A Fully Automatic *hp*-Adaptive Refinement Strategy

Zur Erlangung des Grades eines

Doktors der Naturwissenschaften

von der Fakultät für Mathematik
des Karlsruher Instituts für Technologie (KIT)
genehmigte

Dissertation

von

Dipl.-Math. Markus Bürg

aus

Lahr

Tag der mündlichen Prüfung: 18. Juli 2012

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Acknowledgements

At this point I would like to take the opportunity to thank several people for their continuing support during the time I have worked on this dissertation project.
First, I want to thank my supervisor Willy Dörfler for giving me the opportunity to work on this exciting topic. Without his ongoing support and patience this work would not have been possible in this form. He also gave me the possibility to get in contact with various other researchers and discuss several aspects of my ideas with them. The various fruitful and inspiring discussions with my co-supervisor Christian Wieners made my working days much more interesting and have influenced this disseration very much. Without his valuable comments and hints this work would not be in this final form.
I would also like to thank Dominik Schötzau, who introduced me into the very interesting topic of discontinuous Galerkin finite elements. Without his effort and support an important part of this work would not have been possible.
The Research Training Group 1294 *Analysis, Simulation and Design of Nanotechnological Processes* gave me the opportunity to learn a lot about very interesting applications of my rather academic research.
At the end I want to mention the financial support of the Karlsruhe House of Young Scientists under the grant *KHYS Networking Scholarship 2011* leading to Section 3.2 of this work.

Contents

Chapter 0

Introduction

In natural sciences many processes and phenomenas can be described mathematically by partial differential equations. Although there is a vast amount of literature considering the existence and uniqueness of solutions for many different kinds of partial differential equations (see e.g. [110] and the references therein) an explicit representation of the analytical solution often remains an open question.
One tool to overcome this limitation is the numerical solution of partial differential equations. In this field a lot of different methods for approximating the analytical solution, e.g. the finite difference method [160], the finite element method [64, 202], the finite volume method [111], the boundary element method [141, 189] and isogeometric analysis [90], have been developed.
In this work we consider the finite element method (FEM) in more detail. It is the most widely used tool for engineering design and numerical analysis of partial differential equations. The finite element method gives an approximative function to the analytical solution of a partial differential equation and, thus, a huge machinery of functional analytic tools can be applied.
Only in the context of the finite element method the principle of a posteriori error estimation is known. With this tool at hand one can decide how accurate the computed solution is without actually knowing the analytic solution. This knowledge is good for finding a suitable point to stop the computation and one can identify the parts of the computational domain, where the numerical solution is still not accurate enough.
The performance of the finite element method can be improved by either decreasing the mesh size (h-FEM) or the use of higher-order ansatz spaces (p-FEM). By knowing the parts of the domain, where the error is still large, we can adapt the mesh to the specific problem we are solving and, thus, require much less degrees of freedom than one would need, if the mesh was always refined globally. The gain becomes much bigger, if one combines the h-FEM and p-FEM by decreasing the mesh size, where the analytic solution is singular, and increasing the polynomial degree, where the solution is smooth. This method is called hp-FEM.
Since one usually does not know much about the analytic solution , the basic question in hp-FEM is, when to do h-refinement and when to do p-refinement. There have been proposed a lot of different strategies to support this decision, e.g. [9, 10, 11, 74, 96, 100, 104, 107, 157, 174, 180, 215]. Most of them are based on some local heuristics, but there are also a few which try to minimize the error globally. In times of many-core clusters the local strategies seem to be advantageous, because they can be parallelized almost perfectly.
In this work we present a generalization of a strategy from [104], which requires the solution of local boundary value problems. This refinement strategy can be adapted to different types of partial differ-

ential equations quite easily. First we apply the refinement strategy to the classical academic model problem of the Poisson equation and show two convergence results of the fully automatic *hp*-adaptive refinement algorithm resulting from it. After that we tend to a more recent class of partial differential equations, namely Maxwell's equations. Also for this class we present an adaptation of the refinement strategy and prove its convergence.
This dissertation is organized as follows: In Chapter 1 we introduce the function spaces, which we face throughout this work. The finite element method is presented in Chapter 2. In Chapter 3 we derive the refinement strategy for the Poisson problem in arbitrary space-dimensions. Finally the strategy is adapted to Maxwell's equations in the electric field formulation in Chapter 4. Chapter 5 gives a global round-up of the results obtained in this work and highlights its major similarities and differences.

Chapter 1

Function Spaces

In this chapter we shortly introduce the function spaces, which we use later on. We begin with the Lebesgue spaces L^p for $p \in [1, \infty]$. Here especially the space L^2 plays a significant role in the finite element method. Then we consider the standard Sobolev spaces H^r for $r \geq 0$, where the space H^1 is very import, since it is strongly connected to the "standard" (also called H^1-conforming) finite element approximation. With the space H^1 in mind we are able to introduce the curl-conforming space $H(\mathrm{curl})$. This space plays an essential role in the mathematical consideration of electromagnetics. Linked with all the spaces we also have a look at some important results, which we will use throughout this work. For a more in-depth view into this theory we refer to the book of Rudin [188], which is an excellent monograph about many topics of functional analysis, and the book of Monk [163], which focusses on the mathematical theory of the finite element approximation for Maxwell's equations. To conclude this chapter we present the theory of the de Rham complex, which gives a good insight into the interaction of these spaces.

1.1 The Lebesgue Spaces

Now we shortly review the basics of Lebesgue integration theory. It is an essential part of classical and modern functional analysis and can be considered as the basement of the finite element method.
Let $d \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^d$. For $p \in [1, \infty)$ we set

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}$$

and for the special case $p = \infty$ we define

$$\|u\|_{L^\infty(\Omega)} := \sup_{x \in \Omega} |u(x)|,$$

where $u : \overline{\Omega} \to \mathbb{R}$ denotes some function. Then the Lebesgue space $L^p(\Omega)$, $p \in [1, \infty]$, is given by

$$L^p(\Omega) := \{u : \|u\|_{L^p(\Omega)} < \infty\}.$$

For short notice we write $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. Then it can be shown that the mapping $\|\cdot\|_{L^p(\Omega)} : L^p(\Omega) \to \mathbb{R}_+$ is a norm (see e.g. [188]). Two functions $u, v \in L^p(\Omega)$ are identified, if and only if they satisfy

$$\|u - v\|_{L^p(\Omega)} = 0.$$

Now let us state some well-known results for these spaces. The proofs of the following two theorems can be found in [188].
The first inequality we state is *Minkowski's inequality.* It can be considered as the triangle inequality for L^p-spaces. This inequality was first derived from Riesz [183] in 1910 as a direct consequence of its analogue for sums from Minkowski [159].

Theorem 1.1 (Minkowski's Inequality)**.** *For $p \in [1, \infty]$ and $u, v \in L^p(\Omega)$ it holds*

$$\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.$$

From this result it follows immediately that the space $L^p(\Omega)$ is a vector space. Then it can be shown that the pairing $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ induces a Banach space. In the special case $p = 2$ we can easily verify that the mapping $(\cdot, \cdot) : L^p(\Omega) \times L^p(\Omega) \to \mathbb{R}$ given by

$$(f, g) := \int_\Omega fg$$

is an inner product. Since it holds $\|\cdot\|^2_{L^2(\Omega)} = (\cdot, \cdot)$, the pairing $(L^2(\Omega), (\cdot, \cdot))$ even introduces a Hilbert space.
The next result is called *Hölder's inequality.* It was discovered independently by Rogers [185] in 1888 and Hölder [130] in 1889.

Theorem 1.2 (Hölder's Inequality)**.** *Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. Then $fg \in L^1(\Omega)$ and it holds*

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

For the special case $p = q = 2$ this inequality conincides with the *Cauchy-Schwarz inequality* for integrals, which was shown by Bunyakovsky [71] in 1859 and rediscovered by Schwarz [197] in 1888. Therefore we do not give the Cauchy-Schwarz inequality for integrals here and refer to Hölder's inequality instead. However, we will need the Cauchy-Schwarz inequality for sums, too. This inequality was proven by Cauchy [77] in 1821.

Theorem 1.3 (Cauchy-Schwarz Inequality)**.** *For $x, y \in \mathbb{R}^d$ it holds*

$$\sum_{i=1}^d |x_i y_i| \leq \left(\sum_{i=1}^d x_i^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^d y_i^2 \right)^{\frac{1}{2}}.$$

The proof of this theorem can be found in [187].
For later needs we define the space of functions in $L^2(\Omega)$ with zero mean value by

$$L^2_0(\Omega) := \left\{ u \in L^2(\Omega) : \int_\Omega u = 0 \right\}.$$

With this definition we conclude the section on Lebesgue spaces, because now we have tied together all the results from this topic, which we need. Of course the collection of these few facts is far away from providing a complete overview of this area. For this purpose we refer the interested reader to the excellent monograph of Rudin [188].

1.2 The Sobolev Spaces

In this section we present the standard Sobolev spaces H^r for $r \geq 0$. These spaces can be understood as subspaces of the space L^2 with some additional regularity properties. The notion of boundary conditions is strongly associated with these spaces, too. The standard Sobolev space H^1 is also one of the most often used spaces for finite element approximation. Another important Sobolev space is the space $H(\mathrm{curl})$ of curl-conforming functions. These functions play an essential role in the mathematical modelling of electromagnetics, since they correspond to the finite-energy solutions of Maxwell's equations.
As already mentioned in Section 1.1 the space L^2 can be considered as the foundation of the finite element method. However, in its construction in Section 1.1 we did not demand any regularity properties at all. Thus we cannot expect to deal with differentiable functions in general. To overcome this dilemma let us introduce another notion of differentiability.
Let $\alpha \in \mathbb{N}_0^d$ be some multi-index and set

$$|\alpha|_1 := \sum_{i=1}^{d} \alpha_i.$$

For $k \in \mathbb{N}_0 \cup \{\infty\}$ we denote the space of k-times continuously differentiable functions on Ω by $C^k(\Omega)$ and the space of all functions in $C^k(\Omega)$, which have compact support in Ω, by $C_c^k(\Omega)$. Then we define the multi-dimensional derivative $\frac{d^\alpha u}{dx^\alpha}$ by

$$\frac{d^\alpha u}{dx^\alpha} := \frac{d^{|\alpha|_1} u}{dx^{\alpha_1} dx^{\alpha_2} \dots dx^{\alpha_d}}$$

for all $u \in C^k(\Omega)$. With this notation at hand we now can define the notion of a *distribution*. This definition was formulated independently by Sobolev [200] in 1936 and Schwartz [196] in 1944.

Definition 1.1 (Distribution). *The linear functional $T : C_c^\infty(\Omega) \to \mathbb{C}$ is called a* distribution, *if and only if for every compact set $K \subset \Omega$ there exist some constant $C > 0$ and some integer $k \in \mathbb{N}$ such that*

$$|T(\phi)| \leq C \sum_{|\alpha|_1 \leq k} \sup_{x \in K} \left| \frac{d^\alpha \phi}{dx^\alpha}(x) \right|$$

for all $\phi \in C_c^\infty(\Omega)$. We denote the set of all distributions by

$$\mathcal{D}(\Omega) := \{T : C_c^\infty(\Omega) \to \mathbb{C} : T \textit{ is a distribution}\}.$$

Then we identify two distributions $T_1, T_2 \in \mathcal{D}(\Omega)$, if and only if

$$T_1(\phi) = T_2(\phi) \qquad \forall \phi \in C_c^\infty(\Omega).$$

For more information on distributions we refer the interested reader to the monographs of Gelfand and Shilov [117], Hörmander [132] and Wloka [217]. We go on with the definition of a *weak derivative* from Sobolev [200].

Definition 1.2 (Weak Derivative). *Let $\phi \in C_c^\infty(\Omega)$ and $\alpha \in \mathbb{N}_0^d$. Then the* weak derivative $\partial^\alpha \phi \in \mathcal{D}(\Omega)$ *is defined as the unique distribution that satisfies*

$$\int_\Omega \psi \partial^\alpha \phi = (-1)^{|\alpha|_1} \int_\Omega \frac{d^\alpha \psi}{dx^\alpha} \phi \qquad \forall \psi \in C_c^\infty(\Omega).$$

We observe that for $\phi \in C^k(\Omega)$ and $|\alpha|_1 \leq k$ the weak derivative coincides with the strong derivative known from basic calculus.
Now we are ready to define the Sobolev spaces $H^r(\Omega)$ for $r \geq 0$. For $u \in L^2(\Omega)$ we set

$$\|u\|_{H^r(\Omega)} := \left(\sum_{|\alpha|_1 \leq \lfloor r \rfloor} \|\partial^\alpha u\|^2_{L^2(\Omega)} + \sum_{|\alpha|_1 = \lfloor r \rfloor} \int_\Omega \int_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{d+2(r-\lfloor r \rfloor)}} \, dx \, dy \right)^{\frac{1}{2}}.$$

Then the space $H^r(\Omega)$ is simply defined by

$$H^r(\Omega) := \{u \in L^2(\Omega) : \|u\|_{H^r(\Omega)} < \infty\}.$$

It can be easily shown that the mapping $\|\cdot\|_{H^r(\Omega)} : H^r(\Omega) \to \mathbb{R}_+$ is a norm. From its definition we see immediately that it holds $\|\cdot\|^2_{H^r(\Omega)} = (\cdot,\cdot)$ with

$$(u,v) := \sum_{|\alpha|_1 \leq \lfloor r \rfloor} \int_\Omega \partial^\alpha u \partial^\alpha v + \sum_{|\alpha|_1 = \lfloor r \rfloor} \int_\Omega \int_\Omega \frac{(\partial^\alpha u(x) - \partial^\alpha u(y))(\partial^\alpha v(x) - \partial^\alpha v(y))}{|x-y|^{d+2(r-\lfloor r \rfloor)}} \, dx \, dy.$$

A simple computation shows that the mapping $(\cdot,\cdot) : H^r(\Omega) \times H^r(\Omega) \to \mathbb{R}$ is an inner product and it can be proven that the pairing $(H^r(\Omega), (\cdot,\cdot))$ induces a Hilbert space. Note that for the special case $r = 1$ the norm $\|\cdot\|_{H^r}$ reduces to

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)^d}\right)^{\frac{1}{2}}.$$

For $\Omega \subset \mathbb{R}^d$ bounded with Lipschitz-continuous boundary we use the trace theorem (see Lemma A6.6 in [12]) to describe the *homogeneous Dirichlet boundary conditions*

$$u = 0 \qquad \text{on } \partial\Omega.$$

Then the space of all functions $u \in H^1(\Omega)$, which additionally satisfy these homogeneous Dirichlet boundary conditions, is denoted by

$$H^1_0(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\}.$$

The way above is not the only possible derivation of the spaces $H^r(\Omega)$ and $H^1_0(\Omega)$. One could also consider $H^r(\Omega)$ as a subspace of the space of distributions $\mathcal{D}(\Omega)$ and then define the space $H^1_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ under the norm $\|\cdot\|_{H^1(\Omega)}$ (cf. [163]). A third possibility is to use Fourier transforms (see [153]). Nevertheless it can be shown that the spaces obtained from these three different approaches coincide for Ω with $C^{0,1}$-boundary, say, and thus we stay with the definitions from above, since these fit our needs best.
Now we get to the vector-valued Sobolev spaces. For completeness we introduce the space $H(\mathrm{div},\Omega)$, which consists of functions in $L^2(\Omega)^d$ with square-integrable divergence. It is defined by

$$H(\mathrm{div},\Omega) := \{u \in L^2(\Omega)^d : \mathrm{div}(u) \in L^2(\Omega)\}.$$

Let $n : \partial\Omega \to \mathbb{R}^d$ be the outward-pointing unit normal vector to Ω. Then the *homogeneous Dirichlet boundary conditions*

$$n^T u = 0 \qquad \text{on } \partial\Omega$$

are well-defined by the trace theorem (cf. Lemmas 1.1 and 1.2 in [65]). We denote the space of all function $u \in H(\mathrm{div},\Omega)$, which additionally satisfy these homogeneous Dirichlet boundary conditions, by

$$H_0(\mathrm{div},\Omega) = \{u \in H(\mathrm{div},\Omega) : n^T u = 0 \text{ on } \partial\Omega\}.$$

Since we do not rely on any special properties of these spaces, we do not investigate them here any further but refer the interested reader to the book of Girault and Raviart [121].
The next space, which we want to consider, is the vector-valued space $H(\mathrm{curl}, \Omega)$. For this purpose we have to restrict the dimension of the underlying vector space $\mathbb{R}^d$. From now on let $\Omega \subset \mathbb{R}^d$, where $d \in \{2, 3\}$. Before we begin with the derivation of the space we have to specify the definition of the operator $\nabla\times$, because the curl looks completely different for $d = 2$ and $d = 3$. Therefore let us denote the space of all functions mapping from Ω to $\mathbb{R}^d$ by $M\left(\Omega, \mathbb{R}^d\right)$.
We start with the case $d = 2$. Here we define the operator $\nabla\times : M(\Omega, \mathbb{R}) \to M\left(\Omega, \mathbb{R}^2\right)$ by

$$\nabla \times u := \begin{pmatrix} \frac{du}{dx_2} \\ -\frac{du}{dx_1} \end{pmatrix},$$

where $u : \Omega \to \mathbb{R}$ denotes some suffciently regular scalar-valued function. The operator $\nabla\times : M\left(\Omega, \mathbb{R}^2\right) \to M(\Omega, \mathbb{R})$ is defined by

$$\nabla \times u := \frac{du_2}{dx_1} - \frac{du_1}{dx_2}$$

with $u : \Omega \to \mathbb{R}^2$ denoting some sufficiently regular vector-valued function. We do not introduce different notations for these two operators, because it becomes immediately clear from the context which one is used. For $d = 3$ life is a little bit easier. Here only the operator $\nabla\times : M\left(\Omega, \mathbb{R}^3\right) \to M\left(\Omega, \mathbb{R}^3\right)$ is needed. It is defined by

$$\nabla \times u := \begin{pmatrix} \frac{du_2}{dx_3} - \frac{du_3}{dx_2} \\ \frac{du_1}{dx_3} - \frac{du_3}{dx_1} \\ \frac{du_2}{dx_1} - \frac{du_1}{dx_2} \end{pmatrix},$$

where $u : \Omega \to \mathbb{R}^3$ denotes some sufficiently regular function.
The space $H(\mathrm{curl}, \Omega)$ collects the curl-conforming functions of $L^2(\Omega)^d$. To state the definition of the space $H(\mathrm{curl})$ we have to distinguish between the cases $d = 2$ and $d = 3$.
We begin with the case $d = 2$ and set

$$H(\mathrm{curl}, \Omega) := \{u \in L^2(\Omega)^2 : \nabla \times u \in L^2(\Omega)\}.$$

Let us denote a unit tangential vector to Ω by $t : \partial\Omega \to \mathbb{R}^2$. Then we can write

$$H_0(\mathrm{curl}, \Omega) := \{u \in H(\mathrm{curl}, \Omega) : t^T u = 0 \text{ on } \partial\Omega\}$$

for the space of all functions $u \in H(\mathrm{curl}, \Omega)$, which additionally satisfy the *homogeneous Dirichlet boundary conditions*

$$t^T u = 0 \qquad \text{on } \partial\Omega.$$

This definition is justified by the trace theorem (see e.g. [67, 68, 69, 78]). The space $H(\mathrm{curl}, \Omega)$ is equipped with the norm

$$\|u\|_{H(\mathrm{curl},\Omega)} := \left(\|u\|^2_{L^2(\Omega)^2} + \|\nabla \times u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

In the case $d = 3$ we set

$$H(\mathrm{curl}, \Omega) := \left\{u \in L^2(\Omega)^3 : \nabla \times u \in L^2(\Omega)^3\right\}.$$

Here the *homogeneous Dirichlet boundary conditions* are given by

$$n \times u = 0 \qquad \text{on } \partial\Omega. \tag{1.1}$$

Thus we define the space of all functions in $H(\mathrm{curl},\Omega)$, which additionally satisfy condition (1.1), by

$$H_0(\mathrm{curl},\Omega) := \{u \in H(\mathrm{curl},\Omega) : n \times u = 0 \text{ on } \partial\Omega\}.$$

In this case the space $H(\mathrm{curl},\Omega)$ is equipped with the norm

$$\|u\|_{H(\mathrm{curl},\Omega)} := \left(\|u\|^2_{L^2(\Omega)^3} + \|\nabla \times u\|^2_{L^2(\Omega)^3}\right)^{\frac{1}{2}}.$$

From this construction it follows immediately that also the pairing $(H(\mathrm{curl},\Omega), (\cdot,\cdot)_{\mathrm{curl}})$ with

$$(u,v)_{\mathrm{curl}} := (u,v) + (\nabla \times u, \nabla \times v)$$

induces a Hilbert space.
If $\Omega \subset \mathbb{R}^3$ and $\partial\Omega$ is Lipschitz-continuous, then there exist two important decompositions of the space $H(\mathrm{curl},\Omega)$, which split each function from $H(\mathrm{curl},\Omega)$ into a divergence-free part and a gradient of some scalar potential. The first decomposition is called *Helmholtz decomposition.* Its proof can be found in [105] and the reference therein.

Theorem 1.4 (Helmholtz Decomposition)**.** *Let $\Omega \subset \mathbb{R}^3$ be bounded and connected with Lipschitz-continuous boundary and $u \in H_0(\mathrm{curl},\Omega)$. Then there exist $z \in H_0(\mathrm{curl},\Omega)$ satisfying*

$$\mathrm{div}(z) = 0 \qquad \text{in } \Omega \tag{1.2}$$

and $q \in H^1(\Omega)$ being constant on every connected part of $\partial\Omega$ such that

$$u = z + \nabla q \qquad \text{in } \Omega \tag{1.3}$$

and

$$\|z\|_{H(\mathrm{curl},\Omega)} + \|\nabla q\|_{L^2(\Omega)^3} \leq \|u\|_{H(\mathrm{curl},\Omega)}.$$

Proof. See Theorem 1.2.3 in [105]. □

In [14] Amrouche, Bernardi, Dauge and Girault showed that the space $H_0(\mathrm{curl},\Omega) \cap H(\mathrm{div},\Omega)$ is continuously embedded in $H^1(\Omega)$ for Ω convex or $\partial\Omega$ of class $C^{1,1}$. This result was extended to a direct splitting of $H_0(\mathrm{curl},\Omega) \cap H(\mathrm{div},\Omega)$ in [88]. This splitting is called *regular decomposition.*

Theorem 1.5 (Regular Decomposition)**.** *Let $\Omega \subset \mathbb{R}^3$ be bounded and connected with connected, Lipschitz-continuous boundary and $u \in H_0(\mathrm{curl},\Omega) \cap H(\mathrm{div},\Omega)$. Then there exist $z \in H_0(\mathrm{curl},\Omega) \cap H^1(\Omega)^3$ satisfying (1.2) and $q \in H^1_0(\Omega)$ such that decomposition (1.3) holds. Further there exists some constant $C_{reg} > 0$ such that*

$$\|z\|_{H^1(\Omega)^3} + \|\nabla q\|_{L^2(\Omega)^3} \leq C_{reg}\|u\|_{H(\mathrm{curl},\Omega)}.$$

Proof. See Theorem 3.4 in [88]. □

With this result we conclude the section on Sobolev spaces. Although these few facts are not even the tip of the iceberg, we have brought together all statements which we need throughout this work. For a more detailed insight we refer the interested reader to the monographs of Adams and Fournier [2] and Monk [163].

1.3 The de Rham Complex

In the previous sections we have introduced all spaces we need. Now we want to see how they interact with each other. For this purpose the *de Rham complex* is a very powerful tool. It was introduced by de Rham [95] in 1931 for the analyis of smooth manifolds and has become a standard tool to express the various properties of mixed finite element approximations (see e.g. [65]). Usually the theory of the de Rham complex is closely linked to differential forms. However, for simplicity we do not derive the results of this section in the context of differential forms, but prove the relations rather directly. For that reason we refer the interested reader to the monographs of Abraham and Marsden [1] and Choquet-Bruhat [80] for more information about the de Rham complex and differential forms.
We observe easily that for $u \in H^1(\Omega)$ it holds

$$\nabla u = 0 \qquad \text{in } \Omega, \tag{1.4}$$

if and only if u is piecewise constant. Then it follows immediately that for $u \in H_0^1(\Omega)$ equation (1.4) holds, if and only if $u = 0$. Hence we have shown the following result:

Lemma 1.1 (Kernel of ∇)**.** *For $\nabla : H^1(\Omega) \to L^2(\Omega)^d$ and $\nabla : H_0^1(\Omega) \to L^2(\Omega)^d$ it holds*

$$\ker(\nabla) \cong \mathbb{R}$$

and

$$\ker(\nabla) \cong \{0\},$$

respectively.

A bit more involved is the relation between the spaces $H(\mathrm{curl}, \Omega)$ and $H^1(\Omega)$. This is partially investigated in the next theorem, whose proof can be found in [121].

Theorem 1.6 (Kernel of $\nabla\times$)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be bounded and simply-connected with Lipschitz-continuous boundary.*

1. *Let $d = 2$. For $\nabla\times : H(\mathrm{curl}, \Omega) \to L^2(\Omega)$ and $\nabla\times : H_0(\mathrm{curl}, \Omega) \to L^2(\Omega)$ it holds*

   $$\ker(\nabla\times) = \nabla H^1(\Omega) \tag{1.5}$$

   and

   $$\ker(\nabla\times) = \nabla H_0^1(\Omega), \tag{1.6}$$

   respectively. For $\nabla\times : H^1(\Omega) \to H(\mathrm{div}, \Omega)$ and $\nabla\times : H_0^1(\Omega) \to H_0(\mathrm{div}, \Omega)$ it holds

   $$\ker(\nabla\times) \cong \mathbb{R}$$

   and

   $$\ker(\nabla\times) \cong \{0\},$$

   respectively.

2. *Let $d = 3$. For $\nabla\times : H(\mathrm{curl}, \Omega) \to L^2(\Omega)^3$ and $\nabla\times : H_0(\mathrm{curl}, \Omega) \to L^2(\Omega)^3$ it holds (1.5) and (1.6), respectively.*

Proof. See Theorem 2.9 in [121] and Lemma 1.1. □

This theorem can also be formulated for arbitrary open sets, but, since the finite element method requires even more rigorous assumptions on the domain than stated above, we do not give the general result here. It can be found in [92, 93].
Also between the spaces $H(\mathrm{div})$ and $H(\mathrm{curl})$ a similar result to the one above holds.

Theorem 1.7 (Kernel of div)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be bounded, connected and open.*

1. *Let $d = 2$. For $\mathrm{div} : H(\mathrm{div}, \Omega) \to L^2(\Omega)$ and $\mathrm{div} : H_0(\mathrm{div}, \Omega) \to L^2(\Omega)$ it holds*

$$\ker(\mathrm{div}) = \nabla \times H^1(\Omega)$$

and

$$\ker(\mathrm{div}) = \nabla \times H_0^1(\Omega),$$

respectively.

2. *Let $d = 3$. For $\mathrm{div} : H(\mathrm{div}, \Omega) \to L^2(\Omega)$ and $\mathrm{div} : H_0(\mathrm{div}, \Omega) \to L^2(\Omega)$ it holds*

$$\ker(\mathrm{div}) = \nabla \times H(\mathrm{curl}, \Omega)$$

and

$$\ker(\mathrm{div}) = \nabla \times H_0(\mathrm{curl}, \Omega),$$

respectively.

Proof.

1. See Theorem 3.6 in [121].

2. See Theorem 3.4 in [121].

□

This result can be generalized to multiply-connected sets and can be found in [92, 93], too.
Now we are missing only two simple results to state the de Rham complex. The following lemma determines the image of the operator div. It was shown in [219].

Lemma 1.2 (Range of div)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be bounded with Lipschitz-continuous boundary. For $\mathrm{div} : H(\mathrm{div}, \Omega) \to L^2(\Omega)$ and $\mathrm{div} : H_0(\mathrm{div}, \Omega) \to L_0^2(\Omega)$ it holds*

$$\mathrm{div}\,(H(\mathrm{div}, \Omega)) = L^2(\Omega)$$

and

$$\mathrm{div}\,(H_0(\mathrm{div}, \Omega)) = L_0^2(\Omega),$$

respectively.

Proof. See Lemma 3.15 in [219]. □

The next lemma determines the image of the operator $\nabla\times : H(\mathrm{curl}, \Omega) \to L^2(\Omega)$ in the case $d = 2$.

Lemma 1.3 (Range of $\nabla\times$)**.** *Let $\Omega \subset \mathbb{R}^2$ be bounded with Lipschitz-continuous boundary. For $\nabla\times : H(\mathrm{curl}, \Omega) \to L^2(\Omega)$ and $\nabla\times : H_0(\mathrm{curl}, \Omega) \to L_0^2(\Omega)$ it holds*

$$\nabla \times H(\mathrm{curl}, \Omega) = L^2(\Omega)$$

and

$$\nabla \times H_0(\mathrm{curl}, \Omega) = L_0^2(\Omega),$$

respectively.

Proof. Let $u \in H_0^1(\Omega)$ and $f \in L^2(\Omega)$ such that u solves the homogeneous Dirichlet boundary value problem

$$\int_\Omega (\nabla \times \phi)^T \nabla \times u = \int_\Omega \phi f \qquad \forall \phi \in H_0^1(\Omega).$$

Then it holds $\nabla \times u \in L^2(\Omega)^2$ and integration by parts yields

$$\int_\Omega \phi \nabla \times (\nabla \times u) = \int_\Omega \phi f \qquad \forall \phi \in H_0^1(\Omega).$$

This implies $\nabla \times (\nabla \times u) = f$ a.e. in Ω and, hence, $\nabla \times u \in H(\text{curl}, \Omega)$.
Now let $u \in H^1(\Omega)$ and $f \in L_0^2(\Omega)$ such that u solves the homogeneous Neumann boundary value problem

$$\int_\Omega (\nabla \times \phi)^T \nabla \times u = \int_\Omega \phi f \qquad \forall \phi \in H^1(\Omega).$$

Note that this problem is well-posed due to the ellipticity of the bilinear form

$$A(u, v) := \int_\Omega (\nabla \times u)^T \nabla \times v$$

for $u, v \in H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega)$ (see e.g. Section 3.4.2 in [219] for a detailed outline of the arguments). Then it holds $\nabla \times u \in L^2(\Omega)$ and as above integration by parts yields

$$\int_\Omega \phi \nabla \times (\nabla \times u) = \int_\Omega \phi f \qquad \forall \phi \in H^1(\Omega).$$

This implies $\nabla \times (\nabla \times u) = f$ a.e. in Ω and, hence, $\nabla \times u \in H_0(\text{curl}, \Omega)$. □

Before we now state the de Rham complex let us first introduce the notion of an *exact sequence.*

Definition 1.3 (Exact Sequence)**.** *For $n \in \mathbb{N}$ with $n \geq 2$ let $G_0, \dots, G_n$ be groups and $f_i : G_{i-1} \to G_i$, $i \in \{1, \dots, n\}$, be group homomorphisms. Then the sequence*

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n$$

is called exact*, if and only if it holds*

$$\text{im}(f_i) = \ker(f_{i+1})$$

for all $i \in \{1, \dots, n-1\}$.

For more information about sequences and its exactness we refer the interested reader to [201]. We go on and consider the de Rham complex, which gives us a nice overview of the spaces we have introduced above. Let the operator $I : \mathbb{R} \to H^1(\Omega)$ be defined by

$$Ic := x \mapsto c$$

for all $c \in \mathbb{R}$. Further we define the operator $\mathcal{O} : L^2(\Omega) \to \{0\} \subset \mathbb{R}$ by

$$\mathcal{O}u := 0$$

for all $u \in L^2(\Omega)$. Then it follows directly from Lemmas 1.1–1.3 and Theorems 1.6 and 1.7:

Theorem 1.8 (de Rham Complex)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be bounded, simply-connected and open with Lipschitz-continuous boundary.*

1. *Let $d = 2$. Then the de Rham complexes*

$$\mathbb{R} \xrightarrow{I} H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} L^2(\Omega) \xrightarrow{\mathcal{O}} \{0\},$$

$$\mathbb{R} \xrightarrow{I} H^1(\Omega) \xrightarrow{\nabla\times} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

   and

$$\{0\} \xrightarrow{I} H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} L_0^2(\Omega) \xrightarrow{\mathcal{O}} \{0\},$$

$$\{0\} \xrightarrow{I} H_0^1(\Omega) \xrightarrow{\nabla\times} H_0(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L_0^2(\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

   form exact sequences.

2. *Let $d = 3$. Then the de Rham complexes*

$$\mathbb{R} \xrightarrow{I} H^1(\Omega) \xrightarrow{\nabla} H(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

   and

$$\{0\} \xrightarrow{I} H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} H_0(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L_0^2(\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

   form exact sequences.

The occurence of two shorter sequences in the case $d = 2$ is a direct concequence of the presence of two different operators $\nabla\times$. Here one can see immediately that this case has not a physical origin, but rather has arisen from some mathematical simplification.
For less restrictive domains, e.g. only multiply-connected, the sequences from above need not to be exact anymore. Then the sequence is not called a complex but a *cohomology*. For more information about this topic we refer the reader to the monographs of Bossavit [57, 58] and Hiptmair [127].
We end this section with a note on Section 2.5, where we derive similar sequences for the finite dimensional approximation spaces of the Sobolev spaces from above. Then the two complexes are connected with the help of a special choice of interpolation operators such that a commuting diagram exists. For more information about the de Rham complexes from this section we refer the interested reader to the dissertation of Zaglmayr [219].

Chapter 2

The Finite Element Method

In this chaper we put everything, which has to do with the finite element method. First we motivate its general idea and present the basics of it. For more information on this topic we refer the interested reader to the excellent monograph of Brenner and Scott [64]. After that we introduce the finite element spaces, which we require for the finite-dimensional approximation of the Poisson and the Maxwell boundary value problem in the following chapters. In detail these are the discontinuous Galerkin elements, which define an L^2-conforming approximation space, the continuous Galerkin elements, which define an H^1-conforming approximation space, and the Nédélec elements, which define an $H(\mathrm{curl})$-conforming approximation space. In this work we restrict ourselves to quadrilaterals and hexahedrals. For other types of reference cells, e.g. triangles, tetrahedras and pyramids, we refer to the books of Demkowicz [97], Šolín, Segeth and Doležel [209] and the dissertation of Zaglmayr [219]. For the finite element spaces, which we introduce, we present a family of interpolation operators, which commute in a special sense. This commuting behaviour is reviewed in the context of the de Rham complex from Section 1.3. To conclude this chapter we motivate the use of adaptive finite elements and give an overview of the various kinds of adaptivity.

2.1 Basic Concepts

In this section we present the basic concepts of the finite element method. We begin with a simple model problem and derive its weak formulation. Here the Lax-Milgram theorem plays an essential role in proving the existence and uniqueness of a solution of the weak problem. Then we see how to get from the weak formulation to a discrete finite-dimensional problem, whose solution approximates the analytical solution of the weak problem. Here especially the choice of a correct finite-dimensional approximation space plays an important role. Another important topic is the triangulation of the domain on which the problem is posed, because this should be easily accessible to reduce the effort of the solution process. Note that the aim of this section is rather the introduction of various notions in the context of the finite element method than a rigorous mathematical development of the finite element method itself. For a mathematically more sophisticated introduction into the finite element method with various a priori error estimates for the approximation error we refer the interested reader to the monographs of Braess [60], Brenner and Scott [64], Ciarlet [81], Ern and Guermond [109] and Szabó and Babuška [203].
Let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be open and bounded with polygonal (in the case $d = 2$) or polyhedral (in the case $d = 3$), Lipschitz-continuous boundary. Then our model problem is given as: Find $u : \overline{\Omega} \to \mathbb{R}$ such

that

$$
(2.1) \qquad \begin{aligned} -\Delta u &= \widetilde{f} && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega, \end{aligned}
$$

where $\widetilde{f} : \Omega \to \mathbb{R}$ is some right-hand side function and $g : \partial\Omega \to \mathbb{R}$ denotes some boundary function. This problem is called *Poisson problem* and is probably one of the most analyzed partial differential equations. Whereas the first line is the actual partial differential equation, the second line prescribes the boundary conditions of the problem. In this case we have chosen simple *nonhomogeneous Dirichlet boundary conditions*, which already appeared in Section 1.2 in its homogeneous form. In Section 1.2 we have also seen that Dirichlet boundary conditions do not necessarily always have exactly the form as above, but can also prescribe the tangential or the normal components of a function only. This depends on the function space in which the solution u is sought. We will comment on this issue later, when we consider the different finite element spaces in detail. Another very popular choice of boundary conditions are the *Neumann boundary conditions.* In its most simple form they read

$$
\frac{du}{dn} = g \qquad \text{on } \partial\Omega.
$$

In contrast to Dirichlet boundary conditions, where the actual values of the solution u are prescribed, Neumann boundary conditions only prescribe the normal derivatives of the solution at the boundary. This can lead to the situation that the solution of the problem

$$
\begin{aligned} -\Delta u &= \widetilde{f} && \text{in } \Omega \\ \frac{du}{dn} &= g && \text{on } \partial\Omega \end{aligned}
$$

cannot be determined uniquely but only up to an arbitrary additive constant, because only derivatives of the solution occur in this formulation. However, since we do not make any use of Neumann boundary conditions in this work, we do not dig into this topic any further, but refer the interested reader to the book of Evans [110] for more information.

Thus let us return to problem (2.1). In general it is very cumbersome to deploy a (theoretical) finite element framework for problems with nonhomogeneous Dirichlet boundary conditions. Therefore a usual way to overcome this uncomfortableness is to assume that there exists a *lifting function* $u_g \in H^2(\Omega)$ such that $u_g = g$ on $\partial\Omega$. Then it suffices to consider the homogeneous Dirichlet boundary value problem to find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$
(2.2) \qquad \begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}
$$

with $f := \widetilde{f} + \Delta u_g$, because the solution of the original problem (2.1) can be obtained by adding up the lifting function u_g and the solution of problem (2.2).

In general it is rather involved to derive existence and uniqueness results for the *strong formulation* of problem (2.2). We refer to the monograph of Gilbarg and Trudinger [120] for a detailed coverage of this topic. Here we go in another direction. We multiply the first equation of problem (2.2) with a test function $\phi \in H_0^1(\Omega)$ and integrate over Ω to obtain

$$
-\int_\Omega \phi \Delta u = \int_\Omega \phi f \qquad \forall \phi \in H_0^1(\Omega).
$$

Then integration by parts and using the homogeneous Dirichlet boundary conditions yields the *weak formulation*

$$\int_\Omega (\nabla\phi)^T \nabla u = \int_\Omega \phi f \qquad \forall \phi \in H_0^1(\Omega) \tag{2.3}$$

of problem (2.2). Note that this approach is closely linked to the topic of distributions and weak derivatives from Section 1.2. Especially the derivatives appearing in (2.3) do not have to hold in a strong sense anymore, but in distributional sense only. This reduces the assumptions on the solution u from two-times differentiable to one-time weakly differentiable.

Of course, now the question arises how the solutions of (2.2) and (2.3) are related to each other. The answer was given in the following theorem in [64].

Theorem 2.1. *Let $f \in C(\Omega)$ and $u \in C^2(\Omega)$ be a solution of (2.3). Then u is a solution of (2.2).*

Proof. For the case $d = 1$ and $\Omega = (0, 1)$ see Theorem 0.1.4 in [64].
For the cases $d \in \{2, 3\}$ the proof is almost the same and, thus, we do not repeat it here. □

This result is very important, because it makes sure that we really find practically relevant solutions of (2.2) by solving (2.3). However we still do not know, if there exists a solution of (2.3) at all. Therefore let us state the following remarkable result of Lax and Milgram [146] from 1954.

Theorem 2.2 (Lax-Milgram Theorem)**.** *Let $(H, (\cdot,\cdot))$ be a Hilbert space, $a : H \times H \to \mathbb{R}$ be a continuous, elliptic bilinear form and $F : H \to \mathbb{R}$ be continuous and linear. Then there exists a unique $u \in H$ such that*

$$a(u, v) = F(v) \qquad \forall v \in H.$$

Proof. See Theorem 2.1 in [146]. □

Although there have been proven more general versions of this theorem by Babuška (see [21]) and Lions (see e.g. [92, 93]), we stay with the original result from above, since it suffices for our needs.

Hence, from the Lax-Milgram Theorem we know that there exists a unique solution $u \in H_0^1(\Omega)$ of problem (2.3). This is the last step to ensure that it really makes sense to consider weak problem (2.3) instead of the original model problem (2.2).

However it still is not a trivial task to find the analytic solution of problem (2.3). That is why we want to approximate this solution numerically. But, since the space $H_0^1(\Omega)$ is not finite-dimensional, it is impossible to actually construct a solution from a basis of $H_0^1(\Omega)$. The finite element method tackles this problem by using a finite-dimensional subspace $V(\Omega)$ of $H_0^1(\Omega)$, which consists of piecewise polynomials. Then the solution can be computed easily by choosing a suitable basis of $V(\Omega)$. This approach is called *Ritz-Galerkin approximation* and was introduced by Ritz [184] in 1909 and Galerkin [114] in 1915. By replacing $H_0^1(\Omega)$ by $V(\Omega)$ in weak problem (2.3) we arrive at the *discrete problem* to find $u_{\mathrm{FE}} \in V(\Omega)$ such that

$$\int_\Omega (\nabla\phi)^T \nabla u_{\mathrm{FE}} = \int_\Omega \phi f \qquad \forall \phi \in V(\Omega). \tag{2.4}$$

Since $V(\Omega)$ is a subspace of $H_0^1(\Omega)$ and, thus, also a Hilbert space, it follows immediately from the Lax-Milgram Theorem that also discrete problem (2.4) has a unique solution $u_{\mathrm{FE}} \in V(\Omega)$.

Now we are left with the question on how to construct such a space $V(\Omega)$ in a clever way. There have been developed many different approaches to do so, e.g. the finite element method, isogeometric analysis [90] and mesh free methods [149]. The finite element method defines a mesh, which covers Ω, and –

depending on this mesh – the finite-dimensional space $V(\Omega)$ is constructed. The *mesh* or *triangulation* $\mathcal{K}$ of Ω is an indexed collection of closed sets K_i, $i \in \{0, \dots, N\}$ for some $N \in \mathbb{N}$, with $\mathrm{meas}_d(K_i \cap K_j) = 0$ for $i \neq j$ such that

$$\overline{\Omega} = \bigcup_{i=1}^{N} K_i. \tag{2.5}$$

These sets are usually called *cells* and can be obtained as the image of a *reference cell* (see Figure 2.1). Of course one wants to choose the reference cell as simple as possible. That is why the usual

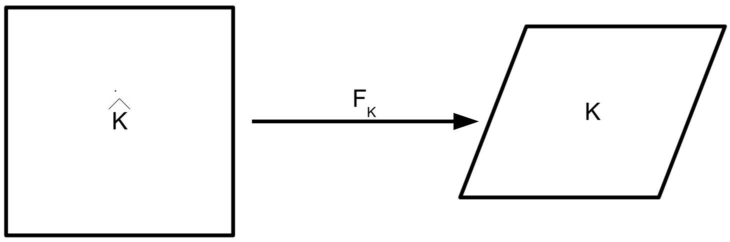

Figure 2.1: Mapping $F_K : \widehat{K} \to K$

choices are intervals in the case $d = 1$, quadrilaterals and triangles in the case $d = 2$ (see Figure 2.2) and hexahedrals, tetrahedrons, prisms and pyramids in the case $d = 3$ (see Figure 2.3). For a rigorous mathematical definition of these geometrical objects we refer to Section 2.2. Note that due to our

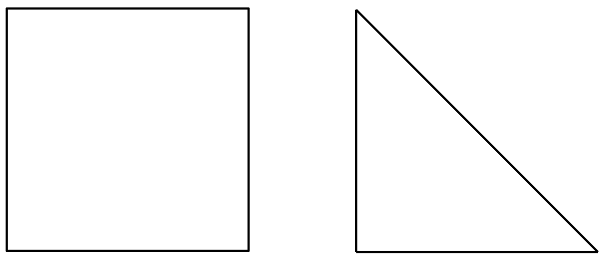

Figure 2.2: Left: Quadrilateral. Right: Triangle

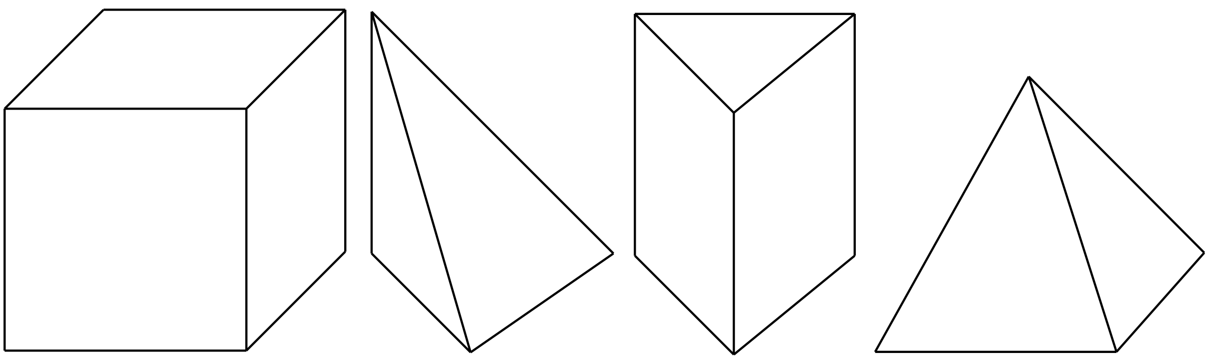

Figure 2.3: Outer left: Hexahedral. Center left: Triangle. Center right: Prism. Outer right: Pyramid

assumption on the boundary of Ω it is guaranteed that Ω can be triangulated. However, it can be quite involved to find such a mapping $F_K : \widehat{K} \to K$ such that (2.5) holds. To face this problem one can for example include some information about the boundary into the polynomial space. For more information about this topic we refer to [202].
Now we can choose an arbitrary polynomial space and represent its basis on the reference cell. Usually the basis elements are associated with a point in the cell, an edge, a face or the interior of the cell. This association is done by a linear functional. All the terms introduced above can be grouped together to the notion of a *finite element*.

Definition 2.1 (Finite Element). *Let*

1. *$\widehat{K} \subset \mathbb{R}^d$ be compact and connected with Lipschitz-continuous boundary and $\widehat{K} \neq \emptyset$.*
2. *P be a vector space of functions p with* $\operatorname{dom}(p) = \widehat{K}$.
3. *$\Sigma := \{N_0, \dots, N_{\dim(P)-1}\}$ be a set of linear functionals $N_i : P \to \mathbb{R}$, $i \in \{0, \dots, \dim(P) - 1\}$, such that Σ is a basis of the space of linear functionals $\mathcal{L}(P, \mathbb{R})$.*

Then the triplet $\left(\widehat{K}, P, \Sigma\right)$ is called a finite element. *The linear functionals N_i, $i \in \{0, \dots, \dim(P) - 1\}$, are called* degrees of freedom.

With this definition we see easily that for some $n \in \mathbb{N}$ the solution u_{FE} of problem (2.4) can be written in the form

$$u_{\mathrm{FE}} = \sum_{i=0}^{n} u_i \phi_i,$$

where $u_0, \dots, u_n \in \mathbb{R}$ and $\{\phi_i\}_{i \in \{0,\dots,n\}}$ denotes a basis of $V(\Omega)$. With this ansatz and the fact that it suffices to consider test problem (2.4) with all basis functions of $V(\Omega)$, problem (2.4) reads

$$\sum_{i=0}^{n} u_i \int_{\Omega} (\nabla \phi_j)^T \nabla \phi_i = \int_{\Omega} \phi_j f \qquad \forall j \in \{0, \dots, n\}.$$

But this is just a linear system of equations and can be solved by using standard techniques.
Now we have introduced all the vocabulary, which we will use throughout the following chapters extensively, and presented some ideas behind the finite element method. Let us conclude this section with a short remark on the history of the finite element method. It first appeared in works of Schellbach [190] in 1851, Trefftz [205] in 1926, Hrennikoff [137] in 1941 and Courant [91] in 1943 and, then, became more and more popular quite fast. But not earlier than in 1960 the finite element method got its name from Clough, who introduced this notion in [83]. A more detailed historical overview can be found in [172]. For more information about the basics of the finite element method we refer the interested reader to the references, which we already mentioned in the beginning of this section.

2.2 Finite Element Spaces

In this section we introduce the finite element spaces, which we use in the following chapters. In detail these are L^2-, H^1- and $H(\mathrm{curl})$-conforming finite element spaces. Further we present for each space a set of finite elements, which satisfies its continuity requirements, respectively. Therefore a rigorous definition of the reference cells is necessary to fix some notations. However, here we restrict ourselves to quadrilaterals in the case $d = 2$ and hexahedra in the case $d = 3$, because the finite element library deal.II

[41, 42], which we are using, currently only supports these types of reference cells. For other types we refer the interested reader to the books of Demkowicz [97], Demkowicz, Kurtz, Pardo, Paszyński, Rachowicz and Zdunek [100] and Šolín, Segeth and Doležel [209] and the dissertation of Zaglmayr [219]. For the construction of the finite elements we also need some orthogonal polynomials, which are introduced in this section, too.

2.2.1 The Reference Cells

In this subsection we define the reference cells. The reference cells are the geometrical basis to constuct the shape functions and degrees of freedom later on. For the reference cells we basically use the definitions from the finite element library deal.II [41, 42]. As already mentioned above deal.II currently supports only quadrilaterals and hexahedra. Thus we can define the reference cell $\widehat{K} \in \mathbb{R}^d$ by

$$\widehat{K} := [0,1]^d$$

for $d \in \{1,2,3\}$. Now we consider these cells in more detail and determine which geometrical subobjects, e.g. vertices, edges, faces, are present in each cell.
Let us begin with the case $d = 1$. Here the only subdimensional objects, which the reference cell $\widehat{K} = [0,1]$ has, are two vertices $v_0, v_1 \in \mathbb{R}$. These are given by

$$v_0 := 0 \qquad \text{and} \quad v_1 := 1.$$

In Figure 2.4 on the left-hand side $\widehat{K}$ is plotted.
Next we consider the case $d = 2$. Here the cell is given by $\widehat{K} = [0,1]^2$. It has four vertices $v_0, \dots, v_3 \in \mathbb{R}^2$

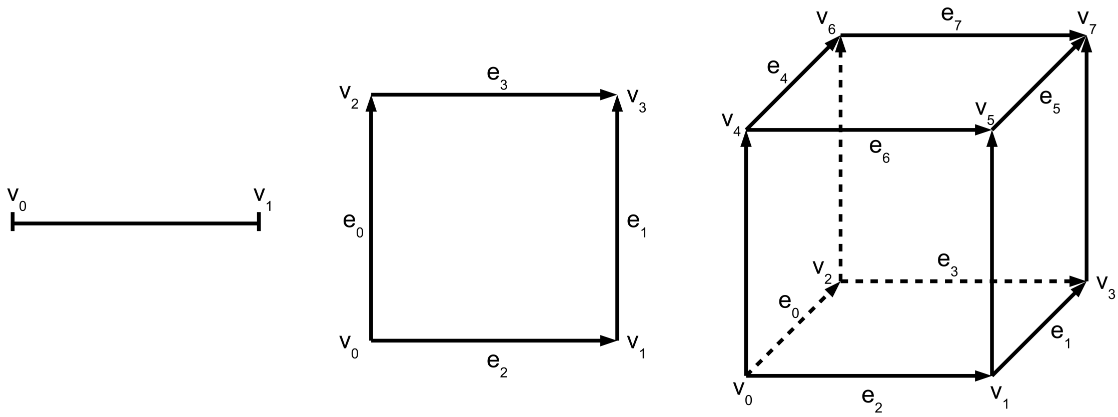

Figure 2.4: Reference cells. Left: $d = 1$. Center: $d = 2$. Right: $d = 3$.

and four edges $e_0, \dots, e_3$. The vertices are defined as follows:

$$v_0 := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad v_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and the edges are given by

$$e_0 := [v_0, v_2], \quad e_1 := [v_1, v_3], \quad e_2 := [v_0, v_1], \quad e_3 := [v_2, v_3].$$

This reference cell is illustrated in the center of Figure 2.4.
The final case is $d = 3$. In this case the reference cell $\widehat{K} = [0,1]^3$ has 8 vertices $v_0, \ldots, v_7 \in \mathbb{R}^3$, 12 edges $e_0, \ldots, e_{11}$ and six faces $f_0, \ldots, f_5$. The vertices are defined as follows:

$$v_0 := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$
$$v_4 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_5 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_6 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_7 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Then the edges are given by

$$\begin{aligned} &e_0 := [v_0, v_2], \quad e_1 := [v_1, v_3], \quad e_2 := [v_0, v_1], \quad e_3 := [v_2, v_3], \quad e_4 := [v_4, v_6], \quad e_5 := [v_5, v_7], \\ &e_6 := [v_5, v_6], \quad e_7 := [v_7, v_8], \quad e_8 := [v_0, v_4], \quad e_9 := [v_1, v_5], \quad e_{10} := [v_2, v_6], \quad e_{11} := [v_3, v_7], \end{aligned}$$

and the faces can be defined as follows:

$$\begin{aligned} &f_0 := [v_0, v_2, v_4, v_6], \quad f_1 := [v_1, v_3, v_5, v_7], \quad f_2 := [v_0, v_1, v_4, v_5], \\ &f_3 := [v_2, v_3, v_6, v_7], \quad f_4 := [v_0, v_1, v_2, v_3], \quad f_5 := [v_4, v_5, v_6, v_7]. \end{aligned}$$

For this case a graphical representation is available in Figure 2.4 on the right-hand side. With these definitions we have clarified the orientations and positions of vertices, edges and faces in the reference cell. For other types of reference cells, e.g. triangles, tetrahedra, prisms and pyramids, we refer to [97, 100, 209, 219].

2.2.2 Polynomials with Orthonormality Relations

In this subsection we present the families of polynomials, which we use in the construction of the finite elements in the following subsections. These are the Legendre polynomials and the integrated Legendre polynomials. For more information about orthogonal polynomials we refer to the standard textbook of Szegö [204].
Let us start with the family of *Legendre polynomials*. Legendre polynomials are a special case of Jacobi polynomials with certain weights chosen to be zero (cf. [204]). Usually they are defined on the interval $[-1, 1]$, but, since our reference cells are given by $[0, 1]^d$, we also define the polynomials on the interval $[0, 1]$.

Definition 2.2 (Legendre Polynomials)**.** *The* Legendre polynomials $L_0, L_1, \ldots : [0,1] \to \mathbb{R}$ *are defined by the recursion formula*

$$\begin{aligned} L_0(x) &:= 1, \\ L_1(x) &:= \sqrt{3}(2x-1), \\ L_n(x) &:= \frac{\sqrt{2n+1}}{n}\left(\sqrt{2n-1}(2x-1)L_{n-1}(x) + \frac{1-n}{\sqrt{2n-3}}L_{n-2}(x)\right) \qquad \text{for } n \geq 2. \end{aligned}$$

For a graphical representation of the first six Legendre polynomials $L_0, \ldots, L_5$ see Figure 2.5. Another possible way of defining the Legendre polynomials is the differential relation

$$L_n(x) = \frac{\sqrt{2n+1}}{n!}\frac{d^n}{dx^n}\left(x^n(x-1)^n\right)$$

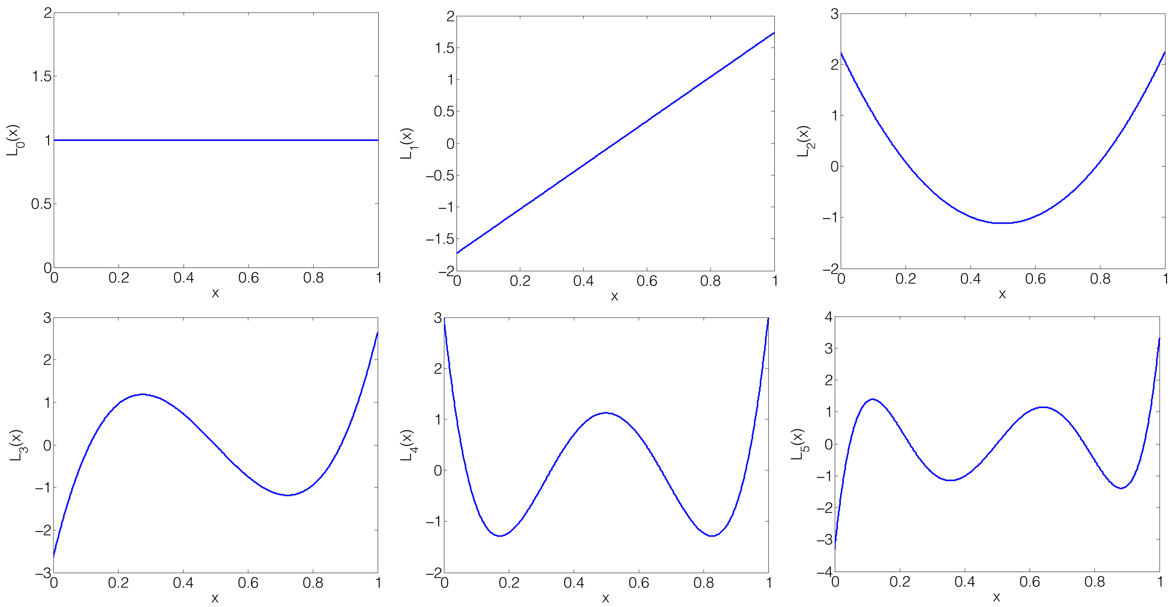

Figure 2.5: Legendre polynomials. Upper left: L_0. Upper center: L_1. Upper right: L_2. Lower left: L_3. Lower center: L_4. Lower right: L_5.

for $n \in \mathbb{N}_0$. These polynomials satisfy $L_n(0) = (-1)^n\sqrt{2n+1}$ and $L_n(1) = \sqrt{2n+1}$ for all $n \in \mathbb{N}_0$. Further it can be shown that the set $\{L_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis of $L^2([0,1])$, i.e.

$$\int_0^1 L_i L_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}. \tag{2.6}$$

The second family of polynomials we introduce are the *integrated Legendre polynomials*. Sometimes they are also called *Lobatto polynomials*.

Definition 2.3 (Integrated Legendre Polynomials)**.** *The* integrated Legendre polynomials $l_0, l_1, \ldots : [0,1] \to \mathbb{R}$ *are defined by*

$$\begin{aligned} l_0(x) &:= 1 - x, \\ l_1(x) &:= x, \\ l_n(x) &:= \int_0^x L_{n-1}(t)\,dt \qquad \textit{for } n \geq 2. \end{aligned}$$

In Figure 2.6 we plot the first six integrated Legendre polynomials $l_0, \ldots, l_5$. Obviously it holds $l_n(0) = 0$ for $n \in \mathbb{N}$. Further we have $l_n(1) = 0$ for $n \in \mathbb{N}_0 \setminus \{1\}$, since the Legendre polynomials $L_1, L_2, \ldots$ are L^2-orthogonal to L_0. It can be proven that for all $p \in \mathbb{N}_0$ the set $\{l_0, \ldots, l_p\}$ is a basis of $P_p([0,1])$, where

$$P_p([0,1]) := \operatorname{span}\left\{x^i : x \in [0,1], 0 \leq i \leq p\right\}$$

denotes the space of all polynomials of degree less or equal than p on the interval $[0,1]$. Although the integrated Legendre polynomials itself are not L^2-orthogonal, they satisfy an orthonormality relation, which is very important for the finite element method. Since there one often has to deal with the

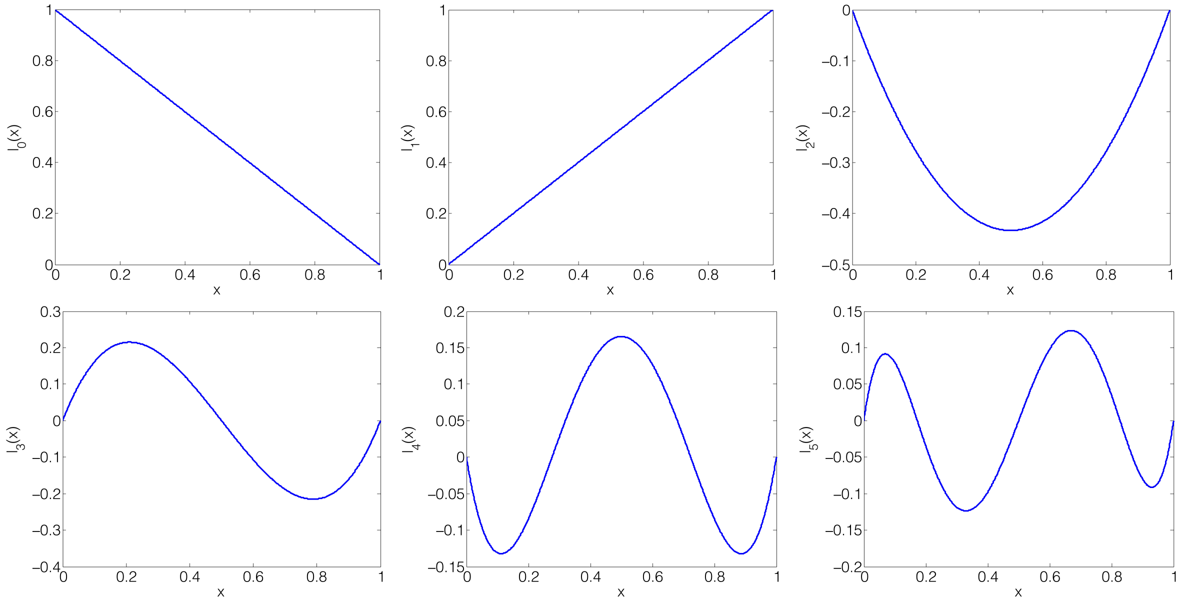

Figure 2.6: Integrated Legendre polynomials. Upper left: l_0. Upper center: l_1. Upper right: l_2. Lower left: l_3. Lower center: l_4. Lower right: l_5.

derivatives of polynomials, we emphasize the orthonormality relation

$$\int_0^1 l_i' l_j' = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases},$$

which immediately follows from the L^2-orthonormality of the Legendre polynomials (2.6).
These two families of polynomials with orthonormality relations suffice to define the shape functions of the finite elements we want to consider. Therefore we refer the interested reader to the monograph of Szegö [204] for more information about orthogonal polynomials in general and to the book of Šolín, Segeth and Doležel [209] and the dissertation of Zaglmayr [219] for more information about Legendre and integrated Legendre polynomials (on the interval $[-1, 1]$).

2.2.3 The L^2-Conforming Finite Element Space

In this subsection we introduce the L^2-conforming finite element space. These finite elements are commonly known as *discontinuous Galerkin elements*. As it can already be derived from its name this type of elements does not satisfy any continuity requirements across the boundaries of the cell. For its construction we follow the ideas of a hierarchical finite element of Šolín, Segeth and Doležel [209]. For nodal approaches we refer to the monograph of Kanschat [138]. Note that we do not support polynomial anisotropy here, because the finite element library deal.II [41, 42], which we are using, currently does not support this. Therefore we refer to [209] for more information about this topic.
Let $\widehat{K}$ be the reference cell and $p \in \mathbb{N}_0$ be arbitrary. Then we can define the polynomial space $P_p\left(\widehat{K}\right)$

by

$$(2.7) \qquad P_p\left(\widehat{K}\right) := \begin{cases} \mathrm{span}\{x^i : x \in [0,1], 0 \le i \le p\}, & \text{if } \widehat{K} = [0,1] \\ \mathrm{span}\{x_1^i x_2^j : x \in [0,1]^2, 0 \le i,j \le p\}, & \text{if } \widehat{K} = [0,1]^2 \\ \mathrm{span}\{x_1^i x_2^j x_3^k : x \in [0,1]^3, 0 \le i,j,k \le p\}, & \text{if } \widehat{K} = [0,1]^3 \end{cases}.$$

The next step is to contruct a basis for the polynomial space $P_p\left(\widehat{K}\right)$. Let us begin with the case $d = 1$. We set $B_1 := \{\phi_i\}_{i\in\{0,\dots,p\}}$, where the *shape functions* ϕ_i are just the Legendre polynomials:

$$\phi_i(x) := L_i(x)$$

for $i \in \{0,\dots,p\}$. Then it can be shown that B_1 is a basis of the polynomial space $P_p([0,1])$.

Lemma 2.1 (Basis ($d = 1$))**.** *For all $p \in \mathbb{N}_0$ the set B_1 is a basis of $P_p([0,1])$.*

Proof. It holds $\dim(P_p([0,1])) = p+1$ and $\phi_0,\dots,\phi_p \in P_p([0,1])$ are linearly independent. □

For the case $d = 2$ we choose the tensor-product of the basis in the one-dimensional case to construct a basis for $P_p\left([0,1]^2\right)$. Let $B_2 := \{\phi_{i,j}\}_{i,j\in\{0,\dots,p\}}$, where we set

$$\phi_{i,j}(x) := L_i(x_1)L_j(x_2)$$

for $i,j \in \{0,\dots,p\}$. Then it can be shown analogously to the case $d = 1$ that B_2 is a basis of $P_p\left([0,1]^2\right)$.

Lemma 2.2 (Basis ($d = 2$))**.** *For all $p \in \mathbb{N}_0$ the set B_2 is a basis of $P_p\left([0,1]^2\right)$.*

In the last case $d = 3$ again a tensor-product construction is employed. For $i,j,k \in \{0,\dots,p\}$ set

$$\phi_{i,j,k}(x) := L_i(x_1)L_j(x_2)L_k(x_3).$$

Then a basis of the polynomial space $P_p\left([0,1]^3\right)$ is given by $B_3 := \{\phi_{i,j,k}\}_{i,j,k\in\{0,\dots,p\}}$.

Lemma 2.3 (Basis ($d = 3$))**.** *For all $p \in \mathbb{N}_0$ the set B_3 is a basis of $P_p\left([0,1]^3\right)$.*

To complete the definition of the finite element we are only missing the degrees of freedom. Therefore we follow the spirit of Demkowicz [97] and define the linear functional $N_i : P_p\left(\widehat{K}\right) \to \mathbb{R}$ by

$$N_i(\phi) := \int_{\widehat{K}} \phi\phi_i \qquad \forall \phi \in P_p\left(\widehat{K}\right)$$

for $i \in \{0,\dots,\dim(B_d)-1\}$, where ϕ_i is the i-th element of basis B_d. Now we have fixed all components and can define the L^2-conforming finite element.

Definition 2.4 (Discontinuous Galerkin Element)**.** *The* discontinuous Galerkin element *is defined by the triplet $\left(\widehat{K}, P_p\left(\widehat{K}\right), \Sigma\right)$, where Σ is given by*

$$\Sigma := \left\{N_0,\dots,N_{\dim(B_d)-1}\right\}.$$

The number of shape functions and, hence, the number of degrees of freedom of this finite element can be computed easily. We see:

Remark 2.1 (Number of Shape Functions)**.** *The discontinuous Galerkin element of degree p has*

$$\dim (B_d) = (p+1)^d$$

shape functions.

Let $\mathcal{K}$ be some triangulation of Ω. For $K \in \mathcal{K}$ we denote the mapping from reference cell $\widehat{K}$ to K by $F_K : \widehat{K} \to K$. Further let $p := (p_K)_{K\in\mathcal{K}}$, $p_K \in \mathbb{N}_0$, be the polynomial degree vector of triangulation $\mathcal{K}$. Then we define the finite element space $U^p(\mathcal{K}, \Omega)$ of L^2-conforming elements by

$$U^p(\mathcal{K}, \Omega) := \left\{ u \in L^2(\Omega) : u|_K \circ F_K \in P_{p_K}\left(\widehat{K}\right) \ \forall K \in \mathcal{K} \right\}.$$

With this definition we conclude this subsection about the discontinuous Galerkin element. For more information we refer to the books of Cockburn, Karniadakis and Shu [84] and Kanschat [138]. In [84] one can also find discontinuous Galerkin elements for other types of reference cells, e.g. triangles and tetrahedra. Therefore we may now assume that the L^2-conforming finite element space $U^p(\mathcal{K}, \Omega)$ can be constructed with triangulations $\mathcal{K}$, which consist of any of these reference cells.

