 to the covering $f : X \to \mathbb{P}^1$ yields

$$\begin{aligned} 2g-2 &= \deg(f)(2g(\mathbb{P}^1)-2) + \sum_{P\in X}(\mathrm{mult}_P(f)-1) \\ &= -4 + \sum_{P\in X}(\mathrm{mult}_P(f)-1) \end{aligned}$$

Hence,

$$\sum_{P\in X}(\mathrm{mult}_P(f)-1) = 2g+2,$$

and since f is a two-sheeted covering, one has $\mathrm{mult}_P(f) \in \{1,2\}$. This shows that there are precisely $2g+2$ points, where f is ramified.

Now, let $P \in X$ be a ramification point of f. Then P is a Weierstraß point. Indeed, if $f(P) = \infty$, then f has a double pole at P. Otherwise, if $f(P) \neq \infty$, then the function

$$\frac{1}{f - f(P)}$$

has a double pole at P. So in each case, there exists a non-constant function in $L(2P)$, which means that 2 is not a gap at P and the gap sequence $(n_1, \ldots, n_g) \neq (1, 2, \ldots, g)$.

Moreover, since the sum of two non-gaps is again a non-gap, all the numbers $2, 4, \ldots, 2g$ are non-gaps at P. These are already g numbers, which means, that the sequence of gaps at P must be

$$(n_1, \ldots, n_g) = (1, 3, 5, \ldots, 2g-1).$$

Hence, the weight of P is

$$\begin{aligned} \tau(P) &= \sum_{i=1}^{g}(n_i - i) = \sum_{i=1}^{g}((2i-1)-i) \\ &= \left(\sum_{i=1}^{g} i\right) - g = \frac{g(g+1)}{2} - g = \frac{g(g-1)}{2}. \end{aligned}$$

Therefore, the sum of the weights of the $(2g+2)$ ramification points of f is equal to $(g-1)g(g+1)$. Thus by Theorem II.6.5, there are no other Weierstraß points, and each Weierstraß point is already a ramification point of f. □

The next proposition shows that the two-sheeted covering of a hyperelliptic surface is in some sense unique.

II.6.10 Proposition. *Let X be a hyperelliptic Riemann surface and let both $f : X \to \mathbb{P}^1$ and $h : X \to \mathbb{P}^1$ be two-sheeted covering maps. Then there exists a Möbius transformation γ, such that*

$$f = \gamma \circ h$$

Hence, the two-sheeted covering map is unique up to fractional linear transformations.

Proof: Let $P \in X$ be a Weierstraß point. Then P is a ramification point of f. We will show that the polar divisor $(f)_\infty$ of f is linearly equivalent to $2P$. If $f(P) = \infty$, there is nothing to prove. Otherwise, let $f(P) = c \in \mathbb{C}$. Then $f - c$ has a double zero at P, and $2P = (f - c)_0$. Since

$$(f)_0 - (f-c)_0 = (f)_0 - (f)_\infty + \underbrace{(f)_\infty}_{=(f-c)_\infty} - (f-c)_0 = (f) - (f-c),$$

the divisors $(f)_0$ and $(f - c)_0$ are linearly equivalent. From $(f)_0 \sim (f)_\infty$, it follows that $2P \sim (f)_\infty$. Similarly, $(h)_\infty \sim 2P$, which means that $(h)_\infty \sim (f)_\infty$.

Now let $(f)_\infty = Q + R$ and $(h)_\infty = Q' + R'$. Since $Q + R \sim Q' + R'$, there exists a function $g \in \mathcal{M}(X)$ such that $Q' + R' = (g) + Q + R$, and g induces an isomorphism

$$\begin{array}{rcl} L(Q' + R') & \to & L(Q + R) \\ \omega & \mapsto & g\,\omega \end{array}$$

Since $\ell(Q + R) = \ell(Q' + R') = 2$, the set $\{1, f\}$ is a $\mathbb{C}$-basis for $L(Q + R)$ and $\{1, h\}$ is a $\mathbb{C}$-basis for $L(Q' + R')$. Thus there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, such that

$$\begin{array}{rcl} 1 & = & a \cdot g1 + b \cdot gh \\ f & = & c \cdot g1 + d \cdot gh \end{array}$$

Therefore,

$$f = \frac{dh + c}{bh + a},$$

and $\gamma := (z \mapsto \frac{dz+c}{bz+a})$ is a Möbius transformation satisfying $f = \gamma \circ h$. $\square$

Next we will show that hyperelliptic surfaces are always endowed with a particular automorphism that characterizes them.

II.6.11 Theorem. *Let X be a compact Riemann surface of genus $g \geq 2$. Then, X is hyperelliptic, if and only if there exists a holomorphic involution $\sigma \in \mathrm{Aut}(X)$ with $2g+2$ fixed points. Moreover, the fixed points of σ are precisely the Weierstraß points of X.*

Proof: Let X be hyperelliptic and let $f : X \to \mathbb{P}^1$ be a two-sheeted covering map. Let $A \subset X$ be the set of ramification points of f. We define a map $\sigma : X \to X$ as follows: Let $P \in X \setminus A$. Then $|f^{-1}(\{f(P)\}| = 2$, so let $Q_P \in X$ be the unique point satisfying $f(P) = f(Q_P)$ and $P \neq Q_P$. We set

$$\begin{aligned} \sigma : X &\longrightarrow X \\ P &\longmapsto \sigma(P) = \begin{cases} Q_P & , P \in X \setminus A \\ P & , P \in A \end{cases} \end{aligned}$$

We have to show that σ is a holomorphic map. Let $P \in X \setminus A$. Then f is unramified at P, and there exists an open neighborhood $V \subset \mathbb{P}^1$ of $f(P)$ and open sets $U_1, U_2 \subset X$ such that

$$f^{-1}(V) = U_1 \cup U_2 \text{ and } U_1 \cap U_2 = \emptyset$$

and such that $f|U_i \to V$ is biholomorphic ($i = 1, 2$). We assume that $P \in U_1$. Thus by the definition of σ, we have

$$\sigma|U_1 = (f|V \to U_2)^{-1} \circ (f|U_1 \to V)$$

Hence, $\sigma|U_1$ is holomorphic as a composition of two holomorphic maps.

It remains the case $P \in A$. Since f has multiplicity 2 at P, there exists a chart $\phi : U \to \mathbb{D}$ on X at P with $\phi(P) = 0$ and a chart $\psi : U' \to \mathbb{D}$ on $\mathbb{P}^1$ at $f(P)$ with $\psi(f(P)) = 0$, such that $f(U) \subset U'$ and

$$\psi \circ f \circ \phi^{-1} = (\omega \mapsto \omega^2)$$

In particular, there is no other ramification point in U except P. Let $Q \in U$, $Q \neq P$ and let $\omega_1 = \phi(Q)$, $\omega_2 = \phi(\sigma(Q))$. Then

$$\omega_1^2 = \psi \circ f(Q) = \psi \circ f(\sigma(Q)) = \omega_2^2,$$

and $\omega_1 \neq \omega_2$, because $Q \neq \sigma(Q)$. Hence $\omega_2 = -\omega_1$, which means that

$$\phi \circ \sigma \circ \phi^{-1}(\omega) = -\omega \ , \quad \text{for all } \omega \in \phi(U) = \mathbb{D}.$$

Thus there is a chart around P, such that σ is holomorphic. Therefore σ is a holomorphic involution of X and its fixed points are the $2g+2$ ramification points of f, which coincide with the Weierstraß points of X.

On the other hand, if there is an involution $\sigma \in \mathrm{Aut}(X)$, fixing $2g+2$ points, then the quotient map $\phi : X \to X/_{<\sigma>}$ is a two-sheeted covering map, ramified at $2g+2$ points. Thus by Riemann-Hurwitz, we get

$$\begin{aligned} 2g(X) - 2 &= \deg(\phi)(2g(X/_{<\sigma>}) - 2) + \sum_{P \in X} (\mathrm{mult}_P(\phi) - 1) \\ 2g - 2 &= 4g(X/_{<\sigma>}) - 4 + 2g + 2. \end{aligned}$$

Therefore, $X/_{<\sigma>}$ has genus 0 and is biholomorphic to $\mathbb{P}^1$. This shows that X is hyperelliptic. Moreover, the ramification points of $\phi : X \to \mathbb{P}^1$ are the Weierstraß points of X, as we have seen in Proposition II.6.9. □

II.6.12 Remark. The proof of Theorem II.6.11 shows that every two-sheeted covering $f : X \to \mathbb{P}^1$ of a hyperelliptic surface is a quotient map for the action of the subgroup $\{\mathrm{id}, \sigma\} \subset \mathrm{Aut}(X)$ on X.

II.6.13 Corollary. *On a hyperelliptic surface X of genus $g \geq 2$, there is only one involution that fixes $2g+2$ points. It will therefore be called the* hyperelliptic involution *of X.*

PROOF: Let σ be the involution constructed in Theorem II.6.11 and let $\vartheta \in \mathrm{Aut}(X)$ be another involutive automorphism with $2g+2$ fixed points. Again by the proof of Theorem II.6.11, the fixed points of ϑ must be the Weierstraß points of X. Let $f : X \to \mathbb{P}^1$ be a two-sheeted covering map. Then $f \circ \vartheta$ is also a two-sheeted covering map and by Proposition II.6.10, there exists a Möbius transformation γ such that $f \circ \vartheta = \gamma \circ f$. Let $P \in X$ be a Weierstraß point. Then $\gamma \circ f(P) = f \circ \vartheta(P) = f(P)$ and $f(P)$ is a fixed point of γ. Thus γ fixes $2g+2 \geq 3$ distinct points and hence is the identity map. So $f \circ \vartheta = f$. For a point $Q \in X$, it follows that either $\vartheta(Q) = Q$ or $\vartheta(Q) = \sigma(Q)$. One of the sets $\{Q \in X \mid \vartheta(Q) = Q\}$, $\{Q \in X \mid \vartheta(Q) = \sigma(Q)\}$ has an accumulation point. From the identity theorem for holomorphic functions and from $\vartheta \neq \mathrm{id}$, it follows that $\vartheta = \sigma$. □

II.6.14 Corollary. *On a hyperelliptic surface X, the hyperelliptic involution σ lies in the center of* $\mathrm{Aut}(X)$.

PROOF: Let $\rho \in \mathrm{Aut}(X)$. Then $\rho\,\sigma\,\rho^{-1}$ is also an involution that fixes the $2g+2$ points $\rho(P)$, where P is a fixed point of σ. Thus by Corollary II.6.13, it is the hyperelliptic involution, and it follows that $\rho\,\sigma = \sigma\,\rho$. $\square$

II.7 The Moduli Space and the Teichmüller Space

In this section we will present the moduli space of compact Riemann surfaces of genus g and the corresponding Teichmüller space. We will merely concentrate on the definitions and the properties of these spaces without going into details.

II.7.1 The Moduli Space M_g

II.7.1 Definition. Let $g \geq 0$. The *moduli space* M_g is the set of all equivalence classes of compact Riemann surfaces of genus g. In other terms, we define M_g as

$$M_g = \{X \mid X \text{ is a compact Riemann surface of genus } g\}/\sim,$$

where we set $X \sim Y$ for two Riemann surfaces X, Y, if and only if there exists a biholomorphic map $f : X \to Y$.

Thus M_g is the classification space for the different complex structures that can be put on a topological surface of genus g. Equivalently, one can describe M_g as the classification space for projective, regular algebraic curves of genus g, using the equivalence of theses two categories (cf. Theorem II.4.2). Interestingly, M_g can itself be endowed with a topology and turned into an algebraic variety. However, the geometry of M_g is very difficult to understand, at least if the genus $g \geq 2$. A detailed discussion of moduli spaces can for example be found in [HM98].

Let us take a look at M_g for $g = 0$ and $g = 1$. A compact Riemann surface of genus 0 is always biholomorphically equivalent to the Riemann sphere $\mathbb{P}^1$. Thus the moduli space M_0 is simply a point. If the genus $g = 1$, every compact Riemann surface in M_1 is biholomorphically equivalent to a torus of the form $\mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice. Thus the classification problem consists in deciding, when two lattices in $\mathbb{C}$ lead to the same complex structure, and M_1 turns out to be isomorphic to $\mathbb{C}$.

In general, for $g \geq 2$, one can show that M_g is a complex space of dimension $3g - 3$.

II.7.2 The Teichmüller Space T_g

Frequently, an attempt to handle a classification problem is to endow the objects with an additional structure or marking in order to separate them artificially. One now has to classify the marked objects and in a second step to understand the equivalence relation that forgets the marking. This was the underlying idea that led to the discovery of the Teichmüller space.

In our situation, we will mark Riemann surfaces by using orientation preserving diffeomorphisms from a reference surface R to an arbitrary Riemann surface S. The set of equivalence classes of marked Riemann surfaces is then the Teichmüller space. In this section, we present some properties of the Teichmüller space without proofs. They can be found for example in [IT92].

II.7.2 Definition. Let $g \geq 1$ and let R be a compact Riemann surface of genus g. For two Riemann surfaces S, S' and orientation preserving diffeomorphisms $f : R \to S$, $f' : R \to S'$, we define an equivalence relation by $(S, f) \sim (S', f')$, if

$$f' \circ f^{-1} : S \to S' \text{ is homotopic to a biholomorphic map } h : S \to S'.$$

Let

$$\begin{aligned} T(R) \quad := \quad & \{ [S, f] \mid S \text{ is a Riemann surface}, f : R \to S \\ & \quad \text{is an orientation preserving diffeomorphism}\} \end{aligned}$$

be the set of equivalence classes of $\sim$. We call $T(R)$ the *Teichmüller space of R.*

The Teichmüller space of compact Riemann surfaces of genus one plays a special role. Most of the properties of Teichmüller spaces hold only if the genus satisfies $g \geq 2$.

II.7.3 Remark.

a) Let R, S be compact Riemann surfaces of genus $g(R)$ and $g(S)$. There exists an orientation preserving diffeomorphism $f : R \to S$, if and only if $g(R) = g(S)$.

b) Let R be a compact Riemann surface. Then the Teichmüller space $T(R)$ is a complex manifold. If R has genus $g \geq 2$, then the dimension of $T(R)$ is equal to $3g - 3$.

c) Let R, R' be compact Riemann surfaces. Then the Teichmüller spaces $T(R)$ and $T(R')$ are isomorphic as complex manifolds.

d) Again let R be a compact Riemann surface of genus $g \geq 2$. There exists a distance on $T(R)$, the Teichmüller distance, which turns $T(R)$ into a metric space.

Proof: Part a) follows from the fact that two surfaces that admit a smooth structure are homeomorphic, if and only if they are diffeomorphic. Thus if R and S have the same genus, there exists a diffeomorphism $f : R \to S$. By composing with an orientation reversing diffeomorphism of R we can assume that f preserves the orientation. Part c) is then a consequence of a): Let R, R' be compact Riemann surfaces of genus g. Then there exists an orientation preserving diffeomorphism $d : R \to R'$. Let $P = [S, f] \in T(R)$, then $P' = [S, f \circ d^{-1}]$ is a point in $T(R')$. This induces an isomorphism $T(R) \to T(R')$.
For a general discussion of Part b) and d), we refer to [IT92]. □

Remark II.7.3 c) tells us that the following definition makes sense.

II.7.4 Definition. Let $g \geq 0$. We define T_g, the *Teichmüller space of genus* g, as the set $T(R)$, where R is an arbitrary compact Riemann surface of genus g.

II.7.5 Definition. Let R be a compact Riemann surface of genus g. Let $\mathrm{Diffeo}^+(R)$ be the group of orientation preserving diffeomorphisms of R and let $\mathrm{Diffeo}^0(R) \subset \mathrm{Diffeo}^+(R)$ be the subgroup of such diffeomorphisms that are homotopic to the identity map. Clearly, $\mathrm{Diffeo}^0(R)$ is a normal subgroup. The quotient group

$$\mathrm{Diffeo}^+(R)/\mathrm{Diffeo}^0(R)$$

is called the *Teichmüller modular group* or the *mapping class group* of T_g. We denote it by $\mathrm{Mod}(g)$.

Let $T_g = T(R)$ be the Teichmüller space of genus g. Let $d \in \mathrm{Diffeo}^+(R)$ and let $P = [S, f]$ be a point in $T(R)$. We set

$$d \circ [S, f] := [S, f \circ d^{-1}]$$

and obtain a new point $[S, f \circ d^{-1}] \in T_g$.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "d"] & S \\ R \arrow[ur, "f \circ d^{-1}"'] & \end{tikzcd}$$

Since for any element $d \in \mathrm{Diffeo}^0(R)$ we have $d \circ [S, f] = [S, f]$, the map

$$\rho(d) : T_g \to T_g, \ [S, f] \mapsto [S, f \circ d^{-1}]$$

depends only on the equivalence class of d in $\mathrm{Mod}(g)$. Moreover, $\rho(d \circ d') = \rho(d) \circ \rho(d')$. Thus, we have a group action of $\mathrm{Mod}(g)$ on T_g.

II.7.6 Theorem. *Let T_g be the Teichmüller space of genus g and let $d \in$ Mod(g). Then, $\rho(d)$ is a holomorphic isometry of T_g and the group* Mod(g) *acts properly discontinuously on T_g via $d \mapsto \rho(d)$ as a group of holomorphic isometries. Its quotient $T_g/\mathrm{Mod}(g)$ is isomorphic to M_g. We denote the quotient map by* $\mathrm{proj} : T_g \to M_g$.

PROOF: We refer to [IT92] for a proof of this theorem. □

II.7.3 The Spaces $T_{g,n}$ and $M_{g,n}$

Another concept of marking besides the one used for the Teichmüller space, is to mark a finite number of points on a Riemann surface. We consequently allow only morphisms that respect the marked points, i.e. that map a marked point to another. This concept leads to the moduli space $M_{g,n}$ and the corresponding Teichmüller space $T_{g,n}$.

In the following, let g and n always be non-negative integers, satisfying $3g - 3 + n > 0$.

II.7.7 Definition. Let X, Y be compact Riemann surfaces of genus g. Let $P_1, \ldots, P_n \in X$ and $Q_1, \ldots, Q_n \in Y$ be n marked points on X and Y respectively. We set

$$(X; P_1, \ldots, P_n) \sim (Y; Q_1, \ldots, Q_n),$$

if there exists a biholomorphic map $f : X \to Y$ satisfying

$$f(P_i) = Q_i, \quad \text{for all } i \in \{1, \ldots, n\}.$$

Then $\sim$ is an equivalence relation and the corresponding set of equivalence classes is denoted by $M_{g,n}$. We call $M_{g,n}$ the *moduli space of compact Riemann surfaces of genus g with n marked points*.

II.7.8 Definition. The *Teichmüller space $T_{g,n}$ of compact Riemann surfaces of genus g with n marked points* is defined in a similar way. Let R be a closed Riemann surface of genus g. Let $P_1, \ldots, P_n$ be n marked points on R. Let S, S' be Riemann surfaces and let $Q_1 \ldots, Q_n$, $Q'_1, \ldots, Q'_n$ be n marked points on S and S' respectively. Moreover, let $f : R \to S$, $f' : R \to S'$ be orientation preserving diffeomorphisms, satisfying

$$f(P_i) = Q_i \quad \text{and} \quad f'(P_i) = Q'_i \quad \text{for all } i \in \{1, \ldots, n\}.$$

Then, $(S, f; Q_1, \ldots, Q_n) \sim (S', f'; Q'_1, \ldots, Q'_n)$, if

$$f' \circ f^{-1} : S \to S' \qquad \text{is homotopic to a biholomorphic map } h : S \to S' \text{ with } h(P_i) = P'_i \quad (1 \leq i \leq n)$$

Again, $\sim$ is an equivalence relation and we define $T_{g,n}$ to be the set of equivalence classes for $\sim$.

Similarly to the unmarked case, we define the *Teichmüller modular group* $\mathrm{Mod}(g, n)$ to be the quotient group

$$\mathrm{Diffeo}^+(R)_n / \mathrm{Diffeo}^0(R)_n,$$

where $\mathrm{Diffeo}^+(R)_n$ denotes the orientation preserving diffeomorphims of R that fix each of the n points $P_1, \ldots, P_n$, and $\mathrm{Diffeo}^0(R)_n$ is the normal subgroup of those homotopic to the identity map.

II.7.9 Remark.

a) The analogue of Theorem II.7.6 holds respectively.

b) The analogues of parts a), c) and d) of Remark II.7.3 hold respectively.

c) $M_{g,n}$ and $T_{g,n}$ are both complex spaces of dimension $3g - 3 + n$.

d) There are natural projections

$$M_{g,n} \to M_g \qquad \text{and} \qquad T_{g,n} \to T_g$$

that forget the marked points. In the first case, this is a morphism between algebraic varieties, in the second case, it is a holomorphic map.

Proof: We refer to [Abi89] and to [Gar87]. □

Chapter III

Origamis

Origamis, as we have defined them in I.1.1 are so far only combinatorial objects. The purpose of this chapter is to explain how we can turn them into Riemann surfaces. Moreover, we describe how an origami gives rise to a curve in the moduli space. The main results of this chapter come from the works of Veech [Vee89], McMullen [McM03], Earle and Gardiner [EG97] and Schmithüsen [Sch04]. In our presentation, we closely follow Schmithüsen [Sch05] and Herrlich and Schmithüsen [HS06].

III.1 Translation Surfaces

An origami is a special case of a translation surface. These are surfaces that are endowed with an atlas, for which the transition maps are locally translations. We will present some properties of translation surfaces in this section.

III.1.1 Definition & Remark.

a) Let X be a topological surface, i.e. a connected topological manifold of real dimension 2. An atlas $\mathcal{A}$ on X is called a *translation atlas*, if for any two charts (U,ϕ), (U',ϕ') with $U \cap U' \neq \emptyset$ the transition map

$$\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$$

is locally a translation of $\mathbb{R}^2$, i.e. for every point $P \in U \cap U'$, there exists an open neighborhood $V \subset U \cap U'$ of P such that $\phi' \circ \phi^{-1} : \phi(V) \to \phi'(V)$ is a translation of $\mathbb{R}^2$.

b) As for the case of Riemann surfaces, one defines a translation structure on X in the following way.

We say that two charts (U,ϕ), (U',ϕ') on X are compatible, if the transition map $\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ is locally a translation. Two translation atlases $\mathcal{A}$,$\mathcal{A}'$ on X are said to be equivalent, if each chart of $\mathcal{A}$ is compatible with each chart of $\mathcal{A}'$. This clearly is an equivalence relation. An equivalence class of translation atlases is called a *translation structure* on X, and it contains a unique maximal translation atlas. In the following, we will often identify a translation structure with the unique maximal translation atlas that it contains.

c) If ν is a translation structure on X, then the pair $X_\nu := (X, \nu)$ is called a *translation surface*.

