DOMAIN DECOMPOSITION METHODS IN OPTIMAL FLOW CONTROL FOR HIGH PERFORMANCE COMPUTING

Zur Erlangung des akademischen Grades eines

DOKTOR DER NATURWISSENSCHAFTEN

von der Fakultät für Mathematik des
Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von
Dipl.-Math. techn. Eva Ketelaer
aus
Emmerich am Rhein

Tag der mündlichen Prüfung: 08. Mai 2013

Referent: Prof. Dr. Vincent Heuveline
Korreferent: Prof. Dr. Götz Alefeld
Korreferent: Prof. Dr. Matthias Heinkenschloss

Für Oma Kleve und meine Eltern

ABSTRACT

This thesis is concerned with linear and non-linear optimal flow control problems which are modeled by systems of partial differential equations. The numerical treatment of such problems, especially in the context of flow problems, is often very expensive and challenging. To tackle this complexity, we present parallel approaches based on non-overlapping domain decomposition methods that exploit the computational power provided by modern high performance computing technologies. On the algebraic level, we derive parallel solvers and Neumann-Neumann type preconditioners which enable us to solve these sophisticated problems in an efficient way. By means of investigating the cost for the preconditioners and of studying both scalability and efficiency of the domain decomposition approaches, we analyze the developed methods in the framework of high performance computing.

ACKNOWLEDGMENTS

First, I want to thank Prof. Heuveline for supporting me as my supervisor and for offering me the great opportunity to be part of the EMCL-team. Especially, I want to thank him for his guidance and encouragement in many fruitful discussions during my time at his institute. Furthermore, I am grateful to Prof. Heinkenschloss for his support, in particular for inviting me for a research stay of three weeks to the Rice University in Houston. During this stay at his department, I achieved important steps towards the completion of this thesis. I also want to thank Prof. Alefeld for his support as one of my second supervisors.

I am really grateful to all my great colleagues at EMCL. The working atmosphere and team spirit have been a great backup and motivation. Besides the insightful discussions, I especially enjoyed the coffee breaks, having lunch together and all the great birthday cakes. I thank all my colleagues who helped me to correct my thesis. In particular, I want to mention Staffan and Michael. Michael is a great listener. Many times, he helped me to find solutions for my problems, often without saying a word but instead simply by listening to me. Staffan always had an answer to my C++ programming issues and especially at the end of my work, he supported me a lot.

Furthermore, I want to thank my friends Britta and Katrin from my hometown Kleve. Even though they try their best to keep away from math, they always listened to me, motivated me during the lows of my PhD and shared my joys during the highs.

Particularly, I am deeply grateful to my family. My parents always supported me and had faith in me. Without their support, patience and love, I never would have had the motivation and ability to write this thesis.

Finally, I would like to deeply thank Manuel. He has been and still is the best support I can imagine. Even though lots of times many kilometers were inbetween us, it always felt like that he was right next to me.

CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ALGORITHMS

NOMENCLATURE

d	dimension ($d = 2, 3$)
s	total number of subdomains
Ω	global Lipschitz-domain
Ω_0	subdomain of Ω; the desired state $\hat{\mathbf{u}}$ has support on Ω_0
Ω_C	subdomain of Ω; the distributed control has support on Ω_C
Ω_{C_i}	control subdomain
Ω_i	subdomain i
$\partial\Omega$	global boundary
$\partial\Omega_D$	global Dirichlet boundary

$\partial\Omega_{out}$ global outflow boundary

Γ Lipschitz $(d-1)$ dimensional manifold; global interface (continuous); global skeleton (discrete)

Γ_i local interface (continous); local skeleton (discrete)

Γ_C global control boundary; subset of $\partial\Omega_D$

$\mathcal{N}$ set of outflow subdomains

$\mathcal{C}_\Omega$ set of control subdomains

$\mathbb{1}_{\Omega_0}$ indicator function for the subdomain Ω_0

$\mathbb{1}_{\Omega_C}$ indicator function for the subdomain Ω_C

$\mathbb{1}_{\Omega_{C_i}}$ indicator function for the control subdomain Ω_{C_i}

$\mathbf{u}$ velocity (state); vector variable of dimension d

$\widehat{\mathbf{u}}$ desired state, vector variable of dimension d

$\mathbf{u}_D$ extension of Dirichlet data; vector variable of dimension d

$\mathbf{d}$ Dirichlet data; vector variable of dimension d

p pressure (state); scalar

$\mathbf{a}$ advection; vector variable of dimension d

μ dynamic viscosity; scalar

$\sigma(\mathbf{u}, p)$ stress tensor; tensor of dimenesion d

$\varepsilon(\mathbf{u})$ strain or deformation tensor; tensor of dimenesion d

$\mathbf{f}$ right hand side; external force on Ω; vector variable of dimension d

$\mathbf{h}$ right hand side on $\partial\Omega_{out}$; external force on $\partial\Omega_{out}$; vector variable of dimension d

$\mathbf{n}$ normal outward normal vector on $\partial\Omega$; vector variable of dimension d

$\mathbf{n}_i$ normal outward normal vector on $\partial\Omega_i$; vector variable of dimension d

$\mathbf{v}$ test function for velocity and adjoint velocity; vector variable of dimension d

q test function for pressure and adjoint pressure; scalar

q_Ω test function for globally constant pressure and adjoint pressure; scalar

$H^1(\Omega)$ one dimensional Sobolev space of $L^2(\Omega)$ functions with two times integrable derivations of first order

$H_0^1(\Omega)$ one dimensional Sobolev space; subpace of $H^1(\Omega)$ with generalized zero boundary conditions

$H^{1/2}(\Gamma)$ one dimensional Sobolev trace space

$\mathbf{H}^1(\Omega)$ d dimensional Sobolev space of $L^2(\Omega)$ functions with two times integrable derivations of first order

$\mathbf{H}^1(\Omega_i)$ d local dimensional Sobolev space of $L^2(\Omega_i)$ functions with two times integrable derivations of first order

$\mathbf{H}_0^1(\Omega)$ d dimensional Sobolev space; subpace of $\mathbf{H}^1(\Omega)$ with generalized zero boundary conditions

$\mathbf{H}_0^1(\Omega_i)$ d local dimensional Sobolev space of $L^2(\Omega_i)$ functions with two times integrable derivations of first order with generalized zero boundary conditions on $\partial\Omega_i$; local velocity space on non-outflow subdomains

$\mathbf{H}_D^1(\Omega)$ d dimensional Sobolev space; subpace of $\mathbf{H}^1(\Omega)$ with generalized zero boundary conditions on $\partial\Omega_D$

$\mathbf{H}_D^1(\Omega_i)$ d dimensional Sobolev space on Ω_i; subspace of $\mathbf{H}^1(\Omega_i)$ with generalized zero boundary conditions on $\partial\Omega_i \cap \partial\Omega_D$

$\mathbf{H}_N^1(\Omega_i)$ d local dimensional Sobolev space of $L^2(\Omega_i)$ functions with two times integrable derivations of first order with generalized zero boundary conditions on $\partial\Omega_i \setminus \partial\Omega_{out}$; local velocity space on outflow subdomains

$\mathbf{V}_i$ local velocity space

$\widetilde{\mathbf{V}_i}$ local velocity space on subdomain Ω_i

$\mathbf{H}^{1/2}(\partial\Omega)$ d dimensional Sobolev trace space on $\partial\Omega$; global space of inhomogeneous Dirichlet data $\mathbf{d}$ on $\partial\Omega$ in case of SWF and AWF

$\mathbf{H}^{1/2}(\Gamma)$ d dimensional Sobolev trace space on Γ

$\mathbf{H}_{00}^{1/2}(\Gamma)$ d dimensional Sobolev trace space on Γ with general zero boundary conditions

$\mathbf{H}_{00}^{1/2}(\Gamma_i)$ d dimensional Sobolev trace space on Γ_i with generalized zero boundary conditions; local velocity space on subdomains intersecting global Dirichlet boundary $\partial\Omega_D$

$\mathbf{V}_{\Gamma_i}$ local velocity space on interface Γ_i

$L^2(\Omega)$ one dimensional Lebesgue space of two times integrable function on Ω; global pressure space (except for SWF)

$L_0^2(\Omega)$ one dimensional Lebesgue space; normalized subspace of $L^2(\Omega)$; global pressure space for SWF

$\mathbf{L}^2(\Omega)$ d dimensional Lebesgue space of two times integrable function on Ω; global space of right hand side

$\mathbf{L}(\Omega_C)$ d dimensional Lebesque space; global space of distributed control

$\mathbf{L}(\Omega_{C_i})$ d dimensional Lebesque space; local space of distributed control

$\mathbf{V}^h$ global finite element velocity space on Ω

$\mathbf{V}_i^h$ general local finite element velocity space on Ω_i

$\mathbf{V}_{i,N}^h$ local finite element velocity space on Ω_i with zero boundary conditions on $\partial\Omega_i \setminus \partial\Omega_{out}$

$\mathbf{V}_{i,0}^h$ local finite element velocity space on Ω_i with zero boundary conditions

$\mathbf{V}_\Gamma^h$ finite element velocity space on global skeleton Γ

$\mathbf{V}_{\Gamma_i}^h$ finite element velocity space on local skeleton Γ_i

Q^h global finite element pressure space on Ω

$Q_{i,0}^h$ normalized finite element pressure space on subdomain Ω_i

$a(\cdot,\cdot)$ global bilinear form; Note different definitions for Oseen and Navier-Stokes model problems

$a_i(\cdot,\cdot)$ local bilinear form

$b(\cdot,\cdot)$ global bilinear form

$b_i(\cdot,\cdot)$ local bilinear form

$c(\cdot,\cdot)$ global bilinear form

$c_i(\cdot,\cdot)$ local bilinear form

$d(\cdot,\cdot)$ global bilinear form

$d_i(\cdot,\cdot)$ local bilinear form

$m(\cdot,\cdot)$ global bilinear form

$m_i(\cdot,\cdot)$ local bilinear form

$\widetilde{f}(\cdot)$ global linear form for SWF and AWF case

$\widetilde{f}_i(\cdot)$ local linear form

$f(\cdot)$ global linear form for all cases with outflow boundary conditions

$f_i(\cdot)$ local linear form

$\mathcal{R}$ global extension operator

$\mathcal{R}_i$ local extension operator

$\mathcal{R}_c$ extension operator for boundary control

γ_d trace operator for Dirichlet data

$\mathcal{T}_h$ global triangulation

$\mathcal{T}_{h,i}$ local triangulation

K element of triangluation

$\mathbf{R}_{D_i}$ restriction matrix from global Dirichlet boundary to corresponding local Dirichlet boundary

$\mathbf{R}_{\Omega_i}$ restriction matrix from global pressure space to subdomain pressure constant (all outflow cases)

$\widehat{\mathbf{R}}_{\Omega_i}$ restriction matrix from global pressure space to subdomain pressure constant (AWF)

$\widetilde{\mathbf{R}}_{\Omega_i}$ restriction matrix from global pressure space to subdomain pressure constant (SWF)

$\mathbf{R}_{\Gamma_i}$ restriction matrix from global skeleton to local skeleton

$\mathbf{R}_{\Gamma_{0,i}}$ projection matrix from local skeleton to local skeleton which also lies in Ω_0

SWF standard weak formulation

AWF alternative weak formulation

OWF outflow weak formulation

1

INTRODUCTION

Optimal flow control deals with complex problems which appear in many areas of science and engineering, see, e.g., [5]. The physical dynamics of flow problems are usually modeled by systems of partial differential equations (PDEs). The numerical treatment of flow problems is typically already very expensive and challenging. Beyond that the optimization of systems constrained by PDEs increases the cost and complexity of the numerical treatment significantly, which turns them into substantially more challenging problems. Applying an adequate discretization approach, such as a finite element method, to these PDE-constraint optimization problems results mathematically in fully coupled and large scale linear systems with up to several millions of degrees of freedom. These systems are normally too large and expensive to be solved even on a modern desktop workstation. One promising approach to address this challenge of providing efficient solvers is to exploit the computational power available today in high performance computing. The nowadays underlying technology requires intrinsically parallel approaches. In this work, we develop parallel scalable and efficient numerical solvers and preconditioners based on non-overlapping domain decomposition methods in the context of optimal flow control problems. Making full use of available hardware architectures by providing parallel numerical methods for optimal flow problems also opens up new opportunities for better scientific insights and significant improvements in engineering applications.

One typical configuration is a channel flow around an obstacle behind which an eddy is formed. The aim is not only to simulate the flow, but to optimize the flow with respect to a given objective. The goal could be to reduce the eddy by optimally controlling the flow. For example, considering a ferromagnetic fluid, the flow can be controlled by applying an external distributed force in form of an electromagnetic field. The challenge is not to find any force but a force which controls the flow optimally regarding to the given objective.

First, we introduce the topic of optimal flow control, secondly we present the issue of the domain decomposition method and finally we cover the subject of high performance computing.

OPTIMAL FLOW CONTROL

Optimal flow control aims at influencing a flow with the subject of minimization or maximization of a certain objective, see, e.g., [6, 16, 20, 31, 39], and references therein. In this work, we are interested in optimally controlling the flow by matching it to a given flow profile which is modeled by a tracking type cost functional. While different type of controls exist [23], we assume distributed or boundary control. For example, distributed control can be realized by electromagnetic induction applied to a part of the domain. Boundary control means for instance to influence the flow by injection or suction of fluid on a part of the boundary. As constraints for the optimal control problem, we consider both the linear Oseen equations and the non-linear Navier-Stokes equations modeling an incompressible Newtonian fluid.

To solve an optimal control problem numerically, it is necessary to first derive the optimality system. Here, this step is done on the continuous level by applying a Lagrangian based adjoint approach [33]. Secondly, we discretize the fully coupled optimality system by using an appropriate finite element method. In addition to the constraint equations, the optimality system includes adjoint equations and an optimality condition. Thus, the system of PDEs for optimal control problems includes more variables than the flow problem itself and therefore becomes essentially larger. Additional difficulty arises from the fact that this optimality system is fully coupled. While the simulation of flow problems may already be challenging, the aim of controlling the flow increases the complexity significantly. The development of sophisticated, parallel methods is a very promising way to tackle this complexity. We accept this challenge by proposing a domain decomposition method as parallelization approach that can even be adapted to the available hardware resources.

DOMAIN DECOMPOSITION METHOD

In a non-overlapping domain decomposition method, the global domain is split into non-overlapping subdomains. The partition into subdomains yields smaller and locally independent problems, which can be solved intrinsically in parallel. Furthermore, they have the same structure as the global problem such that well established numerical methods suitable for solving the global problems can be reused for the local problems on the subdomains. Flow problems feature globally a saddle point structure. To obtain this saddle point structure also for the local problems, we establish the domain decomposition method on the continuous level and decompose already the continuous spaces adequately. The local subproblems are then coupled through the interface. Besides, it is possible to reduce the global problem to an interface equation that implicitly requires the solution of the local problems.

The interface equation, whose algebraic counterpart is known as the Schur-complement equation, is typically much smaller than the global problem. However, the direct computation of the Schur-complement operator is numerically very expensive. Therefore, the equation is usually solved by an iterative method. The main challenge in the context of non-overlapping domain decomposition methods is to develop adequate and in particular parallel preconditioners for the interface solver. One appropriate choice is to use Neumann-Neumann type preconditioners that were originally developed for the Poisson equation [46, 51, 56] and have been adapted to the saddle point structure of the Stokes equation [45]. In [28, 25**?** , 26, 27, 44] they have been transferred to the context of optimal control problems constraint by scalar linear and non-linear PDEs. In this work, we extend them to the scope of linear and non-linear optimal flow control problems. Especially, we are interested to provide parallel methods which fulfill the requirements of high performance computing. Therefore it is crucial that the domain decomposition methods are scalable and efficient to exploit the available computational power. Another important aspect in particular analyzing the preconditioners is the trade off between the cost to apply and to build the preconditioner and the effectiveness in terms of number of iterations.

HIGH PERFORMANCE COMPUTING

In high performance computing it is essential to provide scalable and efficient parallel numerical methods. These are crucial requirements to fully take advantage of the computing power provided by the different hardware technologies available, see, e.g., [11]. Another important aspect when developing a method for high performance computing is to take into account the trade off between the computing and memory capacities of each processor and the bandwidth of the communication network. The limiting factor of parallel numerical methods, solving large scale systems resulting from the discretization with the finite element methods, is usually the bandwidth of the network.

The parallelization of non-overlapping domain decomposition methods is realized by mapping each subdomain to one processor. Therefore, the size of a subdomain needs to be chosen in such way that it fits into the local memory of a processor or even better into its cache. Otherwise a processor needs to access the storage of the disk which usually slows down the algorithm notably. Another consideration is the size of the interface, coupling the local independent subdomains, that directly reflects the cost of the global communication. The smaller the size of the subdomains is chosen the more subdomains are obtained which clearly links to an increasing size of the interface. Its size directly has an impact on the cost for the communication but also on the difficulty to solve the corresponding interface equation.

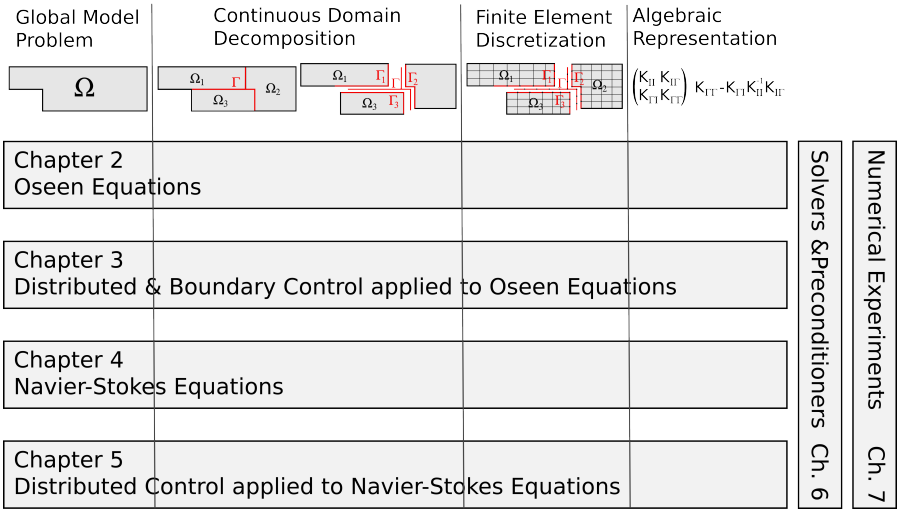


Figure 1: Chapter overview.

To overcome this difficulty, the effectiveness of preconditioners in reducing the number of iterations plays an important role.

Based on numerical experiments, we study domain decomposition methods applied to optimal flow control problems in the context of high performance computing. On the one hand, we investigate both scalability and efficiency of the methods and on the other hand, the effectiveness of the preconditioners to reduce the number of iterations. Since we use iterative solvers, their efficiency strongly corresponds to the performance of the preconditioner. Therefore, we study the cost for the set-up of the preconditioner and the cost per iteration. Analyzing the relation between the number of subdomains and the size of the skeleton, we give insight in to how the method can be adapted to the underlying hardware.

OUTLINE

In each of the Chapters 2-5, we derive a domain decomposition method for a different model problem. Based on each model problem, we focus on a distinct aspect of the method. The structure of these chapters as illustrated in Fig. 1 is similar: After introducing the global model problem, we derive the domain decomposition method on the continuous level in two sub-steps: First, we derive a globally coupled weak formulation on subdomains and secondly, we decouple the problem into local problems on subdomains and one system of coupling equations on the interface Γ. Then, we discretize employing a finite element method and close up with the algebraic formulation. In Chapter 2, we study the saddle point structure of flow problems by means of the Oseen equations. We outline the differences between inhomogeneous Dirichlet and mixed outflow and Dirichlet boundary conditions. We extend the domain decomposition method in Chapter

3 to the context of linear optimal flow control. Therefore, we study a linear quadratic optimal control problem assuming distributed or boundary control constrained by the Oseen equations. In Chapter 4, we consider a non-linear flow problem modeled by the Navier-Stokes equations. In particular, we address the treatment of the non-linearity in relation with the domain decomposition method. In Chapter 5, we combine the results of the previous three chapters and derive a domain decomposition method for a non-linear quadratic distributed optimal flow control problem constraint by the Navier-Stokes equations. Chapter 6 is dedicated to the derivation of two different parallel solution algorithms and appropriate preconditioners for all considered model problems. The main emphasis is the development of one- and two-level Neumann-Neumann preconditioners. In Chapter 7, we study the domain decomposition method for optimal flow control problems in the context of high performance computing based on a parallel implementation. We conclude with a summary and an outlook in Chapter 8.

2

DOMAIN DECOMPOSITION METHOD FOR THE OSEEN EQUATIONS

Our aim is this work is to derive a domain decomposition method for a non-linear quadratic optimal control problem constrained by the full Navier-Stokes equations. We reduce the complexity of the model problem and first derive a non-overlapping domain decomposition method for the Oseen equations. This approach enables us to focus on the challenges already emerging for linear flow problems when deriving this method.

The derivation of the non-overlapping domain decomposition method is done in four main steps. First, we present a strong and weak formulation of the global system of partial differential equations modeling the Oseen equations. Secondly, we derive a domain decomposition method applied on the continuous level in two sub-steps. In the first sub-step, we replace the global weak formulation by an equivalent weak formulation on subdomains which is still globally fully coupled. In the second sub-step, we decouple this formulation, which leads to locally independent weak formulations on the subdomains and one system of coupling conditions on the interface. In the third main step, we discretize the decoupled weak formulation applying an appropriate finite element method. Last, we derive an algebraic representation of the global linear system corresponding to the globally coupled weak formulation on subdomains and the Schur-complement equation which is the algebraic counterpart of the coupling condition on the interface. The following three chapters are organized in the same way but focus on a different aspect in the derivation of the domain decomposition method with respect to the model problem.

One of the main ideas of a non-overlapping domain decomposition is to obtain local problems of the same structure as the global problems. All flow problems treated in this work result on the global level mathematically in a saddle point structure. In this chapter, one focus lies on the treatment of this characterizing structure. The approach to derive the domain decomposition on the continuous level directly yields independent local problems on the subdomains which feature the same saddle point structure. Also for the coupling conditions, we obtain a global saddle point structure. The main ideas of the derivation are based on [12] and [45].

Figure 2: This figure refers to the first main step of the derivation of the domain decomposition method for the case with only Dirichlet boundary conditions applied on $\partial\Omega$. The figures shows exemplarily a global domain Ω.

The second focus lies on the treatment of two different type of boundary conditions. On the one hand, we consider the Oseen equations with inhomogeneous Dirichlet boundary conditions on the global boundary. On the other hand, we equip the Oseen equations with mixed outflow and inhomogeneous Dirichlet boundary conditions. For the former case, we derive two weak formulations, the standard weak formulation (SWF) and an alternative weak formulation (AWF). We discuss the AWF due to the fact that it is more convenient for implementation. For the latter outflow case, we establish the domain decomposition method based on the alternative weak formulation (OWF).

2.1 GENERAL DEFINITIONS

In this work, let $\Omega \in \mathbb{R}^d$ $(d = 2, 3)$ be a Lipschitz-domain, i.e. an open, bounded and connected set, where $\partial\Omega$ is a piecewise Lipschitz boundary, see Fig. 2. Furthermore, we use the $L^2(\Omega)$ space, and the Sobolev spaces $H^1(\Omega)$, $H^1_0(\Omega)$ and $H^{1/2}(\Gamma)$, which are defined in the usual way [2]. Here, $\Gamma \subset \overline{\Omega}$ denotes a Lipschitz $(d-1)$-dimensional manifold. We set

$$L^2_0(\Omega) := \left\{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \right\}, \; \mathbf{L}^2(\Omega) := \left[L^2(\Omega)\right]^d,$$
$$\mathbf{H}^1(\Omega) := \left[H^1(\Omega)\right]^d, \; \mathbf{H}^1_0(\Omega) = \left[H^1_0(\Omega)\right]^d \text{ and } \mathbf{H}^{1/2}(\Gamma) := \left[H^{1/2}(\Gamma)\right]^d.$$

2.2 INHOMOGENEOUS DIRICHLET BOUNDARY CONDITIONS

In the first main step, we present a strong and weak formulation of the Oseen equations on the global domain Ω equipped with inhomogeneous Dirichlet boundary conditions as depicted in Fig. 2.

Therefore, we assume that the following functions and values

$$\begin{aligned} &\mathbf{a} \in \mathbf{H}^1(\Omega) \text{ with } \boldsymbol{\nabla} \cdot \mathbf{a} = 0 \text{ a.e. in } \Omega, \\ &\mathbf{f} \in \mathbf{L}^2(\Omega),\ \mathbf{d} \in \mathbf{H}^{1/2}(\partial\Omega) \text{ and } \mu > 0 \end{aligned} \tag{2.1}$$

are given. $\boldsymbol{\nabla} \cdot \mathbf{a} = 0$ means that the given advection $\mathbf{a}$ is divergence free. The Dirichlet data $\mathbf{d}$ also fulfills the compatibility condition to be divergence free, which guarantees the conservation of mass, i.e.

$$\int_{\partial\Omega} \mathbf{d} \cdot \mathbf{n} \, d\mathbf{s} = 0. \tag{2.2}$$

$\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. Under the assumptions (2.1) and (2.2), we want to solve the following boundary value problem for the Oseen equations modeling a Newtonian fluid equipped with inhomogeneous Dirichlet boundary conditions:

$$(\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla} \cdot \sigma(\mathbf{u}, p) = \mathbf{f} \qquad \text{in } \Omega, \tag{2.3a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2.3b}$$

$$\mathbf{u} = \mathbf{d} \qquad \text{on } \partial\Omega. \tag{2.3c}$$

The Oseen equations model creeping flow with additional given advection. Creeping flows are characterized by Reynolds numbers tending to zero. The viscosity of the fluid is normally very high at low velocities such that the diffusion is the dominating part [38]. Since we model Newtonian fluids, the primitive variables $\mathbf{u}$ and p and the other variables are defined as, see, e.g., [38, Chapter 5]:

- $\mathbf{u} : \Omega \to \mathbb{R}^d$ the velocity,
- $p : \Omega \to \mathbb{R}$ the pressure,
- $\sigma(\mathbf{u}, p) : \Omega \to \mathbb{R}^{d \times d}$ the stress tensor with $\sigma(\mathbf{u}, p) = -p\mathbf{I} + 2\mu\boldsymbol{\varepsilon}(\mathbf{u})$,
- $\mathbf{I} \in \mathbb{R}^{d \times d}$ the identity tensor,
- $\boldsymbol{\varepsilon}(\mathbf{u}) : \Omega \to \mathbb{R}^{d \times d}$ the strain or deformation tensor with $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\boldsymbol{\nabla}\mathbf{u} + \boldsymbol{\nabla}\mathbf{u}^T)$,
- μ the dynamic viscosity,
- $\mathbf{a} : \Omega \to \mathbb{R}^d$ a given advection,
- $\mathbf{f} : \Omega \to \mathbb{R}^d$ a given external force and
- $\mathbf{d} : \partial\Omega \to \mathbb{R}^d$ given Dirichlet boundary conditions.

Let $\mathbf{u}_D \in \mathbf{H}^1(\Omega)$ be the extension of the Dirichlet data $\mathbf{d}$ from $\mathbf{H}^{1/2}(\partial\Omega)$ to $\mathbf{H}^1(\Omega)$ such that $\gamma_d(\mathbf{u}_D) = \mathbf{d}$, where $\gamma_d : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\partial\Omega)$ denotes a trace operator.

Since we are interested in solving the Oseen equations with the finite element method, we derive a weak formulation for (2.3). On the one hand, for a solution of a weak formulation less regularity is required and on the other hand the framework of the finite element method is based on the concept of a weak formulation.

First, we introduce the following notations:

$$\begin{aligned}
[\boldsymbol{\nabla}\mathbf{u}]_{i,j} &= \frac{\partial u_j}{\partial x_i} && i,j = 1,\dots,d,\\
\boldsymbol{\nabla}\mathbf{u}\colon\boldsymbol{\nabla}\mathbf{v} &= \sum_{i,j}^{d} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i},\\
((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u})\mathbf{v} &= \sum_{i,j=1}^{d} a_j \frac{\partial u_i}{\partial x_j} v_i,\\
(\boldsymbol{\nabla}\cdot\mathbf{a})\mathbf{u}\mathbf{v} &= \sum_{i=1}^{d} (\boldsymbol{\nabla}\cdot\mathbf{a}) u_i v_i,\\
((\mathbf{a}\cdot\mathbf{n})\mathbf{u})\mathbf{v} &= \sum_{i=1}^{d} (\mathbf{a}\cdot\mathbf{n}) u_i v_i.
\end{aligned}$$

Note that the fluid is incompressible, which is modeled by (2.3b) and μ is constant, since we only model one fluid. Under these two assumptions, the equality

$$2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u})\colon\boldsymbol{\varepsilon}(\mathbf{v})\,d\mathbf{x} = \mu \int_\Omega \boldsymbol{\nabla}\mathbf{u}\colon\boldsymbol{\nabla}\mathbf{v}\,d\mathbf{x} \tag{2.4}$$

holds, see, e.g., [14, Chapter 6].

Then, we define the bilinear form $a(\cdot,\cdot)$:

$$\begin{aligned}
&a : \mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\to\mathbb{R},\\
&a(\mathbf{u},\mathbf{v}) = 2\mu\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u})\colon\boldsymbol{\varepsilon}(\mathbf{v})\,d\mathbf{x} + \int_\Omega ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u})\mathbf{v}\,d\mathbf{x}\\
&\qquad\overset{(2.4)}{=} \mu\int_\Omega \boldsymbol{\nabla}\mathbf{u}\colon\boldsymbol{\nabla}\mathbf{v}\,d\mathbf{x} + \int_\Omega ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u})\mathbf{v}\,d\mathbf{x}.
\end{aligned}$$

We also introduce the bilinear forms $b(\cdot,\cdot)$, $c(\cdot,\cdot)$ and the linear form $\widetilde{f}(\cdot)$:

$$b : \mathbf{H}^1(\Omega)\times L^2(\Omega)\to\mathbb{R}, \qquad b(\mathbf{u},q) = -\int_\Omega q\boldsymbol{\nabla}\cdot\mathbf{u}\,d\mathbf{x}, \tag{2.5a}$$

$$c : \mathbb{R}\times L^2(\Omega)\to\mathbb{R}, \qquad c(\eta,q) = \eta\int_\Omega q\,dx, \tag{2.5b}$$

$$\widetilde{f} : \mathbf{H}^1(\Omega)\to\mathbb{R}, \qquad \widetilde{f}(\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,d\mathbf{x}. \tag{2.5c}$$

Before we introduce the weak formulation, we show the consequences of the compatibility condition (2.2) for the extension of the Dirichlet boundary $\mathbf{u}_D$ in the next proposition.

Proposition 1. *It holds that*

$$b(\mathbf{u}_D, q) = 0 \qquad \forall q \in L^2(\Omega).$$

Proof. From the divergence theorem, see, e.g., [38, Chapter 2, Lemma 4], it holds for the extension $\mathbf{u}_D$ that

$$0 \overset{(2.2)}{=} \int_{\partial\Omega} \mathbf{d} \cdot \mathbf{n} \, d\mathbf{s} = \int_{\Omega} \boldsymbol{\nabla} \cdot \mathbf{u}_D \, d\mathbf{x}.$$

Using [19, Lemma 2.2], it follows that $\mathbf{u}_D$ with $\boldsymbol{\nabla} \cdot \mathbf{u}_D = 0$ in Ω and $\gamma_d(\mathbf{u}_D) = \mathbf{d}$ exists. Thus, we get

$$b(\mathbf{u}_D, q) = -\int_{\Omega} \underbrace{\boldsymbol{\nabla} \cdot \mathbf{u}_D}_{=0} q \, d\mathbf{x} = 0 \qquad \forall q \in L^2(\Omega).$$

□

2.2.0.1 *Standard Weak Formulation (SWF)*

Using these definitions, we get the standard weak formulation (SWF) for the system of partial differential equations (2.3), see also [18, 52, 55]: Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $\widetilde{p} \in L_0^2(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}) = \widetilde{f}(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}), \tag{2.6a}$$

$$b(\mathbf{u}, \widetilde{q}) = \underbrace{-b(\mathbf{u}_D, \widetilde{q})}_{=0, \text{(Prop. 1)}} \tag{2.6b}$$

for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $\widetilde{q} \in L_0^2(\Omega)$. Even though the right hand side of (2.6b) $-b(\mathbf{u}_D, \widetilde{q}) = 0$, we keep the term, since it is not negligible in the domain decomposition approach, as we see later on in Section 2.4.

It is well known, see, e.g., [19, Chapter I, § 5.1.], that due to the Dirichlet conditions being imposed on the velocities everywhere on the boundary, the pressure is only defined up to a constant. Thus, to obtain a unique solution, we enforce the normalization condition $\int_{\Omega} p(\mathbf{x}) d\mathbf{x} = 0$ on the pressure. Note that we use a mixed type formulation or velocity-pressure formulation [19, Chapter I, §5.1.], which models the physical variables velocity and pressure in adequate but different spaces. This is a commonly used choice when implementing a finite element method for flow problems. The mixed type formulation only requires finite element with local support which leads to sparse matrices.

Remark. We want to mention, that there also exists another "classical" weak formulation in the divergence free space

$$\mathbf{H}_{div} := \left\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : b(\mathbf{v}, q) = 0 \quad \forall q \in L_0^2(\Omega) \right\} :$$

Find $\mathbf{u} \in \mathbf{H}_{div}$ such that

$$a(\mathbf{u}, \mathbf{v}) = f(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_{div}.$$

This formulation is often used in numerical analysis, e.g., to proof the existence of a unique formulation and to study the regularity of the solutions. There also exist divergence free finite elements, see, e.g., [40] and references therein, which we do not apply in this work.

Next, we proof the existence and uniqueness of (2.6):

Theorem 2. *If the assumptions (2.1) and (2.2) hold, then the SWF (2.6) has a unique solution* $(\mathbf{u}^*, p^*) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$.

Proof. We only have to proof that $a(\cdot, \cdot)$ is $\mathbf{H}_{div}-$elliptic. Then we can directly apply the proof of [19, Chapter I, § 5.1., Theorem 5.1.]. Applying the divergence theorem , see, e.g., [38, Chapter 2, Lemma 4], to the term

$$\int_\Omega ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{v})\mathbf{v} d\mathbf{x},$$

we can show that the bilinear form $a(\cdot, \cdot)$ is $\mathbf{H}_{div}$-elliptic:

$$\begin{aligned} a(\mathbf{v}, \mathbf{v}) &= \mu \int_\Omega \boldsymbol{\nabla}\mathbf{v} \colon \boldsymbol{\nabla}\mathbf{v} \, d\mathbf{x} + \int_\Omega ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{v})\mathbf{v} \, d\mathbf{x} \\ &= \mu \int_\Omega \boldsymbol{\nabla}\mathbf{v} \colon \boldsymbol{\nabla}\mathbf{v} \, d\mathbf{x} + \underbrace{\frac{1}{2} \int_{\partial\Omega} (\mathbf{a} \cdot \mathbf{n})\mathbf{v}\mathbf{v} \, d\mathbf{s}}_{=0,\, \mathbf{v} \in \mathbf{H}_0^1(\Omega)} - \frac{1}{2} \int_\Omega (\underbrace{(\boldsymbol{\nabla} \cdot \mathbf{a})}_{=0,\, (2.1)} \mathbf{v})\mathbf{v} \, d\mathbf{x} \\ &\quad \underbrace{-\frac{1}{2} \int_\Omega ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{v})\mathbf{v} \, d\mathbf{x} + \frac{1}{2} \int_\Omega ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{v})\mathbf{v} \, d\mathbf{x}}_{=0} \\ &= \mu \left\| \mathbf{v} \right\|_{\mathbf{H}_{div}}^2 . \end{aligned}$$

The existence and uniqueness of the solution $\mathbf{u}_D + \mathbf{u} \in \mathbf{u}_D + \mathbf{H}_0^1(\Omega)$ and $\widetilde{p} \in L_0^2(\Omega)$ follows from the Lax-Milgram theorem and the fact that the bilinear form $b(\cdot, \cdot)$ fulfills the inf-sup condition, which states

$$\sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v}, q)}{\left\| \mathbf{v} \right\|_{\mathbf{H}_0^1(\Omega)}} \geq \beta \left\| q \right\|_{L_0^2(\Omega)} \quad \forall q \in L_0^2(\Omega).$$

For more details, see Girault and Raviart [19, Theorem 5.1]. □

2.2.0.2 *Alternative Weak Formulation (AWF)*

Next, we discuss an alternative mixed type weak formulation (AWF), that is equivalent to the SWF (2.6). The AWF is more convenient from the numerical and implementational point of view, since the pressure variable p can be discretized by "standard" piecewise linear basis functions. The term "standard" is explained in Section 2.6.2.

Lemma 3. *Assuming (2.1) and (2.2), an equivalent alternative weak formulation (AWF) of the partial differential equation (2.6) is given as follows: Find* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p \in L^2(\Omega)$ *and* $\eta \in \mathbb{R}$ *such that*

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = \widetilde{f}(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}), \tag{2.7a}$$

$$b(\mathbf{u}, q) + c(\eta, q) = \underbrace{-b(\mathbf{u}_D, q)}_{=0,\,(\textit{Prop. } 1)}, \tag{2.7b}$$

$$c(\xi, p) = 0 \tag{2.7c}$$

for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, $q \in L^2(\Omega)$ *and* $\xi \in \mathbb{R}$.

Proof. We observe that for any function $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and any constant $q_\Omega \in \mathbb{R}$, it holds that

$$\begin{aligned} b(\mathbf{v}, q_\Omega) &= -\int_\Omega \nabla \cdot \mathbf{v}\, q_\Omega \, dx \\ &= \int_\Omega \mathbf{v} \underbrace{\nabla q_\Omega}_{=0} dx - \int_{\partial\Omega} \underbrace{\mathbf{v}}_{=0} \cdot \mathbf{n}\, q_\Omega \, dx = 0. \end{aligned} \tag{2.8}$$

Moreover, for any real number $\xi \in \mathbb{R}$ and any $\widetilde{q} \in L_0^2(\Omega)$,

$$c(\xi, \widetilde{q}) = \int_\Omega \xi\, \widetilde{q}\, dx = \xi \underbrace{\int_\Omega \widetilde{q}\, dx}_{=0} = 0. \tag{2.9}$$

Furthermore, we note that every $q \in L^2(\Omega)$ can be written as

$$q = \widetilde{q} + q_\Omega \text{ with } \widetilde{q} \in L_0^2(\Omega) \text{ and } q_\Omega = \frac{1}{|\Omega|} \int_\Omega q\, dx.$$

Then, we can reformulate the AWF (2.7) equivalently as:

Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p = \widetilde{p} + p_\Omega \in L^2(\Omega)$ with $\widetilde{p} \in L_0^2(\Omega)$, $p_\Omega \in \mathbb{R}$ and $\eta \in \mathbb{R}$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}) + \underbrace{b(\mathbf{v}, p_\Omega)}_{=0,\,(2.8)} = \widetilde{f}(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}), \tag{2.10a}$$

$$b(\mathbf{u}, \widetilde{q}) + \underbrace{b(\mathbf{u}, q_\Omega)}_{=0,\,(2.8)} + \underbrace{c(\eta, \widetilde{q})}_{=0,\,(2.9)} + c(\eta, q_\Omega) = 0, \tag{2.10b}$$

$$\underbrace{c(\xi, \widetilde{p})}_{=0,\,(2.9)} + c(\xi, p_\Omega) = 0 \tag{2.10c}$$

for all $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, $\widetilde{q} \in L_0^2(\Omega)$, $q_\Omega \in \mathbb{R}$ and $\xi \in \mathbb{R}$.

"$\Rightarrow$"

Let $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $\widetilde{p} \in L_0^2(\Omega)$ solve (2.6). Then $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p = \widetilde{p} \in L_0^2(\Omega)$, $p_\Omega = \frac{1}{|\Omega|}\int_\Omega p dx = 0 \in \mathbb{R}$ and $\eta = 0 \in \mathbb{R}$ solve (2.10) since:

$(2.10a)$		trivially fulfilled;
$p_\Omega = 0$	$\Rightarrow$	$(2.10c)$ is trivially fulfilled;
$(2.6b)$	$\Rightarrow$	$b(\mathbf{u}, \widetilde{q}) = 0$
	$\Rightarrow$	$c(\eta, q_\Omega) = 0 \quad \forall q_\Omega \in \mathbb{R}$
	$\Rightarrow$	$\eta = 0$
	$\Rightarrow$	$(2.10b)$ is fulfilled.

"$\Leftarrow$" Let $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$, $p = \widetilde{p} + p_\Omega \in L^2(\Omega)$ with $\widetilde{p} \in L_0^2(\Omega)$ and $p_\Omega \in \mathbb{R}$ and $\eta \in \mathbb{R}$ solve (2.10). To proof that $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $\widetilde{p} \in L_0^2(\Omega)$ solve (2.6), we have to show that $p_\Omega = 0$ and $\eta = 0$:

$$c(\xi, p_\Omega) = 0 \quad \forall \xi \in \mathbb{R} \qquad \Rightarrow \qquad p_\Omega = 0.$$

Furthermore, (2.10b) holds for all $\widetilde{q} \in L_0^2(\Omega)$ and $q_\Omega \in \mathbb{R}$. Hence, it also must hold for $\widetilde{q} = 0 \in L_0^2(\Omega)$:

$$\widetilde{q} = 0 \qquad \Rightarrow \qquad c(\eta, q_\Omega) = 0 \quad \forall q_\Omega \in \mathbb{R} \qquad \Rightarrow \qquad \eta = 0.$$

□

Remark. Both formulations (SWF and AWF) can also be formulated in terms of an optimization problem [19, Chapter I, §4.2] which yields the structure of a saddle point. Therefore, the pressure can be interpreted as a Lagrange parameter. Analogously, we can refer to the parameter η as a Lagrange parameter. Then we interpret the requirement $p \in L_0^2(\Omega)$ explicitly as an extra constraint for the optimization problem. This is done in the AWF whereas in the SWF it is implicitly required by the set-up of the pressure space.

2.3 OUTFLOW BOUNDARY CONDITIONS

Next, we discuss the Oseen equations with mixed outflow and inhomogeneous Dirichlet boundary conditions. In this subsection, we apply the first main step of the derivation of the domain decomposition method. Furthermore, we point out the differences to the case with

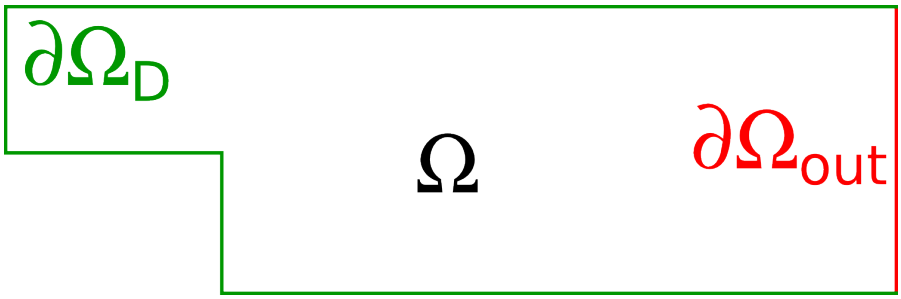


Figure 3: This figure corresponds to the first main step of the derivation in case with mixed outflow and Dirichlet boundary conditions. The figure shows a global domain Ω. On $\partial\Omega_D$ we equip the Oseen equations with inhomogeneous Dirichlet and on $\partial\Omega_{out}$ with outflow boundary conditions.

inhomogeneous boundary conditions, see Fig 3. Here, we assume that the boundary $\partial\Omega$ is decomposed into

$$\partial\Omega = \partial\Omega_D \cup \partial\Omega_{out} \text{ with } \partial\Omega_D \cap \partial\Omega_{out} = \emptyset.$$

$\partial\Omega_D$ and $\partial\Omega_{out}$ are sets with nonempty relative interior, see Fig. 3.

For given functions and values

$$\begin{aligned} &\mathbf{a} \in \mathbf{H}^1(\Omega) \text{ with } \boldsymbol{\nabla}\cdot\mathbf{a} = 0 \text{ a.e. in } \Omega \text{ and } \mathbf{n}\cdot\mathbf{a} = 0 \text{ on } \partial\Omega_{out}, \\ &\mathbf{f} \in \mathbf{L}^2(\Omega), \mathbf{h} \in \mathbf{L}^2(\partial\Omega_{out}), \mathbf{d} \in \mathbf{H}^{1/2}(\partial\Omega_D) \text{ and } \mu > 0, \end{aligned} \tag{2.11}$$

we want to solve the following boundary value problem for the Oseen equations modeling a Newtonian fluid equipped with outflow boundary conditions on $\partial\Omega_{out}$ and Dirichlet boundary conditions on $\partial\Omega_D$:

$$(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}\cdot\sigma(\mathbf{u},p) = \mathbf{f} \quad \text{in } \Omega, \tag{2.12a}$$
$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.12b}$$
$$\mathbf{u} = \mathbf{d} \quad \text{on } \partial\Omega_D, \tag{2.12c}$$
$$\sigma(\mathbf{u},p)\mathbf{n} = \mathbf{h} \quad \text{on } \partial\Omega_{out}. \tag{2.12d}$$

The right hand side $\mathbf{h}$ of (2.12d) models a given external force on the outflow boundary. Again, we derive a weak formulation. Therefore, we define

$$\mathbf{H}^1_D(\Omega) = \left\{\mathbf{v} \in \mathbf{H}^1(\Omega) \,:\, \mathbf{v} = \mathbf{0} \text{ on } \partial\Omega_D\right\},$$

which accounts for the outflow conditions.

As in the inhomogeneous case, let $\mathbf{u}_D \in \mathbf{H}^1(\Omega)$ denote the extension of the Dirichlet data $\mathbf{d} \in \mathbf{H}^{1/2}(\partial\Omega_D)$ such that $\gamma_d(\mathbf{u}_D) = \mathbf{d}$ where $\gamma_d : \mathbf{H}^1(\Omega) \to \mathbf{H}^{1/2}(\partial\Omega_D)$ denotes a trace operator. Note that the space for the Dirichlet data is slightly different, because now we

impose Dirichlet boundary conditions only on a part of the global boundary $\partial\Omega_D \subsetneq \partial\Omega$.

We use the bilinear forms introduced before (2.5) and define the linear form $f(\cdot)$ which also accounts for the outflow boundary conditions.

$$f : \mathbf{H}^1(\Omega) \to \mathbb{R}, \quad f(\mathbf{v}) = \int_\Omega \mathbf{f}\mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_{out}} \mathbf{h}\mathbf{v}\, d\mathbf{s}.$$

The weak form (OWF) of the system of partial differential equations (2.12) is given as follows: Find $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$ and $p \in L^2(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}), \tag{2.13a}$$

$$b(\mathbf{u}, q) = -\, b(\mathbf{u}_D, q) \tag{2.13b}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$ and $q \in L^2(\Omega)$.

Remark. If a solution $(\mathbf{u}, p) \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega)$ for the outflow problem exists, then the velocity is uniquely defined, because we impose Dirichlet boundary condition on $\partial\Omega_D$. The pressure is then uniquely defined, due to the fact that we impose outflow boundary conditions on $\partial\Omega_{out}$. Therefore, in contrast to the case with only inhomogeneous Dirichlet boundary conditions on $\partial\Omega$, no normalization condition must be imposed for the pressure.

Remark. In difference to the inhomogeneous case, $b(\mathbf{u}_D, q)$ must not be 0 because the compatibility condition (2.2) must not hold.

The next theorem states under which conditions a unique solution of (2.13) exists.

Theorem 4. *If the assumptions (2.11) hold, then the Oseen equations (2.13) have a unique solution* $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$ *and* $p \in L^2(\Omega)$.

Proof. Due to the outflow conditions, the proof becomes very technical. Therefore, we only sketch the ideas. We need to construct a solenoidal extension of the pre-described Dirichlet boundary data into the domain Ω. Therefore we need to construct a flux carrier which must carry the incoming flux from inflow boundary across the domain to the outflow boundary. By splitting the solution into an inhomogeneous part and the solenoidal extension, we can apply the same arguments as in the proof of Theorem 2. For more details, see, e.g., [30, 53]. □

Remark 5. At a first glance, the weak formulation of the SWF and OWF seem to be the same. One has to closely look at the used spaces. For the SWF, we look for a solution $(\mathbf{u}, p) \in \mathbf{H}^1_0(\Omega) \times L^2_0(\Omega)$ and for the OWF, we seek a solution $(\mathbf{u}, p) \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega)$. The right hand side of the OWF also differs from the one of the SWF and the AWF.

2.4 CONTINUOUS DOMAIN DECOMPOSITION

Next, we derive a non-overlapping domain decomposition method on the continuous level for the Oseen equations by applying the following steps: Firstly, we partition the domain, the spaces, and the bilinear forms. Secondly, an equivalent weak formulation on subdomains, which is still globally coupled is derived. Finally, we decouple the subdomain formulation into s independent subdomain problems and one system of global coupling conditions.

2.4.1 *Decomposition of the Domain*

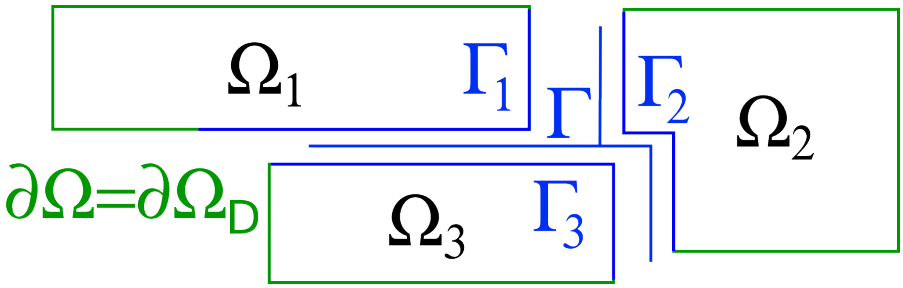


Figure 4: The figure depicts the decomposition of the global domain Ω into non-overlapping subdomains and the interface Γ. This figure corresponds to the case with inhomogeneous Dirichlet conditions on the global boundary and refers to the second sub-step of the second main step of the derivation of the domain decomposition method on the continuous level.

As outlined in Fig. 4, the domain Ω is divided into s non-overlapping subdomains $\Omega_i \subset \Omega$, such that

$$\overline{\Omega} = \bigcup_{i=1}^{s} \overline{\Omega}_i \qquad \text{with} \qquad \Omega_i \cap \Omega_j = \emptyset,\ i \neq j,\ i,j \in \{1,2,\ldots,s\}\,.$$

We set the local interfaces Γ_i and Γ_{ij}

$$\begin{aligned} \Gamma_i &:= \partial\Omega_i \setminus \partial\Omega, \\ \Gamma_{ij} &:= \partial\Omega_i \cap \partial\Omega_j, \end{aligned}$$

and we define the global interface Γ by

$$\Gamma := \bigcup_{i,j=1,\, j>i}^{s} \Gamma_{ij}.$$

The splitting of Ω is done such that Γ_{ij} are Lipschitz $(d-1)$-dimensional manifolds. To be able to use the same notation for the case of only inhomogeneous Dirichlet and the case with mixed outflow and Dirich-

let boundary conditions, we also denote the global Dirichlet boundary $\partial\Omega$ in the inhomogeneous case with $\partial\Omega_D$, see also Fig. 2.

In the case of outflow boundary conditions, we need to distinguish between the set of non-outflow subdomains and the set of outflow subdomains denoted by $\mathcal{N}$ and defined as

$$\mathcal{N} := \{i \in \{1, \dots, s\} : \partial\Omega_i \cap \partial\Omega_{out} \neq \emptyset\},$$

where the relative interior of $\partial\Omega_i \cap \partial\Omega_{out}$ must be non-empty.

2.4.2 *Decomposition of the Velocity Space*

Before we partition the velocity space $\mathbf{H}_0^1(\Omega)$ in case of inhomogeneous Dirichlet boundary conditions, and $\mathbf{H}_D^1(\Omega)$ in case of outflow boundary conditions, we need to define several subspaces for the velocity.

First, we define the velocity spaces on the global interface Γ:

$$\mathbf{H}_{00}^{1/2}(\Gamma) := \{\mathbf{v} \in \mathbf{H}^{1/2}(\Gamma) : \mathbf{v}|_{\partial\Omega_D \cap \overline{\Gamma}} = 0\}.$$

Then, we define the following spaces for the local velocities on Ω_i:

$$\begin{aligned} \mathbf{H}_N^1(\Omega_i) &:= \{\mathbf{v} \in \mathbf{H}^1(\Omega_i) : v|_{\partial\Omega_i \setminus \partial\Omega_{out}} = 0\}, \\ \mathbf{V}_i &:= \begin{cases} \mathbf{H}_N^1(\Omega_i) & \text{if } i \in \mathcal{N}, \\ \mathbf{H}_0^1(\Omega_i) & \text{if } i \notin \mathcal{N}. \end{cases} \end{aligned} \tag{2.14}$$

Using these definitions, we decompose the global velocity spaces $\mathbf{H}_0^1(\Omega)$ and $\mathbf{H}_D^1(\Omega)$ as follows:

$$\begin{aligned} \mathbf{H}_0^1(\Omega) &= \bigoplus_{i=1}^{s} \mathbf{H}_0^1(\Omega_i) \oplus \mathbf{H}_{00}^{1/2}(\Gamma), \\ \mathbf{H}_D^1(\Omega) &= \bigoplus_{i \notin \mathcal{N}} \mathbf{H}_0^1(\Omega_i) \bigoplus_{i \in \mathcal{N}} \mathbf{H}_N^1(\Omega_i) \oplus \mathbf{H}_{00}^{1/2}(\Gamma) = \bigoplus_{i=1}^{s} \mathbf{V}_i \oplus \mathbf{H}_{00}^{1/2}(\Gamma). \end{aligned}$$

Moreover, we define velocity spaces on the local interfaces Γ_i:

$$\begin{aligned} \mathbf{H}_{00}^{1/2}(\Gamma_i) &:= \{\mathbf{v}_i \in \mathbf{H}^{1/2}(\Gamma_i) : \mathbf{v}_i|_{\overline{\Gamma}_i \cap \partial\Omega_D} = 0\}, \\ \mathbf{V}_{\Gamma_i} &:= \begin{cases} \mathbf{H}^{1/2}(\Gamma_i) & \text{if } \overline{\Gamma}_i \cap \partial\Omega_D = \emptyset, \\ \mathbf{H}_{00}^{1/2}(\Gamma_i) & \text{if } \overline{\Gamma}_i \cap \partial\Omega_D \neq \emptyset. \end{cases} \end{aligned}$$

We also set the following local velocity spaces:

$$\mathbf{H}_D^1(\Omega_i) := \{\mathbf{v} \in \mathbf{H}^1(\Omega) : \mathbf{v}|_{\partial\Omega_i \cap \partial\Omega_D} = 0\},$$

$$\widetilde{\mathbf{V}}_i := \begin{cases} \mathbf{H}^1(\Omega_i) & \text{if } \partial\Omega_i \cap \partial\Omega_D \neq \emptyset, \\ \mathbf{H}^1_D(\Omega_i) & \text{if } \partial\Omega_i \cap \partial\Omega_D = \emptyset. \end{cases} \tag{2.15}$$

Finally, we define trace operators γ, γ_i and extension operators $\mathcal{R}$, $\mathcal{R}_i$ as follows

$$\gamma : \mathbf{H}^1_D(\Omega) \to \mathbf{H}^{1/2}_{00}(\Gamma), \tag{2.16}$$

$$\mathcal{R} : \mathbf{H}^{1/2}_{00}(\Gamma) \to \mathbf{H}^1_D(\Omega), \text{ such that } \gamma(\mathcal{R}(\mathbf{v}_\Gamma)) = \mathbf{v}_\Gamma \quad \forall \mathbf{v}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma), \tag{2.17}$$

$$\gamma_i : \widetilde{\mathbf{V}}_i \to \mathbf{V}_{\Gamma_i},$$

$$\mathcal{R}_i : \mathbf{V}_{\Gamma_i} \to \widetilde{\mathbf{V}}_i, \text{ such that } \gamma_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) = \mathbf{v}_{\Gamma_i} \quad \forall \mathbf{v}_{\Gamma_i} \in \mathbf{V}_{\Gamma_i}. \tag{2.18}$$

The global operator $\mathcal{R}$ extends a function defined in the global interface space $\mathbf{H}^{1/2}_{00}(\Gamma)$ onto the global space $\mathbf{H}^1_D(\Omega)$ and similarly, the local operator $\mathcal{R}_i$ extends a function defined on the local interface into the local space $\widetilde{\mathbf{V}}_i$. This is needed when evaluating a bilinear or linear form for an interface variable, since these forms are defined on the global domain or subdomains. The global or local interfaces are zero sets with respect to the global or local domains.