2.2.4 The H^1-Conforming Finite Element Space

The H^1-conforming finite elements are well studied. Hence there have been proposed various approaches for defining shape functions and finite elements, which are continuous across the boundaries of the cell. In a small overview on the literature we name [9, 22, 35, 60, 64, 65, 81, 97, 100, 109, 140, 194, 203, 209, 219]. In this work we follow the approach of Šolín, Segeth and Doležel [209] and construct a hierarchical finite element. Again, polynomial anisotropy is excluded and we refer the interested reader to [209, 219] for more information about this topic. For convenience we group the H^1-conforming shape functions and degrees of freedom in a geometrical meaning, i.e. we speak of vertex, edge, face and interior shape functions and vertex, edge, face and interior degrees of freedom, respectively, if they can be associated with the corresponding geometrical object of the cell (and if $d \in \{1, 2, 3\}$ is big enough such that these geometrical objects exist). The H^1-conforming finite elements provide continuity across cell boundaries and, thus, are often called *continuous Galerkin elements* in contrast to the discontinuous Galerkin elements from the previous subsection.
For $p \in \mathbb{N}$ the polynomial space $P_p\left(\widehat{K}\right)$ is given by (2.7) again. However we have to constuct new bases, since the shape functions from B_1, B_2 and B_3 are not continuous across cell boundaries and, thus, cannot be used here.

Example. *Let $d = 1$ and $\mathcal{K} := \{[0, 1], [1, 2]\}$. Then the lowest order shape functions $\phi_{K_0,0} := \phi_0 \circ F_{K_0}$ and $\phi_{K_1,0} := \phi_0 \circ F_{K_1}$ of the discontinuous Galerkin element on cells $K_0 := [0, 1]$ and $K_1 := [1, 2]$ are continuous across vertex 1, but shape functions $\phi_{K_0,1} := \phi_1 \circ F_{K_0}$ and $\phi_{K_1,1} := \phi_1 \circ F_{K_1}$ are not. See Figure 2.7 for a graphical representation.*

Let $d = 1$. Then two groups of shape functions are present on reference cell $\widehat{K} = [0, 1]$. Namely these are the vertex shape functions and the interior shape functions. Let us begin with the *vertex shape functions.* For vertices $v_0 = 0$ and $v_1 = 1$ these are defined by

$$\phi_{v_0}(x) := l_0(x) \qquad \text{and} \quad \phi_{v_1}(x) := l_1(x),$$

where l_0 and l_1 denote the first two integrated Legendre polynomials. The *interior shape functions* are just the higher-order integrated Legendre polynomials: For $i \in \{2, \dots, p\}$ we set

$$\phi_i(x) := l_i(x).$$

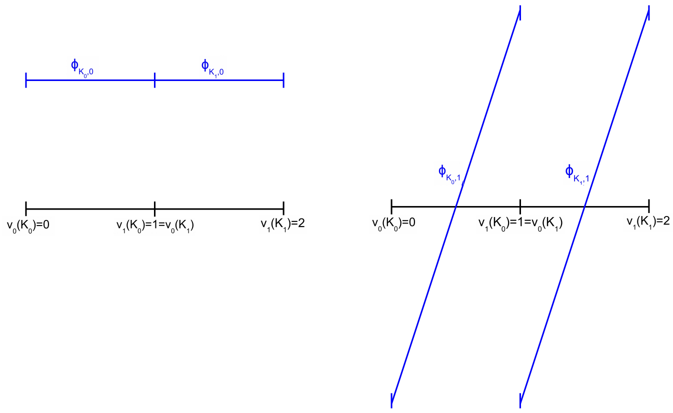

Figure 2.7: Example. Left: Shape functions $\phi_{K_0,0}, \phi_{K_1,0}$. Right: Shape functions $\phi_{K_0,1}, \phi_{K_1,1}$.

Then it can be shown that the set $\widetilde{B}_1 := \{\phi_{v_0}, \phi_{v_1}\} \cup \{\phi_i\}_{i\in\{2,\ldots,p\}}$ is a basis of $P_p\left([0,1]\right)$.

Lemma 2.4 (Basis $(d=1)$)**.** *For all $p \in \mathbb{N}$ the set $\widetilde{B}_1$ is a basis of $P_p\left([0,1]\right)$.*

Proof. See Proposition 2.1 in [209]. □

In the case $d = 2$ there are edge shape functions in addition to the vertex and interior shape functions, which are also present for $d = 1$. To define the shape functions for $d = 2$ we use a tensor-product structure similar to the one employed in Section 2.2.3. Let us start with the *vertex shape functions* again. These four shape functions are defined by

$$\phi_{v_0}(x) := l_0(x_1)l_0(x_2), \quad \phi_{v_1}(x) := l_1(x_1)l_0(x_2), \quad \phi_{v_2}(x) := l_0(x_1)l_1(x_2), \quad \phi_{v_3}(x) := l_1(x_1)l_1(x_2).$$

The next group of shape functions we consider is the group of *edge shape functions*. For $i \in \{2,\ldots,p\}$ these shape functions are given by

$$\phi_{e_0,i}(x) := l_0(x_1)l_i(x_2), \quad \phi_{e_1,i}(x) := l_1(x_1)l_i(x_2), \quad \phi_{e_2,i}(x) := l_i(x_1)l_0(x_2), \quad \phi_{e_3,i}(x) := l_i(x_1)l_1(x_2).$$

Of course, there are also some *interior shape functions*. These are defined by

$$\phi_{i,j}(x) := l_i(x_1)l_j(x_2)$$

for $i, j \in \{2,\ldots,p\}$. Then we can set $\widetilde{B}_2 := \{\phi_{v_0},\ldots,\phi_{v_3}\} \cup \{\phi_{e_0,i},\ldots,\phi_{e_3,i}\}_{i\in\{2,\ldots,p\}} \cup \{\phi_{i,j}\}_{i,j\in\{2,\ldots,p\}}$. In [209] it was shown that $\widetilde{B}_2$ is a basis of $P_p\left([0,1]^2\right)$.

Lemma 2.5 (Basis $(d=2)$)**.** *For all $p \in \mathbb{N}$ the set $\widetilde{B}_2$ is a basis of $P_p\left([0,1]^2\right)$.*

Proof. See Proposition 2.2 in [209]. □

Now we are left with the case $d = 3$. Here we have the full variety of shape functions, namely vertex, edge, face and interior shape functions. As in the case $d = 2$ we employ a tensor-product structure for the construction of these shape functions. Then the *vertex shape functions* are defined by

$$\begin{aligned}
&\phi_{v_0}(x) := l_0(x_1)l_0(x_2)l_0(x_3), && \phi_{v_1}(x) := l_1(x_1)l_0(x_2)l_0(x_3), && \phi_{v_2}(x) := l_0(x_1)l_1(x_2)l_0(x_3),\\
&\phi_{v_3}(x) := l_1(x_1)l_1(x_2)l_0(x_3), && \phi_{v_4}(x) := l_0(x_1)l_0(x_2)l_1(x_3), && \phi_{v_5}(x) := l_1(x_1)l_0(x_2)l_1(x_3),\\
&\phi_{v_6}(x) := l_0(x_1)l_1(x_2)l_1(x_3), && \phi_{v_7}(x) := l_1(x_1)l_1(x_2)l_1(x_3).
\end{aligned}$$

For $i \in \{2, \dots, p\}$ we denote the *edge shape functions* by

$$\begin{aligned}
&\phi_{e_0,i}(x) := l_0(x_1)l_i(x_2)l_0(x_3), \quad \phi_{e_1,i}(x) := l_1(x_1)l_i(x_2)l_0(x_3), \quad \phi_{e_2,i}(x) := l_i(x_1)l_0(x_2)l_0(x_3),\\
&\phi_{e_3,i}(x) := l_i(x_1)l_1(x_2)l_0(x_3), \quad \phi_{e_4,i}(x) := l_0(x_1)l_i(x_2)l_1(x_3), \quad \phi_{e_5,i}(x) := l_1(x_1)l_i(x_2)l_1(x_3),\\
&\phi_{e_6,i}(x) := l_i(x_1)l_0(x_2)l_1(x_3), \quad \phi_{e_7,i}(x) := l_i(x_1)l_1(x_2)l_1(x_3), \quad \phi_{e_8,i}(x) := l_0(x_1)l_0(x_2)l_i(x_3),\\
&\phi_{e_9,i}(x) := l_1(x_1)l_0(x_2)l_i(x_3), \quad \phi_{e_{10},i}(x) := l_0(x_1)l_1(x_2)l_i(x_3), \quad \phi_{e_{11},i}(x) := l_1(x_1)l_1(x_2)l_i(x_3).
\end{aligned}$$

Then we consider the group of *face shape functions*. These shape functions are given by

$$\begin{aligned}
&\phi_{f_0,i,j}(x) := l_0(x_1)l_i(x_2)l_j(x_3), \quad \phi_{f_1,i,j}(x) := l_1(x_1)l_i(x_2)l_j(x_3), \quad \phi_{f_2,i,j}(x) := l_i(x_1)l_0(x_2)l_j(x_3),\\
&\phi_{f_3,i,j}(x) := l_i(x_1)l_1(x_2)l_j(x_3), \quad \phi_{f_4,i,j}(x) := l_i(x_1)l_j(x_2)l_0(x_3), \quad \phi_{f_5,i,j}(x) := l_i(x_1)l_j(x_2)l_1(x_3)
\end{aligned}$$

for $i, j \in \{2, \dots, p\}$. Last but not least we also introduce the *interior shape functions* $\phi_{i,j,k}$ for $i, j, k \in \{2, \dots, p\}$ by

$$\phi_{i,j,k}(x) := l_i(x_1)l_j(x_2)l_k(x_3).$$

It can be shown that the set

$$\widetilde{B}_3 := \{\phi_{v_0}, \dots, \phi_{v_7}\} \cup \{\phi_{e_0,i}, \dots, \phi_{e_{11},i}\}_{i\in\{2,\dots,p\}} \cup \{\phi_{f_0,i,j}, \dots, \phi_{f_5,i,j}\}_{i,j\in\{2,\dots,p\}} \cup \{\phi_{i,j,k}\}_{i,j,k\in\{2,\dots,p\}}$$

is a basis of $P_p\left([0,1]^3\right)$.

Lemma 2.6 (Basis ($d = 3$))**.** *For all $p \in \mathbb{N}$ the set $\widetilde{B}_3$ is a basis of $P_p\left([0,1]^3\right)$.*

Proof. See Proposition 2.4 in [209]. □

We note that in this subsection the name *interior shape function* becomes clear: Interior shape functions vanish on $\partial\widehat{K}$, whereas vertex, edge and face shape functions do not. Further we see that only for higher orders ($p \geq 2$) there exist other shape functions than the vertex shape functions. With this element one can also see the advantages of the careful construction of the shape functions. In [225] it was shown that by using integrated Legendre polynomials the condition number of the finite element matrix

$$A := \left(\int_\Omega (\nabla\psi_i)^T \nabla\psi_j\right)_{i,j}$$

can be reduced significantly in comparison to many other families of polynomials.
To complete the definition of the finite element we have to construct some degrees of freedom. Again we make use of the geometrical grouping and define the degrees of freedom for each group of shape functions separately. For these definitions we follow the idea of Demkowicz [97]. We begin with the vertex shape functions. Here we define the associated degrees of freedom as follows: For $i \in \left\{0, 2^d - 1\right\}$ we set

$$N_{v_i}(\phi) := \phi(v_i) \qquad \forall\phi \in P_p\left(\widehat{K}\right).$$

For $d \in \{2, 3\}$ there might also exist some edge shape functions. The degrees of freedom $N_{e_i,j} : P_p\left(\widehat{K}\right) \to \mathbb{R}$ corresponding to these shape functions are defined by

$$N_{e_i,j}(\phi) := \int_{e_i} \frac{d\phi}{ds}\frac{dl_j}{ds} \qquad \forall\phi \in P_p\left(\widehat{K}\right),$$

where s denotes the parametrization of edge e_i, $i \in \{0, \dots, 3\}$ in the case $d = 2$ and $i \in \{0, \dots, 11\}$ in the case $d = 3$, and $j \in \{2, \dots, p\}$. Only in the case $d = 3$ face shape functions may occur. The face degrees of freedom $N_{f_i,j,k} : P_p\left(\widehat{K}\right) \to \mathbb{R}$ for $i \in \{0, \dots, 5\}$ and $j, k \in \{2, \dots, p\}$ are given by

$$N_{f_i,j,k}(\phi) := \int_{f_i} \left((n_{f_i} \times \nabla\phi|_{f_i}) \times n_{f_i}\right)^T (n_{f_i} \times \nabla\phi_{f_i,j,k}|_{f_i}) \times n_{f_i} \qquad \forall \phi \in P_p\left(\widehat{K}\right),$$

where n_{f_i} denotes the outward-pointing unit normal vector to reference cell $\widehat{K}$ on face f_i. Interior shape functions may occur for every $d \in \{1, 2, 3\}$. The associated degrees of freedom $N_i : P_p\left(\widehat{K}\right) \to \mathbb{R}$ are defined by

$$N_i(\phi) := \int_{\widehat{K}} (\nabla\phi)^T \nabla\phi_i \qquad \forall \phi \in P_p\left(\widehat{K}\right),$$

where for $i \in \{0, \dots, (p-1)^d - 1\}$ the i-th interior shape function is denoted by ϕ_i. Now we have presented all necessary definitions to give a rigorous introduction of the H^1-conforming finite element.

Definition 2.5 (Continuous Galerkin Element)**.** *The* continuous Galerkin Element *is defined by the triplet* $\left(\widehat{K}, P_p\left(\widehat{K}\right), \Sigma\right)$*, where* Σ *is given by*

$$\Sigma := \{N_{v_0}, N_{v_1}\} \cup \{N_i\}_{i\in\{2,\dots,p\}}$$

in the case $d = 1$*,*

$$\Sigma := \{N_{v_0}, \dots, N_{v_3}\} \cup \{N_{e_0,i}, \dots, N_{e_3,i}\}_{i\in\{2,\dots,p\}} \cup \{N_{i,j}\}_{i,j\in\{2,\dots,p\}}$$

in the case $d = 2$ *and*

$$\Sigma := \{N_{v_0}, \dots, N_{v_7}\} \cup \{N_{e_0,i}, \dots, N_{e_{11},i}\}_{i\in\{2,\dots,p\}} \cup \{N_{f_0,i,j}, \dots, N_{f_5,i,j}\}_{i,j\in\{2,\dots,p\}} \cup \{N_{i,j,k}\}_{i,j,k\in\{2,\dots,p\}}$$

in the case $d = 3$*, respectively.*

The number of shape functions of this finite element can be computed quite easily. Therefore we see:

Remark 2.2 (Number of Shape Functions)**.** *The continuous Galerkin element of degree* p *has*

$$\dim\left(\widetilde{B}_d\right) = p^d + d\sum_{i=1}^{d-1} p^i + 1 = \begin{cases} p^d + d\left(\frac{p^d-1}{p-1} - 1\right), & \text{if } p \in \mathbb{N} \setminus \{1\} \\ \frac{d^2(d-1)(2d-1)}{6} + 1 - d, & \text{if } p = 1 \end{cases}$$

shape functions. Asymptotically we have $p^d + dp^{d-1}$ *shape functions for large* p*.*

Now let $\mathcal{K}$ be some triangulation of $\Omega \subset \mathbb{R}^d$ for $d \in \{1, 2, 3\}$ and $p := (p_K)_{K\in\mathcal{K}}$, $p_K \in \mathbb{N}$, be the polynomial degree vector of $\mathcal{K}$. Let $F_K : \widehat{K} \to K$ be the reference mapping from $\widehat{K}$ to K. Then we define the finite element space $V^p(\mathcal{K}, \Omega)$ of H^1-conforming elements by

$$V^p(\mathcal{K}, \Omega) := \left\{u \in H_0^1(\Omega) : u|_K \circ F_K \in P_{p_K}\left(\widehat{K}\right)\ \forall K \in \mathcal{K}\right\}.$$

With this definition we conclude this subsection about the continuous Galerkin element. For more information we refer the interested reader to the books of Brenner and Scott [64], Ciarlet [81], Demkowicz [97], Demkowicz, Kurtz, Pardo, Paszyński, Rachowicz and Zdunek [100], Schwab [194] and Šolín, Segeth and Doležel [209] and the dissertation of Zaglmayr [219]. In [209] one can also find continuous Galerkin elements for other types of reference cells, e.g. triangles, tetrahedra and prisms. Therefore we may now assume that the H^1-conforming finite element space $V^p(\mathcal{K}, \Omega)$ can be constructed with triangulations $\mathcal{K}$, which consist of any of these reference cells.

2.2.5 The $H(\mathrm{curl})$-Conforming Finite Element Space

The $H(\mathrm{curl})$-conforming finite element spaces gained attentation after it turned out that vector-valued finite elements built from H^1-conforming elements are inappropriate for the approximation of electromagnetic fields (c.f. [59, 124, 165]), because functions from the space $H(\mathrm{curl}, \Omega)$ may have discontinuous normal components. This led to the construction of *Whitney elements*, which have been discovered by Whitney [214] in 1957 and many others [3, 47, 53, 147] independently. However these elements are lowest order only and thus cannot satisfy the wish of an efficient numerical approximation with high accuracy. This observation kicked off a stepwise construction of quadratic and cubic $H(\mathrm{curl})$-conforming finite elements [3, 119, 163, 210] until Nédélec [166, 167] gave a definition of this type of elements for arbitrary polynomial degree in 1980. Therefore the $H(\mathrm{curl})$-conforming finite elements are often called *Nédélec elements*. Since then there have been proposed many approaches to improve these finite elements (c.f. [4, 6, 15, 65, 97, 100, 209, 211, 212, 218]). Because the conditioning number of the finite element matrix

$$A := \left(\int_\Omega (\nabla \times \phi_i)^T \nabla \times \phi_j^T + \int_\Omega \phi_i^T \phi_j \right)_{ij}$$

is a very critical issue for this type of finite elements [5, 54, 219], this problem has been investigated quite often in recent years. In this field we want to name the considerable improvements obtained by Beuchler, Pillwein and Zaglmayr [54], Xin and Cai [218] and Zaglmayr [219].
However, to bring together a real familiy of elements we follow the ideas of Šolín, Segeth and Doležel [209] for the construction of a hierarchial finite element again. As above we refer the interested reader to [209, 219] for the construction of polynomial anisotropic finite elements. Also in this subsection we group the shape functions and degrees of freedom in a geometrical meaning and call them edge, face or interior shape functions and edge, face and interior degrees of freedom, respectively, if they can be associated with the corresponding object (and if $d \in \{2, 3\}$ is big enough such that these geometrical objects exist). Due to the continuity properties of the space $H(\mathrm{curl}, \Omega)$ it is not required to prescribe any values at the vertices of the cells. Therefore there are no vertex shape functions or vertex degrees of freedom in this element.
Let $p \in \mathbb{N}_0$ be arbitrary. Then we define the polynomial space $P_p\left(\widehat{K}\right)$ by

$$P_p\left(\widehat{K}\right) := \begin{cases} Q_{p,p+1}\left(\widehat{K}\right) \times Q_{p+1,p}\left(\widehat{K}\right), & \text{if } \widehat{K} = [0,1]^2 \\ Q_{p,p+1,p+1}\left(\widehat{K}\right) \times Q_{p+1,p,p+1}\left(\widehat{K}\right) \times Q_{p+1,p+1,p}\left(\widehat{K}\right), & \text{if } \widehat{K} = [0,1]^3 \end{cases},$$

where the scalar polynomial spaces $Q_{p,q}\left([0,1]^2\right)$ and $Q_{p,q,r}\left([0,1]^3\right)$ are given by

$$Q_{p,q}\left([0,1]^2\right) := \mathrm{span}\{x_1^i x_2^j : x \in [0,1]^2, 0 \leq i \leq p, 0 \leq j \leq q\}$$

and

$$Q_{p,q,r}\left([0,1]^3\right) := \mathrm{span}\{x_1^i x_2^j x_3^k : x \in [0,1]^3, 0 \leq i \leq p, 0 \leq j \leq q, 0 \leq k \leq r\}$$

for $p, q, r \in \mathbb{N}_0$.
Now we have to construct a basis for the polynomial space $P_p\left(\widehat{K}\right)$. We start with the case $d = 2$. Here two types of shape functions are present on reference cell $\widehat{K} = [0,1]^2$. Namely, these are the edge shape functions and the interior shape functions. To define the shape functions we use a tensor-product structure similar to the one employed in Section 2.2.3. However for this finite element we have to take special care of the polynomials, which we choose, to meet the continuity properties of the space $H(\mathrm{curl}, \Omega)$

correctly. Let us begin with the *edge shape functions*. For $i \in \{0,1\}$ and $j \in \{0,\dots,p\}$ these are defined by

$$\phi_{e_i,j}(x) := \begin{pmatrix} 0 \\ l_i(x_1)L_j(x_2) \end{pmatrix}, \quad \phi_{e_{i+2},j}(x) := \begin{pmatrix} L_j(x_1)l_i(x_2) \\ 0 \end{pmatrix}.$$

The *interior shape functions* are given by

$$\phi_{1,i,j}(x) := \begin{pmatrix} L_i(x_1)l_j(x_2) \\ 0 \end{pmatrix}$$

and

$$\phi_{2,j,i}(x) := \begin{pmatrix} 0 \\ l_j(x_1)L_i(x_2) \end{pmatrix}$$

for $i \in \{0,\dots,p\}$ and $j \in \{2,\dots,p+1\}$. In [209] it was shown that the set

$$B_2 := \{\phi_{e_0,i},\dots,\phi_{e_3,i}\}_{i\in\{0,\dots,p\}} \cup \{\phi_{1,i,j},\phi_{2,j,i}\}_{\substack{i\in\{0,\dots,p\}\\ j\in\{2,\dots,p+1\}}}$$

is a basis of $P_p\left([0,1]^2\right)$.

Lemma 2.7 (Basis ($d=2$))**.** *For all $p \in \mathbb{N}_0$ the set B_2 is a basis of $P_p\left([0,1]^2\right)$.*

Proof. See Proposition 2.7 in [209]. □

Now let us continue with the case $d=3$. Here face shape functions might occur, too. As in the case $d=2$ we employ a tensor-product structure to define the shape functions. Let us begin with the *edge shape functions* again. For $i,j \in \{0,1\}$ and $k \in \{0,\dots,p\}$ these are defined by

$$\phi_{e_{i+4j},k}(x) := \begin{pmatrix} 0 \\ l_i(x_1)L_k(x_2)l_j(x_3) \\ 0 \end{pmatrix}, \quad \phi_{e_{i+4j+2},k}(x) := \begin{pmatrix} L_k(x_1)l_i(x_2)l_j(x_3) \\ 0 \\ 0 \end{pmatrix},$$

$$\phi_{e_{i+2j+8},k}(x) := \begin{pmatrix} 0 \\ 0 \\ l_i(x_1)l_j(x_2)L_k(x_3) \end{pmatrix}.$$

The group of *face shape functions* is given by

$$\phi_{f_i,1,j,k}(x) := \begin{pmatrix} 0 \\ l_i(x_1)L_j(x_2)l_k(x_3) \\ 0 \end{pmatrix}, \quad \phi_{f_i,2,k,j}(x) := \begin{pmatrix} 0 \\ 0 \\ l_i(x_1)l_k(x_2)L_j(x_3) \end{pmatrix},$$

$$\phi_{f_{i+2},1,j,k}(x) := \begin{pmatrix} L_j(x_1)l_i(x_2)l_k(x_3) \\ 0 \\ 0 \end{pmatrix}, \quad \phi_{f_{i+2},2,k,j}(x) := \begin{pmatrix} 0 \\ 0 \\ l_k(x_1)l_i(x_2)L_j(x_3) \end{pmatrix},$$

$$\phi_{f_{i+4},1,j,k}(x) := \begin{pmatrix} L_j(x_1)l_k(x_2)l_i(x_3) \\ 0 \\ 0 \end{pmatrix}, \quad \phi_{f_{i+4},2,k,j}(x) := \begin{pmatrix} 0 \\ l_k(x_1)L_j(x_2)l_i(x_3) \\ 0 \end{pmatrix}$$

for $i \in \{0, 1\}, j \in \{0, \dots, p\}$ and $k \in \{2, \dots, p+1\}$. Finally we introduce the *interior shape functions* $\phi_{1,i,j,k}, \phi_{2,j,i,k}$ and $\phi_{3,j,k,i}$ for $i \in \{0, \dots, p\}$ and $j, k \in \{2, \dots, p+1\}$ by

$$\phi_{1,i,j,k}(x) := \begin{pmatrix} L_i(x_1)l_j(x_2)l_k(x_3) \\ 0 \\ 0 \end{pmatrix}, \quad \phi_{2,j,i,k}(x) := \begin{pmatrix} 0 \\ l_j(x_1)L_i(x_2)l_k(x_3) \\ 0 \end{pmatrix},$$

$$\phi_{3,j,k,i}(x) := \begin{pmatrix} 0 \\ 0 \\ l_j(x_1)l_k(x_2)L_i(x_3) \end{pmatrix}.$$

Then we set

$$\begin{aligned} B_3 := \{\phi_{e_0,i}, \dots, \phi_{e_{11},i}\}_{i \in \{0,\dots,p\}} &\cup \{\phi_{f_0,1,i,j}, \phi_{f_0,2,j,i}, \dots, \phi_{f_5,1,i,j}, \phi_{f_5,2,j,i}\}_{\substack{i \in \{0,\dots,p\} \\ j \in \{2,\dots,p+1\}}} \\ &\cup \{\phi_{1,i,j,k}, \phi_{2,j,i,k}, \phi_{3,j,k,i}\}_{\substack{i \in \{0,\dots,p\} \\ j,k \in \{2,\dots,p+1\}}} \end{aligned}$$

and it can be shown that B_3 is a basis of $P_p\left([0,1]^3\right)$.

Lemma 2.8 (Basis ($d = 3$))**.** *For all $p \in \mathbb{N}_0$ the set B_3 is a basis of $P_p\left([0,1]^3\right)$.*

Proof. See Proposition 2.9 in [209]. □

Note that these sets of shape functions are implemented in the finite element library deal.II [41, 42] by the author. In [5] it was shown that the bases B_2 and B_3 lead to good conditioning numbers of the finite element matrix A.

To complete the definition of the $H(\text{curl})$-conforming finite element we have to define some degrees of freedom. Here we can take advantage of the geometrical grouping again and define degrees of freedom for each group of shape functions separately. Again this part is inspired by the work of Demkowicz [97]. Let us begin with the edge shape functions. They exist for both, $d = 2$ and $d = 3$. For $j \in \{0, \dots, p\}$ we define the corresponding edge degrees of freedom $N_{e_i,j} : P_p\left(\widehat{K}\right) \to \mathbb{R}$ by

$$N_{e_i,j}(\phi) := \int_{e_i} \phi^T t L_j \qquad \forall \phi \in P_p\left(\widehat{K}\right),$$

where t denotes the unit tangential vector to cell $\widehat{K}$ determined by the orientation of edge e_i, $i \in \{0, \dots, 3\}$ in the case $d = 2$ and $i \in \{0, \dots, 11\}$ in the case $d = 3$. Now let us proceed to the face degrees of freedom. These degrees of freedom only exist in the case $d = 3$. Here we have to provide two sets of degrees of freedom $N_{f_i,1,j,k}, N_{f_i,2,k,j} : P_p\left(\widehat{K}\right) \to \mathbb{R}$, since we also have two sets of shape functions corresponding to the two different tangential directions of face f_i. For $i \in \{0, \dots, 5\}, j \in \{0, \dots, p\}$ and $k \in \{2, \dots, p+1\}$ we set

$$N_{f_i,1,j,k}(\phi) := \int_{f_i} \left((\nabla \times \phi)^T n (\nabla \times \phi_{f_i,1,j,k})^T n + \phi^T \nabla\left(l_j l_k\right)\right)$$

and

$$N_{f_i,2,k,j}(\phi) := \int_{f_i} \left((\nabla \times \phi)^T n (\nabla \times \phi_{f_i,2,k,j})^T n + \phi^T \nabla\left(l_k l_j\right)\right)$$

for all $\phi \in P_p\left(\widehat{K}\right)$, respectively. The interior degrees of freedom are present in both cases $d = 2$ and $d = 3$ again. Let us consider the case $d = 2$ first. Here the two sets of degrees of freedom $N_{1,i,j}, N_{2,j,i} : P_p\left([0,1]^2\right) \to \mathbb{R}$ are given by

$$N_{1,i,j}(\phi) := \int_{\widehat{K}} \left((\nabla \times \phi)^T \nabla \times \phi_{1,i,j} + \phi^T \nabla\left(l_i l_j\right)\right)$$

and

$$N_{2,j,i}(\phi) := \int_{\widehat{K}} \left((\nabla \times \phi)^T \nabla \times \phi_{2,j,i} + \phi^T \nabla \left(l_j l_i\right)\right)$$

for $i \in \{0, \ldots, p\}, j \in \{2, \ldots, p+1\}$ and all $\phi \in P_p\left(\widehat{K}\right)$, respectively. In the case $d = 3$ we define the interior degrees of freedom $N_{1,i,j,k}, N_{2,j,i,k}, N_{3,j,k,i} : P_p\left([0,1]^3\right) \to \mathbb{R}$ for $i \in \{0, \ldots, p\}$ and $j, k \in \{2, \ldots, p+1\}$ as follows:

$$\begin{aligned} N_{1,i,j,k}(\phi) &:= \int_{\widehat{K}} \left((\nabla \times \phi)^T \nabla \times \phi_{1,i,j,k} + \phi^T \nabla \left(l_i l_j l_k\right)\right), \\ N_{2,j,i,k}(\phi) &:= \int_{\widehat{K}} \left((\nabla \times \phi)^T \nabla \times \phi_{2,j,i,k} + \phi^T \nabla \left(l_j l_i l_k\right)\right) \end{aligned}$$

and

$$N_{3,j,k,i}(\phi) := \int_{\widehat{K}} \left((\nabla \times \phi)^T \nabla \times \phi_{3,j,k,i} + \phi^T \nabla \left(l_j l_k l_i\right)\right)$$

for all $\phi \in P_p\left(\widehat{K}\right)$, respectively. Now we have introduced all necessary notions to give a rigorous definition of the $H(\text{curl})$-conforming finite element.

Definition 2.6 (Nédélec Element)**.** *The* Nédélec element *is defined by the triplet* $\left(\widehat{K}, P_p\left(\widehat{K}\right), \Sigma\right)$, *where* Σ *is given by*

$$\Sigma := \left\{N_{e_0,i}, \ldots, N_{e_3,i}\right\}_{i \in \{0,\ldots,p\}} \cup \left\{N_{1,i,j}, N_{2,j,i}\right\}_{\substack{i \in \{0,\ldots,p\} \\ j \in \{2,\ldots,p+1\}}}$$

in the case $d = 2$ *and*

$$\begin{aligned} \Sigma := \left\{N_{e_0,i}, \ldots, N_{e_{11},i}\right\}_{i \in \{0,\ldots,p\}} &\cup \left\{N_{f_0,1,i,j}, N_{f_0,2,j,i}, \ldots, N_{f_5,1,i,j}, N_{f_5,2,j,i}\right\}_{\substack{i \in \{0,\ldots,p\} \\ j \in \{2,\ldots,p+1\}}} \\ &\cup \left\{N_{1,i,j,k}, N_{2,j,i,k}, N_{3,j,k,i}\right\}_{\substack{i \in \{0,\ldots,p\} \\ j,k \in \{2,\ldots,p+1\}}} \end{aligned}$$

in the case $d = 3$, *respectively.*

Again we compute the number of shape functions of this element.

Remark 2.3 (Number of Shape Functions)**.** *The Nédélec element of degree p has*

$$\dim\left(B_d\right) = \left(2^{d-1} d + 12(d-2)p + dp^{d-1}\right)(p+1)$$

shape functions. Asympotitically we have $dp^d + 12(d-2)p^2$ *shape functions for large p.*

Now let $\mathcal{K}$ be some triangulation of $\Omega \subset \mathbb{R}^d$ with $d \in \{2,3\}$ and $p := (p_K)_{K \in \mathcal{K}}$, $p_K \in \mathbb{N}_0$, be the polynomial degree vector of $\mathcal{K}$. Let $F_K : \widehat{K} \to K$ be some orientation preserving affine mapping from $\widehat{K}$ to K. Then we define the finite element space $W^p(\mathcal{K}, \Omega)$ of $H(\text{curl})$-conforming elements by

$$W^p(\mathcal{K}, \Omega) := \left\{u \in H_0(\text{curl}, \Omega) : \left((\nabla F_K)^{-T} u|_K\right) \circ F_K \in P_{p_K}\left(\widehat{K}\right) \; \forall K \in \mathcal{K}\right\}. \tag{2.8}$$

Let $K \in \mathcal{K}$ be arbitrary and $e \subset \partial K$ denote some edge of cell K. Then the change of variables formula reads

$$\int_e u^T t = \int_{\widehat{e}} \frac{\operatorname{sign}\left(\det\left(\nabla F_K\right)\right)}{\left|\nabla F_K \widehat{t}\right|} \left(u \circ F_K\right)^T \left(t \circ F_K\right),$$

where $\widehat{e}$ denotes the reference edge of cell $\widehat{K}$ such that $F_K\widehat{e} = e$. There exists some $\widehat{u} \in H_0(\mathrm{curl}, \Omega)$ such that

$$\left((\nabla F_K)^{-T}\,\widehat{u}\right) \circ F_K = u \qquad \text{in } K$$

and it follows

$$\begin{aligned}\int_e u^T t &= \int_{\widehat{e}} \frac{\operatorname{sign}\left(\det\left(\nabla F_K\right)\right)}{\left|\nabla F_K \widehat{t}\right|} \widehat{u}^T \left(\nabla F_K\right)^{-1} \nabla F_K \widehat{t} \\ &= \int_{\widehat{e}} \frac{\operatorname{sign}\left(\det\left(\nabla F_K\right)\right)}{\left|\nabla F_K \widehat{t}\right|} \widehat{u}^T \widehat{t}.\end{aligned}$$

Now let $f \subset \partial K$ denote some face of cell K. Then the change of variables formula reads

$$\int_f u^T n = \int_{\widehat{f}} \frac{\operatorname{sign}\left(\det\left(\nabla F_K\right)\right)}{\left|\left(\nabla F_K\right)^{-T} \widehat{n}\right|} \left(u \circ F_K\right)^T \left(n \circ F_K\right)$$

where $\widehat{f}$ denotes the reference face of cell $\widehat{K}$ such that $F_K\widehat{f} = f$. Then it follows

$$\begin{aligned}\int_f u^T n &= \int_{\widehat{f}} \frac{\operatorname{sign}\left(\det\left(\nabla F_K\right)\right)}{\left|\left(\nabla F_K\right)^{-T} \widehat{n}\right|} \widehat{u}^T \left(\nabla F_K\right)^{-1} \left(\nabla F_K\right)^{-T} \widehat{n} \\ &= \int_{\widehat{f}} \frac{\operatorname{sign}\left(\det\left(\nabla F_K\right)\right)}{\left|\left(\nabla F_K\right)^{-T} \widehat{n}\right|} \widehat{u}^T \widehat{n}.\end{aligned}$$

Thus, definition (2.8) is justified.
With this statement we close this subsection about the Nédélec element. For more information we refer the interested reader to the book of Šolín, Segeth and Doležel [209] and the dissertation of Zaglmayr [219]. In both references one can also find Nédélec elements for other types of reference cells, e.g. triangles, tetrahedra and prisms. Therefore we may now assume that the $H(\mathrm{curl})$-conforming finite element space $W^p(\mathcal{K}, \Omega)$ can be constructed with triangulations $\mathcal{K}$, which consist of any of these reference cells.

2.3 Adaptivity

In its early years the classical finite element method was used with a fixed triangulation consisting of approximately equally-sized cells and polynomial degree vector $p = 1$ (see e.g. [17, 83, 206, 223]). If the computed finite element solution was not accurate enough, the whole grid was rebuild with smaller cells. It did not take very long until it became clear that this procedure is not very efficient, because the global error is usually dominated by the local error in only a few small subdomains on which the solution is singular. This led to adaptive refinement of the cells around known singularities like e.g. corners of the domain (see e.g. [20]). The h-adaptive finite element method was born and its application was boosted by the development of the a posteriori error estimation [30, 31, 32, 33, 34]. Another approach for increasing the accuracy of the finite element solution is the use of higher-order ansatz spaces on some or all cells of the triangulation [38]. The adaptive application of this method is called the p-adaptive finite element method nowadays. Shortly after the work of Babuška, Szabó and Katz [38] in 1981 the h- and the p-adaptive finite element method were combined in the paper of Babuška and Dorr [23]. This was the birth of the hp-adaptive finite element method. It received further consideration in the works of Babuška and Guo [24, 25] from 1986. For various applications it could be shown that this method can achieve exponential rates of convergence with respect to the number of degrees of freedom [89, 113, 156, 192, 193, 194, 195].

Another adaptive finite element method, which received some consideration, is the r-adaptive finite element method. This method adapts the location of the cells in the triangulation with respect to local features of the solution. Since we do not apply this method in our work, we refer the interested reader to the book of Baines [39] for more information about this topic. For a graphical overview of these four adaptive methods see Figure 2.8.

Strongly connected to the area of adaptivity is the *a posteriori error estimation* [8, 208]. Since one usually does not know the analytic solution of the problem, one has to estimate the error of the finite element solution in terms of the computed solution. Therefore a local a posteriori error estimator, which admits a splitting of the form

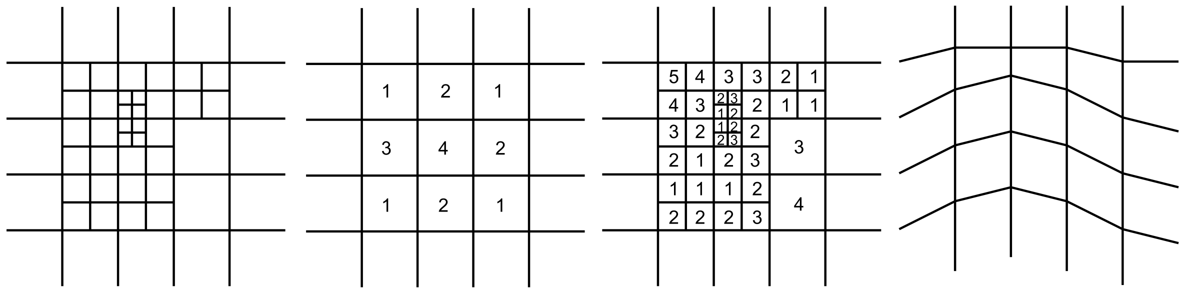

Figure 2.8: Adaptive methods. Outer left: h-adaptivity. Center left: p-adaptivity. Center right: hp-adaptivity. Outer right: r-adaptivity.

$$\eta^2 = \sum_{K \in \mathcal{K}} \eta_K^2$$

for the estimated error η, is used. Then a fully automatic algorithm can be construced for the h- and the p-adaptive finite element method by refining the cells where the estimated error η_K is large (see e.g. [7, 8, 31, 32, 43, 102, 103, 207, 208, 220, 224]). However for the hp-adaptive finite element method this information alone is not enough. Here we also need an indication which refinement pattern – h-refinement or p-refinement – performs best. There have been various trials to tackle this task, e.g. in [107, 136, 215] the analyticity of the solution is estimated, in [10, 74, 104, 126] local boundary value problems are solved and in [101, 180, 181] the global interpolation error is minimized.

To conclude this short comment on the adaptive finite element method we want to point out the two main parts of every adaptive algorithm again – the a posteriori error estimation and the refinement strategy. For more information about the a posteriori error estimation we refer the interested reader to the work of Becker and Rannacher [52] and the books of Ainsworth and Oden [8] and Verfürth [208]. In the survey articles of Babuška and Suri [37], Eriksson, Estep, Hansbo and Johnson [108], Mitchell and McClain [162] and Rannacher [182] and the book of Schwab [194] one can find more information about the principles of adaptive finite element methods.

2.4 Interpolation

The goal of this section is to present some interpolation operators for the hp-adaptive finite element method, which map functions from the continuous spaces L^2, H^1 and $H(\mathrm{curl})$ into the corresponding discrete finite element spaces. Of course, for each pair of continuous and discrete space there are many different interpolation operators in literature. The most simple choice are nodal-based interpolation operators, which can be found in e.g. [194]. These operators simply evaluate the function at some

nodal points and use these values as the value of the corresponding degree of freedom of the finite element function. However, the drawback of this procedure is the need of some extra regularity for the point evaluation. To overcome this drawback local averaging operators have been used in e.g. [82, 198]. Unfortunately, these Clément- or Scott-Zhang-type interpolation operators loose the locality of the interpolation estimates. In the works of Demkowicz and Babuška [98] and Demkowicz and Buffa [99] one can find another approach – the so called *projection-based interpolation*. This type of interpolation operators determines the values of the degrees of freedom by solving some local minimization problems on edges, faces and cells. Also here some extra regularity is required to ensure that all operations are well-defined.
Throughout this section let $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$, be open and bounded with polygonal (in the case $d = 2$) or polyhedral (in the case $d = 3$), Lipschitz-continuous boundary. Further let $\mathcal{K}$ be some regular triangulation of Ω consisting of simplices. Then we start with the L^2-conforming interpolation. This interpolation operator maps functions from the space $L^2(\Omega)$ into the space of discontinuous Galerkin finite elements $U^p(\mathcal{K}, \Omega)$. It is the most simple one, since it comes from the heart of the finite element method, and thus we just state the best approximation property of the Ritz-Galerkin method here:

Theorem 2.3 (L^2-Conforming Interpolation)**.** *Let $n \in \mathbb{N}$ and $u \in L^2(\Omega)^n$. Then there exists some linear operator $\Pi : L^2(K)^n \to U^{p_K}(\{K\}, K)^n$ such that*

$$\int_K \phi^T(u - \Pi u) = 0 \qquad \forall \phi \in U^{p_K}(\{K\}, K).$$

Proof. See [64], Theorem 0.3.3. □

For interpolation error estimates see e.g. Houston, Schwab and Süli [134].

2.4.1 The H^1-Conforming Interpolation

The H^1-conforming interpolation maps functions from the space $H^1(\Omega)$ into the space of continuous Galerkin finite elements $V^p(\mathcal{K}, \Omega)$. For the *hp*-adaptive finite element method one can find a nodal-based interpolation operator in the book of Schwab [194]. In [157] Melenk and Wohlmuth derived a Clément-type interpolation operator, which follows the ideas of Scott and Zhang [198] and does not require any additional regularity. Melenk investigated this operator even further in his later work [155]. Demkowicz and co-authors proposed a projection-based interpolation operator for the case $d = 2$ in [98] and for the case $d = 3$ in [99]. Also these operators use nodal-based interpolation and thus require some additional regularity. However, one can replace the nodal-based parts by some Clément-type interpolation to weaken the regularity assumptions. In this subsection we present a Clément-type interpolation operator with minimal regularity assumptions.
Before we start with the interpolation results let us define two regularity properties, which will become part of our basic assumptions later on. The first notion describes the limited variation of the diameter of two neighbouring cells. This property is called *shape regularity* and the following definition can be found in the books of Schwab [194] and Szabó and Babuška [203].

Definition 2.7 (Shape Regularity)**.** *Let $K \in \mathcal{K}$ be the image of reference cell $\widehat{K}$ under some orientation preserving diffeomorphism $F_K : \widehat{K} \to K$ and set $h_K := \operatorname{diam}(K)$. Then $\mathcal{K}$ is called γ-*shape regular*, if and only if there exists some constant $\gamma > 0$ such that for all $K \in \mathcal{K}$ it holds*

$$\frac{\|\nabla F_K\|_{L^\infty(\widehat{K})^d}}{h_K} + h_K \left\| (\nabla F_K)^{-1} \circ F_K \right\|_{L^\infty(\widehat{K})^d} \leq \gamma. \tag{2.9}$$

A similar property exists for the polynomial degrees distributed over triangulation $\mathcal{K}$. We call this property *polynomial regularity.*

Definition 2.8 (Polynomial Regularity)**.** *$\mathcal{K}$ is called* γ-polynomial regular*, if and only if there exists some constant $\gamma > 0$ such that for all $K_1, K_2 \in \mathcal{K}$ with $K_1 \cap K_2 \neq \emptyset$ it holds*

$$\frac{p_{K_1}}{\gamma} \leq p_{K_2} \leq \gamma p_{K_1}. \tag{2.10}$$

In the following we will use these two notions almost always together. Therefore we define a third notion, which collects these two different regularity properties to only one notion.

Definition 2.9 (Regularity)**.** *Let $\gamma_1, \gamma_2 > 0$. Then $\mathcal{K}$ is called* (γ_1, γ_2)-regular*, if and only if $\mathcal{K}$ is γ_1-shape regular and γ_2-polynomial regular.*

Now we are ready to present the H^1-conforming interpolation operator. It is a Clément-type interpolation, which replaces the point evaluation of the interpolated function by some local average. This procedure does not require the extra regularity of a point evaluation, but is also well-defined for functions from the space H^1. However, since we do not rely on its construction, but only use its interpolation estimates in this work, we refer to the work of Melenk [155] for a more detailed insight into this topic. Another way to obtain such an interpolation operator is to use the projection-based interpolation operator from Demkowicz and Babuška [98] for $d = 2$ and Demkowicz and Buffa [99] for $d = 3$, respectively, and replace the interpolations on the cell boundary by Clément-type interpolations as proposed in [191].
Let $h := (h_K)_{K \in \mathcal{K}}$ be the mesh size vector of triangulation $\mathcal{K}$. For $K \in \mathcal{K}$ we define the local patch $\omega_{K,1} \subset \mathbb{R}^d$ by

$$\omega_{K,1} := \bigcup_{L \in \mathcal{K}} \{L : K \text{ and } L \text{ share a common edge}\}.$$

Then we can iteratively define

$$\omega_{K,i} := \bigcup_{L \in \mathcal{K}} \left\{L : \exists \widetilde{K} \in \mathcal{K} : \widetilde{K} \text{ and } L \text{ share a common edge}, \widetilde{K} \subset \omega_{K,i-1}\right\} \cup \omega_{K,i-1}$$

for $i \geq 2$. For simplicity we write $\omega_K := \omega_{K,1}$ in the case $i = 1$. Then we get the following result, which gives us an optimal estimate for the interpolation error in terms of the gradient of the interpolated function.

Theorem 2.4 (H^1-Conforming Interpolation)**.** *Let $\mathcal{K}$ be (γ_1, γ_2)-regular and $K \in \mathcal{K}$ be arbitrary. Further let $u \in H_0^1(\Omega)$. Then there exists some linear operator $\Pi^1 : H_0^1(\Omega) \to V^p(\mathcal{K}, \Omega)$ and some constant $C_{grad} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left\|u - \Pi^1 u\right\|_{L^2(\omega_K)} + \frac{h_K}{p_K} \left\|\nabla \left(u - \Pi^1 u\right)\right\|_{L^2(\omega_K)^d} + \sqrt{\frac{h_e}{p_e}} \left\|u - \Pi^1 u\right\|_{L^2(e)} \leq C_{grad} \frac{h_K}{p_K} \|\nabla u\|_{L^2(\omega_{K,7})^d}.$$

Here $e \subset \partial K \cap \partial\Omega$ denotes an interior edge (in the case $d = 2$) or face (in the case $d = 3$) of cell K and $h_e := \operatorname{diam}(e)$ is the edge or face diameter. The edge or face polynomial degree p_e is given by

$$p_e := \max\{p_{K_1}, p_{K_2}\}$$

for $K_1, K_2 \in \mathcal{K}$ with $e = K_1 \cap K_2$.

Proof. Let $\mathcal{V}_K$ be the set of all vertices of cell K and $v \in \mathcal{V}_K$ be arbitrary. Then we define the patch $\omega_{v,1}$ by

$$\omega_{v,1} := \bigcup_{L \in \mathcal{K}} \{L : v \in L\}$$

and for $i \geq 2$ we set

$$\omega_{v,i} := \bigcup_{L \in \mathcal{K}} \{L : \omega_{v,i-1} \cap L \neq \emptyset\}.$$

Further let $e \subset \partial K \cap \partial\Omega$ denote some interior edge (in the case $d = 2$) or face (in the case $d = 3$) of K. In Theorem 2.2 in [157] it was shown that there exists a linear operator $\Pi^1 : H_0^1(\Omega) \to V^p(\mathcal{K}, \Omega)$ such that

$$\begin{aligned} &\sum_{v \in \mathcal{V}_K} \left(\left\|u - \Pi^1 u\right\|^2_{L^2(\omega_{v,1})} + C\frac{h_v}{p_v}\left(\frac{h_v}{p_v}\left\|\nabla\left(u - \Pi^1 u\right)\right\|^2_{L^2(\omega_{v,1})^3} + \left\|u - \Pi^1 u\right\|^2_{L^2(e)}\right)\right) \\ &\quad \leq C^2 \sum_{v \in \mathcal{V}_K} \frac{h_v^2}{p_v^2}\|\nabla u\|^2_{L^2(\omega_{v,4})^d} \end{aligned} \tag{2.11}$$

for some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p. With the (γ_1, γ_2)-regularity of $\mathcal{K}$ we see easily

$$\begin{aligned} &\left\|u - \Pi^1 u\right\|^2_{L^2(\omega_K)} + \frac{h_K^2}{p_K^2}\left\|\nabla\left(u - \Pi^1 u\right)\right\|^2_{L^2(\omega_K)^3} + \frac{h_e}{p_e}\left\|u - \Pi^1 u\right\|^2_{L^2(e)} \\ &\quad \leq \sum_{v \in \mathcal{V}_K} \left(\left\|u - \Pi^1 u\right\|^2_{L^2(\omega_{v,1})} + C\frac{h_v}{p_v}\left(\frac{h_v}{p_v}\left\|\nabla\left(u - \Pi^1 u\right)\right\|^2_{L^2(\omega_{v,1})^3} + \left\|u - \Pi^1 u\right\|^2_{L^2(e)}\right)\right), \end{aligned}$$

where $C > 0$ denotes some constant independent of mesh size vector h and polynomial degree vector p,

$$h_v := \max_{\substack{L \in \mathcal{K} \\ L \subset \omega_{v,1}}} (h_L)$$

and

$$p_v := \max_{\substack{L \in \mathcal{K} \\ L \subset \omega_{v,1}}} (p_L).$$

Then, estimate (2.11) implies

$$\left\|u - \Pi^1 u\right\|^2_{L^2(\omega_K)} + \frac{h_K^2}{p_K^2}\left\|\nabla\left(u - \Pi^1 u\right)\right\|^2_{L^2(\omega_K)^3} + \frac{h_e}{p_e}\left\|u - \Pi^1 u\right\|^2_{L^2(e)} \leq C^2 \sum_{v \in \mathcal{V}_K} \frac{h_v^2}{p_v^2}\|\nabla u\|^2_{L^2(\omega_{v,4})^d}$$

and by using regularity assumptions (2.9) and (2.10) we get

$$\begin{aligned} \left\|u - \Pi^1 u\right\|^2_{L^2(\omega_K)} + \frac{h_K^2}{p_K^2}\left\|\nabla\left(u - \Pi^1 u\right)\right\|^2_{L^2(\omega_K)^3} + \frac{h_e}{p_e}\left\|u - \Pi^1 u\right\|^2_{L^2(e)} &\leq C^2_{\text{grad}}\frac{h_K^2}{p_K^2}\sum_{v \in \mathcal{V}_K}\|\nabla u\|^2_{L^2(\omega_{v,4})^d} \\ &\leq C^2_{\text{grad}}\frac{h_K^2}{p_K^2}\|\nabla u\|^2_{L^2(\omega_{K,7})^d}, \end{aligned}$$

since

$$\bigcup_{v \in \mathcal{V}_K} \omega_{v,4} \subset \omega_{K,7}.$$

The result follows by taking the square root on both sides. $\square$

Another way to derive such an interpolation error estimate can be found in Theorem 2 in [98] for the case $d = 2$ and Theorem 4 in [99] for the case $d = 3$, respectively, together with Theorem 5 and Corollary 6 in [191] and the Bramble-Hilbert Lemma (see e.g. [63]).
With this remark we conclude the subsection about H^1-conforming interpolation. For more information about such operators we refer the interested reader to the publications of Demkowicz and Babuška [98], Demkowicz and Buffa [99], Melenk [155], Melenk and Wohlmuth [157] and Schöberl [191], and the books of Demkowicz [97], Demkowicz, Kurtz, Pardo, Paszyński, Rachowicz and Zdunek [100] and Šolín, Segeth and Doležel [209]. Especially [98, 99, 155, 191, 209] provide a detailed insight into the construction of such operators.

2.4.2 The $H(\mathrm{curl})$-Conforming Interpolation

In this subsection we want to consider the $H(\mathrm{curl})$-conforming interpolation. It maps functions from the space $H(\mathrm{curl}, \Omega)$ into the space of Nédélec elements $W^p(\mathcal{K}, \Omega)$. Demkowicz and co-authors presented the following projection-based $H(\mathrm{curl})$-conforming interpolation operator for the *hp*-adaptive finite element method for the case $d = 2$ in [98] and for the case $d = 3$ in [99]. This interpolation requires some extra regularity of the interpolated function, because the interpolation error estimate is bounded in the H^1-seminorm $\|\nabla \cdot\|_{L^2}$ instead of the $H(\mathrm{curl})$-norm $\|\cdot\|_{H(\mathrm{curl})}$.

Theorem 2.5 ($H(\mathrm{curl})$-Conforming Interpolation)**.** *Let $d \in \{2, 3\}$ and $\mathcal{K}$ be (γ_1, γ_2)-regular. Further let $K \in \mathcal{K}$ and $u \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^d$. Then there exists some linear operator $\Pi^{\mathrm{curl}} : H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^d \to W^p(\mathcal{K}, \Omega)$ and some constant $C_{\mathrm{curl}} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left\|u - \Pi^{\mathrm{curl}} u\right\|_{L^2(\omega_K)^d} + \frac{\sqrt{h_e}}{(p_e + 1)^{\frac{1}{2}(1-\varepsilon)}} \left\|u - \Pi^{\mathrm{curl}} u\right\|_{L^2(e)^d} \leq C_{\mathrm{curl}} \frac{h_K}{(p_K + 1)^{1-\varepsilon}} \|\nabla u\|_{L^2(\omega_{K,5})^d}$$

for all $\varepsilon > 0$. Here $e \subset \partial K \cap \Omega$ denotes an interior edge (in the case $d = 2$) or face (in the case $d = 3$) of cell K.

Proof. From Theorem 3 in [98] and Theorem 5 in [99], respectively, and the Bramble-Hilbert Lemma (see [63]) we know that there exists a linear operator $\Pi^{\mathrm{curl}} : H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^d \to W^p(\mathcal{K}, \Omega)$ such that

$$\left\|u - \Pi^{\mathrm{curl}} u\right\|_{L^2(\omega_K)^d} \leq C_1 \frac{h_K}{(p_K + 1)^{1-\varepsilon}} \|\nabla u\|_{L^2(\omega_{K,5})^d} \tag{2.12}$$

for some constant $C_1 > 0$ independent of mesh size vector h and polynomial degree vector p.
Now let $e \subset \partial K \cap \Omega$ be some interior edge (in the case $d = 2$) or face (in the case $d = 3$) of cell K. Then the trace theorem (cf. [67, 68, 69, 78]) implies

$$\left\|u - \Pi^{\mathrm{curl}} u\right\|^2_{L^2(e)^d} \leq \begin{cases} C_2 \left\|u - \Pi^{\mathrm{curl}} u\right\|_{L^2(\omega_K)^d} \left\|\nabla \times \left(u - \Pi^{\mathrm{curl}} u\right)\right\|_{L^2(\omega_K)}, & \text{if } d = 2 \\ C_2 \left\|u - \Pi^{\mathrm{curl}} u\right\|_{L^2(\omega_K)^d} \left\|\nabla \times \left(u - \Pi^{\mathrm{curl}} u\right)\right\|_{L^2(\omega_K)^d}, & \text{if } d = 3 \end{cases}$$

for some constant $C_2 > 0$ independent of mesh size vector h and polynomial degree vector p. Then using Theorem 3 in [98] and Theorem 5 in [99], respectively, estimate (2.12) and the (γ_1, γ_2)-regularity of $\mathcal{K}$ gives

$$\frac{\sqrt{h_e}}{(p_e + 1)^{\frac{1}{2}(1-\varepsilon)}} \left\|u - \Pi^{\mathrm{curl}} u\right\|_{L^2(e)^d} \leq \widetilde{C}_2 \frac{h_K}{(p_K + 1)^{1-\varepsilon}} \|\nabla u\|_{L^2(\omega_{K,5})^d}$$

for some constant $\widetilde{C}_2 > 0$ independent of mesh size vector h and polynomial degree vector p. This concludes the proof. □

Similar to the H^1-conforming projection-based interpolation operator also the projection-based $H(\mathrm{curl})$-conforming interpolation has some drawbacks. The major restriction is the extra regularity requirement $u \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^d$ instead of the desired minimal regularity assumption $u \in H_0(\mathrm{curl}, \Omega)$. Further, the order of the polynomial degree $p_K + 1$ in the interpolation error estimate is suboptimal by ε. As for the H^1-conforming interpolation also here the estimate itself is not fully local, but extends to some bigger domain $\omega_{K,5}$.
This concludes the subsection about the $H(\mathrm{curl})$-conforming interpolation. For more information on this kind of interpolation we refer the interested reader to the publications of Demkowicz and Babuška [98] and Demkowicz and Buffa [99] and the books of Demkowicz [97], Demkowicz, Kurtz, Pardo, Paszyński, Rachowicz and Zdunek [100] and Šolín, Segeth and Doležel [209].

2.5 The de Rham Complex Reviewed

In Theorem 1.8 we have derived the exact sequences

$$\{0\} \xrightarrow{I} H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} L_0^2(\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

and

$$\{0\} \xrightarrow{I} H_0^1(\Omega) \xrightarrow{\nabla\times} H_0(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L_0^2(\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

in the case $d = 2$ and

$$\{0\} \xrightarrow{I} H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl}, \Omega) \xrightarrow{\nabla\times} H_0(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L_0^2(\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

in the case $d = 3$. In this section we derive similar de Rham complexes for the discrete spaces $U^p(\mathcal{K}, \Omega)$, $V^p(\mathcal{K}, \Omega)$ and $W^p(\mathcal{K}, \Omega)$ from Section 2.2. Finally, the interpolation operators from Section 2.4 come into play and connect the de Rham complexes for the continuous and the discrete spaces.
Throughout this section let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be open and bounded with polygonal (in the case $d = 2$) or polyhedral (in the case $d = 3$), Lipschitz-continuous boundary. Further let $\mathcal{K}$ be some regular triangulation of Ω. We observe easily that for $u \in V^p(\mathcal{K}, \Omega)$ it holds

$$\nabla u = 0 \text{ in } \Omega \quad \Longleftrightarrow \quad u = 0.$$

Hence we have shown the following result:

Lemma 2.9 (Kernel of ∇). *For $\nabla : V^p(\mathcal{K}, \Omega) \to \mathbb{R}^d$ it holds*

$$\ker(\nabla) = \{0\}.$$

For the next result we require an $H(\mathrm{div})$-conforming finite element space. However, we do not use this finite element space later one and, thus, did not construct it in Section 2.2. Therefore we simply define it here and refer the interested reader to the books of Brezzi and Fortin [65], Girault and Raviart [121] and Šolín, Segeth and Doležel [209] and the dissertation of Zaglmayr [219] for more information about $H(\mathrm{div})$-conforming finite elements.
For $p \in \mathbb{N}_0$ arbitrary let the polynomial space $P_p\left(\widehat{K}\right)$ be given by

$$P_p\left(\widehat{K}\right) := \begin{cases} Q_{p+1,p}\left(\widehat{K}\right) \times Q_{p,p+1}\left(\widehat{K}\right), & \text{if } \widehat{K} = [0,1]^2 \\ Q_{p+1,p,p}\left(\widehat{K}\right) \times Q_{p,p+1,p}\left(\widehat{K}\right) \times Q_{p,p,p+1}\left(\widehat{K}\right), & \text{if } \widehat{K} = [0,1]^3 \end{cases},$$

where the scalar polynomial spaces $Q_{p,q}\left([0,1]^2\right)$ and $Q_{p,q,r}\left([0,1]^3\right)$ are given by

$$Q_{p,q}\left([0,1]^2\right) := \operatorname{span}\left\{x_1^i x_2^j : x \in [0,1]^2, 0 \le i \le p, 0 \le j \le q\right\}$$

and

$$Q_{p,q,r}\left([0,1]^3\right) := \operatorname{span}\left\{x_1^i x_2^j x_3^k : x \in [0,1]^2, 0 \le i \le p, 0 \le j \le q, 0 \le k \le r\right\}$$

for $p, q, r \in \mathbb{N}_0$. Then we denote the $H(\operatorname{div})$-conforming finite element space $X^p(\mathcal{K}, \Omega)$ by

$$X^p(\mathcal{K}, \Omega) := \left\{u \in H_0(\operatorname{div}, \Omega) : \left(\frac{1}{\det\left(\nabla F_K\right)} \nabla F_K u|_K\right) \circ F_K \in P_{p_K}\left(\widehat{K}\right) \ \forall K \in \mathcal{K}\right\}.$$

Again this definition can be generalized to any type of reference cells, i.e. triangles, tetrahedra, prisms, etc. For more information we refer the interested reader to the book of Šolín, Segeth and Doležel [209] and the dissertation of Zaglmayr [219]. Therefore we may now assume that the $H(\operatorname{div})$-conforming finite element space $X^p(\mathcal{K}, \Omega)$ can be constructed with triangulations $\mathcal{K}$, which consist of any of these reference cells.

A bit more involved is the relation between the spaces $W^p(\mathcal{K}, \Omega)$ and $V^p(\mathcal{K}, \Omega)$. This is partially investigated in the following theorem, whose proof can be found in [56, 57, 58].

Theorem 2.6 (Kernel of $\nabla\times$)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be simply-connected.*

1. *Let $d = 2$. For $\nabla\times : W^p(\mathcal{K}, \Omega) \to U^p(\mathcal{K}, \Omega)$ it holds*

$$\ker(\nabla\times) = \nabla V^p(\mathcal{K}, \Omega). \tag{2.13}$$

For $\nabla\times : V^p(\mathcal{K}, \Omega) \to X^p(\mathcal{K}, \Omega)$ it holds

$$\ker(\nabla\times) = \{0\}.$$

2. *Let $d = 3$. For $\nabla\times : W^p(\mathcal{K}, \Omega) \to U^p(\mathcal{K}, \Omega)^3$ it holds (2.13).*

Proof. See Proposition 5.5 in [58] and Lemma 2.9. □

Between the spaces $X^p(\mathcal{K}, \Omega)$ and $V^p(\mathcal{K}, \Omega)$ (in the case $d = 2$) and $X^p(\mathcal{K}, \Omega)$ and $W^p(\mathcal{K}, \Omega)$ (in the case $d = 3$) it holds a result similar to Theorem 2.6.

Theorem 2.7 (Kernel of div)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be connected.*

1. *Let $d = 2$. For $\operatorname{div} : X^p(\mathcal{K}, \Omega) \to U^p(\mathcal{K}, \Omega)$ it holds*

$$\ker(\operatorname{div}) = \nabla \times V^p(\mathcal{K}, \Omega).$$

2. *Let $d = 3$. For $\operatorname{div} : X^p(\mathcal{K}, \Omega) \to U^p(\mathcal{K}, \Omega)$ it holds*

$$\ker(\operatorname{div}) = \nabla \times W^p(\mathcal{K}, \Omega).$$

Proof. See Proposition 5.5 in [58]. □

Now we are missing only two simple results to state the de Rham complex for the discrete finite element spaces $U^p(\mathcal{K}, \Omega), V^p(\mathcal{K}, \Omega), W^p(\mathcal{K}, \Omega)$ and $X^p(\mathcal{K}, \Omega)$. The following lemma determines the image of the operator div. It was shown in [219].

Lemma 2.10 (Range of div)**.** *Let* $d \in \{2, 3\}$*. For* $\operatorname{div} : X^p(\mathcal{K}, \Omega) \to U^p(\mathcal{K}, \Omega)$ *it holds*

$$\operatorname{div}\left(X^p(\mathcal{K}, \Omega)\right) = U^p(\mathcal{K}, \Omega).$$

Proof. From Proposition 5.4 in [58] we know

$$\operatorname{div}\left(X^p(\mathcal{K}, \Omega)\right) \subset U^p(\mathcal{K}, \Omega).$$

Then the result follows from

$$\dim\left(\operatorname{div}\left(X^p(\mathcal{K}, \Omega)\right)\right) = |\mathcal{K}|(p+1)^d = \dim\left(U^p(\mathcal{K}, \Omega)\right),$$

where $|\cdot|$ denotes the cardinality. □

The next lemma determines the image of the operators $\nabla\times : W^p(\mathcal{K}, \Omega) \to U^p(\mathcal{K}, \Omega)$ and $\nabla\times : V^p(\mathcal{K}, \Omega) \to X^p(\mathcal{K}, \Omega)$ in the case $d = 2$.

Lemma 2.11 (Range of $\nabla\times$)**.** *Let* $d = 2$*.*

1. *For* $\nabla\times : W^p(\mathcal{K}, \Omega) \to U^p(\mathcal{K}, \Omega)$ *it holds*

$$\nabla \times W^p(\mathcal{K}, \Omega) = U^p(\mathcal{K}, \Omega).$$

2. *For* $\nabla\times : V^p(\mathcal{K}, \Omega) \to X^p(\mathcal{K}, \Omega)$ *it holds*

$$\nabla \times V^p(\mathcal{K}, \Omega) = X^p(\mathcal{K}, \Omega).$$

Proof. 1. We see easily

$$\nabla \times W^p(\mathcal{K}, \Omega) \subset U^p(\mathcal{K}, \Omega).$$

Then the result follows from

$$\dim\left(\nabla \times W^p(\mathcal{K}, \Omega)\right) = \dim\left(U^p(\mathcal{K}, \Omega)\right).$$

2. From

$$\operatorname{div}(\nabla \times u) = 0 \qquad \forall u \in V^p(\mathcal{K}, \Omega)$$

we see easily

$$\nabla \times V^p(\mathcal{K}, \Omega) \subset X^p(\mathcal{K}, \Omega).$$

Then the result follows from

$$\dim\left(\nabla \times V^p(\mathcal{K}, \Omega)\right) = \dim\left(X^p(\mathcal{K}, \Omega)\right),$$

which can be seen easily for polynomial spaces based on a tensor-product structure. For other polynomial spaces see Proposition 5.5 in [57].

□

Let the operator $\mathcal{O} : U^p(\mathcal{K}, \Omega) \to \{0\}$ be defined by

$$\mathcal{O}u := 0$$

for all $u \in U^p(\mathcal{K}, \Omega)$. Then it follows directly from Lemmas 2.9–2.11 and Theorems 2.6 and 2.7:

Theorem 2.8 (de Rham Complex)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be connected.*

1. *Let $d = 2$. Then the de Rham complexes*

$$\{0\} \xrightarrow{I} V^p(\mathcal{K},\Omega) \xrightarrow{\nabla} W^p(\mathcal{K},\Omega) \xrightarrow{\nabla\times} U^p(\mathcal{K},\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

   and

$$\{0\} \xrightarrow{I} V^p(\mathcal{K},\Omega) \xrightarrow{\nabla\times} X^p(\mathcal{K},\Omega) \xrightarrow{\mathrm{div}} U^p(\mathcal{K},\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

   form exact sequences.

2. *Let $d = 3$. Then the de Rham complex*

$$\{0\} \xrightarrow{I} V^p(\mathcal{K},\Omega) \xrightarrow{\nabla} W^p(\mathcal{K},\Omega) \xrightarrow{\nabla\times} X^p(\mathcal{K},\Omega) \xrightarrow{\mathrm{div}} U^p(\mathcal{K},\Omega) \xrightarrow{\mathcal{O}} \{0\}$$

   forms an exact sequence.

Together with Theorem 1.8 we now have derived exact sequences for the continuous spaces $H_0^1(\Omega)$, $H_0(\mathrm{curl},\Omega)$, $H_0(\mathrm{div},\Omega)$ and $L_0^2(\Omega)$ and the discrete spaces $V^p(\mathcal{K},\Omega)$, $W^p(\mathcal{K},\Omega)$, $X^p(\mathcal{K},\Omega)$ and $U^p(\mathcal{K},\Omega)$. Now we want to consider how these two sequences relate to each other. Here the interpolation operators from Section 2.4 come into play. Thus, we have to assume again that triangulation $\mathcal{K}$ is regular and consists of simplices only. By the correct choice of interpolation operators we can derive a commuting diagram connecting the de Rham complexes from Theorems 1.8 and 2.8. However, for a complete diagram we also need an $H(\mathrm{div})$-conforming interpolation operator. Since we did not construct an $H(\mathrm{div})$-conforming finite element space in Section 2.2, we neither constructed an $H(\mathrm{div})$-conforming interpolation operator in Section 2.4. Therefore we have to do it here. We follow the ideas for the construction of projection-based interpolation operators in Section 2.4 and replace the lowest-order interpolation by the Clément-type interpolation from Schöberl [191]. Since we do not use this interpolation operator later on, we do not derive any error estimates here.