In the following, we always identify $\mathbb{C}$ with $\mathbb{R}^2$ by sending 1 to $(1,0)^t$ and i to $(0,1)^t$. In this way, translations of $\mathbb{R}^2$ become translations of $\mathbb{C}$ and vice versa. Moreover, a translation is a biholomorphic map. This shows the following remark.

III.1.2 Remark. Let X_ν be a translation surface. Then the translation structure ν induces a unique complex structure on X. Thus a translation surface is also a Riemann surface.

III.1.3 Examples.

a) $\mathbb{C}$ itself with the translation structure given by the atlas $\mathcal{A}$ consisting of the single chart $(\mathbb{C}, \mathrm{id})$ is a translation surface, because the condition on the transition maps is trivially satisfied.

   Which other maps are compatible to the chart $(\mathbb{C}, \mathrm{id})$? Surely, any translation $\mathbb{C} \to \mathbb{C}, z \mapsto z + c$ $(c \in \mathbb{C})$ and any restriction of a translation to an open subset of $\mathbb{C}$ satisfies this property. But since we require the transition maps to be only locally translations, any homeomorphism $\phi : U \to V$ of open sets of $\mathbb{C}$ is compatible to $(\mathbb{C}, \mathrm{id})$, as long as on each connected component, it is the restriction of a translation of $\mathbb{C}$.

b) Let $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and let $\Lambda_B \subset \mathbb{C}$ be the lattice generated by $\omega_1 = a + ic$ and $\omega_2 = b + id$, i.e. $\Lambda_B = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. Then $\mathbb{C}/\Lambda_B$ is a torus and it is naturally endowed with a translation structure, which descends from $\mathbb{C}$.

   Indeed, let $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_B$, $z \mapsto z + \Lambda_B$ denote the quotient map. Recall that π is a topological covering and that a chart on $\mathbb{C}/\Lambda_B$ is always a local inverse of π. More precisely, if $P \in \mathbb{C}/\Lambda_B$, let $z_0 \in \pi^{-1}(P)$ be any preimage and let $V \subset \mathbb{C}$ be an open neighborhood of z_0 not containing two points z, w that are equivalent modulo Λ_B (i.e. $\forall z, w \in V : z - w \notin \Lambda_B$). Then $\pi|V \to U := \pi(V)$ is a homeomorphism and its inverse $\phi : U \to V$ is a chart on $\mathbb{C}/\Lambda_B$ for its complex structure.

   We claim that these charts also form a translation atlas for $\mathbb{C}/\Lambda_B$. Let $\phi : U \to V$, $\phi' : U' \to V'$ be two charts on $\mathbb{C}/\Lambda_B$ such that $U \cap U' \neq \emptyset$.

Let
$$\psi := \phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U').$$
For $z \in \phi(U \cap U')$, we have $\pi(\psi(z)) = \pi(z)$, thus $\psi(z) - z \in \Lambda_B$. Since the map $\psi - \mathrm{id}$ is continuous and takes values in a discrete set, it must be locally constant. Hence ψ is locally a translation and $\mathbb{C}/\Lambda_B$ is endowed with a translation structure, which we call ν_B.

We will refer to the translation surface $(\mathbb{C}/\Lambda_B, \nu_B)$ as E_B.

Note that every torus is biholomorphic to a torus of the form $\mathbb{C}/\Lambda_B$ with a lattice Λ_B, $B \in \mathrm{SL}_2(\mathbb{R})$, defined as above.

III.1.1 Translations

We want to define the category $\mathfrak{Tr}$ of all translation surfaces. First of all, we must say what a morphism between translation surfaces is.

III.1.4 Definition. Let X_ν, Y_ω be translation surfaces and let $f : X_\nu \to Y_\omega$ be a continuous map. We say that f is a *translation*, if for any two charts $(U, \phi) \in \nu$, $(V, \psi) \in \omega$, satisfying $f(U) \subset V$ and for any point $P \in U$, there exists an open neighborhood $W \subset U$ of P, such that on $\phi(W)$ we have
$$\psi \circ f \circ \phi^{-1} = (z \mapsto z + c), \qquad (c \in \mathbb{C}).$$

III.1.5 Remark. Let $f : X_\nu \to Y_\omega$ be a map between translation surfaces. The following statements are equivalent:

a) The map f is a translation.

b) For every point $P \in X_\nu$, there exist charts $(U, \phi) \in \nu$ at P and $(V, \psi) \in \omega$ at $f(P)$, such that $\psi \circ f \circ \phi^{-1}$ is locally a translation of $\mathbb{C}$.

c) For every point $P \in X_\nu$, there exist charts $(U, \phi) \in \nu$ at P and $(V, \psi) \in \omega$ at $f(P)$, such that $\psi \circ f \circ \phi^{-1}$ is a translation of $\mathbb{C}$.

Proof: The implications a) $\Rightarrow$ b) and b) $\Rightarrow$ c) are immediate. We have just to take into account that a map is locally a translation, if and only if it is a translation on every connected component of its domain. The proof of the implication c) $\Rightarrow$ a) is slightly tedious and we will not carry it out here. $\square$

III.1.6 Remark.

a) If X_ν, Y_ω, Z_μ are translation surfaces and $f : X_\nu \to Y_\omega$ and $g : Y_\omega \to Z_\mu$ are translations, then $g \circ f$ is a translation.

b) The category $\mathfrak{Tr}$, whose objects are translation surfaces and whose morphisms are translations, is a subcategory of the category $\mathfrak{Riem}$.

c) If $f : X_\nu \to Y_\omega$ is a translation, then $f : X_\nu \to f(X_\nu)$ is locally bijective.

d) Let X_ν, Y_ω be translation surfaces. Then $X_\nu \cong Y_\omega$, if and only if there exists a bijective translation $f : X_\nu \to Y_\omega$.

Proof: The assertion in a) can easily be shown by expressing f and g in local coordinates. Part b) follows directly from a) and Remark III.1.2. Part c) follows immediately from the definition. To show d) it suffices to prove that the inverse of f is again a translation, which is easy to see. $\square$

III.1.7 Definition. We denote the set of all bijective translations of X_ν by $\mathrm{Trans}(X_\nu)$. Clearly, $\mathrm{Trans}(X_\nu)$ is a group and we call it the *group of translations* of X_ν.

III.1.8 Proposition. *Let X_ν be a translation surface. Let Z be a Hausdorff space and let $p : Z \to X$ be a topological covering of X. Then there is a unique translation structure η on Z such that p is a translation.*

Proof: Let $z \in Z$ and let $\phi : U \to V$ be a chart of ν at the point $p(z)$. Since p is a topological covering, there exists an open neighborhood $W \subset Z$ of z and an open neighborhood $U' \subset X$ of $p(z)$ such that $p|W \to U'$ is a homeomorphism. By adjusting W and U' if necessary, we can assume that $U' \subset U$. Let $\psi := \phi \circ p|_W$ and let $\mathcal{A}$ be the set of all maps obtained in this way. It is easy to see that $\mathcal{A}$ is a translation atlas. Let η be the translation structure that it defines. Then $p : Z_\eta \to X_\nu$ is a translation.

If η' is another translation structure on Z, such that p is a translation, then $\mathrm{id} : Z_\eta \to Z_{\eta'}$ is a translation. Thus $Z_\eta \cong Z_{\eta'}$. $\square$

III.1.9 Example. Let $B \in \mathrm{SL}_2(\mathbb{R})$, let Λ_B be the lattice defined by B and let us again consider the torus $E_B = (\mathbb{C}/\Lambda_B, \nu_B)$ as in Example III.1.3 b). The map $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_B$ is a topological covering and we can lift the structure ν_B via π to $\mathbb{C}$. The resulting translation structure η then coincides with the natural translation structure $\nu_\mathbb{C}$ of $\mathbb{C}$, generated by the single chart $(\mathbb{C}, \mathrm{id})$.

Making use of Proposition III.1.8, we only have to show that π is a translation for the respective translation structures $\nu_\mathbb{C}$ and ν_B. So let $z \in \mathbb{C}$.

Following Remark III.1.5, we can choose a chart of the form $(W, \mathrm{id}|_W)$ of $\nu_{\mathbb{C}}$ at z such that W does not contain two points z, w that are equivalent modulo Λ_B. Furthermore, let (U, ϕ) be a chart of ν_B at $\pi(z)$. Then we have

$$\phi \circ \pi \circ \mathrm{id}|_W = \phi \circ \pi|_W$$

and $(\pi|_W)^{-1}$ is already a chart of ν_B, thus $\phi \circ \pi|_W$ is a transition map of ν_B and hence locally a translation.

III.1.2 Translation Surfaces from Holomorphic 1-Forms

A translation surface leads to a Riemann surface. But we can also proceed in the opposite direction. As this brings in some new aspects, we shortly describe how this is done.

For a Riemann surface X, we denote by $\Omega(X)$ the vector space of holomorphic 1-forms (or holomorphic abelian differentials) on X, i.e. an element $\omega \in \Omega(X)$ is locally an expression of the form $f\,dz$, where (U, z) is a chart on X and $f \in \mathcal{O}(U)$ is a holomorphic function. If Y is a Riemann surface and $p : Y \to X$ is a holomorphic map, then $p^*\omega$, the pullback of ω, is a holomorphic 1-form in $\Omega(Y)$, that is locally an expression of the form

$$p^*(f\,dz) := (p^*f)d(p^*z) := (f \circ p)d(z \circ p),$$

if ω is locally given by $f\,dz$.

III.1.10 Example. Let X be a Riemann surface, and let $\omega \in \Omega(X)$ be a holomorphic 1-form. Let Z be the set of zeros of ω and let $X^* = X \setminus Z$. We define charts on X^* in the following way:

Let $P_0 \in X$ and let U be an open neighborhood of P that is homeomorphic to $\mathbb{D}$. We set

$$\phi : U \to \mathbb{C},\ P \mapsto \int_{P_0}^{P} \omega\ ,$$

where we integrate over an arbitrary path that connects P_0 to P. Note that ϕ is well-defined, because U is simply connected. The map ϕ is a local primitive of ω. As ω has no zeros in U, ϕ is locally injective. Thus if we restrict ϕ to an open subset $U' \subset U$, we get a biholomorphic map

$$\phi|U' \to \phi(U').$$

The set of all charts obtained in this way is a translation atlas $\mathcal{A}_\omega$ on X^*.

To see this, let $\phi : U \to \mathbb{C}$, $\psi : V \to \mathbb{C}$ be two charts of $\mathcal{A}_\omega$. Then ϕ and ψ are both local primitives of ω, thus their difference is locally constant.

Next we see about the points that we took out. The following remark is well-known and can for instance be found in [Zor06].

III.1.11 Remark. A zero P of the holomorphic 1-form ω leads to a singularity of the translation structure. Around the point P the translation surface locally looks like a cone; the angle is no longer equal to 2π, but it mesures $2\pi(d+1)$, where d is the zero order of ω at P. We call a singularity of this type a *cusp* or a conical point.

With the help of Example III.1.10, we get another description of the translation structure on the torus.

III.1.12 Remark. Let $B \in \mathrm{SL}_2(\mathbb{R})$. The 1-form dz on $\mathbb{C}$ descends to a 1-form ω on $\mathbb{C}/\Lambda_B$ as dz is invariant under the group Λ_B. Moreover, ω has no zeros, as this is true for dz, and thus induces a translation structure ν_ω on $\mathbb{C}/\Lambda_B$. Then the translation surfaces $E_B = (\mathbb{C}/\Lambda_B, \nu_B)$ and $E_\omega = (\mathbb{C}/\Lambda_B, \nu_\omega)$ are isomorphic.

PROOF: A chart ϕ of ν_ω is a local primitive of ω. So we have $d\phi = \omega$. If we pull back ω to $\mathbb{C}$ via $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_B$ we recover the 1-form dz, which has primitives of the form $z + c = \mathrm{id}_\mathbb{C} + c$, where $c \in \mathbb{C}$ is a constant. Then,

$$dz = \pi^*\omega = \pi^* d\phi = d(\phi \circ \pi).$$

Hence $d(z - \phi \circ \pi) = 0$, which implies that locally, we have

$$\phi \circ \pi = \phi \circ \pi \circ \mathrm{id}_\mathbb{C} = z + c$$

for a constant $c \in \mathbb{C}$. Therefore π is a translation for the translation structures $\nu_\mathbb{C}$ and ν_ω, and by Example III.1.9, π is a translation for $\nu_\mathbb{C}$ and ν_B. As π is locally invertible, the map $\mathrm{id}_{\mathbb{C}/\Lambda_B}$ is locally a composition of translations. $\square$

III.1.13 Remark. Let $p : Y \to X$ be an unramified holomorphic map between Riemann surfaces. Let $\omega \in \Omega(X)$ be a holomorphic 1-form, and let $X^* = X \setminus Z$, where Z is the set of zeros of ω. Let ν_ω be the translation structure on X^*. Then the pullback $p^*\omega$ defines a translation structure $p^*\nu_\omega$ on $p^{-1}(X^*) \subset Y$, such that p is a translation.

PROOF: There are no zeros of $p^*\omega$ in $p^{-1}(X^*)$. This holds, since locally $p^*\omega = p^*(f\,dz) = (f \circ p)d(z \circ p)$ and since ω has no zeros in X^*. Thus, it defines a translation structure $p^*\nu_\omega$ on $p^{-1}(X^*)$. Let (U,ϕ) be a chart of $p^*\nu_\omega$, and let (V,ψ) be a chart of ν_ω, such that $p(U) \subset V$. Then $d\phi = p^*\omega$ and $d\psi = \omega$. One has

$$d(\psi \circ p) = dp^*\psi = p^*(d\psi) = p^*\omega = d\phi,$$

which implies $d(\psi \circ p - \phi) = 0$. Thus for $P \in U$, we get

$$\psi \circ p(P) + c = \phi(P), \qquad c \in \mathbb{C}.$$

If we write $z = \phi(P)$, it follows that

$$\psi \circ p \circ \phi^{-1}(z) = z + c,$$

hence p is a translation. $\square$

III.2 The Affine Group and the Veech Group

For a fixed translation surface, we will now consider maps that respect the given translation structure of the surface. Naturally, they will locally look like affine maps of $\mathbb{C}$. The affine group of a translation surface is the group of all orientation preserving affine diffeomorphisms. To each element of the affine group, we can associate a matrix. The set of these matrices form a group, which is called the Veech group of the translation surface. The Veech group will play a prominent role in our discussion of Teichmüller curves.

III.2.1 Notation. We will introduce a convenient notation for real affine maps of the complex plane. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$. For a complex number $z = x + iy$, $x, y \in \mathbb{R}$, we set

$$A \cdot z := (ax + by) + i(cx + dy).$$

Then every real affine map $\mathbb{R}^2 \to \mathbb{R}^2$, $\binom{x}{y} \mapsto A\binom{x}{y} + \binom{t_1}{t_2}$, $\binom{t_1}{t_2} \in \mathbb{R}^2$ can be written in the form

$$f : \mathbb{C} \to \mathbb{C},\ z \mapsto A \cdot z + t$$

with $t = t_1 + it_2 \in \mathbb{C}$.

III.2.2 Definition & Remark.

a) Let X_ν, Y_ω be translation surfaces and let $f : X \to Y$ be a continuous map. Then f is called *affine* (with respect to the translation structures ν and ω), if it can locally be described as a real affine map. More precisely, let $(U,\phi) \in \nu$, $(V,\psi) \in \omega$ be charts, such that $f(U) \subset V$. Then for every

$z \in \phi(U)$, there exists an open neighborhood $W \subset \phi(U)$ of z and there exists a matrix $A \in \mathbb{R}^{2\times 2}$ and an element $t \in \mathbb{C}$ such that

$$\psi \circ f \circ \phi^{-1}|_W = (z \mapsto A \cdot z + t). \tag{III.1}$$

b) With the notations from a), the following statements are equivalent:

   (1) The map f is affine.

   (2) For every point $P \in X_\nu$, there exists a chart (U, ϕ) of ν at P and a chart (V, ψ) of ω at $f(P)$ such that $\psi \circ f \circ \phi^{-1}$ is locally an affine map of $\mathbb{R}^2$.

   (3) For every point $P \in X_\nu$, there exists a chart (U, ϕ) of ν at P and a chart (V, ψ) of ω at $f(P)$ such that $\psi \circ f \circ \phi^{-1}$ is an affine map of $\mathbb{R}^2$.

   This is the analogue of Remark III.1.5. Again we omit the proof.

c) An affine map $f : X_\nu \to Y_\omega$ that is a diffeomorphism is called *affine diffeomorphism*.

d) Notice that f is an affine diffeomorphism, if and only if f is bijective. In this case, the matrix A in (III.1) is always in $\mathrm{GL}_2(\mathbb{R})$.

III.2.3 Example. If X_ν is a translation surface and $f : X_\nu \to X_\nu$ is a bijective translation, then f is an affine diffeomorphism and the matrix in (III.1) is equal to $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Notice that a translation surface is an orientable manifold. So we can speak of orientation preserving maps. In particular, if $f : X_\nu \to X_\nu$ is an affine diffeomorphism of a translation surface X_ν, then f preserves the orientation, if and only if the matrix A in (III.1) lies in $\mathrm{GL}_2^+(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det(A) > 0\}$. We will mainly be interested in those affine diffeomorphisms, which are orientation preserving.

III.2.4 Definition. Let X_ν be a translation surface. The *affine group* of X_ν is defined as

$$\begin{aligned} \mathrm{Aff}^+(X_\nu) \quad = \quad & \{f : X \to X \mid f \text{ orientation preserving diffeomorphism,} \\ & \quad \text{affine with respect to } \nu\} \end{aligned}$$

III.2.5 Remark. $\mathrm{Aff}^+(X_\nu)$ is a group.

PROOF: We have to show that the composition of two elements $f, g \in \mathrm{Aff}^+(X_\nu)$ is well-defined. It suffices to see that $g \circ f$ is again affine and to calculate its matrix at a point $P \in X$ in terms of the matrices of f and g. Let $P \in X$. By Remark III.2.2 b), there exist charts $(U, \phi), (V, \psi), (W, \vartheta) \in \nu$ at P, $f(P)$ and $g \circ f(P)$ respectively, such that $f(U) \subset V$ and $g(V) \subset W$ and such that

$$\begin{aligned}\psi \circ f \circ \phi^{-1} &= (z \mapsto A \cdot z + t)\\ \vartheta \circ g \circ \psi^{-1} &= (z \mapsto B \cdot z + s),\end{aligned}$$

with $A, B \in \mathrm{GL}_2^+(\mathbb{R})$ and $s, t \in \mathbb{C}$. Thus,

$$\begin{aligned}\vartheta \circ g \circ f \circ \phi^{-1} &= \vartheta \circ g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1}\\ &= (z \mapsto B \cdot z + s) \circ (z \mapsto A \cdot z + t)\\ &= (z \mapsto B\,A \cdot z + B \cdot t + s).\end{aligned}$$

Thus $g \circ f$ is affine with respect to ν. Since $B\,A$ is again in $\mathrm{GL}_2^+(\mathbb{R})$, it follows that $g \circ f \in \mathrm{Aff}^+(X_\nu)$. □

III.2.6 Lemma. *Let X_ν, Y_ω be translation surfaces. If $f : X_\nu \to Y_\omega$ is an affine map, then the matrix in equation (III.1) is globally the same, i.e. it is independent of the choice of charts and of the point P.*

PROOF: First we show the independence of the choice of charts. Let $P \in X$ and let (U, ϕ), (U', ϕ') be two charts of ν at P and let (V, ψ), (V', ψ') be two charts of ω at $f(P)$. There exists a neighborhood $W \subset U \cap U'$ of P such that

$$\psi \circ f \circ \phi^{-1}|_{\phi(W)} = (z \mapsto A_1 \cdot z + t_1)$$

and

$$\psi' \circ f \circ \phi'^{-1}|_{\phi'(W)} = (z \mapsto A_2 \cdot z + t_2),$$

where $A_1, A_2 \in \mathbb{R}^{2\times 2}$, $t_1, t_2 \in \mathbb{C}$.

The transition maps of two charts of ν and ω are locally translations. Thus we have $\phi' \circ \phi^{-1}|_{\phi(W)} = (z \mapsto z + c)$ for an element $c \in \mathbb{C}$, if we assume, that W is connected (we shrink W if necessary). Similarly, $\psi \circ \psi'^{-1}|_{\psi'(W)} = (z \mapsto z + \tilde{c})$, $\tilde{c} \in \mathbb{C}$, where $W' \subset V \cap V'$ is a connected neighborhood of $f(P)$. Hence,

$$\begin{aligned}A_1 \cdot z + t_1 &= \psi \circ f \circ \phi^{-1}(z)\\ &= \psi \circ (\psi'^{-1} \circ \psi') \circ f \circ (\phi'^{-1} \circ \phi') \circ \phi^{-1}(z)\\ &= (\psi \circ \psi'^{-1}) \circ (\psi' \circ f \circ \phi'^{-1}) \circ (\phi' \circ \phi^{-1})(z)\\ &= A_2 \cdot z + A_2 \cdot c + t_2 + \tilde{c}\end{aligned}$$

for all $z \in \phi(W \cap f^{-1}(W'))$. Taking the derivative yields $A_1 = A_2 =: A$.

It remains to show that the matrix is independent of the point $P \in X$. From now on, we assume that each chart has a connected domain. Let

$$\begin{aligned} M &= \{Q \in X \mid \exists \text{ a chart } (U,\phi) \in \nu \text{ at } Q \text{ and a chart } (V,\psi) \in \omega \text{ at } f(Q) \\ &\quad \text{such that } \psi \circ f \circ \phi^{-1} = (z \mapsto A \cdot z + t),\ t \in \mathbb{C}\}. \end{aligned}$$

Since $P \in M$, the set M is not empty. Clearly, M is an open subset of X. Let $\bar{M} = X \setminus M$. Then $\bar{M}$ is also open. Indeed, if $Q \in \bar{M}$, there exists a chart $(U', \phi') \in \nu$ at Q and a chart $(V', \psi') \in \omega$ at $f(Q)$ such that

$$\psi' \circ f \circ \phi'^{-1} = (z \mapsto A' \cdot z + t')$$

for a matrix $A' \neq A$ and $t' \in \mathbb{C}$. Thus every point in $U' \cap f^{-1}(V')$ is also in $\bar{M}$. Using the fact that X is connected, we conclude that $\bar{M}$ is empty. This achieves the proof. □

III.2.7 Definition & Remark.

a) Lemma III.2.6 allows us to define a map

$$\mathrm{der} : \mathrm{Aff}^+(X_\nu) \to \mathrm{GL}_2^+(X_\nu), \quad f \mapsto A,$$

where A is the matrix from equation (III.1). We say that $A = \mathrm{der}(f)$ is the *derivative* of f. The map der is called the *derived map*.

b) The map der is a group homomorphism.

c) The image $\mathrm{der}(\mathrm{Aff}^+(X_\nu)) \subset \mathrm{GL}_2^+(\mathbb{R})$ is called the *Veech group* of X_ν and we denote it by $\Gamma(X_\nu)$.

d) The kernel of der is precisely the subgroup $\mathrm{Trans}(X_\nu) \subset \mathrm{Aff}^+(X_\nu)$.

