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ON THE LINEARIZATION OF OPERATORS RELATED TO THE FULL WAVEFORM INVERSION IN SEISMOLOGY

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ABSTRACT. In this work we analyze the parameter-to-solution map of the acoustic wave equation with respect to its parameters wave speed and mass density. This map is a mathematical model for the seismic inverse problem where one wants to recover the parameters from measurements of the acoustic potential. We show its complete continuity and Fréchet differentiability. To this end we provide necessary existence, stability and regularity results. Moreover, we discuss various implications of our findings on the inverse problem and comment on the Born series.

1. INTRODUCTION

Our goal is to study Fréchet differentiability of the map $\mathcal{F}: (\nu, \varrho) \mapsto u$ in appropriate function spaces where u solves the scalar wave equation

$$(1) \quad \frac{1}{\varrho \nu^2} \partial_t^2 u - \nabla_{\mathbf{x}} \cdot \left(\frac{1}{\varrho} \nabla_{\mathbf{x}} u \right) = f(\mathbf{x}, t), \quad u|_{\partial\Omega} = 0.$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary and (1) is furnished with Cauchy data

$$(2) \quad u(\cdot, 0) = u_0; \quad \partial_t u(\cdot, 0) = u_1.$$

The coefficients ν and ϱ are the wave speed and the mass density of the medium, respectively. They are real valued and allowed to be spatially varying: $\nu = \nu(\mathbf{x})$, $\varrho = \varrho(\mathbf{x})$.

The map \mathcal{F} plays a prominent role in the so-called full waveform inversion in seismic imaging, see, e.g., Symes [12]. Here ν and ϱ have to be recovered from the acoustic potential $u|_M$ where M is a certain subset of $\bar{\Omega} \times [0, \infty[$ (on M the measurements are taken). Although Newton-like solvers are used in practice, sound investigations of the Fréchet differentiability of \mathcal{F} under realistic assumptions are rare. In his PhD thesis Stolk [11] proved that \mathcal{F} possesses a Gâteaux (directional) derivative which is also its Fréchet derivative as we will validate in this paper. Indeed, the solution u' of the wave equation

$$\begin{aligned} \frac{1}{\varrho \nu^2} \partial_t^2 u' - \nabla_{\mathbf{x}} \cdot \left(\frac{1}{\varrho} \nabla_{\mathbf{x}} u' \right) &= \left(\frac{2h_1}{\varrho \nu^3} + \frac{h_2}{\varrho^2 \nu^2} \right) \partial_t^2 u - \nabla_{\mathbf{x}} \cdot \left(\frac{h_2}{\varrho^2} \nabla_{\mathbf{x}} u \right), \quad u'|_{\partial\Omega} = 0, \\ u'(\cdot, 0) &= \partial_t u'(\cdot, 0) = 0, \end{aligned}$$

is the Fréchet derivative of \mathcal{F} about (ν, ϱ) in the direction (h_1, h_2) (underlying spaces and further requirements will be formulated below).

First differentiability results have been reported by Bao and Symes [2]. They studied stability properties of a formal derivative of the parameter-to-solution map of a variant of (1) (only the density determination problem was considered, however, in a half space). Bao [1] extended this line of research and showed that the formal derivative is a Fréchet derivative for densities in a certain smoothness class. Very recently – during our work on

the present article – Blazek, Stolk, and Symes [3] came up with an elaborate consideration of first order hyperbolic systems which cover the first order formulation of the above wave equation as a special case. Thus, some of our results can be obtained from [3] under slightly stronger assumptions (we explain this in detail in the last section). Since their approach is very general and rather abstract an independent, self-contained, concrete study of the acoustic wave equation is justified, the more so as (1) is still a widely used model in seismology. Besides, we add many new aspects to this subject.

In the next section we lay the analytic foundation of this paper: We formulate the wave equation weakly, recall and provide existence and uniqueness results as well as stability (energy) estimates. Finally, we prove regularity of the acoustic potential with respect to time under regularity of the source term and the Cauchy data. Section 3 contains preparatory results to validate Fréchet differentiability of \mathcal{F} which is the main topic of Section 4. We also show compactness of the Fréchet derivative and discuss its implication on the seismic inverse problem. Further, we comment on the Born series and why our results do not yield its convergence. Finally, in Section 5 we relate the work of Blazek et al. [3] to ours.

2. THE ANALYTIC SETTING

In a first step we consider

$$(3) \quad c \partial_t^2 u - \nabla_{\mathbf{x}} \cdot (r \nabla_{\mathbf{x}} u) = f(\mathbf{x}, t), \quad u|_{\partial\Omega} = 0,$$

with initial data (2). Let $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. We search for the solution u in the space

$$X := \mathcal{C}^0([0, T], V) \cap \mathcal{C}^1([0, T], H)$$

equipped with the canonical norm

$$\|u\|_X := \left(\max_{0 \leq t \leq T} \|u(t)\|_V^2 + \max_{0 \leq t \leq T} \|\dot{u}(t)\|_H^2 \right)^{1/2}.$$

The dot on top of a time dependent function indicates the time derivative. With this norm X is a Banach space.

The weak formulation of (3) reads:

Given $c, r \in L^\infty(\Omega)$ and $u_0 \in V$ and $u_1 \in H$ and $f \in L^2([0, T] \times \Omega) = L^2((0, T), H)$ find $u \in X$ with $u(0) = u_0$ and $\dot{u}(0) = u_1$ such that

$$(4) \quad \int_0^T \left(a_r(u(t), v(t)) - \langle c \dot{u}(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T \langle f(t), v(t) \rangle_H dt$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$,

see e.g. Lions and Magenes [7, Chap. 3.8] or Stolk [11, Sec. 2.4]. Here, $\langle \cdot, \cdot \rangle_H$ denotes the inner product in H and

$$a_r: V \times V \rightarrow \mathbb{R}, \quad a_r(\psi, \varphi) = \int_\Omega r \nabla_{\mathbf{x}} \psi \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x}.$$

We assume that the coefficients $c, r \in L^\infty(\Omega)$ are bounded away from zero; that is,

$$(5) \quad c(\mathbf{x}) \geq c_- \text{ a.e. and } r(\mathbf{x}) \geq r_- \text{ a.e.}$$

for positive constants c_- and r_- .

Under the above hypotheses Stolk [11, Lemma 2.4.1] has shown an energy estimate for u resulting in¹

$$(6) \quad \|u(t)\|_V^2 + \|\dot{u}(t)\|_H^2 \lesssim \|u_0\|_V^2 + \|u_1\|_H^2 + \int_0^T \|f(\tau)\|_H^2 d\tau,$$

that is,

$$(7) \quad \|u\|_X^2 \lesssim \|u_0\|_V^2 + \|u_1\|_H^2 + \int_0^T \|f(\tau)\|_H^2 d\tau.$$

By this estimate he proved that (4) admits a unique solution $u \in X$ which depends continuously on u_0 , u_1 , and f , see also [7, Chap. 3.8] or the proof of Theorem 2.3 below.

Moreover, for almost all $s \in]0, T[$ we have that

$$(8) \quad a_r(u(s), w) + \langle c\ddot{u}(s), w \rangle_{V' \times V} = \langle f(s), w \rangle_H \quad \text{for all } w \in V,$$

where $\langle \cdot, \cdot \rangle_{V' \times V}$ is the duality pairing in $V' \times V$. In particular

$$c\ddot{u} \in L^2([0, T], V') \text{ and } \ddot{u} \in L^2([0, T], V') \text{ provided } c \in W^{1, \infty}(\Omega).$$

We will now verify (8) following Stolk [11, p. 23]: Let $\{\phi_m\}_{m \in \mathbb{N}} \subset \mathcal{C}_0^\infty(-1, 1)$ be a regularizing sequence, that is, $\phi_m \geq 0$, $\int \phi_m(t) dt = 1$, and $\text{supp } \phi_m = [-1/m, 1/m]$. Let $\varepsilon > 0$, $1/m < \varepsilon$, and $s \in [\varepsilon, T - \varepsilon]$. Then, $v(\cdot) = \phi_m(s - \cdot)w \in \mathcal{C}_0^\infty([0, T], V)$ for any $w \in V$. Plugged into (4) and taking into account that $\int \phi_m(s - t)u(t) dt = (\phi_m \star u)(s)$ yields

$$a_r((\phi_m \star u)(s), w) + \langle c(\dot{\phi}_m \star \dot{u})(s), w \rangle_{V' \times V} = \langle (\phi_m \star f)(s), w \rangle_H$$

which results in (8) for $m \rightarrow \infty$.