2.4.3 *Decomposition of the Pressure Space*

We define the spaces $\widetilde{D}(\Omega)$, $\widehat{D}(\Omega)$ and $D(\Omega)$. All are used for the decomposition of the global pressure space. The first one for the SWF (2.6), the second for the AWF (2.7) and the last one for the OWF (2.13).

$$\widetilde{D}(\Omega) := \{q \in L^2_0(\Omega) : q|_{\Omega_i} \text{ const.} \quad \forall i\},$$

$$\widehat{D}(\Omega) := \{q \in L^2(\Omega) : q|_{\Omega_i} \text{ const.} \quad \forall i\},$$

$$D(\Omega) := \left\{ q \in L^2(\Omega) : q|_{\Omega_i} = \begin{cases} \text{const} & \text{if } i \notin \mathcal{N}, \\ 0 & \text{if } i \in \mathcal{N} \end{cases} \right\}.$$

For a more simple notation in case of outflow boundary conditions, we define the local pressure space Q_i:

$$Q_i := \begin{cases} L^2_0(\Omega_i) & \text{if } i \notin \mathcal{N}, \\ L^2(\Omega_i) & \text{if } i \in \mathcal{N}. \end{cases}$$

Based on these spaces, we decompose $L^2_0(\Omega)$ and $L^2(\Omega)$:

$$L^2_0(\Omega) = \bigoplus_{i=1}^{s} L^2_0(\Omega_i) \oplus \widetilde{D}(\Omega) \quad (SWF),$$

$$L^2(\Omega) = \bigoplus_{i=1}^{s} L_0^2(\Omega_i) \oplus \widehat{D}(\Omega) \quad (AWF).$$

For the outflow case (OWF), we decompose $L^2(\Omega)$ as follows:

$$L^2(\Omega) = \bigoplus_{i \notin \mathcal{N}} L_0^2(\Omega_i) \bigoplus_{i \in \mathcal{N}} L^2(\Omega_i) \oplus D(\Omega) = \bigoplus_{i=1}^{s} Q_i \oplus D(\Omega).$$

As we see later on in Section 2.4.6, this partitioning leads to local subdomain problems with the same saddle point structure as the global problem. Additionally, we get a global saddle point structure for the coupling conditions due to the global coupling spaces $\widetilde{D}(\Omega)$, $\widehat{D}(\Omega)$ or $D(\Omega)$.

2.4.4 *Decomposition of the Bilinear Forms*

Firstly, we define the local bilinear form $a_i(\cdot,\cdot)$:

$$\begin{aligned}
a_i &: \mathbf{H}^1(\Omega_i) \times \mathbf{H}^1(\Omega_i) \to \mathbb{R} \\
a_i(\mathbf{u}_i, \mathbf{v}_i) &= 2\mu \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}_i) : \boldsymbol{\varepsilon}(\mathbf{v}_i)\, d\mathbf{x} \\
&\quad + \frac{1}{2} \int_{\Omega_i} ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i\, d\mathbf{x} - \frac{1}{2} \int_{\Omega_i} ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{v}_i)\mathbf{u}_i\, d\mathbf{x} \\
&= \mu \int_{\Omega_i} \boldsymbol{\nabla}\mathbf{u}_i : \boldsymbol{\nabla}\mathbf{v}_i\, d\mathbf{x} \\
&\quad + \frac{1}{2} \int_{\Omega_i} ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i\, d\mathbf{x} - \frac{1}{2} \int_{\Omega_i} ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{v}_i)\mathbf{u}_i\, d\mathbf{x}.
\end{aligned}$$

Secondly, we define the local bilinear forms $b_i(\cdot,\cdot)$ and $c_i(\cdot,\cdot)$:

$$\begin{aligned}
b_i &: \mathbf{H}^1(\Omega_i) \times L^2(\Omega_i) \to \mathbb{R} \qquad & b_i(\mathbf{u}_i, q_i) &= -\int_{\Omega_i} \boldsymbol{\nabla} \cdot \mathbf{u}_i\, q_i\, d\mathbf{x}, \\
c_i &: \mathbb{R} \times L^2(\Omega_i) \to \mathbb{R} & c(\eta_i, q_i) &= \eta_i \int_{\Omega_i} q_i\, dx.
\end{aligned}$$

Finally, we define the local linear forms $\widetilde{f}_i(\cdot)$ for the inhomogeneous case and $f_i(\cdot)$ for the outflow case:

$$\begin{aligned}
\widetilde{f}_i &: \mathbf{H}^1(\Omega_i) \to \mathbb{R} \qquad & \widetilde{f}_i(\mathbf{v}_i) &= \int_{\Omega_i} \mathbf{f} \cdot \mathbf{v}_i\, d\mathbf{x}, \\
f_i &: \mathbf{H}^1(\Omega_i) \to \mathbb{R} \qquad & f_i(\mathbf{v}_i) &= \int_{\Omega_i} \mathbf{f} \cdot \mathbf{v}_i\, d\mathbf{x} + \int_{\partial\Omega_{out} \cap \partial\Omega_i} \mathbf{h} \cdot \mathbf{v}_i\, d\mathbf{s}.
\end{aligned}$$

The canonical restriction of the global bilinear form $a(\cdot,\cdot)$ is given by

$$\widetilde{a}_i(\mathbf{u}_i, \mathbf{v}_i) = 2\mu \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}_i) : \boldsymbol{\varepsilon}(\mathbf{v}_i)\, d\mathbf{x} + \int_{\Omega_i} ((\mathbf{a} \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i\, d\mathbf{x}$$

$$
\begin{aligned}
&= 2\mu \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}_i) : \boldsymbol{\varepsilon}(\mathbf{v}_i)\, d\mathbf{x} + \frac{1}{2}\int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i\, d\mathbf{x} + \\
&\underbrace{-\frac{1}{2}\int_{\Omega_i} (\boldsymbol{\nabla}\cdot\mathbf{a})\mathbf{v}_i\mathbf{u}_i\, d\mathbf{x}}_{=0,\,\boldsymbol{\nabla}\cdot\mathbf{a}=0 \text{ a.e. in } \Omega} + \\
&- \frac{1}{2}\int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{v}_i)\mathbf{u}_i\, d\mathbf{x} + \frac{1}{2}\int_{\Gamma_i} ((\mathbf{a}\cdot\mathbf{n}_i)\mathbf{u}_i)\mathbf{v}_i\, d\mathbf{s} \\
&= a_i(\mathbf{u}_i,\mathbf{v}_i) - \frac{1}{2}\int_{\Gamma_i} ((\mathbf{a}\cdot\mathbf{n}_i)\mathbf{u}_i)\mathbf{v}_i\, d\mathbf{s},
\end{aligned}
$$

where $\mathbf{n}_i$ denotes the outer unit normal vector on Ω_i. We do not consider the canonical restriction $\widetilde{a}_i(\cdot,\cdot)$, because the $\mathbf{V}_i$-ellipticity might not be guaranteed due to the boundary integral. But note that for $\mathbf{u}_i \in \widetilde{\mathbf{V}}_i$ and $\mathbf{v}_i \in \mathbf{H}^1_0(\Omega_i)$, we get

$$a_i(\mathbf{u}_i,\mathbf{v}_i) = \widetilde{a}_i(\mathbf{u}_i,\mathbf{v}_i).$$

Proposition 6 states the $\mathbf{V}_i$-ellipticity for $a_i(\cdot,\cdot)$ needed to show that the local subdomain problems on Ω_i, see Section 2.4.6, have a unique solution.

Proposition 6. *The bilinear form $a_i(\cdot,\cdot)$ is $\mathbf{V}_i$-elliptic.*

Proof.

$$
\begin{aligned}
a_i(\mathbf{v}_i,\mathbf{v}_i) &= \mu \int_{\Omega_i} \boldsymbol{\nabla}\mathbf{v}_i : \boldsymbol{\nabla}\mathbf{v}_i d\mathbf{x} + \\
&\underbrace{+\frac{1}{2}\int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{v}_i)\mathbf{v}_i\, d\mathbf{x} - \frac{1}{2}\int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{v}_i)\mathbf{v}_i\, d\mathbf{x}}_{=0} \\
&\geq c\,\|\mathbf{v}_i\|_{\mathbf{H}^1(\Omega_i)}\ \forall \mathbf{v}_i \in \mathbf{V}_i,\ c>0,
\end{aligned}
$$

see also [1]. □

To be able to decouple the global weak formulations, we need to proof that global bilinear forms equal the sum over the corresponding local bilinear forms which is shown in Proposition 7. This proposition is used, e.g., in the proof of Lemma 9.

Proposition 7. *For the local bilinear forms $a_i(\cdot,\cdot)$, $\widetilde{a}_i(\cdot,\cdot)$ and $b_i(\cdot,\cdot)$, it holds that*

$$
\begin{aligned}
&\sum_{i=1}^{s} a_i(\mathbf{u}|_{\Omega_i},\mathbf{v}|_{\Omega_i}) = \sum_{i=1}^{s} \widetilde{a}_i(\mathbf{u}|_{\Omega_i},\mathbf{v}|_{\Omega_i}) = a(\mathbf{u},\mathbf{v}) \\
&\qquad\qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\ \forall \mathbf{v} \in \begin{cases} \mathbf{H}^1_0(\Omega), & (SWF,\ AWF) \\ \mathbf{H}^1_D(\Omega), & (OWF) \end{cases} \\
&\sum_{i=1}^{s} b_i(\mathbf{u}|_{\Omega_i}, q|_{\Omega_i}) = b(\mathbf{u},q) \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\ q \in L^2(\Omega).
\end{aligned}
$$

Proof.

$$\begin{aligned}\sum_{i=1}^{s} a_i(\mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i}) = &\sum_{i=1}^{s} \Big(\mu \int_{\Omega_i} \boldsymbol{\nabla}\mathbf{u}_i : \boldsymbol{\nabla}\mathbf{v}_i \, d\mathbf{x} + \\ &+\frac{1}{2}\int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{x} - \frac{1}{2}\int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{v}_i)\mathbf{u}_i \, d\mathbf{x}\Big) \\ = &\mu \int_{\Omega} \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{v} \, d\mathbf{x} + \sum_{i=1}^{s}\Big(\int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{x} + \\ &- \frac{1}{2}\int_{\partial\Omega_i} ((\mathbf{a}\cdot\mathbf{n}_i)\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{s}\Big) \\ = &a(\mathbf{u},\mathbf{v}) - \frac{1}{2}\underbrace{\int_{\partial\Omega} ((\mathbf{a}\cdot\mathbf{n})\mathbf{u})\mathbf{v} \, d\mathbf{s}}_{=0,\ (*)} + \\ &- \frac{1}{2}\sum_{\substack{i=1\\ j>i}}^{s} \int_{\Gamma_{ij}} ((\mathbf{a}\cdot\underbrace{(\mathbf{n}_i + \mathbf{n}_j)}_{=0})\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{s} \\ = &\, a(\mathbf{u},\mathbf{v})\end{aligned}$$

(*) In case of global Dirichlet boundary, the integral is 0, since $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$. In case of outflow boundary:

$$(*) = -\frac{1}{2}\underbrace{\int_{\Omega_D} ((\mathbf{a}\cdot\mathbf{n})\mathbf{u})\mathbf{v} \, d\mathbf{s}}_{=0,\, \mathbf{v}\in\mathbf{H}_D^1(\Omega)} - \frac{1}{2}\int_{\Omega_{out}} (\underbrace{(\mathbf{a}\cdot\mathbf{n})}_{=0,\,(2.11)}\mathbf{u})\mathbf{v} \, d\mathbf{s} = 0$$

We do not give the proof for $\sum_{i=1}^{s} \widetilde{a}_i(\mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i})$, it works analogously.

$$\sum_{i=1}^{s} b_i(\mathbf{u}|_{\Omega_i}, q|_{\Omega_i}) = \sum_{i=1}^{s} -\int_{\Omega_i} \boldsymbol{\nabla}\cdot(\mathbf{u}|_{\Omega_i})\, q|_{\Omega_i} d\mathbf{x} = -\int_{\Omega} \boldsymbol{\nabla}\cdot\mathbf{u}\, q \, d\mathbf{x}.$$

□

The next proposition states that one specific term which occurs when decomposing the bilinear form $b(\cdot,\cdot)$ canonically, can be neglected. The proposition is used when deriving the local weak formulation on subdomains.

Proposition 8. *It holds that*

$$b_i(\mathbf{v}_i, q_{\Omega_i}) = 0 \quad \forall \mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i),\ q_{\Omega_i} \textit{ constant}.$$

Proof. We use the definition of the bilinear form and apply the divergence theorem, see, e.g., [38, Chapter 2, Lemma 4]:

$$b_i(\mathbf{v}_i, q_{\Omega_i}) = -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\mathbf{v}_i \, q_{\Omega_i} d\mathbf{x}$$

$$= \int_{\Omega_i} \mathbf{v}_i \underbrace{\boldsymbol{\nabla} \cdot q_{\Omega_i}}_{=0} d\mathbf{x} - \underbrace{\int_{\partial\Omega_i} \mathbf{v}_i \cdot \mathbf{n}_i q_{\Omega_i} d\mathbf{x}}_{=0,\, \mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)} = 0$$

□

2.4.5 *Weak Formulation on Subdomains*

Using these definitions, we derive weak formulations on subdomains for the SWF, the AWF and the OWF. We use these fully coupled formulations, when we are interested in solving directly the global linear system. In that case, we apply the domain decomposition method only as parallelization method without decoupling the global system. For more details, see Section 6.5 in Chapter 6.

Remark. For sake of better readability, we use the following notation:

- $\mathbf{v}_{\Gamma_i} := \mathbf{v}_\Gamma|_{\Gamma_i} \in \mathbf{V}_{\Gamma_i}$ denotes the restriction of $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$,
- $\mathbf{u}_{D_i} = \mathbf{u}_D|_{\Omega_i} \in \mathbf{H}^1(\Omega_i)$ the restriction of $\mathbf{u}_D \in \mathbf{H}^1(\Omega)$,
- $q_{\Omega_i} := q_\Omega|_{\Omega_i}$ the restriction of $q_\Omega \in D(\Omega)$ with $q_{\Omega_i} = const$,
- $\widetilde{q}_{\Omega_i} := \widetilde{q}|_{\Omega_i}$ the restriction of $\widetilde{q}_\Omega \in D(\Omega)$ with $\widetilde{q}_{\Omega_i} = const$,
- $\widehat{q}_{\Omega_i} := \widehat{q}_\Omega|_{\Omega_i}$ the restriction of $\widehat{q}_\Omega \in D(\Omega)$ with $\widehat{q}_{\Omega_i} = const$.

Standard Weak Formulation (SWF)

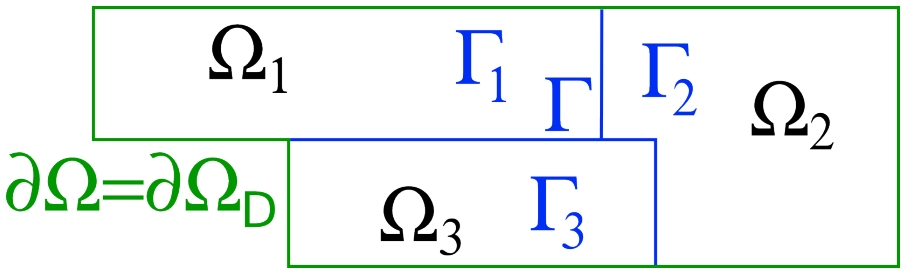


Figure 5: This figure illustrates the first sub-step of the second main step of the derivation. The global domain Ω is partitioned into subdomains but still coupled analogously to the weak formulation on subdomains.

The next lemma states an equivalent weak formulation on subdomains corresponding the SWF of the global weak formulation (2.6). This weak formulation is still globally coupled and is the result of the first sub-step as sketched in Fig. 5. Recall that $\mathcal{R}$ and $\mathcal{R}_i$ are the global and local extension operators defined before, see (2.17) and (2.18), respectively.

Lemma 9. *(2.6) is equivalent to the following formulation on subdomains: Find*

$$\mathbf{u} = \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}_0^1(\Omega) \qquad \text{with } \mathbf{u}_i \in \mathbf{H}_0^1(\Omega_i),\ \mathbf{u}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma),$$

$$\widetilde{p} = \sum_{i=1}^{s} \widetilde{p}_i + \widetilde{p}_\Omega \in L_0^2(\Omega) \qquad \text{with } \widetilde{p}_i \in L_0^2(\Omega_i),\ \widetilde{p}_\Omega \in \widetilde{D}(\Omega)$$

such that

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i,\mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i) + a_i(\mathbf{u}_i,\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathbf{v}_i,\widetilde{p}_i) + \\
&\quad + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}),\widetilde{p}_i)\Big) + a(\mathcal{R}(\mathbf{u}_\Gamma),\mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma),\widetilde{p}_\Omega) \\
&= \sum_{i=1}^{s}\Big(\widetilde{f}_i(\mathbf{v}_i) - a_i(\mathbf{u}_D|_{\Omega_i},\mathbf{v}_i)\Big) + \widetilde{f}(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D,\mathcal{R}(\mathbf{v}_\Gamma)), \qquad (2.19a) \\
&\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_i,\widetilde{q}_i) + b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\widetilde{q}_i)\Big) + b(\mathcal{R}(\mathbf{u}_\Gamma),\widetilde{q}_\Omega) \\
&= -\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_D|_{\Omega_i},\widetilde{q}_i)\Big) - b(\mathbf{u}_D,\widetilde{q}_\Omega) \qquad (2.19b)
\end{aligned}$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$, $\widetilde{q}_i \in L_0^2(\Omega_i)$, $(i = 1,\dots,s)$, $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$ *and* $\widetilde{q}_\Omega \in \widetilde{D}(\Omega)$.

Proof. The equivalence can be shown by using Proposition 7 to partition the global bilinear forms onto the subdomains and then decomposing the ansatz and test functions based on the splitting of the corresponding global spaces. We exemplary show the partitioning of the global bilinear form $a(\cdot,\cdot)$ into the local bilinear forms $a_i(\cdot,\cdot)$. The splitting of the other bilinear and linear forms works analogously.

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}) &\overset{\text{(Prop. 7)}}{=} \sum_{i=1}^{s} a_i(\mathbf{u}|_{\Omega_i},\mathbf{v}|_{\Omega_i}) \\
&= \sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i + \mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i + \mathcal{R}_i(\mathbf{v}_{\Gamma_i}))\Big) \\
&= \sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i,\mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i) + a_i(\mathbf{u}_i,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))\Big) + \\
&\quad + \underbrace{\sum_{i=1}^{s}\Big(a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))\Big)}_{=a(\mathcal{R}(\mathbf{u}_\Gamma),\mathcal{R}(\mathbf{v}_\Gamma))} \\
&= \sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i,\mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i) + a_i(\mathbf{u}_i,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))\Big) + \\
&\quad + a(\mathcal{R}(\mathbf{u}_\Gamma),\mathcal{R}(\mathbf{v}_\Gamma)).
\end{aligned}$$

For the bilinear form $b(\cdot,\cdot)$, we further apply Proposition 8. □

Remark 10. We note that the global bilinear form $b(\cdot,\cdot)$ for all $\mathbf{v}_\Gamma \in \mathbf{H}^1_{00}(\Gamma)$ and $\widetilde{q}_\Omega \in D(\Omega)$ are zero:

$$\begin{aligned} b(\mathcal{R}(\mathbf{v}_\Gamma),\widetilde{q}_\Omega) &= -\int_\Omega \boldsymbol{\nabla}\cdot\mathcal{R}(\mathbf{v}_\Gamma)\widetilde{q}_\Omega\,dx \\ &= \int_\Omega \mathcal{R}(\mathbf{v}_\Gamma)\underbrace{\boldsymbol{\nabla}\widetilde{q}_\Omega}_{=0,\in\widetilde{D}(\Omega)}\,dx - \int_{\partial\Omega}\underbrace{\mathcal{R}(\mathbf{v}_\Gamma)|_{\partial\Omega}}_{=0,\in\mathbf{H}^1_0(\Omega)}\widetilde{q}_\Omega\cdot\mathbf{n}\,ds = 0. \end{aligned}$$

But we do not neglect this global term because when decomposing the bilinear form onto the subdomains, the single summands must not be zero:

$$\begin{aligned} 0 &= b(\mathcal{R}(\mathbf{v}_\Gamma),\widetilde{q}_\Omega) \\ &= \sum_{i=1}^{s} b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}),\widetilde{q}_{\Omega_i}) \\ &= \sum_{i=1}^{s} -\int_{\Omega_i}\boldsymbol{\nabla}\cdot\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\widetilde{q}_{\Omega_i}dx \\ &= \sum_{i=1}^{s}\Big(\int_{\Omega_i}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\underbrace{\boldsymbol{\nabla}\widetilde{q}_\Omega}_{=0,\in\widetilde{D}(\Omega)}\,dx - \underbrace{\int_{\partial\Omega_i}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})|_{\partial\Omega_i}\widetilde{q}_{\Omega_i}\cdot\mathbf{n}_i\,ds}_{=(*)}\Big) \\ &= -\int_{\partial\Omega}\underbrace{\mathcal{R}(\mathbf{v}_\Gamma)|_{\partial\Omega}}_{=0,\in\mathbf{H}^1_0(\Omega)}\widetilde{q}_\Omega\cdot\mathbf{n}\,ds - \sum_{\substack{i=1\\ j>i}}^{s}\int_{\Gamma_{ij}}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})|_{\partial\Omega_i}\widetilde{q}_{\Omega_i}\cdot\underbrace{(\mathbf{n}_i+\mathbf{n}_j)}_{=0}\,d\mathbf{s} \end{aligned}$$

(*) each of this terms is not zero, but the sum over all equals zero.

Alternative Weak Formulation (AWF)

Lemma 11 presents an equivalent weak formulation on subdomains for the global weak formulation (2.7) corresponding to the AWF. We have the same situation as for the SWF case that this formulation is still globally coupled, see Fig. 5.

Lemma 11. *(2.7) is equivalent to the following formulation on subdomains: Find*

$$\begin{aligned} \mathbf{u} &= \sum_{i=1}^{s}\mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}^1_0(\Omega) && \text{with } \mathbf{u}_i \in \mathbf{H}^1_0(\Omega_i),\ \mathbf{u}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma), \\ p &= \sum_{i=1}^{s}\widetilde{p}_i + p_\Omega \in L^2(\Omega) && \text{with } \widetilde{p}_i \in L^2_0(\Omega_i),\ p_\Omega \in \widehat{D}(\Omega), \end{aligned}$$

such that

$$\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i,\mathbf{v}_i)+a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i)+a_i(\mathbf{u}_i,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))+b_i(\mathbf{v}_i,\widetilde{p}_i)+$$
$$+b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}),\widetilde{p}_i)\Big)+a(\mathcal{R}(\mathbf{u}_\Gamma),\mathcal{R}(\mathbf{v}_\Gamma))+b(\mathcal{R}(\mathbf{v}_\Gamma),p_\Omega)$$
$$=\sum_{i=1}^{s}\Big(\widetilde{f}_i(\mathbf{v}_i)-a_i(\mathbf{u}_D|_{\Omega_i},\mathbf{v}_i)\Big)+\widetilde{f}(\mathcal{R}(\mathbf{v}_\Gamma))-a(\mathbf{u}_D,\mathcal{R}(\mathbf{v}_\Gamma)), \tag{2.20a}$$
$$\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_i,\widetilde{q}_i)+b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\widetilde{q}_i)\Big)+b(\mathcal{R}(\mathbf{u}_\Gamma),q_\Omega)+c(\eta,q_\Omega)$$
$$=-\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_D|_{\Omega_i},\widetilde{q}_i)\Big)-b(\mathbf{u}_D,q_\Omega) \tag{2.20b}$$
$$c(\xi,p_\Omega)=0 \tag{2.20c}$$

for all $\mathbf{v}_i\in\mathbf{H}_0^1(\Omega_i)$, $\mathbf{v}_\Gamma\in\mathbf{H}_{00}^{1/2}(\Gamma)$, $\widetilde{q}_i\in L_0^2(\Omega_i)$, $q_\Omega\in D(\Omega)$ *and* $\xi\in\mathbb{R}$.

Proof. Use the same arguments as in proof for Lemma 9. □

Outflow Weak Formulation (OWF)

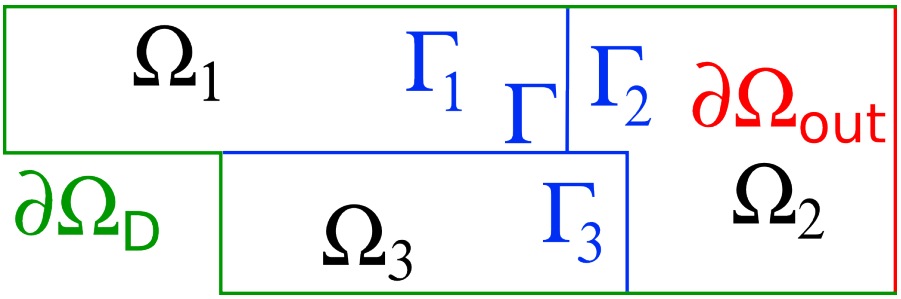


Figure 6: The figure shows the global domain Ω, which is already partitioned into subdomains but still coupled. We apply outflow boundary conditions on $\partial\Omega_{out}$. It also corresponds to the first sub-step of the second main step for the outflow case.

The next lemma introduces an equivalent weak formulations on subdomains for the OWF (2.13). Analogously to the previous cases the formulation is still globally coupled which is depicted in Fig 6 and refers to the first sub-step of the second main step of the derivation of the method. The formulation changes slightly due to the different boundary conditions.

Lemma 12. *(2.13) is equivalent to the following weak formulation on subdomains:*

$$\mathbf{u}=\sum_{i=1}^{s}\mathbf{u}_i+\mathcal{R}(\mathbf{u}_\Gamma)\in\mathbf{H}_D^1(\Omega) \qquad \textit{with } \mathbf{u}_i\in\mathbf{V}_i,\ \mathbf{u}_\Gamma\in\mathbf{H}_{00}^{1/2}(\Gamma),$$
$$p=\sum_{i=1}^{s}p_i+p_\Omega\in L^2(\Omega) \qquad \textit{with } p_i\in Q_i,\ p_\Omega\in D(\Omega)$$

such that

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i,\mathbf{v}_i)+a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i)+a_i(\mathbf{u}_i,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))+b_i(\mathbf{v}_i,p_i)+\\
&\quad+b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}),p_i)\Big)+a(\mathcal{R}(\mathbf{u}_\Gamma),\mathcal{R}(\mathbf{v}_\Gamma))+b(\mathcal{R}(\mathbf{v}_\Gamma),p_\Omega)\\
&=\sum_{i=1}^{s}\Big(f_i(\mathbf{v}_i)-a_i(\mathbf{u}_{D_i},\mathbf{v}_i)\Big)+f(\mathcal{R}(\mathbf{v}_\Gamma))-a(\mathbf{u}_D,\mathcal{R}(\mathbf{v}_\Gamma)), \qquad (2.21a)\\
&\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_i,q_i)+b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),q_i)\Big)+b(\mathcal{R}(\mathbf{u}_\Gamma),q_\Omega)\\
&=-\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_{D_i},q_i)\Big)-b(\mathbf{u}_D,q_\Omega) \qquad (2.21b)
\end{aligned}$$

for all $\mathbf{v}_i \in \mathbf{V}_i$, $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, $q_i \in Q_i$ *and* $q_\Omega \in D(\Omega)$.

Proof. Note that on non-outflow subdomains for any function $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$ and any constant q_{Ω_i}, $b_i(\mathbf{v}_i, q_{\Omega_i}) = 0$, see Proposition 8. On outflow subdomains, it holds that $q_{\Omega_i} = p_{\Omega_i} = 0$, which directly comes from the definition of $D(\Omega)$. Consequently, it also holds on outflow domains that $b_i(\mathbf{v}_i, q_{\Omega_i}) = 0 \quad \forall \mathbf{v}_i \in \mathbf{H}_N^1$, $q_{\Omega_i} = 0$. Moreover, we use the same arguments as in the proof of Lemma 9. □

2.4.6 *Decoupling of Weak Formulation on Subdomains*

In the next step also referred as the second sub-step, we decouple the weak formulations on subdomains for the SWF (2.19), the AWF (2.20) and the OWF (2.21). In all cases, we gain s independent local subdomain problems, which are coupled through one system of interface equations. Fig. 4 sketches exemplary the case of the SWF and the AWF. In that example, we get three independent weak formulations on the subdomains and one system of coupling equations on the interface Γ. On the subdomains, we apply Dirichlet boundary conditions on the local interfaces Γ_i and on the part of the subdomain boundary which intersects the global boundary $\partial\Omega_D$.

Standard Weak Formulation (SWF)

For a given $\mathbf{u}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i \in \mathbf{H}_0^1(\Omega_i)$, $\widetilde{p}_i \in L_0^2(\Omega_i)$ $(i = 1, \ldots, s)$ such that

$$\begin{aligned}
a_i(\mathbf{u}_i,\mathbf{v}_i)+b_i(\mathbf{v}_i,\widetilde{p}_i) &= \widetilde{f}_i(\mathbf{v}_i)-a_i(\mathbf{u}_{D_i},\mathbf{v}_i)-a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i), \qquad (2.22a)\\
b_i(\mathbf{u}_i,\widetilde{q}_i) &= -b_i(\mathbf{u}_{D_i},\widetilde{q}_i)-b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\widetilde{q}_i) \qquad (2.22b)
\end{aligned}$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$ and $\widetilde{q}_i \in L_0^2(\Omega_i)$. We directly see that the decoupled local problem on subdomain Ω_i does not depend on the global

pressure variable $\widetilde{p} \in \widetilde{D}(\Omega)$. We also note that we obtain the same saddle point structure on the local level as on the global level.

Remark. The existence and uniqueness can be shown by applying the Theorem 2 on the local level. In Proposition 6, we already proofed the $\mathbf{V}_i-$ellipticity of the bilinear form $a_i(\cdot,\cdot)$.

Our aim is to solve the global linear system (2.19) by solving coupling conditions on the interface. Therefore, the next lemma states under which conditions the local and the global formulations are equivalent.

Lemma 13. *It holds that*

$$\mathbf{u} = \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}_0^1(\Omega) \text{ and } \widetilde{p} = \sum_{i=1}^{s} \widetilde{p}_i + \widetilde{p}_\Omega \in L_0^2(\Omega)$$

with $(\mathbf{u}_i, \widetilde{p}_i) \in \mathbf{H}_0^1(\Omega_i) \times L_0^2(\Omega_i)$ solutions of (2.22) solve (2.19) if and only if the following coupling conditions hold for $(\mathbf{u}_\Gamma, \widetilde{p}_\Omega) \in \mathbf{H}_{00}^{1/2}(\Gamma) \times \widetilde{D}(\Omega)$:

$$\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \widetilde{p}_i)\Big) + \tag{2.23a}$$

$$+a(\mathcal{R}(\mathbf{u}_\Gamma), \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), \widetilde{p}_\Omega) = \widetilde{f}(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)),$$

$$b(\mathcal{R}(\mathbf{u}_\Gamma), \widetilde{q}_\Omega) = -b(\mathbf{u}_D, \widetilde{q}_\Omega) \tag{2.23b}$$

for all $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$ and $\widetilde{q}_\Omega \in \widetilde{D}(\Omega)$.

Proof. This can be shown by summing up the coupling conditions (2.23) and the weak formulations (2.22) on each subdomain Ω_i. □

Alternative Weak Formulation (AWF)

In a first step, the decoupling of (2.20) leads to a the same local subdomain formulation (2.22) as for the SWF. We use the same arguments as in the global case to give an alternative weak formulation.

Proposition 14. *Assuming that*

$$c_i(\eta_i, \widehat{q}_{\Omega_i}) = b_i(\mathbf{u}_{D_i}, \widehat{q}_{\Omega_i}) + b(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \widehat{q}_{\Omega_i}) \tag{2.24}$$

holds, (2.22) is equivalent to the following subdomain formulation: For given $\mathbf{u}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i \in \mathbf{H}_0^1(\Omega_i)$, $p_i \in L^2(\Omega_i)$ and $\eta_i \in \mathbb{R}$ $(i = 1, \ldots, s)$ such that

$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) = \widetilde{f}_i(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \tag{2.25a}$$

$$b_i(\mathbf{u}_i, q_i) + c_i(\eta_i, q_i) = - b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i), \tag{2.25b}$$

$$c_i(\xi_i, p_i) = 0 \tag{2.25c}$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$, $q_i \in L^2(\Omega_i)$ and $\xi_i \in \mathbb{R}$.

Proof. Analogously to the proof of Lemma 3, we equivalently reformulate (2.25) as follows:

$$
\begin{aligned}
&a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, \widetilde{p}_i) + \underbrace{b_i(\mathbf{v}_i, \widetilde{p}_{\Omega_i})}_{=0,\,(2.8)} \\
&= \widetilde{f}_i(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \\
&b_i(\mathbf{u}_i, \widetilde{q}_i) + \underbrace{b_i(\mathbf{u}_i, \widetilde{q}_{\Omega_i})}_{=0,\,(2.8)} + \underbrace{c_i(\eta_i, \widetilde{q}_i)}_{=0,\,(2.9)} + c_i(\eta_i, \widetilde{q}_{\Omega_i}) \\
&= -b_i(\mathbf{u}_{D_i}, \widetilde{q}_i) - b_i(\mathbf{u}_{D_i}, \widetilde{q}_{\Omega_i}) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \widetilde{q}_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \widetilde{q}_{\Omega_i}), \\
&\underbrace{c_i(\xi_i, \widetilde{p}_i)}_{=0,\,(2.9)} + c_i(\xi_i, \widetilde{p}_{\Omega_i}) = 0
\end{aligned}
$$

Under the assumptions (2.24), we can analogously apply the argumentation of the proof of Lemma 3. □

Similarly to the SWF case, the next lemma explains the conditions under which the local and global formulations are equivalent.

Lemma 15. *It holds that*

$$\mathbf{u} = \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}_0^1(\Omega) \text{ and } p = \sum_{i=1}^{s} p_i + \widehat{p}_\Omega \in L^2(\Omega)$$

with $(\mathbf{u}_i, p_i) \in \mathbf{H}_0^1(\Omega_i) \times L^2(\Omega_i)$ *solutions of (2.25) solve (2.20) if and only if the following coupling conditions hold for* $(\mathbf{u}_\Gamma, \widehat{p}_\Omega) \in \mathbf{H}_{00}^{1/2}(\Gamma) \times \widehat{D}(\Omega)$ *and* $\eta \in \mathbb{R}$*:*

$$
\begin{aligned}
\sum_{i=1}^{s} \Big(a_i(\mathbf{u}_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), p_i) \Big) + & \qquad (2.26a) \\
+ a(\mathcal{R}(\mathbf{u}_\Gamma), \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), \widehat{p}_\Omega) &= \widetilde{f}(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)), \\
b(\mathcal{R}(\mathbf{u}_\Gamma), \widehat{q}_\Omega) + c(\eta, \widehat{q}_\Omega) &= -b(\mathbf{u}_D, \widehat{q}_\Omega), \qquad (2.26b) \\
c(\xi, \widehat{p}_\Omega) &= 0 \qquad (2.26c)
\end{aligned}
$$

for all $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, $\widehat{q}_\Omega \in \widehat{D}(\Omega)$ *and* $\xi \in \mathbb{R}$.

Proof. Use the same arguments as in proof for Lemma 13. □

2.4.6.1 *Outflow Weak Formulation (OWF)*

In the outflow case, we have to distinguish two different local independent subdomain problems: One for the set of outflow subdomains $\mathcal{N}$ and another one for non-outflow subdomains, which is sketched in Fig. 7. For outflow subdomains, we apply Dirichlet boundary conditions on the local skeleton and on the subdomain boundary which intersects the global boundary $\partial\Omega_D$ and outflow boundary conditions on subdomain boundary which intersects the outflow boundary $\partial\Omega_{out}$. Thus,

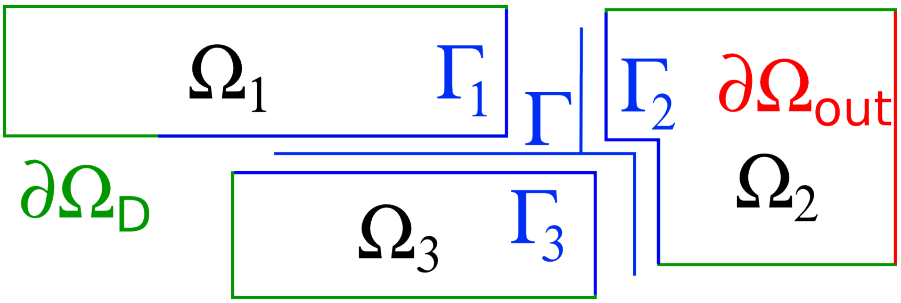


Figure 7: This figure refers to the second sub-step in which we obtain decoupled subdomain problems and one system of coupling conditions on the interface Γ. It shows three decoupled subdomains, the global interface Γ and the local interfaces Γ_i. In this example, the subdomain Ω_2 is an outflow subdomain while the subdomains Ω_1 and Ω_3 are non-outflow subdomains.

we obtain the following weak formulation for outflow subdomains: For given $\mathbf{u}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i \in \mathbf{H}_N^1(\Omega_i)$, $p_i \in L^2(\Omega_i)$ $(i \in \mathcal{N})$ such that

$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) = f_i(\mathbf{v}_i) - a_i(\mathbf{u}_D|_{\Omega_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \quad (2.27a)$$
$$b_i(\mathbf{u}_i, q_i) = - b_i(\mathbf{u}_D|_{\Omega_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i) \quad (2.27b)$$

for all $\mathbf{v}_i \in \mathbf{V}_i$ and $q_i \in L^2(\Omega_i)$.

For non-outflow subdomains, i.e. subdomains not intersecting an outflow boundary $\partial\Omega_{out}$, we apply Dirichlet boundary conditions on the whole subdomain boundary $\partial\Omega_i$. This yields the following weak formulation: For a given $\mathbf{u}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i \in \mathbf{H}_0^1(\Omega_i)$, $\widetilde{p}_i \in L_0^2(\Omega_i)$ $(i \notin \mathcal{N})$ such that

$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, \widetilde{p}_i) = f_i(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \quad (2.28a)$$
$$b_i(\mathbf{u}_i, \widetilde{q}_i) = - b_i(\mathbf{u}_{D_i}, \widetilde{q}_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \widetilde{q}_i) \quad (2.28b)$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$ and $\widetilde{q}_i \in L_0^2(\Omega_i)$.

We note that the local formulation on outflow subdomains has the same structure as the global problem with outflow conditions whereas the local problem on non-outflow subdomains has the same structure as the global problems with inhomogeneous boundary conditions. Again, we give an alternative weak formulations for the non-outflow subdomains.

Proposition 16. *Assuming (2.24), then (2.28) is equivalent to the following weak formulation. For given $\mathbf{u}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i \in \mathbf{H}_0^1(\Omega_i)$, $p_i \in L^2(\Omega_i)$, $\eta_i \in \mathbb{R}$ $(i \notin \mathcal{N})$ such that*

$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) = f_i(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \quad (2.29a)$$
$$b_i(\mathbf{u}_i, q_i) + c_i(\eta_i, q_i) = - b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i), \quad (2.29b)$$
$$c_i(\xi_i, p_i) = 0 \quad (2.29c)$$

for all $\mathbf{v}_i \in \mathbf{V}_i$, $q_i \in L^2(\Omega_i)$ *and* $\xi_i \in \mathbb{R}$.

Proof. The only differences between formulation (2.29) and (2.25) are the definitions of the right hand sides. Consequently, we can use the same proof as for Proposition 14. □

In the next lemma, we state under which conditions the global and local formulations are equivalent for the outflow case.

Lemma 17. *It holds that*

$$\mathbf{u} = \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}^1_D(\Omega) \text{ and } p = \sum_{i=1}^{s} p_i + p_\Omega \in L^2(\Omega)$$

with $(\mathbf{u}_i, p_i) \in \mathbf{V}_i \times Q_i$ *solutions of (2.27) or (2.29) (depending on the subdomain type) solve (2.21) if and only if the following coupling conditions for* $(\mathbf{u}_\Gamma, p_\Omega) \in \mathbf{H}^{1/2}_{00}(\Gamma) \times D(\Omega)$ *hold:*

$$\sum_{i=1}^{s} \Big(a_i(\mathbf{u}_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) \Big) + a(\mathcal{R}(\mathbf{u}_\Gamma), \mathcal{R}(\mathbf{v}_\Gamma)) + \tag{2.30a}$$

$$+ \sum_{i=1}^{s} \Big(b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), p_i) \Big) + b(\mathcal{R}(\mathbf{v}_\Gamma), p_\Omega) = f(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)),$$

$$b(\mathcal{R}(\mathbf{u}_\Gamma), q_\Omega) = -b(\mathbf{u}_D, q_\Omega) \tag{2.30b}$$

for all $\mathbf{v}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$ *and* $q_\Omega \in D(\Omega)$.

Proof. Use the same arguments as in proof for Lemma 13. □

We note that all three systems of coupling conditions (2.23), (2.26) and (2.30) feature a global saddle point structure analogously to the local formulations. This structure is due to the splitting of the pressure space.

Interpretation of the Coupling Conditions

For a better understanding of the coupling conditions (2.23, 2.26, 2.30), we give an interpretation in strong form. Therefore, we assume that the functions are smooth enough. The first coupling conditions can be interpreted in the same way for all three cases (SWF, AWF and OWF). We denote by $\mathbf{u}|_{\Omega_i} := \mathbf{u}_i + \mathcal{R}_i(\mathbf{u}_{\Gamma_i}) + \mathbf{u}_D|_{\Omega_i}$ the "complete" solution of the velocity on subdomain Ω_i , by $\mathbf{u} := \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) + \mathbf{u}_D$ the "complete" global solution of the velocity, and by $p|_{\Omega_i} := \widetilde{p}_i + p_{\Omega_i}$ the "complete" solution of the pressure on subdomain Ω_i. Exemplarily, we reformulate (2.19a) to obtain an interpretation of the first coupling conditions. Analogously it works for the AWF and OWF case.

$(2.19a) \Leftrightarrow$

$$
\begin{aligned}
&\sum_{i=1}^{s}\Big(\mu\int_{\Omega_i}\boldsymbol{\nabla}\mathbf{u}_i:\boldsymbol{\nabla}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,d\mathbf{x}+\\
&\quad+\frac{1}{2}\int_{\Omega_i}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}_i)\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,d\mathbf{x}-\frac{1}{2}\int_{\Omega_i}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))\mathbf{u}_i\,d\mathbf{x}+\\
&\quad-\int_{\Omega_i}\boldsymbol{\nabla}\cdot\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,\tilde{p}_i\,d\mathbf{x}\Big)+\\
&\quad+\mu\int_{\Omega}\boldsymbol{\nabla}\mathcal{R}(\mathbf{u}_\Gamma):\boldsymbol{\nabla}\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\int_{\Omega}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathcal{R}(\mathbf{u}_\Gamma))\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\\
&\quad-\int_{\Omega}\boldsymbol{\nabla}\cdot\mathcal{R}(\mathbf{v}_\Gamma)\,\tilde{p}_\Omega\,d\mathbf{x}\\
&=\int_{\Omega}f\cdot\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\\
&\quad-\mu\int_{\Omega}\boldsymbol{\nabla}\mathbf{u}_D:\boldsymbol{\nabla}\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}-\int_{\Omega}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}_D)\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}\\
&\Leftrightarrow\\
&\sum_{i=1}^{s}\Big(-\mu\int_{\Omega_i}\Delta\mathbf{u}_i\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,d\mathbf{x}+\int_{\partial\Omega_i}\frac{\partial}{\partial\mathbf{n}_i}\mathbf{u}_i\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,d\mathbf{s}+\\
&\quad+\int_{\Omega_i}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}_i)\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,d\mathbf{x}-\frac{1}{2}\int_{\partial\Omega_i}((\mathbf{a}\cdot\mathbf{n}_i)\mathbf{u}_i)\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,d\mathbf{s}+\\
&\quad+\int_{\Omega_i}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\,\boldsymbol{\nabla}\tilde{p}_i\,d\mathbf{x}-\int_{\partial\Omega_i}(\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\cdot\mathbf{n}_i\tilde{p}_i\,d\mathbf{s}\Big)+\\
&\quad-\mu\int_{\Omega}\Delta\mathcal{R}(\mathbf{u}_\Gamma)\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\underbrace{\int_{\partial\Omega}\frac{\partial}{\partial\mathbf{n}}\mathcal{R}(\mathbf{u}_\Gamma)\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{s}}_{=0,\,\mathcal{R}(\mathbf{v}_\Gamma)\in\mathbf{H}_0^1(\Omega)}+\\
&\quad+\int_{\Omega}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathcal{R}(\mathbf{u}_\Gamma))\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\\
&\quad+\underbrace{\int_{\Omega}\mathcal{R}(\mathbf{v}_\Gamma)\,\boldsymbol{\nabla}\tilde{p}_\Omega\,d\mathbf{x}}_{=0,\,\boldsymbol{\nabla}\tilde{p}_\Omega=0}-\underbrace{\int_{\partial\Omega}(\mathcal{R}(\mathbf{v}_\Gamma)\cdot\mathbf{n}\tilde{p}_\Omega\,d\mathbf{s}}_{=0,\,\mathcal{R}(\mathbf{v}_\Gamma)\in\mathbf{H}_0^1(\Omega)}\\
&=\int_{\Omega}f\cdot\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}-\int_{\Omega}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}_D)\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\\
&\quad+\mu\int_{\Omega}\Delta\mathbf{u}_D\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}-\underbrace{\int_{\partial\Omega}\frac{\partial}{\partial\mathbf{n}}\mathbf{u}_D\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{s}}_{=0,\,\mathcal{R}(\mathbf{v}_\Gamma)\in\mathbf{H}_0^1(\Omega)}\\
&\Leftrightarrow\\
&\quad-\mu\int_{\Omega}\Delta\mathbf{u}\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\int_{\Omega}((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u})\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}+\\
&\quad+\int_{\Omega}\mathcal{R}(\mathbf{v}_\Gamma)\,\boldsymbol{\nabla}\tilde{p}\,d\mathbf{x}\\
&\quad+\sum_{\substack{i=1\\j>i}}^{s}\int_{\Gamma_{ij}}\Big(\Big(\frac{\partial}{\partial\mathbf{n}_i}\mathbf{u}_i-(\mathbf{a}\cdot\mathbf{n}_i)\mathbf{u}_i\Big)\mathcal{R}_i(\mathbf{v}_{\Gamma_i})-(\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\cdot\mathbf{n}_i)\tilde{p}_i+\\
&\qquad\Big(\frac{\partial}{\partial\mathbf{n}_j}\mathbf{u}_j-(\mathbf{a}\cdot\mathbf{n}_j)\mathbf{u}_j\Big)\mathcal{R}_j(\mathbf{v}_{\Gamma_j})-(\mathcal{R}_j(\mathbf{v}_{\Gamma_j})\cdot\mathbf{n}_j)\tilde{p}_j\,d\mathbf{s}\Big)+
\end{aligned}
$$

$$+\sum_{i=1}^{s}\int_{\partial\Omega_i\cap\partial\Omega}\underbrace{\left(\frac{\partial}{\partial\mathbf{n}_i}\mathbf{u}_i-(\mathbf{a}\cdot\mathbf{n}_i)\mathbf{u}_i\right)\mathcal{R}_i(\mathbf{v}_{\Gamma_i})-(\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\cdot\mathbf{n}_i)\tilde{p}_i\,d\mathbf{s}}_{=0,\,\mathcal{R}_i(\mathbf{v}_{\Gamma_i})\in\mathbf{H}^1_D(\Omega_i)}$$
$$=\int_\Omega f\cdot\mathcal{R}(\mathbf{v}_\Gamma)\,d\mathbf{x}$$

Since for the solution the volume integrals over Ω vanish, we get for the first coupling condition:

$$\begin{aligned}&\frac{\partial(\mathbf{u}|_{\Omega_i})}{\partial\mathbf{n}_i}-(\mathbf{a}\cdot\mathbf{n}_i)\mathbf{u}|_{\Omega_i}-(p_i|_{\Omega_i})\mathbf{I}\cdot\mathbf{n}_i\\&=-\frac{\partial(\mathbf{u}|_{\Omega_j})}{\partial\mathbf{n}_j}+(\mathbf{a}\cdot\mathbf{n}_j)\mathbf{u}|_{\Omega_j}+(p_j|_{\Omega_j})\mathbf{I}\cdot\mathbf{n}_j\qquad\text{on }\Gamma_{ij}.\end{aligned}$$

It can be interpreted that on Γ_{ij} we have outflow-outflow type coupling conditions.

The second coupling condition for the SWF and the OWF guaranties that the interface component of the velocity defined on the skeleton is globally divergence free. It is given by

$$\boldsymbol{\nabla}\cdot\mathcal{R}\mathbf{u}_\Gamma+\underbrace{\boldsymbol{\nabla}\cdot\mathbf{u}_D}_{=0,\ (2.2)}=0\qquad\text{in }\Omega.$$

The second condition of the AWF can be represented strongly by

$$\boldsymbol{\nabla}\cdot\mathcal{R}\mathbf{u}_\Gamma+\underbrace{\boldsymbol{\nabla}\cdot\mathbf{u}_D}_{=0,\ (2.2)}+\underbrace{\eta}_{=0,\,(*)}=0\qquad\text{in }\Omega.$$

(*) follows from the proof of Lemma 3. As for the SWF and the OWF, it guaranties that the skeleton velocity fulfills the incompressibility condition.

Third coupling, in case of the AWF, explicitly requires that $p\in L^2_0(\Omega)$.

2.5 DISCRETIZATION BASED ON A FINITE ELEMENT METHOD

We use a conforming mixed type Galerkin finite element method to discretize our system of partial differential equations on the local subdomains and the coupling condition on the skeleton as well as for the globally coupled formulations on subdomains. For more details, see, e.g., [10, 14, 19, 38].

- Conforming means that the discrete finite element spaces are subspaces of the continuous spaces: $V^h\subset V$.

- Galerkin describes that we use the same spaces for the ansatz functions as well as the test functions [10, Chapter II, §4] .
- Mixed type denotes that we discretize the velocity and the pressure in different adequate finite element spaces [10, Chapter III, §4]. We carefully have to choose numerically stable finite element spaces, which fulfill the discrete inf-sup condition (2.31) [19, Chapter II, §4].

2.5.1 *Definitions for the Finite Element Methods*

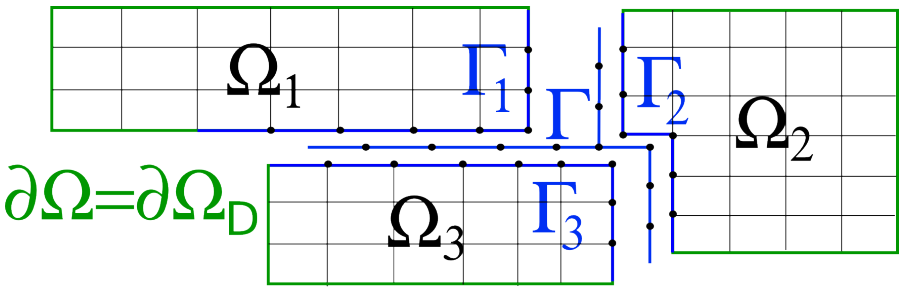


Figure 8: This figure shows a triangulation of the subdomains and the global skeleton Γ for the case of global inhomogeneous boundary conditions. It corresponds to the third main step of the derivation in which we discretize the decoupled formulation with an appropriate finite element approach.

For simplicity, we assume that Ω is polygonal, such that an exact and uniform triangulation $\mathcal{T}_h$ of Ω exists. h characterizes the discretization and describes the diameter of K, where $K \in \mathcal{T}_h$ denotes one element of $\mathcal{T}_h$. Depending on the dimension and shape of Ω, an element K is in two dimensions a triangle or a quadrilateral and in three dimensions a tetrahedron or a hexahedron. It holds, that each $\mathring{K}$ belongs to exactly one subdomain $\overline{\Omega}_i$. Consequently, the triangulation of a subdomain Ω_i is the restriction of the global triangulation to the subdomain:

$$\mathcal{T}_{h,i} := \mathcal{T}_h|_{\Omega_i}.$$

The triangulation of the interface Γ is given by the degrees of freedom which lie on the interface. The term skeleton refers to the triangulation on the interface and is also denoted by Γ. Fig. 8 outlines a triangulation for the case with inhomogeneous boundary conditions on $\partial\Omega$.

On each subdomain, we use inf-sup stable Taylor-Hood finite elements to discretize the Oseen equations. The following definitions and statements about Taylor-Hood elements, Taylor-Hood spaces, and the inf-sup conditions can be found with more details in several standard references, including [10, Chapter III], [19, Chapter II], [22, Part I], [15, Part II], [29], and [54].

We define for the two dimensional case $\mathcal{P}_m(K) :=$

$$\left\{ v \in C(K) : v(x_1, x_2) = \begin{cases} \sum_{\substack{i,j\geq 0,\\ i+j\leq m,}} \alpha_{ij} x_1^i x_2^j & \text{if } K \text{ triangle} \\ \sum_{i,j=0}^{m} \alpha_{ij} x_1^i x_2^j & \text{if } K \text{ quadrilateral} \end{cases} \right\},$$

and for the three dimensional case $\mathcal{P}_m(K) :=$

$$\left\{ v \in C(K) : v(x_1, x_2, x_3) = \begin{cases} \sum_{\substack{i,j,k\geq 0,\\ i+j+k\leq m}} \alpha_{ijk} x_1^i x_2^j x_3^k & \text{if } K \text{ tetraeder} \\ \sum_{i,j.k=0}^{m} \alpha_{ijk} x_1^i x_2^j x_3^k & \text{if } K \text{ hexaeder} \end{cases} \right\}.$$

In general, Taylor-Hood spaces of dimension m are defined as follows:

$$\begin{aligned} \mathbf{V}_m^h &= \{\mathbf{v}^h \in [C(\overline{\Omega})]^d \cap \mathbf{H}_0^1(\Omega) : \mathbf{v}^h|_K \in [\mathcal{P}_m(K)]^d, \forall K \in \mathcal{T}_h\}, \\ Q_{m-1}^h &= \{q^h \in C(\Omega) \cap L^2(\Omega) : q^h|_K \in \mathcal{P}_{m-1}(K), \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Taylor-Hood elements fulfill the so called "inf-sup"- or LBB-condition and are therefore numerically stable:

$$\exists \beta > 0 : \sup_{\mathbf{v}^h \in \mathbf{V}_m^h \setminus \{0\}} \frac{b(q^h, \mathbf{v}^h)}{\|\mathbf{v}^h\|} \geq \beta \left\| q^h \right\| \quad \forall q^h \in Q_{m-1}^h \setminus \{0\}. \tag{2.31}$$

Next, we give the definitions of the finite element spaces for the velocity. The global velocity space is given by:

$$\mathbf{V}^h = \{\mathbf{v}^h \in [C(\overline{\Omega})]^d : \mathbf{v}^h|_K \in [\mathcal{P}_2(K)]^d, \forall K \in \mathcal{T}_h\} \subset \mathbf{H}^1(\Omega).$$

For the local velocity spaces on Ω_i, we get:

$$\begin{aligned} \widehat{\mathbf{V}}_i^h &= \{\mathbf{v}_i^h \in [C(\overline{\Omega}_i)]^d : \mathbf{v}_i^h = \mathbf{v}^h|_{\Omega_i}, \mathbf{v}^h \in \mathbf{V}^h\} \subset \mathbf{H}^1(\Omega_i), \\ \mathbf{V}_{i,0}^h &= \{\mathbf{v}^h \in \widehat{\mathbf{V}}_i^h : \mathbf{v}^h|_{\partial\Omega_i} = 0\} \subset \mathbf{H}_0^1(\Omega_i), \\ \mathbf{V}_{i,N}^h &= \{\mathbf{v}^h \in \widehat{\mathbf{V}}_i^h : \mathbf{v}^h|_{\partial\Omega_i \setminus \partial\Omega_{out}} = 0\} \subset \mathbf{H}_N^1(\Omega_i), \\ \mathbf{V}_i^h &= \begin{cases} \mathbf{V}_{i,N}^h & \text{if } i \in \mathcal{N} \\ \mathbf{V}_{i,0}^h & \text{if } i \notin \mathcal{N} \end{cases}. \end{aligned}$$

The finite element space on the local and global skeleton for the velocity are defined as follows:

$$\begin{aligned} \mathbf{V}_\Gamma^h &= \{\mathbf{v}_\Gamma^h \in \mathbf{V}^h : \mathbf{v}_\Gamma^h(x_{v_j}) = 0 \, \forall x_{v_j} \notin \Gamma\}, \\ \mathbf{V}_{\Gamma_i}^h &= \{\mathbf{v}_{\Gamma_i}^h \in \mathbf{V}_i^h : \mathbf{v}_{\Gamma_i}^h(x_{v_j}) = 0 \, \forall x_{v_j} \notin \Gamma_i\}. \end{aligned}$$

Remark 18. By defining the global and local skeleton space as a restriction of the global and local discrete velocity spaces, the interface functions are implicitly extended to the neighboring elements of the triangulation which are directly connected to the interface. Thus, we implicitly use the finite element functions as extension operator into the subdomain. Therefore, we do not need to explicitly define discrete extension operators which correspond to $\mathcal{R}$ and $\mathcal{R}_i$.

The finite element spaces for the pressure are given by:

$$\begin{aligned} Q_i^h &= \{q_i^h \in C(\Omega_i) : q_i^h|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_{h,i}\} \subset L^2(\Omega_i), \\ Q_{i,0}^h &= \{q_i^h \in Q_i^h : q_i^h \in L_0^2(\Omega_i)\} \subset L_0^2(\Omega), \\ Q^h &= \{q^h \in L^2(\Omega) : q^h|\Omega_i \in Q_i^h\} \subset L^2(\Omega). \end{aligned}$$

We use the bilinear forms $a_i(\cdot,\cdot)$, $b_i(\cdot,\cdot)$ and $c_i(\cdot,\cdot)$ and the linear forms $\widetilde{f}_i(\cdot)$ and $f_i(\cdot)$ as defined before in Section 2.4.4.

