On the Spectral Properties of Dispersive Photonic Crystals

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

Dipl.-Math. Philipp Schmalkoke
aus Dudenhofen

Tag der mündlichen Prüfung:	17. April 2013
Referent:	Prof. Dr. Michael Plum
Korreferent:	Prof. Dr. Roland Schnaubelt
Externer Korreferent:	Professor B. Malcolm Brown, PhD, Cardiff University, Vereinigtes Königreich

"It was a bright cold day in April, and the clocks were striking thirteen."
— GEORGE ORWELL, 1984

Abstract

This thesis is concerned with a parameter-nonlinear Helmholtz-type spectral problem on $\mathbb{R}^2$, which describes light propagation in certain two-dimensional, dispersive photonic crystals. The overall aim of the work is to enhance the basic understanding of the problem, given that it has not yet received much attention in the mathematical literature. A realization of the equation in a suitably weighted variant of $L^2(\mathbb{R}^2)$ leads to the analysis of an operator pencil with a periodic coefficient depending on the spectral variable. It is readily shown that the corresponding spectrum is related to a family of eigenvalue equations posed on a bounded periodicity cell Ω. This generalizes a well-known result of the classical Floquet-Bloch theory. Under a monotonicity assumption on the parameter-nonlinearity, the spectra of the problems on Ω are shown to be purely discrete. Further, the associated eigenfunctions are analyzed in detail. The main theorems proved in this dissertation establish their Riesz basicity and completeness, respectively, in the underlying function space $L^2(\Omega)$. At this, the specific form of the result depends on additional assumptions on the parameter-dependence of the coefficient of the problem.

Acknowledgments

The present dissertation came into being during my time as a scholar of the Research Training Group 1294 "Analysis, Simulation and Design of Nanotechnological Processes" of the German Research Foundation (DFG) at the Karlsruhe Institute of Technology (KIT). The scholarship I received is greatly appreciated.

Besides, I would like to thank several people that supported me during my time as a doctoral student and without whom this thesis would not exist. First and foremost, I most sincerely thank Prof. Dr. Michael Plum for supervising my research project. The countless hours we spent discussing my progress, mathematics in general, and sometimes even personal matters were most helpful to me. I learned a great deal during these meetings and I especially appreciate the freedom I was given to follow my interests. In addition, I give thanks to Prof. Dr. Roland Schnaubelt for the discussions we had as well as his approval to be the co-examiner of my work. Professor B. Malcolm Brown, PhD from Cardiff University, Wales, United Kingdom deserves my thanks for kindly serving as an external referee. Moreover, I thank Prof. Dr. Marlis Hochbruck for her guidance concerning career options and for always dealing with organizational affairs in a very efficient way. Further, I thank Marion Ewald for being incredibly friendly.

On a more personal level, I am grateful for my supporting parents and my admirable grandmother, who always believe in me. Also, I thank Eva's whole family for being so warmhearted and understanding. Moreover, I give thanks to my encouraging friends Steffen Fischer, Isabell Graf, Bernhard Konrad, Laura Munzig, Matthias Munzig, Karl Piskorek, Dennis Prill, Evelina Schüle, and Liwang Shen as well as my former colleagues at the Research Training Group. In particular, I would like to mention Bernhard Barth, who reliably provided me with lots of coffee and mathematical insight, as well as Hans-Jürgen Freisinger, Hannes Gerner, and Dominik Müller, who are always fun to have around.

Finally, but most importantly, I thank Eva for being my loving partner and best friend. She keeps me grounded, yet always knows when to keep me afloat.

Karlsruhe, January 2013 Philipp Schmalkoke

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CHAPTER 1

INTRODUCTION

In recent years, mathematicians and physicists alike became evermore interested in the study of problems related to light propagation in photonic crystals (see [Dör11], [Joa08], and [Kuc01]). This is due to the promising features concerning applications which these artificial materials have shown in experiments. For instance, the guiding of light around sharp corners in a waveguide, the localization of light within a photonic crystal, and the frequency-dependent reflection or transmission of light impinging on such a structure, i.e., the fabrication of an optical filter, have been demonstrated (see [Joa08, Chapt. 10] and [Mek96]).

The distinguishing feature of a photonic crystal is its spatially periodic structure in up to three dimensions, with the repeating units consisting of two or more different materials. Commonly, the underlying pattern repeats itself every few hundred nanometers, where the precise length depends on the specific application. Experiments have shown that it should approximately correspond to the wavelength of the light that is intended to be affected. In any case, the scale is large enough to be considered "classical" in the sense that effects on the atomic level may be neglected when problems of light propagation in photonic crystals are analyzed mathematically.

In view of this, the natural starting point when setting up a model to describe the related physics are Maxwell's equations. With their help—specializing to nanostructures periodic in only two dimensions and under certain assumptions on the involved electromagnetic fields and the materials themselves—an eigenvalue problem of the form

$$-\Delta u = \frac{\omega^2}{c_0^2}\varepsilon_{\mathrm{r}} u \qquad \text{in } \mathbb{R}^2 \tag{1.1}$$

can be deduced (see Section 3.2). Here, the spectral variable ${}^{\omega^2}/_{c_0^2}$ incorporates the non-negative frequency ω of the electromagnetic field of a propagating light wave and c_0 denotes the vacuum speed of light. Further, the function u corresponds to one component of the associated electric field and is spatially- but not time-dependent. The latter holds likewise for the coefficient function ε_{r} of the problem, which is a material-dependent quantity called the photonic crystal's relative permittivity and constitutes the input data. It is in terms of ε_{r} that the periodicity of the studied structure enters the mathematical model. More precisely, this function is periodic with respect to a bounded so-called primitive cell in $\mathbb{R}^2$. Knowing the relative permittivity on this set is sufficient to determine the relevant properties of the material at any point in space. Naturally, this leads to the question whether the spectral problem (1.1) can be reduced to one posed just on the primitive cell.

An answer thereto is provided by the spectral theory of periodic partial differential equations—often condensed under the name Floquet-Bloch theory (see [Kuc93]). Referring to Subsection 4.1.2 below for details, we indicate that a reduction of the problem is indeed possible, albeit this does not result in a single equation but rather in a family of eigenvalue problems with parameter-dependent quasi-periodic boundary conditions. Through the main result of the Floquet-Bloch theory, these problems are related to the spectrum of a self-adjoint operator realizing equation (1.1) in a suitably chosen L^2-space.

Besides being mathematically interesting, knowing said spectrum is highly relevant in applications. That is because a light wave with a frequency ω such that ${\omega^2}/{c_0^2}$ is an element of the respective resolvent set cannot propagate inside the considered material. By way of example, if there even exists a whole interval of such "forbidden" frequencies, called a band-gap, then the above-mentioned wave guide can be constructed. This is done by altering the periodic structure of the photonic crystal along a predefined path in such a way that wave propagation for a frequency in a band-gap is there again possible. The light is then confined to the modified region and can in this way be guided (see [AS04]).

As was already said, several simplifying assumptions led to the spectral problem (1.1) in the first place. In this work we forgo just one of them, but nevertheless obtain an equation with an entirely different structure and one which is, to this date, nearly unaccounted for in the mathematical literature (see our related review in Section 5.5): The subject of our interest is the eigenvalue problem

$$-\Delta u = \frac{\omega^2}{c_0^2}\varepsilon_{\mathrm{r}}(\cdot,\omega)u \qquad \text{in } \mathbb{R}^2, \tag{1.2}$$

which, contrary to before, features a frequency-dependent relative permittivity. This equation thereby allows for the study of light wave propagation in photonic crystals having properties varying with the respective wave's frequency. Such nanostructures are referred to as being dispersive. In contrast, the first-mentioned eigenvalue problem in this introduction governs only nondispersive media. Since nearly all materials used in the manufacturing of photonic crystals exhibit dispersiveness (see Subsection 3.2.2), we are motivated to study equation (1.2) as an enhanced, more realistic model than what is given by equation (1.1).

Mathematically, the differences between the two introduced eigenvalue problems are far greater than they might at first appear. While equation (1.1) is linear in the parameter ω, we observe that the assumption of a dispersive material leads to what is called a parameter-nonlinear problem. As such, no single corresponding operator whose spectrum can be analyzed exists and it is in fact not obvious what concept of "spectrum" is actually appropriate here. Moreover—assuming that term has been given a proper meaning—the classical Floquet-Bloch theory, valid only for parameter-linear problems, is not applicable anymore. In particular, a connection with eigenvalue equations posed on a primitive cell is a priori not at hand and, if at all applicable, well-known properties of said "cell-problems" in the nondispersive setting have to be reestablished under the more relaxed assumptions.

Of course, by virtue of this being a doctoral thesis, the already mentioned fact that the spectral problem for dispersive photonic crystals in the form (1.2) has not yet received considerable attention also has some merit. As a result, we are able to concern ourselves with the related fundamental issues and can pursue our search for mathematical insight in a free and unbiased way. To be exact, what we intend to contribute with our work are answers regarding the following questions, which we were guided by during the course of our research:

- How can equation (1.2) be realized operator-theoretically and what notion of "spectrum" applies?
- In what way does the spectrum of the dispersive problem, once defined, differ from that of its well-studied nondispersive counterpart?
- Can a connection with spectral problems posed on a primitive cell be established?
- If so, is the spectral structure of some specific form, e.g., purely discrete?
- If so, are the corresponding eigenfunctions in some sense complete or do they even form a basis of the underlying function space?

A reader with prior knowledge in the field surely realizes that having positive answers to the last three questions in this list would imply that the essential characteristics of the spectral problem for nondispersive photonic crystals carry over to the dispersive case. To come straight to the point, we shall see that there are physically reasonable assumptions on the data, i.e., on the relative permittivity ε_{r} and its frequency-dependence, such that this resemblance indeed holds. However, demanding too little of the coefficient function can also result in noticeable distinctions. Making this precise is the overall aim of the remainder of this work, which we now outline to close this introduction.

Each of the following chapters starts with a brief overview of what material is covered therein. It is either there or at the beginning of sections that we provide the reader with references and further readings. In particular, additional literature covering all topics we merely sketched so far will be mentioned in later parts of this work, wherein we revisit these issues again in great detail.

Subsequent to this introduction, we present in Chapter 2 the mathematical tools and concepts that we rely on in the main part of our work. In addition, standard notation is reviewed and fixed therein. Chapter 3 is then devoted to further physical, and some historical, information regarding photonic crystals. In addition, it introduces the reader to related terms and the way that these structures are abstractly modeled. Furthermore, an extensive deduction of the eigenvalue problems (1.1) and (1.2), starting from the full system of Maxwell's equations, is included. Thereafter, in Chapter 4, the mathematically in-depth part of this work commences. The chapter is exclusively concerned with the spectral problem for nondispersive photonic crystals and as such assembles known facts. The final section therein is a literature review on related work and more general problems in periodic partial differential equations. It summarizes the state-of-the-art in this field and mentions important open questions. Finally, Chapter 5 contains the outcomes of our research project. Therein, we successfully answer the questions stated before by proving our main theorems on the spectral structure of the problem (1.2) and on the completeness and basicity of the eigenfunctions of associated problems on a primitive cell. The chapter is once more finished with a related literature review, but, as was indicated before, this can only be brief. The bibliography, a list of symbols we introduce, and the curriculum vitae of the author constitute the back matter of this dissertation.

CHAPTER 2

SELECTED PRELIMINARIES

In this chapter, we introduce our notation and review certain mathematical concepts that are of relevance later on. At this, we assume that the reader is familiar with the most important results and notions of functional analysis. If needed, however, the material we presuppose can be looked up in [RS80] or [Kat95] and, regarding function spaces, in [AF03]. The preliminaries we do present below are chosen by their relevance to the body of the text and such that a reasonably self-contained dissertation emerges. Literature for further reading, and so as to look up the omitted proofs, is cited individually in each section.

The structure of this chapter is as follows: We start in Section 2.1 with some remarks about the conventions we adhere to. Thereafter, in Section 2.2, we introduce and compare several types of bases in Banach and Hilbert spaces. The subsequent paragraph, Section 2.3, is concerned with important theorems concerning holomorphic operator-valued functions. Finally, Section 2.4 deals with various basic topics in spectral theory.

2.1 NOTATION AND CONVENTIONS

We adopt a notation that is standard in the mathematical literature and is as such easily accessible to the reader. The table hereafter provides symbols and abbreviations that are widely-known and only require very little explanation. Besides, an index of symbols that are specific to our work, e.g., the spaces and operators we define, is included at the very end of this document for added convenience (see p. 159).

$\mathbb{N}$ The set of natural numbers

$\mathbb{N}_0$ The set of non-negative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{Z}$ The set of integers

$\mathbb{R}$	The field of real numbers (the real axis)
$\mathbb{R}_{\geq 0}$	The set of non-negative real numbers
$\mathbb{C}$	The field of complex numbers (the complex plane)
$\mathbb{H}$	The upper complex half-plane
$\overline{z}$	The complex conjugate of $z \in \mathbb{C}$
$\operatorname{Re} z$, $\operatorname{Im} z$	The real and imaginary part of $z \in \mathbb{C}$
$\operatorname{Re} u$, $\operatorname{Im} u$	The real and imaginary part of a complex-valued function u
$\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{C}^d$	The d-tuples of elements of $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, where $d \in \mathbb{N}$
$z \cdot w$	The dot product on $\mathbb{C}^d$, i.e., $z \cdot w = \sum_{i=1}^d z_i w_i$ for $z, w \in \mathbb{C}^d$
$\lvert z \rvert$	The Euclidean norm on $\mathbb{C}^d$, i.e., $\lvert z \rvert = \sqrt{z \cdot \overline{z}}$ for $z \in \mathbb{C}^d$
A_{ij}	The entry in the ith row and the jth column of a matrix A
A^T	The transpose of a matrix A
$u\vert_U$	The restriction of a function u to a subset U of its domain
dx	The Lebesgue measure of appropriate dimension
$d\sigma$	The Lebesgue surface measure of appropriate dimension
a. a.	(Lebesgue-) "almost all"
a. e.	(Lebesgue-) "almost every(where)"
T^*	The adjoint of an operator T on a Hilbert space
$T\vert_U$	The restriction of an operator T to a subset U of its domain
$\nabla\times$, $\nabla\cdot$, ∇, Δ	The curl, divergence, gradient, and Laplace operator
$\mathcal{O}$, o	The Landau symbols
δ_{ij}	The Kronecker delta, i.e., $\delta_{ij} = 1$, if $i = j$, $\delta_{ij} = 0$, otherwise
χ_M	The characteristic function of a set M, i.e., $\chi_M(x) = 1$, if $x \in M$, $\chi_M(x) = 0$, otherwise
$M \times N$	The cartesian product of two sets M and N
$\operatorname{dist}(M, N)$	The distance between two subsets M and N of a metric space, i.e., $\operatorname{dist}(M, N) = \inf\{d(x, y) \mid x \in M, y \in N\}$
$\operatorname{vol}(M)$	The volume of a Lebesgue measurable set $M \subset \mathbb{R}^d$
$U_r(p)$, $\dot{U}_r(p)$	The open and punctured open r-neighborhood of a point p in a metric space
$\overline{M}$	The closure of a subset M of a topological space
∂M	The boundary of a subset M of a topological space
$V \oplus W$	The direct sum of two subspaces V and W of a vector space, i.e., $V \oplus W = \{v + w \mid v \in V, w \in W\}$ and $V \cap W = \emptyset$
$V^\perp$	The orthogonal complement of a closed subspace V of a Hilbert space
C	A constant which may change its value at successive appearances
$\iff$	"if and only if"

An important role in the study of problems of mathematical physics is played by operators and function spaces. In their regard we stipulate the following principles: All vector spaces we will be concerned with are assumed to be complex and nontrivial; we denote them by capital Latin letters. For a normed space X we write $\|\cdot\|_X$ for its norm and sometimes, when no confusion can arise, also $\|\cdot\|$. The same rule applies if X is a (pre-)Hilbert space, i.e., the respective inner product is denoted by $\langle\cdot,\cdot\rangle_X$ or $\langle\cdot,\cdot\rangle$.

If X and Y are normed spaces and $A : X \to Y$ is a linear operator, then we denote by $D(A) \subseteq X$ the domain of definition of A and we write $\operatorname{Ker} A$ and $\operatorname{Ran} A$ for the kernel and the range of A, respectively. Moreover, $\mathcal{B}(X,Y)$ is the space of all bounded linear operators from X to Y and, if $X = Y$, we abbreviate $\mathcal{B}(X) := \mathcal{B}(X,X)$. These spaces shall always be equipped with the usual operator norm given by

$$\|A\|_{\mathcal{B}(X,Y)} := \sup_{x\in X\setminus\{0\}} \frac{\|Ax\|_Y}{\|x\|_X} \qquad \text{for all } A \in \mathcal{B}(X,Y).$$

Here, as with the norm of functions, we sometimes omit the subscript and write $\|A\|$ for the norm of $A \in \mathcal{B}(X,Y)$.

Although bounded operators occur many times in what follows, our main objects of study will be densely-defined, closed linear operators between normed spaces X and Y. The set of all such operators is written as $\mathcal{C}(X,Y)$, where we write $\mathcal{C}(X) := \mathcal{C}(X,X)$ in the important special case $X = Y$.

The function spaces that are of importance to this work are well-known Lebesgue and Sobolev spaces or certain subspaces thereof. Given a domain $U \subseteq \mathbb{R}^d$, where $d \in \mathbb{N}$, they are denoted in the usual manner as, e.g., $L^2(U)$, $L^\infty(U)$, $H^s(U)$, and $H^s_0(U)$, respectively, where only $s \in \{1,2\}$ occurs in this work. The functions in these spaces are understood to have values in $\mathbb{C}$. Otherwise we explicitly specify the target space, Y say, and write $L^2(U;Y)$ and so forth.

If nothing else is said, we consider the canonical inner products in the aforementioned Hilbert spaces. Quite often, though, our analysis benefits from using a weighted, and thus equivalent, variant of such a space. Then, for a *weight function* $w \in L^\infty(U;\mathbb{R})$ that is bounded below by some positive constant, we write, e.g., $L^2_w(U)$ if the space $L^2(U)$ is endowed with the inner product

$$\langle v,v\rangle_{L^2_w(U)} := \int_U w(x)u(x)\overline{v(x)}\,dx \qquad \text{for all } u,v \in L^2(U). \tag{2.1}$$

In accordance with what we wrote above, the norm induced by this inner product is generally denoted by $\|\cdot\|_{L^2_w(U)}$. However, we also write $\|\cdot\|_w$ when we can avoid a bulky notation by doing so. It will then always be clear from the context what underlying space is weighted.

2.2 BASES IN BANACH AND HILBERT SPACES

The main theorems we establish in this thesis concern the question of whether the eigenfunctions of the spectral problem we study exhibit some sort of completeness or basis property. Thus, in the current section, we aim to provide an introduction to these concepts in infinite-dimensional spaces. Our presentation follows [Chr03, Chapt. 3], [Hei11, Chapts. 1, 4, and 7], and [GK69, Chapt. VI]. Besides the mentioned references, the monographs [Sin70] and [Sin81] are particularly suitable to find further details.

2.2.1 SCHAUDER BASES

In this subsection, let X be a separable Banach space. We begin by reviewing definitions related to sequences in X.

Definition 2.2.1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and let $\operatorname{span}\{x_n\}_{n\in\mathbb{N}}$ denote the set of all finite linear combinations of elements of the sequence. We say that

(a) $\{x_n\}_{n\in\mathbb{N}}$ is *linearly independent* if every finite subset of $\{x_n\}_{n\in\mathbb{N}}$ is linearly independent in the sense of linear algebra.

(b) $\{x_n\}_{n\in\mathbb{N}}$ is *ω-independent* if $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ and $\sum_{n=1}^{\infty} c_n x_n = 0$ imply $c_n = 0$ for all $n \in \mathbb{N}$.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is *complete* in X if $\operatorname{span}\{x_n\}_{n\in\mathbb{N}}$ is dense in X.

(d) $\{x_n\}_{n\in\mathbb{N}}$ is a *Schauder basis* (or simply *basis*) of X if for all $x \in X$ there exists a unique sequence of coefficients $\{c_n(x)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ such that

$$x = \sum_{n=1}^{\infty} c_n(x) x_n, \tag{2.2}$$

where the series converges in the norm of X. We refer to (2.2) as the *expansion* of x in the basis $\{x_n\}_{n\in\mathbb{N}}$ and call the mappings $X \ni x \mapsto c_n(x) \in \mathbb{C}$ the *coefficient functionals* of the basis.

Note that the equality (2.2) holds with respect to the chosen order of the summands. Upon reordering them, the series may become divergent, which motivates part (i) of the next definition.

Definition 2.2.2. Let $\{x_n\}_{n\in\mathbb{N}}$ be a basis of X. We say that

(a) $\{x_n\}_{n\in\mathbb{N}}$ is *unconditional* if the convergence of the series (2.2) is unconditional for each $x \in X$.

(b) $\{x_n\}_{n\in\mathbb{N}}$ is *bounded* if $0 < \inf_{n\in\mathbb{N}} \|x_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| < \infty$.

A basis of X is obviously ω-independent and complete in X. The converse statement, however, is not true (see [Hei11, Expl. 1.29]) and it is very unfortunate that "completeness" and "basis property" are rather often incorrectly used interchangeably in the literature. The crucial difference between these concepts lies therein that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is complete in X if and only if for all $x \in X$ there exist coefficients $c_n^{(s)}(x) \in \mathbb{C}$, where $n = 1, \dots, N_s$ and $s \in \mathbb{N}$, such that there holds

$$\sum_{n=1}^{N_s} c_n^{(s)}(x) x_n \to x \quad \text{as } s \to \infty,$$

while, on the other hand, $\{x_n\}_{n\in\mathbb{N}}$ being a basis of X requires definite coefficients (and not sequences thereof) as in (2.2). Another characterization of this relationship is given through the next result.

Theorem 2.2.3 ([Chr03, Thm. 3.1.4])**.** *A sequence of nonzero vectors $\{x_n\}_{n\in\mathbb{N}}$ in X is a basis of X if and only if the sequence is complete and there exists a constant K such that for all $M, N \in \mathbb{N}$ with $M \leq N$,*

$$\left\| \sum_{n=1}^{M} c_n x_n \right\| \leq K \left\| \sum_{n=1}^{N} c_n x_n \right\| \qquad \text{for all sequences } \{c_n\}_{n\in\mathbb{N}} \subset \mathbb{C}.$$

Remark. Requiring the sequence in the statement of this theorem to consist of nonzero vectors is necessary, since otherwise expansions in it cannot be unique.

Theorem 2.2.3 allows for an easy proof of the next result which is often the starting point when proving properties of bases in Banach spaces.

Theorem 2.2.4 ([Hei11, Thm. 4.13])**.** *The coefficient functionals $\{c_n(\cdot)\}_{n\in\mathbb{N}}$ of a basis of X are linear and continuous and thus elements of the dual space of X.*

Finally in this paragraph, we mention a relatively easy way to gain new bases from existing ones. Hereto, let Y be another separable Banach space.

Theorem 2.2.5 ([Hei11, Lem. 4.18])**.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be a basis of X. If $T : X \to Y$ is an isomorphism, i.e., a bounded and bijective linear operator, then $\{Tx_n\}_{n\in\mathbb{N}}$ is a basis of Y.*

This connection suggests the next definition.

Definition 2.2.6. A basis $\{x_n\}_{n\in\mathbb{N}}$ of X is said to be *equivalent* to a basis $\{y_n\}_{n\in\mathbb{N}}$ of Y if there exists an isomorphism $T : X \to Y$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$.

Remark. In the special case $X = Y$ we can define an equivalence relation on the set of all bases of X, where $\{x_n\}_{n\in\mathbb{N}} \sim \{y_n\}_{n\in\mathbb{N}}$ if and only if $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are equivalent in the sense of the last definition.

2.2.2 Bessel Sequences, Orthonormal and Riesz Bases

Let us now consider bases in a separable Hilbert space H. Due to the additional structure, there hold many results for these bases that are not true for their counterparts in general Banach spaces. This applies, in particular, to orthonormal bases, which are widely known and used. For the sake of completeness, we recall their definition.

Definition 2.2.7. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in H. We say that

(a) $\{x_n\}_{n\in\mathbb{N}}$ is a *Bessel sequence* if there exists a constant $B > 0$ such that

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq B\,\|x\|^2 \qquad \text{for all } x \in H.$$

(b) $\{x_n\}_{n\in\mathbb{N}}$ is *orthonormal* if $\langle x_n, x_m\rangle = \delta_{nm}$ for all $n, m \in \mathbb{N}$.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is an *orthonormal basis* of H if it is orthonormal and a basis of H.

While a Bessel sequence certainly need not have the basis property, expansions in it can be unconditionally convergent for suitable coefficient sequences:

Theorem 2.2.8 ([Chr03, Cor. 3.2.5])**.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be a Bessel sequence in H. Then $\sum_{n=1}^{\infty} c_n x_n$ converges unconditionally for all $\{c_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{N})$.*

Regarding orthonormal bases we only mention the following two noteworthy theorems, since we assume that the reader is familiar with the matter.

Theorem 2.2.9 ([Hei11, Thm. 1.50] and [Chr03, Cor. 3.4.3])**.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be an orthonormal sequence in H. Then $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H if and only if $\{x_n\}_{n\in\mathbb{N}}$ is complete in H.*

Remark. The equivalence stated in the last result follows from the orthogonality of the involved vectors and does not need to hold for an arbitrary complete sequence in H (see [Hei11, Expl. 1.46 (b)]). Hence, also in Hilbert spaces "completeness" and "basis property" are different notions.

Given one orthonormal basis, the next result characterizes all other orthonormal bases of the same Hilbert space.

Theorem 2.2.10 ([Chr03, Thm. 3.4.7])**.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of H. Then $\{y_n\}_{n\in\mathbb{N}}$ is likewise an orthonormal basis of H if and only if $y_n = Ux_n$ for all $n \in \mathbb{N}$ where $U : H \to H$ is a unitary operator.*

Of course, the application of a non-unitary isomorphism to an orthonormal basis of H still yields a basis of that space due to Theorem 2.2.5. Such bases are given a special name.

Definition 2.2.11. A basis $\{x_n\}_{n\in\mathbb{N}}$ of H that is equivalent to an orthonormal basis $\{y_n\}_{n\in\mathbb{N}}$ of H is called a *Riesz basis*. The isomorphism $T : H \to H$ such that $Tx_n = y_n$ for all $n \in \mathbb{N}$ is called an *orthogonalizer* of $\{x_n\}_{n\in\mathbb{N}}$.

Remark. Due to Theorem 2.2.10 a Riesz basis $\{x_n\}_{n\in\mathbb{N}}$ of H is equivalent to all orthonormal bases of H and the corresponding orthogonalizers only differ by a unitary transformation. For that reason $\max\left\{\|T\|, \left\|T^{-1}\right\|\right\}$, where T is any orthogonalizer of $\{x_n\}_{n\in\mathbb{N}}$, is a unique constant which can be seen as a measure of the non-orthogonality of the Riesz basis.

Several equivalent characterizations of Riesz bases are known. We mention some of them below.

Theorem 2.2.12 ([Hei11, Thm. 7.13] and [Chr03, Prop. 3.6.4])**.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in H. Then the following statements are equivalent:*

(a) $\{x_n\}_{n\in\mathbb{N}}$ is a Riesz basis of H.

(b) $\{x_n\}_{n\in\mathbb{N}}$ is a bounded and unconditional basis of H.

(c) $\{x_n\}_{n\in\mathbb{N}}$ is a basis of H and $\sum_{n=1}^{\infty} c_n x_n$ converges if and only if $\{c_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{N})$.

(d) $\{x_n\}_{n\in\mathbb{N}}$ is complete in H and there exist positive constants b and B such that

$$b\sum_{n=1}^{N}|c_n|^2 \leq \left\|\sum_{n=1}^{N} c_n x_n\right\|^2 \leq B\sum_{n=1}^{N}|c_n|^2 \qquad \text{for all } c_1,\dots,c_N \in \mathbb{C} \text{ and all } N \in \mathbb{N}.$$

(e) There exists an equivalent inner product $(\cdot,\cdot)$ on H such that $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H with respect to $(\cdot,\cdot)$.

Remark. If $\{x_n\}_{n\in\mathbb{N}}$ is a Riesz basis of H which turns into the orthonormal basis $\{y_n\}_{n\in\mathbb{N}}$ of H after an application of the orthogonalizer T, then we can supplement the last result as follows:

(a) We have $\left\|T^{-1}\right\|^{-1} \leq \|x_n\| \leq \|T\|$ for all $n \in \mathbb{N}$.

(b) The (unconditionally convergent) expansion of an element $x \in H$ in the basis $\{x_n\}_{n\in\mathbb{N}}$ has the form

$$x = \sum_{n=1}^{\infty} \langle x, T^* y_n\rangle\, x_n,$$

where T^* denotes the adjoint of the operator T.

(c) The constants b and B mentioned in part (d) are optimal for the choice $b = \|T\|^{-2}$ and $B = \left\|T^{-1}\right\|^2$.

(d) The inner product $(\cdot,\cdot)$ introduced in part (e) is given by $(x,y) := \langle Tx, Ty\rangle$ for all $x, y \in H$.

To finish this brief treatise, we note that it is a hard problem to find a bounded basis of a Hilbert space that is not a Riesz basis. A difficult example showing that this is indeed possible can be found in [Bab48].

2.3 HOLOMORPHIC OPERATOR-VALUED FUNCTIONS

This section is devoted to the analogs for operator-valued functions of some well-known theorems from complex function theory. By a consequence of the Hahn-Banach theorem, most of the classical scalar results can be easily generalized to the operator-valued setting. Sometimes, though, a direct imitation of a proof that is valid in the complex-valued case is not possible. This is rigorously elaborated in [GL09, Chapt. 1], which is also our main reference for this section. Besides, we occasionally rely on [Loc00, Sect. 1.3] and, for some results on Riemann integrals for operator-valued functions, on [HP57, Sect. 3.3].

In the whole section, X denotes a Banach space and $F : U \to \mathcal{B}(X)$, where U is an open set in $\mathbb{C}$, is an operator-valued function.[1] Here, the Banach space $\mathcal{B}(X)$ is always considered under the uniform operator topology.

2.3.1 DEFINITIONS AND PREPARATORY MATERIAL

Origin of our considerations is, of course, the notion of holomorphicity. Its definition in the operator-valued case does not differ from the scalar version.

Definition 2.3.1. A function $F : U \to \mathcal{B}(X)$ is called *differentiable* at a point $w \in U$ if the limit

$$F'(w) := \lim_{z \to w} \frac{F(z) - F(w)}{z - w}$$

exists. If F is differentiable at every point in U, we say that F is *holomorphic* in U and call the function $F' : U \to \mathcal{B}(X)$ the *derivative* of F.

Remarks.

(a) Clearly, a holomorphic operator-valued function is also continuous.

(b) Although, as was already said, the limit occurring in the last definition is understood with respect to the operator norm, we note that strong and even weak limit processes yield equivalent notions of holomorphicity (see [GL09, Thms. 1.6.1 and 1.7.1]).

(c) Even for unbounded-operator-valued functions there exist concepts of holomorphicity (see [Kat95, Chapt. 7] and [RS78, p. 14ff.]). In the language of the cited authors, what we have just defined is the property of being *bounded-holomorphic*.

[1] Note that all definitions and results can be transferred to functions $f : U \to X$ by means of only slight refinements.

Several important theorems in complex analysis are concerned with the evaluation of contour integrals in the complex plane. We briefly remind the reader of a related definition.

Definition 2.3.2. A set $\Gamma \subset \mathbb{C}$ is said to be a *connected contour* if there exist real numbers $a < b$ and a continuous function $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma([a,b]) = \Gamma$. The mapping γ is called a *parametrization* of Γ. Lastly, a union of a finite number of pairwise disjoint connected contours is called a (*non-connected*) *contour*.

In order for integration along a contour to make sense, its parametrization needs to be suitably smooth. We clarify this as a part of the next definition.

Definition 2.3.3. Let Γ be a connected contour with parametrization γ. We say that

(a) Γ is *closed* if $\gamma(a) = \gamma(b)$.

(b) Γ is *simple* if $\gamma(s) \neq \gamma(t)$ for all $a \leq s < t \leq b$ such that $(s,t) \neq (a,b)$.

(c) Γ is *smooth* if γ is continuously differentiable.

(d) Γ is *piecewise smooth* if there is a finite partition $a = a_1 < a_2 < \cdots < a_{m+1} = b$ of the interval $[a,b]$ such that the restrictions $\gamma_n := \gamma|_{[a_n,a_{n+1}]}$ are smooth for $1 \leq n \leq m$. We write $\gamma = \gamma_1 \dotplus \gamma_2 \dotplus \cdots \dotplus \gamma_m$ in this case.

A non-connected contour $\Gamma = \cup_{n=1}^{N} \Gamma_n$ is called *closed*, *simple*, *smooth* or *piecewise smooth* if each of its components has the respective property.

With this at hand, we can now proceed to define a contour integral for operator-valued functions. There, and in the rest of this subsection, we are concerned with a simple piecewise smooth contour $\Gamma = \cup_{n=1}^{N} \Gamma_n$. Without further mentioning, the parametrization of the component Γ_n, where $1 \leq n \leq N$, shall be given by $\gamma_{n,1} \dotplus \gamma_{n,2} \dotplus \cdots \dotplus \gamma_{n,m_n} : [a_n, b_n] \to \mathbb{C}$ and with respect to the partition $a_n = a_{n,1} < a_{n,2} < \cdots < a_{n,m_n+1} = b_n$ of the interval $[a_n, b_n]$.

Definition 2.3.4. Let $F : U \to \mathcal{B}(X)$ be a continuous function and let $\Gamma = \cup_{n=1}^{N} \Gamma_n$ be a simple piecewise smooth contour with $\Gamma \subset U$. We define the *contour integral* of F along Γ by

$$\int_\Gamma F(z)\, dz := \sum_{n=1}^{N} \sum_{l=1}^{m_n} \int_{a_{n,l}}^{a_{n,l+1}} \gamma'_{n,l}(t)\, F(\gamma_{n,l}(t))\, dt, \tag{2.3}$$

where the right-hand side is a sum of Riemann integrals.[1]

[1] As in the scalar case, this definition is seen to be independent of both the chosen parametrization (as long as the orientation is kept the same) and the chosen partition.

Remarks.

(a) Note that we write $\gamma'_{n,l}(t)$ to the left of the operator $F(\gamma_{n,l}(t))$ in (2.3) so as to highlight that the first-mentioned quantity is just a complex number.

(b) There is no difficulty in considering a Riemann integral for operator-valued functions as it occurs above. Similar to the case of a scalar-valued function it is defined as the limit of suitable Riemann sums. Of course, convergence has to be understood with respect to the operator norm, so that, if existent, the integral of a function with values in $\mathcal{B}(X)$ is itself an element of that space. We refer to [HP57, Sect. 3.3] for details, which also contains a proof of the existence of the integrals on the right-hand side of (2.3) under the given assumptions on F and Γ (see [ibid., Thm. 3.3.4]). Further, the cited theorem gives that

$$\int_\Gamma F(z)\,dz\,x = \int_\Gamma F(z)x\,dz \qquad \text{for all } x \in X,$$

where the integral on the right-hand side is a Riemann integral for vector-valued functions.

The usual characteristics of the scalar contour integral carry over to the operator-valued case. Two properties that are of particular importance to us are stated as the proposition that finishes this paragraph. First, though, we need a definition.

Definition 2.3.5. Let $\Gamma = \cup_{n=1}^N \Gamma_n$ be a simple piecewise smooth contour. Then

$$l(\Gamma) := \sum_{n=1}^N \sum_{l=1}^{m_n} \int_{a_{n,l}}^{a_{n,l+1}} |\gamma'_{n,l}(t)|\,dt$$

is called the *length* of Γ.[1]

Proposition 2.3.6 ([HP57, Thm. 3.3.2][2]). *Let $F : U \to \mathcal{B}(X)$ be a continuous function and let $\Gamma = \cup_{n=1}^N \Gamma_n$ be a simple piecewise smooth contour with $\Gamma \subset U$.*

(a) There holds the estimate

$$\left\| \int_\Gamma F(z)\,dz \right\| \leq \sum_{n=1}^N \sum_{l=1}^{m_n} \int_{a_{n,l}}^{a_{n,l+1}} \|F(\gamma_{n,l}(t))\|\,|\gamma'_{n,l}(t)|\,dt \leq l(\Gamma) \max_{z\in\Gamma} \|F(z)\|.$$

(b) If A is a closed operator on X such that $D(A) \supseteq \operatorname{Ran} F(z)$ for all $z \in \Gamma$ and if the mapping $z \mapsto A[F(z)]$ is continuous on Γ, then

$$A\left[\int_\Gamma F(z)\,dz\right] = \int_\Gamma A[F(z)]\,dz.$$

[1] The first footnote on the previous page likewise applies here.

[2] The cited reference only contains a proof of the assertion in part (b). However, the estimate stated in part (a) is clear from the definition of the contour integral as a limit of Riemann sums.

2.3.2 ANALOGS OF THEOREMS FROM COMPLEX ANALYSIS

For later reference, we now provide the operator-valued versions of several well-known results from complex function theory. They will be very important to our reasoning in Subsection 5.4.2 in which we prove our main theorem.

First, we make precise what type of contours we are concerned with here.

Definition 2.3.7. Let $D \subset \mathbb{C}$ be a bounded open set with boundary ∂D that is given by a simple, closed, and piecewise smooth contour, i.e., $\partial D = \cup_{n=1}^{N} \Gamma_n$ for some $N \in \mathbb{N}$. If the parametrization of each component Γ_n is chosen clockwise whenever the region bounded by Γ_n does not contain points in D which are arbitrarily close to Γ_n and counter-clockwise otherwise[1], then D is called an (*oriented*) *Cauchy domain* and ∂D is said to be a *Cauchy contour*.

With this we can state the operator-valued versions of Cauchy's integral theorem and formula.

Theorem 2.3.8 ([GL09, Thms. 1.4.2 and 1.5.1])**.** *Let $F : U \to \mathcal{B}(X)$ be a holomorphic function. If D is a Cauchy domain with $\overline{D} \subset U$, then*

$$\int_{\partial D} F(z)\, dz = 0 \qquad \text{and} \qquad F(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{z-w}\, dz \qquad \text{for all } w \in D.$$

Inductively—and with the help of the corresponding result for a complex-valued holomorphic function—we also obtain Cauchy's integral formula for derivatives.

Theorem 2.3.9 ([GL09, Cor. 1.5.3])**.** *Let $F : U \to \mathcal{B}(X)$ be a holomorphic function. Then F is arbitrarily often complex differentiable on U. If D is a Cauchy domain with $\overline{D} \subset U$, and if we denote by $F^{(n)}$ the nth complex derivative of F, where $n \in \mathbb{N}_0$, then*

$$F^{(n)}(w) = \frac{n!}{2\pi i} \int_{\partial D} \frac{F(z)}{(z-w)^{n+1}}\, dz \qquad \text{for all } w \in D.$$

Not only are holomorphic operator-valued functions of class C^∞, but also locally expandable in a power series (or *analytic*). This is formulated in the next theorem.

Theorem 2.3.10 ([GL09, Thm. 1.8.5])**.** *Let $F : U \to \mathcal{B}(X)$ be a holomorphic function and let $z_0 \in U$ and $r > 0$ be such that $U_r(z_0) \subseteq U$. Then*

$$F(z) = \sum_{n=0}^{\infty} (z-z_0)^n F_n \quad \text{for all } z \in U_r(z_0), \quad \text{where } F_n = \frac{F^{(n)}(z_0)}{n!} \quad \text{for all } n \in \mathbb{N}_0.$$

[1] This orientation is also known as the *orientation defined by D* in the literature. Some authors just say that D shall be "on the left" of Γ.

Remark. Due to Theorem 2.3.9, the coefficient operators of the above power series expansion have the integral representation

$$F_n = \frac{1}{2\pi i} \int_{\partial D} \frac{F(z)}{(z - z_0)^{n+1}}\, dz \qquad \text{for all } n \in \mathbb{N}_0,$$

where D is any Cauchy domain with $z_0 \in D$ and $\overline{D} \subset U$.

The next definition is well-known and unchanged from the scalar case.

Definition 2.3.11. Let $F : U \to \mathcal{B}(X)$ be a holomorphic function.

(a) A point $z_0 \in \mathbb{C}$ is called an *isolated singularity* of F if $z_0 \notin U$ and for some sufficiently small $r > 0$ the punctured disc $\dot{U}_r(z_0)$ is contained in U.

(b) An isolated singularity z_0 of F is called *removable* if there exists a holomorphic function $\widetilde{F} : U \cup \{z_0\} \to \mathbb{C}$ such that $\widetilde{F}|_U = F$. The function $\widetilde{F}$ is referred to as the *holomorphic extension* of F to $U \cup \{z_0\}$.

Remark. If it exists, the holomorphic extension in part (b) of this definition is unique (see [GL09, Thm. 1.1.3]).

The result hereafter is known as Riemann's theorem on removable singularities. It provides a helpful criterion for checking whether an isolated singularity of some operator-valued function is removable.

Theorem 2.3.12 ([GL09, Thm. 1.10.3][1]). *Let $F : U \to \mathcal{B}(X)$ be a holomorphic function. An isolated singularity z_0 of F is removable if and only if there exists a punctured disc $\dot{U}_r(z_0) \subseteq U$ in which F is bounded.*

Finally, we present the generalization of the residue theorem and a corollary. To state these results, we need a definition.

Definition 2.3.13. Let $F : U \to \mathcal{B}(X)$ be a holomorphic function. If z_0 is an isolated singularity of F, then for $r > 0$ so small that $\partial U_r(z_0) \subset U$ we call

$$\operatorname{Res}_{z_0}(F) := \frac{1}{2\pi i} \int_{\partial U_r(z_0)} F(z)\, dz$$

the *residue* of F at the point z_0.

Remark. Cauchy's integral theorem implies that the definition of the residue does not depend on the radius of the circular contour integrated along. In fact, a non-circular Cauchy contour enclosing the isolated singularity may likewise be used, but for simplicity we chose the boundary of a circle here.

[1] The reference only contains a proof of the "if" part of the theorem. The "only if" part, however, follows readily from the holomorphicity of $\widetilde{F}$.

Theorem 2.3.14 ([GL09, Thm. 1.10.5]). *Let $D \subset \mathbb{C}$ be a Cauchy domain, and let $z_1, \ldots, z_N \in D$ be finitely many distinct points. Further, let $F : D \setminus \{z_1, \ldots, z_N\} \to \mathcal{B}(X)$ be a holomorphic function. Then*

$$\int_{\partial D} F(z)\, dz = 2\pi i \sum_{n=1}^{N} \operatorname{Res}_{z_n}(F).$$

Since residues at removable singularities vanish due to the Cauchy integral theorem, we obtain the following important consequence.

Corollary. *If F and D are as in the last theorem, and if the isolated singularities $z_1, \ldots, z_n$ of F are removable, then*

$$\int_{\partial D} F(z)\, dz = 0.$$

This finishes our compilation of results on holomorphic operator-valued functions. The reader will find them used extensively later on in this work.

2.4 SOME CONCEPTS OF SPECTRAL THEORY

In this section, we are concerned with various aspects of spectral theory. Divided in three subsections, our treatise is restricted to providing basic definitions and some few results related to closed operators, Riesz projections, and operator pencils.

2.4.1 SPECTRUM AND RESOLVENT OF CLOSED OPERATORS

The material covered below is clearly well-known. We nevertheless include it here since there is some ambiguity surrounding spectral notions in the literature (see [Kat95, fn. 2 on p. 517]). For further reading, classical books such as [RS80], [GG81], and [Kat95] can be consulted.

We begin with a definition. There and throughout, X is a Banach space and A denotes a closed operator on X with dense domain $D(A)$.

Definition 2.4.1.

(a) The *resolvent set* of A is given by

$$\rho(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I : D(A) \to X \text{ is bijective}\},$$

and the *spectrum* of A is defined as $\sigma(A) := \mathbb{C} \setminus \rho(A)$.

(b) For $\lambda \in \rho(A)$ the bounded operator

$$R_A(\lambda) := (A - \lambda I)^{-1} : X \to D(A) \tag{2.4}$$

is called the *resolvent* of A (*at the point* λ).

Remark. It is standard in the mathematical literature to also refer to the mapping $\lambda \mapsto R_A(\lambda)$ as the *resolvent* of A. We adopt this convention as well.

If the space X is finite-dimensional, then $\sigma(A)$ is simply the set of all eigenvalues of (the matrix) A. In general, however, the situation is more sophisticated. This gives rise to a finer subdivision of the set $\sigma(A)$.

Definition 2.4.2.

(a) The *point spectrum* of A is given by

$$\begin{aligned}\sigma_{\mathrm{p}}(A) &:= \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective}\} \\ &= \{\lambda \in \mathbb{C} \mid \mathrm{Ker}(A - \lambda) \neq \{0\}\}.\end{aligned}$$

A complex number $\lambda \in \sigma_{\mathrm{p}}(A)$ is called an *eigenvalue* of A and any nonzero $u \in \mathrm{Ker}(A - \lambda)$ is a corresponding *eigenvector*. The null space $\mathrm{Ker}(A - \lambda)$ is referred to as the *eigenspace* of A corresponding to λ and $\dim(\mathrm{Ker}(A - \lambda))$ is the *(geometric) multiplicity* of that eigenvalue.

(b) The *continuous spectrum* of A is the set

$$\begin{aligned}\sigma_{\mathrm{c}}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I &\text{ is injective, } \mathrm{Ran}(A - \lambda I) \text{ is dense in } X, \\ &\text{but } (A - \lambda I)^{-1} \text{ is unbounded}\}.\end{aligned}$$

(c) The *residual spectrum* of A is defined as

$$\begin{aligned}\sigma_{\mathrm{r}}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I &\text{ is injective, but } \mathrm{Ran}(A - \lambda I) \text{ is not} \\ &\text{dense in } X\}.\end{aligned}$$

Remark. There holds $\sigma(A) = \sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{c}}(A) \cup \sigma_{\mathrm{r}}(A)$ with pairwise disjoint sets on the right-hand side of this equality. To realize this it is important to note that

$$\begin{aligned}\{\lambda \in \mathbb{C} \mid A - \lambda I &\text{ is injective, } \mathrm{Ran}(A - \lambda I) \text{ is dense in } X \text{ but not equal to } X, \\ &\text{and } (A - \lambda I)^{-1} \text{ is bounded}\}\end{aligned}$$

is the empty set if A is a closed operator.

Whereas in linear algebra the multiplicity of an eigenvalue is always finite, there might be infinitely many eigenvectors corresponding to some $\lambda \in \sigma_{\mathrm{p}}(A)$ if A acts on an infinite-dimensional space. This is accounted for in yet another decomposition of the spectrum of A.

Definition 2.4.3. The *discrete spectrum* of A is the set

$$\sigma_{\mathrm{d}}(A) := \{\lambda \in \mathbb{C} \mid \lambda \text{ is an isolated eigenvalue of } A \text{ that has finite multiplicity}\}.$$

Its complement $\sigma_{\mathrm{ess}}(A) := \mathbb{C} \setminus \sigma_{\mathrm{d}}(A)$ is referred to as the *essential spectrum* of A.

Remark. The term "isolated" in the definition of $\sigma_{\mathrm{d}}(A)$ is meant in the sense that the respective eigenvalue is an isolated point of $\sigma(A)$ and not just in the set of all eigenvalues.

Several other decompositions of the spectrum that we did not cover as yet are sometimes of interest. In our work, however, we will only be concerned with subsets introduced so far. We thus refer the reader to the already mentioned literature and [Dav96] for a broader treatment. What we do need, though, are some basic results on the resolvent of a closed operator. We now recall them from the literature.

Proposition 2.4.4 ([RS80, Thm. VIII.2])**.**

(a) The resolvent set $\rho(A)$ is open (thus the spectrum $\sigma(A)$ is closed) and the resolvent is a holomorphic operator-valued function on $\rho(A)$.

(b) For $\lambda, \mu \in \rho(A)$ there holds the equality

$$R_A(\lambda) - R_A(\mu) = (\mu - \lambda) R_A(\lambda) R_A(\mu),$$

called the first resolvent identity. *In particular, $R_A(\lambda)$ and $R_A(\mu)$ commute.*

Assertion (b) of this proposition establishes a relation between the resolvents of a fixed operator evaluated at two points in its resolvent set. It is also possible to compare the resolvents of two operators at a point for which they both exist:

Proposition 2.4.5 ([HS96, Prop. 1.9])**.** *Let B be another closed operator on X such that $D(B) = D(A)$ and let $\lambda \in \rho(A) \cap \rho(B)$. Then*

$$R_A(\lambda) - R_B(\lambda) = -R_A(\lambda)(A - B) R_B(\lambda),$$

which is named the second resolvent identity*.*

In the main part of this work we will mostly be concerned with densely-defined self-adjoint operators on Hilbert spaces. Our preceding definitions and results apply, since operators with these properties are necessarily closed (see [Yos95, Prop. VII.3.2]). Due to the additional self-adjointness, more detailed information about their spectra and resolvents can be deduced. Some are stated in the next theorem, which, in particular, provides a norm equality that will be used quite frequently in our work.

Theorem 2.4.6 ([HS96, Thm. 5.5] and [Kat95, Sect. V.3.5])**.** *Let A be a self-adjoint operator on a Hilbert space. Then $\sigma(A) \subseteq \mathbb{R}$ and $\sigma_{\mathrm{r}}(A) = \emptyset$. Moreover, for $\lambda \in \rho(A)$,*

$$\|R_A(\lambda)\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(A))}.$$

With this we finish our short account on the spectral theory of closed operators. More advanced results will be cited in the body of this work and for the specific operators that are of interest to us.

2.4.2 RIESZ PROJECTIONS

Above we introduced the decomposition of the spectrum of a closed operator in its discrete and essential part. We shall now devote our attention to the first-mentioned component, i.e., we are concerned with isolated eigenvalues that have a finite multiplicity. For each such eigenvalue we define the so-called Riesz projection, which is helpful in further studying the spectral properties of the corresponding operator. Our presentation is based on [GGK93, Sect. I.2] and [HS96, Chapt. 6]. Moreover, the material is briefly discussed in [GK69, Sect. I.1.3].

In all of this paragraph, let A be a closed operator on a Banach space X with dense domain $D(A)$. We suppose that $\sigma(A)$ is purely discrete, i.e., $\sigma_{\text{ess}}(A) = \emptyset$, as this is given later in the setting in which we intend to apply the material discussed below. To begin with, we state a definition.

Definition 2.4.7. Let σ_0 be a set of finitely many eigenvalues of A. We call a Cauchy contour Γ (see Definition 2.3.7) *admissible* for A and σ_0 if $\Gamma \subset \rho(A)$ and

$$n(\Gamma, z) = \begin{cases} 1, & \text{if } z \in \sigma_0, \\ 0, & \text{if } z \in \sigma(A) \setminus \sigma_0, \end{cases} \tag{2.5}$$

where $n(\Gamma, z) := \frac{1}{2\pi i} \int_\Gamma \frac{dw}{w-z}$ denotes the *winding number* of Γ for $z \in \mathbb{C} \setminus \Gamma$.

Remark. The second condition for a contour to be admissible requires two things: First, that eigenvalues which lie in one of the regions bounded by the components of Γ lie in no other such region, and second, that they are precisely the eigenvalues in the set σ_0.

Part (a) of Proposition 2.4.4 states that $\lambda \mapsto R_A(\lambda) = (A - \lambda I)^{-1} \in \mathcal{B}(X)$ is a holomorphic operator-valued function on $\rho(A)$. Clearly, under our assumption that $\sigma(A)$ is purely discrete, the eigenvalues of A are isolated singularities of this mapping. Since these singularities are not removable (see [HS96, Prop. 1.8]), the contour integral introduced in the next definition does not vanish.

Definition 2.4.8. Let σ_0 be a set of finitely many eigenvalues of A, and let Γ be an admissible contour for σ_0 and A. Then the bounded operator

$$P_{\sigma_0} := -\frac{1}{2\pi i} \int_\Gamma R_A(\lambda)\, d\lambda : X \to X, \tag{2.6}$$

which is a Riemann integral in the sense of Definition 2.3.4, is called the *Riesz integral* for A and σ_0.

Remark. A standard argument of complex function theory and the operator-valued variant of the Cauchy integral theorem (Theorem 2.3.8) show that P_{σ_0} is independent of the chosen admissible contour (see [HS96, Lem. 6.1]).

Usually P_{σ_0} is introduced as the *Riesz projection* in the literature. This is justified by part (a) of the next result, which summarizes some important properties of this operator.

Proposition 2.4.9 ([GGK93, Lem. 2.1] for (a), [Kat95, Sect. III.6.4] for the rest)**.**

(a) There holds $P_{\sigma_0}^2 = P_{\sigma_0}$, i.e., P_{σ_0} is a projection.

(b) We have $\operatorname{Ran} P_{\sigma_0} \subseteq D(A)$ and the operators P_{σ_0} and A commute on $D(A)$.

(c) If $\sigma_1 \subseteq \sigma(A)$ is another set of finitely many eigenvalues of A with $\sigma_0 \cap \sigma_1 = \emptyset$, then P_{σ_0} and P_{σ_1} are disjoint*, i.e., $P_{\sigma_0} P_{\sigma_1} = P_{\sigma_1} P_{\sigma_0} = 0$.*

(d) For $X_0 := \operatorname{Ran} P_{\sigma_0}$ and $A_0 := A|_{X_0}$ we have $A_0 \in \mathcal{B}(X_0)$ and $\sigma(A_0) = \sigma_0$.

Under certain assumptions, more details are known about the structure of the range of the Riesz projection:

Theorem 2.4.10 ([GK69, Thm. I.2.2][1] for (a), [HS96, Prop. 6.3][2] for (b))**.**

(a) If $\operatorname{Ran} P_{\sigma_0}$ is finite-dimensional, then

$$\operatorname{Ran} P_{\sigma_0} = \bigoplus_{\lambda \in \sigma_0} \{u \in X \mid u \in \operatorname{Ker}(A - \lambda I)^n \text{ for some } n \in \mathbb{N}\}.$$

(b) If X is a Hilbert space and A is self-adjoint, then P_{σ_0} is an orthogonal projection and

$$\operatorname{Ran} P_{\sigma_0} = \bigoplus_{\lambda \in \sigma_0} \operatorname{Ker}(A - \lambda I),$$

where the direct sum is an orthogonal one.

Remark. The subspaces which constitute the direct sum in part (a) are called the *generalized eigenspaces* of A for the eigenvalues in σ_0. If A has a compact resolvent, then their dimension is finite (see [Kat95, Thm. III.6.29]) and if A is a self-adjoint operator on a Hilbert space, then they coincide with the "usual" eigenspaces (see [ibid., Sect. V.5]). The existence of an infinite-dimensional generalized eigenspace—in the case of an eigenvalue of infinite multiplicity, for instance—is one, but not the only reason for a likewise infinite-dimensional range of a Riesz projection (see the characterization of $\operatorname{Ran} P_{\sigma_0} < \infty$ provided by the above-mentioned Theorem in [GK69]). Recall, however, that only eigenvalues of finite multiplicity will be of interest in what follows.

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[1] In this reference a bounded operator on a Hilbert space is considered. However, the generalization to our setting causes no difficulties since the crucial properties of the resolvent and the Riesz projection are the same for a closed operator on a Banach space.

[2] The cited result only covers the simpler case wherein σ_0 contains only one eigenvalue. However, if there are $m > 1$ isolated eigenvalues in σ_0, we can choose a Cauchy contour consisting of m disjoint circles, each enclosing precisely one eigenvalue. The corresponding operator P_{σ_0} is then just a sum of m Riesz projections which are pairwise disjoint by part (c) of Proposition 2.4.9. Clearly, the range of such a sum of projections is given by the direct sum of the ranges of the individual summands.

To finish our treatise of Riesz projections, let us comment on the restriction $\sigma(A) = \sigma_{\mathrm{d}}(A)$ which we imposed on the considered operator A for simplicity. It is by no means necessary to have this spectral structure so as to define the Riesz integral. For instance, if A is a closed operator on a Banach space satisfying $\sigma(A) = \sigma_0 \cup \sigma_1$, where $\sigma_0 \cap \sigma_1 = \emptyset$, σ_0 is compact, and σ_1 is closed, then P_{σ_0} exists for any Cauchy contour satisfying a similar "winding condition" as above. Moreover, the projection property of P_{σ_0} is still valid in this setting. Details in this direction can be found in any of the above-mentioned books.

2.4.3 Operator Pencils and Related Spectral Notions

The spectral problem that we study in the body of this dissertation is nonlinear in the spectral parameter (see also the introduction to our work). Problems of this kind can be dealt with by studying suitable operator-valued functions, which, in the spectral context, are also called *operator pencils*. For those we collect the definitions of appropriate generalizations of the basic spectral theoretic notions below. At this, X denotes a Banach space and $\mathscr{A} : U \to \mathcal{C}(X)$ is an operator pencil[1] defined on some set $U \subseteq \mathbb{C}$. Here, the domain of the closed operator $\mathscr{A}(\lambda)$ is allowed to vary with $\lambda \in U$.

Particularly well-studied variants of operator pencils are so-called *operator polynomials* (or *polynomial pencils*) of the form

$$U \ni \lambda \mapsto \mathscr{A}(\lambda) := \sum_{n=0}^{N} \lambda^n A_n,$$

where $A_n \in \mathcal{B}(X)$ and $N \in \mathbb{N}$ (see [Mar88] and [Rod89]). Coefficients $A_n \in \mathcal{C}(X)$ are likewise possible here, but the large majority of results in the literature is concerned with the bounded case. For that reason, a polynomial pencil with coefficients that are merely closed is usually tried to be reduced to a certain pencil with bounded coefficients (see [Mar88, Chapt. II, §20]).[2] Note that in the unbounded case the domain of any operator $\mathscr{A}(\lambda)$ is given by the intersection of the domains of all its coefficient operators.

Although we shall not be concerned with polynomial pencils in the main part of our work, let us nevertheless briefly consider an easy example of this class given by

$$\mathbb{C} \ni \lambda \mapsto \mathscr{A}(\lambda) := A - \lambda I,$$

where $A \in \mathcal{C}(X)$. Here, we readily see that $\lambda \in \sigma(A)$ if and only if $0 \in \sigma(\mathscr{A}(\lambda))$. This observation motivates the following two definitions in which $\mathscr{A} : U \to \mathcal{C}(X)$ denotes an arbitrary operator pencil again.

[1] Capital letters in script font always denote operator pencils in this thesis.

[2] This is not possible for arbitrary closed coefficients but means that they need to fulfill further assumptions (see ibid.).

Definition 2.4.11.

(a) The *resolvent set* of $\mathscr{A}$ is given by

$$\rho(\mathscr{A}) := \{\lambda \in U \mid 0 \in \rho(\mathscr{A}(\lambda))\}$$

and the *spectrum* of $\mathscr{A}$ is defined as

$$\sigma(\mathscr{A}) := U \setminus \rho(\mathscr{A}) = \{\lambda \in U \mid 0 \in \sigma(\mathscr{A}(\lambda))\}.$$

(b) The operator pencil $\mathscr{R}_{\mathscr{A}} : \rho(\mathscr{A}) \to \mathcal{B}(X)$ given through

$$\mathscr{R}_{\mathscr{A}}(\lambda) := \mathscr{A}(\lambda)^{-1} \tag{2.7}$$

is called the *resolvent* of $\mathscr{A}$.

Remark. Observe that $\rho(\mathscr{A}) \cup \sigma(\mathscr{A}) = U$ and thus a point $\lambda \in \mathbb{C} \setminus U$ is, reasonably, neither in the resolvent set nor in the spectrum of $\mathscr{A}$.

Similar to the case of a closed operator we now introduce important subsets of the spectrum of an operator pencil and generalize some notions known from linear algebra.

Definition 2.4.12.

(a) The *point spectrum* of $\mathscr{A}$ is given by

$$\sigma_{\mathrm{p}}(\mathscr{A}) := \{\lambda \in U \mid 0 \in \sigma_{\mathrm{p}}(\mathscr{A}(\lambda))\}.$$

Following the very same pattern, we also define the spectral subsets $\sigma_{\mathrm{c}}(\mathscr{A})$, $\sigma_{\mathrm{r}}(\mathscr{A})$, $\sigma_{\mathrm{d}}(\mathscr{A})$, and $\sigma_{\mathrm{ess}}(\mathscr{A})$. They are called, as in the operator case, the *continuous*, *residual*, *discrete* and *essential spectrum* of $\mathscr{A}$, respectively.

(b) A complex number $\lambda \in \sigma_{\mathrm{p}}(\mathscr{A})$ is called an *eigenvalue* of $\mathscr{A}$ and any nonzero $u \in \mathrm{Ker}(\mathscr{A}(\lambda))$ is a corresponding *eigenvector*. The null space $\mathrm{Ker}(\mathscr{A}(\lambda))$ is referred to as the *eigenspace* of $\mathscr{A}$ corresponding to λ and $\dim(\mathrm{Ker}(\mathscr{A}(\lambda)))$ is the (*geometric*) *multiplicity* of that eigenvalue.

Remarks.

(a) Note that we have

$$\sigma(\mathscr{A}) = \sigma_{\mathrm{p}}(\mathscr{A}) \cup \sigma_{\mathrm{c}}(\mathscr{A}) \cup \sigma_{\mathrm{r}}(\mathscr{A}) \quad \text{and} \quad \sigma(\mathscr{A}) = \sigma_{\mathrm{d}}(\mathscr{A}) \cup \sigma_{\mathrm{ess}}(\mathscr{A}),$$

where the sets on the right-hand side of these equalities are pairwise disjoint. This follows immediately from the last definition and the identical decompositions that hold for each operator $\mathscr{A}(\lambda)$ with $\lambda \in U$.

(b) Based on the construction of an explicit example we shall see in Section 5.3 that $\lambda \in \sigma_{\mathrm{d}}(\mathscr{A})$ does not necessarily imply that λ is isolated in $\sigma(\mathscr{A})$. More precisely,

$$\sigma_{\mathrm{d}}(\mathscr{A}) = \{\lambda \in U \mid \lambda \text{ is an eigenvalue of } \mathscr{A} \text{ that has finite multiplicity}\},$$

which some authors actually use as the definition of the discrete spectrum of an operator pencil. This clearly contrasts the case of a closed operator discussed before (see Definition 2.4.3).

Given additional information about the λ-dependence of a pencil $\mathscr{A}$, more results similar to those known from the spectral theory of a closed operator can be deduced. For instance, $\sigma(\mathscr{A})$ is closed in U if $\mathscr{A}$ is a polynomial pencil or if it is a holomorphic function of λ. In the latter case it can also be shown that the resolvent $\mathscr{R}_{\mathscr{A}}$ is holomorphic on $\rho(\mathscr{A})$. A resolvent identity resembling that in part (b) of Proposition 2.4.4, however, is not to be expected unless the λ-dependence of the pencil is of a rather simple form. Further details can be obtained in the literature mentioned above and, for the specific pencils that we shall work with, in Chapter 5.

CHAPTER 3

PHYSICAL BACKGROUND AND MATHEMATICAL MODELING

As mentioned in the introduction to this work, the eigenvalue problem we study later on appears in the mathematical treatment of light propagation in photonic crystals. The present chapter aims to elaborate on this connection by providing additional information regarding physical as well as mathematical aspects.

Our presentation is organized in three sections: The first of these slightly complements our discussion of photonic crystals in Chapter 1 with further physical and some historical details. Mostly, however, it serves to familiarize the reader with technical terms as well as aspects of the modeling of such structures in theoretical studies and thereby allows for the comprehension of the subsequent mathematical assumptions. Section 3.2 is then concerned with Maxwell's equations, which govern the phenomena we intend to analyze. It is also here that we rigorously deduce the above-mentioned eigenvalue problem. The differential operator appearing therein has a coefficient function of physical relevance—the relative permittivity known from the introduction—and thus, to obtain a reasonable model, our assumptions on it are subject to certain limitations. This aspect is finally discussed in Section 3.3, which closes the chapter.

3.1 AN INTRODUCTION TO PHOTONIC CRYSTALS

In this section, we are concerned with the physical properties of photonic crystals and elaborate on how their periodic structure is usually modeled. Besides, the concept of a "reciprocal lattice" is also specified here. As we shall see later, it plays an important role in the spectral theory of photonic crystals.

