Gaussian Filtering using State Decomposition Methods

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Abstract – *State estimation for nonlinear systems generally requires approximations of the system or the probability densities, as the occurring prediction and filtering equations cannot be solved in closed form. For instance, Linear Regression Kalman Filters like the Unscented Kalman Filter or the considered* Gaussian Filter *propagate a small set of sample points through the system to approximate the posterior mean and covariance matrix. To reduce the number of sample points, special structures of the system and measurement equation can be taken into account. In this paper, two principles of system decomposition are considered and applied to the Gaussian Filter. One principle exploits that only a part of the state vector is directly observed by the measurement. The second principle separates the system equations into linear and nonlinear parts in order to merely approximate the nonlinear part of the state. The benefits of both decompositions are demonstrated on a real-world example.*

Keywords: Estimation, tracking, filtering, Rao-Blackwellization, Linear Regression Kalman Filter.

1 Introduction

Estimation is applied to problems, where the state of a dynamical system has to be calculated based on noisy observations and where uncertainties in modeling have to be taken into account. In case of linear systems with additive Gaussian noise, the well-known Kalman Filter provides optimal estimates of the system state in form of first and second moments. All required calculations can be performed in closed form. The same is not true when nonlinearities arise in the measurement and system equation. Here, approximation has to be applied, where two different approximation approaches can be found in literature. The first one relies on approximating the system equations as done by the Extended Kalman Filter. In the second approach, the probability density representing the state estimate is approximated instead. Here, sample-based approaches like Particle Filters [1] are common. A drawback of Particle Filters is that sampling is performed randomly and thus, many particles have to be used to achieve accurate results. Moreover, Particle Filters suffer from the *curse of dimensionality*, i.e., the number of particles increases exponentially with the number of dimensions [4].

Another class of sample-based estimators are *Linear Regression Kalman Filters* (LRKFs), like the well-known Unscented Kalman Filter (UKF) [6], the Divided Difference Filter [10], the Central Difference Filter [13], or the Gaussian Filter [5]. Here, it is assumed that the state estimate can be sufficiently characterized by its first two moments, i.e., mean and covariance matrix. The regression points (sample points) for exactly capturing these moments can be easily determined in a deterministic fashion. Propagating the regression points through the nonlinear system equations and calculating the first two moments of the posterior estimate implicitly linearizes the nonlinear equations. For this procedure, only a small number of regression points is required and the number of regression points grows only linearly with the dimension of the state space.

In this paper, the latest LRKF, the so-called *Gaussian Filter* introduced in [5], is extended in such a way that the number of regression points can be reduced without a noticeable effect on the estimation quality. For this purpose, two different approaches for state decomposition are discussed. The first one relies on Rao-Blackwellization [14], which is often applied to Particle Filters for attenuating the curse of dimensionality and which has been firstly applied to the unscaled version of the UKF in [9]. Here, the equations are separated into a linear and nonlinear substructure, where only the nonlinear part is processed in an approximate fashion.

The second decomposition is described in [8] for the Kalman Filter. Transferred to LRKFs, only the directly observed state needs to be represented by means of regression points and updated with the current measurement. Based on the correlation between the observed and the indirectly observed part, the indirectly observed state can be updated after the filter step without further approximations. Although both approaches are derived for the Gaussian Filter, they can be directly applied to any estimator belonging to the class of LRKFs.

In Sec. 2, a brief problem formulation is given. The considered Gaussian Filter is explained in Sec. 3. Both decomposition approaches are content of Sec. 4, while in Sec. 5, the estimator equations for both decompositions are derived. In Sec. 6, the example application employed for simulations and experiments is introduced. Here, tracking an object based on reference signals for a large-scale telepresence scenario [12] is considered. Furthermore, a mathematical formulation of the considered example is given. In Sec. 7, the decomposed case and the full state case are compared in a simulation and an experiment. The paper closes with conclusions.

2 Problem Formulation

A nonlinear discrete-time dynamic systems is given by

$$\begin{aligned} \underline{\boldsymbol{x}}_{k+1} &= \underline{a}_k\left(\underline{\boldsymbol{x}}_k, \underline{u}_k, \underline{\boldsymbol{w}}_k\right) \ , \\ \underline{\boldsymbol{y}}_k &= \underline{h}_k\left(\underline{\boldsymbol{x}}_k, \underline{\boldsymbol{v}}_k\right) \ , \end{aligned}$$

where the functions $\underline{a}_k(\cdot)$ and $\underline{h}_k(\cdot)$ are known. The vector $\underline{\boldsymbol{x}}_k$ is the state of the system, $\underline{\boldsymbol{y}}_k$ is the measurement vector, and $\underline{u}_k$ is a known system input at the discrete time k. The measurement and process noise are characterized by $\underline{\boldsymbol{v}}_k$ and $\underline{\boldsymbol{w}}_k$, respectively. Based on the measurements $\underline{\boldsymbol{y}}_k$, the density of the system state $\underline{\boldsymbol{x}}_k$ has to be estimated by using filtering and prediction techniques.

Filtering and prediction for nonlinear systems typically cannot be realized in closed form. To achieve an efficient estimation scheme, the exact density or the system equations have to be approximated in an adequate manner. If the approximation is based on the density, the number of parameters, i.e., the number of regression or sample points[1], for approximating the state density has to be constant for efficient state estimation. Furthermore, the approximation quality should be adjustable in order to improve the estimation accuracy if required.

In many cases, however, parts of the estimation problem can be solved in closed form. Hence, only a part of the density has to be represented by a set of regression points. By exploiting the structure of the underlying system, several decomposition methods can be applied for this purpose.

3 The Gaussian Filter

In contrast to most of the LRKFs like the famous UKF [6], the Gaussian Filter [5] considered in this paper allows varying the number of sample points. In doing so, more information of the nonlinear system model is captured by increasing the number of sample points and thus, one can trade estimation quality for computational demand. Moreover, information of higher-order moments can be explicitly incorporated into the estimation process. But still, as typical for LRKFs, the number of regression points grows only linearly with the dimension of the state space.