Remark. As before we use the following notation for sake of better readability,:

- $\mathbf{v}_{\Gamma_i}^h := \mathbf{v}_\Gamma^h|_{\Gamma_i} \in \mathbf{V}_{\Gamma_i}^h$ denotes the restriction of $\mathbf{v}_\Gamma^h \in \mathbf{V}_\Gamma^h$,
- $\mathbf{u}_{D_i}^h = \mathbf{u}_D^h|_{\Omega_i} \in \widehat{\mathbf{V}}_i^h$ the restriction of $\mathbf{u}_D^h \in \mathbf{V}^h$,
- $q_{\Omega_i}^h := q_\Omega^h|_{\Omega_i}$ the restriction of $q_\Omega^h \in D(\Omega)$ with $q_{\Omega_i}^h = const$,
- $\widetilde{q}_{\Omega_i}^h := \widetilde{q}_\Omega^h|_{\Omega_i}$ the restriction of $\widetilde{q}_\Omega^h \in D(\Omega)$ with $\widetilde{q}_{\Omega_i}^h = const$,
- $\widehat{q}_{\Omega_i}^h := \widehat{q}_\Omega^h|_{\Omega_i}$ the restriction of $\widehat{q}_\Omega^h \in D(\Omega)$ with $\widehat{q}_{\Omega_i}^h = const$.

Remark. We do not give a finite element formulation for the globally coupled formulations (2.19), (2.20) and (2.21). They can be derived analogously as depicted in the next subsection for the decoupled subdomain formulations and coupling conditions.

2.5.2 *Decoupled Finite Element Formulation*

Using these definitions, we discretize the s decoupled subdomain weak formulations and the system of coupling conditions on the interface for SWF (2.22,2.23), AWF (2.25,2.26) and OWF (2.27,2.29,2.30). We refer to this step as the third main step of the derivation of the domain decomposition method. This step is depicted in Fig. 8 for the case in which we only assume inhomogeneous boundary conditions.

Standard Weak Formulation (SWF)

For given $\mathbf{u}_\Gamma^h \in \mathbf{V}_\Gamma^h$, we get the following finite element formulation for the decoupled subdomain problems: For each subdomain Ω_i $(i = 1, \ldots, s)$, find $\mathbf{u}_i^h \in \mathbf{V}_{i,0}^h$ and $\widetilde{p}_i^h \in Q_{i,0}^h$ such that

$$a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, \widetilde{p}_i^h) = \widetilde{f}_i(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h), \quad (2.32a)$$

$$b_i(\mathbf{u}_i^h, \widetilde{q}_i^h) = -b_i(\mathbf{u}_{D_i}^h, \widetilde{q}_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, \widetilde{q}_i^h) \tag{2.32b}$$

for all $\mathbf{v}_i^h \in \mathbf{V}_{i,0}^h$ and $\widetilde{q}_i^h \in Q_{i,0}^h$.

The finite element formulation of the coupling condition reads: Find $\mathbf{u}_\Gamma \in \mathbf{V}_\Gamma^h$ and $\widetilde{p}_\Omega^h \in \widetilde{D}(\Omega)$ such that

$$\begin{aligned}\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i^h, \mathbf{v}_{\Gamma_i}^h) + b_i(\mathbf{v}_{\Gamma_i}^h, \widetilde{p}_i^h)\Big) + \\ + a(\mathbf{u}_\Gamma^h, \mathbf{v}_\Gamma^h) + b(\mathbf{v}_\Gamma^h, \widetilde{p}_\Omega^h) = \widetilde{f}(\mathbf{v}_\Gamma^h) - a(\mathbf{u}_D^h, \mathbf{v}_\Gamma^h), &\qquad (2.33a)\\ b(\mathbf{u}_\Gamma^h, \widetilde{q}_\Omega^h) = -b(\mathbf{u}_D^h, \widetilde{q}_\Omega^h) &\qquad (2.33b)\end{aligned}$$

for all $\mathbf{v}_\Gamma^h \in \mathbf{V}_\Gamma^h$ and $q_\Omega^h \in \widetilde{D}(\Omega)$, where $\mathbf{u}_i^h \in \mathbf{V}_{i,0}^h$ and $\widetilde{p}_i^h \in Q_{i,0}^h$ solve (2.32).

Alternative Weak Formulation (AWF)

In correspondence to the local alternative weak formulation (2.25), the local finite element formulations are given by:

For given $\mathbf{u}_\Gamma^h \in \mathbf{V}_\Gamma^h$, find for each subdomain Ω_i $(i = 1, \dots, s)$ $\mathbf{u}_i^h \in \mathbf{V}_{i,0}^h$, $p_i^h \in Q_i^h$ and $\eta_i^h \in \mathbb{R}$ such that

$$\begin{aligned}a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) = \widetilde{f}_i(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h), &\qquad (2.34a)\\ b_i(\mathbf{u}_i^h, q_i^h) + c_i(\eta_i^h, q_i^h) = - b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h), &\qquad (2.34b)\\ c_i(\xi_i^h, p_i^h) = 0 &\qquad (2.34c)\end{aligned}$$

for all $\mathbf{v}_i^h \in \mathbf{V}_{i,0}^h$, $q_i^h \in Q_i^h$ and $\xi_i^h \in \mathbb{R}$.

The finite element formulation for the coupling condition are given as follows: Find $\mathbf{u}_\Gamma \in \mathbf{V}_\Gamma^h$, $\widehat{p}_\Omega^h \in \widehat{D}(\Omega)$ and $\eta^h \in \mathbb{R}$ such that

$$\begin{aligned}\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i^h, \mathbf{v}_{\Gamma_i}^h) + b_i(\mathbf{v}_{\Gamma_i}^h, p_i^h)\Big) + \\ + a(\mathbf{u}_\Gamma^h, \mathbf{v}_\Gamma^h) + b(\mathbf{v}_\Gamma^h, p_\Omega^h) = \widetilde{f}(\mathbf{v}_\Gamma^h) - a(\mathbf{u}_D^h, \mathbf{v}_\Gamma^h), &\qquad (2.35a)\\ b(\mathbf{u}_\Gamma^h, q_\Omega^h) + c(\eta^h, q_\Omega^h) = -b(\mathbf{u}_D^h, q_\Omega^h), &\qquad (2.35b)\\ c(\xi^h, p_\Omega^h) = 0 &\qquad (2.35c)\end{aligned}$$

for all $\mathbf{v}_\Gamma^h \in \mathbf{V}_\Gamma^h$, $q_\Omega^h \in D(\Omega)$ and $\xi^h \in \mathbb{R}$, where $\mathbf{u}_i^h \in \mathbf{V}_{i,0}^h$ and $p_i^h \in Q_i^h$ are solutions of (2.34).

Outflow Weak Formulation (OWF)

In the outflow case, see also Fig. 9, we get analogously two different local finite element formulations depending on the subdomain type

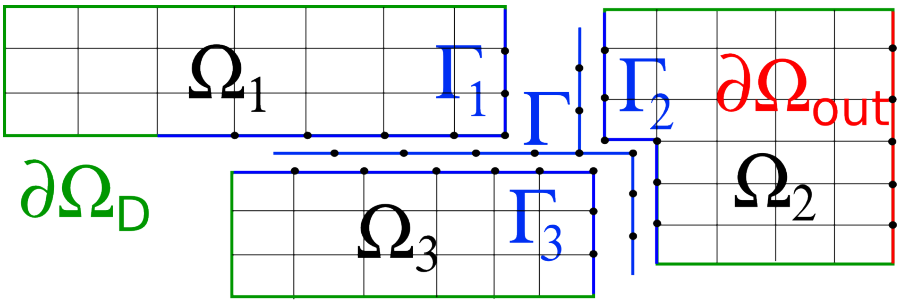


Figure 9: This figure corresponds to the third main step of the derivation. It shows a triangulation of the subdomains and the local and global skeleton for the case with mixed outflow and Dirichlet boundary conditions. In difference to the case with inhomogeneous Dirichlet boundary conditions, we also obtain degrees of freedom on the outflow boundary whereas on the inhomogeneous Dirichlet boundary, we fix the values on that boundary to the given data $\mathbf{d}$ such that they are not treated as degrees of freedom.

for given $\mathbf{u}_\Gamma^h \in \mathbf{V}_\Gamma^h$. For an outflow subdomain, find $\mathbf{u}_i^h \in \mathbf{V}_i^h$, $p_i^h \in Q_i^h$ $(i \in \mathcal{N})$ such that

$$\begin{aligned} a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) =& f_i(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h), && \text{(2.36a)} \\ b_i(\mathbf{u}_i^h, q_i^h) =& - b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h) && \text{(2.36b)} \end{aligned}$$

for all $\mathbf{v}_i^h \in \mathbf{V}_i^h$ and $q_i^h \in Q_i^h$.

For the other non-outflow subdomains, find $\mathbf{u}_i^h \in \mathbf{V}_i^h$, $p_i^h \in Q_i^h$, $\eta_i^h \in \mathbb{R}$ $(i \notin \mathcal{N})$ such that

$$\begin{aligned} a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) =& f_i(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h), && \text{(2.37a)} \\ b_i(\mathbf{u}_i^h, q_i^h) + c_i(\eta_i^h, q_i^h) =& - b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h), && \text{(2.37b)} \\ c_i(\xi_i^h, p_i^h) =& 0 && \text{(2.37c)} \end{aligned}$$

for all $\mathbf{v}_i^h \in \mathbf{V}_i^h$, $q_i^h \in Q_i^h$ and $\xi_i^h \in \mathbb{R}$.

The finite element formulation of the coupling condition reads: Find $\mathbf{u}_\Gamma \in \mathbf{V}_\Gamma^h$, $p_\Omega^h \in D(\Omega)$ such that

$$\begin{aligned} &\sum_{i=1}^{s} \Big(a_i(\mathbf{u}_i^h, \mathbf{v}_{\Gamma_i}^h) + b_i(\mathbf{v}_{\Gamma_i}^h, p_i^h) \Big) + \\ &\qquad + a(\mathbf{u}_\Gamma^h, \mathbf{v}_\Gamma^h) + b(\mathbf{v}_\Gamma^h, p_\Omega^h) = f(\mathbf{v}_\Gamma^h) - a(\mathbf{u}_D^h, \mathbf{v}_\Gamma^h), && \text{(2.38a)} \\ &\qquad\qquad b(\mathbf{u}_\Gamma^h, q_\Omega^h) = -b(\mathbf{u}_D^h, q_\Omega^h) && \text{(2.38b)} \end{aligned}$$

for all $\mathbf{v}_\Gamma^h \in \mathbf{V}_\Gamma^h$ and $q_\Omega^h \in D(\Omega)$. $\mathbf{u}_i^h \in \mathbf{V}_i^h$ and $p_i^h \in Q_i^h$ solve (2.36) and (2.37), respectively.

2.6 ALGEBRAIC FORMULATION

Based on the finite element discretization, we now derive the resulting global linear systems as well as the Schur-complement equations which corresponds to the last main step. The global system is the algebraic counterpart to the still coupled weak formulation on subdomains, while the Schur-complement equation is the algebraic counterpart of the coupling conditions on the skeleton Γ [46, 56].

2.6.1 *Algebraic Formulation for the Velocity*

In this subsection, we derive the algebraic representation for the velocity and the bilinear form $a_i(\cdot,\cdot)$. First, let $\boldsymbol{\psi}_j \in \mathbf{V}_{i,0}^h$ $(j = 1,\dots,n_{v_i})$ and $\boldsymbol{\psi}_j \in \mathbf{V}_{\Gamma_i}^h$ $(j = 1,\dots,n_{v_{\Gamma_i}})$ be the piecewise quadratic basis functions for the velocity, such that

$$\mathbf{V}_{i,0}^h = \operatorname{span}\left\{\boldsymbol{\psi}_j \,:\, j = 1,\dots,n_{v_i}\right\},$$
$$\mathbf{V}_{\Gamma_i}^h = \operatorname{span}\left\{\boldsymbol{\psi}_j \,:\, j = 1,\dots,n_{\Gamma_i}\right\},$$

respectively. n_{v_i} denotes the number of degrees of freedom of the velocity components in Ω_i. n_{v_Γ} and $n_{v_{\Gamma_i}}$ denote the numbers of degrees of freedom of the velocity components on Γ and Γ_i, respectively. Second, we define a restriction matrix $\mathbf{R}_{\Gamma_i} \in \{0,1\}^{n_{v_{\Gamma_i}} \times n_{v_\Gamma}}$ that restricts degrees of freedom of the velocity on the global skeleton Γ on the local skeletons Γ_i and a restriction matrix $\mathbf{R}_{D_i} \in \{0,1\}^{n_{v_{D_i}} \times n_{v_D}}$ that restricts the degrees of freedom on the global Dirichlet boundary to the corresponding subdomain boundary. n_{v_D} and $n_{v_{D_i}}$ are the numbers of degrees of freedom for the velocity on the global Dirichlet boundary and corresponding local Dirichlet boundary.

Using the basis functions $\boldsymbol{\psi}_l$, we define the following block matrices and their entries related to the bilinear form $a_i(\cdot,\cdot)$.

$$(\mathbf{A})_{kl} = a_i(\boldsymbol{\psi}_l, \boldsymbol{\psi}_k) = \mu \int_{\Omega_i} \boldsymbol{\nabla}\boldsymbol{\psi}_l : \boldsymbol{\nabla}\boldsymbol{\psi}_k \, d\mathbf{x} + \int_{\Omega_i} ((\mathbf{a}\cdot\boldsymbol{\nabla})\boldsymbol{\psi}_l)\boldsymbol{\psi}_k \, d\mathbf{x} :$$

$$\begin{aligned}
&\mathbf{A}_{ii} \in \mathbb{R}^{n_{v_i}\times n_{v_i}} && (\mathbf{A})_{kl} && k,l = 1,\dots,n_{v_i},\\
&\mathbf{A}_{i\Gamma_i} \in \mathbb{R}^{n_{v_i}\times n_{v_{\Gamma_i}}} && (\mathbf{A})_{kl} && k = 1,\dots,n_{v_i},\ l = 1,\dots,n_{v_{\Gamma_i}}\\
&\mathbf{A}_{\Gamma_i i} \in \mathbb{R}^{n_{v_{\Gamma_i}}\times n_{v_i}} && (\mathbf{A})_{kl} && k = 1,\dots,n_{v_{\Gamma_i}},\ l = 1,\dots,n_{v_i},\\
&\mathbf{A}_{\Gamma_i\Gamma_i} \in \mathbb{R}^{n_{v_{\Gamma_i}}\times n_{v_{\Gamma_i}}} && (\mathbf{A})_{kl} && k,l = 1,\dots,n_{v_{\Gamma_i}},\\
&\mathbf{A}_{\Gamma\Gamma} \in \mathbb{R}^{n_{v_\Gamma}\times n_{v_\Gamma}} && \mathbf{A}_{\Gamma\Gamma} && = \textstyle\sum_{i=1}^{s} \mathbf{R}_{\Gamma_i}^T \mathbf{A}_{\Gamma_i\Gamma_i} \mathbf{R}_{\Gamma_i}\\
&\mathbf{A}_{iD_i} \in \mathbb{R}^{n_{v_i}\times n_{v_{D_i}}} && (\mathbf{A})_{kl} && k = 1,\dots,n_{v_i},\ l = 1,\dots,n_{v_{D_i}},\\
&\mathbf{A}_{\Gamma_i D_i} \in \mathbb{R}^{n_{v_{\Gamma_i}}\times n_{v_{D_i}}} && (\mathbf{A})_{kl} && k = 1,\dots,n_{v_{\Gamma_i}},\ l = 1,\dots,n_{v_{D_i}},\\
&\mathbf{A}_{\Gamma D} \in \mathbb{R}^{n_{v_\Gamma}\times n_{v_D}} && \mathbf{A}_{\Gamma D} && = \textstyle\sum_{i=1}^{s} \mathbf{R}_{D_i}^T \mathbf{A}_{\Gamma_i D_i} \mathbf{R}_{D_i}.
\end{aligned}$$

Furthermore, we define the following finite element functions for the velocity and the coefficient vectors:

$$\begin{aligned}
\mathbf{u}_i^h(\mathbf{x}) &= \sum_{j=1}^{n_{v_i}} u_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{u}_i = (u_1, \ldots, u_{n_{v_i}})^T \in \mathbb{R}^{n_{v_i}}, \\
\mathbf{u}_\Gamma^h(\mathbf{x}) &= \sum_{j=1}^{n_{v_\Gamma}} u_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{u}_\Gamma = (u_1, \ldots, u_{n_{v_\Gamma}})^T \in \mathbb{R}^{n_{v_\Gamma}}, \\
\mathbf{u}_{\Gamma_i}^h(\mathbf{x}) &= \sum_{j=1}^{n_{v_{\Gamma_i}}} u_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{u}_{\Gamma_i} = (u_1, \ldots, u_{n_{v_{\Gamma_i}}})^T \in \mathbb{R}^{n_{v_{\Gamma_i}}}, \\
\mathbf{u}_{D_i}^h(\mathbf{x}) &= \sum_{j=1}^{n_{v_{D_i}}} u_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{u}_{D_i} = (u_1, \ldots, u_{n_{v_{D_i}}})^T \in \mathbb{R}^{n_{v_{D_i}}}. \\
\mathbf{u}_D^h(\mathbf{x}) &= \sum_{j=1}^{n_{v_D}} u_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{u}_D = (u_1, \ldots, u_{n_{v_D}})^T \in \mathbb{R}^{n_{v_D}}.
\end{aligned}$$

2.6.2 *Algebraic Formulation for the Pressure*

Next, we derive the algebraic representation for the pressure. Let n_{q_i} denote the number of degrees of freedom of the pressure. Let φ_j, $j = 1, \ldots, n_{q_i}$, be the standard piecewise linear basis functions for the pressure such that

$$Q_i^h = \operatorname{span}\left\{\varphi_j \,:\, j = 1, \ldots, n_{q_i}\right\}.$$

We call them "standard", due to the fact that they fulfill the two following characteristics:

First, these linear basis functions satisfy

$$\sum_{j=1}^{n_{q_i}} \varphi_j = 1. \tag{2.39}$$

Second, it holds

$$\varphi_j(x_k) = \delta_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

δ_{jk} is known as the Kronecker delta.

We note that the space $L_0^2(\Omega)$ fulfills a normalization condition with respect to $L^2(\Omega)$. We can transfer this observation to make a statement about the dimension of the corresponding discrete finite element spaces Q_i^h and $Q_{i,0}^h$: The dimension of the space $Q_{i,0}^h$ is $n_{q_i} - 1$.

Let $\widetilde{\varphi}_j$, $j = 1, \ldots, (n_{q_i} - 1)$, be the piecewise linear basis functions for the pressure such that

$$Q^h_{i,0} = \text{span}\left\{ \widetilde{\varphi}_j \ : \ j = 1, \ldots, (n_{q_i} - 1) \right\}.$$

Note that $\widetilde{\varphi}_j$ are not the standard piecewise linear basis functions. For these, it holds that

$$\int_{\Omega_i} \widetilde{\varphi}_j dx = 0\,.$$

Let $\widehat{\varphi}_{\Omega_j}$, $j = 1, \ldots, s$, be basis functions for the global pressure space $\widehat{D}(\Omega)$ such that

$$\widehat{D}(\Omega) = \text{span}\left\{ \widehat{\varphi}_{\Omega_j} \ : \ j = 1, \ldots, s \right\}.$$

Note that each basis function $\widehat{\varphi}_{\Omega_j}$ can be represented by a constant per subdomain Ω_j and zero on the other subdomains. The same observation about the dimension of $Q^h_i$and $Q^h_{i,0}$ can be made for the global spaces $\widehat{D}(\Omega)$ and $\widetilde{D}(\Omega)$. The dimension of $\widetilde{D}(\Omega)$ is consequently $s - 1$.

Furthermore, let $\widetilde{\varphi}_{\Omega_j}$, $j = 1, \ldots, (s-1)$, be basis functions that span the global pressure space $\widetilde{D}(\Omega)$:

$$\widetilde{D}(\Omega) = \text{span}\left\{ \widetilde{\varphi}_{\Omega_j} \ : \ j = 1, \ldots, (s-1) \right\}.$$

Let φ_{Ω_j} be the basis functions for the global pressure space $D(\Omega)$ such that

$$D(\Omega) = \text{span}\left\{ \varphi_{\Omega_j} \ : \ j \notin \mathcal{N} \right\}.$$

The dimension of the space $D(\Omega)$ equals the number of non-outflow subdomains. Again, note that each basis function φ_{Ω_j} can be represented by a constant on a non-outflow subdomain and zero on the other subdomains.

We define the following matrices related to the bilinear forms $b_i(\cdot,\cdot)$ and $b(\cdot,\cdot)$, and their entries for the SWF using the basis functions $\widetilde{\varphi}_j$ and $\widetilde{\varphi}_{\Omega_j}$:

$$(\widetilde{\mathbf{B}})_{kl} = b_i(\boldsymbol{\psi}_l, \widetilde{\varphi}_k) = -\int_{\Omega_i} \boldsymbol{\nabla} \cdot \boldsymbol{\psi}_l \, \widetilde{\varphi}_k d\mathbf{x},$$

$$(\widetilde{\mathbf{B}_0})_{kl} = b(\boldsymbol{\psi}_l, \widetilde{\varphi}_{\Omega_k}) = -\int_{\Omega} \boldsymbol{\nabla} \cdot \boldsymbol{\psi}_l \, \widetilde{\varphi}_{\Omega_k} d\mathbf{x}\,:$$

$$
\begin{aligned}
&\widetilde{\mathbf{B}}_{ii} \in \mathbb{R}^{(n_{q_i}-1)\times n_{v_i}} && (\widetilde{\mathbf{B}})_{kl} && k=1,\dots,(n_{q_i}-1),\ l=1,\dots,n_{v_i},\\
&\widetilde{\mathbf{B}}_{i\Gamma_i} \in \mathbb{R}^{(n_{q_i}-1)\times n_{v_{\Gamma_i}}} && (\widetilde{\mathbf{B}})_{kl} && k=1,\dots,(n_{q_i}-1),\ l=1,\dots,n_{v_{\Gamma_i}},\\
&\widetilde{\mathbf{B}}_{iD_i} \in \mathbb{R}^{(n_{q_i}-1)\times n_{v_{D_i}}} && (\widetilde{\mathbf{B}})_{kl} && k=1,\dots,(n_{q_i}-1),\ l=1,\dots,n_{v_{D_i}},\\
&\widetilde{\mathbf{B}}_{0} \in \mathbb{R}^{(s-1)\times n_{v_\Gamma}} && (\widetilde{\mathbf{B}_0})_{kl} && k=1,\dots,(s-1),\ l=1,\dots,n_{\Gamma_i},\\
&\widetilde{\mathbf{B}}_{0D} \in \mathbb{R}^{(s-1)\times n_{v_D}} && (\widetilde{\mathbf{B}_0})_{kl} && k=1,\dots,(s-1),\ l=1,\dots,n_{v_D}.
\end{aligned}
$$

We define the following matrices related to the bilinear forms $b_i(\cdot,\cdot)$ and $b(\cdot,\cdot)$, and their entries for the AWF and OWF using basis functions φ_j, φ_{Ω_j} and $\widehat{\varphi}_{\Omega_j}$. Let $s_{\overline{\mathcal{N}}} = s - |\mathcal{N}|$ denote the number of subdomains, which are non-outflow subdomains.

$$
\begin{aligned}
(\mathbf{B})_{kl} &= b_i(\boldsymbol{\psi}_l, \varphi_k) = -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_l\, \varphi_k d\mathbf{x},\\
(\mathbf{B}_0)_{kl} &= b(\boldsymbol{\psi}_l, \varphi_{\Omega_k}) = -\int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_l\, \varphi_{\Omega_k} d\mathbf{x},\\
(\widehat{\mathbf{B}}_0)_{kl} &= b(\boldsymbol{\psi}_l, \widehat{\varphi}_{\Omega_k}) = -\int_{\Omega} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_l\, \widehat{\varphi}_{\Omega_k} d\mathbf{x}:
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{B}_{ii} \in \mathbb{R}^{n_{q_i}\times n_{v_i}} && (\mathbf{B})_{kl} && k=1,\dots,n_{q_i},\ l=1,\dots,n_{v_i},\\
&\mathbf{B}_{i\Gamma_i} \in \mathbb{R}^{n_{q_i}\times n_{v_{\Gamma_i}}} && (\mathbf{B})_{kl} && k=1,\dots,n_{q_i},\ l=1,\dots,n_{v_{\Gamma_i}},\\
&\mathbf{B}_{iD_i} \in \mathbb{R}^{n_{q_i}\times n_{v_{D_i}}} && (\mathbf{B})_{kl} && k=1,\dots,n_{q_i},\ l=1,\dots,n_{v_{D_i}},\\
&\widehat{\mathbf{B}}_{0} \in \mathbb{R}^{s\times n_{v_\Gamma}} && (\widehat{\mathbf{B}}_0)_{kl} && k=1,\dots,s,\ l=1,\dots,n_{\Gamma},\\
&\widehat{\mathbf{B}}_{0D} \in \mathbb{R}^{s\times n_{v_D}} && (\widehat{\mathbf{B}}_0)_{kl} && k=1,\dots,s,\ l=1,\dots,n_{v_D},\\
&\mathbf{B}_{0} \in \mathbb{R}^{s_{\overline{\mathcal{N}}}\times n_{v_\Gamma}} && (\mathbf{B}_0)_{kl} && k=1,\dots,s_{\overline{\mathcal{N}}},\ l=1,\dots,n_{\Gamma},\\
&\mathbf{B}_{0D} \in \mathbb{R}^{s_{\overline{\mathcal{N}}}\times n_{v_D}} && (\mathbf{B}_0)_{kl} && k=1,\dots,s_{\overline{\mathcal{N}}},\ l=1,\dots,n_{v_D}.
\end{aligned}
$$

Furthermore, we define the following matrices related to the bilinear forms $c_i(\cdot,\cdot)$ and $c(\cdot,\cdot)$ and their entries:

$$
\begin{aligned}
&\mathbf{c}_i \in \mathbb{R}^{n_{q_i}\times 1} && (\mathbf{c}_i)_k = c_i(\tfrac{1}{|\Omega_i|}, \varphi_k) = \tfrac{1}{|\Omega_i|}\int_{\Omega_i}\varphi_k d\mathbf{x} && k=1,\dots,n_{q_i},\\
&\mathbf{c}_0 \in \mathbb{R}^{s\times 1} && (\mathbf{c}_0)_k = c(1, \varphi_{\Omega_k}) = \int_{\Omega}\varphi_{\Omega_k} d\mathbf{x} && k=1,\dots,s,
\end{aligned}
$$

where we choose $\eta_i = \frac{1}{|\Omega_i|}$ and $\eta = 1$. Then it holds

$$
\sum_{k=1}^{n_{q_i}} (\mathbf{c}_i)_k = 1. \tag{2.40}
$$

We define the following finite element functions for the pressure and the coefficient vectors:

$$
\widetilde{p}_i^h(\mathbf{x}) = \sum_{j=1}^{n_{q_i}-1} p_j \widetilde{\varphi}_j(\mathbf{x}) \qquad \widetilde{\mathbf{p}}_i = (p_1,\dots,p_{n_{q_i}-1})^T \in \mathbb{R}^{n_{q_i}-1},
$$

$$\widetilde{p}^h_\Omega(\mathbf{x}) = \sum_{j=1}^{s-1} p_{\Omega_j} \widetilde{\varphi}_{\Omega_j}(\mathbf{x}) \qquad \widetilde{\mathbf{p}}_\Omega = (p_{\Omega_1}, \ldots, p_{\Omega_{s-1}})^T \in \mathbb{R}^{s-1},$$

$$p^h_i(\mathbf{x}) = \sum_{j=1}^{n_{q_i}} p_j \varphi_j(\mathbf{x}) \qquad \mathbf{p}_i = (p_1, \ldots, p_{n_{q_i}})^T \in \mathbb{R}^{n_{q_i}},$$

$$\widehat{p}^h_\Omega(\mathbf{x}) = \sum_{j=1}^{s} p_{\Omega_j} \widehat{\varphi}_{\Omega_j}(\mathbf{x}) \qquad \widehat{\mathbf{p}}_\Omega = (p_{\Omega_1}, \ldots, p_{\Omega_s})^T \in \mathbb{R}^{s},$$

$$p^h_\Omega(\mathbf{x}) = \sum_{j \notin \mathcal{N}} p_{\Omega_j} \varphi_{\Omega_j}(\mathbf{x}) \qquad \mathbf{p}_\Omega = (p_{\Omega_1}, \ldots, p_{\Omega_{s_{\overline{\mathcal{N}}}}})^T \in \mathbb{R}^{s_{\overline{\mathcal{N}}}}.$$

Furthermore, we define restriction matrices $\widetilde{\mathbf{R}}_{\Omega_i} \in \{0,1\}^{1\times(s-1)}$, $\widehat{\mathbf{R}}_{\Omega_i} \in \{0,1\}^{1\times s}$ and $\mathbf{R}_{\Omega_i} \in \{0,1\}^{1\times s_{\Omega_i}}$ that restrict the global pressure variable from Ω to the corresponding subdomain Ω_i.

Remark. The right hand sides $f^h_i(\mathbf{x})$, $\widetilde{f}^h_i(\mathbf{x})$, $f^h_\Gamma(\mathbf{x})$ and $\widetilde{f}^h_\Gamma$ as well as the coefficient vectors $\mathbf{f}_i$, $\widetilde{\mathbf{f}}_i$, $\mathbf{f}_\Gamma$ and $\widetilde{\mathbf{f}}_\Gamma$ are defined analogously.

2.6.2.1 *Relationship between SWF and AWF*

Now, we have a closer look on the relation between the finite element spaces Q^h_i and $Q^h_{i,0}$ for each subdomain Ω_i. First, we establish a relationship between piecewise linear basis functions φ_j and $\widetilde{\varphi}_j$ by constructing $\widetilde{\varphi}_j$ in the following way:

$$\widetilde{\varphi}_j := \varphi_j - \frac{1}{|\Omega_i|}\int_{\Omega_i} \varphi_j \, dx = \varphi_j - (\mathbf{c}_i)_j \quad j = 1, \ldots, n_{q_i}. \tag{2.41}$$

Then we can easily check that

$$\int_{\Omega_i} \widetilde{\varphi}_j dx = 0 \quad j = 1, \ldots, n_{q_i}.$$

Hence, it holds that

$$\sum_{j=1}^{n_{q_i}} \widetilde{\varphi}_j = \underbrace{\sum_{j=1}^{n_{q_i}} \varphi_j}_{=1,(2.39)} - \underbrace{\sum_{j=1}^{n_{q_i}} (\mathbf{c}_i)_j}_{=1,(2.40)} = 0,$$

which shows that the n_{q_i} functions $\widetilde{\varphi}_j$ are linearly dependent. This also demonstrates that the dimension of $Q^h_{i,0}$ is less than n_{q_i}.

Next, we consider the relation based on the matrices. Using integration by parts and that $\boldsymbol{\psi}^h_k \in \mathbf{V}^h_{i,0}$, we can show that rows of the matrix $\mathbf{B}_{ii}$ are linearly dependent ($k = 1, \ldots, n_{v_i}$):

$$\sum_{j=1}^{n_{q_i}} (\mathbf{B}_{ii})_{jk} = -\sum_{j=1}^{n_{q_i}} \int_{\Omega_i} \boldsymbol{\nabla} \cdot \boldsymbol{\psi}_k(x) \, \varphi_j d\mathbf{x}$$

$$
\begin{aligned}
&= -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_k(x) \underbrace{\sum_{j=1}^{n_{q_i}} \varphi_j}_{=1,(2.39)} \, d\mathbf{x} \\
&= -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_k(x) d\mathbf{x} \\
&= -\underbrace{\int_{\partial\Omega_i} \boldsymbol{\psi}_k(x)\cdot\mathbf{n}(\mathbf{x})d\mathbf{x}}_{=0,\,\boldsymbol{\psi}_k(x)\in\mathbf{V}^h_{i,0}} + \underbrace{\int_{\Omega_i} \boldsymbol{\psi}_k(\mathbf{x})\cdot\boldsymbol{\nabla}1 d\mathbf{x}}_{=0,\,\boldsymbol{\nabla}1=0} = 0. \qquad (2.42)
\end{aligned}
$$

We expected this behavior, since the pressure is not uniquely determined in $L^2(\Omega)$, but in $L^2_0(\Omega)$. However, the matrix $\mathbf{B}_{ii}$ discretizes the space $L^2(\Omega)$. Therefore, when using the standard linear basis functions φ_j, we introduced the Lagrange parameter η_i.

Now, we show how the matrices $\mathbf{B}_{ii}$ and $\widetilde{\mathbf{B}}_{ii}$ are related, when constructing $\widetilde{\varphi}_j$ as shown before, see (2.41):

$$
\begin{aligned}
(\widetilde{\mathbf{B}}_{ii})_{jk} &= -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_k(\mathbf{x})\widetilde{\varphi}_j d\mathbf{x} \\
&= -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_k(\mathbf{x})(\varphi_j - (\mathbf{c}_i)_j) d\mathbf{x} \\
&= -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_k(\mathbf{x})\varphi_j d\mathbf{x} + (\mathbf{c}_i)_j \underbrace{\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_k d\mathbf{x}}_{=0,(2.42)} \\
&= -\int_{\Omega_i} \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_k(\mathbf{x})\varphi_j d\mathbf{x} \\
&= (\mathbf{B}_{ii})_{jk} \quad j = 1,\ldots,(n_{q_i}-1), k = 1,\ldots,n_{v_i}.
\end{aligned}
$$

Last, we study the relationship between $p_i^h(x)$ and $\widetilde{p}_i^h(x)$. Using the construction (2.41), we see that

$$
\widetilde{p}_i^h(\mathbf{x}) = \sum_{j=1}^{(n_{q_i}-1)} \widetilde{p}_j\widetilde{\varphi}_j(\mathbf{x}) = \sum_{j=1}^{(n_{q_i}-1)} \widetilde{p}_j\varphi_j(\mathbf{x}) - \sum_{j=1}^{(n_{q_i}-1)} \widetilde{p}_j(\mathbf{c}_i)_j \qquad (2.43)
$$

Using $\varphi_{n_{q_i}} = 1 - \sum_{j=1}^{(n_{q_i}-1)} \varphi_j$, we can show that

$$
\begin{aligned}
p_i^h(x) &= \sum_{j=1}^{n_{q_i}} p_j\varphi_j(\mathbf{x}) = \sum_{j=1}^{(n_{q_i}-1)} p_j\varphi_j(\mathbf{x}) + p_{n_{q_i}}\varphi_{n_{q_i}} \\
&= \sum_{j=1}^{(n_{q_i}-1)} p_j\varphi_j(\mathbf{x}) + p_{n_{q_i}}(1 - \sum_{j=1}^{(n_{q_i}-1)} \varphi_j) \qquad (2.44)
\end{aligned}
$$

$$= \sum_{j=1}^{(n_{q_i}-1)} \underbrace{(p_j - p_{n_{q_i}})}_{=:\widetilde{p}_j} \varphi_j(\mathbf{x}) + p_{n_{q_i}}.$$

Hence, we see that $\left\{\varphi_1, \ldots, \varphi_{(n_{q_i}-1)}, 1\right\}$ is a basis of Q_i^h. Then (2.43) and (2.44) imply

$$p_{n_{q_i}} = -\sum_{j=1}^{(n_{q_i}-1)} \widetilde{p}_j (\mathbf{c}_i)_j,$$
$$p_j = \widetilde{p}_j + p_{n_{q_i}}.$$

2.6.3 *Global Linear Systems and Schur-complement Equations*

Using the definitions of the Sections 2.6.1 and 2.6.2, we discuss the global linear system for the Oseen equations and the Schur-complement equations for the three different cases SWF, AWF and OWF. Furthermore, we give an interpretation of the different terms occurring in the Schur-complement equation.

Remark. For better readability, we omit the restriction matrices $\mathbf{R}_{\Gamma_i}$ ($\widetilde{\mathbf{R}}_{\Omega_i}$, $\mathbf{R}_{\Omega_i}$, and $\widehat{\mathbf{R}}_{\Omega_i}$) needed when applying $\mathbf{A}_{i\Gamma_i}$ ($\widetilde{\mathbf{B}}_{i\Gamma_i}$, and $\mathbf{B}_{i\Gamma_i}$, respectively) to $\mathbf{u}_\Gamma$ ($\widetilde{\mathbf{p}}_\Omega$, $\mathbf{p}_\Omega$, and $\widehat{\mathbf{p}}_\Omega$) as well as the projection matrices $\mathbf{R}_{\Gamma_i}^T$ ($\widetilde{\mathbf{R}}_{\Omega_i}^T$, $\mathbf{R}_{\Omega_i}^T$, and $\widehat{\mathbf{R}}_{\Omega_i}^T$) needed when applying $\mathbf{A}_{\Gamma_i i}$ ($\widetilde{\mathbf{B}}_{i\Gamma_i}^T$, and $\mathbf{B}_{i\Gamma_i}^T$) to $\mathbf{u}_\Gamma$ ($\widetilde{\mathbf{p}}_\Omega$, $\mathbf{p}_\Omega$, and $\widehat{\mathbf{p}}_\Omega$).

Standard Weak Formulation (SWF)

In this subsection, we describe the global linear system and Schur-complement equation representing the Oseen equations based on the SWF. Even though the standard formulation is used, nonstandard finite element basis function are needed.

For two subdomains, we get the following global linear system, which corresponds to the finite element formulation of (2.19):

$$\left(\begin{array}{cc|cc|cc} \mathbf{A}_{11} & \widetilde{\mathbf{B}}_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1} & \mathbf{0} \\ \widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma_1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{A}_{22} & \widetilde{\mathbf{B}}_{22}^T & \mathbf{A}_{2\Gamma_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{22} & \mathbf{0} & \widetilde{\mathbf{B}}_{2\Gamma_2} & \mathbf{0} \\ \hline \mathbf{A}_{\Gamma_1 1} & \widetilde{\mathbf{B}}_{1\Gamma_1}^T & \mathbf{A}_{\Gamma_2 2} & \widetilde{\mathbf{B}}_{2\Gamma_2}^T & \mathbf{A}_{\Gamma\Gamma} & \widetilde{\mathbf{B}}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_0 & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \mathbf{u}_1 \\ \widetilde{\mathbf{p}}_1 \\ \hline \mathbf{u}_2 \\ \widetilde{\mathbf{p}}_2 \\ \hline \mathbf{u}_\Gamma \\ \widetilde{\mathbf{p}}_\Omega \end{array}\right) = \left(\begin{array}{c} \widetilde{\mathbf{f}}_1 - \mathbf{A}_{1D_1}\mathbf{u}_{D_1} \\ -\widetilde{\mathbf{B}}_{1D_1}\mathbf{u}_{D_1} \\ \hline \widetilde{\mathbf{f}}_2 - \mathbf{A}_{2D_2}\mathbf{u}_{D_2} \\ -\widetilde{\mathbf{B}}_{2D_2}\mathbf{u}_{D_2} \\ \hline \widetilde{\mathbf{f}}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\widetilde{\mathbf{B}}_{0D}\mathbf{u}_D \end{array}\right).$$

We can reduce the global problem to a Schur-complement equation because the local block matrices (with the subindex $\cdot_{ii}$) are invertible.

Firstly, we define the Schur-complement matrix $\widetilde{S}$:

$$\widetilde{S} := \begin{pmatrix} \mathbf{A}_{\Gamma\Gamma} & \widetilde{\mathbf{B}}_0^T \\ \widetilde{\mathbf{B}}_0 & \mathbf{0} \end{pmatrix} - \sum_{i=1}^{s} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \widetilde{\mathbf{B}}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} \\ \widetilde{\mathbf{B}}_{i\Gamma_i} & \mathbf{0} \end{pmatrix}}_{:=L1} \tag{2.45}$$

Secondly, we define the right hand side $\widetilde{r}$:

$$\widetilde{r} := \begin{pmatrix} \widetilde{\mathbf{f}}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\widetilde{\mathbf{B}}_{0D}\mathbf{u}_D \end{pmatrix} - \sum_{i=1}^{s} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \widetilde{\mathbf{B}}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{\mathbf{f}}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\widetilde{\mathbf{B}}_{iD_i}\mathbf{u}_{D_i} \end{pmatrix}}_{:=L2} \tag{2.46}$$

These definitions yield the Schur-complement equation, which algebraically represents the coupling on the skeleton (2.33)

$$\widetilde{S}(\mathbf{u}_\Gamma, \widetilde{\mathbf{p}}_\Omega) = \widetilde{r}.$$

The subproblems $L1$ and $L2$, which we introduced in the definition of (2.45) and (2.46), can be interpreted as solving local Oseen problems on the subdomains.

$$L1 \text{ solve} : \begin{pmatrix} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \widetilde{\mathbf{p}}_i \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_{i\Gamma}\mathbf{u}_{\Gamma_i} \\ -\widetilde{\mathbf{B}}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \end{pmatrix}.$$

$L1$ solves a local Oseen problem with Dirichlet data. For inner subdomains, the Dirichlet data is set to $\mathbf{u}_{\Gamma_i}$ on the local skeleton Γ_i. For subdomains intersecting the global boundary, i.e. where $\partial\Omega_i \cap \partial\Omega \neq \emptyset$ holds, the Dirichlet data on the global boundary is set to zero, whereas on the local skeleton Γ_i, the Dirichlet data is set to $\mathbf{u}_{\Gamma_i}$.

$$L2 \text{ solve} : \begin{pmatrix} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbf{f}}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\widetilde{\mathbf{B}}_{iD_i}\mathbf{u}_{D_i} \end{pmatrix}.$$

$L2$ also solves a local Oseen problem with Dirichlet data. For inner subdomains, a homogeneous Dirichlet boundary problem is solved. For subdomains intersecting the global boundary, the Dirichlet data on the global boundary is set to $\mathbf{u}_{D_i}$, whereas on the local skeleton Γ_i, the Dirichlet data is zero.

Adding up $L1$ and $L2$ leads to

$$L1 + L2 \text{ solve} : \begin{pmatrix} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \widetilde{\mathbf{p}}_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} - \mathbf{A}_{i\Gamma}\mathbf{u}_{\Gamma_i} \\ -\widetilde{\mathbf{B}}_{iD_i}\mathbf{u}_{D_i} - \widetilde{\mathbf{B}}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \end{pmatrix},$$

which is the algebraic representation of the local decoupled subproblems on the subdomain Ω_i (2.32).

Remark. The local subdomain problem $L1$ is solved when computing $\widetilde{\mathbf{r}}$. $L2$ is solved when we apply the Schur-complement operator $\widetilde{S}$ to a skeleton vector $(\mathbf{u}_\Gamma, \widetilde{\mathbf{p}}_\Omega)^T$. We see that $L1$ and $L2$ are independent of $\widetilde{\mathbf{p}}_\Omega$.

Alternative Weak Formulation (AWF)

Analogously, we describe the global linear system and derive the Schur-complement equation for the Oseen equations based on the AWF. For an example with two subdomains, the global linear system which corresponds to the finite element formulation of (2.20) yields:

$$\left(\begin{array}{cc|cc|ccc} \mathbf{A}_{11} & \widetilde{\mathbf{B}}_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma} & \mathbf{0} & \mathbf{0} \\ \widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{A}_{22} & \widetilde{\mathbf{B}}_{22}^T & \mathbf{A}_{2\Gamma} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{22} & \mathbf{0} & \widetilde{\mathbf{B}}_{2\Gamma} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{A}_{\Gamma 1} & \widetilde{\mathbf{B}}_{1\Gamma}^T & \mathbf{A}_{\Gamma 2} & \widetilde{\mathbf{B}}_{2\Gamma}^T & \mathbf{A}_{\Gamma\Gamma} & \widehat{\mathbf{B}}_0^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{B}}_0 & \mathbf{0} & \mathbf{c}_0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_0^T & 0 \end{array}\right) \left(\begin{array}{c} \mathbf{u}_1 \\ \mathbf{p}_1 \\ \hline \mathbf{u}_2 \\ \mathbf{p}_2 \\ \hline \mathbf{u}_\Gamma \\ \mathbf{p}_\Omega \\ \eta_0 \end{array}\right) = \left(\begin{array}{c} \widetilde{\mathbf{f}}_1 - \mathbf{A}_{1D_1}\mathbf{u}_{D_1} \\ -\mathbf{B}_{1D_1}\mathbf{u}_{D_1} \\ \hline \widetilde{\mathbf{f}}_2 - \mathbf{A}_{2D_2}\mathbf{u}_{D_2} \\ -\mathbf{B}_{2D_2}\mathbf{u}_{D_2} \\ \hline \widetilde{\mathbf{f}}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\mathbf{B}_{0D}\mathbf{u}_D \\ 0 \end{array}\right).$$

Due to the fact that the local block matrices (with the subindex $\cdot_{ii}$) of the global matrix are invertible, we can analogously reduce the global problem to a Schur-complement equation. We define the Schur-complement matrix $\widehat{S}$:

$$\begin{aligned} \widehat{S} := & \begin{pmatrix} \mathbf{A}_{\Gamma\Gamma} & \widehat{\mathbf{B}}_0^T & \mathbf{0} \\ \widehat{\mathbf{B}}_0 & \mathbf{0} & \mathbf{c}_0 \\ \mathbf{0} & \mathbf{c}_0^T & 0 \end{pmatrix} \\ & - \sum_{i=1}^{s} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} \end{pmatrix}}_{:=L1}. \end{aligned} \tag{2.47}$$

Since we know that we can equivalently reformulate the decoupled local problems with an alternative local formulation, we define a second variant of the Schur-complement operator. We implemented this second variant.

$$\widehat{S} := \begin{pmatrix} \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T & \mathbf{0} \\ \mathbf{B}_0 & \mathbf{0} & \mathbf{c}_0 \\ \mathbf{0} & \mathbf{c}_0^T & 0 \end{pmatrix}$$

$$-\sum_{i=1}^{s}\begin{pmatrix}\mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 0\end{pmatrix}\underbrace{\begin{pmatrix}\mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0}\\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i\\ \mathbf{0} & \mathbf{c}_i^T & 0\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{A}_{i\Gamma_i} & \mathbf{0} & \mathbf{0}\\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 0\end{pmatrix}}_{:=L1}.$$

For the right hand side $\widehat{r}$, we directly define the variant based on the alternative formulation:

$$\widehat{r} := \begin{pmatrix}\mathbf{f}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D\\ -\mathbf{B}_{0D}\mathbf{u}_D\\ 0\end{pmatrix} - \sum_{i=1}^{s}\begin{pmatrix}\mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & 0\end{pmatrix}\underbrace{\begin{pmatrix}\mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0}\\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i\\ \mathbf{0} & \mathbf{c}_i^T & 0\end{pmatrix}^{-1}\begin{pmatrix}\mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i}\\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i}\\ 0\end{pmatrix}}_{:=L2}.$$

Using these definition, it gives the Schur-complement equation, which as before represents algebraically the coupling on the skeleton (2.35):

$$\widehat{S}(\mathbf{u}_\Gamma, p_\Omega) = \widehat{r},$$

Analogously to the SWF, the subproblems $L1$ and $L2$ can be interpreted as solving local Oseen problems on the subdomains and adding up $L1$ and $L2$ is the algebraic representation of the local decoupled subproblems on the subdomain Ω_i (2.34). Similarly, the local subdomain problems $L1$ and $L2$, which are solved when computing the right hand side $\widetilde{r}$ and when applying the Schur-complement operator $\widehat{S}$ to a skeleton vector $(\mathbf{u}_\Gamma, \widehat{\mathbf{p}}_\Omega)^T$, are independent of $\widehat{\mathbf{p}}_\Omega$. Furthermore, the skeleton vector is independent of η_i, which is only used locally to guarantee that $p_i^h \in L_0^2(\Omega_i)$.

Outflow Weak Formulation (OWF)

Now, we derive the global linear system for the Oseen equation with outflow boundary based on the OWF. Then, we discuss the corresponding Schur-complement equation. For an example with two

subdomains assuming that $1 \notin \mathcal{N}$ and that $2 \in \mathcal{N}$, we obtain the following global system:

$$\left(\begin{array}{cc|cc|cc} \mathbf{A}_{11} & \widetilde{\mathbf{B}}_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma} & \mathbf{0} \\ \widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{A}_{22} & \mathbf{B}_{22}^T & \mathbf{A}_{2\Gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{B}_{2\Gamma} & \mathbf{0} \\ \hline \mathbf{A}_{\Gamma 1} & \widetilde{\mathbf{B}}_{1\Gamma}^T & \mathbf{A}_{\Gamma 2} & \mathbf{B}_{2\Gamma}^T & \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} \end{array}\right) \left(\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{p}_1 \\ \hline \mathbf{u}_2 \\ \hline \mathbf{p}_2 \\ \hline \mathbf{u}_\Gamma \\ \hline \mathbf{p}_\Omega \end{array}\right) = \left(\begin{array}{c} \mathbf{f}_1 - \mathbf{A}_{1D_1}\mathbf{u}_{D_1} \\ \hline -\mathbf{B}_{1D_1}\mathbf{u}_{D_1} \\ \hline \mathbf{f}_2 - \mathbf{A}_{2D_2}\mathbf{u}_{D_2} \\ \hline -\mathbf{B}_{2D_2}\mathbf{u}_{D_2} \\ \hline \mathbf{f}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ \hline -\mathbf{B}_{0D}\mathbf{u}_D \end{array}\right).$$

Analogously to the case with inhomogeneous boundary conditions the local block matrices (with subindex $\cdot_{ii}$) are invertible. Consequently, we can reduce the global problem to a Schur-complement equation. We define the Schur-complement matrix S as:

$$\begin{aligned} S := &\begin{pmatrix} \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T \\ \mathbf{B}_0 & \mathbf{0} \end{pmatrix} - \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix} \\ &- \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix}. \end{aligned}$$

Knowing that we can reformulate the decoupled subproblems equivalently with an alternative formulation, we define a second equivalent variant of the Schur-complement operator S:

$$\begin{aligned} S := &\begin{pmatrix} \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T \\ \mathbf{B}_0 & \mathbf{0} \end{pmatrix} - \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix}}_{:=L1} \\ &- \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}}_{:=L2}. \end{aligned} \tag{2.48}$$

We define the right hand side r, directly using the alternative formulation:

$$\widehat{r} := \begin{pmatrix} \mathbf{f}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\mathbf{B}_{0D}\mathbf{u}_D \end{pmatrix} - \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \end{pmatrix}}_{:=L3} \tag{2.49}$$

$$- \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ 0 \end{pmatrix}}_{:=L4}.$$

These definitions (2.48, 2.49) yield the Schur-complement equation:

$$S(\mathbf{u}_\Gamma, \mathbf{p}_\Omega) = r,$$

which represents algebraically the coupling on the skeleton (2.38).

Analogously to the SWF and the AWF the subproblems $L2$ and $L4$ can be interpreted as solving local Oseen problems with inhomogeneous boundary conditions. $L1$ and $L3$ can be interpreted as solving local Oseen problems with mixed outflow and Dirichlet boundary conditions.

$$L1 \text{ solve}: \begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_{i\Gamma}\mathbf{u}_{\Gamma_i} \\ -\mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \end{pmatrix}.$$

$L1$ solves a local Oseen problem with Dirichlet data $\mathbf{u}_{\Gamma_i}$ on the local skeleton Γ_i, natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$ and zero Dirichlet boundary condition on $\partial\Omega_i \cap \partial\Omega_D$.

$$L3 \text{ solve}: \begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD}\mathbf{u}_D \\ -\mathbf{B}_{iD}\mathbf{u}_D \end{pmatrix}$$

$L3$ solves a local Oseen problem with Dirichlet data and outflow boundary conditions. On the local skeleton Γ_i, zero Dirichlet boundary data is set. On $\partial\Omega_i \cap \partial\Omega_{out}$ the outflow conditions hold and on $\partial\Omega_i \cap \partial\Omega_D$ the Dirichlet data is set to $\mathbf{u}_{D_i}$. Adding up $L1$ and $L3$ leads to

$$L1 + L3 \text{ solve}: \begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} - \mathbf{A}_{i\Gamma}\mathbf{u}_{\Gamma_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} - \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \end{pmatrix},$$

which is the algebraic representation of the local decoupled subproblems on outflow subdomains Ω_i (2.36).

Similarly to the case of SWF and AWF, adding up $L2$ and $L4$ leads to the algebraic representation of the local decoupled subproblems on non-outflow subdomain Ω_i (2.37).

Remark. The local subproblems $L1$ and $L3$ are solved when computing the right hand side r. $L2$ and $L4$ are solved when applying the Schur-

complement operator. The four problems do not depend on the global pressure variable p_Ω. The coupling vector $(\mathbf{u}_\Gamma, p_\Omega)$ does also not depend on the local Lagrange parameter η_i.

2.6.4 *General Operator*

In Chapter 6, we derive a solution algorithm for the Schur-complement equations as well as for the global linear systems, which is independent of the model problem. Therefore, we introduce a generalized operator, which enables us to treat all model problems with a local and global saddle point structure at a time. The general operator matrix corresponds to a subdomain Ω_i and is based on the defined block matrices used in the global linear system. Note that we do not refer to the alternative formulations for the general operator matrix.

On non-outflow subdomains, the general operator matrix $\mathbf{K}_i$ is defined as:

$$\mathbf{K}_i = \left(\begin{array}{cc|cc} \mathbf{A}_{ii} & \widetilde{\mathbf{B}}_{ii}^T & \mathbf{A}_{i\Gamma_i} & \mathbf{0} \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i} & \mathbf{0} \\ \hline \mathbf{A}_{\Gamma_i i} & \widetilde{\mathbf{B}}_{i\Gamma_i}^T & \mathbf{A}_{\Gamma_i\Gamma_i} & \mathbf{B}_{0,i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{0,i} & 0 \end{array}\right) := \left(\begin{array}{cc|cc} \mathcal{A}_{ii} & \mathcal{B}_{ii}^T & \mathcal{A}_{i\Gamma_i} & \mathbf{0} \\ \mathcal{B}_{ii} & \mathbf{0} & \mathcal{B}_{i\Gamma_i} & \mathbf{0} \\ \hline \mathcal{A}_{\Gamma_i i} & \mathcal{B}_{\Gamma_i i}^T & \mathcal{A}_{\Gamma_i\Gamma_i} & \mathcal{B}_{\Omega_i}^T \\ \mathbf{0} & \mathbf{0} & \mathcal{B}_{\Omega_i} & \mathbf{0} \end{array}\right)$$

The upper left block is referred to the local saddle point structure. The lower right block is the local part of the global saddle point structure.

Remark. This definitions also holds for the SWF and AWF replacing the block matrices $\mathbf{B}_0$ with the corresponding definition $\widetilde{\mathbf{B}}_0$ or $\widehat{\mathbf{B}}_0$, respectively.

For outflow subdomains, we get a similar definition. We only have to replace all submatrices of $\widetilde{\mathbf{B}}$ with the corresponding submatrices of $\mathbf{B}$.

SUMMARY

In this chapter, we derived a non-overlapping domain decomposition method for the Oseen equations. By applying the domain decomposition approach on the continuous level, it directly yields local problems of the same structure as the global problem. We focused on the treatment of the saddle point structure which characterizes all flow problems treated in this work. Furthermore, we outlined the difference between inhomogeneous Dirichlet and mixed outflow and Dirichlet boundary conditions. This chapter lays down the mathematical foundation for the forthcoming chapters in which more complex model problems are considered.

3

DOMAIN DECOMPOSITION METHOD FOR AN OPTIMAL CONTROL PROBLEMS CONSTRAINED BY THE OSEEN EQUATIONS

As a next step of our aim to derive a domain decomposition method for a non-linear optimal control problem, we consider a linear optimal control problem constrained by the Oseen equations. We use this model problem to emphasize how to extend the non-overlapping domain decomposition approach to the context of optimal flow control problems. In this chapter, we assume distributed or boundary control for the optimal flow control model problem. The Oseen equations are equipped with mixed outflow and Dirichlet boundary conditions. Analogously to the previous chapter, we derive the domain decomposition method based on the four main steps described in Chapter 2. The chapter is based on ideas presented in [25**?** , 27, 44]. If not stated differently, we use the same definitions, notation and assumptions as in the previous Chapter 2.

3.1 GENERAL DEFINITIONS

Let $\Omega_0, \Omega_C \subseteq \Omega \subset \mathbb{R}^d$ be Lipschitz domains. For simplification in the notation, we define the indicator functions

$$\mathbb{1}_{\Omega_C} : \Omega \to \{0,1\}, \quad \mathbb{1}_{\Omega_C}(x) = \begin{cases} 1 & \text{if } x \in \Omega_C \\ 0 & \text{if } x \notin \Omega_C \end{cases},$$

$$\mathbb{1}_{\Omega_0} : \Omega \to \{0,1\}, \quad \mathbb{1}_{\Omega_0}(x) = \begin{cases} 1 & \text{if } x \in \Omega_0 \\ 0 & \text{if } x \notin \Omega_0 \end{cases}.$$

As in the previous chapter, we decompose the boundary $\partial\Omega = \partial\Omega_{out} \cup \partial\Omega_D$ with $\partial\Omega_D \cap \partial\Omega_{out} = \emptyset$.

3.2 DISTRIBUTED OPTIMAL CONTROL PROBLEM

In the first main step, we introduce the global problem in strong and weak form analogously to Chapter 2. Since we now look at an optimal

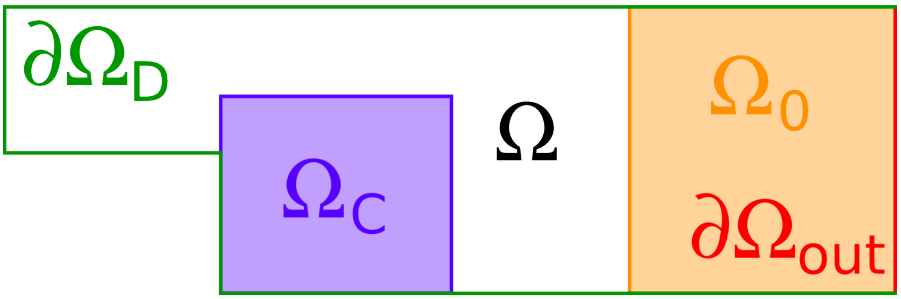


Figure 10: This figure corresponds to the first main step of the derivation of the domain decomposition method and shows exemplary a global domain for which the boundary is split into $\partial\Omega_D$ and $\partial\Omega_{out}$. On $\partial\Omega_D$, we apply inhomogeneous Dirichlet boundary conditions and on $\partial\Omega_{out}$ outflow boundary conditions. The distributed control is applied on the subdomain $\Omega_C \subset \Omega$. A desired state is given on the subdomain Ω_0.

control problem, we do not apply the domain decomposition directly to the model problem but to the corresponding optimality system. This is one main difference when deriving a domain decomposition method in optimal control. Fig. 10 refers to this first main step.