Theorem 2.9 ($H(\mathrm{div})$-Conforming Interpolation)**.** *Let $d \in \{2,3\}$ and $\mathcal{K}$ be (γ_1,γ_2)-regular. Further let $K \in \mathcal{K}$ and $\varepsilon > 0$ be arbitrary. Let $u \in H_0(\mathrm{div},\Omega) \cap H^\varepsilon(\Omega)^d$. Then there exists some linear operator $\Pi^{\mathrm{div}} : H_0(\mathrm{div},\Omega) \cap H^\varepsilon(\Omega)^d \to X^p(\mathcal{K},\Omega)$.*

Proof. The proof follows in the same fashion as the proofs of Theorems 2.4 and 2.5. We take the projection-based interpolation operator from Demkowicz and Buffa [99] and replace the lowest-order interpolation by the Clément-type interpolation from Schöberl [191]. ☐

For more information about the construction of this $H(\mathrm{div})$-conforming interpolation operator and the derivation of an interpolation error estimate we refer the interested to the works of Demkowicz and Buffa [99] and Schöberl [191].
Now we are ready to give the full de Rham diagram, which connects the de Rham complexes from Theorems 1.8 and 2.8.

Theorem 2.10 (de Rham Diagram)**.** *Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, be connected.*

1. *Let $d = 2$. Then the diagrams*

$$\begin{tikzcd}
\{0\} \arrow[r, "I"] \arrow[d, "I"] & H_0^1(\Omega) \arrow[r, "\nabla"] \arrow[d, "\Pi^1"] & H_0(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\Pi^{\mathrm{curl}}"] & L_0^2(\Omega) \arrow[r, "\mathcal{O}"] \arrow[d, "\Pi"] & \{0\} \arrow[d, "I"] \\
\{0\} \arrow[r, "I"] & V^p(\mathcal{K},\Omega) \arrow[r, "\nabla"] & W^p(\mathcal{K},\Omega) \arrow[r, "\nabla\times"] & U^p(\mathcal{K},\Omega) \arrow[r, "\mathcal{O}"] & \{0\}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\{0\} \arrow[r, "I"] \arrow[d, "I"] & H_0^1(\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\Pi^1"] & H_0(\mathrm{div},\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^{\mathrm{div}}"] & L_0^2(\Omega) \arrow[r, "\mathcal{O}"] \arrow[d, "\Pi"] & \{0\} \arrow[d, "I"] \\
\{0\} \arrow[r, "I"] & V^p(\mathcal{K},\Omega) \arrow[r, "\nabla\times"] & X^p(\mathcal{K},\Omega) \arrow[r, "\mathrm{div}"] & U^p(\mathcal{K},\Omega) \arrow[r, "\mathcal{O}"] & \{0\}
\end{tikzcd}$$

commute, i.e. it holds

$$\begin{aligned}
\nabla\Pi^1 &= \Pi^{\mathrm{curl}}\nabla \\
\nabla\times\Pi^{\mathrm{curl}} &= \Pi(\nabla\times) \\
\nabla\times\Pi^1 &= \Pi^{\mathrm{div}}(\nabla\times) \\
\mathrm{div}\left(\Pi^{\mathrm{div}}\right) &= \Pi\,\mathrm{div}\,.
\end{aligned}$$

2. *Let $d = 3$. Then the diagram*

$$\begin{tikzcd}
\{0\} \arrow[r, "I"] \arrow[d, "I"] & H_0^1(\Omega) \arrow[r, "\nabla"] \arrow[d, "\Pi^1"] & H_0(\mathrm{curl},\Omega) \arrow[r, "\nabla\times"] \arrow[d, "\Pi^{\mathrm{curl}}"] & H_0(\mathrm{div},\Omega) \arrow[r, "\mathrm{div}"] \arrow[d, "\Pi^{\mathrm{div}}"] & L_0^2(\Omega) \arrow[r, "\mathcal{O}"] \arrow[d, "\Pi"] & \{0\} \arrow[d, "I"] \\
\{0\} \arrow[r, "I"] & V^p(\mathcal{K},\Omega) \arrow[r, "\nabla"] & W^p(\mathcal{K},\Omega) \arrow[r, "\nabla\times"] & X^p(\mathcal{K},\Omega) \arrow[r, "\mathrm{div}"] & U^p(\mathcal{K},\Omega) \arrow[r, "\mathcal{O}"] & \{0\}
\end{tikzcd}$$

commutes, i.e. it holds

$$\begin{aligned}
\nabla\Pi^1 &= \Pi^{\mathrm{curl}}\nabla \\
\nabla\times\Pi^{\mathrm{curl}} &= \Pi^{\mathrm{div}}(\nabla\times) \\
\mathrm{div}\left(\Pi^{\mathrm{div}}\right) &= \Pi\,\mathrm{div}\,.
\end{aligned}$$

Proof.

1. The proof follows directly from Theorem 1 in [191] and Proposition 3 in [98].
2. The proof follows directly from Theorem 1 in [191] and Theorem 1 in [99].

□

With this result we want to conclude this review of the de Rham complexes. We have derived commuting diagrams consisting of the exact sequences from Theorems 1.8 and 2.8 and the interpolation operators from Section 2.4. Commutativity will be of great importance in the analysis of the following chapters. For more information about de Rham complexes and de Rham diagrams we refer the interested reader to the books of Bossavit [57, 58], the survey article of Hiptmair [127] and the dissertation of Zaglmayr [219].

Chapter 3

The Poisson Problem

In this chapter we want to consider the numerical solution of the *Poisson problem* with the *hp*-adaptive finite element method. This problem has been studied extensively in the past (see e.g. [10, 23, 52, 101, 103, 107, 126, 135, 151, 157, 161, 162, 171, 173, 174, 180, 207, 215] to name only a few references) and, thus, is the perfect candidate for a simple elliptic model problem and results can often be applied easily to other elliptic problems as well. For this model problem we want to develop a fully automatic *hp*-adaptive refinement strategy, whose numerical solution converges to the analytical solution of the problem. The idea for this refinement strategy goes back to the work of Dörfler and Heuveline [104] in the case $d = 1$. Therefore this chapter is split into two parts. In the first part we derive the refinement strategy for the *hp*-adaptive continuous Galerkin finite element method and prove some convergence results. In the second part we derive the refinement strategy for the *hp*-adaptive discontinuous Galerkin finite element method and prove its convergence also for this discretization.

Before we start let us state the Poisson problem, which we consider in this chapter, first. Therefore let $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, be open and bounded with polygonal (in the case $d = 2$) or polyhedral (in the case $d = 3$), Lipschitz-continuous boundary. Then we look for a solution of the problem to find $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{3.1}$$

for $f \in L^2(\Omega)$.

3.1 The Continuous Galerkin Finite Element Method

For the continuous Galerkin finite element method there have been proposed various *hp*-adaptive refinement strategies, e.g. in [107, 136, 215] the analyticity of the solution is estimated, in [10, 74, 104, 126] local boundary value problems are solved and in [101, 180, 181] the global interpolation error is minimized. However most of these strategies lack a rigorous proof of convergence. Thus the goal of this section is to derive an efficient *hp*-adaptive refinement strategy and prove its convergence. Therefore we derive the weak formulation of problem (3.1) first. Then we present the refinement strategy in the case $d = 1$ and prove its convergence to give a simple insight into the basic ideas of the strategy. After this the refinement strategy is extended to higher space-dimensions $d \in \{2, 3\}$.

3.1.1 The Problem Formulation

In this subsection we derive the weak and discrete formulations of problem (3.1). Then the most important properties of the bilinear form resulting from the weak formulation are discussed.
Let us start with the derivation of the weak formulation. Therefore we multiply the first equation of problem (3.1) with some test function $\phi \in H_0^1(\Omega)$ and integrate over Ω. This yields

$$-\int_\Omega \phi \Delta u = \int_\Omega \phi f \qquad \forall \phi \in H_0^1(\Omega)$$

and with integration by parts we obtain the weak formulation

$$\int_\Omega (\nabla\phi)^T \nabla u = \int_\Omega \phi f \qquad \forall \phi \in H_0^1(\Omega) \tag{3.2}$$

of problem (3.1). Then we can define the bilinear form $A : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$A(\phi, \psi) := \int_\Omega (\nabla\phi)^T \nabla\psi.$$

Now we can show that A is continuous and elliptic. The continuity follows immediately from Hölder's inequality.

Lemma 3.1 (Continuity of A)**.** *The bilinear form A is* continuous*, i.e. it holds*

$$A(\phi, \psi) \leq \|\nabla\phi\|_{L^2(\Omega)^d} \|\nabla\psi\|_{L^2(\Omega)^d} \qquad \forall \phi, \psi \in H_0^1(\Omega).$$

The ellipticity of the bilinear form A is trivial.

Lemma 3.2 (Ellipticity of A)**.** *The bilinear form A is* elliptic*, i.e. it holds*

$$A(\phi, \phi) = \|\nabla\phi\|^2_{L^2(\Omega)^d} \qquad \forall \phi \in H_0^1(\Omega).$$

Then it follows immediately from the Lax-Milgram Theorem that weak problem (3.2) has a unique solution $u \in H_0^1(\Omega)$. Thus, it makes sense to consider the discrete formulation of (3.2) to obtain a numerical approximation for the solution of problem (3.1). Therefore let $\mathcal{K}$ be some (γ_1, γ_2)-regular triangulation of Ω. Then the discrete formulation of problem (3.2) reads to find $u_{\mathrm{FE}} \in V^p(\mathcal{K}, \Omega)$ such that

$$A\left(\phi, u_{\mathrm{FE}}\right) = \int_\Omega \phi f \qquad \forall \phi \in V^p(\mathcal{K}, \Omega). \tag{3.3}$$

For this discretization of problem (3.1) the following a priori error estimate was proven by Babuška and Suri [36].

Theorem 3.1 (A Priori Error Estimate)**.** *Let $k \in \mathbb{N}$ be arbitrary and assume that the solution $u \in H_0^1(\Omega)$ of (3.2) has the additional regularity $H^k(\Omega)$. Further let $u_{FE} \in V^p(\mathcal{K}, \Omega)$ be the solution of (3.3) and assume that there exist $\widetilde{h} > 0$ and $\widetilde{p} \in \mathbb{N}$ such that the mesh size vector h is given by $h_K := \widetilde{h}$ for all $K \in \mathcal{K}$ and the polynomial degree vector p is given by $p_K := \widetilde{p}$ for all $K \in \mathcal{K}$, respectively. Then there exists some constant $C > 0$ independent of h and p such that*

$$\|\nabla\left(u - u_{FE}\right)\|_{L^2(\Omega)^d} \leq C \frac{\widetilde{h}^\mu}{\widetilde{p}^{k-1}} \|u\|_{H^k(\Omega)},$$

where $\mu := \min\{\widetilde{p}, k-1\}$.

Proof. See Theorem 5.4 in [36]. □

If the mesh size vector h and the polynomial degree vector p are chosen suitably and the solution u is sufficiently smooth, it can be shown that the error decays exponentially. For $d = 2$ Babuška and Guo [24] have proven the estimate

$$\|\nabla (u - u_{\mathrm{FE}})\|_{L^2(\Omega)^d} \leq C_1 \exp\left(-C_2 N^{\frac{1}{3}}\right),$$

where $N := \dim (V^p(\mathcal{K}, \Omega))$ and $C_1, C_2 > 0$ denote some constants independent of N.

3.1.2 The One-Dimensional Case

In this subsection we derive a fully automatic *hp*-adaptive refinement strategy in the case $d = 1$. This is more or less a repetition of the results from Dörfler and Heuveline [104] and shall only provide a simple start for the higher-dimensional version in the following subsection. Therefore we begin with the standard adaptive loop

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE.} \tag{3.4}$$

Whereas the modules SOLVE and REFINE are usually the same for all adaptive strategies and rather depend on the finite element library and the solver one wants to use, the modules ESTIMATE and MARK are the heart of every refinement strategy. Therefore we discuss these modules in detail for the *hp*-adaptive refinement strategy presented in [104]. Further we prove some convergence results for this fully automatic refinement algorithm. For numerical examples we refer to the work of Dörfler and Heuveline [104].

3.1.2.1 The Refinement Strategy

Here we present the basic ideas of the fully automatic *hp*-adaptive refinement strategy from [104] and discuss the modules ESTIMATE and MARK from the adaptive loop (3.4).
Let us start with the module ESTIMATE. The goal of this module is to determine on every cell how large the error of the computed finite element solution, which we obtain from module SOLVE, is. Since one usually does not know the analytic solution of problem (3.2), the exact energy error $\|u' - u'_{\mathrm{FE}}\|_{L^2(\Omega)}$ cannot be computed. To overcome this difficulty *a posteriori error estimators*, which give an estimation of the exact energy error in terms of the computed solution from module SOLVE and the given data, have been developed. Of course there have been proposed many different methods for this task and we name only a few here. In [33, 170, 207] explicit a posteriori error estimators have been presented. This type of error estimators obtains an estimate for the exact energy error by evaluating some explicit formula. The implicit a posteriori error estimators from [29, 30, 31, 32] require the solution of auxiliary boundary value problems instead. Another approach is to solve the problem of interest on two different discretizations in the module SOLVE and compare these two solutions (see e.g. [44, 45]). The equilibrated a posteriori error estimators from [46, 52, 142, 144] require the solution of some dual problem. Besides the estimation of the energy error $\|u' - u'_{\mathrm{FE}}\|_{L^2(\Omega)}$ one can also estimate and refine according to other quantities of interest (see e.g. [26, 27, 28, 51, 76, 143, 150, 175, 179]). Although all a posteriori error estimators could be combined with the *hp*-adaptive refinement strategy presented here, we restrict ourselves to the class of explicit a posteriori error estimators for simplicity. As an example we consider the residual-based a posteriori error estimator from Schwab [194].

Definition 3.1 (A Posteriori Error Estimator). *Let $u_{FE} \in V^p(\mathcal{K}, \Omega)$ be the solution of (3.3). Then the* residual-based a posteriori error estimator *$\eta\,(u_{FE}, \mathcal{K})$ is given by*

$$\eta\,(u_{FE}, \mathcal{K})^2 := \sum_{K \in \mathcal{K}} \eta_K\,(u_{FE}, \mathcal{K})^2\,,$$

where the cell term $\eta_K\,(u_{FE}, \mathcal{K})$ reads

$$\eta_K\,(u_{FE}, \mathcal{K}) := \frac{1}{\sqrt{p_K\,(p_K+1)}} \left\| \sqrt{d_K}\,(\Pi f + u''_{FE}) \right\|_{L^2(K)}.$$

Here the weight function $d_K : K \to \mathbb{R}_+$ is defined by

$$d_K(x) := (v_{K,1} - x)\,(x - v_{K,0})$$

for $K =: [v_{K,0}, v_{K,1}]$.

For this a posteriori error estimator it was shown in [104] that it is *reliable*, i.e. it always overestimates the energy error $\|u' - u'_{\mathrm{FE}}\|_{L^2(\Omega)}$, and *efficient*, i.e. the overestimation is not too large.

Theorem 3.2 (A Posteriori Error Estimates). *Let $u \in H_0^1(\Omega)$ be the solution of (3.2) and $u_{FE} \in V^p(\mathcal{K}, \Omega)$ be the solution of (3.3). Then:*

1. *It holds*
$$\|u' - u'_{FE}\|^2_{L^2(\Omega)} \leq \eta\,(u_{FE}, \mathcal{K})^2 + \frac{1}{4} \sum_{K \in \mathcal{K}} \frac{h_K^2}{p_K^2} \|f - \Pi f\|^2_{L^2(K)}.$$

2. *There exists some constant $C_{\mathit{eff}} > 1$ independent of mesh size vector h and polynomial degree vector p such that*
$$\eta_K\,(u_{FE}, \mathcal{K})^2 \leq C_{\mathit{eff}} \left(\|u' - u'_{FE}\|^2_{L^2(K)} + \frac{h_K^2}{4p_K^2} \|f - \Pi f\|^2_{L^2(K)} \right)$$
for all $K \in \mathcal{K}$.

Proof. See Theorem 3 in [104]. □

These a posteriori error estimates will play an important role in the proofs of convergence of the *hp*-adaptive refinement algorithm later on.
Now we make a step ahead and consider the module MARK. For the *h*- and the *p*-adaptive finite element method it is sufficient to use the information obtained in the previous module and refine the cells with the biggest local error contribution. However, for the *hp*-adaptive finite element method this is not enough, because one also has to decide which refinement should be performed on the selected cells. In the *hp*-adaptive finite element method there are always at least two different *refinement patterns*. The classical *h*-refinement, where the cell is bisected equally, and the classical *p*-refinement, where the polynomial degree present on the cell is increased by one. A graphical representation can be seen in Figure 3.1. However, there can be even more refinement patterns. For example one can define a weighted bisection or increase the polynomial degree by some integer $k \in \mathbb{N}$. Since we do not provide any numerical examples in this subsection, we refer the interested reader to the work of Dörfler and Heuveline [104] for a broader overview. Here we simply assume that we have $n \in \mathbb{N} \setminus \{1\}$ different refinement patterns to choose from. Now let $j \in \{0, \ldots, n-1\}$ and $K \in \mathcal{K}$ be arbitrary. Then we denote by $V_j^p\,(\{K\}, K)$ the local finite element space consisting of functions from $V^p(\mathcal{K}, \Omega)$ compactly supported in cell K with refinement pattern j applied to cell K. Without loss of generality we may assume that $\eta_K\,(u_{\mathrm{FE}}, \mathcal{K}) > 0$. If this is

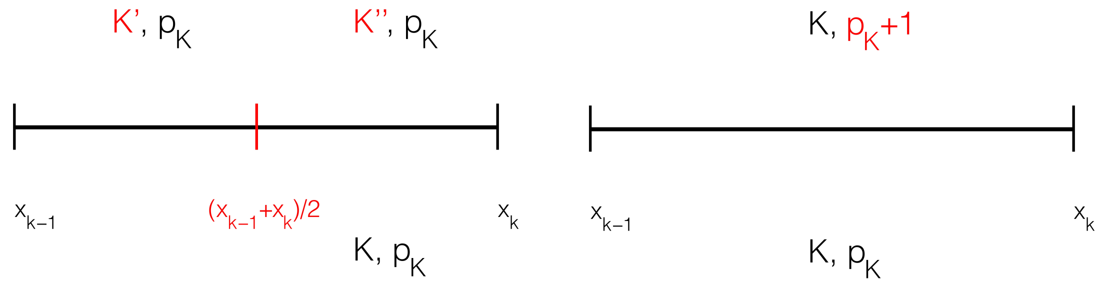

Figure 3.1: Refinement patterns. Left: h-refinement. Right: p-refinement.

not the case, it makes not any sense to refine this cell at all and we can go to the next. Then we define the *convergence indicator* $\kappa_{K,j} \in \mathbb{R}_+$ as the solution of the optimization problem

$$\kappa_{K,j} = \frac{1}{\eta_K\left(u_{\mathrm{FE}}, \mathcal{K}\right)} \sup_{\phi \in V_j^p(\{K\},K)} \left(\frac{\int_K \phi\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|\phi'\|_{L^2(K)}} \right), \tag{3.5}$$

where $\Pi : L^2(K) \to U^p\left(\{K\}, K\right)$ denotes the L^2-conforming interpolation from Section 2.4. To solve problem (3.5) one can use the methods known from numerical optimization [116, 169], of course. However, there is a much simpler way this time.

Lemma 3.3. *Let $v \in V_j^p\left(\{K\}, K\right)$ such that*

$$\int_K \phi' v' = \int_K \phi\left(\Pi f + u''_{FE}\right) \qquad \forall \phi \in V_j^p\left(\{K\}, K\right). \tag{3.6}$$

Then the supremum in (3.5) is obtained for v.

Proof. Let $\phi \in V_j^p\left(\{K\}, K\right)$ be arbitrary. Then we see

$$\begin{aligned} \frac{\int_K \phi\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|\phi'\|_{L^2(K)}} &= \frac{\int_K \phi' v'}{\|\phi'\|_{L^2(K)}} \\ &\leq \|v'\|_{L^2(K)} \end{aligned}$$

with Hölder's inequality and we have

$$\begin{aligned} \frac{\int_K \phi\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|\phi'\|_{L^2(K)}} &\leq \frac{\int_K v' v'}{\|v'\|_{L^2(K)}} \\ &= \frac{\int_K v\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|v'\|_{L^2(K)}} \end{aligned}$$

by (3.6). Since $\phi \in V_j^p\left(\{K\}, K\right)$ was arbitrary, this implies

$$\sup_{\phi \in V_j^p(\{K\},K)} \left(\frac{\int_K \phi\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|\phi'\|_{L^2(K)}} \right) \leq \frac{\int_K v\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|v'\|_{L^2(K)}}.$$

Thus

$$\sup_{\phi \in V_j^p(\{K\},K)} \left(\frac{\int_K \phi\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|\phi'\|_{L^2(K)}} \right) = \frac{\int_K v\left(\Pi f + u''_{\mathrm{FE}}\right)}{\|v'\|_{L^2(K)}},$$

since $v \in V_j^p\left(\{K\}, K\right)$. ☐

We note that problem (3.5) is fully independent for all cells $K \in \mathcal{K}$ and all refinement patterns $j \in \{0, \dots, n-1\}$. Thus this step of the algorithm is almost perfectly parallelizable. Additionally, adding more refinement patterns to the algorithm does not automatically result in a bigger time consumption of this step, if sufficiently many cores are available on the machine.

After solving optimization problem (3.5) for every cell $K \in \mathcal{K}$ and every refinement pattern $j \in \{0, \dots, n-1\}$ we have an indication which refinement pattern provides the biggest error reduction on every cell. However, this information alone does not necessarily lead to an efficient refinement strategy. To see this let us think of the case of two refinement patterns promising almost the same error reduction on some cell. Refinement pattern 0 requires 5 work units and refinement pattern 1 requires two work units. Then, we probably would like to choose refinement pattern 1, because it produces almost the same output as refinement pattern 0, but requires less than half the work. Therefore let us define *workload numbers* $w_{K,j} \in \mathbb{R}_+$ for every cell $K \in \mathcal{K}$ and every refinement pattern $j \in \{0, \dots, n-1\}$. This workload numbers shall indicate the required work of refinement pattern j on cell K. Possible choices for such workload numbers are e.g. the number of degrees of freedom of the local finite element space $V_j^p(\{K\}, K)$, the time required for solving optimization problem (3.5) or the memory used in the solution process of (3.5).

Now we can go ahead and mark the cells for refinement. This is done by looking for a solution $\left(\mathcal{A}, (j_K)_{K \in \mathcal{A}}\right)$, where $\mathcal{A} \subseteq \mathcal{K}$, of the maximization problem

$$\sum_{K \in \mathcal{A}} \frac{\kappa_{K,j_K}}{w_{K,j_K}} = \max \tag{3.7}$$

under the constraint

$$\sum_{K \in \mathcal{A}} \kappa_{K,j_K}^2 \eta_K \left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 \geq \theta^2 \eta \left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 \tag{3.8}$$

for some $\theta \in (0, 1]$ sufficiently small. Condition (3.8) is a modifcation of the *fixed energy fraction strategy*, which was introduced by Dörfler in [103]. It basically controls the portion of cells which shall be refined, whereas (3.7) decides which refinement pattern shall be applied. However, it cannot be guaranteed that problem (3.7), (3.8) has a solution at all. This might be the case, if e.g. the convergence indicators $\kappa_{K,j}$ are too small. We will discuss this point at the end of Section 3.1.2.2. For now let us assume that maximization problem (3.7), (3.8) has a solution. Even then it is not an easy task to find one. We see immediately that problem (3.7), (3.8) is a variation of the Travelling Salesman Problem (see e.g. [87, 158]). This problem is NP-hard [87] and, thus, solution time grows exponentially. Although there has been some success in solving this type of problems for a few thousand nodes (see. e.g. [16, 86, 123]), the solution time of these approaches exceeds the time for solving discrete problem (3.3) by far. Therefore we try a quite simple, but cheap, approach to approximate a solution of maximization problem (3.7), (3.8). In a first step we define for every cell $K \in \mathcal{K}$ the integer $j_K \in \{0, \dots, n-1\}$ by the relation

$$\frac{\kappa_{K,j_K}}{w_{K,j_K}} = \max_{j \in \{0, \dots, n-1\}} \left(\frac{\kappa_{K,j}}{w_{K,j}}\right).$$

Then we construct the set $\mathcal{A} \subseteq \mathcal{K}$ with minimal cardinality. For this task we can sort the cells $K \in \mathcal{K}$ according to the values of $\kappa_{K,j_K} \eta_K \left(u_{\mathrm{FE}}, \mathcal{K}\right)$ by e.g. counting sort, radix sort or bucket sort and then add cells to $\mathcal{A}$ until condition (3.8) is fulfilled. Another approach is sorting into bins as proposed in [103]. Both ways allow the construction of $\mathcal{A}$ in linear time and thus are applicable equally well.

Now we have derived the *hp*-adaptive refinement strategy in detail. To conclude this paragraph let us state the fully automatic *hp*-adaptive refinement algorithm in total:

(S0) Build a coarse grid $\mathcal{K}_0$ and set $N := 0$. Choose $\theta \in (0,1]$ and tolerance TOL.

(S1) SOLVE: Compute the solution $u_N \in V^p(\mathcal{K}_N, \Omega)$ of discrete problem (3.3).

(S2) ESTIMATE: Compute the a posteriori error estimator $\eta(u_N, \mathcal{K}_N)$.

(S3) If $\eta(u_N, \mathcal{K}_N) \leq$ TOL: STOP

(S4) MARK: Compute the convergence indicators $\kappa_{K,j}$ and the workload numbers $w_{K,j}$ for all cells $K \in \mathcal{K}_N$ and all refinement possibilities $j \in \{0, \ldots, n-1\}$.

(S5) MARK: Approximate the solution $\left(\mathcal{A}_N, (j_K)_{K \in \mathcal{A}_N}\right)$ of maximization problem (3.7), (3.8).

(S6) REFINE: Refine the cells contained in $\mathcal{A}_N$ according to refinement vector (j_K).

(S7) Set $N := N + 1$ and continue with step (S1).

We have seen that the interesting parts of the *hp*-adaptive refinement strategy in the adaptive loop (3.4) are the modules ESTIMATE and MARK. In module ESTIMATE we require an a posteriori error estimator, which allows a decomposition of the estimated error $\eta(u_{\mathrm{FE}}, \mathcal{K})$ into local error contributions $\eta_K(u_{\mathrm{FE}}, \mathcal{K})$. For the proof of convergence of the fully automatic refinement algorithm also reliability and efficiency estimates as presented in Theorem 3.2 will be required. In module MARK the crucial part of the *hp*-adaptive refinement strategy is located. Here, for every cell $K \in \mathcal{K}$ the decision, which refinement pattern is favourable, is made. This decision is based on the solution of optimization problem (3.5). The actual marking of the cells is performed by approximating the solution of maximization problem (3.7), (3.8).

3.1.2.2 Convergence Results

Now we prove convergence of the fully automatic *hp*-adaptive refinement strategy. Therefore we derive two results. The first result proves that the exact energy error $\|u' - u'_{\mathrm{FE}}\|_{L^2(\Omega)}$ is reduced in every refinement step of the algorithm. The second result gives us that a weighted sum of exact energy error $\|u' - u'_{\mathrm{FE}}\|_{L^2(\Omega)}$ and estimated error $\eta(u_{\mathrm{FE}}, \mathcal{K})$ is reduced in every refinement step.
Before we prove the main results of this paragraph let us consider the error estimator $\eta(u_{\mathrm{FE}}, \mathcal{K})$, which was introduced in Definition 3.1, in more detail. We want to investigate how the estimated error is influenced by the application of the *hp*-adaptive refinement algorithm presented in Section 3.1.2.1. The various assumptions, which we make in the following lemmas and theorems, will be discussed at the end of this section in detail.

Lemma 3.4 (Error Estimator Reduction). *Let $N \in \mathbb{N}_0$ be arbitrary and assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p(\mathcal{K}_N, \Omega)$ and $u_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. We assume that for all refinement patterns $j \in \{0, \ldots, n-1\}$ there exists some constant $\rho \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\frac{\sqrt{d_K}}{p_K} \leq \rho \frac{\sqrt{d_{\widetilde{K}}}}{p_{\widetilde{K}}} \tag{3.9}$$

for all refined cells $\widetilde{K} \in \mathcal{K}_N$ and all $K \in \mathcal{K}_{N+1}$ with $K \subseteq \widetilde{K}$. Additionally let us assume that there exists some $\tau \in (0,1]$ such that

$$\sum_{K \in \mathcal{K}_N} \frac{h_K^2}{p_K^2} \|f - \Pi f\|^2_{L^2(K)} \leq \tau^2 \eta(u_N, \mathcal{K}_N)^2 . \tag{3.10}$$

Then it holds

$$\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 \leq (1+\delta)^2\left(\left(1+\frac{\rho^2\tau^2}{2\delta}\right)\eta\left(u_N,\mathcal{K}_N\right)^2-\left(1-\rho^2\right)\eta\left(u_N,\mathcal{A}_N\right)^2\right) + \left(1+\frac{1}{\delta}\right)\left\|u'_{N+1}-u'_N\right\|^2_{L^2(\Omega)}$$

for all $\delta > 0$.

Proof. Let $K \in \mathcal{K}_{N+1}$ be arbitrary. Then, by Definition 3.1 it holds

$$\eta_K\left(u_{N+1},\mathcal{K}_{N+1}\right) = \frac{1}{\sqrt{p_K\left(p_K+1\right)}}\left\|\sqrt{d_K}\left(\Pi_{\mathcal{K}_{N+1}}f+u''_{N+1}\right)\right\|_{L^2(K)}$$

with $\Pi_{\mathcal{K}_{N+1}} : L^2(\Omega) \to U^p\left(\mathcal{K}_{N+1},\Omega\right)$ denoting the L^2-conforming interpolation from Section 2.4. The triangle inequality immediately yields

$$\eta_K\left(u_{N+1},\mathcal{K}_{N+1}\right) \leq T_1+T_2+T_3 \tag{3.11}$$

with the terms T_1, T_2 and T_3 given by

$$T_1 := \frac{1}{\sqrt{p_K(p_K+1)}}\left\|\sqrt{d_K}\left(\Pi_{\mathcal{K}_N}f+u''_N\right)\right\|_{L^2(K)},$$
$$T_2 := \frac{1}{\sqrt{p_K(p_K+1)}}\left\|\sqrt{d_K}\left(\Pi_{\mathcal{K}_{N+1}}f-\Pi_{\mathcal{K}_N}f\right)\right\|_{L^2(K)}$$

and

$$T_3 := \frac{1}{\sqrt{p_K(p_K+1)}}\left\|\sqrt{d_K}\left(u''_{N+1}-u''_N\right)\right\|_{L^2(K)}.$$

Let us consider term T_1 first. If there exists some cell $\widetilde{K} \in \mathcal{A}_N$ such that $K \subseteq \widetilde{K}$, then it holds

$$T_1 \leq \frac{\rho}{\sqrt{p_{\widetilde{K}}\left(p_{\widetilde{K}}+1\right)}}\left\|\sqrt{d_{\widetilde{K}}}\left(\Pi_{\mathcal{K}_N}f+u''_N\right)\right\|_{L^2(K)} \tag{3.12}$$

by assumption (3.9). For T_2 it holds

$$\begin{aligned} T_2 &\leq \frac{1}{p_K}\left\|\sqrt{d_K}\left(f-\Pi_{\mathcal{K}_N}f\right)\right\|_{L^2(K)} \\ &\leq \frac{\rho}{p_{\widetilde{K}}}\left\|\sqrt{d_{\widetilde{K}}}\left(f-\Pi_{\mathcal{K}_N}f\right)\right\|_{L^2(K)} \end{aligned}$$

by assumption (3.9) and we obtain

$$T_2 \leq \frac{\rho h_{\widetilde{K}}}{2p_{\widetilde{K}}}\left\|f-\Pi_{\mathcal{K}_N}f\right\|_{L^2(K)}. \tag{3.13}$$

Now let us consider the case that there exists no such $\widetilde{K} \in \mathcal{A}_N$. Then $K \in \mathcal{K}_N$ and it follows

$$T_1 = \eta_K\left(u_N,\mathcal{K}_N\right) \tag{3.14}$$

and

$$T_2 = 0. \tag{3.15}$$

For the term T_3 we get

$$T_3 \le \left\| u'_{N+1} - u'_N \right\|_{L^2(K)} \tag{3.16}$$

by Lemma 2 in [104] in both cases.
Then inserting estimates (3.12)–(3.16) into (3.11) gives

$$\begin{aligned} \eta_K\left(u_{N+1}, \mathcal{K}_{N+1}\right) \le \rho \left(\frac{1}{\sqrt{p_{\widetilde{K}}\left(p_{\widetilde{K}}+1\right)}} \left\| \sqrt{d_{\widetilde{K}}} \left(\Pi_{\mathcal{K}_N} f + u''_N\right) \right\|_{L^2(K)} + \frac{h_{\widetilde{K}}}{2p_{\widetilde{K}}} \left\| f - \Pi_{\mathcal{K}_N} f \right\|_{L^2(K)} \right) \\ + \left\| u'_{N+1} - u'_N \right\|_{L^2(K)}, \end{aligned}$$

if there exists such a cell $\widetilde{K} \in \mathcal{A}_N$ with $K \subseteq \widetilde{K}$, and

$$\eta_K\left(u_{N+1}, \mathcal{K}_{N+1}\right) \le \eta_K\left(u_N, \mathcal{K}_N\right) + \left\| u'_{N+1} - u'_N \right\|_{L^2(K)},$$

else. Squaring the estimates above, summing over all $K \in \mathcal{K}_{N+1}$ and using Young's inequality yields

$$\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \le (1+\delta)\eta\left(u_N, \mathcal{K}_N \setminus \mathcal{A}_N\right)^2 + \rho^2(1+\delta)^2 T + \left(1 + \frac{1}{\delta}\right) \left\| u'_{N+1} - u'_N \right\|^2_{L^2(\Omega)} \tag{3.17}$$

for $\delta > 0$ with T given by

$$T := \eta\left(u_N, \mathcal{A}_N\right)^2 + \frac{1}{2\delta} \sum_{K \in \mathcal{A}_N} \frac{h_K^2}{p_K^2} \left\| f - \Pi_{\mathcal{K}_N} f \right\|^2_{L^2(K)}.$$

By data saturation assumption (3.10) it follows

$$T \le \eta\left(u_N, \mathcal{A}_N\right)^2 + \frac{\tau^2}{2\delta} \eta\left(u_N, \mathcal{K}_N\right)^2$$

and inserting into (3.17) implies

$$\begin{aligned} \eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \le & (1+\delta)^2 \left(\left(1 + \frac{\rho^2\tau^2}{2\delta}\right) \eta\left(u_N, \mathcal{K}_N\right)^2 - \left(1 - \rho^2\right) \eta\left(u_N, \mathcal{A}_N\right)^2 \right) \\ & + \left(1 + \frac{1}{\delta}\right) \left\| u'_{N+1} - u'_N \right\|^2_{L^2(\Omega)}, \end{aligned}$$

which is the assertion. □

Now we show two auxiliary results, which we use in the proofs of the main results of this paragraph. The first result gives a lower bound for the term $\left\| u'_{N+1} - u'_N \right\|_{L^2(\Omega)}$ in terms of the energy error $\left\| u' - u'_N \right\|_{L^2(\Omega)}$ and the estimated error $\eta\left(u_N, \mathcal{K}_N\right)$.

Lemma 3.5. *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H^1_0(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p\left(\mathcal{K}_N, \Omega\right)$ and $u_{N+1} \in V^p\left(\mathcal{K}_{N+1}, \Omega\right)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau \in (0,1]$ such that (3.10) holds. Then we have*

$$\left\| u'_{N+1} - u'_N \right\|^2_{L^2(\Omega)} \ge \delta \left(\frac{4\theta^2}{5(1+\delta)} \left\| u' - u'_N \right\|^2_{L^2(\Omega)} - C^2_{grad} \tau^2 \eta\left(u_N, \mathcal{K}_N\right)^2 \right)$$

for all $\delta > 0$.

Proof. Let $K \in \mathcal{K}_N$ be arbitrary and $\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ with $\operatorname{supp}(\phi_{N+1}) \subseteq K$. Then we see

$$\int_K \phi'_{N+1}\left(u'_{N+1} - u'_N\right) = \int_K \phi_{N+1} f - \int_K \phi'_{N+1} u'_N,$$

since $u_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ solves discrete problem (3.3). This reads

$$\int_K \phi'_{N+1}\left(u'_{N+1} - u'_N\right) = \int_K \phi_{N+1} \Pi f - \int_K \phi'_{N+1} u'_N + \int_K \phi_{N+1}(f - \Pi f),$$

where $\Pi : L^2(\Omega) \to U^p(\mathcal{K}_N, \Omega)$ denotes the L^2-conforming interpolation from Section 2.4. With integration by parts and the L^2-interpolation property this implies

$$\int_K \phi'_{N+1}\left(u'_{N+1} - u'_N\right) = \int_K \phi_{N+1}\left(\Pi f + u''_N\right) + \int_K \left(\phi_{N+1} - \phi_N\right)\left(f - \Pi f\right)$$

for $\phi_N \in V^{p_K}(\{K\}, K)$ and using the inverse triangle inequality yields

$$\left|\int_K \phi'_{N+1}\left(u'_{N+1} - u'_N\right)\right| \geq \left|\int_K \phi_{N+1}\left(\Pi f + u''_N\right)\right| - \left|\int_K \left(\phi_{N+1} - \phi_N\right)\left(f - \Pi f\right)\right|.$$

With Hölder's inequality we have

$$\left|\int_K \phi_{N+1}\left(\Pi f + u''_N\right)\right| \leq \left\|u'_{N+1} - u'_N\right\|_{L^2(K)} \left\|\phi'_{N+1}\right\|_{L^2(K)} + \|f - \Pi f\|_{L^2(K)} \|\phi_{N+1} - \phi_N\|_{L^2(K)} \tag{3.18}$$

and choosing $\phi_N := \Pi^1 \phi_{N+1}$ with $\Pi^1 : H_0^1(\Omega) \to V^p(\mathcal{K}_N, \Omega)$ from Section 2.4.1 implies

$$\begin{aligned}\|\phi_{N+1} - \phi_N\|_{L^2(K)} &= \left\|\phi_{N+1} - \Pi^1 \phi_{N+1}\right\|_{L^2(K)} \\ &\leq C_{\mathrm{grad}} \frac{h_K}{p_K} \left\|\phi'_{N+1}\right\|_{L^2(\omega_{K,4})}\end{aligned}$$

by Theorem 2.4. Since $\operatorname{supp}(\phi_{N+1}) \subseteq K \subset \omega_{K,4}$, it holds

$$\|\phi_{N+1} - \phi_N\|_{L^2(K)} \leq C_{\mathrm{grad}} \frac{h_K}{p_K} \left\|\phi'_{N+1}\right\|_{L^2(K)}$$

and inserting into (3.18) gives

$$\left|\int_K \phi_{N+1}\left(\Pi f + u''_N\right)\right| \leq \left(\left\|u'_{N+1} - u'_N\right\|_{L^2(K)} + C_{\mathrm{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(K)}\right) \left\|\phi'_{N+1}\right\|_{L^2(K)}.$$

Dividing by $\left\|\phi'_{N+1}\right\|_{L^2(K)}$ yields

$$\begin{aligned}\sup_{\phi \in V^p_{j_K}(\{K\}, K)} \left(\frac{\int_K \phi\left(\Pi f + u''_N\right)}{\|\phi'\|_{L^2(K)}}\right) &\leq \sup_{\substack{\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega) \\ \operatorname{supp}(\phi_{N+1}) \subseteq K}} \left(\frac{\int_K \phi_{N+1}\left(\Pi f + u''_N\right)}{\left\|\phi'_{N+1}\right\|_{L^2(K)}}\right) \\ &\leq \left\|u'_{N+1} - u'_N\right\|_{L^2(K)} + C_{\mathrm{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(K)},\end{aligned}$$

since $\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ with $\operatorname{supp}(\phi_{N+1}) \subseteq K$ was arbitrary. Then it follows

$$\kappa_{K, j_K} \eta_K(u_N, \mathcal{K}_N) \leq \left\|u'_{N+1} - u'_N\right\|_{L^2(K)} + C_{\mathrm{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(K)}$$

from optimization problem (3.5). Since $\mathcal{A}_N \subseteq \mathcal{K}_N$, squaring both sides, summing over $K \in \mathcal{K}_N$ and using Young's inequality yields

$$\begin{aligned}\sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N,\mathcal{K}_N\right)^2 \le& \sum_{K\in\mathcal{K}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N,\mathcal{K}_N\right)^2 \\ &\le \left(1+\frac{1}{\delta}\right)\left\|u_{N+1}'-u_N'\right\|_{L^2(\Omega)}^2 + C_{\text{grad}}^2(1+\delta)\sum_{K\in\mathcal{K}_N}\frac{h_K^2}{p_K^2}\|f-\Pi f\|_{L^2(K)}^2\end{aligned}$$

for $\delta > 0$. Finally, from data saturation assumption (3.10) it follows

$$\sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N,\mathcal{K}_N\right)^2 \le \left(1+\frac{1}{\delta}\right)\left\|u_{N+1}'-u_N'\right\|_{L^2(\Omega)}^2 + C_{\text{grad}}^2\tau^2(1+\delta)\eta\left(u_N,\mathcal{K}_N\right)^2. \tag{3.19}$$

From Theorem 3.2 we know

$$\begin{aligned}\|u'-u_N'\|_{L^2(\Omega)}^2 &\le \eta\left(u_N,\mathcal{K}_N\right)^2 + \frac{1}{4}\sum_{K\in\mathcal{K}_N}\frac{h_K^2}{p_K^2}\|f-\Pi f\|_{L^2(K)}^2 \\ &\le \left(1+\frac{\tau^2}{4}\right)\eta\left(u_N,\mathcal{K}_N\right)^2\end{aligned}$$

by assumption (3.10). For $\tau \le 1$ this reads

$$\|u'-u_N'\|_{L^2(\Omega)}^2 \le \frac{5}{4}\eta\left(u_N,\mathcal{K}_N\right)^2$$

and multiplying both sides by θ^2, $\theta \in (0,1]$, yields

$$\theta^2\|u'-u_N'\|_{L^2(\Omega)}^2 \le \frac{5}{4}\sum_{K\in\mathcal{A}_N}\kappa_{K,j_K}^2\eta_K\left(u_N,\mathcal{K}_N\right)^2$$

by constraint (3.8). With (3.19) it follows

$$\frac{4}{5}\theta^2\|u'-u_N'\|_{L^2(\Omega)}^2 \le \left(1+\frac{1}{\delta}\right)\left\|u_{N+1}'-u_N'\right\|_{L^2(\Omega)}^2 + C_{\text{grad}}^2\tau^2(1+\delta)\eta\left(u_N,\mathcal{K}_N\right)^2$$

and this concludes the proof. $\square$

The next result compares the energy error of the finite element approximation on two successive grids of the algorithm.

Lemma 3.6 (Comparison of Errors)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H_0^1(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p\left(\mathcal{K}_N,\Omega\right)$ and $u_{N+1} \in V^p\left(\mathcal{K}_{N+1},\Omega\right)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau \in (0,1]$ such that (3.10) is fulfilled. Then it holds*

$$\left\|u'-u_{N+1}'\right\|_{L^2(\Omega)}^2 \le \left(1-\frac{2\delta\theta^2}{5(1+\delta)}\right)\|u'-u_N'\|_{L^2(\Omega)}^2 + \frac{C_{grad}^2\tau^2\delta}{2}\eta\left(u_N,\mathcal{K}_N\right)^2 - \frac{1}{2}\left\|u_{N+1}'-u_N'\right\|_{L^2(\Omega)}^2$$

for all $\delta > 0$.

Proof. Since $V^p\left(\mathcal{K}_N,\Omega\right) \subset V^p\left(\mathcal{K}_{N+1},\Omega\right)$, we can use the Galerkin orthogonality

$$A\left(u-u_{N+1}, u_{N+1}-u_N\right) = 0$$

to get

$$\begin{aligned}\|u' - u'_N\|^2_{L^2(\Omega)} &= A(u - u_N, u - u_N)\\ &= A(u - u_{N+1}, u - u_{N+1}) + A(u_{N+1} - u_N, u_{N+1} - u_N)\\ &= \|u' - u'_{N+1}\|^2_{L^2(\Omega)} + \left(\frac{1}{2} + \frac{1}{2}\right)\|u'_{N+1} - u'_N\|^2_{L^2(\Omega)}.\end{aligned}$$

By using Lemma 3.5 it follows

$$\left(1 - \frac{2\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|^2_{L^2(\Omega)} \geq \|u' - u'_{N+1}\|^2_{L^2(\Omega)} + \frac{1}{2}\|u'_{N+1} - u'_N\|^2_{L^2(\Omega)} - \frac{C^2_{\text{grad}}\tau^2\delta}{2}\eta(u_N, \mathcal{K}_N)^2$$

and this concludes the proof. □

Now we come to the first main result of this paragraph. It is basically the convergence result from [104] and states that the energy error $\|u' - u'_{\text{FE}}\|_{L^2(\Omega)}$ is reduced uniformly in every refinement step of the fully automatic *hp*-adative refinement algorithm from Section 3.1.2.1.

Theorem 3.3 (Convergence)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H^1_0(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p(\mathcal{K}_N, \Omega)$ and $u_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau \in (0,1]$ sufficiently small such that (3.10) is fulfilled. Then there exists some $\mu \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|u' - u'_{N+1}\|_{L^2(\Omega)} \leq \mu\|u' - u'_N\|_{L^2(\Omega)}.$$

Proof. From Theorem 3.2 we know

$$\begin{aligned}\eta(u_N, \mathcal{K}_N)^2 &\leq C_{\text{eff}}\left(\|u' - u'_N\|^2_{L^2(\Omega)} + \frac{1}{4}\sum_{K\in\mathcal{K}_N}\frac{h_K^2}{p_K^2}\|f - \Pi f\|^2_{L^2(K)}\right)\\ &\leq C_{\text{eff}}\left(\|u' - u'_N\|^2_{L^2(\Omega)} + \frac{\tau^2}{4}\eta(u_N, \mathcal{K}_N)^2\right)\end{aligned}$$

by data saturation assumption (3.10). Hence for $\tau < \frac{2}{\sqrt{C_{\text{eff}}}}$ we have

$$\eta(u_N, \mathcal{K}_N)^2 \leq \frac{4C_{\text{eff}}}{4 - C_{\text{eff}}\tau^2}\|u' - u'_N\|^2_{L^2(\Omega)}$$

and with the even more restrictive assumption $\tau \leq \frac{1}{\sqrt{C_{\text{eff}}}}$ we get

$$\eta(u_N, \mathcal{K}_N)^2 \leq \frac{4C_{\text{eff}}}{3}\|u' - u'_N\|^2_{L^2(\Omega)}. \tag{3.20}$$

From Lemma 3.6 we know

$$\|u' - u'_{N+1}\|^2_{L^2(\Omega)} \leq \left(1 - \frac{2\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|^2_{L^2(\Omega)} + \frac{C^2_{\text{grad}}\tau^2\delta}{2}\eta(u_N, \mathcal{K}_N)^2 - \frac{1}{2}\|u'_{N+1} - u'_N\|^2_{L^2(\Omega)}$$

for $\delta > 0$ and using Lemma 3.5 yields

$$\begin{aligned}\|u' - u'_{N+1}\|^2_{L^2(\Omega)} &\leq \left(1 - \frac{4\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|^2_{L^2(\Omega)} + C^2_{\text{grad}}\tau^2\delta\eta(u_N, \mathcal{K}_N)^2\\ &\leq \left(1 + \frac{4C_{\text{eff}}C^2_{\text{grad}}\tau^2\delta}{3} - \frac{4\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|^2_{L^2(\Omega)}\end{aligned}$$

by (3.20). The result follows with

$$\tau^2 < \frac{3\theta^2}{5C_{\mathrm{eff}}C_{\mathrm{grad}}^2(1+\delta)}.$$

□

The second main result of this paragraph is another convergence result, which states that the weighted sum of energy error $\|u' - u'_{\mathrm{FE}}\|_{L^2(\Omega)}$ and estimated error $\eta(u_{\mathrm{FE}}, \mathcal{K})$ is reduced in every iteration of the fully automatic *hp*-adaptive refinement algorithm from Section 3.1.2.1. We call this convergence property *quasi-convergence*. The proof follows the ideas of Bonito and Nochetto [55].

Theorem 3.4 (Quasi-Convergence). *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H_0^1(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p(\mathcal{K}_N, \Omega)$ and $u_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some constant $\rho \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that assumption (3.9) holds. We assume that there exists some $\tau \in (0,1]$ such that (3.10) is fulfilled. Then there exist constants $\mu \in (0,1)$ and $\nu > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left\|u' - u'_{N+1}\right\|_{L^2(\Omega)}^2 + \nu\eta(u_{N+1}, \mathcal{K}_{N+1})^2 \le \mu\left(\|u' - u'_N\|_{L^2(\Omega)}^2 + \nu\eta(u_N, \mathcal{K}_N)^2\right).$$

Proof. From Lemma 3.6 we know

$$\begin{aligned}&\left\|u' - u'_{N+1}\right\|_{L^2(\Omega)}^2 + \nu\eta(u_{N+1}, \mathcal{K}_{N+1})^2 \\ &\quad\le \left(1 - \frac{2\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|_{L^2(\Omega)}^2 + \frac{C_{\mathrm{grad}}^2\tau^2\delta}{2}\eta(u_N, \mathcal{K}_N)^2 + \nu\eta(u_{N+1}, \mathcal{K}_{N+1})^2 - \frac{1}{2}\left\|u'_{N+1} - u'_N\right\|_{L^2(\Omega)}^2\end{aligned}$$

and applying Lemma 3.4 yields

$$\begin{aligned}&\left\|u' - u'_{N+1}\right\|_{L^2(\Omega)}^2 + \nu\eta(u_{N+1}, \mathcal{K}_{N+1})^2 \\ &\quad\le \left(1 - \frac{2\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|_{L^2(\Omega)}^2 + \left(\frac{C_{\mathrm{grad}}^2\tau^2\delta}{2} + \frac{\delta(1+\delta)}{2}\left(1 + \frac{\rho^2\tau^2}{2\delta}\right)\right)\eta(u_N, \mathcal{K}_N)^2 \\ &\qquad - \frac{\delta(1+\delta)\left(1-\rho^2\right)}{2}\eta(u_N, \mathcal{A}_N)^2\end{aligned} \tag{3.21}$$

by choosing

$$\nu := \frac{\delta}{2(1+\delta)}.$$

We observe

$$C_{\max}^2\eta(u_N, \mathcal{A}_N)^2 \ge \sum_{K\in\mathcal{A}_N}\kappa_{K,j_K}^2\eta_K(u_N, \mathcal{K}_N)^2,$$

where

$$C_{\max} := \max\left\{\max_{K\in\mathcal{A}_N}(\kappa_{K,j_K}), 1\right\}.$$

Thus (3.21) reads

$$\begin{aligned}&\left\|u' - u'_{N+1}\right\|_{L^2(\Omega)}^2 + \nu\eta(u_{N+1}, \mathcal{K}_{N+1})^2 \\ &\quad\le \left(1 - \frac{2\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|_{L^2(\Omega)}^2 + \left(\frac{C_{\mathrm{grad}}^2\tau^2\delta}{2} + \frac{\delta(1+\delta)}{2}\left(1 + \frac{\rho^2\tau^2}{2\delta}\right)\right)\eta(u_N, \mathcal{K}_N)^2 \\ &\qquad - \frac{\delta(1+\delta)\left(1-\rho^2\right)}{2C_{\max}^2}\sum_{K\in\mathcal{A}_N}\kappa_{K,j_K}^2\eta_K(u_N, \mathcal{K}_N)^2,\end{aligned}$$

since $\rho < 1$. Then constraint (3.8) implies

$$\left\|u' - u'_{N+1}\right\|^2_{L^2(\Omega)} + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq \left(1 - \frac{2\delta\theta^2}{5(1+\delta)}\right)\|u' - u'_N\|^2_{L^2(\Omega)} + T\eta\left(u_N, \mathcal{K}_N\right)^2,$$

where the term T is given by

$$T := \frac{\delta}{2}\left(C^2_{\mathrm{grad}}\tau^2 + (1+\delta)\left(1 + \frac{\rho^2\tau^2}{2\delta}\right) - \frac{\theta^2(1+\delta)\left(1-\rho^2\right)}{C^2_{\max}}\right).$$

The result follows for δ sufficiently small. □

To conclude this paragraph let us shortly discuss the various assumptions we made above and see how these affect the main results of this paragraph.
We begin with the assumption that maximization problem (3.7), (3.8) has a solution $\left(\mathcal{A}_N, (j_K)_{K\in\mathcal{A}_N}\right)$ for all $\theta \in (0,1]$ and all $N \in \mathbb{N}_0$. This assumption might not be true for all θ. If the convergence indicators κ_{K,j_K} are too small or θ is chosen too large, then constraint (3.8) cannot be satisfied and, thus, no solution of (3.7), (3.8) exists. Especially this is the case, if

$$\max_{K\in\mathcal{K}_N}\left(\kappa_{K,j_K}\right) < \theta.$$

Then the algorithm continues with global h-refinement to enforce at least some convergence. If κ_{K,j_K} is uniformly bounded from below, θ can be chosen such that convergence is assured due to Theorem 3.3. In Theorem 5 in [104] it was shown that this is true for the equally-weighted bisection of cells. However, it can always be guaranteed that constraint (3.8) can be fulfilled for

$$\theta \in \left(0, \min_{K\in\mathcal{K}_N}\left(\kappa_{K,j_K}\right)\right).$$

Thus a practical approach to ensure solvability of maximization problem (3.7), (3.8) might be to monitor the computed values of κ_{K,j_K} and check convergence in an a posteriori way. Although it might happen that $\kappa_{K,j_K} \to 0$ for $N \to \infty$, we did not observe such a behaviour in our numerical experiments. However, a theoretical consideration of this point would be desirable.
The next assumption we want to consider is assumption (3.9). Here we assume that the quotient $\frac{\sqrt{\omega_K}}{p_K}$ is reduced by a factor $\rho \in (0,1)$ at least, if some refinement pattern $j \in \{0,\ldots,n-1\}$ is applied. This condition is purely theoretical. For h-refinements this assumption is always fulfilled, since it is a direct consequence from shape regularity assumption (2.9). For p-refinements things are a little bit different. For every $\rho \in (0,1)$ there exists some $p_K \in \mathbb{N}$ sufficiently large such that $p_K > \rho(p_K+1)$. Thus, in this case it is not enough to increase the polynomial degree p_K by one, but one has to increase it by some bigger integer k. However, in practice one either has to determine some maximal polynomial degree $p_{\max}$ before running the finite element programme or can almost surely guarantee that it holds $p_K \leq 100$ for all $K \in \mathcal{K}_N$ and all $N \in \{0,\ldots,N_{\max}\}$ for example. Then $\rho \in (0,1)$ can be chosen as e.g. $\rho := \frac{99}{100}$ and assumption (3.9) is always fulfilled.
The last major assumption we want to discuss is data saturation assumption (3.10). This assumption can only be satisfied, if the integrals on the left-hand side are computed with negligible error. To achieve this higher-order quadrature rules have to be used. If this does not suffice to satisfy inequality (3.10), one could perform local refinement according to the local error indicator

$$\frac{h_K}{p_K}\|f - \Pi f\|_{L^2(K)}$$

until (3.10) is fulfilled. Another approach might be to build some data error control into the whole algorithm as proposed in [154, 164].
Now let us consider the two convergence results. Theorem 3.3 gives us uniform convergence of the fully automatic *hp*-adaptive refinement algorithm in the energy error $\|u' - u'_{\text{FE}}\|_{L^2(\Omega)}$. The special assumptions for this result are that data saturation assumption (3.10) has to hold for some $\tau \in (0, 1]$ sufficiently small and the a posteriori error estimator $\eta(u_{\text{FE}}, \mathcal{K})$ has to provide an efficiency estimate like the one shown in Theorem 3.2. As we have seen in the proof of Theorem 3.3 the upper bound for the parameter τ depends on θ only:

$$\tau^2 < \frac{3\theta^2}{5C_{\text{eff}}C_{\text{grad}}^2(1+\delta)}$$

for $\delta > 0$. Thus the data approximation has to be more and more accurate the smaller $\theta \in (0, 1]$ is chosen. The assumption on the a posteriori error estimator might reduce the number of suitable a posteriori error estimators [199]. Altogether one can consider Theorem 3.3 as the theoretical justification for the use of the fully automatic *hp*-adaptive refinement strategy from Section 3.1.2.1.
The second result somewhat considers the practical side. In Theorem 3.4 the convergence of a weighted sum of exact energy error $\|u' - u'_{\text{FE}}\|_{L^2(\Omega)}$ and estimated error $\eta(u_{\text{FE}}, \mathcal{K})$ is shown. In this result the parameter $\tau \in (0, 1]$ in data saturation assumption (3.10) can be chosen arbitrarily. Further no efficiency estimate for the a posteriori error estimator is needed. However, the refinement patterns have to satisfy assumption (3.9). Altogether Theorem 3.4 can be considered as the practical justification of the fully automatic *hp*-adaptive refinement strategy from Section 3.1.2.1, because in practice these assumptions can be satisfied easier than the ones from Theorem 3.3 and it also states some convergence in terms of the – in practice more important – estimated error. Further a larger class of a posteriori error estimators with and without efficiency estimate can be used [199].
For numerical examples and more information about this *hp*-adaptive refinement strategy we refer the interested reader to the work of Dörfler and Heuveline [104].

3.1.3 The Higher-Dimensional Case

In this subsection we generalize the fully automatic *hp*-adaptive refinement strategy from Section 3.1.2 to higher space-dimensions $d \in \{2, 3\}$. This section is based on the results from [74]. Since we already have discussed the basic principles of the fully automatic *hp*-adaptive refinement strategy in Section 3.1.2, we only highlight the differences here. As in Section 3.1.2 our starting point is the adaptive loop (3.4). We consider the modules ESTIMATE and MARK again and prove some convergence results for this fully automatic refinement algorithm.

3.1.3.1 The Refinement Strategy

Here the modules ESTIMATE and MARK are considered again, but this time for the higher-dimensional cases $d \in \{2, 3\}$.
As before we begin with the module ESTIMATE. In this module we want to estimate the error of the computed finite elment solution which we obtain from module SOLVE. Therefore the following a posteriori error estimator was introduced in the work of Melenk and Wohlmuth [157].

Definition 3.2 (A Posteriori Error Estimator)**.** *Let $u_{FE} \in V^p(\mathcal{K}, \Omega)$ be the solution of (3.3). Then the* residual-based a posteriori error estimator *$\eta(u_{FE}, \mathcal{K})$ is given by*

$$\eta(u_{FE}, \mathcal{K})^2 := \sum_{K \in \mathcal{K}} \eta_K(u_{FE}, \mathcal{K})^2$$

with

$$\eta_K\left(u_{FE},\mathcal{K}\right)^2 := \eta_{R,K}\left(u_{FE},\mathcal{K}\right)^2 + \eta_{B,K}\left(u_{FE},\mathcal{K}\right)^2 .$$

Here the residual term $\eta_{R,K}\left(u_{FE},\mathcal{K}\right)$ reads

$$\eta_{R,K}\left(u_{FE},\mathcal{K}\right) := \frac{h_K}{p_K}\left\|\Pi f + \Delta u_{FE}\right\|_{L^2(K)}$$

and the boundary term $\eta_{B,K}\left(u_{FE},\mathcal{K}\right)$ is given by

$$\eta_{B,K}\left(u_{FE},\mathcal{K}\right)^2 := \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)}\frac{h_e}{p_e}\left\|\left[\frac{du_{FE}}{dn_e}\right]\right\|^2_{L^2(e)},$$

where

$$\mathcal{E}_I(K) := \{e\subset\partial K\cap\Omega : e \textit{ is an elemental edge of cell } K\}$$

denotes the set of all interior edges (in the case $d=2$) or faces (in the case $d=3$) of cell K and $h_e := \mathrm{diam}(e)$ is the edge or face diameter. The edge or face polynomial degree p_e is given by

$$p_e := \max\left\{p_{K_1}, p_{K_2}\right\}$$

for $K_1, K_2 \in \mathcal{K}$ with $e = K_1\cap K_2$. $[\cdot]$ denotes the jump over e and n_e is the outward-pointing unit normal vector to cell K on edge e.

For this a posteriori error estimator it was shown in [157] that it is reliable. Further an efficiency estimate, which depends on the polynomial degree vector p, was derived.

Theorem 3.5 (A Posteriori Error Estimates). *Let $u \in H^1_0(\Omega)$ be the solution of (3.2) and $u_{FE} \in V^p\left(\mathcal{K},\Omega\right)$ be the solution of (3.3). Then:*

1. *There exists some constant $C_{rel} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left\|\nabla\left(u-u_{FE}\right)\right\|^2_{L^2(\Omega)^d} \le C_{rel}\left(\eta\left(u_{FE},\mathcal{K}\right)^2 + \sum_{K\in\mathcal{K}}\frac{h_K^2}{p_K^2}\left\|f-\Pi f\right\|^2_{L^2(K)}\right).$$

2. *There exists some constant $C_{\mathit{eff}} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\eta_K\left(u_{FE},\mathcal{K}\right)^2 \le C_{\mathit{eff}}\left(p_K^{2(1+\varepsilon)}\left\|\nabla\left(u-u_{FE}\right)\right\|^2_{L^2(\omega_{K,2})^d} + \frac{h_K^2}{p_K^{1-4\varepsilon}}\left\|f-\Pi f\right\|^2_{L^2(\omega_{K,2})}\right)$$

for all $K\in\mathcal{K}$ and all $\varepsilon > 0$.

Proof. Choose $\alpha = 0$ in Theorem 3.6 in [157]. □

We see that in constrast to the case $d=1$ this error estimator is not *hp*-efficient, because the efficiency estimate is not uniform in p. This is due to the fact that the edge (in the case $d=2$) and face (in the case $d=3$) contributions cannot be bounded uniformly in p.
Now let us make a step ahead and consider the module MARK. From Verfürth [208] it is known that edge (in the case $d=2$) and face (in the case $d=3$) contibutions dominate the error of the finite element approximation. Therefore it does not suffice to consider the refinement patterns on cells $K\in\mathcal{K}$ as we

did in Section 3.1.2, but we have to extend the area of interest to some local patches. The smallest conforming possibility for this is to choose the patch ω_K from Section 2.4.1. Although we still want to determine for every cell $K \in \mathcal{K}$ which refinement pattern performs best, we also have to apply the refinement patterns in some sense to the neighbouring cells of K contained in the patch ω_K such that the error contribution of the boundary terms is reduced appropriately. This can be done by ensuring that no new hanging nodes are produced at the interior edges of cell K. For the h-refinement pattern this means that we have to refine the neighbours of cell K at least anisotropically. For a graphical representation see Figure 3.2 on the left-hand side for the case $d = 2$ and Figure 3.3 on the left-hand side for the case $d = 3$. In case of p-refinement we also increase the polynomial degree on the neighbouring cells. This can be seen in Figure 3.2 in the center and on the right-hand side for the case $d = 2$ and in Figure 3.3 on the right-hand side for the case $d = 3$.

As in Section 3.1.2.1 we assume that we have $n \in \mathbb{N} \setminus \{1\}$ different refinement patterns to choose from.

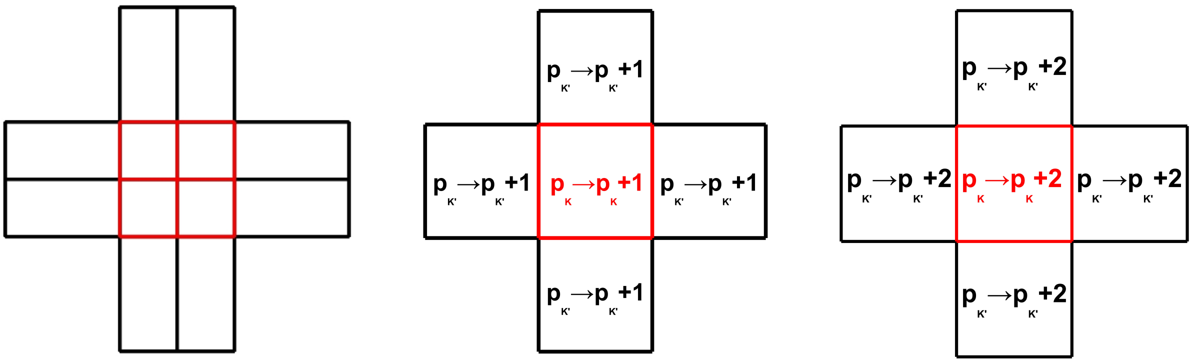

Figure 3.2: Refinement patterns ($d = 2$). Left: Equally-weighted bisection. Center: Increase polynomial degree by one. Right: Increase polynomial degree by two.

Now let $j \in \{0, \ldots, n-1\}$ and $K \in \mathcal{K}$ be arbitrary. Then we denote by $V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ the local finite element space consisting of functions from $V^p(\mathcal{K}, \Omega)$ compactly supported in the local patch ω_K with refinement pattern j applied to cell K. Without loss of generality we may assume that $\eta_K\left(u_{\mathrm{FE}}, \mathcal{K}\right) > 0$. If this is not the case, it makes not any sense to refine this cell at all and we can go to the next. Then we define the convergence indicator $\kappa_{K,j} \in \mathbb{R}_+$ as the solution of the optimization problem

$$\kappa_{K,j} = \frac{1}{\eta_K\left(u_{\mathrm{FE}}, \mathcal{K}\right)} \sup_{\phi \in V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)} \left(\frac{\sum_{K \in \mathcal{K}|_{\omega_K}} \int_K \phi\left(\Pi f + \Delta u_{\mathrm{FE}}\right)}{\|\nabla \phi\|_{L^2(\omega_K)^d}} \right), \tag{3.22}$$

where $\Pi : L^2\left(\omega_K\right) \to U^p\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ denotes the L^2-conforming interpolation from Section 2.4. As in Section 3.1.2.1 we can solve problem (3.22) easily by considering an equivalent local boundary value problem.

Lemma 3.7. *Let $v \in V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ such that*

$$\int_{\omega_K} (\nabla \phi)^T \nabla v = \sum_{K \in \mathcal{K}|_{\omega_K}} \int_K \phi\left(\Pi f + \Delta u_{FE}\right) \qquad \forall \phi \in V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right). \tag{3.23}$$

Then the supremum in (3.22) is obtained for v.

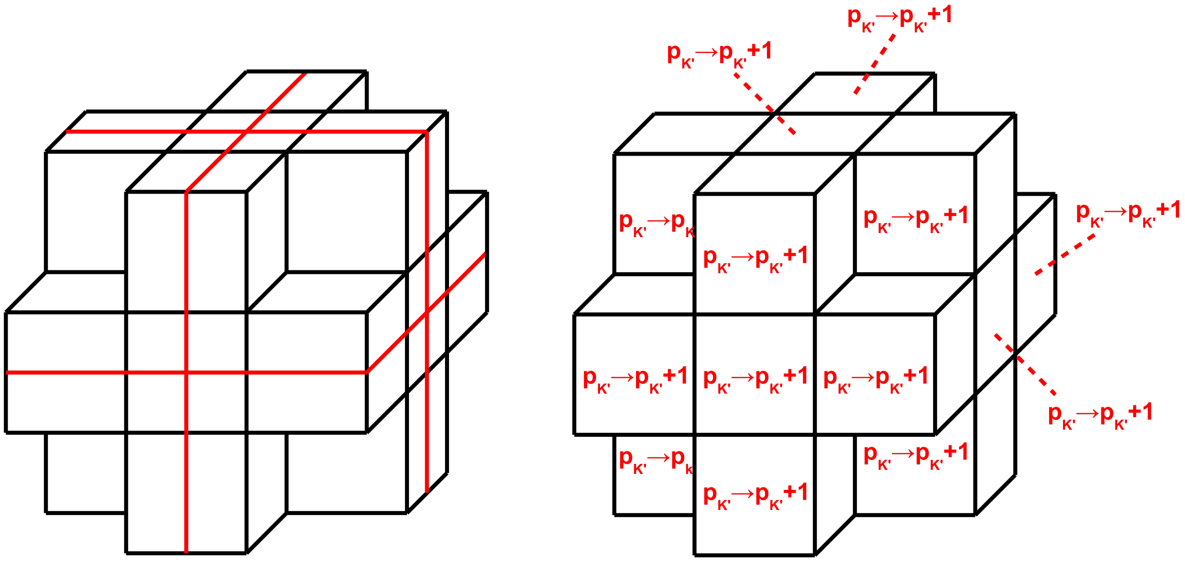

Figure 3.3: Refinement patterns ($d = 3$). Left: Equally-weighted bisection. Right: Increase polynomial degree by one.