Proof: The fact that der is a group homomorphism follows directly from the proof of Remark III.2.5 and from Lemma III.2.6, and we obviously have $\ker(\mathrm{der}) = \mathrm{Trans}(X_\nu)$. □

III.2.8 Examples.

a) **The affine group of $\mathbb{C}$.** Let $\mathbb{C}$ be endowed with the natural translation structure $\nu_{\mathbb{C}}$ as in Example III.1.3. Then f is an element of $\mathrm{Aff}^+(\mathbb{C})$, if and only if there exists $A \in \mathrm{GL}_2^+(\mathbb{R})$, $t \in \mathbb{C}$ such that f can be written as

$$f : \mathbb{C} \to \mathbb{C},\ z \mapsto A \cdot z + t.$$

b) **The affine group of E_B.** Let $B \in \mathrm{SL}_2(\mathbb{R})$ and let E_B be the torus with the translation structure constructed in III.1.3. We wish to determine the group $\mathrm{Aff}^+(E_B)$. Let $f \in \mathrm{Aff}^+(E_B)$. Since $\pi : \mathbb{C} \to \mathbb{C}/\Lambda_B$ is the universal covering, we can lift f to a map $\hat{f}$ such that the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\hat{f}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi"] \\
\mathbb{C}/\Lambda_B \arrow[r, "f"] & \mathbb{C}/\Lambda_B
\end{tikzcd}$$

commutes. Example III.1.9 now tells us that π is a translation. Moreover, π is locally invertible and each local inverse is also a translation. Therefore, $\hat{f}$ is an affine map of $\mathbb{C}$ and $\mathrm{der}(\hat{f}) = \mathrm{der}(f)$.

Thus, it suffices to determine, under which condition an element of the affine group of $\mathbb{C}$ descends to E_B. Let $\hat{f} \in \mathrm{Aff}^+(\mathbb{C})$, $\hat{f}(z) = A \cdot z + t$ with $A \in \mathrm{GL}_2^+(\mathbb{R}), t \in \mathbb{C}$. Let Λ_B be the lattice defined by B, $\Lambda_B = \{B \cdot (n + im) \mid n, m \in \mathbb{Z}\}$. If $\hat{f}$ descends to E_B, then the map

$$\pi \circ \hat{f} = (z \mapsto A \cdot z + t + \Lambda_B)$$

must be Λ_B-invariant. Thus for any $\lambda \in \Lambda_B$ we must have

$$\pi \circ \hat{f}(z + \lambda) = \pi \circ \hat{f}(z) \quad \text{for all } z \in \mathbb{C}.$$

This is equivalent to

$$A \cdot (z + \lambda) + t + \Lambda_B = A \cdot z + t + \Lambda_B,$$

which means that $A \cdot \lambda \in \Lambda_B$. Hence, given any $\lambda = B \cdot (n + im) \in \Lambda_B$, there must exist $n', m' \in \mathbb{Z}$ such that

$$AB \cdot (n + im) = B \cdot (n' + im'),$$

whereby it follows that $B^{-1}AB \in \mathrm{SL}_2(\mathbb{Z})$.

Thus an element $\hat{f}$ descends to E_B, if and only if $\mathrm{der}(\hat{f}) \in B\mathrm{SL}_2(\mathbb{Z})B^{-1}$. Altogether we have shown that

$$\Gamma(E_B) = B\mathrm{SL}_2(\mathbb{Z})B^{-1}.$$

In particular, if we consider the torus $E_I = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, we get

$$\Gamma(E_I) = \mathrm{SL}_2(\mathbb{Z}).$$

In particular, this shows that $\Gamma(E_B) \subset \mathrm{SL}_2(\mathbb{R})$. This fact generalizes to an arbitrary translation surface of finite volume, since every affine diffeomorphism preserves the volume, which is equivalent to its derivative having determinant ± 1. For instance, if the translation surface X_ν is obtained from a holomorphic 1-form on compact Riemann surface $\bar{X}$, then it has finite volume, and its Veech group $\Gamma(X_\nu)$ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$.

III.2.9 Remark. Let X be a topological surface. Then the group $\mathrm{SL}_2(\mathbb{R})$ acts on the set of translation structures on X in the following way. For a matrix $B \in \mathrm{SL}_2(\mathbb{R})$, let

$$\varphi_B : \mathbb{C} \to \mathbb{C}, \; z \mapsto B \cdot z.$$

Let ν be a translation structure on X and let ψ be a chart of ν. We perform an affine deformation of ψ by composing it with φ_B and we denote by $B \cdot \nu$ the resulting translation structure, whose charts are of the form $\varphi_B \circ \psi$. This gives a group action

$$(B, \nu) \mapsto B \cdot \nu.$$

Proof: Let ν be a translation structure on X. If I denotes the identity matrix of $\mathrm{SL}_2(\mathbb{R})$, then we clearly have $I \cdot \nu = \nu$. Moreover, if $B, B' \in \mathrm{SL}_2(\mathbb{R})$, then the charts of $(BB') \cdot \nu$ are of the form $\varphi_{BB'} \circ \psi$ with a chart $\psi \in \nu$. Now $\varphi_{BB'} = \varphi_B \circ \varphi_{B'}$, thus $(BB') \cdot \nu = B \cdot (B' \cdot \nu)$, which completes the proof. $\square$

III.2.10 Proposition. *Let X_ν be a translation surface, let $B \in \mathrm{SL}_2(\mathbb{R})$ and let $X_{B\cdot\nu}$ be the translation surface obtained by the action of B. Then*

$$\mathrm{Aff}^+(X_{B\cdot\nu}) \cong \mathrm{Aff}^+(X_\nu), \quad \mathrm{Trans}(X_{B\cdot\nu}) \cong \mathrm{Trans}(X_\nu)$$

$$\textit{and} \quad \Gamma(X_{B\cdot\nu}) = B\Gamma(X_\nu)B^{-1}.$$

Proof: Let $\mathrm{id} : X_\nu \to X_{B\cdot\nu}$ be the map that is topologically the identity map and that switches over the translation structures. Then id is an affine map with derivative B and induces a group isomorphism

$$\begin{array}{rcl} \Psi : \mathrm{Aff}^+(X_{B\cdot\nu}) & \to & \mathrm{Aff}^+(X_\nu) \\ f & \mapsto & \Psi(f) = \mathrm{id}^{-1} \circ f \circ \mathrm{id}. \end{array}$$

Ψ is well-defined, since $\Psi(f)$ is a composition of affine diffeomorphisms and since $\mathrm{der}(\Psi(f)) = \mathrm{der}(\mathrm{id}^{-1} \circ f \circ \mathrm{id}) = B^{-1}\,\mathrm{der}(f)\,B$ is again in $\mathrm{GL}_2^+(\mathbb{R})$. One can easily verify that Ψ is a group homomorphism and that its inverse is given by $\mathrm{Aff}^+(X_\nu) \to \mathrm{Aff}^+(X_{B\cdot\nu})$, $g \mapsto \mathrm{id}\circ g \circ \mathrm{id}^{-1}$. Moreover, $\mathrm{der}(\Psi(f)) = I$, if and only if $\mathrm{der}(f) = I$, thus $\Psi(\mathrm{Trans}(X_{B\cdot\nu})) = \mathrm{Trans}(X_\nu)$.

This argument also shows that the corresponding Veech groups are conjugated to each other in $\mathrm{GL}_2^+(\mathbb{R})$. For if $A \in \Gamma(X_{B\cdot\nu})$ and $f \in \mathrm{Aff}^+(X_{B\cdot\nu})$ with $\mathrm{der}(f) = A$, then $\mathrm{der}(\Psi(f)) = B^{-1}AB \in \Gamma(X_\nu)$, which implies $\Gamma(X_{B\cdot\nu}) \subset B\Gamma(X_\nu)B^{-1}$. The other inclusion follows similarly. □

III.2.11 Example. Let us once again consider the case of our tori E_B, $B \in \mathrm{SL}_2(\mathbb{R})$. The associated set $\{\nu_B \,|\, B \in \mathrm{SL}_2(\mathbb{R})\}$ furnishes us with a bunch of translation structures on the torus. On the other hand, we let act $\mathrm{SL}_2(\mathbb{R})$ on the set of translation structures and we can determine the orbit of ν_I under this action (where I denotes the identity matrix of $\mathrm{SL}_2(\mathbb{R})$). Our claim is that we get nothing new.

Claim. Let $B \in \mathrm{SL}_2(\mathbb{R})$. Then $\nu_B = B \cdot \nu_I$.

Let $\mathrm{id} : E_I \to E_{B\cdot I} := (\mathbb{C}/\Lambda_I, B\cdot\nu_I)$ be the map that is topologically the identity. Then $\mathrm{der}(\mathrm{id}) = B$. On the other hand, the affine map $\varphi_B : \mathbb{C} \to \mathbb{C}, z \mapsto B\cdot z$ descends to a map $\bar\varphi_B : E_I \to E_B$, since $B\cdot\Lambda_I = \Lambda_B$ and its derivative is also equal to B.

$$\begin{tikzcd} E_I \arrow[r, "\mathrm{id}"] \arrow[dr, "\bar\varphi_B"'] & E_{B\cdot I} \\ & E_B \arrow[u, "\psi"'] \end{tikzcd}$$

Hence the map
$$\psi = \mathrm{id}\circ\bar\varphi_B^{-1} : E_B \to E_{B\cdot I}$$
has derivative $\mathrm{der}(\psi) = BB^{-1} = I$ and is thus a translation. Therefore, $E_B \cong E_{B\cdot I}$.

Thus, the action of an element $B \in \mathrm{SL}_2(\mathbb{R})$ on ν_I corresponds to a shearing of the unit square into a parallelogram.

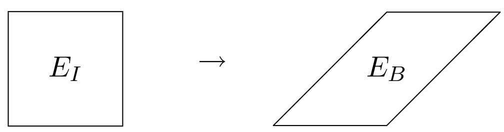


Figure III.1: *The action of* $B = \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ *on* E_I

III.3 Origamis

In this section, we will see another way to define origamis. So far, an origami (as we have presented it in Definition I.1.1) is the topological surface obtained by gluing a finite number of unit squares at their edges. This also means that one has a natural map onto the basic origami that consists of only one unit square. Since the basic origami can be identified with a torus, we may now reformulate the definition of an origami.

III.3.1 Definition. Let E be a fixed torus (i.e. a topological surface of genus one) and let $\bar{P} \in E$ be a point. Then

$$E^* = E \setminus \{\bar{P}\}$$

is a *once-punctured torus*. An *origami* of genus $g \geq 1$ is a finite covering

$$O = (p : X \to E)$$

such that X is a compact surface of genus g and such that for $X^* = X \setminus p^{-1}(\{\bar{P}\})$, the restriction

$$p : X^* \to E^*$$

is a finite, topological covering map (i.e. p is ramified at most over the point $\bar{P}$).

III.3.2 Remark. The definition given in III.3.1 is equivalent to our earlier definition of an origami in I.1.1.

PROOF: An origami as defined in I.1.1 was the result of a gluing process. Let X be the topological space obtained by gluing a finite number of copies of the unit square at their edges. Recall that the basic origami E is a torus and let $\bar{P} \in E$ be the point where the edges meet. Let $p : X \to E$ be the map sending each of the squares to the basic origami. This map is well-defined, since all gluing relations are respected, and for E^* and X^* defined as above, we get a finite, topological covering $p : X^* \to E^*$.

On the other hand, let $O = (p : X \to E)$ as in III.3.1. Let us choose generators $x, y : [0, 1] \to E$ of the fundamental group $\pi_1(E, \bar{P})$. If we cut E along the simple closed curves x and y, we get a patch homeomorphic to a square. Thus, the preimage of $x([0, 1]) \cup y([0, 1])$ under p provides a tiling of X into squares.

These two processes are inverse to each other, thus the two definitions coincide. $\square$

Note that the choice of a fixed torus E in Definiton III.3.1 is not a restriction. If $\tilde{E}$ is another torus and $\tilde{P} \in \tilde{E}$, one always finds a homeomorphism $h : E \to \tilde{E}$ satisfying $h(\bar{P}) = \tilde{P}$. Thus, an origami $O = (p : X \to E)$ defined over E becomes an origami $\tilde{O} = (\tilde{p} : X \to \tilde{E})$ defined over $\tilde{E}$ by setting $\tilde{p} = h \circ p$.

We have to say, when two origamis shall describe the same object. One easily checks that the following is an equivalence relation.

III.3.3 Definition. Let $O = (p : X \to E)$, $O' = (p' : X' \to E)$ be two origamis (defined over the same torus E). Then O and O' are called *equivalent*, if there exists a homeomorphism $f : X \to X'$ such that $p' \circ f = p$.

In the following, we fix the torus $E := \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ and the point $\bar{P} := \bar{0} \in E$ (where $\bar{0}$ is the image of 0 under the projection $\pi : \mathbb{C} \to E$) and we restrict to origamis $O = (p : X \to E)$ defined over E, where p is ramified at most over $\bar{0}$.

III.3.4 Definition. Let $O = (p : X \to E)$ be an origami. Then we can define a translation structure on X^* in the following way:

Let the torus $E = \mathbb{C}/(Z + i\mathbb{Z})$ be equipped with the translation structure ν_I as in Example III.1.3 and let $E_I^* = (E \setminus \{\bar{0}\}, \nu_I)$ be the corresponding punctured translation surface.

By Proposition III.1.8, we can lift the translation structure ν_I on E_I^* via p to a translation structure η_I on X^*. We denote the translation surface that we obtain by $X_I^* := (X^*, \eta_I)$.

Note that equivalent origamis lead to isomorphic translation surfaces.

III.3.5 Consequence. *An origami $O = (p : X \to E)$ defines a bunch of translation surfaces $(X_B^*)_{B \in \mathrm{SL}_2(\mathbb{R})}$ by setting $X_B^* = (X^*, B \cdot \eta_I)$, where $B \cdot \eta_I$ is the image of the translation structure η_I under the action of $B \in \mathrm{SL}_2(\mathbb{R})$.*

Note that one may proceed in another way to obtain the same translation surface X_B^*: One can first vary the translation structure on E_I^* and then lift the deformed translation structure to X^*. By Example III.2.11, we have $(\mathbb{C}/\Lambda_I, B \cdot \nu_I) \cong (\mathbb{C}/\Lambda_B, \nu_B)$. Thus, for any two charts $(U, \phi) \in \nu_I$, $(U', \phi') \in \nu_B$, the diagram

$$\begin{tikzcd} E_I \supset & U \arrow[r, "\bar{\varphi}_B"] \arrow[d, "\phi"'] & U' \arrow[d, "\phi'"] & \subset E_B \\ & \mathbb{C} \arrow[r, "z \mapsto B \cdot z + t"] & \mathbb{C} & \end{tikzcd}$$

is commutative. Here, $t \in \mathbb{C}$ and $\bar{\varphi}_B$ is defined as in Example III.2.11. Hence, if η_B denotes the lift of the translation structure ν_B on $E_B^* := E_B \setminus \{\bar{0}\}$ via $\bar{\varphi}_B \circ p$, we find that $(X^*, B \cdot \eta_I) \cong (X^*, \eta_B)$.

Moreover, if we set $p_B = \bar{\varphi}_B \circ p$ for $B \in \mathrm{SL}_2(\mathbb{R})$, we obtain covering maps $p_B : X_B^* \to E_B^*$.

Let us retain these results in the following proposition.

III.3.6 Proposition. *Let $B \in \mathrm{SL}_2(\mathbb{R})$. Let $\bar{\varphi}_B : E_I \to E_B$ be the map induced by $\mathbb{C} \to \mathbb{C}$, $z \mapsto B \cdot z$. Then there exists an affine diffeomorphism $\psi_B : X_I^* \to X_B^*$ with derivative* $\mathrm{der}(\psi_B) = B$ *such that the diagram*

$$\begin{tikzcd} X_I* \arrow[r, "\psi_B"] \arrow[d, "p_I"'] & X_B^* \arrow[d, "p_B"] \\ E_I^* \arrow[r, "\bar{\varphi}_B"'] & E_B^* \end{tikzcd}$$

commutes. Moreover, ψ_B is unique up to composition with an element of $\mathrm{Trans}(X_I^*)$ *or* $\mathrm{Trans}(X_B^*)$.

Proof: By the above argument, the map $\mathrm{id} : X_I^* \to (X^*, B \cdot \eta_I) \cong (X^*, \eta_B)$ that is topologically the identity and changes the translation structures has the required properties. If ψ_B and ψ'_B are two such affine diffeomorphisms, then $\psi_B^{-1} \circ \psi'_B \in \mathrm{Trans}(X_I^*)$ and $\psi'_B \circ \psi_B^{-1} \in \mathrm{Trans}(X_B^*)$, since they both have derivative I. □

We now view the translation surfaces $(X_B^*)_{B \in \mathrm{SL}_2(\mathbb{R})}$ as Riemann surfaces and get the following proposition.

III.3.7 Proposition. *Let $O = (p : X \to E)$ be an origami of genus g.*

a) The origami O defines a family of Riemann surfaces $(X_B^*)_{B \in \mathrm{SL}_2(\mathbb{R})}$ of finite type (g, n) (where n equals the cardinality of $p^{-1}(\{\bar{0}\})$) together with covering maps $p_B : X_B^* \to E_B^*$.

b) For each $B \in \mathrm{SL}_2(\mathbb{R})$, the covering map $p_B : X_B^* \to E_B^*$ can be extended to a proper, holomorphic covering map $p_B : X_B \to E_B$, where X_B is a compact Riemann surface of genus g, such that there exists a biholomorphic map $i : X_B^* \to X_B \setminus p_B^{-1}(\{\bar{0}\})$.

Proof: Since Part a) is a consequence of b), only Part b) is to show. By Proposition II.2.5, the map $p : X^* \to E^*$ is proper. Therefore, we can apply Proposition II.3.3. This achieves the proof. □

III.3.1 Cusps

Let $O = (p : X \to E)$ be an origami. Now we start with the Riemann surface X_B, $B \in \mathrm{SL}_2(\mathbb{R})$ and go back to a translation surface with the help of a holomorphic 1-form.

From Remark III.1.12, we know that $E_B \cong E_\omega$, where ω is the holomorphic 1-form induced by dz on $\mathbb{C}$. Let $p_B^*\omega$ be the pullback of ω via $p_B : X_B \to E_B$. It follows from Remark III.1.13 and from Proposition III.1.8 that the translation structure induced by $p_B^*\omega$ is the same as the translation structure η_B on X_B^*.

Therefore, we state the following proposition.

III.3.8 Proposition. *Let $O = (p : X \to E)$ be an origami of genus $g \geq 1$. The translation structure η_B and the translation structure defined by $\omega_B := p_B^*\omega$ lead to isomorphic translation surfaces.*

The translation structure η_B extends to $X \setminus Z$, where Z is set of zeros of ω_B, and a point $P \in Z$ is a cusp with cone angle $2\pi(\mathrm{ord}_P(\omega_B) + 1)$. Moreover, there are precisely $2g-2$ zeros of ω_B, counted with multiplicities. In particular, if $g = 1$, then η_B extends to all of X.

Proof: By Corollary II.4.6, it follows that

$$\sum_{P \in Z} \mathrm{ord}_P(\omega_B) = \deg(\omega_B) = 2g - 2.$$

A comparison with Remark III.1.11 shows the remaining assertions. $\square$

In particular, an origami of genus 2 has either one or two cusps.

III.3.9 Example. Our origami $S = (p : X \to E)$ from Example I.1.3 has two cusps in the points $\square$ and $\blacksquare$, the cone angle being equal to 4π in each of them. They correspond to two simple zeros of the associated holomorphic 1-form.

III.3.2 The Veech Group and the Group of Automorphisms of an Origami

Given an origami $O = (p : X \to E)$, we want to study affine diffeomorphisms, respectively biholomorphic maps that exist on the whole family of translation surfaces, respectively Riemann surfaces.

We know from Proposition III.2.10 that the affine groups $(\mathrm{Aff}^+(X_B^*))_B$ are all isomorphic and that the Veech groups of the surfaces $(X_B^*)_B$ are all conjugated to each other. Therefore, we may restrict to the case $B = I$ and examine $\mathrm{Aff}^+(X_I^*)$, $\mathrm{Trans}(X_I^*)$ and $\Gamma(X_I^*)$.

III.3.10 Definition. Let $O = (p : X \to E)$ be an origami. The *affine group of the origami* O is defined as

$$\mathrm{Aff}^+(O) := \mathrm{Aff}^+(X_I^*)$$

and the *Veech group of* O is defined as

$$\Gamma(O) := \Gamma(X^*) := \Gamma(X_I^*).$$

Finally, the group of translations of the origami O is defined as

$$\mathrm{Trans}(O) := \mathrm{Trans}(X_I^*).$$

III.3.11 Proposition. *Let $O = (p : X \to E)$ be an origami. Every $f \in \mathrm{Aff}^+(O)$ descends via p to an element $\bar{f} \in \mathrm{Aff}^+(E_I^*)$. Therefore, the Veech group $\Gamma(O)$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. It is even a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z})$.*

PROOF: See [Sch05, Proposition 2.6] and [Sch05, Corollary 2.9] for a proof of this statement. $\square$

III.3.12 Definition. Let $O = (p : X \to E)$ be an origami and let $\mathrm{Aut}(X_I^*)$ denote the set of biholomorphic maps $X_I^* \to X_I^*$. An *automorphism of* O is an element $f \in \mathrm{Aut}(X_I^*)$, such that for all $B \in \mathrm{SL}_2(\mathbb{R})$ the map

$$\mathrm{id}_B \circ f \circ (\mathrm{id}_B)^{-1} : X_B^* \to X_B^*$$

is a holomorphic map in $\mathrm{Aut}(X_B^*)$. Here, $\mathrm{id}_B : X_I^* \to X_B^*$ is the map that is topologically the identity and that exchanges the translation structures. The *group of automorphisms of* O is denoted by $\mathrm{Aut}(O)$.

III.3.13 Remark. It is easy to see that $\mathrm{Aut}(O)$ is in particular a group. Thus, $\mathrm{Aut}(O)$ is a subgroup of $\mathrm{Aut}(X_I^*)$. By Corollary II.3.6, every biholomorphic automorphism of X_I^* extends uniquely to X_I, therefore $\mathrm{Aut}(O)$ can also be considered as a subgroup of $\mathrm{Aut}(X_I)$.