[1] Throughout this paper, the terms *regression points* and *sample points* are used interchangeably.

D	μ_1	μ_2	μ_3
3	-1.2247	-	-
5	-1.4795	-0.5578	-
7	-1.6346	-0.8275	-0.3788

Table 1: Sample positions for several numbers of samples.

The determination of the sample points for the Gaussian Filter relies on efficiently solving an optimization problem, where a certain distance measure between the Gaussian density representing the state estimate and the Dirac mixture density representing the sample points is minimized. An additional constraint ensures that mean and covariance matrix of the state are captured exactly. In order to calculate the parameters, i.e., the weights and the positions of the regression points, a two step procedure is employed.

In the computationally demanding first step, which can be performed off-line, the samples for an univariate standard Gaussian density are calculated [5]. In this paper, the number of sample points D is assumed to be odd, because in this case one sample point is always located at the mean. In Tab. 1, the sample points with negative positions for various numbers of samples D are given. The mean point as well as the positive positions are calculated according to

$$\mu_{\frac{D+1}{2}} = 0, \ \ \mu_i = -\mu_{D+1-i} \text{ for } i = \tfrac{D+3}{2}, \ldots, D \ .$$

In the second step, for on-line approximating an arbitrary N-dimensional Gaussian density $f\left(\underline{x}\right) = \mathcal{N}\left(\underline{x} - \underline{\mu}^x, \mathbf{C}^{x,x}\right)$, N sets of D sample points are placed along the N coordinate axes. By means of affine operations, these sample points are transformed for exactly capturing the mean $\underline{\mu}^x$ and the covariance matrix $\mathbf{C}^{x,x}$. The transformed sample points are calculated via

$$\underline{\mu}_i^x = \underline{\mu}^x + \mathbf{V} \cdot \sqrt{\mathbf{D}} \cdot \underline{S}_i \text{ for } i = 1, \ldots, L \ , \qquad (1)$$

where $\mathbf{V}$ and $\mathbf{D}$ is the matrix of eigenvectors and diagonal matrix of eigenvalues, respectively, with $\mathbf{C}^{x,x} = \mathbf{V} \cdot \mathbf{D} \cdot (\mathbf{V})^{\mathrm{T}}$. Furthermore, $L = N \cdot (D-1) + 1$ is the total number of sample points, $\underline{S}_i$ is the i-th column of the matrix

$$\mathbf{S} = \Bigg[\underbrace{\underline{1} \cdot \mu_{\frac{D+1}{2}}}_{=\underline{0}} \quad \mathbf{I}_{N,N} \otimes \left[\mu_1, \ldots, \mu_{\frac{D-1}{2}}, \mu_{\frac{D+3}{2}}, \ldots, \mu_D \right] \Bigg]$$

and $\otimes$ is the Kronecker product.

For calculating the mean $\underline{\mu}^x$ by means of set of sample points $\underline{\mu}_i^x$, the same sample weight

$$\omega = \tfrac{1}{L} \qquad (2)$$

is used for all sample points. A different weight is required for ensuring that the sample covariance matrix coincides with given covariance matrix $\mathbf{C}^{x,x}$. Here, the weight $\omega_s = 1/D$ has to be used instead of ω, (2). This is shown in the following.

Proof. To prove that the weight for calculating the sample covariance matrix has to be different, the sample covariance matrix for the N-dimensional Gaussian density is considered

$$\sum_{i=1}^{L} \omega_s \cdot \left(\underline{\mu}^x - \underline{\mu}_i^x\right) \cdot \left(\underline{\mu}^x - \underline{\mu}_i^x\right)^{\mathrm{T}} \equiv \mathbf{C}^{x,x} \quad .$$

Using (1) results in

$$\sum_{i=1}^{L} \omega_s \cdot \left(\mathbf{V} \cdot \sqrt{\mathbf{D}} \cdot \underline{S}_i\right) \cdot \left(\mathbf{V} \cdot \sqrt{\mathbf{D}} \cdot \underline{S}_i\right)^{\mathrm{T}} \equiv \mathbf{C}^{x,x}$$

$$\left(\mathbf{V}\sqrt{\mathbf{D}}\right) \left(\sum_{i=1}^{L} \omega_s \cdot \underbrace{\underline{S}_i \cdot (\underline{S}_i)^{\mathrm{T}}}_{(\mu_i^x)^2 \cdot \mathbf{I}_{N,N}}\right) \left(\mathbf{V}\sqrt{\mathbf{D}}\right)^{\mathrm{T}} \equiv \mathbf{C}^{x,x} \quad .$$

The sum has to be equal to one so that the sample covariance matrix corresponds to the given covariance matrix. Furthermore, the sample points are point-symmetric regarding to the mean of the Gaussian density, which results in

$$\sum_{i=1}^{L} \omega_s \cdot (\mu_i^x)^2 \equiv 1 \text{ and } \sum_{i=1}^{\frac{D-1}{2}} \omega_s \cdot 2 \cdot (\mu_i^x)^2 \equiv 1 \quad .$$

From [5], one nonlinear equation from the optimization problem is given by

$$\sum_{i=1}^{\frac{D-1}{2}} (\mu_i^x)^2 - \tfrac{D}{2} = 0 \quad .$$

Based on this relationship, the weight must be $\omega_s = \frac{1}{D}$. □

4 Decomposition Methods

4.1 Case I: directly observed, indirectly observed

Often, only a part of the state is observed through the measurement model. Based on stochastic dependency, the indirectly observed part is updated. The filter step

$$f(\underline{x}|\underline{y}) = \tfrac{f(\underline{y}|\underline{x}^o) f(\underline{x})}{f(\underline{y})}$$

can be written in the from

$$\begin{aligned} f(\underline{x}|\underline{y}) &= f(\underline{x}^u|\underline{x}^o) \tfrac{f(\underline{y}|\underline{x}^o) f(\underline{x}^o)}{f(\underline{y})} \\ &= f(\underline{x}^u|\underline{x}^o) f(\underline{x}^o|\underline{y}) \quad , \end{aligned}$$

where the state vector is decomposed into an observed and an indirectly observed part

$$\underline{\boldsymbol{x}} = \begin{bmatrix} \underline{\boldsymbol{x}}^o \\ \underline{\boldsymbol{x}}^u \end{bmatrix} \quad .$$