Example 2.1. We include an example to show that – in contrast to elliptic and parabolic boundary value problems – the weaker assumption $f \in L^2([0, T], V')$ in (4) is not sufficient to guarantee $u \in L^2([0, T], V)$. Let $r = 1$ and $c = 1$ and Ω be the square $\Omega =]0, \pi[\times]0, \pi[$ in \mathbb{R}^2 . The functions

$$v_n(\mathbf{x}) = \frac{2}{\pi |n|} \sin(n_1 x_1) \sin(n_2 x_2), \quad \mathbf{x} = (x_1, x_2) \in \Omega, \quad n = (n_1, n_2) \in \mathbb{N}^2,$$

are eigenfunctions of $-\Delta_{\mathbf{x}}$ in Ω with respect to Dirichlet boundary conditions corresponding to the eigenvalues $\lambda_n = |n|^2$. They are normalized such that $\|\nabla v_n\|_{L^2(\Omega)} = 1$ as well as $\|v_n\|_{L^2(\Omega)} = 1/|n|$. Furthermore, $\{v_n\}$ and $\{|n|v_n\}$ are complete orthonormal systems in $(V, \langle \nabla \cdot, \nabla \cdot \rangle_{L^2(\Omega)})$ and $L^2(\Omega)$, respectively. Define the source function

$$f(\mathbf{x}, t) = \sum_{n \in \mathbb{N}^2} \rho_n |n|^2 \cos(|n|t) v_n(\mathbf{x})$$

for any coefficients ρ_n with $\sum_{n \in \mathbb{N}^2} \rho_n^2 < \infty$. Then $f(t) = f(\cdot, t) \in V' = H^{-1}(\Omega)$ as seen from the form

$$\langle f(t), \psi \rangle_{V' \times V} = \sum_{n, m \in \mathbb{N}^2} \rho_n |n|^2 \cos(|n|t) \psi_m \langle v_n, v_m \rangle_{L^2(\Omega)} = \sum_{n \in \mathbb{N}^2} \rho_n \psi_n \cos(|n|t)$$

¹ $A \lesssim B$ indicates the existence of a generic constant m such that $A \leq mB$ uniformly in all relevant parameters of the expressions A and B . The respective context will define the meaning of 'relevant parameters'.

where $\psi_m = \langle \nabla \psi, \nabla v_m \rangle_{L^2(\Omega)}$. The solution of $\partial_t^2 u(t) - \Delta_{\mathbf{x}} u(t) = f(t)$ with $u(0) = \dot{u}(0) = 0$ is given by

$$u(t) = -\frac{1}{2} \sum_{n \in \mathbb{N}^2} \rho_n |n| t \sin(|n|t) v_n.$$

Obviously, ρ_n can be chosen such that $u(t) \in L^2(\Omega)$ fails to be in $V = H_0^1(\Omega)$, for instance, $\rho_n = 1/|n|$.

Remark 2.2. We motivate why the Dirichlet boundary restriction in (1) and (4) is – to a certain extent – physically meaningful for seismic wave propagation in free space. Indeed, due to the finite wave propagation speed homogeneous Dirichlet boundary conditions can be safely assumed in case Ω is sufficiently large: To this end consider the Cauchy problem

$$c \partial_t^2 u - \nabla_{\mathbf{x}} \cdot (r \nabla_{\mathbf{x}} u) = f \quad \text{in } \mathbb{R}^d \times]0, T[,$$

with initial conditions $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$ in \mathbb{R}^d where $f(\cdot, t)$, u_0 , u_1 have compact support in some ball $B(0, R)$ of radius R . Furthermore, $c(\mathbf{x}) = c_0$ and $r(\mathbf{x}) = 1$ for all $|\mathbf{x}| \geq R$ for some constant $c_0 > 0$. Let $\Omega = B(0, \hat{R})$ be the ball of radius \hat{R} which satisfies $\hat{R} > R + T/c_0$. For $\mathbf{z} \in \partial\Omega$ we conclude that $B(\mathbf{z}, T/c_0) \cap B(0, R) = \emptyset$, because $|\mathbf{x}| \geq |\mathbf{z}| - |\mathbf{x} - \mathbf{z}| \geq \hat{R} - T/c_0 > R$ for $\mathbf{x} \in B(\mathbf{z}, T/c_0)$. Therefore, fixing $\mathbf{z} \in \partial\Omega$ and defining the cones

$$C = \{(\mathbf{x}, t) : c_0 |\mathbf{x} - \mathbf{z}| \leq T - t\} \quad \text{and} \quad C_t = \{\mathbf{x} : c_0 |\mathbf{x} - \mathbf{z}| \leq T - t\}$$

we conclude that $u(\mathbf{x}, 0)$ vanishes in $C_0 = B(\mathbf{z}, T/c_0)$. By a well-known result (see Evans [4]) we conclude that u vanishes in all of C and, in particular, $u(\mathbf{z}, t) = 0$ for all $t \in [0, T]$. This holds for all $\mathbf{z} \in \partial\Omega$. Therefore, u is a solution of the boundary–initial value problem (3) with $u(\cdot, 0) = u_0$ and $\partial_t u(\cdot, 0) = u_1$ in Ω .

In our subsequent considerations we encounter a wave equation with a source term different to the one of (4):

Given $g \in \mathcal{C}^1([0, T], V)$, find $w \in X$ such that $w(0) = \dot{w}(0) = 0$ and

$$(9) \quad \int_0^T \left(a_r(w(t), v(t)) - \langle c \dot{w}(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T a_r(g(t), v(t)) dt$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$.

As we are not aware of a reference for existence, uniqueness, and stability for this setting we will provide a proof.

Theorem 2.3. Let $g \in \mathcal{C}^1([0, T], V)$. Then, (9) admits a unique solution $w \in X$ satisfying

$$(10) \quad \|w\|_X \lesssim \|g\|_{\mathcal{C}^1([0, T], V)}$$

where the constant in the estimate depends only on T , r_- , $\|r\|_\infty$, and c_- .

Proof. The argument for uniqueness is as follows. Let d be the difference of two solutions of (9). Then, d solves (4) with data $u_0 = u_1 = f = 0$. Thus, $d = 0$ by (7).

To prove existence of a solution we generalize the approach of Example 2.1 and formulate (9) as a series of ordinary differential equations relying on the spectral theorem: There is a sequence of positive eigenvalues $\{\lambda_k\}_{k \in \mathbb{N}}$ converging to infinity and corresponding eigenfunctions $\{v_k\}_{k \in \mathbb{N}} \subset V$ such that

$$a_r(v_k, \varphi) - \lambda_k \langle c v_k, \varphi \rangle_H = 0 \quad \text{for all } \varphi \in V.$$

Furthermore, the sets $\{v_k\}_{k \in \mathbb{N}}$ and $\{\sqrt{\lambda_k} v_k\}_{k \in \mathbb{N}}$ form complete orthonormal systems in (V, a_r) and $(H, \langle c \cdot, \cdot \rangle_H)$, respectively, see, e.g., [4, Chap. 6.5]. Note that a_r defines an inner product in V which is equivalent to the ordinary inner product by Friedrich's inequality. We denote the norms in (V, a_r) and $(H, \langle c \cdot, \cdot \rangle_H)$ by $\|\cdot\|_{V, a_r}$ and $\|\cdot\|_{H, c}$, respectively. Then we have estimates of the form

$$(11) \quad \|r\|_\infty \|\varphi\|_V^2 \geq \|\varphi\|_{V, a_r}^2 \geq r_- \|\nabla \varphi\|_{L^2(\Omega)}^2 \geq \gamma^2 r_- \|\varphi\|_V^2, \quad \varphi \in V,$$

where $\gamma > 0$ comes from Friedrich's inequality and

$$(12) \quad \|c\|_\infty \|\psi\|_H^2 \geq \|\psi\|_{H, c}^2 \geq c_- \|\psi\|_H^2, \quad \psi \in H.$$

We make the ansatz $w(\cdot) = \sum_{k=1}^\infty \omega_k(\cdot) v_k$ where the ω_k 's are given as unique solutions of the initial value problems

$$(13) \quad \ddot{\omega}_k(t) + \lambda_k^2 \omega_k(t) = \lambda_k^2 g_k(t), \quad \omega_k(0) = \dot{\omega}_k(0) = 0,$$

where $g_k(t) = a_r(g(t), v_k)$. Note that $\sum_{k=1}^\infty g_k(t) v_k = g(t)$ and $\|g(t)\|_{V, a_r}^2 = \sum_{k=1}^\infty |g_k(t)|^2$ and $\|\dot{g}(t)\|_{V, a_r}^2 = \sum_{k=1}^\infty |\dot{g}_k(t)|^2$.