III.3.14 Proposition. *The group $\mathrm{Aut}(O)$ consists precisely of the affine maps in $\mathrm{Aff}^+(X_I^*)$ with derivative I or $-I$.*

Proof: Let $f \in \mathrm{Aff}^+(X_I^*)$ with $\mathrm{der}(f) = \pm I$. Then f is holomorphic and for a matrix $B \in \mathrm{SL}_2(\mathbb{R})$ the map $\mathrm{id}_B \circ f \circ (\mathrm{id}_B)^{-1}$ has derivative $\pm I$, so it is also a holomorphic map. Thus $f \in \mathrm{Aut}(O)$.

Conversely, let $f \in \mathrm{Aut}(O)$. Let $P \in X_I^*$ and let (U, ϕ) and (V, ψ) be charts of the translation structure ν_I at P and $f(P)$ respectively. We consider the map

$$F = \psi \circ f \circ \phi^{-1}$$

which is a map between open sets of $\mathbb{C}$ and we want to compute its derivative at $z_0 = \phi(P)$.

Let $\varphi_B : \mathbb{C} \to \mathbb{C}, z \mapsto B \cdot z$ ($B \in \mathrm{SL}_2(\mathbb{R})$). If ϕ is a chart for ν_I then $\varphi_B \circ \phi$ is a chart for ν_B. Thus if we express the map $\mathrm{id}_B \circ f \circ (\mathrm{id}_B)^{-1}$ in local coordinates, we get

$$\varphi_B \circ \psi \circ f \circ (\varphi_B \circ \phi)^{-1} = \varphi_B \circ F \circ \varphi_{B^{-1}}.$$

By our assumption, the map $\varphi_B \circ F \circ \varphi_{B^{-1}}$ is holomorphic in a neighborhood of $B \cdot z_0$. With the help of the following lemma we conclude that F is affine with derivative $\mathrm{der}(F) = c \cdot I$, $c \in \mathbb{R} \setminus \{0\}$. Thus, f is an affine diffeomorphism of X_I^*, and by Lemma III.2.6, $\mathrm{der}(F) = \mathrm{der}(f) \in \Gamma(O) \subset \mathrm{SL}_2(\mathbb{R})$. Therefore $\mathrm{der}(f) \in \{\pm I\}$. □

III.3.15 Lemma. *Let $f : U \to \mathbb{C}$ be a holomorphic map on a domain $U \subset \mathbb{C}$. For $B \in \mathrm{SL}_2(\mathbb{R})$ let $\varphi_B : \mathbb{C} \to \mathbb{C}, z \mapsto B \cdot z$. If for all $B \in \mathrm{SL}_2(\mathbb{R})$ the map*

$$\varphi_B \circ f \circ \varphi_{B^{-1}}$$

is holomorphic, then there exists $c \in \mathbb{R}$ such that $f'(z) = c$ for all $z \in U$.

Proof: Let $z_0 \in U$ and let $A = A(z_0)$ be the real derivative of f at z_0. By the Cauchy-Riemann differential equations, $A \in \mathbb{R}_{\geq 0} \cdot \mathrm{SO}_2(\mathbb{R})$. The derivative of $\varphi_B \circ f \circ \varphi_{B^{-1}}$ evaluated at $B \cdot z_0$ is then equal to BAB^{-1}. Since $\varphi_B \circ f \circ \varphi_{B^{-1}}$ is holomorphic, it follows that $BAB^{-1} \in \mathbb{R}_{\geq 0} \cdot \mathrm{SO}_2(\mathbb{R})$. We write $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. In particular, for $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ the matrix

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+b & 2b \\ -b & a-b \end{pmatrix}$$

has to be in $\mathbb{R}_{\geq 0} \cdot \mathrm{SO}_2(\mathbb{R})$. It follows that $2b = b$, and hence $b = 0$. Thus, $f'(z_0) = a \in \mathbb{R}$. So $f'(U) \subset \mathbb{R}$, which is not open in $\mathbb{C}$. Since f' is holomorphic on U, this forces f' to be constant. □

III.4 Teichmüller Curves

Our aim is to sketch why and how an origami defines an algebraic curve in the moduli space. This is a special case of a more general concept. Let X be a translation surface that is a finite Riemann surface of genus g with n punctures. Then X induces an embedding ι of the upper half-plane into the Teichmüller space $T(X) = T_{g,n}$. The image of ι composed with the projection $\text{proj} : T_{g,n} \to M_{g,n}$ is sometimes an algebraic curve. In fact, this happens if and only if the Veech group $\Gamma(X)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ (i.e. it has a fundamental domain of finite hyperbolic volume). For an origami O, its Veech group $\Gamma(O)$ always satisfies this condition, thus an origami always induces a curve in the associated moduli space.

III.4.1 Definition. Let $3g - 3 + n > 0$, and let $T_{g,n}$ be the Teichmüller space of marked Riemann surfaces of genus g with n punctures.

a) A map
$$\iota : \mathbb{H} \to T_{g,n}$$
is called *Teichmüller embedding*, provided that ι is a holomorphic, isometric embedding (with respect to the hyperbolic metric on $\mathbb{H}$ and the Teichmüller metric on $T_{g,n}$).

b) Let $\iota : \mathbb{H} \to T_{g,n}$ be a Teichmüller embedding. Then $\Delta = \iota(\mathbb{H})$ is called *Teichmüller (geodesic) disk*. If the image of Δ under the map $\text{proj} : T_{g,n} \to M_{g,n}$ is an algebraic curve $\mathcal{C}$, then $\mathcal{C}$ is called *Teichmüller curve*.

III.4.1 Constructing a Teichmüller embedding

In the following, let $g, n \geq 0$ be integers, satisfying $3g - 3 + n > 0$. Let $X_\nu = (X, \nu)$ denote a fixed translation surface, such that X_ν is a Riemann surface of finite type, having genus g and n punctures. Let $T_{g,n}$ be the corresponding Teichmüller space. We take X_ν as a reference surface and think of $T_{g,n}$ as $T(X_\nu)$.

First, we want to explain, how such a translation surface X_ν defines a Teichmüller embedding and an associated Teichmüller disk.

Recall that the group $\mathrm{SL}_2(\mathbb{R})$ acts on the set of translation structures of X (cf. III.2.9). If we are given a matrix $B \in \mathrm{SL}_2(\mathbb{R})$, let us denote by X_B the translation surface $(X, B \cdot \nu)$ (thus X_ν itself is equal to X_I). Let $\mathrm{id} : X_I \to X_B$ be the map that is topologically the identity. Then,

$$P_B = [X_B, \mathrm{id} : X_I \to X_B]$$

defines another point in $T_{g,n}$ and we get a map

$$\mathrm{SL}_2(\mathbb{R}) \to T_{g,n},\ B \mapsto P_B. \tag{III.2}$$

Let $B, B' \in \mathrm{SL}_2(\mathbb{R})$ and let $B'B^{-1} \in \mathrm{SO}_2(\mathbb{R})$. We claim that $P_B = P_{B'}$. Indeed, the map

$$(X_I \xrightarrow{\mathrm{id}} X_{B'}) \circ (X_I \xrightarrow{\mathrm{id}} X_B)^{-1} : X_B \to X_{B'}$$

has derivative $B'B^{-1} \in \mathrm{SO}_2(\mathbb{R})$ and is thus biholomorphic.

Therefore, the map in (III.2) factors through $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$ and induces a map

$$\hat{\imath} : \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R}) \to T_{g,n}, \quad B \cdot \mathrm{SO}_2(\mathbb{R}) \mapsto P_B.$$

We can identify $\mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R})$ with $\mathbb{H}$ in the following way. The group $\mathrm{SL}_2(\mathbb{R})$ acts on the upper half-plane by Möbius transformations. Namely, for a matrix $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\tau \in \mathbb{H}$, let

$$B(\tau) = \frac{a\tau + b}{c\tau + d}.$$

This action is transitive and the subgroup $\mathrm{SO}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ is the stabilizer of i. With these ingredients we can construct a bijection as follows.

III.4.2 Remark. The map

$$\hat{\jmath} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{H},\ B \mapsto -\overline{B^{-1}(i)}$$

induces a bijection

$$j : \mathrm{SO}_2(\mathbb{R})\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathbb{H}.$$

Proof: First of all, note that $\hat{\jmath}$ is well-defined. Let

$$k : \mathbb{H} \to \mathrm{SL}_2(\mathbb{R}),\ \tau \mapsto \frac{1}{\sqrt{\mathrm{Im}(\tau)}} \begin{pmatrix} 1 & \mathrm{Re}(\tau) \\ 0 & \mathrm{Im}(\tau) \end{pmatrix}.$$

Then $\hat{\jmath} \circ k = \mathrm{id}_{\mathbb{H}}$, hence $\hat{\jmath}$ is surjective. If $\hat{\jmath}(B) = \hat{\jmath}(\tilde{B})$, then $B^{-1}(i) = \tilde{B}^{-1}(i)$, and therefore $\tilde{B}B^{-1}(i) = i$, whereby we have that $\tilde{B}B^{-1} \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbb{R})}(i) = \mathrm{SO}_2(\mathbb{R})$. Hence, $\tilde{B}^{-1}B \in \mathrm{SO}_2(\mathbb{R})$ and $B \in \tilde{B} \cdot \mathrm{SO}_2(\mathbb{R})$. Similarly, $\tilde{B} \in B \cdot \mathrm{SO}_2(\mathbb{R})$, and altogether we have $B \cdot \mathrm{SO}_2(\mathbb{R}) = \tilde{B} \cdot \mathrm{SO}_2(\mathbb{R})$. This shows that j is a bijection. □

Thus by composing $\hat{\imath}$ with j^{-1}, we get a map

$$\iota = \hat{\imath} \circ j^{-1} : \mathbb{H} \to T_{g,n}.$$

III.4.3 Proposition. *The map $\iota : \mathbb{H} \to T_{g,n}$ is an isometric, holomorphic map*[1]*. Thus, ι is a Teichmüller embedding and we denote $\Delta := \iota(\mathbb{H})$ the corresponding Teichmüller disk.*

PROOF: We may refer to [HS06, Proposition 2.8] for a proof of this fact. $\square$

III.4.2 Constructing a Teichmüller curve

For a Teichmüller disk constructed in this way, we want to establish what its image in the moduli space looks like and when it is a Teichmüller curve. To do this, we need to know how the modular group $\mathrm{Mod}(g,n)$ acts on the Teichmüller disk.

First of all, we observe that the affine group $\mathrm{Aff}^+(X_I)$ is a subgroup of $\mathrm{Diffeo}^+(X_I)$. Thus $\mathrm{Aff}^+(X_I)$ acts on $T_{g,n}$. We want to specify what this action does on the points of Δ.

III.4.4 Remark. Let $f \in \mathrm{Aff}^+(X_I)$ and let $P_B \in \Delta$, where $B \in \mathrm{SL}_2(\mathbb{R})$. With the notations of p. 31, we have

$$\rho(f)(P_B) = P_{BA^{-1}}$$

where $A = \mathrm{der}(f)$.

PROOF: We have to show that $\rho(f)(P_B) = [X_B, (\mathrm{id} : X_I \to X_B) \circ f^{-1}]$ is the same point as $P_{BA^{-1}}$. Consider the following commutative diagram.

$$\begin{tikzcd}
X_I \arrow[r, "f^{-1}"] \arrow[dr, "\mathrm{id}"'] & X_I \arrow[r, "\mathrm{id}"] & X_B \\
& X_{BA^{-1}} \arrow[ur, "\psi"'] &
\end{tikzcd}$$

Here

$$X_{BA^{-1}} \xrightarrow{\psi} X_B = (\mathrm{id} : X_I \to X_B) \circ f^{-1} \circ (\mathrm{id} : X_I \to X_{BA^{-1}})^{-1},$$

and the derivative of ψ evaluates to $\mathrm{der}(\psi) = BA^{-1}(BA^{-1})^{-1} = I$. Thus, ψ is a biholomorphic map and $\rho(f)(P_B) = P_{BA^{-1}}$. $\square$

---

[1]The reason why we define the map j in such a complicated way, is that we want ι to become a holomorphic map. Normally, one would define j to be the map $A \mapsto A(i)$, $(A \in \mathrm{SL}_2(\mathbb{R}))$.

Remark III.4.4 shows that $\mathrm{Aff}^+(X_I)$ stabilizes Δ. Thus, it can be mapped by a group homomorphism to the stabilizer of Δ,

$$\mathrm{Stab}(\Delta) = \{f \in \mathrm{Mod}(g,n) \mid f(\Delta) = \Delta\}\,.$$

The following proposition states that this group homomorphism is even an isomorphism.

III.4.5 Proposition. *The group* $\mathrm{Aff}^+(X_I)$ *is equal to* $\mathrm{Stab}(\Delta) \subset \mathrm{Mod}(g,n)$.

PROOF: The group homomorphism $\mathrm{Aff}^+(X_I) \to \mathrm{Stab}(\Delta)$ is injective by [EG97, Lemma 5.2] and it is surjective by [EG97, Theorem 1]. $\square$

Furthermore, Remark III.4.4 shows that the action of $f \in \mathrm{Aff}^+(X_I)$ on Δ depends only on the derivative $\mathrm{der}(f)$. Thus, we can interpret this action as an action of the Veech group $\Gamma(X)$ on Δ. Namely, let $A \in \Gamma(X)$ and $P_B \in \Delta$ $(B \in \mathrm{SL}_2(\mathbb{R}))$. Then $\Gamma(X)$ acts on Δ by

$$\rho(A)(P_B) = P_{BA^{-1}}.$$

Thus, the Veech group $\Gamma(X)$ acts on Δ as a group of holomorphic isometries (since this is the case for $\mathrm{Aff}^+(X_I)$). On the other hand, $\Gamma(X) \subset \mathrm{SL}_2(\mathbb{R})$ and it acts on $\mathbb{H}$ via Moebius transformations, i.e. as a group of holomorphic isometries of $\mathbb{H}$. Veech showed in [Vee89] that this action is always discrete, in other terms, $\Gamma(X)$ is a Fuchsian group.

The two actions fit together via the holomorphic isometry $\iota : \mathbb{H} \hookrightarrow \Delta$ in the following way.

III.4.6 Remark. Let $A \in \Gamma(X)$ and let $R = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then A and RAR^{-1} act on $\mathbb{H}$ via Moebius transformations and the diagram

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "t \mapsto -\bar{t}"] \arrow[d, "A"'] & \mathbb{H} \arrow[r, "\iota"] \arrow[d, "RAR^{-1}"] & \Delta \arrow[d, "\rho(A)"] \\
\mathbb{H} \arrow[r, "t \mapsto -\bar{t}"] & \mathbb{H} \arrow[r, "\iota"] & \Delta
\end{tikzcd}$$

commutes.

PROOF: See [HS06, Remark 2.20]. $\square$

Now we consider the image of the Teichmüller disk Δ in the moduli space $M_{g,n}$. Let $\text{proj} : T_{g,n} \to M_{g,n}$ denote the natural projection (cf. II.7.6).

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "\iota"] \arrow[drr, "\text{proj} \circ \iota"'] & \Delta \arrow[r, hook] & T_{g,n} \arrow[d, "\text{proj}"] \\
 & & M_{g,n}
\end{tikzcd}$$

By the preceding remark, it follows that $\text{proj} \circ \iota$ factors through $\mathbb{H}/\Gamma^*(X)$. This is the quotient by the *mirror Veech group* $\Gamma^*(X)$ defined by

$$\Gamma^*(X) = R\,\Gamma(X)R^{-1},$$

where $R = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

The quotient $\mathbb{H}/\Gamma^*(X)$ is a Riemann surface of finite type, and thus an algebraic curve, if and only if the group $\Gamma^*(X)$ (and consequently $\Gamma(X)$) is a lattice in $\mathrm{SL}_2(\mathbb{R})$, i.e. $\mathbb{H}/\Gamma^*(X)$ has finite hyperbolic volume.

Finally, there is a strong link between $\mathbb{H}/\Gamma^*(X)$ and the image of the Teichmüller disk in the moduli space.

III.4.7 Theorem. *Let X_ν be a translation surface of genus g obtained from a compact surface by removing n points. The construction of the associated Teichmüller disk leads to a Teichmüller curve $\mathcal{C}$ in $M_{g,n}$ if and only if the group $\Gamma^*(X)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$. In this case $\mathbb{H}/\Gamma^*(X)$ is the normalization of $\mathcal{C}$ and is birationally equivalent to $\mathcal{C}$.*

$$\begin{tikzcd}
\mathbb{H} \arrow[r, "\iota"] \arrow[d] & \Delta \arrow[r, hook] \arrow[d, "\text{proj}"] & T_{g,n} \arrow[d, "\text{proj}"] \\
\mathbb{H}/\Gamma^*(X) \arrow[r, "\text{birational}"'] & \mathcal{C} \arrow[r, hook] & M_{g,n}
\end{tikzcd}$$

Proof: See [McM03, Corollary 3.3] $\square$

III.4.3 Origami Curves

Now we specialize the above construction to the case of origamis. Let $O = (p : X \to E)$ be an origami of genus g. Let X_I^* be the corresponding translation surface as constructed in Definition III.3.4 and let n be the number of punctures of X_I^*. Applying the above construction to X_I^* yields a Teichmüller disk in $T_{g,n}$.

By Proposition III.3.11, the Veech group $\Gamma(O)$ is always a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ of finite index and hence a lattice in $\mathrm{SL}_2(\mathbb{R})$: a fundamental domain for $\Gamma(O)$ is given by a finite union of translates of a fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$, which has finite volume. Thus, by Theorem III.4.7, an origami leads to a Teichmüller curve in $M_{g,n}$.

III.4.8 Definition. A Teichmüller curve coming from an origami is called an *origami curve*.

III.4.9 Definition. Let $O = (p : X \to E)$ be an origami of genus $g \geq 1$. By Theorem II.4.2, we can assign an algebraic curve to each of the Riemann surfaces X_B, $B \in \mathrm{SL}_2(\mathbb{R})$. By an *equation of the origami* O, we mean a family of polynomials $(F_\lambda)_{\lambda \in U}$ in $\mathbb{C}[x, y]$, parametrized by a Zariski-open set $U \subset \mathbb{C}$, such that

(1) for each X_B, there exists a $\lambda \in U$ such that X_B is birational to the set of zeros C_λ of F_λ

(2) and such that the projection $U \to M_g$, $\lambda \mapsto C_\lambda$ is finite.

III.4.10 Example. With the help of [Sch05], we can compute the Veech group for our origami S introduced in Example I.1.3. Let

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

be the standard generators of $\mathrm{SL}_2(\mathbb{Z})$. Then $\Gamma(S)$ is generated by

$$S^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, TST^{-2} = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}, STS^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$$T^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T^2ST^{-1} = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix},$$

and coset representatives of $\mathrm{SL}_2(\mathbb{Z})/\Gamma(S)$ are given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Furthermore, we can compute the geometrical type of the affine algebraic curve $\mathbb{H}/\bar{\Gamma}(S)$, where $\bar{\Gamma}(S) = \Gamma(S)/\{\pm I\} \subset \mathrm{PSL}_2(\mathbb{Z})$ is the *projective Veech group* of S. Let $\mathcal{F} = \Delta(P, Q, R)$ be the standard fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$, i.e. $\mathcal{F}$ is the hyperbolic pseudo-triangle with vertices

$$P = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \; Q = \frac{1}{2} + i\frac{\sqrt{3}}{2} \quad \text{and} \quad R = i\infty.$$

The coset representatives of $\bar{\Gamma}(S)$ in $\mathrm{PSL}_2(\mathbb{Z})$ are the projections $\bar{I}$, $\bar{T}$, $\bar{S}$, $\bar{T}^2$ of the respective matrices to $\mathrm{PSL}_2(\mathbb{Z})$. A fundamental domain for $\mathbb{H}/\bar{\Gamma}(S)$ is thus given by

$$\mathcal{F}_S = \mathcal{F} \cup \bar{T}(\mathcal{F}) \cup \bar{S}(\mathcal{F}) \cup \bar{T}^2(\mathcal{F}).$$

Therefore, we can depict $\mathcal{F}_S$ according to [Sch05] and get a picture as in

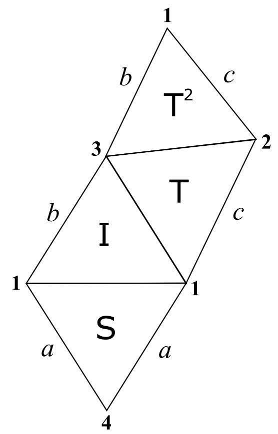


Figure III.2: *Fundamental domain for* $\mathbb{H}/\bar{\Gamma}(S)$

Figure III.2. We even have a triangulation of $\mathbb{H}/\bar{\Gamma}(S)$ with $t = 4$ triangles, $e = 6$ edges and $v = 4$ vertices. Thus the genus of $\mathbb{H}/\bar{\Gamma}(S)$ can be computed with the help of Euler's formula; we have

$$v - e + t = 2 - 2g(\mathbb{H}/\bar{\Gamma}(S)),$$

whereby

$$g(\mathbb{H}/\bar{\Gamma}(S)) = 0.$$

Among the four vertices, 3 and 4 are vertices at ∞. In other terms, $\mathbb{H}/\bar{\Gamma}(S)$ has two cusps.

Altogether, this shows that the origami curve of S, of which $\mathbb{H}/\bar{\Gamma}(S)$ is the normalization, is birational to $\mathbb{P}^1$.

Note that it may happen that two origamis lead to the same Teichmüller disk and consequently to the same origami curve. In fact, this is always the case if the Veech group of an origami is not entirely $\mathrm{SL}_2(\mathbb{Z})$.

III.4.11 Definition. Let $O = (p : X \to E)$, $\tilde{O} = (\tilde{p} : \tilde{X} \to E)$ be origamis. Then O and $\tilde{O}$ are called *affinely equivalent*, if there exist affine diffeomorphisms $\psi : X_I^* \to \tilde{X}_I^*$, and $\bar{\psi} : E_I^* \to E_I^*$ such that

$$\begin{tikzcd} X_I^* \arrow[r, "\psi"] \arrow[d, "p_I"'] & \tilde{X}_I^* \arrow[d, "\tilde{p}_I"] \\ E_I^* \arrow[r, "\bar{\psi}"'] & E_I^* \end{tikzcd}$$

commutes. (Here the respective translation structures on X^* and $\tilde{X}^*$ are induced by lifting ν_I on E_I).

Obviously, this gives rise to an equivalence relation. Moreover, the derivative of $\bar{\psi}$ is in $\mathrm{SL}_2(\mathbb{Z})$ by Example III.2.8. Conversely, given a matrix $B \in \mathrm{SL}_2(\mathbb{Z})$, we get an origami that is affinely equivalent to a given one as a consequence of Proposition III.3.6.