Proof. Let $\phi \in V^p_{K,j}(\mathcal{K}|_{\omega_K}, \omega_K)$ be arbitrary. Then we see

$$\begin{aligned}\frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi\,(\Pi f + \Delta u_{\mathrm{FE}})}{\|\nabla\phi\|_{L^2(\omega_K)^d}} &= \frac{\int_{\omega_K} (\nabla\phi)^T \nabla v}{\|\nabla\phi\|_{L^2(\omega_K)^d}} \\ &\leq \|\nabla v\|_{L^2(\omega_K)^d}\end{aligned}$$

with Hölder's inequality and we have

$$\begin{aligned}\frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi\,(\Pi f + \Delta u_{\mathrm{FE}})}{\|\nabla\phi\|_{L^2(\omega_K)^d}} &\leq \frac{\int_{\omega_K} (\nabla v)^T \nabla v}{\|\nabla v\|_{L^2(\omega_K)^d}} \\ &= \frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K v\,(\Pi f + \Delta u_{\mathrm{FE}})}{\|\nabla v\|_{L^2(\omega_K)^d}}\end{aligned}$$

by (3.23). Since $\phi \in V^p_{K,j}(\mathcal{K}|_{\omega_K}, \omega_K)$ was arbitrary, this implies

$$\sup_{\phi\in V^p_{K,j}(\mathcal{K}|_{\omega_K},\omega_K)} \left(\frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi\,(\Pi f + \Delta u_{\mathrm{FE}})}{\|\nabla\phi\|_{L^2(\omega_K)^d}} \right) \leq \frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K v\,(\Pi f + \Delta u_{\mathrm{FE}})}{\|\nabla v\|_{L^2(\omega_K)^d}}.$$

Thus

$$\sup_{\phi\in V^p_{K,j}(\mathcal{K}|_{\omega_K},\omega_K)} \left(\frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi\,(\Pi f + \Delta u_{\mathrm{FE}})}{\|\nabla\phi\|_{L^2(\omega_K)^d}} \right) = \frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K v\,(\Pi f + \Delta u_{\mathrm{FE}})}{\|\nabla v\|_{L^2(\omega_K)^d}},$$

since $v \in V^p_{K,j}(\mathcal{K}|_{\omega_K}, \omega_K)$. $\square$

Although the local patches ω_K overlap for two neighbouring cells, problem (3.22) is still fully independent for all cells $K \in \mathcal{K}$ and all refinement patterns $j \in \{0, \ldots, n-1\}$, since it only depends on the refinement pattern j applied to cell K. Thus we do not loose the parallelizability from the case $d = 1$ here. Maximization problem (3.7), (3.8) does not depend on the dimension $d \in \{1, 2, 3\}$ and, hence, does not change. Therefore we do not discuss it here again.

To conclude this paragraph let us summarize the changes for the case $d \in \{2, 3\}$ briefly. In contrast to the case $d = 1$ the a posteriori error estimator is not *hp*-efficient anymore, because the efficiency estimate is not uniform in p. For higher dimensions $d \in \{2, 3\}$ the edge (in the case $d = 2$) or face (in the case $d = 3$) contributions dominate the approximation error of the finite element solution from module SOLVE. Therefore it does not suffice to consider local finite element spaces spanned over cells $K \in \mathcal{K}$ for computing the convergence indicators $\kappa_{K,j}$. Here we have to extend the domain to the local patch ω_K, which also includes the jumps over the boundaries of the cell. With these few modifications the fully automatic *hp*-adaptive refinement strategy from Section 3.1.2 can also be used for higher-dimensional situations where $d \in \{2, 3\}$.

3.1.3.2 Convergence Results

Now we prove convergence of the fully automatic *hp*-adaptive refinement strategy. Therefore we derive two results similar to Section 3.1.2.2. As before the first result proves that the exact energy error $\|\nabla (u - u_{\mathrm{FE}})\|_{L^2(\Omega)^d}$ is reduced in every refinement step of the algorithm. The second result gives us that a weighted sum of exact energy error $\|\nabla (u - u_{\mathrm{FE}})\|_{L^2(\Omega)^d}$ and estimated error $\eta(u_{\mathrm{FE}}, \mathcal{K})$ is reduced in every refinement step.

Let us assume that triangulation $\mathcal{K}$ consists of simplices only. Before we prove the main results of this paragraph we state some auxiliary results. The first one is a standard polynomial inverse estimate for the *hp*-adaptive finite element method.

Lemma 3.8 (Polynomial Inverse Estimate)**.** *Let $K \in \mathcal{K}$ be arbitrary and $u \in P_{p_K}(K)$ denote some polynomial. Then there exists some constant $C_{inv} > 0$ independent of h_K and p_K such that*

$$\|\partial^\alpha u\|_{L^2(K)} \leq C_{inv} \frac{p_K^2}{h_K} \|u\|_{L^2(K)}$$

for all multi-indices $\alpha \in \mathbb{N}_0^d$ satisfying $|\alpha|_1 = 1$.

Proof. There exists some $\widehat{u} \in P_{p_K}\left(\widehat{K}\right)$ such that

$$\widehat{u} \circ F_K = u \qquad \text{in } K,$$

where $F_K : \widehat{K} \to K$ denotes the reference mapping. Then we see

$$\begin{aligned}\|\partial^\alpha u\|_{L^2(K)} &= \sqrt{h_K}\, \|\partial^\alpha \widehat{u}\|_{L^2\left(\widehat{K}\right)} \\ &\leq C_{\mathrm{inv}} p_K^2 \sqrt{h_K}\, \|\widehat{u}\|_{L^2\left(\widehat{K}\right)}\end{aligned}$$

by Theorem 4.76 in [194] and the result follows. ☐

We also need a polynomial trace estimate for the *hp*-adaptive finite element method, which gives an upper bound for the trace of a polynomial on the boundary of some cell $K \in \mathcal{K}$.

Lemma 3.9 (Polynomial Trace Estimate)**.** *Let $K \in \mathcal{K}$ be arbitrary and $u \in P_{p_K}(K)$. Then there exists some constant $C_{tr} > 0$ independent of h_K and p_K such that*

$$\|u\|_{L^2(\partial K)} \leq C_{tr} \frac{p_K}{\sqrt{h_K}} \|u\|_{L^2(K)}.$$

Proof. There exists some $\widehat{u} \in P_{p_K}\left(\widehat{K}\right)$ such that

$$\widehat{u} \circ F_K = u \qquad \text{in } K,$$

where $F_K : \widehat{K} \to K$ denotes the reference mapping. Then we see

$$\begin{aligned} \|u\|_{L^2(\partial K)} &= h_K \|\widehat{u}\|_{L^2\left(\partial \widehat{K}\right)} \\ &\leq C_{\mathrm{tr}} h_K p_K \|\widehat{u}\|_{L^2\left(\widehat{K}\right)} \end{aligned}$$

by Theorem 4.76 in [194] and the result follows. □

Now let us consider the error estimator $\eta\left(u_{\mathrm{FE}}, \mathcal{K}\right)$, which was introduced in Definition 3.2, in more detail. Similar to Section 3.1.2.2 we want to investigate how the estimated error is influenced by the application of the *hp*-adaptive refinement algorithm presented in Section 3.1.3.1. As in Section 3.1.2.2 the various assumptions, which we make in the following lemmas and theorems, will be discussed at the end of this section in detail.

Lemma 3.10 (Error Estimator Reduction)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and assume that there exists a solution of maximization problem (3.7), (3.8) for some $\theta \in (0, 1]$. Further let $u_N \in V^p\left(\mathcal{K}_N, \Omega\right)$ and $u_{N+1} \in V^p\left(\mathcal{K}_{N+1}, \Omega\right)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that for all refinement patterns $j \in \{0, \dots, n-1\}$ there exists some constant $\rho \in (0, 1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\frac{h_K}{p_K} \leq \rho \frac{h_{\widetilde{K}}}{p_{\widetilde{K}}} \tag{3.24}$$

for all refined cells $\widetilde{K} \in \mathcal{K}_N$ and all $K \in \mathcal{K}_{N+1}$ with $K \subseteq \widetilde{K}$. We assume that there exists some $\tau \in (0, 1]$ such that data saturation assumption (3.10) is fulfilled. Then there exists some constant $C_{red} > 0$ independent of mesh size vector h and polynomial degree vector p such that

$$\begin{aligned} \eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq (1+\delta)^2 &\left(\left(1 + \frac{\rho^2 \tau^2}{\delta}\right) \eta\left(u_N, \mathcal{K}_N\right)^2 - (1-\rho) \eta\left(u_N, \mathcal{A}_N\right)^2 \right) \\ &+ C_{red} \left(1 + \frac{1}{\delta}\right) \max_{K \in \mathcal{K}_N} \left(p_K\right)^2 \left\|\nabla\left(u_{N+1} - u_N\right)\right\|_{L^2(\Omega)^d}^2 \end{aligned}$$

for all $\delta > 0$.

Proof. By Definition 3.2 it holds

$$\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 = \sum_{K \in \mathcal{K}_{N+1}} \left(\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 + \eta_{B,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \right), \tag{3.25}$$

where the cell term $\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)$ is given by

$$\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right) = \frac{h_K}{p_K} \left\|\Pi_{\mathcal{K}_{N+1}} f + \Delta u_{N+1}\right\|_{L^2(K)}$$

with $\Pi_{\mathcal{K}_{N+1}} : L^2(\Omega) \to U^p(\mathcal{K}_{N+1}, \Omega)$ denoting the L^2-conforming interpolation from Section 2.4. Then the Minkowski inequality immediately yields

(3.26)
$$\eta_{R,K}(u_{N+1}, \mathcal{K}_{N+1}) \leq \frac{h_K}{p_K}\left(\|\Pi_{\mathcal{K}_N} f + \Delta u_N\|_{L^2(K)} + \left\|\Pi_{\mathcal{K}_{N+1}} f - \Pi_{\mathcal{K}_N} f\right\|_{L^2(K)} + \|\Delta(u_{N+1} - u_N)\|_{L^2(K)}\right).$$

Let us consider the term $\frac{h_K}{p_K}\|\Pi_{\mathcal{K}_N} f + \Delta u_N\|_{L^2(K)}$ first. Therefore we introduce the set

$$\mathcal{R}_N := \{K \in \mathcal{K}_N : K \text{ is refined}\}$$

of all elements from triangulation $\mathcal{K}_N$ that are refined in module REFINE of iteration step N. Clearly we have $\mathcal{A}_N \subseteq \mathcal{R}_N$. If there exists some $\widetilde{K} \in \mathcal{R}_N$ such that $K \subseteq \widetilde{K}$, then it holds

$$\frac{h_K}{p_K}\|\Pi_{\mathcal{K}_N} f + \Delta u_N\|_{L^2(K)} \leq \rho \frac{h_{\widetilde{K}}}{p_{\widetilde{K}}}\|\Pi_{\mathcal{K}_N} f + \Delta u_N\|_{L^2(K)} \tag{3.27}$$

by assumption (3.24). For the term $\frac{h_K}{p_K}\left\|\Pi_{\mathcal{K}_{N+1}} f - \Pi_{\mathcal{K}_N} f\right\|_{L^2(K)}$ it holds

$$\begin{aligned} \frac{h_K}{p_K}\left\|\Pi_{\mathcal{K}_{N+1}} f - \Pi_{\mathcal{K}_N} f\right\|_{L^2(K)} &\leq \frac{h_K}{p_K}\|f - \Pi_{\mathcal{K}_N} f\|_{L^2(K)} \\ &\leq \rho \frac{h_{\widetilde{K}}}{p_{\widetilde{K}}}\|f - \Pi_{\mathcal{K}_N} f\|_{L^2(K)} \end{aligned} \tag{3.28}$$

by assumption (3.24).
Now let us consider the case that there exists no such $\widetilde{K} \in \mathcal{R}_N$. Then $K \in \mathcal{K}_N$ and it follows

$$\frac{h_K}{p_K}\|\Pi_{\mathcal{K}_N} f + \Delta u_N\|_{L^2(K)} = \eta_{R,K}(u_N, \mathcal{K}_N) \tag{3.29}$$

and

$$\left\|\Pi_{\mathcal{K}_{N+1}} f - \Pi_{\mathcal{K}_N} f\right\|_{L^2(K)} = 0. \tag{3.30}$$

For the term $\frac{h_K}{p_K}\|\Delta(u_{N+1} - u_N)\|_{L^2(K)}$ we get

$$\frac{h_K}{p_K}\|\Delta(u_{N+1} - u_N)\|_{L^2(K)} \leq C_{\mathrm{inv}} p_K \|\nabla(u_{N+1} - u_N)\|_{L^2(K)^d} \tag{3.31}$$

by Lemma 3.8 in both cases.
Inserting estimates (3.27)–(3.31) into (3.26) gives

(3.32)
$$\eta_{R,K}(u_{N+1}, \mathcal{K}_{N+1}) \leq \rho \frac{h_{\widetilde{K}}}{p_{\widetilde{K}}}\left(\|\Pi_{\mathcal{K}_N} f + \Delta u_N\|_{L^2(K)} + \|f - \Pi_{\mathcal{K}_N} f\|_{L^2(K)}\right) + C_{\mathrm{inv}} p_K \|\nabla(u_{N+1} - u_N)\|_{L^2(K)^d},$$

if there exists such a cell $\widetilde{K} \in \mathcal{R}_N$ with $K \subseteq \widetilde{K}$, and

$$\eta_{R,K}(u_{N+1}, \mathcal{K}_{N+1}) \leq \eta_{R,K}(u_N, \mathcal{K}_N) + C_{\mathrm{inv}} p_K \|\nabla(u_{N+1} - u_N)\|_{L^2(K)^d}, \tag{3.33}$$

else.
Now let us consider the boundary term $\eta_{B,K}(u_{N+1}, \mathcal{K}_{N+1})$. By Definition 3.2 we have

$$\begin{aligned} \eta_{B,K}(u_{N+1}, \mathcal{K}_{N+1})^2 &= \frac{1}{2}\sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e}\left\|\left[\frac{du_{N+1}}{dn_e}\right]\right\|_{L^2(e)}^2 \\ &\leq \frac{1}{2}\sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e}\left\|\left[\frac{du_{N+1}}{dn_e}\right]\right\|_{L^2(e)}\left(\left\|\left[\frac{du_N}{dn_e}\right]\right\|_{L^2(e)} + \left\|\left[\frac{d(u_{N+1} - u_N)}{dn_e}\right]\right\|_{L^2(e)}\right) \end{aligned}$$

from the Minkowski inequality and using the Cauchy-Schwarz inequality gives

$$\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 \leq \eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)\left(T_1+T_2\right), \tag{3.34}$$

where the terms T_1 and T_2 are given by

$$T_1^2 := \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)}\frac{h_e}{p_e}\left\|\left[\frac{du_N}{dn_e}\right]\right\|_{L^2(e)}^2$$

and

$$T_2^2 := \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)}\frac{h_e}{p_e}\left\|\left[\frac{d\left(u_{N+1}-u_N\right)}{dn_e}\right]\right\|_{L^2(e)}^2.$$

If there exists some cell $\widetilde{K}\in\mathcal{R}_N$ such that $K\subseteq\widetilde{K}$, then it holds

$$T_1^2 \leq \frac{\rho}{2}\sum_{e\in\mathcal{E}_I(\widetilde{K})}\frac{h_e}{p_e}\left\|\left[\frac{du_N}{dn_e}\right]\right\|_{L^2(e\cap\partial K)}^2. \tag{3.35}$$

If there exists no such $\widetilde{K}\in\mathcal{R}_N$, then

$$T_1^2 \leq \eta_{B,K}\left(u_N,\mathcal{K}_N\right)^2. \tag{3.36}$$

For the term T_2 we get

$$T_2^2 \leq 2^{d-2}dC_{\mathrm{tr}}^2 p_K\left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(K)^d}^2 \tag{3.37}$$

by Lemma 3.9 in both cases.
Inserting estimates (3.35)–(3.37) into (3.34) gives

$$\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right) \leq \left(\frac{\rho}{2}\sum_{e\in\mathcal{E}_I(\widetilde{K})}\frac{h_e}{p_e}\left\|\left[\frac{du_N}{dn_e}\right]\right\|_{L^2(e\cap\partial K)}^2\right)^{\frac{1}{2}} + C_{\mathrm{tr}}\sqrt{2^{d-2}dp_K}\left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(K)^d}, \tag{3.38}$$

if there exists such a cell $\widetilde{K}\in\mathcal{R}_N$ with $K\subseteq\widetilde{K}$, and

$$\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right) \leq \eta_{B,K}\left(u_N,\mathcal{K}_N\right) + C_{\mathrm{tr}}\sqrt{2^{d-2}dp_K}\left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(K)^d}, \tag{3.39}$$

else. Further inserting estimates (3.32), (3.33), (3.38) and (3.39) into (3.25) and using Young's inequality implies

$$\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 \leq (1+\delta)\eta\left(u_N,\mathcal{K}_N\setminus\mathcal{R}_N\right)^2+\rho(1+\delta)^2T+C_{\mathrm{red}}\left(1+\frac{1}{\delta}\right)\max_{K\in\mathcal{K}_N}\left(p_K\right)^2\left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(\Omega)^d}^2 \tag{3.40}$$

for $\delta>0$ and some constant $C_{\mathrm{red}}>0$ independent of polynomial degree vector p. Here T is given by

$$T := \eta\left(u_N,\mathcal{R}_N\right)^2 + \frac{\rho}{\delta}\sum_{K\in\mathcal{R}_N}\frac{h_K^2}{p_K^2}\left\|f-\Pi_{\mathcal{K}_N}f\right\|_{L^2(K)}^2.$$

By data saturation assumption (3.10) it follows

$$T \leq \eta\left(u_N,\mathcal{R}_N\right)^2 + \frac{\rho\tau^2}{\delta}\eta\left(u_N,\mathcal{K}_N\right)^2$$

and inserting into (3.40) gives

$$\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 \leq (1+\delta)^2\left(\left(1+\frac{\rho^2\tau^2}{\delta}\right)\eta\left(u_N,\mathcal{K}_N\right)^2-(1-\rho)\eta\left(u_N,\mathcal{A}_N\right)^2\right)$$
$$+C_{\mathrm{red}}\left(1+\frac{1}{\delta}\right)\max_{K\in\mathcal{K}_N}\left(p_K\right)^2\left\|\nabla\left(u_{N+1}-u_N\right)\right\|^2_{L^2(\Omega)^d},$$

since $\mathcal{A}_N\subseteq\mathcal{R}_N$. □

Now we show two auxiliary results, which we use in the proofs of the main results of this paragraph. The first result gives a lower bound for the term $\left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(\Omega)^d}$ in terms of the energy error $\left\|\nabla\left(u-u_N\right)\right\|_{L^2(\Omega)^d}$ and the estimated error $\eta\left(u_N,\mathcal{K}_N\right)$.

Lemma 3.11. *Let $N\in\mathbb{N}_0$ be arbitrary and $u\in H^1_0(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta\in(0,1]$. Further let $u_N\in V^p\left(\mathcal{K}_N,\Omega\right)$ and $u_{N+1}\in V^p\left(\mathcal{K}_{N+1},\Omega\right)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau\in(0,1]$ such that (3.10) holds. Then there exist some constants $C_1>1$ and $C_2>0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left\|\nabla\left(u_{N+1}-u_N\right)\right\|^2_{L^2(\Omega)^d}\geq\delta\left(\frac{\theta^2}{C_1(1+\delta)}\left\|\nabla\left(u-u_N\right)\right\|^2_{L^2(\Omega)^d}-C_2\tau^2\eta\left(u_N,\mathcal{K}_N\right)^2\right)$$

for all $\delta>0$.

Proof. Let $K\in\mathcal{K}_N$ be arbitrary and $\phi_{N+1}\in V^p\left(\mathcal{K}_{N+1},\Omega\right)$ with $\operatorname{supp}\left(\phi_{N+1}\right)\subseteq\omega_K$. Then we see

$$\int_{\omega_K}\left(\nabla\phi_{N+1}\right)^T\nabla\left(u_{N+1}-u_N\right)=\int_{\omega_K}\phi_{N+1}f-\int_{\omega_K}\left(\nabla\phi_{N+1}\right)^T\nabla u_N,$$

since $u_{N+1}\in V^p\left(\mathcal{K}_{N+1},\Omega\right)$ solves discrete problem (3.3). This reads

$$\int_{\omega_K}\left(\nabla\phi_{N+1}\right)^T\nabla\left(u_{N+1}-u_N\right)=\int_{\omega_K}\phi_{N+1}\Pi f-\int_{\omega_K}\left(\nabla\phi_{N+1}\right)^T\nabla u_N+\int_{\omega_K}\phi_{N+1}(f-\Pi f),$$

where $\Pi:L^2(\Omega)\to U^p\left(\mathcal{K}_N,\Omega\right)$ denotes the L^2-conforming interpolation from Section 2.4. With integration by parts and the L^2-interpolation property this implies

$$\int_{\omega_K}\left(\nabla\phi_{N+1}\right)^T\nabla\left(u_{N+1}-u_N\right)=\sum_{K\in\mathcal{K}_N|_{\omega_K}}\int_K\phi_{N+1}\left(\Pi f+\Delta u_N\right)+\int_{\omega_K}\left(\phi_{N+1}-\phi_N\right)\left(f-\Pi f\right)$$

for $\phi_N\in V^p\left(\mathcal{K}_N,\Omega\right)$ with $\operatorname{supp}\left(\phi_N\right)\subseteq\omega_K$ and using Minkowski's inequality yields

$$\left|\int_{\omega_K}\left(\nabla\phi_{N+1}\right)^T\nabla\left(u_{N+1}-u_N\right)\right|\geq\left|\sum_{K\in\mathcal{K}_N|_{\omega_K}}\int_K\phi_{N+1}\left(\Pi f+\Delta u_N\right)\right|-\left|\int_{\omega_K}\left(\phi_{N+1}-\phi_N\right)\left(f-\Pi f\right)\right|.$$

With Hölder's inequality we have

$$\left|\sum_{K\in\mathcal{K}_N|_{\omega_K}}\int_K\phi_{N+1}\left(\Pi f+\Delta u_N\right)\right| \tag{3.41}$$
$$\leq\left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(\omega_K)^d}\left\|\nabla\phi_{N+1}\right\|_{L^2(\omega_K)^d}+\left\|f-\Pi f\right\|_{L^2(\omega_K)}\left\|\phi_{N+1}-\phi_N\right\|_{L^2(\omega_K)}$$

and choosing $\phi_N := \Pi^1 \phi_{N+1}$ with $\Pi^1 : H_0^1(\Omega) \to V^p(\mathcal{K}_N, \Omega)$ from Section 2.4.1 implies

$$\begin{aligned}\|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)} &= \left\|\phi_{N+1} - \Pi^1 \phi_{N+1}\right\|_{L^2(\omega_K)} \\ &\leq C_{\text{grad}} \frac{h_K}{p_K} \|\nabla \phi_{N+1}\|_{L^2(\omega_{K,7})^d}\end{aligned}$$

by Theorem 2.4. Since $\operatorname{supp}(\phi_{N+1}) \subseteq \omega_K \subset \omega_{K,7}$, it holds

$$\|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)} \leq C_{\text{grad}} \frac{h_K}{p_K} \|\nabla \phi_{N+1}\|_{L^2(\omega_K)^d}$$

and inserting into (3.41) gives

$$\begin{aligned}&\left| \sum_{K \in \mathcal{K}_N|_{\omega_K}} \int_K \phi_{N+1} (\Pi f + \Delta u_N) \right| \\ &\qquad \leq \left(\|\nabla (u_{N+1} - u_N)\|_{L^2(\omega_K)^d} + C_{\text{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(\omega_K)} \right) \|\nabla \phi_{N+1}\|_{L^2(\omega_K)^d} .\end{aligned}$$

Dividing by $\|\nabla \phi_{N+1}\|_{L^2(\omega_K)^d}$ yields

$$\begin{aligned}\sup_{\phi \in V^p_{K,j_K}(\mathcal{K}_N|_{\omega_K}, \omega_K)} &\left(\frac{\sum_{K \in \mathcal{K}_N|_{\omega_K}} \int_K \phi (\Pi f + \Delta u_N)}{\|\nabla \phi_{N+1}\|_{L^2(\omega_K)^d}} \right) \\ &\leq \sup_{\substack{\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega) \\ \operatorname{supp}(\phi_{N+1}) \subseteq \omega_K}} \left(\frac{\sum_{K \in \mathcal{K}_N|_{\omega_K}} \int_K \phi_{N+1} (\Pi f + \Delta u_N)}{\|\nabla \phi_{N+1}\|_{L^2(\omega_K)^d}} \right) \\ &\leq \|\nabla (u_{N+1} - u_N)\|_{L^2(\omega_K)^d} + C_{\text{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(\omega_K)},\end{aligned}$$

since $\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ with $\operatorname{supp}(\phi_{N+1}) \subseteq \omega_K$ was arbitrary. Then it follows

$$\kappa_{K,j_K} \eta_K (u_N, \mathcal{K}_N) \leq \|\nabla (u_{N+1} - u_N)\|_{L^2(\omega_K)^d} + C_{\text{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(\omega_K)}$$

from optimization problem (3.22). Since $\mathcal{A}_N \subseteq \mathcal{K}_N$, squaring both sides, summing over $K \in \mathcal{K}_N$ and using Young's inequality yields

$$\begin{aligned}&\sum_{K \in \mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K (u_N, \mathcal{K}_N)^2 \\ &\quad \leq \sum_{K \in \mathcal{K}_N} \kappa_{K,j_K}^2 \eta_K (u_N, \mathcal{K}_N)^2 \\ &\quad \leq \left(1 + \frac{1}{\delta}\right) \sum_{K \in \mathcal{K}_N} \|\nabla (u_{N+1} - u_N)\|_{L^2(\omega_K)^d}^2 + C_{\text{grad}}^2 (1 + \delta) \sum_{K \in \mathcal{K}_N} \frac{h_K^2}{p_K^2} \|f - \Pi f\|_{L^2(\omega_K)}^2\end{aligned}$$

for $\delta > 0$. Now let us define the covering constant $C_{\text{cov}} > 0$ by

$$C_{\text{cov}} := \max_{K \in \mathcal{K}_N} |\{L \in \mathcal{K}_N : L \subset \omega_K\}| .$$

Then the (γ_1, γ_2)-regularity of $\mathcal{K}_N$ implies

$$\sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K \left(u_N, \mathcal{K}_N\right)^2 \leq C_{\text{cov}} \left(\left(1+\frac{1}{\delta}\right) \|\nabla \left(u_{N+1}-u_N\right)\|_{L^2(\Omega)^d}^2 + C(1+\delta) \sum_{K\in\mathcal{K}_N} \frac{h_K^2}{p_K^2} \|f-\Pi f\|_{L^2(K)}^2 \right)$$

for some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p. Finally, from data saturation assumption (3.10) it follows

(3.42)

$$\sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K \left(u_N, \mathcal{K}_N\right)^2 \leq C_{\text{cov}} \left(\left(1+\frac{1}{\delta}\right) \|\nabla \left(u_{N+1}-u_N\right)\|_{L^2(\Omega)^d}^2 + C(1+\delta)\tau^2 \eta \left(u_N, \mathcal{K}_N\right)^2 \right).$$

From Theorem 3.5 we know

$$\begin{aligned}\|\nabla \left(u-u_N\right)\|_{L^2(\Omega)^d}^2 &\leq C_{\text{rel}} \left(\eta \left(u_N, \mathcal{K}_N\right)^2 + \sum_{K\in\mathcal{K}_N} \frac{h_K^2}{p_K^2} \|f-\Pi f\|_{L^2(K)}^2 \right) \\ &\leq C_{\text{rel}} \left(1+\tau^2\right) \eta \left(u_N, \mathcal{K}_N\right)^2\end{aligned}$$

by assumption (3.10). For $\tau \leq 1$ this reads

$$\|\nabla \left(u-u_N\right)\|_{L^2(\Omega)^d}^2 \leq 2C_{\text{rel}} \eta \left(u_N, \mathcal{K}_N\right)^2$$

and multiplying both sides by θ^2, $\theta \in (0,1]$, yields

$$\theta^2 \|\nabla \left(u-u_N\right)\|_{L^2(\Omega)^d}^2 \leq 2C_{\text{rel}} \sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K \left(u_N, \mathcal{K}_N\right)^2$$

by constraint (3.8). With (3.42) it follows

$$\frac{\theta^2}{2C_{\text{rel}}} \|\nabla \left(u-u_N\right)\|_{L^2(\Omega)^d}^2 \leq C_{\text{cov}} \left(\left(1+\frac{1}{\delta}\right) \|\nabla \left(u_{N+1}-u_N\right)\|_{L^2(\Omega)^d}^2 + C(1+\delta)\tau^2 \eta \left(u_N, \mathcal{K}_N\right)^2 \right)$$

and this concludes the proof. □

The next result compares the energy error of the finite element approximation on two successive grids of the algorithm.

Lemma 3.12 (Comparison of Errors). *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H_0^1(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p\left(\mathcal{K}_N, \Omega\right)$ and $u_{N+1} \in V^p\left(\mathcal{K}_{N+1}, \Omega\right)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau \in (0,1]$ such that (3.10) is fulfilled. Then there exist some constants $C_1 > 1$ and $C_2 > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\begin{aligned}\|\nabla \left(u-u_{N+1}\right)\|_{L^2(\Omega)^d}^2 \leq \left(1-\frac{\delta\theta^2}{2C_1(1+\delta)}\right) \|\nabla \left(u-u_N\right)\|_{L^2(\Omega)^d}^2 &+ \frac{C_2\delta\tau^2}{2} \eta \left(u_N, \mathcal{K}_N\right)^2 \\ &- \frac{1}{2} \|\nabla \left(u_{N+1}-u_N\right)\|_{L^2(\Omega)^d}^2\end{aligned}$$

for $\delta > 0$.

Proof. Since $V^p\left(\mathcal{K}_N, \Omega\right) \subset V^p\left(\mathcal{K}_{N+1}, \Omega\right)$, we can use the Galerkin orthogonality

$$A\left(u-u_{N+1}, u_{N+1}-u_N\right) = 0$$

to get

$$\begin{aligned}\|\nabla(u-u_N)\|^2_{L^2(\Omega)} &= A(u-u_N, u-u_N)\\ &= A(u-u_{N+1}, u-u_{N+1}) + A(u_{N+1}-u_N, u_{N+1}-u_N)\\ &= \|\nabla(u-u_{N+1})\|^2_{L^2(\Omega)} + \left(\frac{1}{2}+\frac{1}{2}\right)\|\nabla(u_{N+1}-u_N)\|^2_{L^2(\Omega)}.\end{aligned}$$

By using Lemma 3.11 it follows

$$\begin{aligned}&\left(1-\frac{\delta\theta^2}{2C_1(1+\delta)}\right)\|\nabla(u-u_N)\|^2_{L^2(\Omega)^d}\\ &\qquad \geq \|\nabla(u-u_{N+1})\|^2_{L^2(\Omega)^d} + \frac{1}{2}\|\nabla(u_{N+1}-u_N)\|^2_{L^2(\Omega)^d} - \frac{C_2\delta\tau^2}{2}\eta(u_N,\mathcal{K}_N)^2.\end{aligned}$$

This concludes the proof. □

Remark 3.1. *We see easily that the constants C_1 and C_2 in Lemmas 3.11 and 3.12 are the same.*

Now we come to the first main result of this paragraph. It states that the energy error $\|\nabla(u-u_{\mathrm{FE}})\|_{L^2(\Omega)^d}$ is reduced in every refinement step of the fully automatic hp-adaptive refinement algorithm from Section 3.1.3.1.

Theorem 3.6 (Convergence)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H^1_0(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p(\mathcal{K}_N,\Omega)$ and $u_{N+1} \in V^p(\mathcal{K}_{N+1},\Omega)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau \in (0,1]$ sufficiently small (depending on polynomial degree vector p) such that (3.10) is fulfilled. Then there exists some $\mu \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|\nabla(u-u_{N+1})\|_{L^2(\Omega)^d} \leq \mu\|\nabla(u-u_N)\|_{L^2(\Omega)^d}.$$

Proof. From Theorem 3.5 we know

$$\eta(u_N,\mathcal{K}_N)^2 \leq C_{\mathrm{eff}}\sum_{K\in\mathcal{K}_N}\left(p_K^{2(1+\varepsilon)}\|\nabla(u-u_N)\|^2_{L^2(\omega_{K,2})^d} + \frac{h_K^2}{p_K^{1-4\varepsilon}}\|f-\Pi f\|^2_{L^2(\omega_{K,2})}\right)$$

for $\varepsilon > 0$. Now let us define the covering constant $C_{\mathrm{cov}} > 0$ by

$$C_{\mathrm{cov}} := \max_{K\in\mathcal{K}_N}|\{L\in\mathcal{K}_N : L\subset\omega_{K,2}\}|.$$

Then the (γ_1,γ_2)-regularity of $\mathcal{K}_N$ implies

$$\eta(u_N,\mathcal{K}_N)^2 \leq C_{\mathrm{cov}}C_{\mathrm{eff}}\max_{K\in\mathcal{K}_N}(p_K)^{1+4\varepsilon}\left(\max_{K\in\mathcal{K}_N}(p_K)^{1-2\varepsilon}\|\nabla(u-u_N)\|^2_{L^2(\Omega)^d} + C\sum_{K\in\mathcal{K}_N}\frac{h_K^2}{p_K^2}\|f-\Pi f\|^2_{L^2(K)}\right)$$

for some constant

$$C > \frac{2}{\theta^2} \geq 2,$$

which is independent of mesh size vector h and polynomial degree vector p. With data saturation assumption (3.10) this reads

$$\eta(u_N,\mathcal{K}_N)^2 \leq C_{\mathrm{cov}}C_{\mathrm{eff}}\max_{K\in\mathcal{K}_N}(p_K)^{1+4\varepsilon}\left(\max_{K\in\mathcal{K}_N}(p_K)^{1-2\varepsilon}\|\nabla(u-u_N)\|^2_{L^2(\Omega)^d} + C\tau^2\eta(u_N,\mathcal{K}_N)^2\right).$$

Hence, for

$$\tau < \frac{1}{\sqrt{CC_{\text{cov}}C_{\text{eff}}}\max\limits_{K\in\mathcal{K}_N}(p_K)^{\frac{1}{2}+2\varepsilon}}$$

we have

$$\eta(u_N,\mathcal{K}_N)^2 \le \frac{C_{\text{cov}}C_{\text{eff}}\max\limits_{K\in\mathcal{K}_N}(p_K)^{2(1+\varepsilon)}}{1-CC_{\text{cov}}C_{\text{eff}}\max\limits_{K\in\mathcal{K}_N}(p_K)^{1+4\varepsilon}\tau^2}\left\|\nabla(u-u_N)\right\|^2_{L^2(\Omega)^d}$$

and with the even more restrictive assumption

$$\tau \le \frac{1}{\sqrt{2CC_{\text{cov}}C_{\text{eff}}}\max\limits_{K\in\mathcal{K}_N}(p_K)^{\frac{1}{2}+2\varepsilon}}$$

we get

$$\eta(u_N,\mathcal{K}_N)^2 \le 2C_{\text{cov}}C_{\text{eff}}\max_{K\in\mathcal{K}_N}(p_K)^{2(1+\varepsilon)}\left\|\nabla(u-u_N)\right\|^2_{L^2(\Omega)^d}. \tag{3.43}$$

From Lemma 3.12 we know

$$\begin{aligned}\left\|\nabla(u-u_{N+1})\right\|^2_{L^2(\Omega)^d} \le \left(1-\frac{\delta\theta^2}{2C_1(1+\delta)}\right)\left\|\nabla(u-u_N)\right\|^2_{L^2(\Omega)^d} + \frac{C_2\delta\tau^2}{2}\eta(u_N,\mathcal{K}_N)^2 \\ -\frac{1}{2}\left\|\nabla(u_{N+1}-u_N)\right\|^2_{L^2(\Omega)^d}\end{aligned}$$

for $\delta > 0$ and using Lemma 3.11 yields

$$\begin{aligned}\left\|\nabla(u-u_{N+1})\right\|^2_{L^2(\Omega)^d} &\le \left(1-\frac{\delta\theta^2}{C_1(1+\delta)}\right)\left\|\nabla(u-u_N)\right\|^2_{L^2(\Omega)^d} + C_2\delta\tau^2\eta(u_N,\mathcal{K}_N)^2 \\ &\le \left(1+2C_2C_{\text{cov}}C_{\text{eff}}\max_{K\in\mathcal{K}_N}(p_K)^{2(1+\varepsilon)}\delta\tau^2-\frac{\delta\theta^2}{C_1(1+\delta)}\right)\left\|\nabla(u-u_N)\right\|^2_{L^2(\Omega)^d}\end{aligned}$$

by (3.43). By choosing $\varepsilon := \frac{1}{2}$ we obtain

$$\left\|\nabla(u-u_{N+1})\right\|^2_{L^2(\Omega)^d} \le \left(1+\frac{\delta}{C_1}\left(\frac{1}{C}-\frac{\theta^2}{1+\delta}\right)\right)\left\|\nabla(u-u_N)\right\|^2_{L^2(\Omega)^d}$$

for

$$\tau \le \frac{1}{\sqrt{2CC_1C_2C_{\text{cov}}C_{\text{eff}}}\max\limits_{K\in\mathcal{K}_N}(p_K)^{\frac{3}{2}}}.$$

Then the result follows for $\delta < \frac{C\theta^2}{2} - 1$. □

The second main result of this paragraph is another convergence result, which states that the weighted sum of energy error $\left\|\nabla(u-u_{\text{FE}})\right\|_{L^2(\Omega)^d}$ and estimated error $\eta(u_{\text{FE}},\mathcal{K})$ is reduced in every iteration of the fully automatic *hp*-adaptive refinement algorithm from Section 3.1.3.1. The proof follows the ideas of Bonito and Nochetto [55] again.

Theorem 3.7 (Quasi-Convergence)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H^1_0(\Omega)$ be the solution of (3.2). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p(\mathcal{K}_N,\Omega)$ and $u_{N+1} \in V^p(\mathcal{K}_{N+1},\Omega)$ be the solutions of (3.3) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some constant $\rho \in (0,1)$ independent of mesh size vector h and polynomial* *degree vector p such that assumption (3.24) holds. We assume that there exists some $\tau \in (0,1]$ sufficiently small such that (3.10) is fulfilled. Then there exists some constant $\mu \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|\nabla (u - u_{N+1})\|^2_{L^2(\Omega)^d} + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq \mu \left(\|\nabla (u - u_N)\|^2_{L^2(\Omega)^d} + \nu\eta\left(u_N, \mathcal{K}_N\right)^2 \right)$$

for some $\nu > 0$ sufficiently small (depending on polynomial degree vector p).

Proof. From Lemma 3.12 we know

$$\begin{aligned}
&\|\nabla (u - u_{N+1})\|^2_{L^2(\Omega)^d} + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\
&\quad\leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right) \|\nabla (u - u_N)\|^2_{L^2(\Omega)^d} + \frac{C_2\delta\tau^2}{2}\eta\left(u_N, \mathcal{K}_N\right)^2 + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\
&\qquad - \frac{1}{2}\|\nabla (u_{N+1} - u_N)\|^2_{L^2(\Omega)^d}
\end{aligned}$$

for $\delta > 0$ and applying Lemma 3.10 yields

$$\begin{aligned}
&\|\nabla (u - u_{N+1})\|^2_{L^2(\Omega)^d} + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\
&\quad\leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right) \|\nabla (u - u_N)\|^2_{L^2(\Omega)^d} + \left(\frac{C_2\delta\tau^2}{2} + \nu(1+\delta)^2\left(1 + \frac{\rho^2\tau^2}{\delta}\right)\right)\eta\left(u_N, \mathcal{K}_N\right)^2 \\
&\qquad - \nu(1+\delta)^2(1-\rho)\eta\left(u_N, \mathcal{A}_N\right)^2
\end{aligned} \tag{3.44}$$

by choosing

$$\nu \leq \frac{\delta}{2C_{\mathrm{red}}(1+\delta)\max\limits_{K\in\mathcal{K}_N}\left(p_K\right)^2}.$$

We observe

$$C^2_{\max}\eta\left(u_N, \mathcal{A}_N\right)^2 \geq \sum_{K\in\mathcal{A}_N} \kappa^2_{K,j_K}\eta_K\left(u_N, \mathcal{K}_N\right)^2,$$

where

$$C_{\max} := \max\left\{\max_{K\in\mathcal{A}_N}\left(\kappa_{K,j_K}\right), 1\right\}.$$

Thus (3.44) reads

$$\begin{aligned}
&\|\nabla (u - u_{N+1})\|^2_{L^2(\Omega)^d} + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\
&\quad\leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right) \|\nabla (u - u_N)\|^2_{L^2(\Omega)^d} + \left(\frac{C_2\delta\tau^2}{2} + \nu(1+\delta)^2\left(1 + \frac{\rho^2\tau^2}{\delta}\right)\right)\eta\left(u_N, \mathcal{K}_N\right)^2 \\
&\qquad - \frac{\nu(1+\delta)^2(1-\rho)}{C^2_{\max}} \sum_{K\in\mathcal{A}_N} \kappa^2_{K,j_K}\eta_K\left(u_N, \mathcal{K}_N\right)^2,
\end{aligned}$$

since $\rho < 1$. Finally constraint (3.8) implies

$$\|\nabla (u - u_{N+1})\|^2_{L^2(\Omega)^d} + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right) \|\nabla (u - u_N)\|^2_{L^2(\Omega)^d} + T\eta\left(u_N, \mathcal{K}_N\right)^2,$$

where the term T is given by

$$T := \frac{C_2\delta\tau^2}{2} + \nu(1+\delta)^2\left(1 + \frac{\rho^2\tau^2}{\delta}\right) - \frac{\nu\theta^2(1+\delta)^2(1-\rho)}{C^2_{\max}}.$$

For $\tau \to 0$ we have

$$T \to \nu(1+\delta)^2 \left(1 - \frac{\theta^2(1-\rho)}{C_{\max}^2}\right)$$

and, thus, the result follows for δ and τ sufficiently small. $\square$

To conclude this paragraph let us shortly discuss the various assumptions we made above and see how these affect the main results of this paragraph. Most of the assumptions were already discussed in Section 3.1.2.2. Therefore we restrict ourselves to those assumptions, which changed fundamentally or appeared newly.

In Theorem 3.6 the parameter $\tau \in (0,1]$ from data saturation assumption (3.10) depends on the polynomial degree vector p. This is due to the fact that the a posteriori error estimator from Section 3.1.3.1 is not uniform in p. In the proof of Theorem 3.6 we obtained the explicit upper bound

$$\tau \leq \frac{1}{\sqrt{2CC_1C_2C_{\text{cov}}C_{\text{eff}}} \max\limits_{K\in\mathcal{K}_N} (p_K)^{\frac{3}{2}}}$$

and, thus, data saturation assumption (3.10) becomes more and more restrictive for increasing polynomial degree p_K. Since the constant $C \geq 2$ depends on $\theta \in (0,1]$, this assumption gets more restrictive the smaller θ becomes. Although this fact seems a bit surprising at a first sight, it makes perfectly sense. The reason for this is that, if we only refine a few cells, we have to be even more certain about processing the right cells than in the case when we refine a whole bunch of cells. However from Lemma 3.3 in [134] we know

$$\sum_{K\in\mathcal{K}} \frac{h_K^2}{p_K^2} \|f - \Pi f\|_{L^2(K)}^2 \leq \sum_{K\in\mathcal{K}} \frac{h_K^2}{p_K^2\,(2p_K-1)!} \sum_{|\alpha|_1 \leq p_K} \|\partial^\alpha f\|_{L^2(K)}^2$$

for f piecewise analytic. Thus, if there exist some constants $C, \gamma > 0$ independent of polynomial degree vector p such that

$$\sum_{K\in\mathcal{K}} \sum_{|\alpha|_1 \leq p_K} \|\partial^\alpha f\|_{L^2(K)}^2 \leq C\gamma^{|p|}$$

and f is analytic, then from standard approximation theory (see e.g. chapter 4 in [94]) it follows

$$\sum_{K\in\mathcal{K}} \frac{h_K^2}{p_K^2} \|f - \Pi f\|_{L^2(K)}^2 \leq \widetilde{C} \exp(-\sigma|p|)$$

for some constants $\widetilde{C}, \sigma > 0$ independent of polynomial degree vector p.

In [74], Theorem 3, one can find a slightly different proof of convergence for the fully automatic *hp*-adaptive refinement strategy from Section 3.1.3.1. Here the assumptions on parameter τ from data saturation assumption (3.10) are more or less the same, but for θ the lower bound also depends on polynomial degree vector p. Thus, Theorem 3.6 can be seen as a slight generalization of the results from [74].

In Theorem 3.7 the constant $\nu > 0$ depends on the polynomial degree vector p. This is due to the fact that the a posteriori error estimator $\eta\,(u_{\text{FE}}, \mathcal{K})$ from Definition 3.2 cannot be bounded independent of p from above. A way to eliminate this p-dependence might be the use of an equilibrated residual error estimator as proposed by Braess and Schöberl in [62, 61]. Further we assume that there exists some $\rho \in (0,1)$ such that (3.24) holds. This is the higher-dimensional analogue to assumption (3.9). Therefore the same comments as in the discussion of (3.9) in Section 3.1.2.2 apply here, too. In the proof of Theorem 3.7 we have obtained the explicit upper bound

$$\nu \leq \frac{\delta}{2C_{\text{red}}(1+\delta) \max\limits_{K\in\mathcal{K}_N} (p_K)^2}.$$

3.1.3.3 Numerical Results

Now we want to consider the performance of the fully automatic hp-adaptive refinement strategy from Section 3.1.3.1 on the basis of some numerical examples. Therefore we consider some two- and three-dimensional elliptic boundary value problems of the form (3.1). All computations are performed with the finite element library deal.II [41, 42].

Example 1

The first example is a two-dimensional example with a smooth analytic solution. Let $\Omega := (0,1)^2$ and $u : \overline{\Omega} \to \mathbb{R}$ be given by

$$u(x) := x_1 (1 - x_1) x_2 (1 - x_2) (1 - 2x_2) \exp\left(-\frac{5}{2} (2x_1 - 1)^2 \right).$$

The initial triangulation $\mathcal{K}_0$ consists of 64 equally-sized cells and as initial polynomial degree vector we choose $p = 2$. Further we set $\theta := 0.35$.

In this example we perform two different runs of the algorithm. In the first run we provide only two different refinement patterns the algorithm can choose from. The first refinement pattern is classical h-refinement, where the cell is bisected into four equally-sized children. The second refinement pattern is classical p-refinement, where the polynomial degree of the cell is increased by one. In Figure 3.4 we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. In Table 3.1 the marking history of the algorithm is shown. We observe that the hp-adapative refinement

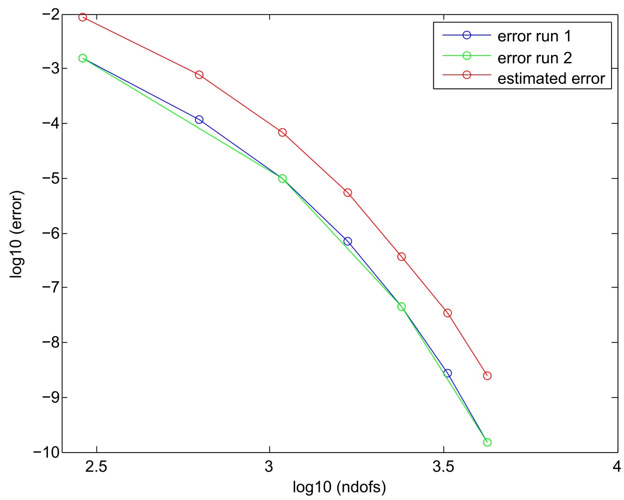

Figure 3.4: Example 1: Number of degrees of freedom vs. error.

Step	#Cells	max(p)	#h	#p	Step	#Cells	max(p)	#h	#p
0	64	2	0	64	4	64	6	0	64
1	64	3	0	64	5	64	7	0	64
2	64	4	0	64	6	64	8	0	64
3	64	5	0	64					

Table 3.1: Example 1: Marking history.

strategy from Section 3.1.3.1 chooses p-refinement on all cells. This is basically what we expect, because the exact solution u is analytic and there are no local features to detect. Thus p-refinement performs best.
In a second run we add a third refinement pattern to the refinement algorithm. Now the strategy can additionally choose to increase the polynomial degree by two. In Figure 3.4 we can see that the algorithm really takes advantage of this new refinement pattern and reaches the same accuracy as in run 1 in only half the number of refinement steps.

Example 2
Also for the second example we stay in the case $d = 2$. However, this time we consider the behaviour of the fully automatic hp-adaptive refinement algorithm for a singular analytic solution. Let $\Omega := (-1,1)^2 \setminus [0,1) \times (-1,0]$ and the exact solution u be given by

$$u(r,\varphi) := r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right),$$

where $r \in [0,1)$ and $\varphi \in [0,2\pi)$ denote the polar coordinates. The initial triangulation $\mathcal{K}_0$ consists of 48 equally-sized cells and as initial polynomial degree vector we choose $p = 2$ again. Further we set $\theta := 0.25$. The algorithm can choose from classical h- and p-refinement again. In Figure 3.5 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. On the right-hand side of Figure 3.5 we can see the final grid produced by the

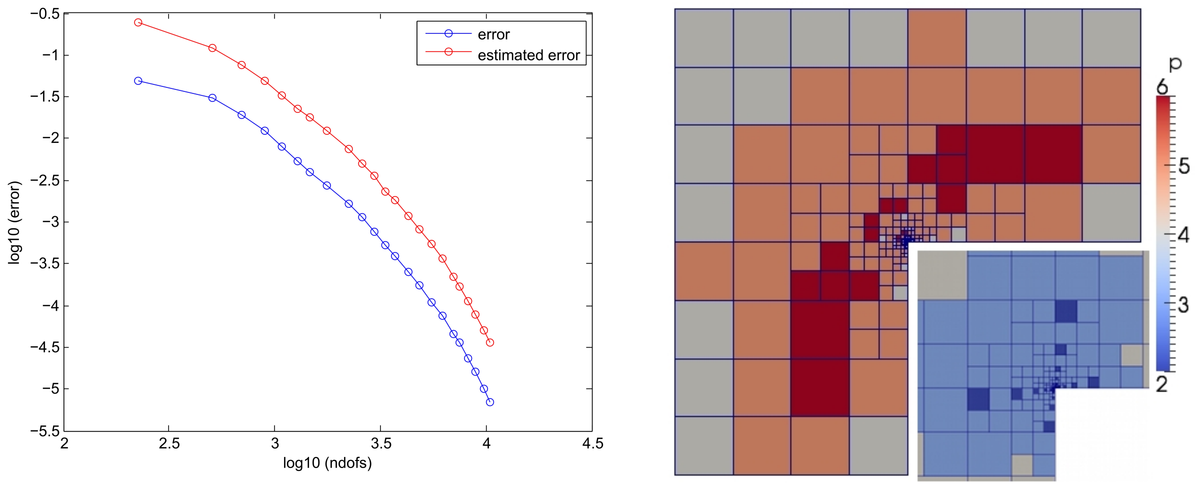

Figure 3.5: Example 2. Left: Number of degrees of freedom vs. error. Right: Final grid.

algorithm, where the fourth quadrant is a zoom into the reentrant corner of the domain. We observe that the final grid basically is linearly graded towards the singularity located at the origin. This means that around the origin cells are small and polynomial degrees are low. The more one goes away from the singularity the larger are the cells and the higher are the polynomial degrees. In Figure 3.6 we plot the distance to the origin vs. the average polynomial degree present on that circle in $\log_{10}$-1-scale. The marking history of the algorithm is shown in Table 3.2. Also in this example we get more or less the result which one expects from the hp-adaptive refinement algorithm. The singularity is identified correctly and the refinement choices are appropriate.

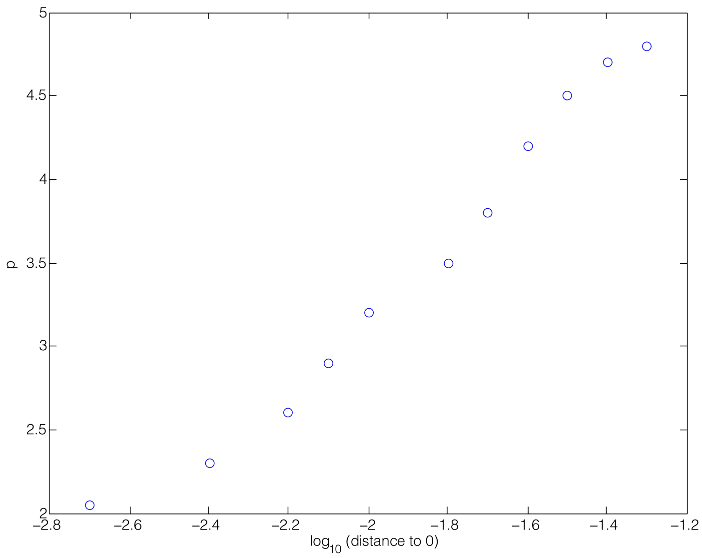

Figure 3.6: Example 2. Left: Distance to 0 vs. average polynomial degree.

Example 3

This is the first three-dimensional example. Again we start with a smooth analytic solution. Let $\Omega := (0,1)^3$ and $u : \overline{\Omega} \to \mathbb{R}$ be given by

$$u(x) := \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3) \,.$$

The initial triangulation $\mathcal{K}_0$ consists of 64 equally-sized cells and as initial polynomial degree vector we choose $p = 2$. Further we set $\theta := 0.2$.

As in Example 1 we perform two different runs of the algorithm. In the first run we provide only two different refinement patterns the algorithm can choose from. As usual the first refinement pattern is classical h-refinement and the second refinement pattern is classical p-refinement. In Figure 3.7 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in a $\log_{10}$-$\log_{10}$-scale. In Table 3.3 on the left-hand side the marking history of the algorithm is shown. We observe that the hp-adaptive refinement strategy from Section 3.1.3.1 chooses p-refinement only. This is basically what we expect, because the exact solution u is analytic and, thus, p-refinement performs best.

In a second run we add a third refinement pattern to the refinement algorithm again. Now the strategy can additionally choose to increase the polynomial degree by two. In Figure 3.7 on the right-hand side we can see that the algorithm really takes advantage of this new refinement pattern and requires only half the number of refinement steps to achieve the desired tolerance TOL $:= 2 \cdot 10^{-7}$. The refinement history of the second run is shown in Table 3.3 on the right-hand side.

Example 4

The last example is for the case $d = 3$ again. As in Example 2 we consider the behaviour of the fully automatic hp-adaptive refinement algorithm for a singular analytic solution. Let $\Omega := (-1,1)^3 \setminus [0,1)^3$ and the exact solution u be given by

$$u(x) := \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{3}} .$$

Step	#Cells	max(p)	h	p	Step	#Cells	max(p)	h	p
0	48	2	3	2	11	345	4	3	10
1	75	3	3	0	12	372	4	3	18
2	102	3	3	0	13	399	4	3	18
3	129	3	3	0	14	426	4	3	22
4	156	3	3	0	15	453	5	3	23
5	183	3	3	0	16	480	5	3	28
6	210	3	3	4	17	507	6	3	12
7	237	3	3	10	18	534	6	3	22
8	264	4	3	10	19	561	6	1	28
9	291	4	3	11	20	576	6	1	32
10	318	4	3	10	21	591	6	1	26

Table 3.2: Example 2: Marking history.

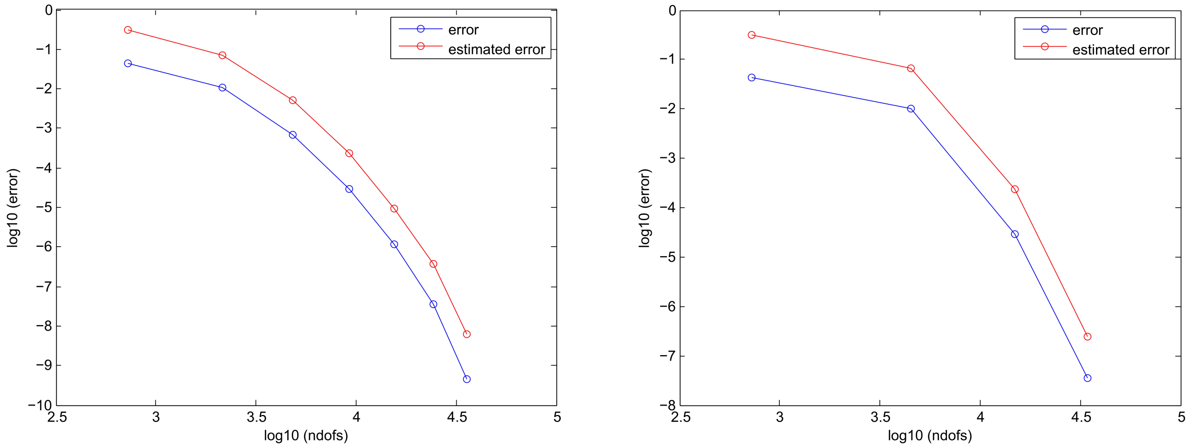

Figure 3.7: Example 3: Number of degrees of freedom vs. error. Left: Run 1. Right: Run 2.

The initial triangulation consists of 56 equally-sized cells and as initial polynomial degree vector we choose $p = 1$. Further we set $\theta := 0.16$. The algorithm can choose from classical h- and p-refinement again. In Figure 3.8 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. On the right-hand side we can see the final grid produced by the algorithm. We observe that the grid basically is linearly graded towards the singularity located at the origin. This means that around the origin cells are small and polynomial degrees are low. The more one goes away from the singularity the larger are the cells and the higher are the polynomial degrees. The marking history of the algorithm is shown in Table 3.4. Also in this example we get more or less the result which one expects from the hp-adaptive refinement algorithm. The singularity is indentified correctly and the refinement choices are appropriate.

3.2 The Discontinuous Galerkin Finite Element Method

In contrast to the continuous Galerkin finite element method there is not *the* discontinuous Galerkin finite element method, but there exist lots of different approaches to obtain an approximated solution of

Step	#Cells	max(p)	$\#h$	$\#p$
0	64	2	0	8
1	64	3	0	24
2	64	4	0	24
3	64	5	0	24
4	64	6	0	24
5	64	7	0	24

Step	#Cells	max(p)	$\#h$	$\#p$
0	64	2	0	8
1	64	4	0	24
2	64	6	0	24
3	64	8	0	24

Table 3.3: Example 3: Marking history. Left: Run 1. Right: Run 2.

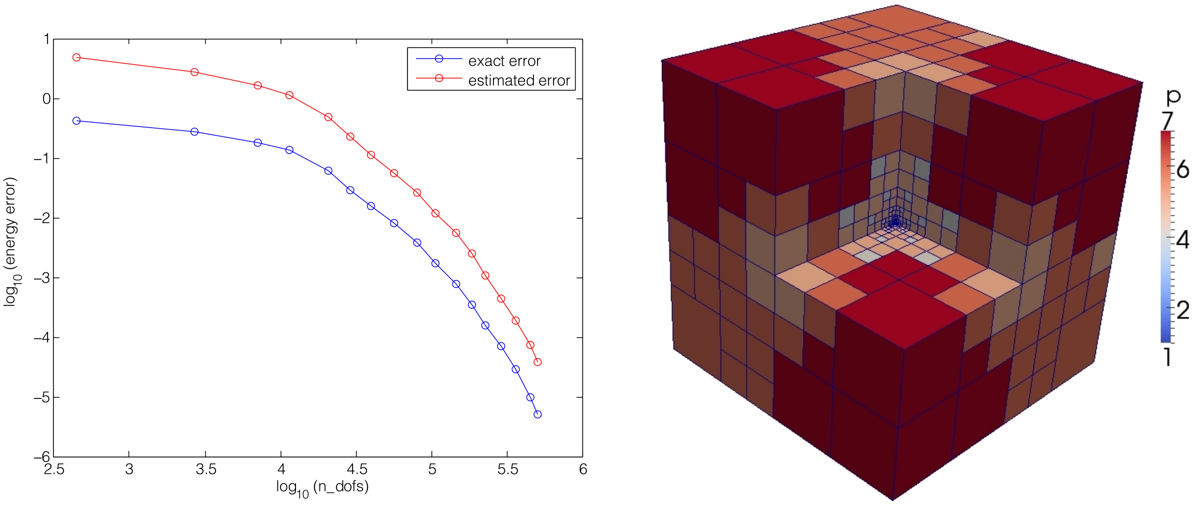

Figure 3.8: Example 4. Left: Number of degrees of freedom vs. error. Right: Final grid.

problem (3.1). E.g. there are the interior penalty method [18, 40, 106, 168, 213], the local discontinuous Galerkin methods [19, 85] and the Bassi-Rebay methods [48, 49, 66] to name only a few. We restrict ourselves to the interior penalty method for simplicity. Similar to the continuous Galerkin finite element method there is much more literature about the convergence of the h-adaptive discontinuous Galerkin finite element method (see e.g. [55, 131, 139]) than the hp-adaptive one. The development of fully automatic hp-adaptive refinement strategies for the discontinuous Galerkin finite element method was initiated only recently in [133, 221], where a residual-based a posteriori error estimator for the hp-adaptive discontinuous Galerkin finite element method was presented. The goal of this section is to extend the hp-adaptive refinement strategy from Section 3.1 to the hp-version of the discontinuous Galerkin finite element method and prove its convergence. Therefore we derive the weak formulation of problem (3.1) first. Then we extend the refinement strategy from Section 3.1 to the discontinuous Galerkin finite element method and prove its convergence. The results of this section are based on [75]. Throughout this subsection we assume $d \in \{2, 3\}$.

3.2.1 The Problem Formulation

In this subsection we derive the weak and discrete formulations of problem (3.1). Then the most important properties of the bilinear form resulting from the weak formulation are discussed.
Before we derive of the weak formulation let us introduce some basic notations first. Let $\mathcal{K}$ be some

Step	#Cells	max(p)	#h	#p	Step	#Cells	max(p)	#h	#p
0	56	1	7	0	8	624	2	7	28
1	99	1	7	0	9	724	3	7	7
2	142	1	7	0	10	825	3	7	53
3	203	1	7	0	11	924	4	7	22
4	264	1	7	26	12	1026	4	7	102
5	343	2	7	4	13	1129	5	6	6
6	422	2	7	0	14	1242	6	7	0
7	520	2	7	0	15	1341	6	7	17

Table 3.4: Example 4: Marking history.

(γ_1, γ_2)-regular triangulation of Ω. We denote the set of all interior edges (in the case $d = 2$) or faces (in the case $d = 3$) of $\mathcal{K}$ by

$$\mathcal{E}_I(\mathcal{K}) := \{e \subset \partial K \cap \Omega : e \text{ is an elemental edge of cell } K\}$$

and the set of all boundary edges or faces by

$$\mathcal{E}_B(\mathcal{K}) := \{e \subset \partial K \cap \partial\Omega : e \text{ is an elemental edge of cell } K\}.$$

Then we let

$$\mathcal{E}(\mathcal{K}) := \mathcal{E}_I(\mathcal{K}) \cup \mathcal{E}_B(\mathcal{K})$$

to be the set of all edges or faces of triangulation $\mathcal{K}$. Further we introduce the *energy space*

$$E(\mathcal{K}) := \prod_{K \in \mathcal{K}} H^1(K).$$

Let $K_1, K_2 \in \mathcal{K}$ be two neighbouring cells and denote the outward-pointing unit normal vectors to these cells by n_{K_1} and n_{K_2} respectively. Then we define the *jumps* $[\![\cdot]\!] : E(\mathcal{K}) \to E(\mathcal{K})^d$ and $[\![\cdot]\!] : E(\mathcal{K})^d \to E(\mathcal{K})$ by

$$[\![u]\!] := u|_{K_1} n_{K_1} + u|_{K_2} n_{K_2} \qquad \forall u \in E(\mathcal{K})$$

and

$$[\![q]\!] := (q|_{K_1})^T n_{K_1} + (q|_{K_2})^T n_{K_2} \qquad \forall q \in E(\mathcal{K})^d,$$

respectively. The *average* $\{\cdot\} : E(\mathcal{K})^d \to E(\mathcal{K})^d$ is defined by

$$\{q\} := \frac{1}{2}(q|_{K_1} + q|_{K_2}) \qquad \forall q \in E(\mathcal{K})^d.$$

On a boundary edge or face $e \in \mathcal{E}_B(\mathcal{K})$ with $e \subset \partial K \cap \partial\Omega$ for some $K \in \mathcal{K}$ we set accordingly

$$[\![u]\!] := u|_K n_K \qquad \forall u \in E(\mathcal{K})$$

and

$$[\![q]\!] := (q|_K)^T n_K, \quad \{q\} := q \qquad \forall q \in E(\mathcal{K})^d.$$

To obtain the weak formulation of problem (3.1) we multiply the first equation of (3.1) with some test function $\phi \in E(\mathcal{K})$ and integrate over Ω. This yields

$$-\int_\Omega \phi \Delta u = \int_\Omega \phi f \qquad \forall \phi \in E(\mathcal{K})$$

and with integration by parts we obtain

$$\sum_{K\in\mathcal{K}}\int_K (\nabla\phi)^T\nabla u - \sum_{e\in\mathcal{E}(\mathcal{K})}\int_e [\![\phi]\!]^T\{\nabla u\} = \int_\Omega \phi f \qquad \forall\phi\in E(\mathcal{K}).$$

However it can be seen easily that this formulation is not symmetric. Following the ideas of Wheeler [213] based on the observation

$$[\![u]\!] = 0 \qquad \text{on } e$$

for all $e\in\mathcal{E}(\mathcal{K})$ we arrive immediately at the problem to find $u\in H_0^1(\Omega)\cap H^{\frac{3}{2}}(\Omega)$ such that

$$\sum_{K\in\mathcal{K}}\int_K (\nabla\phi)^T\nabla u + \sum_{e\in\mathcal{E}(\mathcal{K})}\left(\int_e c[\![\phi]\!]^T[\![u]\!] - \int_e [\![\phi]\!]^T\{\nabla u\} - \int_e [\![u]\!]^T\{\nabla\phi\}\right) = \int_\Omega \phi f \qquad \forall\phi\in E(\mathcal{K}), \tag{3.45}$$

where $c\in L^\infty(\mathcal{E}(\mathcal{K}))$ denotes some weighting function. Now we are left with two open tasks. The first one is the extra regularity requirement $u\in H_0^1(\Omega)\cap H^{\frac{3}{2}}(\Omega)$ instead of $u\in H_0^1(\Omega)$. This is due to the integration of the average $\{\nabla u\}$. To overcome this restriction we follow the ideas of Arnold, Brezzi, Cockburn and Marini [19] and Perugia and Schötzau [178] and define a *lifting operator* $L: E(\mathcal{K})\to U^p(\mathcal{K},\Omega)^d$ such that

$$\int_\Omega L(\phi)^T\psi := \sum_{e\in\mathcal{E}(\mathcal{K})}\int_e [\![\phi]\!]^T\{\psi\} \qquad \forall\psi\in U^p(\mathcal{K},\Omega)^d.$$

Then problem (3.45) reads to find $u\in H_0^1(\Omega)$ such that

$$\sum_{K\in\mathcal{K}}\int_K (\nabla\phi)^T\nabla u - \int_\Omega L(\phi)^T\nabla u - \int_\Omega L(u)^T\nabla\phi + \sum_{e\in\mathcal{E}(\mathcal{K})}\int_e c[\![\phi]\!]^T[\![u]\!] = \int_\Omega \phi f \qquad \forall\phi\in E(\mathcal{K}).$$

In [133] it was shown that the lifting operator L is L^2-stable.