Assume for the moment that we have shown already that $w \in X = \mathcal{C}^0([0, T], V) \cap \mathcal{C}^1([0, T], H)$. Then, w solves (9). Indeed, $w(0) = \dot{w}(0) = 0$ and for any $\varphi \in \mathcal{C}_0^\infty([0, T], V)$ with $\varphi(t) = \sum_\ell \varphi_\ell(t) v_\ell$, we have

$$\begin{aligned} & \int_0^T \left(a_r(w(t), \varphi(t)) - \langle c \dot{w}(t), \dot{\varphi}(t) \rangle_H \right) dt \\ &= \sum_{k, \ell} \int_0^T \left(\omega_k(t) \varphi_\ell(t) a_r(v_k, v_\ell) - \dot{\omega}_k(t) \dot{\varphi}_\ell(t) \langle c v_k, v_\ell \rangle_H \right) dt \\ &= \sum_k \int_0^T \left(\omega_k(t) \varphi_k(t) - \frac{1}{\lambda_k^2} \dot{\omega}_k(t) \dot{\varphi}_k(t) \right) dt = \sum_k \int_0^T \left(\omega_k(t) + \frac{1}{\lambda_k^2} \ddot{\omega}_k(t) \right) \varphi_k dt \\ &= \sum_k \int_0^T g_k(t) \varphi_k(t) dt = \int_0^T a_r(g(t), \varphi(t)) dt. \end{aligned}$$

Finally, we verify the stability estimate which shows that $w \in X = \mathcal{C}^0([0, T], V) \cap \mathcal{C}^1([0, T], H)$. We note that ω_k is explicitly given by

$$\omega_k(t) = \lambda_k \int_0^t \sin(\lambda_k(t-s)) g_k(s) ds, \quad t \in [0, T].$$

By partial integration we rewrite this as

$$\omega_k(t) = g_k(t) - \cos(\lambda_k t) g_k(0) - \int_0^t \cos(\lambda_k(t-s)) \dot{g}_k(s) ds, \quad t \in [0, T].$$

From this we observe that

$$\|\omega_k\|_\infty \leq 2 \|g_k\|_\infty + \sqrt{T} \|\dot{g}_k\|_{L^2(0, T)} \lesssim \|g_k\|_{L^2(0, T)} + \|\dot{g}_k\|_{L^2(0, T)}$$

where the constant depends on T only. Analogously,

$$\begin{aligned} \dot{\omega}_k(t) &= \lambda_k^2 \int_0^t \cos(\lambda_k(t-s)) g_k(s) ds \\ &= \lambda_k \sin(\lambda_k t) g_k(0) + \lambda_k \int_0^t \sin(\lambda_k(t-s)) \dot{g}_k(s) ds, \quad t \in [0, T], \end{aligned}$$

and thus

$$\|\dot{\omega}_k\|_\infty \leq \lambda_k \|g_k\|_\infty + \lambda_k \sqrt{T} \|\dot{g}_k\|_{L^2(0,T)} \lesssim \lambda_k (\|g_k\|_{L^2(0,T)} + \|\dot{g}_k\|_{L^2(0,T)}).$$

Again, the constant depends on T only. Defining the partial sums $w^n(t) = \sum_{k=1}^n \omega_k(t) v_k$ we note that for $m > n$

$$\begin{aligned} \max_{0 \leq t \leq T} \|w^n(t) - w^m(t)\|_{V, a_r}^2 &= \max_{0 \leq t \leq T} \sum_{k=n+1}^m \omega_k(t)^2 \leq \sum_{k=n+1}^m \|\omega_k\|_\infty^2 \\ &\lesssim \sum_{k=n+1}^m (\|g_k\|_{L^2(0,T)}^2 + \|\dot{g}_k\|_{L^2(0,T)}^2). \end{aligned}$$

Analogously, we make use of the fact that $\{\lambda_k v_k : k \in \mathbb{N}\}$ is an orthonormal system in $(H, \langle c \cdot, \cdot \rangle_H)$. Therefore,

$$\begin{aligned} \max_{0 \leq t \leq T} \|\dot{w}^n(t) - \dot{w}^m(t)\|_{H, c}^2 &= \max_{0 \leq t \leq T} \sum_{k=n+1}^m \frac{1}{\lambda_k^2} \dot{\omega}_k(t)^2 \leq \sum_{k=n+1}^m \frac{1}{\lambda_k^2} \|\dot{\omega}_k\|_\infty^2 \\ &\lesssim \sum_{k=n+1}^m (\|g_k\|_{L^2(0,T)}^2 + \|\dot{g}_k\|_{L^2(0,T)}^2). \end{aligned}$$

Again, the constants in the estimates depend on T only. Combining these estimates yields with (11) and (12)

$$\begin{aligned} \|w^n - w^m\|_X^2 &\leq \frac{1}{\gamma \sqrt{r_-}} \max_{0 \leq t \leq T} \|w^n(t) - w^m(t)\|_{V, a_r} + \frac{1}{\sqrt{c_-}} \max_{0 \leq t \leq T} \|\dot{w}^n(t) - \dot{w}^m(t)\|_{H, c} \\ &\lesssim \sum_{k=n+1}^m (\|g_k\|_{L^2(0,T)}^2 + \|\dot{g}_k\|_{L^2(0,T)}^2) \end{aligned}$$

where the constant in the estimate depends on T , r_- , and c_- only. This shows that $\{w^n\}$ is a Cauchy sequence in X because of the convergence

$$\begin{aligned} \sum_{k=1}^{\infty} (\|g_k\|_{L^2(0,T)}^2 + \|\dot{g}_k\|_{L^2(0,T)}^2) &= \int_0^T (\|g(t)\|_{V, a_r}^2 + \|\dot{g}(t)\|_{V, a_r}^2) dt \\ &\leq \|r\|_\infty [\|g\|_{L^2((0,T), V)}^2 + \|\dot{g}\|_{L^2((0,T), V)}^2]. \end{aligned}$$

Therefore, $w \in X$, and the proof is complete. \square

In the remainder of this section we study how higher regularity of u_0 , u_1 , and f transfers to the solution of (4). Similar results can be found, e.g., in [13, §30] or [4, Sec. 7.2.3]. For the sake of completeness and self-containedness we present elementary proofs.

Recall that a_r , $r \in L^\infty(\Omega)$, induces a bounded operator $A_r : V \rightarrow V'$ by $a_r(v, w) = \langle A_r v, w \rangle_{V' \times V} = \langle A_r w, v \rangle_{V' \times V}$ for all $v, w \in V$, that is, $A_r v = -\nabla_x \cdot (r \nabla_x v)$.

Lemma 2.4. *Let $f \in \mathcal{C}^1([0, T], H)$. Further, let u_1 be in V and let $u_0 \in V$ satisfy $A_r u_0 \in H$. Then,*

$$u \in \mathcal{C}^1([0, T], V) \cap \mathcal{C}^2([0, T], H) \quad \text{and} \quad c \ddot{u} \in L^2([0, T], V').$$

In particular, $\dot{u} \in X$. Furthermore,

$$(14) \quad \|u\|_X + \|\dot{u}\|_X \lesssim \|u_1\|_V + \|A_r u_0\|_H + \|f\|_{\mathcal{C}^1([0, T], V)}$$

where the constant in the estimate depends only on T , r_- , $\|r\|_\infty$, c_- , and $\|c\|_\infty$.