III.4.12 Proposition. *Let $O = (p : X \to E)$ be an origami. Then the origamis that are affinely equivalent to O modulo the ones that are equivalent to O, correspond bijectively to the right cosets of* $\Gamma(O)$ *in* $\mathrm{SL}_2(\mathbb{Z})$*.*

Proof: Let $\tilde{O} = (\tilde{p} : \tilde{X} \to E)$ be an origami that is affinely equivalent to O. We continue to use the notations above. If $B = \mathrm{der}(\psi) \in \Gamma(O)$, then there exists $f \in \mathrm{Aff}^+(X_I^*)$ with $\mathrm{der}(f) = B$. The map f descends to $\bar{f} \in \mathrm{Aff}^+(E_I^*)$ by Proposition III.3.11. Then, $\bar{\psi} \circ \bar{f}^{-1} : E_I^* \to E_I^*$ is a translation of E_I^*. Thus it is also a translation of E_I that fixes the point $\bar{0}$. Since a translation has no fixed points, $\mathrm{Trans}(E_I^*) = \{\mathrm{id}\}$, and it follows that

$$p_I = \bar{\psi} \circ \bar{f}^{-1} \circ p_I = \tilde{p}_I \circ \psi \circ f,$$

thus O and $\tilde{O}$ are equivalent.

Conversely, if O and $\tilde{O}$ are equivalent, and if $\psi : X_I^* \to \tilde{X}_I^*$ is an affine diffeomorphism, then surely $\mathrm{der}(\psi) \in \Gamma(O)$. □

III.4.13 Example. To draw pictures of the origamis that are affinely equivalent to a given one, we can proceed as follows. Each of these origamis corresponds to a coset of $\Gamma(O)$ in $\mathrm{SL}_2(\mathbb{Z})$. We take a coset representative $A \in \mathrm{SL}_2(\mathbb{Z})$ and shear the original picture of O with the linear map $z \mapsto A{\cdot}z$. Then we have to subdivide the result into upright squares again and to examine how the gluings have changed.

If we apply these considerations to our example origami S, then we can depict four origamis S_A, $A \in \{I, T, S, T^2\}$ that are affinely equivalent to S (where S itself is S_I).

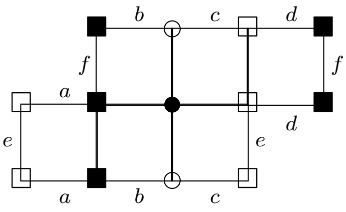


Figure III.3: *The origami* $S = S_I$

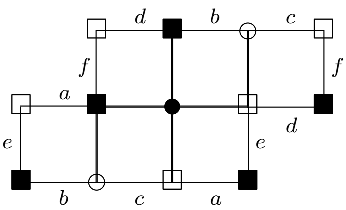


Figure III.4: *The origami* S_T *to* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

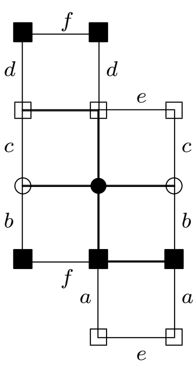


Figure III.5: *The origami* S_S *to* $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

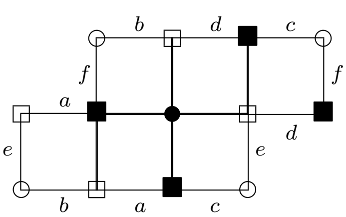


Figure III.6: *The origami* S_{T^2} *to* $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Chapter IV

The Origami S

From now on we focus on our example origami $S = (p : X \to E)$ as presented in Example I.1.3. This is an origami defined over the torus $E = E_I = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. First we examine the automorphisms of S.

As the origami S was defined by a picture (cf. Figure I.2), we will have to transfer properties given by pictures into algebraic terms.

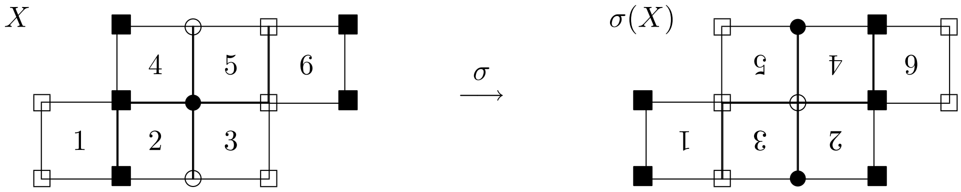
Let X denote the topological space of S and let $p : X \to E$ be the associated covering map. Let X_I^* and X_I be the associated translation surface, respectively Riemann surface (cf. Definition III.3.4). For the sake of simplicity of notations, let us in the following write $X^* := X_I^*$ and $X := X_I$ (hoping that this will not amount to confusion).

IV.1 Automorphisms of S

IV.1.1 The Hyperelliptic Involution on the Origami

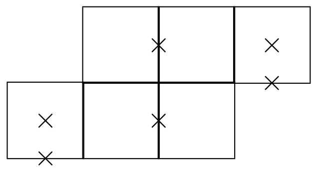
We already saw in Example I.1.3, that S is an origami of genus $g = 2$. Hence, by Proposition II.6.8, the Riemann surface X is a hyperelliptic surface. We wish to identify the hyperelliptic involution on X.

IV.1.1 Definition. Let $\sigma : X \to X$ be the map given by the following picture.



Obviously, restricting σ to X^* yields an automorphism $\sigma : X^* \to X^*$ and this is an affine map with derivative $-I$. Thus, by Proposition III.3.14, σ is an element of $\mathrm{Aut}(S)$.

IV.1.2 Proposition. *The map $\sigma : X \to X$ is a holomorphic involution, having the following six fixed points*



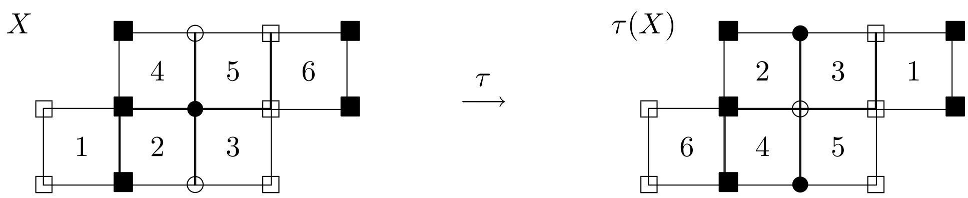
Hence, σ is the hyperelliptic involution of the hyperelliptic Riemann surface X.

Proof: Looking at the definition of σ, one sees that σ is an involution. By Corollary II.6.13, any holomorphic involution that fixes $2g + 2 = 6$ points is already the hyperelliptic involution. The fixed points can easily be found with the help of the following consideration.

The map $\sigma : X^* \to X^*$ descends to the map $\varphi_{-I} : E \to E$ on $E = \mathbb{C}/\Lambda_I$. The image of a fixed point of σ is a fixed point of φ_{-I}. The fixed points of φ_{-I} are precisely the 2-torsion points $0+\Lambda_I$, $\frac{1}{2}+\Lambda_I$, $i\frac{1}{2}+\Lambda_I$ and $\frac{1}{2}+i\frac{1}{2}+\Lambda_I$. Thus, we only have to look at the fibers of these latter points to find fixed points of σ. □

IV.1.2 The Group $\mathrm{Aut}(S)$

IV.1.3 Definition. Let $\tau : X \to X$ be given by the following picture.

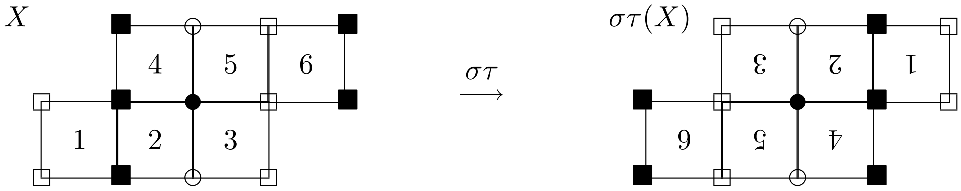


For the same reasons as for σ, the map τ is an element of $\mathrm{Aut}(S)$. Since its derivative is equal to I, τ is an isomorphism for the translation structure on X^*, hence a translation.

IV.1.4 Remark. The map τ is an involution. Hence, the set

$$G = \{\mathrm{id}, \sigma, \tau, \sigma\tau\}$$

is a subgroup of $\mathrm{Aut}(S)$, isomorphic to the Klein four-group V_4. The element $\sigma\tau : X \to X$ is given by the picture



Proof: One checks that $\tau^2 = \mathrm{id}$ by looking sharply at the defining picture. By Corollary II.6.14, one has $\tau\sigma = \sigma\tau$, thus $G \cong V_4$. □

IV.1.5 Proposition. *The group* $\mathrm{Trans}(S)$ *is equal to* $\{\mathrm{id}, \tau\}$ *and the group* $\mathrm{Aut}(S)$ *is equal to* $\{\mathrm{id}, \sigma, \tau, \sigma\tau\} \cong V_4$.

Proof: First we show that the group of translations of X^* is precisely $\{\mathrm{id}, \tau\}$. Let $t \in \mathrm{Trans}(X^*)$. Then the induced biholomorphic automorphism $t : X \to X$ must map singularities of the translation structure η_I to singularities. Thus, $t(\{\square, \blacksquare\}) = \{\square, \blacksquare\}$. Moreover, $t : X^* \to X^*$ is a deck transformation for the covering $p : X^* \to E^*$ and must therefore map squares to squares.

Let Q_i be the square labeled with i in Figure I.2. Then $t(Q_1) = Q_r$ where $r \in \{1, \ldots, 6\}$. The left edge of Q_1 is mapped to the left edge of Q_r. Since the left edge of Q_1 abuts on the vertex $\square$, the possibilities are $r = 1, 2, 4, 6$. The vertices at the right edge of Q_1 are identical, thus only $r = 1, 6$ are left over. By the identity theorem, either $t = \mathrm{id}$ or $t = \tau$. Note that this argument generalizes to any member of the family $S_{n,k}$, $k, n \geq 1$ (cf. Figure I.3).

Now let $f \in \mathrm{Aut}(S)$, i.e. $f \in \mathrm{Aff}^+(X_I^*)$ with $\mathrm{der}(f) = \pm I$. If $\mathrm{der}(f) = I$, then f is already a translation, hence $f \in \{\mathrm{id}, \tau\}$ by the above argument. If $\mathrm{der}(f) = -I$, then $\sigma \circ f$ is a translation and hence $f \in \{\sigma, \sigma\tau\}$. □

To find fixed points of τ and $\sigma\tau$, we proceed as in the proof of Proposition IV.1.2.

IV.1.6 Remark. The fixed points of τ are the points $\square$ and $\blacksquare$. The fixed points of $\sigma\tau$ are the points $\bigcirc$ and $\bullet$.

IV.2 Properties of Riemann Surfaces of Genus Two

In this section, we study the family of Riemann surfaces $(X_B^*)_{B\in\mathrm{SL}_2(\mathbb{R})}$ coming from the origami S from a more general point of view. The automorphism group of any of these surfaces contains the Klein four-group $\mathrm{Aut}(S) \cong V_4$ as a subgroup. Moreover, the sets of fixed points of σ, τ and $\sigma\tau$ are mutually disjoint. In this section, we focus on Riemann surfaces of genus two with this property.

In the following, Y always denotes a compact Riemann surface with the

> **Property (∗).** The genus of Y is two and $\mathrm{Aut}(Y)$ has a subgroup $G = \{\mathrm{id}, \sigma, \tau, \sigma\tau\}$ isomorphic to V_4, where σ is the hyperelliptic involution on Y and σ, τ and $\sigma\tau$ have no common fixed point.

Furthermore, let $\phi : Y \to Y/_{<\sigma>} \cong \mathbb{P}^1$ be a covering map for the action of σ on Y. Let $A \subset Y$ be the set of fixed points of σ, which is at the same time the set of ramification points of ϕ, and let $B = \phi(A) \subset \mathbb{P}^1$ be the set of branch points.

IV.2.1 Remark. The map $\phi|_A$ is injective, and therefore the cardinality of B is equal to 6. Moreover, $\phi^{-1}(B) = A$.

Proof: This is due to the fact that $\sigma^2 = \mathrm{id}$, so the fixed points of σ correspond to orbits of $< \sigma >$ on Y, having only one element. □

IV.2.2 Proposition. *Any automorphism $\theta \in \mathrm{Aut}(Y)$ descends via ϕ to an automorphism $\bar\theta \in \mathrm{Aut}(\mathbb{P}^1)$. The automorphism $\bar\theta$ acts on B, and this action has no fixed point, if and only if the automorphisms σ and θ have no common fixed point.*

Conversely, any automorphism $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$ that satisfies $\gamma(B) = B$ can be lifted to an automorphism $\hat\gamma \in \mathrm{Aut}(Y)$ and there are precisely two possible lifts $\hat\gamma$ and $\sigma\hat\gamma$.

Proof: By Corollary II.6.14, we have $\sigma\theta = \theta\sigma$, and therefore the map $\phi \circ \theta$ is invariant on the orbits of $< \sigma >$. Thus, we get a map $\bar\theta : \mathbb{P}^1 \to \mathbb{P}^1$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\theta} & Y \\ \phi\downarrow & & \downarrow\phi \\ \mathbb{P}^1 & \xrightarrow{\bar\theta} & \mathbb{P}^1 \end{array}$$

commutes. Moreover,

$$\sigma\theta(P) = \theta\sigma(P) = \theta(P)$$

for any fixed point P of σ, so $\theta(P)$ is again a fixed point of σ. Thus, $\theta(A) = A$. By $\bar{\theta}\phi = \phi\theta$, it follows that $\bar{\theta}(B) \subset B$ and even $\bar{\theta}(B) = B$, since $\bar{\theta}$ is bijective.

Let $z \in B$ be a fixed point of $\bar{\theta}$ and let $P \in A$ be its preimage under ϕ. Then, $\phi\theta(P) = \bar{\theta}(z) = z = \phi(P)$. By the above remark, $\phi|_A$ is injective, hence $\theta(P) = P$, and P is a common fixed point of σ and θ. Conversely, for any such fixed point P, $\bar{\theta}\phi(P) = \phi\theta(P) = \phi(P)$, thus $\phi(P) \in B$ is a fixed point of $\bar{\theta}$.

To show the converse, let $U = \mathbb{P}^1 \setminus B$ and let $W = Y \setminus \phi^{-1}(B) = Y \setminus A$. Let $q \in W$ and let $\phi(q) = r \in U$. Let $H = \phi_*(\pi_1(W, q))$ be the subgroup of the fundamental group $\pi_1(U, r)$ coming from the topological covering $\phi|W \to U$. We set $r' := \gamma(r)$ and choose a point $q' \in \phi^{-1}(\{r'\})$. Let $H' = \phi_*(\pi_1(W, q'))$. Our aim is to find $\hat{\gamma} : W \to W$ such that the diagram

$$\begin{tikzcd}
W \arrow[r, dashed, "\hat{\gamma}"] \arrow[d, "\phi"'] & W \arrow[d, "\phi"] \\
U \arrow[r, "\gamma"] & U
\end{tikzcd}$$

commutes. By Theorem II.2.6, it suffices to show that $\gamma_*(H) \subset H'$.

We first choose appropriate generators for $\pi_1(U, r)$. The fundamental group of $\mathbb{P}^1$ with six points removed is isomorphic to the free group on five generators $g_1, \ldots, g_5$, each of the generators being a loop around a point in B. Up to renumbering the elements $g_1, \ldots, g_5$, the element $g_6 = g_1^{-1} g_2^{-1} \cdots g_5^{-1}$ then winds around the remaining sixth point of B. Let us make this a bit more explicit.

Let us write $B = \{b_1, \ldots, b_6\}$ and let us fix an index $i \in \{1, \ldots, 6\}$. Let $a_i \in A$ be the preimage of $b_i \in B$. Since b_i is a branch point of ϕ, there exist charts $z_i : V_i \to \mathbb{D}$ at a_i and $w_i : U_i \to \mathbb{D}$ at b_i, such that $w_i \circ \phi \circ z_i^{-1} = (\mathbb{D} \to \mathbb{D}, z \mapsto z^2)$. We consider the fundamental group of $U_i \setminus \{b_i\}$. Let r_i be a point in $U_i \setminus \{b_i\}$ and let β_i be a generator of $\pi_1(U_i \setminus \{b_i\}, r_i)$. (This fundamental group is isomorphic to $\mathbb{Z}$.) To relate $\pi_1(U_i \setminus \{b_i\}, r_i)$ with $\pi_1(U, r)$, let α_i be a path in U that joins r to r_i. Then $\alpha_i \beta_i \alpha_i^{-1}$ is a path that goes from the base point r to r_i, then winds around b_i once and goes back to r. We set $g_i := \alpha_i \beta_i \alpha_i^{-1}$. Up to replacing β_i by its inverse path and renumbering, the elements $g_1, \ldots, g_6$ generate $\pi_1(U, r)$ and are subject to the single relation

$$g_1 g_2 \cdots g_6 = 1.$$

Since ϕ is a two-sheeted covering map, its monodromy is a group homomorphism

$$\rho : \pi_1(U, r) \to \mathbb{Z}/2\mathbb{Z}.$$

What is the image of g_i under ρ? There are only two possibilities: either the lift of g_i starting at q is a closed path in W, in which case $\rho(g_i) = 0$ or it is not, i.e. $\rho(g_i) = 1$. Since $g_i = \alpha_i \beta_i \alpha_i^{-1}$, it follows that $\rho(g_i) = \rho_i(\beta_i)$, where

$$\rho_i : \pi_1(U_i \setminus \{b_i\}, r_i) \to \mathbb{Z}/2\mathbb{Z},$$

is the monodromy of the covering $\phi|V_i \setminus \{a_i\} \to U_i \setminus \{b_i\}$. Since this covering can be expressed as $z \mapsto z^2$, we have $\rho_i(\beta_i) = 1$, and hence $\rho(g_i) = 1$.

It is easy to see that $H = \ker \rho$. Moreover, an element $h \in \pi_1(U, r)$ can be written as

$$h = g_{l_1}^{\epsilon_1} \cdots g_{l_s}^{\epsilon_s}$$

with $s \in \mathbb{N}$ and $l_j \in \{1, \ldots, 6\}$, $\epsilon_j \in \{\pm 1\}$, $1 \le j \le s$. If $h \in \ker \rho$, then

$$0 = \rho(h) = \rho(g_{l_1}^{\epsilon_1}) + \ldots + \rho(g_{l_s}^{\epsilon_s}) = \underbrace{1 + \ldots + 1}_{s\text{-times}}.$$

It follows that s is even, if and only if $h \in \ker \rho$.

Let $g_i' = \gamma(g_i)$, $1 \le i \le 6$. Since $\gamma(B) = B$ and since γ is an automorphism of $\mathbb{P}^1$, it maps a small neighborhood of b_i to a small neighborhood of $\gamma(b_i) \in B$, $1 \le i \le 6$. Therefore, the monodromy

$$\rho' : \pi_1(U, r') \to \mathbb{Z}/2\mathbb{Z}$$

maps g_i' to 1 for all $i \in \{1, \ldots, 6\}$. Again, we have $H' = \ker \rho'$.

Let us show that $\gamma_*(H) \subset H'$. Let $h \in H$, then

$$h = g_{l_1}^{\epsilon_1} \cdots g_{l_s}^{\epsilon_s}$$

with $l_j \in \{1, \ldots, 6\}$, $\epsilon_j \in \{\pm 1\}$, $1 \le j \le s$ and $s \in 2\mathbb{N}$. Applying γ_* to h yields

$$\gamma_*(h) = \gamma_*(g_{l_1})^{\epsilon_1} \cdots \gamma_*(g_{l_s})^{\epsilon_s} \in \ker \rho' = H',$$

because s is even. This shows the existence of a lift $\hat{\gamma}$ of γ. By Theorem II.2.6, the map $\hat{\gamma}$ is uniquely determined by its value at the point $q \in W$. As $\hat{\gamma}(q) \in \phi^{-1}(\{r'\}) = \{q', \sigma(q')\}$, it follows that γ admits precisely two lifts $\hat{\gamma}$ and $\sigma\hat{\gamma}$. □

Proposition IV.2.2 shows that studying automorphisms of Y essentially means studying Moebius transformations of $\mathrm{Aut}(\mathbb{P}^1)$ that fix the set of branch points of ϕ. The following properties of Moebius transformations will therefore be useful later on.

IV.2.3 Lemma. *An element $\gamma \in \mathrm{Aut}(\mathbb{P}^1), \gamma \neq \mathrm{id}$ of finite order has exactly two fixed points.*

PROOF: We write γ as $z \mapsto \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. Then, γ has either one fixed point or two fixed points since the equation

$$\frac{az+b}{cz+d} = z$$

leads either to a linear or a quadratic polynomial in z. We show that γ has infinite order, if it has only one fixed point. Let z_0 be a fixed point of γ. There exists a Moebius transformation $\beta \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\beta(z_0) = \infty$. Let $\tilde{\gamma} := \beta\gamma\beta^{-1}$. Then the order of $\tilde{\gamma}$ and the number of fixed points are the same as for γ. Moreover, ∞ is a fixed point of $\tilde{\gamma}$. Therefore, $\tilde{\gamma}$ can be written as $z \mapsto rz + s$ with $r, s \in \mathbb{C}$, $r \neq 0$. By our assumption, the equation

$$z = rz + s$$

has no solution in $\mathbb{C}$. This is the case, if and only if $r = 1$ and $s \neq 0$. Hence, $\tilde{\gamma} = (z \mapsto z + s)$, $s \neq 0$, and it has infinite order. $\square$

IV.2.4 Lemma. *Let $\gamma \in \mathrm{Aut}(\mathbb{P}^1)$, $\gamma \neq \mathrm{id}$ and let $z \in \mathbb{P}^1$ be not a fixed point of γ. Then, the cardinality of the orbit $< \gamma > \cdot z$ is equal to the order of γ.*

PROOF: Let $\Gamma =< \gamma >$ and let Γ_z be the stabilizer subgroup of z. Then the cardinality of the orbit $\Gamma \cdot z$ is equal to $(\Gamma : \Gamma_z)$ by the orbit-stabilizer theorem. Thus we have to show that $\Gamma_z = \{\mathrm{id}\}$. First we assume that γ has two fixed points. Let $\tilde{\gamma} \in \Gamma_z$, $\tilde{\gamma} = \gamma^m$, $m \in \mathbb{Z}$. Every fixed point of γ is already a fixed point of γ^m. Thus γ^m has three fixed points, and it follows that $\tilde{\gamma} = \gamma^m = \mathrm{id}$.