Our treatment mostly follows [Joa08] and [Pra09] but we also provide additional references where it is beneficial.

3.1.1 SEMICONDUCTORS OF LIGHT

Most of the important inventions in modern electronics, such as transistors and integrated circuits, are based on the properties of semiconductors. Commonly these materials are *crystalline* solids, i.e., their atomic arrangement is of a periodic nature, and are characterized by having an (*energy*) *band gap* that is nonzero but smaller than that of an insulator (see [Kit04, Chapts. 7 and 8]). An electron within the semiconductor cannot have an energy that falls within the band gap so that this energy region is often also called the *forbidden band*.

In 1987, two researchers, E. Yablonovitch and S. John, independently proposed the creation of materials that allow for the manipulation of light similar to how semiconductors can regulate electrical currents. Their publications [Yab87] and [Joh87], respectively, originated from different motives[1] but both essentially described the same structure: An "optical semiconductor", which they decided to name a *photonic crystal*. Nowadays this term refers to any material that is composed of different *dielectrics*, i.e., insulating materials that can be polarized by an applied electric field, in a periodic manner. Such a material need not be periodic in all three spatial directions, but may also be a simple *multilayer film* consisting of alternating dielectric materials.[2]

Although all photonic crystals are clearly three-dimensional objects, they are commonly called *one-*, *two-*, or *three-dimensional* if they are periodic along just as many spatial directions. This is illustrated in Figure 3.1 below. The leftmost cube therein, for instance, depicts a portion of the mentioned multilayer film.

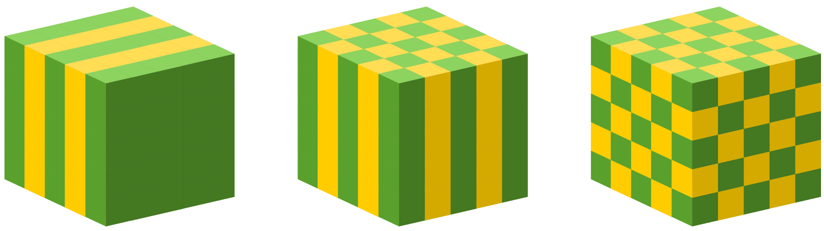

FIGURE 3.1 — Schematic examples of (portions of) one-, two-, and three-dimensional photonic crystals. Each structure is periodic along one or more axes and composed of two different dielectric materials which are distinctly colored.

Periodicity is a characterizing feature of both electrical semiconductors and photonic crystals. However, whereas single atoms or molecules are arranged periodically in silicon, say, the repeated portions of dielectric media in a photonic crystal are usually of the size of several hundred nanometers. Both structures

[1] Yablonovitch worked in industry at the time and wanted to improve telecommunication lasers while John was a professor at Princeton and worked out of pure research interest (see [Yab01] for further historical details).

[2] Optical properties of such structures have already been studied by Lord Rayleigh in 1887 (see [Str87] and [Str17]). However, he did (probably) not know that such an arrangement will allow for the applications known at the present day.

have in common, though, that it is their periodic nature which can prevent the propagation of certain waves—electrons or light, respectively. This is due to occurring diffractions and reflections within the material and, as a result, destructive interference. Here, in analogy to the electrical case, a region of "forbidden" frequencies of a photonic crystal is referred to as a (*photonic*) *band gap*.

Depending on whether an electromagnetic wave propagates parallel or normal to a direction along which a one- or two-dimensional photonic crystal is homogeneous, it may be more or less affected by the periodic structure. Therefore it can happen that a wave's time frequency falls within a band gap for one direction but not for another. If, however, propagation of waves is prohibited in any case, i.e, regardless of polarization and direction of travel, then the respective frequencies constitute a so-called *complete* (*photonic*) *band gap*. The existence of such a complete forbidden region is, with some few exceptions, reserved for materials that are periodic in all three space directions.

Besides the pattern with which the constituents of a photonic crystal are repeated, its (periodically varying) electromagnetic properties have a bearing on whether or not band gaps exist. Experimentalists and theorists alike found out that a high contrast in the *permittivity* of the utilized components and low *absorptivity* favor forbidden frequency regions (see also our literature review in Section 4.2). Roughly speaking, these two quantities measure how resistant a material is to electric fields and how much impinging electromagnetic energy it is taking up, respectively. In view of this, many photonic crystals are manufactured by enclosing air or vacuum in a non-absorptive background material with high permittivity. The periodic arrangement and the air-to-background volume ratio are then tuned to the particular application.

Further details on the physics of photonic crystals, information on their industrial applications, their manufacturing, and the most well-functioning geometries can be found in [Yab01] and our main references cited above.

3.1.2 CRYSTAL LATTICES

So as to mathematically analyze the band gap phenomena of photonic crystals, it is initially necessary to formally describe their periodic geometry. Of course, solid state physicists and material scientists working in crystallography faced this task long before periodic nanostructures were known. We thus employ their established framework here as in [Kit04, Chapt. 1] and [AM76, Chapts. 4–7].

An idealized photonic crystal, which is for simplicity thought of as being infinite in extent, is characterized by its so-called *Bravais lattice*. This discrete set of points in $\mathbb{R}^3$ reflects the crystal's periodicity in that at each lattice point the rest of the lattice has the same arrangement and orientation. Formally, given linearly independent vectors $\{a_1, a_2, a_3\} \subset \mathbb{R}^3$, a Bravais lattice is a discrete set of points of the form

$$\Theta := \{z_1 a_1 + z_2 a_2 + z_3 a_3 \mid z_1, z_2, z_3 \in \mathbb{Z}\}. \tag{3.1}$$

The vectors a_1, a_2, and a_3 are referred to as the *lattice vectors* (or *lattice translations*) of the photonic crystal and span the parallelepiped

$$\Omega_\Theta := \{r_1 a_1 + r_2 a_2 + r_3 a_3 \mid r_1, r_2, r_3 \in [0,1]\}. \tag{3.2}$$

This volume constitutes an example of a so-called *unit cell*, i.e., a "building block" of dielectric material that forms the photonic crystal by being translated through all the vectors in the Bravais lattice Θ. Regarding this, one distinguishes unit cells that are *primitive*, meaning that they cannot be narrowed without loosing the ability to recreate the periodic structure with their copies. Put otherwise, the volume of primitive cells is minimal among all unit cells. In the schematic examples depicted above, for instance, two (four, eight) adjacent layers (columns, cubes) form a primitive cell. This is shown in Figure 3.2 hereunder. Note from the illustration that a primitive cell is not uniquely determined, since non-cuboid portions of the respective crystals generate the same structure by infinite periodic repetition.

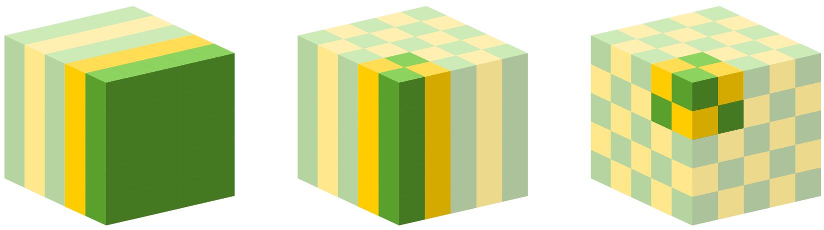

FIGURE 3.2 — Schematic examples of (portions of) one-, two-, and three-dimensional photonic crystals. A possible primitive cell is highlighted in each case.

Media that are only periodic in $d \in \{1,2\}$ dimensions can, without loss of generality, be described with likewise one- or two-dimensional Bravais lattices, which are defined analogously to Θ above. This is possible by choosing lattice vectors in $\mathbb{R}^d$ such that they span the line or plane, respectively, of periodicity of the photonic crystal. With this identification we can treat these structures as if they were lower-dimensional. Once more, this motivates the notions of one- and two-dimensional photonic crystals introduced at the outset of this section.

In Figure 3.3 below we clarify said reduction in dimensionality using the second of the cubes depicted further up on this page as an example. The illustration also shows that different sets of lattice vectors may result in the same Bravais lattice—the one shown being referred to as the *square lattice*. In theoretical considerations it is usually identified with $a\mathbb{Z}^2$ for $a > 0$, i.e., the lattice vectors are chosen as scalar multiples of the cartesian standard basis vectors in $\mathbb{R}^2$. The primitive cell they span is a square with side length a, which is just the horizontal (and vertical) distance between neighboring lattice points. In our mathematical analysis that follows in later chapters we exclusively study structures with this type of two-dimensional Bravais lattice. In fact, we even restrict ourselves to the special case $a = 1$, resulting in the square lattice $\mathbb{Z}^2$ with a unit square as its primitive cell. This choice is a matter of mathematical convenience and all of our results can be transferred to more complicated periodic media with only minimal effort.

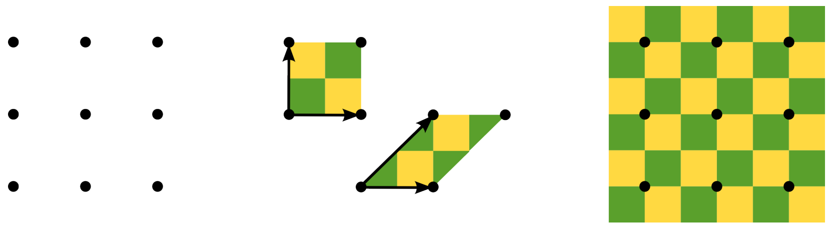

FIGURE 3.3 — From left to right: A portion of the Bravais lattice of a two-dimensional photonic crystal, two possible pairs of lattice vectors and the primitive cells they span, and the structure of the material with indicated lattice points. Note that the rightmost figure arises from our identification of the physically three-dimensional medium with its plane of periodicity.

Given a periodic structure, several *lattice parameters* can be assigned to its Bravais lattice and allow for the definition of so-called *lattice systems*, which group together materials having similar structural properties. As examples of such parameters, we mention the length of the lattice vectors that span a primitive cell, called *lattice constants*, and the *lattice angles* between them. Also, symmetries besides the ever-present translational symmetry, e.g., rotational and reflectional ones, are used as distinguishing features. Since altogether there exist four different lattice systems in two and seven in three dimensions, we once again have to point to the already mentioned books for a complete presentation. Moreover, the connections between lattice systems and group theory are surely worth a look in the corresponding literature, such as [Bor12], [DM07], and [Pow10].

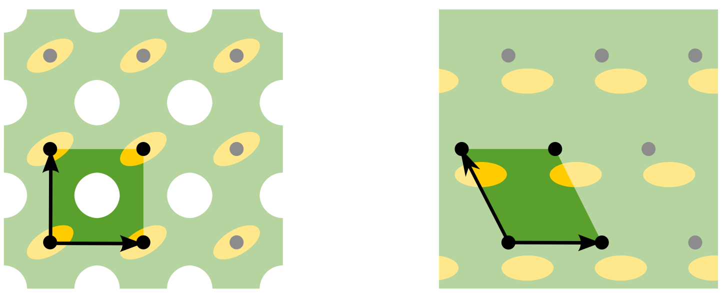

FIGURE 3.4 — Schematic examples of (portions of) two-dimensional photonic crystals with indicated lattice points (square and hexagonal), lattice vectors, and possible primitive cells (highlighted).

As a final example of possible two-dimensional material structures and related Bravais lattices, the figure right above ends this subsection. The photonic crystals schematically depicted therein are not of the checkerboard-type seen up to here, but more similar to the materials manufactured in practice, where white parts would correspond to air or vacuum for instance. Note moreover that the two illustrated media exhibit different Bravais lattices—a square one on the left and the so-called *hexagonal* one on the right—and different symmetry properties beyond translational symmetry. They thus belong to distinct lattice systems.

3.1.3 RECIPROCAL LATTICES

For theoretical purposes, the introduction of a second type of lattice, which is not directly determined by the photonic crystal's geometry, is important. It arises in the study of semiconductor crystal diffraction, i.e., scattering from an array of atoms, and in the Fourier analysis of periodic functions. We present the latter approach as per [Pra09, Sect. 2.2.2] and name [Kit04, Chapt. 2] as a reference for the first-mentioned connection.

Let us consider a function $f : \mathbb{R}^d \to \mathbb{C}$, where $d \in \{1,2,3\}$, which models some property of a d-dimensional photonic crystal that we are interested in. For instance, the permittivity of the structure can often be modeled by such a scalar function. Clearly, f is then periodic with respect to the underlying Bravais lattice Θ (or Θ-*periodic*), i.e.,

$$f(x) = f(x+a) \qquad \text{for all } x \in \mathbb{R}^d \text{ and all } a \in \Theta, \tag{3.3}$$

and is fully determined by its values on a primitive cell Ω_Θ of the lattice.

Being a periodic function we can (formally) expand f in a Fourier series as

$$f(x) = \sum_k c_k e^{ik\cdot x} \qquad \text{for all } x \in \mathbb{R}^d,$$

where c_k denotes the Fourier coefficient corresponding to the plane wave with wave vector k. The periodicity requirement (3.3) now yields a constraint on the set of wave vectors that the last sum is taken over: Since

$$f(x) = \sum_k c_k e^{ik\cdot x} = f(x+a) = \sum_k c_k e^{ik\cdot(x+a)} \qquad \text{for all } x \in \mathbb{R}^d \text{ and all } a \in \Theta,$$

nonvanishing Fourier coefficients exist at most for those k that fulfill $e^{ik\cdot a} = 1$ for all $a \in \Theta$. These wave vectors constitute what is called the *reciprocal lattice* (or *dual lattice*) of Θ, which we denote by Θ^*. More precisely, and with the help of Euler's formula,

$$\Theta^* = \{k \in \mathbb{R}^d \mid k \cdot a \in 2\pi\mathbb{Z} \text{ for all } a \in \Theta\}. \tag{3.4}$$

It can be shown that the reciprocal lattice is itself a Bravais lattice, i.e., there exist linearly independent lattice vectors $b_1, \ldots, b_d$ such that

$$\Theta^* = \{z_1 b_1 + \ldots + z_d b_d \mid z_1, \ldots, z_d \in \mathbb{Z}\},$$

which readily implies that the relation between the lattice vectors of Θ and Θ^* is given by

$$a_i \cdot b_j = 2\pi\delta_{ij} \qquad \text{for } i,j = 1,2,3.$$

Hence, the lattice vectors $b_1, \dots, b_d$ are the columns of the matrix $2\pi(A^T)^{-1}$, where $A \in \mathbb{R}^{d\times d}$ has the columns $a_1, \dots, a_d$. For instance, for the two-dimensional square lattice with side length 1, with which we shall always work in what follows, these relations imply that the corresponding reciprocal lattice is itself square but has side length 2π.

Of course, the concepts introduced in the previous subsection, such as unit and primitive cells, lattice parameters, and lattice systems, can again be used to analyze and categorize reciprocal lattices. In contrast to the rather arbitrary choice of a primitive cell of a photonic crystal's geometric lattice, however, one building block of the reciprocal lattice is of particular relevance. This so-called (*first*) *Brillouin zone* B_{Θ^*} is defined as the closure of the set of all points in $\mathbb{R}^d$ that are closer to the origin than to any other reciprocal lattice point. That is,

$$B_{\Theta^*} := \overline{\{x \in \mathbb{R}^d \mid |x| < |x-k| \text{ for all } k \in \Theta^* \setminus \{0\}\}}. \tag{3.5}$$

The importance of this set lies in its connection to the solutions of partial differential equations with Θ-periodic coefficients as those we study ourselves in Chapters 4 and 5 (see also Subsection 3.2.5 below).

3.2 MAXWELL'S EQUATIONS IN PERIODIC DIELECTRICS

As a problem of classical electromagnetism, the propagation of light in a photonic crystal is governed by the well-known Maxwell equations. In the section at hand, we shall recall this system of equations, whereby we start from a rather general form. By gradually introducing reasonable simplifying assumptions, we eventually arrive at two eigenvalue problems, one of which the reader will recognize as equation (1.2) from the introduction. As a general rule, we neglect questions concerning regularity and function spaces.

Our main references here are the classical treatises [Gri99, Chapts. 7 and 9] and [Jac99, Chapts. 6 and 7]. Whereas the latter texts mainly examine physical aspects, [Mon03] provides a more mathematical approach. With a focus on photonic crystals, the material is given in [Joa08, Chapt. 2].

3.2.1 THE FUNDAMENTAL LAWS OF ELECTROMAGNETISM

Having the application to nanostructures in mind, we start from the *macroscopic*[1] *Maxwell's equations* in SI units[2]. Under the assumption that an electromagnetic

[1] The macroscopic system (3.6) can be seen as describing the average behavior of an electromagnetic field in a volume that is very large compared to the atomic scale. On the other hand, the *microscopic* form of Maxwell's equations rigorously takes phenomena on the atomic level into account. The connection between both formulations is outlined in [Jac99, Sect. 6.6].

[2] See [Mon03, Table 1.1 on p.3].

field occupies the whole three-dimensional space (possibly containing dielectric matter somewhere), they read

$$\begin{cases} \nabla \times E + \dfrac{\partial B}{\partial t} = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \qquad (\textit{Faraday's law of induction}), \\ \nabla \cdot B = 0 & \text{in } \mathbb{R}^3 \times \mathbb{R} \qquad (\textit{Gauss's law for magnetism}), \\ \nabla \times H - \dfrac{\partial D}{\partial t} = J & \text{in } \mathbb{R}^3 \times \mathbb{R} \qquad (\textit{Ampère's circuital law}), \\ \nabla \cdot D = \rho & \text{in } \mathbb{R}^3 \times \mathbb{R} \qquad (\textit{Gauss's law}). \end{cases} \tag{3.6}$$

This system of equations relates the vector fields E, B, D, $H : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$, referred to, in order, as the *electric field*, the *magnetic induction*, the *electric displacement field*, and the *magnetic field*, to given *free current density* $J : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ and *free charge density* $\rho : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$. Here, all six quantities depend on position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. For details on their physical interpretation we refer to the above-mentioned references.

The fields E and B together describe the *electromagnetic field* and in that sense are the fundamental fields.[1] In contrast, the fields D and H are derived and satisfy

$$D = \varepsilon_0 E + P \qquad \text{and} \qquad H = \frac{1}{\mu_0} B - M, \tag{3.7}$$

where P, $M : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ are, in general, spatially- and time-dependent vector fields called the *electric* and the *magnetic polarization density*. The occurring constants ε_0 and μ_0 are referred to as the *vacuum permittivity* and the *vacuum permeability*, respectively.[2] With c_0 denoting the *vacuum speed of light*, there holds

$$\sqrt{\varepsilon_0 \mu_0} = \frac{1}{c_0}. \tag{3.8}$$

The just introduced polarization densities are material-dependent and describe how dielectric matter responds—through induced electric and magnetic dipole moments—to an external electromagnetic field. In all generality P and M depend on both external fields E and B. For many materials used in applications, though, the dependencies reduce to a pair of so-called *constitutive relations*[3]. We emphasize that specifying these relations is necessary in order to even have a chance of solving the otherwise underdetermined system of Maxwell's equations.

[1] For this reason B (instead of H) is sometimes called the magnetic field in the literature. Usually H is then named the *magnetizing field*.

[2] In SI units, their values are given by $\varepsilon_0 \approx 8.854 \times 10^{-12}\ \mathrm{Fm}^{-1}$ and $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{VsA}^{-1}\mathrm{m}^{-1}$ (see [Jac99, Appx. 4]).

[3] There is a certain ambiguity surrounding this term in the literature. Some authors refer to the equations (3.7) as the constitutive relations, although they are actually the definitions of the auxiliary fields D and H. We shall use the expression for any *material-specific* pair of relations of either the form $D = D(E,B)$ and $H = H(E,B)$ or of the form $P = P(E,B)$ and $M = M(E,B)$.

It is beyond the scope of our work, however, to discuss the existence and uniqueness of solutions of the full Maxwell system, so that we refer to [Mon03] in this respect.

Before we come to material assumptions reasonable in the context of photonic crystals below, let us mention that it is convenient for us to transform the equations (3.6) into a simpler form by assuming that the time-dependence of all occurring quantities is of a *harmonic* (or *sinusoidal*) manner. That is, for a fixed frequency $\omega \in [0, \infty)$ there shall hold, e.g.,

$$E(x,t) = \mathrm{Re}\left[E_\omega(x)e^{-i\omega t}\right] \quad \text{for all } x \in \mathbb{R}^3 \text{ and all } t \in \mathbb{R} \tag{3.9}$$

with $E_\omega : \mathbb{R}^3 \to \mathbb{C}^3$ being a complex-valued vector field. Analogously we introduce the variables B_ω, D_ω, H_ω, P_ω, $M_\omega : \mathbb{R}^3 \to \mathbb{C}^3$. For consistency this requires that the data J and ρ be time-harmonic with amplitudes $J_\omega : \mathbb{R}^3 \to \mathbb{C}^3$ and $\rho_\omega : \mathbb{R}^3 \to \mathbb{C}$. This ansatz is justified in many applications, for instance, whenever the considered field is generated by a sinusoidally varying source current, or as an approximation when only a very narrow frequency region is of interest.

Substituting sinusoidal relations as in (3.9) for all unknowns and the two density functions in (3.6) yields the *time-harmonic Maxwell equations*

$$\begin{cases} \nabla \times E_\omega - i\omega B_\omega = 0 & \text{in } \mathbb{R}^3, \\ \nabla \cdot B_\omega = 0 & \text{in } \mathbb{R}^3, \\ \nabla \times H_\omega + i\omega D_\omega = J_\omega & \text{in } \mathbb{R}^3, \\ \nabla \cdot D_\omega = \rho_\omega & \text{in } \mathbb{R}^3. \end{cases} \tag{3.10}$$

Note that formally these equations can also be obtained by a Fourier transform in time of all fields and densities occurring in the system (3.6).

The most important feature of Maxwell's equations in their time-harmonic formulation is the absence of any dependence on time. This has benefits for numerical computations and also simplifies a rigorous analytical treatment. It is, however, important to remark that the physically meaningful components of the electromagnetic field need to be recovered by multiplying a solution $(E_\omega, B_\omega, D_\omega, H_\omega)$ of the system (3.10) by $e^{-i\omega t}$ and taking the real part thereafter (compare to (3.9)).

3.2.2 MATERIAL ASSUMPTIONS AND CONSTITUTIVE RELATIONS

Recall from the initial section of this chapter that the nanostructures we are interested in are comprised of finitely many different dielectric materials. The current subsection discusses assumptions on the properties of these constituents and the respective consequences for Maxwell's equations. The periodic nature of a photonic crystal, however, is not yet incorporated here (see Subsection 3.2.5 in that respect), so that the equations we derive are likewise valid for appropriate media with less spatial structure. Besides the main references for this section we rely on [Joa08, Chapt. 2] and [ST07, Chapt. 5] in our presentation.

Any medium that we study in this thesis shall always be infinite in extent, whereby we neglect boundary effects[1] and can work with Maxwell's equations in all of $\mathbb{R}^3$ as in (3.6) and (3.10) above. This idealization is justified, since the structures that are fabricated are very large compared to the volume of material that constitutes their boundaries. Further, we suppose that there are no free currents or charges present, giving $J = 0$ and $\rho = 0$, and that the structure of the considered medium does not vary with time.

With respect to the magnetic and electric properties of the treated materials we always suppose that there hold constitutive relations of the form

$$\begin{aligned} D(x,t) &= \varepsilon_0 E(x,t) \\ &\quad + \varepsilon_0 \int_{-\infty}^{\infty} G(x,\tau) E(x,t-\tau)\, d\tau \quad \text{for all } x \in \mathbb{R}^3 \text{ and all } t \in \mathbb{R}, \qquad (3.11) \\ H(x,t) &= \frac{1}{\mu_0} B(x,t) \qquad\qquad\qquad\qquad \text{for all } x \in \mathbb{R}^3 \text{ and all } t \in \mathbb{R}, \qquad (3.12) \end{aligned}$$

where $G : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is the so-called *susceptibility kernel* (or *dielectric response function*) satisfying the physically motivated conditions

$$G(x,s) = 0 \qquad \text{for all } x \in \mathbb{R}^3 \text{ and all } s \in (-\infty, 0), \tag{3.13}$$

$$\lim_{s \to \infty} G(x,s) = 0 \qquad \text{for all } x \in \mathbb{R}^3. \tag{3.14}$$

Note here that the existence of the last-mentioned integral is an additional assumption on the function G.

Dependencies as in (3.11) and (3.12) are generally used to model an infinitely extended, linear, isotropic, inhomogeneous, dispersive, and non-magnetic medium.[2] Clearly, the last-mentioned property implies a vanishing magnetic polarization, resulting, by the second equality in (3.7), in the simple proportionality (3.12) between H and B. The remaining material characteristics mentioned refer to the structure of the relation between the electric polarization and the applied electric field. First, for a *linear* material, the dependence of P on E is linear in the mathematical sense. Next, a medium is *inhomogeneous* if the P-E relation varies spatially and *isotropic* if this relation is independent of the direction of the E-field.[3] Finally, a medium is said to be *dispersive* if it does not respond instantaneously to the applied electric field.[4] Its electric polarization is then given by a convolution as on the right-hand side of equation (3.11) and as such depends on the past

[1] Note that there are still interior boundary effects, such as continuity and jump conditions on the propagating fields, at interfaces between the different materials which constitute the infinite medium. These relationships can be deduced from integral equivalents of Maxwell's equations as it is done in [Jac99, Sect. 1.5].

[2] What sounds like a multitude of material restrictions are actually common properties of photonic crystals that are built and experimented upon in practice (see [Joa08, Chapt. 2]).

[3] The latter implies that P and E are parallel at all times and all positions.

[4] Since some delay is always present in physical systems, an instantaneous reaction can only be an idealization. Nevertheless, it can still be modeled with a constitutive relation of the form (3.11) with a susceptibility kernel $G(x,\tau) = g(x)\delta(\tau)$ where δ denotes the Dirac delta.

time behavior of E. At this, the requirements (3.13) and (3.14) guarantee *causality*, i.e., the polarization at some fixed time can neither be determined by the electric field at future time points nor—at least not considerably—by those that are long bygone.

Having all of the above material assumptions in mind, we turn to the consideration of time-harmonic fields again. With the complex-valued amplitudes introduced in the first part of this section (carrying the fixed non-negative frequency ω as a subscript) the constitutive relations (3.11) and (3.12) become

$$D_\omega(x) = \varepsilon_0 \left(1 + \int_{-\infty}^{\infty} G(x,t) e^{i\omega t}\, dt\right) E_\omega(x) \qquad \text{for all } x \in \mathbb{R}^3, \tag{3.15}$$

$$H_\omega(x) = \frac{1}{\mu_0} B_\omega(x) \qquad \text{for all } x \in \mathbb{R}^3, \tag{3.16}$$

which can also be obtained formally by a Fourier transform in time and the convolution theorem.[1] The first of these two relations is usually written differently, namely as

$$D_\omega(x) = \varepsilon_0 \left(1 + \chi_\omega(x)\right) E_\omega(x) \qquad \text{for all } x \in \mathbb{R}^3, \tag{3.17}$$

where we introduced the *electric susceptibility* $\chi : \mathbb{R}^3 \times [0,\infty) \to \mathbb{C}$ and defined $\chi_\omega := \chi(\cdot,\omega)$ for $\omega \in [0,\infty)$. This notation shall highlight once more that we consider a fixed frequency of the time-harmonic field.

Note from the relation (3.17) that χ is the inverse Fourier transform in time of the susceptibility kernel G, which also explains the name of the latter function.[2] The assumed dispersiveness of the modeled medium is responsible for the frequency-dependence of χ and thus also for that of the so-called *permittivity* $\varepsilon : \mathbb{R}^3 \times [0,\infty) \to \mathbb{C}$ given by

$$\varepsilon := \varepsilon_0 \varepsilon_\mathrm{r} := \varepsilon_0 (1 + \chi), \tag{3.18}$$

which is the proportionality function between the fields D and E.[3] We remark that the dimensionless *relative permittivity* (or *dielectric function*) $\varepsilon_\mathrm{r} = \varepsilon/\varepsilon_0 = 1 + \chi$ appearing in (3.18) is sometimes also denoted by ε in the literature. To avoid confusion, we keep the subscript "r" for the relative quantity in this work. Besides, we put $\varepsilon_{\mathrm{r},\omega} := 1 + \chi(\cdot,\omega)$ for $\omega \in [0,\infty)$ with the same motivation as for χ_ω above.

We shall not go into details on the mathematical properties of permittivity functions here, but refer to Section 3.3 for their discussion. Instead, to close this

[1] For equation (3.15) to be correct we obviously have to require $G(x,\cdot) \in L^1(\mathbb{R})$ for all $x \in \mathbb{R}^3$.

[2] At least qualitatively. Of course, whether the occurring integral is regarded as a Fourier transform, its inverse, or a quantity proportional to either of them is just a matter of convention.

[3] For some authors the frequency-dependence of ε is the very definition of the notion "dispersiveness", e.g., for J. D. Jackson in [Jac99]. This author deduces, in a procedure inverse to ours, that dispersive media (in his sense) respond non-instantaneously to applied electric fields (see [ibid., Sect. 7.10]). Thus, our above definition of a dispersive medium and that through a frequency-dependent permittivity are equivalent.

paragraph on consequences of certain material properties, let us exploit the constitutive relations (3.16) and (3.17), the equality (3.8) for the occurring constants, and the definition of the permittivity (3.18) so as to rewrite Maxwell's equations (3.10) once again. In their time-harmonic formulation for an infinitely extended, linear, isotropic, inhomogeneous, dispersive, and non-magnetic medium they read

$$\begin{cases} \nabla \times E_\omega - i\omega B_\omega = 0 & \text{in } \mathbb{R}^3, \\ \nabla \cdot B_\omega = 0 & \text{in } \mathbb{R}^3, \\ \nabla \times B_\omega + \dfrac{i\omega}{c_0^2}\varepsilon_{\mathrm{r},\omega} E_\omega = 0 & \text{in } \mathbb{R}^3, \\ \nabla \cdot (\varepsilon_{\mathrm{r},\omega} E_\omega) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.19}$$

where we stress again that the frequency ω is fixed and the complex-valued amplitudes E_ω and B_ω have real parts corresponding to the physical fields.

From now on, whenever we refer to Maxwell's equations in this thesis, we mean the last-mentioned system. It constitutes the appropriate model for wave propagation in dielectric media having the features assumed above and can thus be applied when we specialize our considerations to photonic crystals in what follows. To do so, we will only have to impose further restrictions on the permittivity, which will, in particular, reflect the periodic structure of the studied materials.

3.2.3 TIME-HARMONIC MAXWELL EIGENVALUE PROBLEMS

The attentive reader has surely noticed that Maxwell's equations in the form (3.19) only relate two, instead of formerly four, vector fields. This allows us to decouple them as follows: Assuming that the dielectric function $\varepsilon_{\mathrm{r},\omega}$ is nonzero everywhere (see our discussion in Section 3.3), we deduce, by applying the curl operator to the first equation of (3.19) and dividing it by $\varepsilon_{\mathrm{r},\omega}$,

$$\frac{1}{\varepsilon_{\mathrm{r},\omega}} \nabla \times (\nabla \times E_\omega) = \frac{i\omega}{\varepsilon_{\mathrm{r},\omega}} \nabla \times B_\omega = \frac{\omega^2}{c_0^2} E_\omega \qquad \text{in } \mathbb{R}^3,$$

where the last equality follows from the third equation of the system. Complemented with the corresponding divergence constraint in Maxwell's equations we arrive at the spectral problem

$$\begin{cases} \dfrac{1}{\varepsilon_{\mathrm{r},\omega}} \nabla \times (\nabla \times E_\omega) = \dfrac{\omega^2}{c_0^2} E_\omega & \text{in } \mathbb{R}^3, \\ \nabla \cdot (\varepsilon_{\mathrm{r},\omega} E_\omega) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.20}$$

which we call the (constrained) *Maxwell eigenvalue problem for the electric field*.

To obtain similar equations for the magnetic induction, we exchange the roles of the two Maxwell equations used in our derivation. That is, we divide the third equation of (3.19) by $\varepsilon_{r,\omega}$, apply the curl operator, and finally employ the first equation of the system to obtain

$$\nabla \times \left(\frac{1}{\varepsilon_{r,\omega}} \nabla \times B_\omega \right) = -\frac{i\omega}{c_0^2} \nabla \times E_\omega = \frac{\omega^2}{c_0^2} B_\omega \qquad \text{in } \mathbb{R}^3.$$

Together with the restriction on the divergence of B_ω this equation constitutes the (constrained) *Maxwell eigenvalue problem for the magnetic induction*[1]

$$\begin{cases} \nabla \times \left(\dfrac{1}{\varepsilon_{r,\omega}} \nabla \times B_\omega \right) = \dfrac{\omega^2}{c_0^2} B_\omega & \text{in } \mathbb{R}^3, \\ \nabla \cdot B_\omega = 0 & \text{in } \mathbb{R}^3. \end{cases} \tag{3.21}$$

For the physically interesting positive frequencies both spectral problems we derived share the characteristic that the occurring divergence constraint is, at least formally, automatically fulfilled for any eigensolution.[2] This is due to the respective first equations in (3.20) and (3.21), giving that B_ω and $\varepsilon_{r,\omega} E_\omega$ are curl fields. Furthermore, likewise only for positive frequencies, we remark that the first and third Maxwell equation in (3.19) can be solved for the amplitudes E_ω and B_ω as

$$E_\omega = \frac{ic_0^2}{\omega \varepsilon_{r,\omega}} \nabla \times B_\omega \quad \text{in } \mathbb{R}^3 \qquad \text{and} \qquad B_\omega = \frac{1}{i\omega} \nabla \times E_\omega \quad \text{in } \mathbb{R}^3, \tag{3.22}$$

so that an eigensolution of either spectral problem determines the respective other one. In particular, if one of the eigenvalue problems has only the trivial solution, then so does the other, which gives information about the (non-)propagation of electromagnetic waves inside the considered material. The latter is further explained in Subsection 3.2.5 as well as in Chapters 4 and 5.

3.2.4 TWO-DIMENSIONAL MEDIA AND POLARIZED FIELDS

To further simplify the Maxwell eigenvalue problems, let us assume that the properties of the medium we study wave propagation in are independent of one spatial coordinate. As the material data in the setting of the preceding paragraph are given by the relative permittivity, this means $\varepsilon_r(x,\omega) = \varepsilon_r(x_1, x_2, \omega)$ for all $(x,\omega) \in \mathbb{R}^3 \times [0,\infty)$. In what follows we refer to such dielectrics as *two-dimensional*.[3]

Due to the homogeneity of the medium in the x_3-direction, it is reasonable to assume that the electromagnetic field is likewise only (x_1, x_2)-dependent. We

[1] Due to the simple relation $B_\omega = \mu_0 H_\omega$ (see (3.16)), this can also be seen as a constrained eigenvalue problem for the magnetic field H (or, strictly speaking, for its amplitude).

[2] For $\omega = 0$ in either problem the respective kernel is enlarged by neglecting the divergence constraint. It consists of longitudinal waves, i.e., gradients of scalar functions (see [Kuc01, Sect. 7.2.2]).

[3] If we are concerned with a photonic crystal, then this notion of "two-dimensionality" and that introduced in the first section of this chapter (see p. 26) agree.

model it with time-harmonic fields having amplitudes $E_\omega, B_\omega : \mathbb{R}^2 \to \mathbb{C}^3$ and fixed frequency $\omega \in [0, \infty)$.

The reduction of dimensionality greatly simplifies the Maxwell eigenvalue problems (3.20) and (3.21). To realize this, we first split E_ω and B_ω as

$$E_\omega = \begin{pmatrix} E_{\omega,1} \\ E_{\omega,2} \\ E_{\omega,3} \end{pmatrix} = \begin{pmatrix} E_{\omega,1} \\ E_{\omega,2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ E_{\omega,3} \end{pmatrix} =: E_\omega^{\mathrm{TE}} + E_\omega^{\mathrm{TM}}, \tag{3.23}$$

and

$$B_\omega = \begin{pmatrix} B_{\omega,1} \\ B_{\omega,2} \\ B_{\omega,3} \end{pmatrix} = \begin{pmatrix} B_{\omega,1} \\ B_{\omega,2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ B_{\omega,3} \end{pmatrix} =: B_\omega^{\mathrm{TM}} + B_\omega^{\mathrm{TE}}, \tag{3.24}$$

where $E_{\omega,j}$, $B_{\omega,j} : \mathbb{R}^2 \to \mathbb{C}$ for $j = 1,2,3$ denote the scalar components of the fields. Here, the superscripts "TE" and "TM" abbreviate *transverse electric* and *transverse magnetic*, respectively. These terms refer to the orientation of the oscillations of the electromagnetic field, which is also known as its *polarization*.[1] In case of wave propagation in two-dimensional media, we speak of a *TE-field* if it is of the form above, i.e., its magnetic induction is parallel and its electric field is normal to the axis that the medium is homogeneous in. Similarly, we are concerned with a *TM-field* if its electric field is parallel and its magnetic induction is normal to the direction of homogeneity.[2]

By calculating the action of the curl-curl operator on the fields E_ω^{TE} and E_ω^{TM}, we see that the Maxwell eigenvalue problem (3.20) for an amplitude E_ω of the form (3.23) turns into the pair of spectral problems

$$\begin{cases} \dfrac{1}{\varepsilon_{\mathrm{r},\omega}} \nabla \times \left(\nabla \times E_\omega^{\mathrm{TE}}\right) = \dfrac{1}{\varepsilon_{\mathrm{r},\omega}} \begin{pmatrix} \frac{\partial^2 E_{\omega,2}}{\partial x_2 \partial x_1} - \frac{\partial^2 E_{\omega,1}}{\partial x_2^2} \\ \frac{\partial^2 E_{\omega,1}}{\partial x_1 \partial x_2} - \frac{\partial^2 E_{\omega,2}}{\partial x_1^2} \\ 0 \end{pmatrix} = \dfrac{\omega^2}{c_0^2} E_\omega^{\mathrm{TE}} & \text{in } \mathbb{R}^2, \\ \nabla \cdot \left(\varepsilon_{\mathrm{r},\omega} E_\omega^{\mathrm{TE}}\right) = 0 & \text{in } \mathbb{R}^2 \end{cases} \tag{3.25}$$

and

$$\begin{cases} \dfrac{1}{\varepsilon_{\mathrm{r},\omega}} \nabla \times \left(\nabla \times E_\omega^{\mathrm{TM}}\right) = \dfrac{1}{\varepsilon_{\mathrm{r},\omega}} \begin{pmatrix} 0 \\ 0 \\ -\Delta E_{\omega,3} \end{pmatrix} = \dfrac{\omega^2}{c_0^2} E_\omega^{\mathrm{TM}} & \text{in } \mathbb{R}^2, \\ \nabla \cdot \left(\varepsilon_{\mathrm{r},\omega} E_\omega^{\mathrm{TM}}\right) = 0 & \text{in } \mathbb{R}^2. \end{cases} \tag{3.26}$$

Regarding this, the linearity of the curl operator and the x_3-independence of all occurring quantities are important. A solution of the original problem (3.20)

[1] This shall not be confused with the polarizations P and M introduced before. We carefully make sure that the context always clarifies what type of polarization is meant in this work.

[2] Note that this is the standard definition found in literature on wave propagation in photonic crystals (see [Kuc01, Sect. 7.2.5]). However, in the context of wave guide problems "transverse electric" and "tranverse magnetic" are sometimes used conversely.

exists, and then has the form (3.23), whenever the reduced problems (3.25) and (3.26) both have a solution corresponding to the same eigenvalue ω^2/c_0^2. Two similar spectral problems—one governing the TM-polarized part and the other the TE-polarized part of the amplitude B_ω—can be obtained from the Maxwell eigenvalue problem (3.21), but we omit their explicit statement for brevity and since the magnetic induction will not be of importance in this work.

What is relevant, though, is that for positive frequencies E_ω and B_ω are related to each other by the equations (3.22), i.e.,

$$\begin{aligned} E_\omega &= \frac{ic_0^2}{\omega\varepsilon_{r,\omega}}\nabla\times B_\omega^{\mathrm{TM}} + \frac{ic_0^2}{\omega\varepsilon_{r,\omega}}\nabla\times B_\omega^{\mathrm{TE}} && \text{in } \mathbb{R}^2, \\ B_\omega &= \frac{1}{i\omega}\nabla\times E_\omega^{\mathrm{TE}} + \frac{1}{i\omega}\nabla\times E_\omega^{\mathrm{TM}} && \text{in } \mathbb{R}^2. \end{aligned}$$

The definition of the curl operator and the uniqueness of the splittings (3.23) and (3.24) yield that there necessarily holds

$$E_\omega^{\mathrm{TE}} = \frac{ic_0^2}{\omega\varepsilon_{r,\omega}}\nabla\times B_\omega^{\mathrm{TE}} \quad \text{in } \mathbb{R}^2 \qquad \text{and} \qquad B_\omega^{\mathrm{TM}} = \frac{1}{i\omega}\nabla\times E_\omega^{\mathrm{TM}} \quad \text{in } \mathbb{R}^2. \tag{3.27}$$

Hence, solving the eigenvalue problems for the polarized parts E_ω^{TM} and B_ω^{TE} is sufficient to recover the "full" electromagnetic field amplitudes E_ω and B_ω as long as the case $\omega = 0$ is not of interest.

Thus far it is not obvious that the splitting of both fields into TM- and TE-polarized part indeed reduced the complexity of our task, since after all we still have two spectral problems to solve. However, the one for E_ω^{TM}, i.e., (3.26), which then determines B_ω^{TM} by means of the second equation in (3.27), is not vector-valued, but "just" a scalar problem of Helmholtz type reading

$$-\frac{1}{\varepsilon_{r,\omega}}\Delta E_{\omega,3} = \frac{\omega^2}{c_0^2}E_{\omega,3} \qquad \text{in } \mathbb{R}^2. \tag{3.28}$$

Here it is important to note that

$$\nabla\cdot\left(\varepsilon_{r,\omega}E_\omega^{\mathrm{TM}}\right) = \frac{\partial(\varepsilon_{r,\omega}E_{\omega,3})}{\partial x_3} = 0 \qquad \text{in } \mathbb{R}^2,$$

i.e., the simple structure of the TM-polarized field and its independence of the third coordinate permits the omission of the divergence constraint.[1] Similarly, the scalar problem of divergence type

$$-\nabla\cdot\left(\frac{1}{\varepsilon_{r,\omega}}\nabla B_{\omega,3}\right) = \frac{\omega^2}{c_0^2}B_{\omega,3} \qquad \text{in } \mathbb{R}^2 \tag{3.29}$$

can be deduced from the eigenvalue problem for B_ω^{TE}, where the corresponding divergence constraint is again automatically satisfied.[2]

[1] Due to the independence of E_0^{TM} on x_3 this is even true in the case $\omega = 0$.
[2] The first footnote on this page likewise applies here (with E_0^{TM} replaced by B_0^{TE}).

In what follows, we refer to the spectral problems (3.28) and (3.29) as the *Maxwell eigenvalue problem for TM-polarized waves* and the *Maxwell eigenvalue problem for TE-polarized waves*, respectively. As we outlined in detail, together they govern the propagation of polarized electromagnetic waves in a two-dimensional, infinitely extended, linear, isotropic, inhomogeneous, dispersive, and non-magnetic dielectric medium. It is crucial to observe here that the dispersiveness accounts for a *parameter-nonlinearity* of the spectral problems. That is, the differential operators on the left-hand sides of (3.28) and (3.29) depend, by means of their coefficient function $\varepsilon_{\mathrm{r},\omega} = \varepsilon_{\mathrm{r}}(\cdot,\omega)$, on the spectral parameter ω. We shall see in Chapter 5 that it is this very frequency-dependence which complicates the analysis of the problems.

To close, we remark that the rest of this dissertation is exclusively concerned with the study of the Maxwell eigenvalue problem for TM-polarized waves, which is, unsurprisingly, known from the introduction to our work. The TE-polarized case shall not be covered here and remains, at least to date and for dispersive materials, still largely untreated in the mathematical literature.

3.2.5 PERIODICITY AND BLOCH'S THEOREM

Recall that we did not require the considered material in our preceding derivation of the Maxwell eigenvalue problems to have a periodic structure. On the one hand our model is thus so far quite general and can, for instance, be of use to study light propagation problems in wave guides or other media that exhibit no or only partial periodicity.[1] On the other hand, however, any effects that the spatial structure of a photonic crystal might have on the eigenvalue problems are not yet taken into account. Changing this is the aim of the present paragraph.

Suppose, as always from now on, that the structure of interest is a two-dimensional photonic crystal with corresponding Bravais lattice Θ and primitive cell Ω_Θ. Clearly, any function modeling material properties of such a medium must be Θ-periodic. That is, for $\omega \in [0,\infty)$,

$$\varepsilon_{\mathrm{r}}(x,\omega) = \varepsilon_{\mathrm{r}}(x+a,\omega) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } a \in \Theta \tag{3.30}$$

and similarly for the susceptibility kernel G and thus the susceptibility χ. Moreover, these functions are naturally discontinuous since the photonic crystal is composed of different dielectric materials in a likewise non-smooth manner.

Due to the property (3.30) the Maxwell eigenvalue problem for TM polarized waves (3.28) is, in the context of photonic crystals, a *periodic eigenvalue problem*. Thus, one may well ask whether the spectrum of this problem is somehow related to that of a similar eigenvalue equation posed on a primitive cell of the underlying Bravais lattice. After all, the periodic coefficient function $\varepsilon_{\mathrm{r},\omega}$ of the problem is fully specified once its values on, e.g., Ω_Θ are known.

An indication as to why one can hope for a positive answer is found in F. Bloch's 1929 Article [Blo29]. He studied electron wave propagation in crystal

[1] As long as the assumptions on the involved materials and fields are still in place.

lattices and showed that any solution of the respective equation of motion (the Schrödinger equation with a periodic potential) is given by products of lattice periodic functions and plane waves with wave vectors in the first Brillouin zone. This result is nowadays known as *Bloch's theorem* and of the form indicated above, i.e., it provides a link between a spectral problem on the whole space and a similar one on a primitive cell.

Assuming that we can proceed in a like fashion to Bloch—and leaving all mathematical rigor aside for the rest of this paragraph—we make a product form ansatz for an eigensolution of the Maxwell eigenvalue problem (3.28). More precisely, we suppose that the TM-polarized wave has a time-harmonic electric field amplitude with third component given by

$$E_{\omega,3}(x) = E_{\omega,k}(x)e^{ik\cdot x} \qquad \text{for all } x \in \mathbb{R}^2. \tag{3.31}$$

Here, $E_{\omega,k}$ is a complex-valued Θ-periodic function and $k \in B_{\Theta^*}$ is a vector in the associated first Brillouin zone (see Subsection 3.1.3) called *quasimomentum vector* (or *crystal momentum vector*). For this so-called *Bloch wave* the spectral problem (3.28) takes the form

$$-\frac{1}{\varepsilon_{\mathrm{r},\omega}}(\nabla + ik)\cdot(\nabla + ik)E_{\omega,k} = \frac{\omega^2}{c_0^2}E_{\omega,k} \qquad \text{in } \mathbb{R}^2$$

which is readily seen by a direct calculation. Since both the coefficient $\varepsilon_{\mathrm{r},\omega}$ and the function $E_{\omega,k}$ are periodic with respect to Θ, it suffices to solve the equation (3.32) on any primitive cell of the lattice. Choosing Ω_Θ for instance, this results in the periodic eigenvalue problem

$$-\frac{1}{\varepsilon_{\mathrm{r},\omega}}(\nabla + ik)\cdot(\nabla + ik)E_{\omega,k} = \frac{\omega^2}{c_0^2}E_{\omega,k} \qquad \text{in } \Omega_\Theta, \tag{3.32}$$

or rather in a family of such eigenproblems parametrized by $k \in B_{\Theta^*}$.

Now suppose that an eigenpair $({}^{\omega^2}/_{c_0^2}, E_{\omega,k})$ of the equation (3.32) exists. Then, by the ansatz (3.31) and Θ-periodic extension of $E_{\omega,k}$, we find the associated TM-polarized electric field amplitude on the whole of $\mathbb{R}^2$. Together with the corresponding TM-polarized magnetic field amplitude obtained by the second equation in (3.27) this constitutes a time-harmonic TM-field which is able to propagate inside the photonic crystal. On the other hand, a nontrivial TM-field of the form (3.31) with prescribed frequency ω can only exist inside the material if ${}^{\omega^2}/_{c_0^2}$ is an eigenvalue of (3.32) for some $k \in B_{\Theta^*}$. If this is not the case, ω lies, at least for such TM-waves, in a band gap of the crystal.

Since the above probably raised several questions for the mathematically acquainted reader, we remark that our treatment of an appropriate operator-theoretical realization of the spectral problem (3.32) will surely answer them. We refer to Chapters 4 and 5 in this respect.

3.3 PERMITTIVITY FUNCTIONS

In the previous section we derived the Maxwell eigenvalue problems for the electric field and the magnetic induction for a certain class of dielectric materials. In both these spectral problems the only appearing coefficient function of the respective operators is the relative permittivity and it is precisely this function by which material properties enter the problems. In this sense the dielectric media permissible under our model are fully described by ε_{r} and we therefore devote the section at hand to examining its properties. Note that the treatise below applies to any linear, isotropic, inhomogeneous, dispersive, and non-magnetic medium modeled by the spectral problems (3.20) and (3.21). Specializing to photonic crystals only results in additional periodicity of the relative permittivity.

The most important references for this section are [LLP84, Chapt. 9][1] and [Jac99, Chapt. 7]. Besides, the second subsection incorporates material that is discussed in [ST07, Chapt. 5].

3.3.1 GENERAL CHARACTERISTICS

Recall from Section 3.2 that under our assumptions for any two-dimensional[2] dispersive medium having the above-mentioned material properties, the relation between the physical fields D and E is given by

$$\begin{aligned} D(x,t) &= \varepsilon_0 E(x,t) \\ &\quad + \varepsilon_0 \int_{-\infty}^{\infty} G(x,\tau) E(x,t-\tau)\, d\tau \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } t \in \mathbb{R}. \end{aligned} \tag{3.33}$$

It shall be important to repeat that the second summand on the right-hand side of this equation is the polarization density P (see (3.7)). It accounts for the induced dipole moments within the considered dielectric due to the applied electric field E. Besides, we remark that the susceptibility kernel G occurring under the integral sign is subject to the causality requirements (3.13) and (3.14) and, after assuming time-harmonic fields, related to the permittivity ε through

$$\begin{aligned} \varepsilon(x,\omega) &= \varepsilon_0 (1 + \chi(x,\omega)) \\ &= \varepsilon_0 \left(1 + \int_0^{\infty} G(x,t) e^{i\omega t}\, dt\right) \quad \text{for all } x \in \mathbb{R}^2 \text{ and all } \omega \in [0,\infty). \end{aligned} \tag{3.34}$$

Note that the last equality is a consequence of the validity of the relations (3.15) and (3.17) at any non-negative frequency. This integral representation is a direct result of the causality relation (3.33) and reveals several important properties of the permittivity.

[1] This book uses CGS units whereas we present all formulae in the SI system. Jackson's work [Jac99] provides an "Appendix on units and Dimensions", which also contains helpful tables that allow for an easy conversion between these different systems.

[2] Knowing what is needed later on, we deliberately fix the dimension to 2 here. With the obvious notational differences, the contents of this section are likewise valid for one- and three-dimensional materials.

Before we elaborate on this connection, we comment on the high-frequency behavior of ε. Since we impose certain mathematical restrictions in this respect later on, knowing what is physically reasonable is of course important. To begin with, a high frequency corresponds to a rapidly changing time-harmonic electromagnetic field, which, in theory, results in a likewise rapid formation of electric dipole moments within the material the field propagates in. The physical processes that occur here, however, cannot go on infinitely fast, so the polarization density P has to vanish in the limit $\omega \to \infty$. With the frequency-dependent form of the relation $D = \varepsilon_0 E + P = \varepsilon E$ (see (3.15), (3.17), and (3.18)) this implies

$$\lim_{\omega\to\infty} \varepsilon(x,\omega) = \varepsilon_0 \qquad \text{for all } x \in \mathbb{R}^2. \tag{3.35}$$

Thus, by (3.34), the frequency components of the susceptibility kernel G also have to vanish as ω tends to infinity. Moreover, from physically more profound discussions of the permittivity at high frequencies (see [Jac99, Sect. 7.10.C] and [LLP84, §78]) we can deduce that there hold asymptotics of the form

$$\operatorname{Re}\varepsilon(x,\omega) = \varepsilon_0 + \left[\mathcal{O}\left(\frac{1}{\omega^2}\right)\right](x) \quad \text{as } \omega \to \infty \qquad \text{for all } x \in \mathbb{R}^2, \tag{3.36}$$

$$\operatorname{Im}\varepsilon(x,\omega) = \left[\mathcal{O}\left(\frac{1}{\omega^3}\right)\right](x) \qquad \text{as } \omega \to \infty \qquad \text{for all } x \in \mathbb{R}^2. \tag{3.37}$$

Let us now revisit the equation (3.34). In order to see its implications for the properties of the permittivity, we consider ω as a complex variable and extend the function in the second argument to the closed upper complex half-plane $\mathbb{H}_0 := \mathbb{H} \cup \mathbb{R}$ through

$$\varepsilon(x,\omega) := \varepsilon_0 \left(1 + \int_0^\infty G(x,t)e^{i\omega t}\, dt\right) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } \omega \in \mathbb{H}_0. \tag{3.38}$$

We remark that this extended function still has a physical interpretation but refer the reader to [LLP84, §82] for details. In view of the discussion there, it is appropriate that we keep using the symbol ε and the name "permittivity" for the function defined in (3.38).

Readily seen consequences of this extension are the following symmetry properties of the permittivity: Writing $\varepsilon = \operatorname{Re}\varepsilon + i \operatorname{Im}\varepsilon$, we obtain

$$\begin{aligned} \operatorname{Re}\varepsilon(x,-\omega) &= \operatorname{Re}\varepsilon(x,\omega) && \text{for all } x \in \mathbb{R}^2 \text{ and all } \omega \in \mathbb{R}, \\ \operatorname{Im}\varepsilon(x,-\omega) &= -\operatorname{Im}\varepsilon(x,\omega) && \text{for all } x \in \mathbb{R}^2 \text{ and all } \omega \in \mathbb{R}, \end{aligned}$$

which show that ε has a real part even and an imaginary part odd in real frequencies. Knowing this is helpful if $\operatorname{Re}\varepsilon(x,\cdot)|_{\mathbb{R}}$ and $\operatorname{Im}\varepsilon(x,\cdot)|_{\mathbb{R}}$ are approximated by power series expansions in applications. The former function must necessarily be modeled with only even powers and the latter with only odd ones.

More sophisticated results on the real and imaginary part of the permittivity can be deduced from the fact that, for any fixed $x \in \mathbb{R}^d$, the relation (3.38) defines a holomorphic function of ω in $\mathbb{H}_0$. This is essentially a consequence of the properties (3.13) and (3.14) of G and outlined in [Jac99, Sect. 7.10.D]. There, based on Cauchy's integral theorem, the so-called *Kramers-Kronig relations*[1]

$$\operatorname{Re}\varepsilon(x,\omega) = \varepsilon_0 + \frac{2}{\pi}\,\mathcal{P}\left[\int_0^\infty \frac{\omega' \operatorname{Im}\varepsilon(x,\omega')}{\omega'^2-\omega^2}\,d\omega'\right] \quad \text{for all } x\in\mathbb{R}^2 \text{ and all } \omega\in\mathbb{H}_0,$$
$$\operatorname{Im}\varepsilon(x,\omega) = -\frac{2\omega\varepsilon_0}{\pi}\,\mathcal{P}\left[\int_0^\infty \frac{\operatorname{Re}\varepsilon(x,\omega')-\varepsilon_0}{\omega'^2-\omega^2}\,d\omega'\right] \quad \text{for all } x\in\mathbb{R}^2 \text{ and all } \omega\in\mathbb{H}_0$$

are derived, where the symbol $\mathcal{P}$ stands for the Cauchy principal value (see [Hac95, Chapt. 7]). Note that these formulae relate real and imaginary part of the permittivity at a point $\omega \in \mathbb{H}_0$ to the values of these functions on the physical frequencies in $[0,\infty)$. In particular, the knowledge of either $\operatorname{Re}\varepsilon$ or $\operatorname{Im}\varepsilon$ is sufficient to calculate the respective other function. Most importantly, though, the Kramers-Kronig relations show that a purely real permittivity can only be an approximation, as they imply that such a function must be identically equal to the vacuum permittivity ε_0.

As a final remark, we note that physically the necessary existence of a non-vanishing imaginary part of the permittivity accounts for the ever-present absorption effects (see [Jac99, Sect. 7.10.B] and, in particular, [LLP84, §80]). That is, as we recall, while an electromagnetic wave propagates inside a medium, part of its energy is taken up by the matter and the wave is eventually damped.

3.3.2 TRANSPARENT MEDIA

Real life experiences teach us that there exist so-called *transparent* (or *lossless*) media, which allow certain electromagnetic waves to propagate through them almost unimpeded. For frequencies in the visible spectrum, glass is an example of such a material, but also most dielectrics used in the manufacturing of photonic crystals are approximately only slightly absorbing (see [Joa08, Chapt. 2]).

When modeling a transparent medium, its permittivity is assumed to be real for frequencies in a *transparency range* $I_{\text{tr}} \subseteq [0,\infty)$, usually taken to be an interval, within which absorption, and thus $\operatorname{Im}\varepsilon$, can be neglected (see [LLP84, §84]). To be precise, by the Kramers-Kronig relation for the real part of the permittivity, we obtain

$$\begin{aligned}\varepsilon(x,\omega) &= \operatorname{Re}\varepsilon(x,\omega) \\ &= \varepsilon_0 + \frac{2}{\pi}\int_{[0,\infty)\setminus I_{\text{tr}}} \frac{\omega' \operatorname{Im}\varepsilon(x,\omega')}{\omega'^2-\omega^2}\,d\omega' \quad \text{for all } x\in\mathbb{R}^2 \text{ and all } \omega\in I_{\text{tr}}.\end{aligned} \tag{3.39}$$

Note that the integrand has no singularity within the domain of integration so that there is no need to take the Cauchy principal value. In the special case $I_{\text{tr}} = [0,\omega_0)$ for some $\omega_0 > 0$, the integrand on the right-hand side of the last equality is even

[1]Named after the authors of the independent articles [Kra27] and [Kro26] in which these integral representations were first mentioned.

known to be a strictly positive function, since the law of increasing entropy implies that $\operatorname{Im}\varepsilon$ is positive at all positive frequencies (and even for all substances; see [LLP84, §80]). Therefore,

$$\varepsilon(x,\omega) \geq \varepsilon_0 \quad \text{and} \quad \varepsilon_{\mathrm{r}}(x,\omega) \geq 1 \quad \text{for all } x \in \mathbb{R}^2 \text{ and all } \omega \in I_{\mathrm{tr}} = [0,\omega_0), \tag{3.40}$$

which particularly holds if, as an approximation, a medium is considered to be transparent for all frequencies.[1]

Another (formal) consequence of this integral representation is of importance to us and motivates our mathematical assumptions later on: Upon assuming that the right-hand side of (3.39) can be differentiated with respect to ω under the integral sign, we find

$$\frac{\partial\varepsilon}{\partial\omega}(x,\omega) = \frac{4\omega}{\pi}\int_{[0,\infty)\setminus I_{\mathrm{tr}}} \frac{\omega' \operatorname{Im}\varepsilon(x,\omega')}{(\omega'^2-\omega^2)^2}\, d\omega' \quad \text{for all } x \in \mathbb{R}^2 \text{ and all } \omega \in I_{\mathrm{tr}},$$

where the transparency range I_{tr} is an arbitrary interval again. Similar to above, the positivity of $\operatorname{Im}\varepsilon$ implies that the occurring integrand is non-negative throughout the domain of integration.[2] From this it follows that for fixed $x \in \mathbb{R}^2$, and if no absorption is present, the permittivity is a monotonically increasing function in the frequency as long as ε and I_{tr} are such that our differentiation above is justified.

Of course, as was the case in the previous subsection, the formulae we just derived can only be seen as approximations. Nevertheless, they allow us to judge the physical reasonability of our technical assumptions on the permittivity in Chapter 5. We remark, however, that a medium having a permittivity with a real part that is an increasing function of the frequency is said to be *normally dispersive* with good reason: Physical experiments show that away from certain isolated reasonance frequencies this behavior of ε is to be expected (see [Jac99, Sect. 7.10.B] and, in particular, Figure 7.8 therein).

3.3.3 THE LORENTZ MODEL

In the mathematical chapters of our work we will nowhere make use of a specific model for the frequency-dependence of the permittivity, but rather impose some abstract assumptions on its properties. In particular, we will only study approximately everywhere transparent materials modeled by a real-valued function ε. Nevertheless, for the sake of completeness, we briefly mention below the most common model for a complex-valued and frequency-dependent permittivity of a dielectric. At this, any spatial dependence of this function is left out for brevity.

Based on the equation of motion for an electron inside a dielectric material acted on by an electric field, the *Lorentz model* (or *classical electron oscillator model*)

[1] See also the last page of [LLP84, §84] where the authors explain that "[...] a literally transparent medium is one in which $\varepsilon(\omega)$ is not only real but also positive; if ε is negative, the wave is damped inside the medium, even though no true dissipation of energy occurs."

[2] In fact, even positive for positive frequencies.

proposes a permittivity of the form

$$\varepsilon(\omega) = \varepsilon_0 + \frac{Ne^2}{m_\mathrm{e}} \sum_{n=1}^{M} \frac{f_n}{\omega_n^2 - \omega^2 - i\omega\gamma_n} \qquad \text{for } \omega \in [0,\infty). \tag{3.41}$$

Here, e denotes the elementary charge, m_e the electron mass, and N the number of molecules per unit volume. Out of the Z electrons per molecule, f_n underly a harmonic force with reasonance frequency $\omega_n \geq 0$ and damping constant $\gamma_n \geq 0$. In total, M different reasonance frequencies of the material are considered but most related mathematical as well as physical publications limit M to be 1 or 2 for simplicity (see our literature review in Section 5.5).

The real and imaginary part of the permittivity in this model read

$$\begin{aligned}
\operatorname{Re}\varepsilon(\omega) &= \varepsilon_0 + \frac{Ne^2}{m_\mathrm{e}} \sum_{n=1}^{M} \frac{f_n(\omega_n^2 - \omega^2)}{(\omega_n^2 - \omega^2)^2 + \omega^2\gamma_n^2} && \text{for } \omega \in [0,\infty), \\
\operatorname{Im}\varepsilon(\omega) &= \frac{Ne^2}{m_\mathrm{e}} \sum_{n=1}^{M} \frac{f_n\gamma_n\omega}{(\omega_n^2 - \omega^2)^2 + \omega^2\gamma_n^2} && \text{for } \omega \in [0,\infty)
\end{aligned}$$

and exhibit, when considered as being extended to frequencies in $\mathbb{R}$, the symmetry properties we deduced in the previous subsection. Furthermore, since the damping constants γ_n can often be assumed very small, ε is approximately real away from the reasonance frequencies ω_n. There, a medium described by this model is seen to be normally dispersive. However, entirely neglecting absorption effects by setting $\gamma_n = 0$ for $j = 1, \ldots, M$ results in a permittivity that has singularities at the reasonance frequencies and besides locally violates the boundedness from below as in (3.40). For this reason this simplistic ansatz is usually only applied in a narrow frequency region free of reasonance frequencies (for instance, this is done and commented on in [ER09] and [Eng10]).

A more detailed discussion and further information on other permittivity models appropriate for metals, liquids, and conducting media can be found in the literature cited at the beginning of this section. Moreover, in the abovementioned section of Chapter 5 we comment in more detail on the sparse existing mathematical literature, in which, as of this writing, dispersive photonic crystals have been almost exclusively modeled using the Lorentz model.

CHAPTER 4

A SPECTRAL PROBLEM FOR NONDISPERSIVE PHOTONIC CRYSTALS

The chapter at hand marks the beginning of our mathematical analysis of the Maxwell eigenvalue problem for TM-polarized waves. Since we only cover nondispersive photonic crystals here, we can make use of preexisting results from the literature. Proofs are therefore generally omitted but can be found in the cited references.