Lemma 3.13 (Stability of Lifting Operator)**.** *Let $u\in E(\mathcal{K})$. Then there exists some constant $C_L>0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|L(u)\|^2_{L^2(\Omega)^d} \le C_L \sum_{e\in\mathcal{E}(\mathcal{K})}\frac{p_e^2}{h_e}\|[\![u]\!]\|^2_{L^2(e)^d}.$$

Proof. See Lemma 4.1 in [133]. ☐

Now we still have to find a suitable choice for the weighting function c. Therefore let $e\in\mathcal{E}(\mathcal{K})$ be arbitrary. Then we set $h_e := \operatorname{diam}(e)$ and

$$p_e := \begin{cases}\max\{p_{K_1}, p_{K_2}\}, & \text{if } e\in\mathcal{E}_I(\mathcal{K})\wedge e = K_1\cap K_2 \text{ with } K_1,K_2\in\mathcal{K}\\ p_K, & \text{if } e\in\mathcal{E}_B(\mathcal{K})\wedge e\subset\partial K \text{ with } K\in\mathcal{K}\end{cases}.$$

The weighting function is chosen as

$$c := \gamma\frac{p_e^2}{h_e}$$

for some constant $\gamma>0$ independent of mesh size vector h and polynomial degree vector p. We define the bilinear form $A_\mathcal{K}: E(\mathcal{K})\times E(\mathcal{K})\to\mathbb{R}$ by

$$A_\mathcal{K}(\phi,\psi) := \sum_{K\in\mathcal{K}}\int_K (\nabla\phi)^T\nabla\psi - \int_\Omega L(\phi)^T\nabla\psi - \int_\Omega L(\psi)^T\nabla\phi + \gamma\sum_{e\in\mathcal{E}(\mathcal{K})}\frac{p_e^2}{h_e}\int_e [\![\phi]\!]^T[\![\psi]\!]$$

and see immediately

$$A_{\mathcal{K}}(\phi, \psi) = \int_\Omega (\nabla\phi)^T \nabla\psi$$

for $\phi, \psi \in H_0^1(\Omega)$. Then the weak formulation of problem (3.1) reads to find $u \in H_0^1(\Omega)$ such that

$$A_{\mathcal{K}}(\phi, u) = \int_\Omega \phi f \qquad \forall \phi \in H_0^1(\Omega). \tag{3.46}$$

Further we endow the energy space $E(\mathcal{K})$ with the norm $\|\cdot\|_{E(\mathcal{K})} : E(\mathcal{K}) \to \mathbb{R}_+$ given by

$$\|u\|_{E(\mathcal{K})}^2 := \sum_{K \in \mathcal{K}} \|\nabla u\|_{L^2(K)^d}^2 + \gamma \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![u]\!]\|_{L^2(e)^d}^2 .$$

We observe immediately that in contrast to the continuous Galerkin finite element method the bilinear form $A_{\mathcal{K}}$ and the energy norm $\|\cdot\|_{E(\mathcal{K})}$ are mesh-dependent.
Now it can be shown that $A_{\mathcal{K}}$ is continuous and elliptic. The continuity was shown in [133, 216].

Lemma 3.14 (Continuity of $A_{\mathcal{K}}$)**.** *Let $\gamma \geq 1$. Then the bilinear form $A_{\mathcal{K}}$ is* continuous *in the energy norm $\|\cdot\|_{E(\mathcal{K})}$, i.e. there exists some constant $C_{cont} > 0$ independent of γ, mesh size vector h and polynomial degree vector p such that*

$$A_{\mathcal{K}}(\phi, \psi) \leq C_{cont} \|\phi\|_{E(\mathcal{K})} \|\psi\|_{E(\mathcal{K})} \qquad \forall \phi, \psi \in E(\mathcal{K}).$$

Proof. See Lemma 4.2 in [133]. □

The ellipticity of the bilinear form $A_{\mathcal{K}}$ was shown in [216].

Lemma 3.15 (Ellipticity of $A_{\mathcal{K}}$)**.** *For $\gamma \geq 1$ sufficiently large the bilinear form $A_{\mathcal{K}}$ is* elliptic*, i.e. there exists some constant $C_{ell} > 0$ independent of γ, mesh size vector h and polynomial degree vector p such that*

$$A_{\mathcal{K}}(\phi, \phi) \geq C_{ell} \|\phi\|_{E(\mathcal{K})}^2 \qquad \forall \phi \in E(\mathcal{K}).$$

Proof. See Proposition 3.8 in [216]. □

In [216], Theorem 3.9, it was shown that weak problem (3.46) has a unique solution $u \in H_0^1(\Omega)$. Thus, it makes sense to consider the discrete formulation of (3.46) to obtain a numerical approximation for the solution of problem (3.1). Therefore let us emphasize the following key property of $A_{\mathcal{K}}$: For all $\phi, \psi \in U^p(\mathcal{K}, \Omega)$ it holds

$$A_{\mathcal{K}}(\phi, \psi) = \sum_{K \in \mathcal{K}} \int_K (\nabla\phi)^T \nabla\psi + \sum_{e \in \mathcal{E}(\mathcal{K})} \left(\gamma \frac{p_e^2}{h_e} \int_e [\![\phi]\!]^T [\![\psi]\!] - \int_e [\![\phi]\!]^T \{\nabla\psi\} - \int_e [\![\psi]\!]^T \{\nabla\phi\} \right).$$

Then the discrete formulation of problem (3.46) reads to find $u_{\mathrm{FE}} \in U^p(\mathcal{K}, \Omega)$ such that

$$A_{\mathcal{K}}(\phi, u_{\mathrm{FE}}) = \int_\Omega \phi f \qquad \forall \phi \in U^p(\mathcal{K}, \Omega). \tag{3.47}$$

For this discretization of problem (3.1) the following a priori error estimate was proven by Georgoulis and Süli [118].

Theorem 3.8 (A Priori Error Estimate)**.** *Let $k \in \mathbb{N}$ be arbitrary and assume that the solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ of (3.46) has the additional regularity $u|_K \in H^{k+1}(K)$ for all $K \in \mathcal{K}$. Further let $u_{FE} \in U^p(\mathcal{K}, \Omega)$ be the solution of (3.47). Then there exists some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|u - u_{FE}\|_{E(\mathcal{K})} \leq C \sum_{K \in \mathcal{K}} \frac{h_K^\mu}{p_K^k} \|\nabla u\|_{H^{\mu+1}(K)},$$

where $\mu := \min\{p_K, k\}$.

Proof. See Theorem 7.2 in [118]. □

If the mesh size vector h and the polynomial degree vector p are chosen suitably and the analytic solution u is sufficiently smooth, it can be shown that the error decays exponentially. For $d = 2$ Wihler, Frauenfelder and Schwab [216] have proven the estimate

$$\|u - u_{\mathrm{FE}}\|_{E(\mathcal{K})} \leq C_1 \exp\left(-C_2 N^{\frac{1}{3}}\right),$$

where $N := \dim\left(U^p(\mathcal{K}, \Omega)\right)$ and $C_1, C_2 > 0$ denote some constants independent of N.
To conclude this subsection let us provide some analytical tools which become important later on. The following interpolation operator was derived by Zhu and Schötzau in [222]. It maps functions from the discontinuous Galerkin finite element space $U^p(\mathcal{K}, \Omega)$ into the continuous Galerkin finite element space $V^p(\mathcal{K}, \Omega)$.

Theorem 3.9 (Averaging Operator)**.** *Let $u_{FE} \in U^p(\mathcal{K}, \Omega)$. Then there exists some linear operator $\Pi_{ZS} : U^p(\mathcal{K}, \Omega) \to V^p(\mathcal{K}, \Omega)$ and some constant $C_{ZS} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|u_{FE} - \Pi_{ZS} u_{FE}\|_{L^2(\Omega)}^2 \leq C_{ZS} \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{h_e}{p_e^2} \|[\![u_{FE}]\!]\|_{L^2(e)^d}^2$$

and

$$\sum_{K \in \mathcal{K}} \|\nabla\left(u_{FE} - \Pi_{ZS} u_{FE}\right)\|_{L^2(K)^d}^2 \leq C_{ZS} \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![u_{FE}]\!]\|_{L^2(e)^d}^2.$$

Proof. See Theorem 4.4 in [222]. □

3.2.2 The Refinement Strategy

In this subsection we adapt the fully automatic *hp*-adaptive refinement strategy from Section 3.1 to the discontinuous Galerkin finite element method. Since we already have discussed the basic principles of the fully automatic *hp*-adaptive refinement strategy in Sections 3.1.2.1 and 3.1.3.1, we only highlight the differences here. As in Sections 3.1.2.1 and 3.1.3.1 our starting point is the adaptive loop (3.4). We consider the modules ESTIMATE and MARK again.
Let us begin with the module ESTIMATE. In this module we want to estimate the error of the computed finite element solution which we obtain from module SOLVE. Therefore we define the following a posteriori error estimator. It is obtained by taking the estimator introduced in [133], but omitting the jump term which is given by the jumps of the approximate solution $u_{\mathrm{FE}} \in U^p(\mathcal{K}, \Omega)$ over cell boundaries. This idea was presented in [55] for the h-adaptive discontinuous Galerkin finite element method.

Definition 3.3 (A Posteriori Error Estimator). *Let $u_{FE} \in U^p(\mathcal{K}, \Omega)$ be the solution of (3.47). Then the* residual-based a posteriori error estimator *$\eta(u_{FE}, \mathcal{K})$ is given by*

$$\eta(u_{FE}, \mathcal{K})^2 := \sum_{K \in \mathcal{K}} \eta_K(u_{FE}, \mathcal{K})^2$$

with

$$\eta_K(u_{FE}, \mathcal{K})^2 := \eta_{R,K}(u_{FE}, \mathcal{K})^2 + \eta_{B,K}(u_{FE}, \mathcal{K})^2 .$$

Here the residual term $\eta_{R,K}(u_{FE}, \mathcal{K})$ reads

$$\eta_{R,K}(u_{FE}, \mathcal{K}) := \frac{h_K}{p_K} \|\Pi f + \Delta u_{FE}\|_{L^2(K)}$$

and the boundary term $\eta_{B,K}(u_{FE}, \mathcal{K})$ is given by

$$\eta_{B,K}(u_{FE}, \mathcal{K})^2 := \frac{1}{2} \sum_{e \in \mathcal{E}_I(\mathcal{K}) \cap \partial K} \frac{h_e}{p_e} \|[\![\nabla u_{FE}]\!]\|^2_{L^2(e)} .$$

For this a posteriori error estimator one can show that it is reliable. Before we prove this let us derive an upper bound for the jump of functions from $U^p(\mathcal{K}, \Omega)$ across the cell boundaries of triangulation $\mathcal{K}$. This estimate is an extension of the result derived in [55] for the h-adaptive discontinuous Galerkin finite element method to the hp-adaptive case.

Lemma 3.16 (Jump Control). *Let $u_{FE} \in U^p(\mathcal{K}, \Omega)$ be the solution of (3.47) and $\gamma \geq 1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p. Then there exists some constant $C_{jump} > 1$ independent of γ, mesh size vector h and polynomial degree vector p such that*

$$\gamma^2 \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![u_{FE}]\!]\|^2_{L^2(e)^d} \leq C_{jump} \sum_{K \in \mathcal{K}} \left(\eta_{R,K}(u_{FE}, \mathcal{K})^2 + \frac{h_K^2}{p_K^2} \|f - \Pi f\|^2_{L^2(K)} \right) .$$

Proof. With $\Pi_{\mathrm{ZS}} : U^p(\mathcal{K}, \Omega) \to V^p(\mathcal{K}, \Omega)$ from Theorem 3.9 we see

$$\gamma \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![u_{\mathrm{FE}}]\!]\|^2_{L^2(e)^d} = \gamma \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}]\!]\|^2_{L^2(e)^d} ,$$

since $[\![\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}]\!] = 0$. Then the definition of the energy norm $\|\cdot\|_{E(\mathcal{K})}$ implies

$$\begin{aligned} \gamma \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![u_{\mathrm{FE}}]\!]\|^2_{L^2(e)^d} &\leq \|u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\|^2_{E(\mathcal{K})} \\ &\leq \frac{1}{C_{\mathrm{cell}}} A_{\mathcal{K}}(u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}, u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}) \end{aligned}$$

for $\gamma \geq 1$ sufficiently large by Lemma 3.15. Since $u_{\mathrm{FE}} \in U^p(\mathcal{K}, \Omega)$ solves (3.47), it follows

$$\gamma \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![u_{\mathrm{FE}}]\!]\|^2_{L^2(e)^d} \leq \frac{1}{C_{\mathrm{cell}}} \left(\int_\Omega (u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}) f - A_{\mathcal{K}}(u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}, \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}) \right) . \tag{3.48}$$

With $\Pi : L^2(\Omega) \to U^p(\mathcal{K}, \Omega)$ denoting the L^2-conforming interpolation from Section 2.4 we see easily

$$\int_\Omega (u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}) f = \int_\Omega (u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}) \Pi f + \int_\Omega (u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}) (f - \Pi f) \tag{3.49}$$

and by the definition of the bilinear form $A_{\mathcal{K}}$ and $L\left(\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)=\llbracket \Pi_{\mathrm{ZS}} u_{\mathrm{FE}} \rrbracket=0$ we have

(3.50)

$$
\begin{aligned}
&A_{\mathcal{K}}\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}, \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right) \\
&\quad=\sum_{K \in \mathcal{K}} \int_K\left(\nabla\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)\right)^T \nabla \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}-\int_{\Omega} L\left(u_{\mathrm{FE}}\right)^T \nabla \Pi_{\mathrm{ZS}} u_{\mathrm{FE}} \\
&\quad=\sum_{K \in \mathcal{K}}\left(\int_K\left(\nabla\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)\right)^T \nabla u_{\mathrm{FE}}-\left\|\nabla\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)\right\|_{L^2(K)^d}^2\right)-\int_{\Omega} L\left(u_{\mathrm{FE}}\right)^T \nabla \Pi_{\mathrm{ZS}} u_{\mathrm{FE}} .
\end{aligned}
$$

Then integration by parts yields

$$
\begin{aligned}
\sum_{K \in \mathcal{K}} \int_K\left(\nabla\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)\right)^T \nabla u_{\mathrm{FE}} &=\sum_{e \in \mathcal{E}(\mathcal{K})} \int_e \llbracket u_{\mathrm{FE}} \rrbracket^T\left\{\nabla u_{\mathrm{FE}}\right\}-\sum_{K \in \mathcal{K}} \int_K\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right) \Delta u_{\mathrm{FE}} \\
&=\int_{\Omega} L\left(u_{\mathrm{FE}}\right)^T \nabla u_{\mathrm{FE}}-\sum_{K \in \mathcal{K}} \int_K\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right) \Delta u_{\mathrm{FE}}
\end{aligned}
$$

by the definition of the lifting operator L and inserting into (3.50) gives

(3.51)

$$
\begin{aligned}
&A_{\mathcal{K}}\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}, \Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right) \\
&\quad=\int_{\Omega} L\left(u_{\mathrm{FE}}\right)^T \nabla\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)-\sum_{K \in \mathcal{K}}\left(\int_K\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right) \Delta u_{\mathrm{FE}}+\left\|\nabla\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)\right\|_{L^2(K)^d}^2\right) .
\end{aligned}
$$

By inserting equations (3.49) and (3.51) into (3.48) we obtain

$$
\gamma \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e}\left\|\llbracket u_{\mathrm{FE}} \rrbracket\right\|_{L^2(e)^d}^2 \leq \frac{1}{C_{\text{ell}}}\left(T_1+T_2+T_3+T_4\right), \tag{3.52}
$$

where the terms $T_1, \ldots, T_4$ are given by

$$
\begin{aligned}
T_1 &:=\sum_{K \in \mathcal{K}} \int_K\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)\left(\Pi f+\Delta u_{\mathrm{FE}}\right), \\
T_2 &:=\sum_{K \in \mathcal{K}}\left\|\nabla\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)\right\|_{L^2(K)^d}^2, \\
T_3 &:=\int_{\Omega} L\left(u_{\mathrm{FE}}\right)^T \nabla\left(\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}-u_{\mathrm{FE}}\right)
\end{aligned}
$$

and

$$
T_4:=\int_{\Omega}\left(u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right)(f-\Pi f).
$$

For the term T_1 we see

$$
T_1 \leq \sum_{K \in \mathcal{K}}\left\|u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right\|_{L^2(K)}\left\|\Pi f+\Delta u_{\mathrm{FE}}\right\|_{L^2(K)}
$$

with Hölder's inequality and the Cauchy-Schwarz inequality implies

$$
\begin{aligned}
T_1 &\leq\left(\sum_{K \in \mathcal{K}} \frac{p_K^2}{h_K^2}\left\|u_{\mathrm{FE}}-\Pi_{\mathrm{ZS}} u_{\mathrm{FE}}\right\|_{L^2(K)}^2\right)^{\frac{1}{2}}\left(\sum_{K \in \mathcal{K}} \eta_{R, K}\left(u_{\mathrm{FE}}, \mathcal{K}\right)^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{C_{\mathrm{ZS}}}\left(\sum_{e \in \mathcal{E}(\mathcal{K})} \frac{1}{h_e}\left\|\llbracket u_{\mathrm{FE}} \rrbracket\right\|_{L^2(e)^d}^2\right)^{\frac{1}{2}}\left(\sum_{K \in \mathcal{K}} \eta_{R, K}\left(u_{\mathrm{FE}}, \mathcal{K}\right)^2\right)^{\frac{1}{2}}
\end{aligned} \tag{3.53}
$$

by Theorem 3.9 and the (γ_1, γ_2)-regularity of $\mathcal{K}$. Applying Theorem 3.9 to T_2 yields

$$T_2 \le C_{\mathrm{ZS}} \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \left\| [\![u_{\mathrm{FE}}]\!] \right\|_{L^2(e)^d}^2 . \tag{3.54}$$

With Hölder's inequality it follows

$$\begin{aligned} T_3 &\le \left\| L\left(u_{\mathrm{FE}}\right) \right\|_{L^2(\Omega)^d} \left(\sum_{K \in \mathcal{K}} \left\| \nabla \left(u_{\mathrm{FE}} - \Pi_{\mathrm{ZS}} u_{\mathrm{FE}} \right) \right\|_{L^2(K)^d}^2 \right)^{\frac{1}{2}} \\ &\le \sqrt{C_L C_{\mathrm{ZS}}} \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \left\| [\![u_{\mathrm{FE}}]\!] \right\|_{L^2(e)^d}^2 \end{aligned} \tag{3.55}$$

by Lemma 3.13 and Theorem 3.9. For the term T_4 we proceed similar to T_1 and obtain

$$T_4 \le \sqrt{C_{\mathrm{ZS}}} \left(\sum_{e \in \mathcal{E}(\mathcal{K})} \frac{1}{h_e} \left\| [\![u_{\mathrm{FE}}]\!] \right\|_{L^2(e)^d}^2 \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{K}} \frac{h_k^2}{p_K^2} \| f - \Pi f \|_{L^2(K)}^2 \right)^{\frac{1}{2}} . \tag{3.56}$$

Inserting estimates (3.53)–(3.56) into (3.52) yields

$$\begin{aligned} &\left(\gamma - \frac{C_{\mathrm{ZS}} + \sqrt{C_L C_{\mathrm{ZS}}}}{C_{\mathrm{ell}}} \right) \left(\sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \left\| [\![u_{\mathrm{FE}}]\!] \right\|_{L^2(e)^d}^2 \right)^{\frac{1}{2}} \\ &\qquad \le \frac{\sqrt{C_{\mathrm{ZS}}}}{C_{\mathrm{ell}}} \left(\left(\sum_{K \in \mathcal{K}} \eta_{R,K}\left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 \right)^{\frac{1}{2}} + \left(\sum_{K \in \mathcal{K}} \frac{h_k^2}{p_K^2} \| f - \Pi f \|_{L^2(K)}^2 \right)^{\frac{1}{2}} \right) \end{aligned}$$

and by squaring both sides and using Young's inequality we get

$$\left(\gamma - \frac{C_{\mathrm{ZS}} + \sqrt{C_L C_{\mathrm{ZS}}}}{C_{\mathrm{ell}}} \right)^2 \sum_{e \in \mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \left\| [\![u_{\mathrm{FE}}]\!] \right\|_{L^2(e)^d}^2 \le \frac{2 C_{\mathrm{ZS}}}{C_{\mathrm{ell}}^2} \sum_{K \in \mathcal{K}} \left(\eta_{R,K}\left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 + \frac{h_k^2}{p_K^2} \| f - \Pi f \|_{L^2(K)}^2 \right)$$

for

$$\gamma > \frac{C_{\mathrm{ZS}} + \sqrt{C_L C_{\mathrm{ZS}}}}{C_{\mathrm{ell}}} .$$

□

Now we are ready to prove reliability of the a posteriori error estimator introduced in Definition 3.3. Further we derive an efficiency estimate which depends on the polynomial degree vector p.

Theorem 3.10 (A Posteriori Error Estimates)**.** *Let $u \in H_0^1(\Omega)$ be the solution of (3.46) and $u_{FE} \in U^p(\mathcal{K}, \Omega)$ be the solution of (3.47). Further let $\gamma \ge 1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p. Then:*

1. *There exists some constant $C_{rel} \ge 1$ independent of γ, mesh size vector h and polynomial degree vector p such that*

$$\| u - u_{FE} \|_{E(\mathcal{K})}^2 \le C_{rel} \left(\eta\left(u_{FE}, \mathcal{K}\right)^2 + \sum_{K \in \mathcal{K}} \frac{h_K^2}{p_K^2} \| f - \Pi f \|_{L^2(K)}^2 \right) .$$

2. There exists some constant $C_{eff} > 0$ independent of γ, mesh size vector h and polynomial degree vector p such that

$$\eta_K\left(u_{FE},\mathcal{K}\right)^2 \leq C_{eff}\left(p_K^{2(1+\varepsilon)}\sum_{L\in\mathcal{K}|_{\omega_K}}\|\nabla\left(u-u_{FE}\right)\|^2_{L^2(L)^d}+\frac{h_K^2}{p_K^{1-2\varepsilon}}\|f-\Pi f\|^2_{L^2(\omega_K)}\right)$$

for all $K\in\mathcal{K}$ and all $\varepsilon>0$.

Proof. 1. From Theorem 3.1 in [133] we obtain

$$\|u-u_{\mathrm{FE}}\|^2_{E(\mathcal{K})}\leq C_{\mathrm{EST}}\left(\eta\left(u_{\mathrm{FE}},\mathcal{K}\right)^2+\frac{\gamma}{2}\sum_{e\in\mathcal{E}(\mathcal{K})}\frac{p_e^2}{h_e}\|[\![u_{\mathrm{FE}}]\!]\|^2_{L^2(e)^d}\right)+C_{\mathrm{APP}}\sum_{K\in\mathcal{K}}\frac{h_K^2}{p_K^2}\|f-\Pi f\|^2_{L^2(K)}$$

and the result follows with Lemma 3.16.

2. See Theorem 3.2 in [133].

□

Now let us make a step ahead and consider the module MARK. Similar to Section 3.1.3.1 we assume that we have $n\in\mathbb{N}\setminus\{1\}$ different refinement patterns to choose from. Let $j\in\{0,\ldots,n-1\}$ and $K\in\mathcal{K}$ be arbitrary. As in Section 3.1.3.1 we denote by $V^p_{K,j}\left(\mathcal{K}_{K,j}|_{\omega_K},\omega_K\right)$ the local finite element space consisting of functions from the continuous Galerkin finite element space $V^p(\mathcal{K},\Omega)$ compactly supported in the local patch ω_K with refinement pattern j applied to cell K. Without loss of generality we may assume that $\eta_K\left(u_{\mathrm{FE}},\mathcal{K}\right)>0$. If this is not the case, it makes not any sense to refine this cell at all and we can go to the next. Then we define the convergence indicator $\kappa_{K,j}\in\mathbb{R}_+$ as the solution of the optimization problem

$$\kappa_{K,j}=\frac{1}{\eta_K\left(u_{\mathrm{FE}},\mathcal{K}\right)}\sup_{\phi\in V^p_{K,j}\left(\mathcal{K}_{K,j}|_{\omega_K},\omega_K\right)}\left(\frac{\int_{\omega_K}\phi\Pi f-A_{\mathcal{K}_{K,j}|_{\omega_K}}\left(\phi,u_{\mathrm{FE}}\right)}{\|\nabla\phi\|_{L^2(\omega_K)^d}}\right),\tag{3.57}$$

where $\Pi:L^2\left(\omega_K\right)\to U^p\left(\mathcal{K}_{K,j}|_{\omega_K},\omega_K\right)$ denotes the L^2-conforming interpolation from Section 2.4. As in Sections 3.1.2.1 and 3.1.3.1 we can solve (3.57) easily by considering an equivalent local boundary value problem.

Lemma 3.17. *Let $v\in V^p_{K,j}\left(\mathcal{K}_{K,j}|_{\omega_K},\omega_K\right)$ such that*

$$\int_{\omega_K}(\nabla\phi)^T\nabla v=\int_{\omega_K}\phi\Pi f-A_{\mathcal{K}_{K,j}|_{\omega_K}}\left(\phi,u_{FE}\right)\qquad\forall\phi\in V^p_{K,j}\left(\mathcal{K}_{K,j}|_{\omega_K},\omega_K\right).\tag{3.58}$$

Then the supremum in (3.57) is obtained for v.

The proof follows exactly the same arguments as the proof of Lemma 3.7 and, thus, we do not repeat it here. Also maximization problem (3.7), (3.8) can be used without modification for the discontinuous Galerkin finite element method.
To conclude this subsection let us discuss the change in optimization problem (3.57). In contrast to Sections 3.1.2.1 and 3.1.3.1 the local finite element test space $V^p_{K,j}\left(\mathcal{K}_{K,j}|_{\omega_K},\omega_K\right)$ is not a simple local enhancement of the global finite element space $U^p(\mathcal{K},\Omega)$, but we choose a conforming subset of the

locally enhanced discontinuous Galerkin finite element space instead. This is due to the fact that for $\phi \in U^p(\mathcal{K},\Omega) \setminus V^p(\mathcal{K},\Omega)$ it usually holds

$$\begin{aligned} A_{\mathcal{K}}(\phi,\phi) &= \sum_{K\in\mathcal{K}} \|\nabla\phi\|^2_{L^2(K)^d} + \sum_{e\in\mathcal{E}(\mathcal{K})} \left(\gamma \frac{p_e^2}{h_e} \|[\![\phi]\!]\|^2_{L^2(e)^d} - 2\int_e [\![\phi]\!]^T \{\nabla\phi\}\right) \\ &\neq \sum_{K\in\mathcal{K}} \|\nabla\phi\|^2_{L^2(K)^d} + \gamma \sum_{e\in\mathcal{E}(\mathcal{K})} \frac{p_e^2}{h_e} \|[\![\phi]\!]\|^2_{L^2(e)^d} \\ &= \|\phi\|^2_{E(\mathcal{K})} \end{aligned}$$

and thus it could only be shown that the solution of the corresponding local boundary value problem is equivalent to the supremum in the optimization problem. But this does not suffice to determine the convergence indicators $\kappa_{K,j}$ explicitly and, hence, the numerical solution of optimization problem (3.57) would probably become much more involved.

3.2.3 Convergence Results

Now we prove convergence of the fully automatic *hp*-adaptive refinement strategy. Therefore we derive two results similar to Sections 3.1.2.2 and 3.1.3.2. As before the first result proves that the exact energy error $\|u-u_{\mathrm{FE}}\|_{E(\mathcal{K})}$ is reduced in every refinement step of the algorithm. The second result gives us that a weighted sum of exact energy error $\|u-u_{\mathrm{FE}}\|_{E(\mathcal{K})}$ and estimated error $\eta(u_{\mathrm{FE}},\mathcal{K})$ is reduced in every refinement step.

Let us assume that triangulation $\mathcal{K}$ consists of simplices only. Before we prove the main results of this subsection let us consider the error estimator $\eta(u_{\mathrm{FE}},\mathcal{K})$, which was introduced in Definition 3.3, in more detail. Similar to Sections 3.1.2.2 and 3.1.3.2 we want to investigate how the estimated error is influenced by the application of the *hp*-adaptive refinement algorithm presented in Section 3.2.2. Again we discuss the new assumptions, which we make in the following lemmas und theorems, will be discussed at the end of this section in detail.

Lemma 3.18 (Error Estimator Reduction)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $\gamma \geq 1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p. We assume that there exists a solution of maximization problem (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in U^p(\mathcal{K}_N,\Omega)$ and $u_{N+1} \in U^p(\mathcal{K}_{N+1},\Omega)$ be the solutions of (3.47) in iteration steps N and $N+1$, respectively. Additionally let us assume that for all refinement patterns $j \in \{0,\ldots,n-1\}$ there exists some constant $\rho \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that (3.24) holds. We assume that there exists some $\tau \in (0,1]$ such that data saturation assumption (3.10) is fulfilled. Then there exists some constant $C_{red} > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\begin{aligned} \eta(u_{N+1},\mathcal{K}_{N+1})^2 \leq (1+\delta)^2 &\left(\left(1+\frac{\rho^2\tau^2}{\delta}\right)\eta(u_N,\mathcal{K}_N)^2 - (1-\rho)\eta(u_N\mathcal{A}_N)^2\right) \\ &+ C_{red}\left(1+\frac{1}{\delta}\right)\sum_{K\in\mathcal{K}_{N+1}} p_K^2 \|\nabla(u_{N+1}-u_N)\|^2_{L^2(K)^d} \end{aligned}$$

for all $\delta > 0$.

Proof. By Definition 3.3 it holds

$$\eta(u_{N+1},\mathcal{K}_{N+1})^2 = \sum_{K\in\mathcal{K}_{N+1}} \left(\eta_{R,K}(u_{N+1},\mathcal{K}_{N+1})^2 + \eta_{B,K}(u_{N+1},\mathcal{K}_{N+1})^2\right). \tag{3.59}$$

We introduce the set

$$\mathcal{R}_N := \{K \in \mathcal{K}_N : K \text{ is refined}\}$$

of all cells from triangulation $\mathcal{K}_N$ that are refined in module REFINE of iteration step N. Clearly we have $\mathcal{A}_N \subseteq \mathcal{R}_N$. Then, in exactly the way as in the proof of Lemma 3.10 we obtain

(3.60)

$$\eta_{R,K}\left(u_{N+1},\mathcal{K}_{N+1}\right) \le \rho \frac{h_{\widetilde{K}}}{p_{\widetilde{K}}} \left(\left\|\Pi_{\mathcal{K}_N} f + \Delta u_N\right\|_{L^2(K)} + \left\|f - \Pi_{\mathcal{K}_N} f\right\|_{L^2(K)} \right) + C_{\mathrm{inv}} p_K \left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(K)^d}$$

with $\Pi_{\mathcal{K}_N} : L^2(\Omega) \to U^p\left(\mathcal{K}_N,\Omega\right)$ denoting the L^2-conforming interpolation from Section 2.4, if there exists some cell $\widetilde{K} \in \mathcal{R}_N$ such that $K \subseteq \widetilde{K}$. If there exists no such cell $\widetilde{K} \in \mathcal{R}_N$, then $K \in \mathcal{K}_N$ and in exactly the same way as in the proof of Lemma 3.10 we get

$$\eta_{R,K}\left(u_{N+1},\mathcal{K}_{N+1}\right) \le \eta_{R,K}\left(u_N,\mathcal{K}_N\right) + C_{\mathrm{inv}} p_K \left\|\nabla\left(u_{N+1}-u_N\right)\right\|_{L^2(K)^d}. \tag{3.61}$$

Now let us consider the boundary term $\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)$. By Definition 3.3 we have

$$\begin{aligned}
\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 &= \frac{1}{2} \sum_{e\in\mathcal{E}_I(\mathcal{K}_{N+1})\cap\partial K} \frac{h_e}{p_e} \left\|[\![\nabla u_{N+1}]\!]\right\|^2_{L^2(e)} \\
&\le \frac{1}{2} \sum_{e\in\mathcal{E}_I(\mathcal{K}_{N+1})\cap\partial K} \frac{h_e}{p_e} \left\|[\![\nabla u_{N+1}]\!]\right\|_{L^2(e)} \left(\left\|[\![\nabla u_N]\!]\right\|_{L^2(e)} + \left\|[\![\nabla\left(u_{N+1}-u_N\right)]\!]\right\|_{L^2(e)} \right)
\end{aligned}$$

from Minkowski's inequality and using the Cauchy-Schwarz inequality gives

$$\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 \le \eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)\left(T_1+T_2\right), \tag{3.62}$$

where the terms T_1 and T_2 are given by

$$T_1^2 := \frac{1}{2} \sum_{e\in\mathcal{E}_I(\mathcal{K}_{N+1})\cap\partial K} \frac{h_e}{p_e} \left\|[\![\nabla u_N]\!]\right\|^2_{L^2(e)}$$

and

$$T_2^2 := \frac{1}{2} \sum_{e\in\mathcal{E}_I(\mathcal{K}_{N+1})\cap\partial K} \frac{h_e}{p_e} \left\|[\![\nabla\left(u_{N+1}-u_N\right)]\!]\right\|^2_{L^2(e)}.$$

If there exists some cell $\widetilde{K} \in \mathcal{R}_N$ such that $K \subseteq \widetilde{K}$, then it holds

$$T_1^2 \le \frac{\rho}{2} \sum_{e\in\mathcal{E}_I(\mathcal{K}_N)} \frac{h_e}{p_e} \left\|[\![\nabla u_N]\!]\right\|^2_{L^2(e\cap\partial K)}. \tag{3.63}$$

If there exists no such $\widetilde{K} \in \mathcal{R}_N$, then

$$T_1^2 \le \eta_{B,K}\left(u_N,\mathcal{K}_N\right)^2. \tag{3.64}$$

For the term T_2 we get

$$\begin{aligned}
T_2^2 &\le \sum_{L\in\mathcal{K}_{N+1}|_{\omega_K}} \frac{h_L}{p_L} \left\|\nabla\left(u_{N+1}-u_N\right)\right\|^2_{L^2(\partial L)^d} \\
&\le C_{\mathrm{tr}}^2 \sum_{L\in\mathcal{K}_{N+1}|_{\omega_K}} p_L \left\|\nabla\left(u_{N+1}-u_N\right)\right\|^2_{L^2(L)^d}
\end{aligned} \tag{3.65}$$

with Lemma 3.9 in both cases.
Inserting estimates (3.63)–(3.65) into (3.62) gives

$$\begin{aligned}
&\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)\\
&\le\left(\frac{\rho}{2}\sum_{e\in\mathcal{E}_I(\mathcal{K}_N)}\frac{h_e}{p_e}\left\|[\![\nabla u_N]\!]\right\|^2_{L^2(e\cap\partial K)}\right)^{\frac{1}{2}}+C_{\mathrm{tr}}\left(\sum_{L\in\mathcal{K}_{N+1}|_{\omega_K}}p_L\left\|\nabla\left(u_{N+1}-u_N\right)\right\|^2_{L^2(L)^d}\right)^{\frac{1}{2}},
\end{aligned}\tag{3.66}$$

if there exists such a cell $\widetilde{K}\in\mathcal{R}_N$ with $K\subseteq\widetilde{K}$, and

$$\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)\le\eta_{B,K}\left(u_N,\mathcal{K}_N\right)+C_{\mathrm{tr}}\left(\sum_{L\in\mathcal{K}_{N+1}|_{\omega_K}}p_L\left\|\nabla\left(u_{N+1}-u_N\right)\right\|^2_{L^2(L)^d}\right)^{\frac{1}{2}},\tag{3.67}$$

else. Further inserting estimates (3.60), (3.61), (3.66) and (3.67) into (3.59) and using Young's inequality implies

$$\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2\le(1+\delta)\eta\left(u_N,\mathcal{K}_N\setminus\mathcal{R}_N\right)^2+\rho(1+\delta)^2\left(\eta\left(u_N,\mathcal{R}_N\right)^2+\frac{\rho}{\delta}\sum_{K\in\mathcal{R}_N}\frac{h_K^2}{p_K^2}\left\|f-\Pi_{\mathcal{K}_N}f\right\|^2_{L^2(K)}\right)+T$$

for $\delta>0$, where the term T is given by

$$T:=C_{\mathrm{red}}\left(1+\frac{1}{\delta}\right)\sum_{K\in\mathcal{K}_{N+1}}p_K^2\left\|\nabla\left(u_{N+1}-u_N\right)\right\|^2_{L^2(K)^d}$$

for some constant $C_{\mathrm{red}}>0$ independent of polynomial degree vector p. By data saturation assumption (3.10) if follows

$$\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2\le(1+\delta)^2\left(\left(1+\frac{\rho^2\tau^2}{\delta}\right)\eta\left(u_N,\mathcal{K}_N\right)^2+(\rho-1)\eta\left(u_N\mathcal{A}_N\right)^2\right)+T,$$

since $\mathcal{A}_N\subseteq\mathcal{R}_N$. □

Now we show three auxiliary results, which we use in the proofs of the main results of this subsection. The first result gives a lower bound for the term $\|u_{N+1}-u_N\|_{E(\mathcal{K}_{N+1})}$ in terms of the energy error $\|u-u_N\|_{E(\mathcal{K}_N)}$ and the estimated error $\eta\left(u_N,\mathcal{K}_N\right)$.

Lemma 3.19. *Let $N\in\mathbb{N}_0$ be arbitrary and $u\in H_0^1(\Omega)$ be the solution of (3.46). We assume that there exists a solution of (3.7), (3.8) for some $\theta\in(0,1]$. Further let $u_N\in U^p\left(\mathcal{K}_N,\Omega\right)$ and $u_{N+1}\in U^p\left(\mathcal{K}_{N+1},\Omega\right)$ be the solutions of (3.47) in iteration steps N and $N+1$, respectively. Let $\gamma\ge1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p. Additionally let us assume that there exists some $\tau\in(0,1]$ such that (3.10) holds. Then there exists some constant $C\in\left(0,\frac{1}{4}\right)$ independent of γ, mesh size vector h and polynomial degree vector p such that*

$$\left\|u_{N+1}-u_N\right\|^2_{E(\mathcal{K}_{N+1})}\ge C\theta^2\left\|u-u_N\right\|^2_{E(\mathcal{K}_N)}-\tau^2\eta\left(u_N,\mathcal{K}_N\right)^2.$$

Proof. Let $K\in\mathcal{K}_N$ be arbitrary and $\phi_{N+1}\in V^p\left(\mathcal{K}_{N+1},\Omega\right)$ with $\operatorname{supp}\left(\phi_{N+1}\right)\subseteq\omega_K$. Then we see

$$A_{\mathcal{K}_{N+1}|_{\omega_K}}\left(\phi_{N+1},u_{N+1}-u_N\right)=\int_{\omega_K}\phi_{N+1}f-A_{\mathcal{K}_{N+1}|_{\omega_K}}\left(\phi_{N+1},u_N\right),$$

since $\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega) \subset U^p(\mathcal{K}_{N+1}, \Omega)$ and $u_{N+1} \in U^p(\mathcal{K}_{N+1}, \Omega)$ solves discrete problem (3.47). This reads

$$A_{\mathcal{K}_{N+1}|_{\omega_K}}(\phi_{N+1}, u_{N+1} - u_N) = T + \int_{\omega_K} \phi_{N+1}(f - \Pi f),$$

where $\Pi : L^2(\Omega) \to U^p(\mathcal{K}_N, \Omega)$ denotes the L^2-conforming interpolation from Section 2.4 and the term T is given by

$$T := \int_{\omega_K} \phi_{N+1}\Pi f - A_{\mathcal{K}_{N+1}|_{\omega_K}}(\phi_{N+1}, u_N).$$

With the L^2-interpolation property this implies

$$A_{\mathcal{K}_{N+1}|_{\omega_K}}(\phi_{N+1}, u_{N+1} - u_N) = T + \int_{\omega_K} (\phi_{N+1} - \phi_N)(f - \Pi f),$$

for $\phi_N \in V^p(\mathcal{K}_N, \Omega)$ with $\operatorname{supp}(\phi_N) \subseteq \omega_K$ and using Minkowski's inequality yields

$$\left| A_{\mathcal{K}_{N+1}|_{\omega_K}}(\phi_{N+1}, u_{N+1} - u_N) \right| \geq |T| - \left| \int_{\omega_K} (\phi_{N+1} - \phi_N)(f - \Pi f) \right|.$$

With Lemma 3.14 and Hölder's inequality we have

$$|T| \leq C_{\text{cont}} \|u_{N+1} - u_N\|_{E\left(\mathcal{K}_{N+1}|_{\omega_K}\right)} \|\nabla\phi_{N+1}\|_{L^2(\omega_K)^d} + \|f - \Pi f\|_{L^2(\omega_K)} \|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)}, \tag{3.68}$$

since $[\![\phi_{N+1}]\!] = 0$. In exactly the same way as in the proof of Lemma 3.11 we obtain

$$\|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)} \leq C_{\text{grad}} \frac{h_K}{p_K} \|\nabla\phi_{N+1}\|_{L^2(\omega_K)^d}$$

and inserting into (3.68) gives

$$|T| \leq \left(C_{\text{cont}} \|u_{N+1} - u_N\|_{E(\mathcal{K}_{N+1})} + C_{\text{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(\omega_K)} \right) \|\nabla\phi_{N+1}\|_{L^2(\omega_K)^d}.$$

Dividing by $\|\nabla\phi_{N+1}\|_{L^2(\omega_K)^d}$ yields

$$\begin{aligned}
\sup_{\phi \in V^p_{K,j_K}\left(\mathcal{K}_{N,K,j_K}|_{\omega_K}, \omega_K\right)} \left(\frac{T}{\|\nabla\phi\|_{L^2(\omega_K)^d}} \right) &\leq \sup_{\substack{\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega) \\ \operatorname{supp}(\phi_{N+1}) \subseteq \omega_K}} \left(\frac{T}{\|\nabla\phi_{N+1}\|_{L^2(\omega_K)^d}} \right) \\
&\leq C_{\text{cont}} \|u_{N+1} - u_N\|_{E\left(\mathcal{K}_{N+1}|_{\omega_K}\right)} + C_{\text{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2(\omega_K)},
\end{aligned}$$

since $\phi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ with $\operatorname{supp}(\phi_{N+1}) \subseteq \omega_K$ was arbitrary. Then optimization problem (3.57) gives

$$\kappa_{K,j_K} \eta_K(u_N, \mathcal{K}_N) \leq C_{\text{cont}} \|u_{N+1} - u_N\|_{E\left(\mathcal{K}_{N+1}|_{\omega_K}\right)} + C_{\text{grad}} \frac{h_K}{p_K} \|f - \Pi f\|_{L^2((\omega_K)}.$$

Since $\mathcal{A}_N \subseteq \mathcal{K}_N$, squaring both sides, summing over $K \in \mathcal{K}_N$ and using Young's inequality yields

$$\begin{aligned}
\sum_{K \in \mathcal{A}_N} \kappa^2_{K,j_K} \eta_K(u_N, \mathcal{K}_N)^2 &\leq \sum_{K \in \mathcal{K}_N} \kappa^2_{K,j_K} \eta_K(u_N, \mathcal{K}_N)^2 \\
&\leq 2 \sum_{K \in \mathcal{K}_N} \left(C^2_{\text{cont}} \|u_{N+1} - u_N\|^2_{E\left(\mathcal{K}_{N+1}|_{\omega_K}\right)} + C^2_{\text{grad}} \frac{h^2_K}{p^2_K} \|f - \Pi f\|^2_{L^2(\omega_K)} \right).
\end{aligned}$$

Then the (γ_1, γ_2)-regularity of $\mathcal{K}_N$ implies

$$\sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N, \mathcal{K}_N\right)^2 \leq 2C\left(\|u_{N+1} - u_N\|_{E(\mathcal{K}_{N+1})}^2 + \sum_{K\in\mathcal{K}_N} \frac{h_K^2}{p_K^2}\|f - \Pi f\|_{L^2(K)}^2\right)$$

for some constant $C > 1$ independent of γ, mesh size vector h and polynomial degree vector p. Finally, from data saturation assumption (3.10) it follows

$$\sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N, \mathcal{K}_N\right)^2 \leq 2C\left(\|u_{N+1} - u_N\|_{E(\mathcal{K}_{N+1})}^2 + \tau^2\eta\left(u_N, \mathcal{K}_N\right)^2\right). \tag{3.69}$$

From Theorem 3.10 we know

$$\begin{aligned}\|u - u_N\|_{E(\mathcal{K}_N)}^2 &\leq C_{\mathrm{rel}}\left(\eta\left(u_N, \mathcal{K}_N\right)^2 + \sum_{K\in\mathcal{K}_N} \frac{h_K^2}{p_K^2}\|f - \Pi f\|_{L^2(K)}^2\right)\\ &\leq C_{\mathrm{rel}}\left(1 + \tau^2\right)\eta\left(u_N, \mathcal{K}_N\right)^2\end{aligned}$$

by assumption (3.10). For $\tau \leq 1$ this reads

$$\|u - u_N\|_{E(\mathcal{K}_N)}^2 \leq 2C_{\mathrm{rel}}\eta\left(u_N, \mathcal{K}_N\right)^2$$

and multiplying both sides by θ^2, $\theta \in (0, 1]$, yields

$$\theta^2\|u - u_N\|_{E(\mathcal{K}_N)}^2 \leq 2C_{\mathrm{rel}}\sum_{K\in\mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N, \mathcal{K}_N\right)^2$$

by constraint (3.8). With (3.69) it follows

$$\frac{\theta^2}{2C_{\mathrm{rel}}}\|u - u_N\|_{E(\mathcal{K}_N)}^2 \leq 2C\left(\|u_{N+1} - u_N\|_{E(\mathcal{K}_{N+1})}^2 + \tau^2\eta\left(u_N, \mathcal{K}_N\right)^2\right)$$

and this concludes the proof. □

Next let us consider the mesh dependence of the bilinear form $A_{\mathcal{K}}$ and the energy norm $\|\cdot\|_{E(\mathcal{K})}$ for two successive triangulations.

Lemma 3.20 (Mesh Perturbation)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $\mathcal{K}_N$ and $\mathcal{K}_{N+1}$ the triangulations in iteration steps N and $N+1$, respectively. Further let $u \in E\left(\mathcal{K}_N\right)$ and $\gamma \geq 1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p. Then there exists some constant $C_{pert} \geq 1$ independent of γ, mesh size vector h and polynomial degree vector p such that*

$$A_{\mathcal{K}_{N+1}}(u, u) \leq (1 + \delta)\|u\|_{E(\mathcal{K}_N)}^2 + \frac{C_{pert}\gamma}{\delta}\sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e}\|[\![u]\!]\|_{L^2(e)^d}^2$$

for $\delta \in (0, 1]$.

Proof. By the definition of the bilinear form $A_{\mathcal{K}_{N+1}}$ we have

$$A_{\mathcal{K}_{N+1}}(u, u) = \|u\|_{E(\mathcal{K}_{N+1})}^2 - 2\int_\Omega L(u)^T\nabla u$$

with $L : E\left(\mathcal{K}_{N+1}\right) \to U^p\left(\mathcal{K}_{N+1}, \Omega\right)$ denoting the lifting operator from Section 3.2.1. Then utilizing Minkowski's inequality this implies

$$A_{\mathcal{K}_{N+1}}(u, u) \leq \|u\|_{E(\mathcal{K}_N)}^2 + 2\left|\int_\Omega L(u)^T\nabla u\right| \tag{3.70}$$

and from Hölder's inequality we get

$$\left|\int_\Omega L(u)^T \nabla u\right| \le \|L(u)\|_{L^2(\Omega)^d} \left(\sum_{K\in\mathcal{K}_{N+1}} \|\nabla u\|^2_{L^2(K)^d}\right)^{\frac{1}{2}}$$
$$\le \frac{1}{2\delta}\|L(u)\|^2_{L^2(\Omega)^d} + \frac{\delta}{2}\sum_{K\in\mathcal{K}_{N+1}} \|\nabla u\|^2_{L^2(K)^d}$$

for $\delta > 0$ by Young's inequality. Applying Lemma 3.13 yields

$$\left|\int_\Omega L(u)^T \nabla u\right| \le \frac{C_L}{2\delta}\sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e}\|[\![u]\!]\|^2_{L^2(e)^d} + \frac{\delta}{2}\sum_{K\in\mathcal{K}_{N+1}} \|\nabla u\|^2_{L^2(K)^d}$$

and inserting into (3.70) gives

$$A_{\mathcal{K}_{N+1}}(u,u) \le (1+\delta)\sum_{K\in\mathcal{K}_{N+1}} \|\nabla u\|^2_{L^2(K)^d} + \left(\gamma + \frac{C_L}{\delta}\right)\sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e}\|[\![u]\!]\|^2_{L^2(e)^d}$$
$$\le (1+\delta)\sum_{K\in\mathcal{K}_{N+1}} \|\nabla u\|^2_{L^2(K)^d} + \frac{C\gamma}{\delta}\left(\delta + \frac{C_L}{\gamma}\right)\sum_{e\in\mathcal{E}(\mathcal{K}_{N})} \frac{p_e^2}{h_e}\|[\![u]\!]\|^2_{L^2(e)^d}$$

for some constant $C \ge 1$ independent of γ, mesh size vector h and polynomial degree vector p, since $\mathcal{K}_N$ is (γ_1,γ_2)-regular and $u \in E(\mathcal{K}_N)$. By choosing $\delta \in (0,1]$ and $\gamma \ge 1$ the result follows. □

Now we compare the energy error of the finite element approximation on two successive grids of the algorithm.

Lemma 3.21 (Comparison of Errors)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H^1_0(\Omega)$ be the solution of (3.46). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in U^p(\mathcal{K}_N,\Omega)$ and $u_{N+1} \in U^p(\mathcal{K}_{N+1},\Omega)$ be the solutions of (3.47) in iteration steps N and $N+1$, respectively. Let $\gamma \ge 1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p such that*

$$\sum_{K\in\mathcal{K}_N} \eta_{R,K}(u_N,\mathcal{K}_N)^2 + \sum_{K\in\mathcal{K}_{N+1}} \eta_{R,K}(u_{N+1},\mathcal{K}_{N+1})^2 \le c_{qe}\gamma \sum_{K\in\mathcal{K}_{N+1}} \|\nabla(u_{N+1}-u_N)\|^2_{L^2(K)^d} \tag{3.71}$$

for some constant $c_{qe} > 0$ independent of γ. Additionally let us assume that there exists some $\tau \in (0,1]$ such that (3.10) is fulfilled. Then there exist some constants $C_1, C_2 > 1$ independent of γ, mesh size vector h and polynomial degree vector p such that

$$(1-C_1\delta)\|u-u_{N+1}\|^2_{E(\mathcal{K}_{N+1})} \le \left(1+\delta-\frac{CC_{ell}}{2}\theta^2\right)\|u-u_N\|^2_{E(\mathcal{K}_N)} + \left(\frac{C_{ell}}{2}+\frac{C_2}{\delta\gamma}\right)\tau^2\eta(u_N,\mathcal{K}_N)^2$$
$$- \frac{C_{ell}}{4}\|u_{N+1}-u_N\|^2_{E(\mathcal{K}_{N+1})}$$

for all $\delta \in (0,1]$.

Proof. By symmetry of the bilinear form $A_{\mathcal{K}_{N+1}}$ we obtain

$$A_{\mathcal{K}_{N+1}}(u-u_N, u-u_N) = T_1 + 2T_2 + T_3, \tag{3.72}$$

where the terms T_1, T_2 and T_3 are given by

$$T_1 := A_{\mathcal{K}_{N+1}}(u-u_{N+1}, u-u_{N+1}),$$
$$T_2 := A_{\mathcal{K}_{N+1}}(u-u_{N+1}, u_{N+1}-u_N)$$

and

$$T_3 := A_{\mathcal{K}_{N+1}}\left(u_{N+1}-u_N, u_{N+1}-u_N\right).$$

For the term T_1 we observe

$$T_1 = \|u-u_{N+1}\|^2_{E(\mathcal{K}_{N+1})} - 2\int_\Omega L\left(u_{N+1}\right)^T \nabla\left(u-u_{N+1}\right)$$

with $L : E\left(\mathcal{K}_{N+1}\right) \to U^p\left(\mathcal{K}_{N+1},\Omega\right)^d$ denoting the lifting operator from Section 3.2.1, since $L(u) = 0$. Then appyling Hölder's inequality implies

$$\begin{aligned} T_1 &\geq \|u-u_{N+1}\|^2_{E(\mathcal{K}_{N+1})} - 2\left\|L\left(u_{N+1}\right)\right\|_{L^2(\Omega)^d} \sum_{K\in\mathcal{K}_{N+1}} \left\|\nabla\left(u-u_{N+1}\right)\right\|^2_{L^2(K)} \\ &\geq (1-\delta)\left\|u-u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} - \frac{1}{\delta}\left\|L\left(u_{N+1}\right)\right\|^2_{L^2(\Omega)^d} \end{aligned}$$

for $\delta > 0$ by Young's inequality and with Lemma 3.13 we get

$$T_1 \geq (1-\delta)\left\|u-u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} - \frac{C_L}{\delta}\sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e}\left\|[\![u_{N+1}]\!]\right\|^2_{L^2(e)^d}. \tag{3.73}$$

Now let us consider the term T_2. With the partial Galerkin orthogonality

$$A_{\mathcal{K}_{N+1}}\left(u-u_{N+1},\phi\right) = 0 \quad \forall\phi\in V^p\left(\mathcal{K}_{N+1},\Omega\right)$$

and Theorem 3.9 it follows

$$\begin{aligned} T_2 &= A_{\mathcal{K}_{N+1}}\left(u-u_{N+1}, u_{N+1}-\Pi_{\mathrm{ZS}}u_{N+1}\right) - A_{\mathcal{K}_{N+1}}\left(u-u_{N+1}, u_N-\Pi_{\mathrm{ZS}}u_N\right) \\ &\geq -C_{\mathrm{cont}}\left\|u-u_{N+1}\right\|_{E(\mathcal{K}_{N+1})}\Big(\left\|u_{N+1}-\Pi_{\mathrm{ZS}}u_{N+1}\right\|_{E(\mathcal{K}_{N+1})} + \left\|u_N-\Pi_{\mathrm{ZS}}u_N\right\|_{E(\mathcal{K}_{N+1})}\Big) \end{aligned}$$

by Lemma 3.14. Then using Young's inequality implies

$$T_2 \geq -C_{\mathrm{cont}}\left(\delta\left\|u-u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} + \frac{1}{2\delta}\left(\left\|u_{N+1}-\Pi_{\mathrm{ZS}}u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} + \left\|u_N-\Pi_{\mathrm{ZS}}u_N\right\|^2_{E(\mathcal{K}_{N+1})}\right)\right) \tag{3.74}$$

for $\delta > 0$.
From Theorem 3.9 we know

$$\left\|u_N-\Pi_{\mathrm{ZS}}u_N\right\|^2_{E(\mathcal{K}_{N+1})} \leq \left(C_{\mathrm{ZS}}+\gamma\right)\sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})}\frac{p_e^2}{h_e}\left\|[\![u_N]\!]\right\|^2_{L^2(e)^d},$$

where $\Pi_{\mathrm{ZS}} : U^p\left(\mathcal{K}_{N+1},\Omega\right) \to V^p\left(\mathcal{K}_{N+1},\Omega\right)$ denotes the averaging operator. Then there exists some constant $C_1 \geq 1$ independent of mesh size vector h and polynomial degree vector p such that

$$\left\|u_N-\Pi_{\mathrm{ZS}}u_N\right\|^2_{E(\mathcal{K}_{N+1})} \leq C_1\left(C_{\mathrm{ZS}}+\gamma\right)\sum_{e\in\mathcal{E}(\mathcal{K}_N)}\frac{p_e^2}{h_e}\left\|[\![u_N]\!]\right\|^2_{L^2(e)^d}$$

by the (γ_1,γ_2)-regularity of $\mathcal{K}_N$ and inserting into (3.74) and using Theorem 3.9 yields

$$T_2 \geq -C_{\mathrm{cont}}\left(\delta\left\|u-u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} + \frac{C_{\mathrm{ZS}}+\gamma}{2\delta}\widetilde{T}_2\right), \tag{3.75}$$

where the term $\widetilde{T}_2$ is given by

$$\widetilde{T}_2 := \sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e} \left\| [\![u_{N+1}]\!] \right\|^2_{L^2(e)^d} + C_1 \sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e} \left\| [\![u_N]\!] \right\|^2_{L^2(e)^d}.$$

For the term T_3 we use Lemma 3.15 to obtain

$$T_3 \geq C_{\text{ell}} \left\| u_{N+1} - u_N \right\|^2_{E(\mathcal{K}_{N+1})} \tag{3.76}$$

and by inserting estimates (3.73), (3.75) and (3.76) into (3.72) we get

$$\begin{aligned} &A_{\mathcal{K}_{N+1}} (u - u_N, u - u_N) \\ &\quad\geq (1 - \delta (2C_{\text{cont}} + 1)) \left\| u - u_{N+1} \right\|^2_{E(\mathcal{K}_{N+1})} - \frac{C_1 C_{\text{cont}} (C_{\text{ZS}} + \gamma)}{\delta} \sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e} \left\| [\![u_N]\!] \right\|^2_{L^2(e)^d} \\ &\qquad + C_{\text{ell}} \left\| u_{N+1} - u_N \right\|^2_{E(\mathcal{K}_{N+1})} - \frac{C_L + C_{\text{cont}} (C_{\text{ZS}} + \gamma)}{\delta} \sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e} \left\| [\![u_{N+1}]\!] \right\|^2_{L^2(e)^d}. \end{aligned} \tag{3.77}$$

Since $[\![u]\!] = 0$, Lemma 3.20 reads

$$A_{\mathcal{K}_{N+1}} (u - u_N, u - u_N) \leq (1 + \delta) \left\| u - u_N \right\|^2_{E(\mathcal{K}_N)} + \frac{C_{\text{pert}}\gamma}{\delta} \sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e} \left\| [\![u_N]\!] \right\|^2_{L^2(e)^d}$$

for $\delta \leq 1$ and using estimate (3.77) implies

$$\begin{aligned} (1 + \delta) \left\| u - u_N \right\|^2_{E(\mathcal{K}_N)} \geq{}& (1 - \delta (2C_{\text{cont}} + 1)) \left\| u - u_{N+1} \right\|^2_{E(\mathcal{K}_{N+1})} + C_{\text{ell}} \left\| u_{N+1} - u_N \right\|^2_{E(\mathcal{K}_{N+1})} \\ &- \frac{C_1 C_{\text{cont}} (C_{\text{ZS}} + \gamma) + C_{\text{pert}}\gamma}{\delta} \sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e} \left\| [\![u_N]\!] \right\|^2_{L^2(e)^d} \\ &- \frac{C_L + C_{\text{cont}} (C_{\text{ZS}} + \gamma)}{\delta} \sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e} \left\| [\![u_{N+1}]\!] \right\|^2_{L^2(e)^d}. \end{aligned}$$

Finally appyling Lemma 3.19 yields

$$\begin{aligned} &\left(1 + \delta - \frac{C C_{\text{ell}} \theta^2}{2}\right) \left\| u - u_N \right\|^2_{E(\mathcal{K}_N)} \\ &\quad\geq (1 - \delta (2C_{\text{cont}} + 1)) \left\| u - u_{N+1} \right\|^2_{E(\mathcal{K}_{N+1})} + \frac{C_{\text{ell}}}{2} \left\| u_{N+1} - u_N \right\|^2_{E(\mathcal{K}_{N+1})} \\ &\qquad - \frac{C_1 C_{\text{cont}} (C_{\text{ZS}} + \gamma) + C_{\text{pert}}\gamma}{\delta} \sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e} \left\| [\![u_N]\!] \right\|^2_{L^2(e)^d} - \frac{C_{\text{ell}}\tau^2}{2} \eta (u_N, \mathcal{K}_N)^2 \\ &\qquad - \frac{C_L + C_{\text{cont}} (C_{\text{ZS}} + \gamma)}{\delta} \sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e} \left\| [\![u_{N+1}]\!] \right\|^2_{L^2(e)^d}. \end{aligned} \tag{3.78}$$

From Lemma 3.16 we know

$$\sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e} \left\| [\![u_N]\!] \right\|^2_{L^2(e)^d} \leq \frac{C_{\text{jump}}}{\gamma^2} \sum_{K\in\mathcal{K}_N} \left(\eta_{R,K} (u_N, \mathcal{K}_N)^2 + \frac{h_K^2}{p_K^2} \left\| f - \Pi_{\mathcal{K}_N} f \right\|^2_{L^2(K)} \right)$$

with $\Pi_{\mathcal{K}_N} : L^2(\Omega) \to U^p (\mathcal{K}_N, \Omega)$ denoting the L^2-conforming interpolation from Section 2.4 and data saturation assumption (3.10) implies

$$\sum_{e\in\mathcal{E}(\mathcal{K}_N)} \frac{p_e^2}{h_e} \left\| [\![u_N]\!] \right\|^2_{L^2(e)^d} \leq \frac{C_{\text{jump}}}{\gamma^2} \left(\sum_{K\in\mathcal{K}_N} \eta_{R,K} (u_N, \mathcal{K}_N)^2 + \tau^2 \eta (u_N, \mathcal{K}_N)^2 \right). \tag{3.79}$$

Further Lemma 3.16 gives

(3.80)

$$\begin{aligned}
\sum_{e\in\mathcal{E}(\mathcal{K}_{N+1})} \frac{p_e^2}{h_e} \|[\![u_{N+1}]\!]\|_{L^2(e)^d}^2 &\le \frac{C_{\text{jump}}}{\gamma^2} \sum_{K\in\mathcal{K}_{N+1}} \left(\eta_{R,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 + \frac{h_K^2}{p_K^2} \left\|f-\Pi_{\mathcal{K}_{N+1}} f\right\|_{L^2(K)}^2 \right) \\
&\le \frac{C_{\text{jump}}}{\gamma^2} \left(\sum_{K\in\mathcal{K}_{N+1}} \eta_{R,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 + \sum_{K\in\mathcal{K}_N} \frac{h_K^2}{p_K^2} \left\|f-\Pi_{\mathcal{K}_N} f\right\|_{L^2(K)}^2 \right) \\
&\le \frac{C_{\text{jump}}}{\gamma^2} \left(\sum_{K\in\mathcal{K}_{N+1}} \eta_{R,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 + \tau^2 \eta\left(u_N,\mathcal{K}_N\right)^2 \right)
\end{aligned}$$

by data saturation assumption (3.10) and inserting (3.79) and (3.80) into (3.78) yields

$$\begin{aligned}
\left(1+\delta-\frac{CC_{\text{ell}}\theta^2}{2}\right) \|u-u_N\|_{E(\mathcal{K}_N)}^2 \ge \left(1-\delta\left(2C_{\text{cont}}+1\right)\right) \|u-u_{N+1}\|_{E(\mathcal{K}_{N+1})}^2 - T_2 \sum_{K\in\mathcal{K}_N} \eta_{R,K}\left(u_N,\mathcal{K}_N\right)^2 \\
- T_1 \eta\left(u_N,\mathcal{K}_N\right)^2 + \frac{C_{\text{ell}}}{2}\left(\frac{1}{2}+\frac{1}{2}\right) \|u_{N+1}-u_N\|_{E(\mathcal{K}_{N+1})}^2 \\
- T_3 \sum_{K\in\mathcal{K}_{N+1}} \eta_{R,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)^2,
\end{aligned}$$

where the terms T_1, T_2 and T_3 are given by

$$\begin{aligned}
T_1 &:= \tau^2 \left(\frac{C_{\text{ell}}}{2} + \frac{C_{\text{jump}}}{\delta\gamma^2} \left(C_{\text{cont}}(C_1+1)\left(C_{\text{ZS}}+\gamma\right) + C_{\text{pert}}\gamma + C_L\right) \right) \\
T_2 &:= \frac{C_{\text{jump}}\left(C_1 C_{\text{cont}}\left(C_{\text{ZS}}+\gamma\right)+C_{\text{pert}}\gamma\right)}{\delta\gamma^2}
\end{aligned}$$

and

$$T_3 := \frac{C_{\text{jump}}\left(C_L + C_{\text{cont}}\left(C_{\text{ZS}}+\gamma\right)\right)}{\delta\gamma^2}.$$

Then the result follows for γ sufficiently large by assumption (3.71). ☐

Now we come to the first main result of this subsection. It states that the energy error $\|u-u_{\text{FE}}\|_{E(\mathcal{K})}$ is reduced in every refinement step of the fully automatic *hp*-adaptive refinement algorithm from Section 3.2.2.

Theorem 3.11 (Convergence)**.** *Let $N\in\mathbb{N}_0$ be arbitrary and $u\in H_0^1(\Omega)$ be the solution of (3.46). We assume that there exists a solution of (3.7), (3.8) for some $\theta\in(0,1]$. Further let $u_N\in V^p\left(\mathcal{K}_N,\Omega\right)$ and $u_{N+1}\in V^p\left(\mathcal{K}_{N+1},\Omega\right)$ be the solutions of (3.47) in iteration steps N and $N+1$, respectively. Let $\gamma\ge 1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p such that (3.71) is fulfilled. Additionally let us assume that there exists some $\tau\in(0,1]$ sufficiently small (depending on polynomial degree vector p) such that (3.10) is fulfilled. Then there exists some $\mu\in(0,1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|u-u_{N+1}\|_{E(\mathcal{K}_{N+1})} \le \mu \|u-u_N\|_{E(\mathcal{K}_N)}.$$

Proof. From Theorem 3.10 we know

$$\eta\left(u_N,\mathcal{K}_N\right)^2 \le C_{\text{eff}} \sum_{K\in\mathcal{K}_N} \left(p_K^{2(1+\varepsilon)} \sum_{L\in\mathcal{K}_N|_{\omega_K}} \|\nabla\left(u-u_N\right)\|_{L^2(L)^d}^2 + \frac{h_K^2}{p_K^{1-2\varepsilon}} \|f-\Pi f\|_{L^2(K)}^2 \right)$$

for $\varepsilon > 0$. Then the (γ_1, γ_2)-regularity of $\mathcal{K}_N$ implies

$$\eta\left(u_N, \mathcal{K}_N\right)^2 \leq \widetilde{C} C_{\text{eff}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{1+2\varepsilon} \left(\max_{K \in \mathcal{K}_N} \left(p_K\right) \sum_{K \in \mathcal{K}_N} \left\|\nabla\left(u - u_N\right)\right\|^2_{L^2(K)^d} + \sum_{K \in \mathcal{K}_N} \frac{h_K^2}{p_K^2} \left\|f - \Pi f\right\|^2_{L^2(K)} \right)$$

for some constant $\widetilde{C} > 1$ independent of mesh size vector h and polynomial degree vector p. With data saturation assumption (3.10) this reads

$$\eta\left(u_N, \mathcal{K}_N\right)^2 \leq \widetilde{C} C_{\text{eff}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{1+2\varepsilon} \left(\max_{K \in \mathcal{K}_N} \left(p_K\right) \sum_{K \in \mathcal{K}_N} \left\|\nabla\left(u - u_N\right)\right\|^2_{L^2(K)^d} + \tau^2 \eta\left(u_N, \mathcal{K}_N\right)^2 \right).$$

Hence for

$$\tau < \frac{1}{\sqrt{\widetilde{C} C_{\text{eff}}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{\frac{1}{2}+\varepsilon}}$$

we have

$$\eta\left(u_N, \mathcal{K}_N\right)^2 \leq \frac{\widetilde{C} C_{\text{eff}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{2(1+\varepsilon)}}{1 - \widetilde{C} C_{\text{eff}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{1+2\varepsilon} \tau^2} \sum_{K \in \mathcal{K}_N} \left\|\nabla\left(u - u_N\right)\right\|^2_{L^2(K)^d}$$

and with the even more restrictive assumption

$$\tau \leq \frac{1}{\sqrt{2 \widetilde{C} C_{\text{eff}}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{\frac{1}{2}+\varepsilon}}$$

we get

$$\eta\left(u_N, \mathcal{K}_N\right)^2 \leq 2 \widetilde{C} C_{\text{eff}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{2(1+\varepsilon)} \sum_{K \in \mathcal{K}_N} \left\|\nabla\left(u - u_N\right)\right\|^2_{L^2(K)^d}. \tag{3.81}$$

From Lemma 3.21 we know

$$\begin{aligned} \left\|u - u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} \leq \frac{2(1+\delta) - C C_{\text{ell}} \theta^2}{2\left(1 - C_1 \delta\right)} \left\|u - u_N\right\|^2_{E(\mathcal{K}_N)} + \frac{\tau^2}{1 - C_1 \delta} \left(\frac{C_{\text{ell}}}{2} + \frac{C_2}{\delta \gamma} \right) \eta\left(u_N, \mathcal{K}_N\right)^2 \\ - \frac{C_{ell}}{4\left(1 - C_1 \delta\right)} \left\|u_{N+1} - u_N\right\|^2_{E(\mathcal{K}_{N+1})} \end{aligned}$$

for $\delta \in \left(0, \frac{1}{C_1}\right)$ and using Lemma 3.19 yields

$$\begin{aligned} &\left\|u - u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} \\ &\quad \leq \frac{1}{1 - C_1 \delta} \left(1 + \delta - \frac{3 C C_{\text{ell}}}{4} \theta^2\right) \left\|u - u_N\right\|^2_{E(\mathcal{K}_N)} + \frac{\tau^2}{1 - C_1 \delta} \left(\frac{3 C_{\text{ell}}}{4} + \frac{C_2}{\delta \gamma} \right) \eta\left(u_N, \mathcal{K}_N\right)^2. \end{aligned}$$

Then, by inserting (3.81) we get

$$\left\|u - u_{N+1}\right\|^2_{E(\mathcal{K}_{N+1})} \leq \frac{T}{1 - C_1 \delta} \left\|u - u_N\right\|^2_{E(\mathcal{K}_N)},$$

where the term T is given by

$$T := 1 + \delta + 2 \widetilde{C} C_{\text{eff}} \max_{K \in \mathcal{K}_N} \left(p_K\right)^{2(1+\varepsilon)} \tau^2 \left(\frac{3 C_{\text{ell}}}{4} + \frac{C_2}{\delta \gamma} \right) - C C_{\text{ell}} \theta^2.$$

With the assumption

$$\tau \leq \frac{\theta^{\frac{3}{2}}}{\sqrt{2\widetilde{C}C_{\mathrm{eff}}} \max\limits_{K\in\mathcal{K}_N} (p_K)^{\frac{1}{2}+\varepsilon}}$$

we obtain

$$\begin{aligned}T &\leq 1+\delta+\theta^2\left(\left(\frac{3C_{\mathrm{ell}}}{4}+\frac{C_2}{\delta\gamma}\right)\theta - CC_{\mathrm{ell}}\right)\\ &\leq 1+\delta+\theta^2\left(\left(\frac{3C_{\mathrm{ell}}}{4}+C_2\right)\theta - CC_{\mathrm{ell}}\right)\end{aligned}$$

for $\gamma \geq \frac{1}{\delta}$ and by assuming

$$\delta < \frac{\theta^2}{1+C_1}\left(CC_{\mathrm{ell}} - \left(\frac{3C_{\mathrm{ell}}}{4}+C_2\right)\theta\right)$$

the result follows. □

The second main result of this subsection is another convergence result, which states that the weighted sum of energy error $\|u-u_{\mathrm{FE}}\|_{E(\mathcal{K})}$ and estimated error $\eta\left(u_{\mathrm{FE}},\mathcal{K}\right)$ is reduced in every iteration of the fully automatic *hp*-adaptive refinement algorithm form Section 3.2.2. The proof follows the ideas of Bonito and Nochetto [55].