The measurement equation can be linear or nonlinear. In the nonlinear case

$$\underline{\boldsymbol{y}} = \underline{h}\left(\underline{\boldsymbol{x}}^o, \underline{\boldsymbol{v}}\right) \quad ,$$

the directly observed state $\underline{\boldsymbol{x}}^o$ is estimated by using a Linear Regression Kalman Filter and after that the indirectly observed state is updated. In the Linear Regression Kalman Filter, the state vector $\underline{\boldsymbol{x}}$ is Gaussian distributed with mean and covariance matrix

$$\underline{\mu}^x = \begin{bmatrix} \underline{\mu}^o \\ \underline{\mu}^u \end{bmatrix} \quad , \qquad \mathbf{C}^{x,x} = \begin{bmatrix} \mathbf{C}^{o,o} & \mathbf{C}^{o,u} \\ \mathbf{C}^{u,o} & \mathbf{C}^{u,u} \end{bmatrix} \quad .$$

To update the indirectly observed state, the estimated mean $\underline{\mu}_e^o$ and estimated covariance matrix $\mathbf{C}_e^{o,o}$ of the density $f(\underline{x}^o|\underline{y})$ are used. According to [8], the mean vector of the indirectly observed state is updated by

$$\underline{\mu}_e^u = \underline{\mu}_p^u + \mathbf{L} \cdot \left(\underline{\mu}_e^o - \underline{\mu}_p^o\right) \quad , \tag{3}$$

the cross-covariance matrix $\mathbf{C}_e^{u,o}$ and the covariance matrix $\mathbf{C}_e^{u,u}$ are calculated according to

$$\mathbf{C}_e^{u,o} = \mathbf{L} \cdot \mathbf{C}_e^{o,o} \text{ and} \tag{4}$$

$$\mathbf{C}_e^{u,u} = \mathbf{C}_p^{u,u} - \mathbf{L} \cdot \left(\mathbf{C}_p^{o,o} - \mathbf{C}_e^{o,o}\right) \cdot (\mathbf{L})^{\mathrm{T}} \quad , \tag{5}$$

with the matrix $\mathbf{L}$

$$\mathbf{L} = \mathbf{C}_p^{u,o} \cdot \left(\mathbf{C}_p^{o,o}\right)^{-1} \quad .$$

4.2 Case II: Linear, Nonlinear

Similar to [14], [9], and [7], conditionally linear models

$$\underline{\boldsymbol{y}} = \underline{g}\left(\underline{\boldsymbol{x}}^n\right) + \mathbf{H}\left(\underline{\boldsymbol{x}}^n\right) \cdot \underline{\boldsymbol{x}}^l$$

are considered. For the Gaussian Filter, the joint density $f\left(\underline{x}, \underline{y}\right)$ has to be approximated by a multivariate Gaussian density, where the joint density is

$$f\left(\underline{x}, \underline{y}\right) = \delta\left(\underline{y} - \underline{g}\left(\underline{x}^n\right) + \mathbf{H}\left(\underline{x}^n\right) \cdot \underline{x}^l\right) \cdot f\left(\underline{x}^n, \underline{x}^l\right) \quad .$$

The state is decomposed into a nonlinear and linear part

$$\underline{\boldsymbol{x}} = \begin{bmatrix} \underline{\boldsymbol{x}}^n \\ \underline{\boldsymbol{x}}^l \end{bmatrix}$$

and the density of the state is Gaussian distributed with mean and covariance matrix

$$\underline{\mu}^x = \begin{bmatrix} \underline{\mu}^n \\ \underline{\mu}^l \end{bmatrix}, \quad \mathbf{C}^{x,x} = \begin{bmatrix} \mathbf{C}^{n,n} & \mathbf{C}^{n,l} \\ \mathbf{C}^{l,n} & \mathbf{C}^{l,l} \end{bmatrix} \quad .$$

The density of the state is separated by using Bayes' law

$$f\left(\underline{x}^n, \underline{x}^l\right) = f\left(\underline{x}^l|\underline{x}^n\right) \cdot f\left(\underline{x}^n\right) \quad .$$

The conditional density is given by

$$f\left(\underline{x}^l|\underline{x}^n\right) = \mathcal{N}\left(\underline{x}^l - \underline{\mu}\left(\underline{x}^n\right), \mathbf{C}^{l|n}\right)$$

with mean and covariance matrix

$$\begin{aligned} \underline{\mu}\left(\underline{x}^n\right) &= \underline{\mu}^l + \mathbf{C}^{l,n} \cdot \left(\mathbf{C}^{n,n}\right)^{-1} \cdot \left(\underline{x}^n - \underline{\mu}^n\right) \\ \mathbf{C}^{l|n} &= \mathbf{C}^{l,l} - \mathbf{C}^{l,n} \cdot \left(\mathbf{C}^{n,n}\right)^{-1} \cdot \mathbf{C}^{n,l} \quad . \end{aligned}$$

The density for the nonlinear part $f\left(\underline{x}^n\right)$ is approximated with a Dirac mixture density based on the deterministic sampling scheme from Sec. 3

$$f\left(\underline{x}^n\right) \approx \tilde{f}\left(\underline{x}^n, \underline{\eta}\right) = \sum_{i=1}^{L} \omega \cdot \delta\left(\underline{x}^n - \underline{\mu}_i^n\right) \quad ,$$

with $L = N \cdot (D-1) + 1$. The approximated joint density $\tilde{f}\left(\underline{x}, \underline{y}\right)$ is given by

$$\begin{aligned}\tilde{f}\left(\underline{x}, \underline{y}\right) =& \delta\left(\underline{y} - \underline{g}\left(\underline{x}^n\right) + \mathbf{H}\left(\underline{x}^n\right) \cdot \underline{x}^l\right) \cdot \\ & \cdot \mathcal{N}\left(\underline{x}^l - \underline{\mu}\left(\underline{x}^n\right), \mathbf{C}^{l|n}\right) \cdot \\ & \cdot \sum_{i=1}^{L} \omega \cdot \delta\left(\underline{x}^n - \underline{\mu}_i^n\right) \quad .\end{aligned}$$