On the other hand, if γ has only one fixed point, then we can proceed as in the proof of the preceding lemma and assume that $\gamma(\infty) = \infty$. Thus, we can write $\gamma : z \mapsto z + b$, $b \in \mathbb{C}$, $b \neq 0$. The orbit of $z \in \mathbb{C}$ under the action of $< \gamma >$ is $\{z, z+b, z+2b, z+3b, \ldots\}$ and its cardinality is infinite. $\square$

IV.2.1 Parametrization

By Theorem II.4.2, the Riemann surface Y can be equivalently described as a projective, regular curve over $\mathbb{C}$. Furthermore, any algebraic curve that has the same function field as Y is birationally equivalent to Y. We now want to find an affine plane curve with this property (this is the most simple object that we can expect). We show that for any Riemann surface Y that satisfies Property $(*)$, there exists an equation of such an affine plane curve depending on two complex parameters λ, μ. This description can be found e.g. in [Gey74], where points in moduli space M_2 are classified according to their automorphism group.

First, we choose appropriate coordinates of $\mathbb{P}^1$ such that B, the set of branch points of ϕ, and the automorphism $\bar{\tau} \in \mathrm{Aut}(\mathbb{P}^1)$ induced by $\tau \in G$ have a very simple form. This is possible since by Proposition II.6.10, the quotient map $\phi : Y \to \mathbb{P}^1$ is unique up to composition with an element $\beta \in \mathrm{Aut}(\mathbb{P}^1)$.

Since $\bar{\tau}^2 = \mathrm{id}$, it follows from Lemma IV.2.3, that $\bar{\tau}$ has two fixed points w_1, w_2. By our assumption on Y, the automorphisms τ and σ have no common fixed point. Therefore, Proposition IV.2.2 implies that $\{w_1, w_2\} \cap B = \emptyset$. Let $b \in B$ be any element. We have three wishes for a Moebius transformation β and we wish that

$$\beta(w_1) = 0, \quad \beta(w_2) = \infty, \quad \beta(b) = 1.$$

(Fortunately, our wishes come true.) Then the map $\beta \circ \phi$ is also a covering map for the action of σ on Y and τ descends to $\tilde{\tau} = \beta\bar{\tau}\beta^{-1}$ via $\beta \circ \phi$, since

$$\beta \circ \phi \circ \tau = \beta \circ \bar{\tau} \circ \phi = \tilde{\tau} \circ \beta \circ \phi.$$

Then we get

$$\tilde{\tau}(0) = 0, \quad \tilde{\tau}(\infty) = \infty,$$

and $\tilde{\tau}$ can be written as $z \mapsto az$ with $a \in \mathbb{C}, a \neq 0$. Since $\tilde{\tau}^2 = \mathrm{id}$, it follows from $\tilde{\tau}^2(1) = 1$ that $a^2 = 1$, which leads to $a = -1$, for $\tilde{\tau} \neq \mathrm{id}$. So $\tilde{\tau}$ is the map

$$\mathbb{P}^1 \to \mathbb{P}^1, \quad z \mapsto -z.$$

Let $\tilde{B} = \beta(B)$. Then $\tilde{B}$ is the set of branch points for the covering map $\beta \circ \phi$. To derive the form of $\tilde{B}$, note that the map $\tilde{\tau}$ acts on this set. Since $1 \in \tilde{B}$, so is $\tilde{\tau}(1) = -1$. Let $\lambda \in \tilde{B} \setminus \{1, -1\}$, then $\tilde{\tau}(\lambda) = -\lambda \in \tilde{B}$. In the same way, let $\mu \in \tilde{B} \setminus \{1, -1, \lambda, -\lambda\}$, then $-\mu \in \tilde{B}$. Hence,

$$\tilde{B} = \{1, -1, \lambda, -\lambda, \mu, -\mu\}.$$

Altogether, we have shown the first half of the following proposition.

IV.2.5 Proposition. *Let Y be a Riemann surface, satisfying Property $(*)$. In particular, we fix an involution $\tau \in \mathrm{Aut}(Y)$ such that τ and the hyperelliptic involution σ have no common fixed point. Then there exists a quotient map $\phi : Y \to \mathbb{P}^1$ for the action of the hyperelliptic involution on Y, such that the automorphism $\tau \in \mathrm{Aut}(Y)$ descends to the map*

$$\bar{\tau} : \mathbb{P}^1 \to \mathbb{P}^1, \ z \mapsto -z,$$

and such that the set of branch points of ϕ is of the form

$$B = \{1, -1, \lambda, -\lambda, \mu, -\mu\},$$

where $\lambda, \mu \in \mathbb{C} \setminus \{0, \pm 1\}$ and $\lambda \neq \pm\mu$.

If $\tilde{\phi} : Y \to \mathbb{P}^1$ is a map with the same properties as ϕ, then $\tilde{\phi} = \delta \circ \phi$, where δ is one of the maps in the set

$$\begin{aligned}
&\{\,\mathrm{id},\ (z \mapsto \lambda^{-1} z),\ (z \mapsto \mu^{-1} z)\,\}\\
&\cup \{\,\bar{\tau},\ (z \mapsto \lambda^{-1} z) \circ \bar{\tau},\ (z \mapsto \mu^{-1} z) \circ \bar{\tau}\,\}\\
&\cup \{\,(z \mapsto z^{-1}),\ (z \mapsto \lambda z^{-1}),\ (z \mapsto \mu z^{-1})\,\}\\
&\cup \{\,(z \mapsto z^{-1}) \circ \bar{\tau},\ (z \mapsto \lambda z^{-1}) \circ \bar{\tau},\ (z \mapsto \mu z^{-1}) \circ \bar{\tau}\,\}.
\end{aligned}$$

Proof: By Proposition II.6.10, there exists $\delta \in \mathrm{Aut}(\mathbb{P}^1)$, such that $\tilde{\phi} = \delta \circ \phi$. The map $\tilde{\phi}$ satisfies $\bar{\tau}\tilde{\phi} = \tilde{\phi}\tau$. Thus

$$\bar{\tau}\delta\phi = \bar{\tau}\tilde{\phi} = \tilde{\phi}\tau = \delta\phi\tau = \delta\bar{\tau}\phi,$$

which leads to $\bar{\tau} = \delta\bar{\tau}\delta^{-1}$, because ϕ is surjective. If we write $\delta = (z \mapsto \frac{az+b}{cz+d})$ with $ad - bc = 1$, then this is equivalent to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The solutions of this equation are $\left(\begin{smallmatrix} a & 0 \\ 0 & \frac{1}{a} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & b \\ \frac{1}{b} & 0 \end{smallmatrix}\right)$. Thus, there exists $r \in \mathbb{C} \setminus \{0\}$ such that $\delta = (z \mapsto rz)$ or $\delta = (z \mapsto rz^{-1})$. Let $\tilde{\phi}(A) = \tilde{B}$ be the set of branch points of $\tilde{\phi}$. Then $\delta(B) = \tilde{B}$. Since $1 \in \tilde{B}$, there exists $b \in B$, such that $\delta(b) = 1$. This determines the factor r, and δ is one of the maps in the list. Conversely, every map in the list induces a covering map $\delta \circ \phi$ of the desired form. $\square$

Note that if Y satisfies Property $(*)$ and if it has another involution τ' of the same kind as τ, then there might be more maps $\delta \in \mathrm{Aut}(\mathbb{P}^1)$ that link two covering maps ϕ, $\tilde{\phi}$ as above.

IV.2.2 An Affine Plane Curve for the Riemann Surface Y

Next, we aim at proving the following theorem (see also [Gey74, I. Case 6)]).

IV.2.6 Theorem. *Any compact Riemann surface Y that satisfies Property $(*)$ is birationally equivalent to an affine plane curve*

$$C_{\lambda,\mu} := \big\{(u, v) \in \mathbb{C} \mid v^2 = (u^2 - 1)(u^2 - \lambda^2)(u^2 - \mu^2)\big\}$$

for some parameters $\lambda, \mu \in \mathbb{C} \setminus \{0, \pm 1\}$, $\lambda \neq \pm\mu$.

Conversely, given $\lambda, \mu \in \mathbb{C} \setminus \{0, \pm 1\}$, $\lambda \neq \pm\mu$, the affine plane curve $C_{\lambda,\mu}$ is a Riemann surface of finite type and can be completed to a compact Riemann surface Y that satisfies Property $(*)$, using the covering map

$$\hat{\phi} : C_{\lambda,\mu} \to \mathbb{C}, \; (u, v) \mapsto u.$$

In order to prove the theorem, we make the following observations. Let Y be a Riemann surface that satisfies Property $(*)$. The parameters (λ, μ) will depend on the chosen covering map $\phi : Y \to \mathbb{P}^1$, so we have to fix one first, and we take a covering map as given by Proposition IV.2.5.

To obtain the equation of an affine plane curve that is birationally equivalent to Y, we naturally have to consider the field of meromorphic functions of Y. The covering map $\phi : Y \to \mathbb{P}^1$ induces a field extension

$$\phi^* : \mathcal{M}(\mathbb{P}^1) \to \mathcal{M}(Y)$$

by setting $\phi^*(f) := f \circ \phi$ for a function $f \in \mathcal{M}(\mathbb{P}^1)$.

If we set $x : \mathbb{P}^1 \to \mathbb{P}^1, z \mapsto z$, then the field $\mathcal{M}(\mathbb{P}^1)$ is canonically isomorphic to $\mathbb{C}(x)$ the field of rational functions in one variable over $\mathbb{C}$. The covering map ϕ has degree two, thus ϕ^* is a finite field extension of the same degree two by Proposition II.4.4. Hence there exists a function $y \in \mathcal{M}(Y)$ and there exist functions $a, b \in \mathcal{M}(\mathbb{P}^1)$ such that

$$y^2 + \phi^*(a)y + \phi^*(b) = 0$$

holds in $\mathcal{M}(Y)$ and such that

$$\mathcal{M}(Y) = \mathcal{M}(\mathbb{P}^1)(y) = \mathbb{C}(x)(y).$$

Note that $\{1, y\}$ is a basis of the two-dimensional $\mathbb{C}(x)$-vector space $\mathcal{M}(Y)$ and that this basis can still be modified by addition or multiplication of y with an element of $\mathcal{M}(\mathbb{P}^1)$. In particular, this means that we can complete the square. Hence, we can assume that there exist $y \in \mathcal{M}(Y)$, $c \in \mathcal{M}(\mathbb{P}^1)$ such that

$$y^2 = \phi^*(c).$$

Furthermore, by multiplying this equation with the square of the denominator of c, we can assume that c is a polynomial. Next, we can write $c = fg^2$, where $f, g \in \mathbb{C}[x]$ are polynomials and f is square-free with leading coefficient 1. Thus, if we divide the equation $y^2 = \phi^*(c)$ by g^2, we find $y \in \mathcal{M}(Y)$, $f \in \mathbb{C}[x]$ square-free, satisfying

$$y^2 = \phi^*(f), \tag{IV.1}$$

while $\mathcal{M}(Y) = \mathcal{M}(\mathbb{P}^1)(y)$ still holds.

IV.2.7 Remark. The locus of zeros of the polynomial $V^2 - f(U) \in \mathbb{C}[U,V]$, coming from Equation (IV.1), defines a regular, affine plane curve

$$C_Y := \left\{(u,v) \in \mathbb{C}^2 \mid v^2 - f(u) = 0\right\} \subset \mathbb{C}^2,$$

and hence a Riemann surface.

PROOF: This is due to the fact, that f is a square-free polynomial: $V^2 - f(U)$ is irreducible and f and its derivative f' have no common zero. Therefore, the partial derivatives

$$\partial_U(V^2 - f(U)) = -f'(U), \quad \partial_V(V^2 - f(U)) = 2V$$

evaluated at $p \in \left\{(u,v) \in \mathbb{C}^2 \mid v^2 - f(u) = 0\right\}$ are not both equal to zero. $\square$

We show that this affine plane curve is biholomorphically equivalent to an open subset of the Riemann surface Y.

Because of the choice of ϕ, the point $\infty \in \mathbb{P}^1$ is not a branch point of ϕ. Hence, there are exactly two points ∞_1, ∞_2 that are mapped to ∞ by ϕ. Let

$$Y_{\mathit{aff}} := Y \setminus \{\infty_1, \infty_2\}.$$

IV.2.8 Proposition. *Let $y \in \mathcal{M}(Y)$ satisfy Equation (IV.1). Then the mapping*

$$\psi : \begin{cases} Y_{\mathit{aff}} & \longrightarrow C_Y = \{(u,v) \in \mathbb{C}^2 \mid v^2 = f(u)\} \\ z & \longmapsto (\phi^*(x)(z), y(z)) \end{cases}$$

is biholomorphic.

PROOF: First of all, we must show that ψ is well-defined. For a meromorphic function g on Y let $P(g) := \{z \in Y \mid g(z) = \infty\}$ be the set of poles of g. Then

$$P(y) = P(y^2) = P(\phi^*(f)) = P(\phi^*(x)) = \phi^{-1}(\infty).$$

Thus for all $z \in Y$ the image $\psi(z) = (\phi^*(x)(z), y(z))$ lies in $\mathbb{C}^2$. Moreover $y^2 = \phi^*(f)$ implies $y(z)^2 = f(x \circ \phi(z)) = f(\phi^*(x)(z))$. This shows that ψ is well-defined.

Next we show that ψ is one-to-one. Let $z_1, z_2 \in Y_{\mathit{aff}}$ with $\psi(z_1) = \psi(z_2)$. Then $\phi^*(x)(z_1) = \phi^*(x)(z_2)$ and $y(z_1) = y(z_2)$. Since every function $g \in \mathcal{M}(Y)$ can be written as a rational function in $\phi^*(x)$ and y, it follows $g(z_1) = g(z_2)$ for all $g \in \mathcal{M}(Y)$. By Corollary II.4.7, we get $z_1 = z_2$.

Finally, we show that ψ is also onto. Let $(u,v) \in \mathbb{C}^2$ with $v^2 = f(u)$. Since $\phi^*(x) = \phi$ and since ϕ is onto, there exists a $z_0 \in Y$ with $\phi^*(x)(z_0) = u$. It remains to show that $v \in \{y(z_0), y(\sigma(z_0))\}$, which in turn means that $y(\sigma(z_0))$ must be equal to $-y(z_0)$, since the equation

$$y^2(z_0) = f(\phi^*(x)(z_0)) = f(u) = v^2$$

holds. To the element $\sigma \in \mathrm{Aut}(Y)$ we assign the automorphism σ^* of $\mathcal{M}(Y)$, defined by $f \mapsto \sigma^*(f) := f \circ \sigma^{-1}$. If $f \in \mathcal{M}(\mathbb{P}^1)$, then

$$\sigma^*(\phi^*(f)) = f \circ \phi \circ \sigma^{-1} = f \circ \phi \circ \sigma = f \circ \phi = \phi^*(f),$$

so σ^* leaves an element of $\mathcal{M}(\mathbb{P}^1)$ invariant. On the other hand σ^* is not the identity map. Otherwise, if $\sigma^*(y) = y$, then for all $z \in Y$ one has $y(\sigma^{-1})(z) = y(\sigma(z)) = y(z)$. Together with $\sigma^*(\phi^*(x)) = \phi^*(x)$ and Corollary II.4.7, we would have $\sigma(z) = z$ for all $z \in Y$, contradicting the fact that $\sigma \neq \mathrm{id}$. Applying σ^* to the equation (IV.1) we get

$$(\sigma^*(y))^2 = \sigma^*(y^2) = \sigma^*(\phi^*(f)) = \phi^*(f) = y^2$$

Since $\sigma^* \neq \mathrm{id}$, this shows $\sigma^*(y) = -y$.

It remains to show that ψ is holomorphic. A chart (U,φ) of C_Y can be choosen to be locally either the projection onto the first or on the second coordinate. This implies that $\varphi \circ \psi = \phi|_{\psi^{-1}(U)}$ or $\varphi \circ \psi = y|_{\psi^{-1}(U)}$, which in turn shows ψ is holomorphic. This completes the proof. □

In particular, we can transfer the covering map ϕ to C_Y by setting $\hat{\phi} := \phi \circ \psi^{-1}$. Then $\hat{\phi}$ is the projection to the first coordinate

$$\hat{\phi} : C_Y \longrightarrow \mathbb{C}, \quad (u,v) \longmapsto u.$$

It remains to gather some information on the polynomial f. By the choice of coordinates we already know that all ramification points of ϕ are in Y_{aff}.

IV.2.9 Lemma. *If $z \in Y$ is a ramification point of ϕ, then $f(\phi^*(x)(z)) = 0$. Conversely, if $u \in \mathbb{C}$ satisfies $f(u) = 0$, then $\phi^{-1}(\{u\})$ is a singleton and thus a ramification point of ϕ.*

Proof: Let $z \in Y$ be a ramification point of ϕ. Then $\sigma(z) = z$ and

$$y(z) = y(\sigma(z)) = y(\sigma^{-1}(z)) = \sigma^*(y(z)) = -y(z),$$

thus $0 = y(z)^2 = f(\phi^*(x)(z))$.

In order to show the converse, let $u \in \mathbb{C}$ be a zero of f and let $\hat{\phi}^{-1}(\{u\}) = \{(u, v_1), (u, v_2)\}$ be the set of preimages of u. Then,

$$v_1^2 = f(u) = 0 = f(u) = v_2^2,$$

so $v_1 = v_2 = 0$ and $\phi^{-1}(\{u\})$ is a singleton. □

Altogether we can now complete the proof of Theorem IV.2.6.

Proof of Theorem IV.2.6: Let $\lambda, \mu \in \mathbb{C} \setminus \{0, \pm 1\}$, $\lambda \neq \pm\mu$ and let $B = \{1, -1, \lambda, -\lambda, \mu, -\mu\}$ be the set of branch points of ϕ. By the preceding lemma,

$$C_Y = \left\{(u, v) \in \mathbb{C}^2 \mid v^2 = (u^2 - 1)(u^2 - \lambda^2)(u^2 - \mu^2)\right\},$$

and C_Y is biholomorphically equivalent to Y_{aff}, which is a Zariski-open subset of Y. Therefore, C_Y is birationally equivalent to Y in the category of algebraic varieties.

Let $\lambda, \mu \in \mathbb{C} \setminus \{0, \pm 1\}$, $\lambda \neq \pm\mu$. Being an affine plane curve, $C_{\lambda,\mu}$ is naturally also a Riemann surface of finite type. The holomorphic map

$$\hat{\phi} : C_{\lambda,\mu} \to \mathbb{C}, (u, v) \to u$$

is a finite, surjective covering map. Let $B = \{1, -1, \lambda, -\lambda, \mu, -\mu\}$ and let $A = \hat{\phi}^{-1}(B)$. If we restrict $\hat{\phi}$ to $C_{\lambda,\mu} \setminus A$, then it is unramified. Thus by Proposition II.2.5, it is proper. Hence, Proposition II.3.3 implies that $\hat{\phi}$ can be extended to a holomorphic map $\phi : Y \to \mathbb{P}^1$, where Y is a compact Riemann surface of genus 2 such that $C_{\lambda,\mu} \setminus A \hookrightarrow Y$. By Proposition II.3.4, the four automorphisms of $C_{\lambda,\mu}$.

$$\mathrm{id}_{C_{\lambda,\mu}},\ (u, v) \mapsto (u, -v),\ (u, v) \mapsto (-u, v),\ \text{and } (u, v) \mapsto (-u, -v)$$

extend uniquely to automorphisms of Y. The map $(u, v) \mapsto (u, -v)$ extends to the hyperelliptic involution on Y, since it is an involution with $|A| = 6$ fixed points. Moreover, we will see later in Corollary IV.2.15 that the condition on the fixed points of the four automorphisms is fulfilled. □

IV.2.10 Remark. Note that the projective closure

$$\overline{\{(u, v) \in \mathbb{C}^2 \mid v^2 - f(u) = 0\}} \subset \mathbb{P}^2$$

has a singularity at ∞. This is due to the general fact, that a regular, projective curve of genus two cannot be embedded in $\mathbb{P}^2$, which is a consequence of Plücker's Formula II.4.1.

IV.2.3 A First Approach to the Moduli Space M_2

Now we want to answer the question how many different choices of parameters (λ, μ) belong to the same Riemann surface Y. In other words, when do (λ, μ) and (λ', μ') give rise to isomorphic curves $C_{\lambda,\mu}, C_{\lambda',\mu'}$, i.e. when do they define the same point in the moduli space M_2? We make a first approach to answer this question. A more detailed description can be found e.g. in [Gey74].

The underlying parameter space P is the set

$$P = (\mathbb{C} \setminus \{0, \pm 1\} \times \mathbb{C} \setminus \{0, \pm 1\}) \setminus (\Delta \cup \Delta'),$$

where $\Delta = \{(z, z) \mid z \in \mathbb{C}\}$ is the diagonal and $\Delta' = \{(z, -z) \mid z \in \mathbb{C}\}$. We get a map

$$\mathrm{pr} : P \to M_2,\ (\lambda, \mu) \mapsto C_{\lambda,\mu},$$

where $C_{\lambda,\mu}$ is the affine plane curve defined in Theorem IV.2.6. (Note that this theorem also permits us to identify $C_{\lambda,\mu}$ with its associated compact counterpart.)

Clearly, (λ, μ) and (μ, λ) lead to isomorphic curves $C_{\lambda,\mu}$ and $C_{\mu,\lambda}$. Because λ and μ only appear squared in the equation to C_Y, the pairs

$$(\lambda, -\mu),\ (-\lambda, \mu) \text{ and } (-\lambda, -\mu)$$

also lead to curves that are isomorphic to $C_{\lambda,\mu}$.

Moreover, we can still vary the covering map by composition with $\delta \in \mathrm{Aut}(\mathbb{P}^1)$ from the list in Proposition IV.2.5 and stay in the same isomorphism class. This leads to changing the set of branch points B and thus to changing (λ, μ). We get the following proposition.