Theorem 3.12 (Quasi-Convergence)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H_0^1(\Omega)$ be the solution of (3.46). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in V^p\left(\mathcal{K}_N,\Omega\right)$ and $u_{N+1} \in V^p\left(\mathcal{K}_{N+1},\Omega\right)$ be the solutions of (3.47) in iteration steps N and $N+1$, respectively. Let $\gamma \geq 1$ be sufficiently large and independent of mesh size vector h and polynomial degree vector p such that (3.71) is fulfilled. Further we assume that there exists some constant $\rho \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that assumption (3.24) holds. Additionally let us assume that there exists some $\tau \in (0,1]$ sufficiently small such that (3.10) is fulfilled. Then there exists some constant $\mu \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\|u-u_{N+1}\|^2_{E(\mathcal{K}_{N+1})} + \nu\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 \leq \mu\left(\|u-u_N\|^2_{E(\mathcal{K}_N)} + \nu\eta\left(u_N,\mathcal{K}_N\right)^2\right)$$

for some $\nu > 0$ sufficiently small (depending on polynomial degree vector p).

Proof. From Lemma 3.21 we know

$$\begin{aligned}&\|u-u_{N+1}\|^2_{E(\mathcal{K}_{N+1})} + \nu\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2\\ &\quad\leq \frac{2(1+\delta)-CC_{\mathrm{ell}}\theta^2}{2\left(1-C_1\delta\right)}\|u-u_N\|^2_{E(\mathcal{K}_N)} + \frac{1}{1-C_1\delta}\left(\frac{C_{\mathrm{ell}}}{2}+\frac{C_2}{\delta\gamma}\right)\tau^2\eta\left(u_N,\mathcal{K}_N\right)^2 + \nu\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2\\ &\qquad - \frac{C_{\mathrm{ell}}}{4\left(1-C_1\delta\right)}\|u_{N+1}-u_N\|^2_{E(\mathcal{K}_{N+1})}\end{aligned}$$

for $\delta \in \left(0,\frac{1}{C_1}\right)$ and applying Lemma 3.18 yields

$$\begin{aligned}\|u-u_{N+1}\|^2_{E(\mathcal{K}_{N+1})} + \nu\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 &\leq \frac{2(1+\delta)-CC_{\mathrm{ell}}\theta^2}{2\left(1-C_1\delta\right)}\|u-u_N\|^2_{E(\mathcal{K}_N)} + T\eta\left(u_N,\mathcal{K}_N\right)^2\\ &\quad -\nu(1+\delta)^2(1-\rho)\eta\left(u_N,\mathcal{A}_N\right)^2\end{aligned} \tag{3.82}$$

for

$$\nu \leq \frac{C_{\text{ell}}\delta}{4C_{\text{red}}\left(1-C_1\delta\right)\left(1+\delta\right)\max\limits_{K\in\mathcal{K}_N}\left(p_K\right)^2}, \tag{3.83}$$

where the term T is given by

$$T := \frac{1}{1-C_1\delta}\left(\frac{C_{\text{ell}}}{2}+\frac{C_2}{\delta\gamma}\right)\tau^2+\nu(1+\delta)^2\left(1+\frac{\rho^2\tau^2}{\delta}\right).$$

We observe

$$C_{\max}^2\eta\left(u_N,\mathcal{A}_N\right)^2 \geq \sum_{K\in\mathcal{A}_N}\kappa_{K,j_K}^2\eta_K\left(u_N,\mathcal{K}_N\right)^2,$$

where

$$C_{\max} := \max\left\{\max_{K\in\mathcal{A}_N}\left(\kappa_{K,j_K}\right),1\right\}.$$

Thus (3.82) reads

$$\begin{aligned}&\left\|u-u_{N+1}\right\|_{E(\mathcal{K}_{N+1})}^2+\nu\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2\\&\quad\leq\frac{2(1+\delta)-CC_{\text{ell}}\theta^2}{2\left(1-C_1\delta\right)}\left\|u-u_N\right\|_{E(\mathcal{K}_N)}^2+T\eta\left(u_N,\mathcal{K}_N\right)^2-\frac{\nu(1+\delta)^2(1-\rho)}{C_{\max}^2}\sum_{K\in\mathcal{A}_N}\kappa_{K,j_K}^2\eta_K\left(u_N,\mathcal{K}_N\right)^2,\end{aligned}$$

since $\rho<1$. Then constraint (3.8) implies

$$\begin{aligned}&\left\|u-u_{N+1}\right\|_{E(\mathcal{K}_{N+1})}^2+\nu\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2\\&\quad\leq\frac{2(1+\delta)-CC_{\text{ell}}\theta^2}{2\left(1-C_1\delta\right)}\left\|u-u_N\right\|_{E(\mathcal{K}_N)}^2+\left(T-\frac{\nu(1+\delta)^2(1-\rho)\theta^2}{C_{\max}^2}\right)\eta\left(u_N,\mathcal{K}_N\right)^2.\end{aligned}$$

For

$$\delta < \frac{CC_{\text{ell}}}{2\left(1+C_1\right)}\theta^2$$

it holds

$$\frac{2(1+\delta)-CC_{\text{ell}}\theta^2}{2\left(1-C_1\right)} < 1$$

and by assuming $\tau\leq\nu$ we obtain

$$T-\frac{\nu(1+\delta)^2(1-\rho)\theta^2}{C_{\max}^2}\leq\left(\frac{1}{1-C_1\delta}\left(\frac{C_{\text{ell}}}{2}+\frac{C_2}{\delta\gamma}\right)\nu+(1+\delta)^2\left(1+\frac{\rho^2\nu^2}{\delta}-\frac{(1-\rho)\theta^2}{C_{\max}^2}\right)\right)\nu. \tag{3.84}$$

For $\delta\leq\frac{3}{4C_1}$ estimate (3.83) reads

$$\nu\leq\frac{C_{\text{ell}}\delta}{C_{\text{red}}}$$

and inserting into (3.84) yields

$$T-\frac{\nu(1+\delta)^2(1-\rho)\theta^2}{C_{\max}^2}\leq\left(\frac{C_{\text{ell}}\delta}{C_{\text{red}}\left(1-C_1\delta\right)}\left(\frac{C_{\text{ell}}}{2}+C_2\right)+(1+\delta)^2\left(1+\frac{C_{\text{ell}}^2\rho^2\delta}{C_{\text{red}}^2}-\frac{(1-\rho)\theta^2}{C_{\max}^2}\right)\right)\nu$$

for $\gamma\geq\frac{1}{\delta}$. By letting $\delta\to0$ we observe

$$\frac{C_{\text{ell}}\delta}{C_{\text{red}}\left(1-C_1\delta\right)}\left(\frac{C_{\text{ell}}}{2}+C_2\right)+(1+\delta)^2\left(1+\frac{C_{\text{ell}}^2\rho^2\delta}{C_{\text{red}}^2}-\frac{(1-\rho)\theta^2}{C_{\max}^2}\right)\to1-\frac{(1-\rho)\theta^2}{C_{\max}^2}$$

and, thus, the result follows for δ small enough. $\square$

To conclude this subsection let us shortly discuss the various assumptions we made above and see how these affect the main result of this subsection. Most of the assumptions were already discussed in Sections 3.1.2.2 and 3.1.3.2. Therefore we restrict ourselves to those assumptions, which changed fundamentally or appeared newly.
In contrast to the continuous Galerkin finite element method we have another basic assumption. In Lemma 3.21 we assume that $\gamma \geq 1$ is sufficiently large and independent of mesh size vector h and polynomial degree vector p such that quasi-efficiency assumption (3.71) is fulfilled. If there was an hp-efficient a posteriori error estimator $\eta(u_{\mathrm{FE}}, \mathcal{K})$, then this assumption would be satisfied trivially. But, since we only have the efficiency estimate from Theorem 3.10 which is not uniform in p, this condition does not hold automatically for γ independent of mesh size vector p. However numerical examples in [133] and Section 3.2.4 indicate that the efficiency estimate from Theorem 3.10 is not sharp and thus we expect that the error estimator from Definition 3.3 satisfies this assumption for γ sufficiently large. The presence of assumption (3.71) in this section is due to the fact that for the discontinuous Galerkin finite element method we only have the partial Galerkin orthogonality

$$A_{\mathcal{K}_{N+1}}(u - u_{N+1}, \phi) = 0 \quad \forall \phi \in V^p(\mathcal{K}_{N+1}, \Omega)$$

instead of the full Galerkin orthogonality from the continuous Galerkin finite element method. Then, in Lemma 3.21 the mixed term $A_{\mathcal{K}_{N+1}}(u - u_{N+1}, u_{N+1} - u_N)$ does not disappear and we are left with these trouble causing terms. However a more detailed analysis of this point would be desirable.
In Theorem 3.11 the parameter $\tau \in (0, 1]$ from data saturation assumption (3.10) depends on the polynomial degree vector p again. In the proof we have obtained the explicit upper bound

$$\tau \leq \frac{\theta^{\frac{3}{2}}}{\sqrt{2\widetilde{C}C_{\mathrm{eff}}} \max\limits_{K \in \mathcal{K}_N} (p_K)^{1+\varepsilon}}$$

for $\varepsilon > 0$.
In Theorem 3.12 the constant $\nu > 0$ depends on the polynomial degree vector p again. We have obtained the explicit upper bound

$$\nu \leq \frac{C_{\mathrm{ell}}\delta}{4C_{\mathrm{red}}(1 - C_1\delta_1)(1 + \delta) \max\limits_{K \in \mathcal{K}_N} (p_K)^2}$$

in the proof.
In Lemma 3.20 we also derived a mesh perturbation result, which did not appear in Sections 3.1.2.2 and 3.1.3.2 for the continuous Galerkin finite element method. The presence of this type of result is due to the mesh-dependence of the bilinear form $A_{\mathcal{K}}$ and the energy norm $\|\cdot\|_{E(\mathcal{K})}$.

3.2.4 Numerical Results

Now we want to consider the performance of the fully automatic hp-adaptive refinement strategy from Section 3.2.2 on the basis of some numerical examples. Therefore we consider the same problems as in Section 3.1.3.3, but this time for the discontinuous Galerkin finite element method. All computations are performed with the finite element library deal.II [41, 42].

Example 1

The first example is a two-dimensional example with a smooth analytic solution. Let $\Omega := (0,1)^2$ and $u : \overline{\Omega} \to \mathbb{R}$ be given by

$$u(x) := x_1\,(1-x_1)\,x_2\,(1-x_2)\,(1-2x_2)\exp\left(-\frac{5}{2}\,(x_1-1)\right).$$

The initial triangulation $\mathcal{K}_0$ consists of four equally-sized cells and as initial polynomial degree vector we choose $p = 1$. Further we set $\theta := 0.6$.
Like in Section 3.1.3.3 we perform two different runs of the algorithm. In the first run we provide only two different refinement patterns the algorithm can choosen from. The first refinement pattern is classical h-refinement and the second refinement pattern is classical p-refinement. In Figure 3.9 we plot the number of degrees of freedom vs. the exact enegy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. In Table 3.5

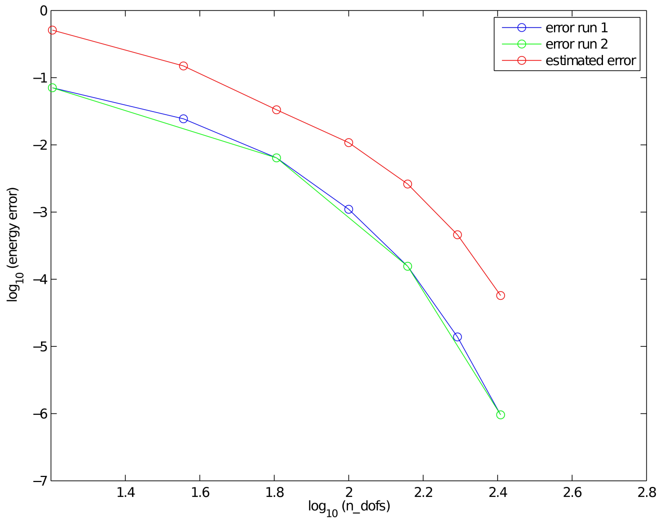

Figure 3.9: Example 1: Number of degrees of freedom vs. error.

the marking history of the algorithm is shown. We observe that the hp-refinement strategy from Section

Step	#Cells	max(p)	#h	#p	Step	#Cells	max(p)	#h	#p
0	4	1	0	4	4	4	5	0	4
1	4	2	0	4	5	4	6	0	4
2	4	3	0	4	6	4	7	0	4
3	4	4	0	4					

Table 3.5: Example 1: Marking history.

3.2.2 chooses p-refinement on all cells. This is basically what we expect, because this refinement scheme already performed best for the continuous Galerkin finite element method in Section 3.1.3.3.
In a second run we add a third refinement pattern to the refinement algorithm. Now the strategy can additionally choose to increase the polynomial degree by two. In Figure 3.9 we can see that the algorithm really takes advantage of this new refinement pattern and reaches the same accuracy as in run 1 in only half the number of refinement steps. Also this behaviour was already observed for the continuous Galerkin finite element method in Section 3.1.3.3.

Example 2

Also for the second example we stay in the case $d = 2$. However, this time we consider the behaviour of the fully automatic hp-adaptive refinement algorithm for a singular analytic solution. Let $\Omega := (-1,1)^2 \setminus [0,1) \times (-1,0]$ and the exact solution u be given by

$$u(r,\varphi) := r^{\frac{2}{3}} \sin\left(\frac{2}{3}\varphi\right),$$

where $r \in [0,1)$ and $\varphi \in [0,2\pi)$ denote the polar coordinates. The initial triangulation $\mathcal{K}_0$ consists of 12 equally-sized cells and as initial polynomial degree vector we choose $p = 2$. Further we set $\theta := 0.15$. The algorithm can choose from classical h- and p-refinement again. In Figure 3.10 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. On the right-hand side we can see the final grid produced by the algorithm. We observe

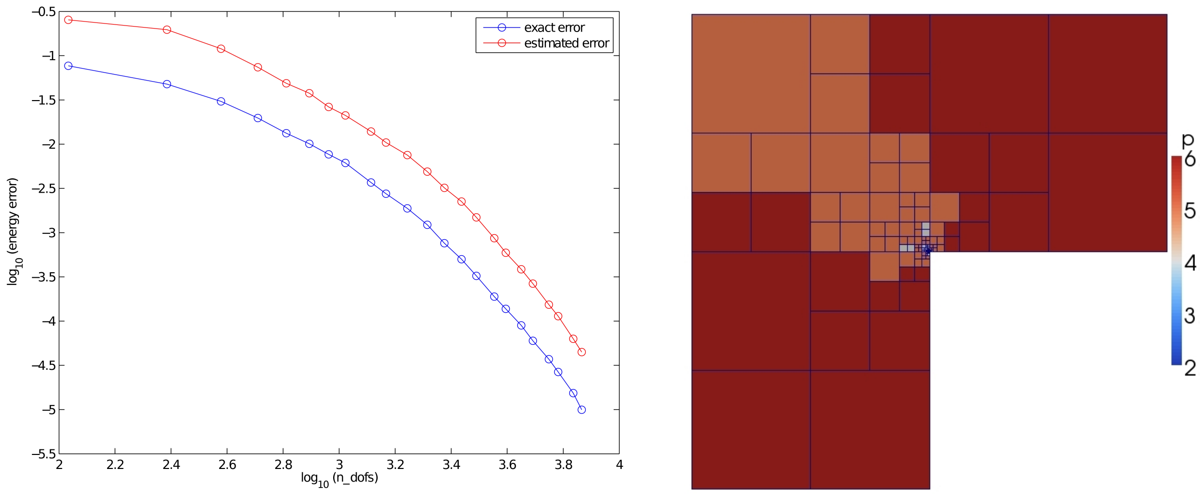

Figure 3.10: Example 2. Left: Number of degrees of freedom vs. error. Right: Final grid.

that the final grid basically is linearly graded towards the singularity located at the origin. This means that around the origin cells are small and polynomial degrees are low. The more one goes away from the singularity the larger are the cells and the higher are the polynomial degrees. The marking history of the algorithm is shown in Table 3.6. Also in this result we get more or less the result which one expects from the hp-adaptive refinement algorithm and which we already obtained for the continuous Galerkin finite element method in Section 3.1.3.3. The singularity is identified correctly and the refinement choices are appropriate.

Example 3

This is the first three-dimensional example. Again we start with a smooth analytic solution. Let $\Omega := (0,1)^3$ and $u : \overline{\Omega} \to \mathbb{R}$ be given by

$$u(x) := \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3).$$

The initial triangulation $\mathcal{K}_0$ consists of 8 equally-sized cells and as initial polynomial degree vector we choose $p = 1$. Further we set $\theta := 0.2$.

Step	#Cells	max(p)	h	p	Step	#Cells	max(p)	h	p
0	12	2	3	0	11	309	4	3	11
1	39	2	3	0	12	336	4	3	13
2	66	2	3	0	13	363	4	3	19
3	93	2	3	5	14	390	5	4	21
4	120	3	3	9	15	419	5	3	15
5	147	3	3	3	16	445	5	3	18
6	174	3	3	0	17	471	5	3	22
7	201	3	3	10	18	498	6	3	0
8	228	3	3	4	19	525	6	1	31
9	255	4	3	15	20	562	6	1	24
10	282	4	3	8	21	589	6	1	25

Table 3.6: Example 2: Marking history

As in Example 1 we perform two different runs of the algorithm. In the first run we provide only two different refinement patterns the algorithm can choose from. As usual the first refinement pattern is classical h-refinement and the second refinement pattern is classical p-refinement. In Figure 3.11 we plot the number of degrees of freedom vs. the exact error and the estimated error in a $\log_{10}$-$\log_{10}$-scale. In

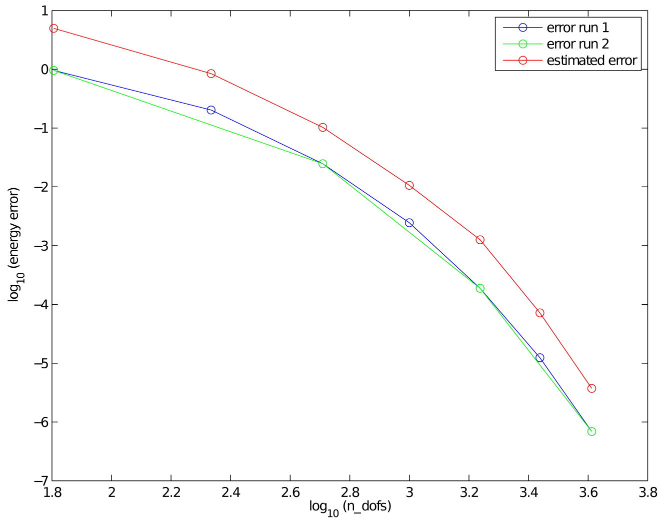

Figure 3.11: Example 3: Number of degrees of freedom vs. error.

Table 3.7 the marking history of the algorithm is shown. We observe that the hp-adaptive refinement strategy from Section 3.2.2 chooses p-refinement only. This is basically what we expect, because this refinement scheme already performed best for the continuous Galerkin finite element method in Section 3.1.3.3.

In a second run we add a third refinement pattern to the refinement algorithm again. Now the strategy can additionally choose to increase the polynomial degree by two. In Figure 3.11 we can see that the algorithm really takes advantage of this new refinement pattern and reaches the same accuracy as in run 1 in only half the number of refinement steps. Also this behaviour was already observed for the continuous Galerkin finite element method in Section 3.1.3.3.

Step	#Cells	max(p)	$\#h$	$\#p$	Step	#Cells	max(p)	$\#h$	$\#p$
0	8	1	0	8	3	8	4	0	8
1	8	2	0	8	4	8	5	0	8
2	8	3	0	8	5	8	6	0	8

Table 3.7: Example 3: Marking history.

Example 4

The last example is for the case $d = 3$ again. As in Example 2 we consider the behaviour of the fully automatic hp-adaptive refinement algorithm for a singular analytic solution. Let $\Omega := (-1,1)^3 \setminus [0,1)^3$ and the exact solution u be given by

$$u(x) := \left(x_1^2 + x_2^2 + x_3^2\right)^{\frac{1}{3}}.$$

The initial triangulation consists of 56 equally-sized cells and as initial polynomial degree vector we choose $p = 1$. Further we set $\theta := 0.15$.
The algorithm can choose from classical h- and p-refinement again. In Figure 3.12 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. On the right-hand side we can see the final grid produced by algorithm. We observe

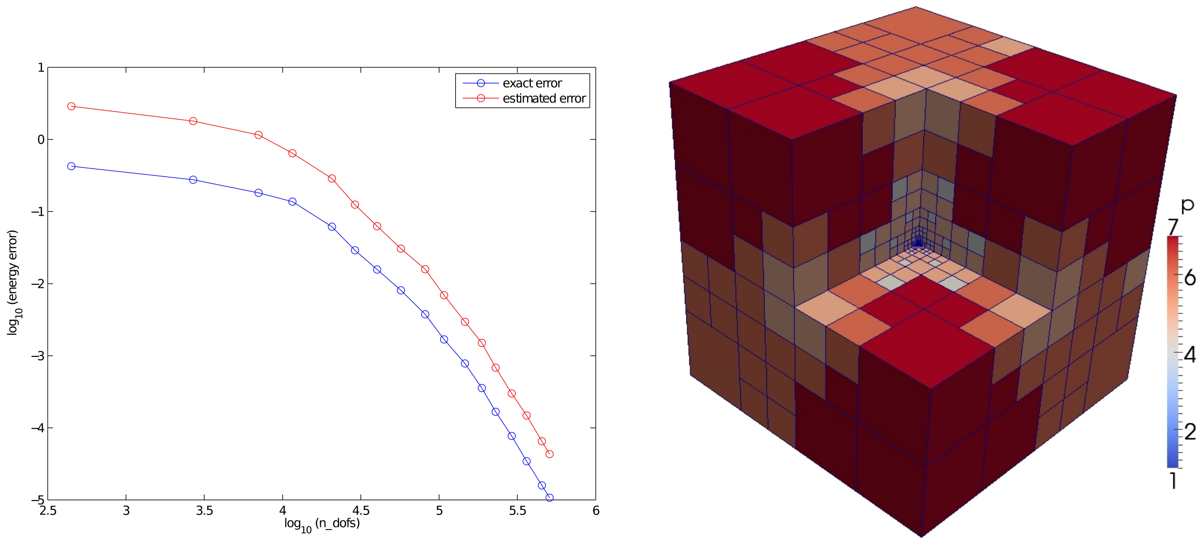

Figure 3.12: Example 4. Left: Number of degrees of freedom vs. error. Right: Final grid.

that the grid basically is linearly graded towards the singularity located at the origin. This means that around the origin cells are small and polynomial degrees are low. The more one goes away from the singularity the larger are the cells and the higher are the polynomial degrees. The marking history of the algorithm is shown in Table 3.8. Also in this example we get more or less the result which one expects from the hp-adaptive refinement algorithm and which we already obtained for the continuous Galerkin finite element method in Section 3.1.3.3. The singularity is identified correctly and the refinement choices are appropriate.

Step	#Cells	max(p)	# h	# p	Step	#Cells	max(p)	# h	# p
0	56	1	7	0	8	624	3	7	18
1	99	1	7	0	9	724	3	7	0
2	142	1	7	0	10	825	3	7	42
3	203	1	7	3	11	924	4	7	31
4	264	2	7	22	12	1026	4	7	113
5	343	2	7	0	13	1129	5	6	11
6	422	2	6	13	14	1242	6	7	27
7	520	2	7	2	15	1341	7	7	3

Table 3.8: Example 4: Marking history.

Chapter 4

The Maxwell Boundary Value Problem

In this chapter we want to consider the numerical solution of the *Maxwell boundary value problem* with the *hp*-adaptive finite element method. Although in recent years there has been growing interest in solving Maxwell's equations numerically, there is only few literature considering the problem-adapted creation of approximation spaces for this system of partial differential equations. The *h*-adaptive finite element method is discussed in e.g. [50, 79, 125]. For the *p*- and the *hp*-adaptive finite element method Demkowicz, Pardo and co-workers have introduced a global optimization scheme in [96, 100, 176, 177]. Our goal is to derive a fully automatic *hp*-adaptive refinement strategy for this problem, which is based on local criteria, and prove its convergence.

Therefore we start with the derivation of a model problem which we want to consider. Then we introduce a residual-based a posteriori error estimator for the *hp*-adaptive finite element method for this system of partial differential equations and prove its reliablity and some efficiency estimate. Further we present a fully automatic *hp*-adaptive refinement strategy for the model problem and prove some convergence results. To conclude this chapter we present the performance of the a posteriori error estimator and the refinement strategy on the basis of some numerical examples.

4.1 The Problem Formulation

In this section we derive the model problem which we want to consider later on. Therefore we begin with the classical definition of Maxwell's equations as introduced by Maxwell [152] in 1864. Then we exploit some basic material properties to arrive at a somewhat simpler system of partial differential equations.

The behaviour of an electromagnetic field can be described by a set of four equations. Three of them are named by their discoverers: Gauß' s law [115], Ampère's circuital law [13] and Faraday's law of induction [112]. The fourth equation simply states that there are no magnetic charges. Unfortunately these equations together do not describe a correct model of the time-varying electromagnetic field, because all of them are derived from stationary observations. In 1865 Maxwell [152] modified Ampère's law in such a way that this system of equations describes a consistent model for the time-varying electromagnetic fields. Therefore this modified system of equations is known as *Maxwell's equations*.

Now let us consider these equations in more detail. Therefore let $\Omega \subset \mathbb{R}^3$ be open and bounded with polyhedral, Lipschitz-continuous boundary. For $T > 0$ we denote the *electric* and *magnetic field intensities* by $E : \overline{\Omega} \times [0,T] \to \mathbb{C}^3$ and $H : \overline{\Omega} \times [0,T] \to \mathbb{C}^3$, respectively. The function $D : \overline{\Omega} \times [0,T] \to \mathbb{C}^3$ describes the *electric displacement* and the *magnetic induction* is denoted by $B : \overline{\Omega} \times [0,T] \to \mathbb{C}^3$. Then the effect of the charge density on the electric displacement is described by Gauß's law

$$\operatorname{div}(D) = 0 \qquad \text{in } \Omega \times (0,T), \tag{4.1}$$

if no free charges are present. Ampère's circuital law as modified by Maxwell states that electric currents are vortices of magnetic induction

$$\frac{dD}{dt} - \nabla \times H = 0 \qquad \text{in } \Omega \times (0,T), \tag{4.2}$$

if there are no electric currents. The effect of a changing magnetic field on the electric field is described by Faraday's law

$$\nabla \times E + \frac{dB}{dt} = 0 \qquad \text{in } \Omega \times (0,T). \tag{4.3}$$

Finally the fourth equation says that there are no magnetic charges:

$$\operatorname{div}(B) = 0 \qquad \text{in } \Omega \times (0,T) \tag{4.4}$$

By putting equations (4.1)–(4.4) together we obtain the following system of equations:

$$\begin{aligned} \frac{dD}{dt} - \nabla \times H &= 0 \qquad \text{in } \Omega \times (0,T) \\ \nabla \times E + \frac{dB}{dt} &= 0 \qquad \text{in } \Omega \times (0,T) \\ \operatorname{div}(D) &= 0 \qquad \text{in } \Omega \times (0,T) \\ \operatorname{div}(B) &= 0 \qquad \text{in } \Omega \times (0,T). \end{aligned} \tag{4.5}$$

This system of partial differential equations is called *Maxwell's equations*. Now let us assume that Ω is occupied by one or more different materials. Then there exist positive-definite functions $\alpha, \widetilde{\beta} : \overline{\Omega} \to \mathbb{C}^{3,3}$ such that

$$D = \widetilde{\beta} E \qquad \text{in } \Omega$$

and

$$B = \alpha H \qquad \text{in } \Omega$$

(see e.g. [145]). Inserting this into system (4.5) yields

$$\begin{aligned} \widetilde{\beta}\frac{dE}{dt} - \nabla \times H &= 0 \qquad \text{in } \Omega \times (0,T) \\ \nabla \times E + \alpha\frac{dH}{dt} &= 0 \qquad \text{in } \Omega \times (0,T) \\ \operatorname{div}\left(\widetilde{\beta} E\right) &= 0 \qquad \text{in } \Omega \times (0,T) \\ \operatorname{div}(\alpha H) &= 0 \qquad \text{in } \Omega \times (0,T). \end{aligned}$$

Differentiating the first equation with respect to the time t and inserting the second equation yields the electric field formulation

$$\begin{aligned} \widetilde{\beta}\frac{d^2 E}{dt^2} - \nabla \times (\alpha \nabla \times E) &= 0 \qquad \text{in } \Omega \times (0,T) \\ \operatorname{div}\left(\widetilde{\beta} E\right) &= 0 \qquad \text{in } \Omega \times (0,T) \end{aligned}$$

and by applying some time stepping-scheme we arrive at the *Maxwell boundary value problem* to find $u : \overline{\Omega} \to \mathbb{C}^3$ such that

$$\begin{aligned} \nabla \times (\alpha \nabla \times u) + \beta u &= f && \text{in } \Omega \\ \operatorname{div}(\beta u) &= 0 && \text{in } \Omega \\ n \times u &= 0 && \text{on } \partial\Omega \end{aligned} \tag{4.6}$$

for some $f : \Omega \to \mathbb{C}^3$ with $\operatorname{div}(f) = 0$ in Ω, where n denotes the outward-pointing unit normal vector to Ω. Here $\beta : \Omega \to \mathbb{C}^{3,3}$ denotes the coefficient $\widetilde{\beta}$ scaled with the length of the time step. For simplicity we restrict ourselves to functions mapping into $\mathbb{R}^3$ instead of $\mathbb{C}^3$ and homogeneous Dirichlet boundary conditions here. Then the first equation of (4.6) implies, assuming enough regularity,

$$\operatorname{div}(\beta u) = 0 \qquad \text{in } \Omega, \tag{4.7}$$

since $\operatorname{div}(\nabla \times \phi) = 0$ for all $\phi : \Omega \to \mathbb{R}^3$ sufficiently smooth. Thus it suffices to consider the reduced boundary value problem to find $u \in H_0(\operatorname{curl}, \Omega)$ such that

$$\begin{aligned} \nabla \times (\alpha \nabla \times u) + \beta u &= f && \text{in } \Omega \\ n \times u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.8}$$

Let $\mathcal{K}$ be some (γ_1, γ_2)-regular triangulation of Ω. Then, for the coefficients $\alpha, \beta : \Omega \to \mathbb{R}^{3,3}$ we assume that it holds $\alpha \in U^{p_\alpha}(\mathcal{K}, \Omega)^{3,3}$ and $\beta \in U^{p_\beta}(\mathcal{K}, \Omega)^{3,3}$ for $p_\alpha, p_\beta \in \mathbb{N}_0$. Further we assume that α and β are *uniformly positive definite*, i.e. there exist constants $0 < \alpha_{\min} < \alpha_{\max}$ and $0 < \beta_{\min} < \beta_{\max}$ such that

$$\alpha_{\min} \|\phi\|^2_{L^2(\Omega)^3} \le \int_\Omega \phi^T \alpha \phi \le \alpha_{\max} \|\phi\|^2_{L^2(\Omega)^3} \tag{4.9}$$

and

$$\beta_{\min} \|\phi\|^2_{L^2(\Omega)^3} \le \int_\Omega \phi^T \beta \phi \le \beta_{\max} \|\phi\|^2_{L^2(\Omega)^3} \tag{4.10}$$

for $\phi : \Omega \to \mathbb{R}^3$. Now let us derive the weak formulation of problem (4.8). For this we multiply the first equation of (4.8) with some test function $\phi \in H_0(\operatorname{curl}, \Omega)$ and integrate over Ω. This yields

$$\int_\Omega \phi^T \nabla \times (\alpha \nabla \times u) + \int_\Omega \phi^T \beta u = \int_\Omega \phi^T f \qquad \forall \phi \in H_0(\operatorname{curl}, \Omega)$$

and with integration by parts we obtain the weak formulation

$$\int_\Omega (\nabla \times \phi)^T \alpha \nabla \times u + \int_\Omega \phi^T \beta u = \int_\Omega \phi^T f \qquad \forall \phi \in H_0(\operatorname{curl}, \Omega) \tag{4.11}$$

of problem (4.8). Then we can define the bilinear form $A : H_0(\operatorname{curl}, \Omega) \times H_0(\operatorname{curl}, \Omega) \to \mathbb{R}$ by

$$A(\phi, \psi) := \int_\Omega (\nabla \times \phi)^T \alpha \nabla \times \psi + \int_\Omega \phi^T \beta \psi.$$

Further we define the energy norm $\| \cdot \|_\Omega : H_0(\operatorname{curl}, \Omega) \to \mathbb{R}_+$ by

$$\|u\|^2_\Omega = \left\| \alpha^{\frac{1}{2}} \nabla \times u \right\|^2_{L^2(\Omega)^3} + \left\| \beta^{\frac{1}{2}} u \right\|^2_{L^2(\Omega)^3}.$$

Now it can be shown that A is continuous and elliptic with respect to $\| \cdot \|_\Omega$. The continuity can be proven easily.

Lemma 4.1 (Continuity of A)**.** *The bilinear form A is* continuous *with respect to $\|\cdot\|_\Omega$, i.e. it holds*

$$A(\phi,\psi) \le \|\phi\|_\Omega \|\psi\|_\Omega \qquad \forall \phi,\psi \in H_0(\mathrm{curl},\Omega).$$

Proof. With Hölder's inequality we have

$$A(\phi,\psi) \le \left\|\alpha^{\frac{1}{2}} \nabla \times \phi\right\|_{L^2(\Omega)^3} \left\|\alpha^{\frac{1}{2}} \nabla \times \psi\right\|_{L^2(\Omega)^3} + \left\|\beta^{\frac{1}{2}} \phi\right\|_{L^2(\Omega)^3} \left\|\beta^{\frac{1}{2}} \psi\right\|_{L^2(\Omega)^3}$$

and the result follows with the Cauchy-Schwarz inequality. □

The ellipticity of the bilinear form A is trivial.

Lemma 4.2. *(Ellipticity of A) The bilinear form A is* elliptic *with respect to $\|\cdot\|_\Omega$, i.e. it holds*

$$A(\phi,\phi) = \|\phi\|_\Omega^2 \qquad \forall \phi \in H_0(\mathrm{curl},\Omega).$$

Then it follows immediately from the Lax-Milgram Theorem that weak problem (4.11) has a unique solution $u \in H_0(\mathrm{curl},\Omega)$ for $f \in L^2(\Omega)^3$ with $\mathrm{div}(f) = 0$ in Ω. Thus, it makes sense to consider the discrete formulation of (4.11) to obtain a numerical approximation for the solution of problem (4.8). Therefore let $\mathcal{K}$ be a regular triangulation of Ω. Then the discrete formulation of problem (4.11) reads to find $u_{\mathrm{FE}} \in W^p(\mathcal{K},\Omega)$ such that

$$A(\phi, u_{\mathrm{FE}}) = \int_\Omega \phi^T f \qquad \forall \phi \in W^p(\mathcal{K},\Omega). \tag{4.12}$$

For this discretization of problem (4.8) it can be shown that the error decays exponentially, if the mesh size vector h and the polynomial degree vector p are chosen suitably. Under some regularization assumptions Costabel, Dauge and Schwab [89] have proven the estimate

$$\|u - u_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)} \le C_1 \exp\left(-C_2 N^{\frac{1}{3}}\right),$$

where $N := \dim\left(W^p(\mathcal{K},\Omega)\right)$ and $C_1, C_2 > 0$ denote some constants independent of N.

4.2 The Error Estimator

In this section we want to derive a residual-based a posteriori error estimator for the *hp*-adaptive finite element method for Maxwell's equations in the electric field formulation. The estimator is quite similar to the FEM-part of the a posteriori error estimator derived in [148], but to the best of our knowledge there has not been any discussion about its *hp*-capabilities yet. Thus we derive a similar residual-based error estimator, which is based on a pure finite element discretization, and prove upper and lower bounds for this estimator in terms of the exact energy error. Therefore we will use the $H(\mathrm{curl})$-conforming finite element space $W^p(\mathcal{K},\Omega)$ from Section 2.2.5. The results of this section are based on [73].
Let us assume that triangulation $\mathcal{K}$ consists of tetrahedra only. We begin with the definition of the residual-based a posteriori error estimator. It is basically an extension of the h-version a posteriori error estimator from Beck, Hiptmair, Hoppe and Wohlmuth [50] to the *hp*-adaptive case.

Definition 4.1 (A Posteriori Error Estimator)**.** *Let $u_{FE} \in W^p(\mathcal{K},\Omega)$ be the solution of (4.12). Then the* residual-based a posteriori error estimator $\eta(u_{FE},\mathcal{K})$ *is given by*

$$\eta(u_{FE},\mathcal{K})^2 := \sum_{K\in\mathcal{K}} \eta_K(u_{FE},\mathcal{K})^2$$

with

$$\eta_K\left(u_{FE},\mathcal{K}\right)^2 := \eta_{R,K}\left(u_{FE},\mathcal{K}\right)^2 + \eta_{B,K}\left(u_{FE},\mathcal{K}\right)^2.$$

Here the residual term $\eta_{R,K}\left(u_{FE},\mathcal{K}\right)$ *reads*

$$\eta_{R,K}\left(u_{FE},\mathcal{K}\right)^2 := \frac{h_K^2}{\left(p_K+1\right)^2}\left(\left\|\Pi f - \nabla\times\left(\alpha\nabla\times u_{FE}\right) - \beta u_{FE}\right\|_{L^2(K)^3}^2 + \left\|\operatorname{div}\left(\beta u_{FE}\right)\right\|_{L^2(K)}^2\right)$$

and the boundary term $\eta_{B,K}\left(u_{FE},\mathcal{K}\right)$ *is given by*

$$\eta_{B,K}\left(u_{FE},\mathcal{K}\right)^2 := \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)}\frac{h_e}{p_e+1}\left(\left\|\left[n_e\times\alpha\nabla\times u_{FE}\right]\right\|_{L^2(e)^3}^2 + \left\|\left[n_e^T\beta u_{FE}\right]\right\|_{L^2(e)}^2\right),$$

where

$$\mathcal{E}_I(K) := \{e\subset\partial K\cap\Omega : e \text{ is a face of cell } K\}$$

denotes the set of all interior faces of cell K *and* $h_e := \operatorname{diam}(e)$ *is the face diameter. The face polynomial degree* p_e *is given by*

$$p_e := \max\left\{p_{K_1}, p_{K_2}\right\}$$

for all $K_1, K_2 \in \mathcal{K}$ *with* $e = K_1\cap K_2$*,* $[\cdot]$ *denotes the jump over* e *and* n_e *is the outward-pointing unit normal vector to cell* K *on face* e*.*

For this a posteriori error estimator we want to derive some reliability and efficiency estimates. Before we do this let us show a polynomial smoothing estimate which allows us to introduce some smoothing function into the L^2-norm of a polynomial.

Lemma 4.3 (Polynomial Smoothing Estimates)**.** *Let* $K\in\mathcal{K}$ *be arbitrary and* $a,b\in\mathbb{R}$ *with* $b>a>-\frac{1}{2}$*. Then:*

1. *Let* $u\in P_{p_K}(K)$ *denote some polynomial and define the smoothing function* $\phi_K : K\to\mathbb{R}_+$ *by*

   $$\phi_K(x) := \frac{1}{h_K}\operatorname{dist}(x,\partial K).$$

   Then there exists some constant $C_s>0$ *independent of polynomial degree vector* p *such that*

   $$\left\|\phi_K^a u\right\|_{L^2(K)} \leq C_s\left(p_K+1\right)^{b-a}\left\|\phi_K^b u\right\|_{L^2(K)}.$$

2. *Let* $e\in\mathcal{E}_I(K)$ *denote some interior face of cell* K *with* $e = K\cap\widetilde{K}$ *for some* $\widetilde{K}\in\mathcal{K}$ *and* $u\in P_{p_e}\left(K|_e\right)$ *denote some polynomial. We define the smoothing function* $\phi_e : K\cup\widetilde{K}\to\mathbb{R}_+$ *by*

   $$\phi_e(x) := \frac{1}{\operatorname{diam}\left(K\cup\widetilde{K}\right)}\operatorname{dist}\left(x,\partial\left(K\cup\widetilde{K}\right)\right).$$

   Then there exists some constant $C_s>0$ *independent of polynomial degree vector* p *such that*

   $$\left\|\phi_e^a u\right\|_{L^2(e)} \leq C_s\left(p_e+1\right)^{b-a}\left\|\phi_e^b u\right\|_{L^2(e)}.$$

   Further there exists some extension $v_e\in H_0^1\left(K\cup\widetilde{K}\right)$ *of* $\phi_e^a u$ *such that:*

   (a) $v_e = \phi_e^a u$ *on* e*.*

(b) There exists some constant $C_{s,tr} > 0$ independent of mesh size vector h and polynomial degree vector p such that

$$\|v_e\|_{L^2\left(K\cup\widetilde{K}\right)} \leq C_{s,tr}\frac{\sqrt{h_e}}{p_e+1}\|\phi_e^a u\|_{L^2(e)}.$$

(c) There exists some constant $C_{s,inv} > 0$ independent of mesh size vector h and polynomial degree vector p such that

$$\|\nabla v_e\|_{L^2\left(K\cup\widetilde{K}\right)^3} \leq C_{s,inv}\frac{(p_e+1)\sqrt{(p_e+1)^{-2a}+1}}{\sqrt{h_e}}\|\phi_e^a u\|_{L^2(e)}.$$

Proof. 1. There exists some $\widehat{u} \in P_{p_K}\left(\widehat{K}\right)$ such that

$$\widehat{u}\circ F_K = u \qquad \text{in } K,$$

where $F_K : \widehat{K} \to K$ denotes the reference mapping. Then we see

$$\begin{aligned}\|\phi_K^a u\|_{L^2(K)} &= h_K^{\frac{3}{2}}\left\|\widehat{\phi}_{\widehat{K}}^a\widehat{u}\right\|_{L^2\left(\widehat{K}\right)}\\ &\leq C_s h_K^{\frac{3}{2}}(p_K+1)^{b-a}\left\|\widehat{\phi}_{\widehat{K}}^b\widehat{u}\right\|_{L^2\left(\widehat{K}\right)}\end{aligned}$$

by Theorem 2.5 in [157] and the result follows.

2. For the first statement replace K by e and Theorem 2.5 by Lemma 2.4 in 1.
In the second statement $v_e \in H_0^1\left(K\cup\widetilde{K}\right)$ can be constructed in exactly the same way as in the proof of Lemma 2.6 in [157]. Then (a) is fulfilled. For (b) we observe that there exists some reference patch $\widehat{\omega}_{\widehat{K}}$ and some reference mapping $F_{K\cup\widetilde{K}} : \widehat{\omega}_{\widehat{K}} \to K\cup\widetilde{K}$ such that

$$F_{K\cup\widetilde{K}} = F_K \qquad \text{in } \widehat{K}.$$

Further there exists some $\widehat{v}_e \in H_0^1\left(\widehat{\omega}_{\widehat{K}}\right)$ such that

$$\widehat{v}_e\circ F_{K\cup\widetilde{K}} = v_e \qquad \text{in } K\cup\widetilde{K}.$$

Then, shape regularity (2.9) implies

$$\|v_e\|_{L^2\left(K\cup\widetilde{K}\right)} \leq C h_K^{\frac{3}{2}}\|\widehat{v}_e\|_{L^2\left(\widehat{\omega}_{\widehat{K}}\right)}$$

for some constant $C > 0$ independent of mesh size vector h. With Lemma 2.6 in [157] and the (γ_1,γ_2)-regularity of $\mathcal{K}$ we obtain

$$\|v_e\|_{L^2\left(K\cup\widetilde{K}\right)} \leq CC_a\frac{h_e^{\frac{3}{2}}}{p_e+1}\left\|\widehat{\phi}_{\widehat{e}}^a\widehat{u}\right\|_{L^2(\widehat{e})}$$

for $\widehat{u} \in P_{p_e}\left(\widehat{K}|_{\widehat{e}}\right)$ such that

$$\widehat{u}\circ F_K = u \qquad \text{on } e.$$

Then the result follows. For assertion (c) we proceed quite similar to the proof of (b). From shape regularity (2.9) we obtain

$$\|\nabla v_e\|_{L^2\left(K\cup\widetilde{K}\right)^3} \leq C\sqrt{h_K}\|\nabla\widehat{v}_e\|_{L^2\left(\widehat{\omega}_{\widehat{K}}\right)}$$

for some constant $C > 0$ independent of mesh size vector h. Then Lemma 2.6 in [157] and the (γ_1, γ_2)-regularity of $\mathcal{K}$ imply

$$\|\nabla v_e\|_{L^2\left(K \cup \widetilde{K}\right)^3} \le C C_a \left(p_e + 1\right) \sqrt{h_e \left(\left(p_e + 1\right)^{-2a} + 1\right)} \left\|\widehat{\phi}^a_{\widehat{e}} \widehat{u}\right\|_{L^2(\widehat{e})}$$

and the result follows.

□

Now we show some auxiliary results which we use in the proof of the main results of this section.

Lemma 4.4. *Let $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of (4.11) and $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Further let $z \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ with $\mathrm{div}(z) = 0$ in Ω such that*

$$u - u_{FE} = z + \nabla q \qquad \text{in } \Omega \tag{4.13}$$

for some $q \in H^1(\Omega)$. Then there exists some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p such that

$$\begin{aligned}
&\left|\sum_{K \in \mathcal{K}} \int_K \left(z - \Pi^{\mathrm{curl}} z\right)^T \left(f - \nabla \times (\alpha \nabla \times u_{FE}) - \beta u_{FE}\right)\right| \\
&\quad \le C \left(\sum_{K \in \mathcal{K}} \frac{h_K^2}{\left(p_K + 1\right)^{2(1-\varepsilon)}} \left(\|\Pi f - \nabla \times (\alpha \nabla \times u_{FE}) - \beta u_{FE}\|^2_{L^2(K)^3} + \|f - \Pi f\|^2_{L^2(K)^3}\right)\right)^{\frac{1}{2}} \|u - u_{FE}\|_\Omega
\end{aligned}$$

for all $\varepsilon > 0$.

Proof. We set

$$T := \sum_{K \in \mathcal{K}} \int_K \left(z - \Pi^{\mathrm{curl}} z\right)^T \left(f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right)$$

and by using Hölder's inequality it follows

$$|T| \le \sum_{K \in \mathcal{K}} \left\|z - \Pi^{\mathrm{curl}} z\right\|_{L^2(K)^3} \|f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\|_{L^2(K)^3}\,. \tag{4.14}$$

For the first term using Minkowski's inequality implies

$$\|f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\|_{L^2(K)^3} \le \|\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\|_{L^2(K)^3} + \|f - \Pi f\|_{L^2(K)^3}$$

with $\Pi : L^2(\Omega) \to U^p(\mathcal{K}, \Omega)$ denoting the L^2-conforming interpolation from Section 2.4. From Theorem 2.5 we obtain

$$\left\|z - \Pi^{\mathrm{curl}} z\right\|_{L^2(K)^3} \le C_{\mathrm{curl}} \frac{h_K}{\left(p_K + 1\right)^{1-\varepsilon}} \|\nabla z\|_{L^2(\omega_{K,2})^3}$$

and inserting these estimates into (4.14) yields

(4.15)

$$\begin{aligned}
|T| &\le C_{\mathrm{curl}} \sum_{K \in \mathcal{K}} \frac{h_K}{\left(p_K + 1\right)^{1-\varepsilon}} \left(\|\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\|_{L^2(K)^3} + \|f - \Pi f\|_{L^2(K)^3}\right) \|\nabla z\|_{L^2(\omega_{K,2})^3} \\
&\le C_{\mathrm{cov}} \left(\sum_{K \in \mathcal{K}} \frac{h_K^2}{\left(p_K + 1\right)^{2(1-\varepsilon)}} \left(\|\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\|^2_{L^2(K)^3} + \|f - \Pi f\|^2_{L^2(K)^3}\right)\right)^{\frac{1}{2}} \|\nabla z\|_{L^2(\Omega)^3}
\end{aligned}$$

with the Cauchy-Schwarz inequality and Young's inequality, where the constant $C_{\mathrm{cov}} > 0$ is given by

$$C_{\mathrm{cov}} := \sqrt{2} C_{\mathrm{curl}} \max_{K \in \mathcal{K}} \left| \{ L \in \mathcal{K} : L \subset \omega_{K,2} \} \right| .$$

From Theorem 1.5 we know

$$\begin{aligned} \|\nabla z\|_{L^2(\Omega)^3} &\leq C_{\mathrm{reg}} \|u - u_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)} \\ &\leq \frac{C_{\mathrm{reg}}}{\min\left\{\sqrt{\alpha}_{\min}, \sqrt{\beta}_{\min}\right\}} \|u - u_{\mathrm{FE}}\|_{\Omega} , \end{aligned}$$

since α and β are uniformly positive definite (4.9), (4.10). Then, inserting into (4.15) gives the result. $\square$

Lemma 4.5. *Let $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of (4.11) and $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Further let $q \in H^1(\Omega)$ such that (4.13) holds for some $z \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$. Then there exists some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left| \sum_{K \in \mathcal{K}} \int_K \left(q - \Pi^1 q\right) \mathrm{div}\left(\beta u_{FE}\right) \right| \leq C \left(\sum_{K \in \mathcal{K}} \frac{h_K^2}{(p_K + 1)^2} \|\mathrm{div}\left(\beta u_{FE}\right)\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \|u - u_{FE}\|_{\Omega} .$$

Proof. From Hölder's inequality it follows

$$\begin{aligned} \left| \sum_{K \in \mathcal{K}} \int_K \left(q - \Pi^1 q\right) \mathrm{div}\left(\beta u_{\mathrm{FE}}\right) \right| &\leq \sum_{K \in \mathcal{K}} \left\|q - \Pi^1 q\right\|_{L^2(K)} \|\mathrm{div}\left(\beta u_{\mathrm{FE}}\right)\|_{L^2(K)} \\ &\leq C_{\mathrm{grad}} \sum_{K \in \mathcal{K}} \frac{h_K}{p_K + 1} \|\mathrm{div}\left(\beta u_{\mathrm{FE}}\right)\|_{L^2(K)} \|\nabla q\|_{L^2(\omega_{K,2})^3} \end{aligned}$$

with Theorem 2.4 and by using the Cauchy-Schwarz inequality we get

$$\left| \sum_{K \in \mathcal{K}} \int_K \left(q - \Pi^1 q\right) \mathrm{div}\left(\beta u_{\mathrm{FE}}\right) \right| \leq C_{\mathrm{cov}} \left(\sum_{K \in \mathcal{K}} \frac{h_K^2}{(p_K + 1)^2} \|\mathrm{div}\left(\beta u_{\mathrm{FE}}\right)\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \|\nabla q\|_{L^2(\Omega)^3},$$

where the constant $C_{\mathrm{cov}} > 0$ is given by

$$C_{\mathrm{cov}} := C_{\mathrm{grad}} \max_{K \in \mathcal{K}} \left| \{ L \in \mathcal{K} : L \subset \omega_{K,2} \} \right| .$$

Then Theorem 1.5 implies

$$\left| \sum_{K \in \mathcal{K}} \int_K \left(q - \Pi^1 q\right) \mathrm{div}\left(\beta u_{\mathrm{FE}}\right) \right| \leq C_{\mathrm{cov}} C_{\mathrm{reg}} \left(\sum_{K \in \mathcal{K}} \frac{h_K^2}{(p_K + 1)^2} \|\mathrm{div}\left(\beta u_{\mathrm{FE}}\right)\|_{L^2(K)}^2 \right)^{\frac{1}{2}} \|u - u_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)}$$

and using assumptions (4.9) and (4.10) gives the result. $\square$

Lemma 4.6. *Let $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of (4.11) and $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Further let $z \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ such that (4.13) holds for some $q \in H^1(\Omega)$. Then there exists some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\begin{aligned} &\left| \sum_{K \in \mathcal{K}} \int_{\partial K} \left(n \times \left(z - \Pi^{\mathrm{curl}} z\right)\right)^T \left(n \times \alpha \nabla \times u_{FE}\right) \times n \right| \\ &\qquad \leq C \left(\frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{(p_e + 1)^{1-\varepsilon}} \|[n_e \times \alpha \nabla \times u_{FE}]\|_{L^2(e)^3}^2 \right)^{\frac{1}{2}} \|u - u_{FE}\|_{\Omega} \end{aligned}$$

for all $\varepsilon > 0$.

Proof. We set

$$T := \sum_{K\in\mathcal{K}} \int_{\partial K} \left(n \times \left(z - \Pi^{\mathrm{curl}} z\right)\right)^T (n \times \alpha \nabla \times u_{\mathrm{FE}}) \times n$$

and see easily

$$T = \frac{1}{2} \sum_{K\in\mathcal{K}} \sum_{e\in\mathcal{E}_I(K)} \int_e \left(n_e \times \left(z - \Pi^{\mathrm{curl}} z\right)\right)^T \left[(n_e \times \alpha \nabla \times u_{\mathrm{FE}}) \times n_e\right].$$

With Hölder's inequality it follows

$$\begin{aligned} |T| &\le \frac{1}{2} \sum_{K\in\mathcal{K}} \sum_{e\in\mathcal{E}_I(K)} \left\| z - \Pi^{\mathrm{curl}} z \right\|_{L^2(e)^3} \left\| [n_e \times \alpha \nabla \times u_{\mathrm{FE}}] \right\|_{L^2(e)^3} \\ &\le \frac{C_{\mathrm{curl}}}{2} \sum_{K\in\mathcal{K}} \|\nabla z\|_{L^2(\omega_{K,2})^3} \sum_{e\in\mathcal{E}_I(K)} \frac{\sqrt{h_e}}{(p_e+1)^{\frac{1}{2}(1-\varepsilon)}} \left\| [n_e \times \alpha \nabla \times u_{\mathrm{FE}}] \right\|_{L^2(e)^3} \end{aligned}$$

by Theorem 2.5 and using the Cauchy-Schwarz inequality and Young's inequality implies

$$|T| \le C_{\mathrm{cov}} C_{\mathrm{curl}} \left(\frac{1}{2} \sum_{K\in\mathcal{K}} \sum_{e\in\mathcal{E}_I(K)} \frac{h_e}{(p_e+1)^{1-\varepsilon}} \left\| [n_e \times \alpha \nabla \times u_{\mathrm{FE}}] \right\|^2_{L^2(e)^3} \right)^{\frac{1}{2}} \|\nabla z\|_{L^2(\Omega)^3},$$

where the constant $C_{\mathrm{cov}} > 0$ is given by

$$C_{\mathrm{cov}} := \max_{K\in\mathcal{K}} \left| \{ L \in \mathcal{K} : L \subset \omega_{K,2} \} \right|.$$

With Theorem 1.5 and the (γ_1, γ_2)-regularity of $\mathcal{K}$ we get

$$|T| \le C_{\mathrm{cov}} C_{\mathrm{curl}} C_{\mathrm{reg}} \left(\frac{1}{2} \sum_{K\in\mathcal{K}} \sum_{e\in\mathcal{E}_I(K)} \frac{h_e}{(p_e+1)^{1-\varepsilon}} \left\| [n_e \times \alpha \nabla \times u_{\mathrm{FE}}] \right\|^2_{L^2(e)^3} \right)^{\frac{1}{2}} \|u - u_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)}$$

and the result follows, since α and β are uniformly positive definite (4.9), (4.10). □

Lemma 4.7. *Let $u \in H_0(\mathrm{curl},\Omega)$ be the solution of (4.11) and $u_{FE} \in W^p(\mathcal{K},\Omega)$ be the solution of (4.12). Further let $q \in H^1(\Omega)$ such that (4.13) holds for some $z \in H_0(\mathrm{curl},\Omega) \cap H^1(\Omega)^3$. Then there exists some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p such that*

$$\left| \sum_{K\in\mathcal{K}} \int_{\partial K} \left(q - \Pi^1 q\right) n^T \beta u_{FE} \right| \le C \left(\frac{1}{2} \sum_{K\in\mathcal{K}} \sum_{e\in\mathcal{E}_I(K)} \frac{h_e}{p_e+1} \left\| \left[n_e^T \beta u_{FE} \right] \right\|^2_{L^2(e)} \right)^{\frac{1}{2}} \|u - u_{FE}\|_{\Omega}.$$

Proof. We set

$$T := \sum_{K\in\mathcal{K}} \int_{\partial K} \left(q - \Pi^1 q\right) n^T \beta u_{\mathrm{FE}}$$

and see easily

$$T = \frac{1}{2} \sum_{K\in\mathcal{K}} \sum_{e\in\mathcal{E}_I(K)} \int_e \left(q - \Pi^1 q\right) \left[n_e^T \beta u_{\mathrm{FE}} \right].$$

With Hölder's inequality it follows

$$\begin{aligned} |T| &\le \frac{1}{2} \sum_{K\in\mathcal{K}} \sum_{e\in\mathcal{E}_I(K)} \left\| q - \Pi^1 q \right\|_{L^2(e)} \left\| \left[n_e^T \beta u_{\mathrm{FE}} \right] \right\|_{L^2(e)} \\ &\le \frac{C_{\mathrm{grad}}}{2} \sum_{K\in\mathcal{K}} \|\nabla q\|_{L^2(\omega_{K,2})^3} \sum_{e\in\mathcal{E}_I(K)} \frac{\sqrt{h_e}}{(p_e+1)^{\frac{1}{2}}} \left\| \left[n_e^T \beta u_{\mathrm{FE}} \right] \right\|_{L^2(e)} \end{aligned}$$

by Theorem 2.4 and using the Cauchy-Schwarz inequality and Young's inequality implies

$$|T| \le C_{\mathrm{cov}} C_{\mathrm{grad}} \left(\frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e + 1} \left\| \left[n_e^T \beta u_{\mathrm{FE}} \right] \right\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \|\nabla q\|_{L^2(\Omega)^3},$$

where the constant $C_{\mathrm{cov}} > 0$ is given by

$$C_{\mathrm{cov}} := \max_{K \in \mathcal{K}} \left| \{ L \in \mathcal{K} : L \subset \omega_{K,2} \} \right|.$$

With Theorem 1.5 and the (γ_1, γ_2)-regularity of $\mathcal{K}$ we get

$$|T| \le C_{\mathrm{cov}} C_{\mathrm{grad}} C_{\mathrm{reg}} \left(\frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e + 1} \left\| \left[n_e^T \beta u_{\mathrm{FE}} \right] \right\|_{L^2(e)}^2 \right)^{\frac{1}{2}} \|u - u_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)}$$

and the result follows, since α and β are uniformly positive definite (4.9), (4.10). □

Lemma 4.8. *Let $K \in \mathcal{K}$ be arbitrary. Further let $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of (4.11) and $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Then there exists some constant $C > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\begin{aligned} &\|\Pi f - \nabla \times (\alpha \nabla \times u_{FE}) - \beta u_{FE}\|_{L^2(K)^3} \\ &\quad \le C \max\{p_\alpha - 2, p_\beta, 1\}^{\frac{7-\varepsilon}{4}} (p_K + 1)^{\frac{1+\varepsilon}{4}} \left(\frac{(p_K + 1)^{\frac{3-\varepsilon}{2}}}{h_K} \|u - u_{FE}\|_K + \|f - \Pi f\|_{L^2(K)^3} \right) \end{aligned}$$

for all $\varepsilon \in (0, 3]$.

Proof. We set

$$\mathrm{res} := \Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}.$$

From Lemma 4.3 it follows

$$\|\mathrm{res}\|_{L^2(K)^3} \le C_s \max\{p_\alpha - 2, p_\beta, 1\}^{\frac{1+\varepsilon}{4}} (p_K + 1)^{\frac{1+\varepsilon}{4}} \left\| \phi_K^{\frac{1+\varepsilon}{4}} \mathrm{res} \right\|_{L^2(K)^3}. \tag{4.16}$$

Then we define the function $v_K : \Omega \to \mathbb{R}^3$ by

$$v_K := \begin{cases} \phi_K^{\frac{1+\varepsilon}{2}} \mathrm{res} & \text{in } K \\ 0 & \text{in } \Omega \setminus K \end{cases}.$$

Since

$$0 \le \phi_K \le \frac{1}{2} \tag{4.17}$$

and

$$|\nabla \phi_K| \le \frac{C}{h_K}$$

for some constant $C > 0$, it follows immediately $v_K \in H_0^1(K)^3$ with Lemma 3.8.
We observe

$$\left\| \phi_K^{\frac{1+\varepsilon}{4}} \mathrm{res} \right\|_{L^2(K)^3}^2 = \int_K v_K^T \left(f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}} \right) + \int_K v_K^T (\Pi f - f). \tag{4.18}$$

For the first term using integration by parts yields

$$\int_K v_K^T \left(f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right) = A\left(v_K, u - u_{\mathrm{FE}}\right), \tag{4.19}$$

since $u \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ solves (4.11). Then it follows

$$\left| \int_K v_K^T \left(f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right) \right| \leq C_{\mathrm{cont}} \left\| v_K \right\|_K \left\| u - u_{\mathrm{FE}} \right\|_K$$

by Lemma 4.1. Since α and β are uniformly positive definite (4.9), (4.10), we have

$$\begin{aligned} \left\| v_K \right\|_K^2 &\leq \max\left\{\alpha_{\max}, \beta_{\max}\right\} \left\| v_K \right\|_{H(\mathrm{curl},K)}^2 \\ &\leq \max\left\{\alpha_{\max}, \beta_{\max}\right\} \left(\left\| \nabla \times v_K \right\|_{L^2(K)^3}^2 + 2^{-\frac{1+\varepsilon}{2}} \left\| \phi_K^{\frac{1+\varepsilon}{4}} \operatorname{res} \right\|_{L^2(K)^3}^2 \right) \end{aligned}$$

by (4.17). Then in exactly the same way as in the proof of Lemma 3.4 in [157] we obtain

$$\left\| v_K \right\|_K^2 \leq \max\left\{\alpha_{\max}, \beta_{\max}\right\} \left(C \max\left\{p_\alpha - 2, p_\beta, 1\right\}^{3-\varepsilon} \frac{\left(p_K + 1\right)^{3-\varepsilon}}{h_K^2} + 2^{-\frac{1+\varepsilon}{2}} \right) \left\| \phi_K^{\frac{1+\varepsilon}{4}} \operatorname{res} \right\|_{L^2(K)^3}^2$$

and inserting into (4.19) yields

$$\begin{aligned} &\left| \int_K v_K^T \left(f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right) \right| \\ &\qquad \leq C \max\left\{p_\alpha - 2, p_\beta, 1\right\}^{\frac{3-\varepsilon}{2}} \frac{\left(p_K + 1\right)^{\frac{3-\varepsilon}{2}}}{h_K} \left\| u - u_{\mathrm{FE}} \right\|_K \left\| \phi_K^{\frac{1+\varepsilon}{4}} \operatorname{res} \right\|_{L^2(K)^3} \end{aligned} \tag{4.20}$$

for $\varepsilon \leq 3$. Here $C > 0$ denotes some constant independent of mesh size vector h and polynomial degree vector p. Further using Hölder's inequality gives

$$\begin{aligned} \left| \int_K v_K^T (f - \Pi f) \right| &\leq \left\| f - \Pi f \right\|_{L^2(K)^3} \left\| v_K \right\|_{L^2(K)^3} \\ &\leq 2^{-\frac{1+\varepsilon}{4}} \left\| f - \Pi f \right\|_{L^2(K)^3} \left\| \phi_K^{\frac{1+\varepsilon}{4}} \operatorname{res} \right\|_{L^2(K)^3} \end{aligned} \tag{4.21}$$

with (4.17) and by inserting estimates (4.20) and (4.21) into (4.18) we get

$$\left\| \phi_K^{\frac{1+\varepsilon}{4}} \operatorname{res} \right\|_{L^2(K)^3} \leq C \max\left\{p_\alpha - 2, p_\beta, 1\right\}^{\frac{3-\varepsilon}{2}} \frac{\left(p_K + 1\right)^{\frac{3-\varepsilon}{2}}}{h_K} \left\| u - u_{\mathrm{FE}} \right\|_K + 2^{-\frac{1+\varepsilon}{4}} \left\| f - \Pi f \right\|_{L^2(K)^3}.$$

Then the result follows with estimate (4.16). □

Lemma 4.9. *Let $K \in \mathcal{K}$ be arbitrary and $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of (4.11). Further let $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Then there exists some constant $C > 0$ independent of mesh size vector h, polynomial degree vector p and p_β such that*

$$\left\| \operatorname{div}\left(\beta u_{FE}\right) \right\|_{L^2(K)} \leq C \max\left\{p_\beta - 1, 1\right\}^{\frac{7-\varepsilon}{4}} \frac{\left(p_K + 1\right)^{\frac{7-\varepsilon}{4}}}{h_K} \left\| \beta^{\frac{1}{2}} \left(u - u_{FE}\right) \right\|_{L^2(K)^3}$$

for all $\varepsilon \in (0, 3]$.