In selecting the covered material, we have had the preparatory character of this chapter in mind. It includes, in Section 4.1.3, just the mathematical results we need to later present our own research outcomes on dispersive photonic crystals in a self-contained manner. However, in order to provide for a more complete picture, Section 4.2, which finishes the chapter, consists of a brief review of further mathematical work on light propagation in nondispersive photonic crystals and other periodic spectral problems.

4.1 OPERATOR-THEORETIC TREATMENT

The aim of this section is the rigorous mathematical analysis of the eigenvalue problem (3.28) for a TM-polarized and time-harmonic wave propagating in a two-dimensional photonic crystal. Our presentation is based on [Dör11, Chapt. 3], which provides an easy access to some of the material originally covered in the more abstract, but also more in-depth, book [Kuc93]. Moreover, we incorporate parts of the survey article [Kuc01] and mention [RS78, Sect. XIII.16] as another valuable reference.

4.1.1 PROBLEM STATEMENT

In order to precisely formulate our spectral problem, we first have to specify the photonic crystal we shall study. Just as was the case in the last chapter, we suppose that the considered medium is infinitely extended and comprised of linear, isotropic, inhomogeneous, and non-magnetic components. Besides, we assume that the absorption of electromagnetic energy by the photonic crystal can be neglected for waves with arbitrary frequencies, i.e., it is made of fully transparent materials. Finally, and crucial to the whole chapter, the nanostructure shall be nondispersive. We discussed earlier that these material properties constitute an approximate, but often reasonable model.

The supposed properties of the considered medium are mathematically reflected in assumptions on its relative permittivity (see Subsections 3.2.5 and 3.3.2 and recall that $\varepsilon = \varepsilon_0 \varepsilon_\mathrm{r}$). Most importantly, ε_r is taken to be independent of the frequency of the applied electromagnetic field in view of the non-dispersiveness of the photonic crystal (see Subsection 3.2.2 and particularly footnote 3 on p. 35). For later reference, we group all necessary requirements right below.

Assumptions 4.1.1 (Basic assumptions on ε_r in the nondispersive case)**.** *We suppose that $\varepsilon_\mathrm{r} \in L^\infty(\mathbb{R}^2;\mathbb{R})$ is such that*

(a) the function is $\mathbb{Z}^2$-periodic, i.e.,

$$\varepsilon_\mathrm{r}(x) = \varepsilon_\mathrm{r}(x+a) \qquad \text{for a. a. } x \in \mathbb{R}^2 \text{ and all } a \in \mathbb{Z}^2. \tag{4.1}$$

(b) there exists a positive constant $\varepsilon_{\mathrm{r,min}}$ with

$$\varepsilon_{\mathrm{r,min}} \le \varepsilon_\mathrm{r}(x) \le \|\varepsilon_\mathrm{r}\|_{L^\infty(\mathbb{R}^2)} =: \varepsilon_{\mathrm{r,max}} \qquad \text{for a. a. } x \in \mathbb{R}^2. \tag{4.2}$$

Remarks.

(a) Note that it is implicit in equation (4.1) that the photonic crystal's underlying Bravais lattice is chosen to be $\Theta = \mathbb{Z}^2$. Non-square two-dimensional geometries are likewise covered by the theory that follows, as long as the respective primitive cells are bounded.

(b) Of course, if Ω denotes a primitive cell of the Bravais lattice $\mathbb{Z}^2$, then supposing $\varepsilon_r \in L^\infty(\Omega;\mathbb{R})$ together with the above requirements (4.1) and (4.2) results in exactly the same class of functions. In particular, there holds the equality $\|\varepsilon_r\|_{L^\infty(\mathbb{R}^2)} = \|\varepsilon_r\|_{L^\infty(\Omega)}$, which we will use frequently for the $\mathbb{Z}^2$-periodic functions occurring in the remainder of this work.

With the assumed properties of its coefficient function in mind, we state the spectral problem (3.28) in its nondispersive form. Adjusted to a notation common in mathematics texts it reads

$$-\frac{1}{\varepsilon_\mathrm{r}} \Delta u = \lambda u \qquad \text{in } \mathbb{R}^2. \tag{4.3}$$

Here, u stands for the component $E_{\omega,3}$ of the electric field amplitude E_ω with time-frequency $\omega \in [0,\infty)$ and $\lambda := \omega^2/c_0^2$ denotes the eigenvalue parameter. It is of crucial importance to note that the non-dispersiveness of the modeled medium eliminates the parameter-nonlinearity of the spectral problem. That is, λ or, strictly speaking, ω does not appear on the left-hand side of the eigenvalue equation.

We study the Helmholtz type spectral problem (4.3) in the complex Hilbert space $L^2_{\varepsilon_r}(\mathbb{R}^2)$ which, as we recall from the preliminaries, denotes $L^2(\mathbb{R}^2)$ when endowed with the weighted inner product given by

$$\langle u,v\rangle_{\varepsilon_r} := \langle u,v\rangle_{L^2_{\varepsilon_r}(\mathbb{R}^2)} := \int_{\mathbb{R}^2} \varepsilon_r(x)u(x)\overline{v(x)}\,dx \qquad \text{for all } u,v \in L^2(\mathbb{R}^2). \tag{4.4}$$

Note that it is due to part (b) of Assumptions 4.1.1 that ε_r can be chosen as an integration weight function here, giving that $\langle\cdot,\cdot\rangle_{\varepsilon_r}$ and the canonical inner product on $L^2(\mathbb{R}^2)$ are equivalent. Besides, it is readily seen that the mapping $u \mapsto \varepsilon_r^{-1/2}u$ is a Hilbert space isometry between $L^2(\mathbb{R}^2)$ and $L^2_{\varepsilon_r}(\mathbb{R}^2)$.

Now, the operator whose eigenvalue problem corresponds to (4.3) is given by $A : D(A) \to L^2_{\varepsilon_r}(\mathbb{R}^2)$ in $L^2_{\varepsilon_r}(\mathbb{R}^2)$, where

$$D(A) := H^2(\mathbb{R}^2), \qquad Au := -\frac{1}{\varepsilon_r}\Delta u. \tag{4.5}$$

Clearly, A is densely-defined and, due to the boundedness assumption (4.2) on ε_r, uniformly elliptic. Note moreover that this operator is associated with the sesquilinear form $a : D(a) \times D(a) \to \mathbb{C}$ in $L^2_{\varepsilon_r}(\mathbb{R}^2)$ given by

$$D(a) := H^1(\mathbb{R}^2), \qquad a[u,v] := \int_{\mathbb{R}^2} \nabla u(x)\cdot\overline{\nabla v(x)}\,dx, \tag{4.6}$$

which is likewise densely-defined and furthermore positive and symmetric. From these properties and the maximality of the domain $D(a)$ it follows, by [Kat95, Thms. VI.2.6 and VI.2.7] (two results on the Friedrichs extension), that A is a positive and self-adjoint operator. Finally, as a consequence of its self-adjointness we obtain that A is closed.[1]

In summary, the eigenvalue problem (4.3) which we intend to study is realized as that of the elliptic, self-adjoint, and periodic operator A. Our subsequent account of the so-called Floquet-Bloch theory provides the results we need to analyze the spectrum of operators having these properties.

4.1.2 FLOQUET-BLOCH THEORY

In Subsection 3.2.5 we argued—far from being mathematically precise—that the spectrum of the Maxwell eigenvalue problem (3.28) is determined by a family of

[1] In fact, the closedness of A also follows directly as the operator is the product of the bounded and invertible multiplication operator $\varepsilon_r I : L^2_{\varepsilon_r}(\mathbb{R}^2) \to L^2_{\varepsilon_r}(\mathbb{R}^2)$ and the closed negative Laplacian $-\Delta : H^2(\mathbb{R}^2) \to L^2_{\varepsilon_r}(\mathbb{R}^2)$ in $L^2_{\varepsilon_r}(\mathbb{R}^2)$. Also, Green's second identity applied to functions in a suitable dense subset of $H^2(\mathbb{R}^2)$, such as $C_0^\infty(\mathbb{R}^2)$, implies that A is symmetric and positive.

spectra of related problems posed on a primitive cell. This relation is the most important outcome of the *Floquet-Bloch theory*, which is, roughly speaking, the spectral theory of periodic differential operators.[1]

An important tool towards our goal of determining the spectrum of the periodic operator A is given by the *Floquet transform*. This analog of the Fourier transform on the lattice of periodicity, here $\mathbb{Z}^2$, is given by

$$(Uu)(x,k) := \sum_{n\in\mathbb{Z}^2} u(x-n)e^{ik\cdot n} \qquad \text{for all } u \in C_0^\infty(\mathbb{R}^2) \text{ and all } x,k \in \mathbb{R}^2 \tag{4.7}$$

and can also be defined for functions in $L^2(\mathbb{R}^2)$ by a standard density argument. Note that Uu depends on the dual variable k but also—in contrast to a Fourier transformed function—on x. Moreover, the Floquet transform has the properties

$$(Uu)(x+a,k) = e^{ik\cdot a}(Uu)(x,k) \qquad \text{for all } x,k \in \mathbb{R}^2 \text{ and all } a \in \mathbb{Z}^2, \tag{4.8}$$

$$(Uu)(x,k+b) = (Uu)(x,k) \qquad \text{for all } x,k \in \mathbb{R}^2 \text{ and all } b \in 2\pi\mathbb{Z}^2. \tag{4.9}$$

Hence, a Floquet transformed function is what is called *k-quasi-periodic* in the variable x with respect to $\mathbb{Z}^2$ and periodic in the variable k with respect to $2\pi\mathbb{Z}^2$. We remark that it is no coincidence that the two involved lattices are reciprocal to each other. If the sum in (4.7) extends over an arbitrary Bravais lattice Θ, then the resulting transform will have the properties (4.8) and (4.9) for all $a \in \Theta$ and all $b \in \Theta^*$, respectively.

The just derived periodicity relations show that all information about the function Uu is contained in its values on the Cartesian product of two primitive cells—one each of the crystal's geometric lattice and the corresponding reciprocal one. We make the standard choices[2]

$$\begin{aligned} \Omega := \Omega_\Theta &:= (0,1)^2 \qquad && \text{for } \Theta = \mathbb{Z}^2, \\ B := B_{\Theta^*} &:= [-\pi,\pi]^2 \qquad && \text{for } \Theta^* = 2\pi\mathbb{Z}^2, \end{aligned} \tag{4.10}$$

where we recall that B is the (first) Brillouin zone determined at the end of Subsection 3.1.3. With this, we can consider Uu as being defined on $\Omega \times B$, keeping in mind that (4.8) and (4.9) determine the function elsewhere.

As was already said, the domain of the Floquet transform can be extended to all of $L^2(\mathbb{R}^2)$ or, more suitable for our purposes, its weighted variant $L^2_{\varepsilon_r}(\mathbb{R}^2)$. In order to state the mapping properties of U on this space, we first have to introduce the space $L^2_{\varepsilon_r}(\Omega)$. Similar to the whole-space case discussed above, it

[1]G. Floquet worked on ordinary differential equations with periodic coefficients. His 1883 work [Flo83], together with the already cited article [Blo29] by F. Bloch, are the foundations of what is nowadays named Floquet-Bloch theory.

[2]Note that the primitive cell of the crystal lattice is here taken to be an open set. This is slightly less in line with the definition as closed sets used in solid state physics (see (3.2)), but reasonable since Sobolev spaces of functions defined on Ω play a role in our analysis. Of course, both choices only differ by a set of zero two-dimensional Lebesgue measure.

is defined to be the complex Hilbert space $L^2(\Omega)$ endowed with the weighted inner product given by

$$\langle u,v\rangle_{\varepsilon_r} := \langle u,v\rangle_{L^2_{\varepsilon_r}(\Omega)} := \int_\Omega \varepsilon_r(x)u(x)\overline{v(x)}\,dx \qquad \text{for all } u,v \in L^2(\Omega).$$

Clearly, the existence of an isometry between the weighted and the unweighted space and the equivalence of the associated inner products can be deduced just as before. Note, however, that our notation $\langle\cdot,\cdot\rangle_{\varepsilon_r}$ is ambiguous, as we also use it to denote the inner product of $L^2_{\varepsilon_r}(\mathbb{R}^2)$. In what follows it will always be clear from the context what mapping is meant.

Having defined the necessary spaces, we can now recall an important Plancherel theorem for the Floquet transform.

Theorem 4.1.2 (A Plancherel theorem – [Dör11, Lem. 3.4.1])**.** *The Floquet transform $U : L^2_{\varepsilon_r}(\mathbb{R}^2) \to L^2(B; L^2_{\varepsilon_r}(\Omega))$ defined by*

$$(Uu)(x,k) := \frac{1}{\sqrt{\mathrm{vol}(B)}} \sum_{n\in\mathbb{Z}^2} u(x-n)e^{ik\cdot n} \qquad \textit{for all } x \in \Omega \textit{ and all } k \in B \tag{4.11}$$

is an isometric isomorphism. Its inverse is given by

$$U^{-1}v = \frac{1}{\sqrt{\mathrm{vol}(B)}} \int_B v(\cdot,k)\,dk,$$

where $v(\cdot,k)$ is extended k-quasi-periodically to the whole of $\mathbb{R}^2$. That is, for $k \in B$,

$$v(x+a,k) := e^{ik\cdot a}v(x,k) \qquad \textit{for all } x \in \Omega \textit{ and all } a \in \mathbb{Z}^2.$$

Remark. Note that in the statement of the theorem we slightly changed the definition of U. The multiplicative factor $\mathrm{vol}(B)^{-1/2} = (2\pi)^{-1}$ is needed to obtain the isometry property and will from now on always be taken into account.

As yet, the importance of the Floquet transform for periodic partial differential equations has not been revealed. A first observation in this respect is that U commutes with the corresponding periodic differential operators. Although this and its consequences can be shown for a large class of operators (see, in particular, [Kuc93, Chapt.2] and [Kuc01, Sect. 3.5]), we restrict our considerations to the operator A as in (4.5) from now on. There holds, for all $k \in B$,

$$\begin{aligned}
[U(Au)](\cdot,k) &= \frac{1}{\sqrt{\mathrm{vol}(B)}} \sum_{n\in\mathbb{Z}^2} (Au)(\cdot-n)e^{ik\cdot n} \\
&= -\frac{1}{\varepsilon_r}\Delta\left[\frac{1}{\sqrt{\mathrm{vol}(B)}} \sum_{n\in\mathbb{Z}^2} u(\cdot-n)e^{ik\cdot n}\right] \\
&= -\frac{1}{\varepsilon_r}\Delta[(Uu)(\cdot,k)] \qquad \text{for all } u \in H^2(\mathbb{R}^2),
\end{aligned} \tag{4.12}$$

where we used the $\mathbb{Z}^2$-periodicity of ε_r and the closedness of the operator A. Observe that on the right-hand side of this equation the differential operator acts on images under U, i.e., functions that satisfy k-quasi-periodic boundary conditions on the primitive cell. Hence, the symbol of this operator is still given by $-\varepsilon_\mathrm{r}^{-1}\Delta$, but its domain changes with $k \in B$. This will be made precise in the statement of the next theorem.

Applying the inverse Floquet transform on both sides of (4.12), we expect, in a sense to be determined, a relation of the form $Au = U^{-1}[A_k(Uu)(\cdot,k)]$ to hold. Here, by writing A_k on the right-hand side, we emphasize the outlined k-dependence of the operator. Indeed, this equality can be made precise and constitutes the main abstract result of the Floquet-Bloch theory. In a form common in the related literature, and again specifically for the operator A, the associated theorem reads as follows:

Theorem 4.1.3 ([RS78, Thm. XIII.97 pt. (b)][1]). *Let $k \in B$ and define the operator $A_k : D(A_k) \to L^2_{\varepsilon_\mathrm{r}}(\Omega)$ in $L^2_{\varepsilon_\mathrm{r}}(\Omega)$ by*

$$\begin{aligned} D(A_k) := H^2_{k\text{-per}}(\Omega) := \{u|_\Omega \mid u \in H^2_{\mathrm{loc}}(\mathbb{R}^2) \textit{ and } u(x+a) = e^{ik\cdot a}u(x) \\ \textit{for all } a \in \mathbb{Z}^2 \textit{ and a.a. } x \in \Omega\}, \\ A_k u := -\frac{1}{\varepsilon_\mathrm{r}}\Delta u. \end{aligned} \tag{4.13}$$

Then the Floquet transform U expands the operator A in $L^2_{\varepsilon_\mathrm{r}}(\mathbb{R}^2)$ into the direct integral of the operators $\{A_k\}_{k\in B}$ in $L^2_{\varepsilon_\mathrm{r}}(\Omega)$, i.e.,

$$UAU^{-1} = \frac{1}{\mathrm{vol}(B)} \int_B^\oplus A_k \, dk. \tag{4.14}$$

Remarks.

(a) The space $H^2_{k\text{-per}}(\Omega)$ introduced in the statement of the theorem is an example of a *Sobolev space of k-quasi-periodic functions*. It is a closed subspace of $H^2(\Omega)$ and thus a Hilbert space in its own right when equipped with the (restriction of the) inner product $\langle\cdot,\cdot\rangle_{H^2(\Omega)}$.

(b) It is important to note that a function $u \in H^2_{k\text{-per}}(\Omega)$ also satisfies the k-quasi-periodic boundary conditions

$$\nabla u(x+a)\cdot n(x) = e^{ik\cdot a}\nabla u(x)\cdot n(x) \quad \text{for all } a \in \mathbb{Z}^2 \text{ and a.a. } x \in \partial\Omega,$$

where $n(x)$ denotes the outward unit normal at $x \in \partial\Omega$. In particular, recalling that Ω is the unit cell in $\mathbb{R}^2$, we have

$$\frac{\partial u}{\partial x_1}(1,x_2) = e^{ik_1}\frac{\partial u}{\partial x_1}(0,x_2) \quad \text{for a.a. } x_2 \in [0,1] \text{ and all } u \in H^2_{k\text{-per}}(\Omega) \tag{4.15}$$

[1]The cited theorem is formulated for Schrödinger operators of the form $-\Delta + V$ where V is a periodic potential. Nevertheless, the there given proof can be transferred to our situation with little effort (see also the more general discussions in [Kuc93, Chapt.2] and [Kuc01, Sect. 3.5]).

and

$$\frac{\partial u}{\partial x_2}(x_1,1) = e^{ik_2}\frac{\partial u}{\partial x_2}(x_1,0) \quad \text{for a. a. } x_1 \in [0,1] \text{ and all } u \in H^2_{k\text{-per}}(\Omega) \quad (4.16)$$

where $k = (k_1,k_2)$.

An important consequence of these boundary conditions is the symmetry of each operator A_k with respect to the weighted inner product $\langle\cdot,\cdot\rangle_{\varepsilon_r}$. This can be seen as follows: By Green's second identity we find, for all $k \in B$,

$$\begin{aligned}
\langle A_k u,v\rangle_{\varepsilon_r} &= -\int_\Omega \Delta u(x)\overline{v(x)}\,dx \\
&= -\int_\Omega u(x)\overline{\Delta v(x)}\,dx + \int_{\partial\Omega}\Big[u(x)\overline{\nabla v(x)} - \nabla u(x)\overline{v(x)}\Big]\cdot n(x)\,d\sigma \\
&= \langle u, A_k v\rangle_{\varepsilon_r} - \int_0^1 u(x_1,0)\overline{\frac{\partial v}{\partial x_2}(x_1,0)} + \frac{\partial u}{\partial x_2}(x_1,0)\overline{v(x_1,0)}\,dx_1 \\
&\qquad + \int_0^1 u(x_1,1)\overline{\frac{\partial v}{\partial x_2}(x_1,1)} - \frac{\partial u}{\partial x_2}(x_1,1)\overline{v(x_1,1)}\,dx_1 \\
&\qquad - \int_0^1 u(0,x_2)\overline{\frac{\partial v}{\partial x_1}(0,x_2)} + \frac{\partial u}{\partial x_1}(0,x_2)\overline{v(0,x_2)}\,dx_2 \\
&\qquad + \int_0^1 u(1,x_2)\overline{\frac{\partial v}{\partial x_1}(1,x_2)} - \frac{\partial u}{\partial x_1}(1,x_2)\overline{v(1,x_2)}\,dx_2 \\
&= \langle u, A_k v\rangle_{\varepsilon_r} \qquad \text{for all } u,v \in H^2_{k\text{-per}}(\Omega),
\end{aligned}$$

using the k-quasi-periodicity of u and v as well as (4.15) and (4.16). In short: Boundary integrals over opposing parts of $\partial\Omega$ cancel.

(c) It is beyond the scope of our work to discuss the direct integral notation in the expansion (4.14). As it will not reappear in this thesis, we refer the reader to [RS78, Sect. XIII.16] for details.

A reasoning similar to that employed in Subsection 4.1.1 above shows that the operators defined in (4.13) share the properties of the operator A. In particular, each A_k is self-adjoint in $L^2_{\varepsilon_r}(\Omega)$ as a consequence of the properties of the associated sesquilinear form $a_k : D(a_k) \times D(a_k) \to L^2_{\varepsilon_r}(\Omega)$ given by[1]

$$D(a_k) := H^1_{k\text{-per}}(\Omega), \qquad a_k[u,v] := \int_\Omega \nabla u(x)\cdot\overline{\nabla v(x)}\,dx, \qquad (4.17)$$

where the space $H^1_{k\text{-per}}(\Omega)$ is defined in the obvious way analogous to $H^2_{k\text{-per}}(\Omega)$ and a Hilbert space with respect to the inner product $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$ (compare to part (a) of the last remark). A crucial difference between A and A_k, however, lies in the fact that the latter operators are defined on functions on the bounded primitive

[1]Note in this respect that the boundary integral in Green's first identity vanishes by the very same arguments we employed right above when we showed the symmetry of the operator A_k.

cell Ω. This allows us to deduce that the resolvent of any operator A_k is compact by using [RS78, Thm. XIII.98] (which gives the compactness of the resolvent of $-\Delta : L^2(\Omega) \supseteq H^2_{k\text{-per}}(\Omega) \to L^2(\Omega)$) and the characterization (iii) of a compact resolvent in [RS78, Thm. XIII.64]. Here, again, the norm equivalence of $L^2(\Omega)$ and $L^2_{\varepsilon_r}(\Omega)$ is important.

In view of its just discussed properties self-adjointness, positivity, and compactness of the resolvent, a standard argument shows that the spectrum of each operator A_k is purely discrete. We elaborate on this in the next subsection, which collects results on the spectral characteristics of A as well as A_k for later reference. Before, let us mention that it is sometimes preferable to work with the variant $V : L^2_{\varepsilon_r}(\mathbb{R}^2) \to L^2(B; L^2_{\varepsilon_r}(\Omega))$ of the Floquet transform given by

$$(Vu)(x,k) := e^{-ik\cdot x}(Uu)(x,k) \qquad \text{for all } x \in \Omega \text{ and all } k \in B.$$

It is outlined in [Kuc01, Sect. 7.3] that a result similar to Theorem 4.1.3 can also be established for V. Therein, for all $k \in B$ the place of A_k is taken by the operator $\widehat{A}_k : D(\widehat{A}_k) \to L^2_{\varepsilon_r}(\Omega)$ given in $L^2_{\varepsilon_r}(\Omega)$ by

$$\begin{aligned} D(\widehat{A}_k) &:= H^2_{\mathrm{per}}(\Omega) := \{u|_\Omega \mid u \in H^2_{\mathrm{loc}}(\mathbb{R}^2) \text{ and } u(x+a) = u(x) \\ &\qquad\qquad \text{for all } a \in \mathbb{Z}^2 \text{ and a. a. } x \in \Omega\}, \\ \widehat{A}_k u &:= -\frac{1}{\varepsilon_r}(\nabla + ik)\cdot(\nabla + ik)u, \end{aligned} \tag{4.18}$$

and corresponds to the sesquilinear form $\widehat{a}_k : D(\widehat{a}_k) \times D(\widehat{a}_k) \to L^2_{\varepsilon_r}(\Omega)$ defined through

$$D(\widehat{a}_k) := H^1_{\mathrm{per}}(\Omega), \qquad \widehat{a}_k[u,v] := \int_\Omega (\nabla + ik)u(x) \cdot \overline{(\nabla + ik)v(x)}\, dx, \tag{4.19}$$

where the space $H^1_{\mathrm{per}}(\Omega)$ is defined in the evident manner analogous to $H^2_{\mathrm{per}}(\Omega)$. Both these spaces are Hilbert spaces with respect to the inner products $\langle\cdot,\cdot\rangle_{H^1(\Omega)}$ and $\langle\cdot,\cdot\rangle_{H^2(\Omega)}$, respectively (arguments similar to those stated in part (a) of the last remark apply). Clearly, for all $k \in B$ the operator $\widehat{A}_k$ shares all the above-mentioned properties of its counterpart A_k.

Remark. Studying the operator A by means of its expansion into the direct integral of the operators $\widehat{A}_k$ is particularly advantageous when one wants to apply results from the theory of operator-valued functions to the mapping $B \ni k \mapsto \widehat{A}_k$. The fact that the operators $\widehat{A}_k$ all have the same domain is of great help here (see, e.g., the concept of a "holomorphic family of type (A)" defined in [Kat95, Chapt. 7] and [ibid., Thm. VII.3.9]).

4.1.3 SOME SPECTRAL-THEORETIC RESULTS

Below we summarize the most important results on the spectra of the operators A, A_k, and $\widehat{A}_k$. First, as a direct consequence of Theorem 4.1.3 we have the following spectral equality.

Theorem 4.1.4 ([Dör11, Thms. 3.6.1 and 3.6.2]). *For the spectrum of the operator A there holds*

$$\sigma(A) = \bigcup_{k \in B} \sigma(A_k).$$

The importance of the last theorem lies in its consequence for the analytical and numerical considerations of the spectrum of A. Instead of solving an eigenvalue problem on the whole of $\mathbb{R}^2$, we may study a family of related problems on the bounded primitive cell Ω parametrized by $k \in B$. As mentioned previously, the spectral structure of the latter problems is rather simple, namely purely discrete. We retain this assertion as part of the next theorem.

Theorem 4.1.5 ([Dör11, Sect. 3.4] and [WS72, Thm. 2.2.1]). *For all $k \in B$ the operators A_k and $\widehat{A}_k$ have the same purely discrete spectrum $\{\lambda_{k,n}\}_{n\in\mathbb{N}}$ and corresponding eigenfunctions $\{\psi_{k,n}\}_{n\in\mathbb{N}}$ and $\{\widehat{\psi}_{k,n}\}_{n\in\mathbb{N}}$, respectively, which each form an orthonormal basis of $L^2_{\varepsilon_r}(\Omega)$. Further,*

$$0 \le \lambda_{k,1} \le \lambda_{k,2} \le \cdots \le \lambda_{k,n} \to \infty \quad \text{as } n \to \infty, \tag{4.20}$$

where the min-max principle *characterizes these eigenvalues. That is, for all $k \in B$ there holds*

$$\begin{aligned} \lambda_{k,n} &= \min_{\substack{U \sqsubseteq H^1_{k\text{-per}}(\Omega) \\ \dim U = n}} \max_{u \in U \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla u(x)|^2\, dx}{\displaystyle\int_\Omega \varepsilon_{\mathrm{r}}(x) |u(x)|^2\, dx} \\ &= \min_{\substack{U \sqsubseteq H^1_{\mathrm{per}}(\Omega) \\ \dim U = n}} \max_{u \in U \setminus \{0\}} \frac{\displaystyle\int_\Omega |(\nabla + ik) u(x)|^2\, dx}{\displaystyle\int_\Omega \varepsilon_{\mathrm{r}}(x) |u(x)|^2\, dx} = \widehat{\lambda}_{k,n} \qquad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{4.21}$$

with $\sqsubseteq$ denoting the subspace relation. Finally, for all $k \in B$ the eigenfunctions of the two operators are related via

$$\psi_{k,n}(x) = e^{ik\cdot x} \widehat{\psi}_{k,n}(x) \quad \text{for a. a. } x \in \Omega \text{ and all } n \in \mathbb{N}. \tag{4.22}$$

Remarks.

(a) In case we want to stress the dependence of an eigenvalue or -function of A_k or $\widehat{A}_k$ on the parameter k, we write it as an argument later on, i.e., $\lambda_n(k)$, $\psi_n(\cdot, k)$ and the like.

(b) Observe that the relation (4.22) is precisely of the form predicted by Bloch's theorem: The eigenfunction $\psi_{k,n}$ of our periodic spectral problem—referred to as a *Bloch wave*—is given by the product of a plane wave and the $\mathbb{Z}^2$-periodic function $\widehat{\psi}_{k,n}$ (see also equation (3.31)). Most importantly, it is but now that we made the contents of Subsection 3.2.5, at least for a nondispersive structure, mathematically precise.

The next lemma provides useful information about the dependence of the eigenvalues and eigenfunctions of the operators A_k and $\widehat{A}_k$ on k.

Lemma 4.1.6 ([Kuc93, Lem. 4.5.7] for (a), [Kat95, Sect. IV.3.5][1] for (b))**.** *Let $n \in \mathbb{N}$.*

(a) The eigenfunctions $\psi_n(\cdot,k)$ and $\widehat{\psi}_n(\cdot,k)$ can be chosen measurably in $k \in B$.

(b) The mapping $B \ni k \mapsto \lambda_n(k)$ is continuous.

(c) The mapping $B \ni k \mapsto \lambda_n(k)$ is even, i.e., $\lambda_n(k) = \lambda_n(-k)$.

Proof of assertion (c). Let $k \in B$ and $n \in \mathbb{N}$ be fixed. First, $u \in H^1_{\text{per}}(\Omega)$ if and only if $\overline{u} \in H^1_{\text{per}}(\Omega)$. Furthermore, for all $x \in \Omega$,

$$\begin{aligned}
|(\nabla + ik)u(x)|^2 &= (\nabla + ik)u(x) \cdot \overline{(\nabla + ik)u(x)} \\
&= \overline{(\nabla - ik)\overline{u(x)}} \cdot \overline{(\nabla + ik)u(x)} \\
&= \overline{(\nabla - ik)\overline{u(x)}} \cdot (\nabla - ik)\overline{u(x)} = |(\nabla + i(-k))\overline{u(x)}|^2,
\end{aligned}$$

since k is real. Moreover, trivially, $|u(x)|^2 = |\overline{u(x)}|^2$ for all $x \in \Omega$ and hence, by the min-max principle (see the second line of (4.21)) it follows $\lambda_n(k) = \lambda_n(-k)$. □

Remark. The evenness of the eigenvalues in k is particularly useful for numerical approximations since it halves the set of relevant values of k. In fact, if a modeled medium has more than just translational symmetry, then the significant part of its Brillouin zone can be reduced even further. For instance, for most structures with a square lattice this so-called *irreducible Billouin zone* is a triangle of only an eighth of the volume of B.

In view of part (b) of the last theorem, and since the Brillouin zone is compact and connected, we can reformulate Theorem 4.1.4 as follows:

Theorem 4.1.7. *For the spectrum of the operator A there holds*

$$\sigma(A) = \bigcup_{k \in B} \sigma(A_k) = \bigcup_{k \in B} \sigma(\widehat{A}_k) = \bigcup_{n \in \mathbb{N}} \Big[\min_{k \in B} \lambda_n(k), \max_{k \in B} \lambda_n(k)\Big]. \tag{4.23}$$

Remark. Since $\sigma(A)$ is a closed set (see part (a) of Proposition 2.4.4), the equality (4.23) can only hold if the union on its right-hand side likewise contains all its limit points. This is indeed the case and a consequence of the compactness of the intervals in this union and the fact that $\min_{k \in B} \lambda_n(k) \to \infty$ as $n \to \infty$. The latter can be shown by means of the min-max principle and a comparison argument with appropriate Neumann eigenvalues (see the techniques used in the proof of Lemma 5.4.10).

[1] The cited perturbation result is applicable since $\widehat{A}_k$ depends continuously on k.

Theorem 4.1.7 constitutes the main spectral statement of the Floquet-Bloch theory for the operator A and shows that $\sigma(A)$ has a *band-gap structure*. That is to say that there might be regions on the positive real axis not covered by the union of the *spectral bands*

$$S_n := \left[\min_{k\in B}\lambda_n(k), \max_{k\in B}\lambda_n(k)\right],$$

which are the ranges of the *band functions* $B \ni k \mapsto \lambda_n(k)$ for $n \in \mathbb{N}$. The ordering of the eigenvalues by magnitude implies that this can happen if and only if

$$\left(\max_{k\in B}\lambda_n(k), \min_{k\in B}\lambda_{n+1}(k)\right) \neq \emptyset \qquad \text{for some } n \in \mathbb{N}. \tag{4.24}$$

An interval of this sort is called a *spectral gap* (or *band gap*) of the operator A. We comment on the existence of such gaps and on how likely they are to appear in Section 4.2. Moreover, we provide a fictitious *band diagram*, i.e., a plot of some band functions and the corresponding spectral bands, in Figure 4.1 below.

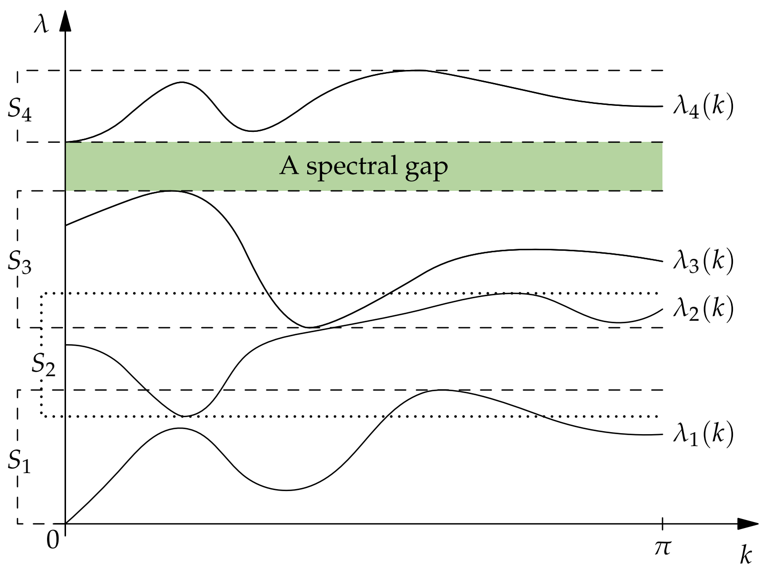

FIGURE 4.1 — A fictitious band diagram of a one-dimensional (for simplicity) photonic crystal having a spectral gap between the third and the fourth spectral band. Note that part (c) of Lemma 4.1.6 allows us to ignore negative values of k in our illustration.

The terms we just introduced are, incidentally, perfectly consistent with those we used in Section 3.1 when discussing the physics of photonic crystals. There, a frequency ω was said to be in a (photonic) band gap if the corresponding electromagnetic wave cannot propagate inside the medium. On the other hand, we found out in Section 3.2 that any "allowed" frequency corresponds via the relation $\lambda = \omega^2/c_0^2$ to a complex number in the spectrum of the operator A and hence, as we now know, to $\lambda_n(k)$ for some $n \in \mathbb{N}$ and some $k \in B$. Therefore, clearly, both definitions of a band gap coincide.

Let us now briefly turn our attention to the spectral properties of the "whole-space operator" A. As it is self-adjoint, it has no residual spectrum (see part (a) of Theorem 2.4.6). Moreover, also the point spectrum of A is empty, which is the assertion of the next result.

Theorem 4.1.8 ([Kuc01, Thm. 7.7][1])**.** *The operator A has no eigenvalues.*

Remarks.

(a) Part of this theorem is easy to show, namely that A has no eigenvalues of finite multiplicity. Thereto a simple proof of the same result for ordinary periodic differential equations can be adapted (see [Dör11, Thm. 3.6.3] and the original source [Eas73, Thm. 5.3.1]).

(b) To be exact, the spectrum of the operator A is purely absolutely continuous. This is stronger than the absence of eigenvalues, but requires knowledge of a decomposition of the spectrum that we do not intend to use in this thesis. We refer to [Kat95, Chapt. 10] in this regard.

Since an isolated point in the spectrum of a self-adjoint operator on a Hilbert space is necessarily an eigenvalue (see [HS96, Prop. 6.4]), we obtain, together with the spectral equality (4.23), a corollary to Theorem 4.1.8.

Corollary. *The spectrum of A contains no isolated points. In particular, there is no $n \in \mathbb{N}$ such that the band function $B \ni k \mapsto \lambda_n(k)$ is constant.*

The final result we present in this section addresses the Bloch waves. It turns out that the Floquet transform U can be used to prove that they are, in a certain sense, complete in $L^2_{\varepsilon_\mathrm{r}}(\mathbb{R}^2)$ when k runs through the Brillouin zone B.

Theorem 4.1.9 ("Completeness" of the Bloch waves – [Dör11, Thm. 3.5.1])**.** *Let $u \in L^2_{\varepsilon_\mathrm{r}}(\mathbb{R}^2)$ and define, for $N \in \mathbb{N}$,*

$$u_N(x) := \sum_{n=1}^{N} \int_B \langle (Uu)(\cdot,k), \psi_n(\cdot,k)\rangle_{\varepsilon_\mathrm{r}}\, \psi_n(x,k)\, dk \qquad \text{for all } x \in \mathbb{R}^2,$$

where the Bloch waves $\{\psi_n(\cdot,k)\}_{n\in\mathbb{N}}$ are k-quasi-periodically extended to the whole of $\mathbb{R}^2$ as in (4.13)*. Then, $u_N \to u$ in $L^2_{\varepsilon_\mathrm{r}}(\mathbb{R}^2)$ as $N \to \infty$.*

To close, we remark that several of the results presented in this section also hold in an appropriate sense for certain classes of dispersive photonic crystals and their parameter-nonlinear eigenvalue problems. Of course, the specific frequency-dependence of a given relative permittivity plays a crucial role as to whether a proof is successful. The reader finds this discussed in Chapter 5.

[1] The cited result can be applied since $\varepsilon_\mathrm{r}|_\Omega \in L^2(\Omega)$ in view of the boundedness of Ω (see footnote 4 in the cited source).

4.2 RELATED WORK IN THE LITERATURE

In our above considerations of the Floquet-Bloch theory we quickly specialized to the operator A that realizes our Maxwell eigenvalue problem for TM-polarized waves. Moreover, we only covered some of the many questions arising naturally when an operator is known to have a spectrum with a band-gap structure. Both of these shortcomings shall be briefly addressed in the section at hand.

First, we mention that a spectral decomposition similar to that in Theorem 4.1.4 can be established for any elliptic linear differential operator A that realizes an mth-order differential expression of the form

$$L(x,D) = \sum_{|\alpha| \leq m} c_\alpha(x) D^\alpha$$

with smooth $\mathbb{Z}^n$-periodic[1] and complex-valued coefficients in $\mathbb{R}^n$ (see [Kuc93, Thm. 4.5.1]). If one further assumes that A is self-adjoint and even uniformly strongly elliptic, then the band-gap structure of $\sigma(A)$ in the form given in Theorem 4.1.7 can be proven (see [Dör11, Thms. 3.6.1 and 3.6.2]). Here, in view of the physically very important Schrödinger operators $-\Delta + V$, it is important to remark that the lowest order coefficient $c_{(0,\dots,0)}$, which then corresponds to the periodic potential V, is allowed to be merely in $L^\infty(\mathbb{R}^n)$. However, even this can be weakened so that Schrödinger operators with periodic potentials having certain types of singularities still have a spectrum with a band-gap structure (see [Kuc93, Sect. 4.5]).

Another important contribution to the Floquet-Bloch theory of periodic operators with non-smooth coefficients is the article [Bro11]. The authors show that a result similar to Theorem 4.1.4 holds for an elliptic differential operator in divergence form with periodic coefficients (of all orders) in $L^\infty(\mathbb{R}^n)$. In contrast to the above-mentioned results, the problem is not studied in $L^2(\mathbb{R}^n)$ but in $H^{-m}(\mathbb{R}^n)$, which, as is also a result of the article, does not change the spectrum. Besides coping easily with discontinuous coefficients, this approach also handles unbounded primitive cells, i.e., periodicity is not required in all spatial directions.

In case it is already known that the spectrum of a considered periodic operator has a band-gap structure, further, more in-depth issues can be raised. Among the first questions that come to mind—and which thus stimulated a vast amount of mathematical research—are the following:

Questions (concerning an operator that has a spectral band-gap structure)**.**

(a) Do spectral gaps exist and, if so, are there finitely or infinitely many of them?

(b) Can certain assumptions on the coefficients of the operator guarantee the existence of spectral gaps?

(c) Does the operator have eigenvalues?

[1] As before, one can easily allow for more general Bravais lattices.

Let us address these questions one by one.

Ad (a): The existence and number of spectral gaps is clearly a central issue of problems motivated by applications. That is because only sufficiently wide gaps at suitable positions on the spectral axis allow for the desired effects. Mathematically, though, it is a hard task to provide the related results—at least for the more important two- and three-dimensional problems. In one space-dimension, on the other hand, it is known that spectral bands can only touch, but do not overlap[1] and the existence of gaps is therefore to be expected. Moreover, for one-dimensional Schrödinger operators, Borg's theorem, established in 1946, states that spectral gaps are absent if and only if the potential of the operator is constant (see [Kuc93, Thm. 4.4.4] and the original source [Bor46]).

In dimensions greater than one the nowadays most important results were conjectured by H. Bethe and A. Sommerfeld in 1933 (see [BS67]). They guessed for certain Schrödinger operators the by now general belief that there can only be a finite number of gaps in the spectrum of any self-adjoint periodic elliptic operator if n, the space dimension, is at least 2. The most general result in this direction known to date is a proof of the Bethe-Sommerfeld conjecture for the Schrödinger operator with periodic magnetic potential in arbitrary dimensions (see [PS10]). Of course, this article was preceded by several significant results for what are now special cases. We shall not go into details on the relevant publications, but rather refer to the survey article [Sob07] and its references.

For the operator $-1/\varepsilon_{\mathrm{r}}\,\Delta$ occurring in the analysis of two-dimensional photonic crystals, a result similar to the Bethe-Sommerfeld conjecture has, to the best of our knowledge, not yet been proven. Based on physical reasoning it "should" hold true (see [Kuc01, Sect. 7.4]), but so far a rigorous proof exists only in the case of a separable and frequency-independent coefficient, i.e., $\varepsilon_{\mathrm{r}} = \varepsilon_{\mathrm{r},1}(x_1) + \varepsilon_{\mathrm{r},2}(x_2)$, where both functions on the right-hand side are periodic, bounded, and C^2-smooth (see [Vor11]).

Ad (b): The question whether a relation between the coefficients of a differential operator and the existence of its spectral gaps exists seems to be covered in the literature only for cases relevant to applications. There, however, this is a key issue because manufacturers are generally interested in knowing in advance, i.e., without actually producing a structure, what materials constituents will favor the opening of spectral gaps. As regards photonic crystals, physical experiments suggest that a high contrast in the relative permittivity of the dielectric components is desirable (see [Joa08]). This has been verified mathematically by A. Figotin and P. Kuchment for certain two-dimensional nondispersive media in their two papers [FK96a] and [FK96b]. The nanostructures that the authors consider consist of narrow strips of a dense dielectric with relative permittivity $\varepsilon_{\mathrm{r}} \gg 1$ which

[1] Note that our illustration in Figure 4.1 is thus clearly inaccurate and should not be understood as depicting the band structure of a realistic medium, but rather as a means to clarify the notions surrounding photonic band structures.

separate square "columns" of air as is depicted in Figure 4.2 right below.

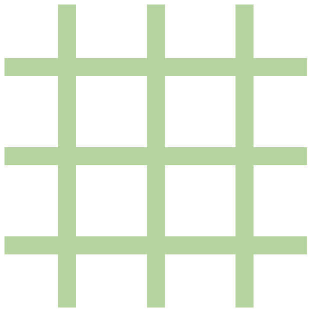

FIGURE 4.2 — A schematic illustration of the two-dimensional photonic crystal studied in the articles [FK96a] and [FK96b]. The white areas represent air, while the colored strips indicate a dielectric with a high relative permittivity.

The essential result in the cited publications is that there exist combinations of material parameters—being the width, the distance, and the relative permittivity of the dielectrically dense strips—such that both the operator $-{}^1\!/\!_{\varepsilon_r}\Delta$ and the operator $-\nabla \cdot ({}^1\!/\!_{\varepsilon_r}\nabla)$, which models TE-polarized waves (see 3.29), have a finite number of spectral gaps. Building up on their results, Figotin and Kuchment also published an asymptotic analysis (the contrast in the permittivity going to infinity) of a wide range of two-dimensional non-square photonic crystals in [FK98] (see also [Fig99]). Similar studies for three-dimensional periodic media still need to be carried out, but seem very challenging. The only result in this direction available as of this writing is [Fil03]. Therein, the existence of gaps is established under the assumption that the contrast in both the relative permittivity and the relative permeability of the material is suitably high. However, as we said before, practically all photonic crystals manufactured in practice are non-magnetic, so that the result is for now "only" of mathematical interest.

Ad (c): The third and last question we raised above, on the absence of eigenvalues, is just as the Bethe-Sommerfeld conjecture a key problem in the spectral theory of periodic operators.[1] It is believed by physicists and mathematicians alike that no periodic elliptic differential operator of second order has eigenvalues, but a full proof of this statement is yet to be found (see the discussion in [KL02]). Important contributions for certain special cases are L. E. Thomas' well-known paper [Tho73] and the article [Mor00] by A. Morame. In the first-mentioned publication the absence of eigenvalues is proven for a class of Schrödinger operators in up to three dimensions, and the second paper establishes the result for the three-dimensional Maxwell problem in case of a sufficiently smooth relative

[1] We remark that a periodic and self-adjoint elliptic differential operator of arbitrary order with sufficiently regular coefficients has empty singular continuous spectrum (see the note on [KL02, p. 538] regarding [Kuc93, Thm. 4.5.9] and [Kuc01, Sect. 7.4]). Thus, the absence of eigenvalues of such an operator is equivalent to the absolute continuity of its spectrum.

permittivity ε_r. Of course, the latter requirement is not fulfilled for a photonic crystal since there ε_r is discontinuous by the nature of the structure. In the two-dimensional TM-polarized case, however, the absence of eigenvalues is again known to be true, which is the statement of Theorem 4.1.8 given in the last section. This is due to the relation between the eigenvalue problem for the associated operator $-{}^1\!/\!_{\varepsilon_r}\Delta$ and the kernel of a suitable Schrödinger operator governed by Thomas' result. Finally, the two-dimensional TE-polarized case, realized by the operator $-\nabla \cdot ({}^1\!/\!_{\varepsilon_r}\nabla)$ with discontinuous ε_r, still needs to be examined for the absence of eigenvalues. For further details on any of the addressed special cases and a broader overview, we refer to the surveys [Sus00], [Kuc04], and their references as well as to the papers [KL99] and [KL02].

By now, the reader has some intuition as to what is (not) known regarding the questions we asked further up in this section. Clearly, though, many other interesting aspects have not even been mentioned by us. For instance, this refers to (locally) perturbed photonic crystals (see [Kuc01, Sect. 7.6]) as well as the thriving field of photonic crystal waveguides (see the references in [Kuc04, Sect. 3], the articles [SW02], [Fri03], [AS04], and the very recent publications [Bro09], [KO10], [HR11], and [Bro12]). Moreover, we also refer to the literature for information on any related numerical approaches. This concerns the approximation, verification, and optimization of photonic band gaps, but also computer-assisted proofs of their existence in certain special cases (see [Kuc01, Sect. 7.7], [HPW09], [Dör11, Chapt. 2], and [Ric10]).

Summarizing, we hope to have made clear that the periodic operators governing light propagation in photonic crystals still deserve a large amount of attention due to many open problems. In particular, dispersive photonic crystals are not covered by any of the references mentioned in this section.

Chapter 5

A Spectral Problem for Dispersive Photonic Crystals

Having discussed mathematical preliminaries, the physics of light propagation in periodic dielectrics, and the Maxwell eigenvalue problem for a frequency-independent relative permittivity, we are finally in a position to present our research outcomes on a similar equation in the case of dispersive photonic crystals. Because we are discussing new results, and naturally include all relevant proofs, the present chapter is the mathematically most demanding, but also the most elaborated in this dissertation.

Our findings are organized in five sections as follows: First of all, in Section 5.1, we introduce the eigenvalue problem we are concerned with, discuss some of its characteristics, and present first thoughts as to how it can be approached. It is also here that we generalize the main result of the Floquet-Bloch theory known from the previous chapter (see Theorem 4.1.4) and thereby relate our spectral problem to similar equations posed on a primitive cell. Thereafter, we present our results regarding the spectral structure of the problem in Section 5.2. It follows Section 5.3, which reveals some spectral features that are unique to the eigenproblem in the dispersive case and clearly absent in the nondispersive setting of Chapter 4. Subsequently, in Section 5.4, we are concerned with the eigenfunctions of the problems on a primitive cell and prove their basicity and completeness, respectively, under certain assumptions on the underlying relative permittivity. In one of these proofs we rely heavily on the properties of an operator resembling the Riesz projection for our parameter-nonlinear eigenvalue problem. Its construction is likewise provided herein. Finally, Section 5.5 briefly covers the few related studies that exist in the mathematical literature to date.

5.1 PROBLEM STATEMENT AND FIRST THOUGHTS

The properties of the photonic crystals we consider in this chapter differ from those governed by the theory in Chapter 4 in one significant aspect: We now allow for dispersive media and are thus, in case of time-harmonic problems, concerned with a relative permittivity ε_{r} that depends on the frequency ω of a propagating electromagnetic wave (see Section 3.2). In order to avoid ambiguities with respect to the remaining permissible material characteristics, we repeat that we study an infinitely extended two-dimensional nanostructure comprised of linear, isotropic, inhomogeneous, non-magnetic, and fully transparent components. These properties lead—as with nondispersive photonic crystals—to certain basic assumptions on the function ε_{r} summarized hereinafter. From now on, they shall hold without further mentioning.

Assumptions 5.1.1 (Basic assumptions on ε_{r} in the dispersive case)**.** *We suppose that ε_{r} is a function of $(x,\omega) \in \mathbb{R}^2 \times [0,\infty)$ such that*

(a) for all $\omega \in [0,\infty)$ we have $\varepsilon_{\mathrm{r}}(\cdot,\omega) \in L^\infty(\mathbb{R}^2;\mathbb{R})$;

(b) for all $\omega \in [0,\infty)$ the function $\varepsilon_{\mathrm{r}}(\cdot,\omega)$ is $\mathbb{Z}^2$-periodic, i.e.,

$$\varepsilon_{\mathrm{r}}(x,\omega) = \varepsilon_{\mathrm{r}}(x+a,\omega) \qquad \text{for a. a. } x \in \mathbb{R}^2 \text{ and all } a \in \mathbb{Z}^2; \tag{5.1}$$

(c) for all $\omega \in [0,\infty)$ and some positive constant $\varepsilon_{\mathrm{r,min}}$ we have

$$\varepsilon_{\mathrm{r,min}} \leq \varepsilon_{\mathrm{r}}(x,\omega) \leq \|\varepsilon_{\mathrm{r}}(\cdot,\omega)\|_{L^\infty(\mathbb{R}^2)} =: \varepsilon_{\mathrm{r,max}}(\omega) \qquad \text{for a. a. } x \in \mathbb{R}^2. \tag{5.2}$$

Remarks.

(a) As in Chapter 4, the periodicity requirement on ε_{r} implies that we model a photonic crystal with underlying Bravais lattice $\Theta = \mathbb{Z}^2$. Again, other two-dimensional geometries with bounded primitive cells are also covered by the results that follow. Furthermore, just as before, if Ω denotes a corresponding primitive cell, then $\|\varepsilon_{\mathrm{r}}(\cdot,\omega)\|_{L^\infty(\mathbb{R}^2)} = \|\varepsilon_{\mathrm{r}}(\cdot,\omega)\|_{L^\infty(\Omega)}$ for all $\omega \in [0,\infty)$.

(b) Assuming the lower bound $\varepsilon_{\mathrm{r,min}}$ in (5.2) to be frequency-dependent does not cause difficulties in most of what follows. However, doing so does not enlarge the scope of our model under physical aspects, since we saw in Subsection 3.3.2 that the relative permittivity of a transparent medium is uniformly bounded below by 1 (see (3.40)).

With our basic assumptions on ε_{r} in mind, we turn again to the Maxwell eigenvalue problem for TM-polarized waves (3.28). In the dispersive case considered here it reads

$$-\frac{1}{\varepsilon_{\mathrm{r}}(\cdot,\omega)}\Delta u = \frac{\omega^2}{c_0^2} u \qquad \text{in } \mathbb{R}^2.$$

This equation can equivalently be formulated as

$$-\frac{1}{\varepsilon_{\mathrm{r}}(\cdot,c_0\sqrt{\lambda})}\Delta u=\lambda u \qquad \text{in } \mathbb{R}^2,$$

where we introduce the spectral variable $\lambda := \omega^2/c_0^2$ in the same way as in the previous chapter, and thus with the same physical interpretation. Moreover, also as before, u denotes the third component of an electric field amplitude with time-frequency $\omega \in [0,\infty)$. Upon defining the coefficient function ξ through

$$\xi(x,\lambda) := \varepsilon_{\mathrm{r}}(x,c_0\sqrt{\lambda}) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } \lambda \in [0,\infty), \tag{5.3}$$

we rewrite our spectral problem once again, in its final form, as

$$-\frac{1}{\xi(\cdot,\lambda)}\Delta u=\lambda u \qquad \text{in } \mathbb{R}^2. \tag{5.4}$$

As was said before, the complicating feature of this equation is its nonlinearity in the spectral parameter, which is commonly referred to as a *λ-nonlinearity*. Owing to this naming convention, equation (5.4) constitutes the *λ-nonlinear Maxwell eigenvalue problem for TM-polarized waves*, but we shall often use shorter terms when addressing it. Note that this eigenvalue problem is not one for a single operator—as it were in the case of a nondispersive medium—since the coefficient ξ varies with λ. As a consequence, we have to carefully think about suitable generalizations of the usual spectral notions in this context. The rest of this section is devoted to outlining our thought process concerning this issue.

As a first observation, we note that for any fixed $\mu \in [0,\infty)$ the function $\xi(\cdot,\mu)$ is an element of $L^\infty(\mathbb{R}^2;\mathbb{R})$ and $\mathbb{Z}^2$-periodic by its definition (5.3) as well as Assumptions 5.1.1. The latter also imply, still keeping μ fixed, that

$$0<\varepsilon_{\mathrm{r,min}} =: \xi_{\min} \leq \xi(x,\mu) \leq \xi_{\max}(\mu) := \varepsilon_{\mathrm{r,max}}(c_0\sqrt{\mu}) \qquad \text{for a.a. } x \in \mathbb{R}^2. \tag{5.5}$$

Hence, $\xi(\cdot,\mu)$ precisely fulfills Assumptions 4.1.1 so that the family of spectral problems

$$-\frac{1}{\xi(\cdot,\mu)}\Delta u=\lambda u \qquad \text{in } \mathbb{R}^2, \tag{5.6}$$

where $\mu \in [0,\infty)$, is governed by the Floquet-Bloch theory that we presented in the previous chapter. To make use of this connection, we introduce, for $\mu \in [0,\infty)$, the elliptic, self-adjoint, and periodic operator A_μ realizing the equation (5.6) in $L^2_{\xi_\mu}(\mathbb{R}^2)$. That is, $A_\mu : D(A_\mu) \to L^2_{\xi_\mu}(\mathbb{R}^2)$ is given by

$$D(A_\mu) := H^2(\mathbb{R}^2), \qquad A_\mu u := -\frac{1}{\xi(\cdot,\mu)}\Delta u, \tag{5.7}$$

where we indicate by the subscript "ξ_μ" that $\xi(\cdot,\mu)$ is the integration weight function in the occurring L^2-space (see (4.4) and the discussion on p. 49).

Having defined the operators A_μ, we can reconsider our λ-nonlinear problem (5.4) again. When thinking about the term "spectrum" in the context of a suitable L^2-realization of this equation, it is quite canonical to demand that a complex number μ belongs to this set, Σ say, if and only if $A_\mu - \mu I$ is defined[1] and not boundedly invertible. More precisely,

$$\Sigma := \{\mu \in [0, \infty) \mid \mu \in \sigma(A_\mu)\}$$

seems to be the proper generalization of $\sigma(A)$ from the λ-linear case (for the operator A as in (4.5)) to the λ-nonlinear one. This becomes even more convincing in connection with the following arguments: Note that we have

$$\Sigma = \{\mu \in [0, \infty) \mid 0 \in \sigma(A_\mu - \mu I)\},$$

which reminds us of the sets we introduced as spectra of certain operator pencils in the preliminaries (see Subsection 2.4.3). Indeed, we obtain $\Sigma = \sigma(\mathscr{A})$ for the operator pencil $\mathscr{A} : [0, \infty) \to \mathcal{C}(L^2(\mathbb{R}^2))$ defined[2] by

$$\mu \mapsto \mathscr{A}(\mu) := A_\mu - \mu I = -\frac{1}{\xi(\cdot, \mu)}\Delta - \mu I, \quad D(\mathscr{A}(\mu)) = H^2(\mathbb{R}^2). \tag{5.8}$$

Since we shall see below that a decomposition result for $\sigma(\mathscr{A})$ analog to that valid in the nondispersive setting (see Theorem 4.1.4) can be proven, we are confident that the operator pencil $\mathscr{A}$ is the right object to study. This is also why we refer to $\sigma(\mathscr{A})$ as the *spectrum* of the λ-nonlinear problem (5.4) (or the *λ-nonlinear spectrum on the whole space*) in what follows.

In order to establish said spectral decomposition for $\mathscr{A}$, we draw on the results of the Floquet-Bloch theory for the operators A_μ, where $\mu \in [0, \infty)$. As is outlined and proven in Section 4.1, we can characterize $\sigma(A_\mu)$ in terms of the spectra of the operators $A_{\mu,k} : D(A_{\mu,k}) \to L^2_{\xi_\mu}(\Omega)$ given in $L^2_{\xi_\mu}(\Omega)$ by

$$D(A_{\mu,k}) := H^2_{k\text{-per}}(\Omega), \qquad A_{\mu,k}u := -\frac{1}{\xi(\cdot, \mu)}\Delta u. \tag{5.9}$$

Here, as before (see (4.10)), $\Omega = (0, 1)^2$ is chosen as the primitive cell of the lattice $\mathbb{Z}^2$, k is a vector in the Brillouin zone $B = [-\pi, \pi]^2$, and $\xi(\cdot, \mu)$ is the weight function of the inner product in $L^2_{\xi_\mu}(\Omega)$.

[1] Recall that we only declared the function ξ for second arguments in $[0, \infty)$ as they correspond to time frequencies.

[2] Note closely that we introduce $\mathscr{A}$ as a map into the closed operators on the unweighted space $L^2(\mathbb{R}^2)$. It is therefore not entirely correct to make use of the operators A_μ in (5.8). On the other hand, if, for $\mu \in [0, \infty)$, we define an operator $B_\mu : D(A_\mu) \to L^2(\mathbb{R}^2)$ in $L^2(\mathbb{R}^2)$ by $B_\mu u := A_\mu u$, then the boundedness of the weight function $\xi(\cdot, \mu)$ as in (5.5) implies $L^2(\mathbb{R}^2) = L^2_{\xi_\mu}(\mathbb{R}^2)$ and therefore $\sigma(B_\mu) = \sigma(A_\mu)$. This spectral equality made us decide that we should avoid unnecessary complexity by being overly rigorous here. What we have to keep in mind, however, is that no operator $\mathscr{A}(\mu)$ is self-adjoint (or even symmetric) unless we weight the space it acts in accordingly.

With the operators $A_{\mu,k}$ at hand, let us reconsider the λ-nonlinear spectrum on the whole space. First, we have

$$\begin{aligned}\sigma(\mathscr{A}) &= \{\mu \in [0,\infty) \mid 0 \in \sigma(\mathscr{A}(\mu))\} \\ &= \{\mu \in [0,\infty) \mid \mu \in \sigma(A_\mu)\} \\ &= \{\mu \in [0,\infty) \mid \mu \in \sigma(A_{\mu,k}) \text{ for some } k \in B\},\end{aligned} \tag{5.10}$$

where we used that

$$\sigma(A_\mu) = \bigcup_{k\in B} \sigma(A_{\mu,k}) \qquad \text{for all } \mu \in [0,\infty)$$

due to Theorem 4.1.4. Now, upon introducing, for $k \in B$, the operators pencils[1] $\mathscr{A}_k : [0,\infty) \to \mathcal{C}(L^2(\Omega))$ by

$$\mu \mapsto \mathscr{A}_k(\mu) := A_{\mu,k} - \mu I = -\frac{1}{\xi(\cdot,\mu)}\Delta - \mu I, \quad D(\mathscr{A}_k(\mu)) = H^2_{k\text{-per}}(\Omega), \tag{5.11}$$

we can rewrite the equality (5.10) as

$$\begin{aligned}\sigma(\mathscr{A}) &= \{\mu \in [0,\infty) \mid 0 \in \sigma(A_{\mu,k} - \mu I) \text{ for some } k \in B\} \\ &= \bigcup_{k\in B} \{\mu \in [0,\infty) \mid 0 \in \sigma(A_{\mu,k} - \mu I)\} \\ &= \bigcup_{k\in B} \sigma(\mathscr{A}_k).\end{aligned}$$

Owing to the last chain of equalities, we have just proved the following result:

Proposition 5.1.2. *For the spectrum of the operator pencil $\mathscr{A}$ there holds*

$$\sigma(\mathscr{A}) = \bigcup_{k\in B} \sigma(\mathscr{A}_k).$$

In retrospect—i.e., once it was clear to us that the operator pencils $\mathscr{A}$ and $\mathscr{A}_k$ are the right mathematical objects to work with—the proof of the just stated result was rather simple. Its implications, however, are far-reaching. Most importantly, the λ-nonlinear spectral problem (5.4) can be analyzed by studying a family of appropriately realized λ-nonlinear equations of the form

$$-\frac{1}{\xi(\cdot,\lambda)}\Delta u = \lambda u \qquad \text{in } \Omega,$$

which are posed on the bounded primitive cell Ω and subject to k-quasi-periodic boundary conditions, where $k \in B$. In the course of this chapter we shall see that this is not the only similarity between the, so to say, λ-linear spectral problem (4.3) and the λ-nonlinear one (5.4). However, in most cases we have to impose additional restrictions on the coefficient function ξ (or rather ε_{r}) to be able to recover further results from the setting of Chapter 4. This is addressed in detail in the subsequent section.

[1]The remarks concerning the inaccuracies in the definition of the map $\mathscr{A}$ above (see footnote 2 on p. 66) apply in a similar manner also to the operator-valued functions $\mathscr{A}_k$. Most noteworthy, no operator $\mathscr{A}_k(\mu)$ is self-adjoint (or even symmetric) unless the appropriate integration weight is paid attention to.

5.2 THE STRUCTURE OF THE SPECTRUM

Knowing now that the main spectral result of the Floquet-Bloch theory can be generalized to our λ-nonlinear problem, we are naturally interested in more detailed information about the spectra of the occurring operator pencils $\mathscr{A}$ and $\mathscr{A}_k$. Similar to our arguments in the previous section, we shall see that results for the related operators A_μ and $A_{\mu,k}$ are of great help here. Thus, in the lemma below, we summarize important outcomes of our treatment of the λ-linear theory from Section 4.1. They apply to A_μ and $A_{\mu,k}$ due to our basic Assumptions 5.1.1 on ε_{r} and the definition of ξ (see (5.3)).

Lemma 5.2.1 ([Sect. 4.1, Thms. 4.1.5, 4.1.7, 4.1.8 (w. Cor.), and Lem. 4.1.6]). *Let $\mu \in [0, \infty)$.*

(a) For all $k \in B$ the operator $A_{\mu,k}$ has purely discrete spectrum $\{\lambda_{\mu,k,n}\}_{n\in\mathbb{N}}$ with corresponding eigenfunctions $\{\psi_{\mu,k,n}\}_{n\in\mathbb{N}}$ which form an orthonormal basis of $L^2_{\xi_\mu}(\Omega)$. Further,

$$0 \le \lambda_{\mu,k,1} \le \lambda_{\mu,k,2} \le \cdots \le \lambda_{\mu,k,n} \to \infty \quad \text{as } n \to \infty, \tag{5.12}$$

where the min-max principle characterizes these eigenvalues. That is, for all $k \in B$ there holds

$$\lambda_{\mu,k,n} = \min_{\substack{U \sqsubseteq H^1_{k\text{-per}}(\Omega) \\ \dim U = n}} \max_{u \in U\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u(x)|^2 \, dx}{\displaystyle\int_\Omega \xi(x,\mu)|u(x)|^2 \, dx} \qquad \text{for all } n \in \mathbb{N}. \tag{5.13}$$

Besides, the mapping $B \ni k \mapsto \lambda_{\mu,n}(k)$ is non-constant and continuous for all $n \in \mathbb{N}$.