For the Gaussian Filter, the joint density is approximated with a Gaussian density according to

$$\tilde{f}\left(\underline{x}, \underline{y}\right) \approx \mathcal{N}\left(\begin{bmatrix}\underline{x} \\ \underline{y}\end{bmatrix} - \begin{bmatrix}\underline{\mu}^x \\ \underline{\mu}^y\end{bmatrix}, \begin{bmatrix}\mathbf{C}^{x,x} & \mathbf{C}^{x,y} \\ \mathbf{C}^{y,x} & \mathbf{C}^{y,y}\end{bmatrix}\right) \quad .$$

For calculating the mean $\underline{\mu}^y$, the covariance matrix $\mathbf{C}^{y,y}$ and the cross-covariance matrix $\mathbf{C}^{x,y}$ first the approximated density $\tilde{f}\left(\underline{y}\right)$ depending on $\underline{y}$ is calculated

$$\begin{aligned}\tilde{f}\left(\underline{y}\right) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \tilde{f}\left(\underline{x}, \underline{y}\right) \mathrm{d}\underline{x}^l \mathrm{d}\underline{x}^n \\ &= \sum_{i=1}^{L} \omega \int \delta\left(\underline{y} - \underline{g}\left(\underline{\mu}_i^n\right) + \mathbf{H}\left(\underline{\mu}_i^n\right) \cdot \underline{x}^l\right) \cdot \\ &\quad \cdot \mathcal{N}\left(\underline{x}^l - \underline{\mu}\left(\underline{\mu}_i^n\right), \mathbf{C}^{l|n}\right) \mathrm{d}\underline{x}^l \quad ,\end{aligned}$$

which results in a Gaussian mixture

$$\tilde{f}\left(\underline{y}\right) = \sum_{i=1}^{L} \omega \cdot \mathcal{N}\left(\underline{y} - \underline{\mu}_i^y, \mathbf{C}_i^{y,y}\right) \quad ,$$

with mean and covariance matrix

$$\begin{aligned}\underline{\mu}_i^y &= \underline{g}\left(\underline{\mu}_i^n\right) + \mathbf{H}\left(\underline{\mu}_i^n\right) \cdot \underline{\mu}\left(\underline{\mu}_i^n\right) \\ \mathbf{C}_i^{y,y} &= \mathbf{H}\left(\underline{\mu}_i^n\right) \cdot \mathbf{C}^{l|n} \cdot \left(\mathbf{H}\left(\underline{\mu}_i^n\right)\right)^{\mathrm{T}} \quad .\end{aligned}$$

The first and the second moments are approximated by

$$\underline{\mu}^y = \omega \cdot \sum_{i=1}^{L} \underline{\mu}_i^y \text{ and} \tag{6}$$

$$\mathbf{C}^{y,y} = \sum_{i=1}^{L} \left(\omega \cdot \mathbf{C}_i^{y,y} + \omega_s \cdot \left(\underline{\mu}_i^y - \underline{\mu}^y\right) \cdot \left(\underline{\mu}_i^y - \underline{\mu}^y\right)^{\mathrm{T}}\right) \quad , \tag{7}$$

where the covariance matrix consists of the sample covariance matrix and the covariance matrix of the linear part. The cross-covariance matrix is approximated with

$$\begin{aligned}\mathbf{C}^{x,y} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\underline{x} - \underline{\mu}^x\right) \cdot \left(\underline{y} - \underline{\mu}^y\right)^{\mathrm{T}} \cdot \tilde{f}\left(\underline{x}, \underline{y}\right) \mathrm{d}\underline{x} \mathrm{d}\underline{y} \\ &= \sum_{i=1}^{L} \left(\omega \cdot \begin{bmatrix}\mathbf{0} \\ \mathbf{C}^{l|n} \mathbf{H}\left(\underline{\mu}_i^n\right)^{\mathrm{T}}\end{bmatrix} + \right. \\ &\quad \left. + \omega_s \cdot \left(\begin{bmatrix}\underline{\mu}_i^n \\ \underline{\mu}\left(\underline{\mu}_i^n\right)\end{bmatrix} - \underline{\mu}^x\right) \cdot \left(\underline{\mu}_i^y - \underline{\mu}^y\right)^{\mathrm{T}}\right) \quad .\end{aligned} \tag{8}$$

5 Estimation

5.1 Prediction Step

The information flow for the prediction step is shown Fig. 1. First, the state has to be separated into a nonlinear and linear part. Then, the system equation has to be converted into the form

$$\underline{\boldsymbol{x}}_{k+1} = \underline{g}_k\left(\underline{\boldsymbol{x}}_k^n, \underline{u}_k, \underline{\boldsymbol{w}}_k^n\right) + \mathbf{H}_k\left(\underline{\boldsymbol{x}}_k^n, \underline{u}_k, \underline{\boldsymbol{w}}_k^n\right) \cdot \begin{bmatrix}\underline{\boldsymbol{x}}_k^l \\ \underline{\boldsymbol{w}}_k^l\end{bmatrix} \quad ,$$

where the system state is augmented with the noise variable $\underline{\boldsymbol{X}}_k = \left[\underline{\boldsymbol{x}}_k^{\mathrm{T}} \quad \underline{\boldsymbol{w}}_k^{\mathrm{T}}\right]^{\mathrm{T}}$ to consider additive and/or multiplicative noise. The mean and covariance matrix is then given by

$$\underline{\mu}_e^X = \begin{bmatrix}\left(\underline{\mu}_e^x\right)^{\mathrm{T}} & \underline{0}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}_e^{X,X} = \begin{bmatrix}\mathbf{C}_e^{x,x} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_k^{w,w}\end{bmatrix} \quad .$$

In the second step, the sample points for the nonlinear part are calculated based on the scheme described in Sec. 3. For calculating the predicted mean and covariance matrix (6) and (7) are used, where the components of the Gaussian Mixture are used.