Under the assumptions (2.11) and given $\hat{\mathbf{u}} \in \mathbf{L}^2(\Omega_0)$, we want to solve a linear quadratic optimal control problem. As constraints, we apply the Oseen equations with mixed outflow and Dirichlet boundary conditions and we assume distributed control with support on Ω_C. We consider the following optimal control problem:

$$\min_{\mathbf{u},\mathbf{c}} \frac{1}{2}\int_{\Omega_0} (\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}})^2 d\mathbf{x} + \frac{\alpha}{2}\int_{\Omega_C} \mathbf{c}^2\, d\mathbf{x} \tag{3.1a}$$

$$\text{subject to (s.t.)}\quad (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}\cdot\sigma(\mathbf{u},p) = \mathbf{f} + \mathbb{1}_{\Omega_C}\mathbf{c} \quad \text{in } \Omega, \tag{3.1b}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{3.1c}$$

$$\mathbf{u} = \mathbf{d} \quad \text{on } \partial\Omega_D, \tag{3.1d}$$

$$\sigma(\mathbf{u},p)\mathbf{n} = \mathbf{h} \quad \text{on } \partial\Omega_{out}. \tag{3.1e}$$

In this set-up, $\hat{\mathbf{u}} : \Omega_0 \to \mathbb{R}^d$ models the desired state of the velocity variable $\mathbf{u}$. We consider a tracking type cost functional, i.e. our aim is to control the velocity $\mathbf{u}$ such that the difference between the optimal solution $\mathbf{u}^*$ for the velocity and the desired state $\hat{\mathbf{u}}$ is minimal, see also [36, 37]. For the Dirichlet boundary function $\mathbf{d} \in H^{1/2}(\partial\Omega_D)$, we again assume that an extension $\mathbf{u}_D \in \mathbf{H}^1(\Omega)$ of $\mathbf{d}$ exists such that $\gamma_d(\mathbf{u}_D) = \mathbf{d}$. Fig. 10 illustrates one possible set up for the domains Ω, Ω_0 and Ω_C and the splitting of the global boundary.

Since our aim is to solve the optimal control problem numerically with a finite element method, we derive a weak formulation. Therefore, we need some more definitions. In addition to the bilinear and linear forms used in Chapter 2, we define the following bilinear forms:

$$m : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \to \mathbb{R}, \qquad m(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}\,\mathbf{v}\,d\mathbf{x},$$
$$d : \mathbf{L}^2(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}, \qquad d(\mathbf{c}, \mathbf{v}) = \int_\Omega \mathbf{c}\,\mathbf{v}\,d\mathbf{x}.$$

Using these definitions, we obtain the following weak formulation corresponding to the distributed control problem (3.1). Find $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$, $p \in L^2(\Omega)$ and $\mathbf{c} \in \mathbf{L}^2(\Omega_c)$ that solve the following linear quadratic distributed optimal control problem:

$$\min_{\mathbf{u},\mathbf{c}} \frac{1}{2} m(\mathbb{1}_{\Omega_0}(\mathbf{u} + \mathbf{u}_D - \hat{\mathbf{u}}), \mathbb{1}_{\Omega_0}(\mathbf{u} + \mathbf{u}_D - \hat{\mathbf{u}})) + \frac{\alpha}{2} m(\mathbb{1}_{\Omega_c}\mathbf{c}, \mathbb{1}_{\Omega_c}\mathbf{c}) \tag{3.2a}$$

$$\text{s.t. } a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - d(\mathbb{1}_{\Omega_c}\mathbf{c}, \mathbf{v}) = f(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}), \tag{3.2b}$$

$$b(\mathbf{u}, q) = -b(\mathbf{u}_D, q) \tag{3.2c}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$.

Remark 19. Assuming that $(\mathbf{u}, p)$ solve (3.2), the velocity is uniquely defined since we impose Dirichlet boundary conditions on $\partial\Omega_D$. The pressure is also uniquely defined due to the outflow boundary conditions implied on $\partial\Omega_{out}$. Therefore, no normalization condition must be imposed for the pressure p. The same arguments hold for the adjoint velocity $\mathbf{z}$ and adjoint pressure r, which we introduce in the next lemma.

Lemma 20. *Under the assumptions that an optimal solution $(\mathbf{u}, p, \mathbf{c})^* \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Omega_c)$ for (3.2) and Lagrange multipliers $(\mathbf{z}, r) \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega)$ exist, the optimal solution is a KKT point. $(\mathbf{u}, p, \mathbf{c})^*$ fulfills the necessary optimality conditions described by the KKT conditions which are given by the following optimality system: Find $\mathbf{u}, \mathbf{z} \in \mathbf{H}^1_D(\Omega)$, $p, r \in L^2(\Omega)$ and $\mathbf{c} \in \mathbf{L}^2(\Omega_c)$ such that*

$$a(\mathbf{v}, \mathbf{z}) + b(\mathbf{v}, r) - m(\mathbb{1}_{\Omega_0}\mathbf{u}, \mathbb{1}_{\Omega_0}\mathbf{v}) = -m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u}_D), \mathbb{1}_{\Omega_0}\mathbf{v}), \tag{3.3a}$$
$$b(\mathbf{z}, q) = 0, \tag{3.3b}$$
$$d(\mathbb{1}_{\Omega_c}\mathbf{e}, \mathbf{z}) + \alpha m(\mathbb{1}_{\Omega_c}\mathbf{c}, \mathbb{1}_{\Omega_c}\mathbf{e}) = 0, \tag{3.3c}$$
$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) - d(\mathbb{1}_{\Omega_c}\mathbf{c}, \mathbf{v}) = f(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}), \tag{3.3d}$$
$$b(\mathbf{u}, q) = -b(\mathbf{u}_D, q) \tag{3.3e}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$ and $\mathbf{e} \in \mathbf{L}^2(\Omega_c)$.

Proof. We apply a Lagrangian based adjoint approach to derive the optimality system (3.3) [33]. Therefore, we define the Lagrange function

$$\begin{aligned}\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c}) = &\frac{1}{2}m(\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}),\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}))+\\ &+\frac{\alpha}{2}m(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbb{1}_{\Omega_c}\mathbf{c})-a(\mathbf{u},\mathbf{z})-b(\mathbf{z},p)+d(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbf{z})+\\ &+f(\mathbf{z})-a(\mathbf{u}_D,\mathbf{z})-b(\mathbf{u},r)-b(\mathbf{u}_D,r),\end{aligned}$$

and derive a stationary condition, which reads:

$$\begin{aligned}\frac{\partial}{\partial\mathbf{u}}\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c})\cdot\mathbf{v} &= m(\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}),\mathbb{1}_{\Omega_0}\mathbf{v})-a(\mathbf{v},\mathbf{z})+,\\ &\quad -b(\mathbf{v},r)=0\\ \frac{\partial}{\partial p}\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c})\cdot q &= -b(\mathbf{z},q)=0,\\ \frac{\partial}{\partial\mathbf{c}}\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c})\cdot\mathbf{e} &= \alpha m(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbb{1}_{\Omega_c}\mathbf{e})+d(\mathbb{1}_{\Omega_c}\mathbf{e},\mathbf{z})=0\end{aligned}$$

for all $\mathbf{v}\in\mathbf{H}^1_D(\Omega)$, $q\in L^2(\Omega)$ and $\mathbf{e}\in\mathbf{L}^2(\Omega_c)$. □

Remark 21. For example, in [23, Chapter 6] or [34, Part III] it is stated which assumptions need to be made, to assure that an optimal solution and Lagrange multipliers exist. This remark holds for Lemma 20, 22 and 37.

In Remark 19, we already mentioned the variable $\mathbf{z}$ and r and denoted them by adjoint velocity and adjoint pressure. The name adjoint is derived from the equations (3.3a) and (3.3b), which are called adjoint equations. We introduced $\mathbf{z}$ and r as Lagrange parameters by deriving the optimality system via the Lagrange functional. $\mathbf{z}$ and r model the sensitivity with respect to perturbation of the velocity $\mathbf{u}$ and the pressure p.

We derive a non-overlapping domain decomposition method for the optimality system (3.3).

3.2.1 *Strong Interpretation of the Optimality System*

Since the strong formulation is easier to read, we formally give a strong interpretation of the optimality condition and the adjoint equations.

For sufficiently smooth functions $\mathbf{z}$ and $\mathbf{c}$, we can interpret the optimality condition (3.3c) in strong sense as:

$$\mathbf{z}+\alpha\mathbf{c}=0 \qquad \text{a.e. in } \Omega_c.$$

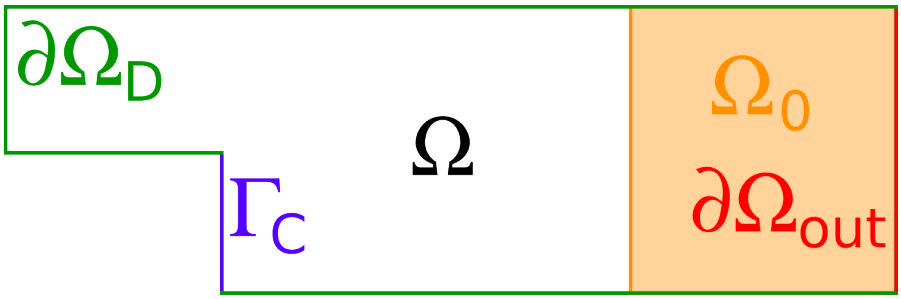


Figure 11: This figure refers to the first main step of the derivation and depicts an example for a global domain where we impose boundary control on Γ_C. Analogously to the distributed control case, we apply different boundary conditions and the desired state is given on the subdomain Ω_0.

Furthermore for sufficiently smooth functions $\mathbf{z}$, r, $\mathbf{u}$ and $\hat{\mathbf{u}}$, the adjoint equations (3.3a, 3.3b) can also be interpreted in strong form:

$$\begin{aligned}
(\mathbf{a}\cdot\nabla)\mathbf{z} - \nabla\cdot\sigma(\mathbf{z},r) &= \mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}) && \text{in } \Omega,\\
\nabla\cdot\mathbf{z} &= 0 && \text{in } \Omega,\\
\mathbf{z} &= 0 && \text{on } \partial\Omega_D,\\
\sigma(\mathbf{z},r)\mathbf{n} &= 0 && \text{on } \partial\Omega_{out}.
\end{aligned}$$

We see, that the adjoint equations have the same structure as the state equations, see constraints in (3.1b) and (3.1c). We observe, that in contrast to the inhomogeneous Dirichlet boundary on $\partial\Omega_D$ for the velocity $\mathbf{u}$ of the state equation, we have homogeneous Dirichlet boundary conditions on $\partial\Omega_D$ for the adjoint velocity $\mathbf{z}$. This comes from the fact, that the adjoint equation models the sensitivity. Since on the Dirichlet boundary the velocity is fixed, the sensitivity must be zero. A similar observation holds for the outflow boundary conditions. The outflow boundary conditions for the velocity and pressure are set to the given function $\mathbf{h}$ on $\partial\Omega_{out}$. For the adjoint velocity and adjoint pressure the outflow conditions are zero.

3.3 OPTIMAL BOUNDARY CONTROL PROBLEM

In contrast to the previous case of distributed control, we now control the flow through the boundary. Similarly in the first main step, we state a strong and weak formulation of the global model problem, the domain decomposition method is then applied to the corresponding optimality system, see also Fig 11. The control is only applied on part of the boundary, denoted by Γ_C. We restrict the control to be applied on a part of the Dirichlet boundary and not on the part with outflow boundary, therefore we set $\Gamma_C \subsetneq \partial\Omega_D$. Under the assumptions (2.11) and given $\hat{\mathbf{u}} \in \mathbf{L}^2(\Omega_0)$, we want to solve an optimal boundary control problem constrained by the Oseen equations equipped with mixed

outflow and Dirichlet boundary conditions. We consider the following problem:

$$\min_{\mathbf{u},\mathbf{c}} \frac{1}{2}\int_{\Omega_0}(\mathbf{u}+\mathbf{u}_D+\mathcal{R}_c(\mathbf{c})-\hat{\mathbf{u}})^2 d\mathbf{x}+\frac{\alpha}{2}\int_{\partial\Omega_c}\boldsymbol{\nabla}\mathbf{c}:\boldsymbol{\nabla}\mathbf{c}d\mathbf{x} \tag{3.4a}$$

$$\text{s.t. } (\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{u}-\boldsymbol{\nabla}\cdot\sigma(\mathbf{u},p)=\mathbf{f} \qquad \text{in } \Omega, \tag{3.4b}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}=0 \qquad \text{in } \Omega, \tag{3.4c}$$

$$\mathbf{u}=\mathbf{d} \qquad \text{on } \partial\Omega_D\setminus\Gamma_C, \tag{3.4d}$$

$$\sigma(\mathbf{u},p)\mathbf{n}=\mathbf{h} \qquad \text{on } \partial\Omega_{out}, \tag{3.4e}$$

$$\mathbf{u}=\mathbf{d}+\mathbf{c} \qquad \text{on } \Gamma_C, \tag{3.4f}$$

For the Dirichlet boundary function $\mathbf{d}\in H^{1/2}(\partial\Omega_D)$, we again assume that a function $\mathbf{u}_D\in\mathbf{H}^1(\Omega)$ exists such that the $\gamma_d(\mathbf{u}_D)=\mathbf{d}$. By restricting the boundary control to a real subset of the part of the boundary with Dirichlet boundary conditions, we guarantee that the velocity is uniquely defined assuming that a solution exists. The same arguments as in Remark 19 hold.

We derive a weak formulation and therefore define additionally to the bilinear forms defined before the bilinear form q:

$$q:\mathbf{H}^1(\Gamma_C)\times\mathbf{H}^1(\Gamma_C)\to\mathbb{R}, \qquad q(\mathbf{c},\mathbf{e})=\int_{\Gamma_C}\nabla\mathbf{c}:\nabla\mathbf{e}\,d\mathbf{x},$$

We also define a trace operator γ_c and an extension operator $\mathcal{R}_c$:

$$\gamma_c:\mathbf{H}^1(\Omega)\to\mathbf{H}^{1/2}_{00}(\Gamma_C),$$
$$\mathcal{R}_c:\mathbf{H}^{1/2}_{00}(\Gamma_C)\to\mathbf{H}^1(\Omega), \text{ such that } \gamma_c(\mathcal{R}_c(\mathbf{c}))=\mathbf{c}\;\forall\mathbf{c}\in\mathbf{H}^{1/2}_{00}(\Gamma_C).$$

Remark. The space $\mathbf{H}^1_{00}(\Gamma_C)$ is analogously defined to the space $\mathbf{H}^1_0(\Omega)$ only that the zero corresponds to the boundary of the boundary. This makes it easier to implement the optimal boundary control $\mathbf{c}$. A higher regularity than $L^2(\Gamma_C)$ is needed, due to the fact, that we need the first derivative of the control in $L^2(\Omega)$ in the bilinear form $q(\cdot,\cdot)$.

These definitions lead to the following weak formulation of the boundary control problem (3.4). Find $\mathbf{u}\in\mathbf{H}^1_D(\Omega)$, $p\in L^2(\Omega)$ and $\mathbf{c}\in\mathbf{H}^1_{00}(\Gamma_C)$ that solve the following linear quadratic optimal control problem with boundary control:

$$\min_{\mathbf{u},\mathbf{c}} \frac{1}{2}m(\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D+\mathcal{R}_c(\mathbf{c})-\hat{\mathbf{u}}),\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D+\mathcal{R}_c(\mathbf{c})-\hat{\mathbf{u}}))+ \tag{3.5a}$$

$$+\frac{\alpha}{2}q(\mathbf{c},\mathbf{c})$$

$$\text{s.t. } a(\mathbf{u},\mathbf{v})+b(\mathbf{v},p)+a(\mathcal{R}_c(\mathbf{c}),\mathbf{v})=f(\mathbf{v})-a(\mathbf{u}_D,\mathbf{v}), \tag{3.5b}$$

$$b(\mathbf{u},q)+b(\mathcal{R}_c(\mathbf{c}),q)=-b(\mathbf{u}_D,q) \tag{3.5c}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$.

Lemma 22. *Under the assumption that an optimal solution* $(\mathbf{u}, p, \mathbf{c})^* \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega) \times \mathbf{H}^1_{00}(\Gamma_C)$ *for (3.5) and the Lagrange multipliers* $(\mathbf{z}, r) \in \mathbf{H}^1_D \times L^2(\Omega)$ *exists, the optimal solution is a KKT point. It fulfills the necessary optimality conditions described by the KKT conditions which are given by the following optimality system: Find* $\mathbf{u}, \mathbf{z} \in \mathbf{H}^1_D(\Omega)$, $p, r \in L^2(\Omega)$ *and* $\mathbf{c} \in \mathbf{H}^1_{00}(\Gamma_C)$ *such that*

$$\begin{aligned} a(\mathbf{w}, \mathbf{z}) + b(\mathbf{w}, r) + & \\ -m(\mathbb{1}_{\Omega_0}(\mathbf{u} + \mathcal{R}_c(\mathbf{c})), \mathbb{1}_{\Omega_0}\mathbf{w}) &= -m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u}_D), \mathbb{1}_{\Omega_0}\mathbf{w}), && \text{(3.6a)} \\ b(\mathbf{z}, q) &= 0, && \text{(3.6b)} \\ a(\mathcal{R}_c(\mathbf{e}), \mathbf{z}) + b(\mathcal{R}_c(\mathbf{e}), r) - \alpha q(\mathbf{c}, \mathbf{e}) + & && \text{(3.6c)} \\ -m(\mathbb{1}_{\Omega_0}(\mathbf{u} - \mathcal{R}_c(\mathbf{c})), \mathbb{1}_{\Omega_0}(\mathcal{R}_c(\mathbf{e}))) &= m(\mathbb{1}_{\Omega_0}(\mathbf{u}_D - \hat{\mathbf{u}}), \mathbb{1}_{\Omega_0}(\mathcal{R}_c(\mathbf{e}))), && \text{(3.6d)} \\ a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + a(\mathcal{R}_c(\mathbf{c}), \mathbf{v}) &= f(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}), && \text{(3.6e)} \\ b(\mathbf{u}, q) + b(\mathcal{R}_c(\mathbf{c}), q) &= -b(\mathbf{u}_D, q) && \text{(3.6f)} \end{aligned}$$

for all $\mathbf{v}, \mathbf{w} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$ *and* $\mathbf{e} \in \mathbf{H}^1_0(\Gamma_c)$.

Proof. Analogously to the case with distributed control, the optimality system (3.6) can be derived by applying a Lagrangian based adjoint approach [33]. Therefore we define the Lagrange function

$$\begin{aligned} &\mathcal{L}(\mathbf{u}, p, \mathbf{z}, r, \mathbf{c}) = \\ &\frac{1}{2} m(\mathbb{1}_{\Omega_0}(\mathbf{u} + \mathbf{u}_D + \mathcal{R}_c(\mathbf{c}) - \hat{\mathbf{u}}), \mathbb{1}_{\Omega_0}(\mathbf{u} + \mathbf{u}_D + \mathcal{R}_c(\mathbf{c}) - \hat{\mathbf{u}})) + \frac{\alpha}{2} q(\mathbf{c}, \mathbf{c}) + \\ &- a(\mathbf{u}, \mathbf{z}) - a(\mathcal{R}_c(\mathbf{c}), \mathbf{z}) - b(\mathbf{z}, p) + \\ &+ f(\mathbf{z}) - a(\mathbf{u}_D, \mathbf{z}) + \\ &- b(\mathbf{u}, r) - b(\mathcal{R}_c(\mathbf{c}), r) - b(\mathbf{u}_D, r), \end{aligned}$$

and derive a stationary condition, which reads:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{u}} \mathcal{L}(\mathbf{u}, p, \mathbf{z}, r, \mathbf{c}) \cdot \mathbf{w} &= m(\mathbb{1}_{\Omega_0}(\mathbf{u} + \mathbf{u}_D + \mathcal{R}_c(\mathbf{c}) - \hat{\mathbf{u}}), \mathbb{1}_{\Omega_0}(\mathbf{w})) + \\ &\quad - a(\mathbf{w}, \mathbf{z}) - b(\mathbf{w}, r) = 0, \\ \frac{\partial}{\partial p} \mathcal{L}(\mathbf{u}, p, \mathbf{z}, r, \mathbf{c}) \cdot q &= -b(\mathbf{z}, q) = 0, \\ \frac{\partial}{\partial \mathbf{c}} \mathcal{L}(\mathbf{u}, p, \mathbf{z}, r, \mathbf{c}) \cdot \mathbf{e} &= \alpha q(\mathbf{c}, \mathbf{e}) - a(\mathcal{R}_c(\mathbf{e}), \mathbf{z}) - b(\mathcal{R}_c(\mathbf{e}), r) + \\ &\quad m(\mathbb{1}_{\Omega_0}(\mathbf{u} + \mathbf{u}_D + \mathcal{R}_c(\mathbf{c}) - \hat{\mathbf{u}}), \mathbb{1}_{\Omega_0}(\mathcal{R}_c(\mathbf{e}))) = 0 \end{aligned}$$

for all $\mathbf{w} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$ and $\mathbf{e} \in \mathbf{H}^1_0(\Gamma_c)$.

For further details, we also refer to [17, 32]. □

3.3.1 *Strong Interpretation of the Optimality System*

For sufficiently smooth functions $\mathbf{z}$ and $\mathbf{c}$, we can reformulate the optimality condition (3.6d):

$$\begin{aligned}
&a(\mathcal{R}_c(\mathbf{e}),\mathbf{z}) + b(\mathcal{R}_c(\mathbf{e}),r) - \alpha q(\mathbf{c},\mathbf{e}) + \\
&- m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u} - \mathbf{u}_D - \mathcal{R}_c(\mathbf{c})), \mathbb{1}_{\Omega_0}(\mathcal{R}_c(\mathbf{e}))) \\
&= -\mu \int_\Omega \Delta \mathbf{z}\, \mathcal{R}_c(\mathbf{e})\, d\mathbf{x} + \mu \int_{\partial\Omega_C} \frac{\partial \mathbf{z}}{\partial \mathbf{n}}\, \mathbf{e}\, d\mathbf{x} + \\
&\quad - \int_\Omega ((\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{z})\mathcal{R}_c(\mathbf{e}) + \int_{\partial\Omega_C} (\mathbf{a}\cdot\mathbf{n})(\mathbf{z}\mathbf{e})\, d\mathbf{x} + \\
&\quad + \int_\Omega \boldsymbol{\nabla} r\, \mathcal{R}_c(\mathbf{e})\, d\mathbf{x} - \int_{\partial\Omega_C} \mathbf{e}\, r\cdot\mathbf{n}\, d\mathbf{x} + \alpha \int_{\partial\Omega_C} \Delta\mathbf{c}\mathbf{e} d\mathbf{x} + \\
&\quad - \int_{\Omega_0} (\hat{\mathbf{u}} - \mathbf{u} - \mathbf{u}_D - \mathcal{R}_c(\mathbf{c}))\mathcal{R}_c(\mathbf{e}) d\mathbf{x}
\end{aligned}$$

This leads to the strong interpretation of the optimality condition (3.6d):

$$\begin{aligned}
\frac{\partial \mathbf{z}}{\partial \mathbf{n}} + (\mathbf{a}\cdot\mathbf{n})\mathbf{z} - rI\mathbf{n} + \alpha\Delta\mathbf{c} = 0 \qquad & \text{on } \Gamma_C \\
\Leftrightarrow & \\
(\mathbf{a}\cdot\mathbf{n})\mathbf{z} + \sigma(\mathbf{z},r)\mathbf{n} + \alpha\Delta\mathbf{c} = 0 \qquad & \text{on } \Gamma_C.
\end{aligned}$$

For sufficiently smooth functions $\mathbf{z}$, r, $\mathbf{u}$ and $\hat{\mathbf{u}}$, the adjoint equations (3.6a, 3.6b) can also be interpreted in strong form:

$$\begin{aligned}
(\mathbf{a}\cdot\boldsymbol{\nabla})\mathbf{z} - \boldsymbol{\nabla}\cdot\sigma(\mathbf{z},r) &= \mathbb{1}_0(\mathbf{u} + \mathbf{u}_D + \mathcal{R}_c(\mathbf{c}) - \hat{\mathbf{u}}) && \text{in } \Omega, \quad (3.7a)\\
\boldsymbol{\nabla}\cdot\mathbf{z} &= 0 && \text{in } \Omega, \quad (3.7b)\\
\mathbf{z} &= 0 && \text{on } \partial\Omega_D, \\
\sigma(\mathbf{z},r)\mathbf{n} &= 0 && \text{on } \partial\Omega_{out}.
\end{aligned}$$

Analogously to the distributed control case, we note that the adjoint equations (3.7a) and (3.7b) have the same structure as the state equations (3.4b) and (3.4c). For the same reasons as for the distributed control case, we have homogeneous Dirichlet boundary conditions on $\partial\Omega_D$ and zero outflow boundary conditions on $\partial\Omega_{out}$.

3.4 CONTINUOUS DOMAIN DECOMPOSITION

In the second main step, we derive a non-overlapping domain decomposition method on the continuous level for the optimality systems with distributed and boundary control. We do not repeat definitions made in Chapter 2. Analogously, in the first sub-step an equivalent weak formulation on subdomains is derived. In the second sub-step, we decouple the subdomain formulation into s independent

subdomain optimality problems and one system of global coupling conditions.

3.4.1 *Definitions for the Distributed Control Case*

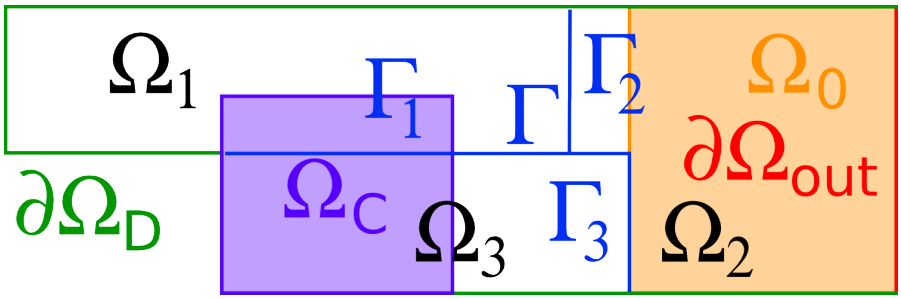


Figure 12: This figure illustrates the first sub-step of the derivation of the non-overlapping domain decomposition on the continuous level for the distributed control case. The global domain is partitioned into subdomains but still coupled. It shows the partition of the global domain into subdomains and the global and local interfaces for the distributed control case. The control domain lies partly inside of Ω_1 and Ω_2 which makes them both control subdomains. The outflow subdomain Ω_2 also contains the subdomain Ω_0 on which the desired state is defined.

We partition the domain as explained in Chapter 2. For the sake of simplicity, we introduce the following notation for the control subdomains

$$\Omega_{C_i} := \Omega_i \cap \Omega_C,$$

and the indicator function

$$\mathbb{1}_{\Omega_{C_i}} : \Omega \to \{0,1\} \quad \mathbb{1}_{\Omega_{C_i}}(x) = \begin{cases} 1 & \text{if } x \in \Omega_{C_i}, \\ 0 & \text{if } x \notin \Omega_{C_i}. \end{cases}$$

In addition to the subset $\mathcal{N}$, we define the subset $\mathcal{C}_\Omega$ containing the indices of the subdomains on which the distributed control is applied:

$$\mathcal{C}_\Omega := \{i \in \{1,\dots,s\} : \Omega_{C_i} \neq \emptyset\}.$$

Next, we decompose the global control space $\mathbf{L}^2(\Omega_C)$:

$$\mathbf{L}^2(\Omega_C) := \bigoplus_{i \in \mathcal{C}_\Omega} \mathbf{L}^2(\Omega_{C_i}).$$

Remark 23. By defining the splitting of the global control space like this, the global control variable is partitioned only into local variables. We do not get an explicit coupling space or coupling variable as for the velocity and pressure.

Furthermore, we define the local bilinear forms $m_i(\cdot,\cdot)$ and $d_i(\cdot,\cdot)$ as follows:

$$m_i : \mathbf{L}^2(\Omega_i) \times \mathbf{L}^2(\Omega_i) \to \mathbb{R}, \qquad m_i(\mathbf{u}_i, \mathbf{v}_i) = \int_{\Omega_i} \mathbf{u}_i \, \mathbf{v}_i \, d\mathbf{x},$$
$$d_i : \mathbf{H}^1(\Omega_i) \times \mathbf{L}^2(\Omega_i) \to \mathbb{R}, \qquad d_i(\mathbf{u}_i, \mathbf{v}_i) = \int_{\Omega_i} \mathbf{u}_i \, \mathbf{v}_i \, d\mathbf{x}.$$

3.4.2 *Weak Formulation on Subdomains for Distributed Control*

Using these definitions, the next lemma states an equivalent weak formulation on subdomains for the optimality system with distributed control (3.3). Similarly to the previous chapter this formulation is still globally coupled. This first sub-step of the second main step, is illustrated in Fig. 12.

Lemma 24. *(3.3) is equivalent to the following weak formulation on subdomains:*
Find

$$\mathbf{u} = \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}^1_D(\Omega) \qquad \textit{with } \mathbf{u}_i \in \mathbf{V}_i,\ \mathbf{u}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma),$$
$$\mathbf{z} = \sum_{i=1}^{s} \mathbf{z}_i + \mathcal{R}(\mathbf{z}_\Gamma) \in \mathbf{H}^1_D(\Omega) \qquad \textit{with } \mathbf{z}_i \in \mathbf{V}_i,\ \mathbf{z}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma),$$
$$\mathbf{c} = \sum_{i=1}^{s} \mathbf{c}_i \in \mathbf{L}^2(\Omega_C) \qquad \textit{with } \mathbf{c}_i \in \mathbf{L}^2(\Omega_{C_i}),$$
$$p = \sum_{i=1}^{s} p_i + p_\Omega \in L^2(\Omega) \qquad \textit{with } p_i \in Q_i,\ p_\Omega \in D(\Omega)$$
$$r = \sum_{i=1}^{s} r_i + r_\Omega \in L^2(\Omega) \qquad \textit{with } r_i \in Q_i,\ r_\Omega \in D(\Omega)$$

such that

$$\begin{aligned}
\sum_{i=1}^{s} \Big(& a_i(\mathbf{v}_i, \mathbf{z}_i) + a_i(\mathbf{v}_i, \mathcal{R}_i(\mathbf{z}_{\Gamma_i})) + a_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{z}_i) + \\
& + b_i(\mathbf{v}_i, r_i) + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), r_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathbf{v}_i) + \\
& - m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbb{1}_{\Omega_0}\mathbf{v}_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) \Big) + \\
& + a(\mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{z}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), r_\Omega) - m(\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{u}_\Gamma), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma))
\end{aligned}$$

$$= \sum_{i=1}^{s} \Big(-m_i((\mathbb{1}_{\Omega_0}\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{v}_i) \Big) - m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u}_D), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma)), \tag{3.8a}$$

$$\sum_{i=1}^{s} \Big(b_i(\mathbf{z}_i, q_i) + b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}), q_i) \Big) + b(\mathcal{R}(\mathbf{z}_\Gamma), q_\Omega) = 0 \tag{3.8b}$$

$$\sum_{i=1}^{s} \Big(d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i, \mathbb{1}_{\Omega_C}\mathbf{z}_i) + d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i, \mathbb{1}_{\Omega_C}\mathcal{R}_i(\mathbf{z}_{\Gamma_i})) + \\ + \alpha m_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathbf{e}_i) \Big) = 0 \tag{3.8c}$$

$$\sum_{i=1}^{s} \Big(a_i(\mathbf{u}_i, \mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i) + a_i(\mathbf{u}_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathbf{v}_i, p_i) + \\ + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), p_i) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathbf{v}_i) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) \Big) + \\ + a(\mathcal{R}(\mathbf{u}_\Gamma), \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), p_\Omega) + \\ = \sum_{i=1}^{s} \Big(f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) \Big) + f(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)),$$

$$\sum_{i=1}^{s} \Big(b_i(\mathbf{u}_i, q_i) + b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i) \Big) + b(\mathcal{R}(\mathbf{u}_\Gamma), q_\Omega) \tag{3.8d}$$

$$= -\sum_{i=1}^{s} \Big(b_i(\mathbf{u}_{D_i}, q_i) \Big) - b(\mathbf{u}_D, q_\Omega) \tag{3.8e}$$

for all $\mathbf{v}_i \in \mathbf{V}_i$, $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, $\mathbf{e}_i \in \mathbf{L}^2(\Omega_i)$ $q_i \in Q_i$ *and* $q_\Omega \in D(\Omega)$.

Proof. Note that on outflow subdomains it holds that $r_{\Omega_i} = 0$. Then we can use the same arguments as in the proof for Lemma 12. For the splitting of the optimality condition (3.3c), we only get a global coupling for the adjoint velocity due to the definition of the decomposition of the control space, see also Remark 23. □

3.4.3 *Decoupling of Weak Formulation on Subdomains for Distributed Control*

Applying the second sub-step of the derivation of the non-overlapping domain decomposition method leads to one system of coupling equations which is defined for the velocity and adjoint velocity in the interface space $\mathbf{H}_{00}^{1/2}(\Gamma)$ and for the pressure and adjoint pressure in the space $D(\Omega)$, and s independent local optimality systems on each subdomain, see also Fig. 13. For the local optimality systems, we have to distinguish between weak formulations for the outflow subdomains and for the non-outflow subdomains, analogously to Chapter 2 for the outflow case.

On a non-outflow subdomain, we obtain the following local optimality system:

For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i$, $\mathbf{z}_i \in \mathbf{H}_0^1(\Omega_i)$, $\mathbf{c}_i \in \mathbf{L}^2(\Omega_i)$ and $\widetilde{p}_i$, $\widetilde{r}_i \in L_0^2(\Omega_i)$, such that

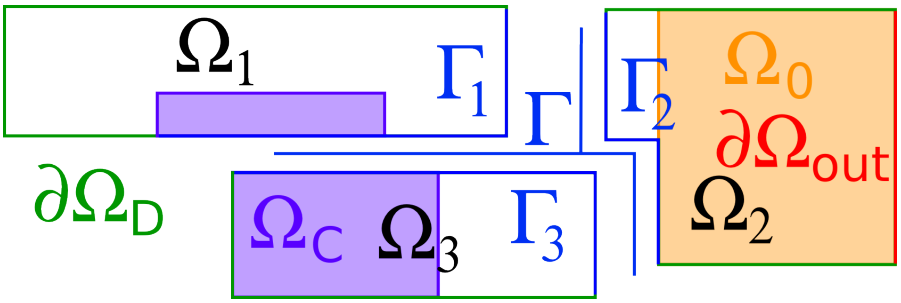


Figure 13: This figure sketches the second sub-step in the derivation of the domain decomposition method in which we derive decoupled local weak formulations on the subdomains and one system of coupling conditions on the global interface Γ. We have to distinguish the different subdomain types. In this example Ω_1 and Ω_3 are non-outflow and at the same time control subdomains. Ω_2 is an outflow subdomain and the desired state is defined on part of that subdomain.

$$
\begin{aligned}
&a_i(\mathbf{v}_i,\mathbf{z}_i)+b_i(\mathbf{v}_i,\widetilde{r}_i)-m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i,\mathbb{1}_{\Omega_0}\mathbf{v}_i)+m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbb{1}_{\Omega_0}\mathbf{v}_i)\\
&=-m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i}-\mathbf{u}_{D_i}),\mathbb{1}_{\Omega_0}\mathbf{v}_i)-a_i(\mathbf{v}_i,\mathcal{R}_i(\mathbf{z}_{\Gamma_i})) && \text{(3.9a)}\\
&b_i(\mathbf{z}_i,\widetilde{q}_i)=-b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}),\widetilde{q}_i) && \text{(3.9b)}\\
&d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i,\mathbb{1}_{\Omega_C}\mathbf{z}_i)+\alpha m_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i,\mathbb{1}_{\Omega_C}\mathbf{e}_i) && \text{(3.9c)}\\
&=-d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i,\mathbb{1}_{\Omega_C}\mathcal{R}_i(\mathbf{z}_{\Gamma_i}))\\
&a_i(\mathbf{u}_i,\mathbf{v}_i)+b_i(\mathbf{v}_i,\widetilde{p}_i)-d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i,\mathbb{1}_{\Omega_C}\mathbf{v}_i) && \text{(3.9d)}\\
&=f(\mathbf{v}_i)-a_i(\mathbf{u}_{D_i},\mathbf{v}_i)-a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i),\\
&b_i(\mathbf{u}_i,\widetilde{q}_i)=-b_i(\mathbf{u}_{D_i},\widetilde{q}_i)-b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\widetilde{q}_i) && \text{(3.9e)}
\end{aligned}
$$

for all $\mathbf{v}_i \in \mathbf{H}^1_0(\Omega_i)$, $\mathbf{e}_i \in \mathbf{L}^2(\Omega_i)$ and $\widetilde{q}_i \in L^2_0(\Omega_i)$. Due to implementation issues we are interested in an alternative formulation for non-outflow subdomains, which is stated in the next proposition.

Proposition 25. *Assuming (2.24) and*

$$\int_{\Omega_i}\theta_i d\mathbf{x}=\int_{\Omega_i}\boldsymbol{\nabla}\cdot\mathcal{R}(\mathbf{z}_{\Gamma_i})d\mathbf{x}, \tag{3.10}$$

then (3.9) and the following local subdomain optimality system are equivalent: For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$, find $\mathbf{u}_i$, $\mathbf{z}_i \in \mathbf{H}^1_0(\Omega_i)$, $\mathbf{c}_i \in \mathbf{L}^2(\Omega_i)$, p_i, $r_i \in L^2(\Omega_i)$ and η_i, $\theta_i \in \mathbb{R}$, such that

$$
\begin{aligned}
&a_i(\mathbf{v}_i,\mathbf{z}_i)+b_i(\mathbf{v}_i,r_i)-m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i,\mathbb{1}_{\Omega_0}\mathbf{v}_i) && \text{(3.11a)}\\
&=-m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i}-\mathbf{u}_{D_i}),\mathbb{1}_{\Omega_0}\mathbf{v}_i)-a_i(\mathbf{v}_i,\mathcal{R}_i(\mathbf{z}_{\Gamma_i}))+\\
&\quad+m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbb{1}_{\Omega_0}\mathbf{v}_i) && \text{(3.11b)}\\
&b_i(\mathbf{z}_i,q_i)+c_i(\theta_i,q_i)=-b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}),q_i) && \text{(3.11c)}\\
&c_i(\xi_i,r_i)=0 && \text{(3.11d)}
\end{aligned}
$$

$$d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i, \mathbb{1}_{\Omega_C}\mathbf{z}_i) + \alpha m_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathbf{e}_i) \tag{3.11e}$$
$$= -d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i, \mathbb{1}_{\Omega_C}\mathcal{R}_i(\mathbf{z}_{\Gamma_i}))$$
$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathbf{v}_i) \tag{3.11f}$$
$$= f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i),$$
$$b_i(\mathbf{u}_i, q_i) + c_i(\eta_i, q_i) = -b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i) \tag{3.11g}$$
$$c_i(\xi_i, p_i) = 0 \tag{3.11h}$$

for all $\mathbf{v}_i \in \mathbf{V}_i$, $\mathbf{e}_i \in \mathbf{L}^2(\Omega_i)$, $q_i \in L^2(\Omega_i)$ *and* $\xi_i \in \mathbb{R}$.

Proof. Analogously to the proof of Proposition 14 in Chapter 2 . □

As we can see, the local optimality systems on non-outflow subdomains are independent of the global constants p_Ω and r_Ω and have the same structure as the global optimality system.

On outflow subdomains, the local optimality system yields:

For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$, find $\mathbf{u}_i$, $\mathbf{z}_i \in \mathbf{H}^1_N(\Omega_i)$,$\mathbf{c}_i \in \mathbf{L}^2(\Omega)$ and p_i, $r_i \in L^2(\Omega)$ such that

$$a_i(\mathbf{v}_i, \mathbf{z}_i) + b_i(\mathbf{v}_i, r_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathbf{v}_i) \tag{3.12a}$$
$$= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{v}_i) - a_i(\mathbf{v}_i, \mathcal{R}(\mathbf{z}_{\Gamma_i})) +$$
$$+ m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbb{1}_{\Omega_0}\mathbf{v}_i), \tag{3.12b}$$
$$b_i(\mathbf{z}_i, q_i) = -b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}), q_i), \tag{3.12c}$$
$$d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i, \mathbb{1}_{\Omega_C}\mathbf{z}_i) + \alpha m_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathbf{e}_i)$$
$$= -d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i, \mathbb{1}_{\Omega_C}\mathcal{R}_i(\mathbf{z}_{\Gamma_i})) \tag{3.12d}$$
$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathbf{v}_i)$$
$$= f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \tag{3.12e}$$
$$b_i(\mathbf{u}_i, q_i) = -b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i) \tag{3.12f}$$

for all $\mathbf{v}_i \in \mathbf{H}^1_N(\Omega_i)$, $\mathbf{e}_i \in \mathbf{L}^2(\Omega_i)$, $q_i \in L^2(\Omega_i)$. Since on outflow subdomains, the pressure p_i and adjoint pressure r_i are uniquely defined as long a solution exists, it is clear that p_{Ω_i} and r_{Ω_i} must be zero.

Since we are interested in solving the global system (3.3) by solving coupling conditions on the interface, the next lemma states, under which conditions we get a solution of the global system by solving the system of interface equations.

Lemma 26. It holds that

$$\mathbf{u} = \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}^1_D(\Omega) \qquad \text{with } \mathbf{u}_i \in \mathbf{V}_i,$$
$$\mathbf{z} = \sum_{i=1}^{s} \mathbf{z}_i + \mathcal{R}(\mathbf{z}_\Gamma) \in \mathbf{H}^1_D(\Omega) \qquad \text{with } \mathbf{z}_i \in \mathbf{V}_i,$$
$$\mathbf{c} = \sum_{i=1}^{s} \mathbf{c}_i \in \mathbf{L}^2(\Omega_C) \qquad \text{with } \mathbf{c}_i \in \mathbf{L}^2(\Omega_{C_i}),$$

$$p = \sum_{i=1}^{s} p_i + p_\Omega \in L^2(\Omega) \qquad \text{with } p_i \in Q_i,$$
$$r = \sum_{i=1}^{s} r_i + r_\Omega \in L^2(\Omega) \qquad \text{with } r_i \in Q_i$$

with $(\mathbf{u}_i, \mathbf{z}_i, \mathbb{1}_C \mathbf{c}_i, p_i, r_i) \in \mathbf{V}_i \times \mathbf{V}_i \times \mathbf{L}^2(\Omega_{C_i}) \times Q_i \times Q_i$ *solutions of (3.11) or (3.12) (depending on the subdomain type) solve (3.3) if and only if the following coupling conditions hold for* $(\mathbf{u}_\Gamma, \mathbf{z}_\Gamma, p_\Omega, r_\Omega) \in \mathbf{H}_{00}^{1/2}(\Gamma) \times \mathbf{H}_{00}^{1/2}(\Gamma) \times D(\Omega) \times D(\Omega)$*: Find* $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$ *and* p_Ω, $r_\Omega \in D(\Omega)$ *such that*

$$\begin{aligned}
&\sum_{i=1}^{s} \Big(a_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{z}_i) + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), r_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) \Big) + \\
&\quad + a(\mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{z}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), r_\Omega) - m(\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{u}_\Gamma), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma)) \\
&= -m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u}_D), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma)), \qquad (3.13) \\
&b(\mathcal{R}(\mathbf{z}_\Gamma), q_\Omega) = 0, \qquad (3.14) \\
&\sum_{i=1}^{s} \Big(a_i(\mathbf{u}_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), p_i) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i, \mathbb{1}_{\Omega_C}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) \Big) + \\
&\quad + a(\mathcal{R}(\mathbf{u}_\Gamma), \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), p_\Omega) \\
&= f(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)), \qquad (3.15) \\
&b(\mathcal{R}(\mathbf{u}_\Gamma), q_\Omega) = -b(\mathbf{u}_D, q_\Omega) \qquad (3.16)
\end{aligned}$$

for all $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$ *and* $q_\Omega \in D(\Omega)$.

Proof. This can be shown by using the same arguments as in Chapter 2 in the proof of Lemma 13. □

3.4.4 *Definitions for the Boundary Control Case*

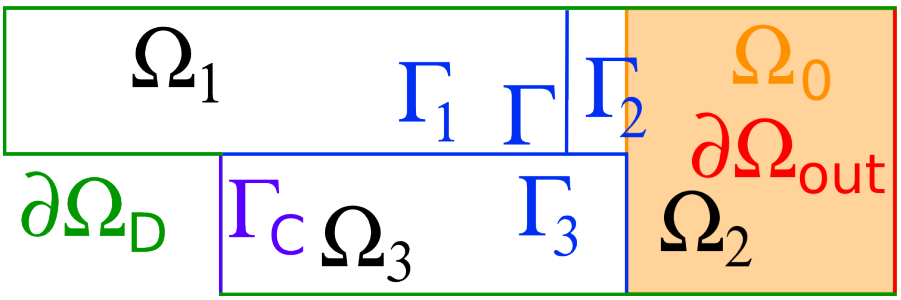


Figure 14: This figure refers to the first sub-step of the second main step of the derivation, in which we obtain a formulation on subdomains which is still globally coupled. It depicts the decomposition of the global domain into subdomains in case of boundary control. In this example, subdomain Ω_3 is the control subdomain.

We partition the domain as explained in Chapter 2 and additionally require that the control boundary belongs to exactly one subdomain named Ω_j:

$$\Gamma_C \subsetneq \partial\Omega_j \setminus \overline{\Gamma}_j.$$

Due to the fact that we have a kind of outflow boundary condition on the control boundary Γ_C for the adjoint velocity and pressure, the pressure is uniquely determined on the control subdomain Ω_j. Therefore, we need to slightly change the definition of the local and global adjoint pressure spaces:

$$Q_i^r := \begin{cases} L_0^2(\Omega_i) & \text{if } i \notin \mathcal{N} \cup j, \\ L^2(\Omega) & \text{if } i \in \mathcal{N} \cup j. \end{cases}$$

$$D^r(\Omega) := \left\{ q \in L^2(\Omega) : q_{\Omega_i} = \begin{cases} const. & \text{if } i \notin \mathcal{N} \cup j, \\ 0 & \text{if } i \in \mathcal{N} \cup j \end{cases} \right\}.$$

These definitions lead to the following decomposition of the space $L^2(\Omega)$ for the adjoint pressure r:

$$L^2(\Omega) = \bigoplus_{i \notin \mathcal{N} \cup j} L_0^2(\Omega_i) \bigoplus_{i \in \mathcal{N} \cup j} L^2(\Omega) \oplus D^r(\Omega) = \bigoplus_{i=1}^{s} Q_i^r \oplus D^r(\Omega).$$

Furthermore, we define the local bilinear form $q_i(\cdot,\cdot)$ as follows:

$$q_i : \mathbf{H}^1(\Gamma_{C_i}) \times \mathbf{H}^1(\Gamma_{C_i}) \to \mathbb{R}, \qquad q_i(\mathbf{c}_i, \mathbf{e}_i) = \int_{\Gamma_{C_i}} \nabla \mathbf{c}_i : \nabla \mathbf{e}_i \, d\mathbf{x}.$$

We define the local extension operator $\mathcal{R}_{c_i} := \mathcal{R}_c|_{\Omega_i}$. We assume that $\mathcal{R}_{c_i}(\mathbf{e}) = 0$ for $i \neq j$ with $\mathbf{e} \in \mathbf{H}_{00}^1(\Gamma_C)$.

3.4.5 *Weak Formulation on Subdomains for Boundary Control*

Using these definitions in the next lemma, we derive an equivalent weak formulation on subdomains for the optimality system with boundary control. This step corresponds to the first sub-step in the derivation of the domain decomposition method on the continuous level and is also sketched in Fig. 14.

Lemma 27. *(3.6) is equivalent to the following weak formulation on subdomains: Find*

$$\mathbf{u} = \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}^1_D(\Omega) \qquad \textit{with } \mathbf{u}_i \in \mathbf{V}_i,\ \mathbf{u}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma),$$

$$\mathbf{z} = \sum_{i=1}^{s} \mathbf{z}_i + \mathcal{R}(\mathbf{z}_\Gamma) \in \mathbf{H}^1_D(\Omega) \qquad \textit{with } \mathbf{z}_i \in \mathbf{V}_i,\ \mathbf{z}_\Gamma \in \mathbf{H}^{1/2}_{00,C}(\Gamma),$$

$$p = \sum_{i=1}^{s} p_i + p_\Omega \qquad \textit{with } p_i \in Q_i,\ p_\Omega \in D(\Omega),$$

$$r = \sum_{i=1}^{s} r_i + r_\Omega \qquad \textit{with } r_i \in Q^r_i,\ r_\Omega \in D^r(\Omega),$$

$$\mathbf{c} \in \mathbf{H}^{1/2}_{00}(\Gamma_C), \qquad \textit{such that}$$

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathbf{w}_i,\mathbf{z}_i) + a_i(\mathbf{w}_i,\mathcal{R}_i(\mathbf{z}_{\Gamma_i})) + a_i(\mathcal{R}_i(\mathbf{w}_{\Gamma_i}),\mathbf{z}_i) + \\
&\quad + b_i(\mathbf{w}_i,r_i) + b_i(\mathcal{R}_i(\mathbf{w}_{\Gamma_i}),r_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i,\mathbb{1}_{\Omega_0}\mathbf{w}_i) + \\
&\quad - m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbb{1}_{\Omega_0}\mathbf{w}_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i,\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{w}_{\Gamma_i}))\Big) + \\
&\quad - m_j(\mathbb{1}_{\Omega_0}\mathcal{R}_{c_j}(\mathbf{c}),\mathbb{1}_{\Omega_0}\mathbf{w}_j) - m_j(\mathbb{1}_{\Omega_0}\mathcal{R}_{c_j}(\mathbf{c}),\mathbb{1}_{\Omega_0}\mathbf{w}_{\Gamma_j}) + \qquad (3.17a)\\
&\quad + a(\mathcal{R}(\mathbf{w}_\Gamma),\mathcal{R}(\mathbf{z}_\Gamma)) + b(\mathcal{R}(\mathbf{w}_\Gamma),r_\Omega) - m(\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{u}_\Gamma),\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{w}_\Gamma)) \\
&= -\sum_{i=1}^{s}\Big(m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}),\mathbb{1}_{\Omega_0}\mathbf{w}_i)\Big) - m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u}_D),\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{w}_\Gamma), \qquad (3.17b)
\end{aligned}$$

$$\sum_{i=1}^{s}\Big(b_i(\mathbf{z}_i,q^r_i) + b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}),q^r_i)\Big) + b(\mathcal{R}(\mathbf{z}_\Gamma),q^r_\Omega) = 0, \qquad (3.17c)$$

$$\begin{aligned}
&a_j(\mathcal{R}_{c_j}(\mathbf{e}),\mathbf{z}_j) + a_j(\mathcal{R}_{c_j}(\mathbf{e}),\mathcal{R}_j(\mathbf{z}_{\Gamma_j})) + \\
&\quad + b_j(\mathcal{R}_{c_j}(\mathbf{e}),r_j) - \alpha q_j(\mathbf{c},\mathbf{e}) + \qquad (3.17d)\\
&\quad - m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_i - \mathbf{u}_{D_i} - \mathcal{R}_{c_j}(\mathbf{c})),\mathbb{1}_{\Omega_0}\mathcal{R}_j(\mathbf{e})) = 0, \qquad (3.17e)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i,\mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),\mathbf{v}_i) + a_i(\mathbf{u}_i,\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathbf{v}_i,p_i) + \\
&\quad + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}),p_i)\Big) + a_j(\mathcal{R}_{c_j}(\mathbf{c}),\mathbf{v}_j) + a_j(\mathcal{R}_{c_j}(\mathbf{c}),\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + \\
&\quad + b(\mathcal{R}(\mathbf{v}_\Gamma),p_\Omega) + a(\mathcal{R}(\mathbf{u}_\Gamma),\mathcal{R}(\mathbf{v}_\Gamma)) \\
&= \sum_{i=1}^{s}\Big(f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i},\mathbf{v}_i)\Big) + f(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D,\mathcal{R}(\mathbf{v}_\Gamma)), \qquad (3.17f)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_i,q_i) + b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}),q_i)\Big) + \\
&\quad + b_j(\mathcal{R}_{c_j}(\mathbf{c}),q_j) + b_j(\mathcal{R}_{c_j}(\mathbf{c}),q_{\Omega_j}) + b(\mathcal{R}(\mathbf{u}_\Gamma),q_\Omega) \\
&= -\sum_{i=1}^{s}\Big(b_i(\mathbf{u}_{D_i},q_i)\Big) - b(\mathcal{R}(\mathbf{u}_D),q_\Omega) \qquad (3.17g)
\end{aligned}$$

for all $\mathbf{v}_i \in \mathbf{V}_i$, $\mathbf{v}_\Gamma, \mathbf{w}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, $\mathbf{w}_i \in \mathbf{V}_i^C$, $\mathbf{e} \in \mathbf{H}_{00}^1(\Gamma_C)$,$q_i \in Q_i$, $q_\Omega \in D(\Omega)$, $q_i^r \in Q_i^r$, $q_\Omega^r \in D^r(\Omega)$ *and* $q_j \in L^2(\Omega_j)$.

Proof. Use the same arguments as for the proof of Lemma 24. □

3.4.6 *Decoupling of Weak Formulation on Subdomains for Boundary Control*

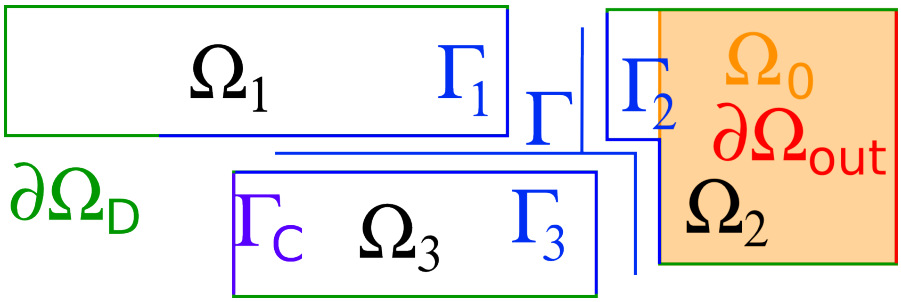


Figure 15: This figure illustrates the second sub-step of the derivation of the non-overlapping domain decomposition method in the boundary control case which yields s locally independent subproblems and one system of global coupling conditions. Subdomains Ω_1 is of the first type, a subdomain which only intersects global Dirichlet boundary, whereas subdomain Ω_2 is of type 2 and as before it contains the support for the desired state. As mentioned before, Ω_3 is the control subdomain and thus of type 3.

In the second sub-step of the second main step, we decouple the weak formulation on subdomains (3.17), which results in s independent local optimality systems coupled by one system of interface equations. Depending on the subdomain type, we have to distinguish four cases, see also Fig. 15 :

1. inner subdomains ($\partial\Omega_i \cap \partial\Omega = \emptyset$) and subdomains, which only intersect global Dirichlet boundary ($\partial\Omega_i \cap (\Gamma_C \cup \partial\Omega_{out}) = \emptyset$ and $\partial\Omega_i \cap \partial\Omega \neq \emptyset$),

2. outflow subdomains ($\partial\Omega_i \cap \partial\Omega_{out} \neq \emptyset$), which do not share control boundary ($\partial\Omega_i \cap \Gamma_C = \emptyset$),

3. control subdomains ($\partial\Omega_i \cap \Gamma_C \neq \emptyset$), which do not share outflow boundary ($\partial\Omega_i \cap \partial\Omega_{out} = \emptyset$),

4. subdomains, which share control and outflow boundary ($\partial\Omega_i \cap \Gamma_C \neq \emptyset$) and ($\partial\Omega_i \cap \partial\Omega_{out} \neq \emptyset$).

Since we made the simplification that the control boundary Γ_C lies completely on the boundary of subdomain Ω_j, either case 3 or case 4 can occur. For the sake of simplicity, we assume that the subdomain Ω_j does not share outflow boundary. Thus, we can neglect case 4.

For a subdomain of type 1 ($i \notin \mathcal{N}$ and $i \neq j$), the local optimality system is given by: For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i$, $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega)$ and $\widetilde{p}_i$, $\widetilde{r}_i \in L_0^2(\Omega_i)$ such that

$$
\begin{aligned}
&a_i(\mathbf{w}_i, \mathbf{z}_i) + b_i(\mathbf{w}_i, \widetilde{r}_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathbf{w}_i) && \text{(3.18a)}\\
&= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) - a_i(\mathbf{w}_i, \mathcal{R}_i(\mathbf{z}_{\Gamma_i})) + \\
&\quad + m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) && \text{(3.18b)}\\
&b_i(\mathbf{z}_i, \widetilde{q}_i) = -b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}), \widetilde{q}_i), && \text{(3.18c)}\\
&a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, \widetilde{p}_i) && \text{(3.18d)}\\
&= f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \\
&b_i(\mathbf{u}_i, \widetilde{q}_i) = -b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i) && \text{(3.18e)}
\end{aligned}
$$

for all $\mathbf{v}_i$, $\mathbf{w}_i \in \mathbf{H}_0^1(\Omega_i)$ and $\widetilde{q}_i \in L_0^2(\Omega_i)$.