Proof. From Lemma 4.3 we know

$$\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)} \leq C_s \max\{p_\beta - 1, 1\}^{\frac{1+\varepsilon}{4}} (p_K+1)^{\frac{1+\varepsilon}{4}} \left\|\phi_K^{\frac{1+\varepsilon}{4}} \operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)} \tag{4.22}$$

and we get

$$\left\|\phi_K^{\frac{1+\varepsilon}{4}} \operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2 = \int_K \phi_K^{\frac{1+\varepsilon}{2}} \operatorname{div}(\beta u_{\mathrm{FE}} - f) \operatorname{div}(\beta u_{\mathrm{FE}}),$$

since $\operatorname{div}(f) = 0$. Then integration by parts and the fact that u solves (4.11) yield

$$\begin{aligned}\left\|\phi_K^{\frac{1+\varepsilon}{4}} \operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2 &= \int_K (\nabla \times u)^T \alpha \nabla \times \nabla\left(\phi_K^{\frac{1+\varepsilon}{2}} \operatorname{div}(\beta u_{\mathrm{FE}})\right) \\ &\quad + \int_K (u - u_{\mathrm{FE}})^T \beta^T \nabla\left(\phi_K^{\frac{1+\varepsilon}{2}} \operatorname{div}(\beta u_{\mathrm{FE}})\right).\end{aligned}$$

Since $\nabla \times \nabla\phi = 0$ for all ϕ sufficiently smooth, it follows

$$\begin{aligned}\left\|\phi_K^{\frac{1+\varepsilon}{4}} \operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2 &= \int_K (u - u_{\mathrm{FE}})^T \beta^T \nabla\left(\phi_K^{\frac{1+\varepsilon}{2}} \operatorname{div}(\beta u_{\mathrm{FE}})\right) \\ &\leq \|\beta(u - u_{\mathrm{FE}})\|_{L^2(K)^3} \left\|\nabla\left(\phi_K^{\frac{1+\varepsilon}{2}} \operatorname{div}(\beta u_{\mathrm{FE}})\right)\right\|_{L^2(K)^3}\end{aligned}$$

with Hölder's inequality and in exactly the same way as in Lemma 3.4 in [157] we obtain

$$\left\|\phi_K^{\frac{1+\varepsilon}{4}} \operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)}^2 \leq C \max\{p_\beta - 1, 1\}^{\frac{3-\varepsilon}{2}} \frac{(p_K+1)^{\frac{3-\varepsilon}{2}}}{h_K} \left\|\phi_K^{\frac{1+\varepsilon}{4}} \operatorname{div}(\beta u_{\mathrm{FE}})\right\|_{L^2(K)} \|\beta(u - u_{\mathrm{FE}})\|_{L^2(K)^3}$$

for $\varepsilon \leq 3$. Then the result follows with estimate (4.22), since β is uniformly positive definite (4.10). □

Lemma 4.10. *Let $K \in \mathcal{K}$ be arbitrary and $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of (4.11). Further let $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Then there exists some constant $C > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\begin{aligned}&\sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e+1} \|[n_e \times \alpha\nabla \times u_{FE}]\|_{L^2(e)^3}^2 \\ &\qquad \leq C \max\{p_\alpha - 1, p_\beta, 1\}^4 \left((p_K+1)^{\frac{3+\varepsilon}{2}} \|u - u_{FE}\|_{\omega_K}^2 + \frac{h_K^2}{(p_K+1)^{2-\varepsilon}} \|f - \Pi f\|_{L^2(\omega_K)^3}^2\right)\end{aligned}$$

for all $\varepsilon \in (0, 3]$.

Proof. From Lemma 4.3 we know

$$\begin{aligned}&\sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e+1} \|[n_e \times \alpha\nabla \times u_{\mathrm{FE}}]\|_{L^2(e)^3}^2 \\ &\qquad \leq C_s \max\{p_\alpha - 1, 1\}^{\frac{1+\varepsilon}{2}} \sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{(p_e+1)^{\frac{1}{2}(1-\varepsilon)}} \left\|\phi_e^{\frac{1+\varepsilon}{4}} [n_e \times \alpha\nabla \times u_{\mathrm{FE}}]\right\|_{L^2(e)^3}^2.\end{aligned} \tag{4.23}$$

Now let $e \in \mathcal{E}_I(K)$ be arbitrary. Then there exists some $\widetilde{K} \in \mathcal{K}$ such that $e = K \cap \widetilde{K}$ and Lemma 4.3 gives the existence of some $v_e \in H_0^1\left(K \cup \widetilde{K}\right)$ such that

$$v_e = \phi_e^{\frac{1+\varepsilon}{2}} [n_e \times \alpha\nabla \times u_{\mathrm{FE}}] \qquad \text{on } e.$$

We observe

$$\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e\times\alpha\nabla\times u_{\mathrm{FE}}\right]\right\|_{L^2(e)^3}^2=\int_e v_e^T\left[n_e\times\alpha\nabla\times u_{\mathrm{FE}}\right]$$
$$=\int_e v_e^T n_e\times\left(\alpha|_K\nabla\times u_{\mathrm{FE}}|_K-\alpha|_{\widetilde{K}}\nabla\times u_{\mathrm{FE}}|_{\widetilde{K}}\right).$$

Then the integration by parts formula implies

$$\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e\times\alpha\nabla\times u_{\mathrm{FE}}\right]\right\|_{L^2(e)^3}^2=A\left(\widetilde{v}_e,u_{\mathrm{FE}}\right)-\int_{K\cup\widetilde{K}}\widetilde{v}_e^T\left(\nabla\times\left(\alpha\nabla\times u_{\mathrm{FE}}\right)-\beta u_{\mathrm{FE}}\right)$$

for $\widetilde{v}_e:\Omega\to\mathbb{R}$ given by

$$\widetilde{v}_e:=\begin{cases}v_e & \text{in } K\cup\widetilde{K}\\ 0 & \text{in } \Omega\setminus\left(K\cup\widetilde{K}\right)\end{cases}.$$

Since $u\in H_0(\mathrm{curl},\Omega)$ solves (4.11), this reads

$$\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e\times\alpha\nabla\times u_{\mathrm{FE}}\right]\right\|_{L^2(e)^3}^2=A\left(\widetilde{v}_e,u_{\mathrm{FE}}-u\right)+T, \tag{4.24}$$

where the term T is given by

$$T:=\int_{K\cup\widetilde{K}}\widetilde{v}_e^T\left(f-\nabla\times\left(\alpha\nabla\times u_{\mathrm{FE}}\right)-\beta u_{\mathrm{FE}}\right).$$

For the term $A\left(\widetilde{v}_e,u_{\mathrm{FE}}-u\right)$ we get

$$A\left(\widetilde{v}_e,u_{\mathrm{FE}}-u\right)\le C_{\mathrm{cont}}\left\|\widetilde{v}_e\right\|_{K\cup\widetilde{K}}\left\|u-u_{\mathrm{FE}}\right\|_{K\cup\widetilde{K}} \tag{4.25}$$

by using Lemma 4.1. Since α and β are uniformly positive definite (4.9), (4.10), we obtain

$$\begin{aligned}\left\|\widetilde{v}_e\right\|_{K\cup\widetilde{K}}^2&\le\max\left\{\alpha_{\max},\beta_{\max}\right\}\left\|\widetilde{v}_e\right\|_{H\left(\mathrm{curl},K\cup\widetilde{K}\right)}^2\\&=\max\left\{\alpha_{\max},\beta_{\max}\right\}\left(\left\|\nabla\times\widetilde{v}_e\right\|_{L^2\left(K\cup\widetilde{K}\right)^3}^2+\left\|\widetilde{v}_e\right\|_{L^2\left(K\cup\widetilde{K}\right)^3}^2\right)\\&\le\max\left\{\alpha_{\max},\beta_{\max}\right\}\left(2C_{s,\mathrm{inv}}^2\max\left\{p_\alpha-1,1\right\}^2\frac{\left(p_e+1\right)^2}{h_e}+C_{s,\mathrm{tr}}^2\frac{h_e}{\left(p_e+1\right)^2}\right)\left\|\phi_e^{-\frac{1+\varepsilon}{4}}v_e\right\|_{L^2(e)^3}^2\end{aligned}$$

by Lemma 4.3 for $\varepsilon>0$. Thus taking the square root on both sides gives

$$\left\|\widetilde{v}_e\right\|_{K\cup\widetilde{K}}\le C_1\max\left\{p_\alpha-1,1\right\}\frac{p_e+1}{\sqrt{h_e}}\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e\times\alpha\nabla\times u_{\mathrm{FE}}\right]\right\|_{L^2(e)^3}$$

for some constant $C_1>0$ independent of mesh size vector h, polynomial degree vector p and p_α. Then inserting into (4.25) yields

$$A\left(\widetilde{v}_e,u_{\mathrm{FE}}-u\right)\le C_1\max\left\{p_\alpha-1,1\right\}\frac{p_e+1}{\sqrt{h_e}}\left\|u-u_{\mathrm{FE}}\right\|_{K\cup\widetilde{K}}\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e\times\alpha\nabla\times u_{\mathrm{FE}}\right]\right\|_{L^2(e)^3}. \tag{4.26}$$

With $\Pi:L^2(\Omega)^3\to U^p(\mathcal{K},\Omega)^3$ denoting the L^2-conforming interpolation from Section 2.4 we have

$$T=\int_{K\cup\widetilde{K}}\widetilde{v}_e^T\left(\Pi f-\nabla\times\left(\alpha\nabla\times u_{\mathrm{FE}}\right)-\beta u_{\mathrm{FE}}\right)+\int_{K\cup\widetilde{K}}\widetilde{v}_e^T(f-\Pi f)$$

and with Hölder's inequality it follows

$$
\begin{aligned}
|T| &\leq \left(\|\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\|_{L^2\left(K \cup \widetilde{K}\right)^3} + \|f - \Pi f\|_{L^2\left(K \cup \widetilde{K}\right)^3} \right) \|\widetilde{v}_e\|_{L^2\left(K \cup \widetilde{K}\right)^3} \\
&\leq C_2 \max\{p_\alpha - 2, p_\beta, 1\}^{\frac{7-\varepsilon}{4}} \frac{\sqrt{h_e}}{(p_e+1)^{\frac{3-\varepsilon}{4}}} \widetilde{T} \left\| \phi_e^{\frac{1+\varepsilon}{4}} \left[n_e \times \alpha \nabla \times u_{\mathrm{FE}} \right] \right\|_{L^2(e)^3}
\end{aligned} \tag{4.27}
$$

for $\varepsilon \leq 3$ by Lemmas 4.3 and 4.8 and the (γ_1, γ_2)-regularity of $\mathcal{K}$. Here the constant $C_2 > 0$ is independent of mesh size vector h, polynomial degree vector p, p_α and p_β and the term $\widetilde{T}$ is given by

$$
\widetilde{T} := \frac{(p_e+1)^{\frac{3-\varepsilon}{2}}}{h_e} \|u - u_{\mathrm{FE}}\|_{K \cup \widetilde{K}} + \|f - \Pi f\|_{L^2\left(K \cup \widetilde{K}\right)^3}.
$$

By inserting estimates (4.26) and (4.27) into (4.24) we get

$$
\begin{aligned}
&\left\| \phi_e^{\frac{1+\varepsilon}{4}} \left[n_e \times \alpha \nabla \times u_{\mathrm{FE}} \right] \right\|_{L^2(e)^3} \\
&\qquad \leq C \max\{p_\alpha - 1, p_\beta, 1\}^{\frac{7-\varepsilon}{4}} \frac{\sqrt{h_e}}{(p_e+1)^{\frac{3-\varepsilon}{4}}} \left(\frac{(p_e+1)^{\frac{7-\varepsilon}{4}}}{h_e} \|u - u_{\mathrm{FE}}\|_{K \cup \widetilde{K}} + \|f - \Pi f\|_{L^2\left(K \cup \widetilde{K}\right)^3} \right)
\end{aligned}
$$

for some constant $C > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β. Squaring both sides and using Young's inequality gives

$$
\begin{aligned}
&\left\| \phi_e^{\frac{1+\varepsilon}{4}} \left[n_e \times \alpha \nabla \times u_{\mathrm{FE}} \right] \right\|_{L^2(e)^3}^2 \\
&\qquad \leq 2C^2 \max\{p_\alpha - 1, p_\beta, 1\}^{\frac{7-\varepsilon}{2}} \frac{h_e}{(p_e+1)^{\frac{3-\varepsilon}{2}}} \left(\frac{(p_e+1)^{\frac{7-\varepsilon}{2}}}{h_e^2} \|u - u_{\mathrm{FE}}\|_{K \cup \widetilde{K}}^2 + \|f - \Pi f\|_{L^2\left(K \cup \widetilde{K}\right)^3}^2 \right)
\end{aligned}
$$

and by inserting into (4.23) and using the (γ_1, γ_2)-regularity of $\mathcal{K}$ the result follows. ☐

Lemma 4.11. *Let $K \in \mathcal{K}$ be arbitrary and $u \in H_0(\mathrm{curl}, \Omega)$ be the solution of (4.11). Further let $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Then there exists some constant $C > 0$ independent of mesh size vector h, polynomial degree vector p and p_β such that*

$$
\sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e+1} \left\| \left[n_e^T \beta u_{FE} \right] \right\|_{L^2(e)}^2 \leq C \max\{p_\beta, 1\}^4 (p_K+1)^{\frac{3+\varepsilon}{2}} \left\| \beta^{\frac{1}{2}} (u - u_{FE}) \right\|_{L^2(\omega_K)^3}^2
$$

for all $\varepsilon \in (0, 3]$.

Proof. From Lemma 4.3 we know

$$
\sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{p_e+1} \left\| \left[n_e^T \beta u_{\mathrm{FE}} \right] \right\|_{L^2(e)}^2 \leq C_s \max\{p_\beta, 1\}^{\frac{1+\varepsilon}{2}} \sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{(p_e+1)^{\frac{1}{2}(1-\varepsilon)}} \left\| \phi_e^{\frac{1+\varepsilon}{4}} \left[n_e^T \beta u_{\mathrm{FE}} \right] \right\|_{L^2(e)}^2 . \tag{4.28}
$$

Now let $e \in \mathcal{E}_I(K)$ be arbitrary. Then there exists some $\widetilde{K} \in \mathcal{K}$ such that $e = K \cap \widetilde{K}$ and Lemma 4.3 gives the existence of some $v_e \in H_0^1\left(K \cup \widetilde{K}\right)$ such that

$$
v_e = \phi_e^{\frac{1+\varepsilon}{2}} \left[n_e^T \beta u_{\mathrm{FE}} \right] \qquad \text{on } e.
$$

We observe

$$\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e^T\beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)}^2 = \int_e v_e\left[n_e^T\beta u_{\mathrm{FE}}\right]$$
$$= \int_e v_e n_e^T\left(f|_K - \beta|_K u_{\mathrm{FE}}|_K - \left(f|_{\widetilde{K}} - \beta|_{\widetilde{K}} u_{\mathrm{FE}}|_{\widetilde{K}}\right)\right).$$

Then the integration by parts formula implies

$$\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e^T\beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)}^2 = \int_{K\cup\widetilde{K}} (\nabla v_e)^T(f-\beta u_{\mathrm{FE}}) + \int_{K\cup\widetilde{K}} v_e \operatorname{div}(f-\beta u_{\mathrm{FE}}). \tag{4.29}$$

Since u solves (4.11), we have

$$\int_{K\cup\widetilde{K}} (\nabla v_e)^T(f-\beta u_{\mathrm{FE}}) = \int_{K\cup\widetilde{K}} (\nabla\times\nabla v_e)^T \alpha\nabla\times u + \int_{K\cup\widetilde{K}} (\nabla v_e)^T\beta(u-u_{\mathrm{FE}})$$
$$= \int_{K\cup\widetilde{K}} (\nabla v_e)^T\beta(u-u_{\mathrm{FE}})$$

with the fact that $\nabla\times\nabla\phi = 0$ for ϕ sufficiently smooth. Then Hölder's inequality implies

$$\begin{aligned}\int_{K\cup\widetilde{K}} &(\nabla v_e)^T(f-\beta u_{\mathrm{FE}})\\ &\le \|\nabla v_e\|_{L^2(K\cup\widetilde{K})^3}\|\beta(u-u_{\mathrm{FE}})\|_{L^2(K\cup\widetilde{K})^3}\\ &\le C_{s,\mathrm{inv}}\sqrt{\beta_{\max}}\max\{p_\beta,1\}\frac{p_e+1}{\sqrt{h_e}}\left\|\beta^{\frac{1}{2}}(u-u_{\mathrm{FE}})\right\|_{L^2(K\cup\widetilde{K})^3}\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e^T\beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)}\end{aligned} \tag{4.30}$$

by Lemma 4.3, since β is uniformly positive definite (4.10).
Since $\operatorname{div}(f)=0$, it follows

$$\left|\int_{K\cup\widetilde{K}} v_e\operatorname{div}(f-\beta u_{\mathrm{FE}})\right| = \left|\int_{K\cup\widetilde{K}} v_e\operatorname{div}(\beta u_{\mathrm{FE}})\right|$$
$$\le \|v_e\|_{L^2(K\cup\widetilde{K})}\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K\cup\widetilde{K})}$$

by Hölder's inequality and with Lemma 4.3 we have

$$\left|\int_{K\cup\widetilde{K}} v_e\operatorname{div}(f-\beta u_{\mathrm{FE}})\right| \le C_{s,\mathrm{tr}}\frac{\sqrt{h_e}}{p_e+1}\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K\cup\widetilde{K})}\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e^T\beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)}. \tag{4.31}$$

Inserting estimates (4.30) and (4.31) into (4.29) yields

$$\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e^T\beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)} \le C_{s,\mathrm{inv}}\sqrt{\beta_{\max}}\max\{p_\beta,1\}\frac{p_e+1}{\sqrt{h_e}}\left\|\beta^{\frac{1}{2}}(u-u_{\mathrm{FE}})\right\|_{L^2(K\cup\widetilde{K})^3}$$
$$+ C_{s,\mathrm{tr}}\frac{\sqrt{h_e}}{p_e+1}\|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K\cup\widetilde{K})}$$

and with Lemma 4.9 and the (γ_1,γ_2)-regularity of $\mathcal{K}$ we get

$$\left\|\phi_e^{\frac{1+\varepsilon}{4}}\left[n_e^T\beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)} \le C\max\{p_\beta,1\}^{\frac{7-\varepsilon}{4}}\frac{p_e+1}{\sqrt{h_e}}\left\|\beta^{\frac{1}{2}}(u-u_{\mathrm{FE}})\right\|_{L^2(K\cup\widetilde{K})^3}$$

for some constant $C>0$ independent of mesh size vector h, polynomial degree vector p and p_β. Then inserting into (4.28) and using the (γ_1,γ_2)-regularity of $\mathcal{K}$ gives the result. □

Now we come to the main result of this section. It gives a reliability and an efficiency estimate for the a posteriori error estimator from Definition 4.1.

Theorem 4.1 (A Posteriori Error Estimates)**.** *Let $u \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ be the solution of (4.11) and $u_{FE} \in W^p(\mathcal{K}, \Omega)$ be the solution of (4.12). Then:*

1. *There exists some constant $C_{rel} > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\|u - u_{FE}\|_\Omega^2 \leq C_{rel} \max_{K\in\mathcal{K}} (p_K + 1)^{2\varepsilon} \left(\eta\left(u_{FE}, \mathcal{K}\right)^2 + \sum_{K\in\mathcal{K}} \frac{h_K^2}{(p_K+1)^2} \|f - \Pi f\|^2_{L^2(K)^3} \right)$$

   for all $\varepsilon > 0$.

2. *There exists some constant $C_{\mathit{eff}} > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\eta_K\left(u_{FE}, \mathcal{K}\right)^2 \leq C_{\mathit{eff}} \max\left\{p_\alpha - 1, p_\beta, 1\right\}^4 \left((p_K+1)^{\frac{3+\varepsilon}{2}} \|u - u_{FE}\|^2_{\omega_K} + \frac{h_K^2}{(p_K+1)^{\frac{3-\varepsilon}{2}}} \|f - \Pi f\|^2_{L^2(\omega_K)^3} \right)$$

   for all $K \in \mathcal{K}$ and all $\varepsilon \in (0, 3]$.

Proof. 1. From Lemma 4.2 we have

$$\begin{aligned} \|u - u_{\mathrm{FE}}\|_\Omega^2 &= A\left(u - u_{\mathrm{FE}}, u - u_{\mathrm{FE}}\right) \\ &= A\left(u - u_{\mathrm{FE}} - \Pi^{\mathrm{curl}}\left(u - u_{\mathrm{FE}}\right), u - u_{\mathrm{FE}}\right) \end{aligned} \tag{4.32}$$

with the Galerkin orthogonality

$$A\left(\Pi^{\mathrm{curl}}\left(u - u_{\mathrm{FE}}\right), u - u_{\mathrm{FE}}\right) = 0,$$

where $\Pi^{\mathrm{curl}} : H_0(\mathrm{curl}\,\Omega) \cap H^\varepsilon(\Omega)^3 \to W^p(\mathcal{K}, \Omega)$ denotes the $H(\mathrm{curl})$-conforming interpolation from Section 2.4.2. From Theorems 1.4 and 1.5 we know that there exist $z \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ and $q \in H^1(\Omega)$ such that decomposition (4.13) holds. Thus equation (4.32) reads

$$\|u - u_{\mathrm{FE}}\|_\Omega^2 = T_1 + T_2, \tag{4.33}$$

where the terms T_1 and T_2 are given by

$$T_1 := \int_\Omega \left(\nabla \times \left(z + \nabla q - \Pi^{\mathrm{curl}}(z + \nabla q)\right)\right)^T \alpha \nabla \times \left(u - u_{\mathrm{FE}}\right)$$

and

$$T_2 := \int_\Omega \left(z + \nabla q - \Pi^{\mathrm{curl}}(z + \nabla q)\right)^T \beta \left(u - u_{\mathrm{FE}}\right).$$

Then Theorem 2.10 implies

$$T_1 = \int_\Omega \left(\nabla \times \left(z - \Pi^{\mathrm{curl}} z + \nabla\left(q - \Pi^1 q\right)\right)\right)^T \alpha \nabla \times \left(u - u_{\mathrm{FE}}\right)$$

and

$$\begin{aligned} T_2 &= \int_\Omega \left(z - \Pi^{\mathrm{curl}} z + \nabla\left(q - \Pi^1 q\right)\right)^T \beta \left(u - u_{\mathrm{FE}}\right) \\ &= \int_\Omega \nabla\left(q - \Pi^1 q\right)^T f + \sum_{K\in\mathcal{K}} \left(\int_K \left(z - \Pi^{\mathrm{curl}} z\right)^T \beta \left(u - u_{\mathrm{FE}}\right) - \int_K \nabla\left(q - \Pi^1 q\right)^T \beta u_{\mathrm{FE}} \right), \end{aligned} \tag{4.34}$$

since u solves (4.11) and $\nabla \times \nabla \phi = 0$ for all ϕ sufficiently smooth. This also implies

$$
\begin{aligned}
T_1 &= \int_\Omega \left(\nabla \times \left(z - \Pi^{\mathrm{curl}} z\right)\right)^T \alpha \nabla \times (u - u_{\mathrm{FE}}) \\
&= \int_\Omega \left(z - \Pi^{\mathrm{curl}} z\right)^T (f - \beta u) - \sum_{K \in \mathcal{K}} \int_K \left(\nabla \times \left(z - \Pi^{\mathrm{curl}} z\right)\right)^T \alpha \nabla \times u_{\mathrm{FE}}
\end{aligned} \tag{4.35}
$$

and by inserting equations (4.34) and (4.35) into (4.33) and using integration by parts and the fact that u solves (4.11) we get

$$
\|u - u_{\mathrm{FE}}\|_\Omega^2 = \sum_{K \in \mathcal{K}} \left(T_{K,1} - T_{K,2} - \frac{T_{K,3} + T_{K,4}}{2}\right),
$$

where the terms $T_{K,1}, \ldots, T_{K,4}$ are given by

$$
\begin{aligned}
T_{K,1} &:= \int_K \left(z - \Pi^{\mathrm{curl}} z\right)^T \left(f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right), \\
T_{K,2} &:= \int_K \left(q - \Pi^1 q\right) \operatorname{div}\left(f - \beta u_{\mathrm{FE}}\right), \\
T_{K,3} &:= \int_{\partial K} \left(n \times \left(z - \Pi^{\mathrm{curl}} z\right)\right)^T (n \times \alpha \nabla \times u_{\mathrm{FE}}) \times n
\end{aligned}
$$

and

$$
T_{K,4} := \int_{\partial K} \left(q - \Pi^1 q\right) n^T \beta u_{\mathrm{FE}}.
$$

Here n denotes outward-pointing unit normal vector to cell K. Since $\operatorname{div}(f) = 0$, appyling Lemmas 4.4–4.7 yields

$$
\|u - u_{\mathrm{FE}}\|_\Omega^2 \leq C \left(T_1 + T_2 + T_3 + T_4\right) \|u - u_{\mathrm{FE}}\|_\Omega
$$

for some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p, where the terms $T_1, \ldots, T_4$ are given by

$$
\begin{aligned}
T_1^2 &:= \sum_{K \in \mathcal{K}} \frac{h_K^2}{(p_K + 1)^{2(1-\varepsilon)}} \left(\|\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\|_{L^2(K)^3}^2 + \|f - \Pi f\|_{L^2(K)^3}^2\right) \\
T_2^2 &:= \sum_{K \in \mathcal{K}} \frac{h_K^2}{(p_K + 1)^{2(1-\varepsilon)}} \|\operatorname{div}(\beta u_{\mathrm{FE}})\|_{L^2(K)}^2 \\
T_3^2 &:= \frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{(p_e + 1)^{1-\varepsilon}} \|[n_e \times \alpha \nabla \times u_{\mathrm{FE}}]\|_{L^2(e)^3}^2
\end{aligned}
$$

and

$$
T_4^2 := \frac{1}{2} \sum_{K \in \mathcal{K}} \sum_{e \in \mathcal{E}_I(K)} \frac{h_e}{(p_e + 1)^{1-\varepsilon}} \left\|\left[n_e^T \beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)}^2.
$$

Then using Young's inquality and the (γ_1, γ_2)-regularity of $\mathcal{K}$ implies

$$
\|u - u_{\mathrm{FE}}\|_\Omega^2 \leq 4C^2 \sum_{K \in \mathcal{K}} (p_K + 1)^{2\varepsilon} \left(\eta_K \left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 + \|f - \Pi f\|_{L^2(K)^3}^2\right)
$$

and the result follows.

2. Let $K \in \mathcal{K}$ be arbitrary. Then we know

$$
\eta_K \left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 = \eta_{R,K} \left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 + \eta_{B,K} \left(u_{\mathrm{FE}}, \mathcal{K}\right)^2 \tag{4.36}
$$

from Definition 4.1. Then Lemmas 4.8 and 4.9 imply

$$\eta_{R,K}^2 \leq C_1 \max\left\{p_\alpha - 2, p_\beta, 1\right\}^{\frac{7-\varepsilon}{2}} \left((p_K+1)^{\frac{3-\varepsilon}{2}} \|u - u_{\mathrm{FE}}\|_K^2 + \frac{h_K^2}{(p_K+1)^{\frac{3-\varepsilon}{2}}} \|f - \Pi f\|_{L^2(K)^3}^2 \right)$$

for some constant $C_1 > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β. Further Lemmas 4.10 and 4.11 give

$$\eta_{B,K}^2 \leq C_2 \max\left\{p_\alpha - 1, p_\beta, 1\right\}^4 \left((p_K+1)^{\frac{3+\varepsilon}{2}} \|u - u_{\mathrm{FE}}\|_{\omega_K}^2 + \frac{h_k^2}{(p_K+1)^{2-\varepsilon}} \|f - \Pi f\|_{L^2(\omega_K)^3}^2 \right)$$

for some constant $C_2 > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β. Inserting these estimates into (4.36) shows the result. $\square$

To conclude this section let us shortly highlight the assumptions we made above and see how these affect the main results of this section.
For proving the first part of Theorem 4.1 we use the projection-based interpolation operators from Sections 2.4.1 and 2.4.2. Therefore we require the additional regularity $u \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ instead of the minimal regularity assumption $u \in H_0(\mathrm{curl}\,\Omega)$, which is sufficient for the boundary value problem (4.8) to be well-posed. We have discussed this issue already extensively in Sections 2.4.1 and 2.4.2.
The residual-based a posteriori error estimator derived in this section looks quite similar to those for the Poisson problem from Sections 3.1.2.1, 3.1.3.1 and 3.2.2. Also here we have the splitting

$$\eta_K^2 = \eta_{R,K}^2 + \eta_{B,K}^2$$

of the local error estimator η_K into a residual term $\eta_{R,K}$ and a boundary term $\eta_{B,K}$. However, in this case we have the additional terms $\|\mathrm{div}\,(\beta u_{\mathrm{FE}})\|_{L^2(K)}$ in $\eta_{R,K}$ and $\left\|\left[n_e^T \beta u_{\mathrm{FE}}\right]\right\|_{L^2(e)}$ in $\eta_{B,K}$. These come into play by the treatment of divergence condition (4.7).
At a first sight also the reliability and efficiency estimates obtained in Theorem 4.1 look quite similar to those for the Poisson problem from Sections 3.1.2.1, 3.1.3.1 and 3.2.2. However, some details have changed. Already the reliability estimate is not uniform in p anymore. This is due to the suboptimality of the projection-based interpolation operators from Sections 2.4.1 and 2.4.2. The efficiency estimate is of the same quality as those for the higher-dimensional version of the Poisson problem in Sections 3.1.3.1 and 3.2.2. This is a major improvement over the result in [73] and probably the best one can expect at the moment, because the loss of locality and the nonuniformity in p are still open challenges for the Poisson problem in the case $d \in \{2, 3\}$.

4.3 The Refinement Strategy

In this section we adapt the fully automatic *hp*-adaptive refinement strategy from Section 3.1.3.1 to Maxwell's equations in the electric field formulation. This section is based on the results of [72]. Since we already have discussed the basic principles of the fully automatic *hp*-adaptive refinement strategy in Section 3.1.3.1, we only highlight the differences here. As in Section 3.1.3.1 our starting point is the adaptive loop (3.4). Again the interesting parts of this loop are the modules ESTIMATE and MARK. Whereas we have considered module ESTIMATE already in Section 4.2, module MARK is investigated here.
Similar to Section 3.1.2.1 we assume that we have $n \in \mathbb{N} \setminus \{1\}$ different refinement patterns to choose from. Let $j \in \{0, \ldots, n-1\}$ and $K \in \mathcal{K}$ be arbitrary. Then we denote by $W_{K,j}^p\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)$ the local

finite element space consisting of functions from $W^p(\mathcal{K},\Omega)$ compactly supported in the local patch ω_K with refinement pattern j applied to cell K, which additionally satisfy the weak local divergence condition

$$\int_{\omega_K} (\nabla\psi)^T \beta\phi = 0 \qquad \forall\psi \in V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right). \tag{4.37}$$

Without loss of generality we may assume that $\eta_K\left(u_{\mathrm{FE}}, \mathcal{K}\right) > 0$. If this is not the case, it dones not make any sense to refine this cell at all and we can go to the next one. Then we define the convergence indicator $\kappa_{K,j} \in \mathbb{R}_+$ as the solution of the optimization problem

$$\kappa_{K,j} = \frac{1}{\eta_K\left(u_{\mathrm{FE}}, \mathcal{K}\right)} \sup_{\phi\in W^p_{K,j}\left(\mathcal{K}|_{\omega_K},\omega_K;\beta\right)} \left(\frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi\left(\Pi f - \nabla\times\left(\alpha\nabla\times u_{\mathrm{FE}}\right) - \beta u_{\mathrm{FE}}\right)}{\|\phi\|_{\omega_K}} \right), \tag{4.38}$$

where $\Pi : L^2\left(\omega_K\right)^3 \to U^p\left(\mathcal{K}|_{\omega_K}, \omega_K\right)^3$ denotes the L^2-conforming interpolation from Section 2.4. As in Section 3.1.3.1 we can solve problem (4.38) easily by considering an equivalent local boundary value problem.

Lemma 4.12. *Let $v \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)$ such that*

$$\int_{\omega_K} (\nabla\times\phi)^T \alpha\nabla\times v + \int_{\omega_K} \phi^T\beta v = \sum_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi^T\left(\Pi f - \nabla\times\left(\alpha\nabla\times u_{FE}\right) - \beta u_{FE}\right) \tag{4.39}$$

for all $\phi \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)$. Then v solves (4.38).

Proof. Let $\phi \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ be arbitrary. Then we see

$$\begin{aligned}
&\frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi^T\left(\Pi f - \nabla\times\left(\alpha\nabla\times u_{\mathrm{FE}}\right) - \beta u_{\mathrm{FE}}\right)}{\|\phi\|_{\omega_K}} \\
&\quad = \frac{\int_{\omega_K} (\nabla\times\phi)^T\alpha\nabla\times v + \int_{\omega_K}\phi^T\beta v}{\|\phi\|_{\omega_K}} \\
&\quad \leq \frac{\left\|\alpha^{\frac12}\nabla\times\phi\right\|_{L^2(\omega_K)^3}\left\|\alpha^{\frac12}\nabla\times v\right\|_{L^2(\omega_K)^3} + \left\|\beta^{\frac12}\phi\right\|_{L^2(\omega_K)^3}\left\|\beta^{\frac12}v\right\|_{L^2(\omega_K)^3}}{\|\phi\|_{\omega_K}}
\end{aligned}$$

with Hölder's inequality and by using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K \phi^T\left(\Pi f - \nabla\times\left(\alpha\nabla\times u_{\mathrm{FE}}\right) - \beta u_{\mathrm{FE}}\right)}{\|\phi\|_{\omega_K}} &\leq \|v\|_{\omega_K} \\
&= \frac{\int_{\omega_K}(\nabla\times v)^T\alpha\nabla\times v + \int_{\omega_K} v^T\beta v}{\|v\|_{\omega_K}} \\
&= \frac{\sum\limits_{K\in\mathcal{K}|_{\omega_K}} \int_K v^T\left(\Pi f - \nabla\times\left(\alpha\nabla\times u_{\mathrm{FE}}\right) - \beta u_{\mathrm{FE}}\right)}{\|v\|_{\omega_K}}
\end{aligned}$$

with (4.39). Since $\phi \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)$ was arbitrary, this implies

$$\sup_{\phi \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)} \left(\frac{\sum_{K \in \mathcal{K}|_{\omega_K}} \int_K \phi^T \left(\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right)}{\|\phi\|_{\omega_K}} \right) \leq \frac{\sum_{K \in \mathcal{K}|_{\omega_K}} \int_K v^T \left(\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right)}{\|v\|_{\omega_K}}.$$

Thus

$$\sup_{\phi \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)} \left(\frac{\sum_{K \in \mathcal{K}|_{\omega_K}} \int_K \phi^T \left(\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right)}{\|\phi\|_{\omega_K}} \right) = \frac{\sum_{K \in \mathcal{K}|_{\omega_K}} \int_K v^T \left(\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right)}{\|v\|_{\omega_K}},$$

since $v \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)$. ☐

The maximization problem (3.7), (3.8) can be used without modification again. Therefore we do not discuss it here a second time.
To conclude this section let us discuss the choice of optimization problem (4.38). Similar to Section 3.2.2 the local finite element test space $W^p_{K,j}\left(\mathcal{K}_{K,j}|_{\omega_K}, \omega_K; \beta\right)$ is not a simple local enhancement of the global finite element space $W^p(\mathcal{K}, \Omega)$, but we choose a subset of weakly divergence-free functions satisfying (4.37). Basically this is the same approach as we followed for deriving the global boundary value problem (4.8), but there $\mathrm{div}(f) = 0$ in Ω immediately implies (4.7). Here we hardly know anything about the divergence of the term $\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}$ and, thus, we cannot expect to get an approximately divergence-free solution of problem (4.39) for free. Therefore we have to enforce weak divergence condition (4.37) explicitly. Since the $H(\mathrm{curl})$-conforming finite element space $W^p(\mathcal{K}, \Omega)$ does not satisfy (4.37), we look for a solution of the mixed problem to find $(z, p) \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right) \times V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ such that

$$\int_{\omega_K} (\nabla \times \phi)^T \alpha \nabla \times z + \int_{\omega_K} \phi^T \beta z + \int_{\omega_K} \phi^T \beta^T \nabla p = \sum_{K \in \mathcal{K}|_{\omega_K}} \int_K \phi^T \left(\Pi f - \nabla \times (\alpha \nabla \times u_{\mathrm{FE}}) - \beta u_{\mathrm{FE}}\right)$$
$$\int_{\omega_K} (\nabla \psi)^T \beta z = 0$$

for all $\phi \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ and all $\psi \in V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$. Here the local finite element spaces $V^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ and $W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K\right)$ are given by all functions from $V^p(\mathcal{K}, \Omega)$ and $W^p(\mathcal{K}, \Omega)$, respectively, having compact support in ω_K with refinement pattern j applied to cell K. Then $v := z \in W^p_{K,j}\left(\mathcal{K}|_{\omega_K}, \omega_K; \beta\right)$ is the solution of problem (4.39). For example in the monographs of Hiptmair [127] and Monk [163] it was shown that this discretization is inf-sup-stable. Thus, this problem is well-posed and can be solved efficiently by the use of the precondition strategies proposed in [122].

4.4 Convergence Results

Now we prove convergence of the fully automatic hp-adaptive refinement strategy. Therefore we derive two results similar to Section 3.1.3.2. As before the first result proves that the exact energy error $\|u - u_{\mathrm{FE}}\|_\Omega$

is reduced in every refinement step of the algorithm. The second result gives us that a weighted sum of exact energy error $\|u - u_{\mathrm{FE}}\|_\Omega$ and estimated error $\eta(u_{\mathrm{FE}}, \mathcal{K})$ is reduced in every refinement step. The results of this section are based on [72].
Let us assume that triangulation $\mathcal{K}$ consists of tetrahedra only. Before we prove the main results of this paragraph let us prove the following discrete version of the Helmholtz decomposition. It follows the ideas of Hiptmair and Xu [128] and Hiptmair and Zheng [129].

Theorem 4.2 (Discrete Helmholtz Decomposition)**.** *Let $\varepsilon > 0$ be arbitrary and $v_{FE} \in W^p(\mathcal{K}, \Omega)$ such that*

$$\int_\Omega (\nabla\psi)^T \beta v_{FE} = 0 \qquad \forall \psi \in V^p(\mathcal{K}, \Omega). \tag{4.40}$$

Then there exist some $z_{FE} \in W^p(\mathcal{K}, \Omega)$, $\xi_{FE} \in V^p(\mathcal{K}, \Omega)^3$ and $q_{FE} \in V^p(\mathcal{K}, \Omega)$ such that

$$v_{FE} = z_{FE} + \Pi^{\mathrm{curl}} \xi_{FE} + \nabla q_{FE}. \tag{4.41}$$

Further there exists some constant $C_H > 0$ independent of mesh size vector h, polynomial degree vector p and p_β such that

$$\|z_{FE}\|_{L^2(\Omega)^3} + \frac{h_K}{(p_K+1)^{1-\varepsilon}} \|\nabla \xi_{FE}\|_{L^2(\Omega)^3} + \|\nabla q_{FE}\|_{L^2(\Omega)^3} \le C_H \frac{h_K}{(p_K+1)^{1-\varepsilon}} \|v_{FE}\|_{H(\mathrm{curl},\Omega)}.$$

Proof. In exactly the same way as in the proofs of Theorems 1.4 and 1.5 one can show that there exist some $z \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ and $q \in H^1(\Omega)$ such that

$$v_{\mathrm{FE}} = z + \nabla q \qquad \text{in } \Omega$$

with

$$\operatorname{div}(\beta z) = 0 \qquad \text{in } \Omega. \tag{4.42}$$

Then weak divergence condition (4.40) reads

$$\begin{aligned} 0 &= \int_\Omega (\nabla\psi)^T \beta v_{\mathrm{FE}} \\ &= \int_\Omega (\nabla\psi)^T \beta (z + \nabla q) \\ &= -\int_\Omega \psi \operatorname{div}(\beta z) + \int_\Omega (\nabla\psi)^T \beta \nabla q \end{aligned}$$

with integration by parts and condition (4.42) implies

$$0 = \int_\Omega (\nabla\psi)^T \beta \nabla q \quad \forall \psi \in V^p(\mathcal{K}, \Omega). \tag{4.43}$$

Since β is uniformly positive definite (4.10), it holds

$$\|\nabla q_{\mathrm{FE}}\|^2_{L^2(\Omega)^3} \le \frac{1}{\beta_{\min}} \int_\Omega (\nabla q_{\mathrm{FE}})^T \beta \nabla q_{\mathrm{FE}}. \tag{4.44}$$

From the proof of Lemma 5.1 in [128] we know that there exists some $\widetilde{q} \in H^1(\Omega)$ such that

$$z = \Pi^{\mathrm{curl}} z + \nabla \widetilde{q} \qquad \text{in } \Omega. \tag{4.45}$$

Then we set $q_{\mathrm{FE}} := q + \widetilde{q}$. Since

$$\begin{aligned}\nabla q_{\mathrm{FE}} &= \nabla(q+\widetilde{q}) \\ &= v_{\mathrm{FE}} - z + z - \Pi^{\mathrm{curl}} z \in W^p(\mathcal{K},\Omega),\end{aligned}$$

it holds $q_{\mathrm{FE}} \in V^p(\mathcal{K},\Omega)$ by Theorem 2.10, indeed. Then estimate (4.44) reads

$$\begin{aligned}\|\nabla q_{\mathrm{FE}}\|^2_{L^2(\Omega)^3} &\leq \frac{1}{\beta_{\min}} \int_\Omega \left(\nabla q_{\mathrm{FE}}\right)^T \beta \nabla\left(q+\widetilde{q}\right) \\ &= \frac{1}{\beta_{\min}} \int_\Omega \left(\nabla q_{\mathrm{FE}}\right)^T \beta \nabla \widetilde{q}\end{aligned}$$

with (4.43). By using Hölder's inequality we obtain

$$\begin{aligned}\|\nabla q_{\mathrm{FE}}\|^2_{L^2(\Omega)^3} &\leq \frac{1}{\beta_{\min}} \|\beta\nabla\widetilde{q}\|_{L^2(\Omega)^3} \|\nabla q_{\mathrm{FE}}\|_{L^2(\Omega)^3} \\ &\leq \frac{\sqrt{\beta_{\max}}}{\beta_{\min}} \|\nabla\widetilde{q}\|_{L^2(\Omega)^3} \|\nabla q_{\mathrm{FE}}\|_{L^2(\Omega)^3},\end{aligned}$$

since β is uniformly positive definite (4.10). Thus

$$\begin{aligned}\|\nabla q_{\mathrm{FE}}\|_{L^2(\Omega)^3} &\leq \frac{\sqrt{\beta_{\max}}}{\beta_{\min}} \|\nabla\widetilde{q}\|_{L^2(\Omega)^3} \\ &= \frac{\sqrt{\beta_{\max}}}{\beta_{\min}} \left\|z - \Pi^{\mathrm{curl}} z\right\|_{L^2(\Omega)^3}\end{aligned}$$

by (4.45) and Theorem 2.5 implies

$$\begin{aligned}\|\nabla q_{\mathrm{FE}}\|_{L^2(\Omega)^3} &\leq C_{\mathrm{curl}} \frac{\sqrt{\beta_{\max}}}{\beta_{\min}} \frac{h_K}{\left(p_K+1\right)^{1-\varepsilon}} \|\nabla z\|_{L^2(\Omega)^3} \\ &\leq C_{\mathrm{reg}} C_{\mathrm{curl}} \frac{\sqrt{\beta_{\max}}}{\beta_{\min}} \frac{h_K}{\left(p_K+1\right)^{1-\varepsilon}} \|v_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)}\end{aligned}$$

with Theorem 1.5.
Further we set $z_{\mathrm{FE}} := \Pi^{\mathrm{curl}}\left(z - \Pi^1 z\right)$ and $\xi_{\mathrm{FE}} := \Pi^1 z$ with $\Pi^1 : H^1_0(\Omega) \to V^p(\mathcal{K},\Omega)$ denoting the H^1-conforming interpolation operator from Theorem 2.4. Then it follows

$$\begin{aligned}\|z_{\mathrm{FE}}\|_{L^2(\Omega)^3} &\leq \left\|z - \Pi^1 z - \Pi^{\mathrm{curl}}\left(z - \Pi^1 z\right)\right\|_{L^2(\Omega)^3} + \left\|z - \Pi^1 z\right\|_{L^2(\Omega)^3} \\ &\leq C_{\mathrm{curl}} \frac{h_K}{\left(p_K+1\right)^{1-\varepsilon}} \left\|\nabla\left(z - \Pi^1 z\right)\right\|_{L^2(\Omega)^3} + \left\|z - \Pi^1 z\right\|_{L^2(\Omega)^3}\end{aligned}$$

with Theorem 2.5 and Theorem 2.4 implies

$$\begin{aligned}\|z_{\mathrm{FE}}\|_{L^2(\Omega)^3} &\leq C_{\mathrm{grad}}\left(C_{\mathrm{curl}}+1\right) \frac{h_K}{\left(p_K+1\right)^{1-\varepsilon}} \|\nabla z\|_{L^2(\Omega)^3} \\ &\leq C_{\mathrm{grad}} C_{\mathrm{reg}} \left(C_{\mathrm{curl}}+1\right) \frac{h_K}{\left(p_K+1\right)^{1-\varepsilon}} \|v_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)}\end{aligned}$$

by Theorem 1.5. Since $\Pi^1 : H^1_0(\Omega) \to V^p(\mathcal{K},\Omega)$ is continuous, we have

$$\begin{aligned}\|\nabla\xi_{\mathrm{FE}}\|_{L^2(\Omega)^3} &= \left\|\nabla\Pi^1 z\right\|_{L^2(\Omega)^3} \\ &\leq \left\|\Pi^1\right\| \|z\|_{H^1(\Omega)^3} \\ &\leq C_{\mathrm{reg}} \left\|\Pi^1\right\| \|v_{\mathrm{FE}}\|_{H(\mathrm{curl},\Omega)}\end{aligned}$$

by Theorem 1.5 and this concludes the proof. □

Now let us consider the error estimator $\eta\left(u_{\mathrm{FE}}, \mathcal{K}\right)$, which was introduced in Definition 4.1, in more detail. We want to investigate how the estimated error is influenced by the application of the hp-adaptive refinement algorithm presented in Section 4.3.

Lemma 4.13 (Error Estimator Reduction)**.** *Let $N \in \mathbb{N}_0$ be arbitrary. We assume that there exists a solution of maximization problem (3.7), (3.8) for some $\theta \in (0,1]$. Let $u_N \in W^p\left(\mathcal{K}_N, \Omega\right)$ and $u_{N+1} \in W^p\left(\mathcal{K}_{N+1}, \Omega\right)$ be the solutions of (4.12) in iteration steps N and $N+1$, respectively. Additionally let us assume that for all refinement patterns $j \in \{0, \ldots, n-1\}$ there exists some constant $\rho \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that*

$$\frac{h_K}{p_K+1} \leq \rho \frac{h_{\widetilde{K}}}{p_{\widetilde{K}}+1} \tag{4.46}$$

for all refined cells $\widetilde{K} \in \mathcal{K}_N$ and all $K \in \mathcal{K}_{N+1}$ with $K \subseteq \widetilde{K}$. Additionally let us assume that there exists some $\tau \in (0,1]$ such that

$$\sum_{K \in \mathcal{K}_N} \frac{h_K^2}{\left(p_K+1\right)^2} \|f - \Pi f\|_{L^2(K)}^2 \leq \tau^2 \eta\left(u_N, \mathcal{K}_N\right)^2 . \tag{4.47}$$

Then there exists some constant $C_{red} > 0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that

$$\begin{aligned}
\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq{} & \left(1 + 3\delta + \left(1 + \delta + \frac{2}{\delta}\right)\rho^2\tau^2\right) \eta\left(u_N, \mathcal{K}_N\right)^2 - (1+3\delta)(1-\rho)\eta\left(u_N, \mathcal{A}_N\right)^2 \\
& + C_{red}\left(1 + \delta + \frac{2}{\delta}\right) \max\left\{p_\alpha - 1, p_\beta, 1\right\}^2 \max_{K \in \mathcal{K}_N} \left(p_K+1\right)^2 \left\|u_{N+1} - u_N\right\|_\Omega^2
\end{aligned}$$

for all $\delta > 0$.

Proof. By Definition 4.1 it holds

$$\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 = \sum_{K \in \mathcal{K}_{N+1}} \left(\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 + \eta_{B,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2\right), \tag{4.48}$$

where the cell term $\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)$ is given by

$$\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 = \frac{h_K^2}{\left(p_K+1\right)^2}\left(\left\|\Pi_{\mathcal{K}_{N+1}} f - \nabla \times \left(\alpha \nabla \times u_{N+1}\right) - \beta u_{N+1}\right\|_{L^2(K)^3}^2 + \left\|\operatorname{div}\left(\beta u_{N+1}\right)\right\|_{L^2(K)}^2\right)$$

with $\Pi_{\mathcal{K}_{N+1}} : L^2(\Omega)^3 \to U^p\left(\mathcal{K}_{N+1}, \Omega\right)^3$ denoting the L^2-conforming interpolation from Section 2.4. Then using Minkowski's and Young's inequality immediately yields

$$\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq (1+3\delta) T_1 + \left(1 + \delta + \frac{2}{\delta}\right)\left(T_2 + T_3 + T_4\right) \tag{4.49}$$

for $\delta > 0$. Here the terms $T_1, \ldots, T_4$ are given by

$$\begin{aligned}
T_1 &:= \frac{h_K^2}{\left(p_K+1\right)^2}\left(\left\|\Pi_{\mathcal{K}_N} f - \nabla \times \left(\alpha \nabla \times u_N\right) - \beta u_N\right\|_{L^2(K)^3}^2 + \left\|\operatorname{div}\left(\beta u_N\right)\right\|_{L^2(K)}^2\right), \\
T_2 &:= \frac{h_K^2}{\left(p_K+1\right)^2}\left\|\Pi_{\mathcal{K}_{N+1}} f - \Pi_{\mathcal{K}_N} f\right\|_{L^2(K)^3}^2, \\
T_3 &:= \frac{h_K^2}{\left(p_K+1\right)^2}\left(\left\|\nabla \times \alpha \nabla \times \left(u_{N+1} - u_N\right)\right\|_{L^2(K)^3}^2 + \left\|\operatorname{div}\left(\beta\left(u_{N+1} - u_N\right)\right)\right\|_{L^2(K)}^2\right)
\end{aligned}$$

and

$$T_4 := \frac{h_K^2}{(p_K+1)^2} \left\| \beta \left(u_{N+1} - u_N \right) \right\|_{L^2(K)^3}^2 .$$

Let us consider the term T_1 first. Therefore we introduce the set

$$\mathcal{R}_N := \{ K \in \mathcal{K}_N : K \text{ is refined} \}$$

of all elements from triangulation $\mathcal{K}_N$ that are refined in module REFINE of iteration step N. Clearly we have $\mathcal{A}_N \subseteq \mathcal{R}_N$. If there exists some $\widetilde{K} \in \mathcal{R}_N$ such that $K \subseteq \widetilde{K}$, then it holds

$$T_1 \leq \rho^2 \frac{h_{\widetilde{K}}^2}{\left(p_{\widetilde{K}}+1\right)^2} \left(\left\| \Pi_{\mathcal{K}_N} f - \nabla \times (\alpha \nabla \times u_N) - \beta u_N \right\|_{L^2(K)^3}^2 + \left\| \operatorname{div} \left(\beta u_N \right) \right\|_{L^2(K)}^2 \right) \tag{4.50}$$

by assumption (4.46). For the term T_2 it holds

$$\begin{aligned} T_2 &\leq \frac{h_K^2}{(p_K+1)^2} \left\| f - \Pi_{\mathcal{K}_N} f \right\|_{L^2(K)^3}^2 \\ &\leq \rho^2 \frac{h_{\widetilde{K}}^2}{\left(p_{\widetilde{K}}+1\right)^2} \left\| f - \Pi_{\mathcal{K}_N} f \right\|_{L^2(K)^3}^2 \end{aligned} \tag{4.51}$$

by assumption (4.46).
Now let us consider the case that there exists no such $\widetilde{K} \subseteq \mathcal{R}_N$. Then $K \in \mathcal{K}_N$ and it follows

$$T_1 = \eta_{R,K} \left(u_N, \mathcal{K}_N \right)^2 \tag{4.52}$$

and

$$T_2 = 0. \tag{4.53}$$

Next in line is the term T_3. In both cases using Lemma 3.8 implies

$$\begin{aligned} T_3 &\leq C_{\text{inv}}^2 \max \{ p_\alpha - 1, p_\beta, 1 \}^2 (p_K+1)^2 \left(\left\| \alpha \nabla \times \left(u_{N+1} - u_N \right) \right\|_{L^2(K)^3}^2 + \left\| \beta \left(u_{N+1} - u_N \right) \right\|_{L^2(K)^3}^2 \right) \\ &\leq C_{\text{inv}}^2 \max \{ \alpha_{\max}, \beta_{\max} \} \max \{ p_\alpha - 1, p_\beta, 1 \}^2 (p_K+1)^2 \left\| u_{N+1} - u_N \right\|_K^2 , \end{aligned} \tag{4.54}$$

since α and β are uniformly positive definite (4.9), (4.10). Finally, since β is uniformly positive definite (4.10), it follows

$$T_4 \leq \sqrt{\beta_{\max}} \left\| \beta^{\frac{1}{2}} \left(u - u_{\text{FE}} \right) \right\|_{L^2(K)^3} \tag{4.55}$$

in both cases.
Inserting estimates (4.50)–(4.55) into (4.49) gives

$$\begin{aligned} &\eta_{R,K} \left(u_{N+1}, \mathcal{K}_{N+1} \right)^2 \\ &\quad \leq \rho^2 \frac{h_{\widetilde{K}}^2}{\left(p_{\widetilde{K}}+1\right)^2} \Bigg((1+3\delta) \left(\left\| \Pi_{\mathcal{K}_N} f - \nabla \times (\alpha \nabla \times u_N) - \beta u_N \right\|_{L^2(K)^3}^2 + \left\| \operatorname{div} \left(\beta u_N \right) \right\|_{L^2(K)}^2 \right) \\ &\qquad\qquad + \left(1 + \delta + \frac{2}{\delta} \right) \left\| f - \Pi_{\mathcal{K}_N} f \right\|_{L^2(K)^3}^2 \Bigg) \\ &\qquad + C \left(1 + \delta + \frac{2}{\delta} \right) \max \{ p_\alpha - 1, p_\beta, 1 \}^2 (p_K+1)^2 \left\| u_{N+1} - u_N \right\|_K^2 , \end{aligned} \tag{4.56}$$

if there exists such a cell $\widetilde{K} \in \mathcal{R}_N$ with $K \subseteq \widetilde{K}$, and

$$
\begin{aligned}
&\eta_{R,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\
&\qquad \leq (1+3\delta)\eta_{R,K}\left(u_N, \mathcal{K}_N\right)^2 + C\left(1+\delta+\frac{2}{\delta}\right)\max\left\{p_\alpha - 1, p_\beta, 1\right\}^2 \left(p_K+1\right)^2 \left\|u_{N+1}-u_N\right\|_K^2,
\end{aligned} \tag{4.57}
$$

else. Here $C > 0$ denotes some constant independent of mesh size vector h, polynomial degree vector p, p_α and p_β.
Now let us consider the boundary term $\eta_{B,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)$. By Definition 4.1 we have

$$
\begin{aligned}
\eta_{B,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 &= \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)} \frac{h_e}{p_e+1}\left(\left\|\left[n_e \times \alpha\nabla\times u_{N+1}\right]\right\|_{L^2(e)^3}^2 + \left\|\left[n_e^T \beta u_{N+1}\right]\right\|_{L^2(e)}^2\right) \\
&\leq \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)} \frac{h_e}{p_e+1}\left(\left\|\left[n_e \times \alpha\nabla\times u_{N+1}\right]\right\|_{L^2(e)^3} T_1 + \left\|\left[n_e^T \beta u_{N+1}\right]\right\|_{L^2(e)} T_2\right)
\end{aligned}
$$

with Minkowski's inequality, where the terms T_1 and T_2 are given by

$$
T_1 := \left\|\left[n_e \times \alpha\nabla\times u_N\right]\right\|_{L^2(e)^3} + \left\|\left[n_e \times \alpha\nabla\times \left(u_{N+1}-u_N\right)\right]\right\|_{L^2(e)^3}
$$

and

$$
T_2 := \left\|\left[n_e^T \beta u_N\right]\right\|_{L^2(e)} + \left\|\left[n_e^T \beta\left(u_{N+1}-u_N\right)\right]\right\|_{L^2(e)}.
$$

Using the Cauchy-Schwarz inequality gives

$$
\eta_{B,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq \eta_{B,K}\left(u_{N+1}, \mathcal{K}_{N+1}\right)\left(\widetilde{T}_1 + \widetilde{T}_2\right), \tag{4.58}
$$

where the terms $\widetilde{T}_1$ and $\widetilde{T}_2$ are given by

$$
\widetilde{T}_1^2 := \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)} \frac{h_e}{p_e+1}\left(\left\|\left[n_e \times \alpha\nabla\times u_N\right]\right\|_{L^2(e)^3}^2 + \left\|\left[n_e^T \beta u_N\right]\right\|_{L^2(e)}^2\right)
$$

and

$$
\widetilde{T}_2^2 := \frac{1}{2}\sum_{e\in\mathcal{E}_I(K)} \frac{h_e}{p_e+1}\left(\left\|\left[n_e \times \alpha\nabla\times \left(u_{N+1}-u_N\right)\right]\right\|_{L^2(e)^3}^2 + \left\|\left[n_e^T \beta\left(u_{N+1}-u_N\right)\right]\right\|_{L^2(e)}^2\right).
$$

If there exists some cell $\widetilde{K} \in \mathcal{R}_N$ such that $K \subseteq \widetilde{K}$, then it holds

$$
\widetilde{T}_1^2 \leq \frac{\rho}{2}\sum_{e\in\mathcal{E}_I(\widetilde{K})} \frac{h_e}{p_e+1}\left(\left\|\left[n_e \times \alpha\nabla\times u_N\right]\right\|_{L^2(e\cap\partial K)^3}^2 + \left\|\left[n_e^T \beta u_N\right]\right\|_{L^2(e\cap\partial K)}^2\right). \tag{4.59}
$$

If there exists no such $\widetilde{K} \in \mathcal{R}_N$, then

$$
\widetilde{T}_1^2 \leq \eta_{B,K}\left(u_N, \mathcal{K}_N\right)^2. \tag{4.60}
$$

Next in line is the term $\widetilde{T}_2$. In both cases using Lemma 3.9 and the (γ_1, γ_2)-regularity of $\mathcal{K}_N$ implies

$$
\begin{aligned}
\widetilde{T}_2^2 &\leq 6C_{\mathrm{tr}}^2 \max\left\{p_\alpha - 1, p_\beta, 1\right\}^2 \left(p_K+1\right)\left(\left\|\alpha\nabla\times\left(u_{N+1}-u_N\right)\right\|_{L^2(K)^3}^2 + \left\|\beta\left(u_{N+1}-u_N\right)\right\|_{L^2(K)}^2\right) \\
&\leq 6C_{\mathrm{tr}}^2 \max\left\{\alpha_{\max}, \beta_{\max}\right\}\max\left\{p_\alpha - 1, p_\beta, 1\right\}^2 \left(p_K+1\right)\left\|u_{N+1}-u_N\right\|_K^2,
\end{aligned} \tag{4.61}
$$

since α and β are uniformly positive definite (4.9), (4.10).
Inserting estimates (4.59)–(4.61) into (4.58) gives
(4.62)

$$\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right) \leq \left(\frac{\rho}{2}\sum_{e\in\mathcal{E}_I(\widetilde{K})}\frac{h_e}{p_e+1}\left(\left\|[n_e\times\alpha\nabla\times u_N]\right\|^2_{L^2(e\cap\partial K)^3}+\left\|\left[n_e^T\beta u_N\right]\right\|^2_{L^2(e\cap\partial K)}\right)\right)^{\frac{1}{2}}+T,$$

if there exists such a cell $\widetilde{K}\in\mathcal{R}_N$ with $K\subseteq\widetilde{K}$, and

$$\eta_{B,K}\left(u_{N+1},\mathcal{K}_{N+1}\right)\leq\eta_{B,K}\left(u_N,\mathcal{K}_N\right)+T, \tag{4.63}$$

else. Here the term T is given by

$$T := C_{\mathrm{tr}}\max\left\{\sqrt{\alpha_{\max}},\sqrt{\beta_{\max}}\right\}\max\left\{p_\alpha-1,p_\beta,1\right\}\sqrt{6\left(p_K+1\right)}\left\|u_{N+1}-u_N\right\|_K.$$

Further inserting estimates (4.56), (4.57), (4.62) and (4.63) into (4.48) and using Young's inequality implies

$$\begin{aligned}\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 &\leq (1+3\delta)\eta\left(u_N,\mathcal{K}_N\setminus\mathcal{R}_N\right)^2+\rho T\\ &\quad + C_{\mathrm{red}}\left(1+\delta+\frac{2}{\delta}\right)\max\left\{p_\alpha-1,p_\beta,1\right\}^2\max_{K\in\mathcal{K}_N}\left(p_K+1\right)^2\left\|u_{N+1}-u_N\right\|^2_\Omega\end{aligned} \tag{4.64}$$

for some constant $C_{\mathrm{red}}>0$ independent of polynomial degree vector p, p_α and p_β. Here the term T is given by

$$T := (1+3\delta)\eta\left(u_N,\mathcal{R}_N\right)^2+\left(1+\delta+\frac{2}{\delta}\right)\rho\sum_{K\in\mathcal{R}_N}\frac{h_K^2}{\left(p_K+1\right)^2}\left\|f-\Pi_{\mathcal{K}_N}f\right\|^2_{L^2(K)^3}.$$

By data saturation assumption (4.47) it follows

$$T\leq(1+3\delta)\eta\left(u_N,\mathcal{R}_N\right)^2+\left(1+\delta+\frac{2}{\delta}\right)\rho\tau^2\eta\left(u_N,\mathcal{K}_N\right)^2$$

and inserting into (4.64) gives

$$\begin{aligned}\eta\left(u_{N+1},\mathcal{K}_{N+1}\right)^2 &\leq \left(1+3\delta+\left(1+\delta+\frac{2}{\delta}\right)\rho^2\tau^2\right)\eta\left(u_N,\mathcal{K}_N\right)^2-(1+3\delta)(1-\rho)\eta\left(u_N,\mathcal{A}_N\right)^2\\ &\quad + C_{\mathrm{red}}\left(1+\delta+\frac{2}{\delta}\right)\max\left\{p_\alpha-1,p_\beta,1\right\}^2\max_{K\in\mathcal{K}_N}\left(p_K+1\right)^2\left\|u_{N+1}-u_N\right\|^2_\Omega,\end{aligned}$$

since $\mathcal{A}_N\subseteq\mathcal{R}_N$. □

Now we show two auxiliary results, which we use in the proofs of the main results of this section. The first result gives a lower bound for the term $\|u_{N+1}-u_N\|_\Omega$ in terms of the energy error $\|u-u_N\|_\Omega$ and the estimated error $\eta\left(u_N,\mathcal{K}_N\right)$.

Lemma 4.14. *Let $N\in\mathbb{N}_0$ be arbitrary and $u\in H_0(\mathrm{curl},\Omega)\cap H^1(\Omega)^3$ be the solution of (4.11). We assume that there exists a solution of (3.7), (3.8) for some $\theta\in(0,1]$. Further let $u_N\in W^p\left(\mathcal{K}_N,\Omega\right)$ and $u_{N+1}\in W^p\left(\mathcal{K}_N,\Omega\right)$ be the solutions of (4.12) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau\in(0,1]$ such that (4.47) holds. Then there exist some constants $C_1>1$ and $C_2>0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\left\|u_{N+1}-u_N\right\|^2_\Omega\geq\delta\left(\frac{\theta^2}{C_1(1+\delta)}\left\|u-u_N\right\|^2_\Omega-C_2\tau^2\eta\left(u_N,\mathcal{K}_N\right)^2\right)$$

for all $\delta>0$.