5.2 Filter Step

In a first step, the directly observed and the indirectly observed state are separated. Then the measurement equation has to be converted, as in the prediction step and the sample points for the nonlinear state are determined. Then, the covariance matrix (7), cross-covariance matrix (8), and the predicted measurement (6) are calculated. Based on the approximated joint density, mean and covariance matrix are estimated according to the conditional density $f\left(\underline{x}|\underline{\hat{y}}\right)$ for a given measurement $\underline{\hat{y}}$

$$\begin{aligned}\underline{\mu}_e^o &= \underline{\mu}_p^o + \mathbf{C}^{x,y} \cdot \left(\mathbf{C}^{y,y}\right)^{-1} \cdot \left(\underline{\hat{y}} - \underline{\mu}^y\right) \quad , \\ \mathbf{C}_e^{o,o} &= \mathbf{C}_p^{o,o} - \mathbf{C}^{x,y} \cdot \left(\mathbf{C}^{y,y}\right)^{-1} \cdot \left(\mathbf{C}^{x,y}\right)^{\mathrm{T}} \quad .\end{aligned}$$

After the mean and the covariance matrix of the directly observed state is updated, the indirectly observed state is calculated based on (3), (4) and (5). In Fig. 2, the information flow for the filter step is shown.

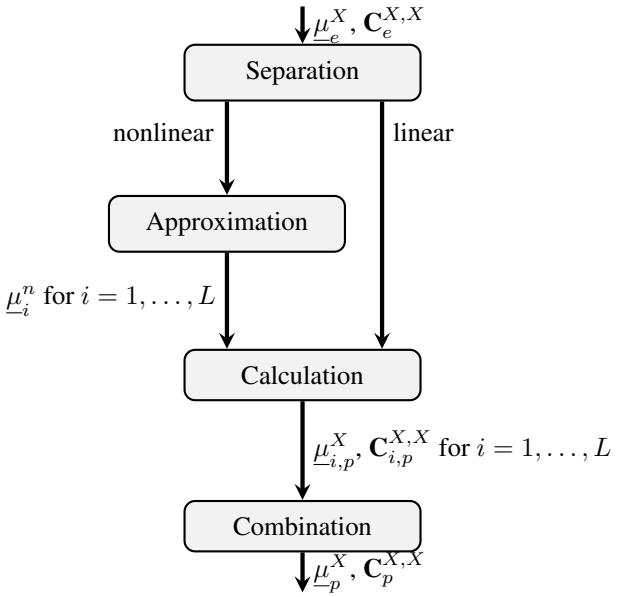


Figure 1: Information flow of the prediction step.

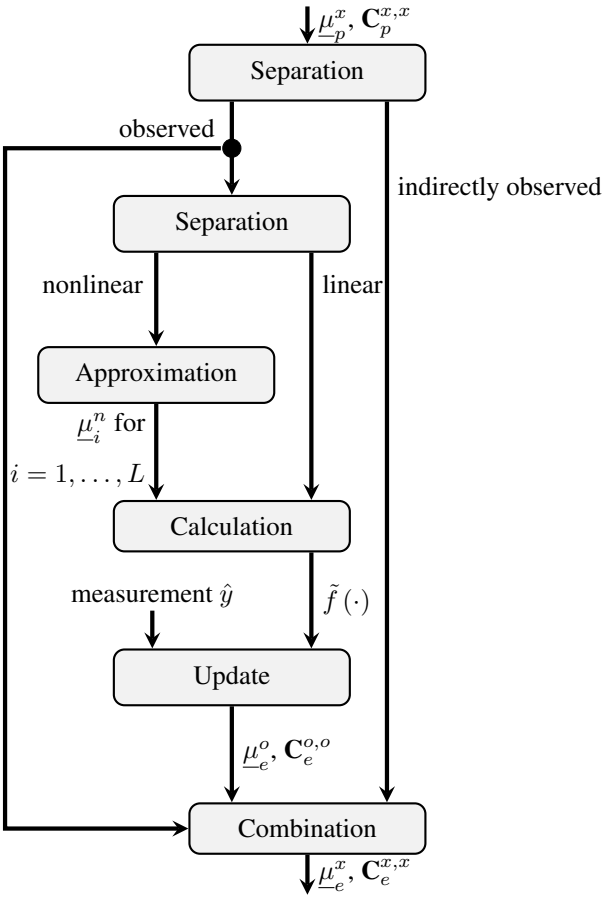


Figure 2: Information flow of the filter step.

6 Considered Example

The proposed algorithm is evaluated in a pose tracking scenario. Based on the observation of two different sensors, the translation and the rotation of an object is estimated. The estimation for the pose with respect to a global coordinate system is performed by known reference signals. Several loudspeakers emit signals, which are received by microphones attached to the tracked object. The other type of sensors measures the inertial angular velocity of the object with respect to the object coordinate system.

The state vector consists of the translation, the rotation, the translation velocity, and the angular velocity in three-dimensional space. Based on the proposed principles, the twelve-dimensional state vector is decomposed to reduce the computational effort. In this example, the sensors only observe a part of the state vector. Furthermore, the kinematics of the object is described by a nonlinear system equation, which can be separated into a linear and a nonlinear part.

6.1 Prediction

The system model describes the evolution of the state over time. For the considered example, the dynamic behavior of the translation and the rotation has to be characterized in a adequate manner.