IV.2.11 Proposition. *The group Γ generated by*

$$A : (\lambda, \mu) \mapsto (\lambda^{-1}, \mu^{-1}), \qquad B : (\lambda, \mu) \mapsto (\mu, \lambda), \qquad C : (\lambda, \mu) \mapsto (\lambda^{-1}, \lambda^{-1}\mu)$$

$$D : (\lambda, \mu) \mapsto (-\lambda, \mu), \quad E : (\lambda, \mu) \mapsto (-\lambda, -\mu)$$

acts on the algebraic variety P as a group of automorphisms. The following holds:

a) *Γ is isomorphic to the semidirect product $V_4 \rtimes_\varphi D_6$, where the dihedral group $D_6 \cong < A, B, C >$ acts on the Klein four group $V_4 \cong < D, E >$ by conjugation.*

b) *The map $\mathrm{pr} : P \to M_2$ induces a surjective morphism*

$$\overline{\mathrm{pr}} : P/\Gamma \to \mathcal{S} \subset M_2,$$

where $\mathcal{S}$ is the set of Riemann surfaces in M_2 that satisfy Property $(*)$.

c) *If we restrict* $\overline{\mathrm{pr}}$ *to* $\overline{\mathrm{pr}}^{-1}(\mathcal{S}')$*, where* $\mathcal{S}' \subset \mathcal{S}$ *is the set of Riemann surfaces, whose automorphism group is precisely* $G \cong V_4$*, then* $\overline{\mathrm{pr}}$ *is an isomorphism.*

PROOF: *Part a).* Note that each of these maps is a well-defined automorphism of P. Clearly,

$$< D, E >= \{\mathrm{id}, D, E, DE = ED\} \cong V_4.$$

Moreover, $A^2 = \mathrm{id}$, and one shows easily that $AB = BA$, $AC = CA$. The elements B and BC generate a subgroup isomorphic to S_3. Surely, $B^2 = \mathrm{id}$, and an easy computation shows that $(BC)^3 = \mathrm{id}$ and that $B(BC) = (BC)^2 B$. Therefore,

$$< A, B, C >\cong (\mathbb{Z}/2\mathbb{Z}) \times S_3 = D_6.$$

It remains to show that $< D, E >$ is a normal subgroup of Γ. This can be verified on the generators:

$$ADA = D,\ AEA = E,\ BDB = DE,\ BEB = E,\ CDC = E,\ CEC = D.$$

Thus, $\varphi :< A, B, C >\to \mathrm{Aut}(< D, E >),\ g \mapsto (h \mapsto ghg^{-1})$ is a well-defined homomorphism and $\Gamma \cong V_4 \rtimes_\varphi D_6$.

Part b). By Theorem IV.2.6, the map $\mathrm{pr} : P \to \mathcal{S}$ is surjective. Next, we justify that $\mathrm{pr} : P \to M_2$ factors through P/Γ. This has already been done above for $< D, E >$. Let Y be a Riemann surface, satisfying Property $(*)$. Let $\phi : Y \to \mathbb{P}^1$ be a covering map such that $Y_{\mathit{aff}} \cong C_{\lambda,\mu}$ for $(\lambda, \mu) \in P$. Then the set of branch points of ϕ can be written as

$$B = \{1, -1, \lambda, -\lambda, \mu, -\mu\}.$$

Applying A to (λ, μ) corresponds to changing the covering map ϕ by $\delta = (z \mapsto z^{-1})$. Thus,

$$C_{\lambda,\mu} \cong C_{\lambda^{-1},\mu^{-1}}.$$

If we change the covering map by composition with $\delta = (z \mapsto \lambda^{-1}z)$, then B is mapped to

$$B' = \{1, -1, \lambda', -\lambda', \mu', -\mu'\} = \{1, -1, \lambda^{-1}, -\lambda^{-1}, \lambda^{-1}\mu, -\lambda^{-1}\mu\},$$

so applying C to (λ, μ) yields isomorphic curves. Note that $\delta = \bar{\tau}$ corresponds to applying E to (λ, μ). By composing the maps A, B, C and E, one sees that all the cases of the list in Proposition IV.2.5 are treated. This shows that we get a map $\overline{\mathrm{pr}} : P/\Gamma \to M_2$.

Part c). To prove that the restriction of $\overline{\mathrm{pr}}$ to $\overline{\mathrm{pr}}^{-1}(\mathcal{S}')$ is injective, let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in P$, and let Y_i be the compact Riemann surface associated to C_{λ_i,μ_i}, $i = 1, 2$ as in Theorem IV.2.6. Suppose that Y_1, $Y_2 \in \mathcal{S}'$, i.e.

$$\mathrm{Aut}(Y_i) = \{\mathrm{id}, \sigma_i, \tau_i, \sigma_i\tau_i\} \cong V_4, \quad i = 1, 2$$

and suppose that they are isomorphic via an isomorphism

$$h : Y_1 \to Y_2.$$

Let $\phi_i : Y_i \to \mathbb{P}^1$ be the associated covering map, coming from the projection onto the first coordinate in C_{λ_i,μ_i}. We show that $\phi_2 \circ h$ is also a quotient map for the hyperelliptic involution σ_1 on Y_1. The map $h \circ \sigma_1 \circ h^{-1}$ is a holomorphic involution on Y_2 with 6 fixed points, and it follows from Proposition II.6.13, that $h \circ \sigma_1 \circ h^{-1} = \sigma_2$ is the hyperelliptic involution on Y_2. Therefore,

$$\phi_2 \circ h \circ \sigma_1 = \phi_2 \circ h \circ \sigma_1 \circ h^{-1} \circ h = \phi_2 \circ \sigma_2 \circ h = \phi_2 \circ h.$$

Before we can apply Proposition IV.2.5, it remains to check, whether τ_1 descends to $z \mapsto -z$ via $\phi_2 \circ h$. This holds, since $h \circ \tau_1 \circ h^{-1}$ is either equal to τ_2 or to $\sigma_2\tau_2$, and both τ_2 and $\sigma_2\tau_2$ descend to $z \mapsto -z$ by the choice of ϕ_2. Hence there exists $\delta \in \mathrm{Aut}(\mathbb{P}^1)$ from the list in Proposition IV.2.5, such that

$$\begin{tikzcd}
Y_1 \arrow[r, "h"] \arrow[d, "\phi_1"'] & Y_2 \arrow[d, "\phi_2"] \\
\mathbb{P}^1 \arrow[r, "\delta"'] & \mathbb{P}^1
\end{tikzcd}$$

But this means that (λ_1, μ_1) and (λ_2, μ_2) are equivalent modulo Γ. $\square$

IV.2.4 Automorphisms of the Affine Curve

Now that we have modeled Y as an affine plane curve, we want to translate $G \subset \mathrm{Aut}(Y)$ into a group of automorphisms of the affine curve. Let us fix parameters $(\lambda, \mu) \in P$, such that

$$Y_{\mathit{aff}} \xrightarrow[\psi]{\sim} C_{\lambda,\mu} = C_Y,$$

where ψ is the isomorphism introduced in Proposition IV.2.8. Note that ψ depends on the choice of parameters.

IV.2.12 Remark. The map $\hat{\sigma} = \psi \circ \sigma \circ \psi^{-1} \in \mathrm{Aut}(C_Y)$, induced by the hyperelliptic involution, is given by

$$\hat{\sigma} : C_Y \to C_Y,\ (u,v) \mapsto (u,-v).$$

PROOF: In the course of the proof of Proposition IV.2.8, we showed that $y(\sigma(z)) = -y(z)$. Moreover, $\phi^*(x)(\sigma(z)) = \phi^*(x)(z)$, which proves our claim. $\square$

IV.2.13 Lemma. *The automorphisms τ and $\sigma\tau \in \mathrm{Aut}(Y)$ have two fixed points.*

PROOF: The quotient map $Y \to Y/_{<\tau>}$ is a two-sheeted covering. Let g denote the genus of $Y/_{<\tau>}$ and B be the number of fixed points of τ. The Riemann-Hurwitz formula II.3.2 implies that

$$2g(Y) - 2 = 2(2g-2) + B,$$

which is equivalent to

$$\frac{6-B}{4} = g.$$

Since $g \geq 0$, the only possible values for B are $2, 6$. As τ is different from the hyperelliptic involution, Corollary II.6.13 implies that $B = 2$. This argument works similarly for $\sigma\tau$. $\square$

IV.2.14 Proposition. *The automorphisms τ and $\sigma\tau$ of Y induce the automorphisms*

$$(u,v) \mapsto (-u,v)$$

and

$$(u,v) \mapsto (-u,-v)$$

of C_Y.

Moreover, there exists a choice of parameters $(\lambda,\mu) \in P$, such that Y_{aff} is isomorphic to $C_{\lambda,\mu} = C_Y$ and such that $\hat{\tau} = \psi \circ \tau \circ \psi^{-1} \in \mathrm{Aut}(C_Y)$ is the map

$$C_Y \to C_Y,\ (u,v) \mapsto (-u,v).$$

PROOF: First we show that $\hat{\tau}$ is well-defined. This is equivalent to proving $\tau(Y_{aff}) = Y_{aff}$. For $i = 1,2$, we have

$$\phi\tau(\infty_i) = \bar{\tau}\phi(\infty_i) = \bar{\tau}(\infty) = \infty.$$

Thus $\tau(\phi^{-1}(\{\infty\})) \subset \phi^{-1}(\{\infty\})$ and even $\tau(\phi^{-1}(\{\infty\})) = \phi^{-1}(\{\infty\})$, since τ is bijective. Hence, $\tau(Y_{aff}) = Y_{aff}$.

Recall that τ is a lift of $\bar{\tau}$ and that the only other lift of $\bar{\tau}$ is $\sigma\tau$. So let t be a lift of $\bar{\tau}|_{\mathbb{C}} : z \mapsto -z$ to $\mathrm{Aut}(C_Y)$. We make the ansatz

$$t : C_Y \to C_Y, (u, v) \mapsto (t_1(u, v), t_2(u, v)).$$

Since $\hat{\phi}t = \bar{\tau}\hat{\phi}$, we get $t_1(u, v) = -u$. As

$$t_2(u, v)^2 = f(t_1(u, v)) = f(-u) = f(u) = v^2,$$

the only possibilities are $t_2(u, v) = \pm v$. Thus, t, and consequently $\hat{\tau}$, is either the map $(u, v) \mapsto (-u, v)$ or $(u, v) \mapsto (-u, -v)$.

Let us assume that $\hat{\tau} = (u, v) \mapsto (-u, -v)$. The map $(u, v) \mapsto (-u, -v)$ has no fixed points in C_Y, since $(u, v) = (0, 0) \notin C_Y$. Therefore, the two fixed points of τ are ∞_1 and ∞_2. On the other hand, solving $(u, v) = (-u, v)$, yields $u = 0$, and $(0, i\lambda\mu), (0, -i\lambda\mu)$ are fixed points of $(u, v) \mapsto (-u, v)$ in C_Y. This means that the fixed points of $\sigma\tau$ lie in $\phi^{-1}(\{0\})$. Changing (λ, μ) to (λ^{-1}, μ^{-1}) corresponds to composing ϕ with $\delta = (z \mapsto z^{-1})$. The map δ exchanges 0 and ∞, thus if we choose the parameters (λ^{-1}, μ^{-1}), then $\hat{\tau} \in \mathrm{Aut}(C_{\lambda^{-1},\mu^{-1}})$ is of the form $(u, v) \mapsto (-u, v)$. □

In the following, let us assume that $C_{\lambda,\mu} = C_Y$ is chosen in such a way that

$$\hat{\tau} = (u, v) \mapsto (-u, v).$$

With this definition, the above proof also shows the following corollary.

IV.2.15 Corollary. *The fixed points of $\sigma\tau$ are the points ∞_1, ∞_2, and the fixed points of τ correspond to the points $\{(0, i\lambda\mu), (0, -i\lambda\mu)\} \in C_Y$.*

Now we investigate the quotient surface $\bar{Y} := Y/_{<\tau>}$. Let $\pi : Y \to \bar{Y}$ be the quotient map.

IV.2.16 Proposition. *The Riemann surface $\bar{Y}$ has genus one. With the above assumption on the form of $\hat{\tau}$, the points $\pi(\infty_1)$ and $\pi(\infty_2)$ satisfy $\pi(\infty_1) = \pi(\infty_2) =: N$, and the elliptic curve $(\bar{Y}, N)$ has the equation*

$$y^2 = (x - 1)(x - \lambda^2)(x - \mu^2).$$

Proof: It follows from the Riemann-Hurwitz formula II.3.2 that $\bar{Y}$ has genus one. Because of our assumption, a quotient map for the action of $< \hat{\tau} >$ on C_Y is given by

$$\hat{\pi} : C_Y \to C_{\bar{Y}},\ (u, v) \mapsto (x, y) = (u^2, v),$$

where $C_{\bar{Y}} := \{(x,y) \in \mathbb{C}^2 \mid y^2 = (x-1)(x-\lambda^2)(x-\mu^2)\}$. This is true, since $\hat{\pi}(u_1,v_1) = \hat{\pi}(u_2,v_2)$, if and only if $v_1 = v_2$ and $(u_1-u_2)(u_1+u_2) = 0$, which in turn is equivalent to $(u_1,v_1) = (u_2,v_2)$ or $(u_1,v_1) = \hat{\tau}(u_2,v_2)$.

The Riemann surfaces $\bar{Y}$ and $C_{\bar{Y}}$ are linked via

$$C_{\bar{Y}} = C_Y/_{<\hat{\tau}>} \cong Y_{\mathit{aff}}/_{<\tau>} = \bar{Y} \setminus \pi(\{\infty_1, \infty_2\}).$$

So $C_{\bar{Y}}$ is a Riemann surface of finite type and can be embedded into the compact Riemann surface $\bar{Y}$. On the other hand, the projective closure of $C_{\bar{Y}}$,

$$\overline{C_{\bar{Y}}} = \{(X:Y:1) \in \mathbb{P}^2 \mid Y^2 = (X-1)(X-\lambda^2)(X-\mu^2)\} \cup \{(0:1:0)\}$$

is also a compact Riemann surface that contains $C_{\bar{Y}}$ as a Zariski-open subset. By Corollary II.3.7, $\overline{C_{\bar{Y}}}$ is isomorphic to $\bar{Y}$. This implies $\pi(\infty_1) = \pi(\infty_2)$, and the elliptic curve (Y,N) has the above equation. □

IV.3 An Affine Equation for the Origami S

Let us return to our origami $S = (p : X \to E)$. We already know from Proposition IV.1.5 and Theorem IV.2.6 that every surface X_B has a parametrization with parameters $(\lambda,\mu) \in P$, such that X_B corresponds to the curve $C_{\lambda,\mu}$. Since the family of Riemann surfaces $(X_B)_{B\in \mathrm{SL}_2(\mathbb{R})}$ describes a curve in the moduli space M_2, there has to be an algebraic relation between λ and μ, which we are going to establish in the following.

First, we consider S again from a topological point of view. In particular, the map $\tau \in \mathrm{Aut}(S)$ is a homeomorphism of X, such that $p \circ \tau = p$, i.e. it is a covering transformation for $S = (p : X \to E)$. Let

$$\pi : X \to X/_{<\tau>},\ x \mapsto <\tau> \cdot x$$

denote the quotient map.

IV.3.1 Proposition. *The map $\pi|X^* \to X^*/_{<\tau>}$ is a topological covering map. Moreover, there exists a unique covering map $\bar{p} : X/_{<\tau>} \to E$ such that $p = \bar{p} \circ \pi$ and such that*

$$\bar{S} = (\bar{p} : X/_{<\tau>} \to E)$$

is an origami of genus one.

A picture for the origami $\bar{S}$ is given by Figure IV.1. Here, the letters indicate, which edges are glued together and numbers indicate, which squares of S are mapped to which squares of $\bar{S}$. The points $\square$ and $\blacksquare$ are the images of the fixed points of τ and the point @ is the image of $\bigcirc$ and $\bullet$ $\in X$.

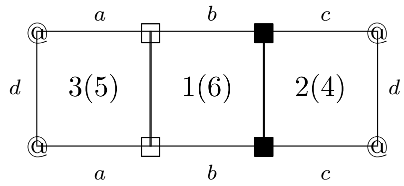


Figure IV.1: *The origami* $\bar{S}$

Proof: For $\bar{x} = <\tau> \cdot x \in X/_{<\tau>}$, we set

$$\bar{p}(\bar{x}) := p(x).$$

This is a well-defined map $\bar{p} : X/_{<\tau>} \to E$, since τ is a covering transformation, which is equivalent to p being constant on every orbit $<\tau> \cdot x$, $x \in X$. Obviously, $\bar{p} \circ \pi = p$.

If we make use of the fact that X is the topological space of the Riemann surface X_I and that $\tau \in \mathrm{Aut}(X_I)$, then $\pi : X_I \to X_I/_{<\tau>}$ is a holomorphic covering map and $X_I/_{<\tau>}$ is a surface of genus one. Moreover, π is proper and unramified on X_I^*. By Proposition II.2.5, π is a topological covering map.

It remains to show that $\bar{S} = (\bar{p} : X/_{<\tau>}) \to E$ is an origami. First, we show that $\bar{p}$ is a finite covering map. Let $U \subset E$ be open, then $p^{-1}(U)$ is open, as well as $\pi(p^{-1}(U))$, since $X/_{<\tau>}$ is endowed with the quotient topology. As π is surjective by Proposition II.2.5, $\pi(p^{-1}(U)) = \bar{p}^{-1}(U)$. Thus, $\bar{p}$ is continuous. In the same way, $\pi^{-1}(V)$ is open for an open set $V \subset X/_{<\tau>}$ and $p(\pi^{-1}(V)) = \bar{p}(V)$ is open in E, since p is open. Finally, $\bar{p}$ is also discrete and finite, since p is discrete and finite and since π is surjective. Therefore, $\bar{p} : X/_{<\tau>} \to E$ is a finite covering map.

Since the genus of $X/_{<\tau>}$ equals one, $\bar{p}$ is unramified. Since $X/_{<\tau>}$ is compact, $\bar{p}$ is proper and it follows by Proposition II.2.5, that it is a topological covering map.

The uniqueness of $\bar{p}$ follows from the fact that π is surjective, and the picture for $\bar{S}$ is derived directly from the picture for the action of τ on S (cf. Definition IV.1.3). □

We abbreviate $X/_{<\tau>}$ by $\bar{X}$ and $X^*/_{<\tau>}$ by $\bar{X}^*$.

IV.3.2 Proposition. *Let ν be a translation structure on E and let η be the lift of ν on X^* via p. There exists a unique translation structure $\bar{\eta}$ on $\bar{X}^*$ such that $\pi : X^* \to \bar{X}^*$ and $\bar{p} : \bar{X}^* \to E^*$ are translations.*

Moreover, the translation structure $\bar{\eta}$ extends uniquely to a translation structure, also called $\bar{\eta}$, on $\bar{X}$.

PROOF: By Proposition III.1.8, we can lift the translation structure ν via p to a unique translation structure $\bar{\eta}$ on $\bar{X}^*$, such that $\bar{p}$ is a translation. Since p and $\bar{p}$ are translations, and since $\bar{p}$ is locally invertible, the map π is also a translation.

Since $\bar{p}$ is a topological covering, Proposition III.1.8 also guarantees that we can extend $\bar{\eta}$ to $\bar{X}$ in a unique way. $\square$

We can apply the preceding proposition to define a translation structure on $\bar{X}$ for every $B \in \mathrm{SL}_2(\mathbb{R})$. As usual, let $E = E_B = (\mathbb{C}/\Lambda_B, \nu_B)$ be the torus associated to B. We denote by $\bar{\eta}_B$ the lift of ν_B to $\bar{X}^*$, respectively $\bar{X}$, and we write $\bar{X}^*_B$, respectively $\bar{X}_B$ for the resulting translation surfaces. Furthermore, $\pi_B : X^*_B \to \bar{X}^*_B$ shall denote the map π with respect to these translation structures.

Next, we compare the origami $\bar{S}$ to the torus $\mathbb{C}/(\mathbb{Z}3 + \mathbb{Z}i)$. As $\mathbb{Z}3 + \mathbb{Z}i$ is a subgroup of $\mathbb{Z} + \mathbb{Z}i$, the canonical map $\mathbb{C} \to \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ factors as in the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r] \arrow[dr] & \mathbb{C}/(\mathbb{Z}3+\mathbb{Z}i) \arrow[d, "q"] \\ & \mathbb{C}/(\mathbb{Z}+\mathbb{Z}i) \end{tikzcd}$$

IV.3.3 Proposition. *Let $C = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ and let Λ_C be the lattice associated to C, i.e. $\Lambda_C = \mathbb{Z}3 + \mathbb{Z}i$. Then $T = (q : \mathbb{C}/\Lambda_C \to E)$ is an origami defined over $E = (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}i), \bar{0})$, and T is isomorphic to $\bar{S} = (\bar{p} : \bar{X} \to E)$. The translation structure ν_C on $\mathbb{C}/\Lambda_C$ descending from $\mathbb{C}$ coincides with the lift of the translation structure ν_I on E via q. Moreover, there exists a bijective translation*

$$f : \bar{X}_I \longrightarrow E_C := (\mathbb{C}/\Lambda_C, \nu_C),$$

such that $f(@) = \bar{0}$, $f(\square) = \bar{1}$ and $f(\blacksquare) = \bar{2}$, and such that the diagram

$$\begin{tikzcd} \bar{X}_I \arrow[rr, "f"] \arrow[dr, "\bar{p}"'] & & E_C \arrow[dl, "q"] \\ & E_I & \end{tikzcd}$$

commutes.

PROOF: The map $q : \mathbb{C}/\Lambda_C \to \mathbb{C}/\Lambda_I,\ z + \Lambda_C \to z + \Lambda_I$ is a finite topological covering map with $q^{-1}(\bar{0}) = \{\bar{0}, \bar{1}, \bar{2}\}$. More precisely, the degree of q equals $(\Lambda_I : \Lambda_C) = 3$ and q is the quotient map for the action of $\Lambda_I/\Lambda_C \cong \mathbb{Z}/3\mathbb{Z}$

on $\mathbb{C}/\Lambda_C$. Since this action is free and since Λ_I/Λ_C is finite, it follows by Proposition II.2.5 that q is a topological covering map. Thus, T is an origami.

From Figure IV.1, we conclude that T and $\bar{S}$ are isomorphic origamis. Hence, there exists a homeomorphism $\tilde{f} : \bar{X} \to \mathbb{C}/\Lambda_C$ such that $\bar{p} = q \circ \tilde{f}$. Next, we endow E and $\bar{X}$ with the respective translation structures ν_I and $\bar{\eta}_I$. An argument similar to the one in the proof of Proposition IV.3.2 shows that we get the same translation structure on $\mathbb{C}/\Lambda_C$, if we either lift ν_I to $\mathbb{C}/\Lambda_C$ or if we consider the translation structure ν_C descending from $\mathbb{C}$.

With this setting, the maps p and q are translations. As q is locally invertible, the map $\tilde{f}$ is a translation as well. By composition of $\tilde{f}$ with a translation of $\mathbb{C}/\Lambda_C$, we get a translation f with the right properties. □

If we consider $\bar{X}_I$ and $\mathbb{C}/\Lambda_C$ as Riemann surfaces, then the preceding proposition implies that they are isomorphic via an isomorphism that maps @ to $\bar{0}$. This means that the elliptic curves $(\bar{X}_I, @)$ and $(\mathbb{C}/\Lambda_C, \bar{0})$ are isomorphic.