Proof. We formulate (8) as

$$(15) \quad c\ddot{u}(t) = f(t) - A_r u(t), \quad t \in]0, T[,$$

where the equality holds in V' . Taking the limit $t \rightarrow 0$ shows that $c\ddot{u}(0) = f(0) - A_r u_0 \in H$ due to our assumptions on f and u_0 and thus also $\ddot{u}(0) \in H$. Therefore, the following problem is well defined with a unique solution $w \in X$:

Find $w \in X$ with $w(0) = u_1 \in V$ and $\dot{w}(0) = \dot{u}(0) \in H$ such that

$$\int_0^T \left(a_r(w(t), v(t)) - \langle c\dot{w}(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T \langle \dot{f}(t), v(t) \rangle_H dt \quad \text{for all } v \in \mathcal{C}_0^\infty([0, T], V).$$

We are done if we can show that $\dot{w}(t) \in V$. Indeed, then \dot{w} solves above problem with the correct initial values which yields $w = \dot{u}$. Also the estimate (14) holds by (7).²

Let $\eta \in \mathbb{R}$ such that $0 < |\eta| < T$. Define $t_{\min} = t_{\min}(\eta) = \max\{0, -\eta\}$ and $t_{\max} = t_{\max}(\eta) = \min\{T, T - \eta\}$. By construction $0 \leq t_{\min} < t_{\max} \leq T$. Further, let $D_\eta \ell(t) = (\ell(t + \eta) - \ell(t))/\eta$ for any time dependent function ℓ . Consider now the problem:

Find $w_\eta \in \mathcal{C}^0([t_{\min}, t_{\max}], V) \cap \mathcal{C}^1([t_{\min}, t_{\max}], H)$ with $w_\eta(t_{\min}) = D_\eta u(t_{\min}) \in V$ and $\dot{w}_\eta(t_{\min}) = D_\eta \dot{u}(t_{\min}) \in H$ such that

$$(16) \quad \int_{t_{\min}}^{t_{\max}} \left(a_r(w_\eta(t), v_\eta(t)) - \langle c\dot{w}_\eta(t), \dot{v}_\eta(t) \rangle_H \right) dt = \int_{t_{\min}}^{t_{\max}} \langle D_\eta f(t), v_\eta(t) \rangle_H dt$$

for all $v_\eta \in \mathcal{C}_0^\infty([t_{\min}, t_{\max}], V)$.

Since $\mathcal{C}_0^\infty([t_{\min}, t_{\max}], V) \subset \mathcal{C}_0^\infty([0, T], V)$ we have $w_\eta(t) = D_\eta u(t)$, $t \in [t_{\min}, t_{\max}]$, which can be verified by the following line of arguments: subtract (8) for $s = t$ from (8) for $s = t + \eta$, divide the difference by η , set $w = v_\eta(t)$ and finally integrate over $[t_{\min}, t_{\max}]$. Thus, $D_\eta u$ satisfies (16) with the right initial values and the corresponding energy estimate (6) reads

$$\|D_\eta u(t)\|_V^2 + \|D_\eta \dot{u}(t)\|_H^2 \lesssim \|D_\eta u(t_{\min})\|_V^2 + \|D_\eta \dot{u}(t_{\min})\|_H^2 + \int_{t_{\min}}^{t_{\max}} \|D_\eta f(\tau)\|_H^2 d\tau$$

where $t \in]t_{\min}, t_{\max}[$. Letting η approach 0 results in $\dot{u}(t) \in V$ and the proof is complete. \square

By an inductive argument we get even higher regularity.

Corollary 2.5. *Let $f \in \mathcal{C}^k([0, T], H)$, $k \geq 2$, and assume that $u_0, u_1 \in V$. Define*

$$u_i := c^{-1}(f^{(i-2)}(0) - A_r u_{i-2}), \quad i = 2, \dots, k+1,$$

and assume that $u_i \in V$ for $i = 2, \dots, k$ and $A_r u_{k-1} \in H$. Then $u_i = u^{(i)}(0)$ for all $i = 0, \dots, k+1$ and

$$u \in \mathcal{C}^k([0, T], V) \cap \mathcal{C}^{k+1}([0, T], H).$$

Proof. We will only sketch the arguments. Assume the result to hold true for k ($k = 1$ is assured by Lemma 2.4 above), that is, $u^{(k)} \in X$ uniquely satisfies $u^{(k)}(0) = u_k$ and $u^{(k+1)}(0) = u_{k+1}$ and

$$\int_0^T \left(a_r(u^{(k)}(t), v(t)) - \langle cu^{(k+1)}(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T \langle f^{(k)}(t), v(t) \rangle_H dt$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$. Now, we are in a position to carry over the proof of Lemma 2.4. \square

²Note that the constant in (7) depends only on T , r_- , $\|r\|_\infty$, c_- , and $\|c\|_\infty$.

3. THE PARAMETER TO SOLUTION OPERATOR AND ITS PROPERTIES

Now we are able to define and study the forward map

$$(17) \quad F: \mathbf{D}(F) \subset L^\infty(\Omega) \times L^\infty(\Omega) \rightarrow X, \quad (c, r) \mapsto u,$$

where $u \in X$ solves (3) in the weak sense (4) and the domain of definition of F is

$$\mathbf{D}(F) = \{(c, r) \in L^\infty(\Omega) \times L^\infty(\Omega) : c(\mathbf{x}) \geq k_-, r(\mathbf{x}) \geq k_-, \text{ a.e.}\}$$

where $k_- := \max\{c_-, r_-\}$. First we show a compactness result.

Lemma 3.1. *Let $f \in \mathcal{C}^1([0, T], H)$, $u_0 = 0$ and $u_1 \in V$. Then, F is uniformly Lipschitz continuous on bounded subsets of $\mathbf{D}(F)$.*

Proof. Let $(r_j, c_j) \in \mathbf{D}(F)$, $j = 1, 2$, and set $u_j = F(r_j, c_j)$. By Lemma 2.4 we note that $u_j \in \tilde{X} = \mathcal{C}^1([0, T], V) \cap \mathcal{C}^2([0, T], H)$. We subtract the variational equations for u_1 and u_2 and have for $\psi \in X$ with $\psi(0) = \psi(T) = 0$:

$$\begin{aligned} & \int_0^T \left[\langle \dot{u}_1(t) - \dot{u}_2(t), c_2 \dot{\psi}(t) \rangle_H - a_{r_2}(u_1(t) - u_2(t)) \right] dt \\ &= \int_0^T \langle (c_2 - c_1) \dot{u}_1(t), \dot{\psi}(t) \rangle_H dt - \int_0^T a_{r_2 - r_1}(u_1(t), \psi(t)) dt \\ &= \int_0^T \langle (c_1 - c_2) \ddot{u}_1(t), \psi(t) \rangle_H dt - \int_0^T a_{r_2 - r_1}(u_1(t), \psi(t)) dt. \end{aligned}$$

Now, for every $t \in [0, T]$ we determine $w(t) \in V$ with

$$a_{r_2}(w, \psi) = a_{r_2 - r_1}(u_1, \psi) \quad \text{for all } \psi \in V.$$

The differentiability properties of u_1 yield that also $w \in \mathcal{C}^1([0, T], V)$ and

$$\|w\|_{\mathcal{C}^1([0, T], V)} \leq r_-^{-1} \|r_2 - r_1\|_\infty \|u_1\|_{\mathcal{C}^1([0, T], V)}.$$

Thus,

$$\begin{aligned} & \int_0^T \left[\langle \dot{u}_1(t) - \dot{u}_2(t), c_2 \dot{\psi}(t) \rangle_H - a_{r_2}(u_1(t) - u_2(t), \psi(t)) \right] dt \\ &= \int_0^T \langle (c_1 - c_2) \ddot{u}_1(t), \psi(t) \rangle_H dt - \int_0^T a_{r_2}(w(t), \psi(t)) dt \end{aligned}$$

for $\psi \in X$ with $\psi(0) = \psi(T) = 0$. Therefore, the difference $u_1 - u_2$ satisfies the inhomogeneous wave equation with a source term being the sum of the source terms of (4) and (9) where $f(t) = (c_1 - c_2) \ddot{u}_1(t)$ and $g = w$, respectively. Furthermore, $u_1 - u_2$ satisfies homogeneous initial conditions. Formulas (7) and (10) yield

$$\begin{aligned} \|u_1 - u_2\|_X &\lesssim \|c_1 - c_2\|_\infty \|\ddot{u}_1\|_{L^2((0, T) \times \Omega)} + \|w\|_{\mathcal{C}^1([0, T], V)} \\ &\lesssim \|c_1 - c_2\|_\infty + \|r_1 - r_2\|_\infty \end{aligned}$$

which proves the assertion since the constant is uniformly bounded on bounded subsets of $\mathbf{D}(F)$. \square