6.1.1 Translation

For the translation, a constant velocity model is assumed. This model is given by a linear differential equation, which is represented in discrete time as

$$\underline{\boldsymbol{z}}_{k+1} = \mathbf{A} \cdot \underline{\boldsymbol{z}}_k + \underline{\boldsymbol{w}}_k^{\boldsymbol{z}} \ ,$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} \mathbf{I}_{3,3} & T \cdot \mathbf{I}_{3,3} \\ \mathbf{0} & \mathbf{I}_{3,3} \end{bmatrix} \ ,$$

and $\underline{\boldsymbol{z}}_k$ consists of the translation $\underline{\boldsymbol{T}}_k$ and the velocity $\underline{\boldsymbol{V}}_k$. T is the sampling time. The process noise $\underline{\boldsymbol{w}}_k^{\boldsymbol{z}}$ is assumed to be white, zero-mean, and Gaussian with covariance matrix

$$\mathbf{Q}_{\underline{w}_k^{\boldsymbol{z}}} = \begin{bmatrix} \frac{T^3}{3} \cdot \mathbf{q} & \frac{T^2}{2} \cdot \mathbf{q} \\ \frac{T^2}{2} \cdot \mathbf{q} & T \cdot \mathbf{q} \end{bmatrix} \ ,$$

when $\mathbf{q}$ is given by

$$\mathbf{q} = \mathrm{diag}\left(\begin{bmatrix} q_{w_{V,x}}^2 & q_{w_{V,y}}^2 & q_{w_{V,z}}^2 \end{bmatrix}^{\mathrm{T}} \right) \ .$$

6.1.2 Rotation

The rotation is described by a rotation vector [2]. In a tracking scenario, the rotation vector is typically time-variant. This dynamic behavior can be described by a nonlinear differential equation [3]

$$\dot{\underline{r}}(t) = \begin{cases} \mathbf{a}(\underline{r}(t)) \cdot \underline{\omega}(t) & \text{for } \|\underline{r}(t)\| \in]0, \pi] \\ \underline{\omega}(t) & \text{for } \|\underline{r}(t)\| = 0 \end{cases} \ , \quad (9)$$

which depends on the angular velocity $\underline{\omega}(t)$. The matrix $\mathbf{a}(\cdot)$ is given by

$$\mathbf{a}(\underline{r}(t)) = \mathbf{I}_{3,3} + \tfrac{1}{2} \cdot \mathbf{C}(\underline{r}(t)) + + \frac{1 - \frac{1}{2} \cdot \|\underline{r}(t)\| \cdot \cot\left(\frac{\|\underline{r}(t)\|}{2}\right)}{\|\underline{r}(t)\|^2} \mathbf{C}(\underline{r}(t)) \cdot \mathbf{C}(\underline{r}(t)) \ ,$$

where $\cot(\cdot)$ is the cotangent and the matrix $\mathbf{C}(\underline{r}(t))$ is a skew-symmetric matrix

$$\mathbf{C}(\underline{r}(t)) = \begin{bmatrix} 0 & -r_z(t) & r_y(t) \\ r_z(t) & 0 & -r_x(t) \\ -r_y(t) & r_x(t) & 0 \end{bmatrix} \ .$$

The nonlinear differential equation (9) is discretized by the Euler formula

$$\dot{\underline{r}}(t) \approx \tfrac{\underline{r}_{k+1} - \underline{r}_k}{T}$$

and the resulting discrete-time difference equation is

$$\underline{r}_{k+1} = \underline{r}_k + T \cdot \mathbf{a}(\underline{r}_k) \cdot \underline{\omega}_k \ .$$

In (9), the range of the norm of the rotation vector lies in the interval zero to π. To achieve this constraint in the estimation procedure, a forward inference is performed by

$$\underline{\boldsymbol{r}}_{k,new} = \underline{\boldsymbol{r}}_k \cdot \left(1 - \tfrac{2\pi}{\|\underline{\boldsymbol{r}}_k\|}\right) \ ,$$

if the norm of the rotation vector is higher than π. The forward inference can be calculated by a prediction step, where no process noise is assumed. The angular velocity is modeled as a random walk according to

$$\underline{\boldsymbol{\omega}}_{k+1} = \underline{\boldsymbol{\omega}}_k + \underline{\boldsymbol{w}}_k^{\omega} \ ,$$

where the process noise $\underline{\boldsymbol{w}}_k^{\omega}$ is white, zero-mean, and Gaussian with covariance matrix

$$\mathbf{Q}_{\underline{w}^{\omega}} = \operatorname{diag}\left(\begin{bmatrix} q_{w_{\omega,x}}^2 & q_{w_{\omega,y}}^2 & q_{w_{\omega,z}}^2 \end{bmatrix}^{\mathrm{T}}\right) \ .$$

6.1.3 System Equation

In this example, the nonlinear part depends on the rotation vector. In the linear part of the state $\underline{\boldsymbol{x}}_k^l$, the translation, the velocity, and the angular velocity is considered. The system equation is written in the form

$$\begin{bmatrix} \underline{\boldsymbol{r}}_{k+1} \\ \underline{\boldsymbol{\omega}}_{k+1} \\ \underline{\boldsymbol{z}}_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \underline{\boldsymbol{r}}_k \\ \underline{0} \\ \underline{0} \end{bmatrix}}_{\underline{g}(\underline{\boldsymbol{r}}_k)} + \underbrace{\begin{bmatrix} T\mathbf{a}(\underline{\boldsymbol{r}}_k) & \mathbf{0} \\ \mathbf{I}_{3,3} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}}_{\mathbf{H}(\underline{\boldsymbol{r}}_k)} \underbrace{\begin{bmatrix} \underline{\boldsymbol{\omega}}_k \\ \underline{\boldsymbol{z}}_k \end{bmatrix}}_{\underline{\boldsymbol{x}}_k^l} + \underline{\boldsymbol{w}}_k \ ,$$

where the covariance matrix of the noise $\underline{\boldsymbol{w}}_k$ is given by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{\underline{w}^{\omega}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{Q}_{\underline{w}_n^{\underline{z}}} \end{bmatrix} \ .$$

6.2 Filtering

6.2.1 Inertial Sensors

The inertial sensors measure the angular velocity with respect to the object coordinate system. The measurement equation is modeled by a linear equation

$$\underline{\boldsymbol{y}}_k^{\omega} = k_G \cdot \mathbf{A} \cdot \underline{\boldsymbol{\omega}}_k + \underline{b} + \underline{\boldsymbol{v}}_k^{\omega} \ ,$$

where k_G is a sensor specific factor, $\mathbf{A}$ a misalignment matrix, $\underline{b}$ the sensor offset, and $\underline{\boldsymbol{v}}_k^{\omega}$ the measurement noise. If a measurement is available, the filtering can be performed by using the Kalman Filter equation, because the measurement equation is linear and the state is approximated with a Gaussian distribution.