We can reformulate (3.18) equivalently with an alternative formulation stated in the next proposition:

Proposition 28. *Under the same assumptions (2.24) and (3.10) as in Proposition 25 , (3.18) and the following local subdomain optimality system are equivalent:*

For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, *find* $\mathbf{u}_i$, $\mathbf{z}_i \in \mathbf{H}_0^1(\Omega)$, p_i, $r_i \in L^2(\Omega_i)$ *and* η_i, $\theta_i \in \mathbb{R}$ *such that*

$$
\begin{aligned}
&a_i(\mathbf{w}_i, \mathbf{z}_i) + b_i(\mathbf{w}_i, r_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathbf{w}_i) && \text{(3.19a)}\\
&= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) - a_i(\mathbf{w}_i, \mathcal{R}_i(\mathbf{z}_{\Gamma_i})) + \\
&\quad + m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) && \text{(3.19b)}\\
&b_i(\mathbf{z}_i, q_i) + c_i(\theta_i, q_i) = -b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}), q_i), && \text{(3.19c)}\\
&c_i(\xi_i, r_i) = 0, && \text{(3.19d)}\\
&a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) && \text{(3.19e)}\\
&= f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i), \\
&b_i(\mathbf{u}_i, q_i) + c_i(\eta_i, q_i) = -b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i), && \text{(3.19f)}\\
&c_i(\xi_i, p_i) = 0 && \text{(3.19g)}
\end{aligned}
$$

for all $\mathbf{v}_i$, $\mathbf{w}_i \in \mathbf{H}_0^1(\Omega_i)$, $q_i \in L^2(\Omega_i)$ *and* $\xi_i \in \mathbb{R}$.

For outflow subdomains ($i \in \mathcal{N}$ and $i \neq j$), the local optimality systems yield: For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{u}_i$, $\mathbf{z}_i \in \mathbf{H}_N^1(\Omega_i)$ and p_i, $r_i \in L^2(\Omega_i)$ such that

$$
\begin{aligned}
&a_i(\mathbf{w}_i, \mathbf{z}_i) + b_i(\mathbf{w}_i, r_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathbf{w}_i) && \text{(3.20a)}\\
&= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) - a_i(\mathbf{w}_i, \mathcal{R}_i(\mathbf{z}_{\Gamma_i})) + \\
&\quad + m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) && \text{(3.20b)}\\
&b_i(\mathbf{z}_i, q_i) = -b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}), q_i), && \text{(3.20c)}
\end{aligned}
$$

$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) \tag{3.20d}$$
$$= f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i),$$
$$b_i(\mathbf{u}_i, q_i) = -b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i), \tag{3.20e}$$

for all $\mathbf{v}_i$, $\mathbf{w}_i \in \mathbf{H}^1_N(\Omega_i)$ and $q_i \in L^2(\Omega_i)$.

On the control subdomain Ω_j, we obtain the following local optimality system: For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$, find $\mathbf{u}_i \in \mathbf{H}^1_0(\Omega)\mathbf{z}_i \in \mathbf{H}^1_C(\Omega_i)$, $\mathbf{c} \in \mathbf{H}^1_{00}(\Gamma_C)$ $\widetilde{p}_i \in L^2_0(\Omega_i)$ and $r_i \in L^2(\Omega_i)$ such that

$$a_i(\mathbf{w}_i, \mathbf{z}_i) + b_i(\mathbf{w}_i, r_i) - m_i(\mathbb{1}_{\Omega_0}(\mathbf{u}_i - \mathcal{R}_{c_i}(\mathbf{c})), \mathbb{1}_{\Omega_0}\mathbf{w}_i) \tag{3.21a}$$
$$= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) - a_i(\mathbf{w}_i, \mathcal{R}_i(\mathbf{z}_{\Gamma_i}))+$$
$$+ m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) \tag{3.21b}$$
$$b_i(\mathbf{z}_i, q_i) = -b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}), q_i), \tag{3.21c}$$
$$a_i(\mathcal{R}_{c_i}(\mathbf{e}), \mathbf{z}_i) + b_i(\mathcal{R}_{c_i}(\mathbf{e}), r_i) - \alpha q_i(\mathbf{c}, \mathbf{e})+$$
$$- m_i(\mathbb{1}_{\Omega_0}(\mathbf{u}_i + \mathcal{R}_{c_i}(\mathbf{c})), \mathbb{1}_{\Omega_0}\mathcal{R}_{c_i}(\mathbf{e})) \tag{3.21d}$$
$$= -a_i(\mathcal{R}_{c_i}(\mathbf{e}), \mathcal{R}_i(\mathbf{z}_{\Gamma_i})) - m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathcal{R}_{c_i}(\mathbf{e})) \tag{3.21e}$$
$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, \widetilde{p}_i) + a_i(\mathcal{R}_{c_i}(\mathbf{c}), \mathbf{v}_i) \tag{3.21f}$$
$$= f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i),$$
$$b_i(\mathbf{u}_i, \widetilde{q}_i) + b(\mathcal{R}_{c_i}(\mathbf{c}), \widetilde{q}_i) = -b_i(\mathbf{u}_{D_i}, \widetilde{q}_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \widetilde{q}_i) \tag{3.21g}$$

for all $\mathbf{v}_i$, $\mathbf{w}_i \in \mathbf{H}^1_0(\Omega_i)$, $\mathbf{e} \in \mathbf{H}^1_{00}(\Gamma_C)$ and $\widetilde{q}_i \in L^2_0(\Omega_i)$.

We can reformulate (3.21) equivalently with an alternative formulation stated in the next proposition:

Proposition 29. *Under the assumption (2.24), (3.21) and the following local subdomain optimality system are equivalent: For given $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$, find $\mathbf{u}_i$, $\mathbf{z}_i \in \mathbf{H}^1_0(\Omega)$, $\mathbf{c} \in \mathbf{H}^1_{00}(\Gamma_C)$ p_i, $r_i \in L^2(\Omega_i)$ and $\eta_i \in \mathbb{R}$ such that*

$$a_i(\mathbf{w}_i, \mathbf{z}_i) + b_i(\mathbf{w}_i, r_i) - m_i(\mathbb{1}_{\Omega_0}(\mathbf{u}_i - \mathcal{R}_{c_i}(\mathbf{c})), \mathbb{1}_{\Omega_0}\mathbf{w}_i) \tag{3.22a}$$
$$= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) - a_i(\mathbf{w}_i, \mathcal{R}_i(\mathbf{z}_{\Gamma_i}))+$$
$$+ m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbb{1}_{\Omega_0}\mathbf{w}_i) \tag{3.22b}$$
$$b_i(\mathbf{z}_i, q_i) = -b_i(\mathcal{R}_i(\mathbf{z}_{\Gamma_i}), q_i), \tag{3.22c}$$
$$a_i(\mathcal{R}_{c_i}(\mathbf{e}), \mathbf{z}_i) + b_i(\mathcal{R}_{c_i}(\mathbf{e}), r_i) - \alpha q_i(\mathbf{c}, \mathbf{e})+$$
$$- m_i(\mathbb{1}_{\Omega_0}(\mathbf{u}_i + \mathcal{R}_{c_i}(\mathbf{c})), \mathbb{1}_{\Omega_0}\mathcal{R}_{c_i}(\mathbf{e})) \tag{3.22d}$$
$$= -a_i(\mathcal{R}_{c_i}(\mathbf{e}), \mathcal{R}_i(\mathbf{z}_{\Gamma_i})) - m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathcal{R}_{c_i}(\mathbf{e})) \tag{3.22e}$$
$$a_i(\mathbf{u}_i, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i) + a_i(\mathcal{R}_{c_i}(\mathbf{c}), \mathbf{v}_i) \tag{3.22f}$$
$$= f(\mathbf{v}_i) - a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) - a_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), \mathbf{v}_i),$$
$$b_i(\mathbf{u}_i, q_i) + c_i(\eta_i, q_i) + b(\mathcal{R}_{c_i}(\mathbf{c}_i), q_i)$$
$$= -b_i(\mathbf{u}_{D_i}, q_i) - b_i(\mathcal{R}_i(\mathbf{u}_{\Gamma_i}), q_i) \tag{3.22g}$$
$$c_i(\xi_i, p_i) = 0 \tag{3.22h}$$

for all $\mathbf{v}_i$, $\mathbf{w}_i \in \mathbf{H}^1_0(\Omega_i)$, $\mathbf{e} \in \mathbf{H}^1_{00}(\Gamma_C)$ *and* $\widetilde{q}_i \in L^2_0(\Omega_i)$.

Analogously to the distributed control case, we are interested in solving the global system (3.6) by solving coupling conditions on the interface (3.23). The next lemma states, under which conditions we get a solution of the global system by solving the interface equations.

Lemma 30. It holds that

$$\begin{aligned}
\mathbf{u} &= \sum_{i=1}^{s} \mathbf{u}_i + \mathcal{R}(\mathbf{u}_\Gamma) \in \mathbf{H}^1_D(\Omega) && \text{with } \mathbf{u}_i \in \mathbf{V}_i,\\
\mathbf{z} &= \sum_{i=1}^{s} \mathbf{z}_i + \mathcal{R}(\mathbf{z}_\Gamma) \in \mathbf{H}^1_D(\Omega) && \text{with } \mathbf{z}_i \in \mathbf{V}_i,\\
p &= \sum_{i=1}^{s} p_i + p_\Omega \in L^2(\Omega) && \text{with } p_i \in Q_i,\\
r &= \sum_{i=1}^{s} r_i + r_\Omega \in L^2(\Omega) && \text{with } r_i \in Q^r_i\\
\mathbf{c} &\in \mathbf{H}^1_{00}(\Gamma_C)
\end{aligned}$$

with $(\mathbf{u}_i, \mathbf{z}_i, \mathbf{c}, p_i, r_i) \in \mathbf{V}_i \times \mathbf{V}^C_i \times \mathbf{H}^1_{00}(\Gamma_C) \times Q_i \times Q^r_i$ *solutions of (3.19), (3.20) or (3.22) (depending on the subdomain type) solve (3.6) if and only if the following coupling conditions for* $(\mathbf{u}_\Gamma, \mathbf{z}_\Gamma, p_\Omega, r_\Omega) \in \mathbf{H}^{1/2}_{00}(\Gamma) \times \mathbf{H}^{1/2}_{00}(\Gamma) \times D(\Omega) \times D^r(\Omega)$ *hold: Find* $\mathbf{u}_\Gamma$, $\mathbf{z}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$, $r_\Omega \in D^r(\Omega)$ *and* $p_\Omega \in D(\Omega)$, *such that*

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathcal{R}_i(\mathbf{w}_{\Gamma_i}), \mathbf{z}_i) + b(\mathcal{R}_i(\mathbf{w}_{\Gamma_i}), r_i) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i, \mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{w}_{\Gamma_i}))\Big) + \\
&\quad - m_j(\mathcal{R}_{c_j}(\mathbf{c}), \mathbb{1}_{\Omega_0}\mathcal{R}_j(\mathbf{w}_{\Gamma_j})) + && \text{(3.23a)}\\
&\quad + a(\mathcal{R}(\mathbf{w}_\Gamma), \mathcal{R}(\mathbf{z}_\Gamma)) + b(\mathcal{R}(\mathbf{w}_\Gamma), r_\Omega) - m(\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{u}_\Gamma), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{w}_\Gamma))\\
&= -m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u}_D), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{w}_\Gamma)), && \text{(3.23b)}\\
&b(\mathcal{R}(\mathbf{z}_\Gamma), q^r_\Omega)\Big) = 0, && \text{(3.23c)}\\
&\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), p_i)\Big) + a_j(\mathcal{R}_{c_j}(\mathbf{c}), \mathcal{R}(\mathbf{v}_{\Gamma_j})) + \\
&\quad + b(\mathcal{R}(\mathbf{v}_\Gamma), p_\Omega) + a(\mathcal{R}(\mathbf{u}_\Gamma), \mathcal{R}(\mathbf{v}_\Gamma))\\
&= f(\mathcal{R}(\mathbf{v}_\Gamma)) - a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)), && \text{(3.23d)}\\
&b(\mathcal{R}(\mathbf{u}_\Gamma), q_\Omega) + b_j(\mathcal{R}_{c_j}(\mathbf{c}), q_\Omega) = -b(\mathbf{u}_D, q_\Omega) && \text{(3.23e)}
\end{aligned}$$

for all $\mathbf{v}_\Gamma$, $\mathbf{w}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$, $q^r_\Omega \in D^r(\Omega)$ *and* $q_\Omega \in D(\Omega)$.

Proof. This can be shown by using the same arguments as in Chapter 2 in the proof of Lemma 13. □

3.5 DISCRETIZATION BASED ON A FINITE ELEMENT METHOD FOR DISTRIBUTED CONTROL

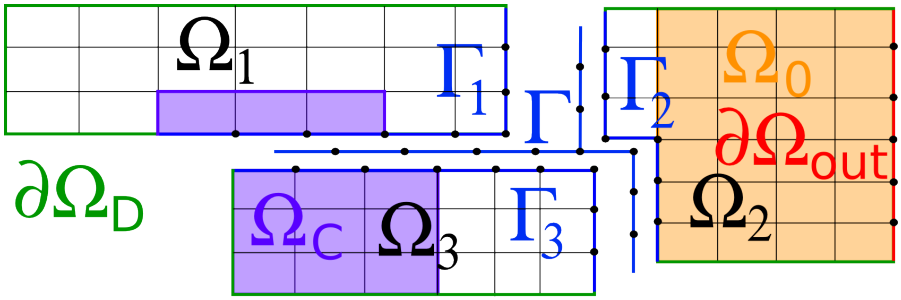


Figure 16: In this figure the third main step of the derivation of the domain decomposition method in case of distributed control is sketched. We discretize the decoupled weak formulations by an appropriate finite element method. The figure also shows the restriction made for the triangulation $\mathcal{T}_h$.

For the triangulation $\mathcal{T}_h$ defined in Chapter 2, we make some further restrictions. By $\mathring{K}$, we denote the interior of an element. It holds that

$$\begin{aligned} K \cap \Omega_0 \neq \emptyset \quad &\Rightarrow \mathring{K} \subset \Omega_0 \quad \forall K \in \mathcal{T}_h, \\ K \cap \Omega_C \neq \emptyset \quad &\Rightarrow \mathring{K} \subset \Omega_C \quad \forall K \in \mathcal{T}_h, \end{aligned}$$

see also Fig. 16 for an illustration.

Analogously to the previous chapter, we use Taylor-Hood elements for the adjoint velocity and adjoint pressure, because the structure of state equations and the adjoint equation is the same. Since the continuous spaces for the distributed control and the pressure only differ in the dimension, we use the same type of discretization for the distributed control as for the pressure. The control is a vector variable of dimension d and the pressure is a scalar variable. To simplify the definition of the finite element space for the distributed control, we denote by $K_C = \{K \in \mathcal{T}_h : K \in \Omega_C\}$ the elements in the triangulation which lie inside the control domain. Hence, we get the following local finite element space $\mathbf{Q}_i^h$ for the distributed control:

$$\mathbf{Q}_i^h = \{\mathbf{e}_i^h \in \left[Q_i^h\right]^d : \mathbf{e}_i^h|_K = 0 \quad \forall K \notin K_C\} \subset \mathbf{L}^2(\Omega_i).$$

3.5.1 *Decoupled Finite Element Formulation*

Using the definitions for the finite element method, we discretize the s decoupled subdomain weak formulations and the coupling condition on the interface. This is the third main step in the derivation of the domain decomposition method, see also Fig. 16.

On non-outflow subdomains, we get the following finite element discretization of the local optimality system: For given $\mathbf{u}_\Gamma^h$, $\mathbf{z}_\Gamma^h \in \mathbf{V}_\Gamma^h$, find $\mathbf{u}_i^h$, $\mathbf{z}_i^h \in \mathbf{V}_{i,0}^h$, $\mathbf{c}_i^h \in \mathbf{Q}_i^h$, p_i^h, $r_i^h \in Q_i^h$ and η_i^h, $\theta_i^h \in \mathbb{R}$, such that

$$
\begin{aligned}
&a_i(\mathbf{v}_i^h, \mathbf{z}_i^h) + b_i(\mathbf{v}_i^h, r_i^h) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i^h, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h) && \text{(3.24a)}\\
&= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h|_{\Omega_i} - \mathbf{u}_{D_i}^h), \mathbb{1}_{\Omega_0}\mathbf{v}_i^h) - a_i(\mathbf{v}_i^h, \mathbf{z}_{\Gamma_i}^h) + m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_{\Gamma_i}^h, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h)\\
&b_i(\mathbf{z}_i^h, q_i^h) + c_i(\theta_i^h, q_i^h) = -b_i(\mathbf{z}_{\Gamma_i}^h, q_i^h) && \text{(3.24b)}\\
&c_i(\xi_i^h, r_i^h) = 0 && \text{(3.24c)}\\
&d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i^h, \mathbb{1}_{\Omega_C}\mathbf{z}_i^h) + \alpha m_i(\mathbf{c}_i^h, \mathbf{e}_i^h) = -d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i^h, \mathbb{1}_{\Omega_C}\mathbf{z}_{\Gamma_i}^h) && \text{(3.24d)}\\
&a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i^h, \mathbb{1}_{\Omega_C}\mathbf{v}_i^h)\\
&= f(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h) && \text{(3.24e)}\\
&b_i(\mathbf{u}_i^h, q_i^h) + c_i(\eta_i^h, q_i^h) = -b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h) && \text{(3.24f)}\\
&c_i(\xi_i^h, p_i^h) = 0 && \text{(3.24g)}
\end{aligned}
$$

for all $\mathbf{v}_i^h \in \mathbf{V}_{i,0}^h$, $\mathbf{e}_i^h \in \mathbf{Q}_i^h$, $q_i^h \in Q_i^h$ and $\xi_i^h \in \mathbb{R}$.

On outflow subdomains, the finite element discretization yields: For given $\mathbf{u}_\Gamma^h$, $\mathbf{z}_\Gamma^h \in \mathbf{V}_\Gamma^h$, find $\mathbf{u}_i^h$, $\mathbf{z}_i^h \in \mathbf{V}_{i,N}^h$,$\mathbf{c}_i^h \in \mathbf{Q}_i^h$ and p_i^h, $r_i^h \in Q_i^h$ such that

$$
\begin{aligned}
&a_i(\mathbf{v}_i^h, \mathbf{z}_i^h) + b_i(\mathbf{v}_i^h, r_i^h) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i^h, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h) && \text{(3.25a)}\\
&= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{v}_i^h) - a_i(\mathbf{v}_i^h, \mathbf{z}_{\Gamma_i}^h) + m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_{\Gamma_i}^h, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h)\\
&b_i(\mathbf{z}_i^h, q_i^h) = -b_i(\mathbf{z}_{\Gamma_i}^h, q_i^h) && \text{(3.25b)}\\
&d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i^h, \mathbb{1}_{\Omega_C}\mathbf{z}_i^h) + \alpha m_i(\mathbf{c}_i^h, \mathbf{e}_i^h) = -d_i(\mathbb{1}_{\Omega_C}\mathbf{e}_i^h, \mathbb{1}_{\Omega_C}\mathbf{z}_{\Gamma_i}^h) && \text{(3.25c)}\\
&a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i^h, \mathbb{1}_{\Omega_C}\mathbf{v}_i^h) && \text{(3.25d)}\\
&= f(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h)\\
&b_i(\mathbf{u}_i^h, q_i^h) = -b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h) && \text{(3.25e)}
\end{aligned}
$$

for all $\mathbf{v}_i^h \in \mathbf{V}_i^h$, $\mathbf{e}_i^h \in \mathbf{Q}_i^h$ and $q_i^h \in Q_i^h$.

For the finite element formulation of the coupling condition, we obtain: Find $\mathbf{u}_\Gamma^h$, $\mathbf{z}_\Gamma^h \in \mathbf{V}_\Gamma^h$ and p_Ω^h, $r_\Omega^h \in D(\Omega)$ such that

$$
\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathbf{v}_{\Gamma_i}^h, \mathbf{z}_i^h) + b_i(\mathbf{v}_{\Gamma_i}^h, r_i^h) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i^h, \mathbb{1}_{\Omega_0}\mathbf{v}_{\Gamma_i}^h)\Big) +\\
&\quad + a(\mathbf{v}_\Gamma^h, \mathbf{z}_\Gamma^h) + b(\mathbf{v}_\Gamma^h, r_\Omega^h) - m(\mathbb{1}_{\Omega_0}\mathbf{u}_\Gamma^h, \mathbb{1}_{\Omega_0}\mathbf{v}_\Gamma^h)\\
&= \sum_{i=1}^{s} -m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h - \mathbf{u}_D^h, \mathbb{1}_{\Omega_0}\mathbf{v}_\Gamma^h), && \text{(3.26a)}\\
&b(\mathbf{z}_\Gamma^h, q_\Omega^h) = 0, && \text{(3.26b)}\\
&\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i^h, \mathbf{v}_{\Gamma_i}^h) + b_i(\mathbf{v}_{\Gamma_i}^h, p_i^h) - d_i(\mathbb{1}_{\Omega_C}\mathbf{c}_i^h, \mathbb{1}_{\Omega_C}\mathbf{v}_{\Gamma_i}^h)\Big) +
\end{aligned}
$$

$$+ a(\mathbf{u}_\Gamma^h, \mathbf{v}_\Gamma^h) + b(\mathbf{v}_\Gamma^h, p_\Omega^h) = f(\mathbf{v}_\Gamma^h) - a(\mathbf{u}_D^h, \mathbf{v}_\Gamma^h), \tag{3.26c}$$

$$b(\mathbf{u}_\Gamma^h, q_\Omega^h) = -b(\mathbf{u}_D^h, q_\Omega^h) \tag{3.26d}$$

for all $\mathbf{v}_\Gamma^h \in \mathbf{V}_\Gamma^h$ and $q_\Omega \in D(\Omega)$. $\mathbf{u}_i^h$, $\mathbf{z}_i^h \in \mathbf{V}_i^h$, $\mathbf{c}_i^h \in \mathbf{Q}_i^h$, p_i^h, $r_i^h \in Q_i^h$ are solutions of (3.24) and (3.25), depending on if a subdomain Ω_i is a non-outflow or an outflow subdomain, respectively.

3.6 DISCRETIZATION BASED ON A FINITE ELEMENT METHOD FOR BOUNDARY CONTROL

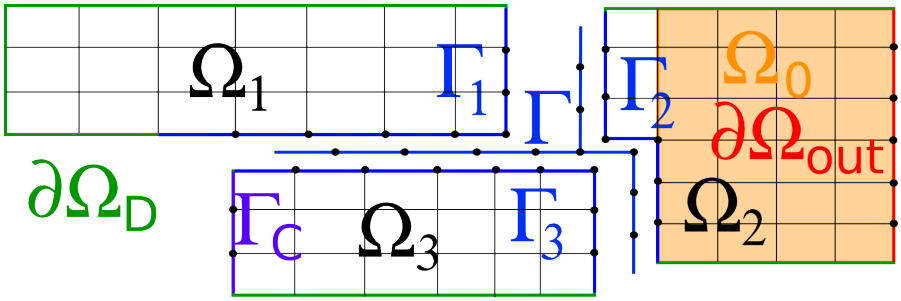


Figure 17: This figure illustrated the triangulation in case of boundary control. It also refers to the third main step of the derivation of the domain decomposition, in which we discretize the decoupled formulation with an appropriate finite element method.

We use the same triangulation $\mathcal{T}_h$ as defined for the distributed control case. Analogously to the distributed control case, we use Taylor-Hood elements for the adjoint velocity and adjoint pressure. For the boundary control, we use the same discretization as for the velocity on the interface, with the difference, that it is applied on the control boundary Γ_C.

Hence, we get the following finite element space for the boundary control:

$$\mathbf{V}_{\Gamma_C}^h = \{\mathbf{e}_j^h \in \widehat{\mathbf{V}}_j^h : \mathbf{e}_j^h(x) = 0 \quad \forall x \notin \Gamma_C\}.$$

Remark 31. By defining local control space as a restriction of the space $\widehat{\mathbf{V}}_j^h$, the control boundary functions are implicitly extended to the neighboring elements of the triangulation which are directly connected to the interface. Thus, we implicitly use the finite element functions as extension operator into the subdomain. Therefore, we do not need to explicitly define a discrete extension operator which corresponds to $\mathcal{R}_{C_i}$.

3.6.1 *Decoupled Finite Element Formulation*

Using the definitions for the finite element method, we discretize the s decoupled subdomain weak formulations and the coupling condition on the interface. This third main step of the derivation is also depicted in Fig. 17.

On inner subdomains or subdomains only intersecting global Dirichlet boundary, we obtain the following finite element discretization of the local optimality system: For given $\mathbf{u}_\Gamma^h, \mathbf{z}_\Gamma^h \in \mathbf{V}_\Gamma^h$, find $\mathbf{u}_i^h$, $\mathbf{z}_i^h \in \mathbf{V}_{i,0}^h$, p_i^h, $r_i^h \in Q_i^h$, and η_i^h, $\theta_i^h \in \mathbb{R}$ such that

$$\begin{aligned} &a_i(\mathbf{w}_i^h, \mathbf{z}_i^h) + b_i(\mathbf{w}_i^h, r_i^h) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i^h, \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) \\ &= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h|_{\Omega_i} - \mathbf{u}_{D_i}^h), \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) - a_i(\mathbf{w}_i^h, \mathbf{z}_{\Gamma_i}^h) + m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_{\Gamma_i}^h, \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) \end{aligned} \tag{3.27a}$$

$$b_i(\mathbf{z}_i^h, q_i^h) + c_i(\theta_i^h, q_i^h) = -b_i(\mathbf{z}_{\Gamma_i}^h, q_i^h), \tag{3.27b}$$

$$c_i(\xi_i^h, r_i^h) = 0, \tag{3.27c}$$

$$a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) = f(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h), \tag{3.27d}$$

$$b_i(\mathbf{u}_i^h, q_i^h) + c_i(\eta_i^h, q_i^h) = -b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h) \tag{3.27e}$$

$$c_i(\xi_i^h, p_i^h) = 0 \tag{3.27f}$$

for all $\mathbf{v}_i^h$, $\mathbf{w}_i^h \in \mathbf{V}_i^h$, $q_i^h \in Q_i^h$, and $\xi_i^h \in \mathbb{R}$.

On outflow subdomains ($i \in \mathcal{N}$, $i \neq j$), the finite element discretization is given by: For given $\mathbf{u}_\Gamma^h, \mathbf{z}_\Gamma^h \in \mathbf{V}_\Gamma^h$, find $\mathbf{u}_i^h$, $\mathbf{z}_i^h \in \mathbf{V}_{i,N}^h$, p_i^h, $r_i^h \in Q_i^h$, and η_i^h, $\theta_i^h \in \mathbb{R}$ such that

$$\begin{aligned} &a_i(\mathbf{w}_i^h, \mathbf{z}_i^h) + b_i(\mathbf{w}_i^h, r_i^h) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i^h, \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) \\ &= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h|_{\Omega_i} - \mathbf{u}_{D_i}^h), \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) - a_i(\mathbf{w}_i^h, \mathbf{z}_{\Gamma_i}^h) + m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_{\Gamma_i}^h, \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) \end{aligned} \tag{3.28a}$$

$$b_i(\mathbf{z}_i^h, q_i^h) = -b_i(\mathbf{z}_{\Gamma_i}^h, q_i^h), \tag{3.28b}$$

$$a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) = f(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h), \tag{3.28c}$$

$$b_i(\mathbf{u}_i^h, q_i^h) = -b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h) \tag{3.28d}$$

for all $\mathbf{v}_i^h$, $\mathbf{w}_i^h \in \mathbf{V}_{i,N}^h$, $q_i^h \in Q_i^h$.

On the control subdomain Ω_j, the finite element discretization of the local optimality system yields: For given $\mathbf{u}_\Gamma^h, \mathbf{z}_\Gamma^h \in \mathbf{V}_\Gamma^h$, find $\mathbf{u}_i^h$, $\mathbf{z}_i^h \in \mathbf{V}_{i,0}^h$, $\mathbf{c}^h \in \mathbf{V}_{\Gamma_C}^h$, p_i^h, $r_i^h \in Q_i^h$ and $\eta_i^h \in \mathbb{R}$ such that

$$\begin{aligned} &a_i(\mathbf{w}_i^h, \mathbf{z}_i^h) + b_i(\mathbf{w}_i^h, r_i^h) - m_i(\mathbb{1}_{\Omega_0}(\mathbf{u}_i^h + \mathbf{c}^h), \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) \\ &= -m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h|_{\Omega_i} - \mathbf{u}_{D_i}^h), \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) - a_i(\mathbf{w}_i^h, \mathbf{z}_{\Gamma_i}^h) + m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_{\Gamma_i}^h, \mathbb{1}_{\Omega_0}\mathbf{w}_i^h) \end{aligned} \tag{3.29a}$$

$$b_i(\mathbf{z}_i^h, q_i^h) = -b_i(\mathbf{z}_{\Gamma_i}^h, q_i^h), \tag{3.29b}$$

$$\begin{aligned} &a_i(\mathbf{e}^h, \mathbf{z}_i^h) + b_i(\mathbf{e}^h, r_i^h) - \alpha q(\mathbf{c}^h, \mathbf{e}^h) - m_i(\mathbb{1}_{\Omega_0}(\mathbf{u}_i^h - \mathbf{c}^h), \mathbf{e}^h) \\ &= -a_i(\mathbf{e}^h, \mathbf{z}_{\Gamma_i}^h) - m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h|_{\Omega_i} - \mathbf{u}_{D_i}^h), \mathbf{e}^h) \end{aligned} \tag{3.29c}$$

$$\begin{aligned} &a_i(\mathbf{u}_i^h, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, p_i^h) + a_i(\mathbf{c}^h, \mathbf{v}_i^h) = f(\mathbf{v}_i^h) - a_i(\mathbf{u}_{D_i}^h, \mathbf{v}_i^h) - a_i(\mathbf{u}_{\Gamma_i}^h, \mathbf{v}_i^h), \\ &b_i(\mathbf{u}_i^h, q_i^h) + c_i(\eta_i^h, q_i^h) + b_i(\mathbf{c}^h, q_i^h) = -b_i(\mathbf{u}_{D_i}^h, q_i^h) - b_i(\mathbf{u}_{\Gamma_i}^h, q_i^h) \end{aligned} \tag{3.29d}$$

$$c_i(\xi_i^h, p_i^h) = 0 \tag{3.29e}$$

for all $\mathbf{v}_i \in \mathbf{V}_{i,0}^h$, $\mathbf{w}_i \in \mathbf{V}_{i,C}^h$, $\mathbf{e} \in \mathbf{V}_{\Gamma_C}^h$, $q_i^h \in Q_i^h$ and $\xi_i^h \in \mathbb{R}$.

The finite element formulation of the coupling condition reads: Find $\mathbf{u}_\Gamma^h$, $\mathbf{z}_\Gamma^h \in \mathbf{V}_\Gamma^h$, $p_\Omega^h \in D(\Omega)$ and $r_\Omega^h \in D^r(\Omega)$ such that

$$\begin{aligned} &\sum_{i=1}^{s}\Big(a_i(\mathbf{w}_{\Gamma_i}^h, \mathbf{z}_i^h) + b_i(\mathbf{w}_{\Gamma_i}^h, r_i^h) - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i^h, \mathbb{1}_{\Omega_0}\mathbf{w}_{\Gamma_i}^h)\Big) + \\ &\quad - m_j(\mathbf{c}^h, \mathbb{1}_{\Omega_0}\mathbf{w}_{\Gamma_i}^h) + \\ &\quad + a(\mathbf{w}_\Gamma^h, \mathbf{z}_\Gamma^h) + b(\mathbf{w}_\Gamma^h, r_\Omega^h) - m(\mathbb{1}_{\Omega_0}\mathbf{u}_\Gamma^h, \mathbb{1}_{\Omega_0}\mathbf{w}_\Gamma^h) \\ &= -m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}^h - \mathbf{u}_D^h), \mathbb{1}_{\Omega_0}\mathbf{w}_\Gamma^h), \end{aligned} \tag{3.30}$$

$$b(\mathbf{z}_\Gamma^h), q_{\Omega,r}^h) = 0, \tag{3.31}$$

$$\begin{aligned} &\sum_{i=1}^{s}\Big(a_i(\mathbf{u}_i^h, \mathbf{v}_{\Gamma_i}^h) + b_i(\mathbf{v}_{\Gamma_i}^h, p_i^h)\Big) + a_j(\mathbf{c}^h, \mathbf{v}_{\Gamma_j}^h) + \\ &\quad + a(\mathbf{u}_\Gamma^h), \mathbf{v}_\Gamma^h) + b(\mathbf{v}_\Gamma^h, p_\Omega^h) = f(\mathbf{v}_\Gamma^h) - a(\mathbf{u}_D^h, \mathbf{v}_\Gamma^h), \end{aligned} \tag{3.32}$$

$$b(\mathbf{u}_\Gamma^h, q_\Omega^h) + b_j(\mathbf{c}^h, q_\Omega^h) = -b(\mathbf{u}_D^h, q_\Omega^h) \tag{3.33}$$

for all $\mathbf{v}_\Gamma^h$, $\mathbf{w}_\Gamma^h \in \mathbf{V}_\Gamma^h$, $q_\Omega^h \in D(\Omega)$ and $q_{\Omega,r}^h \in D^r(\Omega)$. $\mathbf{u}_i^h$, $\mathbf{z}_i^h \in \mathbf{V}_i^h$, $\mathbf{c} \in \mathbf{V}_{\Gamma_C}^h$ and p_i^h, $r_i^h \in Q_i^h$ are solutions of (3.27), (3.28) and (3.29) depending on the subdomain type.

3.7 ALGEBRAIC FORMULATION

Analogously to Chapter 2, we derive the algebraic representation for the velocity and pressure, the adjoint velocity and adjoint pressure, and the control. We give the algebraic representation in the form of a global linear system and a Schur-complement formulation on the skeleton, which is the last main step of the derivation.

First , we give an overview for the numbering of degrees of freedom:

- n_{u_i} denotes the number of the inner degrees of freedom of the velocity $\mathbf{u}$ in the subdomain Ω_i. If Ω_i is an outflow subdomain, it also includes the degrees of freedom on the outflow boundary $\partial\Omega_{out} \cap \Gamma_i$.
- n_{u_Γ} denotes the number of global skeleton degrees of freedom of the velocity,
- $n_{u_{\Gamma_i}}$ the number of local skeleton degrees of freedom of the velocity,

- $n_{u_{c_i}}$ the number of inner velocity degrees of freedom in Ω_{C_i} and
- $n_{u_{c_{\Gamma_i}}}$ the number of local skeleton degrees of freedom of the velocity on Γ_i.
- n_{z_i} is defined analogously to n_{u_i} for the adjoint velocity. It holds that $n_{z_i} = n_{u_i}$.
- n_{q_i} denotes the number of degrees of freedom of the pressure and adjoint pressure variable in Ω_i.
- $n_{u_{0_i}}$ denotes the inner degrees of freedom of the velocity components in $\Omega_0 \cap \Omega_i$ and
- $n_{u_{0_{\Gamma_i}}}$ the local skeleton degrees of freedom of the velocity components in $\Omega_0 \cap \partial\Omega_i$.
- n_{c_i} denotes the number of degrees of freedom of the control variable in $\Omega_C \cap \Omega_i$ and
- n_{c_Γ} the number of degrees of freedom on Γ_C .

3.7.1 *Definitions for the Distributed Control Case*

In the case of distributed control, we use the same ansatz functions for the velocity and adjoint velocity, and for the pressure and adjoint pressure, respectively. Those were already defined in Chapter 2. Hence, we only need the definitions for the distributed control.

Let $\boldsymbol{\mu}_j \in \mathbf{Q}_i^h$, $(j = 1, \ldots, n_{c_i})$ denote the piecewise linear basis functions for the control such that

$$\mathbf{Q}_i^h = \operatorname{span}\left\{\boldsymbol{\mu}_j \,:\, j = 1, \ldots, n_{c_i}\right\}.$$

Moreover, we define the projection matrix $\mathbf{R}_{\Gamma_{0,i}} \in \{0,1\}^{n_{u_{0_{\Gamma_i}}} \times n_{u_{\Gamma_i}}}$ that projects the local skeleton degrees of freedom of the velocity to the local skeleton degrees of freedom which also lie in Ω_0.

In addition to the matrices defined in Chapter 2, we define the following matrices and their entries, which correspond to the bilinear form $m_i(\cdot,\cdot)$ and $d_i(\cdot,\cdot)$, respectively:

$$\begin{aligned}
(\mathbf{M}_0)_{kl} &= m_i(\mathbb{1}_0\boldsymbol{\psi}_l, \mathbb{1}_0\boldsymbol{\psi}_k),\\
(\mathbf{M}_C)_{kl} &= m_i(\mathbb{1}_C\boldsymbol{\mu}_l, \mathbb{1}_C\boldsymbol{\mu}_k),\\
(\mathbf{D}_C)_{kl} &= m_i(\mathbb{1}_C\boldsymbol{\mu}_l, \mathbb{1}_C\boldsymbol{\psi}_k) :
\end{aligned}$$

$$
\begin{aligned}
&(\mathbf{M}_0)_{ii} \in \mathbb{R}^{n_{u_{0_i}} \times n_{u_{0_i}}} && (\mathbf{M}_0)_{kl} && k,l = 1,\dots,n_{u_{0_i}},\\
&(\mathbf{M}_0)_{i\Gamma_i} \in \mathbb{R}^{n_{u_{0_i}} \times n_{u_{0_{\Gamma_i}}}} && (\mathbf{M}_0)_{kl} && k = 1,\dots,n_{u_{0_i}},\ l = 1,\dots,n_{u_{0_{\Gamma_i}}}\\
&(\mathbf{M}_0)_{\Gamma_i i} \in \mathbb{R}^{n_{u_{0_{\Gamma_i}}} \times n_{u_{0_i}}} && (\mathbf{M}_0)_{kl} && k = 1,\dots,n_{u_{0_{\Gamma_i}}},\ l = 1,\dots,n_{u_{0_{\Gamma_i}}},\\
&(\mathbf{M}_0)_{\Gamma_i\Gamma_i} \in \mathbb{R}^{n_{u_{0_{\Gamma_i}}} \times n_{u_{0_{\Gamma_i}}}} && (\mathbf{M}_0)_{kl} && k,l = 1,\dots,n_{u_{0_{\Gamma_i}}},\\
&(\mathbf{M}_0)_{\Gamma\Gamma} \in \mathbb{R}^{n_{u_{0\Gamma}} \times n_{u_{0\Gamma}}} && (\mathbf{M}_0)_{\Gamma\Gamma} && = \textstyle\sum_{i=1}^{s} \mathbf{R}_{\Gamma_{0,i}}^T (\mathbf{M}_0)_{\Gamma_i\Gamma_i} \mathbf{R}_{\Gamma_{0,i}}\\
&(\mathbf{M}_C)_{ii} \in \mathbb{R}^{n_{c_i} \times n_{c_i}} && (\mathbf{M}_C)_{kl} && k,l = 1,\dots,n_{c_i},\\
&(\mathbf{D}_C)_{ii} \in \mathbb{R}^{n_{c_i} \times n_{u_{c_i}}} && (\mathbf{D}_C)_{kl} && k = 1,\dots,n_{c_i},\ l = 1,\dots,n_{u_{c_i}},\\
&(\mathbf{D}_C)_{\Gamma_i i} \in \mathbb{R}^{n_{u_{c_{\Gamma_i}}} \times n_{c_i}} && (\mathbf{D}_C)_{kl} && k = 1,\dots,n_{u_{c_{\Gamma_i}}},\ l = 1,\dots,n_{c_i}.
\end{aligned}
$$

Furthermore, we define the following finite element functions and their coefficient vectors for the adjoint velocity

$$
\begin{aligned}
\mathbf{z}_i^h(\mathbf{x}) &= \sum_{j=1}^{n_{z_i}} z_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{z}_i = (z_1,\dots,z_{n_{z_i}})^T \in \mathbb{R}^{n_{z_i}},\\
\mathbf{z}_\Gamma^h(\mathbf{x}) &= \sum_{j=1}^{n_{z_\Gamma}} z_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{z}_\Gamma = (z_1,\dots,z_{n_{z_\Gamma}})^T \in \mathbb{R}^{n_{z_\Gamma}},\\
\mathbf{z}_{\Gamma_i}^h(\mathbf{x}) &= \sum_{j=1}^{n_{z_{\Gamma_i}}} z_j \boldsymbol{\psi}_j(\mathbf{x}) && \mathbf{z}_{\Gamma_i} = (z_1,\dots,z_{n_{z_{\Gamma_i}}})^T \in \mathbb{R}^{n_{z_{\Gamma_i}}},
\end{aligned}
$$

for the adjoint pressure

$$
\begin{aligned}
r_i^h(\mathbf{x}) &= \sum_{j=1}^{n_{r_i}} r_j \varphi_j(\mathbf{x}) && \mathbf{r}_i = (r_1,\dots,r_{n_{q_i}})^T \in \mathbb{R}^{n_{q_i}},\\
r_\Omega^h(\mathbf{x}) &= \sum_{j \notin \mathcal{N}} r_{\Omega_j} \varphi_{\Omega_j}(\mathbf{x}) && \mathbf{r}_\Omega = (r_{\Omega_1},\dots,r_{\Omega_{s_{\Omega_i}}})^T \in \mathbb{R}^{s_{\Omega_i}},
\end{aligned}
$$

for the desired state

$$
\begin{aligned}
\widehat{\mathbf{u}}_i^h(\mathbf{x}) &= \sum_{j=1}^{n_{0_i}} \widehat{u}_j \boldsymbol{\psi}_j(\mathbf{x}) && \widehat{\mathbf{u}}_i = (\widehat{u}_1,\dots,\widehat{u}_{n_{0_i}})^T \in \mathbb{R}^{n_{0_i}},\\
\widehat{\mathbf{u}}_\Gamma^h(\mathbf{x}) &= \sum_{j=1}^{n_{0_\Gamma}} \widehat{u}_j \boldsymbol{\psi}_j(\mathbf{x}) && \widehat{\mathbf{u}}_\Gamma = (\widehat{u}_1,\dots,\widehat{u}_{n_{0_\Gamma}})^T \in \mathbb{R}^{n_{0_\Gamma}},
\end{aligned}
$$

and for the control

$$
\mathbf{c}_i^h(\mathbf{x}) = \sum_{j=1}^{n_{c_i}} c_j \boldsymbol{\mu}_j(\mathbf{x}) \qquad \mathbf{c}_i = (c_1,\dots,c_{n_{c_i}})^T \in \mathbb{R}^{n_{c_i}}.
$$

3.7.2 *Definitions for the Boundary Control Case*

In the case of boundary control, we also use the same ansatz functions for the velocity and adjoint velocity, and for the pressure and adjoint pressure, respectively. For the boundary control, we use the same ansatz functions as for the velocity on the skeleton. Let $\widehat{\boldsymbol{\mu}}_j \in \mathbf{V}^h_{\Gamma_C}$, $(j = 1, \ldots, n_{c_\Gamma})$ denote the piecewise quadratic basis functions for the control such that

$$\mathbf{V}^h_{\Gamma_C} = \operatorname{span}\left\{\widehat{\boldsymbol{\mu}}_j \,:\, j = 1, \ldots, n_{c_\Gamma}\right\}.$$

Moreover, we define the following matrices and their entries, which correspond to the bilinear forms $a_j(\cdot,\cdot)$, $b_j(\cdot,\cdot)$ and $q_j(\cdot,\cdot)$ needed for the control subdomain Ω_j:

$$\begin{aligned}
(\mathbf{A}_C)_{kl} &= a_j(\mathcal{R}_{c_j}\widehat{\boldsymbol{\mu}}_l, \boldsymbol{\psi}_k),\\
(\mathbf{B}_C)_{kl} &= b_j(\mathcal{R}_{c_j}\widehat{\boldsymbol{\mu}}_l, \varphi_k),\\
(\mathbf{Q})_{kl} &= q_j(\widehat{\boldsymbol{\mu}}_l|_{\Gamma_C}, \widehat{\boldsymbol{\mu}}_k|_{\Gamma_C}) :
\end{aligned}$$

$$\begin{aligned}
&(\mathbf{A}_C)_{jj} \in \mathbb{R}^{n_{u_j}\times n_{c_\Gamma}} \quad (\mathbf{A}_C)_{kl} \quad k = 1, \ldots, n_{u_j},\ l = 1, \ldots, n_{c_\Gamma},\\
&(\mathbf{B}_C)_{jj} \in \mathbb{R}^{n_{q_j}\times n_{c_\Gamma}} \quad (\mathbf{B}_C)_{kl} \quad k = 1, \ldots, n_{q_j},\ l = 1, \ldots, n_{c_\Gamma},\\
&\mathbf{Q}_{jj} \in \mathbb{R}^{n_{c_\Gamma}\times n_{c_\Gamma}} \quad (\mathbf{Q})_{kl} \quad k, l = 1, \ldots, n_{c_\Gamma}.
\end{aligned}$$

Furthermore, we define the following finite element function for the boundary control and its coefficient vector:

$$\mathbf{c}^h_{\Gamma_C}(\mathbf{x}) = \sum_{j=1}^{n_{c_\Gamma}} c_j \boldsymbol{\mu}_j(\mathbf{x}) \qquad \mathbf{c}_i = (c_1, \ldots, c_{\Gamma_C})^T \in \mathbb{R}^{c_{\Gamma_C}}.$$

3.7.3 *Global Linear System and Schur-complement Equation for the Distributed Control Case*

Using the definitions for the algebraic representation, we discuss the global linear system which corresponds to the finite element formulation of (3.8), and the Schur-complement equation corresponding to the coupling conditions (3.26). For simplicity, we assume that $\Omega_0 = \Omega_C = \Omega$. We write the global linear system for an example with two subdomains, where Ω_1 is a non-outflow subdomain and Ω_2 an outflow subdomain:

$$
\left(\begin{array}{ccccc|ccccc|cccc}
\mathbf{A}_{11} & \widetilde{\mathbf{B}}_{11}^T & -(\mathbf{D}_C)_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{11} & (\mathbf{D}_C)_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{\Gamma_1 1}^T & \mathbf{0} \\
-(\mathbf{M}_0)_{11} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{11}^T & \widetilde{\mathbf{B}}_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\mathbf{M}_0)_{1\Gamma_1} & \mathbf{0} & \mathbf{A}_{\Gamma_1 1}^T & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma_1} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{22} & \mathbf{B}_{22}^T & -(\mathbf{D}_C)_{22} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{2\Gamma_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2\Gamma_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{22} & (\mathbf{D}_C)_{22}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{\Gamma_2 2}^T & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\mathbf{M}_0)_{22} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{22}^T & \mathbf{B}_{22}^T & -(\mathbf{M}_0)_{2\Gamma_2} & \mathbf{0} & \mathbf{A}_{\Gamma_2 2}^T & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2\Gamma_2} & \mathbf{0} \\
\hline
\mathbf{A}_{\Gamma_1 1} & \widetilde{\mathbf{B}}_{1\Gamma_1}^T & -(\mathbf{D}_C)_{\Gamma_1 1} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\Gamma_2 2} & \mathbf{B}_{2\Gamma_2}^T & -(\mathbf{D}_C)_{\Gamma_2 2} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-(\mathbf{M}_0)_{\Gamma_1 1} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1}^T & \widetilde{\mathbf{B}}_{1\Gamma_1}^T & -(\mathbf{M}_0)_{\Gamma_2 2} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{2\Gamma_2}^T & \mathbf{B}_{2\Gamma_2}^T & -(\mathbf{M}_0)_{\Gamma\Gamma} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma}^T & \mathbf{B}_0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0}
\end{array}\right)
\left(\begin{array}{c}
\mathbf{u}_1 \\ \widetilde{\mathbf{p}}_1 \\ \mathbf{c}_1 \\ \mathbf{z}_1 \\ \mathbf{r}_1 \\ \hline
\mathbf{u}_2 \\ \mathbf{p}_2 \\ \mathbf{c}_2 \\ \mathbf{z}_2 \\ \mathbf{r}_2 \\ \hline
\mathbf{u}_\Gamma \\ \mathbf{p}_\Omega \\ \mathbf{z}_\Gamma \\ \mathbf{q}_\Omega
\end{array}\right)
=
\left(\begin{array}{c}
\mathbf{f}_1 - \mathbf{A}_{1D}\mathbf{u}_D \\ -\mathbf{B}_{1D}\mathbf{u}_D \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_1 \\ \mathbf{0} \\ \hline
\mathbf{f}_2 - \mathbf{A}_{2D}\mathbf{u}_D \\ -\mathbf{B}_{2D}\mathbf{u}_D \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_2 \\ \mathbf{0} \\ \hline
\mathbf{f}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\mathbf{B}_{0D}\mathbf{u}_D \\ -\widehat{\mathbf{u}}_\Gamma \\ \mathbf{0}
\end{array}\right).
$$

Due to the fact that the inner blocks (denoted with the subindex $\cdot_{ii}$) are invertible, we can derive a Schur-complement equation. We write directly the alternative formulation for the non-outflow subdomains. The derivation of the alternative formulation works analogously to the outflow boundary case of the Oseen equations shown in Section 2.6.3.

The Schur-complement operator S is defined as:

$$
\begin{aligned}
S := &\begin{pmatrix} \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_{\Gamma\Gamma} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma}^T & \mathbf{B}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} \end{pmatrix} \\
&- \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{\Gamma_i i}^T & \mathbf{0} \\ -\mathbf{M}_{i\Gamma_i} & \mathbf{0} & \mathbf{A}_{\Gamma_i i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix}}_{=:L1} \\
&- \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{i\Gamma_i}^T & \mathbf{0} \\ -\mathbf{M}_{i\Gamma_i} & \mathbf{0} & \mathbf{A}_{\Gamma_i i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}}_{=:L2}
\end{aligned}
\tag{3.34}
$$

We define the right hand side r:

$$r := \begin{pmatrix} \mathbf{f}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\widehat{\mathbf{B}}_{0D}\mathbf{u}_D \\ -\widehat{\mathbf{u}}_\Gamma \\ \mathbf{0} \end{pmatrix} - \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD}\mathbf{u}_D \\ -\mathbf{B}_{iD}\mathbf{u}_D \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \end{pmatrix}}_{=:L3} \tag{3.35}$$

$$- \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD}\mathbf{u}_D \\ -\mathbf{B}_{iD}\mathbf{u}_D \\ 0 \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \\ 0 \end{pmatrix}}_{=:L4}$$

These definitions (3.34,3.35) yield the Schur-complement equation for the model problem of distributed control applied to the Oseen equations:

$$S(\mathbf{u}_\Gamma, \mathbf{p}_\Omega, \mathbf{z}_\Gamma, \mathbf{r}_\Omega)^T = r.$$

It represents algebraically the interface coupling corresponding to the finite element formulation (3.26).

$L1 - L4$ can be interpreted as solving local optimality systems of a distributed control problem applied to the Oseen equations.

$L1$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ (\mathbf{D}_C)_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\mathbf{M}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} + \mathbf{A}_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \end{pmatrix}.$$

$L1$ solves a local optimality system of a distributed control problem applied to the Oseen equations with Dirichlet data $\mathbf{u}_{\Gamma_i}$and $\mathbf{z}_{\Gamma_i}$on the local skeleton Γ_i, natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$ and zero Dirichlet boundary data on $\partial\Omega_i \cap \partial\Omega$.

$L2$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ 0 \\ (\mathbf{D}_C)_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\mathbf{M}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} + \mathbf{A}_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \\ 0 \end{pmatrix}.$$

$L2$ solves a local optimality system of a distributed control problem applied to the Oseen equations with Dirichlet data. On the local skeleton Γ_i, the Dirichlet data is set to $\mathbf{u}_{\Gamma_i}$and $\mathbf{z}_{\Gamma_i}$. For non-outflow subdomains intersecting global Dirichlet boundary, the Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ is set to zero.

$L3$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \end{pmatrix}.$$

$L3$ solves a local optimality system of a distributed control problem applied to the Oseen equations with zero Dirichlet data on the local skeleton Γ_i and natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$. On $\partial\Omega_i \cap$

$\partial\Omega_D$ the Dirichlet data for the velocity is set to $\mathbf{u}_{D_i}$ and for the adjoint velocity to zero.

$L4$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ 0 \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \\ 0 \end{pmatrix}.$$

$L4$ solves a local optimality system of a distributed control problem applied to the Oseen equations with Dirichlet data. For inner subdomains a homogeneous Dirichlet boundary problem is solved. For the non-outflow subdomains intersecting global Dirichlet boundary, the Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ is set to $\mathbf{u}_{D_i}$ for the velocity and zero for the adjoint velocity. On the local skeleton Γ_i, homogeneous Dirichlet boundary data is set.

Adding up $L1$ and $L3$ leads to the algebraic representation of the local decoupled subproblems on outflow subdomains Ω_i (3.25):

$L1 + L3$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} - \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} - \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ -(\mathbf{D}_C)_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\widehat{\mathbf{u}}_i + \mathbf{M}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} - \mathbf{A}_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \end{pmatrix},$$

Analogously, adding up $L2$ and $L4$ yields the algebraic representation of the local decoupled subproblems on non-outflow subdomains Ω_i (3.24).

$L2 + L4$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ 0 \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} - \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} - \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ 0 \\ -(\mathbf{D}_C)_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\widehat{\mathbf{u}}_i + \mathbf{M}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} - \mathbf{A}_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \\ 0 \end{pmatrix}.$$

Remark. The local subproblems $L3$ and $L4$ are solved when computing the right hand side r, whereas $L1$ and $L2$ are solved when applying the Schur-complement operator. The four problems do not depend on the global pressure variable p_Ω and global adjoint pressure r_Ω. The coupling vector $(\mathbf{u}_\Gamma, p_\Omega, \mathbf{z}_\Gamma, r_\Omega)$ also does not depend on the local Lagrange parameters η_i and θ_i.

3.7.4 *General Operator for the Distributed Control Case*

In Chapter 6, we derive two algorithms which are independent of the model problem: One to solve the Schur-complement equations and another one to solve the global linear systems. Therefore, we define a generalized operator introduced in Section 2.6.4. Again, the general operator is based on the block matrices used in the global linear system. For non-outflow subdomains, we define:

$$\mathbf{K}_i = \left(\begin{array}{ccccc|cccc} \mathbf{A}_{ii} & \mathbf{0} & -(\mathbf{D}_c)_{ii} & \widetilde{\mathbf{B}}_{ii}^T & \mathbf{0} & \mathbf{A}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{A}_{ii}^T & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{ii}^T & -(\mathbf{M}_0)_{i\Gamma_i} & \mathbf{A}_{\Gamma_i i}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{D}_c)_{ii}^T & \alpha(\mathbf{M}_c)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{D}_c)_{\Gamma_i i}^T & \mathbf{0} & \mathbf{0} \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{B}}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{A}_{\Gamma_i i} & \mathbf{0} & -(\mathbf{D}_c)_{\Gamma_i i} & \widetilde{\mathbf{B}}_{i\Gamma_i}^T & \mathbf{0} & \mathbf{A}_{\Gamma_i\Gamma_i} & \mathbf{0} & \mathbf{B}_{0,i}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{A}_{i\Gamma_i}^T & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i}^T & -(\mathbf{M}_0)_{\Gamma_i\Gamma_i} & \mathbf{A}_{\Gamma_i\Gamma_i}^T & \mathbf{0} & \mathbf{B}_{0,i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{0,i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{0,i} & \mathbf{0} & \mathbf{0} \end{array}\right) := \left(\begin{array}{cc|cc} \mathcal{A}_{ii} & \mathcal{B}_{ii}^T & \mathcal{A}_{i\Gamma_i} & \mathbf{0} \\ \mathcal{B}_{ii} & \mathbf{0} & \mathcal{B}_{i\Gamma_i} & \mathbf{0} \\ \hline \mathcal{A}_{\Gamma_i i} & \mathcal{B}_{\Gamma_i i}^T & \mathcal{A}_{\Gamma_i\Gamma_i} & \mathcal{B}_{\Omega_i}^T \\ \mathbf{0} & \mathbf{0} & \mathcal{B}_{\Omega_i} & \mathbf{0} \end{array}\right)$$

For outflow subdomains, we get a similar operator. We only have to replace the submatrices of $\widetilde{\mathbf{B}}$ with the corresponding submatrices of$\mathbf{B}$.