(b) The spectrum of the operator A_μ has a band-gap structure, i.e.,

$$\sigma(A_\mu) = \bigcup_{k\in B} \sigma(A_{\mu,k}) = \bigcup_{n\in\mathbb{N}} \left[\min_{k\in B} \lambda_{\mu,n}(k), \max_{k\in B} \lambda_{\mu,n}(k)\right]$$

and contains no eigenvalues.

Remarks.

(a) As in the nondispersive case we will at times write $\lambda_{k,n}(\mu)$, $\lambda_{\mu,n}(k)$, and $\lambda_{\mu,k}(n)$, when we want to stress the dependence of the eigenvalues and eigenfunctions on μ, k, or n. With respect to the eigenfunctions $\psi_{\mu,k,n}$ we proceed in a similar fashion.

(b) Recall from Theorem 4.1.7 that the spectrum of the operator A as in (4.5) can be obtained through the operators $\widehat{A}_k$ that have a k-independent domain but a k-dependent symbol. In the above context, defining operators $\widehat{A}_{\mu,k}$ in the obvious way is easily possible and yields a result similar to Lemma 5.2.1. We shall, however, only work with $A_{\mu,k}$ in what follows.

As a first simple consequence of part (a) Lemma 5.2.1 we notice that there holds, for all $\mu \in [0,\infty)$ and all $k \in B$,

$$\mu \in \sigma(\mathscr{A}_k) \iff 0 \in \sigma(A_{\mu,k} - \mu I) \iff \mu \in \sigma(A_{\mu,k}) = \{\lambda_{\mu,k,n} \mid n \in \mathbb{N}\}$$

or, in other words,

$$\mu \in \sigma(\mathscr{A}_k) \iff \mu = \lambda_{k,n}(\mu) \text{ for some } n \in \mathbb{N}. \tag{5.14}$$

Hence, determining the spectra of the operator pencils $\mathscr{A}_k$ is equivalent to finding fixed points of the mappings $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$ and the spectral equality of Proposition 5.1.2 can be restated as

$$\sigma(\mathscr{A}) = \bigcup_{k\in B} \sigma(\mathscr{A}_k) = \bigcup_{k\in B} \{\mu \in (0,\infty) \mid \mu = \lambda_{k,n}(\mu) \text{ for some } n \in \mathbb{N}\}. \tag{5.15}$$

Note that it is a priori not clear whether these fixed points exist at all and, if so, whether they are unique. As a consequence, the spectrum of $\mathscr{A}$ and of any operator pencil $\mathscr{A}_k$ can in general be empty, which is impossible for their λ-linear counterparts A and A_k (see Thms. 4.1.5 and 4.1.7). We shall see below that an additional assumption on the coefficient function ξ guarantees the non-emptiness of $\sigma(\mathscr{A}_k)$ for all $k \in B$ and thus, by (5.15), also of $\sigma(\mathscr{A})$. Before we address this, however, we provide in the next proposition more details on the spectral structure of these pencils. They are deduced solely based on Lemma 5.2.1, i.e., no other than our basic assumptions on ξ need to be made. We remark that the spectral notions used below can be recalled from Subsection 2.4.3 of the preliminaries.

Proposition 5.2.2.

(a) For all $k \in B$ the operator pencil $\mathscr{A}_k$ has purely discrete spectrum.

(b) The operator pencil $\mathscr{A}$ has no eigenvalues.

Proof.

Ad (a): Let $k \in B$ be fixed and $\mu \in [0,\infty)$. Then

$$\begin{aligned} \mu \in \sigma(\mathscr{A}_k) &\iff 0 \in \sigma(A_{\mu,k} - \mu I) \\ &\iff \mu \in \sigma(A_{\mu,k}) = \sigma_{\mathrm{d}}(A_{\mu,k}) \\ &\iff 0 \in \sigma_{\mathrm{d}}(A_{\mu,k} - \mu I) \iff \mu \in \sigma_{\mathrm{d}}(\mathscr{A}_k), \end{aligned}$$

where the set equality on the right-hand side of the second line follows from part (a) of Lemma 5.2.1. This proves the claim.

Ad (b): Let $\mu \in [0,\infty)$. Then

$$\mu \in \sigma_p(\mathscr{A}) \iff 0 \in \sigma_p(A_\mu - \mu I) \iff \mu \in \sigma_p(A_\mu) = \emptyset,$$

where the set equality on the right-hand side follows from part (b) of Lemma 5.2.1. Hence, μ cannot be an eigenvalue of $\mathscr{A}$. □

Remark. Part (a) of the just proved proposition and part (a) of Lemma 5.2.1 show that any eigenvalue μ of $\mathscr{A}_k$, which is then also an eigenvalue of $A_{\mu,k}$, has finite multiplicity. What we cannot conclude, however, is that the eigenvalues of $\mathscr{A}_k$ are necessarily isolated. If, for instance, for some $n \in \mathbb{N}$ there holds $\mu = \lambda_{k,n}(\mu)$ for all $\mu \in [\mu_0, \mu_1]$, where $\mu_0 < \mu_1$, then this whole interval consists of fixed points of the mapping $\mu \mapsto \lambda_{k,n}(\mu)$ an thus of uncountably many eigenvalues, each having finite multiplicity, of $\mathscr{A}_k$.[1] In this respect the discrete spectrum of an operator and that of an operator pencil are fundamentally different (see also part (b) of the remark following Definition 2.4.12).

Although it is clearly satisfying that several characteristics of the nondispersive problem are reappearing for the operator pencils $\mathscr{A}$ and $\mathscr{A}_k$, it is still possible that their spectra are empty. To guarantee that this is not the case, assuming that

$$[0,\infty) \ni \mu \mapsto \xi(\cdot,\mu) = \varepsilon_{\mathrm{r}}(\cdot, c_0\sqrt{\mu}) \in L^\infty(\mathbb{R}^2;\mathbb{R}) \qquad \text{is continuous} \tag{5.16}$$

will be seen to be sufficient. The resulting continuity of the relative permittivity ε_{r} in the frequency variable is physically reasonable in our context: It means, roughly speaking, that small changes in a propagating wave's frequency result in likewise small changes in a studied photonic crystal's material response. Moreover, the condition demands that the partition of the studied medium into different dielectric materials is frequency-independent. This continuity is particularly given within transparency regions of the material (see (3.39)) and, for certain choices of the parameters, also in the widely used Lorentz model discussed in Subsection 3.3.3. We write the requirement (5.16) as $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2))$, meaning that for all $\mu \in [0,\infty)$ we use the notation $\xi(\cdot,\mu)$ instead of the formally more correct $[\xi(\mu)](\cdot)$, which will be the case in all of this work.[2] Further, we remark that the Assumptions 5.1.1 are, of course, still implicitly in place. An important outcome of these stronger conditions is formulated in the next lemma.

Lemma 5.2.3. *Let $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$. Then, for all $k \in B$ and all $n \in \mathbb{N}$, the mapping $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$ is continuous.*

Proof. Let $k \in B$ and $\mu \in [0,\infty)$ be fixed. First, for all nonzero $u \in H^1_{k\text{-per}}(\Omega)$ and all $\nu \in [0,\infty) \setminus \{\mu\}$ we have

$$\begin{aligned}
&\frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_\Omega \xi(x,\nu)|u(x)|^2\,dx} - \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_\Omega \xi(x,\mu)|u(x)|^2\,dx} \\
&\qquad = \frac{\int_\Omega |\nabla u(x)|^2\,dx \cdot \int_\Omega \xi(x,\nu)\left(\frac{\xi(x,\mu)}{\xi(x,\nu)} - 1\right)|u(x)|^2\,dx}{\int_\Omega \xi(x,\nu)|u(x)|^2\,dx \cdot \int_\Omega \xi(x,\mu)|u(x)|^2\,dx}.
\end{aligned} \tag{5.17}$$

[1] A coefficient function ξ for which $\sigma(\mathscr{A}_k)$ shows this phenomenon is explicitly constructed and discussed in Section 5.3.

[2] This also means that when we refer to the "second variable" of ξ or ε_{r} later on, then we always mean the non-spatial argument.

Now let $\delta > 0$. By the continuity assumption on ξ there exists some $\eta = \eta(\delta) > 0$ such that for all $\nu \in U_\eta := \{\tau \in [0,\infty) \mid 0 \le |\mu - \tau| < \eta\}$ we have

$$-\delta \le \left(\frac{\xi(x,\mu)}{\xi(x,\nu)} - 1\right) \le \delta \qquad \text{for a. a. } x \in \Omega.$$

From the equality (5.17) we now deduce that

$$\begin{aligned}(1-\delta)\,\frac{\displaystyle\int_\Omega |\nabla u(x)|^2\,dx}{\displaystyle\int_\Omega \xi(x,\mu)|u(x)|^2\,dx} &\le \frac{\displaystyle\int_\Omega |\nabla u(x)|^2\,dx}{\displaystyle\int_\Omega \xi(x,\nu)|u(x)|^2\,dx} \\ &\le (1+\delta)\,\frac{\displaystyle\int_\Omega |\nabla u(x)|^2\,dx}{\displaystyle\int_\Omega \xi(x,\mu)|u(x)|^2\,dx} \qquad \text{for all } \nu \in U_\eta.\end{aligned}$$

Hence, since u was arbitrary, we obtain by the min-max principle (see (5.13)) the inequalities

$$(1-\delta)\,\lambda_{k,n}(\mu) \le \lambda_{k,n}(\nu) \le (1+\delta)\,\lambda_{k,n}(\mu) \qquad \text{for all } \nu \in U_\eta \text{ and all } n \in \mathbb{N},$$

which can be equivalently rewritten as

$$|\lambda_{k,n}(\mu) - \lambda_{k,n}(\nu)| \le \delta\,\lambda_{k,n}(\mu) \qquad \text{for all } \nu \in U_\eta \text{ and all } n \in \mathbb{N}.$$

This finishes the proof. □

Remark. The continuous dependence of the eigenvalues $\lambda_{\mu,k,n}$ on μ also follows from the continuity of the operator pencils $\mathscr{A}_k$ in the norm-resolvent topology, i.e., $\|(\mathscr{A}_k(\mu) - iI)^{-1} - (\mathscr{A}_k(\nu) - iI)^{-1}\|_{\mathcal{B}(L^2(\Omega))} \to 0$ as $\nu \to \mu$, and several abstract arguments (see [Kat95, Thm. IV.2.25 and Sect. IV.3.5]).

Next, let us proceed with our analysis of the spectra of the pencils $\mathscr{A}_k$, which, as we recall, consist of all fixed points of the mappings $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$ (see (5.15)). Based on Lemma 5.2.3, our next result extends part (a) of Proposition 5.2.2.

Proposition 5.2.4. *Let $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$. Then for all $k \in B$ the spectrum of the operator pencil $\mathscr{A}_k$ contains infinitely many points, but it does not need to be countable. If, furthermore, the upper bound $\xi_{\max}(\mu)$ in* (5.5) *is uniform, i.e.,*

$$\xi_{\max}(\mu) \le \xi_{\mathrm{MAX}} < \infty \qquad \textit{for all } \mu \in [0,\infty) \tag{5.18}$$

and some constant ξ_{MAX}, then these spectra are unbounded above.

Proof. Let $k \in B$ be fixed. To show the assertions, we make use of a comparison argument for eigenvalues which we revisit again later in this thesis: For all nonzero $u \in H^1_{k\text{-per}}(\Omega)$ and all $\mu \in [0,\infty)$ we have, by the boundedness of $\xi(\cdot,\mu)$ (see (5.5)),

$$0 \leq \frac{1}{\xi_{\max}(\mu)} \frac{\displaystyle\int_\Omega |\nabla u(x)|^2 \, dx}{\displaystyle\int_\Omega |u(x)|^2 \, dx} \leq \frac{\displaystyle\int_\Omega |\nabla u(x)|^2 \, dx}{\displaystyle\int_\Omega \xi(x,\mu)|u(x)|^2 \, dx} \leq \frac{1}{\xi_{\min}} \frac{\displaystyle\int_\Omega |\nabla u(x)|^2 \, dx}{\displaystyle\int_\Omega |u(x)|^2 \, dx}.$$

This implies, by the min-max principle (see (5.13)), the inequalities

$$0 \leq \frac{1}{\xi_{\max}(\mu)} \lambda^{\mathbb{1}}_{k,n} \leq \lambda_{k,n}(\mu) \leq \frac{1}{\xi_{\min}} \lambda^{\mathbb{1}}_{k,n} \quad \text{for all } \mu \in [0,\infty) \text{ and all } n \in \mathbb{N}, \tag{5.19}$$

where we denote by $\lambda^{\mathbb{1}}_{k,n}$ the nth eigenvalue of the negative Laplace operator $-\Delta : H^2_{k\text{-per}}(\Omega) \to L^2(\Omega)$ in $L^2(\Omega)$. Hence, for all $n \in \mathbb{N}$ the graph of the function $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$ lies within $[0,\infty) \times \left[0, 1/\xi_{\min} \lambda^{\mathbb{1}}_{k,n}\right]$, which is a semi-infinite strip of finite height, and must therefore cross the angle bisector in the $\mu\lambda$-plane at least once due to the continuity of this mapping (see Lemma 5.2.3). For all $n \in \mathbb{N}$ the corresponding fixed point is then an element of $\sigma(\mathscr{A}_k)$, where only finitely many of them are equal due to part (a) of Proposition 5.2.2. As a consequence, $\sigma(\mathscr{A}_k)$ contains infinitely many points. That this spectrum may indeed be uncountable is shown in Section 5.3 by means of an explicitly constructed example. We refer to part (a) of Theorem 5.3.1 in this respect.

Finally, under the additional uniform boundedness assumption (5.18) on ξ, we obtain from the second inequality in (5.19) that

$$\frac{1}{\xi_{\mathrm{MAX}}} \lambda^{\mathbb{1}}_{k,n} \leq \lambda_{k,n}(\mu) \qquad \text{for all } \mu \in [0,\infty) \text{ and all } n \in \mathbb{N}.$$

Therefore, since for all $k \in B$ there holds $\lambda^{\mathbb{1}}_{k,n} \to \infty$ as $n \to \infty$ (see the proof of [RS78, Thm. XIII.98]), the lower bound on any eigenvalue $\lambda_{k,n}(\mu)$, and thus on any fixed point, increases with n. This implies that $\sigma(\mathscr{A}_k)$ is unbounded above and closes the proof. □

Remark. Explicit formulae stated in the proof of [RS78, Thm. XIII.98] show that $\lambda^{\mathbb{1}}_{k,n} = 0$ if and only if $k = 0$ and $n = 1$. Hence, we readily obtain from (5.19) that $\lambda_{0,1}(\mu) = 0$ for all $\mu \in [0,\infty)$ and therefore $0 \in \sigma(\mathscr{A}_0)$ even without continuity of ξ. Moreover, this implies that our basic Assumptions 5.1.1 are sufficient to deduce the non-emptiness of $\sigma(\mathscr{A})$.

The spectral equality (5.15) allows us to state a corollary to Proposition 5.2.4.

Corollary. *Let $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$. Then $\sigma(\mathscr{A})$ contains infinitely many points. If moreover the uniform boundedness as in* (5.18) *is assumed, $\sigma(\mathscr{A})$ is unbounded above.*

Another property of the spectrum of the operator pencil $\mathscr{A}$ that we can deduce from the continuity of ξ is its closedness. Note that this similarly holds true in the λ-linear setting (see part (a) of Proposition 2.4.4 for the operator A). There, of course, closedness is understood with respect to the topology of $\mathbb{C}$, whereas we have to take into account that the domain of $\mathscr{A}$ is $[0,\infty)$.

Proposition 5.2.5. *Let $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$. Then $\rho(\mathscr{A})$ is open in $[0,\infty)$. Besides, as a consequence, $\sigma(\mathscr{A})$ is closed in $[0,\infty)$.*

Proof. Let $\mu \in \rho(\mathscr{A})$. First, note that the resolvent of $\mathscr{A}$ at μ can be written as

$$\mathscr{R}_{\mathscr{A}}(\mu) = \left(-\frac{1}{\xi(\cdot,\mu)}\Delta - \mu I\right)^{-1} = \big(-\Delta - \mu\xi(\cdot,\mu)I\big)^{-1}\xi(\cdot,\mu)I. \tag{5.20}$$

Now, for all $\nu \in [0,\infty)$ we have

$$\begin{aligned} &-\Delta - \nu\xi(\cdot,\nu)I \\ &\quad= -\Delta - \mu\xi(\cdot,\mu)I - \big(\nu\xi(\cdot,\nu) - \mu\xi(\cdot,\mu)\big)I \\ &\quad= \big(-\Delta - \mu\xi(\cdot,\mu)I\big)\Big[I - \big(-\Delta - \mu\xi(\cdot,\mu)I\big)^{-1}\big(\nu\xi(\cdot,\nu) - \mu\xi(\cdot,\mu)\big)I\Big] \\ &\quad= \big(-\Delta - \mu\xi(\cdot,\mu)I\big)\Big[I - \mathscr{R}_{\mathscr{A}}(\mu)\frac{1}{\xi(\cdot,\mu)}\big(\nu\xi(\cdot,\nu) - \mu\xi(\cdot,\mu)\big)I\Big]. \end{aligned} \tag{5.21}$$

By the standard Neumann series argument the operator in square brackets, which we denote by $I - B_\mu(\nu)$, is bijective if $\|B_\mu(\nu)\|_{\mathcal{B}(L^2(\mathbb{R}^2))} < 1$. This inequality is valid, in particular, if

$$\|\nu\xi(\cdot,\nu) - \mu\xi(\cdot,\mu)\|_{L^\infty(\Omega)} < \left(\|\mathscr{R}_{\mathscr{A}}(\mu)\|_{\mathcal{B}(L^2(\mathbb{R}^2))}\left\|\frac{1}{\xi(\cdot,\mu)}\right\|_{L^\infty(\Omega)}\right)^{-1}, \tag{5.22}$$

where we used that the norm of a multiplication operator equals the essential supremum of its multiplier function.

The continuity assumption on ξ now gives that there is some $\delta > 0$ such that the inequality (5.22) holds for $\nu \in U_\delta(\mu) := \{\tau \in [0,\infty) \mid 0 \le |\mu - \tau| < \delta\}$. Hence, for these ν, the operator $I - B_\mu(\nu) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is a bijection. Thus, clearly, the restriction $\big(I - B_\mu(\nu)\big)|_{H^2(\mathbb{R}^2)} : H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ is injective and, in fact, even bijective since for all $u \in H^2(\mathbb{R}^2)$ there is some $v \in L^2(\mathbb{R}^2)$ such that $(I - B_\mu(\nu))v = u$, giving $v = B_\mu(\nu)v + u$, which is a sum of elements of $H^2(\mathbb{R}^2)$. Finally, (5.21) yields that the resolvent $(-\Delta - \nu\xi(\cdot,\nu)I)^{-1} : L^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ exists for all $\nu \in U_\delta(\mu)$ and is bounded as a closed and everywhere-defined operator. Since the multiplication operator $\xi(\cdot,\nu)I : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is likewise bounded for these ν, we obtain from an identity analogous to (5.20) the inclusion $U_\delta(\mu) \subset \rho(\mathscr{A})$. Thus, $\rho(\mathscr{A})$ is open in $[0,\infty)$. This, of course, allows us to conclude that $\sigma(\mathscr{A})$ is closed in $[0,\infty)$, whereby the proof is complete. $\square$

The results we established up to now show that under continuous dependence of ξ on the spectral variable the λ-nonlinear problem (5.4) shares some of the characteristics of its λ-linear counterpart (4.3). However, $\sigma(\mathscr{A})$ need not have a proper band-gap structure, i.e., consist of a union of non-degenerate intervals, and $\sigma(\mathscr{A}_k)$ can be uncountable for some $k \in B$ under our assumptions so far (see the examples and the related discussion in Section 5.3). Clearly, studying the eigenfunctions of an operator pencil $\mathscr{A}_k$ is problematic if countability of its spectrum cannot be guaranteed. It turns out that we can cope with this issue, and also obtain the band-gap structure of $\sigma(\mathscr{A})$, by assuming, in addition to the continuity of ξ as in (5.16), that

$$\begin{aligned}&(0,\infty) \ni \mu \mapsto \mu\xi(x,\mu) = \mu\varepsilon_{\mathrm{r}}(x, c_0\sqrt{\mu})\\ &\text{is strictly monotonically increasing for almost all } x \in \Omega.\end{aligned} \tag{5.23}$$

Before we formulate the respective theorem, let us briefly reflect about the consequences which the latter assumption has on the media that our model applies to. At last, we should not impose a restriction on ε_{r} that disagrees with the properties of this function known from physical experiments. To see the connection, recall that we introduced the spectral variable $\lambda = \omega^2/c_0^2$ at the outset of this chapter and immediately renamed it to μ. Hence, since for $\omega_1, \omega_2 \in (0,\infty)$ we have $\omega_1 > \omega_2 \iff \omega_1^2 > \omega_2^2$, our requirement (5.23) can be rewritten as

$$\begin{aligned}&(0,\infty) \ni \omega \mapsto \omega^2\varepsilon_{\mathrm{r}}(x,\omega)\\ &\text{is strictly monotonically increasing for almost all } x \in \Omega.\end{aligned}$$

Now, in Subsection 3.3.2 we deduced that it is reasonable to assume that the relative permittivity is a monotonically increasing function of the frequency ω in any transparency interval[1]. On the other hand, fully transparent media, as we model them here, can only be seen as approximations and it is to be expected that the relative permittivity of a realistic material is also decreasing for certain frequencies. It is therefore reassuring that even though an everywhere monotonically increasing function ε_{r} clearly fulfills assumption (5.23), this requirement is actually weaker.

With this in mind, we state our theorem governing the spectral properties of the operator pencils $\mathscr{A}$ and $\mathscr{A}_k$ under our so far strongest assumptions on the coefficient function ξ. For later reference, we also include details which stem from already proved assertions of this chapter.

Theorem 5.2.6. *Let $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$ satisfy the strict monotonicity requirement* (5.23). *Then for all $k \in B$ the operator pencil $\mathscr{A}_k$ has purely discrete spectrum consisting of a sequence $\{\mathring{\mu}_{k,n}\}_{n\in\mathbb{N}}$ of isolated eigenvalues which satisfy*

$$0 \le \mathring{\mu}_{k,1} \le \mathring{\mu}_{k,2} \le \cdots \le \mathring{\mu}_{k,n} \to \infty \quad \text{as } n \to \infty \tag{5.24}$$

and are the unique fixed points of the mappings $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$. Further, the dependence of $\mathring{\mu}_{k,n}$ on k is continuous for all $n \in \mathbb{N}$. Finally, the spectrum of the operator pencil $\mathscr{A}$ is unbounded above, closed in $[0,\infty)$, and has a band-gap structure, i.e.,

$$\sigma(\mathscr{A}) = \bigcup_{k\in B} \sigma(\mathscr{A}_k) = \bigcup_{n\in\mathbb{N}} \left[\min_{k\in B} \mathring{\mu}_{k,n}, \max_{k\in B} \mathring{\mu}_{k,n}\right], \tag{5.25}$$

where the intervals on the right-hand side are non-degenerate for all $n \in \mathbb{N}$.

[1] Recall that this is a range of frequencies for which $\operatorname{Im}\varepsilon_r$ can be neglected.

Proof. The first equality in (5.25) follows from Proposition 5.1.2. Note that this also implies that $\sigma(\mathscr{A})$ is unbounded above once we have established that all spectra $\sigma(\mathscr{A}_k)$ consist of a sequence tending to infinity. Further, the discreteness of the latter spectra for all $k \in B$ has been proven as part (a) of Proposition 5.2.2 and in our discussion in between the two mentioned results we established in (5.14) the fixed point character of the spectral points of any operator pencil $\mathscr{A}_k$. Finally, the closedness of $\sigma(\mathscr{A})$ in $[0,\infty)$ is the statement of Proposition 5.2.5.

For a better readability of our proof, we divide the remaining assertions in the following parts, each, as we will show, valid for all $k \in B$ and all $n \in \mathbb{N}$:

(a) The mapping $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$ has a unique fixed point $\overset{*}{\mu}_{k,n}$.

(b) The sequence $\{\overset{*}{\mu}_{k,n}\}_{n\in\mathbb{N}}$ is monotonically increasing and tends to infinity.

(c) The eigenvalue $\overset{*}{\mu}_{k,n}$ is isolated in $\sigma(\mathscr{A}_k)$.[1]

(d) The mapping $B \ni k \mapsto \overset{*}{\mu}_{k,n}$ is continuous.

(e) The set $S_n = \left\{\overset{*}{\mu}_{k,n} \mid k \in B\right\}$ is a non-degenerate compact interval.

Ad (a): First, from our remark immediately below the proof of Proposition 5.2.4 we deduce that $\lambda_{k,n}(0) = 0$ if and only if $k = 0$ and $n = 1$. Hence, $\overset{*}{\mu}_{0,1} = 0$ and this fixed point is also unique since said remark even gives $\lambda_{0,1}(\mu) = 0$ for all $\mu \in [0,\infty)$. Furthermore, for all $k \in B$ and all $n \in \mathbb{N}$ with $(k,n) \neq (0,1)$ the strict monotonicity assumption (5.23) yields, by the min-max principle (see (5.13)), that

$$(0,\infty) \ni \mu \mapsto \frac{\lambda_{k,n}(\mu)}{\mu} \qquad \text{is strictly monotonically decreasing.} \tag{5.26}$$

Note here that both the minimum and the maximum in the min-max formula are attained (see [WS72, Chapts. 1–3]), so that the strict monotonicity is really preserved. Now, by the continuity of the functions $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$ (see Lemma 5.2.3), and since k and n are such that $\lambda_{k,n}(0) > 0$, we find $\lambda_{k,n}(\mu) > \mu$ or, equivalently, $\frac{1}{\mu}\lambda_{k,n}(\mu) > 1$ for sufficiently small $\mu > 0$. On the other hand, we have $\frac{1}{\mu}\lambda_{k,n}(\mu) \leq 1$ for sufficiently large μ (note the uniform upper bound for $\lambda_{k,n}(\mu)$ in (5.19)). Thus, the remaining fixed points exist and their uniqueness is a consequence of (5.26).

Ad (b): To obtain a contradiction, we suppose that $\overset{*}{\mu}_{k,n+1} < \overset{*}{\mu}_{k,n}$ for some $k \in B$ and some $n \in N$. Since only the fixed point $\overset{*}{\mu}_{0,1}$ is zero (see part (a) of this proof), we may assume $0 < \overset{*}{\mu}_{k,n+1}$ and find, using (5.26),

$$\frac{\lambda_{k,n}(\overset{*}{\mu}_{k,n})}{\overset{*}{\mu}_{k,n}} = 1 = \frac{\lambda_{k,n+1}(\overset{*}{\mu}_{k,n+1})}{\overset{*}{\mu}_{k,n+1}} > \frac{\lambda_{k,n+1}(\overset{*}{\mu}_{k,n})}{\overset{*}{\mu}_{k,n}},$$

[1]This is, of course, a trivial consequence of part (b) since $\sigma(\mathscr{A}_k) = \{\overset{*}{\mu}_{k,n} \mid n \in \mathbb{N}\}$. We nevertheless included it here separately so has to highlight once more that the definition of the discrete spectrum of an operator pencil does not guarantee that the points in this set are isolated (see the remark following the proof of Proposition 5.2.2).

giving $\lambda_{k,n+1}(\overset{*}{\mu}_{k,n}) < \lambda_{k,n}(\overset{*}{\mu}_{k,n})$, which contradicts the increasing ordering of the eigenvalues of the λ-linear problem for the operator $A_{\mu,k}$ with $\mu = \overset{*}{\mu}_{k,n}$. This proves the claimed monotonicity of the sequence $\{\overset{*}{\mu}_{k,n}\}_{n\in\mathbb{N}}$ for all $k \in B$. If we can also show that this sequence is unbounded above for all $k \in B$, assertion (b) readily follows.

To achieve this, we again assume the contrary, i.e., that for some $k \in B$ and some $M > 0$ we have $\overset{*}{\mu}_{k,n} < M$ for all $n \in N$. Then, however, (5.26) implies

$$1 = \frac{\lambda_{k,n}(\overset{*}{\mu}_{k,n})}{\overset{*}{\mu}_{k,n}} > \frac{\lambda_{k,n}(M)}{M} \qquad \text{for all } n \in \mathbb{N},$$

giving $\lambda_{k,n}(M) < M$ for all $n \in N$. Thus, the eigenvalues of the operator $A_{M,k}$ are bounded above, which contradicts part (a) of Lemma 5.2.1.

Ad (c): If for some $k \in B$ a point in $\sigma(\mathscr{A}_k)$ were not isolated, then it were an accumulation point of that set and thus also of the sequence $\{\overset{*}{\mu}_{k,n}\}_{n\in\mathbb{N}}$. The latter is monotonically increasing by part (b) of the theorem we just prove and hence can have at most one accumulation point, which is then its limit. However, $\overset{*}{\mu}_{k,n} \to \infty$ as $n \to \infty$, giving assertion (c).

Ad (d): Let $n \in \mathbb{N}$ be fixed, $k \in B$, and $\delta > 0$. Then, there holds[1]

$$\frac{\lambda_{k,n}(\overset{*}{\mu}_{k,n} + \delta)}{\overset{*}{\mu}_{k,n} + \delta} < 1, \tag{5.27}$$

since otherwise the continuity of the mapping in (5.26) and the convergence $\frac{1}{\mu}\lambda_{k,n}(\mu) \to 0$ as $\mu \to \infty$ (again, this follows from the uniform upper bound for $\lambda_{k,n}(\mu)$ in (5.19)) would yield the existence of some $\overset{*}{\mu} \in [\overset{*}{\mu}_{k,n} + \delta, \infty)$ with $\lambda_{k,n}(\overset{*}{\mu}) = \overset{*}{\mu}$, which contradicts part (a) of this proof. In addition, we deduce from (5.26) and the fixed point character of $\overset{*}{\mu}_{k,n}$, that

$$\frac{\lambda_{k,n}(\overset{*}{\mu}_{k,n} - \delta)}{\overset{*}{\mu}_{k,n} - \delta} > 1, \quad \text{if } \overset{*}{\mu}_{k,n} - \delta > 0.$$

Hence, due to the continuity of the mapping $B \ni k \mapsto \lambda_{k,n}(\mu)$ for all $\mu \in [0,\infty)$ and all $n \in \mathbb{N}$ (see part (a) of Lemma 5.2.1) there exists some $\eta = \eta(\delta) > 0$ such that for all $k' \in U_\eta(k) := \{\widetilde{k} \in B \mid 0 \le |k - \widetilde{k}| < \eta\}$ we have

$$\frac{\lambda_{k',n}(\overset{*}{\mu}_{k,n} + \delta)}{\overset{*}{\mu}_{k,n} + \delta} < 1 \qquad \text{and} \qquad \frac{\lambda_{k',n}(\overset{*}{\mu}_{k,n} - \delta)}{\overset{*}{\mu}_{k,n} - \delta} > 1, \quad \text{if } \overset{*}{\mu}_{k,n} - \delta > 0. \tag{5.28}$$

Now, if $\overset{*}{\mu}_{k,n} - \delta \le 0$ the first of these inequalities and, again, (5.26) (with k replaced by k') yield

$$\frac{\lambda_{k',n}(\mu)}{\mu} \le \frac{\lambda_{k',n}(\overset{*}{\mu}_{k,n} + \delta)}{\overset{*}{\mu}_{k,n} + \delta} < 1 \qquad \text{for all } \mu \in [\overset{*}{\mu}_{k,n} + \delta, \infty).$$

[1]Note that the inequality (5.27) also follows from (5.26) as long as $\overset{*}{\mu}_{k,n}$ is positive, i.e., as long as $(k,n) \neq (0,1)$ (see part (a) of this proof).

Hence, $\overset{*}{\mu}_{k',n} \notin [\overset{*}{\mu}_{k,n} + \delta, \infty)$, i.e., $\overset{*}{\mu}_{k',n} < \overset{*}{\mu}_{k,n} + \delta$ and, trivially, $\overset{*}{\mu}_{k',n} \geq 0 \geq \overset{*}{\mu}_{k,n} - \delta$. All in all, we find

$$|\overset{*}{\mu}_{k,n} - \overset{*}{\mu}_{k',n}| \leq \delta \qquad \text{for all } k' \in U_\eta(k). \tag{5.29}$$

It remains to consider the case $\overset{*}{\mu}_{k,n} - \delta > 0$. Then, however, the continuity of the mapping in (5.26) (with k replaced by k') and the two inequalities in (5.28) imply that for all $k' \in U_\eta(k)$ there is some $\mu_{k'} \in (\overset{*}{\mu}_{k,n} - \delta, \overset{*}{\mu}_{k,n} + \delta)$ such that $\lambda_{k',n}(\mu_{k'}) = \mu_{k'}$. Due to part (a) of this proof, this fixed point equals $\overset{*}{\mu}_{k',n}$, meaning that (5.29) is again valid. Since $k \in B$ was arbitrary, the asserted continuity follows.

Ad (e): Suppose first that for some $n \in \mathbb{N}$ the set $S_n = \{\overset{*}{\mu}_{k,n} \mid k \in B\}$ were a singleton, i.e., $S_n = \{\overset{*}{\mu}_n\}$. Then the mapping $B \ni k \mapsto \lambda_{\overset{*}{\mu}_n,n}(k)$ were constant, which contradicts part (a) of Lemma 5.2.1. Further, since the Brillouin zone B is compact and connected and since $B \ni k \mapsto \overset{*}{\mu}_{k,n}$ has just been proven to be continuous for all $n \in \mathbb{N}$, we find $S_n = \left[\min_{k\in B} \overset{*}{\mu}_{k,n}, \max_{k\in B} \overset{*}{\mu}_{k,n}\right]$ as claimed. This concludes the proof. □

Remarks.

(a) For $k \in B$ and $n \in \mathbb{N}$ the notation $\overset{*}{\mu}_{k,n}$ (sometimes also $\overset{*}{\mu}_n(k)$) that we introduced in the statement of the last theorem shall from now on always be used to denote the nth eigenvalue of the operator pencil $\mathscr{A}_k$. We also call it a *λ-nonlinear eigenvalue* and refer to $\sigma(\mathscr{A}_k) = \{\overset{*}{\mu}_{n,k} \mid n \in \mathbb{N}\}$ as the *λ-nonlinear spectrum* (*on the primitive cell*) in what follows.

(b) Note that contrary to Proposition 5.2.4 we obtain the unboundedness of $\sigma(\mathscr{A}_k)$, and thus of $\sigma(\mathscr{A})$, in Theorem 5.2.6 without assuming uniform boundedness of ξ as in (5.18). This is, of course, an outcome of the additional strict monotonicity requirement (5.23).

(c) We referred to the spectral equality (5.25), particularly to its right-hand side, as a *band-gap structure* of $\sigma(\mathscr{A})$ although no proper definition of the term "spectral gap" has been given for $\mathscr{A}$ so far. From now on, this notion shall stand for any interval of the form

$$\Big(\max_{k\in B} \overset{*}{\mu}_{k,n}, \min_{k\in B} \overset{*}{\mu}_{k,n+1}\Big) \neq \emptyset \qquad \text{for some } n \in \mathbb{N},$$

which is the obvious λ-nonlinear generalization of the definition given in Chapter 4 (compare to (4.24)).

Besides, the physical interpretation of the band-gap structure of $\sigma(\mathscr{A})$ remains unchanged from that in the nondispersive case: A real number λ in a spectral gap of $\mathscr{A}$ corresponds via $\lambda = \omega^2/c_0^2$ to a non-negative frequency ω for which time-harmonic TM-polarized wave propagation is prohibited within the photonic crystal modeled by the (frequency-dependent) relative permittivity ε_r.

The just proved theorem constitutes our main and last result on the spectral structure of our eigenvalue problem for dispersive photonic crystals. It shows that relatively mild, and physically reasonable, assumptions on ξ guarantee that all spectral properties of the operators A and A_k mentioned in Chapter 4 also hold for their λ-nonlinear counterparts $\mathscr{A}$ and $\mathscr{A}_k$.

In the discovery of Theorem 5.2.6 we were often guided by geometrical arguments, which is hardly surprising in view of the fixed point nature of several of our arguments. Since these graphs were so valuable to us, we find it important to include them here. With their help, we analyze the spectral problem (5.4) for a fictitious relative permittivity ε_r such that the associated coefficient function ξ satisfies the assumptions of said theorem. For fixed $\widetilde{k} \in B$ we find the λ-nonlinear eigenvalues as the fixed points of the mappings $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$, which have to be such that $(0,\infty) \ni \mu \mapsto \frac{1}{\mu}\lambda_{k,n}(\mu)$ is monotonically decreasing (see (5.26)). Assuming $k \neq 0$, a possible sketch of these functions for $1 \leq n \leq 4$ in a portion of the $\mu\lambda$-plane is given in Figure 5.1 below, wherein the λ-nonlinear eigenvalues $\overset{*}{\mu}_{\widetilde{k},1}, \ldots, \overset{*}{\mu}_{\widetilde{k},4}$ are assumed to be simple.

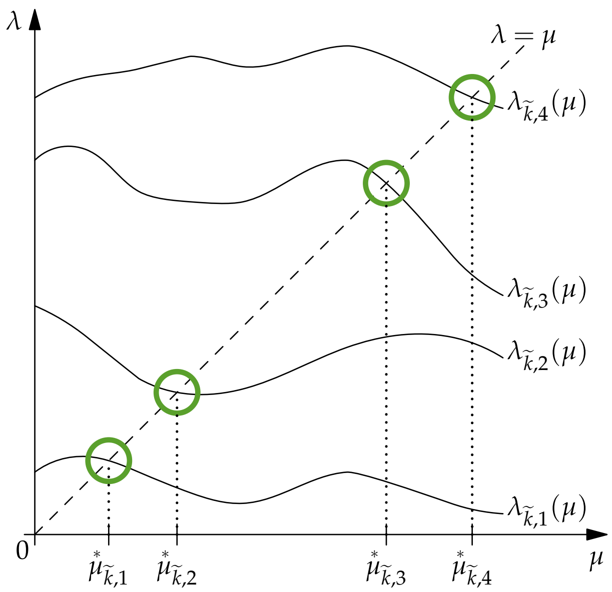

FIGURE 5.1 — A possible sketch of the mappings $\mu \mapsto \lambda_{k,n}(\mu)$ for fixed $\widetilde{k} \in B \setminus \{0\}$ and $1 \leq n \leq 4$ under the assumptions of Theorem 5.2.6 on ε_r. The first four λ-nonlinear eigenvalues are given by the μ-coordinates of the unique fixed-points (marked by colored circles).

So as to graphically find the λ-nonlinear spectrum on the whole space, or, strictly speaking, only a subset thereof, we first fix μ at $\overset{*}{\mu}_{\widetilde{k},3}$ and highlight the spectrum of the corresponding λ-linear problem on $\mathbb{R}^2$, i.e., that of the operator A_μ, where $\mu = \overset{*}{\mu}_{\widetilde{k},3}$. Note that this means that while μ is kept fixed, we let k run through all of B. The result is depicted in Figure 5.2 and, exemplarily, exhibits two (λ-linear) band gaps within the considered λ-range. Proceeding similarly for

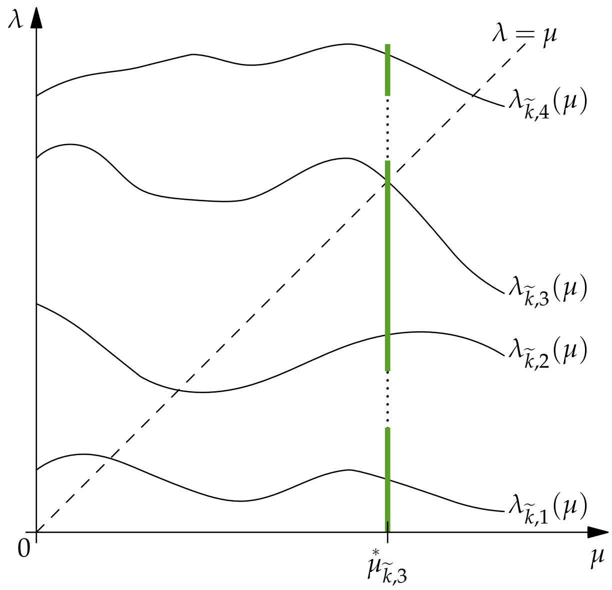

FIGURE 5.2 — The continuation of Figure 5.1. Fixing μ at $\overset{*}{\mu}_{\tilde{k},3}$, the spectrum of the related λ-linear problem on $\mathbb{R}^2$ is highlighted by varying k over B.

all remaining values of μ, any point (μ, λ) in the first quadrant of the $\mu\lambda$-plane can be classified into one of two categories. Either there holds $\lambda \in \sigma(A_\mu)$ (if and only if (μ, λ) is highlighted) or $\lambda \notin \sigma(A_\mu)$. The possible outcomes of this

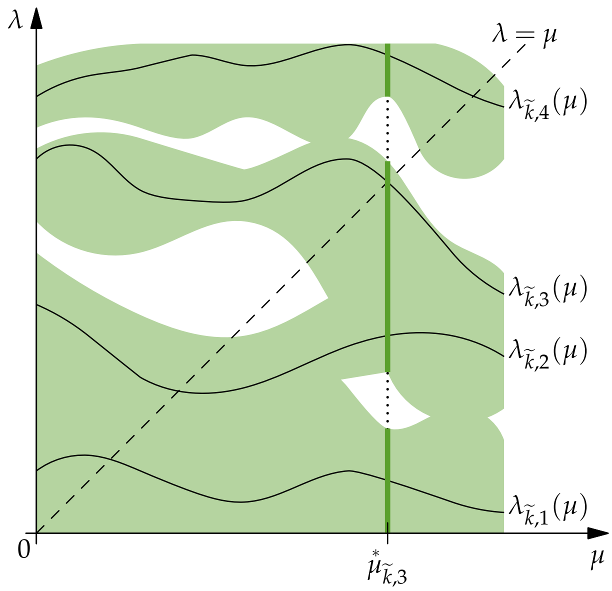

FIGURE 5.3 — The continuation of Figure 5.2. Upon highlighting the λ-linear spectra of the operators A_μ for all shown values of μ, the considered quadrant gets two-colored. Here, a point (μ, λ) is highlighted if and only if $\lambda \in \sigma(A_\mu)$.

procedure are subject to the following restrictions: Firstly, the sketch must be such that the spectrum $\sigma(A_\mu)$ has a band-gap structure consisting of a union of closed intervals for all $\mu \in [0,\infty)$ (see part (b) of Lemma 5.2.1). Secondly, the region $\{(\mu,\lambda) \in \mathbb{R}^2 \mid 0 \leq \mu < \infty, 0 \leq \max_{k\in B} \lambda_{\mu,1}(k)\}$ has to be highlighted entirely, since we know from our remark following the proof of Proposition 5.2.4 that $\lambda_{0,1}(\mu) = 0$ for all $\mu \in [0,\infty)$. Figure 5.3 on the previous page shows what the final result might look like in our example.

Note from the last illustration that the angle bisector in the $\mu\lambda$-plane is partly highlighted. These are precisely the points (μ,μ) for which $\mu \in \sigma(A_\mu)$ and thus, by (5.10), $\mu \in \sigma(\mathscr{A})$. Hence, the λ-nonlinear spectrum is finally found by projecting the highlighted regions onto the λ-axis. This is shown in Figure 5.4.

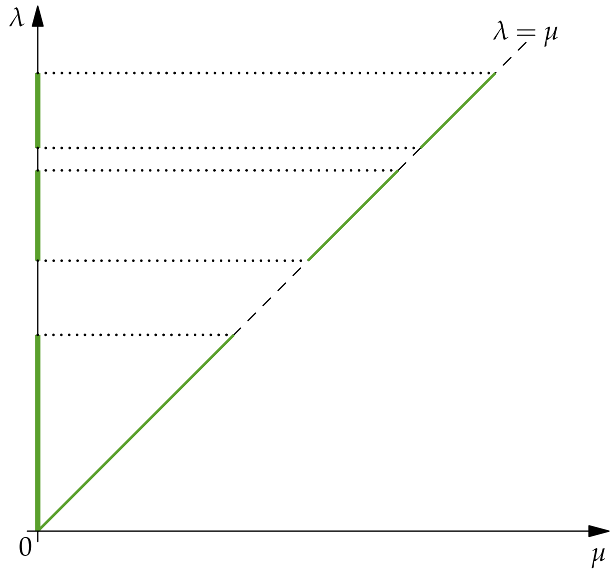

FIGURE 5.4 — The continuation of Figure 5.3. Finally, the λ-nonlinear spectrum on $\mathbb{R}^2$ within the shown quadrant is found by projecting the highlighted parts of the angle bisector onto the λ-axis.

Certainly, for a given photonic crystal, i.e., a relative permittivity ε_r, the information needed to explicitly compute the associated λ-nonlinear spectrum by means of the graphical method we just presented is not known. Nevertheless, we think—and experienced it ourselves—that the thorough understanding of the shown illustrations is helpful in determining spectral properties of the dispersive eigenvalue problem. Moreover, the development of suitable numerical methods might benefit from this way of seeing the problem.

5.3 SPECTRAL PHENOMENA ABSENT FOR THE NONDISPERSIVE PROBLEM

The aim of this section is to show that even a rather simple λ-nonlinearity of the eigenvalue problem (5.4) can result in various spectral effects, referred to as *spectral phenomena*, that are known to be absent for the λ-linear counterpart (4.3). To be more precise, we study the spectra of the operator pencils $\mathscr{A}$ and $\mathscr{A}_k$, where $k \in B$, from the previous section (see (5.8) and (5.11)) for a coefficient function $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$ in product form. That is,

$$\xi(x,\mu) = \xi^{(s)}(x)\eta(\mu) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } \mu \in [0,\infty), \tag{5.30}$$

where the superscript "(s)" is attached to the only spatially-dependent factor of ξ. Here, ξ and $\xi^{(s)}$ shall satisfy the basic Assumptions 4.1.1 and 5.1.1 that we also imposed on the relative permittivity in Section 4.1 and 5.1, respectively. Note that this implies in particular that η is supposed to be a continuous function bounded below by some positive constant $\eta_{\min}$. Naturally, we do not impose the monotonicity requirement (5.23) on ξ, since otherwise Theorem 5.2.6 applies, meaning that $\sigma(\mathscr{A})$ and $\sigma(\mathscr{A}_k)$ do not reveal spectral phenomena.

With our assumptions for this section in place, we turn to the analysis of the spectral consequences of the product form of ξ. First, note that the spectral problems

$$-\frac{1}{\xi^{(s)}}\Delta u = \lambda u \quad \text{in } \mathbb{R}^2 \qquad \text{and} \qquad -\frac{1}{\xi(\cdot,\mu)}\Delta u = \lambda u \quad \text{in } \mathbb{R}^2, \tag{5.31}$$

where $\mu \in [0,\infty)$, are governed by the λ-linear Floquet-Bloch theory. Consistent with the notation used in Chapters 4 and (so far) 5, we denote by A and A_μ the operators realizing the equations (5.31) in appropriately weighted L^2-spaces. Moreover, for $k \in B$, we again introduce the operators A_k and $A_{\mu,k}$ which govern the corresponding k-quasi-periodic problems posed on the primitive cell. We are brief here since this is virtually unchanged from our previous treatises.

Now, for $k \in B$ and $n \in N$ we write $\lambda^{(s)}_{k,n}$ for the nth eigenvalue of the operator A_k and, as before, $\lambda_{k,n}(\mu)$ for the nth eigenvalue of the operator $A_{\mu,k}$. They are related, due to the min-max principle (5.13) and the product form (5.30) of ξ, by the simple equation

$$\lambda_{k,n}(\mu) = \frac{1}{\eta(\mu)}\lambda^{(s)}_{k,n} \qquad \text{for all } \mu \in [0,\infty). \tag{5.32}$$

This shows, in particular, that upper and lower bounds on η restrict the possible eigenvalues of $A_{\mu,k}$. Furthermore, we find that the λ-nonlinear eigenvalues $\overset{*}{\mu}_{k,n}$ of the operator pencil $\mathscr{A}_k$ satisfy

$$\overset{*}{\mu}_{k,n} = \frac{1}{\eta(\overset{*}{\mu}_{k,n})}\lambda^{(s)}_{k,n} \qquad \text{for all } n \in \mathbb{N}, \tag{5.33}$$

owing to their fixed point nature (see (5.14)). In view of this equation it seems plausible that specific functions η exist which unleash certain spectral phenomena. Given that this section is a purely mathematical one, i.e., we are not interested in physically relevant spectral problems per se[1], we are very free in choosing

[1] Although they might very well be some, since, after all, the necessary physically-motivated assumptions are made in this section.

this function, as long as it satisfies our above requirements. By means of three different such choices, we eventually proof the subsequent result.

Theorem 5.3.1. *There exist coefficient functions $\xi = \xi^{(s)}\eta \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$ satisfying the assumptions of this section such that the following occurs:*

(a) The spectrum of the operator pencil $\mathscr{A}_k$ is uncountable for some $k \in B$.

(b) The set $\bigcup_{k\in B} \{\mu \in [0,\infty) \mid \mu = \lambda_{k,n}(\mu)\}$ is not an interval for some $n \in \mathbb{N}$.

(c) The spectrum of the operator pencil $\mathscr{A}$ contains an isolated point.

Moreover, if ξ is only spatially-dependent, i.e., $\eta \equiv 1$, or if ξ additionally satisfies the strict monotonicity requirement (5.23)*, then neither of (a), (b), or (c) arises.*

Proof. Let us first address the last sentence in the statement of the theorem. Its validity follows in the case $\eta \equiv 1$ from Theorems 4.1.5 and 4.1.7 as well as the corollary to Theorem 4.1.8. Besides, under the additional monotonicity assumption (5.23) it holds on account of Theorem 5.2.6.

Before we individually construct the examples that generate the three listed effects, we discuss their similarities. They all share a common structure in so far as the λ-nonlinearity of the problem (5.4) is only given locally as follows: We assume that the function $\xi^{(s)}$ is such that for some $N \in \mathbb{N}$, which is fixed for the rest of the current proof, the λ-linear problem on the left-hand side of (5.31) has a spectral gap

$$G_N := \Big(\max_{k\in B} \lambda_N^{(s)}(k), \min_{k\in B} \lambda_{N+1}^{(s)}(k) \Big) \neq \emptyset. \tag{5.34}$$

Here, the used notation follows our tradition of writing, e.g., $\lambda_N^{(s)}(k)$ instead of $\lambda_{k,N}^{(s)}$. Note that a function $\xi^{(s)}$ that causes the opening of a spectral gap does indeed exist, as we discussed in our literature review (see Section 4.2). For a better readability, we next introduce some notation to be used below. We set

$$\begin{aligned} B_{\min} &:= \operatorname*{arg\,min}_{k\in B} \lambda_{N+1}^{(s)}(k), & B_{\max} &:= \operatorname*{arg\,max}_{k\in B} \lambda_N^{(s)}(k), \\ \mu_{\min} &:= \max_{k\in B} \lambda_N^{(s)}(k), & \mu_{\max} &:= \min_{k\in B} \lambda_{N+1}^{(s)}(k), \end{aligned}$$

where we remark that $B_{\min}$ and $B_{\max}$ are, in general, sets containing more than one element. By definition, there holds

$$\begin{aligned} \lambda_N^{(s)}(k) &= \mu_{\min} \qquad \text{for all } k \in B_{\max}, \\ \lambda_{N+1}^{(s)}(k) &= \mu_{\max} \qquad \text{for all } k \in B_{\min}, \end{aligned} \tag{5.35}$$

and moreover we have $G_N = (\mu_{\min}, \mu_{\max})$. Finally, the above-mentioned local λ-nonlinearity is a consequence of the relation

$$\eta(\mu) = 1 \qquad \text{for all } \mu \notin (\mu_{\min}, \mu_{\max}),$$

which our three constructions below all fulfill. Most importantly, this implies

$$\lambda_{k,n}(\mu) = \lambda_{k,n}^{(s)} \qquad \text{for all } \mu \notin (\mu_{\min}, \mu_{\max})$$

by equation (5.32). Thus, in the indicated μ-region the λ-nonlinear eigenvalues and the λ-linear ones coincide (see also (5.33)), as is depicted in Figure 5.5 below.

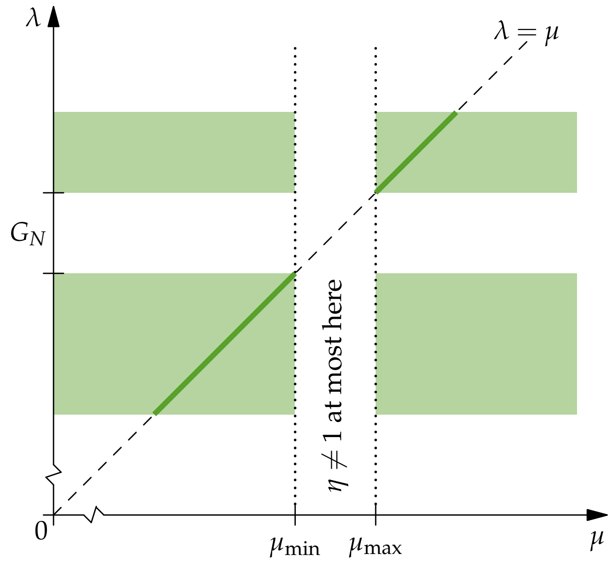

FIGURE 5.5 — A visualisation in the style of Section 5.2 of a part of $\sigma(\mathscr{A})$ (highlighted on the angle bisector). The underlying coefficient function is of the product form given by $\xi(x,\mu) = \xi^{(s)}(x)\eta(\mu)$, where $\eta \neq 1$ at most for $\mu \in G_N$, which is a spectral gap for the λ-linear problem corresponding to $\xi^{(s)}$ (see (5.34)). By varying the values of η for $\mu \in G_n$, the illustration can change at most within the area confined by the two dotted lines.

Note that this implies $\sigma(A) \subseteq \sigma(\mathscr{A})$ and that these spectra can at most differ by points in $G_N = (\mu_{\min}, \mu_{\max})$. That difference however, even if just locally present, is sufficient to reveal the spectral phenomena mentioned in the statement of the theorem, which we shall now finally approach.

Ad (a): For the purpose of showing that $\sigma(\mathscr{A}_k)$ can be uncountable for some $k \in B$ in the given setting, we set $\mu_{\text{mid}} := \frac{1}{2}(\mu_{\min} + \mu_{\max})$ and define the function $\eta_{(a)} : [0,\infty) \to \mathbb{R}$ as

$$\eta_{(a)}(\mu) := \begin{cases} \frac{\mu_{\min}}{\mu}, & \text{if } \mu \in (\mu_{\min}, \mu_{\text{mid}}], \\ \frac{\mu_{\min}}{2\mu_{\text{mid}} - \mu}, & \text{if } \mu \in (\mu_{\text{mid}}, \mu_{\max}), \\ 1, & \text{otherwise.} \end{cases} \tag{5.36}$$

A qualitative plot of this function and its reciprocal is provided in Figure 5.6. Recall in this respect that it is the latter quantity that is most important to us due to equation (5.32).

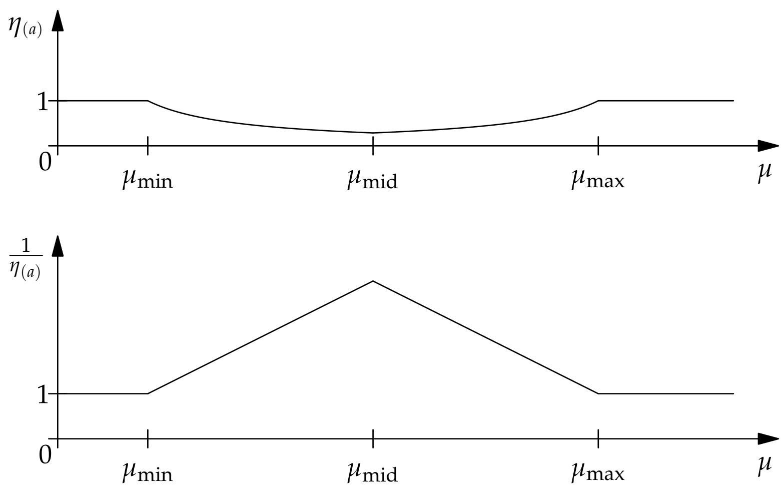

FIGURE 5.6 — Qualitative plots of the functions $\eta_{(a)}$ and $1/\eta_{(a)}$ defined in (5.36). For the purpose of this illustration, we set $\mu_{\min} = 2$ and $\mu_{\max} = 12$.

It is readily seen that $\eta_{(a)}$ is a continuous function and satisfies the estimates $0 < \frac{\mu_{\min}}{\mu_{\text{mid}}} \leq \eta_{(a)}(\mu) \leq 1$ for all $\mu \in [0, \infty)$. Thus, $\xi = \xi^{(s)}\eta_{(a)}$ fulfills the assumptions of the theorem and, due to equation (5.32), the eigenvalues of the associated operators $A_{\mu,k}$ are given by

$$\lambda_{k,n}(\mu) = \begin{cases} \frac{\mu}{\mu_{\min}} \lambda_{k,n}^{(s)}, & \text{if } \mu \in (\mu_{\min}, \mu_{\text{mid}}], \\ \frac{2\mu_{\text{mid}} - \mu}{\mu_{\min}} \lambda_{k,n}^{(s)}, & \text{if } \mu \in (\mu_{\text{mid}}, \mu_{\max}), \\ \lambda_{k,n}^{(s)}, & \text{otherwise}, \end{cases} \qquad \text{for all } k \in B \text{ and all } n \in \mathbb{N}. \tag{5.37}$$

In particular, by the first equation in (5.35),

$$\lambda_{k,N}(\mu) = \begin{cases} \mu, & \text{if } \mu \in (\mu_{\min}, \mu_{\text{mid}}], \\ 2\mu_{\text{mid}} - \mu, & \text{if } \mu \in (\mu_{\text{mid}}, \mu_{\max}), \\ \mu_{\min}, & \text{otherwise}, \end{cases} \qquad \text{for all } k \in B_{\max}, \tag{5.38}$$

giving that for all $k \in B_{\max}$ the non-degenerate interval $(\mu_{\min}, \mu_{\text{mid}}] \subset G_N$ consists entirely of fixed points of the mapping $[0, \infty) \ni \mu \mapsto \lambda_{k,N}(\mu)$. Hence, for these k,

$$(\mu_{\min}, \mu_{\text{mid}}] \subseteq \sigma(\mathscr{A}_k),$$

which shows that these spectra are uncountable.

To conclude this example, we mention that there exist no further λ-nonlinear eigenvalues in G_N, since for all $k \in B$ and all $\mu \in (\mu_{\text{mid}}, \mu_{\max})$ we have the chain of inequalities

$$\begin{aligned} \lambda_{k,1}(\mu) \leq \cdots \leq \lambda_{k,N}(\mu) \\ \leq 2\mu_{\text{mid}} - \mu < \mu_{\text{mid}} < \mu < \mu_{\max} \leq \lambda_{k,N+1}^{(s)} \\ \leq \lambda_{k,N+1}(\mu) \leq \lambda_{k,N+2}(\mu) \leq \cdots, \end{aligned} \tag{5.39}$$

using the increasing ordering of the eigenvalues $\lambda_{k,n}(\mu)$ in n, the definition of the sets $B_{\min}$ and $B_{\max}$, as well as the equations (5.35), (5.37) (in particular, $\eta_{(a)}(\mu) \leq 1$ for all $\mu \in [0,\infty)$), and (5.38). Therefore, the spectral equality (5.15) gives

$$\sigma(\mathscr{A}) = \sigma(A) \cup (\mu_{\min}, \mu_{\text{mid}}],$$

meaning that the λ-nonlinearity generated by the function $\eta_{(a)}$ results in a spectral gap of the operator pencil $\mathscr{A}$ located below $\mu_{\max}$ which is considerably smaller than that of the operator A.

Ad (b): In this second part of the proof we construct an example such that

$$S_n := \bigcup_{k \in B} \{\mu \in [0,\infty) \mid \mu = \lambda_{k,n}(\mu)\}$$

is not an interval for some $n \in \mathbb{N}$. We shall show this for the set S_N and, so as to achieve this, introduce the function $\eta_{(b)} : [0,\infty) \to \mathbb{R}$, using the notation from part (a) above, by

$$\eta_{(b)}(\mu) := \begin{cases} \frac{\mu_{\min}}{\mu}, & \text{if } \mu \in \left(\mu_{\min}, \frac{3}{4}\mu_{\min} + \frac{1}{4}\mu_{\text{mid}}\right], \\ \frac{\mu_{\min}}{\frac{3}{4}\mu_{\min}+\frac{1}{4}\mu_{\text{mid}}}, & \text{if } \mu \in \left(\frac{3}{4}\mu_{\min} + \frac{1}{4}\mu_{\text{mid}}, \frac{1}{2}\mu_{\min} + \frac{1}{2}\mu_{\text{mid}}\right], \\ \frac{\mu_{\min}}{-\frac{1}{4}\mu_{\min}-\frac{3}{4}\mu_{\text{mid}}+2\mu}, & \text{if } \mu \in \left(\frac{1}{2}\mu_{\min} + \frac{1}{2}\mu_{\text{mid}}, \frac{1}{4}\mu_{\min} + \frac{3}{4}\mu_{\text{mid}}\right), \\ \frac{\mu_{\min}}{\mu}, & \text{if } \mu \in \left[\frac{1}{4}\mu_{\min} + \frac{3}{4}\mu_{\text{mid}}, \mu_{\text{mid}}\right], \\ \frac{\mu_{\min}}{2\mu_{\text{mid}}-\mu}, & \text{if } \mu \in (\mu_{\text{mid}}, \mu_{\max}), \\ 1, & \text{otherwise}. \end{cases} \tag{5.40}$$

Similar to the last example we complement this definition by a qualitative plot. It is given in Figure 5.7 further below.

The continuity of the function $\eta_{(b)}$ and the estimate $0 < \frac{\mu_{\min}}{\mu_{\text{mid}}} \leq \eta_{(b)}(\mu) \leq 1$ for all $\mu \in [0,\infty)$ are easily verified, so that $\xi = \xi^{(s)}\eta_{(b)}$ satisfies the assumptions stated in the theorem. Besides, note that

$$\eta_{(b)}(\mu) = \eta_{(a)}(\mu) \qquad \text{for all } \mu \notin \left(\frac{3}{4}\mu_{\min} + \frac{1}{4}\mu_{\text{mid}}, \frac{1}{4}\mu_{\min} + \frac{3}{4}\mu_{\text{mid}}\right),$$

which allows us to deduce from earlier arguments (see (5.38) and the reasoning below that equation) that for all $k \in B_{\max}$ we have

$$\lambda_{k,N}(\mu) = \mu \qquad \text{for all } \mu \in \left(\mu_{\min}, \frac{3}{4}\mu_{\min} + \frac{1}{4}\mu_{\text{mid}}\right] \cup \left[\frac{1}{4}\mu_{\min} + \frac{3}{4}\mu_{\text{mid}}, \mu_{\text{mid}}\right].$$

In other words,

$$\left(\mu_{\min}, \frac{3}{4}\mu_{\min} + \frac{1}{4}\mu_{\text{mid}}\right] \cup \left[\frac{1}{4}\mu_{\min} + \frac{3}{4}\mu_{\text{mid}}, \mu_{\text{mid}}\right] \subseteq S_N. \tag{5.41}$$

Now, so as to find out whether there are any other elements of S_N, we have to consider the Nth eigenvalue of each operator $A_{\mu,k}$ for the remaining values of μ.