6.2.2 Acoustic Sensor

The second sensor measures the wave field. The measurement equation depends on the translation and rotation. The directly observed state is estimated by using the Gaussian Filter. Based on the estimated mean and covariance matrix, the indirectly observed state is updated. The nonlinear measurement equation is given by

$$\boldsymbol{y}_k^j = \sum_{i=1}^{N} \tfrac{1}{4\pi \cdot \left\|\mathbf{D}(\underline{\boldsymbol{r}}_k) \cdot \underline{\tilde{p}}^j + \underline{\boldsymbol{T}}_k - \underline{x}^i\right\|} \cdot \sum_{m=0}^{k} s_m^i \cdot \operatorname{sinc}\left((k-m) \cdot \pi - \tfrac{\left\|\mathbf{D}(\underline{\boldsymbol{r}}_k) \cdot \underline{\tilde{p}}^j + \underline{\boldsymbol{T}}_k - \underline{x}^i\right\|}{c \cdot T}\right) + \boldsymbol{v}_k^j,$$

where $\mathbf{D}\left(\underline{\boldsymbol{r}}_k\right)$ is the rotation matrix

$$\mathbf{D}\left(\underline{\boldsymbol{r}}_k\right) = \mathbf{I}_{3,3} + \tfrac{\sin\left(\|\underline{\boldsymbol{r}}_k\|\right)}{\|\underline{\boldsymbol{r}}_k\|} \mathbf{C}(\underline{\boldsymbol{r}}_k) + \tfrac{1 - \cos\left(\|\underline{\boldsymbol{r}}_k\|\right)}{\|\underline{\boldsymbol{r}}_k\|^2} \mathbf{C}(\underline{\boldsymbol{r}}_k) \mathbf{C}(\underline{\boldsymbol{r}}_k).$$

The measurement equation describes the wave propagation of the N sources to the jth microphone. The signal of each source is characterized by the symbols s_m^i, where m is the discrete time of the source and k of the sensor. The sinc-interpolation [11] is used to get a continuous time signal in order to achieve subsample resolution. T is the sampling interval, c the velocity of sound, and $\boldsymbol{v}_k^j$ the measurement noise of the sensor j.

7 Results

The proposed approach is evaluated in a simulation and in an experiment.

7.1 Simulation

In the simulation setup, a moving target object is considered. The trajectory of the object is simulated by piecewise constant translational and angular velocities. Four microphones are attached to the target, which are receiving MC-CDMA (multi carrier-code division multiple access) signals from four loudspeakers in order to achieve a distinguishable mapping. For the simulation, the signals are delayed depending on the time-variant distance between microphone

and loudspeaker. Furthermore, an inertial measurement unit consisting of three gyroscopes measures the angular velocity with respect to the target coordinate system. The measurement frequency of the microphones and the gyroscopes are 48000 Hz and 480 Hz, respectively. The signals received by the microphones are corrupted with noise. This noise is generated by mirror image sources in order to model reverberations. The signals emitted by the four loudspeakers are reflected at walls of the room, which is modeled by 24 mirror image sources. The attenuation factor of the walls is set to be 0.5, which results in an SNR of 4.247 dB. The angular velocity is corrupted by additive zero-mean Gaussian noise with covariance matrix $\mathbf{R}_{\underline{v}^{\omega}} = \mathbf{I}_{3,3} \cdot 10^{-4}$.

In the simulation, a constant position model is used due to the fact that the simulated velocity has points of discontinuity. The initial state and covariance matrix is set to

$$\underline{\mu}^{r} = \begin{bmatrix} 0 & -0.0245 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}_0^{r,r} = \mathbf{I}_{3,3} \cdot (10^{-3} \cdot \pi/180)^2$$
$$\underline{\mu}^{T} = \begin{bmatrix} -0.9535 & -0.9969 & 2 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}_0^{T,T} = \mathbf{I}_{3,3} \cdot 10^{-6}$$
$$\underline{\mu}^{\omega} = \begin{bmatrix} 0 & -0.6911 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}_0^{\omega,\omega} = \mathbf{I}_{3,3} \cdot 10^{-3}$$

and the process and measurement noise to

$$\mathbf{Q}_{\underline{w}^{\omega}} = \mathbf{I}_{3,3} \cdot 5.1404 \cdot 10^{-5}$$
$$\mathbf{Q}_{\underline{w}_n^z} = \mathrm{diag}\begin{bmatrix} 5.0 \cdot 10^{-7} & 5.0 \cdot 10^{-7} & 5.0 \cdot 10^{-8} \end{bmatrix}$$
$$\mathbf{R}_{\underline{v}} = \mathbf{I}_{4,4} \cdot 10^{-3}, \quad \mathbf{R}_{\underline{v}^{\omega}} = \mathbf{I}_{3,3} \cdot 10^{-4} \ .$$

For the approximation, five sample points for each dimension are used. In the first simulation run, no decomposition is performed. Furthermore, the measurement and process noise is augmented in the state vector. In this case, a total of 89 sample points are used for approximation.

The second simulation run takes advantages of the structure of the system and measurement equation in order to reduce the computational effort. For the prediction step, the state vector is separated into the linear and nonlinear part, where only 13 sample points are used. For the filtering step, the Kalman Filter equations are used when measurements from the inertial measurement unit are available. In the other case when the microphones measures the wave field, the state vector is separated into the directly observed and indirectly observed part, which results in an approximation with 25 sample points.