3.7.5 *Global Linear System and Schur-complement Equation for the Boundary Control Case*

We write the global linear system which corresponds to the finite element formulation of (3.8) for an example with two subdomains. We assume that the boundary control is applied on Ω_1 and that Ω_2 is an outflow subdomain. Then we can neglect the global pressure variable $\mathbf{r}_\Omega$ which is zero due to the boundary conditions which uniquely define the adjoint pressure on all subdomains.

$$
\left(\begin{array}{ccccc|ccccc|ccc}
\mathbf{A}_{11} & \widetilde{\mathbf{B}}_{11}^T & (\mathbf{A}_C)_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1} & \mathbf{0} & \mathbf{0} \\
\widetilde{\mathbf{B}}_{11} & \mathbf{0} & (\mathbf{B}_C)_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma_1} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -\alpha\mathbf{Q}_{11} & (\mathbf{A}_C)_{11}^T & (\mathbf{B}_C)_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{A}_C)_{\Gamma_1 1}^T \\
-(\mathbf{M}_0)_{11} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{11}^T & \mathbf{B}_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\mathbf{M}_0)_{1\Gamma_1} & \mathbf{0} & \mathbf{A}_{\Gamma_1 1}^T \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{1\Gamma_1} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{22} & \mathbf{B}_{22}^T & \mathbf{0} & \mathbf{0} & \mathbf{A}_{2\Gamma_2} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2\Gamma_2} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\mathbf{M}_0)_{22} & \mathbf{0} & \mathbf{A}_{22}^T & \mathbf{B}_{22}^T & -(\mathbf{M}_0)_{2\Gamma_2} & \mathbf{0} & \mathbf{A}_{\Gamma_2 2}^T \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2\Gamma_2} \\
\hline
\mathbf{A}_{\Gamma_1 1} & \widetilde{\mathbf{B}}_{1\Gamma_1}^T & (\mathbf{A}_C)_{\Gamma_1 1} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\Gamma_2 2} & \mathbf{B}_{2\Gamma_2}^T & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} & \mathbf{0} \\
-(\mathbf{M}_0)_{\Gamma_1 1} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1}^T & \mathbf{B}_{1\Gamma_1}^T & -(\mathbf{M}_0)_{\Gamma_2 2} & \mathbf{0} & \mathbf{A}_{2\Gamma_2}^T & \mathbf{B}_{2\Gamma_2}^T & -\mathbf{M}_{\Gamma\Gamma} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma}^T
\end{array}\right)
\left(\begin{array}{c}
\mathbf{u}_1 \\ \mathbf{p}_1 \\ \mathbf{c} \\ \mathbf{z}_1 \\ \mathbf{r}_1 \\ \hline
\mathbf{u}_2 \\ \mathbf{p}_2 \\ \mathbf{z}_2 \\ \mathbf{r}_2 \\ \hline
\mathbf{u}_\Gamma \\ \mathbf{p}_\Omega \\ \mathbf{z}_\Gamma
\end{array}\right)
=
\left(\begin{array}{c}
\mathbf{f}_1 - \mathbf{A}_{1D}\mathbf{u}_D \\ -\mathbf{B}_{1D}\mathbf{u}_D \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_1 \\ \mathbf{0} \\ \hline
\mathbf{f}_2 - \mathbf{A}_{2D}\mathbf{u}_D \\ -\mathbf{B}_{2D}\mathbf{u}_D \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_2 \\ \mathbf{0} \\ \hline
\mathbf{f}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\mathbf{B}_{0D}\mathbf{u}_D \\ -\widehat{\mathbf{u}}_\Gamma
\end{array}\right).
$$

Knowing that the inner block matrices (with the subindex $\cdot_{ii}$) are invertible, we can derive the Schur-complement equation. Again, we write directly the alternative formulation for the non-outflow subdomains. The derivation of the alternative formulation works analogously to the outflow boundary case of the Oseen equations shown in Section 2.6.3.

The Schur-complement S is defined as:

$$
\begin{aligned}
S := &\begin{pmatrix} \mathbf{A}_{\Gamma\Gamma} & \mathbf{B}_0^T & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{M}_{\Gamma\Gamma} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma}^T & \mathbf{B}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} \end{pmatrix} - \sum_{\substack{i\in\mathcal{N} \\ i\neq j}} \begin{pmatrix} \mathbf{A}_{\Gamma i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{i\Gamma_i} & \mathbf{0} & \mathbf{A}_{\Gamma_i i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix}}_{=:L1} \\
&- \sum_{\substack{i\notin\mathcal{N} \\ i\neq j}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i} & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{i\Gamma_i} & \mathbf{0} & \mathbf{A}_{\Gamma_i i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}}_{=:L2} \\
&- \begin{pmatrix} \mathbf{A}_{\Gamma_j j} & \mathbf{B}_{j\Gamma_j}^T & \mathbf{0} & (\mathbf{A}_C)_{\Gamma_j j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_j j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{j\Gamma_j}^T & \mathbf{B}_{j\Gamma_j}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{jj} & \mathbf{B}_{jj}^T & \mathbf{0} & (\mathbf{A}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{jj} & \mathbf{0} & \mathbf{c}_j & (\mathbf{B}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_j^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\alpha\mathbf{Q}_{jj} & (\mathbf{A}_C)_{jj}^T & (\mathbf{B}_C)_{jj}^T \\ -(\mathbf{M}_0)_{jj} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{jj}^T & \mathbf{B}_{jj}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{jj} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{j\Gamma_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{j\Gamma_j} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{A}_C)_{\Gamma_j j}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{j\Gamma_j} & \mathbf{0} & \mathbf{A}_{\Gamma_j j}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{j\Gamma_j} & \mathbf{0} \end{pmatrix}}_{=:L3}
\end{aligned}
$$

For the right hand side r, we obtain:

$$
\begin{aligned}
r := &\begin{pmatrix} \mathbf{f}_\Gamma - \mathbf{A}_{\Gamma D}\mathbf{u}_D \\ -\widehat{\mathbf{B}}_{0D}\mathbf{u}_D \\ -\widehat{\mathbf{u}}_\Gamma \\ \mathbf{0} \end{pmatrix} - \sum_{\substack{i\in\mathcal{N}\\ i\neq j}} \begin{pmatrix} \mathbf{A}_{\Gamma i} & \mathbf{B}_{i\Gamma}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma i} & \mathbf{0} & \mathbf{A}_{i\Gamma}^T & \mathbf{B}_{i\Gamma}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \end{pmatrix}}_{=:L4} \\
&- \sum_{\substack{i\notin\mathcal{N}\\ i\neq j}} \begin{pmatrix} \mathbf{A}_{\Gamma i} & \mathbf{B}_{i\Gamma}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma i} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma}^T & \mathbf{B}_{i\Gamma}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ 0 \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \\ 0 \end{pmatrix}}_{=:L5} \\
&- \begin{pmatrix} \mathbf{A}_{\Gamma_j j} & \mathbf{B}_{j\Gamma_j}^T & \mathbf{0} & (\mathbf{A}_C)_{\Gamma_j j} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_j j} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{j\Gamma_j}^T & \mathbf{B}_{j\Gamma_j}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{jj} & \mathbf{B}_{jj}^T & \mathbf{0} & (\mathbf{A}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{jj} & \mathbf{0} & \mathbf{c}_j & (\mathbf{B}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_j^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\alpha\mathbf{Q}_{jj} & (\mathbf{A}_C)_{jj}^T & (\mathbf{B}_C)_{jj}^T \\ -(\mathbf{M}_0)_{jj} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{jj}^T & \mathbf{B}_{jj}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{jj} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ 0 \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \\ 0 \end{pmatrix}}_{=:L6}
\end{aligned}
$$

Analogously, the subproblems $L1 - L6$ can be interpreted as solving local optimality systems of a boundary control problem applied to the Oseen equations.

$L1$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} + (\mathbf{A}_C)_{i\Lambda_i}\mathbf{c}_{\Lambda_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} + (\mathbf{B}_C)_{i\Lambda_i}\mathbf{c}_{\Lambda_i} \\ -(\mathbf{M}_0)_{i\Gamma_i}\mathbf{u}_{\Gamma_i} + \mathbf{A}_{\Gamma_i i}^T\mathbf{z}_{\Gamma_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \end{pmatrix}.$$

$L1$ solves a local optimality system of a boundary control problem applied to the Oseen equations with Dirichlet data $\mathbf{u}_{\Gamma_i}$ and $\mathbf{z}_{\Gamma_i}$ on the local skeleton Γ_i, natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$ and zero Dirichlet boundary data on $\partial\Omega_i \cap \partial\Omega$.

$L2$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ 0 \\ -(\mathbf{M}_0)_{i\Gamma_i}\mathbf{u}_{\Gamma_i} + \mathbf{A}_{\Gamma_i i}^T\mathbf{z}_{\Gamma_i} \\ \mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \\ 0 \end{pmatrix}.$$

$L2$ solves a local optimality system of a boundary control problem applied to the Oseen equations. On the local skeleton Γ_i, the Dirichlet data is set to $\mathbf{u}_{\Gamma_i}$and $\mathbf{z}_{\Gamma_i}$. For non-outflow subdomains intersecting global Dirichlet boundary, the Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ is set to zero.

$L3$ solve:

$$\begin{pmatrix} \mathbf{A}_{jj} & \mathbf{B}_{jj}^T & \mathbf{0} & (\mathbf{A}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{jj} & \mathbf{0} & \mathbf{c}_j & (\mathbf{B}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_j^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\alpha\mathbf{Q}_{jj} & (\mathbf{A}_C)_{jj}^T & (\mathbf{B}_C)_{jj}^T \\ -(\mathbf{M}_0)_{jj} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{jj}^T & \mathbf{B}_{jj}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{jj} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{p}_j \\ \eta_i \\ \mathbf{c} \\ \mathbf{z}_j \\ \mathbf{r}_j \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{j\Gamma_j}\mathbf{u}_{\Gamma_j} \\ \mathbf{B}_{j\Gamma_j}\mathbf{u}_{\Gamma_j} \\ 0 \\ (\mathbf{A}_C) \\ -(\mathbf{M}_0)_{j\Gamma_j}\mathbf{u}_{\Gamma_j} + \mathbf{A}_{\Gamma_j j}^T\mathbf{z}_{\Gamma_j} \\ \mathbf{B}_{j\Gamma_j}\mathbf{z}_{\Gamma_j} \end{pmatrix}.$$

$L3$ solves a local optimality system of a boundary control problem applied to the Oseen equations on the control subdomain Ω_j. On the local skeleton Γ_i, the Dirichlet data is set to $\mathbf{u}_{\Gamma_i}$and $\mathbf{z}_{\Gamma_i}$. The Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ for the velocity is set to zero for the adjoint velocity, outflow boundary conditions are applied on Γ_C.

$L4$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \end{pmatrix}.$$

$L4$ solves a local optimality system of a boundary control problem applied to the Oseen equations with zero Dirichlet data on the local skeleton Γ_i and natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$. On $\partial\Omega_i \cap \partial\Omega$ the Dirichlet data for the velocity is set to $\mathbf{u}_{D_i}$ and for the adjoint velocity to zero.

$L5$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} \\ 0 \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_i \\ \mathbf{0} \\ 0 \end{pmatrix}.$$

$L5$ solves a local optimality system of a boundary control problem applied to the Oseen equations with Dirichlet data. For inner subdomains a homogeneous Dirichlet boundary problem is solved. For the non-outflow subdomains intersecting global Dirichlet boundary, the Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ is set to $\mathbf{u}_{D_i}$ for the velocity and zero for the adjoint velocity. On the skeleton Γ_i, homogeneous Dirichlet boundary data is set.

$L6$ solve:

$$\begin{pmatrix} \mathbf{A}_{jj} & \mathbf{B}_{jj}^T & \mathbf{0} & (\mathbf{A}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{jj} & \mathbf{0} & \mathbf{c}_j & (\mathbf{B}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_j^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\alpha\mathbf{Q}_{jj} & (\mathbf{A}_C)_{jj}^T & (\mathbf{B}_C)_{jj}^T \\ -(\mathbf{M}_0)_{jj} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{jj}^T & \mathbf{B}_{jj}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{jj} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{p}_j \\ \eta_i \\ \mathbf{c} \\ \mathbf{z}_j \\ \mathbf{r}_j \end{pmatrix} = \begin{pmatrix} \mathbf{f}_j - \mathbf{A}_{jD_j}\mathbf{u}_{D_j} \\ -\mathbf{B}_{jD_j}\mathbf{u}_{D_j} \\ 0 \\ \mathbf{0} \\ -\widehat{\mathbf{u}}_j \\ \mathbf{0} \end{pmatrix}.$$

$L6$ solves a local optimality system of a boundary control problem applied to the Oseen equations on the control subdomain Ω_j. On the local skeleton Γ_i, homogeneous Dirichlet data is set. The Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ for the velocity is set to $\mathbf{u}_{D_i}$. For the adjoint velocity outflow boundary conditions are applied on Γ_C.

Adding up $L1$ and $L4$, we obtain the algebraic representation of the local decoupled subproblems on outflow subdomains Ω_i (3.28).

$L1 + L4$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} - \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} - \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ -(\mathbf{D}_C)_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\widehat{\mathbf{u}}_i + \mathbf{M}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} - \mathbf{A}_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \end{pmatrix}.$$

Adding up $L2$ and $L5$ leads to

$L2 + L5$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii} & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{f}_i - \mathbf{A}_{iD_i}\mathbf{u}_{D_i} - \mathbf{A}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ -\mathbf{B}_{iD_i}\mathbf{u}_{D_i} - \mathbf{B}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} \\ 0 \\ -(\mathbf{D}_C)_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\widehat{\mathbf{u}}_i + \mathbf{M}_{i\Gamma_i}\mathbf{u}_{\Gamma_i} - \mathbf{A}_{\Gamma_i i}^T \mathbf{z}_{\Gamma_i} \\ -\mathbf{B}_{i\Gamma_i}\mathbf{z}_{\Gamma_i} \\ 0 \end{pmatrix},$$

which is the algebraic representation of the local decoupled optimality systems on non-outflow subdomains Ω_i (3.27).

Adding up $L3$ and $L6$ yields the algebraic representation of the local decoupled optimal control system on the control domain Ω_j (3.29):

$L3 + L6$ solve:

$$\begin{pmatrix} \mathbf{A}_{jj} & \mathbf{B}_{jj}^T & \mathbf{0} & (\mathbf{A}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{jj} & \mathbf{0} & \mathbf{c}_j & (\mathbf{B}_C)_{jj} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_j^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\alpha\mathbf{Q}_{jj} & (\mathbf{A}_C)_{jj}^T & (\mathbf{B}_C)_{jj}^T \\ -(\mathbf{M}_0)_{jj} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{jj}^T & \mathbf{B}_{jj}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{jj} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_j \\ \mathbf{p}_j \\ \eta_i \\ \mathbf{c} \\ \mathbf{z}_j \\ \mathbf{r}_j \end{pmatrix} = \begin{pmatrix} \mathbf{f}_j - \mathbf{A}_{jD_j}\mathbf{u}_{D_j} - \mathbf{A}_{j\Gamma_j}\mathbf{u}_{\Gamma_j} \\ -\mathbf{B}_{iD_j}\mathbf{u}_{D_j} - \mathbf{B}_{j\Gamma_j}\mathbf{u}_{\Gamma_j} \\ 0 \\ -(\mathbf{A}_C) \\ -\widehat{\mathbf{u}}_j + (\mathbf{M}_0)_{j\Gamma_j}\mathbf{u}_{\Gamma_j} - \mathbf{A}_{\Gamma_j j}^T \mathbf{z}_{\Gamma_j} \\ -\mathbf{B}_{j\Gamma_j}\mathbf{z}_{\Gamma_j} \end{pmatrix}.$$

SUMMARY

In this chapter, we derived a non-overlapping domain decomposition method for two linear quadratic optimal flow control problems. As constraints, we applied the Oseen equations equipped with mixed outflow and Dirichlet boundary conditions. For one problem, we assumed distributed and for the other boundary control. The local problems, we obtained have the same structure as the global problems. This main idea of the domain decomposition is therefore transferable to the optimal flow control case. We emphasized on the differences between the distributed and boundary control case. This chapter builds a foundation for Chapter 5, in which we analyze a non-linear optimal flow control problem assuming distributed control.

4

DOMAIN DECOMPOSITION METHOD FOR THE NAVIER-STOKES EQUATIONS

Before we consider a non-linear optimal flow control problem in the forthcoming chapter, we consider the Navier-Stokes equations as a non-linear model problem. Based on this flow problem, we concentrate on how to deal with the non-linearity in the context of a non-overlapping domain decomposition method. We discuss how to solve the Navier-Stokes equations combining the Newton-method with the domain decomposition approach. Analogously to the former chapters, we derive a domain decomposition method for the non-linear Navier-Stokes equations equipped with mixed outflow and Dirichlet boundary conditions. The derivation of the domain decomposition method is based on the description given in Chapter 2.

If not stated differently, we use the same definitions, notation and assumptions as in Chapter 2.

4.1 NAVIER-STOKES EQUATIONS WITH OUTFLOW BOUNDARY CONDITIONS

Analogously to the previous chapters, in the first main step, we present a global problem formulation in strong and weak form. In difference to the first chapter, the domain decomposition is applied to the linearized weak formulation. Again, let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a Lipschitz domain. We decompose the boundary $\partial\Omega = \partial\Omega_D \cup \partial\Omega_{out}$ with $\partial\Omega_D \cap \partial\Omega_{out} = \emptyset$ as before in Chapter 2 for the outflow boundary case, see also Fig. 3. For given functions and values

$$\mathbf{f} \in \mathbf{L}^2(\Omega),\ \mathbf{h} \in \mathbf{L}^2(\partial\Omega_{out}),\ \mathbf{d} \in \mathbf{H}^{1/2}(\partial\Omega_D) \text{ and } \mu > 0, \tag{4.1}$$

we want to solve the following boundary value problem for the Navier-Stokes equations, modeling a Newtonian fluid equipped with mixed outflow and Dirichlet boundary conditions:

$$(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla} \cdot \sigma(\mathbf{u}, p) = \mathbf{f} \qquad \text{in } \Omega, \tag{4.2a}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{4.2b}$$

$$\mathbf{u} = \mathbf{d} \qquad \text{on } \partial\Omega_D, \tag{4.2c}$$

$$\sigma(\mathbf{u}, p)\mathbf{n} = \mathbf{h} \qquad \text{on } \partial\Omega_{out}. \tag{4.2d}$$

The Navier-Stokes equations compared to the Oseen equations model flow at higher Reynolds numbers that means the viscosity tends to be smaller at higher velocities. Because of this the non-linear term $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$ is not negligible and becomes dominant.

For the Dirichlet boundary data $\mathbf{d}$, we again assume that there exists an extension $\mathbf{u}_D \in \mathbf{H}^1(\Omega)$ such that $\gamma_d(\mathbf{u}_D) = \mathbf{d}$.

Since we want to discretize the Navier-Stokes equations with a finite element method, we derive a weak formulation. Therefore, we use the bilinear form $b(\cdot,\cdot)$ and the linear form $f(\cdot)$ as defined in Chapter 2. Moreover, we redefine the bilinear form $a(\cdot,\cdot)$ and define the trilinear form $n(\cdot,\cdot,\cdot)$:

$$\begin{aligned} a : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}, \qquad a(\mathbf{u}, \mathbf{v}) &= 2\mu \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, d\mathbf{x}, \\ &= \mu \int_\Omega \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{v}\, d\mathbf{x}, \\ n : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R}, \quad n(\mathbf{w}, \mathbf{u}, \mathbf{v}) &= \int_\Omega ((\mathbf{w} \cdot \boldsymbol{\nabla})\mathbf{u})\mathbf{v}\, d\mathbf{x}, \end{aligned}$$

Using these definitions, the weak form of the system of partial differential equation (4.2) is given as follows: Find $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$, $p \in L^2(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) + n(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = f(\mathbf{v}) - a(\mathbf{u}_D, \mathbf{v}) - n(\mathbf{u}_D, \mathbf{u}_D, \mathbf{v}), \tag{4.3a}$$

$$b(\mathbf{u}, q) = -b(\mathbf{u}_D, q) \tag{4.3b}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$.

Remark 32. The existence and uniqueness of solutions for the Navier-Stokes equations equipped with different types of boundary conditions have been treated in many publication. See, e.g., [20, Part III, Remark 9.1] for further literature. Since the existence and uniqueness is not the focus of this work, we assume that for our set-up a solution exists.

Due to the term $n(\mathbf{u}, \mathbf{u}, \mathbf{v})$ this is a non-linear problem, which we solve using the Newton method. Therefore, we write the problem as a root problem ($F(x) = 0$) and approximate the solution based on the Taylor series expansion. We assume that an appropriate start solution x^0 is given. The steps of the Newton method are given as in as in Alg. 4.1.

In step 4, we solve the linearized problem, with the residual $F(\cdot)$ evaluated at the linearization point x^k and get the correction c^k. In step 5, also called update step, we calculate the next iteration x^{k+1}. Since we apply the Newton method to solve the Navier-Stokes problem, we need to linearize its variational equations (4.3). First, we define the residuals at the linearization point $(\mathbf{u}^k, p^k) \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega)$ for the test functions $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$ and $q \in L^2(\Omega)$:

Algorithm 4.1 Newton Method

1: Input: Appropriate start solution x^0
2: $k = 0$
3: **repeat**
4: $\quad \nabla F[x^k]c^k = F(x^k)$
5: $\quad x^{k+1} = x^k - c^k$
6: $\quad k = k + 1$
7: **until** convergence

$$\begin{aligned} F^1(\mathbf{u}^k, p^k, \mathbf{v}) &= a(\mathbf{u}^k, \mathbf{v}) + n(\mathbf{u}^k, \mathbf{u}^k, \mathbf{v}) + b(\mathbf{v}, p^k) + \\ &\quad - f(\mathbf{v}) + a(\mathbf{u}_D, \mathbf{v}) + n(\mathbf{u}_D, \mathbf{u}_D, \mathbf{v}), \\ F^2(\mathbf{u}^k, q) &= b(\mathbf{u}^k, q) + b(\mathbf{u}_D, q). \end{aligned}$$

We set

$$\mathbf{F}(\mathbf{u}^k, p^k, \mathbf{v}, q) = \begin{pmatrix} F^1(\mathbf{u}^k, p^k, \mathbf{v}) \\ F^2(\mathbf{u}^k, q) \end{pmatrix}.$$

Our root problem then reads

$$\mathbf{F}(\mathbf{u}, p, \mathbf{v}, q) = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_D(\Omega),\, q \in L^2(\Omega).$$

Next, we linearize (4.3) at the linearization point $(\mathbf{u}^k, p^k)$ and obtain the following weak form for the corrections for the kth Newton step: Find the corrections $\mathbf{c}_u \in \mathbf{H}^1_D(\Omega)$, $c_p \in L^2(\Omega)$ such that

$$a(\mathbf{c}_u, \mathbf{v}) + n(\mathbf{c}_u, \mathbf{u}^k, \mathbf{v}) + n(\mathbf{u}^k, \mathbf{c}_u, \mathbf{v}) + b(\mathbf{v}, c_p) = F^1(\mathbf{u}^k, p^k, \mathbf{v}), \quad (4.4a)$$

$$b(\mathbf{c}_u, q) = F^2(\mathbf{u}^k, p^k, q) \quad (4.4b)$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$.

4.2 CONTINUOUS DOMAIN DECOMPOSITION

In the second main step, we derive a non-overlapping domain decomposition method on the continuous level for the linearized weak Navier-Stokes equations (4.4). Analogously to the previous chapters, we first derive a fully coupled equivalent formulation on subdomains to which we refer as the first sub-step of the second main step. This sub-step is sketched in Fig. 6. In the second sub-step, we decouple that formulation which then leads to s independent linearized subdomain problems and one linearized system of coupling conditions, see Fig. 7.

We partition the domain as explained in Chapter 2 and use the same decomposition of the spaces for the velocity and pressure.

4.2.1 *Decomposition of the Bilinear Forms*

We define the local bilinear forms $a_i(\cdot,\cdot)$ and $n_i(\cdot,\cdot,\cdot)$. For the local bilinear form $b(\cdot,\cdot)$, we use the definition from Chapter 2. Before we define the local bi- and trilinear forms, we define the partitioning of the linearization point. Recall that $\mathbf{V}_i$ is the local velocity space with mixed outflow and generalized zero boundary conditions, see definition (2.14). On the other hand, $\widetilde{\mathbf{V}}_i$ is the local velocity space with generalized zero boundary conditions only on the local boundary which intersects the global Dirichlet boundary, i.e. on $\partial\Omega_i \cap \partial\Omega_D$, see also definition (2.16). We set

$$\mathbf{u}_i^k := \mathbf{u}^k|_{\Omega_i}, \qquad \mathbf{u}^k = (\mathbf{u}_1^k, \mathbf{u}_2^k, \dots, \mathbf{u}_s^k) \in \bigotimes_{i=1}^{s} \widetilde{\mathbf{V}}_i.$$

For the pressure, we have to be more careful. On inner subdomains the pressure is split into a normalized part in $L_0^2(\Omega)$ and a constant. On outflow subdomains the local pressure directly lies in $L^2(\Omega)$. Hence, it gives

$$p_i^k = p^k|_{\Omega_i} \in \begin{cases} L^2(\Omega_i) & \text{if } i \in \mathcal{N} \\ L_0^2(\Omega_i) & \text{if } i \notin \mathcal{N} \end{cases},$$

which results in the definition

$$p^k = (p_1^k, p_2^k, \dots, p_s^k, p_\Omega^k) \in \bigoplus_{i=1}^{s} Q_i \oplus D(\Omega).$$

We redefine the local bilinear forms $a_i(\cdot,\cdot)$:

$$\begin{aligned} a_i : \mathbf{H}^1(\Omega_i) \times \mathbf{H}^1(\Omega_i) \to \mathbb{R}, \qquad a_i(\mathbf{u}_i, \mathbf{v}_i) &= 2\mu \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}_i) : \boldsymbol{\varepsilon}(\mathbf{v}_i)\, d\mathbf{x} \\ &= \mu \int_{\Omega_i} \boldsymbol{\nabla}\mathbf{u}_i : \boldsymbol{\nabla}\mathbf{v}_i\, d\mathbf{x}, \end{aligned}$$

For a fixed $\mathbf{u}_i^k$, we define the local trilinear form n_i:

$$\begin{aligned} &n_i : \mathbf{H}^1(\Omega_i) \times \mathbf{H}^1(\Omega_i) \times \mathbf{H}^1(\Omega_i) \to \mathbb{R}, \\ &n_i(\mathbf{u}_i^k, \mathbf{u}_i, \mathbf{v}_i) = \frac{1}{2}\int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i\, d\mathbf{x} - \frac{1}{2}\int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{v}_i)\mathbf{u}_i\, d\mathbf{x} + \\ &\qquad - \frac{1}{2}\int_{\Omega_i} ((\boldsymbol{\nabla} \cdot \mathbf{u}_i^k)\mathbf{v}_i)\mathbf{u}_i\, d\mathbf{x} + \frac{1}{2}\int_{\partial\Omega_i \cap \partial\Omega_{out}} (\mathbf{u}_i^k \cdot \mathbf{n}_i)(\mathbf{u}_i\mathbf{v}_i)\, d\mathbf{s}. \\ &n_i : \mathbf{H}^1(\Omega_i) \times \mathbf{H}^1(\Omega_i) \times \mathbf{H}^1(\Omega_i) \to \mathbb{R}, \end{aligned}$$

$$n_i(\mathbf{u}_i, \mathbf{u}_i^k, \mathbf{v}_i) = \int_{\Omega_i} ((\mathbf{u}_i \cdot \boldsymbol{\nabla})\mathbf{u}_i^k)\mathbf{v}_i \, d\mathbf{x}.$$

For a fixed $\mathbf{u}_i^k$, the canonical decomposition of $n(\mathbf{u}^k, \mathbf{u}, \mathbf{v})$ denoted by $\widetilde{n}_i(\mathbf{u}_i^k, \mathbf{u}_i, \mathbf{v}_i)$ leads to

$$\begin{aligned}
\widetilde{n}_i(\mathbf{u}_i^k, \mathbf{u}_i, \mathbf{v}_i) &= \int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{x} \\
&= \frac{1}{2}\int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{x} - \frac{1}{2}\int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{v}_i)\mathbf{u}_i \, d\mathbf{x} + \\
&\quad - \frac{1}{2}\int_{\Omega_i} ((\boldsymbol{\nabla} \cdot \mathbf{u}_i^k)\mathbf{v}_i)\mathbf{u}_i \, d\mathbf{x} + \frac{1}{2}\int_{\partial\Omega_i} (\mathbf{u}_i^k \cdot \mathbf{n}_i)(\mathbf{u}_i\mathbf{v}_i) \, d\mathbf{s}
\end{aligned}$$

$$\Rightarrow \quad n_i(\mathbf{u}_i^k, \mathbf{u}_i, \mathbf{v}_i) = \widetilde{n}_i(\mathbf{u}_i^k, \mathbf{u}_i, \mathbf{v}_i) - \frac{1}{2}\int_{\partial\Omega_i} (\mathbf{u}_i^k \cdot \mathbf{n}_i)(\mathbf{u}_i\mathbf{v}_i) \, d\mathbf{x}.$$

Remark. Note that $n_i(\mathbf{u}_i^k, \mathbf{v}_i, \mathbf{v}_i) = 0$.

Remark. The decomposition of $n_i(\mathbf{u}_i^k, \mathbf{u}_i, \mathbf{v}_i)$ works analogously to the decomposition of the advection part in the Oseen equations.

Proposition 33. *For the local bilinear and trilinear forms $a(\cdot,\cdot)$, $n_i(\mathbf{u}_i^k, \cdot, \cdot)$, and $n_i(\cdot, \mathbf{u}_i^k, \cdot)$ it holds that*

$$\begin{aligned}
\sum_{i=1}^{s} a_i(\mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i}) &= a(\mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\, \mathbf{v} \in \mathbf{H}^1_D(\Omega), \\
\sum_{i=1}^{s} n_i(\mathbf{u}_i^k, \mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i}) &= \sum_{i=1}^{s} \widetilde{n}_i(\mathbf{u}_i^k, \mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i}) \\
&= n(\mathbf{u}^k, \mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\, \mathbf{v} \in \mathbf{H}^1_D(\Omega), \\
\sum_{i=1}^{s} n_i(\mathbf{u}|_{\Omega_i}, \mathbf{u}_i^k, \mathbf{v}|_{\Omega_i}) &= n(\mathbf{u}, \mathbf{u}^k, \mathbf{v}) \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\, \mathbf{v} \in \mathbf{H}^1_D(\Omega).
\end{aligned}$$

Proof. The first and third equation can be computed trivially. We show the equality of

$$\sum_{i=1}^{s} n_i(\mathbf{u}_i^k, \mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i}) = n(\mathbf{u}^k, \mathbf{u}, \mathbf{v}) \qquad \forall \mathbf{u} \in \mathbf{H}^1(\Omega),\, \mathbf{v} \in \mathbf{H}^1_D(\Omega).$$

The remaining part can be shown analogously.

$$\begin{aligned}
&\sum_{i=1}^{s} n_i(\mathbf{u}_i^k, \mathbf{u}|_{\Omega_i}, \mathbf{v}|_{\Omega_i}) \\
&= \sum_{i=1}^{s} \Big(\frac{1}{2}\int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{x} - \frac{1}{2}\int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{v}_i)\mathbf{u}_i \, d\mathbf{x} + \\
&\qquad - \frac{1}{2}\int_{\Omega_i} ((\boldsymbol{\nabla} \cdot \mathbf{u}_i^k)\mathbf{v}_i)\mathbf{u}_i \, d\mathbf{x} + \frac{1}{2}\int_{\partial\Omega_i \cap \partial\Omega_{out}} (\mathbf{u}_i^k \cdot \mathbf{n}_i)(\mathbf{u}_i\mathbf{v}_i) \, d\mathbf{s} \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{s} \Big(\int_{\Omega_i} ((\mathbf{u}_i^k \cdot \boldsymbol{\nabla})\mathbf{u}_i)\mathbf{v}_i \, d\mathbf{x} - \frac{1}{2} \int_{\partial\Omega_i} (\mathbf{u}_i^k \cdot \mathbf{n}_i)(\mathbf{u}_i \mathbf{v}_i) \, d\mathbf{s} + \\
&\quad + \frac{1}{2} \int_{\partial\Omega_i \cap \partial\Omega_{out}} (\mathbf{u}_i^k \cdot \mathbf{n}_i)(\mathbf{u}_i \mathbf{v}_i) \, d\mathbf{s} \Big) \\
&= \int_{\Omega} ((\mathbf{u}^k \cdot \boldsymbol{\nabla})\mathbf{u})\mathbf{v} \, d\mathbf{x} - \frac{1}{2} \sum_{\substack{i=1 \\ j>i}}^{s} \underbrace{\int_{\Gamma_{ij}} (\mathbf{u}_i^k \cdot (\mathbf{n}_i + \mathbf{n}_j))(\mathbf{u}_i \mathbf{v}_i) \, d\mathbf{s}}_{=0,\, (\mathbf{n}_i + \mathbf{n}_j) = 0} + \\
&\quad \underbrace{-\frac{1}{2} \int_{\partial\Omega_{out}} (\mathbf{u}^k \cdot \mathbf{n})(\mathbf{u}\mathbf{v}) \, d\mathbf{s} + \frac{1}{2} \int_{\partial\Omega_{out}} (\mathbf{u}^k \cdot \mathbf{n})(\mathbf{u}\mathbf{v}) \, d\mathbf{s}}_{=0} + \\
&\quad \underbrace{+ \frac{1}{2} \int_{\partial\Omega_D} (\mathbf{u}^k \cdot \mathbf{n})(\mathbf{u}\mathbf{v}) \, d\mathbf{s}}_{=0,\, \mathbf{v} \in \mathbf{H}_D^1(\Omega)} \\
&= n(\mathbf{u}^k, \mathbf{u}, \mathbf{v})
\end{aligned}
$$

□

4.2.2 *Decomposition of the Residuals*

We define the local residuals

$$
\begin{aligned}
F_i^1(\mathbf{u}_i^k, p_i^k, \mathbf{v}_i) :&= a_i(\mathbf{u}_i^k, \mathbf{v}_i) + n_i(\mathbf{u}_i^k, \mathbf{u}_i^k, \mathbf{v}_i) + b_i(\mathbf{v}_i, p_i^k) + \\
&\quad + a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) + n_i(\mathbf{u}_{D_i}, \mathbf{u}_{D_i}, \mathbf{v}_i) - f_i(v_i), \\
F_i^2(\mathbf{u}_i^k, q_i) :&= b_i(\mathbf{u}_i^k, q_i) + b_i(\mathbf{u}_{D_i}, q_i), \\
F_i^3(p_i^k, \xi_i) :&= c_i(p_i^k, \xi_i)
\end{aligned}
$$

with $\mathbf{v}_i \in \mathbf{V}_i$, $q_i \in Q_i$ and $\xi_i \in \mathbb{R}$.

Moreover, we define the residuals on the interface:

$$
\begin{aligned}
F_\Gamma^1(\mathbf{u}^k, p^k, \mathbf{v}_\Gamma) :&= a(\mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + n(\mathbf{u}^k, \mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), p^k) + \\
&\quad + a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)) + n(\mathbf{u}_D, \mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)) - f(\mathcal{R}(\mathbf{v}_\Gamma)), \\
F_\Gamma^2(\mathbf{u}^k, q_\Omega) :&= b(\mathbf{u}^k, q_\Omega) + b(\mathbf{u}_D, q_\Omega),
\end{aligned}
$$

with $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$ and $q_\Omega \in D(\Omega)$.

4.2.3 *Weak Formulation on Subdomains*

Using these definitions, we apply the first sub-step to derive an equivalent weak formulation on subdomains that is still globally coupled.

Lemma 34. *(4.4) is equivalent to the following weak formulation on subdomains: At linearization point* $(\mathbf{u}^k, p^k) \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega)$*, find the corrections*

$$\mathbf{c}_u = \sum_{i=1}^{s} \mathbf{c}_{u_i} + \mathbf{c}_{u_\Gamma} \in \mathbf{H}^1_D(\Omega) \qquad \text{with } \mathbf{c}_{u_i} \in \mathbf{V}_i,\ \mathbf{c}_{u_\Gamma} \in \mathbf{H}^{1/2}_{00}(\Gamma),$$
$$c_p = \sum_{i=1}^{s} c_{p_i} + c_{p_\Omega} \in L^2(\Omega) \qquad \text{with } c_{p_i} \in Q_i,\ c_{p_\Omega} \in D(\Omega)$$

such that

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathbf{c}_{u_i}, \mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}), \mathbf{v}_i) + a_i(\mathbf{c}_{u_i}, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + n_i(\mathbf{c}_{u_i}, \mathbf{u}_i^k, \mathbf{v}_i) + \\
&\quad + n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}), \mathbf{u}_i^k, \mathbf{v}_i) + n_i(\mathbf{c}_{u_i}, \mathbf{u}_i^k, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}, \mathbf{v}_i) + \\
&\quad + n_i(\mathbf{u}_i^k, \mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}), \mathbf{v}_i) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathbf{v}_i, c_{p_i}) + \\
&\quad + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), c_{p_i})\Big) + a(\mathcal{R}(\mathbf{c}_{u_\Gamma}), \mathcal{R}(\mathbf{v}_\Gamma)) + n(\mathcal{R}(\mathbf{c}_{u_\Gamma}), \mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + \\
&\qquad + n(\mathbf{u}^k, \mathcal{R}(\mathbf{c}_{u_\Gamma}), \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), c_{p_\Omega}) \\
&= \sum_{i=1}^{s}\Big(F_i^1(\mathbf{u}_i^k, p_i^k, \mathbf{v}_i)\Big) + F_\Gamma^1(\mathbf{u}^k, p^k, \mathbf{v}_\Gamma), \qquad (4.5a) \\
&\sum_{i=1}^{s}\Big(b_i(\mathbf{c}_{u_i}, q_i) + b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}), q_i)\Big) + b(\mathcal{R}(\mathbf{c}_{u_\Gamma}), q_\Omega) \\
&= \sum_{i=1}^{s}\Big(F_i^2(\mathbf{u}_i^k, q_i)\Big) + F_\Gamma^2(\mathbf{u}^k, q_\Omega) \qquad (4.5b)
\end{aligned}$$

for all $\mathbf{v}_i \in \mathbf{V}_i$, $\mathbf{v}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$, $q_i \in Q_i$ *and* $q_\Omega \in D(\Omega)$.

Proof. Use the same arguments as in the proof of Lemma 9 in Chapter 2. □

4.2.4 *Decoupling of Weak Formulation on Subdomains*

Then in the next sub-step, we decouple (4.5) into s independent local weak formulations on subdomains and one system of coupling equations. For the local subproblems, we have to distinguish two formulations, one for outflow subdomains and one for non-outflow subdomains.

On a non-outflow subdomain, the local linearized Navier-Stokes problem is given as follows:

For a given $\mathbf{c}_{u_\Gamma} \in \mathbf{H}^{1/2}_{00}(\Gamma)$, find $\mathbf{c}_{u_i} \in \mathbf{H}^1_0(\Omega_i)$ and $\widetilde{c}_{p_i} \in L^2_0(\Omega_i)$ at the linearization point $(\mathbf{u}_i^k, p_i^k)$, such that

$$a_i(\mathbf{c}_{u_i}, \mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}), \mathbf{v}_i) + b_i(\mathbf{v}_i, \widetilde{c}_{p_i}) +$$

$$
\begin{aligned}
&+n_i(\mathbf{c}_{u_i},\mathbf{u}_i^k,\mathbf{v}_i)+n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{u}_i^k,\mathbf{v}_i)+\\
&+n_i(\mathbf{u}_i^k,\mathbf{c}_{u_i},\mathbf{v}_i)+n_i(\mathbf{u}_i^k,\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{v}_i)=F_i^1(\mathbf{u}_i^k,p_i^k,\mathbf{v}_i), \qquad (4.6a)\\
&b_i(\mathbf{c}_{u_i},\widetilde{q}_i)+b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\widetilde{q}_i)=F_i^2(\overline{\mathbf{u}}_i^k,\widetilde{q}_i) \qquad (4.6b)
\end{aligned}
$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega)$ and $\widetilde{q}_i \in L_0^2(\Omega_i)$.

In the next proposition, we give an equivalent alternative formulation. Analogously to the previous chapters, we prefer this alternative formulation due to the used finite element implementation.

Proposition 35. *Under the assumption (2.24), (4.6) is equivalent to the following subdomain problem.*

For a given $\mathbf{c}_{u_\Gamma} \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{c}_{u_i} \in \mathbf{H}_0^1(\Omega_i)$, $c_{p_i} \in L^2(\Omega_i)$ and $c_{\eta_i} \in \mathbb{R}$ at the linearization point $(\mathbf{u}_i^k, p_i^k)$, such that

$$
\begin{aligned}
&a_i(\mathbf{c}_{u_i},\mathbf{v}_i)+a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{v}_i)+b_i(\mathbf{v}_i,c_{p_i})+\\
&+n_i(\mathbf{c}_{u_i},\mathbf{u}_i^k,\mathbf{v}_i)+n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{u}_i^k,\mathbf{v}_i)+\\
&+n_i(\mathbf{u}_i^k,\mathbf{c}_{u_i},\mathbf{v}_i)+n_i(\mathbf{u}_i^k,\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{v}_i)=F_i^1(\mathbf{u}_i^k,p_i^k,\mathbf{v}_i), \qquad (4.7a)\\
&b_i(\mathbf{c}_{u_i},q_i)+b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),q_i)+c_i(c_{\eta_i},q_i)=F_i^2(\mathbf{u}_i^k,q_i), \qquad (4.7b)\\
&c_i(\xi_i,c_{p_i})=F^3(p_i^k,\xi_i) \qquad (4.7c)
\end{aligned}
$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$, $q_i \in L^2(\Omega_i)$ and $\xi_i \in \mathbb{R}$.

Proof. Use the same arguments as in the proof of Lemma 14 in Chapter 2. □

As we can see, the local Navier-Stokes problems on non-outflow subdomains are independent of the global constants c_{p_Ω}.

Similarly on outflow subdomains, we obtain the following formulation:

For a given $\mathbf{c}_{u_\Gamma} \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{c}_{u_i} \in \mathbf{H}_N^1(\Omega_i)$ and $c_{p_i} \in L^2(\Omega_i)$ at the linearization point $(\mathbf{u}_i^k, p_i^k)$, such that

$$
\begin{aligned}
&a_i(\mathbf{c}_{u_i},\mathbf{v}_i)+a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{v}_i)+b_i(\mathbf{v}_i,c_{p_i})+\\
&+n_i(\mathbf{c}_{u_i},\mathbf{u}_i^k,\mathbf{v}_i)+n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{u}_i^k,\mathbf{v}_i)+ \qquad (4.8a)\\
&+n_i(\mathbf{u}_i^k,\mathbf{c}_{u_i},\mathbf{v}_i)+n_i(\mathbf{u}_i^k,\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),\mathbf{v}_i)=F_i^1(\mathbf{u}_i^k,p_i^k,\mathbf{v}_i),\\
&b_i(\mathbf{c}_{u_i},q_i)+b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}),q_i)=F_i^2(\mathbf{u}_i^k,q_i), \qquad (4.8b)
\end{aligned}
$$

for all $\mathbf{v}_i \in \mathbf{H}_N^1(\Omega_i)$ and $q_i \in L^2(\Omega_i)$. Since on outflow subdomains, the pressure p_i is uniquely defined as long a solution exists, it is clear that the solution is independent of the known global constant c_{p_Ω}.

Since we are interested in solving the linearized global system (4.4) by solving coupling conditions on the interface, the next lemma states under which conditions the local and the global formulations are equivalent.

Lemma 36. It holds that

$$\mathbf{c}_u = \sum_{i=1}^{s} \mathbf{c}_{u_i} + \mathcal{R}(\mathbf{c}_{u_\Gamma}) \in \mathbf{H}^1_D(\Omega),$$
$$c_p = \sum_{i=1}^{s} c_{p_i} + c_{p_\Omega} \in L^2(\Omega)$$

with $(\mathbf{c}_{u_i}, c_{p_i}) \in \mathbf{V}_i \times Q_i$ *solutions of (4.7) or (4.8) (depending on the subdomain type) solve (4.4) if and only if the following coupling conditions hold for* $(\mathbf{c}_{u_\Gamma}, c_{p_\Omega}) \in \mathbf{H}^{1/2}_{00}(\Gamma) \times D(\Omega)$ *at the linearization point* $(\mathbf{u}^k, p^k)$*:*
Find $\mathbf{c}_{u_\Gamma} \in \mathbf{H}^{1/2}_{00}(\Gamma)$ *and* $c_{p_\Omega} \in D(\Omega)$ *such that*

$$\sum_{i=1}^{s} \Big(a(\mathbf{c}_{u_i}, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + n_i(\mathbf{c}_{u_i}, \mathbf{u}^k_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + n_i(\mathbf{u}^k_i, \mathbf{c}_{u_i}, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + \\ + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), c_{p_i}) \Big) + a(\mathcal{R}(\mathbf{c}_{u_\Gamma}), \mathcal{R}(\mathbf{v}_\Gamma)) + n(\mathcal{R}(\mathbf{c}_{u_\Gamma}), \mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + \\ + n(\mathbf{u}^k, \mathcal{R}(\mathbf{c}_{u_\Gamma}), \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), c_{p_\Omega}) = F^1_\Gamma(\mathbf{u}^k, p^k, \mathbf{v}_\Gamma), \tag{4.9}$$
$$b(\mathcal{R}(\mathbf{c}_{u_\Gamma}), q_\Omega) = F^2_\Gamma(\mathbf{u}^k, q_\Omega) \tag{4.10}$$

for all $\mathbf{v}_\Gamma \in \mathbf{H}^{1/2}_{00}(\Gamma)$ *and* $q_\Omega \in D(\Omega)$.

Proof. This can be shown by using the same arguments as in Chapter 2 in the proof of Lemma 13. □

4.3 DISCRETIZATION BASED ON A FINITE ELEMENT METHOD

We use the same type of finite element method as described in Chapter 2. Consequently, we use the same definitions for the triangulation, Taylor-Hood spaces and Taylor-Hood finite elements.

For a more readable notation of the linearization point and corrections, we leave out the superscript h, indicating a discrete function. In the following, we denote by $(\mathbf{u}^k_i, p^k_i) \in \widetilde{\mathbf{V}}^h_i \times Q^h_i$ a discrete local linearization point, and by $(\mathbf{u}^k, p^k) \in \mathbf{V}^h \times Q^h$ a discrete global linearization point.

4.3.1 *Decoupled Finite Element Formulation*

Using the definitions for the finite element method, we discretize the s decoupled subdomain weak formulations and the system of coupling conditions. That is the third main step of the derivation, which is also illustrated in Fig. 9.

On non-outflow subdomains, we get the following finite element discretization for the local linearized Navier-Stokes systems corresponding to the alternative formulation (4.7): For a given $\mathbf{c}^k_{u_\Gamma} \in \mathbf{V}^h_\Gamma$,

find $\mathbf{c}_{u_i}^k \in \mathbf{V}_i^h$, $c_{p_i}^k \in Q_i^h$ and $c_{\eta_i}^k \in \mathbb{R}$ at the linearization point $(\mathbf{u}_i^k, p_i^k)$, such that

$$\begin{aligned} a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + a_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, c_{p_i}^k) + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + \\ + n_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) = F_i^1(\mathbf{u}_i^k, p_i^k, \mathbf{v}_i^h), \end{aligned} \tag{4.11a}$$

$$b_i(\mathbf{c}_{u_i}^k, q_i^h) + b_i(\mathbf{c}_{u_{\Gamma_i}}^k, q_i^h) + c_i(c_{\eta_i}^k, q_i^h) = F_i^2(\mathbf{u}_i^k, q_i^h), \tag{4.11b}$$

$$c_i(\xi_i^h, c_{p_i}^k) = F_i^3(p_i^k, \xi_i^h) \tag{4.11c}$$

for all $\mathbf{v}_i^h \in \mathbf{V}_i^h$, $q_i^h \in Q_i^h$ and $\xi_i^h \in \mathbb{R}$.

Analogously the finite element formulation on outflow subdomains yields:

For a given $\mathbf{c}_{u_\Gamma}^k \in \mathbf{V}_\Gamma^h$, find $\mathbf{c}_{u_i}^k \in \mathbf{V}_i^h$ and $c_{p_i}^k \in Q_i^h$ at the linearization point $(\mathbf{u}_i^k, p_i^k)$ such that

$$\begin{aligned} a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + a_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, c_{p_i}^k) + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + \\ + n_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) = F_i^1(\mathbf{u}_i^k, p_i^k, \mathbf{v}_i^h), \end{aligned} \tag{4.12a}$$

$$b_i(\mathbf{c}_{u_i}^k, q_i^h) + b_i(\mathbf{c}_{u_{\Gamma_i}}^k, q_i^h) = F_i^2(\mathbf{u}_i^k, q_i^h), \tag{4.12b}$$

for all $\mathbf{v}_i^h \in \mathbf{V}_i^h$ and $q_i^h \in Q_i^h$.

The finite element formulation of the coupling condition reads:

At the linearization point $(\mathbf{u}^k, p^k)$, find $\mathbf{c}_{u_\Gamma}^k \in \mathbf{V}_\Gamma^h$ and $c_{p_\Omega}^k \in D(\Omega)$ such that

$$\begin{aligned} &\sum_{i=1}^{s} \Big(a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_{\Gamma_i}^h) + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_{\Gamma_i}^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_{\Gamma_i}^h) + b_i(\mathbf{v}_{\Gamma_i}^h, c_{p_i}^k) \Big) + \\ &\quad + a(\mathbf{c}_{u_\Gamma}^k, \mathbf{v}_\Gamma^h) + n(\mathbf{c}_{u_\Gamma}^k, \mathbf{u}^k, \mathbf{v}_\Gamma^h) + n(\mathbf{u}^k, \mathbf{c}_{u_\Gamma}^k, \mathbf{v}_\Gamma^h) + b(\mathbf{v}_\Gamma^h, c_{p_\Omega}^k) + \\ &= F_\Gamma^1(\mathbf{u}^k, p^k, \mathbf{v}_\Gamma^h), \end{aligned} \tag{4.13a}$$

$$b(\mathbf{c}_{u_\Gamma}^k, q_\Omega^h) = F_\Gamma^2(\mathbf{u}^k, q_\Omega^h) \tag{4.13b}$$

for all $\mathbf{v}_\Gamma^h \in \mathbf{V}_\Gamma^h$ and $q_\Omega^h \in D(\Omega)$. $\mathbf{c}_{u_i}^h \in \mathbf{V}_i^h$ and $c_{p_i}^h \in Q_i^h$ are solutions of (4.11) and (4.12) depending on whether a subdomain Ω_i is a non-outflow or an outflow subdomain, respectively.

4.4 ALGEBRAIC FORMULATION

Based on the finite element discretization, we derive the resulting global linear system corresponding to (4.5) and the Schur-complement equations corresponding to (4.13). This is the last main step of the

derivation of the domain decomposition method. We use the same definition as in Chapter 2. Furthermore, we define the finite element function for the linearization point $\mathbf{u}_i^k$ and the resulting coefficient vector:

$$\mathbf{u}_i^k(\mathbf{x}) = \sum_{j=1}^{n_{u_i}+n_{u_{\Gamma_i}}} u_j^k \boldsymbol{\psi}_j(\mathbf{x}) \qquad \mathbf{u}_i^k = (u_1, \ldots, u_{n_{u_i}+n_{u_{\Gamma_i}}})^T \in \mathbb{R}^{n_{u_i}+n_{u_{\Gamma_i}}}.$$

We define the following matrices related to the bilinear form $a_i(\cdot,\cdot)$ and their entries

$$\begin{aligned}(\mathbf{A}\left[\mathbf{u}_i^k\right])_{tl} = (\mathbf{A})_{tl} &= a_i(\boldsymbol{\psi}_l, \boldsymbol{\psi}_t) + n_i(\mathbf{u}_i^k, \boldsymbol{\psi}_l, \boldsymbol{\psi}_t) + n_i(\boldsymbol{\psi}_l, \mathbf{u}_i^k, \boldsymbol{\psi}_t) \\ &= \mu \int_{\Omega_i} \boldsymbol{\nabla}\boldsymbol{\psi}_l : \boldsymbol{\nabla}\boldsymbol{\psi}_t \, d\mathbf{x} + \\ &\quad + \int_{\Omega_i} (\mathbf{u}_i^k(\mathbf{x}) \cdot \boldsymbol{\nabla})\boldsymbol{\psi}_l)\boldsymbol{\psi}_t d\mathbf{x} + \int_{\Omega_i} ((\boldsymbol{\psi}_l \cdot \boldsymbol{\nabla})\mathbf{u}_i^k(\mathbf{x}))\boldsymbol{\psi}_t d\mathbf{x}.\end{aligned}$$

The submatrices $\mathbf{A}_{ii}\left[\mathbf{u}_i^k\right]$, $\mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right]$, $\mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right]$, $\mathbf{A}_{\Gamma_i\Gamma_i}\left[\mathbf{u}_i^k\right]$, $\mathbf{A}_{\Gamma\Gamma}\left[\mathbf{u}^k\right]$, $\mathbf{A}_{iD_i}\left[\mathbf{u}_i^k\right]$ and $\mathbf{A}_{\Gamma D}\left[\mathbf{u}^k\right]$ are defined analogously to the definitions in Chapter 2.

Furthermore, we define the finite element functions for the corrections of the velocity and the resulting coefficient vectors:

$$\begin{aligned}\mathbf{c}_{\mathbf{u}_i}^h(\mathbf{x}) &= \sum_{j=1}^{n_{u_i}} c_{u_j}\boldsymbol{\psi}_j(\mathbf{x}) & \mathbf{c}_{\mathbf{u}_i} &= (c_{u_1}, \ldots, c_{u_{n_{u_i}}})^T \in \mathbb{R}^{n_{u_i}}, \\ \mathbf{c}_{\mathbf{u}_\Gamma}^h(\mathbf{x}) &= \sum_{j=1}^{n_{u_\Gamma}} c_{u_j}\boldsymbol{\psi}_j(\mathbf{x}) & \mathbf{c}_{\mathbf{u}_\Gamma} &= (c_{u_1}, \ldots, c_{u_{n_{u_\Gamma}}})^T \in \mathbb{R}^{n_{u_\Gamma}}, \\ \mathbf{c}_{\mathbf{u}_{\Gamma_i}}^h(\mathbf{x}) &= \sum_{j=1}^{n_{u_{\Gamma_i}}} c_{u_j}\boldsymbol{\psi}_j(\mathbf{x}) & \mathbf{c}_{\mathbf{u}_{\Gamma_i}} &= (c_{u_1}, \ldots, c_{n_{u_{\Gamma_i}}})^T \in \mathbb{R}^{n_{u_{\Gamma_i}}}.\end{aligned}$$

Finally, we define the finite element functions for the corrections of the pressure and the resulting coefficient vectors:

$$\begin{aligned}c_{p_i}^h(\mathbf{x}) &= \sum_{j=1}^{n_{q_i}} c_{p_j}\varphi_j(\mathbf{x}) & \mathbf{c}_{p_i} &= (c_{p_1}, \ldots, c_{p_{n_{q_i}}})^T \in \mathbb{R}^{n_{q_i}}, \\ c_{p_\Omega}^h(\mathbf{x}) &= \sum_{j=1}^{s_{\overline{\mathcal{N}}}} c_{p_{\Omega_j}}\varphi_{\Omega_j}(\mathbf{x}) & \mathbf{c}_{p_\Omega} &= (c_{p_{\Omega_1}}, \ldots, c_{p_{s_{\overline{\mathcal{N}}}}})^T \in \mathbb{R}^{s_{\overline{\mathcal{N}}}}.\end{aligned}$$

4.4.1 *Global Linear System and Schur-complement Equation*

Due to better readability, we write the global linear system corresponding to the finite element formulation of (4.5) for an example with two

subdomains. We assume that Ω_1 is a non-outflow subdomain and that Ω_2 is an outflow subdomain.

$$\left(\begin{array}{cc|cc|cc} \mathbf{A}_{11}\left[\mathbf{u}_1^k\right] & \widetilde{\mathbf{B}}_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1}\left[\mathbf{u}_1^k\right] & \mathbf{0} \\ \widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma_1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{A}_{22}\left[\mathbf{u}_2^k\right] & \mathbf{B}_{22}^T & \mathbf{A}_{2\Gamma_2}\left[\mathbf{u}_2^k\right] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{B}_{2\Gamma_2} & \mathbf{0} \\ \hline \mathbf{A}_{\Gamma_1 1}\left[\mathbf{u}_1^k\right] & \widetilde{\mathbf{B}}_{1\Gamma_1}^T & \mathbf{A}_{\Gamma_2 2}\left[\mathbf{u}_2^k\right] & \mathbf{B}_{2\Gamma_2}^T & \mathbf{A}_{\Gamma\Gamma}\left[\mathbf{u}^k\right] & \mathbf{B}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} \end{array}\right)\left(\begin{array}{c} \mathbf{c}_{u_1} \\ \mathbf{c}_{p_1} \\ \hline \mathbf{c}_{u_2} \\ \mathbf{c}_{p_2} \\ \hline \mathbf{c}_{u_\Gamma} \\ \mathbf{c}_{p_\Omega} \end{array}\right) = \left(\begin{array}{c} \mathbf{F}_1^1\left[\mathbf{u}_1^k, \mathbf{p}_1^k\right] \\ \mathbf{F}_1^2\left[\mathbf{u}_1^k\right] \\ \hline \mathbf{F}_2^1\left[\mathbf{u}_2^k, \mathbf{p}_2^k\right] \\ \mathbf{F}_2^2\left[\mathbf{u}_2^k\right] \\ \hline \mathbf{F}_\Gamma^1\left[\mathbf{u}^k, \mathbf{p}^k\right] \\ \mathbf{F}_\Gamma^2\left[\mathbf{u}^k\right] \end{array}\right)$$

Knowing that the local block matrices (with the subindex $\cdot_{ii}$) are invertible, we can reduce the global problem to a Schur-complement equation. We write the alternative formulation directly using the Schur-complement matrix S which is defined as follows:

$$\begin{aligned} S[\mathbf{u}^k] := & \begin{pmatrix} \mathbf{A}_{\Gamma\Gamma}\left[\mathbf{u}^k\right] & \mathbf{B}_0^T \\ \mathbf{B}_0 & \mathbf{0} \end{pmatrix} + \\ & - \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix}}_{:=L1} + \\ & - \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & 0 \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix}}_{:=L2}, \end{aligned} \tag{4.14}$$

and the right hand side r:

$$\begin{aligned} r\left[\mathbf{u}^k, \mathbf{p}^k\right] := & \begin{pmatrix} \mathbf{F}_\Gamma^1\left[\mathbf{u}^k, \mathbf{p}^k\right] \\ \mathbf{F}_\Gamma^2\left[\mathbf{u}^k\right] \end{pmatrix} + \\ & + \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{F}_i^1\left[\mathbf{u}_i^k, \mathbf{p}_i^k\right] \\ \mathbf{F}_i^2\left[\mathbf{u}_i^k\right] \end{pmatrix}}_{:=L3} + \\ & + \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & 0 \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & 0 \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ 0 & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{F}_i^1\left[\mathbf{u}_i^k, \mathbf{p}_i^k\right] \\ \mathbf{F}_i^2\left[\mathbf{u}_i^k\right] \\ \mathbf{F}_i^3\left[\mathbf{p}_i^k\right] \end{pmatrix}}_{:=L4}. \end{aligned} \tag{4.15}$$

These definitions lead to the linearized Schur-complement equation, which represents algebraically the coupling on the skeleton (4.13):

$$S\left[\mathbf{u}^k\right](\mathbf{c}_{u_\Gamma}^k, \mathbf{c}_{p_\Omega}^k) = r\left[\mathbf{u}^k, \mathbf{p}^k\right].$$

The subproblems $L1$ - $L4$, defined in (4.14) and (4.15), can be interpreted as solving local linearized Navier-Stokes problems on the subdomains.

$$L1 \text{ solve} : \begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c}_{u_i} \\ \mathbf{c}_{p_i} \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right]\mathbf{c}_{u_{\Gamma_i}} \\ -\mathbf{B}_{i\Gamma_i}\mathbf{c}_{u_{\Gamma_i}} \end{pmatrix}.$$

$L1$ solves a local linearized Navier-Stokes problem with Dirichlet data $\mathbf{c}_{u_{\Gamma_i}}$ on the local skeleton Γ_i, natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$, and zero Dirichlet boundary condition on $\partial\Omega_i \cap \partial\Omega_D$.

$$L2 \text{ solve} : \begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}_{u_i} \\ \mathbf{c}_{p_i} \\ \eta_i \end{pmatrix} = \begin{pmatrix} -\mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right]\mathbf{c}_{u_{\Gamma_i}} \\ -\mathbf{B}_{i\Gamma_i}\mathbf{c}_{u_{\Gamma_i}} \\ 0 \end{pmatrix}.$$

$L2$ solves a local linearized Navier-Stokes problem with Dirichlet data. For inner subdomains, the Dirichlet data is set to $\mathbf{c}_{\mathbf{u}_{\Gamma_i}}$ on Γ_i. For subdomains intersecting the global Dirichlet boundary, for which $\partial\Omega_i \cap \partial\Omega_D \neq \emptyset$, the Dirichlet data on the global boundary is set to zero, whereas on the skeleton Γ_i, the Dirichlet data is set to $\mathbf{c}_{\mathbf{u}_{\Gamma_i}}$.

$$L3 \text{ solve} : \begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c}_{u_i} \\ \mathbf{c}_{p_i} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^1\left[\mathbf{u}_i^k, \mathbf{p}_i^k\right] \\ \mathbf{F}_i^2\left[\mathbf{u}_i^k\right] \end{pmatrix}.$$

$L3$ solves a local linearized Navier-Stokes problem with Dirichlet data and outflow conditions. On the local skeleton Γ_i, zero Dirichlet data is set. On $\partial\Omega_i \cap \partial\Omega_{out}$, the outflow conditions hold, and on $\partial\Omega_i \cap \partial\Omega_D$ the Dirichlet data is set to $\mathbf{u}_{D_i}$.

$$L4 \text{ solve} : \begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}_{u_i} \\ \mathbf{c}_{p_i} \\ \eta_i \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^1\left[\mathbf{u}_i^k, \mathbf{p}_i^k\right] \\ \mathbf{F}_i^2\left[\mathbf{u}_i^k\right] \\ \mathbf{F}_i^3\left[\mathbf{p}_i^k\right] \end{pmatrix}.$$

$L4$ solves a local linearized Navier-Stokes problem with Dirichlet data. For inner subdomains, a homogeneous Dirichlet boundary problem is solved. For subdomains intersecting the global boundary, the Dirichlet data on the global boundary is set to $\mathbf{u}_{D_i}$, whereas on the skeleton boundary Γ_i, the Dirichlet data is zero.