Proof. Let $K \in \mathcal{K}_N$ be arbitrary and $\phi_{N+1} \in W^p(\mathcal{K}_{N+1}, \Omega; \beta)$ with $\operatorname{supp}(\phi_{N+1}) \subseteq \omega_K$. Then we see

$$A(\phi_{N+1}, u_{N+1} - u_N) = \int_{\omega_K} \phi_{N+1}^T f - \int_{\omega_K} (\nabla \times \phi_{N+1})^T \alpha \nabla \times u_N - \int_{\omega_K} \phi_{N+1}^T \beta u_N,$$

since $W^p(\mathcal{K}_{N+1}, \Omega; \beta) \subset W^p(\mathcal{K}_{N+1}, \Omega)$ and $u_{N+1} \in W^p(\mathcal{K}_{N+1}, \Omega)$ solves discrete problem (4.12). This reads

$$A(\phi_{N+1}, u_{N+1} - u_N) = \int_{\omega_K} \phi_{N+1}^T \Pi f - \int_{\omega_K} (\nabla \times \phi_{N+1})^T \alpha \nabla \times u_N - \int_{\omega_K} \phi_{N+1}^T \beta u_N + \int_{\omega_K} \phi_{N+1}^T (f - \Pi f),$$

where $\Pi : L^2(\Omega)^3 \to U^p(\mathcal{K}_N, \Omega)^3$ denotes the L^2-conforming interpolation from Section 2.4. With integration by parts and the L^2-interpolation property we have

$$A(\phi_{N+1}, u_{N+1} - u_N) = T(\phi_{N+1}) + \int_{\omega_K} (\phi_{N+1} - \phi_N)^T (f - \Pi f)$$

for $\phi_N \in W^p(\mathcal{K}_N, \Omega)$ with $\operatorname{supp}(\phi_N) \subseteq \omega_K$. Here the function $T : W^p(\mathcal{K}_{N+1}, \Omega) \to \mathbb{R}$ is given by

$$T(\phi_{N+1}) := \sum_{K \in \mathcal{K}_N|_{\omega_K}} \int_K \phi_{N+1}^T (\Pi f - \nabla \times (\alpha \nabla \times u_N) - \beta u_N).$$

Then using the inverse triangle inequality yields

$$|A(\phi_{N+1}, u_{N+1} - u_N)| \geq |T(\phi_{N+1})| - \left| \int_{\omega_K} (\phi_{N+1} - \phi_N)^T (f - \Pi f) \right|$$

and with Lemma 4.1 and Hölder's inequality we have

$$|T(\phi_{N+1})| \leq \|u_{N+1} - u_N\|_{\omega_K} \|\phi_{N+1}\|_{\omega_K} + \|f - \Pi f\|_{L^2(\omega_K)^3} \|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)^3}. \tag{4.65}$$

From Theorem 4.2 we know that there exist some $z_{N+1} \in W^p(\mathcal{K}_{N+1}, \Omega)$, $\xi_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)^3$ and $q_{N+1} \in V^p(\mathcal{K}_{N+1}, \Omega)$ with $\operatorname{supp}(z_{N+1}), \operatorname{supp}(\xi_{N+1}), \operatorname{supp}(q_{N+1}) \subseteq \omega_K$ such that

$$\phi_{N+1} = z_{N+1} + \Pi_{N+1}^{\mathrm{curl}} \xi_{N+1} + \nabla q_{N+1} \qquad \text{in } \Omega$$

with $\Pi_{N+1}^{\mathrm{curl}} : H_0(\mathrm{curl}, \Omega) \cap H^\varepsilon(\Omega) \to W^p(\mathcal{K}_{N+1}, \Omega)$ denoting the $H(\mathrm{curl})$-conforming interpolation operator from Section 2.4.2 for $\varepsilon > 0$. Choosing $\phi_N := \Pi_N^{\mathrm{curl}} \xi_{N+1}$ implies

$$\begin{aligned} \|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)^3} &= \left\| z_{N+1} + \Pi_{N+1}^{\mathrm{curl}} \xi_{N+1} + \nabla q_{N+1} - \Pi_N^{\mathrm{curl}} \xi_{N+1} \right\|_{L^2(\omega_K)^3} \\ &\leq \|z_{N+1}\|_{L^2(\omega_K)^3} + \left\| \Pi_{N+1}^{\mathrm{curl}} \xi_{N+1} - \Pi_N^{\mathrm{curl}} \xi_{N+1} \right\|_{L^2(\omega_K)^3} + \|\nabla q_{N+1}\|_{L^2(\omega_K)^3} \end{aligned}$$

by Minkowski's inequality and it follows

$$\begin{aligned} \|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)^3} &\leq \|z_{N+1}\|_{L^2(\omega_K)^3} + \left\| \xi_{N+1} - \Pi_N^{\mathrm{curl}} \xi_{N+1} \right\|_{L^2(\omega_K)^3} + \|\nabla q_{N+1}\|_{L^2(\omega_K)^3} \\ &\leq \|z_{N+1}\|_{L^2(\omega_K)^3} + C_{\mathrm{curl}} \frac{h_K}{(p_K + 1)^{1-\varepsilon}} \|\nabla \xi_{N+1}\|_{L^2(\omega_{K,5})^3} + \|\nabla q_{N+1}\|_{L^2(\omega_K)^3} \end{aligned}$$

with Theorem 2.5. Since $\operatorname{supp}(\xi_{N+1}) \subseteq \omega_K \subset \omega_{K,5}$, this reads

$$\|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)^3} \leq \|z_{N+1}\|_{L^2(\omega_K)^3} + 2C_{\mathrm{curl}} \frac{h_K}{p_K + 1} \|\nabla \xi_{N+1}\|_{L^2(\omega_K)^3} + \|\nabla q_{N+1}\|_{L^2(\omega_K)^3}$$

for $\varepsilon \leq \log_2 (\max\{p_K+1, 2\})$. Then Theorem 4.2 yields

$$\begin{aligned}\|\phi_{N+1} - \phi_N\|_{L^2(\omega_K)^3} &\leq C_H \frac{h_K}{p_K+1} \|\phi_{N+1}\|_{H(\mathrm{curl},\omega_K)} \\ &\leq \frac{C_H}{\min\left\{\sqrt{\alpha_{\min}}, \sqrt{\beta_{\min}}\right\}} \frac{h_K}{p_K+1} \|\phi_{N+1}\|_{\omega_K},\end{aligned}$$

since α and β are uniformly positive definite (4.9), (4.10). Inserting into (4.65) gives

$$|T(\phi_{N+1})| \leq \left(\|u_{N+1} - u_N\|_{\omega_K} + \frac{C_H}{\min\left\{\sqrt{\alpha_{\min}}, \sqrt{\beta_{\min}}\right\}} \frac{h_K}{p_K+1} \|f - \Pi f\|_{L^2(\omega_K)^3} \right) \|\phi_{N+1}\|_{\omega_K}$$

and dividing by $\|\phi_{N+1}\|_{\omega_K}$ implies

$$\begin{aligned}\sup_{\phi \in W^p_{K,j_K}(\mathcal{K}_N|_{\omega_K}, \omega_K; \beta)} \left(\frac{T(\phi)}{\|\phi\|_{\omega_K}} \right) &\leq \sup_{\substack{\phi_{N+1} \in W^p(\mathcal{K}_{N+1}, \Omega; \beta) \\ \mathrm{supp}(\phi_{N+1}) \subseteq \omega_K}} \left(\frac{T(\phi_{N+1})}{\|\phi_{N+1}\|_{\omega_K}} \right) \\ &\leq \|u_{N+1} - u_N\|_{\omega_K} + \frac{C_H}{\min\left\{\sqrt{\alpha_{\min}}, \sqrt{\beta_{\min}}\right\}} \frac{h_K}{p_K+1} \|f - \Pi f\|_{L^2(\omega_K)^3},\end{aligned}$$

since $\phi_{N+1} \in W^p(\mathcal{K}_{N+1}, \Omega; \beta)$ with $\mathrm{supp}(\phi_{N+1}) \subseteq \omega_K$ was arbitrary. With optimization problem (4.38) we get

$$\kappa_{K,j_K} \eta_K (u_N, \mathcal{K}_N) \leq \|u_{N+1} - u_N\|_{\omega_K} + \frac{C_H}{\min\left\{\sqrt{\alpha_{\min}}, \sqrt{\beta_{\min}}\right\}} \frac{h_K}{p_K+1} \|f - \Pi f\|_{L^2(\omega_K)^3}.$$

Since $\mathcal{A}_N \subseteq \mathcal{K}_N$, squaring both sides, summing over $K \in \mathcal{K}_N$ and using Young's inequality yields

$$\begin{aligned}&\sum_{K \in \mathcal{A}_N} \kappa^2_{K,j_K} \eta_K (u_N, \mathcal{K}_N)^2 \\ &\quad \leq \sum_{K \in \mathcal{K}_N} \kappa^2_{K,j_K} \eta_K (u_N, \mathcal{K}_N)^2 \\ &\quad \leq \left(1 + \frac{1}{\delta}\right) \sum_{K \in \mathcal{K}_N} \|u_{N+1} - u_N\|^2_{\omega_K} + \frac{C_H^2 (1+\delta)}{\min\{\alpha_{\min}, \beta_{\min}\}} \sum_{K \in \mathcal{K}_N} \frac{h_K^2}{(p_K+1)^2} \|f - \Pi f\|_{L^2(\omega_K)^3}\end{aligned}$$

for $\delta > 0$. Now let us define the covering constant $C_{\mathrm{cov}} > 0$ by

$$C_{\mathrm{cov}} := \max_{K \in \mathcal{K}_N} |\{L \in \mathcal{K}_N : L \subset \omega_K\}|.$$

Then the (γ_1, γ_2)-regularity of $\mathcal{K}_N$ implies

$$\sum_{K \in \mathcal{A}_N} \kappa^2_{K,j_K} \eta_K (u_N, \mathcal{K}_N)^2 \leq C_{\mathrm{cov}} \left(\left(1 + \frac{1}{\delta}\right) \|u_{N+1} - u_N\|^2_\Omega + C(1+\delta) \sum_{K \in \mathcal{K}_N} \frac{h_K^2}{(p_K+1)^2} \|f - \Pi f\|^2_{L^2(K)^3} \right)$$

for some constant $C > 0$ independent of mesh size vector h and polynomial degree vector p. Finally, from data saturation assumption (4.47) it follows

$$\sum_{K \in \mathcal{A}_N} \kappa^2_{K,j_K} \eta_K (u_N, \mathcal{K}_N)^2 \leq C_{\mathrm{cov}} \left(\left(1 + \frac{1}{\delta}\right) \|u_{N+1} - u_N\|^2_\Omega + C(1+\delta) \tau^2 \eta (u_N, \mathcal{K}_N)^2 \right). \tag{4.66}$$

From Theorem 4.1 we know

$$\begin{aligned}\|u-u_N\|_\Omega^2 &\le C_{\text{rel}} \max_{K\in\mathcal{K}_N} (p_K+1)^{2\varepsilon} \left(\eta\left(u_N,\mathcal{K}_N\right)^2 + \sum_{K\in\mathcal{K}_N} \frac{h_K^2}{\left(p_K+1\right)^2}\|f-\Pi f\|_{L^2(K)^3}^2\right)\\ &\le 2C_{\text{rel}}\left(\eta\left(u_N,\mathcal{K}_N\right)^2 + \sum_{K\in\mathcal{K}_N} \frac{h_K^2}{\left(p_K+1\right)^2}\|f-\Pi f\|_{L^2(K)^3}^2\right)\end{aligned}$$

for

$$\varepsilon \le \min\left\{\frac{1}{2}\log_2\left(\max\left\{\max_{K\in\mathcal{K}_N}\left(p_K+1\right),2\right\}\right),3\right\}$$

and using data saturation assumption (4.47) gives

$$\begin{aligned}\|u-u_N\|_\Omega^2 &\le 2C_{\text{rel}}\left(1+\tau^2\right)\eta\left(u_N,\mathcal{K}_N\right)^2\\ &\le 4C_{\text{rel}}\eta\left(u_N,\mathcal{K}_N\right)^2\end{aligned}$$

for $\tau \le 1$. Multiplying both sides by θ^2, $\theta\in(0,1]$, and using constraint (3.8) yields

$$\theta^2\|u-u_N\|_\Omega^2 \le 4C_{\text{rel}}\sum_{K\in\mathcal{A}_N}\kappa_{K,j_K}^2\eta_K\left(u_N,\mathcal{K}_N\right)^2$$

and with (4.66) it follows

$$\frac{\theta^2}{4C_{\text{rel}}}\|u-u_N\|_\Omega^2 \le C_{\text{cov}}\left(\left(1+\frac{1}{\delta}\right)\|u_{N+1}-u_N\|_\Omega^2 + C(1+\delta)\tau^2\eta\left(u_N,\mathcal{K}_N\right)^2\right).$$

This concludes the proof. □

The next result compares the energy error of the finite element approximation on two successive grids of the algorithm.

Lemma 4.15 (Comparison of Errors)**.** *Let $N\in\mathbb{N}_0$ be arbitrary and $u\in H_0(\text{curl},\Omega)\cap H^1(\Omega)$ be the solution of (4.11). We assume that there exists a solution of (3.7), (3.8) for some $\theta\in(0,1]$. Further let $u_N\in W^p\left(\mathcal{K}_N,\Omega\right)$ and $u_{N+1}\in W^p\left(\mathcal{K}_{N+1},\Omega\right)$ be the solutions of (4.12) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau\in(0,1]$ such that (4.47) is fulfilled. Then there exist some constants $C_1>1$ and $C_2>0$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\|u-u_{N+1}\|_\Omega^2 \le \left(1-\frac{\delta\theta^2}{2C_1(1+\delta)}\right)\|u-u_N\|_\Omega^2 + \frac{C_2\delta\tau^2}{2}\eta\left(u_N,\mathcal{K}_N\right)^2 - \frac{1}{2}\|u_{N+1}-u_N\|_\Omega^2$$

for all $\delta>0$.

Proof. As in the proof of Lemma 3.6 we obtain

$$\|u-u_N\|_\Omega^2 = \|u-u_{N+1}\|_\Omega^2 + \left(\frac{1}{2}+\frac{1}{2}\right)\|u_{N+1}-u_N\|_\Omega^2$$

and by using Lemma 4.14 it follows

$$\left(1-\frac{\delta\theta^2}{2C_1(1+\delta)}\right)\|u-u_N\|_\Omega^2 \ge \|u-u_{N+1}\|_\Omega^2 + \frac{1}{2}\|u_{N+1}-u_N\|_\Omega^2 - \frac{C_2\delta\tau^2}{2}\eta\left(u_N,\mathcal{K}_N\right)^2.$$

This concludes the proof □

Remark 4.1. *We see easily that the constants C_1 and C_2 in Lemmas 4.14 and 4.15 are the same.*

Now we come to the first main result of this section. It states that the energy error $\|u - u_{\mathrm{FE}}\|_\Omega$ is reduced in every refinement step of the fully automatic *hp*-adaptive refinement algorithm from Section 4.3.

Theorem 4.3 (Convergence)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)$ be the solution of (4.11). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0, 1]$. Further let $u_N \in W^p(\mathcal{K}_N, \Omega)$ and $u_{N+1} \in W^p(\mathcal{K}_{N+1}, \Omega)$ be the solutions of (4.12) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some $\tau \in (0, 1]$ sufficiently small (depending on polynomial degree vector p) such that (4.47) is fulfilled. Then there exists some $\mu \in (0, 1)$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\|u - u_{N+1}\|_\Omega \leq \mu \|u - u_N\|_\Omega .$$

Proof. From Theorem 4.1 we know

$$\eta(u_N, \mathcal{K}_N)^2 \leq C_{\mathrm{eff}} \max\{p_\alpha - 1, p_\beta, 1\}^4 \sum_{K \in \mathcal{K}_N} \left((p_K + 1)^{\frac{3+\varepsilon}{2}} \|u - u_N\|_{\omega_K}^2 + \frac{h_K^2}{(p_K + 1)^{\frac{3-\varepsilon}{2}}} \|f - \Pi f\|_{L^2(\omega_K)^3}^2 \right)$$

for $\varepsilon \in (0, 3]$. Now let us define the covering constant $C_{\mathrm{cov}} > 0$ by

$$C_{\mathrm{cov}} := \max_{K \in \mathcal{K}_N} |\{L \in \mathcal{K}_N : L \subset \omega_K\}| .$$

Then the (γ_1, γ_2)-regularity of $\mathcal{K}_N$ implies

$$\eta(u_N, \mathcal{K}_N)^2 \leq T \left(\max_{K \in \mathcal{K}_N} (p_K + 1) \|u - u_N\|_\Omega^2 + C \sum_{K \in \mathcal{K}_N} \frac{h_K^2}{(p_K + 1)^2} \|f - \Pi f\|_{L^2(K)^3}^2 \right)$$

for some constant

$$C > \frac{2}{\theta^2} \geq 2,$$

which is independent of mesh size vector h and polynomial degree vector p. Here the term T is given by

$$T := C_{\mathrm{cov}} C_{\mathrm{eff}} \max\{p_\alpha - 1, p_\beta, 1\}^4 \max_{K \in \mathcal{K}_N} (p_K + 1)^{\frac{1+\varepsilon}{2}} .$$

With data saturation assumption (4.47) this reads

$$\eta(u_N, \mathcal{K}_N)^2 \leq T \left(\max_{K \in \mathcal{K}_N} (p_K + 1) \|u - u_N\|_\Omega^2 + C\tau^2 \eta(u_N, \mathcal{K}_N)^2 \right).$$

Hence, for $\tau < \frac{1}{\sqrt{CT}}$ we have

$$\eta(u_N, \mathcal{K}_N)^2 \leq \frac{T \max_{K \in \mathcal{K}_N} (p_K + 1)}{1 - CT\tau^2} \|u - u_N\|_\Omega^2$$

and with the even more restrictive assumption $\tau \leq \frac{1}{\sqrt{2CT}}$ we get

$$\eta(u_N, \mathcal{K}_N)^2 \leq 2T \max_{K \in \mathcal{K}_N} (p_K + 1) \|u - u_N\|_\Omega^2 . \tag{4.67}$$

From Lemma 4.15 we know

$$\|u - u_{N+1}\|_\Omega^2 \leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right) \|u - u_N\|_\Omega^2 + \frac{C_2\delta\tau^2}{2} \eta(u_N, \mathcal{K}_N)^2 - \frac{1}{2} \|u_{N+1} - u_N\|_\Omega^2$$

for $\delta > 0$ and using Lemma 4.14 yields

$$\begin{aligned}\|u - u_{N+1}\|_\Omega^2 &\le \left(1 - \frac{\delta\theta^2}{C_1(1+\delta)}\right)\|u - u_N\|_\Omega^2 + C_2\delta\tau^2\eta\left(u_N, \mathcal{K}_N\right)^2 \\ &\le \left(1 + 2C_2T \max_{K\in\mathcal{K}_N}\left(p_K+1\right)\delta\tau^2 - \frac{\delta\theta^2}{C_1(1+\delta)}\right)\|u - u_N\|_\Omega^2\end{aligned}$$

by (4.67). For

$$\tau \le \frac{1}{\sqrt{2CC_1C_2T\max_{K\in\mathcal{K}}\left(p_K+1\right)}}$$

we obtain

$$\|u - u_{N+1}\|_\Omega^2 \le \left(1 + \frac{\delta}{C_1}\left(\frac{1}{C} - \frac{\theta^2}{1+\delta}\right)\right)\|u - u_N\|_\Omega^2$$

and the result follows for $\delta < \frac{C\theta^2}{2} - 1$. □

The second main result of this paragraph is another convergence result, which states that the weighted sum of energy error $\|u - u_{\mathrm{FE}}\|_\Omega$ and estimated error $\eta\left(u_N, \mathcal{K}_N\right)$ is reduced in every iteration of the fully automatic hp-adaptive refinement algorithm from Section 4.3. The proof follows the ideas of Bonito and Nochetto [55].

Theorem 4.4 (Quasi-Convergence)**.** *Let $N \in \mathbb{N}_0$ be arbitrary and $u \in H_0(\mathrm{curl}, \Omega) \cap H^1(\Omega)^3$ be the solution of (4.11). We assume that there exists a solution of (3.7), (3.8) for some $\theta \in (0,1]$. Further let $u_N \in W^p\left(\mathcal{K}_N, \Omega\right)$ and $u_{N+1} \in W^p\left(\mathcal{K}_{N+1}, \Omega\right)$ be the solutions of (4.12) in iteration steps N and $N+1$, respectively. Additionally let us assume that there exists some constant $\rho \in (0,1)$ independent of mesh size vector h and polynomial degree vector p such that assumption (4.46) holds. We assume that there exists some $\tau \in (0,1]$ sufficiently small such that (4.47) is fulfilled. Then there exists some constant $\mu \in (0,1)$ independent of mesh size vector h, polynomial degree vector p, p_α and p_β such that*

$$\|u - u_{N+1}\|_\Omega^2 + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \le \mu\left(\|u - u_N\|_\Omega^2 + \nu\eta\left(u_N, \mathcal{K}_N\right)^2\right)$$

for some $\nu > 0$ sufficiently small (depending on polynomial degree vector p).

Proof. From Lemma 4.15 we know

$$\begin{aligned}&\|u - u_{N+1}\|_\Omega^2 + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\ &\quad\le \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right)\|u - u_N\|_\Omega^2 + \frac{C_2\delta\tau^2}{2}\eta\left(u_N, \mathcal{K}_N\right)^2 + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 - \frac{1}{2}\|u_{N+1} - u_N\|_\Omega^2\end{aligned}$$

for $\delta > 0$ and applying Lemma 4.13 yields

(4.68)

$$\begin{aligned}&\|u - u_{N+1}\|_\Omega^2 + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\ &\quad\le \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right)\|u - u_N\|_\Omega^2 + \left(\frac{C_2\delta\tau^2}{2} + \nu\left(1 + 3\delta + \left(1 + \delta + \frac{2}{\delta}\right)\rho^2\tau^2\right)\right)\eta\left(u_N, \mathcal{K}_N\right)^2 \\ &\qquad - \nu(1+3\delta)(1-\rho)\eta\left(u_N, \mathcal{A}_N\right)^2\end{aligned}$$

by choosing

$$\nu \le \frac{\delta}{2C_{\mathrm{red}}\left(2 + \delta + \delta^2\right)\max\left\{p_\alpha - 1, p_\beta, 1\right\}^2 \max_{K\in\mathcal{K}_N}\left(p_K+1\right)^2}.$$

We observe

$$C_{\max}^2 \eta\left(u_N, \mathcal{A}_N\right)^2 \geq \sum_{K \in \mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N, \mathcal{K}_N\right)^2,$$

where

$$C_{\max} := \max\left\{\max_{K \in \mathcal{A}_N}\left(\kappa_{K,j_K}\right), 1\right\}.$$

Then (4.68) reads

$$\begin{aligned}
&\|u - u_{N+1}\|_\Omega^2 + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \\
&\quad \leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right)\|u - u_N\|_\Omega^2 + \left(\frac{C_2\delta\tau^2}{2} + \nu\left(1 + 3\delta + \left(1 + \delta + \frac{2}{\delta}\right)\rho^2\tau^2\right)\right)\eta\left(u_N, \mathcal{K}_N\right)^2 \\
&\qquad\qquad - \frac{\nu(1+3\delta)(1-\rho)}{C_{\max}^2}\sum_{K \in \mathcal{A}_N} \kappa_{K,j_K}^2 \eta_K\left(u_N, \mathcal{K}_N\right)^2,
\end{aligned}$$

since $\rho < 1$. Finally constraint (3.8) implies

$$\|u - u_{N+1}\|_\Omega^2 + \nu\eta\left(u_{N+1}, \mathcal{K}_{N+1}\right)^2 \leq \left(1 - \frac{\delta\theta^2}{2C_1(1+\delta)}\right)\|u - u_N\|_\Omega^2 + T\eta\left(u_N, \mathcal{K}_N\right)^2,$$

where the term T is given by

$$T := \frac{C_2\delta\tau^2}{2} + \nu\left(1 + 3\delta + \left(1 + \delta + \frac{2}{\delta}\right)\rho^2\tau^2\right) - \frac{\nu\theta^2(1+3\delta)(1-\rho)}{C_{\max}^2}.$$

For $\tau \to 0$ we have

$$T \to \nu(1+3\delta)\left(1 - \frac{\theta^2(1-\rho)}{C_{\max}^2}\right)$$

and, thus, the result follows for δ and τ sufficiently small. $\square$

To conclude this section let us shortly discuss the various assumptions we made above and see how these affect the main results of this section. Most of the assumptions were already discussed in Sections 3.1.2.2 and 3.1.3.2. Therefore we restrict ourselves to those assumptions, which changed fundamentally or appeared newly.

In Theorem 4.3 the parameter $\tau \in (0, 1]$ from data saturation assumption (4.47) depends on the polynomial degree vector p. This is due to the fact that the a posteriori error estimator from Section 4.2 is not uniform in p. Further τ also depends on p_α and p_β. In the proof of Theorem 4.3 we obtained the explicit upper bound

$$\tau \leq \frac{1}{\sqrt{2CC_1C_2C_{\mathrm{cov}}C_{\mathrm{eff}}}\max\left\{p_\alpha - 1, p_\beta, 1\right\}^2 \max_{K \in \mathcal{K}}\left(p_K + 1\right)^{1+\frac{\varepsilon}{2}}}$$

for $\varepsilon \in (0, 3]$.

In [72], Theorem 3, one can find a slightly different proof of convergence for the fully automatic *hp*-adaptive refinement strategy from Section 4.3. Here the assumptions on parameter τ from data saturation assumption (4.47) are more or less the same, but for θ the lower bound and constraint (3.8) also depend on polynomial degree vector p. Thus, Theorem 4.3 can be seen as a slight generalization of the results from [72].

In Theorem 4.4 the constant $\nu > 0$ depends on the polynomial degree vector p. This is due to the fact that the inverse estimates from Lemmas 3.8 and 3.9 are not optimal in p. Further ν also depends on p_α and p_β. In the proof of Theorem 4.4 we obtained the explicit upper bound

$$\nu \leq \frac{\delta}{2C_{\mathrm{red}}\left(2+\delta+\delta^2\right)\max\left\{p_\alpha - 1, p_\beta, 1\right\}^2 \max\limits_{K\in\mathcal{K}_N}\left(p_K+1\right)^2}.$$

A way to eliminate the dependence of ν on the polynomial degree vector p might be the use of an equilibrated residual error estimator as proposed by Braess and Schöberl in [61, 62].

4.5 Numerical Results

Now we want to consider the performance of the fully automatic hp-adaptive refinement strategy from Section 4.3 on the basis of some numerical examples. Therefore we consider some academic and also some more realistic examples of the form (4.8). All computations are performed with the finite element library deal.II [41, 42]. The linear system of equations have been solved by a rather traditional approach using the conjugate gradient method with SSOR preconditioning.

Example 1

In our first example we consider an academic problem with a smooth solution. Let $\Omega := (0,1)^3$, $\alpha := I$ and $\beta \in \left\{10^{-4}, 10^{-2}, 1, 10^2, 10^4\right\} I$. The solution $u : \overline{\Omega} \to \mathbb{R}^3$ is given by

$$u(x) := \begin{pmatrix} 0 \\ 0 \\ \sin\left(\pi x_1\right) \end{pmatrix}. \tag{4.69}$$

The initial triangulation $\mathcal{K}_0$ consists of 8 equally-sized cells and as initial polynomial degree vector we choose $p = 0$. Further we set $\theta := 0.8$.
In this example we perform two different runs of the algorithm. In the first run we provide only two different refinement patterns the algorithm can choose from. The first refinement pattern is classical h-refinement and the second refinement pattern is classical p-refinement, where the polynomial degree of the cell is increased by one. In Figure 4.1 we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. We observe that there is not much difference in the behaviour of the a posteriori error estimator from Section 4.2 for different values of β. Thus we can expect some robustness of the estimator with respect to β, if $f \sim \beta$. This is an important feature for an a posteriori error estimator for Maxwell's equations, because in time-dependent problems β is scaled by the length of the time-step. But the time-step size should not effect the performance of the error estimator too much. In Table 4.1 the marking history of the refinement algorithm is shown. We observe

Step	#Cells	max(p)	$\#h$	$\#p$	Step	#Cells	max(p)	$\#h$	$\#p$
0	8	0	0	8	4	8	4	0	8
1	8	1	0	8	5	8	5	0	8
2	8	2	0	8	6	8	6	0	8
3	8	3	0	8	7	8	7	0	8

Table 4.1: Example 1: Marking history for $\beta \in \left\{10^{-4}, 10^{-2}, 1, 10^2, 10^4\right\} I$.

that for all β the hp-adaptive refinement strategy from Section 4.3 chooses p-refinement on all cells. This

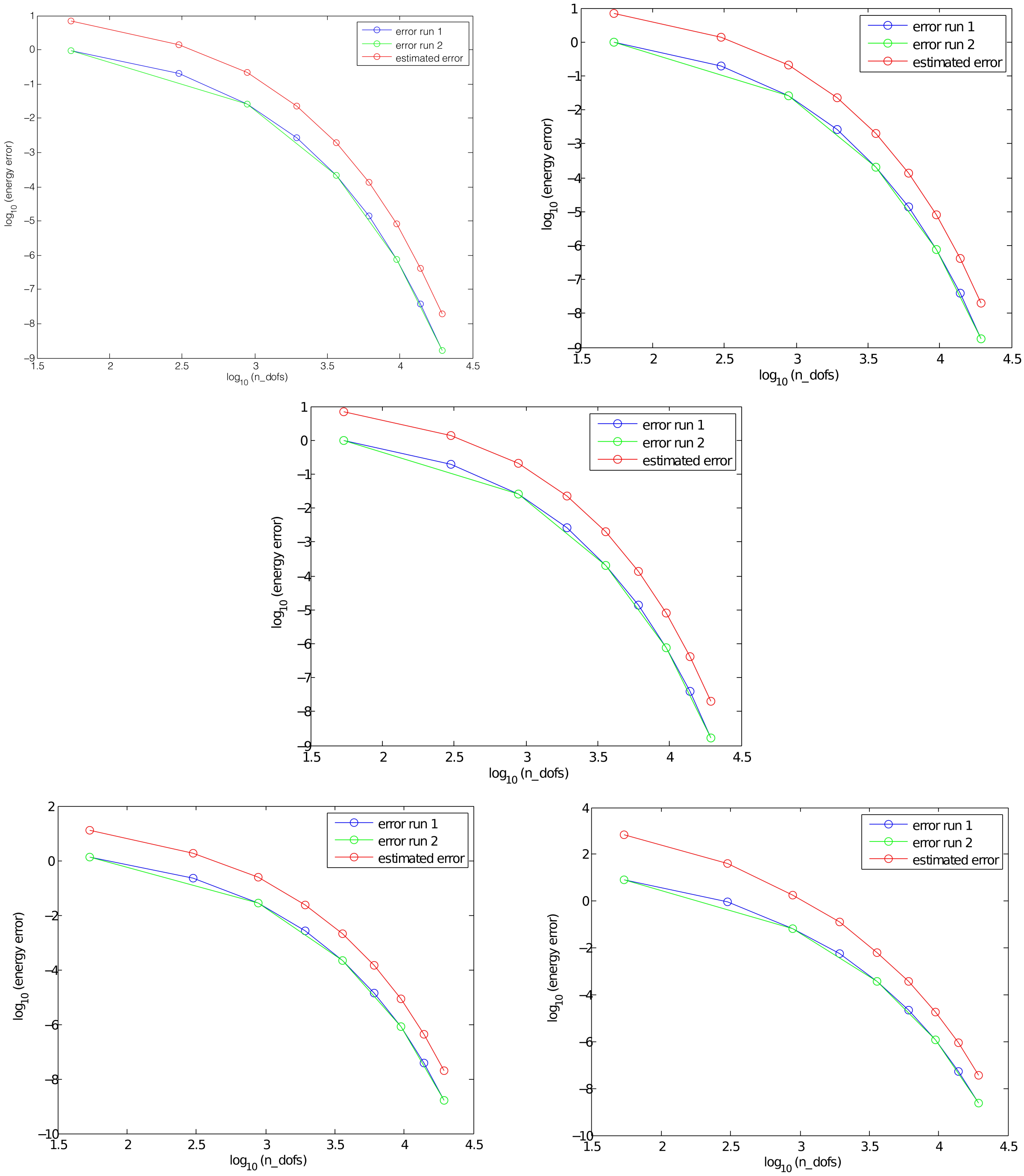

Figure 4.1: Example 1: Number of degrees of freedom vs. error. Upper left: $\beta = 10^{-4} I$. Upper right: $\beta = 10^{-2} I$. Center: $\beta = I$. Lower left: $\beta = 10^{2} I$. Lower right: $\beta = 10^{4} I$.

is basically what we expect, because the exact solution u is smooth and there are no local features to detect. Thus p-refinement performs best.

In a second run we add a third refinement pattern to the refinement algorithm. Now the strategy can additonally choose to increase the polynomial degree by two. In Figure 4.1 we can see that the algorithm really takes advantage of this new refinement pattern and reaches the same accuracy as in run 1 in only half the number of refinement steps.

Example 2

In this example we change the role of the coefficients. Now

$$\alpha(x) := (\sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi x_3) + 1.5)\, I$$

is varying and $\beta := I$ is kept constant. Further let $\Omega := (0,1)^3$ and u be given by (4.69) again. Also the initial triangulation $\mathcal{K}_0$ and the initial polynomial degree vector $p = 0$ are the same as in Example 1. We set $\theta := 0.75$

As in Example 1 we perform two different runs of the algorithm. Let us begin with the first run. As before the algorithm can choose from classical h- and p-refinement again. In Figure 4.2 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. Also in this situation the a posteriori error estimator seems to perform quite well. The

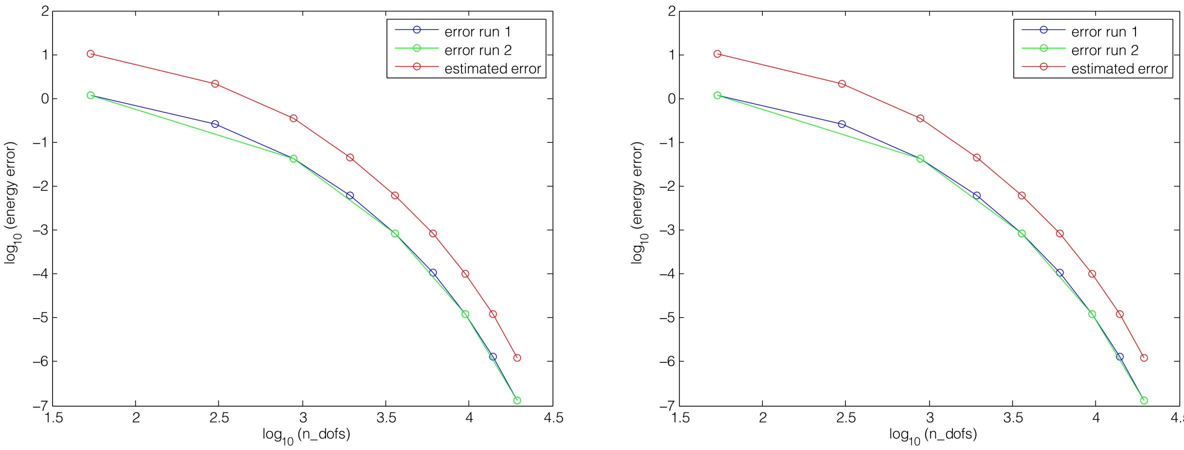

Figure 4.2: Number of degrees of freedom vs. error. Left: Example 2. Right: Example 3.

estimated error approaches the exact energy error as the number of degrees of freedom increases. In Table 4.2 the marking history of the refinement algorithm from Section 4.3 is shown. We observe that

Step	#Cells	max(p)	#h	#p	Step	#Cells	max(p)	#h	#p
0	8	0	0	8	4	8	4	0	8
1	8	1	0	8	5	8	5	0	8
2	8	2	0	8	6	8	6	0	8
3	8	3	0	8	7	8	7	0	8

Table 4.2: Example 2: Marking history.

the hp-adaptive refinement strategy chooses p-refinement only. This is basically what we expect, because this strategy already performed best in Example 1.

As before the refinement algorithm can choose from three refinement possibilities – bisection in every coordinate direction, increase the polynomial degree by one and increase the polynomial degree by two – in the second run. In Figure 4.2 on the left-hand side we can see that similar to Example 1 the fully automatic *hp*-adaptive refinement strategy from Section 4.3 takes advantage of the new refinement possibility and reaches the same accuracy as in the first run in only half the number of refinement steps. This also pays out in the total computation time. Whereas the first run took 11:12 minutes the second one took only 5:01 minutes on one node with 24 cores.

Example 3

In this experiment we consider a more realistic configuration than in the previous examples. Let $\Omega := (0,1)^3$ and $\alpha := I$. We choose

$$\beta(x) := \begin{cases} I, & \text{if } \max\limits_{i\in\{1,2,3\}} |x_i - 0.5| \leq 0.25 \\ 0, & \text{else} \end{cases}$$

to be discontinuous. This is a common situation in realistic applications, where we have a conducting region ($\beta = I$) and an outer space ($\beta = 0$). However β does not fit into our analytical setting, because it is not uniformly positive definite. To overcome this difficulty we replace β by some cut-off function $\chi : \Omega \to \mathbb{R}^3$ given by

$$\chi(x) := \begin{cases} \delta I, & \text{if } \beta(x) < \delta \\ \beta(x), & \text{else} \end{cases}$$

for some $\delta > 0$ with $\delta \ll 1$. With this modification we are back in our analytical background, since χ is uniformly positive definite. The exact solution u is given by (4.69) again. The initial triangulation $\mathcal{K}_0$ consists of 8 equally-sized cells and, hence, does not resolve the geometry exactly. Thus β and χ, respectively, are discontinuous inside the cells and we have to use high-order quadrature rules (i.e. order $p_K + 12$) to approximate the integrals sufficiently accurate. As initial polynomial degree vector we choose $p = 0$ as usual. We set $\theta := 0.9$ and $\delta := 10^{-10}$.

Since the exact solution u is smooth, we perform two different runs of the algorithm again. As in Example 1 we provide two different refinement patterns the fully automatic *hp*-adaptive refinement strategy can choose from in the first run. The first one is classical *h*-refinement and the second one is classical *p*-refinement, where the polynomial degree is increased by one. In Figure 4.2 on the right-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. We observe that the a posteriori error esimator from Section 4.2 also handles this more realistic setting very well. In Table 4.3 the marking history of the refinement algorithm from Section 4.3 is shown. We

Step	#Cells	max(p)	$\#h$	$\#p$	Step	#Cells	max(p)	$\#h$	$\#p$
0	8	0	0	8	4	8	4	0	8
1	8	1	0	8	5	8	5	0	8
2	8	2	0	8	6	8	6	0	8
3	8	3	0	8	7	8	7	0	8

Table 4.3: Example 3: Marking history.

see that the *hp*-adaptive refinement stratagy chooses *p*-refinement only. This is basically what we expect, because the solution u is smooth and this strategy already performed best in Examples 1 and 2.

Now let us consider the second run. Here the algorithm can choose from three different refinement possibilities again. In addition to the two refinement possibilities from the first run we also provide the possibility to increase the polynomial degree by two. We expect that – as in Examples 1 and 2 – the fully automatic hp-adaptive refinement strategy recognizes the smoothness of the solution and takes advantage of the new refinement possiblity. In Figure 4.2 on the right-hand side we can see that this is indeed the case and the algorithm reaches the same accuracy as in run 1 in only half the number of refinement steps.

Example 4

In this example we choose almost the same setting as in Example 3. Let Ω, α, β and χ be as above. However this time we set $f := 1$ to obtain a singular, but unknown, solution u. This example was already considered in [50], where it turned out that even with the use of an h-adaptive finite element method it is very difficult to obtain an accurate numerical approximation of u. Here we want to improve the accuracy by using the fully automatic hp-adaptive refinement strategy from Section 4.3. We start with an intial triangulation $\mathcal{K}_0$ consisting of 64 equally-sized cells and initial polynomial degree vector $p = 0$. Further we set $\theta := 0.275$ and $\delta := 10^{-10}$. The algorithm can choose from classical h- and p-refinement again. In Figure 4.3 we plot the number of degrees of freedom vs. the estimated error in $\log_{10}$-$\log_{10}$-scale. The

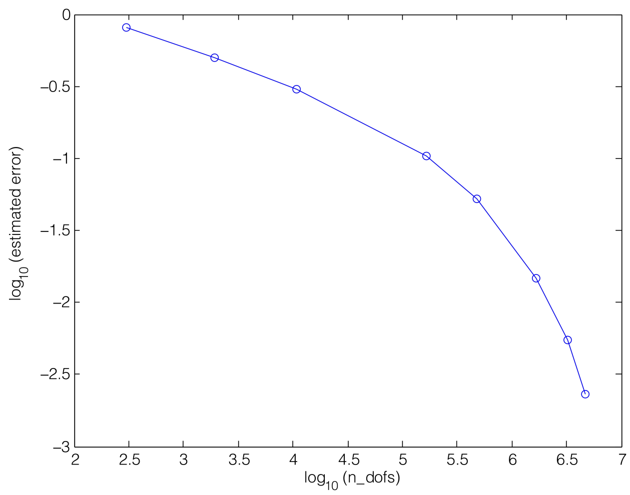

Figure 4.3: Example 4: Number of degrees of freedom vs. estimated error.

computed finite element solution is shown in Figure 4.4 and in Figure 4.5 one can see the final grid. To improve visibility we have grouped the cells by its polynomial degree. In Table 4.4 the marking history of the algorithm is shown. We observe that the fully automatic hp-adaptive refinement strategy from

Step	#Cells	max(p)	$\#h$	$\#p$	Step	#Cells	max(p)	$\#h$	$\#p$
0	64	0	64	0	4	8232	2	4056	32
1	512	0	256	64	5	38812	3	15128	725
2	2304	1	988	156	6	136436	3	57112	3703
3	8064	2	36	296					

Table 4.4: Example 4: Marking history.

Section 4.3 captures the edge singularities quite well and performs h-refinement around the edges of the nonconducting region only.

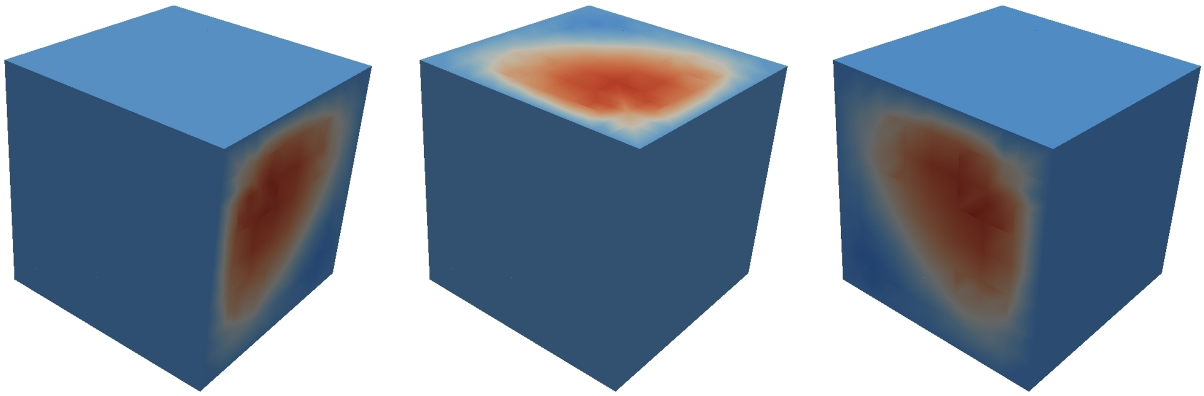

Figure 4.4: Example 4: Computed solution. Left: x_1-component. Center: x_2-component. Right: x_3-component.

Example 5

Now let us consider a classical academic problem with a singular solution. Let $\alpha := \beta := I$ and $\Omega := (-1,1)^3 \setminus ([0,1) \times (-1,0] \times (-1,1))$. The analytic solution $u : \overline{\Omega} \to \mathbb{R}^3$ is given by

$$u\left(r, \phi, x_3\right) := \frac{2}{3} r^{-\frac{1}{3}} \begin{pmatrix} -\sin\left(\frac{\phi}{3}\right) \\ \cos\left(\frac{\phi}{3}\right) \\ 0 \end{pmatrix},$$

where $r \in \mathbb{R}_+$ and $\phi \in [0, 2\pi)$ denote the polar coordinates. The intial triangulation $\mathcal{K}_0$ consists of 48 equally-sized cells and as inital polynomial degree vector we choose $p = 0$. Further we set $\theta := 0.2$. As in Example 4 the algorithm can choose from two refinement possibilities – classical h- and classical p-refinement. In Figure 4.6 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. We observe that the a posteriori error estimator from Section 4.2 yields quite good results and behaves exactly like the exact error as refinement proceeds. On the right-hand side of Figure 4.6 we can see an (x_1, x_2)-cut of the final grid produced by the fully automatic hp-adaptive refinement strategy from Section 4.3. We observe that the final grid basically has a linear structure towards the singularity located at the edge $(0, 0, x_3)$. This means that around this edge cells are small and polynomial degrees are low. The more one goes away from the singularity the larger are the cells and the higher are the polynomial degrees. The marking history of the algorithm is shown in Table 4.5. Also in this example we get more or less the result which one expects from the hp-adaptive

Step	#Cells	max(p)	#h	#p	Step	#Cells	max(p)	#h	#p
0	48	0	4	0	4	3408	2	62	232
1	272	0	8	8	5	6992	3	128	672
2	720	1	15	24	6	14160	4	255	73
3	1616	2	32	126					

Table 4.5: Example 5: Marking history.

refinement algorithm. The singularity is identified correctly and the refinement choices are appropriate.

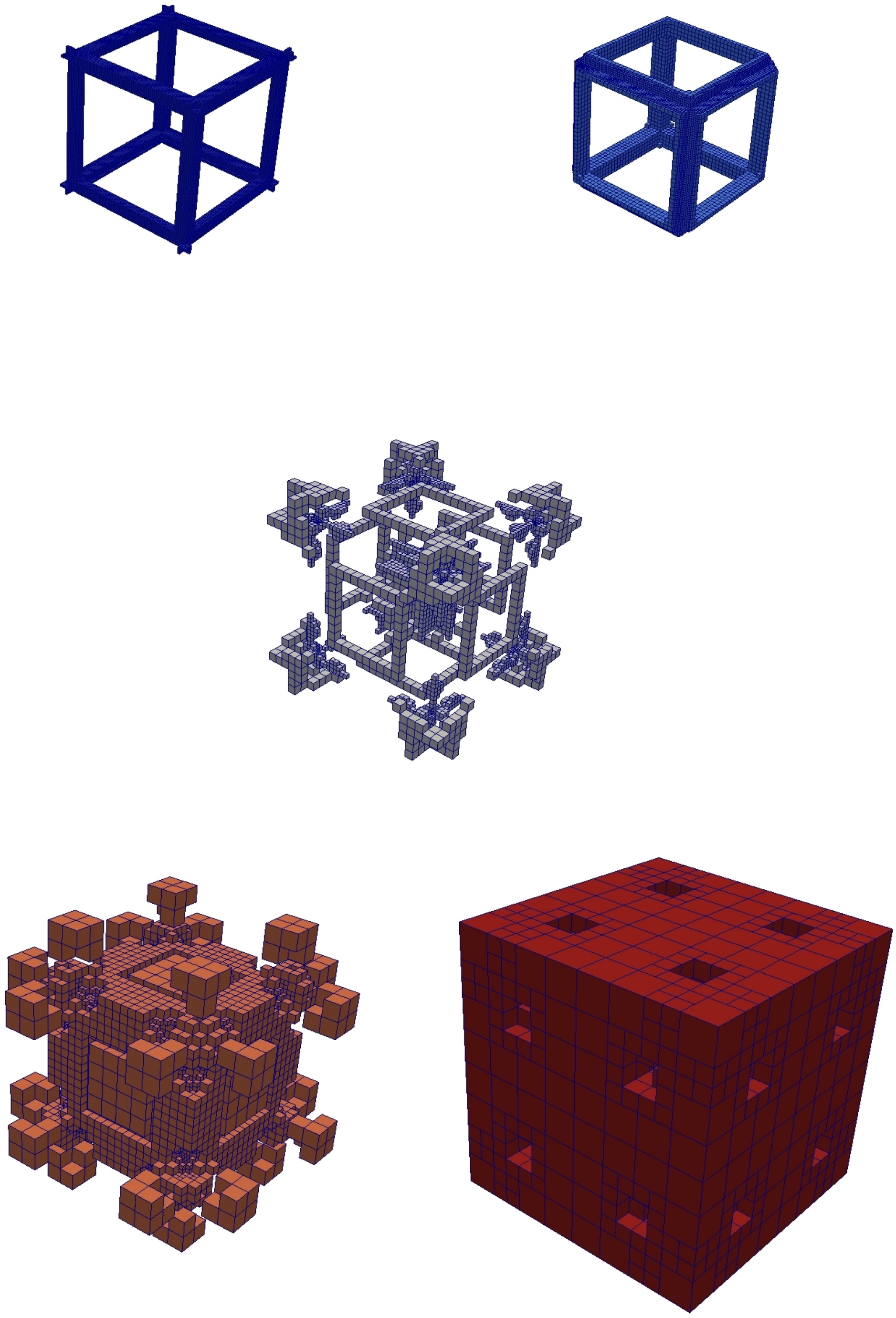

Figure 4.5: Example 4: Final grid. Upper left: $p = 0$. Upper right: $p = 1$. Center: $p = 2$. Lower left: $p = 3$. Lower right: $p = 4$.

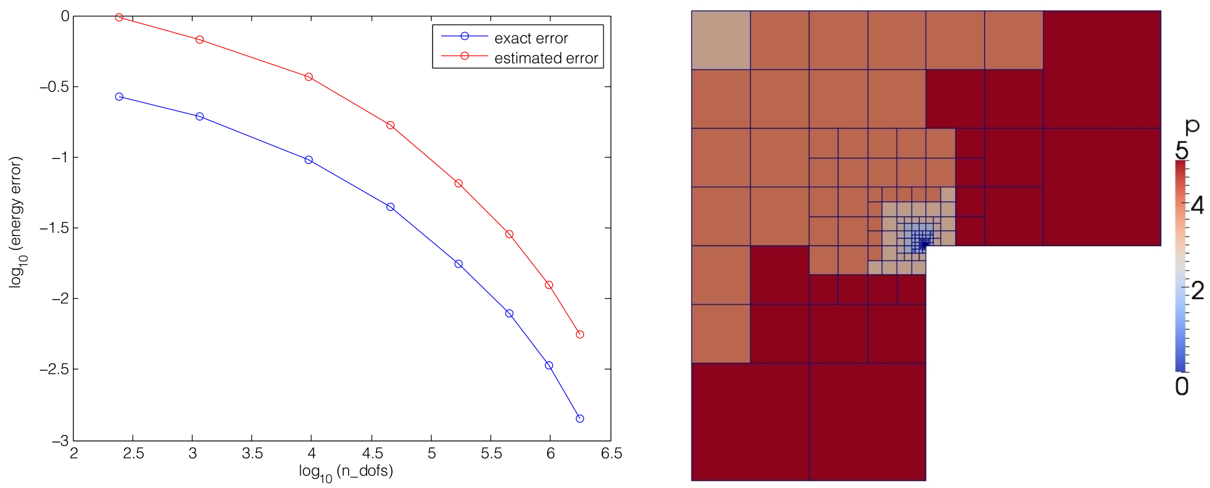

Figure 4.6: Example 5: Left: Number of degrees of freedom vs. error. Right: Final grid.

Example 6

In this example we consider a problem from [79]. Let $\Omega := (-1,1)^3$, $\alpha := I$ and

$$\beta(x) := \begin{cases} 5.82842712474619 07I, & \text{if } x \in \left((-1,0]^2 \cup [0,1)^2\right) \times (-1,1) \\ I, & \text{else} \end{cases}.$$

The analytic solution $u : \overline{\Omega} \to \mathbb{R}^3$ is given by

$$u\left(r, \phi, x_3\right) := \nabla\left(r^{\frac{1}{2}} \widetilde{u}(\phi)\right),$$

where $r \in \mathbb{R}_+$ and $\phi \in [0, 2\pi)$ denote the polar coordinates and $\widetilde{u} : [0, 2\pi) \to \mathbb{R}$ is given by

$$\widetilde{u}(\phi) := \begin{cases} \cos\left(\frac{\pi}{4} + 1.1780972450961724\right)\cos\left(\frac{\phi}{2} - \frac{\pi}{8}\right), & \text{if } \phi \in \left[0, \frac{\pi}{2}\right] \\ \cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\phi}{2} - \frac{\pi}{2} - 1.1780972450961724\right), & \text{if } \phi \in \left(\frac{\pi}{2}, \pi\right] \\ \cos(1.1780972450961724)\cos\left(\frac{\phi}{2} - \frac{5}{8}\pi\right), & \text{if } \phi \in \left(\pi, \frac{3}{2}\pi\right] \\ \cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\phi}{2} + 1.1780972450961724 - \frac{3}{4}\pi\right), & \text{else} \end{cases}.$$

Since u has a strong singularity along the edge $(0,0,x_3)$, we have to use high-order quadrature rules to compute the exact energy error $\|u - u_{\mathrm{FE}}\|_\Omega$ accurate enough. The initial triangulation $\mathcal{K}_0$ consists of 8 equally-sized cells and as initial polynomial degree vector we choose $p = 0$. As before we run our fully automatic *hp*-adaptive refinement algorithm with the two different refinement possibilities bisection in every coordinate direction and increase of the polynomial degree by one. In Figure 4.7 on the left-hand side we plot the number of degrees of freedom vs. the exact energy error and the estimated error in $\log_{10}$-$\log_{10}$-scale. We observe that the a posteriori error estimator from Section 4.2 also handles this difficult setting very well. On the right-hand side of Figure 4.7 we show an (x_1, x_2)-cut of the final grid produced by the algorithm. We observe that as in Example 5 the final grid has a linear structure towards the singularity located at $(0,0,x_3)$. The marking history of the fully automatic *hp*-adaptive refinement strategy from Section 4.3 is shown in Table 4.6. Again the refinement algorithm performs very well. The singularity is identified correctly and the refinement strategy performs lots of *h*-refinements around it.

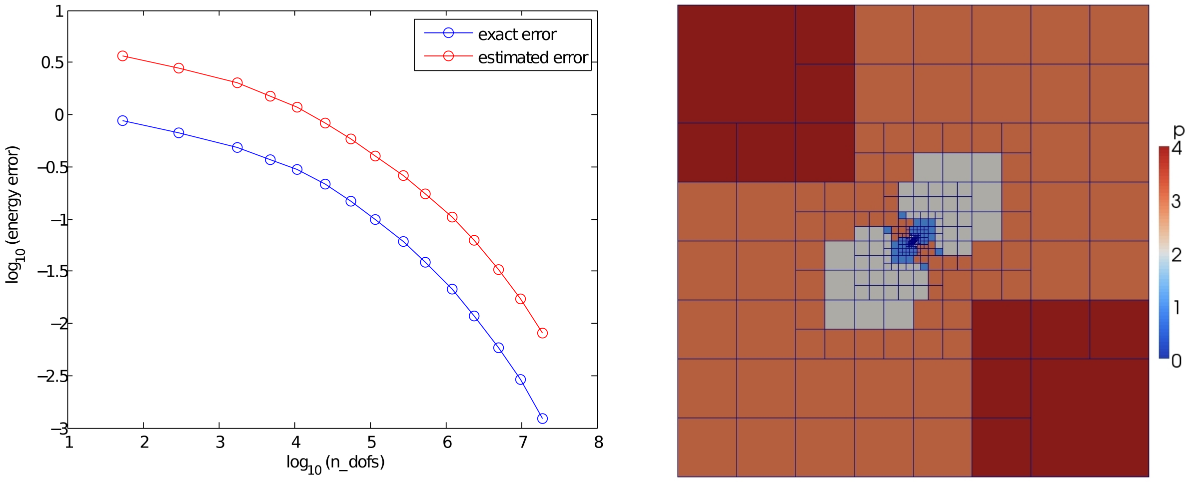

Figure 4.7: Example 6: Left: Number of degrees of freedom vs. error. Right: Final grid.

Step	# Cells	max(p)	$\#h$	$\#p$	# Degrees of Freedom
0	8	0	8	0	54
1	64	0	16	0	300
2	512	0	32	8	1808
3	1408	1	63	0	4916
4	3200	1	128	0	11024
5	6784	1	256	12	25932
6	13952	2	512	2	56336
7	28288	2	1020	0	119328
8	56960	2	2048	45	276528
9	114304	3	4086	4	547136
10	228992	3	8192	10	1212696
11	458386	3	16380	12	2360644
12	917120	3	32768	162	5051384
13	1834624	4	64356	16	9758212
14	3145891	4	129332	45	19034322

Table 4.6: Example 6: Marking history.

Example 7

Now we come to an example with a realistic geometry. We consider a three-dimensional waveguide with 24 reentrant edges. The geometry can be seen in Figure 4.8. A two-dimensional version of a similar problem has been considered in [176, 177]. Let $\alpha := \beta := I$ and $f := 1$. The analytic solution u is unknown. The initial triangulation $\mathcal{K}_0$ consists of 2368 equally-sized cells. As initial polynomial degree vector we choose $p = 0$ and set $\theta := 0.4$. As in Example 6 the hp-adaptive refinement strategy from Section 4.3 can choose from classical h- and p-refinement. In Figure 4.9 we plot the number of degrees of freedom vs. the estimated error in $\log_{10}$-$\log_{10}$-scale. The computed solution is plotted in Figure 4.10 and in Figure 4.11 we show an (x_1, x_2)-cut of the final grid produced by the algorithm. In Table 4.7 the marking history of the fully automatic hp-adaptive refinement strategy can be seen. Basically we observe

Figure 4.8: Example 7: Geometry.

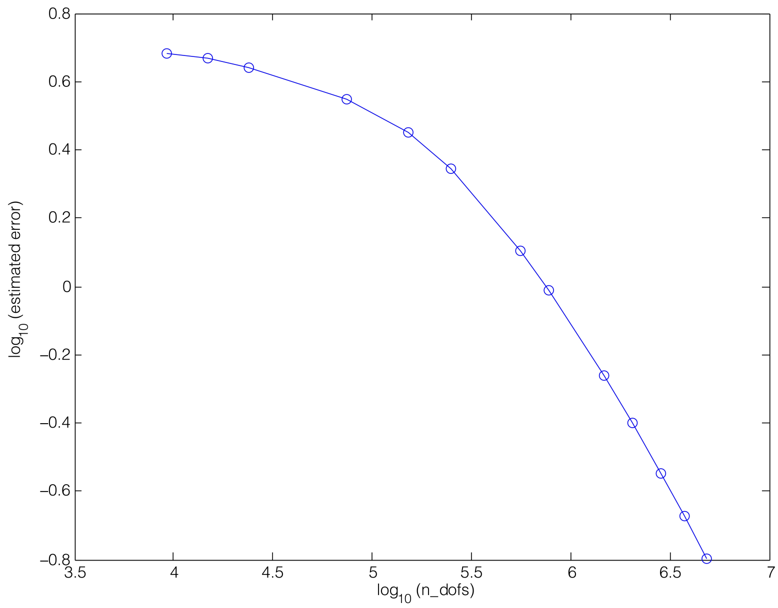

Figure 4.9: Example 7: Number of degrees of freedom vs. estimated error.

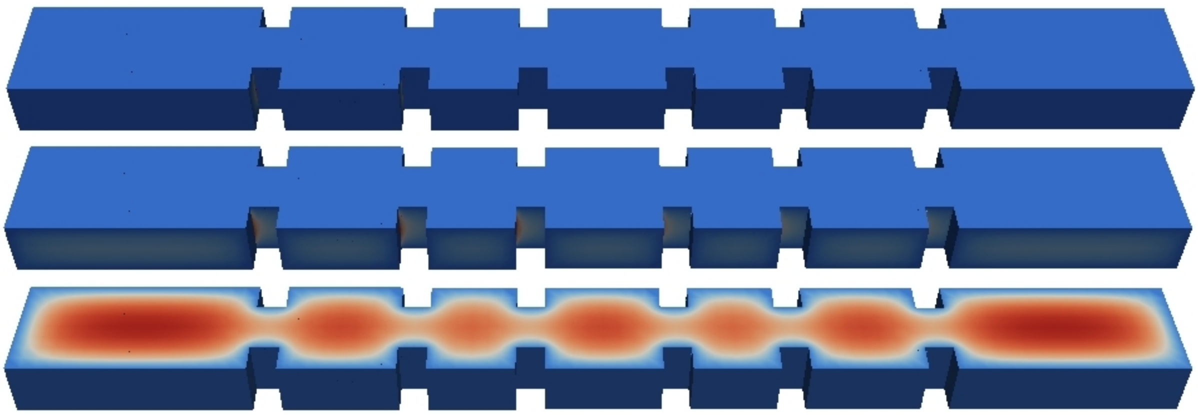

Figure 4.10: Example 7: Computed solution. Upper: x_1-component. Center: x_2-component. Lower: x_3-component.

that the refinement algorithm chooses h-refinement around the reentrant edges and p-refinement else.

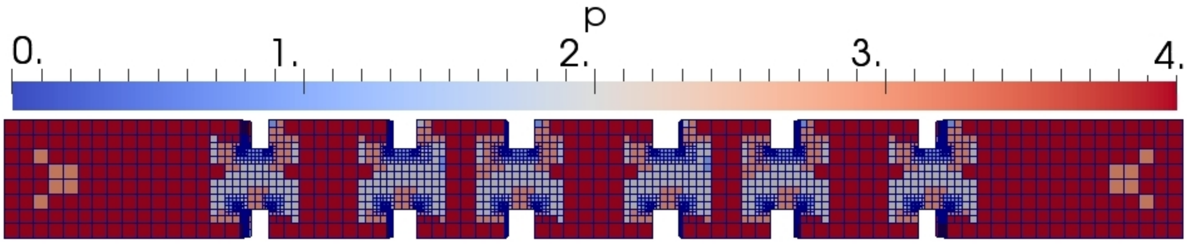

Figure 4.11: Example 7: Final grid.

Step	#Cells	max(p)	$\#h$	$\#p$	Step	#Cells	max(p)	$\#h$	$\#p$
0	2368	0	46	0	6	35930	2	378	823
1	5056	0	72	0	7	42688	2	594	320
2	8437	0	42	183	8	58739	3	924	1294
3	10432	1	193	1012	9	70292	3	1568	1832
4	16932	1	128	345	10	93214	3	14023	12832
5	21184	2	245	404	11	174392	4	3847	5810

Table 4.7: Example 7: Marking history.

Example 8

In the last example we consider a problem from [70]. The geometry is a scaffold structure consisting of a silicon frame and air. Let $\Omega := (0,2)^3$ and $\beta := I$. Further we have $\alpha := \frac{1}{13}I$ in silicon and $\alpha := I$ in air. The distribution of the two materials can be found in Figure 4.12 on the left-hand side. Further we set $f := 1$. The analytic solution u is unknown. The initial triangulation $\mathcal{K}_0$ consists of 4096 cells and as initial polynomial degree vector we choose $p = 0$. We set $\theta := 0.25$. As in the previous examples the refinement algorithm from Section 4.3 can choose from bisection in every coordinate direction and increase the polynomial degree by one. On the right-hand side of Figure 4.12 we plot the number of degrees of freedom vs. the estimated error in $\log_{10}$-$\log_{10}$-scale. The computed finite element solution is shown in Figure 4.13 and in Figure 4.14 one can see the final grid. To improve visibility we have grouped the cells by its polynomial degree. In Table 4.8 the marking history of the algorithm is shown. Also in

Step	#Cells	max(p)	$\#h$	$\#p$	Step	#Cells	max(p)	$\#h$	$\#p$
0	4096	0	123	0	10	40811	2	522	456
1	5860	0	214	0	11	59634	2	674	230
2	7480	0	198	64	12	74355	2	480	873
3	9024	1	134	131	13	87928	3	856	1102
4	10886	1	139	212	14	129298	3	1045	1891
5	12874	1	380	532	15	169107	3	2584	2244
6	17382	1	124	472	16	232730	3	2891	3012
7	19244	2	98	394	17	388375	3	5131	1696
8	20098	2	458	318	18	450619	3	10746	6480
9	26307	2	586	128					

Table 4.8: Example 8: Marking history.

this example we can observe that the major amount of h-refinements is performed at the interfaces of the

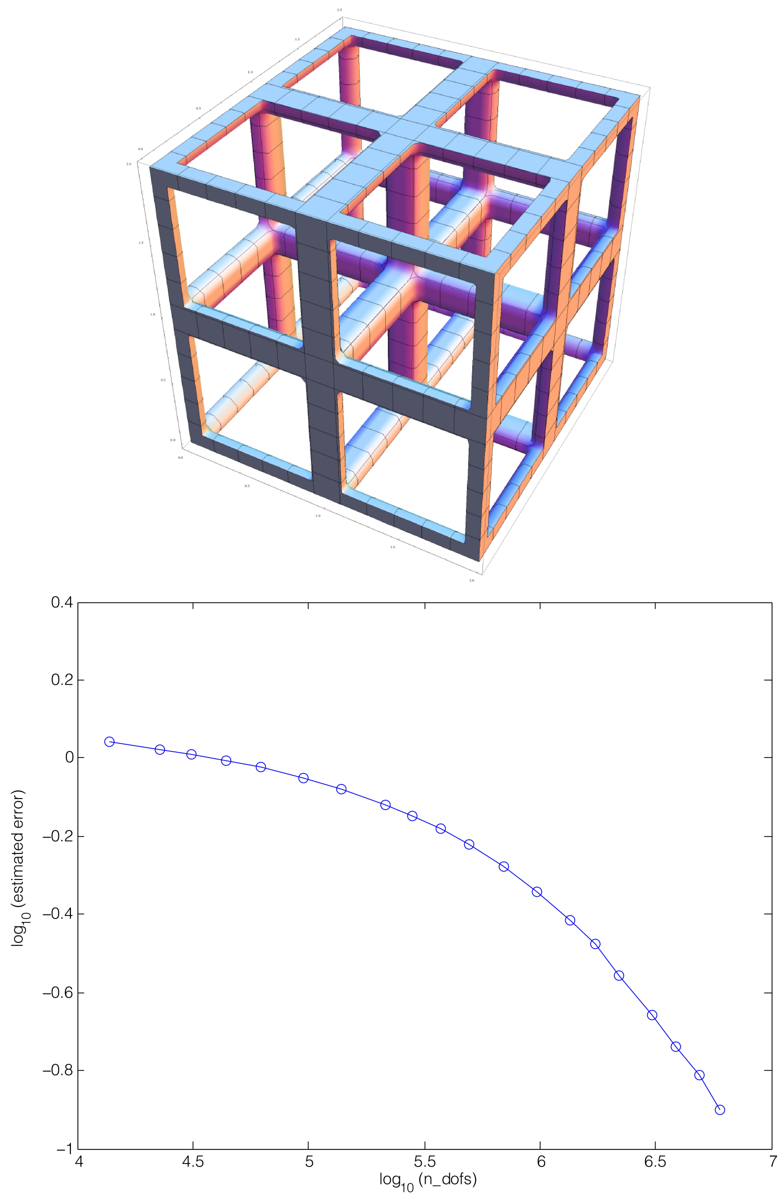

Figure 4.12: Example 8: Left: Geometry. Right: Number of degrees of freedom vs. estimated error.

two materials. Especially in the corners of such interfaces the algorithm chooses h-refinement. Almost all cells with high polynomial degree can be found in the center of those areas, which are occupied by air. Although the distinction is not as clear as in Example 4, we still can see the boundary of the scaffold structure in the group of cells with low polynomial degree. Since we do not know anything about the analytic solution and its singularities, we cannot evaluate the results in more depth here.

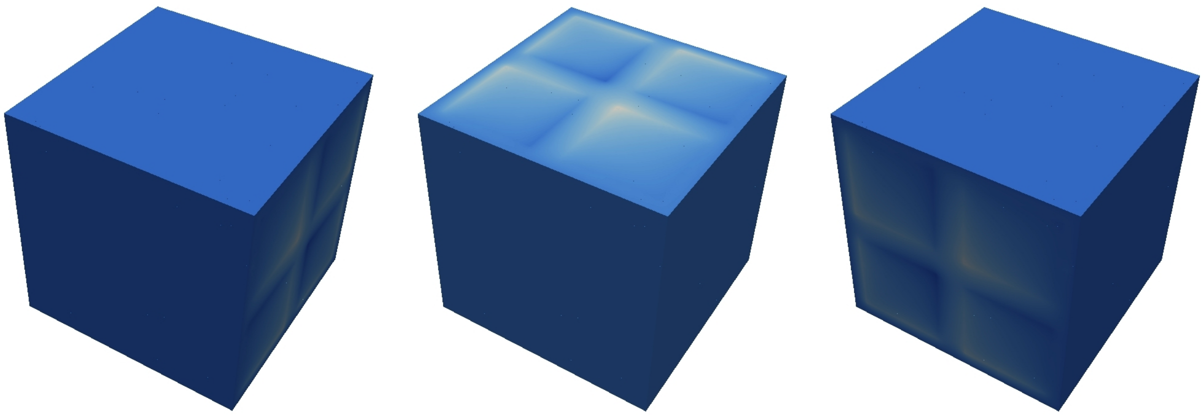

Figure 4.13: Example 8: Computed solution. Left: x_1-component. Center: x_2-component. Right: x_3-component.

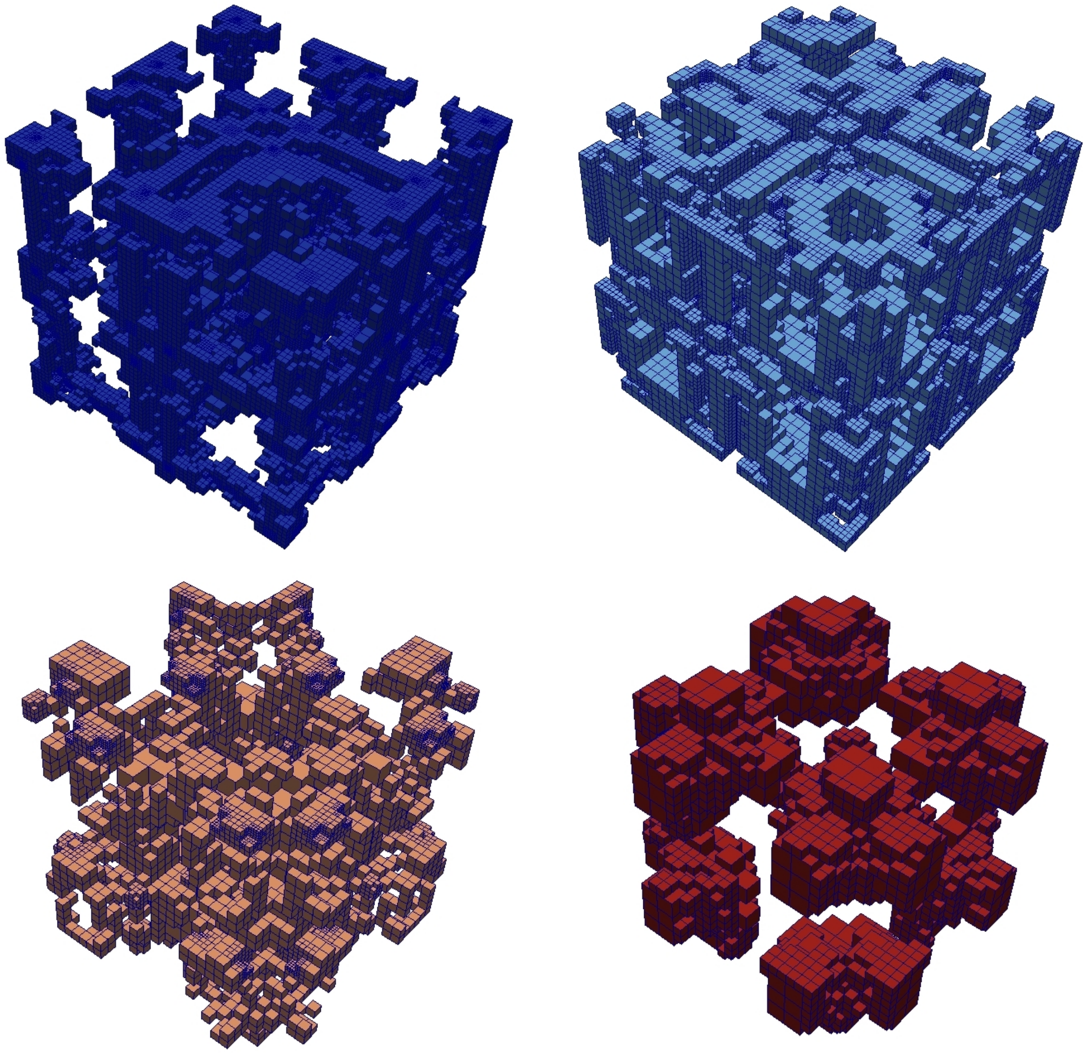

Figure 4.14: Example 8: Final grid. Upper left: $p = 0$. Upper right: $p = 1$. Lower left: $p = 2$. Lower right: $p = 3$.

Chapter 5

Conclusion

We conlude this work with a few comments on the results. We have derived fully automatic *hp*-adaptive refinement strategies for the continuous Galerkin finite element methods for the Poisson and the Maxwell boundary value problem and for the discontinuous Galerkin finite element method for the Poisson problem. Further convergence of the algorithm was proven in all cases. All three *hp*-adaptive refinement algorithms are based on the one-dimensional version from Dörfler and Heuveline [104] and solve local boundary value problems to decide which refinement possiblity promises the biggest reduction of the energy error. It is quite remarkable that the structure of these local boundary value problems does not change that much among the various application cases. All local optimization problems are formulated in an locally enhanced finite element (sub)space. Most of the differences occur in the proofs of convergence when the specific properties of the finite element spaces, e.g. Galerkin orthogonality and Helmholtz decomposition, come into play. Based on this observation there is some hope that this kind of fully automatic *hp*-adaptive refinement strategy could be generalized to a unified framework for a large class of problems. In contrast to the approach of Rognes and Logg [186], this strategy would also guarantee convergence of the algorithm.

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