IV.3.4 Consequence. *If $N = @$ is the zero element of the elliptic curve $(\bar{X}_I, N)$, then the points □ and ■ are 3-torsion points and their sum equals N.*

We now want to generalize this consequence to an arbitrary element in the family $(X_B)_{B\in \mathrm{SL}_2(\mathbb{R})}$. First, we introduce some notations. For $B \in \mathrm{SL}_2(\mathbb{R})$, let $\psi_B : X_I^* \to X_B^*$ be an affine diffeomorphism as in Proposition III.3.6. Then,

$$\mathrm{Aut}(X_B^*) = \psi_B \circ \mathrm{Aut}(S) \circ \psi_B^{-1} = \{\mathrm{id}, \tau_B, \sigma_B, \sigma_B\tau_B\},$$

where $\tau_B = \psi_B \circ \tau \circ \psi_B^{-1}$ and $\sigma_B = \psi_B \circ \sigma \circ \psi_B^{-1}$, and these are affine, holomorphic automorphisms of X_B^*. As usual, they extend to biholomorphic maps $X_B \to X_B$ by Corollary II.3.6. Furthermore, $@_B$ denotes the image of the fixed points of $\sigma_B\tau_B$ under $\pi_B : X_B \to \bar{X}_B = X_B/_{<\tau_B>}$ and $\square_B$ and $\blacksquare_B$ denote the images of the fixed points of τ_B.

IV.3.5 Proposition. *Let $\bar{\varphi}_B : E_I \to E_B$, $z + \Lambda_I \mapsto B \cdot z + \Lambda_B$. There exists a unique affine map $\bar{\psi}_B : \bar{X}_I \to \bar{X}_B$ with derivative B such that the diagrams*

$$\begin{tikzcd} \bar{X}_I \arrow[r, "\bar{\psi}_B"] \arrow[d, "\bar{p}_I"'] & \bar{X}_B \arrow[d, "\bar{p}_B"] \\ E_I \arrow[r, "\bar{\varphi}_B"'] & E_B \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} X_I^* \arrow[r, "\psi_B"] \arrow[d, "\pi_I"'] & X_B^* \arrow[d, "\pi_B"] \\ \bar{X}_I^* \arrow[r, "\bar{\psi}_B"'] & \bar{X}_B^* \end{tikzcd}$$

commute. Moreover, $\bar{\psi}_B(@_I) = @_B$, $\bar{\psi}_B(\Box_I) = \Box_B$ *and* $\bar{\psi}_B(\blacksquare_I) = \blacksquare_B$.

Proof: We define $\bar{\psi}_B$ as

$$\bar{\psi}_B : \bar{X}_I \to \bar{X}_B,\ < \tau_I > \cdot x \longmapsto < \tau_B > \cdot \psi_B(x).$$

This yields a well-defined map, since $\tau_B = \psi_B \tau_I \psi_B^{-1}$, and $\bar{\psi}_B$ is affine with derivative B as $\bar{\psi}_B$ can locally be described as a composition of a local inverse of π_I with $\pi_B \circ \psi_B$. Thus we found a map $\bar{\psi}_B$ such that the right diagram commutes. This also implies that $\bar{\psi}_B(@_I) = @_B$, $\bar{\psi}_B(\Box_I) = \Box_B$ and $\bar{\psi}_B(\blacksquare_I) = \blacksquare_B$.

It remains to show that $\bar{\varphi}_B \circ \bar{p}_I = \bar{p}_B \circ \bar{\psi}_B$. This holds, since

$$\begin{aligned} \bar{\varphi}_B(\bar{p}_I(< \tau_I > \cdot x)) &= \bar{\varphi}_B(p_I(x)) = p_B(\psi_B(x)) = \bar{p}_B(< \tau_B > \cdot \psi_B(x)) \\ &= \bar{p}_B(\bar{\psi}_B(< \tau_I > \cdot x)). \end{aligned}$$

$\Box$

Note that $B \cdot \Lambda_C = \Lambda_{BC}$. We define a map χ_B by

$$\chi_B : \mathbb{C}/\Lambda_C \to \mathbb{C}/\Lambda_{BC},\ z + \Lambda_C \mapsto B \cdot z + \Lambda_{BC}.$$

If we endow $\mathbb{C}/\Lambda_C$ and $\mathbb{C}/\Lambda_{BC}$ with their translation structures ν_C and ν_{BC}, then χ_B is an affine diffeomorphism with derivative B. At the same time it is a group homomorphism.

By Proposition IV.3.3, we know that

$$(\mathbb{C}/\Lambda_C, B \cdot \nu_C) \cong (\bar{X}, B \cdot \bar{\eta}_I) \cong (\bar{X}, \bar{\eta}_B),$$

and Example III.2.11 tells us that

$$(\mathbb{C}/\Lambda_C, B \cdot \nu_C) \cong (\mathbb{C}/\Lambda_{BC}, \nu_{BC}).$$

Thus,

$$\bar{X}_B \cong E_{BC} := (\mathbb{C}/\Lambda_{BC}, \nu_{BC})$$

in the category of translation surfaces. Moreover, we can choose a bijective translation $f_B : \bar{X}_B \to E_{BC}$ with $f_B(@_B) = B \cdot \bar{0} = \bar{0}$, $f_B(\Box_B) = B \cdot \bar{1}$ and $f_B(\blacksquare_B) = B \cdot \bar{2}$.

IV.3.6 Proposition. *The map* $\bar{\psi}_B : (\bar{X}_I, @_I) \to (\bar{X}_B, @_B)$ *is a group isomorphism between the elliptic curves* $(\bar{X}_I, @_I)$ *and* $(\bar{X}_B, @_B)$*. The points* $\Box_B$ *and* $\blacksquare_B$ *are 3-torsion points of the elliptic curve* $(\bar{X}_B, @_B)$*, and their sum equals* $@_B$*.*

PROOF: We have the following commutative diagram

$$\begin{tikzcd}
(\bar{X}_I, @_I) \arrow[r, "\bar{\psi}_B"] \arrow[d, "f_I"'] & (\bar{X}_B, @_B) \arrow[d, "f_B"] \\
(E_C, \bar{0}) \arrow[r, "\chi_B"'] & (E_{BC}, \bar{0})
\end{tikzcd}$$

where f_I and f_B are isomorphisms of elliptic curves and χ_B is a group isomorphism. This shows that $\bar{\psi}_B$ is also a group isomorphism and we conclude with the help of Consequence IV.3.4. $\square$

From now on, let us consider a compact Riemann surface X_B, $B \in \mathrm{SL}_2(\mathbb{R})$, coming from the origami S, thus an arbitrary element in the family $(X_B)_{B\in\mathrm{SL}_2(\mathbb{R})}$. By Theorem IV.2.6, we find a covering map $\phi_B : X_B \to \mathbb{P}^1$ and parameters $(\lambda, \mu) \in P$ such that the affine part $X_B \setminus \phi_B^{-1}(\{\infty\})$ of X_B is isomorphic to the affine plane curve $C_{\lambda,\mu}$. Moreover, we can assume by Proposition IV.2.14 that $(\lambda, \mu) \in P$ are chosen such that $\tau_B \in \mathrm{Aut}(X_B)$ corresponds to the morphism

$$(u, v) \mapsto (-u, v)$$

of $C_{\lambda,\mu}$. From Remark IV.1.6 and from Corollary IV.2.15, we obtain information on the fixed points of τ_B and $\sigma_B\tau_B$, namely

$$\{\bigcirc, \bullet\} = \phi_B^{-1}(\{\infty\}) = \{\infty_1, \infty_2\}$$

and

$$\{\square, \blacksquare\} = \{(0, i\lambda\mu), (0, -i\lambda\mu)\}.$$

From Proposition IV.2.16, it follows that the elliptic curve

$$(\bar{X}_B, @_B)$$

has the equation

$$y^2 = (x-1)(x-\lambda^2)(x-\mu^2),$$

since $@_B$ is the image of the fixed points of $\sigma_B\tau_B$ under π_B. From Proposition IV.3.6, it follows that

$$\{\square_B, \blacksquare_B\} = \pi_B(\{(0, i\lambda\mu), (0, -i\lambda\mu)\}) = \{(0, i\lambda\mu), (0, -i\lambda\mu)\} =: \{P_B, Q_B\}$$

are 3-torsion points of $(\bar{X}_B, @_B)$, such that $P_B + Q_B = @_B$. The latter relation can be rewritten as

$$[2]P_B = Q_B. \tag{IV.2}$$

IV.3.7 Theorem. *The origami curve $\mathcal{C}$ of the origami $S = (p : X \to E)$ is equal to the projection of the affine curve $C \subset P$ to the moduli space M_2, where C is defined by*

$$C := \left\{ (\lambda, \mu) \in P \mid \mu = \frac{\lambda}{\lambda + 1} \right\}.$$

In particular, the origami curve $\mathcal{C}$ consists of those curves that are birationally equivalent to

$$y^2 = (x^2 - 1)(x^2 - \lambda^2)(x^2 - \left(\frac{\lambda}{\lambda+1}\right)^2)$$

with $\lambda \in \mathbb{C} \setminus \{0, \pm 1, -\frac{1}{2}, -2\}$. Moreover, the curve $\mathcal{C}$ is an affine curve of genus zero.

Proof: We compute the double of P_B with respect to the group structure on $(\bar{X}_B, @_B)$ and compare it with Q_B. As the equation for $(\bar{X}_B, @_B)$ is given by

$$\begin{aligned} y^2 &= (x-1)(x-\lambda^2)(x-\mu^2) \\ &= x^3 + (-1 - \lambda^2 - \mu^2)x^2 + (\lambda^2 + \mu^2 + \lambda^2\mu^2)x - \lambda^2\mu^2, \end{aligned}$$

a comparison with Proposition II.5.3 shows that $(\bar{X}_B, @_B)$ has the data

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ a_2 &= -1 - \lambda^2 - \mu^2 \\ a_4 &= \lambda^2 + \mu^2 + \lambda^2\mu^2 \\ a_6 &= -\lambda^2\mu^2 \end{aligned}$$

We use the addition formula on p. 21 to compute $[2]P_B$: Let $P_B = (x_1, y_1) = (0, i\lambda\mu)$ and $Q_B = (x_2, y_2) = (0, -i\lambda\mu)$. Then,

$$\begin{aligned} \alpha &= \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3} \\ &= \frac{3 \cdot 0^2 + 2(-1 - \lambda^2 - \mu^2) \cdot 0 + \lambda^2 + \mu^2 + \lambda^2\mu^2 - 0 \cdot i\lambda\mu}{2i\lambda\mu + 0 \cdot 0 + 0} \\ &= \frac{\lambda^2 + \mu^2 + \lambda^2\mu^2}{2i\lambda\mu} \end{aligned}$$

and

$$\begin{aligned}
\beta &= \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3} \\
&= \frac{-0^3 + (\lambda^2 + \mu^2 + \lambda^2\mu^2) \cdot 0 + 2(-\lambda^2\mu^2) - 0 \cdot i\lambda\mu}{2i\lambda\mu + 0 \cdot 0 + 0} \\
&= \frac{-2\lambda^2\mu^2}{2i\lambda\mu} = i\lambda\mu,
\end{aligned}$$

and the first coordinate of $[2]P_B$ is given by

$$\begin{aligned}
x([2]P_B) &= \alpha^2 + a_1\alpha - a_2 - x_1 - x_1 = \alpha^2 + 0 \cdot \alpha - a_2 - 0 - 0 \\
&= \left(\frac{\lambda^2 + \mu^2 + \lambda^2\mu^2}{2i\lambda\mu}\right)^2 + 1 + \lambda^2 + \mu^2 \\
&= \frac{\lambda^4 + \mu^4 + \lambda^4\mu^4 + 2\lambda^2\mu^2 + 2\lambda^4\mu^2 + 2\lambda^2\mu^4}{-4\lambda^2\mu^2} + 1 + \lambda^2 + \mu^2 \\
&= \frac{1}{-4\lambda^2\mu^2}\Big(\lambda^4 + \mu^4 + \lambda^4\mu^4 + 2\lambda^2\mu^2 + 2\lambda^4\mu^2 + 2\lambda^2\mu^4 - \\
&\quad - 4\lambda^2\mu^2 - 4\lambda^4\mu^2 - 4\lambda^2\mu^4\Big) \\
&= \frac{1}{-4\lambda^2\mu^2}\Big(\lambda^4 + \mu^4 + \lambda^4\mu^4 - 2\lambda^2\mu^2 - 2\lambda^4\mu^2 - 2\lambda^2\mu^4\Big) \\
&= \frac{1}{-4\lambda^2\mu^2}\Big((-\lambda^2 - \mu^2 + \lambda^2\mu^2)^2 - 4\lambda^2\mu^2\Big) \\
&= \left(\frac{-\lambda^2 - \mu^2 + \lambda^2\mu^2}{2i\lambda\mu}\right)^2 + 1
\end{aligned}$$

Since $x([2]P_B) = x_2 = 0$, it follows that

$$0 = \left(\frac{-\lambda^2 - \mu^2 + \lambda^2\mu^2}{2i\lambda\mu}\right)^2 + 1,$$

and therefore

$$\frac{-\lambda^2 - \mu^2 + \lambda^2\mu^2}{2i\lambda\mu} = \pm i,$$

or equivalently

$$-\lambda^2 - \mu^2 + \lambda^2\mu^2 = \mp 2\lambda\mu.$$

We distinguish two cases:

Case 1: $-\lambda^2 - \mu^2 + \lambda^2\mu^2 = -2\lambda\mu$
Then one has

$$-\lambda^2 - \mu^2 + 2\lambda\mu = -(\lambda - \mu)^2 = -\lambda^2\mu^2,$$

hence

$$\lambda - \mu = \pm\lambda\mu,$$

and thus

$$\mu = \frac{\lambda}{\pm\lambda + 1}.$$

<u>Case 2:</u> $-\lambda^2 - \mu^2 + \lambda^2\mu^2 = +2\lambda\mu$
Then one has

$$-\lambda^2 - \mu^2 - 2\lambda\mu = -(\lambda + \mu)^2 = -\lambda^2\mu^2,$$

which implies

$$\lambda + \mu = \pm\lambda\mu,$$

and therefore

$$\mu = \frac{\lambda}{\pm\lambda - 1}.$$

Since $y([2]P_B)$ evaluates to

$$\begin{aligned} y([2]P_B) &= -(\alpha + a_1)x([2]P_B) - \beta - a_3 \\ &= -(\alpha + 0) \cdot 0 - \beta - 0 \\ &= -\beta \\ &= -i\lambda\mu = y_2, \end{aligned}$$

the equation $[2]P_B = Q_B$ is fulfilled, if μ is one of the four numbers

$$\mu_1(\lambda) = \frac{\lambda}{\lambda + 1}, \quad \mu_2(\lambda) = \frac{-\lambda}{\lambda - 1}, \quad \mu_3(\lambda) = \frac{\lambda}{\lambda - 1}, \quad \mu_4(\lambda) = \frac{-\lambda}{\lambda + 1}.$$

It remains to explain, why these four cases reduce to a single one. We define

$$C_i = \{(\lambda, \mu) \in P \mid \mu = \mu_i(\lambda)\},$$

and we claim that they are all mapped to the same set in M_2.

Claim. We have $\mathrm{pr}(C_i) = \mathrm{pr}(C_j)$ for all $i, j = 1, \ldots, 4$.

Indeed, the pair $(\lambda, \mu_1(\lambda)) = (\lambda, \frac{\lambda}{\lambda+1})$ is equivalent to $(\lambda, \mu_4(\lambda)) = (\lambda, \frac{-\lambda}{\lambda+1})$ under the action of Γ, hence $\mathrm{pr}(C_1) = \mathrm{pr}(C_4)$. Similarly, $\mathrm{pr}(C_2) = \mathrm{pr}(C_3)$. Moreover,

$$(\lambda, \mu_3(\lambda)) = \left(\lambda, \frac{\lambda}{\lambda - 1}\right) = \left(\lambda, \frac{-\lambda}{-\lambda + 1}\right)$$

is equivalent modulo Γ to

$$\left(-\lambda, \frac{-\lambda}{-\lambda + 1}\right) = (-\lambda, \mu_1(-\lambda)),$$

so $\mathrm{pr}(C_3) = \mathrm{pr}(C_1)$. This shows the above claim.

Next, we define a map

$$F : \mathbb{C} \setminus \{0, \pm 1, -\frac{1}{2}, -2\} \to P,\ \lambda \mapsto \left(\lambda, \frac{\lambda}{\lambda + 1}\right).$$

This yields a well-defined map: given $\lambda \in \mathbb{C} \setminus \{0, \pm 1, -\frac{1}{2}, -2\}$, we have to check that $\frac{\lambda}{\lambda+1} \in \mathbb{C} \setminus \{0, \pm 1, \pm\lambda\}$. The condition $\frac{\lambda}{\lambda+1} \neq 0$ requires $\lambda \neq 0$, from $\frac{\lambda}{\lambda+1} \neq -1$, we get $\lambda \neq -\frac{1}{2}$ and finally $\frac{\lambda}{\lambda+1} \neq -\lambda$ requires $\lambda \neq -2$. Moreover, F is an injective morphism with image $C := C_1$.

Therefore, every Riemann surface coming from the origami S is birationally equivalent to a curve $C_{\lambda,\mu}$ such that $(\lambda, \mu) \in C$. A Riemann surface coming from S corresponds to a point in the Teichmüller disk $\Delta_S \subset T_{2,4}$ and from Theorem III.4.7, we know that the image of the projection of Δ_S to the moduli space M_2 is an algebraic curve $\mathcal{C}$, which is closed in M_2. Thus, $\mathcal{C} \subset \mathrm{pr}(C)$, where $\mathrm{pr} : P \to M_2$. Since F and pr are continuous for the Zariski-topology and since $\mathbb{C} \setminus \{0, \pm 1, -\frac{1}{2}, -2\}$ is irreducible, we conclude that $\mathcal{C} = \mathrm{pr}(C)$. □

Having obtained this result, one could be tempted to ask further questions. In the very beginning, we have introduced the family $S_{n,k}$, $n, k \geq 1$ of origamis of genus 2 that all have a translation and of which our origami S is a member (cf. Figure I.3). If we strip down the argument for S, we see that it was crucial that S carries a translation τ, whose quotient is an elliptic curve, so we are able to find an equation for the two ramification points of τ. Therefore we can generalize the computation of the equation for S to any member of the family $S_{n,k}$, $n, k \in \mathbb{N}_{>0}$. The only flaw is the increasing complexity to obtain a relation between the two parameters λ, μ for larger n, k, because of the addition formula on an elliptic curve.

More generally, we have even remarked that any (normal) covering of degree 2 of a trivial origami, ramified over precisely 2 points leads to an origami of genus 2 with a translation. This is a consequence of the Riemann-Hurwitz formula II.3.2. So we can proceed in the following way. We take a trivial origami $O = (p : E_0 \to E)$ and choose two distinct points P, Q in the fiber over $\bar{P} \in E$. We construct an origami $O' = (p' : X \to E)$ with the help of a monodromy representation

$$\rho : \pi_1(E_0 \setminus \{P, Q\}) \to \mathbb{Z}/2\mathbb{Z}.$$

as in [Mir95, Proposition 4.9,p. 91]. Then one could examine the automorphism group $\mathrm{Aut}(O')$. The origami O' should probably lead again to a bunch of Riemann surfaces that satisfy Property $(*)$ with O playing the role of $\bar{S}$. If this is the case, then Theorem IV.2.6 holds for these surfaces, i.e. they are described by affine curves $C_{\lambda,\mu}$ for some parameters $(\lambda, \mu) \in P$. The zero element of E_0 with respect to the equation

$$y^2 = (x-1)(x-\lambda^2)(x-\mu^2)$$

is again the image of the ramification points of $\sigma\tau$, and this leads to a homogeneous linear equation for P and Q on the elliptic curve E_0, whereby we can possibly determine the origami curve of O'.

These generalizations will be carried out in our future work.

Another aspect, which remains to be studied, is whether there exist points on the origami curve $\mathcal{C}$ of S, where the group of biholomorphic automorphisms is bigger. Geyer [Gey74] gives a fairly explicit description of M_2 in terms of the automorphism groups of its points. We have seen in Proposition IV.2.2, that studying automorphisms of curves of genus 2 corresponds to studying Moebius transformations that permute 6 points. Instead of examining the group $\mathrm{Aut}(X)$ for a curve X, we can hence study its reduced automorphism group $\overline{\mathrm{Aut}}(X) = \mathrm{Aut}(X)/_{<\sigma>}$. In particular, a curve that satisfies Property $(*)$ has $\overline{\mathrm{Aut}}(X) \supset \mathbb{Z}/2\mathbb{Z}$.

Geyer classifies the curves in M_2 by giving an individual parametrization depending on the highest order of an element in $\overline{\mathrm{Aut}}(X)$. Altogether, he obtains the following result (see [Gey74, Satz 3, Satz 4]).

IV.3.8 Theorem. *The moduli space M_2 of algebraic curves of genus 2 over $\mathbb{C}$ is a 3-dimensional, rational, normal, affine variety V with one singular point P that corresponds to the curve X with $\overline{\mathrm{Aut}}(X) \cong \mathbb{Z}/5\mathbb{Z}$.*

The curves satisfying Property $(*)$ form a rational surface $\mathcal{S}$ in M_2. The surface $\mathcal{S}$ is not normal, its singular points form a rational curve $\mathcal{C}_1$ and parametrize the curves X of genus 2 with $\overline{\mathrm{Aut}}(X) \supset V_4$.

Another rational curve $\mathcal{C}_2$ on $\mathcal{S}$ describes the curves X of genus 2 with $\overline{\mathrm{Aut}}(X) \supset S_3$. In both cases, one has equality, except at the two intersection points P_1, P_2 of $\mathcal{C}_1$ and $\mathcal{C}_2$, which correspond to curves X in M_2 with $\overline{\mathrm{Aut}}(X) \cong S_4$ and $\overline{\mathrm{Aut}}(X) \cong D_6$ respectively.

To find the intersection of the origami curve $\mathcal{C}$ with the curves $\mathcal{C}_1$ and $\mathcal{C}_2$, where the automorphism group is bigger, one would have to adapt the individual parametrizations of $\mathcal{C}_1$ and $\mathcal{C}_2$ to the one of $\mathcal{C}$.

An answer to this question would also provide information whether the restriction of the projection $\overline{\mathrm{pr}} : P/\Gamma \to M_2$ to $\overline{\mathrm{pr}}^{-1}(\mathcal{C})$ is in fact an isomorphism, in which case $\mathcal{C}$ would be isomorphic to its normalization $\mathbb{H}/\bar{\Gamma}(S)$, or if it is not an isomorphism, in which case there would exist singularities on $\mathcal{C}$.

This is another interesting problem, which we are going to deal with in the future.

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