Adding $L1$ and $L3$ leads to

$$L1 + L3 \text{ solve} :$$
$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{c}_{u_i} \\ \mathbf{c}_{p_i} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^1\left[\mathbf{u}_i^k, \mathbf{p}_i^k\right] - \mathbf{A}_{i\Gamma}\left[\mathbf{u}_i^k\right]\mathbf{c}_{u_{\Gamma_i}} \\ \mathbf{F}_i^2\left[\mathbf{u}_i^k\right] - \mathbf{B}_{i\Gamma_i}\mathbf{c}_{u_{\Gamma_i}} \end{pmatrix},$$

which is the algebraic representation of the local decoupled subproblems on subdomain Ω_i (4.12).

Similarly, adding $L2$ and $L4$ leads to

$L2 + L4$ solve :

$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}_{u_i} \\ \mathbf{c}_{p_i} \\ \eta_i \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^1\left[\mathbf{u}_i^k, \mathbf{p}_i^k\right] - \mathbf{A}_{i\Gamma}\left[\mathbf{u}_i^k\right]\mathbf{c}_{u_{\Gamma_i}} \\ \mathbf{F}_i^2\left[\mathbf{u}_i^k\right] - \mathbf{B}_{i\Gamma_i}\mathbf{c}_{u_{\Gamma_i}} \\ \mathbf{F}_i^3\left[\mathbf{p}_i^k\right] \end{pmatrix},$$

which is the algebraic representation of the local decoupled subproblems on subdomain Ω_i (4.11).

4.4.2 *Solution Algorithm*

Algorithm 4.2 Newton-Schur-complement Algorithm

1: Input: Appropriate start solution $x^0 \in \mathbf{V}^h \times Q^h$
2: $k = 0$
3: Assemble (on each process) subdomain matrix $\mathbf{K}_i[x^k]$
4: Compute $r[x^k]$ (implies solving $L3$ or $L4$) ▷ Ch. 6, Alg. 6.6
5: Solve Schur-complement equation $S[x^k](c_{u_\Gamma}^k, c_{p_\Omega}^k) = r[x^k]$ (implies solving $L1$ or $L2$) ▷ Ch. 6, Alg. 6.7
6: Compute global correction c^k based on interface corrections $(c_{u_\Gamma}^k, c_{p_\Omega}^k)$ (implies solving $L1 + L3$ or $L2 + L4$) ▷ Ch. 6, Alg. 6.8
7: Update global iteration $x^{k+1} = x^k - c^k$
8: **if** Stopping criteria fulfilled **then**
9: **return** x^{k+1}
10: **else**
11: $k = k + 1$
12: Go to step 3
13: **end if**

Algorithm 4.2 describes how the outer loop of the Newton method is combined with the inner loop of the domain decomposition method to solve the Schur-complement equation, see also [7, 8]. To compute a new global iterate of the Newton method, we have to first compute a new right hand side, then solve a Schur-complement equation and finally compute the global correction. More detailed Algorithms for some of the steps described here are given in Chapter 6.

4.4.3 *General Operator*

In Chapter 6, we derive a general solution algorithm for the global linear system and the Schur-complement equation, which is independent of the model problem. To be able to treat all derived model problems at a time, we derive a general subdomain operator, which is based

on the block matrices of the global linear system. For non-outflow subdomains, we get:

$$\mathbf{K}_i = \left(\begin{array}{cc|cc} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \widetilde{\mathbf{B}}_{ii}^T & \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{0} \\ \widetilde{\mathbf{B}}_{ii} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i} & \mathbf{0} \\ \hline \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \widetilde{\mathbf{B}}_{i\Gamma_i}^T & \mathbf{A}_{\Gamma_i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{0,i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{0,i} & 0 \end{array}\right)$$
$$:= \left(\begin{array}{cc|cc} \mathcal{A}_{ii} & \mathcal{B}_{ii}^T & \mathcal{A}_{i\Gamma_i} & \mathbf{0} \\ \mathcal{B}_{ii} & \mathbf{0} & \mathcal{B}_{i\Gamma_i} & \mathbf{0} \\ \hline \mathcal{A}_{\Gamma_i i} & \mathcal{B}_{\Gamma_i i}^T & \mathcal{A}_{\Gamma_i\Gamma_i} & \mathcal{B}_{\Omega_i}^T \\ \mathbf{0} & \mathbf{0} & \mathcal{B}_{\Omega_i} & \mathbf{0} \end{array}\right).$$

For outflow subdomains, we get almost the same definition. We only need to replace all submatrices of $\widetilde{\mathbf{B}}$ with the corresponding submatrices of $\mathbf{B}$.

SUMMARY

In this chapter, we derived a non-overlapping domain decomposition method for the Navier-Stokes equations equipped with mixed outflow and Dirichlet boundary conditions. The main focus of this chapter was how to treat the non-linearity by combining the Newton-method with the domain decomposition approach. We applied the Newton-method to the linearization of the model problem. The Newton-method is then applied as an outer loop.

In Chapter 3, we extended the domain decomposition method to the context of optimal flow control problem and in this chapter, we dealt with the non-linearity of a flow problem. In the next chapter, we combine the methods derived in those two chapters to finally apply a domain decomposition method to a non-linear optimal flow control problem constrained by the full Navier-Stokes equations.

5

DOMAIN DECOMPOSITION METHOD FOR AN OPTIMAL CONTROL PROBLEM CONSTRAINED BY THE NAVIER-STOKES EQUATIONS

In this chapter, we finally treat a non-linear quadratic optimal flow control problem constrained by the full Navier-Stokes equations. Therefore, we combine all the insights gained in the previous chapters: In Chapter 2, we focused on the saddle point structure of a flow control problems in the context of a domain decomposition method, in Chapter 3, we emphasized on the extension of the domain decomposition approach to a linear optimal flow control problem and in Chapter 4, we dealt with the non-linearity. Using this knowledge, we derive a non-overlapping domain decomposition method for a non-linear quadratic optimal flow control problem. We assume distributed control and equip the Navier-Stokes equations with mixed outflow and Dirichlet boundary conditions. Analogously to the non-linear flow problem, the domain decomposition approach is applied to the linearization of the optimality system.

If not stated differently, we use the same definitions, notation and assumptions as in Chapters 2-4.

5.1 DISTRIBUTED OPTIMAL CONTROL PROBLEM

In the first step, we introduce the global model problem and derive the optimality system. As in Chapter 3, let $\Omega_C, \Omega_0 \subseteq \Omega \subset \mathbb{R}^d$ be Lipschitz domains. Again, we assume that the boundary $\partial\Omega$ is decomposed into $\partial\Omega = \partial\Omega_{out} \cup \partial\Omega_D$ with $\partial\Omega_{out} \cap \partial\Omega_D = \emptyset$. This first step is also depicted in Fig. 10.

Under the assumptions (4.1) and given $\widehat{\mathbf{u}} \in \mathbf{L}^2(\Omega_0)$, our aim is to solve an optimal control problem constrained by the non-linear Navier-Stokes equations modeling a Newtonian fluid equipped with mixed outflow and Dirichlet boundary conditions. We apply distributed control with support on Ω_C and consider the following optimal control problem:

$$\min_{\mathbf{u},\mathbf{c}} \frac{1}{2} \int_{\Omega_0} (\mathbf{u} + \mathbf{u}_D - \hat{\mathbf{u}})^2 d\mathbf{x} + \frac{\alpha}{2} \int_{\Omega_C} \mathbf{c}^2 \, d\mathbf{x} \tag{5.1a}$$

$$\text{subject to (s.t.) } (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} - \boldsymbol{\nabla}\cdot\sigma(\mathbf{u},p) = \mathbf{f} + \mathbb{1}_{\Omega_c}\mathbf{c} \qquad \text{in } \Omega, \tag{5.1b}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{5.1c}$$

$$\mathbf{u} = \mathbf{d} \qquad \text{on } \partial\Omega_D, \tag{5.1d}$$

$$\sigma(\mathbf{u},p)\mathbf{n} = \mathbf{h} \qquad \text{on } \partial\Omega_{out}. \tag{5.1e}$$

As in the previous chapters, we assume that an extension $\mathbf{u}_D \in \mathbf{H}^1(\Omega)$ of $\mathbf{d}$ exists, such that $\gamma_d(\mathbf{u}_D) = \mathbf{d}$. We present a weak formulation for (5.1), using the definitions from the previous Chapters 2-4. In this chapter, we refer to the definition of the bilinear form $a(\cdot,\cdot)$ as stated in Chapter 4.

With this notation, we seek a solution of the following non-linear quadratic distributed optimal control problem: Find $\mathbf{u} \in \mathbf{H}^1_D(\Omega)$, $p \in L^2(\Omega)$ and $\mathbf{c} \in \mathbf{L}^2(\Omega_c)$ such that:

$$\min_{\mathbf{u},\mathbf{c}} \frac{1}{2} m(\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}),\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}})) + \frac{\alpha}{2} m(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbb{1}_{\Omega_c}\mathbf{c}) \tag{5.2a}$$

$$\begin{aligned} \text{s.t.} \quad & a(\mathbf{u},\mathbf{v}) + n(\mathbf{u},\mathbf{u},\mathbf{v}) + \\ & + b(\mathbf{v},p) - d(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbf{v}) = f(\mathbf{v}) - a(\mathbf{u}_D,\mathbf{v}) - n(\mathbf{u}_D,\mathbf{u}_D,\mathbf{v}), \end{aligned} \tag{5.2b}$$

$$b(\mathbf{u},q) = -b(\mathbf{u}_D,q) \tag{5.2c}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$ and $q \in L^2(\Omega)$.

Lemma 37. *Under the assumption that an optimal solution* $(\mathbf{u}, p, \mathbf{c})^* \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega) \times \mathbf{L}^2(\Omega_c)$ *for (5.2) and the Lagrange multipliers* $(\mathbf{z}, r) \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega)$ *exist, the optimal solution is a KKT point. It fulfills the necessary optimality conditions described by the KKT conditions which are given by the following optimality system: Find* $\mathbf{u}, \mathbf{z} \in \mathbf{H}^1_D(\Omega)$, $p, r \in L^2(\Omega)$ *and* $\mathbf{c} \in \mathbf{L}^2(\Omega_c)$ *such that*

$$\begin{aligned} a(\mathbf{v},\mathbf{z}) + n(\mathbf{v},\mathbf{u},\mathbf{z}) + n(\mathbf{u},\mathbf{v},\mathbf{z}) + & \\ + b(\mathbf{v},r) - m(\mathbb{1}_{\Omega_0}\mathbf{u},\mathbb{1}_{\Omega_0}\mathbf{v}) &= -m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}-\mathbf{u}_D),\mathbb{1}_{\Omega_0}\mathbf{v}), && (5.3a)\\ b(\mathbf{z},q) &= 0, && (5.3b)\\ d(\mathbb{1}_{\Omega_c}\mathbf{e},\mathbf{z}) + \alpha m(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbb{1}_{\Omega_c}\mathbf{e}) &= 0, && (5.3c)\\ a(\mathbf{u},\mathbf{v}) + n(\mathbf{u},\mathbf{u},\mathbf{v}) + & && (5.3d)\\ + b(\mathbf{v},p) - d(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbf{v}) &= f(\mathbf{v}) - a(\mathbf{u}_D,\mathbf{v}) - n(\mathbf{u}_D,\mathbf{u}_D,\mathbf{v}), & \\ b(\mathbf{u},q) &= -b(\mathbf{u}_D,q) && (5.3e) \end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$ *and* $\mathbf{e} \in \mathbf{L}^2(\Omega_c)$.

Proof. We derive the optimality system (5.3) by applying a Lagrangian based adjoint approach [33]. Therefore we define the Lagrange function

$$\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c}) =$$

$$\frac{1}{2}m(\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}),\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}))+\frac{\alpha}{2}m(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbb{1}_{\Omega_c}\mathbf{c})+$$
$$-a(\mathbf{u},\mathbf{z})-n(\mathbf{u},\mathbf{u},\mathbf{z})-b(\mathbf{z},p)+d(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbf{z})+$$
$$+f(\mathbf{z})-a(\mathbf{u}_D,\mathbf{z})-n(\mathbf{u}_D,\mathbf{u}_D,\mathbf{z})+-b(\mathbf{u},r)+b(\mathbf{u}_D,r),$$

and derive a stationary condition, which reads:

$$\begin{aligned}
\frac{\partial}{\partial\mathbf{u}}\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c})\cdot\mathbf{v} &= m(\mathbb{1}_{\Omega_0}(\mathbf{u}+\mathbf{u}_D-\hat{\mathbf{u}}),\mathbb{1}_{\Omega_0}\mathbf{v})-a(\mathbf{v},\mathbf{z})+\\
&\quad -n(\mathbf{v},\mathbf{u},\mathbf{z})-n(\mathbf{u},\mathbf{v},\mathbf{z})-b(\mathbf{v},r)=0,\\
\frac{\partial}{\partial p}\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c})\cdot q &= -b(\mathbf{z},q)=0,\\
\frac{\partial}{\partial\mathbf{c}}\mathcal{L}(\mathbf{u},p,\mathbf{z},r,\mathbf{c})\cdot\mathbf{e} &= \alpha m(\mathbb{1}_{\Omega_c}\mathbf{c},\mathbb{1}_{\Omega_c}\mathbf{e})+d(\mathbb{1}_{\Omega_c}\mathbf{e},\mathbf{z})=0
\end{aligned}$$

for all $\mathbf{v}\in\mathbf{H}^1_D(\Omega)$, $q\in L^2(\Omega)$ and $\mathbf{e}\in\mathbf{L}^2(\Omega_c)$.

For more details, we also refer to [24]. □

We solve this non-linear optimality system by applying the Newton method, see Algorithm 4.1 in Chapter 4 for more details. Therefore, we define the residuals at the linearization point $(\mathbf{u}^k,p^k,\mathbf{c}^k,\mathbf{z}^k,r^k)\in\mathbf{H}^1_D(\Omega)\times L^2(\Omega)\times L^2(\Omega_C)\times\mathbf{H}^1_D(\Omega)\times L^2(\Omega)$:

$$\begin{aligned}
F^z(\mathbf{u}^k,\mathbf{z}^k,r^k,\mathbf{v}) &:= a(\mathbf{v},\mathbf{z}^k)+n(\mathbf{v},\mathbf{u}^k,\mathbf{z}^k)+n(\mathbf{u}^k,\mathbf{v},\mathbf{z}^k)+b(\mathbf{v},r^k)+\\
&\quad -m(\mathbb{1}_{\Omega_0}\mathbf{u}^k,\mathbb{1}_{\Omega_0}\mathbf{v})+m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}-\mathbf{u}_D),\mathbb{1}_{\Omega_0}\mathbf{v}),\\
F^r(\mathbf{z}^k,q) &:= b(\mathbf{z}^k,q),\\
F^c(\mathbf{c}^k,\mathbf{z}^k,\mathbf{e}) &:= d(\mathbb{1}_{\Omega_c}\mathbf{e},\mathbf{z}^k)+\alpha m(\mathbb{1}_{\Omega_c}\mathbf{c}^k,\mathbb{1}_{\Omega_c}\mathbf{e}),\\
F^u(\mathbf{u}^k,p^k,\mathbf{c}^k,\mathbf{v}) &:= a(\mathbf{u}^k,\mathbf{v})+n(\mathbf{u}^k,\mathbf{u}^k,\mathbf{v})+b(\mathbf{v},p^k)-d(\mathbb{1}_{\Omega_c}\mathbf{c}^k,\mathbf{v})+\\
&\quad -f(\mathbf{v})+a(\mathbf{u}_D,\mathbf{v})+n(\mathbf{u}_D,\mathbf{u}_D,\mathbf{v}),\\
F^p(\mathbf{u}^k,q) &:= b(\mathbf{u}^k,q)+b(\mathbf{u}_D,q).
\end{aligned}$$

We set

$$\mathbf{F}(\mathbf{u}^k,p^k,\mathbf{c}^k,\mathbf{z}^k,r^k,\mathbf{v},q,\mathbf{e})=\begin{pmatrix}F^z(\mathbf{u}^k,\mathbf{z}^k,r^k,\mathbf{v})\\F^r(\mathbf{z}^k,q)\\F^c(\mathbf{c}^k,\mathbf{z}^k,\mathbf{e})\\F^u(\mathbf{u}^k,p^k,\mathbf{c}^k,\mathbf{v})\\F^p(\mathbf{u}^k,q)\end{pmatrix}.$$

Our root problem then reads

$$\mathbf{F}(\mathbf{u},p,\mathbf{c},\mathbf{z},r,\mathbf{v},q)=0\quad\forall\mathbf{v}\in\mathbf{H}^1_D(\Omega),\,q\in L^2(\Omega).$$

Next, we linearize (5.3) at the linearization point $(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{z}^k, r^k)$ and obtain the following weak formulation for the kth Newton step: Find the corrections $\mathbf{c}_u \in \mathbf{H}^1_D(\Omega)$, $c_p \in L^2(\Omega)$, $\mathbf{c}_c \in L^2(\Omega_C)$, $\mathbf{c}_z \in \mathbf{H}^1_D(\Omega)$ and $c_r \in L^2(\Omega)$ such that

$$\begin{aligned}
a(\mathbf{v}, \mathbf{c}_z^k) + n(\mathbf{v}, \mathbf{c}_u^k, \mathbf{z}^k) + n(\mathbf{v}, \mathbf{u}^k, \mathbf{c}_z^k) + \\
+ n(\mathbf{c}_u^k, \mathbf{v}, \mathbf{z}^k) + n(\mathbf{u}^k, \mathbf{v}, \mathbf{c}_z^k) + \\
+ b(\mathbf{v}, c_r^k) - m(\mathbb{1}_{\Omega_0}\mathbf{c}_u^k, \mathbb{1}_{\Omega_0}\mathbf{v}) &= F^z(\mathbf{u}^k, \mathbf{z}^k, r^k, \mathbf{v}), && \text{(5.4a)} \\
b(\mathbf{c}_z^k, q) &= F^r(\mathbf{z}^k, q), && \text{(5.4b)} \\
d(\mathbb{1}_{\Omega_c}\mathbf{e}, \mathbf{c}_z^k) + \alpha m(\mathbb{1}_{\Omega_c}\mathbf{c}_c^k, \mathbb{1}_{\Omega_c}\mathbf{e}) &= F^c(\mathbf{c}^k, \mathbf{z}^k, \mathbf{e}), && \text{(5.4c)} \\
a(\mathbf{c}_u^k, \mathbf{v}) + n(\mathbf{c}_u^k, \mathbf{u}^k, \mathbf{v}) + n(\mathbf{u}^k, \mathbf{c}_u^k, \mathbf{v}) + \\
+ b(\mathbf{v}, c_p^k) - d(\mathbb{1}_{\Omega_c}\mathbf{c}_c^k, \mathbf{v}) &= F^u(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{v}), && \text{(5.4d)} \\
b(\mathbf{c}_u^k, q) &= F^p(\mathbf{u}^k, q) && \text{(5.4e)}
\end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}^1_D(\Omega)$, $q \in L^2(\Omega)$ and $\mathbf{e} \in L^2(\Omega_C)$.

5.2 CONTINUOUS DOMAIN DECOMPOSITION

In the second main step, we derive a non-overlapping domain decomposition method on the continuous level for the linearized weak optimality system (5.4). As in the previous chapter in the first sub-step, we derive a fully coupled equivalent formulation on subdomains, see also Fig. 12. In the second sub-step, we decouple the system, which leads to s locally independent linearized optimality systems and one system of linearized coupling conditions, see Fig. 13. We partition the domain as explained in Chapter 2 with the additional restrictions as described in Chapter 3 for the control domain Ω_C. We also use the same definitions as in the previous chapters for the decomposition of the spaces and the bilinear forms.

5.2.1 *Decomposition of the Residuals*

Additionally to the decomposition of $\mathbf{u}^k$ and p^k in the last Chapter 4, we analogously partition $\mathbf{z}^k$, p^k and $\mathbf{c}^k$ as follows:

$$\begin{aligned}
\mathbf{z}_i^k &:= \mathbf{z}^k|_{\Omega_i}, \qquad \mathbf{z}^k := (\mathbf{z}_1^k, \mathbf{z}_2^k, \dots, \mathbf{z}_s^k) \in \bigotimes_{i=1}^{s} \widetilde{\mathbf{V}}_i, \\
r_i^k &:= r^k|_{\Omega_i} \in \begin{cases} L^2(\Omega_i) & \text{if } i \in \mathcal{N} \\ L^2_0(\Omega_i) & \text{if } i \notin \mathcal{N} \end{cases}, \\
r^k &:= (r_1^k, r_2^k, \dots, r_s^k, r_\Omega^k) \in \bigoplus_{i=1}^{s} Q_i \oplus D(\Omega),
\end{aligned}$$

$$\mathbf{c}_i^k := \mathbf{c}^k|_{\Omega_i}, \qquad \mathbf{c}^k := (\mathbf{c}_1^k, \mathbf{c}_2^k, \ldots, \mathbf{c}_s^k) \in \bigoplus_{i=1}^{s} L^2(\Omega_{C_i}).$$

Furthermore, we define the local residuals:

$$\begin{aligned}
F_i^z(\mathbf{u}_i^k, \mathbf{z}_i{}^k, r_i^k, \mathbf{v}_i) :&= a_i(\mathbf{v}_i, \mathbf{z}_i{}^k) + b_i(\mathbf{v}_i, r_i^k) + \\
&\quad + n_i(\mathbf{v}_i, \mathbf{u}_i^k, \mathbf{z}_i{}^k) + n_i(\mathbf{u}_i^k, \mathbf{v}_i, \mathbf{z}_i{}^k) + \\
&\quad - m_i(\mathbb{1}_{\Omega_0}\mathbf{u}_i^k, \mathbb{1}_{\Omega_0}\mathbf{v}_i) + m_i(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}}|_{\Omega_i} - \mathbf{u}_{D_i}), \mathbb{1}_{\Omega_0}\mathbf{v}_i), \\
F_i^r(\mathbf{z}_i{}^k, q_i) :&= b_i(\mathbf{z}_i{}^k, q_i), \\
F_i^\theta(r_i^k, \xi_i) :&= c_i(\xi_i, r_i^k), \\
F_i^c(\mathbf{c}_i^k, \mathbf{z}_i{}^k, \mathbf{e}_i) :&= d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i, \mathbf{z}_i{}^k) + \alpha m_i(\mathbb{1}_{\Omega_c}\mathbf{c}_i^k, \mathbb{1}_{\Omega_c}\mathbf{e}_i), \\
F_i^u(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{v}_i) :&= a_i(\mathbf{u}_i^k, \mathbf{v}_i) + n_i(\mathbf{u}_i^k, \mathbf{u}_i^k, \mathbf{v}_i) + \\
&\quad + b_i(\mathbf{v}_i, p_i^k) - d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_i^k, \mathbf{v}_i) + \\
&\quad - f_i(\mathbf{v}_i) + a_i(\mathbf{u}_{D_i}, \mathbf{v}_i) + n_i(\mathbf{u}_{D_i}, \mathbf{u}_{D_i}, \mathbf{v}_i), \\
F_i^p(\mathbf{u}_i^k, q_i) :&= b_i(\mathbf{u}_i^k, q_i) + b_i(\mathbf{u}_{D_i}, q_i), \\
F_i^\eta(p_i^k) :&= c_i(\xi_i, p_i^k).
\end{aligned}$$

with $\mathbf{v}_i \in \mathbf{V}_i$, $q_i \in Q_i$, $\mathbf{e}_i \in L^2(\Omega_{C_i})$ and $\xi_i \in \mathbb{R}$.

Moreover, we define the residuals on the interface:

$$\begin{aligned}
F_\Gamma^z(\mathbf{u}^k, \mathbf{z}^k, r^k, \mathbf{v}_\Gamma) :&= a(\mathcal{R}(\mathbf{v}_\Gamma), \mathbf{z}^k) + b(\mathcal{R}(\mathbf{v}_\Gamma), r^k) + \\
&\quad + n(\mathcal{R}(\mathbf{v}_\Gamma), \mathbf{u}^k, \mathbf{z}^k) + n(\mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma), \mathbf{z}^k) + \\
&\quad - m(\mathbb{1}_{\Omega_0}\mathbf{u}^k, \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma)) + m(\mathbb{1}_{\Omega_0}(\hat{\mathbf{u}} - \mathbf{u}_D), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma)), \\
F_\Gamma^r(\mathbf{z}^k, q_\Omega) :&= b(\mathbf{z}^k, q_\Omega), \\
F_\Gamma^u(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{v}_\Gamma) :&= a(\mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + n(\mathbf{u}^k, \mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + \\
&\quad + b(\mathcal{R}(\mathbf{v}_\Gamma), p^k) - d(\mathbb{1}_{\Omega_c}\mathbf{c}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + \\
&\quad - f(\mathcal{R}(\mathbf{v}_\Gamma)) + a(\mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)) + n(\mathbf{u}_D, \mathbf{u}_D, \mathcal{R}(\mathbf{v}_\Gamma)), \\
F_\Gamma^p(\mathbf{u}^k, q_\Omega) :&= b(\mathbf{u}^k, q_\Omega) + b(\mathbf{u}_D, q_\Omega).
\end{aligned}$$

with $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$ and $q_\Omega \in D(\Omega)$.

5.2.2 *Weak Formulation on Subdomains*

In the first sub-step of the second step, we derive an equivalent weak formulation on subdomains. Analogously to the previous chapters, this formulation is still globally coupled.

Lemma 38. *(5.4) is equivalent to the following weak formulation on subdomains: At the linearization point* $(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{z}^k, r^k) \in \mathbf{H}^1_D(\Omega) \times L^2(\Omega) \times L^2(\Omega_C) \times \mathbf{H}^1_D(\Omega) \times L^2(\Omega)$*, find the corrections*

$$
\begin{aligned}
\mathbf{c}_u &= \sum_{i=1}^{s} \mathbf{c}_{u_i} + \mathbf{c}_{u_\Gamma} \in \mathbf{H}^1_D(\Omega) && \text{with } \mathbf{c}_{u_i} \in \mathbf{V}_i,\ \mathbf{c}_{u_\Gamma} \in \mathbf{H}^{1/2}_{00}(\Gamma),\\
c_p &= \sum_{i=1}^{s} c_{p_i} + c_{p_\Omega} \in L^2(\Omega) && \text{with } c_{p_i} \in L^2(\Omega_i),\ c_{p_\Omega} \in D(\Omega),\\
\mathbf{c}_c &= \sum_{i=1}^{s} \mathbf{c}_{c_i} \in \mathbf{L}^2(\Omega_C) && \text{with } \mathbf{c}_{c_i} \in L^2(\Omega_{C_i}),\\
\mathbf{c}_z &= \sum_{i=1}^{s} \mathbf{c}_{z_i} + \mathbf{c}_{z_\Gamma} \in \mathbf{H}^1_D(\Omega) && \text{with } \mathbf{c}_{z_i} \in \mathbf{V}_i,\ \mathbf{c}_{z_\Gamma} \in \mathbf{H}^{1/2}_{00}(\Gamma),\\
c_r &= \sum_{i=1}^{s} c_{r_i} + c_{r_\Omega} \in L^2(\Omega) && \text{with } c_{r_i} \in L^2(\Omega_i),\ c_{r_\Omega} \in D(\Omega).
\end{aligned}
$$

such that

$$
\begin{aligned}
&\sum_{i=1}^{s} \Big(a_i(\mathbf{v}_i, \mathbf{c}^k_{z_i}) + a_i(\mathbf{v}_i, \mathcal{R}_i(\mathbf{c}^k_{z_{\Gamma_i}})) + a_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{c}^k_{z_i}) + n_i(\mathbf{v}_i, \mathbf{c}^k_{u_i}, \mathbf{z}^k_i) + \\
&\quad + n_i(\mathbf{v}_i, \mathcal{R}_i(\mathbf{c}^k_{u_{\Gamma_i}}), \mathbf{z}^k_i) + n_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{c}^k_{u_i}, \mathbf{z}^k_i) + n_i(\mathbf{v}_i, \mathbf{u}^k_i, \mathbf{c}^k_{z_i}) + \\
&\quad + n_i(\mathbf{v}_i, \mathbf{u}^k_i, \mathcal{R}_i(\mathbf{c}^k_{z_{\Gamma_i}})) + n_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{u}^k_i, \mathbf{c}^k_{z_i}) + n_i(\mathbf{c}^k_{u_i}, \mathbf{v}_i, \mathbf{z}^k_i) + \\
&\quad + n_i(\mathcal{R}_i(\mathbf{c}^k_{u_{\Gamma_i}}), \mathbf{v}_i, \mathbf{z}^k_i) + n_i(\mathbf{c}^k_{u_i}, \mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{z}^k_i) + n_i(\mathbf{u}^k_i, \mathbf{v}_i, \mathbf{c}^k_{z_i}) + \\
&\quad + n_i(\mathbf{u}^k_i, \mathbf{v}_i, \mathcal{R}_i(\mathbf{c}^k_{z_{\Gamma_i}})) + n_i(\mathbf{u}^k_i, \mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{c}^k_{z_i}) + b_i(\mathbf{v}_i, c^k_{r_i}) + \\
&\quad + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), c^k_{r_i}) - m_i(\mathbb{1}_{\Omega_0}\mathbf{c}^k_{u_i}, \mathbb{1}_{\Omega_0}\mathbf{v}_i) - m_i(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{c}^k_{u_{\Gamma_i}}), \mathbb{1}_{\Omega_0}\mathbf{v}_i) + \\
&\quad - m_i(\mathbb{1}_{\Omega_0}\mathbf{c}^k_{u_i}, \mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) \Big) + a(\mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{c}^k_{z_\Gamma})) + \\
&\quad + n(\mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{c}^k_{u_\Gamma}), \mathbf{z}^k) + n(\mathcal{R}(\mathbf{v}_\Gamma), \mathbf{u}^k, \mathcal{R}(\mathbf{c}^k_{z_\Gamma})) + \\
&\quad + n(\mathcal{R}(\mathbf{c}^k_{u_\Gamma}), \mathcal{R}(\mathbf{v}_\Gamma), \mathbf{z}^k) + n(\mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{c}^k_{z_\Gamma})) + \\
&\quad + b(\mathcal{R}(\mathbf{v}_\Gamma), c^k_{r_\Omega}) - m(\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{c}^k_{u_\Gamma}), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma)) \\
&= \sum_{i=1}^{s} F^z_i(\mathbf{u}^k_i, \mathbf{z}_i{}^k, r^k_i, \mathbf{v}_i) + F^z_\Gamma(\mathbf{u}^k, \mathbf{z}^k, r^k, \mathbf{v}_\Gamma) \qquad (5.5a)\\
&\sum_{i=1}^{s} \Big(b_i(\mathbf{c}^k_{z_i}, q_i) + b_i(\mathcal{R}_i(\mathbf{c}^k_{z_{\Gamma_i}}), q_i) \Big) + b(\mathcal{R}(\mathbf{c}^k_{z_\Gamma}), q_\Omega) \\
&= \sum_{i=1}^{s} \Big(F^r_i(\mathbf{z}_i{}^k, q_i) \Big) + F^r_\Gamma(\mathbf{z}^k, q_\Omega) \qquad (5.5b)\\
&\sum_{i=1}^{s} \Big(d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i, \mathbf{c}^k_{z_i}) + d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i, \mathcal{R}_i(\mathbf{c}^k_{z_{\Gamma_i}})) + \alpha m_i(\mathbb{1}_{\Omega_c}\mathbf{c}^k_{c_i}, \mathbb{1}_{\Omega_c}\mathbf{e}_i) \Big) \\
&= \sum_{i=1}^{s} \Big(F^c_i(\overline{\mathbf{c}}^k_i, \overline{\mathbf{z}}_i{}^k, \mathbf{e}_i) \Big) \qquad (5.5c)
\end{aligned}
$$

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathbf{c}_{u_i}^k,\mathbf{v}_i)+a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k),\mathbf{v}_i)+a_i(\mathbf{c}_{u_i}^k,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))+n_i(\mathbf{c}_{u_i}^k,\mathbf{u}_i^k,\mathbf{v}_i)+\\
&\quad+n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k),\mathbf{u}_i^k,\mathbf{v}_i)+n_i(\mathbf{c}_{u_i}^k,\mathbf{u}_i^k,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))+n_i(\mathbf{u}_i^k,\mathbf{c}_{u_i}^k,\mathbf{v}_i)+\\
&\quad+n_i(\mathbf{u}_i^k,\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k),\mathbf{v}_i)+n_i(\mathbf{u}_i^k,\mathbf{c}_{u_i}^k,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))-d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k,\mathbf{v}_i)+\\
&\quad-d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k,\mathcal{R}_i(\mathbf{v}_{\Gamma_i}))+b_i(\mathbf{v}_i,c_{p_i}^k)+b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}),c_{p_i}^k)\Big)+\\
&\quad+a(\mathcal{R}(\mathbf{c}_{u_\Gamma}^k),\mathcal{R}(\mathbf{v}_\Gamma))+n(\mathcal{R}(\mathbf{c}_{u_\Gamma}^k),\mathbf{u}^k,\mathcal{R}(\mathbf{v}_\Gamma))+\\
&\quad+n(\mathbf{u}^k,\mathcal{R}(\mathbf{c}_{u_\Gamma}^k),\mathcal{R}(\mathbf{v}_\Gamma))+b(\mathcal{R}(\mathbf{v}_\Gamma),c_{p_\Omega}^k)\\
&=\sum_{i=1}^{s}F_i^u(\mathbf{u}_i^k,p_i^k,\mathbf{c}_i^k,\mathbf{v}_i)+F_\Gamma^u(\mathbf{u}^k,p^k,\mathbf{c}^k,\mathbf{v}_\Gamma),
\end{aligned}\tag{5.5d}$$

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(b_i(\mathbf{c}_{u_i}^k,q_i)+b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k),q_i)\Big)+b(\mathcal{R}(\mathbf{c}_{u_\Gamma}^k),q_\Omega)\\
&=\sum_{i=1}^{s}\Big(F_i^p(\mathbf{u}_i^k,q_i)\Big)+F_\Gamma^p(\mathbf{u}^k,q_\Omega)
\end{aligned}\tag{5.5e}$$

for all $\mathbf{v}_i \in \mathbf{V}_i$, $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$, $\mathbf{e}_i \in L^2(\Omega_{C_i})$, $q_i \in Q_i$ *and* $q_\Omega \in D(\Omega)$.

Proof. We can use the same arguments as in Lemma 24 of Chapter 3. □

5.2.3 *Decoupling of Weak Formulation on Subdomains*

Decoupling the weak formulation on subdomains leads to s independent weak subdomain optimality systems and one system of coupling conditions which is the second sub-step. We have to distinguish between subdomain optimality systems on non-outflow and on outflow subdomains.

On a non-outflow subdomain, the local linearized optimality system is given by: For given $\mathbf{c}_{u_\Gamma}^k$, $\mathbf{c}_{z_\Gamma}^k \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{c}_{u_i}^k$, $\mathbf{c}_{z_i}^k \in \mathbf{H}_0^1(\Omega_i)$, $\mathbf{c}_{c_i}^k \in \mathbf{L}^2(\Omega_{C_i})$ and $\widetilde{c}_{p_i}^k$, $\widetilde{c}_{r_i}^k \in L_0^2(\Omega_i)$ at the linearization point $(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{z}_i^k, r_i^k)$, such that

$$\begin{aligned}
&a_i(\mathbf{v}_i,\mathbf{c}_{z_i}^k)+a_i(\mathbf{v}_i,\mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k))+b_i(\mathbf{v}_i,\widetilde{c}_{r_i}^k)+\\
&\quad+n_i(\mathbf{v}_i,\mathbf{c}_{u_i}^k,\mathbf{z}_i^k)+n_i(\mathbf{v}_i,\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k),\mathbf{z}_i^k)+\\
&\quad+n_i(\mathbf{v}_i,\mathbf{u}_i^k,\mathbf{c}_{z_i}^k)+n_i(\mathbf{v}_i,\mathbf{u}_i^k,\mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k))+\\
&\quad+n_i(\mathbf{c}_{u_i}^k,\mathbf{v}_i,\mathbf{z}_i^k)+n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k),\mathbf{v}_i,\mathbf{z}_i^k)+\\
&\quad+n_i(\mathbf{u}_i^k,\mathbf{v}_i,\mathbf{c}_{z_i}^k)+n_i(\mathbf{u}_i^k,\mathbf{v}_i,\mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k))+\\
&-m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_i}^k,\mathbb{1}_{\Omega_0}\mathbf{v}_i)-m(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k),\mathbb{1}_{\Omega_0}\mathbf{v}_i)=F_i^z(\mathbf{u}_i^k,\mathbf{z}_i{}^k,r_i^k,\mathbf{v}_i)
\end{aligned}\tag{5.6a}$$

$$b_i(\mathbf{c}_{z_i}^k,\widetilde{q}_i)+b_i(\mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k),\widetilde{q}_i)=F_i^r(\mathbf{z}_i{}^k,\widetilde{q}_i)\tag{5.6b}$$

$$d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i,\mathbf{c}_{z_i}^k)+d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i,\mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k))+$$

$$+\alpha m_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbb{1}_{\Omega_c}\mathbf{e}_i) = F_i^c(\mathbf{c}_i^k, \mathbf{z}_i{}^k, \mathbf{e}_i) \quad (5.6c)$$

$$\begin{aligned} a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i) + \\ + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_i) + n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{u}_i^k, \mathbf{v}_i) + \\ + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_i) + n_i(\mathbf{u}_i^k, \mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i) + \end{aligned} \quad (5.6d)$$

$$-d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbf{v}_i) + b_i(\mathbf{v}_i, \widetilde{c}_{p_i}^k) = F_i^u(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{v}_i),$$

$$b_i(\mathbf{c}_{u_i}^k, \widetilde{q}_i) + b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \widetilde{q}_i) = F_i^p(\mathbf{u}_i^k, \widetilde{q}_i) \quad (5.6e)$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$, $q_i \in L^2(\Omega_i)$ and $\mathbf{e}_i \in \mathbf{L}^2(\Omega_{C_i})$.

Proposition 39. *Assuming*

$$\int_{\Omega_i} \theta_i dx = \int_{\Omega_i} \boldsymbol{\nabla} \cdot \mathbf{z}_i^k dx,$$
$$\int_{\Omega_i} \eta_i dx = \int_{\Omega_i} \boldsymbol{\nabla} \cdot \mathbf{u}_i^k dx$$

holds, the equation system (5.6) is equivalent to the following local linearized subdomain optimality system:

For given $\mathbf{c}_{u_\Gamma}^k$, $\mathbf{c}_{z_\Gamma}^k \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{c}_{u_i}^k$, $\mathbf{c}_{z_i}^k \in \mathbf{H}_N^1(\Omega_i)$, $\mathbf{c}_{c_i}^k \in \mathbf{L}^2(\Omega_{C_i})$, $c_{p_i}^k$, $c_{r_i}^k \in L^2(\Omega_i)$ and η_i, $\theta_i \in \mathbb{R}$ at the linearization point $(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{z}_i^k, r_i^k)$, such that

$$\begin{aligned} a_i(\mathbf{v}_i, \mathbf{c}_{z_i}^k) + a_i(\mathbf{v}_i, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + b_i(\mathbf{v}_i, c_{r_i}^k) + \\ + n_i(\mathbf{v}_i, \mathbf{c}_{u_i}^k, \mathbf{z}_i^k) + n_i(\mathbf{v}_i, \mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{z}_i^k) + \\ + n_i(\mathbf{v}_i, \mathbf{u}_i^k, \mathbf{c}_{z_i}^k) + n_i(\mathbf{v}_i, \mathbf{u}_i^k, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + \\ + n_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i, \mathbf{z}_i^k) + n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i, \mathbf{z}_i^k) + \\ + n_i(\mathbf{u}_i^k, \mathbf{v}_i, \mathbf{c}_{z_i}^k) + n_i(\mathbf{u}_i^k, \mathbf{v}_i, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + \end{aligned} \quad (5.7a)$$

$$-m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_i}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_i) - m(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbb{1}_{\Omega_0}\mathbf{v}_i) = F_i^z(\mathbf{u}_i^k, \mathbf{z}_i{}^k, r_i^k, \mathbf{v}_i)$$

$$b_i(\mathbf{c}_{z_i}^k, q_i) + b_i(\mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k), q_i) + c_i(\theta_i, q_i) = F_i^r(\mathbf{z}_i{}^k, q_i) \quad (5.7b)$$

$$c_i(\xi_i, c_{r_i}) = F_i^\theta(r_i^k, \xi_i) \quad (5.7c)$$

$$\begin{aligned} d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i, \mathbf{c}_{z_i}^k) + d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + \\ + \alpha m_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbb{1}_{\Omega_c}\mathbf{e}_i) = F_i^c(\mathbf{c}_i^k, \mathbf{z}_i{}^k, \mathbf{e}_i) \end{aligned} \quad (5.7d)$$

$$\begin{aligned} a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i) + \\ + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_i) + n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{u}_i^k, \mathbf{v}_i) + \\ + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_i) + n_i(\mathbf{u}_i^k, \mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i) + \end{aligned} \quad (5.7e)$$

$$-d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbf{v}_i) + b_i(\mathbf{v}_i, c_{p_i}^k) = F_i^u(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{v}_i),$$

$$b_i(\mathbf{c}_{u_i}^k, q_i) + b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), q_i) + c_i(\eta_i, q_i) = F_i^p(\mathbf{u}_i^k, q_i), \quad (5.7f)$$

$$c_i(\xi_i, c_{p_i}) = F_i^\eta(p_i^k, \xi_i) \quad (5.7g)$$

for all $\mathbf{v}_i \in \mathbf{H}_0^1(\Omega_i)$, $q_i \in L^2(\Omega_i)$, $\mathbf{e}_i \in \mathbf{L}^2(\Omega_{C_i})$ *and* $\xi_i \in \mathbb{R}$.

Proof. Use the same arguments as in proof of Lemma 14 in Chapter 2. □

As we can see, the local optimality systems on non-outflow subdomains are independent of the global constants c_{p_Ω}, c_{r_Ω}.

Analogously, on an outflow subdomain, we obtain the following local linearized optimality system: For given $\mathbf{c}_{u_\Gamma}^k$, $\mathbf{c}_{z_\Gamma}^k \in \mathbf{H}_{00}^{1/2}(\Gamma)$, find $\mathbf{c}_{u_i}^k$, $\mathbf{c}_{z_i}^k \in \mathbf{H}_N^1(\Omega_i)$, $\mathbf{c}_{c_i}^k \in \mathbf{L}^2(\Omega_{C_i})$ and $c_{p_i}^k$, $c_{r_i}^k \in L^2(\Omega_i)$ at the linearization point $(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{z}_i^k, r_i^k)$, such that

$$
\begin{aligned}
a_i(\mathbf{v}_i, \mathbf{c}_{z_i}^k) + a_i(\mathbf{v}_i, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + b_i(\mathbf{v}_i, c_{r_i}^k) + \\
+ n_i(\mathbf{v}_i, \mathbf{c}_{u_i}^k, \mathbf{z}_i^k) + n_i(\mathbf{v}_i, \mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{z}_i^k) + \\
+ n_i(\mathbf{v}_i, \mathbf{u}_i^k, \mathbf{c}_{z_i}^k) + n_i(\mathbf{v}_i, \mathbf{u}_i^k, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + \\
+ n_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i, \mathbf{z}_i^k) + n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i, \mathbf{z}_i^k) + \\
+ n_i(\mathbf{u}_i^k, \mathbf{v}_i, \mathbf{c}_{z_i}^k) + n_i(\mathbf{u}_i^k, \mathbf{v}_i, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + \qquad (5.8a) \\
-m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_i}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_i) - m(\mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbb{1}_{\Omega_0}\mathbf{v}_i) = F_i^z(\mathbf{u}_i^k, \mathbf{z}_i{}^k, r_i^k, \mathbf{v}_i) \\
b_i(\mathbf{c}_{z_i}^k, q_i) + b_i(\mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k), q_i) = F_i^r(\mathbf{z}_i{}^k, q_i) \qquad (5.8b) \\
d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i, \mathbf{c}_{z_i}^k) + d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i, \mathcal{R}_i(\mathbf{c}_{z_{\Gamma_i}}^k)) + \\
+ \alpha m_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbb{1}_{\Omega_c}\mathbf{e}_i) = F_i^c(\mathbf{c}_i^k, \mathbf{z}_i{}^k, \mathbf{e}_i) \qquad (5.8c) \\
a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i) + a_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i) + \\
+ n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_i) + n_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{u}_i^k, \mathbf{v}_i) + \\
+ n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_i) + n_i(\mathbf{u}_i^k, \mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), \mathbf{v}_i) + \qquad (5.8d) \\
-d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbf{v}_i) + b_i(\mathbf{v}_i, c_{p_i}^k) = F_i^u(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{v}_i), \\
b_i(\mathbf{c}_{u_i}^k, q_i) + b_i(\mathcal{R}_i(\mathbf{c}_{u_{\Gamma_i}}^k), q_i) = F_i^p(\mathbf{u}_i^k, q_i), \qquad (5.8e)
\end{aligned}
$$

for all $\mathbf{v}_i \in \mathbf{H}_N^1(\Omega_i)$, $q_i \in L^2(\Omega_i)$ and $\mathbf{e}_i \in \mathbf{L}^2(\Omega_{C_i})$. Since on outflow subdomains, the corrections of the pressure c_{p_i} and the adjoint pressure c_{r_i} are uniquely defined as long as a solution exists, it is clear, that the constants $c_{p_{\Omega_i}}$ and $c_{r_{\Omega_i}}$must be zero.

Since we are interested in solving the linearized global system (5.4) by solving coupling conditions on the interface, the next lemma states, under which conditions the local and the global formulations are equivalent.

Lemma 40. It holds that

$$
\begin{aligned}
\mathbf{c}_u^k &= \sum_{i=1}^{s} \mathbf{c}_{u_i}^k + \mathcal{R}(\mathbf{c}_{u_\Gamma}^k) \in \mathbf{H}_D^1(\Omega), \\
c_p^k &= \sum_{i=1}^{s} c_{p_i}^k + c_{p_\Omega}^k \in L^2(\Omega),
\end{aligned}
$$

$$\mathbf{c}_c^k = \sum_{i=1}^{s} \mathbf{c}_{c_i}^k \in \mathbf{L}^2(\Omega_C),$$
$$\mathbf{c}_z^k = \sum_{i=1}^{s} \mathbf{c}_{z_i}^k + \mathcal{R}(\mathbf{c}_{z_\Gamma}^k) \in \mathbf{H}_D^1(\Omega),$$
$$c_r^k = \sum_{i=1}^{s} c_{r_i}^k + c_{r_\Omega}^k \in L^2(\Omega),$$

with $(\mathbf{c}_{u_i}^k, c_{p_i}^k, \mathbf{c}_{c_i}^k, \mathbf{c}_{z_i}^k, c_{r_i}^k) \in \mathbf{V}_i \times Q_i \times \mathbf{L}^2(\Omega_{C_i}) \times \mathbf{V}_i \times Q_i$ *solution of (5.7) or (5.8) (depending on the subdomain type) solve (5.4) if and only if the following coupling conditions hold for* $\mathbf{c}_{u_\Gamma}^k, \mathbf{c}_{z_\Gamma}^k \in \mathbf{H}_{00}^{1/2}(\Gamma)$ *and* $c_{p_\Omega}^k, c_{r_\Omega}^k \in D(\Omega)$ *at the linearization point* $(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{z}^k, r^k) \in \mathbf{H}_D^1(\Omega) \times L^2(\Omega) \times L^2(\Omega_C) \times \mathbf{H}_D^1(\Omega) \times L^2(\Omega)$. *Find* $\mathbf{c}_{u_\Gamma}, \mathbf{c}_{z_\Gamma} \in \mathbf{H}_{00}^{1/2}(\Gamma)$ *and* $c_{p_\Omega}, c_{r_\Omega} \in D(\Omega)$ *such that*

$$\begin{aligned}
&\sum_{i=1}^{s}\Big(a_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{c}_{z_i}^k) + n_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{c}_{u_i}^k, \mathbf{z}_i^k) + n_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{u}_i^k, \mathbf{c}_{z_i}^k)\Big) + \\
&\quad + n_i(\mathbf{c}_{u_i}^k, \mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{z}_i^k) + n_i(\mathbf{u}_i^k, \mathcal{R}_i(\mathbf{v}_{\Gamma_i}), \mathbf{c}_{z_i}^k) - m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_i}^k, \mathbb{1}_{\Omega_0}\mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + \\
&\quad + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), c_{r_i}^k)\Big) + a(\mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{c}_{z_\Gamma}^k)) + n(\mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{c}_{u_\Gamma}^k), \mathbf{z}^k) + \\
&\quad + n(\mathcal{R}(\mathbf{v}_\Gamma), \mathbf{u}^k, \mathcal{R}(\mathbf{c}_{z_\Gamma}^k)) + n(\mathcal{R}(\mathbf{c}_{u_\Gamma}^k), \mathcal{R}(\mathbf{v}_\Gamma), \mathbf{z}^k) + \\
&\quad + n(\mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma), \mathcal{R}(\mathbf{c}_{z_\Gamma}^k)) - m(\mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{c}_{u_\Gamma}^k), \mathbb{1}_{\Omega_0}\mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), c_{r_\Omega}^k) \\
&= F_\Gamma^z(\mathbf{u}^k, \mathbf{z}^k, r^k, \mathbf{v}_\Gamma) \\
&b(\mathcal{R}(\mathbf{c}_{z_\Gamma}^k), q_\Omega) = F_\Gamma^r(\mathbf{z}^k, q_\Omega) \\
&\sum_{i=1}^{s}\Big(a_i(\mathbf{c}_{u_i}^k, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + \\
&\quad - d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathcal{R}_i(\mathbf{v}_{\Gamma_i})) + b_i(\mathcal{R}_i(\mathbf{v}_{\Gamma_i}), c_{p_i}^k)\Big) + a(\mathcal{R}(\mathbf{c}_{u_\Gamma}^k), \mathcal{R}(\mathbf{v}_\Gamma)) + \\
&\quad + n(\mathcal{R}(\mathbf{c}_{u_\Gamma}^k), \mathbf{u}^k, \mathcal{R}(\mathbf{v}_\Gamma)) + n(\mathbf{u}^k, \mathcal{R}(\mathbf{c}_{u_\Gamma}^k), \mathcal{R}(\mathbf{v}_\Gamma)) + b(\mathcal{R}(\mathbf{v}_\Gamma), c_{p_\Omega}^k) \\
&= F_\Gamma^u(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{v}_\Gamma), \\
&b(\mathcal{R}(\mathbf{c}_{u_\Gamma}^k), q_\Omega) = F_\Gamma^p(\mathbf{u}^k, q_\Omega)
\end{aligned}$$

for all $\mathbf{v}_\Gamma \in \mathbf{H}_{00}^{1/2}(\Gamma)$ *and* $q_\Omega \in D(\Omega)$.

Proof. This can be shown by using the same arguments as in Chapter 2 in the proof of Lemma 13. □

5.3 DISCRETIZATION BASED ON A FINITE ELEMENT METHOD

We use the same type of finite element method as described in Chapter 2 as well as the same spaces and finite element functions as described in Chapter 3.

For better readability, we leave out the super index h for the linearization point and the corrections.

5.3.1 *Decoupled Finite Element Formulation*

In the third main step, we use the definitions of the previous chapters for the finite element method and discretize the s decoupled weak formulations of the subdomain optimality systems and the system of the coupling condition. This step is illustrated in Fig. 16.

On a non-outflow subdomain, we obtain the following local linearized optimality system: For given $\mathbf{c}_{u_\Gamma}^k$, $\mathbf{c}_{z_\Gamma}^k \in \mathbf{V}_\Gamma^h$, find $\mathbf{c}_{u_i}^k$, $\mathbf{c}_{z_i}^k \in \mathbf{V}_{i,0}^h$, $\mathbf{c}_{c_i}^k \in \mathbf{Q}_i^h$, $c_{p_i}^k$, $c_{r_i}^k \in Q_i^h$ and η_i^h, $\theta_i^h \in \mathbb{R}$ at the linearization point $(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{z}_i^k, r_i^k)$, such that

$$
\begin{aligned}
&a_i(\mathbf{v}_i^h, \mathbf{c}_{z_i}^k) + a_i(\mathbf{v}_i^h, \mathbf{c}_{z_{\Gamma_i}}^k) + b_i(\mathbf{v}_i^h, c_{r_i}^k) + n_i(\mathbf{v}_i^h, \mathbf{c}_{u_i}^k, \mathbf{z}_i^k) + \\
&+ n_i(\mathbf{v}_i^h, \mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{z}_i^k) + n_i(\mathbf{v}_i^h, \mathbf{u}_i^k, \mathbf{c}_{z_i}^k) + n_i(\mathbf{v}_i^h, \mathbf{u}_i^k, \mathbf{c}_{z_{\Gamma_i}}^k) + \\
&+ n_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i^h, \mathbf{z}_i^k) + n_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h, \mathbf{z}_i^k) + n_i(\mathbf{u}_i^k, \mathbf{v}_i^h, \mathbf{c}_{z_i}^k) + \\
&+ n_i(\mathbf{u}_i^k, \mathbf{v}_i^h, \mathbf{c}_{z_{\Gamma_i}}^k) - m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_i}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h) - m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_{\Gamma_i}}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h) \\
&= F_i^z(\mathbf{u}_i^k, \mathbf{z}_i{}^k, r_i^k, \mathbf{v}_i^h), \qquad (5.9a)\\
&b_i(\mathbf{c}_{z_i}^k, q_i^h) + b_i(\mathbf{c}_{z_{\Gamma_i}}^k, q_i^h) + c_i(\theta_i^h, q_i^h) = F_i^r(\mathbf{z}_i{}^k, q_i^h), \qquad (5.9b)\\
&c_i(\xi_i^h, c_{r_i}^k) = F_i^\theta(r_i^k, \xi_i), \qquad (5.9c)\\
&d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i^h, \mathbf{c}_{z_i}^k) + d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i^h, \mathbf{c}_{z_{\Gamma_i}}^k) + \alpha m_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbb{1}_{\Omega_c}\mathbf{e}_i^h) \\
&= F_i^c(\mathbf{c}_i^k, \mathbf{z}_i{}^k, \mathbf{e}_i^h) \qquad (5.9d)\\
&a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + a_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + n_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + \\
&+ n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) - d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, c_{p_i}^k) \\
&= F_i^u(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{v}_i^h), \\
&b_i(\mathbf{c}_{u_i}^k, q_i^h) + b_i(\mathbf{c}_{u_{\Gamma_i}}^k, q_i^h) + c_i(\eta_i^h, q_i^h) = F_i^p(\mathbf{u}_i^k, q_i^h), \qquad (5.9e)\\
&c_i(\xi_i^h, c_{p_i}^k) = F_i^\eta(p_i^k, \xi_i) \qquad (5.9f)
\end{aligned}
$$

for all $\mathbf{v}_i \in \mathbf{V}_{i,0}^h$, $q_i \in Q_i^h$ and $\xi_i \in \mathbb{R}$. As we can see, the local optimality systems on non-outflow subdomains are independent of the global constants c_{p_Ω}, c_{r_Ω}.