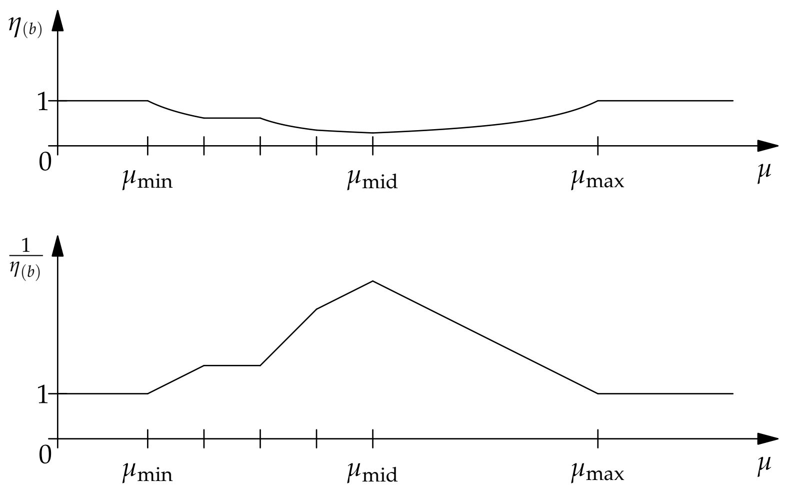

FIGURE 5.7 — Qualitative plots of the functions $\eta_{(b)}$ and $1/\eta_{(b)}$ defined in (5.40). For the purpose of this illustration, we set $\mu_{\min} = 2$ and $\mu_{\max} = 12$. The tick marks between $\mu_{\min}$ and μ_{mid} indicate the beginning and end of the subintervals of $(\mu_{\min}, \mu_{\text{mid}})$ in which the definitions of the plotted functions change from one subfunction to another.

First, for all $k \in B_{\max}$ we obtain from equation (5.32), the first equation in (5.35), and the definition of the function $\eta_{(b)}$ that

$$\lambda_{k,N}(\mu) = \begin{cases} \frac{3}{4}\mu_{\min} + \frac{1}{4}\mu_{\text{mid}}, & \text{if } \mu \in \left(\frac{3}{4}\mu_{\min} + \frac{1}{4}\mu_{\text{mid}}, \frac{1}{2}\mu_{\min} + \frac{1}{2}\mu_{\text{mid}}\right], \\ -\frac{1}{4}\mu_{\min} - \frac{3}{4}\mu_{\text{mid}} + 2\mu, & \text{if } \mu \in \left(\frac{1}{2}\mu_{\min} + \frac{1}{2}\mu_{\text{mid}}, \frac{1}{4}\mu_{\min} + \frac{3}{4}\mu_{\text{mid}}\right), \\ 2\mu_{\text{mid}} - \mu, & \text{if } \mu \in (\mu_{\text{mid}}, \mu_{\max}), \\ \mu_{\min}, & \text{if } \mu \in [0, \mu_{\min}] \cup [\mu_{\max}, \infty). \end{cases}$$

Two consequences of this equation are the relations

$$\mu_{\min} \in S_N \quad \text{and} \quad \left(\tfrac{3}{4}\mu_{\min} + \tfrac{1}{4}\mu_{\text{mid}}, \tfrac{1}{2}\mu_{\min} + \tfrac{1}{2}\mu_{\text{mid}}\right] \not\subseteq S_N, \tag{5.42}$$

the latter of which holds true since for all μ in that set we have $\lambda_{k,N}(\mu) < \mu$. As far as the remaining two μ-regions are concerned, we can deduce that they are not subsets of S_N from the equivalences

$$-\tfrac{1}{4}\mu_{\min} - \tfrac{3}{4}\mu_{\text{mid}} + 2\mu < \mu \iff \mu < \tfrac{1}{4}\mu_{\min} + \tfrac{3}{4}\mu_{\text{mid}}$$

and

$$2\mu_{\text{mid}} - \mu < \mu \iff \mu_{\text{mid}} < \mu.$$

It remains to show that the so far disregarded values of k, i.e., $k \in B \setminus B_{\max}$, cannot account for further elements of S_N. Regarding this, note that by definition of the set $B_{\max}$ we have

$$\lambda_N^{(s)}(k') < \lambda_N^{(s)}(k) \qquad \text{for all } k' \in B \setminus B_{\max} \text{ and all } k \in B_{\max}.$$

Hence, from equation (5.32) and our previous reasoning in this part of the proof we obtain

$$\lambda_{k',N}(\mu) < \lambda_{k,N}(\mu) \leq \mu \quad \text{for all } k' \in B \setminus B_{\max}, \text{ all } k \in B_{\max}, \text{ and all } \mu \in [\mu_{\min}, \infty).$$

This allows us to finally conclude, by (5.41) and the relation on the left-hand side of (5.42), that

$$S_N = \left[\mu_{\min}, \tfrac{3}{4}\mu_{\min} + \tfrac{1}{4}\mu_{\mathrm{mid}}\right] \cup \left[\tfrac{1}{4}\mu_{\min} + \tfrac{3}{4}\mu_{\mathrm{mid}}, \mu_{\mathrm{mid}}\right],$$

which is not an interval.

Ad (c): This third and final part of our proof is concerned with the existence of an isolated point in the spectrum of the operator pencil $\mathscr{A}$. A λ-nonlinearity resulting in this phenomenon is given by that of the function $\eta_{(c)} : [0,\infty) \to \mathbb{R}$ defined by

$$\eta_{(c)}(\mu) := \begin{cases} \left[1 + \frac{(\mu-\mu_{\min})^2}{\mu_{\min}(\mu_{\mathrm{mid}}-\mu_{\min})}\right]^{-1}, & \text{if } \mu \in (\mu_{\min}, \mu_{\mathrm{mid}}], \\ \frac{\mu_{\min}}{2\mu_{\mathrm{mid}}-\mu}, & \text{if } \mu \in (\mu_{\mathrm{mid}}, \mu_{\max}), \\ 1, & \text{otherwise}, \end{cases} \tag{5.43}$$

where, as before, $\mu_{\mathrm{mid}} = \frac{1}{2}(\mu_{\min} + \mu_{\max})$. Note that this function is continuous and satisfies the estimate $0 < \frac{\mu_{\min}}{\mu_{\mathrm{mid}}} \leq \eta_{(c)}(\mu) \leq 1$ for all $\mu \in [0,\infty)$, meaning that for $\xi = \xi^{(s)}\eta_{(c)}$ the assumptions of the theorem are fulfilled. In the tradition of the above, Figure 5.8 provides a qualitative plot of the functions $\eta_{(c)}$ and $1/\eta_{(c)}$.

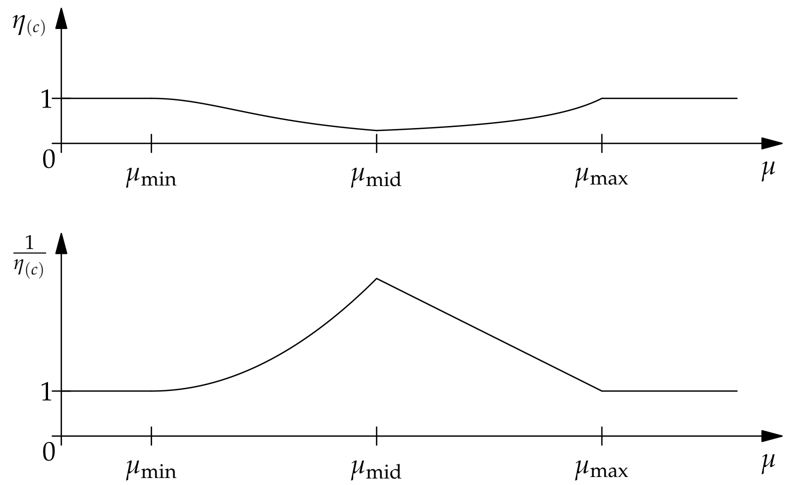

FIGURE 5.8 — Qualitative plots of the functions $\eta_{(c)}$ and $1/\eta_{(c)}$ defined in (5.43). For the purpose of this illustration, we set $\mu_{\min} = 2$ and $\mu_{\max} = 12$.

As a first observation, we remark that an isolated point of $\sigma(\mathscr{A})$ cannot lie in the region within which $\eta_{(c)}$ equals 1, i.e., $[0, \mu_{\min}] \cup [\mu_{\max}, \infty)$, since it were then also an isolated point of $\sigma(A)$, thereby contradicting the corollary to Theorem 4.1.8. Furthermore, the equality

$$\eta_{(c)}(\mu) = \eta_{(a)}(\mu) \qquad \text{for all } \mu \in (\mu_{\text{mid}}, \mu_{\text{max}}), \tag{5.44}$$

allows us to build upon the reasoning from the first part of this proof (see, in particular, the inequalities (5.39) and their derivation) and implies that

$$(\mu_{\text{mid}}, \mu_{\text{max}}) \nsubseteq \sigma(\mathscr{A}_k) \subseteq \sigma(\mathscr{A}) \qquad \text{for all } k \in B. \tag{5.45}$$

Thus, it only remains to consider the interval $(\mu_{\text{min}}, \mu_{\text{mid}}]$ or rather the eigenvalues of the operators $A_{\mu,k}$ for μ in this region. They are given by

$$\lambda_{k,n}(\mu) = \left[1 + \frac{(\mu - \mu_{\text{min}})^2}{\mu_{\text{min}}(\mu_{\text{mid}} - \mu_{\text{min}})}\right] \lambda_{k,n}^{(s)} \qquad \text{for all } k \in B \text{ and all } n \in \mathbb{N}, \tag{5.46}$$

due to equation (5.32). This implies, in particular, that

$$\lambda_{k,N}(\mu) = \mu_{\text{min}} + \frac{(\mu - \mu_{\text{min}})^2}{\mu_{\text{mid}} - \mu_{\text{min}}} \qquad \text{for all } k \in B_{\text{max}}, \tag{5.47}$$

using the first equation in (5.35). Here, the segment of the rescaled parabola $\mu \mapsto \frac{(\mu - \mu_{\text{min}})^2}{\mu_{\text{mid}} - \mu_{\text{min}}}$ which is added to the constant μ_{min} is precisely what is causing an isolated point in $\sigma(\mathscr{A})$.

This can be seen as follows: First, for all $k \in B_{\text{max}}$ we have $\lambda_{k,N}(\mu) = \mu$ if and only if $\mu \in \{\mu_{\text{min}}, \mu_{\text{mid}}\}$, as an easy consequence of solving the corresponding quadratic equation or directly from (5.47). This relation implies

$$\{\mu_{\text{min}}, \mu_{\text{mid}}\} \subset \sigma(\mathscr{A}_k) \qquad \text{for all } k \in B_{\text{max}}$$

and thus, by the spectral equality (5.15),

$$\{\mu_{\text{mid}}\} \cup \sigma(A) \subseteq \sigma(\mathscr{A}) \tag{5.48}$$

where we took into account that μ_{min} is an element of $\sigma(A)$ by virtue of it being the lower bound of the (open) spectral gap G_N.

Next, we show that the previous inclusion is in fact an equality and that μ_{mid} is an isolated point in $\sigma(\mathscr{A})$. To prove the former, in view of (5.45), we may restrict ourselves to looking for further spectrum of the operator pencils $\mathscr{A}_k$ within the interval $(\mu_{\text{min}}, \mu_{\text{mid}})$. In this μ-range, however, we have $\lambda_{k,N}(\mu) < \mu$ for all $k \in B_{\text{max}}$ by equation (5.47) and parabolicity. Therefore, we obtain from equation (5.46) for all $k \in B$ and all $\mu \in (\mu_{\text{min}}, \mu_{\text{mid}})$ the chain of inequalities

$$\lambda_{k,1}(\mu) \leq \cdots \leq \lambda_{k,N}(\mu) < \mu < \mu_{\text{max}} \leq \lambda_{k,N+1}^{(s)} \leq \lambda_{k,N+1}(\mu) \leq \lambda_{k,N+2}(\mu) \leq \cdots,$$

based on the very same reasoning that led to (5.39) in part (a) of this proof. Hence, for no $k \in B$ and no $n \in \mathbb{N}$ does a mapping $(\mu_{\text{min}}, \mu_{\text{mid}}) \ni \mu \mapsto \lambda_{k,n}(\mu)$ possess a fixed point and we can conclude the asserted equality in (5.48), i.e.,

$$\sigma(\mathscr{A}) = \sigma(A) \cup \{\mu_{\text{mid}}\}. \tag{5.49}$$

This also implies that the point μ_{mid} is isolated in $\sigma(\mathscr{A})$ since, as it lies in the open spectral gap of the operator A, for some $\delta > 0$ we have $U_\delta(\mu_{\mathrm{mid}}) \subset \rho(A)$ and thus, note the equality (5.49), $U_\delta(\mu_{\mathrm{mid}}) \setminus \{\mu_{\mathrm{mid}}\} \not\subset \sigma(\mathscr{A})$. This finishes our consideration of the third example, and thereby completes the proof of Theorem 5.3.1. $\square$

To close this section, we remark that more sophisticated examples than those constructed above can also generate all three mentioned spectral phenomena at once. Their exact configuration can be easily derived from the discussed examples, but shall not be included here. Nevertheless, it surely became apparent that a seemingly simple λ-nonlinearity of our eigenvalue problem (5.4) can result in noticeable spectral consequences.

5.4 BASICITY AND COMPLETENESS OF THE EIGENFUNCTIONS IN CERTAIN SETTINGS

Thus far in this chapter we focused on studying the spectrum of the operator pencils $\mathscr{A}_k$. Knowing by now that they are purely discrete under appropriate assumptions, we move our focus towards the corresponding eigenfunctions. Specifically, we are interested in the question under what conditions their completeness or basicity in $L^2(\Omega)$ can be shown. We remark that this is not only of mathematical interest, but also important in applications. For instance, it allows to efficiently deduce certain (local) density of states functions for the studied photonic crystals (see [McP04] and the references therein).

In the two subsections that follow, we present different sets of assumptions on the properties of the considered material that allow us to prove a completeness result in one and even a basicity theorem in the other case. At this, unsurprisingly, the dispersiveness of the λ-nonlinear Maxwell eigenvalue problem for TM-polarized waves (5.4) generates difficulties. What both of these settings have in common are the following basic assumptions which shall hold in the whole of this section. We formulate them right below for the coefficient function ξ, but remind the reader that the relation to the underlying physical quantity, the relative permittivity ε_{r}, is given by

$$\xi(x,\mu) = \varepsilon_{\mathrm{r}}(x, c_0\sqrt{\mu}) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } \mu \in [0,\infty),$$

where μ is the renamed spectral variable $\lambda = \omega^2/c_0^2$ and incorporates the frequency of the studied time-harmonic wave.

Assumptions 5.4.1 (Basic assumptions on ξ if eigenfunctions of $\mathscr{A}_k$ are studied). *We suppose that $\xi \in C([0,\infty); L^\infty(\mathbb{R}^2;\mathbb{R}))$ is such that*

(a) for all $\mu \in [0,\infty)$ the function $\xi(\cdot,\mu)$ is $\mathbb{Z}^2$-periodic, i.e.,

$$\xi(x,\mu) = \xi(x+a,\mu) \qquad \textit{for a. a. } x \in \mathbb{R}^2 \textit{ and all } a \in \mathbb{Z}^2;$$

(b) *for all* $\mu \in [0,\infty)$ *and some positive constant* $\xi_{\min}$ *we have*

$$\xi_{\min} \leq \xi(x,\mu) \leq \|\xi(\cdot,\mu)\|_{L^\infty(\mathbb{R}^2)} =: \xi_{\max}(\mu) \qquad \textit{for a. a. } x \in \mathbb{R}^2;$$

(c) *the function* $\mu \mapsto \mu\xi(x,\mu)$ *is strictly monotonically increasing on* $(0,\infty)$ *for almost all* $x \in \Omega$.

Certainly, these are exactly the requirements of Theorem 5.2.6, so that, within this section and for all $k \in B$, the spectrum of the operator pencil $\mathscr{A}_k$ is purely discrete and consists of a sequence $\{\overset{*}{\mu}_{k,n}\}_{n\in\mathbb{N}}$ of non-negative and increasingly ordered λ-nonlinear eigenvalues tending to infinity. The corresponding sequence $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}} \subset H^2_{k\text{-per}}(\Omega) \subset H^2_{\mathrm{loc}}(\mathbb{R}^2)$ of eigenfunctions, from now on also referred to as *λ-nonlinear Bloch waves*, satisfies

$$0 = \mathscr{A}_k(\overset{*}{\mu}_{k,n})\overset{*}{\psi}_{k,n} = -\frac{1}{\xi(\cdot,\overset{*}{\mu}_{k,n})}\Delta\overset{*}{\psi}_{k,n} - \overset{*}{\mu}_{k,n}\overset{*}{\psi}_{k,n} \qquad \text{for all } n \in \mathbb{N}. \tag{5.50}$$

Rephrasing the above, we are interested in finding out whether $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}}$ is complete in $L^2(\Omega)$ or even a basis of that space. The only easily established result in this direction is stated next.

Proposition 5.4.2. *For all $k \in B$ eigenfunctions of the operator pencil $\mathscr{A}_k$ which correspond to different eigenvalues are pairwise linearly independent.*

Proof. Let $k \in B$ be fixed and consider for $n, m \in \mathbb{N}$ the eigenvalues $\overset{*}{\mu}_{k,n}$ and $\overset{*}{\mu}_{k,m}$ of $\mathscr{A}_k$ with associated eigenfunctions $\overset{*}{\psi}_{k,n}$ and $\overset{*}{\psi}_{k,m}$. Without loss of generality let $\overset{*}{\mu}_{k,n} < \overset{*}{\mu}_{k,m}$. Now, if $0 = \alpha_n\,\overset{*}{\psi}_{k,n} + \alpha_m\,\overset{*}{\psi}_{k,m}$ almost everywhere in Ω for some $\alpha_n, \alpha_m \in \mathbb{C}$, not both being zero, then an application of the Laplace operator yields

$$\begin{aligned}
0 &= \alpha_n\,\Delta\overset{*}{\psi}_{k,n} + \alpha_m\,\Delta\overset{*}{\psi}_{k,m}\\
&= -\alpha_n\,\overset{*}{\mu}_{k,n}\,\xi(\cdot,\overset{*}{\mu}_{k,n})\,\overset{*}{\psi}_{k,n} - \alpha_m\,\overset{*}{\mu}_{k,m}\,\xi(\cdot,\overset{*}{\mu}_{k,m})\,\overset{*}{\psi}_{k,m}\\
&= -\alpha_n\,\overset{*}{\psi}_{k,n}\left[\overset{*}{\mu}_{k,n}\,\xi(\cdot,\overset{*}{\mu}_{k,n}) - \overset{*}{\mu}_{k,m}\,\xi(\cdot,\overset{*}{\mu}_{k,m})\right] \qquad \text{a.e. in } \Omega.
\end{aligned}$$

Hence,

$$\alpha_n = 0 \qquad \text{or} \qquad \overset{*}{\mu}_{k,n}\,\xi(\cdot,\overset{*}{\mu}_{k,n}) = \overset{*}{\mu}_{k,m}\,\xi(\cdot,\overset{*}{\mu}_{k,m}) \quad \text{a.e. in } \{x \in \Omega \mid \overset{*}{\psi}_{k,n}(x) \neq 0\}.$$

Note that the equality on the right-hand side is supposed to hold on a set of positive measure since $\overset{*}{\psi}_{k,n}$ is an eigenfunction. Therefore, it contradicts the monotonicity requirement stated in part (c) of Assumptions 5.4.1 in view of the inequality $\overset{*}{\mu}_{k,n} < \overset{*}{\mu}_{k,m}$. This implies $\alpha_n = 0$, giving $\alpha_m = 0$ and thus the claim. □

Pairwise linear independence of the λ-nonlinear Bloch waves is, of course, still a lot less than what we are aiming for in this section. To understand what exactly is so complicated about our task, we rewrite, for $k \in B$ and $n \in \mathbb{N}$, the eigenvalue equation (5.50) as

$$-\frac{1}{\xi(\cdot,\overset{*}{\mu}_{k,n})}\Delta\overset{*}{\psi}_{k,n} = \overset{*}{\mu}_{k,n}\overset{*}{\psi}_{k,n}.$$

This is clearly trivial, but reminds us that $\overset{*}{\psi}_{k,n}$ is an eigenfunction of the operator $A_{\mu,k}$, where $\mu = \overset{*}{\mu}_{k,n}$, introduced in the beginning of this chapter (see (5.9)). Therefore, by the λ-linear theory (see Lemma 5.2.1) $\overset{*}{\psi}_{k,n}$ is an element of the $\langle\cdot,\cdot\rangle_{\xi(\cdot,\overset{*}{\mu}_{k,n})}$-orthonormal basis

$$\left\{\psi_{k,1}(\overset{*}{\mu}_{k,n}), \psi_{k,2}(\overset{*}{\mu}_{k,n}), \ldots, \psi_{k,n-1}(\overset{*}{\mu}_{k,n}), \overset{*}{\psi}_{k,n}, \psi_{k,n+1}(\overset{*}{\mu}_{k,n}), \ldots\right\}$$

of $L^2(\Omega)$. The important consequence here is that any other λ-nonlinear eigenfunction $\overset{*}{\psi}_{k,m}$, say, which corresponds to an eigenvalue $\overset{*}{\mu}_{k,m} \neq \overset{*}{\mu}_{k,n}$, is part of an orthonormal basis with respect to a generally differently weighted inner product. In particular,

$$\left\|\overset{*}{\psi}_{k,n}\right\|_{\xi(\cdot,\overset{*}{\mu}_{k,n})} = 1 \qquad \text{but, possibly,} \qquad \left\|\overset{*}{\psi}_{k,n}\right\|_{\xi(\cdot,\overset{*}{\mu}_{k,m})} \neq 1.$$

Summing up, each λ-nonlinear eigenfunction is uniquely assigned an integration weight function, which depends itself on the particular λ-nonlinear eigenvalue. Thus, in general, there is no a priori knowledge about terms such as $\langle\overset{*}{\psi}_{k,n},\overset{*}{\psi}_{k,m}\rangle_{\xi(\cdot,\overset{*}{\mu}_{k,n})}$ or any differently weighted variant thereof. In view of this, unsurprisingly, the arguments that allow us to conclude the basis property of the eigenfunctions in the λ-linear setting[1]—heavily making use of their orthonormality—are of no avail to us. For further restricted coefficient functions ξ, and by means of more advanced techniques, however, we shall be successful in spite of these obstacles. This is presented next within this section.

5.4.1 THE HIGH-FREQUENCY NONDISPERSIVE CASE (BASICITY)

The first class of λ-nonlinear problems whose eigenfunctions we shall analyze model what we call *high-frequency nondispersive* media. This notion is motivated by the underlying physics and expresses that the relative permittivity of the studied dispersive photonic crystal has a frequency-dependence that is negligible above a certain threshold. Hence, such a nanostructure can be considered as nondispersive for high frequencies. Formally, i.e., translated into a requirement on the coefficient function ξ, we thus demand the following:

Assumptions 5.4.3 (Entire assumptions on ξ in the high-frequency nondispersive case)**.** *Let Assumptions 5.4.1 hold and suppose further that for some $\widetilde{\mu} > 0$ we have*

$$\xi(x,\mu) = \xi(x,\widetilde{\mu}) =: \widetilde{\xi}(x) \qquad \textit{for a.a. } x \in \mathbb{R}^2 \textit{ and all } \mu \in [\widetilde{\mu},\infty). \tag{5.51}$$

[1] Which is then, but not in the λ-nonlinear case, the same as their completeness (see the treatise in the preliminaries; particularly Theorem 2.2.9).

Remark. The above-mentioned threshold frequency is given by $\widetilde{\omega} = c_0\sqrt{\widetilde{\mu}}$, which can be seen by recalling the definition of ξ (see (5.3)).

To begin our analysis of the high-frequency nondispersive case, we note that only finitely many eigenvalues of any operator pencil $\mathscr{A}_k$ can be smaller than $\widetilde{\mu}$. That is because they are of finite multiplicity and tend to infinity by Theorem 5.2.6. Thus, for all $k \in B$ there exists some $N_k \in \mathbb{N}$ such that

$$\overset{*}{\mu}_{k,1} \leq \cdots \leq \overset{*}{\mu}_{k,N_k} \leq \widetilde{\mu} < \overset{*}{\mu}_{k,N_k+1} \leq \overset{*}{\mu}_{k,N_k+2} \leq \cdots. \tag{5.52}$$

Note that this implicitly requires that $\widetilde{\mu}$ is sufficiently large, so that at least the first λ-nonlinear eigenvalue is smaller than this threshold for all $k \in B$.[1]

An important fact to note here is that the eigenvalues which are greater than or equal to $\widetilde{\mu}$ are, by our assumption (5.51), those of the λ-linear spectral problem for the operator $\widetilde{A}_k : D(\widetilde{A}_k) \to L^2_{\widetilde{\xi}}(\Omega)$ defined in $L^2_{\widetilde{\xi}}(\Omega)$, and for all $k \in B$, by

$$D(\widetilde{A}_k) := H^2_{k\text{-per}}(\Omega), \qquad \widetilde{A}_k u := -\frac{1}{\widetilde{\xi}}\Delta u. \tag{5.53}$$

If we denote the eigenvalues and -functions of $\widetilde{A}_k$ by $\{\widetilde{\lambda}_{k,n}\}_{n\in\mathbb{N}}$ and $\{\widetilde{\psi}_{k,n}\}_{n\in\mathbb{N}}$, respectively, we find

$$\overset{*}{\mu}_{k,n} = \widetilde{\lambda}_{k,n} \quad \text{and} \quad \overset{*}{\psi}_{k,n} = \widetilde{\psi}_{k,n} \qquad \text{for all } n \geq N_k + 1. \tag{5.54}$$

Observe also that the set $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}}$ of λ-nonlinear eigenfunctions, for which we aim to show the basis property, can be rewritten as

$$W_k := \{\overset{*}{\psi}_{k,n} \mid 1 \leq n \leq N_k\} \cup \{\widetilde{\psi}_{k,n} \mid n \geq N_k + 1\},$$

where we know from Theorem 4.1.5 that

$$\langle \widetilde{\psi}_{k,n}, \widetilde{\psi}_{k,m}\rangle_{\widetilde{\xi}} = \delta_{nm} \qquad \text{for all } n, m \in \mathbb{N}. \tag{5.55}$$

This orthonormality property shows that only finitely many elements of W_k, namely those that are no eigenfunctions of $\widetilde{A}_k$, are not orthonormal with respect to the inner product $\langle\cdot,\cdot\rangle_{\widetilde{\xi}}$. In other words, our assumption of a high-frequency nondispersive behavior of the studied medium, i.e., (5.51), results in an only finite-dimensional perturbation of the orthonormal basis $\{\widetilde{\psi}_{k,n}\}_{n\in\mathbb{N}}$ of $L^2_{\widetilde{\xi}}(\Omega)$.

So as to obtain a shorter and more comprehensible proof of our basicity theorem further below, we next provide three important lemmata. The first of them gives a characterization of the basis property of W_k.

Lemma 5.4.4. *For all $k \in B$ the λ-nonlinear eigenfunctions W_k of the operator pencil $\mathscr{A}_k$ form an unconditional basis of $L^2_{\widetilde{\xi}}(\Omega)$ if and only if the matrix*

$$\mathbf{M}_k := \left(\overline{\langle \widetilde{\psi}_{k,n}, \overset{*}{\psi}_{k,m}\rangle_{\widetilde{\xi}}} \right)_{n,m=1}^{N_k} \tag{5.56}$$

is invertible.

[1]This assumption is justified, since otherwise the considered spectral problem is fully λ-linear for some $k \in B$ (see (5.54)) and not of interest to us.

Proof. Let $k \in B$ be fixed. To begin with, we address the "only if" part of the statement. Thereto, let us suppose that W_k is an unconditional basis of the space $L^2_{\tilde{\xi}}(\Omega)$. Then, any function u in that space can be expanded as

$$u = \sum_{n=1}^{N_k} c_{k,n}(u)\, \overset{*}{\psi}_{k,n} \; + \sum_{n=N_k+1}^{\infty} c_{k,n}(u)\, \widetilde{\psi}_{k,n},$$

where the sequence of coefficients $\{c_{k,n}(u)\}_{n\in\mathbb{N}} \subset \mathbb{C}$ is unique and the convergence of the series (in norm) is unconditional. An inner product multiplication of this equation with the elements of the $\langle\cdot,\cdot\rangle_{\tilde{\xi}}$-orthonormal basis $\{\widetilde{\psi}_{k,n}\}_{n\in\mathbb{N}}$ of $L^2_{\tilde{\xi}}(\Omega)$ equivalently yields

$$\left\langle u, \widetilde{\psi}_{k,m}\right\rangle_{\tilde{\xi}} = \sum_{n=1}^{N_k} c_{k,n}(u) \left\langle \overset{*}{\psi}_{k,n}, \widetilde{\psi}_{k,m}\right\rangle_{\tilde{\xi}} + \sum_{n=N_k+1}^{\infty} c_{k,n}(u)\delta_{nm} \qquad \text{for all } m \in \mathbb{N}$$

using the relation (5.55). In other words,

$$\left\langle u, \widetilde{\psi}_{k,m}\right\rangle_{\tilde{\xi}} = \begin{cases} \displaystyle\sum_{n=1}^{N_k} c_{k,n}(u) \left\langle \overset{*}{\psi}_{k,n}, \widetilde{\psi}_{k,m}\right\rangle_{\tilde{\xi}}, & \text{for } 1 \le m \le N_k, \\ \displaystyle\sum_{n=1}^{N_k} c_{k,n}(u) \left\langle \overset{*}{\psi}_{k,n}, \widetilde{\psi}_{k,m}\right\rangle_{\tilde{\xi}} + c_{k,m}(u), & \text{for } m \ge N_k + 1. \end{cases} \tag{5.57}$$

The equations for $1 \le m \le N_k$ can equivalently be written as

$$\begin{pmatrix} \langle u, \widetilde{\psi}_{k,1}\rangle_{\tilde{\xi}} \\ \vdots \\ \langle u, \widetilde{\psi}_{k,N_k}\rangle_{\tilde{\xi}} \end{pmatrix} = \begin{pmatrix} \overline{\langle \widetilde{\psi}_{k,1}, \overset{*}{\psi}_{k,1}\rangle_{\tilde{\xi}}} & \cdots & \overline{\langle \widetilde{\psi}_{k,1}, \overset{*}{\psi}_{k,N_k}\rangle_{\tilde{\xi}}} \\ \vdots & \ddots & \vdots \\ \overline{\langle \widetilde{\psi}_{k,N_k}, \overset{*}{\psi}_{k,1}\rangle_{\tilde{\xi}}} & \cdots & \overline{\langle \widetilde{\psi}_{k,N_k}, \overset{*}{\psi}_{k,N_k}\rangle_{\tilde{\xi}}} \end{pmatrix} \begin{pmatrix} c_{k,1}(u) \\ \vdots \\ c_{k,N_k}(u) \end{pmatrix}. \tag{5.58}$$

Hence, the coefficients $c_{k,1}(u), \ldots, c_{k,N_k}(u)$, and with them also the remaining ones (see the second line of (5.57)), can only be uniquely determined if the matrix on the right-hand side, which is precisely $\mathbf{M}_k$, is invertible. This finishes the first part of our proof.

Next, to show the "if" part of the assertion, we assume the invertibility of $\mathbf{M}_k$ and denote again by u an arbitrary element of $L^2_{\tilde{\xi}}(\Omega)$. Then, there holds the estimate

$$\sum_{n=1}^{N_k} |\langle u, \overset{*}{\psi}_{k,n}\rangle_{\tilde{\xi}}|^2 \; + \sum_{n=N_k+1}^{\infty} |\langle u, \widetilde{\psi}_{k,n}\rangle_{\tilde{\xi}}|^2 \le \left(1 + \sum_{n=1}^{N_k} \|\overset{*}{\psi}_{k,n}\|^2_{\tilde{\xi}}\right) \|u\|^2_{\tilde{\xi}},$$

using the Cauchy-Schwarz and the Bessel inequality. Therefore, the elements of W_k form a Bessel sequence (see part (a) of Definition 2.2.7) and expansions in these functions are unconditionally convergent for all coefficient sequences in

$l^2(\mathbb{N})$ by Theorem 2.2.8 in the preliminaries. Keeping this in mind, we set

$$c_{k,n}(u) := \begin{cases} \left[\mathbf{M}_k^{-1} \left(\langle u, \widetilde{\psi}_{k,m} \rangle_{\widetilde{\xi}} \right)_{m=1}^{N_k} \right]_n, & \text{for } 1 \le n \le N_k, \\ \langle u, \widetilde{\psi}_{k,n} \rangle_{\widetilde{\xi}} - \left\langle \sum_{m=1}^{N_k} c_{k,m}(u) \overset{*}{\psi}_{k,m}, \widetilde{\psi}_{k,n} \right\rangle_{\widetilde{\xi}}, & \text{for } n \ge N_k + 1, \end{cases} \tag{5.59}$$

where $[v]_n$ denotes the nth entry of the column vector v. Observe here that for $n \ge N_k + 1$ we defined $c_{k,n}(u)$ as the difference of the nth Fourier coefficients of the functions u and $\sum_{m=1}^{N_k} c_{k,m}(u) \overset{*}{\psi}_{k,m}$ with respect to the orthonormal basis $\{\widetilde{\psi}_{k,n}\}_{n \in \mathbb{N}}$. Therefore, by Parseval's identity, we find $\{c_{k,n}(u)\}_{n \in \mathbb{N}} \in l^2(\mathbb{N})$. This allows us to make use of the above-mentioned theorem, which gives the unconditional norm-convergence of the series

$$\sum_{n=1}^{N_k} c_{k,n}(u) \overset{*}{\psi}_{k,n} + \sum_{n=N_k+1}^{\infty} c_{k,n}(u) \widetilde{\psi}_{k,n}. \tag{5.60}$$

To see that this is really an expansion of u, call the limit of the series v. Then, proceeding as in the first part of our proof we obtain equations for the Fourier coefficients of v with respect to the orthonormal basis $\{\widetilde{\psi}_{k,n}\}_{n \in \mathbb{N}}$. They, however, are then equal to the right-hand sides of (5.57) and therefore, by our definition of the coefficient sequence $\{c_{k,n}(u)\}_{n \in \mathbb{N}}$, equal to the Fourier coefficients $\{\langle u, \widetilde{\psi}_{k,n} \rangle_{\widetilde{\xi}}\}_{n \in \mathbb{N}}$ of u. The basis property of $\{\widetilde{\psi}_{k,n}\}_{n \in \mathbb{N}}$ then necessarily gives $v = u$. Hence, (5.60) is an unconditionally convergent expansion of u and also unique, which is again a consequence of the reasoning of the first part of our proof and the invertibility of $\mathbf{M}_k$ (see (5.57) and (5.58)). □

Our second preliminary lemma concerns the eigenvalues of the operators $A_{\mu,k}$ defined earlier in this chapter (see (5.9)). To recall the employed notation, see also Lemma 5.2.1, part (a) of the remark thereafter, and the line below (5.7).

Lemma 5.4.5 ([WS72, Thm. 1.3.1])**.** *Let $k \in B$ and $\mu \in [0, \infty)$. Furthermore, define $U_{k,0}(\mu) := H^1_{k\text{-per}}(\Omega) \setminus \{0\}$ and, for all $n \in \mathbb{N}$,*

$$U_{k,n}(\mu) := \left\{ u \in H^1_{k\text{-per}}(\Omega) \setminus \{0\} \mid \langle u, \psi_{k,m}(\mu) \rangle_{\xi_\mu} = 0 \text{ for } 1 \le m \le n \right\}.$$

Then the eigenvalues of the operator $A_{\mu,k}$ can be characterized as

$$\lambda_{k,n}(\mu) = \min_{u \in U_{k,n-1}(\mu)} \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega \xi(x,\mu) |u(x)|^2 \, dx} \qquad \text{for all } n \in \mathbb{N}. \tag{5.61}$$

Remark. A similar characterization of eigenvalues holds for a wide class of operators that are of interest in mathematical physics. In particular, it is valid for any operator acting on functions in $L^2(\Omega)$ (with arbitrary integration weight function) that we introduced so far. This is made precise in the first three chapters of the cited source, which also contain interesting historical information. Note that in contrast to the min-max principle presented as part of Theorem 4.1.5 (which is, of course, also valid for the operators considered here), the equations (5.61) characterize the nth eigenvalue explicitly in terms of the first $n-1$ eigenfunctions. This dependence can sometimes be disadvantageous, but we will benefit greatly from it right below.

The third and final preparing lemma which we present in this subsection links, in a sense, its two predecessors and provides us with an important integral estimate to be used later on in our work. Besides, this result finds immediate application in the proof of the main theorem of the high-frequency nondispersive case presented thereafter.

Lemma 5.4.6. *Let $k \in B$ and denote by $\mu_1, \mu_2, \ldots, \mu_R, \mu_{R+1}$ all the different λ-nonlinear eigenvalues of the operator pencil $\mathscr{A}_k$ within the interval $[\overset{*}{\mu}_{k,1}, \overset{*}{\mu}_{k,N_k+1}]$, i.e.,*

$$\overset{*}{\mu}_{k,1} =: \mu_1 < \mu_2 < \cdots < \mu_R := \overset{*}{\mu}_{k,N_k} < \overset{*}{\mu}_{k,N_k+1} =: \mu_{R+1}. \tag{5.62}$$

Further, set $S_0 := 1$ and, for $1 \leq r \leq R$, let s_r be the multiplicity of μ_r as well as

$$S_r := \sum_{\rho=1}^{r} s_\rho \quad \text{and} \quad \mathbf{M}_{k,r} := \left(\overline{\langle \psi_{k,n}(\mu_{r+1}), \overset{*}{\psi}_{k,m} \rangle_{\xi_{\mu_{r+1}}}} \right)_{n,m=1}^{S_r}.$$

Then, for $1 \leq r \leq R$ the matrix $\mathbf{M}_{k,r}$ is invertible and

$$\int_\Omega |\nabla \Psi(x)|^2 \, dx \leq \mu_r \int_\Omega \xi(x, \mu_r) \, |\Psi(x)|^2 \, dx, \tag{5.63}$$

where Ψ is an arbitrary linear combination of the functions $\overset{*}{\psi}_{k,1}, \ldots, \overset{*}{\psi}_{k,S_r}$.

Proof. We will show both assertions of the lemma with a single proof by induction on r. First, though, let us note that the strict ordering of the λ-nonlinear eigenvalues in (5.62) and part (c) of Assumptions 5.4.1 imply

$$\mu_1 \xi(\cdot, \mu_1) < \mu_2 \xi(\cdot, \mu_2) < \ldots < \mu_R \xi(\cdot, \mu_R) < \mu_{R+1} \xi(\cdot, \mu_{R+1}) \quad \text{a.e. on } \Omega, \tag{5.64}$$

which is crucial to this proof.

As for the induction basis, let $r = 1$ and suppose $\mathbf{M}_{k,1}$ were singular, i.e., $\mathbf{M}_{k,1} c = 0$ for some $c = (c_1, \ldots, c_{S_1})^{\mathrm{T}} \in \mathbb{C}^{S_1} \setminus \{0\}$. Then, using the notation of Lemma 5.4.5,

$$\Psi := \sum_{m=1}^{S_1} c_m \overset{*}{\psi}_{k,m} \in U_{k,S_1}(\mu_2), \tag{5.65}$$

which can be seen as follows: Firstly, $\overset{*}{\psi}_{k,m} = \psi_{k,m}(\mu_1)$ for $1 \leq m \leq S_1$ and thus, since the eigenfunctions $\psi_{k,1}(\mu_1), \ldots, \psi_{k,S_1}(\mu_1)$ are linearly independent, $\Psi \neq 0$.

Secondly, $\mathbf{M}_{k,1}c = 0$ implies $\langle \Psi, \psi_{k,n}(\mu_2)\rangle_{\xi_{\mu_2}}$ for $1 \leq n \leq S_1$, giving, as claimed, (5.65). Therefore, the just-mentioned lemma yields

$$\begin{aligned}
\mu_2 &= \overset{*}{\mu}_{k,S_1+1} \\
&= \lambda_{k,S_1+1}(\overset{*}{\mu}_{k,S_1+1}) \\
&= \lambda_{k,S_1+1}(\mu_2) = \min_{u \in U_{k,S_1}(\mu_2)} \frac{\int_\Omega |\nabla u(x)|^2\,dx}{\int_\Omega \xi(x,\mu_2)|u(x)|^2\,dx} \leq \frac{\int_\Omega |\nabla \Psi(x)|^2\,dx}{\int_\Omega \xi(x,\mu_2)|\Psi(x)|^2\,dx}.
\end{aligned} \tag{5.66}$$

On the other hand, by the eigenvalue equations for the λ-nonlinear eigenfunctions we have

$$\begin{aligned}
-\Delta\Psi = \sum_{m=1}^{S_1} c_m(-\Delta\overset{*}{\psi}_{k,m}) &= \sum_{m=1}^{S_1} c_m \overset{*}{\mu}_{k,m}\xi(\cdot,\overset{*}{\mu}_{k,m})\overset{*}{\psi}_{k,m} \\
&= \mu_1\xi(\cdot,\mu_1)\sum_{m=1}^{S_1} c_m\overset{*}{\psi}_{k,m} = \mu_1\xi(\cdot,\mu_1)\Psi,
\end{aligned} \tag{5.67}$$

giving, by means of partial integration (compare to footnote 1 on p. 53 and to part (b) of the remarks following Theorem 4.1.3 regarding the vanishing boundary terms),

$$\begin{aligned}
\int_\Omega |\nabla \Psi(x)|^2\,dx &= \int_\Omega \big[-\Delta\Psi(x)\big]\,\overline{\Psi(x)}\,dx \\
&= \int_\Omega \mu_1\xi(x,\mu_1)|\Psi(x)|^2\,dx < \mu_2\int_\Omega \xi(x,\mu_2)|\Psi(x)|^2\,dx,
\end{aligned} \tag{5.68}$$

where the inequality is a consequence of (5.64) and $\Psi \neq 0$. This, however, contradicts (5.66), meaning that $\mathbf{M}_{k,1}$ is invertible. In addition, note that (5.67) and the equalities in (5.68) still hold if Ψ is taken to be an arbitrary linear combination of the functions $\overset{*}{\psi}_{k,1},\ldots,\overset{*}{\psi}_{k,S_1}$ instead of one having a coefficient vector in the kernel of $\mathbf{M}_{k,1}$. Hence, the induction basis for the second claim we intend to prove, i.e., (5.63) in the case $r = 1$, is also shown.

To proceed, we assume that both assertions of the lemma hold for some $r \in \{1,\ldots,R-1\}$ and first prove that

$$\overset{*}{\psi}_{k,1},\ldots,\overset{*}{\psi}_{k,S_{r+1}} \quad \text{are linearly independent.} \tag{5.69}$$

Thereto, suppose $\sum_{m=1}^{S_{r+1}} c_m\overset{*}{\psi}_{k,m} = 0$ for some $c_1,\ldots,c_{S_{r+1}} \in \mathbb{C}$. Then, we have

$$\sum_{m=1}^{S_r} c_m\overset{*}{\psi}_{k,m} = -\sum_{m=S_r+1}^{S_{r+1}} c_m\overset{*}{\psi}_{k,m} = -\sum_{m=S_r+1}^{S_{r+1}} c_m\psi_{k,m}(\mu_{r+1}). \tag{5.70}$$

Inner product multiplications of this equation with $\psi_{k,1}(\mu_{r+1}),\ldots,\psi_{k,S_r}(\mu_{r+1})$ yield

$$\sum_{m=1}^{S_r} c_m\langle\overset{*}{\psi}_{k,m},\psi_{k,n}(\mu_{r+1})\rangle_{\xi_{\mu_{r+1}}} = 0 \qquad \text{for } 1 \leq n \leq S_r,$$

using that $\{\psi_{k,n}(\mu_{r+1})\}_{n\in\mathbb{N}}$ is an orthonormal system with respect to $\langle\cdot,\cdot\rangle_{\xi_{\mu_{r+1}}}$. In other words, $\mathbf{M}_{k,r}(c_1,\ldots,c_{S_r})^{\mathrm{T}} = 0$ and hence, by our induction hypothesis, $c_1 = \ldots = c_{S_r} = 0$. The right-hand side of (5.70)—which is a linear combination of linearly independent eigenfunctions—therefore vanishes so that we also get $c_{S_r+1} = \ldots = c_{S_{r+1}} = 0$ and, as a consequence, (5.69).

In order to show that $\mathbf{M}_{k,r+1}$ is invertible, we proceed similar to the case $r = 1$ and suppose that there were some $c = (c_1,\ldots,c_{S_{r+1}})^{\mathrm{T}} \in \mathbb{C}^{S_{r+1}} \setminus \{0\}$ such that $\mathbf{M}_{k,r+1}c = 0$. Together with (5.69) this allows us to conclude that

$$\Psi := \sum_{m=1}^{S_{r+1}} c_m \overset{*}{\psi}_{k,m} \in U_{k,S_{r+1}}(\mu_{r+2}). \tag{5.71}$$

Hence, due to Lemma 5.4.5,

$$\begin{aligned}\mu_{r+2} = \overset{*}{\mu}_{k,S_{r+1}+1} &= \lambda_{k,S_{r+1}+1}(\overset{*}{\mu}_{k,S_{r+1}+1}) \\ &= \lambda_{k,S_{r+1}+1}(\mu_{r+2}) \\ &= \min_{u\in U_{k,S_{r+1}}(\mu_{r+2})} \frac{\int_\Omega |\nabla u(x)|^2\, dx}{\int_\Omega \xi(x,\mu_{r+2})|u(x)|^2\, dx} \\ &\leq \frac{\int_\Omega |\nabla \Psi(x)|^2\, dx}{\int_\Omega \xi(x,\mu_{r+2})|\Psi(x)|^2\, dx}.\end{aligned} \tag{5.72}$$

Just as in the first part of this proof, the last inequality will eventually result in a contradiction. To see this, we first note that, as before, integration by parts gives

$$\begin{aligned}&\int_\Omega \nabla\overset{*}{\psi}_{k,n}(x)\cdot\overline{\nabla\overset{*}{\psi}_{k,m}(x)}\, dx \\ &\quad= \int_\Omega \big[-\Delta\overset{*}{\psi}_{k,n}(x)\big]\,\overline{\overset{*}{\psi}_{k,m}(x)}\, dx \\ &\quad= \overset{*}{\mu}_{k,n}\int_\Omega \xi(x,\overset{*}{\mu}_{k,n})\,\overset{*}{\psi}_{k,n}(x)\,\overline{\overset{*}{\psi}_{k,m}(x)}\, dx \\ &\quad= \overset{*}{\mu}_{k,m}\int_\Omega \xi(x,\overset{*}{\mu}_{k,m})\,\overset{*}{\psi}_{k,n}(x)\,\overline{\overset{*}{\psi}_{k,m}(x)}\, dx \qquad \text{for } 1\leq n,m\leq S_{r+1}.\end{aligned} \tag{5.73}$$

Here, so as to arrive at the last equality, we used the symmetric structure of the term on the left-hand side and the real-valuedness of both ξ and the λ-nonlinear eigenvalues. Immediately utilizing (5.73), combined with the fact that $\overset{*}{\mu}_{k,m} = \mu_{r+1}$ for $S_r + 1 \leq m \leq S_{r+1}$, we get

$$\begin{aligned}\int_\Omega |\nabla\Psi(x)|^2\, dx &= \sum_{n,m=1}^{S_{r+1}} c_n\overline{c_m}\int_\Omega \nabla\overset{*}{\psi}_{k,n}(x)\cdot\overline{\nabla\overset{*}{\psi}_{k,m}(x)}\, dx \\ &= \sum_{n,m=1}^{S_r} c_n\overline{c_m}\int_\Omega \nabla\overset{*}{\psi}_{k,n}(x)\cdot\overline{\nabla\overset{*}{\psi}_{k,m}(x)}\, dx\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\substack{n,m=1 \\ n>S_r \text{ or } m>S_r}}^{S_{r+1}} c_n \overline{c_m} \int_\Omega \nabla \overset{*}{\psi}_{k,n}(x) \cdot \overline{\nabla \overset{*}{\psi}_{k,m}(x)}\, dx \\
&= \sum_{n,m=1}^{S_r} c_n \overline{c_m} \int_\Omega \nabla \overset{*}{\psi}_{k,n}(x) \cdot \overline{\nabla \overset{*}{\psi}_{k,m}(x)}\, dx \qquad (5.74) \\
&\quad + \sum_{\substack{n,m=1 \\ n>S_r \text{ or } m>S_r}}^{S_{r+1}} c_n \overline{c_m}\, \mu_{r+1} \int_\Omega \xi(x,\mu_{r+1})\, \overset{*}{\psi}_{k,n}(x)\, \overline{\overset{*}{\psi}_{k,m}(x)}\, dx.
\end{aligned}
$$

Next, the so far unused part of our induction hypothesis, i.e., the validity of (5.63), comes into play and allows us to estimate the first sum on the last right-hand side as follows:

$$
\begin{aligned}
\sum_{n,m=1}^{S_r} c_n \overline{c_m} \int_\Omega \nabla \overset{*}{\psi}_{k,n}(x) \cdot \overline{\nabla \overset{*}{\psi}_{k,m}(x)}\, dx &= \int_\Omega \left| \nabla \left(\sum_{m=1}^{S_r} c_m \overset{*}{\psi}_{k,m}(x) \right) \right|^2 dx \qquad (5.75) \\
&\le \mu_r \int_\Omega \xi(x,\mu_r) \left| \sum_{m=1}^{S_r} c_m \overset{*}{\psi}_{k,m}(x) \right|^2 dx \\
&\le \mu_{r+1} \int_\Omega \xi(x,\mu_{r+1}) \left| \sum_{m=1}^{S_r} c_m \overset{*}{\psi}_{k,m}(x) \right|^2 dx,
\end{aligned}
$$

the last inequality being a consequence of (5.64). Together with (5.74) we therefore find

$$
\begin{aligned}
\int_\Omega |\nabla \Psi(x)|^2\, dx &\le \mu_{r+1} \int_\Omega \xi(x,\mu_{r+1}) \left| \sum_{m=1}^{S_r} c_m \overset{*}{\psi}_{k,m}(x) \right|^2 dx \\
&\quad + \sum_{\substack{n,m=1 \\ n>S_r \text{ or } m>S_r}}^{S_{r+1}} c_n \overline{c_m}\, \mu_{r+1} \int_\Omega \xi(x,\mu_{r+1})\, \overset{*}{\psi}_{k,n}(x)\, \overline{\overset{*}{\psi}_{k,m}(x)}\, dx \\
&= \mu_{r+1} \int_\Omega \xi(x,\mu_{r+1}) \left| \sum_{m=1}^{S_{r+1}} c_m \overset{*}{\psi}_{k,m}(x) \right|^2 dx \qquad (5.76) \\
&< \mu_{r+2} \int_\Omega \xi(x,\mu_{r+2}) |\Psi(x)|^2\, dx,
\end{aligned}
$$

where, once more, (5.64) and $\Psi \neq 0$ have been used in the last step. This inequality gives the desired contradiction to (5.72) and thus the invertibility of the matrix $\mathbf{M}_{k,r+1}$. Furthermore, similar to the case $r = 1$, from (5.73) onwards we nowhere used the fact that the coefficient vector c of Ψ (see (5.71)) lies in the kernel of $\mathbf{M}_{k,r+1}$. Hence, all but the last line of (5.76) also hold if Ψ is replaced by an arbitrary linear combination of the functions $\overset{*}{\psi}_{k,1}, \ldots, \overset{*}{\psi}_{k,S_{r+1}}$, which finally yields the asserted integral estimate (5.63). This closes the proof. □

With the necessary auxiliary results at hand, we formulate our basicity theorem for the λ-nonlinear Maxwell eigenvalue problem for TM-polarized waves.

Theorem 5.4.7 (Riesz basicity in the high-frequency nondispersive case)**.** *For all $k \in B$ the λ-nonlinear eigenfunctions $W_k = \{\overset{*}{\psi}_{k,n} \mid 1 \le n \le N_k\} \cup \{\widetilde{\psi}_{k,n} \mid n \ge N_k + 1\}$ of the operator pencil $\mathscr{A}_k$ form a Riesz basis of $L^2_{\widetilde{\xi}}(\Omega)$.*

Proof. Let $k \in B$ be fixed. By Theorem 2.2.12 the assertion follows if we can prove that W_k is bounded in the sense of part (b) of Definition 2.2.2 and an unconditional basis of $L^2_{\widetilde{\xi}}(\Omega)$. The former property is readily seen to hold since firstly

$$0 < \frac{\xi_{\min}}{\max\limits_{1 \le n \le N_k} \xi_{\max}(\overset{*}{\mu}_{k,n})} \le \left\| \overset{*}{\psi}_{k,n} \right\|^2_{\widetilde{\xi}} \le \frac{\xi_{\max}(\widetilde{\mu})}{\xi_{\min}} \qquad \text{for } 1 \le n \le N_k$$

by part (b) of Assumptions 5.4.1, and secondly $\|\widetilde{\psi}_{k,n}\|_{\widetilde{\xi}} = 1$ for $n \ge N_k + 1$ as the latter functions are elements of an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_{\widetilde{\xi}}$. Hence, due to Lemma 5.4.4 it only remains to show that the matrix $\mathbf{M}_k$ is invertible. This, however, is readily seen to be a consequence of Lemma 5.4.6. In its notation, and by the definition of $\widetilde{\xi}$ (see (5.51)), we find $\widetilde{\xi} = \xi(\cdot, \widetilde{\mu}) = \xi(\cdot, \overset{*}{\mu}_{k,N_k+1}) = \xi(\cdot, \mu_{R+1})$ and therefore $\widetilde{\psi}_{k,n} = \psi_{k,n}(\widetilde{\mu}) = \psi_{k,n}(\mu_{R+1})$ for all $n \in \mathbb{N}$. Hence, since furthermore $N_k = S_R$, it follows that

$$\mathbf{M}_k = \left(\overline{\langle \widetilde{\psi}_{k,n}, \overset{*}{\psi}_{k,m} \rangle_{\widetilde{\xi}}} \right)_{n,m=1}^{N_k} = \left(\overline{\langle \psi_{k,n}(\mu_{R+1}), \overset{*}{\psi}_{k,m} \rangle_{\xi_{\mu_{R+1}}}} \right)_{n,m=1}^{S_R} = \mathbf{M}_{k,R}$$

and, as was already suggested, this matrix is invertible on account of the last lemma. Therewith, the proof is finished. $\square$

Remarks.

(a) Of course, by the equivalence of $L^2_{\widetilde{\xi}}(\Omega)$ and $L^2(\Omega)$ the λ-nonlinear eigenfunctions of any operator pencil $\mathscr{A}_k$ also form a Riesz basis of the latter space.

(b) By the just proved result and the remark following Theorem 2.2.12 there exists for all $k \in B$ an isomorphism $T_k : L^2_{\widetilde{\xi}}(\Omega) \to L^2_{\widetilde{\xi}}(\Omega)$ such that

$$T_k \overset{*}{\psi}_{k,n} = \widetilde{\psi}_{k,n} \quad \text{for } 1 \le n \le N_k \qquad \text{and} \qquad T_k \widetilde{\psi}_{k,n} = \widetilde{\psi}_{k,n} \quad \text{for } n \ge N_k + 1$$

as well as an inner product on $L^2(\Omega)$, equivalent to $\langle \cdot, \cdot \rangle_{\widetilde{\xi}}$, with respect to which the λ-nonlinear eigenfunctions W_k constitute an orthonormal basis. Moreover, the unconditionally convergent and unique expansion of a function $u \in L^2_{\widetilde{\xi}}(\Omega)$ in the Riesz basis W_k has the form

$$u = \sum_{n=1}^{N_k} \langle u, T_k^* \widetilde{\psi}_{k,n} \rangle_{\widetilde{\xi}} \, \overset{*}{\psi}_{k,n} + \sum_{n=N_k+1}^{\infty} \langle u, T_k^* \widetilde{\psi}_{k,n} \rangle_{\widetilde{\xi}} \, \widetilde{\psi}_{k,n}.$$

On the other hand, by means of the expansion coefficients which we derived in the second part of the proof of Lemma 5.4.4 (see (5.59)),

$$u = \sum_{n=1}^{N_k} c_{k,n}(u) \overset{*}{\psi}_{k,n} + \sum_{n=N_k+1}^{\infty} c_{k,n}(u) \widetilde{\psi}_{k,n}.$$

Hence, $c_{k,n}(u) = \langle u, T_k^* \widetilde{\psi}_{k,n} \rangle_{\widetilde{\xi}} = \langle T_k u, \widetilde{\psi}_{k,n} \rangle_{\widetilde{\xi}}$ for all $n \in \mathbb{N}$ and therefore, by Parseval's identity,

$$\|T_k\|_{\mathcal{B}(L^2_{\widetilde{\xi}}(\Omega))} = \sup_{\substack{u \in L^2(\Omega) \\ \|u\|_{\widetilde{\xi}} = 1}} \left(\sum_{n=1}^{\infty} |\langle T_k u, \widetilde{\psi}_{k,n} \rangle_{\widetilde{\xi}}|^2 \right)^{1/2} = \sup_{\substack{u \in L^2(\Omega) \\ \|u\|_{\widetilde{\xi}} = 1}} \left(\sum_{n=1}^{\infty} |c_{k,n}(u)|^2 \right)^{1/2},$$

meaning that the coefficient functionals $u \mapsto c_{k,n}(u)$, as expected, measure the non-orthogonality of the Riesz basis $\{\overset{*}{\psi}_{k,n} \mid 1 \le n \le N_k\} \cup \{\widetilde{\psi}_{k,n} \mid n \ge N_k + 1\}$.

Having established the basis property of the λ-nonlinear Bloch waves in the high-frequency nondispersive case, we close this subsection and move on to a more general setting in which the last theorem shall play the role of an auxiliary tool (see the proof of part (a) of Proposition 5.4.13).

5.4.2 THE ASYMPTOTICALLY NONDISPERSIVE CASE (COMPLETENESS)

While the basicity result we have just proved is certainly of interest in its own right, we are, of course, interested in obtaining similar results for more general λ-nonlinear spectral problems for dispersive photonic crystals. A quite canonical idea here is to essentially consider the limit $\widetilde{\mu} \to \infty$ in (5.51). In other words, instead of assuming that a modeled structure is behaving nondispersively for frequencies greater than a finite threshold, we impose such a behavior only in the limit as the frequency tends to infinity. Expressed in terms of the coefficient function ξ of the operator pencils $\mathscr{A}_k$ our precise requirements are stated right below. If the relative permittivity of a given photonic crystal is such that these assumptions are fulfilled, we call the medium *asymptotically nondispersive*.

Assumptions 5.4.8 (Basic assumptions on ξ in the asymptotically nondispersive case)**.** *Let Assumptions 5.4.1 hold and suppose further that for some $\widetilde{\xi} \in L^\infty(\mathbb{R}^2;\mathbb{R})$, which shall satisfy Assumptions 4.1.1, we have*

$$\left\| 1 - \frac{\xi(\cdot,\mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } \mu \to \infty. \tag{5.77}$$

Remark. The requirement on $\widetilde{\xi}$ simply means that the spectral problem (4.3) with this function in place of the relative permittivity is governed by the theory presented in Chapter 4. In particular, $\widetilde{\xi}$ is $\mathbb{Z}^2$-periodic and essentially bounded below and above by positive constants $\widetilde{\xi}_{\min}$ and $\widetilde{\xi}_{\max}$. Hence, the requirement (5.77) and $\|\widetilde{\xi} - \xi(\cdot,\mu)\|_{L^\infty(\Omega)} \to 0$ as $\mu \to \infty$ are equivalent, but we chose to impose the stated condition as it appears more naturally in our analysis. Note also that this is in line with the discussion presented in Section 3.3. Therein, we argued on physical grounds that the relative permittivity ε_{r}, thus also ξ, converges to the constant vacuum permittivity ε_0 as the frequency tends to infinity (see

(3.35)). Since $\widetilde{\xi}$ is allowed to be a function of space, our model here is more general, but clearly also more interesting mathematically.

As their name suggests, Assumptions 5.4.8 do not yet constitute the entire set of restrictions which we impose on ξ in the course of the current subsection. In order to successfully prove a completeness result for the λ-nonlinear eigenfunctions of the operator pencils $\mathscr{A}_k$, stronger assumptions, mostly on the rate of the convergence in (5.77), will become necessary later on. Before we go into details, we present in the subsequent paragraph important preliminary results.

5.4.2.1 AUXILIARIES AND AN ABSTRACT COMPLETENESS THEOREM

In properly asymptotically nondispersive cases no threshold $\widetilde{\mu}$ as in (5.51) exists. Nevertheless, we shall reuse the corresponding "tilde-notation" introduced in Subsection 5.4.1. This seems appropriate, since again all quantities that carry such an accent mark stem from a λ-linear eigenvalue problem, namely that for the operator $\widetilde{A}_k : D(\widetilde{A}_k) \to L^2_{\widetilde{\xi}}(\Omega)$ in $L^2_{\widetilde{\xi}}(\Omega)$ given for $k \in B$ by

$$D(\widetilde{A}_k) := H^2_{k\text{-per}}(\Omega), \qquad \widetilde{A}_k u := -\frac{1}{\widetilde{\xi}}\Delta u.$$

Clearly, this is formally the same operator as the equally denoted one in the high-frequency nondispersive case (see (5.53)). However, for the eigenvalues and -functions of this operator, in relation to those of the operator pencil $\mathscr{A}_k$, we now generally obtain

$$\overset{*}{\mu}_{k,n} \neq \widetilde{\lambda}_{k,n} \quad \text{and} \quad \overset{*}{\psi}_{k,n} \neq \widetilde{\psi}_{k,n} \qquad \text{for all } n \in \mathbb{N},$$

in contrast to the equalities (5.54). This is, of course, complicating our analysis and we cannot expect to address the question of completeness of the λ-nonlinear Bloch waves $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}}$ by similar means as in the previous subsection.

What we intend to do instead is based on the reasonable assumption that if μ is sufficiently large, and thus the norm-distance between $\xi(\cdot,\mu)$ and $\widetilde{\xi}$ is small in view of (5.77), then the same is expected to be true for large enough eigenvalues $\overset{*}{\mu}_{k,n}$ and $\widetilde{\lambda}_{k,n}$ and the corresponding eigenfunctions. We postpone specifying the rather vague terms "small" and "closeness of eigenfunctions" to further below. The part of our conjecture that concerns eigenvalues, on the other hand, shall be clarified just yet:

Lemma 5.4.9. *For all $k \in B$ the eigenvalues of $\mathscr{A}_k$ and $\widetilde{A}_k$ satisfy*

$$|\overset{*}{\mu}_{k,n} - \widetilde{\lambda}_{k,n}| \leq \overset{*}{\mu}_{k,n} \left\| 1 - \frac{\xi(\cdot,\overset{*}{\mu}_{k,n})}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)}$$

for all sufficiently large $n \in \mathbb{N}$ in the sense that the norm on the right-hand side is less than or equal to one.