Theorem 3.2. *Let $f \in \mathcal{C}^1([0, T], H)$, $u_0 = 0$ and $u_1 \in V$. The map $F: (c, r) \mapsto u$ is completely continuous from $\mathbf{D}(F)$ into $\mathcal{C}([0, T], H)$, that is, it is continuous and maps bounded sets of $\mathbf{D}(F)$ into relative compact sets in $\mathcal{C}([0, T], H)$. Here, $u \in X$ solves (3) in the weak sense (4).*

Proof. First we show that F maps bounded sets of $\mathbf{D}(F)$ into bounded sets of $\tilde{X} = \mathcal{C}^1([0, T], V) \cap \mathcal{C}^2([0, T], H)$. Let $Q \subset \mathbf{D}(F)$ be bounded, that is, there exist $k_+ > 0$ with $k_- \leq c(\mathbf{x}) \leq k_+$ and $k_- \leq r(\mathbf{x}) \leq k_+$ for almost all $\mathbf{x} \in \Omega$. Let $F(c, r) = u \in \tilde{X}$ be the corresponding solution. Note that $u \in \tilde{X}$ by Lemma 2.4. Formula (14) yields

$$\|u\|_X + \|\dot{u}\|_X \lesssim \|u_1\|_V + \|f\|_{\mathcal{C}^1([0, T], V)}$$

where the constant in the estimate depends only on T , k_- , and k_+ . This shows boundedness of $F(Q)$ in \tilde{X} . We show now that $F(Q)$ is relatively compact in $\mathcal{C}([0, T], H)$ by the (general) theorem of Arzelà–Ascoli, see, e.g., [6, Theorem 3.1]. Indeed, for any $t \in [0, T]$ we have that $\{u(t) : u \in F(Q)\} \subset \{v \in V : \|\nabla_{\mathbf{x}} v\|_{L^2(\Omega)^d} \leq \hat{c}\}$ for some \hat{c} , and the latter set is relatively compact in $H = L^2(\Omega)$. Furthermore, $F(Q)$ is equi-continuous because for $u \in F(Q)$

$$\begin{aligned} \|u(t_2) - u(t_1)\|_H &= \sup_{\|\psi\|_H=1} \langle u(t_2) - u(t_1), \psi \rangle_H = \sup_{\|\psi\|_H=1} \int_{t_1}^{t_2} \frac{d}{ds} \langle u(s), \psi \rangle_H ds \\ &= \sup_{\|\psi\|_H=1} \int_{t_1}^{t_2} \langle \dot{u}(s), \psi \rangle_H ds \leq |t_2 - t_1| \|\dot{u}\|_{\mathcal{C}([0, T], H)} \leq \tilde{c} |t_2 - t_1|. \end{aligned}$$

The continuity of F is due to Lemma 3.1. \square

3.1. Partial derivative with respect to c . Under regularity assumptions on the data u_0 , u_1 , and f we can show Fréchet differentiability of F as defined in (17).

Theorem 3.3. *In (4) let $u_0 = 0$, $u_1 \in V$ with $A_r u_1 \in H$, and $f \in \mathcal{C}^2([0, T], H)$ with $f(0) = 0$. Then, at every $(c, r) \in \mathbf{D}(F)$ the Fréchet derivative of $F(\cdot, r)$ with respect to c is the bounded linear operator $\partial_1 F(c, r) \in \mathcal{L}(L^\infty(\Omega), X)$ defined by $\partial_1 F(c, r)h = u'$ where $u' \in X$ uniquely solves*

$$(18) \quad \int_0^T \left(a_r(u'(t), v(t)) - \langle c\dot{u}'(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T \langle h\dot{u}(t), \dot{v}(t) \rangle_H dt$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$ with initial data $u'(0) = 0$, $\dot{u}'(0) = 0$, where u is the solution of (4).

Proof. The hypotheses of Corollary 2.5 are met for $k = 2$. Hence, $\ddot{u}(t) \in H$ (even in V) and the right hand side of (18) can be formulated as

$$\int_0^T \langle h\dot{u}(t), \dot{v}(t) \rangle_H dt = - \int_0^T \langle h\ddot{u}(t), v(t) \rangle_H dt.$$

From (7) we get

$$\|u'\|_X^2 \lesssim \|h\|_\infty^2 \int_0^T \|\ddot{u}(\tau)\|_H^2 d\tau.$$

Recall that \dot{u} satisfies (4) with f replaced by \dot{f} and Cauchy data $\dot{u}(0) = u_1$ and $\ddot{u}(0) = c^{-1}(f(0) - A_r u_0) = 0$, that is, an energy estimate like (6) holds and yields

$$\|\ddot{u}(\tau)\|_H^2 \lesssim \|u_1\|_V^2 + \int_0^T \|\dot{f}(s)\|_H^2 ds \quad \text{for all } \tau.$$

Hence, $\partial_1 F(c, r) \in \mathcal{L}(L^\infty(\Omega), X)$. Up to here we only made use of the regularity assumptions $k = 1$ of Corollary 2.5.

Next we verify that $\partial_1 F(c, r)$ is the Fréchet derivative of F at c . Let $h \in L^\infty(\Omega)$ be such that $c + h \geq k_-$ a.e. and denote by u^+ the solution of (4) where c is replaced by

$c + h$. Note that the data u_0 , u_1 , and f are such that u^+ has the same regularity as u (Corollary 2.5). Then, $d := u^+ - u - u' \in X$ satisfies $d(0) = \dot{d}(0) = 0$ as well as

$$\begin{aligned} \int_0^T \left(a_r(d(t), v(t)) - \langle c\dot{d}(t), \dot{v}(t) \rangle_H \right) dt &= \int_0^T \langle h(\dot{u}^+(t) - \dot{u}(t)), \dot{v}(t) \rangle_H dt \\ &= - \int_0^T \langle h(\ddot{u}^+(t) - \ddot{u}(t)), v(t) \rangle_H dt \quad \text{for all } v \in \mathcal{C}_0^\infty([0, T], V). \end{aligned}$$

From (7) we obtain

$$\|d\|_X^2 \lesssim \|h\|_\infty^2 \int_0^T \|\ddot{\delta}(\tau)\|_H^2 d\tau \quad \text{where } \delta(t) = u^+(t) - u(t).$$

By (8),

$$(19) \quad a_r(\delta(s), w) + \langle c\ddot{\delta}(s), w \rangle_{V' \times V} = -\langle h\ddot{u}^+(s), w \rangle_H \quad \text{for all } w \in V \text{ and almost all } s.$$

Differentiating (19) and subsequent integration we see that $\dot{\delta} \in X$ is the unique solution of

$$\int_0^T \left(a_r(\dot{\delta}(t), v(t)) - \langle c\ddot{\delta}(t), \dot{v}(t) \rangle_H \right) dt = - \int_0^T \langle h\ddot{u}^+(t), v(t) \rangle_H dt \quad \text{for all } v \in \mathcal{C}_0^\infty([0, T], V)$$

with Cauchy data $\dot{\delta}(0) = \ddot{\delta}(0) = 0$. An application of (6)

$$\|\dot{\delta}(t)\|_H^2 \lesssim \|h\|_\infty^2 \int_0^T \|\ddot{u}^+(\tau)\|_H^2 d\tau \lesssim \|h\|_\infty^2 \quad \text{as } h \rightarrow 0$$

ends the proof. \square

The assumptions $f(0) = 0$ and $u_0 = 0$ on the data in Theorem 3.3 have been necessary to guarantee that both $u = F(c, r)$ and $u^+ = F(c + h, r)$ have the same regularity for all $h \in L^\infty(\Omega)$ with $c + h \geq k_-$:

$$(c + h)^{-1}(f(0) - A_r u_0) = 0 \in V.$$

If we allow higher regularity of c then we can weaken these assumptions.