The computation time decreases by a factor of four compared to the first simulation run. However, the accuracy decreased for the rotation vector, when the change of the angular velocity is high. In contrast, the mean of the root square error (RMSE) for the decomposed algorithm (GF) is lower than form the standard algorithm (GF, UKF). The average of the RMSE for the rotation and the translation is shown in Table 2. The RMSE for the rotation and the translation is shown in Fig. 3, respectively.

7.2 Experiment

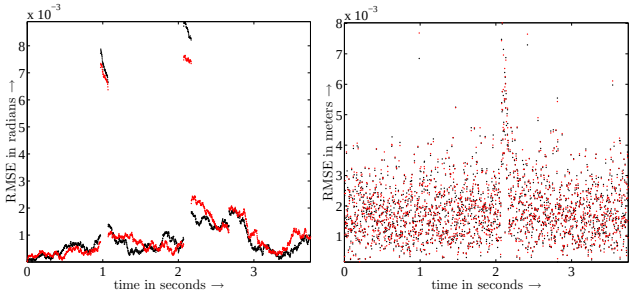
An object is moved ten times from a starting point to an end point on a straight line. The experimental setup is similar to the simulation, where in the experiment a constant velocity model is assumed. Furthermore, five loudspeakers are periodically emitting the signals and the sampling frequency of the gyroscopes are 200 Hz. The initial state and covariance matrix is set to

$$\underline{\mu}^{r} = \begin{bmatrix} 0 & 0 & \pi/2 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}_0^{r,r} = \mathbf{I}_{3,3}(1 \cdot \pi/180)^2$$
$$\underline{\mu}^{T} = \begin{bmatrix} -0.05 & -0.75 & 1.16 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}_0^{T,T} = \mathbf{I}_{3,3} \cdot 10^{-4}$$
$$\underline{\mu}^{\omega} = \underline{\mu}^{V} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \quad \mathbf{C}_0^{\omega,\omega} = \mathbf{C}_0^{V,V} = \mathbf{I}_{3,3} \cdot 9 \cdot 10^{-6},$$

respectively. The covariance matrix of process and measurement noise was

$$\mathbf{Q}_{\underline{w}^{\omega}} = \mathbf{I}_{3,3} \cdot 6.3462 \cdot 10^{-6}$$
$$\mathbf{Q}_{\underline{w}_n^x} = \mathbf{Q}_{\underline{w}_n^y} = \begin{bmatrix} 3.014 \cdot 10^{-14} & 2.170 \cdot 10^{-10} \\ 2.170 \cdot 10^{-10} & 2.083 \cdot 10^{-6} \end{bmatrix}$$
$$\mathbf{Q}_{\underline{w}_n^z} = \mathbf{Q}_{\underline{w}_n^y} \cdot 10^{-4}, \quad \mathbf{R}^{M} = \mathbf{I}_{4,4} \cdot 0.09, \quad \mathbf{R}^{\omega} = \mathbf{I}_{3,3} \cdot 0.01.$$

In Fig. 4, the results for the five test run are shown. The measured end point and the distance between start and end point is given in Table 3. In addition, the average over the ten test runs and the standard deviation is listed.



(a) RMSE for the rotation vector. (b) RMSE for the translation vector.

Figure 3: RMSE for the rotation and translation vector, where the black points represent the decomposed algorithm and the red points result from the full state case.

	Trans. in 10^{-4} m	Rot. in 10^{-4} rad
Dec. GF	19 ± 9.8515	11 ± 17
Full state GF	19 ± 9.9196	12 ± 16
Dec. UKF	19 ± 10.3126	20 ± 35
Full state UKF	18 ± 10.0988	19 ± 14

Table 2: The average and the standard deviation of the RMSE for the rotation and the translation. The Gaussian Filter is compared to the UKF.

8 Conclusions

In this paper, two principles for reducing the computational effort for state estimation in nonlinear systems are discussed and exploited for the Gaussian Filter. These decompositions exploit the structure of the nonlinear system equations and facilitate to reduce the number of sample points for approximating the state density. In doing so, the computation time

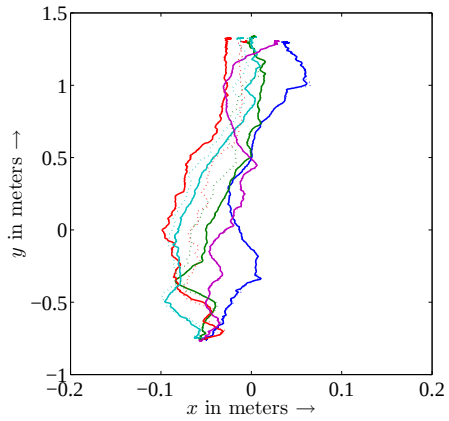


Figure 4: Results from five test runs. The solid lines are the estimates from the full state case. The dashed lines are the results from the decomposed case.

Measured	x in m	y in m	z in m
End point	-0.05	1.33	1.16
Distance		2.08	
Full state	x in m	y in m	z in m
End point mean	0.0161	1.3102	1.0482
End point std.	0.0189	0.0218	0.0397
Distance mean		2.0659	
Distance std.		0.0189	
Decomp.	x in m	y in m	z in m
End point mean	0.01640	1.3057	1.0362
End point std.	0.01655	0.0264	0.0552
Distance mean		2.0620	
Distance std.		0.0252	

Table 3: The average results from ten test runs.

can be significantly decreased without significantly affecting the estimation quality. The advantages are shown in simulations and experiments.

In contrast to a comparable linear/nonlinear decomposition approach presented in [9] for the unscaled version of the UKF, the proposed approaches are more generally applicable. For instance, scaling the sample points can be considered, which is essential for very high-dimensional problems. Furthermore, the decomposition into directly/indirectly observed states is also taken into account.

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