Proof. Let $k \in B$ be fixed and let $\mu_0 \in [0, \infty)$ be such that

$$\left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \leq 1 \qquad \text{for all } \mu \in [\mu_0, \infty). \tag{5.78}$$

Note that such a number μ_0 exists due to Assumptions 5.4.8 (see (5.77)). Now, for all $\mu \in [0, \infty)$ and all nonzero $u \in H^1_{k\text{-per}}(\Omega)$ we have

$$\begin{aligned}
&\frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega \xi(x,\mu)|u(x)|^2 \, dx} - \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega \widetilde{\xi}(x)|u(x)|^2 \, dx} \\
&\qquad = \frac{\int_\Omega |\nabla u(x)|^2 \, dx \cdot \int_\Omega \widetilde{\xi}(x) \left(1 - \frac{\xi(x,\mu)}{\widetilde{\xi}(x)}\right) |u(x)|^2 \, dx}{\int_\Omega \xi(x,\mu)|u(x)|^2 \, dx \cdot \int_\Omega \widetilde{\xi}(x)|u(x)|^2 \, dx} \\
&\qquad \leq \left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega \xi(x,\mu)|u(x)|^2 \, dx}.
\end{aligned} \tag{5.79}$$

Hence, for μ and u unchanged,

$$\frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega \xi(x,\mu)|u(x)|^2 \, dx} \left(1 - \left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \right) \leq \frac{\int_\Omega |\nabla u(x)|^2 \, dx}{\int_\Omega \widetilde{\xi}(x)|u(x)|^2 \, dx}.$$

Restricting our considerations to $\mu \in [\mu_0, \infty)$ from now on, the inequality (5.78) implies that the term in round brackets on the left-hand side is non-negative. Therefore, for these μ and all $n \in \mathbb{N}$ the min-max-principles for the nth eigenvalues of the operators $A_{\mu,k}$ and $\widetilde{A}_k$ (see (5.13) and Theorem 4.1.5, recalling that $\widetilde{\xi}$ satisfies Assumptions 4.1.1) yield

$$\lambda_{k,n}(\mu) \left(1 - \left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \right) \leq \widetilde{\lambda}_{k,n}$$

or, equivalently,

$$\lambda_{k,n}(\mu) - \widetilde{\lambda}_{k,n} \leq \lambda_{k,n}(\mu) \left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)}.$$

In fact, starting again from the negative of the left-hand side of (5.79) and essentially repeating our arguments so far, we even obtain

$$|\lambda_{k,n}(\mu) - \widetilde{\lambda}_{k,n}| \leq \lambda_{k,n}(\mu) \left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)},$$

which is, of course, only valid for $\mu \in [\mu_0,\infty)$. Finally, if $n \in \mathbb{N}$ is such that the nth eigenvalue of the operator pencil $\mathscr{A}_k$, i.e., the unique fixed point $\mathring{\mu}_{k,n}$ of the mapping $[0,\infty) \ni \mu \mapsto \lambda_{k,n}(\mu)$, satisfies $\mathring{\mu}_{k,n} \in [\mu_0,\infty)$, then the last inequality reads

$$|\mathring{\mu}_{k,n} - \widetilde{\lambda}_{k,n}| \leq \mathring{\mu}_{k,n} \left\| 1 - \frac{\xi(\cdot,\mathring{\mu}_{k,n})}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)},$$

which finishes the proof. □

Recalling that $\mathring{\mu}_{k,n} \to \infty$ as $n \to \infty$ for all $k \in B$ we have the following corollary to the last result:

Corollary. *Suppose*

$$\left\| 1 - \frac{\xi(\cdot,\mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \in o\left(\frac{1}{\mu}\right) \quad \textit{as } \mu \to \infty. \tag{5.80}$$

Then for all $k \in B$ we have $\mathring{\mu}_{k,n} - \widetilde{\lambda}_{k,n} \to 0$ as $n \to \infty$.

Remark. The importance of the previous statement will become apparent later on when we prove our completeness theorem for the asymptotically nondispersive case. By assumption the condition (5.80) on the rate of the convergence (5.77) will then be satisfied (see part (c.iv) of Assumptions 5.4.25) and we will therefore be able to satisfy a certain closeness requirement between large enough eigenvalues of $\mathscr{A}_k$ and $\widetilde{A}_k$.

Whereas we were concerned with both the λ-nonlinear and the λ-linear eigenvalues simultaneously so far, we next provide additional information regarding only the spectrum of the "limit problem" for the operator $\widetilde{A}_k$. It will only be used rather late in this thesis, but it seems appropriate to include it already here so that our preliminaries on eigenvalues and -functions are kept separate.

Lemma 5.4.10. *For all $k \in B$, all $\eta \in (0, 4\pi\widetilde{\xi}_{\max}^{-1})$, and all $M > 0$ there exists some $\nu \geq M$ such that $[\nu,\nu+\eta] \subset \rho(\widetilde{A}_k)$, i.e., there repeatedly occur gaps of width at least η in the spectrum of any operator $\widetilde{A}_k$.*

Proof. Let $k \in B$ and η in the specified range be fixed. To perform an indirect proof, we assume that there exists some $M > 0$ such that for all $\nu \geq M$ we have $[\nu,\nu+\eta] \cap \sigma(\widetilde{A}_k) \neq \emptyset$. Then, as $\sigma(\widetilde{A}_k)$ is purely discrete, we can find a subsequence $\{\widetilde{\lambda}_{k,n_j}\}_{j\in\mathbb{N}}$ of the eigenvalues of $\widetilde{A}_k$ such that

$$M + j\eta \leq \widetilde{\lambda}_{k,n_j} \leq M + (j+1)\eta \qquad \text{for all } j \in \mathbb{N}.$$

This easily implies

$$\frac{M}{j} + \eta \leq \frac{\widetilde{\lambda}_{k,n_j}}{j} \leq \frac{M+\eta}{j} + \eta \quad \text{for all } j \in \mathbb{N} \quad \text{and} \quad \frac{\widetilde{\lambda}_{k,n_j}}{j} \to \eta \quad \text{as } j \to \infty.$$

The stated convergence will give a contradiction. To see this, we denote for $n \in \mathbb{N}$ the nth eigenvalue of the operator $-\Delta : H^2_{k\text{-per}}(\Omega) \to L^2(\Omega)$ in $L^2(\Omega)$ by $\lambda^{\mathbb{1}}_{k,n}$ and similarly write $\lambda^{\mathbb{1}}_{N,n}$ for that of the operator $-\Delta : H^2(\Omega) \to L^2(\Omega)$ in $L^2(\Omega)$ subject to vanishing Neumann boundary conditions.[1] The associated forms act in exactly the same way as that corresponding to the operator $\widetilde{A}_k$ and have domains $H^1_{k\text{-per}}(\Omega)$ and $H^1(\Omega)$, respectively (see [Dav96, Chapt. 7]). Hence, we obtain from the min-max principle (see (4.21))

$$\frac{\widetilde{\lambda}_{k,n_j}}{j} \geq \frac{\widetilde{\lambda}_{k,n_j}}{n_j} \geq \frac{1}{\widetilde{\xi}_{\max}} \frac{\lambda^{\mathbb{1}}_{k,n_j}}{n_j} \geq \frac{1}{\widetilde{\xi}_{\max}} \frac{\lambda^{\mathbb{1}}_{N,n_j}}{n_j} \qquad \text{for all } j \in \mathbb{N}, \tag{5.81}$$

using $j \leq n_j$ for all $j \in N$ and the inclusion $H^1_{k\text{-per}}(\Omega) \subset H^1(\Omega)$. These inequalities, however, cannot hold since on the one hand

$$\frac{\widetilde{\lambda}_{k,n_j}}{j} \to \eta < \frac{4\pi}{\widetilde{\xi}_{\max}} \quad \text{as } j \to \infty$$

by our reasoning above, but on the other hand

$$\frac{1}{\widetilde{\xi}_{\max}} \frac{\lambda^{\mathbb{1}}_{N,n_j}}{n_j} \to \frac{4\pi}{\widetilde{\xi}_{\max}} > \eta \quad \text{as } j \to \infty$$

as a consequence of the well-known Weyl asymptotics for Neumann eigenvalues (see [CH04, Sect. VI.4.1] and the classical article [Wey12]). This gives the desired contradiction and thereby closes the proof. □

Remark. The lower bound on the eigenvalues of the operator $\widetilde{A}_k$ used to derive the inequalities (5.81) can be complemented by an upper one: Denote for $n \in \mathbb{N}$ the nth eigenvalue of the operator $-\Delta : H^2(\Omega) \to L^2(\Omega)$ in $L^2(\Omega)$ subject to vanishing Dirichlet boundary conditions by $\lambda^{\mathbb{1}}_{D,n}$. Then, again as a consequence of the min-max principle,

$$\frac{1}{\widetilde{\xi}_{\max}} \lambda^{\mathbb{1}}_{N,n} \leq \widetilde{\lambda}_{k,n} \leq \frac{1}{\widetilde{\xi}_{\min}} \lambda^{\mathbb{1}}_{D,n} \qquad \text{for all } n \in \mathbb{N},$$

since the form domain for the Dirichlet problem is $H^1_0(\Omega) \subset H^1_{k\text{-per}}(\Omega)$ (see [Dav96, Chapt. 6]). For obvious reasons, two-sided inequalities of this type are called *Dirichlet-Neumann-bracketings*. In order to obtain valuable information about eigenvalues of periodic problems these estimates are usually too rough. They have, however, proved helpful for an asymptotic spectral analysis of certain periodic media (see [HL00]).

Having discussed only eigenvalues so far, we now turn our attention to the λ-nonlinear Bloch waves $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}}$. The abstract theorem which we prove next

[1] As before in this work, the superscript "$\mathbb{1}$" reminds us that both operators have a coefficient function identically equal to one.

eventually allows us to show that they are complete in $L^2(\Omega)$ and, in fact, even in $H^1_{k\text{-per}}(\Omega)$. We state this result in a form detached from our specific problem, for it is also of interest in its own right.

Theorem 5.4.11. *Let H be a Hilbert space with orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ and let $\{\phi_n\}_{n\in\mathbb{N}}$ be a sequence in H. For some $N \in \mathbb{N}$ set $V := \operatorname{span}\{\phi_n \mid 1 \le n \le N\}$ as well as $U := \operatorname{span}\{\phi_n \mid n \ge N+1\}$ and suppose that*

$$\phi_1, \dots, \phi_N \text{ are linearly independent} \qquad \text{and} \qquad V \cap \overline{U} = \{0\}. \tag{5.82}$$

Further, let $P^{(N)} : H \to H$ be the orthogonal projection onto $W := \overline{\operatorname{span}\{\psi_n \mid n \ge N+1\}}$ and let $Q^{(N)} : H \to H$ be a linear operator such that

$$\operatorname{Ran} Q^{(N)} \subseteq \overline{U} \qquad \text{and} \qquad \left\|P^{(N)} - Q^{(N)}\right\| < 1. \tag{5.83}$$

Then $H = V \oplus \overline{U}$. In particular, the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ is then complete in H.

Proof. We first show that

$$\Phi : \left\{ \begin{aligned} U^\perp &\to \mathbb{R}^N \\ u &\mapsto \left(\langle u, \psi_n\rangle\right)_{n=1}^N \end{aligned} \right\} \quad \text{is injective.} \tag{5.84}$$

Thereto, let $u \in U^\perp$ with $\langle u, \psi_n \rangle = 0$ for $n = 1, \dots, N$. So, $u \in W^{\perp\perp} = W = \operatorname{Ran} P^{(N)}$ and therefore $u = P^{(N)}u$, since $P^{(N)}$ is a projection. Keeping this in mind, we deduce $U^\perp = \left(\overline{U}\right)^\perp \subseteq \left(\operatorname{Ran} Q^{(N)}\right)^\perp = \operatorname{Ker}\left(Q^{(N)}\right)^*$ from the assumption on $\operatorname{Ran} Q^{(N)}$ and find $\left(Q^{(N)}\right)^* u = 0$. Together with the self-adjointness of $P^{(N)}$ (as an orthogonal projection onto a closed subspace of H) this implies

$$\begin{aligned} \|u\| = \left\|P^{(N)}u\right\| = \left\|\left(P^{(N)}\right)^* u\right\| &= \left\|\left(P^{(N)} - Q^{(N)}\right)^* u\right\| \\ &\le \left\|\left(P^{(N)} - Q^{(N)}\right)^*\right\| \|u\| = \left\|P^{(N)} - Q^{(N)}\right\| \|u\|, \end{aligned} \tag{5.85}$$

giving $u = 0$ by the inequality in (5.83) and thus, as claimed, (5.84). As a consequence of this property, we readily obtain $\dim U^\perp \le N$ so that $U^\perp$ is a closed subspace of H.

Next, let S denote the orthogonal projection onto $U^\perp$. Then $S|_V : V \to U^\perp$ is likewise an injective mapping, since $Sv = 0$ for some $v \in V$ gives $v \in U^{\perp\perp} = \overline{U}$, i.e., $v \in V \cap \overline{U}$, and therefore $v = 0$ due to the second statement in (5.82). As before we now deduce $\dim V \le \dim U^\perp \le N$, which yields $\dim U^\perp = N$ by the assumed linear independence of $\phi_1, \dots, \phi_N$ (see, again, (5.82)). In view of this, $S|_V : V \to U^\perp$ is a bijection.

Finally, for an arbitrary $w \in H$ set $v := \left(S|_V\right)^{-1} Sw \in V$ and $u := w - v$. Then $Su = Sw - Sv = 0$, giving $u \in U^{\perp\perp} = \overline{U}$ and $w = v + u \in V \oplus \overline{U}$. This concludes the proof. $\square$

Remarks.

(a) The equality $H = V \oplus \overline{U}$, which is the outcome of the last theorem, implies that for all $u \in H$ there exist coefficients $c_1(u), \ldots, c_N(u) \in \mathbb{C}$ and $c_n^{(s)}(u) \in \mathbb{C}$, where $n = N+1, \ldots, M_s$ and $s \in \mathbb{N}$, such that

$$\sum_{n=1}^{N} c_n(u)\,\phi_n \;+\; \sum_{n=N+1}^{M_s} c_n^{(s)}(u)\,\phi_n \to u \quad \text{as } s \to \infty.$$

Note that this is stronger than the completeness of $\{\phi_n\}_{n\in\mathbb{N}}$ in H because the first N coefficients do not depend on s.

(b) It is well-known that the projection $P^{(N)}$ of Theorem 5.4.11 is given by

$$P^{(N)}u = \sum_{n=N+1}^{\infty} \langle u, \psi_n \rangle \psi_n \qquad \text{for all } u \in H.$$

On the other hand, if $\{J_m\}_{m\in\mathbb{N}}$ is a partition[1] of the set $\{n \in \mathbb{N} \mid n \geq N+1\}$, then

$$P^{(N)} = \sum_{m=1}^{\infty} P_m \qquad \text{with} \qquad P_m u := \sum_{n \in J_m} \langle u, \psi_n \rangle \psi_n \quad \text{for all } u \in H. \tag{5.86}$$

Here, the sequence of operators $\{P_m\}_{m\in\mathbb{N}}$ consists of pairwise disjoint orthogonal projections with

$$\operatorname{Ran} P_m = \operatorname{span}\{\psi_n \mid n \in J_m\} \qquad \text{for all } m \in \mathbb{N}.$$

Motivated by (5.86), we will later on choose the operator $Q^{(N)}$ occurring in the statement of Theorem 5.4.11 as a convergent series $Q^{(N)} = \sum_{m=1}^{\infty} Q_m$, where $\{Q_m\}_{m\in\mathbb{N}}$ is a sequence of linear operators on H such that

$$\operatorname{Ran} Q_m \subseteq U \quad \text{for all } m \in \mathbb{N} \qquad \text{and} \qquad \left\| \sum_{m=1}^{\infty} (P_m - Q_m) \right\| < 1. \tag{5.87}$$

Note that then $\operatorname{Ran} Q^{(N)} = \operatorname{Ran}\left(\sum_{m=1}^{\infty} Q_m\right) \subseteq \overline{U}$, as it is required by the result we intend to apply (see (5.83)). The exact benefit of working with this form of the operator $Q^{(N)}$ in the context of our specific problem shall soon become apparent.

To proceed, we outline how we intend to apply Theorem 5.4.11 to show the completeness of the λ-nonlinear eigenfunctions of the operator pencils $\mathscr{A}_k$. Furthermore, we provide preparatory results concerning the requirements (5.82).

Keeping $k \in B$ fixed, the obvious choices for the sequences $\{\phi_n\}_{n\in\mathbb{N}}$ and $\{\psi_n\}_{n\in\mathbb{N}}$ introduced in the statement of the theorem are, in that order, the

[1] That is, pairwise disjoint, non-empty subsets of $\mathbb{N}$ such that $\cup_{m\in\mathbb{N}} J_m = \{n \in \mathbb{N} \mid n \geq N+1\}$.

λ-nonlinear eigenfunctions $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}}$ and the $\langle\cdot,\cdot\rangle_{\widetilde{\xi}}$-orthonormal basis $\{\widetilde{\psi}_{k,n}\}_{n\in\mathbb{N}}$ of $L^2(\Omega)$. However, in the topology of the latter space we did not succeed in verifying the second of the requirements (5.82), i.e., that for some $N \in \mathbb{N}$

$$\operatorname{span}\{\overset{*}{\psi}_{k,n} \mid 1 \leq n \leq N\} \cap \overline{\operatorname{span}\{\overset{*}{\psi}_{k,n} \mid n \geq N+1\}} = \{0\}.$$

Fortunately, though, by applying Theorem 5.4.11 in a smaller space, namely $H^1_{k\text{-per}}(\Omega)$, we can hold on to the just-mentioned canonical choices of the necessary sequences and eventually even gain a stronger theorem. In order to be able to follow this approach, we first have to establish that the eigenfunctions of the operator $\widetilde{A}_k$ actually form an orthonormal basis of $H^1_{k\text{-per}}(\Omega)$. This is assured by our next result.

Proposition 5.4.12. *For all $k \in B$ the renormalized eigenfunctions $\big\{1/\sqrt{1+\widetilde{\lambda}_{k,n}}\ \widetilde{\psi}_{k,n}\big\}_{n\in\mathbb{N}}$ of the operator $\widetilde{A}_k$ form an orthonormal basis of $H^1_{k\text{-per}}(\Omega)$ with respect to the inner product $\langle\cdot,\cdot\rangle_{H^1_k,\widetilde{\xi}}$ given by*

$$\langle u,v\rangle_{H^1_k,\widetilde{\xi}} := \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} + \langle u,v\rangle_{\widetilde{\xi}} \qquad \text{for all } u,v \in H^1_{k\text{-per}}(\Omega). \tag{5.88}$$

Proof. Let $k \in B$ be fixed and $u \in H^1_{k\text{-per}}(\Omega)$. Then

$$\begin{aligned} \langle \widetilde{\psi}_{k,n}, u\rangle_{H^1_k,\widetilde{\xi}} &= \langle \nabla\widetilde{\psi}_{k,n}, \nabla u\rangle_{L^2(\Omega)} + \langle \widetilde{\psi}_{k,n}, u\rangle_{\widetilde{\xi}} \\ &= \langle -\Delta\widetilde{\psi}_{k,n}, u\rangle_{L^2(\Omega)} + \langle \widetilde{\psi}_{k,n}, u\rangle_{\widetilde{\xi}} \\ &= \big(1+\widetilde{\lambda}_{k,n}\big)\langle \widetilde{\psi}_{k,n}, u\rangle_{\widetilde{\xi}} \qquad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{5.89}$$

using integration by parts and the eigenvalue equation for $\widetilde{\psi}_{k,n} \in H^2_{k\text{-per}}(\Omega)$. Hence,

$$\left\langle \frac{1}{\sqrt{1+\widetilde{\lambda}_{k,n}}}\ \widetilde{\psi}_{k,n}, \frac{1}{\sqrt{1+\widetilde{\lambda}_{k,m}}}\ \widetilde{\psi}_{k,m} \right\rangle_{H^1_k,\widetilde{\xi}} = \frac{\sqrt{1+\widetilde{\lambda}_{k,n}}}{\sqrt{1+\widetilde{\lambda}_{k,m}}}\ \delta_{nm} = \delta_{nm} \qquad \text{for all } n,m \in \mathbb{N},$$

so that $\big\{1/\sqrt{1+\widetilde{\lambda}_{k,n}}\ \widetilde{\psi}_{k,n}\big\}_{n\in\mathbb{N}}$ is a $\langle\cdot,\cdot\rangle_{H^1_k,\widetilde{\xi}}$-orthonormal sequence. By (5.89) it is even a basis of $H^1_{k\text{-per}}(\Omega)$, because $\langle \widetilde{\psi}_{k,n}, u\rangle_{H^1_k,\widetilde{\xi}} = 0$ for all $n \in \mathbb{N}$ implies $u = 0$ as a consequence of the orthonormal basis property of $\{\widetilde{\psi}_{k,n}\}_{n\in\mathbb{N}}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\widetilde{\xi}}$ and the non-negativity of the corresponding eigenvalues. This proves the claim. $\square$

Remarks.

(a) In view of the lower and upper boundedness of $\widetilde{\xi}$ (see Assumptions 5.4.8) the inner product in (5.88) is equivalent to the canonical one on $H^1_{k\text{-per}}(\Omega)$. In fact, for $k \neq 0$ the second summand in its definition could even be omitted, whereas this is not possible if $k = 0$ since then $\langle \nabla\cdot, \nabla\cdot\rangle_{L^2(\Omega)}$ only induces a seminorm.

(b) Observe from (5.89) that for all $k \in B$ and all $u \in H^1_{k\text{-per}}(\Omega)$ we have

$$\left\langle u, \tfrac{1}{\sqrt{1+\widetilde{\lambda}_{k,n}}}\, \widetilde{\psi}_{k,n} \right\rangle_{H^1_k,\widetilde{\xi}} \tfrac{1}{\sqrt{1+\widetilde{\lambda}_{k,n}}}\, \widetilde{\psi}_{k,n} = \langle u, \widetilde{\psi}_{k,n}\rangle_{\widetilde{\xi}}\, \widetilde{\psi}_{k,n} \qquad \text{for all } n \in \mathbb{N}.$$

Hence, given a finite index set $J \subset \mathbb{N}$ the $\langle\cdot,\cdot\rangle_{H^1_k,\widetilde{\xi}}$-orthogonal projection onto $\operatorname{span}\{1/\sqrt{1+\widetilde{\lambda}_{k,n}}\, \widetilde{\psi}_{k,n} \mid n \in J\}$ in $H^1_{k\text{-per}}(\Omega)$ and the restriction to $H^1_{k\text{-per}}(\Omega)$ of the $\langle\cdot,\cdot\rangle_{\widetilde{\xi}}$-orthogonal projection onto $\operatorname{span}\{\widetilde{\psi}_{k,n} \mid n \in J\}$ in $L^2(\Omega)$ coincide.

While the last result allows us to choose $H^1_{k\text{-per}}(\Omega)$ as the Hilbert space H and $\{1/\sqrt{1+\widetilde{\lambda}_{k,n}}\, \widetilde{\psi}_{k,n}\}_{n\in\mathbb{N}}$ as the orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ in Theorem 5.4.11, it still remains to be shown that the requirements (5.82) can be fulfilled in the norm $\|\cdot\|_{H^1_k,\widetilde{\xi}}$. We do so by means of the following proposition:

Proposition 5.4.13.

(a) For all $k \in B$ and all $N \in \mathbb{N}$ the λ-nonlinear eigenfunctions $\overset{*}{\psi}_{k,1}, \dots, \overset{*}{\psi}_{k,N}$ are linearly independent.

(b) For all $k \in B$ and all $N \in \mathbb{N}$ such that $\overset{*}{\mu}_{k,N} < \overset{*}{\mu}_{k,N+1}$ we have

$$\operatorname{span}\{\overset{*}{\psi}_{k,n} \mid 1 \le n \le N\} \cap \overline{\operatorname{span}\{\overset{*}{\psi}_{k,n} \mid n \ge N+1\}} = \{0\},$$

the closure being taken in the norm $\|\cdot\|_{H^1_k,\widetilde{\xi}}$ of $H^1_{k\text{-per}}(\Omega)$.

Proof.

Ad (a): Let $k \in B$ and $N \in \mathbb{N}$ be fixed. For some $\widetilde{\mu} > \overset{*}{\mu}_{k,N}$ set

$$\Xi(\cdot,\mu) := \begin{cases} \xi(\cdot,\mu), & \text{if } \mu \in [0,\widetilde{\mu}), \\ \xi(\cdot,\widetilde{\mu}), & \text{if } \mu \in [\widetilde{\mu},\infty). \end{cases}$$

Then Assumptions 5.4.1 on ξ ensure that Ξ is a coefficient function falling within the scope of the high-frequency nondispersive case (see Assumptions 5.4.3). The eigenfunctions of the corresponding operator pencil $\mathscr{A}_k$ are therefore linearly independent by Theorem 5.4.7 (giving that they even form a Riesz basis of $L^2(\Omega)$). However, owing to the choice of $\widetilde{\mu}$, the first N of these eigenfunctions are just $\overset{*}{\psi}_{k,1}, \dots, \overset{*}{\psi}_{k,N}$, which gives the assertion.

Ad (b): Let $k \in B$ and $N \in \mathbb{N}$ with $\overset{*}{\mu}_{k,N} < \overset{*}{\mu}_{k,N+1}$ be fixed. Further, let

$$u \in \operatorname{span}\{\overset{*}{\psi}_{k,n} \mid 1 \le n \le N\} \cap \overline{\operatorname{span}\{\overset{*}{\psi}_{k,n} \mid n \ge N+1\}}.$$

Then there exist coefficients $c_1, \dots, c_N \in \mathbb{C}$ and $c_n^{(s)} \in \mathbb{C}$, where $n = N+1, \dots, M_s$ and $M_s \in \mathbb{N}$ for all $s \in \mathbb{N}$, such that

$$u = \sum_{n=1}^{N} c_n \overset{*}{\psi}_{k,n} \qquad \text{and} \qquad u^{(s)} := \sum_{n=N+1}^{M_s} c_n^{(s)} \overset{*}{\psi}_{k,n} \to u \quad \text{as } s \to \infty,$$

the convergence being in the norm $\|\cdot\|_{H^1_k,\widetilde{\xi}}$. Therefore,

$$\left\|\nabla u^{(s)}\right\|_{L^2(\Omega)} \to \|\nabla u\|_{L^2(\Omega)} \quad \text{and} \quad \left\|u^{(s)}\right\|_{\widetilde{\xi}} \to \|u\|_{\widetilde{\xi}} \quad \text{as } s \to \infty. \tag{5.90}$$

Next, we estimate the $L^2(\Omega)$-norm of the gradients of u and $u^{(s)}$. Given that both of these functions are finite linear combinations of λ-nonlinear eigenfunctions, we are reminded of the very same task we faced as part of the proof of Lemma 5.4.6 (see (5.63)). By notationally adapting the arguments given there (compare to (5.73)–(5.76)) we obtain

$$\begin{aligned}
\|\nabla u\|^2_{L^2(\Omega)} &= \sum_{n,m=1}^{N} c_n \overline{c_m} \int_\Omega \nabla \overset{*}{\psi}_{k,n}(x) \cdot \overline{\nabla \overset{*}{\psi}_{k,m}(x)}\, dx \\
&= \sum_{n,m=1}^{N} c_n \overline{c_m}\, \overset{*}{\mu}_{k,\max\{n,m\}} \int_\Omega \xi\big(x, \overset{*}{\mu}_{k,\max\{n,m\}}\big)\, \overset{*}{\psi}_{k,n}(x)\, \overline{\overset{*}{\psi}_{k,m}(x)}\, dx \\
&\le \overset{*}{\mu}_{k,N} \int_\Omega \xi(x, \overset{*}{\mu}_{k,N}) |u(x)|^2\, dx
\end{aligned} \tag{5.91}$$

and, similarly,

$$\begin{aligned}
\left\|\nabla u^{(s)}\right\|^2_{L^2(\Omega)} &= \sum_{n,m=N+1}^{M_s} c_n^{(s)} \overline{c_m^{(s)}} \int_\Omega \nabla \overset{*}{\psi}_{k,n}(x) \cdot \overline{\nabla \overset{*}{\psi}_{k,m}(x)}\, dx \\
&= \sum_{n,m=N+1}^{M_s} c_n^{(s)} \overline{c_m^{(s)}}\, \overset{*}{\mu}_{k,\min\{n,m\}} \int_\Omega \xi\big(x, \overset{*}{\mu}_{k,\min\{n,m\}}\big)\, \overset{*}{\psi}_{k,n}(x)\, \overline{\overset{*}{\psi}_{k,m}(x)}\, dx \\
&\ge \overset{*}{\mu}_{k,N+1} \int_\Omega \xi(x, \overset{*}{\mu}_{k,N+1}) |u^{(s)}(x)|^2\, dx.
\end{aligned}$$

In the limit $s \to \infty$ the last inequality, (5.90), and the equivalence of the norms $\|\cdot\|_{\xi(\cdot,\overset{*}{\mu}_{k,N+1})}$ and $\|\cdot\|_{\widetilde{\xi}}$ on $L^2(\Omega)$ give

$$\|\nabla u\|^2_{L^2(\Omega)} \ge \overset{*}{\mu}_{k,N+1} \int_\Omega \xi(x, \overset{*}{\mu}_{k,N+1}) |u(x)|^2\, dx.$$

Hence, combining this with (5.91) we deduce that

$$\int_\Omega \left[\overset{*}{\mu}_{k,N+1}\, \xi(x, \overset{*}{\mu}_{k,N+1}) - \overset{*}{\mu}_{k,N}\, \xi(x, \overset{*}{\mu}_{k,N})\right] |u(x)|^2\, dx \le 0.$$

The latter inequality now implies $u = 0$, since the expression in square brackets is positive almost everywhere on Ω as a consequence of the inequality $\overset{*}{\mu}_{k,N} < \overset{*}{\mu}_{k,N+1}$ and our monotonicity assumption on ξ (see Assumptions 5.4.1). The proof is finished. □

Our newly found knowledge, i.e., part (b) of the remarks following Theorem 5.4.11, Proposition 5.4.12, part (b) of the remarks thereafter, and Proposition 5.4.13, can now be incorporated into a restatement of Theorem 5.4.11. Naturally, this is then a version specific to our problem.

Theorem 5.4.14 (A problem-specific variant of Theorem 5.4.11)**.** *Let $k \in B$ and and let $N \in \mathbb{N}$ be such that $\mathring{\mu}_{k,N} < \mathring{\mu}_{k,N+1}$. Further, let $\{J_m\}_{m\in\mathbb{N}}$ be a partition of the set $\{n \in \mathbb{N} \mid n \geq N+1\}$. For all $m \in \mathbb{N}$ denote by P_m the orthogonal projection in $H^1_{k\text{-per}}(\Omega)$, equipped with the norm $\|\cdot\|_{H^1_k,\widetilde{\xi}}$, onto the subspace*[1] $\operatorname{span}\{\widetilde{\psi}_{k,n} \mid n \in J_m\}$*. If $\{Q_m\}_{m\in\mathbb{N}}$ is a sequence of linear operators on $H^1_{k\text{-per}}(\Omega)$ such that*

$$\operatorname{Ran} Q_m \subseteq \operatorname{span}\{\mathring{\psi}_{k,n} \mid n \geq N+1\} \qquad \textit{for all } m \in \mathbb{N} \tag{5.92}$$

and

$$\left\| \sum_{m=1}^{\infty} (P_m - Q_m) \right\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} < 1, \tag{5.93}$$

where, implicitly, the convergence of the series is demanded, then

$$H^1_{k\text{-per}}(\Omega) = \operatorname{span}\{\mathring{\psi}_{k,n} \mid 1 \leq n \leq N\} \oplus \overline{\operatorname{span}\{\mathring{\psi}_{k,n} \mid n \geq N+1\}}. \tag{5.94}$$

In particular, the λ-nonlinear eigenfunctions $\{\mathring{\psi}_{k,n}\}_{n\in\mathbb{N}}$ are then complete in $H^1_{k\text{-per}}(\Omega)$.

Remark. Is is important to observe that the index N occurring in the statement of the theorem can be chosen arbitrarily large, because the λ-nonlinear eigenvalues $\{\mathring{\mu}_{k,n}\}_{n\in\mathbb{N}}$ are isolated and tend to infinity by Theorem 5.2.6. Clearly, this should be of help in showing the inequality (5.93), since, intuitively, the λ-nonlinear eigenfunctions $\{\mathring{\psi}_{k,n} \mid n \geq N+1\}$—which are the only ones of interest here (see (5.92))—become ever closer to $\{\widetilde{\psi}_{k,n} \mid n \geq N+1\}$ as N increases (note our assumption (5.77)).

The following corollary is an easy consequence of the embedding of $H^1_{k\text{-per}}(\Omega)$ into $L^2(\Omega)$ and the denseness of the former in the latter space.

Corollary. *Under the conditions of Theorem 5.4.14, the λ-nonlinear eigenfunctions $\{\mathring{\psi}_{k,n}\}_{n\in\mathbb{N}}$ are complete in $L^2(\Omega)$.*

Next, let us analyze Theorem 5.4.14 concerning what is left to work out so as to establish the completeness of the λ-nonlinear eigenfunctions as in (5.94). Obviously, keeping $k \in B$ fixed, this amounts to finding the linear operators Q_m and making sure that they are sufficiently close—in the sense that the estimate (5.93) holds—to the orthogonal projections P_m. Since the latter operators map onto subspaces spanned by eigenfunctions of $\widetilde{A}_k$, we are reminded of their representation as Riesz projections (see Subsection 2.4.2). Here, two observations are crucial: First, by part (b) of the remarks following Proposition 5.4.12, each of the orthogonal projections P_m in $H^1_{k\text{-per}}(\Omega)$ equals the restriction to $H^1_{k\text{-per}}(\Omega)$ of the $\langle\cdot,\cdot\rangle_{\widetilde{\xi}}$-orthogonal projection in $L^2(\Omega)$ onto the same subspace. Second,

[1]Of course, $\operatorname{span}\{\widetilde{\psi}_{k,n} \mid n \in J_m\} = \operatorname{span}\{1/\sqrt{1+\widetilde{\lambda}_{k,n}}\, \widetilde{\psi}_{k,n} \mid n \in J_m\}$ for all $m \in \mathbb{N}$.

orthogonal projections are uniquely determined and thus, as anticipated, the operators P_m can be represented by Riesz integrals for appropriately chosen Cauchy contours.[1,2] In view of this, the idea of choosing "λ-nonlinear Riesz projections" for the operators Q_m immediately comes to mind. However, a literature review has shown that such a concept has not yet received any attention. Fortunately, we are ourselves successful in contributing some results in this direction—at least for our specific λ-nonlinear problem. This material is presented below and requires our assumptions on ξ to be slightly strengthened.

5.4.2.2 A λ-NONLINEAR RIESZ PROJECTION AND RELATED OPERATORS

As was just outlined, we aim to construct operators that map $H^1_{k\text{-per}}(\Omega)$, for $k \in B$, into the span of all but the first finitely many λ-nonlinear eigenfunctions of the operator pencil $\mathscr{A}_k$. Further, we require these mappings to be close to certain Riesz projections for the operator $\widetilde{A}_k$. Motivated by the results presented in Subsection 2.4.2, a natural candidate is of the form

$$-\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{A}_k}(\nu)\, d\nu, \tag{5.95}$$

where Γ is taken to be a suitable Cauchy contour in the complex plane that encloses some of the (large) λ-nonlinear eigenvalues $\{\overset{*}{\mu}_{k,n}\}_{n\in\mathbb{N}}$. Obviously, in order for this integral to be defined, requirements beyond those we so far imposed on the coefficient function ξ have to be met. Note, however, that the application we have in mind for such an operator, i.e., Theorem 5.4.14, does not require it to be a projection. As we shall see further below, this is crucial to our success in proving a completeness theorem in the asymptotically nondispersive case.

Initially, without commenting on their precise application just yet, we list our extended assumptions on ξ. They are referred to as "high-frequency assumptions", for, as the term suggests, they only have to hold for second arguments of ξ larger than some threshold.

Assumptions 5.4.15 (High-frequency assumptions on ξ in the asymptotically nondispersive case)**.** *Let Assumptions 5.4.8 hold and suppose further that there exist $\widehat{\mu} > 0$ and $\kappa > 0$ such that $(\widehat{\mu},\infty) \ni \mu \mapsto \xi(\cdot,\mu)|_\Omega \in L^\infty(\Omega;\mathbb{R})$ has a holomorphic extension, which we again call ξ, to the semi-infinite strip*

$$\mathcal{S} := \{\mu + i\tau \in \mathbb{C} \mid \mu > \widehat{\mu}, |\tau| < \kappa\} \tag{5.96}$$

with $\xi(\cdot,\nu) \in L^\infty(\Omega;\mathbb{C})$ for all $\nu \in \mathcal{S}$. Moreover, we require that

[1] Note that the operator $\widetilde{A}_k$ is not self-adjoint in $H^1_{k\text{-per}}(\Omega)$ so that we cannot simply employ a $\langle\cdot,\cdot\rangle_{H^1_k,\widetilde{\xi}}$-orthogonal Riesz projection.

[2] This requires the partition $\{J_m\}_{m\in\mathbb{N}}$ in Theorem 5.4.14 to be such that indices of any eigenvalue and, if any, its "copies" (accounting for a higher multiplicity) belong to the same set J_m.

(a) for some $q > 0$ we have

$$\mu \left\| \frac{1}{\xi(\cdot,\mu)} \frac{\partial[\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\mu) \right\|_{L^\infty(\Omega)} \le q < 1 \qquad \textit{for all } \mu > \widehat{\mu}; \tag{5.97}$$

(b) for some $C > 0$ we have

$$\mu \left\| \frac{\partial^2[\xi(\cdot,\nu)]}{\partial\nu^2}(\cdot,\mu + i\tau) \right\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } \mu + i\tau \in \mathcal{S}. \tag{5.98}$$

Remark. Note that part (a) of these assumptions demands the convergence in (5.77) of $\xi(\cdot,\mu)$ to $\widetilde{\xi}$ as $\mu \to \infty$ to be sufficiently fast. It is equivalent to the requirement

$$\limsup_{\mu\to\infty} \left(\mu \left\| \frac{1}{\xi(\cdot,\mu)} \frac{\partial[\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\mu) \right\|_{L^\infty(\Omega)} \right) < 1.$$

By definition the holomorphic extension of ξ to $\mathcal{S}$ in its second variable is again a mapping into the essentially bounded functions on Ω. Off of the real axis, though, it is not required to be nonzero anymore. Thus, we introduce for all $k \in B$ the operator pencil $\mathscr{L}_k : [0,\widehat{\mu}] \cup \mathcal{S} \to \mathcal{C}(L^2(\Omega))$ by

$$D(\mathscr{L}_k(\nu)) := H^2_{k\text{-per}}(\Omega) \qquad \text{and} \qquad \mathscr{L}_k(\nu) := -\Delta - \nu\xi(\cdot,\nu)I, \tag{5.99}$$

which is well-defined also if $\xi(\cdot,\nu)$ vanishes (somewhere) on Ω for some (then non-real) $\nu \in \mathcal{S}$. Obviously, we have[1]

$$\mathscr{L}_k(\mu) = \xi(\cdot,\mu)\mathscr{A}_k \qquad \text{for all } \mu \in [0,\infty) \tag{5.100}$$

and, in particular,

$$\mathscr{R}_{\mathscr{L}_k}(\mu) = \mathscr{R}_{\mathscr{A}_k}(\mu)\frac{1}{\xi(\cdot,\mu)}I \qquad \text{for all } \mu \in \rho(\mathscr{L}_k) \cap [0,\infty) = \rho(\mathscr{A}_k) \tag{5.101}$$

as well as

$$\sigma(\mathscr{L}_k) \cap [0,\infty) = \sigma(\mathscr{A}_k). \tag{5.102}$$

Hence, the eigenvalues and -functions of the pencils $\mathscr{L}_k$ and $\mathscr{A}_k$ corresponding to real eigenvalues agree. However, we cannot a priori rule out that $\mathscr{L}_k$ has complex spectrum, which, if existent, might even contain points that are no eigenvalues.

To obtain further insights in this direction, we recall Taylor's theorem for continuously differentiable Banach space-valued mappings from the literature. As to that, note that the function $\mathcal{S} \ni \nu \mapsto \xi(\cdot,\nu) \in L^\infty(\Omega)$ and therewith the operator-valued variant $\mathcal{S} \ni \nu \mapsto \xi(\cdot,\nu)I \in \mathcal{B}(L^2(\Omega))$ are arbitrarily often continuously differentiable in $\mathcal{S}$ as a consequence of our holomorphicity assumption (see [GL09, Cor. 1.5.3] and Theorem 2.3.9).

[1] For consistency with our notation so far, and in order to avoid confusion, μ will always denote a real number in $[0,\infty)$ in what follows.

Lemma 5.4.16 ([BT03, Thm. 3.3.1][1])**.** *Let X be a Banach space and let $F \in C^{n+1}(\mathcal{S};X)$, where $n \in \mathbb{N}_0$. Then, for any $\nu_0 \in \mathcal{S}$ there holds*

$$F(\nu) = \sum_{j=0}^{n} \frac{(\nu-\nu_0)^j}{j!} \frac{d^j[F(\nu')]}{d\nu'^j}(\nu_0) + R_n^{(F)}(\nu,\nu_0) \qquad \text{for all } \nu \in \mathcal{S}. \tag{5.103}$$

In addition, the convexity of $\mathcal{S}$ allows the remainder term to be bounded above as

$$\left\| R_n^{(F)}(\nu,\nu_0) \right\|_X \leq \frac{|\nu-\nu_0|^{n+1}}{(n+1)!} \max_{0 \leq t \leq 1} \left\| \frac{d^{n+1}[F(\nu')]}{d\nu'^{n+1}}((1-t)\nu_0 + t\nu) \right\|_X .$$

Remark. If in the last lemma F is even a holomorphic function (as it will be in our applications), then so is $R_n^{(F)}(\cdot,\nu_0)$ for all $\nu_0 \in \mathcal{S}$ and all $n \in \mathbb{N}_0$. This can be seen from (5.103), whereby the remainder can then be written as a difference of two holomorphic functions on $\mathcal{S}$.

Let us get back to the question whether the operator pencil $\mathscr{L}_k$ can have complex spectrum. The next theorem shows that sufficiently close to the real axis this is not possible.

Lemma 5.4.17. *There exists some $\kappa_0 \in (0,\kappa]$ (with 2κ being the width of $\mathcal{S}$ as in (5.96)) such that for $\mathcal{S}_0 := \{\mu + i\tau \in \mathbb{C} \mid \mu > \widehat{\mu}, |\tau| < \kappa_0\} \subseteq \mathcal{S}$ we have*

$$(\mathcal{S}_0 \setminus \sigma(\mathscr{A}_k)) \subset \rho(\mathscr{L}_k) \qquad \text{for all } k \in B. \tag{5.104}$$

Further, for some $C > 0$ there holds the estimate

$$\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \leq \sqrt{\xi_{\max}(\mu)} \frac{C}{|\tau|} \quad \text{for all non-real } \nu = \mu + i\tau \in \mathcal{S}_0 \text{ and all } k \in B.$$

Proof. Let $\nu = \mu + i\tau \in \mathcal{S}$. Since $\mathbb{R} \cap (\mathcal{S} \setminus \sigma(\mathscr{A}_k)) \subset \rho(\mathscr{L}_k)$ by (5.101), we may restrict ourselves to the case $\tau \neq 0$ in the rest of this proof. For all $k \in B$, since μ is real, the operator $A_{\mu,k}$ (see (5.9)) is self-adjoint on the weighted space $L^2_{\xi_\mu}(\Omega)$. Hence, its spectrum is real, $\nu \in \rho(A_{\mu,k})$, and

$$\left\| R_{A_{\mu,k}}(\nu) \right\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} = \frac{1}{\operatorname{dist}(\nu,\sigma(A_{\mu,k}))} \leq \frac{1}{|\nu-\mu|} = \frac{1}{|\tau|} \qquad \text{for all } k \in B \tag{5.105}$$

by Theorem 2.4.6 in the preliminaries. With this at hand, we fix $k \in B$ and rearrange:

$$\begin{aligned}
\mathscr{L}_k(\nu) &= -\Delta - \nu\xi(\cdot,\mu)I - \big(\nu\xi(\cdot,\nu) - \nu\xi(\cdot,\mu)\big)I \\
&= \big(-\Delta - \nu\xi(\cdot,\mu)I\big)\left[I - \big(-\Delta - \nu\xi(\cdot,\mu)I\big)^{-1}\big(\nu\xi(\cdot,\nu) - \nu\xi(\cdot,\mu)\big)I\right] \\
&= \xi(\cdot,\mu)\big(A_{\mu,k} - \nu I\big)\left[I - R_{A_{\mu,k}}(\nu)\frac{\nu}{\xi(\cdot,\mu)}\big(\xi(\cdot,\nu) - \xi(\cdot,\mu)\big)I\right].
\end{aligned} \tag{5.106}$$

[1]The cited theorem is more general in that it covers functions acting between two Banach spaces. However, the there occurring Fréchet derivative is equivalent to the complex norm-derivative which we work with here so that the result is applicable (see [Muj85, Prop. 13.7]).

Denoting the operator in square brackets by $I - B_\mu(\nu)$, we find, using (5.105) and a Taylor expansion of the function $\mathcal{S} \ni \nu \mapsto \xi(\cdot,\nu)$ about μ as in Lemma 5.4.16,

$$\begin{aligned}
\left\|B_\mu(\nu)\right\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} &\le \left|\frac{\nu}{\nu-\mu}\right| \left\|\frac{\xi(\cdot,\nu)-\xi(\cdot,\mu)}{\xi(\cdot,\mu)}\right\|_{L^\infty(\Omega)} \\
&= \left|\frac{\nu}{\nu-\mu}\right| \left\|\frac{1}{\xi(\cdot,\mu)}\left[(\nu-\mu)\frac{\partial[\xi(\cdot,\mu')]}{\partial\mu'}(\cdot,\mu) + R_1^{(\xi)}(\nu,\mu)\right]\right\|_{L^\infty(\Omega)} \\
&\le |\nu| \left\|\frac{1}{\xi(\cdot,\mu)}\frac{\partial[\xi(\cdot,\mu')]}{\partial\mu'}(\cdot,\mu)\right\|_{L^\infty(\Omega)} \\
&\quad + \frac{|\nu||\nu-\mu|}{2\xi_{\min}} \max_{0\le t\le 1} \left\|\frac{\partial^2[\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,(1-t)\mu+t\nu)\right\|_{L^\infty(\Omega)} \\
&\le \left(1+\frac{|\tau|}{\mu}\right)\mu \left\|\frac{1}{\xi(\cdot,\mu)}\frac{\partial[\xi(\cdot,\mu')]}{\partial\mu'}(\cdot,\mu)\right\|_{L^\infty(\Omega)} \qquad (5.107) \\
&\quad + \frac{(\mu+|\tau|)|\tau|}{2\xi_{\min}} \max_{0\le t\le 1} \left\|\frac{\partial^2[\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,(1-t)\mu+t\nu)\right\|_{L^\infty(\Omega)}.
\end{aligned}$$

At this point both part (a) and (b) of our Assumptions 5.4.15 come into play. With them and the inequality $\mu > \widehat{\mu} > 0$ we get

$$\begin{aligned}
\left\|B_\mu(\nu)\right\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} &\le \left(1+\frac{|\tau|}{\widehat{\mu}}\right) q + \frac{1}{2\xi_{\min}}|\tau|\left(1+\frac{|\tau|}{\widehat{\mu}}\right) C \\
&= q + |\tau|\left[\frac{q}{\widehat{\mu}} + \frac{C}{2\xi_{\min}}\left(1+\frac{|\tau|}{\widehat{\mu}}\right)\right]
\end{aligned} \tag{5.108}$$

for some positive constants q and C where $q < 1$. Here it is important to note that the holomorphicity of ξ in $\mathcal{S}$ implies the continuity of its second derivative. Thus, the maximum on the right-hand side of (5.107) is attained on the compact line segment connecting μ and ν, i.e., at a point having real part equal to μ. Eventually, (5.108) shows that there is some positive $\kappa_0 \le \kappa$, which is independent of both $\operatorname{Re}\nu = \mu$ and k, with

$$\left\|B_\mu(\nu)\right\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} \le q_0 < 1 \qquad \text{for all } \nu = \mu + i\tau \in \mathcal{S} \text{ such that } 0 < |\tau| < \kappa_0.$$

Therefore, by the well-known theorem on the Neumann series,

$$\begin{aligned}
&I - B_\mu(\nu) : L^2_{\xi_\mu}(\Omega) \to L^2_{\xi_\mu}(\Omega) \text{ is bijective, } \left\|(I-B_\mu(\nu))^{-1}\right\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} \le \frac{1}{1-q_0} \\
&\text{for all } \nu = \mu + i\tau \in \mathcal{S} \text{ such that } 0 < |\tau| < \kappa_0.
\end{aligned} \tag{5.109}$$

For the remainder of the proof we only consider ν as in the second line of (5.109), i.e., non-real $\nu \in \mathcal{S}_0$ (see the definition in the statement of the theorem we just prove). In order to obtain the invertibility of $\mathscr{L}_k(\nu)$, we first note that by definition $B_\mu(\nu)$ is a mapping into $D(A_{\mu,k}) = H^2_{k\text{-per}}(\Omega) = D(\mathscr{L}_k(\nu))$. Thus, we get from (5.109) that the restriction $\big(I - B_\mu(\nu)\big)\,|_{D(\mathscr{L}_k(\nu))} : D(\mathscr{L}_k(\nu)) \to D(\mathscr{L}_k(\nu))$ is bijective (for more detailed arguments, compare to the proof of Proposition 5.2.5). Finally, due to (5.105), (5.106), (5.109), and the lower bound on ξ, we find that $\mathscr{L}_k(\nu) : D(\mathscr{L}_k(\nu)) \to L^2_{\xi_\mu}(\Omega) = L^2(\Omega)$ is bijective and

$$\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} = \left\|(I - B_\mu(\nu))^{-1} R_{A_{\mu,k}}(\nu)\frac{1}{\xi(\cdot,\mu)}\right\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} \le \frac{1}{1-q_0}\frac{1}{\xi_{\min}}\frac{1}{|\tau|}$$

Together with the upper bound on $\xi(\cdot,\mu)$ this allows us to similarly estimate the unweighted operator norm of $\mathscr{R}_{\mathscr{L}_k}(\nu)$ as

$$\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \le \frac{\sqrt{\xi_{\max}(\mu)}}{\sqrt{\xi_{\min}}}\,\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} \le \sqrt{\xi_{\max}(\mu)}\frac{1}{1-q_0}\frac{1}{\xi_{\min}^{3/2}}\frac{1}{|\tau|},$$

whereby our proof is complete. □

The previous lemma provides an upper bound for the operator norm of the resolvent of $\mathscr{L}_k$ at non-real points in the semi-infinite strip $\mathcal{S}_0 \subseteq \mathcal{S}$. A slightly different such estimate, namely one that is valid in neighborhoods of the (real) λ-nonlinear eigenvalues of $\mathscr{L}_k$ (or $\mathscr{A}_k$ for that matter), is proved next.

Lemma 5.4.18. *For all $k \in B$ and all $n \in \mathbb{N}$ such that $\overset{*}{\mu}_{k,n} > \widehat{\mu}$ there exists some $\theta > 0$ with $\dot{U}_\theta(\overset{*}{\mu}_{k,n}) \subset \rho(\mathscr{L}_k)$. Moreover, for some $C > 0$ there holds the estimate*

$$\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \le \sqrt{\xi_{\max}(\overset{*}{\mu}_{k,n})}\frac{C}{|\nu - \overset{*}{\mu}_{k,n}|} \qquad \text{for all } \nu \in \dot{U}_\theta(\overset{*}{\mu}_{k,n}). \tag{5.110}$$

Proof. Let $k \in B$ and $n \in \mathbb{N}$ as specified in the statement be fixed. Then, in particular, $\overset{*}{\mu}_{k,n} \in \mathcal{S}$. To simplify our notation and reveal parallels with how we showed Lemma 5.4.17, we omit sub- and superscript(s) of the λ-nonlinear eigenvalue and write μ instead of $\overset{*}{\mu}_{k,n}$ in the current proof. By its definition, μ is an eigenvalue of the operator $A_{\mu,k}$, which is self-adjoint in the weighted space $L^2_{\xi_\mu}(\Omega)$. Hence, similar to the previous proof we obtain

$$\left\|R_{A_{\mu,k}}(\nu)\right\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} = \frac{1}{\operatorname{dist}(\nu,\sigma(A_{\mu,k}))} = \frac{1}{|\nu-\mu|} \qquad \text{for all } \nu \in \dot{U}_\eta(\mu), \tag{5.111}$$

where $\eta > 0$ is sufficiently small and particularly chosen less than μ. Observe that this is qualitatively and even notationally the same as (5.105) and therefore allows us to repeat our proof of Lemma 5.4.17 word-for-word to a great extent. First, for all $\nu \in \dot{U}_\eta(\mu)$ we readily recover the equality (5.106) and, setting $\tau := |\nu - \mu|$, the estimate (5.107), which reads

$$\begin{aligned}\|B_\mu(\nu)\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))} \le \left(1 + \frac{|\tau|}{\mu}\right)\mu\left\|\frac{1}{\xi(\cdot,\mu)}\frac{\partial[\xi(\cdot,\mu')]}{\partial\mu'}(\cdot,\mu)\right\|_{L^\infty(\Omega)} \\ + \frac{(\mu+|\tau|)|\tau|}{2\xi_{\min}}\max_{0\le t\le 1}\left\|\frac{\partial^2[\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,(1-t)\mu+t\nu)\right\|_{L^\infty(\Omega)}.\end{aligned} \tag{5.112}$$

As before, the first summand on the right-hand side can be bounded above as

$$\left(1+\frac{|\tau|}{\mu}\right)\mu\left\|\frac{1}{\xi(\cdot,\mu)}\frac{\partial[\xi(\cdot,\mu')]}{\partial\mu'}(\cdot,\mu)\right\|_{L^\infty(\Omega)}\leq\left(1+\frac{|\tau|}{\mu}\right)q \tag{5.113}$$

for some $q<1$. However, in contrast to the situation in the proof of the last lemma, we cannot conclude that the point at which the maximum in the second summand on the right-hand side of (5.112) is attained has real part equal to μ. Nevertheless, given that this point lies on the line segment connecting μ and $\nu\in\dot{U}_\eta(\mu)$, we know its real part to be in the interval $(\mu-\eta,\mu+\eta)$. Hence, likewise as before, we find

$$\frac{(\mu+|\tau|)|\tau|}{2\xi_{\min}}\max_{0\leq t\leq 1}\left\|\frac{\partial^2[\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,(1-t)\mu+t\nu)\right\|_{L^\infty(\Omega)}\leq\frac{(\mu+|\tau|)|\tau|}{2\xi_{\min}}\frac{C}{\mu-\eta},$$

recalling that $\mu-\eta$ is positive. Together with the inequalities (5.112), (5.113), and $\tau<\eta$ this yields

$$\begin{aligned}\|B_\mu(\nu)\|_{\mathcal{B}(L^2_{\xi_\mu}(\Omega))}&\leq\left(1+\frac{|\tau|}{\mu}\right)q+\frac{(\mu+|\tau|)|\tau|}{2\xi_{\min}}\frac{C}{\mu-\eta}\\&\leq q+\eta\left[\frac{q}{\mu}+\frac{C}{2\xi_{\min}}\frac{\mu+\eta}{\mu-\eta}\right].\end{aligned}$$

Note here that the right-hand side converges to $q<1$ as $\eta\to 0$ (compare to (5.108)). Continuing to follow our previous arguments, we therefore eventually find some positive $\theta\leq\eta$ such that $\mathscr{R}_{\mathscr{L}_k}(\nu)$ exists for all $\nu\in\dot{U}_\theta(\mu)=\dot{U}_\theta(\overset{*}{\mu}_{k,n})$ and satisfies the estimate (5.110). This closes the proof. □

Remark. The radius θ appearing in the last lemma generally depends on the considered λ-nonlinear eigenvalue, i.e., on both $k\in B$ and $n\in\mathbb{N}$, since η (see (5.111)) does so. Furthermore, it is not necessarily less than or equal to κ_0 (which is independent of any parameters; see Lemma 5.4.17). Hence, locally an operator pencil $\mathscr{L}_k$ may be invertible in a region that even overlaps the complement of the semi-infinite strip $\mathcal{S}_0$ in the imaginary direction.

As a consequence of Lemma 5.4.17 we are able to find Cauchy contours in the complex plane that lie entirely in the resolvent set of an operator pencil $\mathscr{L}_k$. Calling such a path Γ, this is certainly a prerequisite for the existence of an operator of the form

$$-\frac{1}{2\pi i}\int_\Gamma\mathscr{R}_{\mathscr{L}_k}(\nu)\,d\nu, \tag{5.114}$$

which we similarly proposed before (see (5.95)).[1] In addition, it is of course important to verify that the appearing integrand is indeed holomorphic on

[1]Note that compared with the referenced operator we replaced $\mathscr{R}_{\mathscr{A}_k}$ with $\mathscr{R}_{\mathscr{L}_k}$ here. This is due to the reasoning presented on p. 112.

the chosen contour and also in its interior.[1] This issue shall be discussed next, beginning with the proofs of an important identity and a series representation for the resolvents we are concerned with. Therein and in the following we set

$$\mathcal{S}_k := S_0 \setminus \sigma(\mathscr{A}_k) \subset \rho(\mathscr{L}_k) \qquad \text{for all } k \in B, \tag{5.115}$$

where the inclusion holds on account of Lemma 5.4.17.

Proposition 5.4.19. *For all $k \in B$ and all $\nu, \nu_0 \in \mathcal{S}_k$ there holds*

$$\mathscr{R}_{\mathscr{L}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu_0) = \mathscr{R}_{\mathscr{L}_k}(\nu)\big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)\mathscr{R}_{\mathscr{L}_k}(\nu_0), \tag{5.116}$$

which we refer to as the first resolvent identity *for the operator pencil $\mathscr{L}_k$. Moreover, if*

$$\|\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\|_{L^\infty(\Omega)} < \|\mathscr{R}_{\mathscr{L}_k}(\nu_0)\|^{-1}_{\mathcal{B}(L^2(\Omega))}, \tag{5.117}$$

then

$$\mathscr{R}_{\mathscr{L}_k}(\nu) = \mathscr{R}_{\mathscr{L}_k}(\nu_0)\left\{I + \sum_{n=1}^{\infty}\Big[\big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)\mathscr{R}_{\mathscr{L}_k}(\nu_0)\Big]^n\right\}. \tag{5.118}$$

Proof. First, the identity (5.116) follows readily from the equality

$$\mathscr{L}_k(\nu_0) - \mathscr{L}_k(\nu) = \big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)I$$

by applying the operators $\mathscr{R}_{\mathscr{L}_k}(\nu)$ and $\mathscr{R}_{\mathscr{L}_k}(\nu_0)$ from the left and from the right, respectively. Therefore,

$$\mathscr{R}_{\mathscr{L}_k}(\nu_0) = \mathscr{R}_{\mathscr{L}_k}(\nu)\Big[I - \big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)\mathscr{R}_{\mathscr{L}_k}(\nu_0)\Big].$$

If $|\nu - \nu_0|$ satisfies the smallness condition mentioned in the statement of the proposition, the last equality can be rewritten as

$$\mathscr{R}_{\mathscr{L}_k}(\nu) = \mathscr{R}_{\mathscr{L}_k}(\nu_0)\Big[I - \big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)\mathscr{R}_{\mathscr{L}_k}(\nu_0)\Big]^{-1},$$

giving (5.118) by a Neumann series expansion of the inverse on the right-hand side. This finishes the proof. □

Remark. Note that the last result does not allow us to conclude that the two resolvents $\mathscr{R}_{\mathscr{L}_k}(\nu)$ and $\mathscr{R}_{\mathscr{L}_k}(\nu_0)$ commute. This is clearly different from the λ-linear case, in which the commutativity of two resolvents at different points in the resolvent set is an immediate consequence of part (b) of Proposition 2.4.4.

With the just proved resolvent identity at hand, we are able to show the desired holomorphicity of the associated resolvent mappings.

[1] In fact, the integral (5.114) exists if the mapping $\Gamma \ni \nu \mapsto \mathscr{R}_{\mathscr{L}_k}(\nu)$ is merely continuous, but only the outlined holomorphicity allows us to establish the desired mapping properties of the operator.

Proposition 5.4.20. *For all $k \in B$ the mapping $\mathcal{S}_k \ni \nu \mapsto \mathscr{R}_{\mathscr{L}_k}(\nu) \in \mathcal{B}(L^2(\Omega))$ is holomorphic with norm-derivative given by*

$$\frac{d\mathscr{R}_{\mathscr{L}_k}}{d\nu}(\nu_0) = \mathscr{R}_{\mathscr{L}_k}(\nu_0)\left[\frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0)\right]\mathscr{R}_{\mathscr{L}_k}(\nu_0) \qquad \textit{for all } \nu_0 \in \mathcal{S}_k.$$

Proof. Let $k \in B$ be fixed and let $\nu_0 \in \mathcal{S}_k$. We first show the norm-continuity of $\mathscr{R}_{\mathscr{L}_k}$ at ν_0. Thereto, let $\nu \in \mathcal{S}_k$ be sufficiently close to ν_0 in the sense that the inequality (5.117) holds. Such ν exist, since the mapping $\mathcal{S}_k \ni \nu \mapsto \nu\xi(\cdot,\nu)$ is continuous by our holomorphicity assumption on ξ. This also implies, using the series representation (5.118),

$$\begin{aligned}
&\|\mathscr{R}_{\mathscr{L}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu_0)\|_{\mathcal{B}(L^2(\Omega))} \\
&\quad = \left\|\mathscr{R}_{\mathscr{L}_k}(\nu_0)\sum_{n=1}^{\infty}\left[\big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)\mathscr{R}_{\mathscr{L}_k}(\nu_0)\right]^n\right\|_{\mathcal{B}(L^2(\Omega))} \\
&\quad \le \|\mathscr{R}_{\mathscr{L}_k}(\nu_0)\|_{\mathcal{B}(L^2(\Omega))}\left[\frac{1}{1 - \left\|\big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)\mathscr{R}_{\mathscr{L}_k}(\nu_0)\right\|_{\mathcal{B}(L^2(\Omega))}} - 1\right] \\
&\quad \to 0 \quad \text{as } \nu \to \nu_0.
\end{aligned}$$

Hence, $\mathscr{R}_{\mathscr{L}_k}$ is norm-continuous at ν_0 and thus everywhere in S_k since that point was arbitrary. To see that the resolvent mapping is even holomorphic, we use the first resolvent identity (5.116) for $\mathscr{L}_k$ and rearrange, for $\nu \in \mathcal{S}_k$,

$$\begin{aligned}
&\frac{\mathscr{R}_{\mathscr{L}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu_0)}{\nu - \nu_0} - \mathscr{R}_{\mathscr{L}_k}(\nu_0)\left[\frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0)\right]\mathscr{R}_{\mathscr{L}_k}(\nu_0) \\
&= \frac{\mathscr{R}_{\mathscr{L}_k}(\nu)\big(\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)\big)\mathscr{R}_{\mathscr{L}_k}(\nu_0)}{\nu - \nu_0} - \mathscr{R}_{\mathscr{L}_k}(\nu_0)\left[\frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0)\right]\mathscr{R}_{\mathscr{L}_k}(\nu_0) \\
&= \left\{\mathscr{R}_{\mathscr{L}_k}(\nu)\left[\frac{\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)}{\nu - \nu_0}\right]I - \mathscr{R}_{\mathscr{L}_k}(\nu_0)\left[\frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0)\right]I\right\}\mathscr{R}_{\mathscr{L}_k}(\nu_0) \\
&= \left\{\mathscr{R}_{\mathscr{L}_k}(\nu)\left[\frac{\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)}{\nu - \nu_0}\right]I\right. \\
&\qquad \left. - \big(\mathscr{R}_{\mathscr{L}_k}(\nu_0) + \mathscr{R}_{\mathscr{L}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu)\big)\left[\frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0)\right]I\right\}\mathscr{R}_{\mathscr{L}_k}(\nu_0).
\end{aligned}$$

Therefore, taking the operator norm yields

$$\begin{aligned}
&\left\|\frac{\mathscr{R}_{\mathscr{L}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu_0)}{\nu - \nu_0} - \mathscr{R}_{\mathscr{L}_k}(\nu_0)\left[\frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0)\right]\mathscr{R}_{\mathscr{L}_k}(\nu_0)\right\|_{\mathcal{B}(L^2(\Omega))} \\
&\qquad \le \left[\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))}\left\|\frac{\nu\xi(\cdot,\nu) - \nu_0\xi(\cdot,\nu_0)}{\nu - \nu_0} - \frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0)\right\|_{L^\infty(\Omega)}\right.
\end{aligned}$$

$$+ \|\mathscr{R}_{\mathscr{L}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu_0)\|_{\mathcal{B}(L^2(\Omega))} \left\| \frac{\partial[\nu\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\nu_0) \right\|_{L^\infty(\Omega)} \Bigg] \|\mathscr{R}_{\mathscr{L}_k}(\nu_0)\|_{\mathcal{B}(L^2(\Omega))}$$

$$\to 0 \quad \text{as } \nu \to \nu_0.$$

Here, the convergence to zero of both summands follows from the holomorphicity of ξ in the second variable and the already established norm-continuity of $\mathscr{R}_{\mathscr{L}_k}$. Note in this respect that the latter property and the reverse triangle inequality imply that also $\mathcal{S}_k \ni \nu \mapsto \|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))}$ is continuous. Again, the considered point was arbitrary so that the proof is complete. □

The results proved so far in this paragraph now finally allow us to address the mapping properties of our candidate for a λ-nonlinear Riesz projection specified in (5.114). As a preliminary, we adapt the definitions known from the Riesz projection of a closed operator to our purpose (compare to Definitions 2.4.7 and 2.4.8).