Corollary 3.4. *Let $u_0 \in V$, $u_1 \in V$, and $f \in \mathcal{C}^2([0, T], H)$ satisfy $f(0) - A_r u_0 \in V$ and $A_r u_1 \in H$. Define*

$$\tilde{F}: \mathbf{D}(\tilde{F}) \subset W^{1,\infty}(\Omega) \times L^\infty(\Omega) \rightarrow X, \quad (c, r) \mapsto u,$$

where u solves (4) and

$$\mathbf{D}(\tilde{F}) = \{(\lambda, \kappa) \in W^{1,\infty}(\Omega) \times L^\infty(\Omega) : \lambda(\mathbf{x}) \geq k_-, \kappa(\mathbf{x}) \geq k_- \text{ a.e.}\}.$$

Then, \tilde{F} is Fréchet differentiable with respect to c and $\partial_1 \tilde{F}(c, r) \in \mathcal{L}(W^{1,\infty}(\Omega), X)$ where $\partial_1 \tilde{F}(c, r)h = u'$ with u' from (18).

Proof. Both, u and u^+ have the regularity to carry over the above proof to the present situation. \square

3.2. Partial derivative with respect to r .

Theorem 3.5. *In (4) let $u_0 = 0$, $u_1 \in V$, and let $f \in \mathcal{C}^1([0, T], H)$. Then, for every $(c, r) \in \mathbf{D}(F)$ the Fréchet derivative of $F(c, \cdot)$ with respect to r exists and is the bounded linear operator $\partial_2 F(c, r) \in \mathcal{L}(L^\infty(\Omega), X)$ defined by $\partial_2 F(c, r)h = u'$ where $u' \in X$ uniquely solves*

$$(20) \quad \int_0^T \left(a_r(u'(t), v(t)) - \langle cu'(t), \dot{v}(t) \rangle_H \right) dt = - \int_0^T a_h(u(t), v(t)) dt$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$ with homogeneous initial data $u'(0) = \dot{u}'(0) = 0$. In the above right hand side, u is the solution of (4).

Proof. First we show that (20) has a unique solution yielding the well-definedness of $\partial_2 F(c, r): L^\infty(\Omega) \rightarrow X$. To this end we reformulate the right hand side of (20) to fit formulation (9). To each $t \in [0, T]$ we therefore determine the auxiliary function $g(t) \in V$ as unique solution of the elliptic problem

$$a_r(g(t), v) = a_h(-u(t), v) \quad \text{for all } v \in \mathcal{C}_0^\infty([0, T], V).$$

The smoothness of u (Lemma 2.4) transfers to g ; that is, $g, \dot{g} \in X$. Further,

$$\|g(t)\|_V \lesssim \|h\|_\infty \|u(t)\|_V \quad \text{and} \quad \|\dot{g}(t)\|_V \lesssim \|h\|_\infty \|\dot{u}(t)\|_V.$$

In view of Theorem 2.3 problem (20) admits a unique solution with

$$(21) \quad \|u'\|_X \lesssim \|g\|_{\mathcal{C}^1([0, T], V)} \lesssim \|h\|_\infty \|u\|_{\mathcal{C}^1([0, T], V)}$$

which settles boundedness of $\partial_2 F$.

Next we validate $\partial_2 F(c, r)$ as Fréchet derivate of F with respect to $r > k_-$. Let $h \in L^\infty$ be so that $r+h \geq k_-$ a.e. and denote by u^+ the solution of (4) where r is replaced by $r+h$. The data u_0 , u_1 , and f are such that u^+ and u have the same regularity (Lemma 2.4). Then, $d := u^+ - u - u' \in X$ satisfies $d(0) = \dot{d}(0) = 0$ as well as

$$\int_0^T \left(a_r(d(t), v(t)) - \langle cd(t), \dot{v}(t) \rangle_H \right) dt = \int_0^T a_h(u(t) - u^+(t), v(t)) dt$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$. Our arguments leading to (21) yield

$$\|d\|_X \lesssim \|h\|_\infty \|\delta\|_{\mathcal{C}^1([0, T], V)}$$

where $\delta(t) = u^+(t) - u(t)$ satisfies $\delta(0) = \dot{\delta}(0) = 0$ as well as

$$\int_0^T \left(a_r(\delta(t), v(t)) - \langle c\dot{\delta}(t), \dot{v}(t) \rangle_H \right) dt = - \int_0^T a_h(u^+(t), v(t)) dt$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$. Again, arguing as above leads to

$$\|\delta(t)\|_V \lesssim \|h\|_\infty \|u^+\|_{\mathcal{C}^1([0, T], V)}$$

where $\|u^+\|_{\mathcal{C}^1([0, T], V)}$ stays bounded as $h \rightarrow 0$. Altogether, $\|d\|_X \lesssim \|h\|_\infty^2$ and we are done. \square

As in the previous section additional regularity of r allows to weaken the assumptions on the data.

Corollary 3.6. *Let $u_0 \in H^2(\Omega) \cap V$, $u_1 \in V$, and $f \in \mathcal{C}^1([0, T], H)$. Define*

$$\tilde{F}: \mathcal{D}(\tilde{F}) \subset L^\infty(\Omega) \times W^{1,\infty}(\Omega) \rightarrow L^2([0, T], V), \quad (c, r) \mapsto u,$$

where u solves (4) and

$$\mathcal{D}(\tilde{F}) = \{(\lambda, \kappa) \in L^\infty(\Omega) \times W^{1,\infty}(\Omega) : \lambda(\mathbf{x}) \geq k_-, \kappa(\mathbf{x}) \geq k_- \text{ a.e.}\}.$$

Then, \tilde{F} is Fréchet differentiable with respect to r and $\partial_2 \tilde{F}(c, r) \in \mathcal{L}(W^{1,\infty}(\Omega), X)$ where $\partial_2 \tilde{F}(c, r)h = u'$ with u' from (20).

Proof. The data are such that the hypotheses of Lemma 2.4 are met independently of $h \in W^{1,\infty}(\Omega)$. Indeed, $A_{r+h}u_0 \in H$ for all h with $r+h \geq k_-$. Thus, $u = \tilde{F}(c, r)$ and $u^+ = \tilde{F}(c, r+h)$ have the required regularity to carry over the proof of Theorem 3.5 to the present setting. \square

4. FRÉCHET DIFFERENTIABILITY OF THE FULL OPERATOR

The operator \mathcal{F} mentioned in the Introduction will be defined exactly with the help of the auxiliary map F from (17). Then, the differentiability of \mathcal{F} follows immediately from the differentiability of F .

Let

$$\mathcal{F}: \mathcal{D}(\mathcal{F}) \subset L^\infty(\Omega)^2 \rightarrow X, \quad (\nu, \varrho) \mapsto F(\varrho^{-1}\nu^{-2}, \varrho^{-1}),$$

where the domain of definition of \mathcal{F} is

$$\mathcal{D}(\mathcal{F}) = \{\lambda \in L^\infty(\Omega) : l_- \leq \lambda(\mathbf{x}) \leq l_+ \text{ a.e.}\}^2$$

with positive constants $l_- < l_+$.

The mapping $G: \mathcal{D}(\mathcal{F}) \subset L^\infty(\Omega)^2 \rightarrow L^\infty(\Omega)^2$, $\begin{pmatrix} \nu \\ \varrho \end{pmatrix} \mapsto \begin{pmatrix} \varrho^{-1}\nu^{-2} \\ \varrho^{-1} \end{pmatrix}$, is Fréchet differentiable with

$$G'(\nu, \varrho) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = - \begin{pmatrix} 2\varrho^{-1}\nu^{-3} & \varrho^{-2}\nu^{-2} \\ 0 & \varrho^{-2} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 2\varrho^{-1}\nu^{-3}h_1 + \varrho^{-2}\nu^{-2}h_2 \\ \varrho^{-2}h_2 \end{pmatrix}.$$

Further, $G(\mathcal{D}(\mathcal{F})) \subset \mathcal{D}(F)$ whenever $k_- \max\{l_+, l_+^3\} \leq 1$.