Definition 5.4.21. Let $k \in B$ and let σ_0 be a set of finitely many λ-nonlinear eigenvalues of $\mathscr{L}_k$ (and thus of $\mathscr{A}_k$; see (5.102)) each being greater than $\widehat{\mu}$ (with $\widehat{\mu}$ as in Assumptions 5.4.15).

(a) We call a Cauchy contour Γ (see Definition 2.3.7) *admissible* for $\mathscr{L}_k$ and σ_0 if $\Gamma \subset \mathcal{S}_k$ and the winding number $n(\Gamma, z)$ of Γ, for $z \in \mathbb{C} \setminus \Gamma$, satisfies

$$n(\Gamma, z) = \begin{cases} 1, & \text{if } z \in \sigma_0, \\ 0, & \text{if } z \in \sigma(\mathscr{L}_k) \setminus \sigma_0. \end{cases}$$

(b) Given an admissible contour Γ for $\mathscr{L}_k$ and σ_0 and a holomorphic operator-valued function $F : \mathcal{S}_0 \to \mathcal{B}(L^2(\Omega))$ we call the bounded operator

$$Q^{(F)}_{k,\sigma_0} := -\frac{1}{2\pi i} \int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) F(\nu)\, d\nu : L^2(\Omega) \to L^2(\Omega) \tag{5.119}$$

a *λ-nonlinear Riesz integral* for $\mathscr{L}_k$ and σ_0.

(c) Given an admissible contour Γ for $\mathscr{L}_k$ and σ_0, where the latter set only contains a single λ-nonlinear eigenvalue of $\mathscr{L}_k$, i.e., $\sigma_0 = \{\mathring{\mu}_{k,n}\}$ for some $\mathring{\mu}_{k,n} > \widehat{\mu}$, we call the bounded operator

$$P_{\mathring{\mu}_{k,n}} := -\frac{1}{2\pi i} \int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu) \right] I\, d\nu : L^2(\Omega) \to L^2(\Omega) \tag{5.120}$$

the *λ-nonlinear Riesz projection* for $\mathscr{L}_k$ and $\mathring{\mu}_{k,n}$.

Remarks.

(a) Similar to the Riesz projection for a closed operator, the integrals in both part (b) and (c) of the last definition are understood as Riemann integrals in the sense of Definition 2.3.4. By the holomorphicity of the respective integrands (see Assumptions 5.4.15 and Proposition 5.4.20), the Cauchy integral theorem in the form of Theorem 2.3.8, and a standard argument of complex function theory they are independent of the chosen admissible contour (compare to [HS96, Lem. 6.1]).

(b) From part (b) of the remarks following Definition 2.3.4 we know that when applied to a function in $u \in L^2(\Omega)$ the operators $Q^{(F)}_{k,\sigma_0}$ and $P_{\overset{*}{\mu}_{k,n}}$ satisfy

$$Q^{(F)}_{k,\sigma_0} u = -\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) F(\nu) u \, d\nu,$$

$$P_{\overset{*}{\mu}_{k,n}} u = -\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial \nu'}(\cdot,\nu)\right] u \, d\nu.$$

That is, u may be taken under the integral sign.

(c) Note that our definition of a λ-nonlinear Riesz projection really generalizes the respective λ-linear operator. For if ξ is an only spatially-dependent function satisfying Assumptions 4.1.1 of the nondispersive case, then

$$\mathscr{R}_{\mathscr{L}_k}(\nu) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial \nu'}(\cdot,\nu)\right] I = \big(-\Delta - \nu\xi I\big)^{-1}\xi I = \left(-\frac{1}{\xi}\Delta - \nu I\right)^{-1}$$

and thus $P_{\overset{*}{\mu}_{k,n}}$ is the λ-linear Riesz projection for the operator A_k (as in (4.13) with ε_r replaced by ξ) and its nth eigenvalue.

Our next theorem provides important information concerning the mapping properties of the λ-nonlinear Riesz integrals.

Theorem 5.4.22. *Let $k \in B$ and let $Q^{(F)}_{k,\sigma_0}$ be a λ-nonlinear Riesz integral for $\mathscr{L}_k$ and σ_0 as in part (b) of Definition 5.4.21.*

(a) We have

$$\operatorname{Ran} Q^{(F)}_{k,\sigma_0} \subseteq \bigoplus_{\mu\in\sigma_0} \operatorname{Ker}(\mathscr{L}_k(\mu)) = \bigoplus_{\mu\in\sigma_0} \operatorname{Ker}(\mathscr{A}_k(\mu)).$$

(b) If σ_0 consists of a single λ-nonlinear eigenvalue, i.e., $\sigma_0 = \{\overset{*}{\mu}_{k,n}\}$ for some $n \in \mathbb{N}$ such that $\overset{*}{\mu}_{k,n} > \widehat{\mu}$, and if the operator $F(\overset{*}{\mu}_{k,n})$ is invertible, then the inclusion stated in part (a) is an equality, i.e.,

$$\operatorname{Ran} Q^{(F)}_{k,\sigma_0} = \operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,n})) = \operatorname{Ker}(\mathscr{A}_k(\overset{*}{\mu}_{k,n})).$$

Proof.

Ad (a): After a renumbering, thereby abusing our notation for the rest of this proof, we may assume that σ_0 contains the λ-nonlinear eigenvalues $\overset{*}{\mu}_{k,1},\dots,\overset{*}{\mu}_{k,N}$. By Lemma 5.4.18 there exist corresponding radii $\theta_{k,1},\dots,\theta_{k,N}$ and constants $C_{k,1},\dots C_{k,N}$, all being positive, such that

$$\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \le \frac{C_{k,n}}{|\nu - \overset{*}{\mu}_{k,n}|} \qquad \text{for all } \nu \in \dot{U}_{\theta_{k,n}}(\overset{*}{\mu}_{k,n}) \text{ and } 1 \le n \le N. \quad (5.121)$$

Now let $r < \min\{\theta_{k,n} \mid 1 \le n \le N\}$ be so small that $\Gamma := \cup_{n=1}^{N} \Gamma_n$, where Γ_n is a circular contour around $\mathring{\mu}_{k,n}$ with radius r for $1 \le n \le N$, is admissible for σ_0. Then for $u \in \operatorname{Ran} Q_{k,\sigma_0}^{(F)}$ there is some $w \in L^2(\Omega)$ such that

$$u = -\frac{1}{2\pi i}\int_{\Gamma} \mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w\, d\nu = \sum_{n=1}^{N} -\frac{1}{2\pi i}\int_{\Gamma_n} \mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w\, d\nu =: \sum_{n=1}^{N} u_n,$$

taking parts (a) and (b) of our previous remarks into account. Since $\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w$ is an element of $D(\mathscr{L}_k(\nu)) = H^2_{k\text{-per}}(\Omega)$ for all $\nu \in \Gamma$, and since the operator pencil $\mathscr{L}_k$ maps into the closed operators on $L^2(\Omega)$, we obtain from part (b) of Proposition 2.3.6 and the remark thereafter

$$\mathscr{L}_k(\mathring{\mu}_{k,n})u_n = -\frac{1}{2\pi i}\int_{\Gamma_n} \mathscr{L}_k(\mathring{\mu}_{k,n})\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w\, d\nu \qquad \text{for } 1 \le n \le N, \tag{5.122}$$

i.e., the operators on the left-hand side may be taken under the integral sign here. To rearrange this further, note that

$$\begin{aligned}
&\mathscr{L}_k(\mathring{\mu}_{k,n})\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w \\
&\qquad = \big(-\Delta - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})I\big)\big(-\Delta - \nu\xi(\cdot,\nu)I\big)^{-1}F(\nu)w \\
&\qquad = F(\nu)w + \big(\nu\xi(\cdot,\nu) - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})\big)\big(-\Delta - \nu\xi(\cdot,\nu)I\big)^{-1}F(\nu)w \\
&\qquad = F(\nu)w + \big(\nu\xi(\cdot,\nu) - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})\big)\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w
\end{aligned}$$

for all $\nu \in \Gamma_n$ and $1 \le n \le N$. Therefore, the equations (5.122) can be rewritten as

$$\begin{aligned}
&\mathscr{L}_k(\mathring{\mu}_{k,n})u_n \qquad\qquad (5.123)\\
&= -\frac{1}{2\pi i}\int_{\Gamma_n} F(\nu)w\, d\nu - \frac{1}{2\pi i}\int_{\Gamma_n} \big(\nu\xi(\cdot,\nu) - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})\big)\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w\, d\nu \\
&= -\frac{1}{2\pi i}\int_{\Gamma_n} \big(\nu\xi(\cdot,\nu) - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})\big)\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)w\, d\nu \qquad \text{for } 1 \le n \le N,
\end{aligned}$$

where the last equality follows from the holomorphicity of F and Theorem 2.3.8 (Cauchy's integral theorem).

Next, we intend to show that the integrals on the right-hand side of (5.123) all vanish. To see this, we estimate the norm of the operators in the respective integrands. By a Taylor expansion of the function $\mathcal{S} \ni \nu \mapsto \nu\xi(\cdot,\nu)$ about $\mathring{\mu}_{k,n}$ and the estimate for the remainder term $R_0^{(\nu\xi)}(\nu,\mathring{\mu}_{k,n})$ provided by Lemma 5.4.16, together with the estimates (5.121), we find, for all $\nu \in \dot{U}_r(\mathring{\mu}_{k,n})$ and $1 \le n \le N$,

$$\begin{aligned}
&\big\|\big(\nu\xi(\cdot,\nu) - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})\big)\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)\big\|_{\mathcal{B}(L^2(\Omega))} \\
&\qquad \le \big\|R_0^{(\nu\xi)}(\nu,\mathring{\mu}_{k,n})\big\|_{L^\infty(\Omega)} \|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \|F(\nu)\|_{\mathcal{B}(L^2(\Omega))}
\end{aligned}$$

$$\begin{aligned}
&\leq |\nu - \overset{*}{\mu}_{k,n}| \max_{0\leq t\leq 1} \left\| \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,(1-t)\overset{*}{\mu}_{k,n}+t\nu) \right\|_{L^\infty(\Omega)} \\
&\qquad \cdot \frac{C_{k,n}}{|\nu-\overset{*}{\mu}_{k,n}|} \|F(\nu)\|_{\mathcal{B}(L^2(\Omega))} \\
&\leq C_{k,n} \max_{0\leq|\nu-\overset{*}{\mu}_{k,n}|\leq r} \left\| \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu) \right\|_{L^\infty(\Omega)} \\
&\qquad \cdot \max_{0\leq|\nu-\overset{*}{\mu}_{k,n}|\leq r} \|F(\nu)\|_{\mathcal{B}(L^2(\Omega))} = \text{const.},
\end{aligned} \tag{5.124}$$

using also the continuity of the involved functions. Hence, on the punctured disk $\dot{U}_r(\overset{*}{\mu}_{k,n})$ the holomorphic function $\nu \mapsto \big(\nu\xi(\cdot,\nu) - \overset{*}{\mu}_{k,n}\xi(\cdot,\overset{*}{\mu}_{k,n})\big)\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)$ is bounded for $1 \leq n \leq N$. Riemann's theorem on removable singularities and the residue theorem (see Theorem 2.3.12 and the corollary to Theorem 2.3.14) therefore imply that the integrals on the right-hand side of (5.123) are equal to zero, giving, as desired, $\mathscr{L}_k(\overset{*}{\mu}_{k,n})u_n = 0$ for $1 \leq n \leq N$. This finishes our proof, since we now obtain

$$u = \sum_{n=1}^{N} u_n \in \bigoplus_{n=1}^{N} \operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,n})) = \bigoplus_{\mu\in\sigma_0} \operatorname{Ker}(\mathscr{L}_k(\mu)) = \bigoplus_{\mu\in\sigma_0} \operatorname{Ker}(\mathscr{A}_k(\mu)),$$

as it is asserted in part (a) of the theorem. Note here that the last equality is a consequence of the definition of the operator pencil $\mathscr{L}_k$ (see (5.100) and (5.102)).

Ad (b): Similar to what we did right above, we renumber the λ-nonlinear eigenvalue contained in σ_0 as $\overset{*}{\mu}_{k,1}$. Besides, we reuse the radii $0 < r < \theta_{k,1}$, the constant $C_{k,1}$, and the admissible circular contour Γ for $\mathscr{L}_k$ and $\sigma_0 = \{\overset{*}{\mu}_{k,1}\}$, which we introduced in the first part of this proof (just set $N := 1$ there). In particular, we may employ the estimate (5.121) for $n = 1$.

In view of what we established in part (a), it remains to show the inclusion $\operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,1})) \subseteq \operatorname{Ran} Q^{(F)}_{k,\sigma_0}$. Thereto, let $\overset{*}{\psi} \in \operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,1}))$. For this eigenfunction we have

$$\begin{aligned}
\mathscr{L}_k(\nu)\overset{*}{\psi} &= \big(-\Delta - \nu\xi(\cdot,\nu)\big)\overset{*}{\psi} \\
&= \big(-\Delta - \overset{*}{\mu}_{k,1}\xi(\cdot,\overset{*}{\mu}_{k,1}) + \overset{*}{\mu}_{k,1}\xi(\cdot,\overset{*}{\mu}_{k,1}) - \nu\xi(\cdot,\nu)\big)\overset{*}{\psi} \\
&= \big(\overset{*}{\mu}_{k,1}\xi(\cdot,\overset{*}{\mu}_{k,1}) - \nu\xi(\cdot,\nu)\big)\overset{*}{\psi} \qquad \text{for all } \nu \in \Gamma,
\end{aligned}$$

giving, since $F(\overset{*}{\mu}_{k,1})$ is invertible by assumption,

$$\frac{1}{\nu - \overset{*}{\mu}_{k,1}}\overset{*}{\psi} = -\mathscr{R}_{\mathscr{L}_k}(\nu)F(\overset{*}{\mu}_{k,1})F(\overset{*}{\mu}_{k,1})^{-1}\frac{\nu\xi(\cdot,\nu) - \overset{*}{\mu}_{k,1}\xi(\cdot,\overset{*}{\mu}_{k,1})}{\nu - \overset{*}{\mu}_{k,1}}\overset{*}{\psi} \qquad \text{for all } \nu \in \Gamma.$$

Dividing both sides of this equality by $2\pi i$ and integrating counterclockwise along Γ, we obtain from Cauchy's integral formula

$$\overset{*}{\psi} = -\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu)F(\overset{*}{\mu}_{k,1})F(\overset{*}{\mu}_{k,1})^{-1}\,\frac{\nu\xi(\cdot,\nu) - \overset{*}{\mu}_{k,1}\xi(\cdot,\overset{*}{\mu}_{k,1})}{\nu - \overset{*}{\mu}_{k,1}}\,\overset{*}{\psi}\,d\nu. \tag{5.125}$$

To proceed, we make use of the identities

$$\begin{aligned}\frac{\nu\xi(\cdot,\nu)-\overset{*}{\mu}_{k,1}\xi(\cdot,\overset{*}{\mu}_{k,1})}{\nu-\overset{*}{\mu}_{k,1}} &= \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\overset{*}{\mu}_{k,1})+\frac{R_1^{(\nu\xi)}(\nu,\overset{*}{\mu}_{k,1})}{\nu-\overset{*}{\mu}_{k,1}} && \text{for all } \nu\in\Gamma,\\ F(\overset{*}{\mu}_{k,1}) &= F(\nu)+R_0^{(F)}(\overset{*}{\mu}_{k,1},\nu) && \text{for all } \nu\in\Gamma,\end{aligned}$$

which both follow from Lemma 5.4.16 on Taylor expansions of holomorphic Banach space-valued functions. Utilizing them, (5.125) reads

$$\begin{aligned}\overset{*}{\psi} &= -\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)F(\overset{*}{\mu}_{k,1})^{-1}\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\overset{*}{\mu}_{k,1})\overset{*}{\psi}\,d\nu\\ &\quad -\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)F(\overset{*}{\mu}_{k,1})^{-1}\frac{R_1^{(\nu\xi)}(\nu,\overset{*}{\mu}_{k,1})}{\nu-\overset{*}{\mu}_{k,1}}\overset{*}{\psi}\,d\nu\\ &\quad -\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu)R_0^{(F)}(\overset{*}{\mu}_{k,1},\nu)F(\overset{*}{\mu}_{k,1})^{-1}\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\overset{*}{\mu}_{k,1})\overset{*}{\psi}\,d\nu\\ &\quad -\frac{1}{2\pi i}\int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu)R_0^{(F)}(\overset{*}{\mu}_{k,1},\nu)F(\overset{*}{\mu}_{k,1})^{-1}\frac{R_1^{(\nu\xi)}(\nu,\overset{*}{\mu}_{k,1})}{\nu-\overset{*}{\mu}_{k,1}}\overset{*}{\psi}\,d\nu\\ &=: \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4.\end{aligned} \tag{5.126}$$

Let us now show that the integrals $\mathcal{I}_2$, $\mathcal{I}_3$, and $\mathcal{I}_4$ in fact vanish. To do so, we argue in the exact same way as in part (a) of this proof, namely by showing that the operator-valued functions in the integrands are uniformly bounded on $\dot{U}_r(\overset{*}{\mu}_{k,1})$ (see our precise reasoning above and note the remark following Lemma 5.4.16). As this is essentially the same as (5.124), we are a bit brief.

First, by the estimate (5.121) for $n=1$ and that proved as part of Lemma 5.4.16 we obtain for the integrand of $\mathcal{I}_2$

$$\begin{aligned}&\left\|\mathscr{R}_{\mathscr{L}_k}(\nu)F(\nu)F(\overset{*}{\mu}_{k,1})^{-1}\frac{R_1^{(\nu\xi)}(\nu,\overset{*}{\mu}_{k,1})}{\nu-\overset{*}{\mu}_{k,1}}I\right\|_{\mathcal{B}(L^2(\Omega))}\\ &\le \frac{C_{k,1}}{|\nu-\overset{*}{\mu}_{k,1}|}\|F(\nu)\|_{\mathcal{B}(L^2(\Omega))}\left\|F(\overset{*}{\mu}_{k,1})^{-1}\right\|_{\mathcal{B}(L^2(\Omega))}\\ &\qquad\cdot\frac{|\nu-\overset{*}{\mu}_{k,1}|}{2}\max_{0\le t\le 1}\left\|\frac{\partial^2[\nu'\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,(1-t)\overset{*}{\mu}_{k,1}+t\nu)\right\|_{L^\infty(\Omega)}\\ &\le \frac{C_{k,1}}{2}\left\|F(\overset{*}{\mu}_{k,1})^{-1}\right\|_{\mathcal{B}(L^2(\Omega))}\max_{0\le|\nu-\overset{*}{\mu}_{k,1}|\le r}\|F(\nu)\|_{\mathcal{B}(L^2(\Omega))}\\ &\qquad\cdot\max_{0\le|\nu-\overset{*}{\mu}_{k,1}|\le r}\left\|\frac{\partial^2[\nu'\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,\nu)\right\|_{L^\infty(\Omega)} = \text{const.} \qquad \text{for all } \nu\in\dot{U}_r(\overset{*}{\mu}_{k,1}),\end{aligned}$$

where, again, the continuity of the involved functions comes into play. This implies $\mathcal{I}_2 = 0$. Similarly, we deduce $\mathcal{I}_3 = \mathcal{I}_4 = 0$, since we have

$$\begin{aligned}
&\left\| \mathscr{R}_{\mathscr{L}_k}(\nu) R_0^{(F)}(\mathring{\mu}_{k,1},\nu) F(\mathring{\mu}_{k,1})^{-1} \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\mathring{\mu}_{k,1}) I \right\|_{\mathcal{B}(L^2(\Omega))} \\
&\quad \le \frac{C_{k,1}}{|\nu - \mathring{\mu}_{k,1}|} |\mathring{\mu}_{k,1} - \nu| \max_{0\le t\le 1} \left\| \frac{\partial[F(\nu')]}{\partial\nu'}((1-t)\nu + t\mathring{\mu}_{k,1}) \right\|_{L^\infty(\Omega)} \\
&\qquad \cdot \left\| F(\mathring{\mu}_{k,1})^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} \left\| \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\mathring{\mu}_{k,1}) \right\|_{L^\infty(\Omega)} \\
&\quad \le C_{k,1} \left\| F(\mathring{\mu}_{k,1})^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} \left\| \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\mathring{\mu}_{k,1}) \right\|_{L^\infty(\Omega)} \\
&\qquad \cdot \max_{0\le|\nu-\mathring{\mu}_{k,1}|\le r} \left\| \frac{\partial[F(\nu')]}{\partial\nu'}(\nu) \right\|_{L^\infty(\Omega)} = \text{const.} \qquad \text{for all } \nu \in \dot{U}_r(\mathring{\mu}_{k,1}),
\end{aligned}$$

as well as

$$\begin{aligned}
&\left\| \mathscr{R}_{\mathscr{L}_k}(\nu) R_0^{(F)}(\mathring{\mu}_{k,1},\nu) F(\mathring{\mu}_{k,1})^{-1} \frac{R_1^{(\nu\xi)}(\nu,\mathring{\mu}_{k,1})}{\nu - \mathring{\mu}_{k,1}} I \right\|_{\mathcal{B}(L^2(\Omega))} \\
&\le \frac{C_{k,1}}{|\nu - \mathring{\mu}_{k,1}|} |\mathring{\mu}_{k,1} - \nu| \max_{0\le t\le 1} \left\| \frac{\partial[F(\nu')]}{\partial\nu'}((1-t)\nu + t\mathring{\mu}_{k,1}) \right\|_{L^\infty(\Omega)} \left\| F(\mathring{\mu}_{k,1})^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} \\
&\quad \cdot \frac{|\nu - \mathring{\mu}_{k,1}|}{2} \max_{0\le t\le 1} \left\| \frac{\partial^2[\nu'\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,(1-t)\mathring{\mu}_{k,1} + t\nu) \right\|_{L^\infty(\Omega)} \\
&\le C_{k,1} \frac{r}{2} \left\| F(\mathring{\mu}_{k,1})^{-1} \right\|_{\mathcal{B}(L^2(\Omega))} \max_{0\le|\nu-\mathring{\mu}_{k,1}|\le r} \left\| \frac{\partial[F(\nu')]}{\partial\nu'}(\nu) \right\|_{L^\infty(\Omega)} \\
&\quad \cdot \max_{0\le|\nu-\mathring{\mu}_{k,1}|\le r} \left\| \frac{\partial^2[\nu'\xi(\cdot,\nu')]}{\partial\nu'^2}(\cdot,\nu) \right\|_{L^\infty(\Omega)} = \text{const.} \qquad \text{for all } \nu \in \dot{U}_r(\mathring{\mu}_{k,1}).
\end{aligned}$$

All in all, equation (5.126) now reads

$$\begin{aligned}
\mathring{\psi} &= -\frac{1}{2\pi i} \int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) F(\nu) F(\mathring{\mu}_{k,1})^{-1} \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\mathring{\mu}_{k,1}) \mathring{\psi} \, d\nu \\
&= Q_{k,\sigma_0}^{(F)} F(\mathring{\mu}_{k,1})^{-1} \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\mathring{\mu}_{k,1}) \mathring{\psi}.
\end{aligned}$$

Hence, $\mathring{\psi} \in \operatorname{Ran} Q_{k,\sigma_0}^{(F)}$ with preimage $F(\mathring{\mu}_{k,1})^{-1} \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\mathring{\mu}_{k,1}) \mathring{\psi} \in L^2(\Omega)$ and the proof is finished. □

Part (a) of the just proved result suggests to choose a sequence of λ-nonlinear Riesz integrals as $\{Q_m\}_{m\in\mathbb{N}}$ in our abstract completeness theorem (see Theorem 5.4.14 and our discussion on p. 110f.). Most importantly, in order to fulfill the closeness requirement (5.93) for this sequence, we are still free in choosing the operator-valued function F appearing in the corresponding integrands. However, we shall see later that even the constant mapping $\mathcal{S}_0 \ni \nu \mapsto F(\nu) := \widetilde{\xi} I$ is sufficient here. Nevertheless, we decided to include Theorem 5.4.22 in its given form, i.e., for general holomorphic functions F and with the additional part (b), since it is, to the best of our knowledge, a new contribution and should be of interest when other λ-nonlinear spectral problems are studied. This likewise applies to the result hereafter, which is solely provided for the interest of the reader and will not be applied in the remainder of this thesis. It addresses the mapping properties of the λ-nonlinear Riesz projections and also shows that their naming is justified.

Theorem 5.4.23. *Let $k \in B$ and let $P_{\overset{*}{\mu}_{k,n}}$ be the λ-nonlinear Riesz projection for $\mathscr{A}_k$ and $\overset{*}{\mu}_{k,n}$ as in part (c) of Definition 2.4.7. Then*

$$P^2_{\overset{*}{\mu}_{k,n}} = P_{\overset{*}{\mu}_{k,n}} \qquad \textit{and} \qquad \operatorname{Ran} P_{\overset{*}{\mu}_{k,n}} = \operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,n})) = \operatorname{Ker}(\mathscr{A}_k(\overset{*}{\mu}_{k,n})).$$

Remark. Note carefully that the first of the equalities concerning the range of the λ-nonlinear Riesz projection is not a consequence of part (b) of Theorem 5.4.22. This is due the invertibility requirement in the statement of said result, which is not necessarily satisfied here.

Proof of Theorem 5.4.23. We first show the asserted projection property. Let Γ_1 and Γ_2 be two admissible circular contours for $\mathscr{L}_k$ and $\sigma_0 = \{\overset{*}{\mu}_{k,n}\}$ with corresponding radii $0 < r_1 < r_2 < \theta_{k,n}$ (with $\theta_{k,n} = \theta$ as in Lemma 5.4.18), both being centered at $\overset{*}{\mu}_{k,n}$. Then, since the operator $P_{\overset{*}{\mu}_{k,n}}$ does not depend on the chosen admissible contour (see part (a) of the remarks following Definition 5.4.21),

$$\begin{aligned} P_{\overset{*}{\mu}_{k,n}} &= -\frac{1}{2\pi i}\int_{\Gamma_1} \mathscr{R}_{\mathscr{L}_k}(\nu_1)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I \, d\nu_1 \\ &= -\frac{1}{2\pi i}\int_{\Gamma_2} \mathscr{R}_{\mathscr{L}_k}(\nu_2)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_2)\right] I \, d\nu_2. \end{aligned} \tag{5.127}$$

A Taylor expansion as we used it several times before (see Lemma 5.4.16) yields

$$\begin{aligned} \mathscr{L}_k(\nu_2) - \mathscr{L}_k(\nu_1) &= \big(\nu_1\xi(\cdot,\nu_1) - \nu_2\xi(\cdot,\nu_2)\big) I \\ &= (\nu_1-\nu_2)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_2)\right] I + R_1^{(\nu\xi)}(\nu_1,\nu_2) I \end{aligned}$$

for all $\nu_1 \in \Gamma_1$ and all $\nu_2 \in \Gamma_2$. Therefore, by applying $\mathscr{R}_{\mathscr{L}_k}(\nu_2)$ from the left and $\mathscr{R}_{\mathscr{L}_k}(\nu_1)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I$ from the right, we find, for these ν_1 and ν_2,

$$\begin{aligned} &\mathscr{R}_{\mathscr{L}_k}(\nu_1)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I - \mathscr{R}_{\mathscr{L}_k}(\nu_2)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I \\ &\quad = (\nu_1-\nu_2)\left\{\mathscr{R}_{\mathscr{L}_k}(\nu_2)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_2)\right] I\right\}\left\{\mathscr{R}_{\mathscr{L}_k}(\nu_1)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I\right\} \\ &\qquad + \mathscr{R}_{\mathscr{L}_k}(\nu_2) R_1^{(\nu\xi)}(\nu_1,\nu_2)\mathscr{R}_{\mathscr{L}_k}(\nu_1)\left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I. \end{aligned}$$

Dividing both sides of the last equality by $\nu_1 - \nu_2$ and integrating counterclockwise along Γ_1 and Γ_2, we obtain with the help of Fubini's theorem (note that all integrands are continuous on $\Gamma_1 \times \Gamma_2$)

$$\begin{aligned}
&\int_{\Gamma_1} \mathscr{R}_{\mathscr{L}_k}(\nu_1) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] \left(\int_{\Gamma_2} \frac{1}{\nu_1 - \nu_2}\, d\nu_2\right) I\, d\nu_1 \\
&\qquad - \int_{\Gamma_2} \mathscr{R}_{\mathscr{L}_k}(\nu_2) \left(\int_{\Gamma_1} \frac{1}{\nu_1 - \nu_2} \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I\, d\nu_1\right)\, d\nu_2 \\
&= \left\{\int_{\Gamma_2} \mathscr{R}_{\mathscr{L}_k}(\nu_2) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_2)\right] I\, d\nu_2\right\} \qquad (5.128) \\
&\qquad\qquad \cdot \left\{\int_{\Gamma_1} \mathscr{R}_{\mathscr{L}_k}(\nu_1) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I\, d\nu_1\right\} \\
&\qquad + \int_{\Gamma_1} \left(\int_{\Gamma_2} \mathscr{R}_{\mathscr{L}_k}(\nu_2) \frac{R_1^{(\nu\xi)}(\nu_1,\nu_2)}{\nu_1 - \nu_2} I\, d\nu_2\right) \mathscr{R}_{\mathscr{L}_k}(\nu_1) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I\, d\nu_1.
\end{aligned}$$

We now consider the two integrals in round brackets on the left-hand side. First, Cauchy's integral formula yields

$$\int_{\Gamma_2} \frac{1}{\nu_1 - \nu_2}\, d\nu_2 = -2\pi i \qquad \text{for all } \nu_1 \in \Gamma_1,$$

since Γ_1 is in the inner domain of Γ_2. Similarly, Cauchy's integral theorem (as it is given in Theorem 2.3.8) implies

$$\int_{\Gamma_1} \frac{1}{\nu_1 - \nu_2} \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I\, d\nu_1 = 0 \qquad \text{for all } \nu_2 \in \Gamma_2,$$

since Γ_2 is in the outer domain of Γ_1. The latter two identities and (5.127) allow us to rewrite equation (5.128), divided by $(-2\pi i)^2$, as

$$\begin{aligned}
P_{\overset{*}{\mu}_{k,n}} = P^2_{\overset{*}{\mu}_{k,n}} + \int_{\Gamma_1} &\left(\int_{\Gamma_2} \mathscr{R}_{\mathscr{L}_k}(\nu_2) \frac{R_1^{(\nu\xi)}(\nu_1,\nu_2)}{\nu_1 - \nu_2} I\, d\nu_2\right) \qquad (5.129) \\
&\qquad \cdot \mathscr{R}_{\mathscr{L}_k}(\nu_1) \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial\nu'}(\cdot,\nu_1)\right] I\, d\nu_1.
\end{aligned}$$

Hence, in order to establish the claimed projection property, we have to show that the double integral on the right-hand side vanishes. This is once more an outcome of Riemann's theorem on removable singularities (Theorem 2.3.12) and a consequence—by the same arguments and estimates employed in the prior proof—of the following: Choose some $r < \min\{r_1, \theta_{k,n} - r_1\}$. Then, there holds $U_r(\nu_1) \subset U_{\theta_{k,n}}(\overset{*}{\mu}_{k,n})$ for all $\nu_1 \in \Gamma_1$, which allows us to make use of the estimate provided by Lemma 5.4.18 hereinafter. Furthermore, keeping $\nu_1 \in \Gamma_1$ fixed we have $|\nu - \overset{*}{\mu}_{k,n}| \geq r_1 - r > 0$ for all $\nu \in U_r(\nu_1)$. With this at hand, we estimate

$$\begin{aligned}
&\left\| \mathscr{R}_{\mathscr{L}_k}(\nu) \frac{R_1^{(\nu\xi)}(\nu_1,\nu)}{\nu_1 - \nu} I \right\|_{\mathcal{B}(L^2(\Omega))} \\
&\qquad \leq \frac{C_{k,n}}{|\nu - \overset{*}{\mu}_{k,n}|} \frac{|\nu_1 - \nu|}{2} \max_{0 \leq t \leq 1} \left\| \frac{\partial^2[\nu'\xi(\cdot,\nu')]}{\partial \nu'^2}(\cdot,(1-t)\nu + t\nu_1) \right\|_{L^\infty(\Omega)} \\
&\qquad \leq \frac{C_{k,n}}{r_1 - r} \frac{r}{2} \max_{0 \leq |\nu - \nu_1| \leq r} \left\| \frac{\partial^2[\nu'\xi(\cdot,\nu')]}{\partial \nu'^2}(\cdot,\nu) \right\|_{L^\infty(\Omega)} = \text{const.}
\end{aligned}$$

for all $\nu \in \dot{U}_r(\nu_1)$.

Since $\nu_1 \in \Gamma_1$ was arbitrary, this implies that the integral in round brackets on the right-hand side of (5.129) always vanishes and thus $P_{\overset{*}{\mu}_{k,n}} = P^2_{\overset{*}{\mu}_{k,n}}$.

Regarding the second assertion of the theorem, i.e.,

$$\operatorname{Ran} P_{\overset{*}{\mu}_{k,n}} = \operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,n})) = \operatorname{Ker}(\mathscr{A}_k(\overset{*}{\mu}_{k,n})),$$

we only have to show the inclusion $\operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,n})) \subseteq \operatorname{Ran} P_{\overset{*}{\mu}_{k,n}}$. That is because the equality of the two kernels follows from the definition of the respective operator pencils and furthermore because $\operatorname{Ran} P_{\overset{*}{\mu}_{k,n}} \subseteq \operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,n}))$ is a consequence of part (a) of Theorem 5.4.22. Note in this respect that for $\sigma_0 = \{\overset{*}{\mu}_{k,n}\}$ and

$$F(\nu) := \left[\frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial \nu'}(\cdot,\nu) \right] I \qquad \text{for all } \nu \in \mathcal{S}_0$$

we have $Q^{(F)}_{k,\sigma_0} = P_{\overset{*}{\mu}_{k,n}}$. So as to complete the proof, let $\overset{*}{\psi} \in \operatorname{Ker}(\mathscr{L}_k(\overset{*}{\mu}_{k,n}))$ and set $\Gamma := \Gamma_1$. Then, repeating our arguments given in the proof of part (b) of Theorem 5.4.22, we find

$$\begin{aligned}
\overset{*}{\psi} &= -\frac{1}{2\pi i} \int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) \frac{\nu\xi(\cdot,\nu) - \overset{*}{\mu}_{k,n}\xi(\cdot,\overset{*}{\mu}_{k,n})}{\nu - \overset{*}{\mu}_{k,n}} \overset{*}{\psi} \, d\nu \\
&= -\frac{1}{2\pi i} \int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial \nu'}(\cdot,\nu) \overset{*}{\psi} \, d\nu \\
&\qquad + \frac{1}{2\pi i} \int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) \frac{R_1^{(\nu\xi)}(\overset{*}{\mu}_{k,n},\nu)}{\nu - \overset{*}{\mu}_{k,n}} \overset{*}{\psi} \, d\nu \\
&= P_{\overset{*}{\mu}_{k,n}} \overset{*}{\psi} + \frac{1}{2\pi i} \int_\Gamma \mathscr{R}_{\mathscr{L}_k}(\nu) \frac{R_1^{(\nu\xi)}(\overset{*}{\mu}_{k,n},\nu)}{\nu - \overset{*}{\mu}_{k,n}} \overset{*}{\psi} \, d\nu
\end{aligned} \tag{5.130}$$

(compare to (5.125), (5.126), and the surroundings). We remark that the identity obtained by Taylor's theorem which we used here reads

$$\frac{\nu\xi(\cdot,\nu) - \overset{*}{\mu}_{k,1}\xi(\cdot,\overset{*}{\mu}_{k,1})}{\nu - \overset{*}{\mu}_{k,1}} = \frac{\partial[\nu'\xi(\cdot,\nu')]}{\partial \nu'}(\nu) - \frac{R_1^{(\nu\xi)}(\overset{*}{\mu}_{k,1},\nu)}{\nu - \overset{*}{\mu}_{k,1}} \qquad \text{for all } \nu \in \Gamma$$

and is thus slightly different, namely with ν and $\mathring{\mu}_{k,n}$ interchanged, than what we just referenced. Nevertheless, our by now well-known arguments imply that the second summand on the right-hand side of (5.130) vanishes, giving the desired equality $\mathring{\psi} = P_{\mathring{\mu}_{k,n}} \mathring{\psi}$ and thereby the completion of the proof. □

Remarks.

(a) In the previous two proofs we showed, always by means of the same method, that several contour integrals of holomorphic operator-valued functions vanish. This could also have been conducted differently and without the theorems from Subsection 2.3.2. For instance, so as to show that $\mathscr{L}_k(\mathring{\mu}_{k,n})u_n = 0$ for $1 \leq n \leq N$ in the proof of part (a) of Theorem 5.4.22, we can make use of the estimate stated as part (a) of Proposition 2.3.6 and otherwise proceed as before to obtain from equation (5.123) that

$$\begin{aligned}
&\left\| \mathscr{L}_k(\mathring{\mu}_{k,n})u_n \right\|_{L^2(\Omega)} \\
&\quad = \left\| \frac{1}{2\pi i} \int_{\Gamma_n} \big(\nu\xi(\cdot,\nu) - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})\big) \mathscr{R}_{\mathscr{L}_k}(\nu) F(\nu) w \, d\nu \right\|_{L^2(\Omega)} \\
&\quad \leq \frac{l(\Gamma_n)}{2\pi} \max_{\nu\in\Gamma_n} \left\| \big(\nu\xi(\cdot,\nu) - \mathring{\mu}_{k,n}\xi(\cdot,\mathring{\mu}_{k,n})\big) \mathscr{R}_{\mathscr{L}_k}(\nu) F(\nu) w \right\|_{L^2(\Omega)} \\
&\quad \leq rCr\frac{C_{k,n}}{r} \max_{\nu\in\Gamma_n} \|F(\nu)w\|_{L^2(\Omega)} \in \mathcal{O}(r) \quad \text{as } r \to 0 \qquad \text{for } 1 \leq n \leq N.
\end{aligned}$$

Since the right-hand side here can be made arbitrarily small by choosing r sufficiently small, we obtain, as desired, $\mathscr{L}_k(\mathring{\mu}_{k,n})u_n = 0$ for $1 \leq n \leq N$. Clearly, the independence of the λ-nonlinear Riesz integral on the chosen admissible contour is crucial for this argumentation to be valid.

(b) Under certain assumptions it is possible to generalize the λ-nonlinear Riesz projection addressed in the last theorem to eigenvalue problems of the form $Au = B(\lambda)u$. Here, A denotes a closed operator between Banach spaces and $U \ni \lambda \mapsto B(\lambda)$ is a bounded-operator-valued function which is holomorphic on some open set $U \subseteq \mathbb{C}$. For a λ-nonlinear eigenvalue λ_0 of this problem and a suitable surrounding contour $\Gamma \subset U$, the operator

$$P_{\lambda_0} := -\frac{1}{2\pi i} \int_\Gamma (A - B(\lambda))^{-1} B'(\lambda) \, d\lambda$$

is then a projection onto $\operatorname{Ker}\big(A - B(\lambda_0)\big)$, as it is to be expected from our own result. This is joint work with Prof. Dr. Michael Plum and a publication is in preparation as of this writing.

(c) It is easy to see that the Riesz projection of a closed operator deserves that name, i.e., is a projection, even for admissible contours that surround more than one eigenvalue (see Proposition 2.4.9). However, for the λ-nonlinear

counterpart as in the previous part of this remark (and also for our problem-specific variant) this is not the case. In fact, even an eigenvalue equation on a two-dimensional space suffices to demonstrate this phenomenon:
Let $a, b \in \mathbb{C} \setminus \{0\}$ and set

$$A := \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \qquad \text{and} \qquad B(\lambda) := \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} \quad \text{for all } \lambda \in \mathbb{C}.$$

Then, the eigenvalues of the spectral problem $Au = B(\lambda)u$ for $u \in \mathbb{C}^2$ are given by

$$\lambda_1 = -a, \quad \lambda_2 = +a, \quad \lambda_3 = -b, \quad \lambda_4 = +b$$

with corresponding eigenvectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Notably, there are different eigenvalues having the same eigenvector, which is clearly a consequence of the λ-nonlinear nature of the problem. Now, it is readily calculated that

$$\big(A - (B(\lambda)\big)^{-1} B'(\lambda) = \begin{pmatrix} \frac{2\lambda}{a^2 - \lambda^2} & 0 \\ 0 & \frac{2\lambda}{b^2 - \lambda^2} \end{pmatrix} \qquad \text{for all } \lambda \in \mathbb{C}$$

and an integration of this function along a Cauchy contour Γ_1, surrounding λ_1 once and no other eigenvalue, yields

$$P_1 := -\frac{1}{2\pi i} \int_{\Gamma_1} \big(A - (B(\lambda)\big)^{-1} B'(\lambda)\, d\lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, P_1 is a projection onto the eigenspace spanned by the eigenvector u_1. Similarly, with the obvious definitions of the occurring operators, we obtain

$$P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where, as expected, P_n is a projection onto the eigenspace spanned by the eigenvector u_n for $n = 2,3,4$. However, for a Cauchy contour Γ_σ surrounding the whole spectrum, i.e., all four eigenvalues $\lambda_1, \dots, \lambda_4$, once, we find

$$P_\sigma := -\frac{1}{2\pi i} \int_{\Gamma_\sigma} \big(A - (B(\lambda)\big)^{-1} B'(\lambda)\, d\lambda = \sum_{n=1}^{4} P_n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix},$$

which is not a projection. In particular, P_σ is not the identity, which is in contrast to what holds for a λ-linear problem.

With these remarks we end our discussion of the generalization of the Riesz projection to the λ-nonlinear Maxwell eigenvalue problem for TM-polarized waves. Right below, we move our considerations to the question of the completeness of the corresponding eigenfunctions again.

5.4.2.3 STATEMENT AND PROOF OF THE COMPLETENESS THEOREM

Since it was nearly 20 pages earlier that we outlined our strategy for proving the completeness of the λ-nonlinear Bloch waves in the asymptotically nondispersive case, we first repeat the essentials here. We do so by utilizing the newly found information from the previous paragraph and the discussion on p. 110f. to once more reformulate the theorem we intend to apply (Theorem 5.4.14) as follows:

Theorem 5.4.24 (An updated problem-specific variant of Theorem 5.4.11)**.** *Let $k \in B$ and let $N \in \mathbb{N}$ be such that $\mathring{\mu}_{k,N} < \mathring{\mu}_{k,N+1}$. Further, let $\{J_m\}_{m\in\mathbb{N}}$ be a partition of the set $\{n \in \mathbb{N} \mid n \geq N+1\}$ such that $\{\widetilde{\lambda}_{k,n} \mid n \in J_{m_1}\} \cap \{\widetilde{\lambda}_{k,n} \mid n \in J_{m_2}\} = \emptyset$ for all $m_1, m_2 \in \mathbb{N}$.*[1] *For all $m \in \mathbb{N}$ denote moreover by P_m the orthogonal projection in $H^1_{k\text{-per}}(\Omega)$, equipped with the norm $\|\cdot\|_{H^1_k,\widetilde{\xi}}$, onto the subspace $\operatorname{span}\{\widetilde{\psi}_{k,n} \mid n \in J_m\}$, i.e.,*

$$P_m := -\frac{1}{2\pi i}\int_{\Gamma_m^{(P)}} R_{\widetilde{A}_k}(\nu)\, d\nu\Big|_{H^1_{k\text{-per}}(\Omega)} : H^1_{k\text{-per}}(\Omega) \to \bigoplus_{n\in J_m} \operatorname{Ker}(\widetilde{A}_k - \widetilde{\lambda}_{k,n} I)$$

with an admissible contour $\Gamma_m^{(P)}$ for $\widetilde{A}_k$ and $\{\widetilde{\lambda}_{k,n} \mid n \in J_m\}$. Finally, for all $m \in \mathbb{N}$ let Q_m be the λ-nonlinear Riesz integral mapping into the subspace $\operatorname{span}\{\mathring{\psi}_{k,n} \mid n \geq N+1\}$ given by

$$Q_m := -\frac{1}{2\pi i}\int_{\Gamma_m^{(Q)}} \mathscr{R}_{\mathscr{L}_k}(\nu)\widetilde{\xi} I\, d\nu\Big|_{H^1_{k\text{-per}}(\Omega)} : H^1_{k\text{-per}}(\Omega) \to \bigoplus_{n\in J_m} \operatorname{Ker}(\mathscr{L}_k(\mathring{\mu}_{n,k}))$$

with an admissible contour $\Gamma_m^{(Q)}$ for $\mathscr{L}_k$ and $\{\mathring{\mu}_{k,n} \mid n \in J_m\}$. If

$$\left\| \sum_{m=1}^{\infty} (P_m - Q_m) \right\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} < 1, \tag{5.131}$$

where, implicitly, the convergence of the series is demanded, then

$$H^1_{k\text{-per}}(\Omega) = \operatorname{span}\{\mathring{\psi}_{k,n} \mid 1 \leq n \leq N\} \oplus \overline{\operatorname{span}\{\mathring{\psi}_{k,n} \mid n \geq N+1\}}.$$

In particular, the λ-nonlinear eigenfunctions $\{\mathring{\psi}_{k,n}\}_{n\in\mathbb{N}}$ are then complete in $H^1_{k\text{-per}}(\Omega)$.

When we apply the just stated result shortly afterwards, we only have to verify the closeness requirement (5.131), since our knowledge about the mapping properties of λ-nonlinear Riesz integrals, i.e., Theorem 5.4.22, has already been accounted for. Thereby, we are still free in choosing an arbitrarily large index N (see the remark following Theorem 5.4.14), the partition $\{J_m\}_{m\in\mathbb{N}}$, and the sequences $\{\Gamma_m^{(P)}\}_{m\in\mathbb{N}}$ and $\{\Gamma_m^{(Q)}\}_{m\in\mathbb{N}}$ of admissible contours. While these are a lot of parameters to work with, we will nevertheless have to further restrict the properties of the coefficient function ξ of our problem to be successful.

[1] If this were not required there could hold $\operatorname{span}\{\widetilde{\psi}_{k,n} \mid n \in J_m\} \subsetneq \bigoplus_{n\in J_m} \operatorname{Ker}(\widetilde{A}_k - \widetilde{\lambda}_{k,n} I)$ for some $m \in \mathbb{N}$, meaning that the orthogonal projection onto the left subspace would then not equal the Riesz projection P_m.

For our full assumptions on ξ are by now widely spread over the current section, we collect them conveniently in one place below. Note here that only parts (c.iv) and (c.v) have been newly added, whereas everything else is a plain word-by-word repetition of aforementioned requirements.

Assumptions 5.4.25 (Entire assumptions on ξ in the asymptotically nondispersive case)**.** *We suppose that $\xi \in C([0,\infty);L^\infty(\mathbb{R}^2;\mathbb{R}))$ is such that*

(a) the following physically-motivated assumptions *hold:*

(a.i) For all $\mu \in [0,\infty)$ the function $\xi(\cdot,\mu)$ is $\mathbb{Z}^2$-periodic, i.e.,

$$\xi(x,\mu) = \xi(x+a,\mu) \qquad \text{for a. a. } x \in \mathbb{R}^2 \text{ and all } a \in \mathbb{Z}^2.$$

(a.ii) For all $\mu \in [0,\infty)$ and some positive constant $\xi_{\min}$ we have

$$\xi_{\min} \leq \xi(x,\mu) \leq \|\xi(\cdot,\mu)\|_{L^\infty(\Omega)} =: \xi_{\max}(\mu) \qquad \text{for a. a. } x \in \mathbb{R}^2.$$

(a.iii) The function $\mu \mapsto \mu\xi(x,\mu)$ is strictly monotonically increasing on $(0,\infty)$ for almost all $x \in \Omega$.

(b) the following name-giving assumption of the asymptotically nondispersive case *holds:*

For some $\widetilde{\xi} \in L^\infty(\mathbb{R}^2;\mathbb{R})$ with

$$\widetilde{\xi}(x) = \widetilde{\xi}(x+a) \qquad \text{for a. a. } x \in \mathbb{R}^2 \text{ and all } a \in \mathbb{Z}^2$$

and, for some positive constant $\widetilde{\xi}_{\min}$,

$$\widetilde{\xi}_{\min} \leq \widetilde{\xi}(x) \leq \|\widetilde{\xi}\|_{L^\infty(\Omega)} =: \widetilde{\xi}_{\max} \qquad \text{for a. a. } x \in \mathbb{R}^2$$

we have

$$\left\| 1 - \frac{\xi(\cdot,\mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } \mu \to \infty.$$

(c) there exist $\widehat{\mu} > 0$ and $\kappa > 0$ such that the following high-frequency assumptions *hold:*

(c.i) The function $(\widehat{\mu},\infty) \ni \mu \mapsto \xi(\cdot,\mu)|_\Omega \in L^\infty(\Omega;\mathbb{R})$ has a holomorphic extension, which we again call ξ, to the semi-infinite strip

$$\mathcal{S} := \{\mu + i\tau \in \mathbb{C} \mid \mu > \widehat{\mu}, |\tau| < \kappa\}$$

with $\xi(\cdot,\nu) \in L^\infty(\Omega;\mathbb{C})$ for all $\nu \in \mathcal{S}$.

(c.ii) For some $q > 0$ we have

$$\mu \left\| \frac{1}{\xi(\cdot,\mu)} \frac{\partial[\xi(\cdot,\nu)]}{\partial\nu}(\cdot,\mu) \right\|_{L^\infty(\Omega)} \leq q < 1 \qquad \text{for all } \mu > \widehat{\mu}.$$

(c.iii) For some $C > 0$ we have

$$\mu \left\| \frac{\partial^2[\xi(\cdot,\nu)]}{\partial \nu^2}(\cdot,\mu+i\tau) \right\|_{L^\infty(\Omega)} \leq C \qquad \textit{for all } \mu+i\tau \in \mathcal{S}.$$

(c.iv) For some $D > 0$ we have

$$|\nu|^{3/2} \left\| 1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \leq D \qquad \textit{for all } \nu \in \mathcal{S}.$$

(c.v) There holds

$$\int_{\widehat{\mu}}^{\infty} \sup_{0<\tau<\kappa} \left(|\mu+i\tau|^{3/2} \left\| 1 - \frac{\xi(\cdot,\mu+i\tau)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \right) d\mu < \infty.$$

Remarks.

(a) An easy example of a function satisfying the entire assumptions of the asymptotically nondispersive case is given by

$$\xi(x,\mu) := \widetilde{\xi}(x)\left(1 - \frac{e^{-\mu}}{2}\right) \qquad \text{for all } x \in \mathbb{R}^2 \text{ and all } \mu \in [0,\infty) \qquad (5.132)$$

where $\widetilde{\xi} \in L^\infty(\mathbb{R}^2;\mathbb{R})$ is $\mathbb{Z}^2$-periodic as well as bounded away from zero and from above by positive constants $\xi_{\min}$ and $\xi_{\max}$, respectively. This is readily seen as follows:

For all $\mu \in [0,\infty)$ the function $\xi(\cdot,\mu)$ is $\mathbb{Z}^2$-periodic and $\frac{1}{2}\widetilde{\xi}_{\min} \leq \xi(x,\mu) \leq \widetilde{\xi}_{\max}$ for almost all $x \in \mathbb{R}^2$. Furthermore, the dependence of ξ on μ is continuous, the mapping $\mu \mapsto \mu\xi(x,\mu)$ is strictly monotonically increasing on $(0,\infty)$ for all $x \in \mathbb{R}^2$, and

$$\left\| 1 - \frac{\xi(\cdot,\mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} = \frac{e^{-\mu}}{2} \to 0 \qquad \text{as } \mu \to \infty.$$

Hence, parts (a) and (b) of Assumptions 5.4.25 hold. Besides, a holomorphic extension of ξ clearly exists—even to the whole complex plane—and retains the form (5.132) where the non-spatial argument can now be taken as an arbitrary complex number. For simplicity, and since this is all that is required, we nevertheless consider this extended function only on a semi-infinite strip $\mathcal{S}$ of width 2κ as before. With $\kappa := 1$ and $\widehat{\mu} > 0$ sufficiently large we find $\xi(\cdot,\nu) \in L^\infty(\Omega;\mathbb{C})$ for all $\nu \in \mathcal{S}$ and furthermore

$$\mu \left\| \frac{1}{\xi(\cdot,\mu)} \frac{\partial[\xi(\cdot,\nu)]}{\partial \nu}(\cdot,\mu) \right\|_{L^\infty(\Omega)} \leq \frac{2\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}} \mu \frac{e^{-\mu}}{2} \leq \frac{1}{2} < 1 \qquad \text{for all } \mu > \widehat{\mu}$$

as well as

$$\mu \left\| \frac{\partial^2[\xi(\cdot,\nu)]}{\partial\nu^2}(\cdot,\mu+i\tau) \right\|_{L^\infty(\Omega)} \\ \leq \frac{\widetilde{\xi}_{\max}}{2}\mu \left| e^{-(\mu+i\tau)} \right| = \frac{\widetilde{\xi}_{\max}}{2}\mu e^{-\mu} \leq \frac{\widetilde{\xi}_{\max}}{2}\widehat{\mu} e^{-\widehat{\mu}} =: C \qquad \text{for all } \mu+i\tau \in \mathcal{S}.$$

Finally, there holds

$$|\nu|^{3/2} \left\| 1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \\ \leq |\nu|^{3/2} \frac{|e^{-\nu}|}{2} \leq (\mathrm{Re}\,\nu+1)^{3/2} \frac{e^{-\mathrm{Re}\,\nu}}{2} \leq (\widehat{\mu}+1)^{3/2} \frac{e^{-\widehat{\mu}}}{2} =: D \qquad \text{for all } \nu \in \mathcal{S},$$

meaning that all but part (c.v) of the assumptions have now been verified. This remaining requirement, however, follows from

$$|\mu+i\tau|^{3/2} \left\| 1 - \frac{\xi(\cdot,\mu+i\tau)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \\ \leq (\mu+\kappa)^{3/2} \frac{\left| e^{-(\mu+i\tau)} \right|}{2} \leq (2\mu)^{3/2} \frac{e^{-\mu}}{2} \qquad \text{for all } \mu \geq \widehat{\mu} \text{ and all } \tau \in (0,\kappa)$$

together with the existence of the improper integral $\int_{\widehat{\mu}}^{\infty} \mu^{3/2} e^{-\mu}\, d\mu$ and thereby concludes our example.

(b) It is by no means necessary that a function satisfying Assumptions 5.4.25 is as simple as that presented in the previous part of this remark, i.e., of a product form, entire, and exponentially converging in the non-spatial variable. For instance, take for some $M \in \mathbb{N}$ functions $\widetilde{\xi}, \xi_1, \ldots, \xi_M \in L^\infty(\mathbb{R}^2;\mathbb{R})$ which are $\mathbb{Z}^2$-periodic and bounded away from zero and from above by positive constants $\widetilde{\xi}_{\min}$, $\xi_{n,\min}$ and $\widetilde{\xi}_{\max}$, $\xi_{n,\max}$, respectively, where $n = 1, \ldots, M$. In certain cases, which we shall specify in the course of this part of our remark, a function of the form

$$\xi(x,\mu) := \widetilde{\xi}(x) - \sum_{n=1}^{M} \frac{S_n(\mu)}{T_n(\mu)} \xi_n(x) \quad \text{for all } x \in \mathbb{R}^2 \text{ and all } \mu \in [0,\infty), \tag{5.133}$$

where S_n and T_n are real-valued polynomials on $[0,\infty)$ for $n = 1, \ldots, M$, does then satisfy the entire assumptions of the asymptotically nondispersive case.[1] Sufficient conditions for this to be true can be derived as follows:

Obviously, for all $\mu \in [0,\infty)$ the function $\xi(\cdot,\mu)$ is $\mathbb{Z}^2$-periodic and its dependence on μ is continuous as long as we require the polynomials T_n in the

[1] An important example of this type of function is given for the choice $\widetilde{\xi} \equiv \varepsilon_0$ (the vacuum permittivity) and $\xi_n|_\Omega(x) = \chi_{\Omega_n}$ for $n = 1, \ldots, M$, where $\Omega = \cup_{n=1}^{M} \Omega_n$ is a partition of the primitive cell Ω of a given photonic crystal (see Section 5.5). This spatially piecewise constant form accounts for the way in which the materials are usually built in practice (see Section 3.1).

denominators to have no zeros in $[0, \infty)$. The lower boundedness of ξ as in part (a.ii) of our assumptions has to be checked in concrete cases since it can only be satisfied for certain coefficients, signs, and degrees of the involved polynomials. The respective μ-dependent upper bounds, however, are easily seen to be themselves less than or equal to

$$\widetilde{\xi}_{\max} + M \max_{1 \le n \le M} \left(\frac{S_n(\mu)}{T_n(\mu)} \xi_{n,\max} \right).$$

Now, stating general conditions leading to a strictly monotonic dependence as it is required in part (a.iii) of Assumptions 5.4.25 is again not possible. In view of the differentiability of the involved polynomials, though, it has to be verified that

$$\widetilde{\xi}(x) > \sum_{n=1}^{M} \left[\frac{S_n(\mu)}{T_n(\mu)} + \mu \left(\frac{S_n}{T_n} \right)'(\mu) \right] \xi_n(x) \quad \text{for all } \mu \in (0, \infty) \text{ and a. a. } x \in \mathbb{R}^2.$$

Supposing that so far all requirements can be satisfied (see our specific example at the end of this remark), we turn our attention to the name-giving assumption of the asymptotically nondispersive case. For all $\mu \in [0, \infty)$ we have

$$\begin{aligned} \left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} &\le \sum_{n=1}^{M} \left| \frac{S_n(\mu)}{T_n(\mu)} \right| \left\| \frac{\xi_n}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \\ &\le \frac{1}{\widetilde{\xi}_{\min}} M \max_{1 \le n \le M} \left(\left| \frac{S_n(\mu)}{T_n(\mu)} \right| \xi_{n,\max} \right). \end{aligned} \tag{5.134}$$

Therefore, a sufficient condition for the left-hand side to converge to zero as $\mu \to \infty$ is

$$\frac{S_n(\mu)}{T_n(\mu)} \to 0 \quad \text{as } \mu \to \infty \qquad \text{for } 1 \le n \le M.$$

Of course, this is equivalent to requiring that the degrees of the involved polynomials satisfy $\deg S_n < \deg T_n$ for $1 \le n \le M$. While this finishes our discussion of parts (a) and (b) of our assumptions, part (c) remains. Note in this respect that real and imaginary parts of complex zeros of the polynomials T_n in the right half-plane determine the constants $\widehat{\mu}$ and κ that define the semi-infinite strip $\mathcal{S}$. In any case, it exists and $(\widehat{\mu}, \infty) \ni \mu \mapsto \xi(\cdot, \mu)|_\Omega \in L^\infty(\Omega; \mathbb{R})$ has an equally named extension, still of the form given in (5.133), satisfying $\xi(\cdot, \nu) \in L^\infty(\Omega; \mathbb{C})$ for all $\nu \in \mathcal{S}$. With this, let us address the requirements (c.ii)–(c.v) of our assumptions. First, for all $\mu > \widehat{\mu}$,

$$\mu \left\| \frac{1}{\xi(\cdot, \mu)} \frac{\partial[\xi(\cdot, \nu)]}{\partial \nu}(\cdot, \mu) \right\|_{L^\infty(\Omega)} \le \frac{1}{\xi_{\min}} M\mu \max_{1 \le n \le M} \left(\left| \left(\frac{S_n}{T_n} \right)'(\mu) \right| \xi_{n,\max} \right)$$

and the right-hand side is required to be bounded by a constant less than 1. Potentially enlarging $\widehat{\mu}$, this is particularly possible if

$$\mu\left(\frac{S_n}{T_n}\right)'(\mu) = \frac{\mu S_n'(\mu)}{T_n(\mu)} - \frac{\mu T_n'(\mu)}{T_n(\mu)}\frac{S_n(\mu)}{T_n(\mu)} \to 0 \quad \text{as } \mu \to \infty \qquad \text{for } 1 \le n \le M,$$

which holds under the same degree condition already derived above. This is likewise true in case of the requirement (c.iii), i.e.,

$$\mu\left|\left(\frac{S_n}{T_n}\right)''(\mu + i\tau)\right| \le C_n \quad \text{for all } \mu + i\tau \in \mathcal{S} \text{ and } 1 \le n \le M,$$

where C_n, for $n = 1, \dots, M$, are constants. To see this, note that

$$\nu\left(\frac{S_n}{T_n}\right)''(\nu) = \frac{\nu S_n''(\nu)}{T_n(\nu)} - 2\frac{\nu S_n'(\nu)T_n'(\nu)}{T_n^2(\nu)} - \frac{\nu T_n''(\nu)S_n(\nu)}{T_n^2(\nu)} + 2\frac{\nu(T_n')^2(\nu)S_n(\nu)}{T_n^3(\nu)}$$
$$\text{for all } \nu \in \mathcal{S} \text{ and } 1 \le n \le M$$

and carefully compare the degrees of the occurring polynomials. Next, so as to satisfy part (c.iv) of Assumptions 5.4.25, the so far strongest condition $\deg S_n \le \deg T_n - 2$ for $1 \le n \le M$ has to be imposed. Then, obviously,

$$|\nu|^{3/2}\left|\frac{S_n(\nu)}{T_n(\nu)}\right| \le D_n \quad \text{for all } \nu \in \mathcal{S} \text{ and } 1 \le n \le M,$$

where D_n, for $n = 1, \dots, M$, are constants. Finally, regarding the integrability condition in part (c.v) of Assumptions 5.4.25 we note that for all $\mu > \widehat{\mu}$

$$\begin{aligned}
&\sup_{0<\tau<\kappa}\left(|\mu + i\tau|^{3/2}\left\|1 - \frac{\xi(\cdot, \mu + i\tau)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)}\right) \\
&\qquad \le \frac{1}{\widetilde{\xi}_{\min}} M(\mu + \kappa)^{3/2} \sup_{0<\tau<\kappa} \max_{1\le n\le M}\left(\left|\frac{S_n(\mu + i\tau)}{T_n(\mu + i\tau)}\right| \xi_{n,\max}\right) \\
&\qquad \le C\mu^{3/2} \max_{1\le n\le M}\left(\xi_{n,\max} \sup_{0<\tau<\kappa}\left|\frac{S_n(\mu + i\tau)}{T_n(\mu + i\tau)}\right|\right)
\end{aligned}$$

for some positive constant C and sufficiently small κ (compare to (5.134)). Hence, in order for the left-hand side to be integrable over $[\widehat{\mu}, \infty)$, a fast enough decrease of the suprema on the right-hand side is sufficient, i.e.,

$$\sup_{0<\tau<\kappa}\left|\frac{S_n(\mu + i\tau)}{T_n(\mu + i\tau)}\right| \in \mathcal{O}\left(\frac{1}{\mu^3}\right) \quad \text{as } \mu \to \infty \qquad \text{for } 1 \le n \le M. \tag{5.135}$$

Here, as with all other requirements of part (c), it is important to remark that $\widehat{\mu}$ can be increased and κ can be decreased if need be. On account of (5.135), we see that the principal conditions that need to be satisfied such that a function of the form (5.133) fulfills the entire assumptions of the asymptotically nondispersive case read

$$\deg S_n \leq \deg T_n - 3 \quad \text{and} \quad T_n \neq 0 \text{ on } [0,\infty) \qquad \text{for } 1 \leq n \leq M.$$

For instance, a very simple suitable example is given by

$$\xi(x,\mu) := \widetilde{\xi}(x)\left(1 - \frac{1}{2+\mu^3}\right) \quad \text{for all } x \in \mathbb{R}^2 \text{ and all } \mu \in [0,\infty),$$

which is readily checked with the help of the comments made before. This shall close our remark.

Having clarified that Assumptions 5.4.25 are not too strong, we move on to state and proof the main theorem of this thesis.

Theorem 5.4.26 (Completeness in the asymptotically nondispersive case)**.** *Let Assumptions 5.4.25 hold. Then for all $k \in B$ the λ-nonlinear eigenfunctions $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}}$ of the operator pencil $\mathscr{A}_k$ are complete in $H^1_{k\text{-per}}(\Omega)$.*

Proof. Let $k \in B$ be fixed and, for a more readable notation in what follows, set

$$K_1 := \sqrt{3}E\frac{\widetilde{\xi}^2_{\max}}{\widetilde{\xi}_{\min}}\sqrt{\widetilde{\xi}_{\max} + \frac{\pi}{2}} \quad \text{and} \quad K_2 := \sqrt{\frac{4}{\pi^3}\frac{\widetilde{\xi}^{17/2}_{\max}}{\widetilde{\xi}^{11/2}_{\min}} + \frac{8}{\pi^4}\frac{\widetilde{\xi}^{11}_{\max}}{\widetilde{\xi}^7_{\min}}} \tag{5.136}$$

with E denoting the positive constant appearing on the right-hand side of the inequality in Lemma 5.4.17. In addition, define $\eta_m := \frac{\eta}{2^m}$ for all $m \in \mathbb{N}$ and a fixed number η such that

$$0 < \eta < \min\left\{1, \kappa_0, \frac{\pi}{4K_2 D}\right\}, \tag{5.137}$$

where $2\kappa_0$ is the width of $\mathcal{S}_0$ as in Lemma 5.4.17 (in particular, $\kappa_0 \leq \kappa$) and the remaining constants stem from Assumptions 5.4.25. Finally, without loss of generality, we suppose

$$\widehat{\mu} \geq \max\left\{1, D^{2/3}, \frac{4D^2}{\pi^2}\widetilde{\xi}^2_{\max}, \frac{4D^2}{\pi^2}\frac{\widetilde{\xi}^5_{\max}}{\widetilde{\xi}^3_{\min}}\right\}. \tag{5.138}$$

Keeping these preliminaries in mind, we employ Lemma 5.4.10 to obtain a sequence $\{G_m\}_{m\in\mathbb{N}}$ of intervals such that

$$G_m = [g_{m,0}, g_{m,1}] \subset \rho(\widetilde{A}_k), \quad g_{m,1} - g_{m,0} \geq \frac{2\pi}{\widetilde{\xi}_{\max}}, \quad g_{m+1,0} > g_{m,1} \quad \text{for all } m \in \mathbb{N},$$
$$g_{m,0} \to \infty \quad \text{as } m \to \infty. \tag{5.139}$$

Here we may assume that these spectral gaps are maximally extended in the sense that a gap of width $5\pi\widetilde{\xi}_{\max}^{-1}$, say, is considered as a single gap and not broken up into two neighboring ones. In particular, this allows us to infer that there is always at least one eigenvalue of $\widetilde{A}_k$ between two of these intervals.

Setting $h_m := \frac{1}{2}(g_{m,0} + g_{m,1})$ for all $m \in \mathbb{N}$ we now choose a subsequence of $\{G_m\}_{m\in\mathbb{N}}$, denoted the same, with $g_{1,0} > \widehat{\mu}$ such that the corresponding midpoints satisfy

$$\int_{h_m}^{\infty} |t + i\eta_m|^{3/2} \left\| 1 - \frac{\xi(\cdot, t + i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} dt \leq \frac{\pi\eta^2}{8K_1} \frac{1}{6^m} \quad \text{for all } m \in \mathbb{N}. \tag{5.140}$$

Note that this is possible by part (c.v) of Assumptions 5.4.25 and the Cauchy criterion for improper integrals. By possibly dropping the first finitely many gaps in this subsequence (and renaming the rest so as to start with G_1 again), we may furthermore assume that for some $N \in \mathbb{N}$ there holds $\widehat{\mu} < \mathring{\mu}_{k,N}, \widetilde{\lambda}_{k,N}$ and $\widetilde{\lambda}_{k,N} < g_{1,0} < g_{1,1} < \widetilde{\lambda}_{k,N+1}$. Recall in this respect that all eigenvalues in this proof are isolated, have finite multiplicities, and tend to infinity by Theorem 4.1.5 (with ε_r replaced by $\widetilde{\xi}$) and Theorem 5.2.6, respectively.

Keeping the gaps and the index N fixed for the rest of the proof, we first obtain by part (c.iv) of our assumptions and (5.138) that

$$\mu \left\| 1 - \frac{\xi(\cdot, \mu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \leq \frac{D}{\sqrt{\mu}} < \frac{D}{\sqrt{\widehat{\mu}}} \leq \frac{\pi}{2\widetilde{\xi}_{\max}} \qquad \text{for all } \mu \geq \mathring{\mu}_{k,N}. \tag{5.141}$$

In particular, since $\mathring{\mu}_{k,N} > \widehat{\mu} \geq D^{2/3}$, the norm on the left-hand side of (5.141) is less than or equal to one for all $\mu \geq \mathring{\mu}_{k,N}$. This implies, using Lemma 5.4.9, that the considered eigenvalues of the operator pencil $\mathscr{A}_k$ are close to those of the operator $\widetilde{A}_k$. More precisely,

$$|\mathring{\mu}_{k,n} - \widetilde{\lambda}_{k,n}| \leq \frac{\pi}{2\widetilde{\xi}_{\max}} \leq \frac{g_{m,1} - g_{m,0}}{4} \qquad \text{for all } n \geq N \text{ and all } m \in \mathbb{N}, \tag{5.142}$$

using the intermediate statement in (5.139). Since $\widetilde{\lambda}_{k,N}$ and $\widetilde{\lambda}_{k,N+1}$ are separated by the gap G_1 (as per our above choice), this yields $\mathring{\mu}_{k,N} < \mathring{\mu}_{k,N+1}$ and thereby the first of the assumptions of Theorem 5.4.24 which we intend to apply.

To proceed, we outline our selection of the partition and the sequences of admissible contours occurring in the statement of said theorem. In the associated notation, we put $J_m := \{n \in \mathbb{N} \mid g_{m,1} < \widetilde{\lambda}_{k,n} < g_{m+1,0}\}$ for all $m \in \mathbb{N}$. In other words, J_m contains the indices of all eigenvalues of the operator $\widetilde{A}_k$ which lie between the spectral gaps G_m and G_{m+1}. By the assumption mentioned right below (5.139) we have $J_m \neq \emptyset$ for all $m \in \mathbb{N}$, where these sets are finite, pairwise disjoint, and contain only eigenvalues greater than $\widetilde{\lambda}_{k,N}$. Furthermore, there holds $\cup_{m=1}^{\infty} J_m = \{n \in \mathbb{N} \mid n \geq N+1\}$ in view of the above-mentioned fact that $g_{m,0} \to \infty$ as $m \to \infty$. Clearly, we also have $\{\widetilde{\lambda}_{k,n} \mid n \in J_{m_1}\} \cap \{\widetilde{\lambda}_{k,n} \mid n \in J_{m_2}\} = \emptyset$ for all $m_1, m_2 \in \mathbb{N}$, meaning that the partition $\{J_m\}_{m\in\mathbb{N}}$ fulfills the requirements of Theorem 5.4.24.

Regarding the necessary sequences of admissible contours, we introduce for all $m \in \mathbb{N}$ the parametrizations

$$\begin{aligned} \gamma_{m,1}(t) &:= h_m - it && \text{for all } t \in [-\eta_m, \eta_m], \\ \gamma_{m,2}(t) &:= t - i\eta_m && \text{for all } t \in [h_m, h_{m+1}], \\ \gamma_{m,3}(t) &:= h_{m+1} + it && \text{for all } t \in [-\eta_m, \eta_m], \\ \gamma_{m,4}(t) &:= h_{m+1} + h_m - t + i\eta_m && \text{for all } t \in [h_m, h_{m+1}] \end{aligned} \tag{5.143}$$

and denote the associated rectangular Cauchy contours by $\Gamma_m := \cup_{l=1}^4 \Gamma_{m,l} \subset \mathcal{S}_k$. As is indicated, the latter all lie in the region $\mathcal{S}_k$, which follows from our above choice of η (see (5.137)) and the evident inequality $\eta_m < \eta$ for all $m \in \mathbb{N}$. For further clarification, we also provide a schematic illustration of Γ_1 and (partly) Γ_2 together with information on the adjacent eigenvalues in Figure 5.9 below.

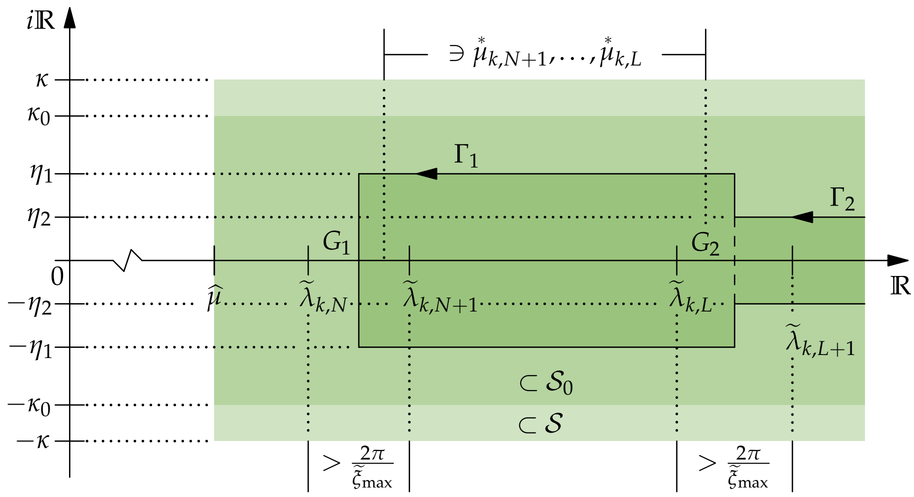

FIGURE 5.9 — A schematic illustration of the setting in the proof of Theorem 5.4.26. It depicts the first two spectral gaps as in (5.139) after the renumbering explained in the proof and also shows the corresponding contours Γ_1 and (partly) Γ_2, which overlap along the dashed line. In the region between the midpoints h_1 and h_2 of the marked spectral gaps G_1 and G_2 there are one or more eigenvalues of $\widetilde{A}_k$, i.e., $L \geq N+2$, and just as many of $\mathscr{L}_k$ (and thus of $\mathscr{A}_k$).

Obviously, the just introduced parametrizations satisfy, for all $m \in \mathbb{N}$,

$$\begin{aligned} \gamma'_{m,1}(t) &= -i, \quad \gamma'_{m,3}(t) = i && \text{for all } t \in [-\eta_m, \eta_m], \\ \gamma'_{m,2}(t) &= 1, \quad \gamma'_{m,4}(t) = -1 && \text{for all } t \in [h_m, h_{m+1}]. \end{aligned} \tag{5.144}$$

Furthermore, the contour Γ_m is admissible for $\widetilde{A}_k$ and $\{\widetilde{\lambda}_{k,n} \mid n \in J_m\}$ and, in fact, likewise admissible for $\mathscr{L}_k$ and $\{\overset{*}{\mu}_{k,n} \mid n \in J_m\}$, both for all $m \in \mathbb{N}$. That is, any of these contours encloses λ-linear and λ-nonlinear eigenvalues carrying the same indices n. To see this, it suffices to observe that (5.142) and

$$\operatorname{dist}\left(h_m, \sigma(\widetilde{A}_k)\right) > \frac{\pi}{\widetilde{\xi}_{\max}} \qquad \text{for all } m \in \mathbb{N} \tag{5.145}$$

imply that $h_m < \widetilde{\lambda}_{k,n} < h_{m+1}$ if and only if $h_m < \overset{*}{\mu}_{k,n} < h_{m+1}$ for all $n \geq N$ and all $m \in \mathbb{N}$. Hence, still in the notation of Theorem 5.4.24, the therein employed Riesz projections and λ-nonlinear Riesz integrals take the form

$$\left.\begin{aligned} P_m &= -\frac{1}{2\pi i}\int_{\Gamma_m} R_{\widetilde{A}_k}(\nu)\, d\nu, \\ Q_m &= -\frac{1}{2\pi i}\int_{\Gamma_m} \mathscr{R}_{\mathscr{L}_k}(\nu)\widetilde{\xi} I\, d\nu, \end{aligned}\right\} \qquad \text{for all } m \in \mathbb{N},$$

where these operators are defined on $H^1_{k\text{-per}}(\Omega)$ equipped with the norm $\|\cdot\|_{H^1_k,\widetilde{\xi}}$.

With these extensive preparations at hand, it remains to show the closeness requirement (5.131), i.e., as we repeat,

$$\left\| \sum_{m=1}^{\infty} (P_m - Q_m) \right\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} < 1. \tag{5.146}$$

In fact, we will even prove that the series is absolutely convergent, wherefore the fact that we were able to choose just one sequence of admissible contours for both the λ-linear and the λ-nonlinear problem is of great help. Before we come to this point, however, we introduce for all $m \in \mathbb{N}$ the abbreviations $\Gamma_m^{\perp} := \Gamma_{m,1} \cup \Gamma_{m,3}$ as well as $\Gamma_m^{\parallel} := \Gamma_{m,2} \cup \Gamma_{m,4}$ and furthermore set

$$\Gamma_\infty := \bigcup_{m\in\mathbb{N}} \Gamma_m, \quad \Gamma_\infty^{\perp} := \bigcup_{m\in\mathbb{N}} \Gamma_m^{\perp}, \quad \text{and} \quad \Gamma_\infty^{\parallel} := \bigcup_{m\in\mathbb{N}} \Gamma_m^{\parallel},$$

thereby splitting the set of all points on the admissible contours Γ_m into two (non-disjoint) subsets: One containing the points on the line segments parallel to the real axis and the other containing the points on the (open) line segments of length $2\eta_m$ perpendicular to the real axis. We shall see below why this separation is necessary and for now only note that (5.145) and $\sigma(\widetilde{A}_k) \subset \mathbb{R}$ imply

$$\begin{aligned} &\operatorname{dist}\big(\nu, \sigma(\widetilde{A}_k)\big) \geq \eta_m && \text{for all } \nu \in \Gamma_m^{\parallel} \text{ and all } m \in \mathbb{N}, \\ &\operatorname{dist}\big(\nu, \sigma(\widetilde{A}_k)\big) > \frac{\pi}{\widetilde{\xi}_{\max}} && \text{for all } \nu \in \Gamma_m^{\perp} \text{ and all } m \in \mathbb{N}. \end{aligned} \tag{5.147}$$

To begin with our estimation of the norm in (5.146), we put

$$W(\nu) := R_{\widetilde{A}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu)\widetilde{\xi} I \qquad \text{for all } \nu \in \Gamma_\infty$$

and use the properties (5.144) of the parametrizations defined in (5.143) to deduce that for all $m \in \mathbb{N}$

$$\begin{aligned} P_m - Q_m &= -\frac{1}{2\pi i}\sum_{l=1}^{4}\int_{\Gamma_{m,l}} W(\nu)\, d\nu \\ &= -\frac{1}{2\pi i}\left[\int_{-\eta_m}^{\eta_m} \gamma_{m,1}'(t)\, W(\gamma_{m,1}(t))\, dt + \int_{h_m}^{h_{m+1}} \gamma_{m,2}'(t)\, W(\gamma_{m,2}(t))\, dt\right. \end{aligned}$$

$$
\begin{aligned}
&\qquad + \int_{-\eta_m}^{\eta_m} \gamma_{m,3}'(t)\, W(\gamma_{m,3}(t))\, dt + \int_{h_m}^{h_{m+1}} \gamma_{m,4}'(t)\, W(\gamma_{m,4}(t))\, dt \Bigg] \\
&= \frac{1}{2\pi} \left(\int_{-\eta_m}^{\eta_m} W(h_m - it)\, dt - \int_{-\eta_m}^{\eta_m} W(h_{m+1} + it)\, dt \right) \\
&\qquad - \frac{1}{2\pi i} \left(\int_{h_m}^{h_{m+1}} W(t - i\eta_m)\, dt - \int_{h_m}^{h_{m+1}} W(t + i\eta_m)\, dt \right).
\end{aligned}
$$

Here, the very last integral is obtained by means of a simple substitution of the integration variable. From this it follows, again for all $m \in \mathbb{N}$,

$$
\begin{aligned}
&\|P_m - Q_m\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \\
&\le \frac{1}{2\pi} \Bigg(\int_{-\eta_m}^{\eta_m} \|W(h_m - it)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))}\, dt + \int_{-\eta_m}^{\eta_m} \|W(h_{m+1} + it)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))}\, dt \\
&\qquad + \int_{h_m}^{h_{m+1}} \|W(t - i\eta_m)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))}\, dt + \int_{h_m}^{h_{m+1}} \|W(t + i\eta_m)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))}\, dt \Bigg) \\
&\le \frac{1}{2\pi} \Bigg[2\eta_m \left(\max_{\nu \in \Gamma_{m,1}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} + \max_{\nu \in \Gamma_{m,3}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \right) \\
&\qquad + \int_{h_m}^{h_{m+1}} \|W(t - i\eta_m)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))}\, dt + \int_{h_m}^{h_{m+1}} \|W(t + i\eta_m)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))}\, dt \Bigg],
\end{aligned}
\tag{5.148}
$$

owing to part (a) of Proposition 2.3.6. In order to further estimate the right-hand side, we note that for all $\nu \in \Gamma_\infty$ the identity

$$
\big(-\Delta - \nu\xi(\cdot,\nu) I \big) - \left(-\Delta - \nu\widetilde{\xi} I \right) = \big(\nu\widetilde{\xi} - \nu\xi(\cdot,\nu) \big) I,
$$

which is valid on $H^2_{k\text{-per}}(\Omega)$, gives

$$
\begin{aligned}
&\left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} - \big(-\Delta - \nu\xi(\cdot,\nu) I \big)^{-1} \\
&\qquad = \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} \big(\nu\widetilde{\xi} - \nu\xi(\cdot,\nu) \big) \big(-\Delta - \nu\xi(\cdot,\nu) I \big)^{-1}
\end{aligned}
$$

by an application of the operators $\left(-\Delta - \nu\widetilde{\xi} I \right)^{-1}$ and $\big(-\Delta - \nu\xi(\cdot,\nu) I \big)^{-1}$ from the left and from the right, respectively. This yields

$$
\begin{aligned}
W(\nu) &= R_{\widetilde{A}_k}(\nu) - \mathscr{R}_{\mathscr{L}_k}(\nu)\widetilde{\xi} I \\
&= \left[\left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} - \big(-\Delta - \nu\xi(\cdot,\nu) I \big)^{-1} \right] \widetilde{\xi} I \\
&= \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} \big(\nu\widetilde{\xi} - \nu\xi(\cdot,\nu) \big) \big(-\Delta - \nu\xi(\cdot,\nu) I \big)^{-1} \widetilde{\xi} I \\
&= R_{\widetilde{A}_k}(\nu) \left[\nu \left(1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}} \right) \right] \mathscr{R}_{\mathscr{L}_k}(\nu) \widetilde{\xi} I \qquad \text{for all } \nu \in \Gamma_\infty.
\end{aligned}
\tag{5.149}
$$

While, as stated, this chain of equalities is valid for all points in Γ_∞, we will not make use of it on the subset $\Gamma_\infty^\perp$, the precise reason for this becoming apparent further below.[1]

Either way, the following preliminary information regarding the operator norm in $H^1_{k\text{-per}}(\Omega)$ is important: We obtain by means of partial integration that

$$\begin{aligned}
\|u\|^2_{H^1_k,\widetilde{\xi}} &= \langle \nabla u, \nabla u\rangle_{L^2(\Omega)} + \langle u,u\rangle_{\widetilde{\xi}} \\
&= \langle -\Delta u, u\rangle_{L^2(\Omega)} - \left\langle \nu\widetilde{\xi}u - \nu\widetilde{\xi}u, u\right\rangle_{L^2(\Omega)} + \left\langle \widetilde{\xi}u, u\right\rangle_{L^2(\Omega)} \\
&= \left\langle \big(-\Delta - \nu\widetilde{\xi}I\big)u, u\right\rangle_{L^2(\Omega)} + \left\langle (1+\nu)\widetilde{\xi}u, u\right\rangle_{L^2(\Omega)}
\end{aligned} \tag{5.150}$$

for all $u \in H^2_{k\text{-per}}(\Omega)$ and all $\nu \in \Gamma_\infty$.

Now, for all $u \in H^1_{k\text{-per}}(\Omega)$ (in fact, even for all $u \in L^2(\Omega)$) and all $\nu \in \Gamma_\infty$ we have $W(\nu)u \in H^2_{k\text{-per}}(\Omega)$ by the mapping properties of the involved operators (see (5.149)). Hence, using the last-mentioned identity we obtain

$$\begin{aligned}
&\|W(\nu)u\|^2_{H^1_k,\widetilde{\xi}} \qquad (5.151)\\
&= \left\langle (-\Delta - \nu\widetilde{\xi}I)W(\nu)u, W(\nu)u\right\rangle_{L^2(\Omega)} + \left\langle (1+\nu)\widetilde{\xi}W(\nu)u, W(\nu)u\right\rangle_{L^2(\Omega)} \\
&\le \left(\left\|(-\Delta - \nu\widetilde{\xi}I)W(\nu)u\right\|_{L^2(\Omega)} + |1+\nu| \left\|\widetilde{\xi}W(\nu)u\right\|_{L^2(\Omega)}\right) \|W(\nu)u\|_{L^2(\Omega)}.
\end{aligned}$$

for all $u \in H^1_{k\text{-per}}(\Omega)$ and all $\nu \in \Gamma_\infty$.

Let us use the right-hand side of (5.149) to address the three occurring norms separately. There holds, for u and ν as in the line above,

$$\begin{aligned}
&\left\|(-\Delta - \nu\widetilde{\xi}I)W(\nu)u\right\|_{L^2(\Omega)} \\
&\quad = \left\|(-\Delta - \nu\widetilde{\xi}I)R_{\widetilde{A}_k}(\nu)\left[\nu\left(1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right)\right]\mathscr{R}_{\mathscr{L}_k}(\nu)\widetilde{\xi}u\right\|_{L^2(\Omega)} \\
&\quad = \left\|\widetilde{\xi}\left[\nu\left(1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right)\right]\mathscr{R}_{\mathscr{L}_k}(\nu)\widetilde{\xi}u\right\|_{L^2(\Omega)} \\
&\quad \le \widetilde{\xi}^2_{\max}|\nu| \left\|1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)} \|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))}\|u\|_{L^2(\Omega)}
\end{aligned}$$

as well as

$$\begin{aligned}
\left\|\widetilde{\xi}W(\nu)u\right\|_{L^2(\Omega)} \le \widetilde{\xi}^2_{\max}|\nu| \left\|1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)} &\left\|R_{\widetilde{A}_k}(\nu)\right\|_{\mathcal{B}(L^2(\Omega))} \\
&\cdot \|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))}\|u\|_{L^2(\Omega)}
\end{aligned}$$

[1] Strictly speaking, we will not make use of it on the subset $\Gamma_\infty^\perp \setminus \Gamma_\infty^\parallel$ (recall that these sets are not disjoint).

and, finally,

$$\|W(\nu)u\|_{L^2(\Omega)} \le \widetilde{\xi}_{\max}|\nu| \left\|1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)} \left\|R_{\widetilde{A}_k}(\nu)\right\|_{\mathcal{B}(L^2(\Omega))} \cdot \|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \|u\|_{L^2(\Omega)} .$$

Combining the latter three estimates with (5.151) and

$$\begin{aligned}\left\|R_{\widetilde{A}_k}(\nu)\right\|_{\mathcal{B}(L^2(\Omega))} &\le \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \left\|R_{\widetilde{A}_k}(\nu)\right\|_{\mathcal{B}(L^2_{\widetilde{\xi}}(\Omega))} \\ &= \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \frac{1}{\operatorname{dist}(\nu,\sigma(\widetilde{A}_k))} \qquad \text{for all } \nu \in \Gamma_\infty,\end{aligned} \tag{5.152}$$

which is a consequence of Theorem 2.4.6, the self-adjointness of $\widetilde{A}_k$ on the weighted space $L^2_{\widetilde{\xi}}(\Omega)$, and the upper and lower bounds on $\widetilde{\xi}$, yields

$$\begin{aligned}&\|W(\nu)u\|^2_{H^1_k,\widetilde{\xi}} \\ &\le \widetilde{\xi}^3_{\max}|\nu|^2 \left\|1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|^2_{L^\infty(\Omega)} \|\mathscr{R}_{\mathscr{L}_k}(\nu)\|^2_{\mathcal{B}(L^2(\Omega))} \left\|R_{\widetilde{A}_k}(\nu)\right\|_{\mathcal{B}(L^2(\Omega))} \\ &\qquad \cdot \left(1 + |1+\nu| \left\|R_{\widetilde{A}_k}(\nu)\right\|_{\mathcal{B}(L^2(\Omega))}\right) \|u\|^2_{L^2(\Omega)} \\ &\le \widetilde{\xi}^3_{\max} \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \frac{1}{\operatorname{dist}(\nu,\sigma(\widetilde{A}_k))} |\nu|^2 \left\|1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|^2_{L^\infty(\Omega)} \|\mathscr{R}_{\mathscr{L}_k}(\nu)\|^2_{\mathcal{B}(L^2(\Omega))} \\ &\qquad \cdot \left(1 + |1+\nu| \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \frac{1}{\operatorname{dist}(\nu,\sigma(\widetilde{A}_k))}\right) \|u\|^2_{L^2(\Omega)}\end{aligned} \tag{5.153}$$

for all $u \in H^1_{k\text{-per}}(\Omega)$ and all $\nu \in \Gamma_\infty$.

As a next step, we restrict our considerations to $\nu \in \Gamma^{\|}_\infty$. Using the first line of (5.147) as well as the obvious inequalities $\widetilde{\xi}_{\min} \le \widetilde{\xi}_{\max}$, $\eta_m < 1$ for all $m \in \mathbb{N}$, and $|\nu| \ge \operatorname{Re}\nu > \widehat{\mu} > 1$ for all $\nu \in \Gamma_\infty$ (note the inequality (5.138)) we get for all $\nu \in \Gamma^{\|}_m$ and all $m \in \mathbb{N}$

$$\begin{aligned}\left(1 + |1+\nu| \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \frac{1}{\operatorname{dist}(\nu,\sigma(\widetilde{A}_k))}\right) &\le \left(1 + |1+\nu| \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \frac{1}{\eta_m}\right) \\ &\le (2+|\nu|) \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \frac{1}{\eta_m} \le 3|\nu| \sqrt{\frac{\widetilde{\xi}_{\max}}{\widetilde{\xi}_{\min}}} \frac{1}{\eta_m}.\end{aligned}$$

Along with

$$\|u\|^2_{L^2(\Omega)} \le \frac{1}{\widetilde{\xi}_{\min}} \|u\|^2_{\widetilde{\xi}} \le \frac{1}{\widetilde{\xi}_{\min}} \|u\|^2_{H^1_k,\widetilde{\xi}} \qquad \text{for all } u \in H^1_{k\text{-per}}(\Omega) \tag{5.154}$$

we thus deduce from (5.153), again employing the first line of (5.147),

$$\|W(\nu)u\|^2_{H^1_k,\widetilde{\xi}} \leq 3\frac{\widetilde{\xi}^4_{\max}}{\widetilde{\xi}^2_{\min}}\frac{1}{\eta_m^2}|\nu|^3\left\|1-\frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|^2_{L^\infty(\Omega)}\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|^2_{\mathcal{B}(L^2(\Omega))}\|u\|^2_{H^1_k,\widetilde{\xi}}$$

$$\text{for all } u \in H^1_{k\text{-per}}(\Omega),\text{ all } \nu \in \Gamma^{\|}_m, \text{ and all } m \in \mathbb{N}. \tag{5.155}$$

A similar estimate holds likewise for all $\nu \in \Gamma^{\perp}_{\infty}$, where only the constant differs due to the second line of (5.147). This, however, is of no use to us since for the yet to be estimated operator norm of $\mathscr{R}_{\mathscr{L}_k}(\nu)$ we unfortunately cannot go back to an upper bound in terms of the distance of ν to the spectrum of $\mathscr{L}_k$. More precisely, we cannot assure that the estimate provided by Lemma 5.4.18, which is in fact of the mentioned form, is applicable for $\nu \in \Gamma_\infty$, as it only holds locally around λ-nonlinear eigenvalues. Nevertheless, what we do have at our disposal is Lemma 5.4.17, which states that for the positive constant E appearing in (5.136) we have

$$\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \leq \sqrt{\xi_{\max}(\operatorname{Re}\nu)}\frac{E}{|\operatorname{Im}\nu|} \qquad \text{for all non-real } \nu \in \Gamma_\infty. \tag{5.156}$$

Of course, on $\Gamma^{\|}_\infty$ this estimate is employable. For such ν we have by (5.141)

$$\begin{aligned}\xi_{\max}(\operatorname{Re}\nu) = \|\xi(\cdot,\operatorname{Re}\nu)\|_{L^\infty(\Omega)} &= \left\|\widetilde{\xi}-\widetilde{\xi}\left(1-\frac{\xi(\cdot,\operatorname{Re}\nu)}{\widetilde{\xi}}\right)\right\|_{L^\infty(\Omega)}\\ &\leq \widetilde{\xi}_{\max}+\widetilde{\xi}_{\max}\left\|1-\frac{\xi(\cdot,\operatorname{Re}\nu)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)}\\ &\leq \widetilde{\xi}_{\max}+\frac{\pi}{2\operatorname{Re}\nu} \leq \widetilde{\xi}_{\max}+\frac{\pi}{2},\end{aligned}$$

using, as on the previous page, $\operatorname{Re}\nu > 1$ for all $\nu \in \Gamma_\infty$. Hence,

$$\|\mathscr{R}_{\mathscr{L}_k}(\nu)\|_{\mathcal{B}(L^2(\Omega))} \leq \sqrt{\widetilde{\xi}_{\max}+\frac{\pi}{2}}\,\frac{E}{\eta_m} \qquad \text{for all } \nu \in \Gamma^{\|}_m \text{ and all } m \in \mathbb{N},$$

which implies, together with (5.155),

$$\begin{aligned}&\|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))}\\ &\quad\leq \sqrt{3}E\frac{\widetilde{\xi}^2_{\max}}{\widetilde{\xi}_{\min}}\sqrt{\widetilde{\xi}_{\max}+\frac{\pi}{2}}\frac{1}{\eta_m^2}|\nu|^{3/2}\left\|1-\frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)}\\ &\quad= K_1\frac{1}{\eta_m^2}|\nu|^{3/2}\left\|1-\frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)} \qquad \text{for all } \nu \in \Gamma^{\|}_m \text{ and all } m \in \mathbb{N},\end{aligned} \tag{5.157}$$

where we remind the reader of the definition of the constant K_1 in (5.136).

With this, let us get back to our previous estimate (5.148) for the norm-difference between the mth Riesz projection and the mth λ-nonlinear Riesz integral. Upon using (5.157) and the definition of the sequence $\{\eta_m\}_{m\in\mathbb{N}}$, we find, for all $m \in \mathbb{N}$,

$$\begin{aligned}
&\|P_m - Q_m\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \\
&\quad \le \frac{\eta_m}{\pi} \max_{\nu\in\Gamma_{m,1}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} + \frac{\eta_m}{\pi} \max_{\nu\in\Gamma_{m,3}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \\
&\qquad + \frac{K_1}{2\pi}\frac{1}{\eta_m^2}\int_{h_m}^{h_{m+1}} |t - i\eta_m|^{3/2} \left\| 1 - \frac{\xi(\cdot, t - i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} dt \\
&\qquad + \frac{K_1}{2\pi}\frac{1}{\eta_m^2}\int_{h_m}^{h_{m+1}} |t + i\eta_m|^{3/2} \left\| 1 - \frac{\xi(\cdot, t + i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} dt \qquad (5.158) \\
&\quad \le \frac{\eta}{\pi}\frac{1}{2^m} \max_{\nu\in\Gamma_{m,1}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} + \frac{\eta}{\pi}\frac{1}{2^m} \max_{\nu\in\Gamma_{m,3}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \\
&\qquad + \frac{K_1}{2\pi\eta^2}4^m \int_{h_m}^{\infty} |t - i\eta_m|^{3/2} \left\| 1 - \frac{\xi(\cdot, t - i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} dt \\
&\qquad + \frac{K_1}{2\pi\eta^2}4^m \int_{h_m}^{\infty} |t + i\eta_m|^{3/2} \left\| 1 - \frac{\xi(\cdot, t + i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} dt.
\end{aligned}$$

To proceed, we note that, obviously, $|t - i\eta_m|^{3/2} = |t + i\eta_m|^{3/2}$ for all $t \in [h_m, \infty)$ and all $m \in \mathbb{N}$ as well as, likewise for these t and m,

$$\left\| 1 - \frac{\xi(\cdot, t - i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} = \left\| 1 - \frac{\overline{\xi(\cdot, t - i\eta_m)}}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} = \left\| 1 - \frac{\xi(\cdot, t + i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)}.$$

These equalities hold since $\widetilde{\xi}$ is a real-valued function and since $\overline{\xi(\cdot,\nu)} = \xi(\cdot,\overline{\nu})$ for all $\nu \in \mathcal{S}$ as a consequence of $\xi(\cdot,\mu) \in L^\infty(\mathbb{R}^2;\mathbb{R})$ for all $\mu \in \mathbb{R}$.[1] Therefore, both last-mentioned integrals are equal and can be bounded from above with the help of (5.140), i.e., our choice of the sequence $\{G_m\}_{m\in\mathbb{N}}$ of spectral gaps. With the just referenced inequality we get from (5.158), for all $m \in \mathbb{N}$,

$$\begin{aligned}
&\|P_m - Q_m\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \\
&\quad \le \frac{\eta}{\pi}\frac{1}{2^m} \max_{\nu\in\Gamma_{m,1}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} + \frac{\eta}{\pi}\frac{1}{2^m} \max_{\nu\in\Gamma_{m,3}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \\
&\qquad + \frac{K_1}{\pi\eta^2}4^m \int_{h_m}^{\infty} |t + i\eta_m|^{3/2} \left\| 1 - \frac{\xi(\cdot, t + i\eta_m)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} dt \qquad (5.159) \\
&\quad \le \frac{\eta}{\pi}\frac{1}{2^m} \max_{\nu\in\Gamma_{m,1}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} + \frac{\eta}{\pi}\frac{1}{2^m} \max_{\nu\in\Gamma_{m,3}} \|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} + \frac{1}{8}\left(\frac{2}{3}\right)^m.
\end{aligned}$$

[1] Note that $U_\eta(\mu) \subset \mathcal{S}$ for all $\mu \in \mathcal{S} \cap \mathbb{R}$ and consider the power series expansion (having real-valued coefficients) of $\xi(\cdot,\nu)$ about $\operatorname{Re}\nu$.

Next, we devote ourselves to the yet untouched first two terms on the right-hand side of the last inequality. Here we are concerned with $\nu \in \Gamma_\infty^\perp$, meaning, as outlined before, that our previous estimate of the operator norm of $W(\nu)$ is of no help here (note the right-hand side of (5.156)). Instead, we have to reconsider the operator itself, which requires some preliminary considerations: With the help of the upper and lower bounds on $\widetilde{\xi}$ and the estimate

$$\left\| R_{\widetilde{A}_k}(\nu) \right\|_{\mathcal{B}(L^2(\Omega))} \leq \frac{1}{\pi} \frac{\widetilde{\xi}_{\max}^{3/2}}{\sqrt{\widetilde{\xi}_{\min}}} \qquad \text{for all } \nu \in \Gamma_\infty^\perp, \tag{5.160}$$

which is a consequence of (5.152) and the second line of (5.147), we arrive at

$$\begin{aligned} &\left\| \nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right\|_{\mathcal{B}(L^2(\Omega))} \\ &\quad \leq \frac{1}{\widetilde{\xi}_{\min}} |\nu| \left\| \xi(\cdot,\nu) - \widetilde{\xi} \right\|_{L^\infty(\Omega)} \left\| R_{\widetilde{A}_k}(\nu) \right\|_{\mathcal{B}(L^2(\Omega))} \\ &\quad \leq \frac{1}{\pi} \frac{\widetilde{\xi}_{\max}^{5/2}}{\widetilde{\xi}_{\min}^{3/2}} |\nu| \left\| 1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \\ &\quad \leq \frac{1}{\pi} \frac{\widetilde{\xi}_{\max}^{5/2}}{\widetilde{\xi}_{\min}^{3/2}} \frac{D}{\sqrt{|\nu|}} \leq \frac{1}{2} \qquad \text{for all } \nu \in \Gamma_\infty^\perp, \end{aligned} \tag{5.161}$$

where the last line is due to part (c.iv) of Assumptions 5.4.25, $|\nu| > \widehat{\mu}$ for all $\nu \in \Gamma_\infty^\perp$, and the lower bound for $\widehat{\mu}$ on the right-hand side of (5.138).

The just derived inequality justifies the expansion in a Neumann series right below. Starting from the second line of (5.149), we find

$$\begin{aligned} &W(\nu) \\ &\quad = \left[\left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} - \left(-\Delta - \nu\xi(\cdot,\nu) I \right)^{-1} \right] \widetilde{\xi} I \\ &\quad = \left\{ \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} - \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} \left[I - \nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} \right]^{-1} \right\} \widetilde{\xi} I \\ &\quad = \left\{ \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} - \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} \left[I - \nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^{-1} \right\} \widetilde{\xi} I \\ &\quad = \left\{ \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} - \left(-\Delta - \nu\widetilde{\xi} I \right)^{-1} \sum_{j=0}^{\infty} \left[\nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^j \right\} \widetilde{\xi} I \\ &\quad = -R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} \sum_{j=1}^{\infty} \left[\nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^j \widetilde{\xi} I \qquad \text{for all } \nu \in \Gamma_\infty^\perp. \end{aligned}$$

An upper bound for the operator norm of the series on the right-hand side shall be important for us. Using (5.160) and (5.161), it can be obtained as follows:

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} \left[\nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^j \right\|_{\mathcal{B}(L^2(\Omega))} \\
&\qquad \le \sum_{j=1}^{\infty} \left\| \nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right\|_{\mathcal{B}(L^2(\Omega))}^j \\
&\qquad = \frac{\left\| \nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right\|_{\mathcal{B}(L^2(\Omega))}}{1 - \left\| \nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right\|_{\mathcal{B}(L^2(\Omega))}} \\
&\qquad \le \frac{\left\| \nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) \right\|_{L^\infty(\Omega)} \left\| R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right\|_{\mathcal{B}(L^2(\Omega))}}{1 - \frac{1}{2}} \\
&\qquad \le \frac{2}{\pi} \frac{\widetilde{\xi}_{\max}^{5/2}}{\widetilde{\xi}_{\min}^{3/2}} |\nu| \left\| 1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}} \right\|_{L^\infty(\Omega)} \qquad \text{for all } \nu \in \Gamma_\infty^\perp.
\end{aligned}$$

The last estimate is helpful in order to bound images of functions in $H^1_{k\text{-per}}(\Omega)$ under $W(\nu)$ in norm. Together with (5.151) and (5.160) it allows us to deduce an important estimate:

$$\begin{aligned}
&\|W(\nu)u\|^2_{H^1_k,\widetilde{\xi}} \\
&\le \left(\left\| (-\Delta - \nu\widetilde{\xi} I) W(\nu) u \right\|_{L^2(\Omega)} + |1+\nu| \left\| \widetilde{\xi} W(\nu) u \right\|_{L^2(\Omega)} \right) \|W(\nu)u\|_{L^2(\Omega)} \\
&= \Bigg(\left\| \sum_{j=1}^{\infty} \left[\nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^j \widetilde{\xi} u \right\|_{L^2(\Omega)} \\
&\qquad + |1+\nu| \left\| \widetilde{\xi} R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} \sum_{j=1}^{\infty} \left[\nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^j \widetilde{\xi} u \right\|_{L^2(\Omega)} \Bigg) \\
&\qquad\quad \cdot \left\| R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} \sum_{j=1}^{\infty} \left[\nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^j \widetilde{\xi} u \right\|_{L^2(\Omega)} \\
&\le \left\| \sum_{j=1}^{\infty} \left[\nu\big(\xi(\cdot,\nu) - \widetilde{\xi}\big) R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right]^j \right\|^2_{\mathcal{B}(L^2(\Omega))} \\
&\qquad \cdot \left(1 + |1+\nu| \left\| \widetilde{\xi} R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right\|_{\mathcal{B}(L^2(\Omega))} \right) \left\| R_{\widetilde{A}_k}(\nu) \frac{1}{\widetilde{\xi}} I \right\|_{\mathcal{B}(L^2(\Omega))} \left\| \widetilde{\xi} u \right\|^2_{L^2(\Omega)} \\
&\le \frac{4}{\pi^2} \frac{\widetilde{\xi}_{\max}^5}{\widetilde{\xi}_{\min}^3} |\nu|^2 \left\| 1 - \frac{\xi(\cdot,\nu)}{\widetilde{\xi}} \right\|^2_{L^\infty(\Omega)} \left(1 + |1+\nu| \frac{1}{\pi} \frac{\widetilde{\xi}_{\max}^{5/2}}{\widetilde{\xi}_{\min}^{3/2}} \right) \frac{1}{\pi} \frac{\widetilde{\xi}_{\max}^{7/2}}{\widetilde{\xi}_{\min}^{3/2}} \|u\|^2_{L^2(\Omega)}
\end{aligned}$$

$$\leq \frac{4}{\pi^2}\frac{\widetilde{\xi}^5_{\max}}{\widetilde{\xi}^3_{\min}}|\nu|^2\left\|1-\frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|^2_{L^\infty(\Omega)}\left(\frac{1}{\pi}\frac{\widetilde{\xi}^{7/2}_{\max}}{\widetilde{\xi}^{5/2}_{\min}}+|1+\nu|\frac{1}{\pi^2}\frac{\widetilde{\xi}^6_{\max}}{\widetilde{\xi}^4_{\min}}\right)\|u\|^2_{H^1_k,\widetilde{\xi}}$$

$$\leq \frac{4}{\pi^2}\frac{\widetilde{\xi}^5_{\max}}{\widetilde{\xi}^3_{\min}}|\nu|^3\left\|1-\frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|^2_{L^\infty(\Omega)}\left(\frac{1}{\pi}\frac{\widetilde{\xi}^{7/2}_{\max}}{\widetilde{\xi}^{5/2}_{\min}}+\frac{2}{\pi^2}\frac{\widetilde{\xi}^6_{\max}}{\widetilde{\xi}^4_{\min}}\right)\|u\|^2_{H^1_k,\widetilde{\xi}}$$

$$= \left(\frac{4}{\pi^3}\frac{\widetilde{\xi}^{17/2}_{\max}}{\widetilde{\xi}^{11/2}_{\min}}+\frac{8}{\pi^4}\frac{\widetilde{\xi}^{11}_{\max}}{\widetilde{\xi}^7_{\min}}\right)|\nu|^3\left\|1-\frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|^2_{L^\infty(\Omega)}\|u\|^2_{H^1_k,\widetilde{\xi}}$$

for all $u \in H^1_{k\text{-per}}(\Omega)$ and all $\nu \in \Gamma^\perp_\infty$,

employing (5.154) and $|\nu| > 1$ for all $\nu \in \Gamma^\perp_\infty$ so as to obtain the last two inequalities. Moreover, on the very right-hand side we recognize the constant K_2 defined at the beginning of our proof (see (5.136)) and therefore find, by part (c.iv) of our assumptions,

$$\|W(\nu)\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \leq K_2|\nu|^{3/2}\left\|1-\frac{\xi(\cdot,\nu)}{\widetilde{\xi}}\right\|_{L^\infty(\Omega)} \leq K_2 D \qquad \text{for all } \nu \in \Gamma^\perp_\infty.$$

Finally, by applying the latter estimate to the right-hand side of (5.159) it follows

$$\|P_m - Q_m\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} \leq 2K_2 D\frac{\eta}{\pi}\frac{1}{2^m}+\frac{1}{8}\left(\frac{2}{3}\right)^m \qquad \text{for all } m \in \mathbb{N}.$$

Hence, by our choice of η (see (5.137)),

$$\begin{aligned}\sum_{m=1}^\infty \|P_m - Q_m\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} &\leq 2K_2 D\frac{\eta}{\pi}\sum_{m=1}^\infty\frac{1}{2^m}+\frac{1}{8}\sum_{m=1}^\infty\left(\frac{2}{3}\right)^m \\ &= 2K_2 D\frac{\eta}{\pi}+\frac{1}{4} \leq \frac{3}{4} < 1,\end{aligned}$$

giving

$$\left\|\sum_{m=1}^\infty (P_m - Q_m)\right\|_{\mathcal{B}(H^1_{k\text{-per}}(\Omega))} < 1.$$

As outlined before, this finishes our proof with the help of Theorem 5.4.24. □

By means of the corollary to Theorem 5.4.14 and its surrounding arguments, we readily obtain the final result in this work.

Corollary. *Let Assumptions 5.4.25 hold. Then for all $k \in B$ the λ-nonlinear eigenfunctions $\{\overset{*}{\psi}_{k,n}\}_{n\in\mathbb{N}}$ of the operator pencil $\mathscr{A}_k$ are complete in $L^2(\Omega)$.*

With our completeness theorem for the λ-nonlinear Bloch waves in certain asymptotically nondispersive cases we end our discussion of the eigenfunctions of the λ-nonlinear Maxwell eigenvalue problem. A brief literature review on spectral problems for dispersive photonic crystals in general shall round out this dissertation.

5.5 RELATED WORK IN THE LITERATURE

Contrary to the λ-linear eigenvalue problem (4.3) for a nondispersive photonic crystal, its λ-nonlinear counterpart (5.4) has not received considerable attention in the literature to date. While this makes for an excellent starting point of a dissertation discussing the basic properties of this spectral problem, an extensive literature review can naturally not be given. Nevertheless, some few related publications exist and shall be addressed in the current section. However, before we discuss these articles we note that the questions posed in Section 4.2 (see p. 59) are likewise worthwhile to be asked in the dispersive case. More precisely, under assumptions such as those of Theorem 5.2.6 it is clearly interesting to know if the then-existing spectral band-gap structure of the operator pencil $\mathscr{A}$ actually features any gaps and, if so, whether there are finitely or infinitely many of them. Besides, it remains to be examined under what assumptions on the λ-nonlinearity of the problem spectral gaps of $\mathscr{A}$ can be guaranteed. Both these issues are highly important to applications and have to be addressed in further research.

Regarding the relevant contributions that do already exist in the literature it stands out that often frequency-dependencies of the relative permittivity ε_{r} (and thus of its rescaled variant ξ we worked with), which are not covered by what we discussed above, are assumed. In mathematical as well as physical texts this usually means that the Lorentz model discussed in Subsection 3.3.3 is applied in a spatially piecewise constant form. More precisely, the corresponding authors use the ansatz

$$\varepsilon_{\mathrm{r}}(x,\omega) := 1 + \frac{C}{\varepsilon_0} \sum_{n=1}^{N} \left(\sum_{m=1}^{M_n} \frac{f_{nm}}{\omega_{nm}^2 - \omega^2 - i\omega\gamma_{nm}} \right) \chi_{\Omega_n}(x) \tag{5.162}$$

for all ω in a considered range of frequencies and all $x \in \Omega = \cup_{n=0}^{N} \Omega_n \subseteq \mathbb{R}^d$, which is the partitioned underlying d-dimensional primitive cell. Here, the set Ω_0 models air or vacuum, i.e., $\varepsilon_{\mathrm{r}}(x,\cdot) \equiv 1$ for x therein, and the real-valued constants C, f_{nm}, ω_{nm}, and γ_{nm} are material-dependent (see the paragraph right below (3.41) for further details). In fact, this is often even further simplified by restricting the analysis to two-component media having one reasonance frequency only, i.e., $N = M_1 = 1$. In any case, this function does not fall within the scope of our model, since it is either complex-valued, or, if all γ_{ij} vanish and the studied material is lossless, it has singularities contradicting our boundedness assumption on ε_{r}.

A Lorentzian permittivity of the just-mentioned form is, in particular, considered in the articles [EHS09], [EW10], [EEK12], [ER09], and [Eng10]—all authored by C. Engström and collaborators. The first three of these papers focus solely on suitable numerical approaches to calculate approximate eigenvalues and shall thus not be covered in our review here. The remaining two publications, on the other hand, also provide analytical insights, which we now briefly summarize. We remark that, to the best of our knowledge, there exist no other non-numerical studies concerning an eigenvalue problem of the form (5.4) in the mathematical literature as of this writing.

In both [ER09] and [Eng10] it attracts attention that the spectrum of the λ-nonlinear problem (5.4) is said to equal the union of parameter-dependent spectra of corresponding problems posed on the primitive cell. In other words, the validity of Proposition 5.1.2 is assumed, but instead of a proof it is referred to [Kuc93, Thm. 4.5.1], which actually only applies to λ-linear problems and cannot be simply adapted to the generalized case. It is unclear whether the authors overlooked this subtlety or deemed the transfer of the presumed spectral equality to the λ-nonlinear setting obvious.[1] In any case, in view of our own result we know that their reduction of the problem to the family of equations

$$-(\nabla + ik) \cdot (\nabla + ik)u = \omega^2 \varepsilon_r(\cdot, \omega)u \qquad \text{in } \Omega, \tag{5.163}$$

where $k \in B$ and a solution is supposed to be in $H^2_{\mathrm{per}}(\Omega)$, is valid.[2] However, the papers differ in the way that these equations are approached:

In [ER09] a rather novel way of studying (5.163) is pursued by understanding the equation as being parametrized by the time frequency ω and not by the quasimomentum vector k. So as to work with a scalar spectral variable, the authors introduce the amplitude λ of k, i.e., $k = \lambda k_0$ for some unit vector $k_0 \in \mathbb{R}^2$, and readily reformulate the problem as

$$\left[\lambda^2 I - \lambda(2ik_0 \cdot \nabla) - \left(\Delta + \omega^2 \varepsilon_{\mathrm{r}}(\cdot, \omega) I\right)\right] u = 0.$$

Hence, regardless of the actual frequency-dependence of the relative permittivity, eigenvalue problems for a family of quadratic operator pencils arise as a result. The main theorem in [ER09] states that their spectrum is purely discrete, provided that the coefficient function ε_{r} satisfies slightly more than our basic Assumptions 5.1.1.[3] In addition, it is explained in the article how the particular choice of the spectral variable can provide information about spectral gaps (as per the common definition) and the mentioned theoretical findings are backed up by numerical results for a simple real-valued permittivity of the form (5.162) with $N = M_1 = 1$.

Finally, in the remaining of the above-mentioned papers, [Eng10], the choice of the spectral variable again agrees with ours, i.e., $\lambda = {\omega^2}/{c_0^2}$. On the other hand, the restrictions on ε_{r} are a lot stronger. The function is assumed to fulfill our basic Assumptions 5.1.1, has to be holomorphic for fixed $x \in \mathbb{R}^2$ in some domain of the complex plane, and needs to be spatially piecewise constant. It is important to remark that the holomorphicity requirement is crucial to the arguments presented in the article. With its help, the author models the weak formulation of the problem by a family of operators depending holomorphically on the frequency

[1] After all, the proof is not that hard (see the referenced proposition on p. 67).

[2] Note that the occurring operators are k-dependent while the boundary conditions are not. In the λ-linear setting of Chapter 4 we also briefly discussed such an equivalent formulation (see (4.18) and the surrounding discussion).

[3] Due to the difference in the studied spectral variable this result is not connected to Theorem 5.2.6 (which gives the discreteness of the spectrum of the operator pencils $\mathscr{A}_k$).

and deduces the discreteness of their spectra using a Fredholm-type theorem from the literature. This result is complemented by an interesting lemma giving the physically reasonable absence of nonzero real eigenvalues whenever the material is absorptive on a set of positive measure, i.e., the imaginary part of ε_r is positive there. Similar to [ER09] the paper concludes with a section in which the permittivity is further restricted to the form (5.162) with $N = M_1 = 1$. In contrast, though, both the real- and the complex-valued case are analyzed. Since the focus of said section lies on certain numerical aspects again, we refer the reader to the original source for further details.

While our review shows that a few results concerning the spectral structure of the eigenvalue problem (5.4)—or rather of special cases of the related problems on the primitive cell—existed before we started working on this topic ourselves, we did not comment on studies discussing the eigenfunctions. This is simply due to the fact that nothing related seems to have been published yet. The general treatises of the properties of eigenfunctions of polynomial, rational, and holomorphic operator pencils in [Rod89] and [Mar88] might be applicable for permittivity functions of the above-mentioned Lorentzian form. However, this has likewise not been rigorously analyzed and thus remains an open problem. Similarly, nothing is known so far regarding a completeness result for the λ-nonlinear Bloch waves in $L^2(\mathbb{R}^2)$, which would generalize Theorem 4.1.9 to the dispersive case. All in all, there remain several interesting problems to hopefully be tackled in further research on the spectral properties of dispersive photonic crystals.

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SYMBOLIC NOTATIONS

In the table below we list (mostly) nonstandard symbols which are used throughout this work. The indicated page and equation numbers guide the reader as close as possible to a suitable explanation or a formal definition of a symbol, but not necessarily to the place of its first usage.[1] The parameters appearing below satisfy $s \in \{1,2\}$, $n \in \mathbb{N}$, $k \in [-\pi,\pi]^2$, and $\mu \in [0,\infty)$.

A	The operator realizing the spectral problem for nondispersive photonic crystals on $\mathbb{R}^2$	see (4.5) on p. 49
A_k	The operator realizing the spectral problem for nondispersive photonic crystals on the primitive cell with k-quasi periodic boundary conditions	see (4.13) on p. 52
$\widetilde{A}_k$	The operator realizing the spectral problem for nondispersive photonic crystals on the primitive cell with coefficient function $\widetilde{\xi}$ and k-quasi periodic boundary conditions	see (5.53) on p. 92
$\widehat{A}_k$	The "k-shifted" variant of the operator A_k which acts on periodic functions	see (4.18) on p. 54
A_μ	The operator realizing the spectral problem for dispersive photonic crystals on $\mathbb{R}^2$ with coefficient function $\xi(\cdot,\mu)$	see (5.7) on p. 65
$A_{\mu,k}$	The operator realizing the spectral problem for nondispersive photonic crystals on the primitive cell with coefficient function $\xi(\cdot,\mu)$ and k-quasi periodic boundary conditions	see (5.9) on p. 66

[1] This is particularly important in the case of A and $\mathscr{A}$, since these two symbols are only fixed in their meaning from Chapter 4 onwards.

$\mathscr{A}$	The operator pencil realizing the spectral problem for dispersive photonic crystals on $\mathbb{R}^2$	see (5.8) on p. 66
$\mathscr{A}_k$	The operator pencil realizing the spectral problem for dispersive photonic crystals on the primitive cell with k-quasi periodic boundary conditions	see (5.11) on p. 67
a	The sesquilinear form associated with the operator A	see (4.6) on p. 49
a_k	The sesquilinear form associated with the operator A_k	see (4.17) on p. 53
$\widehat{a}_k$	The sesquilinear form associated with the operator $\widehat{A}_k$	see (4.19) on p. 54
B	The Brillouin zone $[-\pi,\pi]^2$ of the reciprocal lattice corresponding to the Bravais lattice $\mathbb{Z}^2$	see (4.10) on p. 50
B_{Θ^*}	The Brillouin zone of a reciprocal lattice Θ^*	see (3.5) on p. 31
$H^s_{\mathrm{per}}(\Omega)$	The Sobolev space of functions in $H^s_{\mathrm{loc}}(\mathbb{R}^2)$ which satisfy periodic boundary conditions on $\partial\Omega$	see (4.18) on p. 54
$H^s_{k\text{-per}}(\Omega)$	The Sobolev space of functions in $H^s_{\mathrm{loc}}(\mathbb{R}^2)$ which satisfy k-quasi-periodic boundary conditions on $\partial\Omega$	see (4.13) on p. 52
$L^2_w(O)$	The space $L^2(O)$ with inner product weighted by w ($O \in \{\Omega, \mathbb{R}^2\}$ and $w \in \{\varepsilon_r, \xi, \widetilde{\xi}, \xi(\cdot,\mu)\}$)	see (2.1) on p. 7
$\mathscr{L}_k$	The variant of the operator pencil $\mathscr{A}_k$ defined upon holomorphically extending the coefficient function ξ	see (5.99) on p. 112
$n(\Gamma, z)$	The winding number of a Cauchy contour Γ for $z \in \mathbb{C} \setminus \Gamma$	see (2.5) on p. 20
P_{σ_0}	The Riesz projection for a subset σ_0 of the spectrum of a closed operator	see (2.6) on p. 20
$P_{\overset{*}{\mu}_{k,n}}$	The λ-nonlinear Riesz projection onto the eigenspace corresponding to the λ-nonlinear eigenvalue $\overset{*}{\mu}_{k,n}$ of $\mathscr{L}_k$	see (5.120) on p. 119
$Q^{(F)}_{k,\sigma_0}$	The λ-nonlinear Riesz integral for $\mathscr{L}_k$, a holomorphic operator-valued function F, and a finite subset σ_0 of the spectrum of $\mathscr{L}_k$	see (5.119) on p. 119

$R_A(\lambda)$	The resolvent of a closed operator A at a point λ in its resolvent set	see (2.4) on p. 17
$\mathscr{R}_{\mathscr{A}}(\lambda)$	The resolvent of an operator pencil $\mathscr{A}$ at a point λ in its resolvent set	see (2.7) on p. 23
$\mathcal{S}$	The semi-infinite strip to which $\mu \mapsto \xi(\cdot,\mu)\|_{\Omega}$ is assumed to be holomorphically extendable in the asymptotically nondispersive case	see (5.96) on p. 111
$\mathcal{S}_0$	The semi-infinite strip on which a certain norm-estimate for the resolvent of $\mathscr{L}_k$ holds for all $k \in B$ (a subset of $\mathcal{S}$)	see (5.104) on p. 113
$\mathcal{S}_k$	The semi-infinite strip $\mathcal{S}_0$ excluding the eigenvalues of $\mathscr{A}_k$	see (5.115) on p. 117
ε_{r}	The relative permittivity of a given photonic crystal (the data of all considered spectral problems in this work)	see (3.18) on p. 35
$\varepsilon_{\mathrm{r,min}}$	The essential lower bound on the relative permittivity ε_r	see (4.2) on p. 48
$\varepsilon_{\mathrm{r,max}}$	The essential upper bound on the relative permittivity ε_r	see (4.2) on p. 48
Θ	A Bravais lattice of a photonic crystal	see (3.1) on p. 27
Θ^*	The reciprocal lattice of a Bravais lattice Θ	see (3.4) on p. 30
$\lambda_{k,n}$	The nth smallest eigenvalue of the operator A_k	see (4.20) on p. 55
$\widetilde{\lambda}_{k,n}$	The nth smallest eigenvalue of the operator $\widetilde{A}_k$	see (5.54) on p. 92
$\widehat{\lambda}_{k,n}$	The nth smallest eigenvalue of the operator $\widehat{A}_k$	see (4.20) on p. 55
$\lambda_{\mu,k,n}$	The nth smallest eigenvalue of the operator $A_{\mu,k}$	see (5.12) on p. 68
$\overset{*}{\mu}_{k,n}$	The nth smallest eigenvalue of the operator pencil $\mathscr{A}_k$ (with k fixed, the nth λ-nonlinear eigenvalue)	see (5.24) on p. 74
$\widetilde{\mu}$	The threshold value in the high-frequency nondispersive case such that the studied photonic crystal acts nondispersively for frequencies greater than or equal to $c_0\sqrt{\widetilde{\mu}}$	see (5.51) on p. 91
$\widehat{\mu}$	The threshold value in the asymptotically nondispersive case such that the holomorphic extension of $\mu \mapsto \xi(\cdot,\mu)\|_{\Omega}$ exists sufficiently close to $(\widehat{\mu},\infty)$	see (5.96) on p. 111

ξ	The coefficient function of all spectral problems studied in Chapter 5 (a rescaled variant of a given frequency-dependent relative permittivity)	see (5.3) on p. 65
$\xi_{\min}$	The essential lower bound on the coefficient function ξ (uniform in its second argument)	see (5.5) on p. 65
$\xi_{\max}(\mu)$	The essential upper bound on the coefficient function $\xi(\cdot,\mu)$	see (5.5) on p. 65
$\widetilde{\xi}$	In Subsection 5.4.1: The high-frequency coefficient function of the spectral problem for dispersive photonic crystals (a rescaled variant of a given frequency-independent relative permittivity)	see (5.51) on p. 91
$\widetilde{\xi}$	In Subsection 5.4.2: The asymptotic coefficient function of the spectral problem for dispersive photonic crystals (a rescaled variant of a given frequency-independent relative permittivity)	see (5.77) on p. 100
$\widetilde{\xi}_{\min}$	The essential lower bound on the coefficient function $\widetilde{\xi}$	see (5.77) on p. 100
$\widetilde{\xi}_{\max}$	The essential upper bound on the coefficient function $\widetilde{\xi}$	see (5.77) on p. 100
$\psi_{k,n}$	The eigenfunction corresponding to the eigenvalue $\lambda_{k,n}$ of the operator A_k	see (4.22) on p. 55
$\widetilde{\psi}_{k,n}$	The eigenfunction corresponding to the eigenvalue $\widetilde{\lambda}_{k,n}$ of the operator $\widetilde{A}_k$	see (5.54) on p. 92
$\widehat{\psi}_{k,n}$	The eigenfunction corresponding to the eigenvalue $\widehat{\lambda}_{k,n}$ of the operator $\widehat{A}_k$	see (4.22) on p. 55
$\psi_{\mu,k,n}$	The eigenfunction corresponding to the eigenvalue $\lambda_{\mu,k,n}$ of the operator $\widetilde{A}_{\mu,k}$	see (5.12) on p. 68
$\overset{*}{\psi}_{k,n}$	The eigenfunction corresponding to the eigenvalue $\overset{*}{\mu}_{k,n}$ of the operator pencil $\mathscr{A}_k$ (with k fixed, the nth λ-nonlinear eigenfunction)	see (5.50) on p. 90
Ω	The primitive cell $(0,1)^2$ of the Bravais lattice $\mathbb{Z}^2$	see (4.10) on p. 50
Ω_Θ	A primitive cell of a Bravais lattice of a photonic crystal	see (3.2) on p. 28
$\langle\cdot,\cdot\rangle_{H^1_k,\widetilde{\xi}}$	The inner product in $H^1(\Omega)$ with $L^2(\Omega)$-term weighted by $\widetilde{\xi}$	see (5.88) on p. 107

Curriculum Vitae Auctoris

Philipp Schmalkoke was born in the winter of 1984 in Speyer, Germany. Having visited the primary school in the nearby municipality Berghausen, he went on to attend the Friedrich-Magnus-Schwerd-Gymnasium, a grammar school in the town of his birth. There, in 2004, he received his Abitur certificate. Following his civilian service at the local Caritas center, Philipp decided to enroll in the business mathematics program at the University of Karlsruhe and commenced his studies in the summer semester of 2005. Two years later, just having received his Vordiplom certificate, he decided to slightly alter his study path and began his graduate studies in the pure mathematics program with a minor in economics. This allowed him to specialize in mathematical physics and partial differential equations, where he obtained parts of his education while being a student at Cardiff University, Wales, United Kingdom in the winter semester of 2007. Once back in Germany, Philipp attended the relevant classes for his final thesis entitled "Spectral Floquet-Bloch Theory for Maxwell's Equations with Periodic and Discontinuous Coefficients", which he wrote under the supervision of Prof. Dr. Michael Plum and Prof. Dr. Andreas Kirsch. Following his final examinations in the autumn of 2009, he graduated with distinction with the degree of Diplom-Mathematiker and immediately decided to move on to doctoral studies in mathematics in parallel with selected courses in theoretical physics. He became a scholar of the Research Training Group 1294 "Analysis, Simulation and Design of Nanotechnological Processes" of the German Research Foundation (DFG) at the Department of Mathematics, Karlsruhe Institute of Technology (KIT). His doctoral advisers, Prof. Dr. Michael Plum and Prof. Dr. Roland Schnaubelt, supported him while working on this dissertation.

During nearly his whole period of education, Philipp worked as a research assistant at the mathematical institutes of his university. Besides, in 2010 he interned at a research and development center of the Robert Bosch GmbH in Bühl, Germany working on mathematical modeling of electric motors. In the spring of 2013, close to his final exam for the doctoral degree, Philipp began his professional life by joining the Robert Bosch GmbH near Stuttgart as a development engineer.