The chain rule applied to $\mathcal{F} = F \circ G$ yields

$$\begin{aligned} \mathcal{F}'(\nu, \varrho) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} &= F'(G(\nu, \varrho))G'(\nu, \varrho) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \\ &= -\partial_1 F(G(\nu, \varrho))(2\varrho^{-1}\nu^{-3}h_1 + \varrho^{-2}\nu^{-2}h_2) - \partial_2 F(G(\nu, \varrho))(\varrho^{-2}h_2) \end{aligned}$$

which immediately proves the following theorem.

Theorem 4.1. *Let $u_0 = u_1 = 0$ and $f \in \mathcal{C}^2([0, T], H)$ with $f(0) = 0$. Then, the Fréchet derivative of \mathcal{F} about $(\nu, \varrho) \in \text{int}(\mathcal{D}(\mathcal{F}))$ exists and is the bounded linear operator $\mathcal{F}'(\nu, \varrho) \in \mathcal{L}(L^\infty(\Omega)^2, X)$ defined by $\mathcal{F}'(\nu, \varrho)(h_1, h_2)^\top = u'$ where $u' \in X$ uniquely solves*

$$\begin{aligned} (22) \quad & \int_0^T \left(a_{\varrho^{-1}}(u'(t), v(t)) - \langle \varrho^{-1}\nu^{-2}\dot{u}'(t), \dot{v}(t) \rangle_H \right) dt \\ &= \int_0^T \left(a_{h_2\varrho^{-2}}(u(t), v(t)) - \langle (2\varrho^{-1}\nu^{-3}h_1 + \varrho^{-2}\nu^{-2}h_2)\dot{u}(t), \dot{v}(t) \rangle_H \right) dt \end{aligned}$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$ with initial data $u'(0) = \dot{u}'(0) = 0$. In the above right hand side, u is the weak solution of the wave equation (1) with Cauchy data (2), that is, $u = \mathcal{F}(\nu, \varrho)$.

Moreover, $\mathcal{F}'(\nu, \varrho): L^\infty(\Omega)^2 \rightarrow \mathcal{C}([0, T], H)$ is compact.

Proof. The hypotheses on the data of Theorem 3.3 and 3.5 are satisfied. It remains only to prove the compactness of $\mathcal{F}'(\nu, \varrho) = F'(G(\nu, \varrho))G'(\nu, \varrho)$. As the Fréchet derivative of a completely continuous mapping is a compact operator, see, e.g. [14, Proposition 7.33], the assertion follows by Theorem 3.2. \square

The formulation of (22) as a classical partial differential equation is given in the Introduction.

Remark 4.2. *The setting of Theorem 4.1 seems quite natural for seismic exploration. Sound speed and mass density cannot be modelled to be smoother than $L^\infty(\Omega)$. Further, before firing the energy source f we can reliably assume the environment to be at rest: $u_0 = u_1 = f(0) = 0$.*

In the above remark we justified the strong assumptions on the data u_0 , u_1 and f . Nevertheless, if we allow smoother ν and ρ we can weaken the assumptions on u_1 and f a little bit.

Corollary 4.3. *Let $u_0 = 0$, $u_1 \in V \cap H^2(\Omega)$, and $f \in \mathcal{C}^2([0, T], H)$ with $f(0) \in V$. Define*

$$\tilde{\mathcal{F}}: \mathcal{D}(\tilde{\mathcal{F}}) \subset W^{1,\infty}(\Omega)^2 \rightarrow X, \quad (\nu, \varrho) \mapsto F(\varrho^{-1}\nu^{-2}, \varrho^{-1}),$$

where

$$\mathcal{D}(\tilde{\mathcal{F}}) = \{\lambda \in W^{1,\infty}(\Omega) : 1_- \leq \lambda(\mathbf{x}) \leq 1_+ \text{ a.e.}\}^2.$$

Then, $\tilde{\mathcal{F}}$ is Fréchet differentiable and $\tilde{\mathcal{F}}'(\nu, \varrho) \in \mathcal{L}(W^{1,\infty}(\Omega)^2, X)$ where $\tilde{\mathcal{F}}'(\nu, \varrho)(h_1, h_2) = u'$ with u' from (22).

Remark 4.4. *In this remark we discuss the implication of the compactness of $\mathcal{F}'(\nu, \varrho): L^\infty(\Omega)^2 \rightarrow \mathcal{C}([0, T], H)$ on solving the inverse problem of seismology.*

Let $M \subset \Omega$ be the (smooth) measurement submanifold and let the measurement process be modeled by a bounded linear operator $\Psi: \mathcal{C}([0, T], H) \rightarrow L^2([0, T] \times M)$. For instance, Ψ could be the trace map $u \mapsto u|_M$. In this setting the seismic inverse problem reads:

Given measurements $u^\delta \in L^2([0, T] \times M)$ find $(\nu, \varrho) \in \mathcal{D}(\mathcal{F})$ such that

$$\Psi\mathcal{F}(\nu, \varrho) \approx u^\delta.$$

Typical methods to solve above problem are Newton-like iterations which involve repeated approximate solution of local linearizations to obtain the Newton update. Due to the compactness of $\Psi\mathcal{F}'(\nu, \varrho): L^\infty(\Omega)^2 \rightarrow L^2([0, T] \times M)$ these linear systems are ill-posed. Indeed, compactness yields that the range of $\Psi\mathcal{F}'(\nu, \varrho)$ is non-closed which, in turn, yields the stated ill-posedness, see [9]. As a consequence the computation of the Newton update needs to be regularized adequately. Regularization of ill-posed problems in a Banach space setting is a highly topical research issue. For an state of the art overview we refer to the monograph [10] and to the special section Tackling inverse problems in a Banach space environment (Inverse Problems, 28(10), 2012). As $L^\infty(\Omega)$ is not reflexive most of the known theory does, unfortunately, not directly apply to our situation. One pragmatic remedy is described in [8, Section 5] but more mathematical research is urgently needed.

Remark 4.5. *The Born series is known to converge for the time-harmonic wave equation (Helmholtz equation) provided the frequency is sufficiently small, see, e.g., [5, Chap. 5.2]. Unfortunately, Theorem 4.1 does not provide a rigorous justification of the Born series in time domain for the following reason. We consider the Born series only with respect*

to c . Let $r \in L^\infty(\Omega)$ and $f \in \mathcal{C}^1([0, T], H)$ be fixed and $u, u^+ \in X$ be the solutions of (1), (2) for c and $c + h$, respectively. The difference $\delta = u^+ - u$ solves $\delta(0) = \dot{\delta}(0) = 0$ and

$$\begin{aligned} \int_0^T \left(a_r(\delta(t), v(t)) - \langle c\dot{\delta}(t), \dot{v}(t) \rangle_H \right) dt &= \int_0^T \langle h\dot{u}^+(t), \dot{v}(t) \rangle_H dt \\ &= - \int_0^T \langle h\ddot{u}^+(t), v(t) \rangle_H dt \end{aligned}$$

for all $v \in \mathcal{C}_0^\infty([0, T], V)$. Define the operator $\mathcal{L}: L^2([0, T], H) \rightarrow X$ by $f \mapsto u$ where $u \in X$ solves (1) for c and r with $u(0) = \dot{u}(0) = 0$. Then the previous variational equation can be written as $u^+ - u = -\mathcal{L}(h\ddot{u}^+)$. This is a fixed point equation for u^+ . The first step in the iteration procedure yields the Born approximation, that is, $u_B = u - \mathcal{L}(h\ddot{u})$. Convergence of the iteration sequence in some space Y is proven if the norm of the linear operator $v \mapsto \mathcal{L}(h\ddot{v})$ from Y into itself is less than one. However, this operator does not seem to be bounded – or even well defined – in any reasonable norm because \mathcal{L} does not compensate for the second derivative. The fundamental reason for this shortcoming is the fact that (4) does not admit a solution $u \in L^2([0, T], V)$ for $f \in L^2([0, T], V')$ as we have demonstrated in Example 2.1.

5. COMPARISON TO THE WORK OF BLAZEK, STOLK, AND SYMES

Blazek, Stolk, and Symes [3] considered inverse wave problems in the framework of hyperbolic systems of integro-differential equations with bounded and measurable coefficients. Among other things they showed Fréchet differentiability with respect to the coefficients. In this section we relate their results to ours.

To this end we formulate a first order system leading to (1). Let $p = p(\mathbf{x}, t) \in \mathbb{R}$ and $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^d$ be functions defined on $\Omega \times \mathbb{R}$ where $\Omega \subset \mathbb{R}^d$ is a domain. Let p and \mathbf{v} satisfy $p(\cdot, 0) = 0$, $\mathbf{v}(\cdot, 0) = \mathbf{0}$, $p|_{\partial\Omega} = 0$, and

$$(23) \quad \begin{aligned} c \partial_t p &= -\nabla_{\mathbf{x}} \cdot \mathbf{v} + f, \\ b \partial_t \mathbf{v} &= -\nabla_{\mathbf{x}} p + \mathbf{g}, \end{aligned}$$

in $\Omega \times [0, \infty[$. Further, $\mathbf{g} = \mathbf{g}(\mathbf{x}, t)$ and $f = f(\mathbf{x}, t)$ are external forces. The bounded inhomogeneities $c = c(\mathbf{x})$ and $b = b(\mathbf{x})$ are also bounded away from zero.

Define $\mathbf{G}(\cdot, t) = \int_0^t \mathbf{g}(\cdot, s) ds$ and assume the scalar-valued function $w = w(\mathbf{x}, t)$ solves the wave equation

$$(24) \quad c \partial_t^2 w - \nabla_{\mathbf{x}} \cdot \left(\frac{1}{b} \nabla_{\mathbf{x}} w \right) = f - \nabla_{\mathbf{x}} \cdot \left(\frac{1}{b} \mathbf{G} \right), \quad w|_{\partial\Omega} = 0$$

with $w(\cdot, 0) = \partial_t w(\cdot, 0) = 0$. Then, $p = \partial_t w$ and $\mathbf{v} = -\frac{1}{b}(\nabla_{\mathbf{x}} w - \mathbf{G})$ solve the above first order system, and vice versa.

Blazek, Stolk, and Symes [3] studied abstract evolution problems like the following: Let U be a real separable Hilbert space. Find an U -valued function $\xi = \xi(t)$ which solves (in a suitable sense)

$$(25) \quad A\dot{\xi} + P\xi = \ell, \quad \xi(0) = 0,$$

in which the right hand side $\ell = \ell(t)$ is also U -valued. Further, P is a skew-symmetric operator with domain V being a dense subspace of U with a topology that is stronger than the one induced by U . The operator $A \in \mathcal{L}(U)$ is self-adjoint and coercive.

System (23) can be written in the abstract form when setting $U = L^2(\Omega)^{1+d}$, $V = H_0^1(\Omega) \times H_{\text{div}}^1(\Omega)$,

$$(26) \quad \xi = \begin{pmatrix} p \\ \mathbf{v} \end{pmatrix}, \quad A(c, b) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix}, \quad P = \begin{pmatrix} 0 & \partial_{x_1} & \partial_{x_2} & \partial_{x_3} \\ \partial_{x_1} & 0 & 0 & 0 \\ \partial_{x_2} & 0 & 0 & 0 \\ \partial_{x_3} & 0 & 0 & 0 \end{pmatrix}, \quad \text{and } \ell = \begin{pmatrix} f \\ \mathbf{g} \end{pmatrix},$$

see [3, Appendix A] for the skew-symmetry of P .

We now paraphrase Theorem 9 of [3] concerning the Fréchet-differentiability of the map $\mathcal{S}: \mathbf{D}(\mathcal{S}) \subset \mathcal{L}_{\text{sym}}(U) \rightarrow \mathcal{C}^0([0, T], V)$, $A \mapsto \xi$, where $\mathcal{L}_{\text{sym}}(U)$ is the Banach space of self-adjoint operators acting on U and $\mathbf{D}(\mathcal{S}) = \{B \in \mathcal{L}_{\text{sym}}(U) : \beta_1 I \leq B \leq \beta_2 I\}$ with constants $0 < \beta_1 \leq \beta_2$.

If $\ell \in H_{\text{loc}}^2(\mathbb{R}, U)$, $\text{supp } \ell \subset [0, \infty[$, then \mathcal{S} is (continuously) Fréchet-differentiable with $\mathcal{S}'(A)H = \xi'$ and ξ' is the unique solution of

$$A\dot{\xi}' + P\xi' = -H\dot{\xi}, \quad \xi'(0) = 0,$$

where ξ uniquely solves (25).

We apply this result to (23) via (26). Let $S: \mathbf{D}(S) \subset L^\infty(\Omega)^2 \rightarrow \mathcal{C}^0([0, T], H_0^1(\Omega) \times H_{\text{div}}^1(\Omega))$, $(c, b) \mapsto (p, \mathbf{v})$, where $\mathbf{D}(S) = \{q \in L^\infty(\Omega) : \beta_1 \leq q(\mathbf{x}) \leq \beta_2\}^2$. Since $S(c, b) = \mathcal{S}(A(c, b))$ we have that

$$S'(c, b)(h_1, h_2) = \mathcal{S}'(A(c, b))A'(c, b)(h_1, h_2) = \mathcal{S}'(A(c, b))H$$

where $H = \text{diag}(h_1, h_2, h_2, h_2)$. In other words: $S'(c, b)(h_1, h_2) = (p', \mathbf{v}')$ where $p'(\cdot, 0) = 0$, $\mathbf{v}'(\cdot, 0) = \mathbf{0}$, $p'|_{\partial\Omega} = 0$, and

$$(27) \quad \begin{aligned} c \partial_t p' &= -\nabla_{\mathbf{x}} \cdot \mathbf{v}' - h_1 \partial_t p, \\ b \partial_t \mathbf{v}' &= -\nabla_{\mathbf{x}} p' - h_2 \partial_t \mathbf{v}, \end{aligned}$$

which is equivalent to $w'(\cdot, 0) = \partial_t w'(\cdot, 0) = 0$, $w'|_{\partial\Omega} = 0$,

$$(28) \quad \begin{aligned} c \partial_t^2 w' - \nabla_{\mathbf{x}} \cdot \left(\frac{1}{b} \nabla_{\mathbf{x}} w' \right) &= -h_1 \partial_t p + \nabla_{\mathbf{x}} \cdot \left(\frac{h_2}{b} \mathbf{v} \right) \\ &= -h_1 \partial_t^2 w - \nabla_{\mathbf{x}} \cdot \left(\frac{h_2}{b^2} \nabla_{\mathbf{x}} w \right) + \nabla_{\mathbf{x}} \cdot \left(\frac{h_2}{b^2} \mathbf{G} \right) \end{aligned}$$

by $p' = \partial_t w'$ and $\mathbf{v}' = -\frac{1}{b}(\nabla_{\mathbf{x}} w' + h_2 \mathbf{v})$.

Finally, let $\Phi: \mathbf{D}(\Phi) \subset L^\infty(\Omega)^2 \rightarrow \mathcal{C}^0([0, T], H_0^1(\Omega))$ be defined by $\Phi(c, b) = w(\cdot, t) = \int_0^t p(\cdot, s) ds$ where p and w solve (23) and (24), respectively, with $\mathbf{G} = \mathbf{g} = \mathbf{0}$ and $f \in H_{\text{loc}}^2(\mathbb{R}, L^2(\Omega))$, $\text{supp } f \subset [0, \infty[$. As $p = S_1(c, b)$ (first component of S) we obtain

$$(29) \quad \Phi'(c, b)(h_1, h_2) = \int_0^t p'(\cdot, s) ds = w'(\cdot, t)$$

where w' and p' are the solutions of (28) and (27), respectively ($\mathbf{G} = \mathbf{0}$).

We see that (29) would also have been obtained by our findings from the previous sections (Theorems 3.3 and 3.5). Indeed, let F denote the operator defined in (17). Then, $\Phi(c, b) = F(c, 1/b)$ and

$$\Phi'(c, b)(h_1, h_2) = F' \left(c, \frac{1}{b} \right) \left(h_1, -\frac{h_2}{b^2} \right).$$

Observe, that our assumption on f is a little bit less restrictive: $f \in H_{\text{loc}}^2(\mathbb{R}, L^2(\Omega))$, $\text{supp } f \subset [0, T]$ (Blazek et al.) vs. $f \in \mathcal{C}^2([0, T], L^2(\Omega))$, $f(0) = 0$. In view of Remark 4.2 this difference does not matter though for seismic applications.

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