Similarly, on an outflow subdomain, the local linearized optimality system yields: For given $\mathbf{c}_{u_\Gamma}^k$, $\mathbf{c}_{z_\Gamma}^k \in \mathbf{V}_\Gamma^h$, find $\mathbf{c}_{u_i}^k$, $\mathbf{c}_{z_i}^k \in \mathbf{V}_{i,N}^h$, $\mathbf{c}_{c_i}^k \in \mathbf{Q}_i^h$ and $c_{p_i}^k$, $c_{r_i}^k \in Q_i^h$ at the linearization point $(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{z}_i^k, r_i^k)$, such that

$$
\begin{aligned}
&a_i(\mathbf{v}_i^h, \mathbf{c}_{z_i}^k) + a_i(\mathbf{v}_i^h, \mathbf{c}_{z_{\Gamma_i}}^k) + b_i(\mathbf{v}_i^h, c_{r_i}^k) + n_i(\mathbf{v}_i^h, \mathbf{c}_{u_i}^k, \mathbf{z}_i^k) + \\
&+ n_i(\mathbf{v}_i^h, \mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{z}_i^k) + n_i(\mathbf{v}_i^h, \mathbf{u}_i^k, \mathbf{c}_{z_i}^k) + n_i(\mathbf{v}_i^h, \mathbf{u}_i^k, \mathbf{c}_{z_{\Gamma_i}}^k) + \\
&+ n_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i^h, \mathbf{z}_i^k) + n_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h, \mathbf{z}_i^k) + n_i(\mathbf{u}_i^k, \mathbf{v}_i^h, \mathbf{c}_{z_i}^k) \\
&+ n_i(\mathbf{u}_i^k, \mathbf{v}_i^h, \mathbf{c}_{z_{\Gamma_i}}^k) - m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_i}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h) - m(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_{\Gamma_i}}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_i^h)
\end{aligned}
$$

$$= F_i^z(\mathbf{u}_i^k, \mathbf{z}_i{}^k, r_i^k, \mathbf{v}_i^h), \tag{5.10a}$$

$$b_i(\mathbf{c}_{z_i}^k, q_i^h) + b_i(\mathbf{c}_{z_{\Gamma_i}}^k, q_i^h) = F_i^r(\mathbf{z}_i{}^k, q_i^h), \tag{5.10b}$$

$$\begin{aligned} &d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i^h, \mathbf{c}_{z_i}^k) + d_i(\mathbb{1}_{\Omega_c}\mathbf{e}_i^h, \mathbf{c}_{z_{\Gamma_i}}^k) + \alpha m_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbb{1}_{\Omega_c}\mathbf{e}_i^h) \\ &= F_i^c(\mathbf{c}_i^k, \mathbf{z}_i{}^k, \mathbf{e}_i^h) \end{aligned} \tag{5.10c}$$

$$\begin{aligned} &a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + a_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + n_i(\mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{u}_i^k, \mathbf{v}_i^h) + \\ &+ n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_i^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_{\Gamma_i}}^k, \mathbf{v}_i^h) - d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbf{v}_i^h) + b_i(\mathbf{v}_i^h, c_{p_i}^k) \\ &= F_i^u(\mathbf{u}_i^k, p_i^k, \mathbf{c}_i^k, \mathbf{v}_i^h), \end{aligned} \tag{5.10d}$$

$$b_i(\mathbf{c}_{u_i}^k, q_i^h) + b_i(\mathbf{c}_{u_{\Gamma_i}}^k, q_i^h) = F_i^p(\mathbf{u}_i^k, q_i^h) \tag{5.10e}$$

for all $\mathbf{v}_i \in \mathbf{V}_{i,N}^h$ and $q_i \in Q_i^h$.

The finite element formulation of the coupling conditions reads:

At the linearization point $(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{z}^k, r^k) \in \mathbf{V}^h \times Q^h \times \mathbf{Q}^h \times \mathbf{V}^h \times Q^h$, find $\mathbf{c}_{u_\Gamma}^k$, $\mathbf{c}_{z_\Gamma}^k \in \mathbf{V}_\Gamma^h$ and $c_{p_\Omega}^k$, $c_{r_\Omega}^k \in D(\Omega)$ such that

$$\begin{aligned} &\sum_{i=1}^{s}\Big(a_i(\mathbf{v}_{\Gamma_i}^h, \mathbf{c}_{z_i}^k) + n_i(\mathbf{v}_{\Gamma_i}^h, \mathbf{c}_{u_i}^k, \mathbf{z}_i^k) + n_i(\mathbf{v}_{\Gamma_i}^h, \mathbf{u}_i^k, \mathbf{c}_{z_i}^k) + \\ &\quad n_i(\mathbf{c}_{u_i}^k, \mathbf{v}_{\Gamma_i}^h, \mathbf{z}_i^k) + n_i(\mathbf{u}_i^k, \mathbf{v}_{\Gamma_i}^h, \mathbf{c}_{z_i}^k) + b_i(\mathbf{v}_{\Gamma_i}^h, c_{r_i}^k) - m_i(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_i}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_{\Gamma_i}^h)\Big) + \\ &\quad + a(\mathbf{v}_\Gamma^h, \mathbf{c}_{z_\Gamma}^k) + n(\mathbf{v}_\Gamma^h, \mathbf{c}_{u_\Gamma}^k, \mathbf{z}^k) + n(\mathbf{v}_\Gamma^h, \mathbf{u}^k, \mathbf{c}_{z_\Gamma}^k) + n(\mathbf{c}_{u_\Gamma}^k, \mathbf{v}_\Gamma^h, \mathbf{z}^k) + \\ &\quad + n(\mathbf{u}^k, \mathbf{v}_\Gamma^h, \mathbf{c}_{z_\Gamma}^k) + b(\mathbf{v}_\Gamma^h, c_{r_\Omega}^k) - m(\mathbb{1}_{\Omega_0}\mathbf{c}_{u_\Gamma}^k, \mathbb{1}_{\Omega_0}\mathbf{v}_\Gamma^h) \\ &= F_\Gamma^z(\mathbf{u}^k, \mathbf{z}^k, r^k, \mathbf{v}_\Gamma^h) \end{aligned} \tag{5.11a}$$

$$b(\mathbf{c}_{z_\Gamma}^k, q_\Omega^h) = F_\Gamma^r(\mathbf{z}^k, q_\Omega^h) \tag{5.11b}$$

$$\begin{aligned} &\sum_{i=1}^{s}\Big(a_i(\mathbf{c}_{u_i}^k, \mathbf{v}_{\Gamma_i}^h) + n_i(\mathbf{c}_{u_i}^k, \mathbf{u}_i^k, \mathbf{v}_{\Gamma_i}^h) + n_i(\mathbf{u}_i^k, \mathbf{c}_{u_i}^k, \mathbf{v}_{\Gamma_i}^h) - d_i(\mathbb{1}_{\Omega_c}\mathbf{c}_{c_i}^k, \mathbf{v}_{\Gamma_i}^h) + \\ &\quad + b_i(\mathbf{v}_{\Gamma_i}^h, c_{p_i}^k)\Big) + a(\mathbf{c}_{u_\Gamma}^k, \mathbf{v}_\Gamma^h) + n(\mathbf{c}_{u_\Gamma}^k, \mathbf{u}^k, \mathbf{v}_\Gamma^h) + n(\mathbf{u}^k, \mathbf{c}_{u_\Gamma}^k, \mathbf{v}_\Gamma^h) + \\ &\quad + b(\mathbf{v}_\Gamma^h, c_{p_\Omega}^k) = F_\Gamma^u(\mathbf{u}^k, p^k, \mathbf{c}^k, \mathbf{v}_\Gamma^h), \end{aligned} \tag{5.11c}$$

$$b(\mathbf{c}_{u_\Gamma}^k, q_\Omega^h) = F_\Gamma^p(\mathbf{u}^k, q_\Omega^h) \tag{5.11d}$$

for all $\mathbf{v}_\Gamma^h \in \mathbf{V}_\Gamma^h$ and $q_\Omega^h \in D(\Omega)$. $\mathbf{c}_{u_i}^k$, $\mathbf{c}_{z_i}^k \in \mathbf{V}_i^h$, $\mathbf{c}_{c_i}^k \in \mathbf{Q}_i^h$ and $c_{p_i}^k$, $c_{r_i}^k \in Q_i^h$ are solutions of (5.9) and (5.10) depending on whether a subdomain Ω_i is a non-outflow or an outflow subdomain, respectively.

5.4 ALGEBRAIC FORMULATION

Based on the finite element discretization, we derive the resulting global linear system corresponding to (5.5) and the Schur-complement equations corresponding to (5.11). This is the last main step of the derivation of the domain decomposition method. For the algebraic representation, we use the same definitions as in Chapter 2-4. For the matrix corresponding to the bilinear form $a(\cdot,\cdot)$, we use the def-

inition from Chapter 4. Additionally, we define the finite element functions for the linearization point $\mathbf{z}_i^k$ and the resulting coefficient vector analogously to the definition of $\mathbf{u}_i^k$:

$$\mathbf{z}_i^k(\mathbf{x}) = \sum_{j=1}^{n_{z_i}+n_{z_{\Gamma_i}}} u_j^k \boldsymbol{\psi}_j(\mathbf{x}), \quad \mathbf{u}_i^k = \quad (u_1, \ldots, u_{n_{z_i}+n_{z_{\Gamma_i}}})^T \in \mathbb{R}^{(n_{z_i}+n_{z_{\Gamma_i}})}.$$

Furthermore, we define the finite element functions and the resulting coefficient vectors for the corrections of the adjoint velocity

$$\mathbf{c}_{\mathbf{z}_i}^k(\mathbf{x}) = \sum_{j=1}^{n_{z_i}} c_{z_j} \boldsymbol{\psi}_j(\mathbf{x}), \qquad \mathbf{c}_{\mathbf{z}_i} = (c_{z_1}, \ldots, c_{z_{n_{z_i}}})^T \in \mathbb{R}^{n_{z_i}},$$

$$\mathbf{c}_{\mathbf{z}_\Gamma}^k(\mathbf{x}) = \sum_{j=1}^{n_{z_\Gamma}} c_{z_j} \boldsymbol{\psi}_j(\mathbf{x}), \qquad \mathbf{c}_{\mathbf{z}_\Gamma} = (c_{z_1}, \ldots, c_{z_{n_{z_\Gamma}}})^T \in \mathbb{R}^{n_{z_\Gamma}},$$

$$\mathbf{c}_{\mathbf{z}_{\Gamma_i}}^k(\mathbf{x}) = \sum_{j=1}^{n_{z_{\Gamma_i}}} c_{z_j} \boldsymbol{\psi}_j(\mathbf{x}), \qquad \mathbf{c}_{\mathbf{z}_{\Gamma_i}} = (c_{z_1}, \ldots, c_{z_{n_{\Gamma_i}}})^T \in \mathbb{R}^{n_{z_{\Gamma_i}}},$$

for the corrections of the adjoint pressure

$$c_{r_i}^k(\mathbf{x}) = \sum_{j=1}^{n_{q_i}} c_{r_j} \varphi_j(\mathbf{x}), \qquad \mathbf{c}_{r_i} = (c_{r_1}, \ldots, c_{r_{nq_i}})^T \in \mathbb{R}^{n_{q_i}},$$

$$c_{r_\Omega}^k(\mathbf{x}) = \sum_{j=1}^{s_{\overline{\mathcal{N}}}} c_{r_{\Omega_j}} \varphi_{\Omega_j}(\mathbf{x}), \qquad \mathbf{c}_{r_\Omega} = (c_{r_{\Omega_1}}, \ldots, c_{r_{s_{\overline{\mathcal{N}}}}})^T \in \mathbb{R}^{s_{\overline{\mathcal{N}}}},$$

and the corrections of the control

$$\mathbf{c}_{\mathbf{c}_i}^k(\mathbf{x}) = \sum_{j=1}^{n_{c_i}} c_{c_j} \boldsymbol{\mu}_j(\mathbf{x}), \qquad \mathbf{c}_{\mathbf{c}_i} = (c_{c_1}, \ldots, c_{c_{n_{c_i}}})^T \in \mathbb{R}^{n_{c_i}}.$$

Global Linear System and Schur-complement Equation

We write the global linear system which corresponds to the finite element formulation of (5.5), for an example with two subdomains. We assume that Ω_1 is a non-outflow subdomain and that Ω_2 is an outflow subdomain.

$$
\left(\begin{array}{ccccc|ccccc|ccccc}
\mathbf{A}_{11}\left[\mathbf{u}_1^k\right] & \widetilde{\mathbf{B}}_{11}^T & -(\mathbf{D}_C)_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1}\left[\mathbf{u}_1^k\right] & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{11} & (\mathbf{D}_C)_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{\Gamma_1 1}^T & \mathbf{0} \\
-(\mathbf{M}_0)_{11} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{11}^T\left[\mathbf{z}_1^k\right] & \widetilde{\mathbf{B}}_{11}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\mathbf{M}_0)_{1\Gamma_1} & \mathbf{0} & \mathbf{A}_{\Gamma_1 1}^T\left[\mathbf{z}_1^k\right] & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{11} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{1\Gamma_1} & \mathbf{0} \\
\hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{22}\left[\mathbf{u}_2^k\right] & \mathbf{B}_{22}^T & -(\mathbf{D}_C)_{22} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{2\Gamma_2}\left[\mathbf{u}_2^k\right] & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2\Gamma_2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{22} & (\mathbf{D}_C)_{22}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{\Gamma_2 2}^T & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -(\mathbf{M}_0)_{22} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{22}^T\left[\mathbf{z}_2^k\right] & \mathbf{B}_{22}^T & -(\mathbf{M}_0)_{2\Gamma_2} & \mathbf{0} & \mathbf{A}_{\Gamma_2 2}^T\left[\mathbf{z}_2^k\right] & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{22} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{2\Gamma_2} & \mathbf{0} \\
\hline
\mathbf{A}_{\Gamma_1 1}\left[\mathbf{u}_1^k\right] & \widetilde{\mathbf{B}}_{1\Gamma_1}^T & -(\mathbf{D}_C)_{\Gamma_1 1} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\Gamma_2 2}\left[\mathbf{u}_2^k\right] & \mathbf{B}_{2\Gamma_2}^T & -(\mathbf{D}_C)_{\Gamma_2 2} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma}\left[\mathbf{u}^k\right] & \mathbf{B}_0^T & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-(\mathbf{M}_0)_{\Gamma_1 1} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{1\Gamma_1}^T\left[\mathbf{z}_1^k\right] & \widetilde{\mathbf{B}}_{1\Gamma_1}^T & -(\mathbf{M}_0)_{\Gamma_2 2} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{2\Gamma_2}^T\left[\mathbf{z}_2^k\right] & \mathbf{B}_{2\Gamma_2}^T & -(\mathbf{M}_0)_{\Gamma\Gamma} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma}^T\left[\mathbf{z}^k\right] & \mathbf{B}_0^T \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0}
\end{array}\right)
\left(\begin{array}{c}
\mathbf{c}_{u_1} \\ \widetilde{\mathbf{c}}_{p_1} \\ \mathbf{c}_{c_1} \\ \mathbf{c}_{z_1} \\ \widetilde{\mathbf{c}}_{r_1} \\ \hline
\mathbf{c}_{u_2} \\ \mathbf{c}_{p_2} \\ \mathbf{c}_{c_2} \\ \mathbf{c}_{z_2} \\ \mathbf{c}_{r_2} \\ \hline
\mathbf{c}_{u_\Gamma} \\ \mathbf{c}_{p_\Omega} \\ \mathbf{c}_{z_\Gamma} \\ \mathbf{c}_{r_\Omega}
\end{array}\right)
=
\left(\begin{array}{c}
\mathbf{F}_1^u\left[\mathbf{u}_1^k, \mathbf{p}_1^k, \mathbf{c}_1^k\right] \\ \mathbf{F}_1^p\left[\mathbf{u}_1^k\right] \\ \mathbf{F}_1^c\left[\mathbf{c}_1^k, \mathbf{z}_1^k\right] \\ \mathbf{F}_1^z\left[\mathbf{u}_1^k, \mathbf{z}_1^k, \mathbf{r}_1^k\right] \\ \mathbf{F}_1^r\left[\mathbf{z}_1^k\right] \\ \hline
\mathbf{F}_2^u\left[\mathbf{u}_2^k, \mathbf{p}_2^k, \mathbf{c}_2^k\right] \\ \mathbf{F}_2^p\left[\mathbf{u}_2^k\right] \\ \mathbf{F}_2^c\left[\mathbf{c}_2^k, \mathbf{z}_2^k\right] \\ \mathbf{F}_2^z\left[\mathbf{u}_2^k, \mathbf{z}_2^k, \mathbf{r}_2^k\right] \\ \mathbf{F}_2^r\left[\mathbf{z}_2^k\right] \\ \hline
\mathbf{F}_\Gamma^u\left[\mathbf{u}^k, \mathbf{p}^k, \mathbf{c}^k\right] \\ \mathbf{F}_\Gamma^p\left[\mathbf{u}^k\right] \\ \mathbf{F}_\Gamma^z\left[\mathbf{u}^k, \mathbf{z}^k, \mathbf{r}^k\right] \\ \mathbf{F}_\Gamma^r\left[\mathbf{z}^k\right]
\end{array}\right).
$$

Knowing that the local blocks (with sub index $\cdot_{ii}$) are invertible, we can reduce the problem to a Schur-complement equation. Analogously to the previous chapters, we directly write the alternative formulation. The Schur-complement S is defined as:

$$
\begin{aligned}
S\left[\mathbf{u}^k, \mathbf{z}^k\right] := & \begin{pmatrix} \mathbf{A}_{\Gamma\Gamma}\left[\mathbf{u}^k\right] & \mathbf{B}_0^T & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma\Gamma} & \mathbf{0} & \mathbf{A}_{\Gamma\Gamma}^T\left[\mathbf{z}^k\right] & \mathbf{B}_0^T \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_0 & \mathbf{0} \end{pmatrix} \\
& - \sum_{i \in \mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{\Gamma_i i}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{i\Gamma_i} & \mathbf{0} & \mathbf{A}_{\Gamma_i i}^T\left[\mathbf{z}_i^k\right] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix}}_{=:L1} \\
& - \sum_{i \notin \mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{D}_C)_{i\Gamma_i}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{i\Gamma_i} & \mathbf{0} & \mathbf{A}_{\Gamma_i i}^T\left[\mathbf{z}_i^k\right] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_{i\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}}_{=:L2}.
\end{aligned}
$$

and the right hand side r as follows:

$$r[\mathbf{u}^k,\mathbf{p}^k,\mathbf{c}^k,\mathbf{z}^k,\mathbf{r}^k] := \begin{pmatrix} \mathbf{F}_\Gamma^u\left[\mathbf{u}^k,\mathbf{p}^k,\mathbf{c}^k\right] \\ \mathbf{F}_\Gamma^p\left[\mathbf{u}^k\right] \\ \mathbf{F}_\Gamma^z\left[\mathbf{u}^k,\mathbf{z}^k,\mathbf{r}^k\right] \\ \mathbf{F}_\Gamma^r\left[\mathbf{z}^k\right] \end{pmatrix} - \sum_{i\in\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{F}_i^u\left[\mathbf{u}_i^k,\mathbf{p}_i^k,\mathbf{c}_i^k\right] \\ \mathbf{F}_i^p\left[\mathbf{u}_i^k\right] \\ \mathbf{F}_i^c\left[\mathbf{c}_i^k,\mathbf{z}_i^k\right] \\ \mathbf{F}_i^z\left[\mathbf{u}_i^k,\mathbf{z}_i^k,\mathbf{r}_i^k\right] \\ \mathbf{F}_i^r\left[\mathbf{z}_i^k\right] \end{pmatrix}}_{=:L3}$$

$$- \sum_{i\notin\mathcal{N}} \begin{pmatrix} \mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} & -(\mathbf{D}_C)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{i\Gamma_i}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{i\Gamma_i}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{F}_i^u\left[\mathbf{u}_i^k,\mathbf{p}_i^k,\mathbf{c}_i^k\right] \\ \mathbf{F}_i^p\left[\mathbf{u}_i^k\right] \\ \mathbf{F}_i^\eta\left[\mathbf{p}_i^k\right] \\ \mathbf{F}_i^c\left[\mathbf{c}_i^k,\mathbf{z}_i^k\right] \\ \mathbf{F}_i^z\left[\mathbf{u}_i^k,\mathbf{z}_i^k,\mathbf{r}_i^k\right] \\ \mathbf{F}_i^r\left[\mathbf{z}_i^k\right] \\ \mathbf{F}_i^\theta\left[\mathbf{r}_i^k\right] \end{pmatrix}}_{=:L4}.$$

These definitions yield the linearized Schur-complement equation for the distributed optimal control problem applied to the Navier-Stokes equations:

$$S[\mathbf{u}^k, \mathbf{z}^k](\mathbf{u}_\Gamma, \mathbf{p}_\Omega, \mathbf{z}_\Gamma, \mathbf{r}_\Omega)^T = r[\mathbf{u}^k, \mathbf{p}^k, \mathbf{c}^k, \mathbf{z}^k, \mathbf{r}^k].$$

It represents algebraically the interface coupling corresponding to the finite element formulation (5.11).

The subproblems $L1 - L4$ can be interpreted as solving local linearized optimality systems of a distributed control problem constraint by the Navier-Stokes equations.

$L1$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right]\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ (\mathbf{D}_C)_{\Gamma_i i}^T\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ -\mathbf{M}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} + \mathbf{A}_{\Gamma_i i}^T\left[\mathbf{z}_i^k\right]\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \end{pmatrix}.$$

$L1$ solves a local linearized optimality system of a distributed control problem applied to the Navier-Stokes equations with Dirichlet data $\mathbf{u}_{\Gamma_i}$ and $\mathbf{z}_{\Gamma_i}$ on the local skeleton Γ_i , natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$ and zero Dirichlet boundary data on $\partial\Omega_i \cap \partial\Omega$.

$L2$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right]\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ 0 \\ (\mathbf{D}_C)_{\Gamma_i i}^T\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ -\mathbf{M}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} + \mathbf{A}_{\Gamma_i i}^T\left[\mathbf{z}_i^k\right]\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ 0 \end{pmatrix}.$$

$L2$ solves a local linearized optimality system of a distributed control problem applied to the Navier-Stokes equations with Dirichlet data. On the local skeleton Γ_i, the Dirichlet data is set to $\mathbf{u}_{\Gamma_i}$ and $\mathbf{z}_{\Gamma_i}$. For non-outflow subdomains intersecting global Dirichlet boundary, the Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ is set to zero.

$L3$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^u\left[\mathbf{u}_i^k, \mathbf{p}_i^k, \mathbf{c}_i^k\right] \\ \mathbf{F}_i^p\left[\mathbf{u}_i^k\right] \\ \mathbf{F}_i^c\left[\mathbf{c}_i^k, \mathbf{z}_i^k\right] \\ \mathbf{F}_i^z\left[\mathbf{u}_i^k, \mathbf{z}_i^k, \mathbf{r}_i^k\right] \\ \mathbf{F}_i^r\left[\mathbf{z}_i^k\right] \end{pmatrix}.$$

$L3$ solves a local linearized optimality system of a distributed control problem applied to the Navier-Stokes equations with zero Dirichlet data on the local skeleton Γ_i and natural outflow conditions on $\partial\Omega_i \cap \partial\Omega_{out}$. On $\partial\Omega_i \cap \partial\Omega_D$ the Dirichlet data for the velocity is set to $\mathbf{u}_{D_i}$ and for the adjoint velocity to zero.

$L4$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^u\left[\mathbf{u}_i^k, \mathbf{p}_i^k, \mathbf{c}_i^k\right] \\ \mathbf{F}_i^p\left[\mathbf{u}_i^k\right] \\ \mathbf{F}_i^\eta\left[\mathbf{p}_i^k\right] \\ \mathbf{F}_i^c\left[\mathbf{c}_i^k, \mathbf{z}_i^k\right] \\ \mathbf{F}_i^z\left[\mathbf{u}_i^k, \mathbf{z}_i^k, \mathbf{r}_i^k\right] \\ \mathbf{F}_i^r\left[\mathbf{z}_i^k\right] \\ \mathbf{F}_i^\theta\left[\mathbf{r}_i^k\right] \end{pmatrix}.$$

$L4$ solves a local linearized optimality system of a distributed control problem applied to the Navier-Stokes equations with Dirichlet data. For inner subdomains a homogeneous Dirichlet boundary problem is solved. For the non-outflow subdomains intersecting global Dirichlet boundary, the Dirichlet data on the global boundary $\partial\Omega_i \cap \partial\Omega_D$ is set to $\mathbf{u}_{D_i}$ for the velocity and zero for the adjoint velocity. On the skeleton Γ_i, homogeneous Dirichlet boundary data is set.

Adding up $L1$ and $L3$ leads to

$L1 + L3$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^u\left[\mathbf{u}_i^k, \mathbf{p}_i^k, \mathbf{c}_i^k\right] - \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right]\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ \mathbf{F}_i^p\left[\mathbf{u}_i^k\right] - \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ \mathbf{F}_i^c\left[\mathbf{c}_i^k, \mathbf{z}_i^k\right] - (\mathbf{D}_C)_{\Gamma_i i}^T\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ \mathbf{F}_i^z\left[\mathbf{u}_i^k, \mathbf{z}_i^k, \mathbf{r}_i^k\right] + \mathbf{M}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} - \mathbf{A}_{\Gamma_i i}^T\left[\mathbf{z}_i^k\right]\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ \mathbf{F}_i^r\left[\mathbf{z}_i^k\right] - \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \end{pmatrix},$$

which is the algebraic representation of the local decoupled subproblems (5.10) on outflow subdomains Ω_i.

Similarly, adding up $L2$ and $L4$ yields the algebraic representation of the local decoupled subproblems (5.9) on non-outflow subdomains Ω_i .

$L2 + L4$ solve:

$$\begin{pmatrix} \mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} & -(\mathbf{D}_C)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{c}_i^T & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha(\mathbf{M}_C)_{ii} & (\mathbf{D}_C)_{ii}^T & \mathbf{0} & \mathbf{0} \\ -(\mathbf{M}_0)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_{ii}^T\left[\mathbf{z}_i^k\right] & \mathbf{B}_{ii}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{ii} & \mathbf{0} & \mathbf{c}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{c}_i^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i \\ \mathbf{p}_i \\ \eta_i \\ \mathbf{c}_i \\ \mathbf{z}_i \\ \mathbf{r}_i \\ \theta_i \end{pmatrix} = \begin{pmatrix} \mathbf{F}_i^u\left[\mathbf{u}_i^k, \mathbf{p}_i^k, \mathbf{c}_i^k\right] - \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right]\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ \mathbf{F}_i^p\left[\mathbf{u}_i^k\right] - \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} \\ \mathbf{F}_i^\eta\left[\mathbf{p}_i^k\right] \\ \mathbf{F}_i^c\left[\mathbf{c}_i^k, \mathbf{z}_i^k\right] - (\mathbf{D}_C)_{\Gamma_i i}^T\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ \mathbf{F}_i^z\left[\mathbf{u}_i^k, \mathbf{z}_i^k, \mathbf{r}_i^k\right] + \mathbf{M}_{i\Gamma_i}\mathbf{c}_{\mathbf{u}_{\Gamma_i}} - \mathbf{A}_{\Gamma_i i}^T\left[\mathbf{z}_i^k\right]\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ \mathbf{F}_i^r\left[\mathbf{z}_i^k\right] - \mathbf{B}_{i\Gamma_i}\mathbf{c}_{\mathbf{z}_{\Gamma_i}} \\ \mathbf{F}_i^\theta\left[\mathbf{r}_i^k\right] \end{pmatrix}.$$

Remark. The local subproblems $L3$ and $L4$ are solved when computing the right hand side r, whereas $L1$ and $L2$ are solved when applying the Schur-complement operator. The four problems do not depend on the global pressure variable p_Ω and global adjoint pressure r_Ω.

The coupling vector $(\mathbf{u}_\Gamma, p_\Omega, \mathbf{z}_\Gamma, r_\Omega)$ does also not depend on the local Lagrange parameters η_i and θ_i.

5.4.1 *General Operator*

In Chapter 6, we derive two algorithms, which are independent of the model problem. One to solve the Schur-complement equations and another one to solve the global linear systems. Therefore, we define a generalized operator introduced in Chapter 2 in Section 2.6.4. Again, the general operator is based on the block matrices used in the global linear system. For non-outflow subdomain, we get:

$$
\mathbf{K}_i = \left(\begin{array}{ccccc|ccc}
\mathbf{A}_{ii}\left[\mathbf{u}_i^k\right] & \mathbf{0} & -(\mathbf{D}_c)_{ii} & \widetilde{\mathbf{B}}_{ii}^T & \mathbf{0} & \mathbf{A}_{i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-(\mathbf{M}_0)_{ii} & \mathbf{A}_{ii}^T\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{ii}^T & -(\mathbf{M}_0)_{i\Gamma_i} & \mathbf{A}_{\Gamma_i i}^T\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & (\mathbf{D}_c)_{ii}^T & \alpha(\mathbf{M}_c)_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{D}_c)_{\Gamma_i i}^T & \mathbf{0} & \mathbf{0} \\
\widetilde{\mathbf{B}}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \widetilde{\mathbf{B}}_{ii} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i} & \mathbf{0} & \mathbf{0} \\
\hline
\mathbf{A}_{\Gamma_i i}\left[\mathbf{u}_i^k\right] & \mathbf{0} & -(\mathbf{D}_c)_{\Gamma_i i} & \widetilde{\mathbf{B}}_{i\Gamma_i}^T & \mathbf{0} & \mathbf{A}_{\Gamma_i\Gamma_i}\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{B}_{0,i}^T & \mathbf{0} \\
-(\mathbf{M}_0)_{\Gamma_i i} & \mathbf{A}_{i\Gamma_i}^T\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{0} & \widetilde{\mathbf{B}}_{i\Gamma_i}^T & -(\mathbf{M}_0)_{\Gamma_i\Gamma_i} & \mathbf{A}_{\Gamma_i\Gamma_i}^T\left[\mathbf{u}_i^k\right] & \mathbf{0} & \mathbf{B}_{0,i}^T \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{0,i} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{B}_{0,i} & \mathbf{0} & \mathbf{0}
\end{array}\right)
$$

$$
:= \left(\begin{array}{cc|cc}
\mathcal{A}_{ii} & \mathcal{B}_{ii}^T & \mathcal{A}_{i\Gamma_i} & \mathbf{0} \\
\mathcal{B}_{ii} & \mathbf{0} & \mathcal{B}_{i\Gamma_i} & \mathbf{0} \\
\hline
\mathcal{A}_{\Gamma_i i} & \mathcal{B}_{\Gamma_i i}^T & \mathcal{A}_{\Gamma_i\Gamma_i} & \mathcal{B}_{\Omega_i}^T \\
\mathbf{0} & \mathbf{0} & \mathcal{B}_{\Omega_i} & \mathbf{0}
\end{array}\right).
$$

For outflow subdomains, we get a similar operator. We only have to replace the submatrices of $\widetilde{\mathbf{B}}$ with the corresponding submatrices of $\mathbf{B}$.

SUMMARY

In this chapter, we derived a non-overlapping domain decomposition method for a non-linear quadratic optimal flow control problems assuming distributed control. As constraints, we applied Navier-Stokes equations equipped with mixed outflow and Dirichlet boundary conditions. As in the linear optimal control case, the domain decomposition method was applied to the optimality system. Analogously to the non-linear model problem treated in the previous chapter, the Newton-method was applied as an outer loop. In the next chapter, we derive solution algorithms and preconditioners for all model problems considered in Chapter 2 - 5.

6

PRECONDITIONED SOLVING ROUTINES BASED ON DOMAIN DECOMPOSITION METHODS

In this chapter, we derive two different solution algorithms and appropriate preconditioners: One solution algorithm for the Schur-complement equation and the other one for the coupled global linear system. The preconditioners are derived on the algebraic level and independent of the model problems considered in the previous four chapters. They exploit the structure of the domain decomposition methods and take into account the basic phenomenon of a saddle point structure which is a characteristic of all model problems considered in this work. First, we propose a one level Neumann-Neumann preconditioner and a two level balancing Neumann-Neumann preconditioner for the Schur-complement equations based on [45, Section 4]. Using the ideas of these interface preconditioners, we derive a global Schur-complement preconditioner for the global linear system. This global preconditioner consists of a subdomain and a skeleton block. The structure of this chapter is also outlined in Fig. 18.

We use the following notation throughout this chapter. By the terms velocity and pressure, we refer in case of an optimal control problem to the state and adjoint velocity as well as state and adjoint pressure. Due to the saddle point structure of the model problems, we get global couplings for the velocity on the skeleton and as global couplings for the pressure, we get one constant per subdomain. If necessary, we

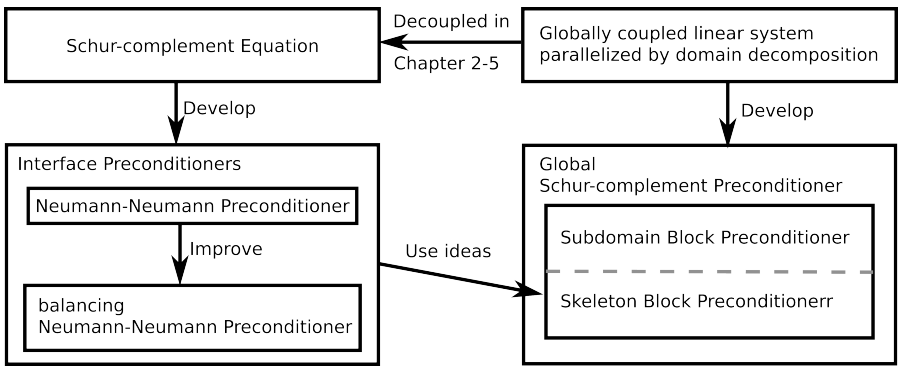


Figure 18: Structure of Chapter 6.

point out, to which coupling we refer, otherwise, we always refer by the term "couplings" to the combination of both coupling types.

6.1 GENERALIZATION OF THE SCHUR-COMPLEMENT EQUATION

In this section, we generalize the Schur-complement equation for the four model problems described in Chapter 2-5.

First, we give a general form of the saddle point Schur-complement operator. Therefore, we repeat the definition of the general local subdomain operator matrix $\mathbf{K}_i$ based on the definitions made for each model problem in Chapters 2-5:

$$\mathbf{K}_i = \begin{pmatrix} \mathbf{K}_{ii} & \mathbf{K}_{i\Gamma_i} \\ \mathbf{K}_{\Gamma_i i} & \mathbf{K}_{\Gamma_i\Gamma_i} \end{pmatrix} = \left(\begin{array}{cc|cc} \mathcal{A}_{ii} & \mathcal{B}_{ii}^T & \mathcal{A}_{i\Gamma_i} & \mathbf{0} \\ \mathcal{B}_{ii} & \mathbf{0} & \mathcal{B}_{i\Gamma_i} & \mathbf{0} \\ \hline \mathcal{A}_{\Gamma_i i} & \mathcal{B}_{\Gamma_i i}^T & \mathcal{A}_{\Gamma_i\Gamma_i} & \mathcal{B}_{\Omega_i}^T \\ \mathbf{0} & \mathbf{0} & \mathcal{B}_{\Omega_i} & \mathbf{0} \end{array}\right). \tag{6.1}$$

Next, we define the general projection matrix

$$\mathbf{R}_{L_i} := \begin{pmatrix} \mathbf{R}_{\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\Omega_i} \end{pmatrix} \in \{0,1\}^{(m_\Gamma + m_\Omega)\times(m_{\Gamma_i}+m_{\Omega_i})},$$

which projects entries of a local coupling vector corresponding to one subdomain into the global coupling vector. $\mathbf{R}_{L_i}$ can be split into the projection matrix $\mathbf{R}_{\Gamma_i} \in \{0,1\}^{m_\Gamma \times m_{\Gamma_i}}$ for the skeleton couplings and the projection matrix for the pressure couplings $\mathbf{R}_{\Omega_i} \in \{0,1\}^{m_\Omega \times m_{\Omega_i}}$. m_Γ and m_{Γ_i} denote the number of degrees of freedom for the velocity on the global and local skeletons, respectively. m_Ω and m_{Ω_i} denote the number of degrees of freedom of the global and local pressure spaces, respectively. In case of a two dimensional optimization problem, m_Γ includes the number of degrees of freedom for the state and adjoint velocity on the global skeleton and $m_{\Omega_i} = 2$, one constant for the state and one for the adjoint pressure. The (restriction) matrix $\mathbf{R}_{L_i}^T$ restricts the global couplings (the velocity degrees of freedom on the global skeleton and the global pressure constants) to the local couplings (the velocity degrees of freedom on the local skeleton and the pressure constants corresponding to the subdomain). Let the vector $(\mathbf{x}_\Gamma, \mathbf{x}_\Omega)^T$ denote a general global coupling vector. First, we number the degrees of freedom on the global skeleton denoted by the subindex Γ, second, we number the degrees of freedom of the pressure denoted by the subindex Ω. Using these definitions, we write the general or global Schur-complement operator $\mathbf{S}$ in two different variants:

$$\begin{aligned}\mathbf{S} :&= \sum_{i=1}^{s} \mathbf{R}_{L_i} \underbrace{\left(\mathbf{K}_{\Gamma_i\Gamma_i} - \mathbf{K}_{\Gamma_i i}\mathbf{K}_{ii}^{-1}\mathbf{K}_{i\Gamma_i}\right)}_{=:\mathbf{S}_i} \mathbf{R}_{L_i}^T \\ &= \sum_{i=1}^{s} \mathbf{R}_{L_i}\mathbf{S}_i\mathbf{R}_{L_i}^T \in \mathbb{R}^{(m_\Gamma+m_\Omega)\times(m_\Gamma+m_\Omega)}\end{aligned}$$

The first variant emphasizes the general block structure based on the local block matrices of $\mathbf{K}_i$ and defines the local Schur-complement operator $\mathbf{S}_i$.

$$\begin{aligned}\mathbf{S} = &\sum_{i=1}^{s} \begin{pmatrix} \mathbf{R}_{\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\Omega_i} \end{pmatrix} \underbrace{\begin{pmatrix} \mathcal{A}_{\Gamma_i\Gamma_i} & \mathcal{B}_{\Omega_i}^T \\ \mathcal{B}_{\Omega_i} & \mathbf{0} \end{pmatrix}}_{(*)} \begin{pmatrix} \mathbf{R}_{\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\Omega_i} \end{pmatrix}^T \\ &- \begin{pmatrix} \mathbf{R}_{\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\Omega_i} \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\Gamma_i i} & \mathcal{B}_{\Gamma_i i}^T \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \underbrace{\begin{pmatrix} \mathcal{A}_{ii} & \mathcal{B}_{ii}^T \\ \mathcal{B}_{ii} & \mathbf{0} \end{pmatrix}^{-1}}_{(**)} \begin{pmatrix} \mathcal{A}_{i\Gamma_i} & \mathbf{0} \\ \mathcal{B}_{i\Gamma_i} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{R}_{\Gamma_i} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\Omega_i} \end{pmatrix}^T. \\ = &\begin{pmatrix} \mathcal{S}_\Gamma & \mathcal{B}_\Omega^T \\ \mathcal{B}_\Omega & \mathbf{0} \end{pmatrix}\end{aligned}$$

The second formulation points out the global $(*)$ and local $(**)$ saddle point structure.

In this work, we do not compute the matrix of the Schur-complement operator $\mathbf{S}$ or the local Schur-complement operator $\mathbf{S}_i$ explicitly. Computing this operator matrix means to compute on each subdomain the inverse of the matrix $\mathbf{K}_{ii}$. Numerically that is too expensive [56, Chapter 4]. In our approach, we compute the result by applying the operator S to a vector $(\mathbf{x}_\Gamma, \mathbf{x}_\Omega)^T$, see Algorithm 6.1.

Let the vector $(\mathbf{b}_1, \ldots, \mathbf{b}_s, \mathbf{b}_\Gamma, \mathbf{b}_\Omega)^T$ denote the right hand side of the global linear system, where we first number the inner degrees of freedom of each subdomain, then the degrees of freedom of the global skeleton and finally the degrees of freedom of the pressure constants. Based on this vector, we consider the following formulation for the general right hand side of the Schur-complement equation:

$$\mathbf{r} := (\mathbf{b}_\Gamma, \mathbf{b}_\Omega)^T - \sum_{i=1}^{s} \mathbf{R}_{L_i}\mathbf{K}_{\Gamma_i i}\mathbf{K}_{ii}^{-1}\mathbf{b}_i.$$

One objective in this section is to derive a preconditioned solution algorithm to solve the general Schur-complement equation:

$$\mathbf{S}(\mathbf{x}_\Gamma, \mathbf{x}_\Omega) = \mathbf{r}.$$

Algorithm 6.1 Apply Schur-complement

1: **function** $S(\mathbf{x}_\Gamma, \mathbf{x}_\Omega)$
2: Restrict: $(\mathbf{x}_{\Gamma_i}, \mathbf{x}_{\Omega_i}) \leftarrow (\mathbf{R}_{\Gamma_i}^T \mathbf{x}_\Gamma, \mathbf{R}_{\Omega_i}^T \mathbf{x}_\Omega)$
3: **if** exact **then**
4: Solve (preconditioned): $K_{ii}\mathbf{x}_i = K_{i\Gamma_i}\mathbf{x}_{\Gamma_i}$
5: **else if** inexact **then**
6: Solve l_Γ (preconditioned) steps: $K_{ii}\mathbf{x}_i = K_{i\Gamma_i}\mathbf{x}_{\Gamma_i}$
7: **end if**
8: Compute: $(\mathbf{y}_{\Gamma_i}, \mathbf{y}_{\Omega_i}) \leftarrow K_{\Gamma_i i}\mathbf{x}_i + K_{\Gamma_i\Gamma_i}(\mathbf{x}_{\Gamma_i}, \mathbf{x}_{\Omega_i})$
9: Allreduce: $(\mathbf{y}_\Gamma, \mathbf{y}_\Omega)^T \leftarrow \sum_i (\mathbf{R}_{\Gamma_i}\mathbf{y}_{\Gamma_i}, \mathbf{R}_{\Omega_i}\mathbf{y}_{\Omega_i})$
10: **return** $(\mathbf{y}_\Gamma, \mathbf{y}_\Omega)^T$
11: **end function**

The "exact" variant of Algorithm 6.1 describes the steps, when applying the the Schur-complement operator $\mathbf{S}$ to a global coupling vector $(\mathbf{x}_\Gamma, \mathbf{x}_\Omega)$. In step 2, we restrict the global data to the subdomains. On each subdomain a local problem with Dirichlet data on the local skeleton is solved in step 4. To solve the local problem, we use an iterative Krylov subspace solver. Since the problem is non-symmetric, a GMRES-method of Saad is our appropriate choice [48, Chapter 6 + 9] and [50]. To improve the solver, we use an incomplete LU factorization as a right preconditioner to accelerate the GMRES-method [48, Chapter 10] and [49]. In the next step 8, the resulting Dirichlet data is projected to natural boundary (outflow) data by computing the described matrix-vector operation. Steps 2, 4 and 8 are all local and can be computed in parallel. Only in step 9, when the local result is projected to the global data structure and added up, global communication is needed. Fig 19 also sketches the application of the Schur-operator.

Remark 41. By solving the local problems in step 4 with an iterative solver, the Schur-complement operator is not computed exactly, only up to the given stopping tolerance of the solver. In consequence, the Schur-complement equation is algebraically not equivalent to the global problem. But our numerical experiments in the next chapter 7 show that we nevertheless obtain accurate results, when solving the local problems in step 4 very accurately.

6.2 PRECONDITIONERS FOR THE SCHUR-COMPLEMENT EQUATION

One of our aims is to develop a parallel method which is scalable and efficient. These two characteristics form the basis that a method is suitable for high performance computing. In this subsection, we describe two interface preconditioners for the Schur-complement equations. In the next Chapter 7, we study their applicability with respect to high performance computing. Consequently, one main requirement is that the preconditioners must run in parallel. Both preconditioner

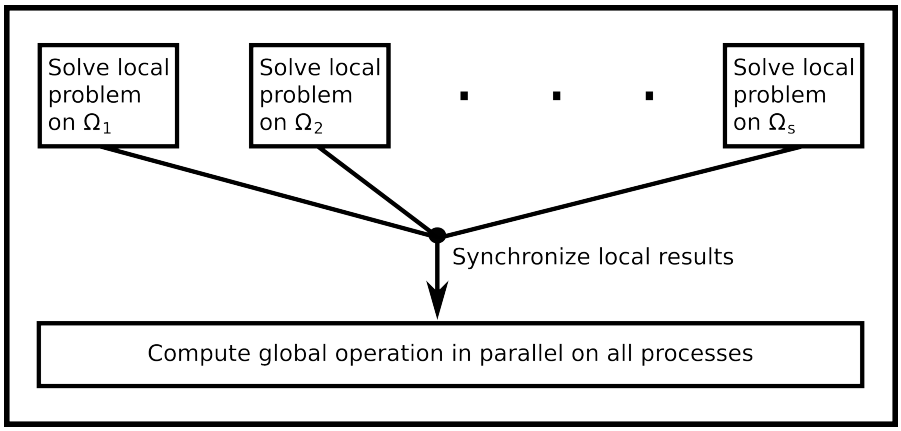


Figure 19: The figure shows schematically how to apply the Schur-complement operator, how to compute the right hand side and final solution in case of solving the global system via the Schur-complement equation, and how to apply the Neumann-Neumann preconditioner. In all cases first, local problems are solved on the subdomain in parallel. The results are collected and distributed to all processes. Then the same global operation is computed on all processes in parallel. Like this all processes know the result of the global computation and no further communication is needed.

are by construction parallel. They exploit the local structure of the global Schur-complement. The Neumann-Neumann preconditioner is based on the ideas of the Neumann-Neumann preconditioner for the Poisson equation. Both preconditioners consider the local saddle point structure and exchange information with the direct neighboring subdomains. The balancing Neumann-Neumann preconditioner is based on the ideas for a preconditioner with the same name for the Poisson equation. It extends the Neumann-Neumann preconditioner by also considering the global saddle point structure and interchanging information globally through a coarse problem. For more details on the preconditioners for the Poisson equation, see [43, 46, 51] and references therein.

6.2.1 *Neumann-Neumann Preconditioner*

The idea of the Neumann-Neumann preconditioner is based on two steps. In the first step, local independent subdomain problems with natural boundary conditions on the skeleton are solved. In the second step, the result in form of Dirichlet data on the local skeleton is weighted and exchanged with the neighboring subdomains. Since the local subproblems are locally independent, the first step can be done in parallel for all subdomains. In the second step, communication with the neighboring subdomains is needed to exchange the Dirichlet data, see also Fig. 19.

The operator form of the Neumann-Neumann preconditioner is given as follows:

$$P^{NN} := \sum_{i=1}^{s} \mathbf{R}_{\Gamma_i} \mathbf{D}_{\Gamma_i} P_i^{NN} \mathbf{D}_{\Gamma_i} \mathbf{R}_{\Gamma_i}^T.$$

It is an additive preconditioner, see, e.g., [51, Chapter 5]. $\mathbf{D}_{\Gamma_i}$ are diagonal matrices of dimension m_{Γ_i} which weight the local skeleton entries. The weights can be chosen arbitrarily as long as it holds that [**?**]

$$\mathbf{I}_\Gamma = \sum_{i=1}^{s} \mathbf{R}_{\Gamma_i} \mathbf{D}_{\Gamma_i} \mathbf{R}_{\Gamma_i}^T$$

with $\mathbf{I}_\Gamma$ the identity matrix of dimension m_Γ. A natural choice for the diagonal entries are the reciprocals of the *counting functions* [**?**]. The counting functions of each degree of freedom is the number of subdomains on which the degree of freedom lies or, in other words, the number of subdomains which share a degree of freedom. Hence, the counting function of an inner degree of freedom is always one. On the skeleton Γ, the counting function is at least two. The local operator P_i^{NN} of the preconditioner corresponds to the local subdomain matrix $\mathbf{K}_i$. It implies solving on each subdomain a local problem with natural boundary conditions on the local skeleton Γ_i. On the other part of the subdomain boundary either outflow conditions or homogeneous Dirichlet boundary conditions are applied. That depends on the type of the subdomain. On inner subdomains, only natural boundary conditions are implied.

Remark. Since the pressure is uniquely defined on all subdomains, no normalization condition must be implied. On the continuous level, it means that the pressure lies in $L^2(\Omega_i)$. This has impact on the implementation.

Algorithm 6.2 describes the steps, when applying the preconditioner to a global skeleton vector. In step 2 of Algorithm 6.2, we restrict the global skeleton to the local skeleton. Since the global skeleton is known to every process (or at least the needed part of the global skeleton), this is a local operation, which can be done in parallel. In step 3 and 9, we weight the entries on the local skeleton. Step 5 and 7 correspond to solving a local subproblem with natural boundary conditions on the local skeleton. Since P^{NN} is used as a preconditioner, we do not need to solve the local problems exactly as in the case when applying the exact Schur-complement operator. In the "exact" variant (step 5), we solve the local problem up to a given tolerance. In the "inexact" variant (step 7), we compute a fixed number l_Γ of iterations. As for the Schur-complement operator, we use as a solver a preconditioned

Algorithm 6.2 Apply Neumann-Neumann preconditioner

1: **function** $P^{NN}(\mathbf{v}_\Gamma)$
2: Restrict: $\mathbf{v}_{\Gamma_i} \leftarrow \mathbf{R}_{\Gamma_i}^T \mathbf{v}_\Gamma$
3: Weight: $\mathbf{v}_{\Gamma_i} \leftarrow \mathbf{D}_{\Gamma_i} \mathbf{v}_{\Gamma_i}$
4: **if** exact **then**
5: Solve (preconditioned): $\mathbf{K}_i(\mathbf{w}_i, \mathbf{w}_{\Gamma_i})^T = (\mathbf{0}_i, \mathbf{v}_{\Gamma_i})^T$
6: **else if** inexact **then**
7: Solve l_Γ (preconditioned) steps: $\mathbf{K}_i(\mathbf{w}_i, \mathbf{w}_{\Gamma_i})^T = (\mathbf{0}_i, \mathbf{v}_{\Gamma_i})^T$
8: **end if**
9: Weight: $\mathbf{w}_{\Gamma_i} \leftarrow \mathbf{D}_{\Gamma_i} \mathbf{v}_{\Gamma_i}$
10: Project and Allreduce: $\mathbf{w}_\Gamma^{k+1} \leftarrow \sum_i \mathbf{R}_{\Gamma_i} \mathbf{w}_{\Gamma_i}^{k+1}$
11: **return** $\mathbf{w}_\Gamma^{k+1}$
12: **end function**

GMRES-method and an incomplete LU-factorization as preconditioner, see, e.g., [48, Chapters 6, 9 + 10]. Step 10 is the only global operation, where global communication is needed. To get Dirichlet data on the global skeleton as the result of applying the Neumann-Neumann preconditioner, we need to add up local Dirichlet data projected to the global skeleton.

Note that the Neumann-Neumann preconditioner is independent of the global pressure couplings $\mathbf{x}_\Omega$. Consequently, it only considers the local saddle point structure of the Schur-complement operator. We also recognize that the step 2, 5, 7 and 10 of applying the Neumann-Neumann preconditioner are similar to the steps 2, 4, 6 and 8 of applying the Schur-complement operator. Therefore, the cost of the application of both operators are comparable. They may differ with respect to the accuracy of the stopping criteria for the local solvers.

6.2.2 *Balancing Neumann-Neumann Preconditioner*

The Neumann-Neumann preconditioner only exchanges information between subdomains which are direct neighbors and neglects the information provided by the global saddle point structure. Hence, we propose as an improvement the balancing Neumann-Neumann preconditioner. In addition to solving local subdomain problems with natural boundary conditions, a global but very coarse problem is solved. By solving the coarse problem, the preconditioner not only considers the global saddle point structure but also provides a global exchange of information. For the Stokes problem, the preconditioner was proposed and studied in detail in [45].

By adding the coarse level, we also add a global coupling which at a first glance contradicts the requirement to provide a parallel preconditioner but we expect that the improvement of the preconditioner with respect to the reduction of iterations makes up for the added sequential step. In numerical experiments, we study and compare

the Neumann-Neumann and balancing Neumann-Neumann preconditioner not only regarding the number of iterations, but also with respect to the cost of the set up of the preconditioner and the cost to apply the preconditioner in each iteration, see Chapter 7.

As stated in [45] by using inf-sup stable spaces for the coarse problem, it is even possible to provide a balancing Neumann-Neumann preconditioner which is independent of the number of subdomains or in other words stable with respect to the increasing number of subdomains.

Before we define the operator for the balancing Neumann-Neumann preconditioner, we define the discrete coarse space. The coarse problem also has a saddle point structure. Since our aim is to provide a very coarse problem, we model the velocity and the pressure only by a few degrees of freedom per subdomain. For the coarse pressure, we use the same discretization as for the global pressure and therefore use the space $\mathbf{D}(\Omega)$:

$$\mathbf{D}(\Omega) := \begin{cases} D(\Omega) \times D(\Omega) & \text{if optimal control problem} \\ D(\Omega) & \text{otherwise} \end{cases}.$$

In case that the model problem are the Oseen equations with inhomogeneous global boundary conditions, we replace $D(\Omega)$ with one of the spaces $\widehat{D}(\Omega)$ or $\widetilde{D}(\Omega)$ depending on the weak formulation.

We give two definitions for the discrete coarse velocity space. The first definition $\mathbf{V}_0^1$ is purely based on the counting functions. Although we know that this proposal does not yield an inf-sup stable coarse space $\mathbf{V}_0^1 \times \mathbf{D}(\Omega)$, we consider it because $\mathbf{V}_0^1$ is very easy to implement and the implementation is independent of how the domain is partitioned. Furthermore, numerical experiments in Chapter 7 show significant improvement comparing the Neumann-Neumann and the balancing Neumann-Neumann preconditioner.

For each subdomain and each velocity component, we define the vector $\mu_i^j \in \mathbb{R}^{m_\Gamma}$. The subindex i indicates the subdomain and the super index j indicates the velocity component. When considering as an example a two dimensional optimal control problem, the index j runs from 1 to 4, where 1 and 2 correspond to the two components of the state velocity, and analogously, 3 and 4 to the two components of the adjoint velocity. All entries of the vector μ_i^j which do not belong to degrees of freedom on the local skeleton Γ_i are set to 0. All entries belonging the to the local subdomain but not to the degrees of freedom of the velocity component j are also 0. The remaining entries contain the reciprocal of the counting function of that degree of freedom. The first coarse velocity space is then defined as

$$\mathbf{V}_0^1 := \left\{ \mathbf{v} \in \mathbf{V}_\Gamma^h : \mathbf{v} \in span\{\mu_i^j\} \right\}.$$

The second proposal $\mathbf{V}_0^2$ for the coarse velocity space leads to an inf-sup stable coarse space. Motivated by inf-sup stable Taylor-Hood finite elements on the fine level, we enrich the first coarse space by adding, depending on the triangulation, continuous coarse piecewise bi-or triquadratic functions or continuous coarse piecewise quadratic functions. This leads to the second coarse velocity space

$$\mathbf{V}_0^2 := \left\{ \mathbf{v} \in \mathbf{V}_\Gamma^h : \mathbf{v} \in span\{\mu_i^j\} \cup Q_2(\Gamma) \text{ (or } P_2(\Gamma)) \right\}.$$

Lemma 42. *The coarse space* $\mathbf{V}_0^2 \times \mathbf{D}(\Omega)$ *satisfies the inf-sup condition.*

Proof. See [45, Lemma 5.2]. □

Remark. The implementation of $\mathbf{V}_0^2$ may be tricky. The definitions of the piecewise quadratic or bi-or triquadratic functions for the coarse space are based on the elements of a triangulation. Aiming at a very coarse velocity space which is formed of only a few basis functions, it is necessary that each coarse triangulation of one subdomain only consists of a few elements. Thus, we need to assume a restriction with respect to the way of splitting the global domain into subdomains. Depending on the element type, the coarse triangulation of the subdomains must be formed in two dimension by a few coarse triangles or quadrilaterals and in three dimensions be a few hexahedra or tetrahedra. The more coarse elements are needed to define a coarse triangulation of one subdomain, the larger gets the coarse velocity space. The smallest coarse space can be achieved if each coarse triangulation of a subdomain contains exactly one element.

Next, we define the coarse operator. Therefore, let $\mathbf{L}_0 \in \mathbb{R}^{m_\Gamma \times m_0^v}$ be the matrix whose columns are the vectors spanning the coarse velocity space, either $\mathbf{V}_0^1$ or $\mathbf{V}_0^2$. m_0^v is the number of vectors which span the coarse velocity space. Then, we can define the projection matrix $\mathbf{R}_0 \in \mathbb{R}^{(m_\Gamma + m_\Omega) \times (m_0^v + m_\Omega)}$, which projects the entries of the coarse space to the coupling space by

$$\mathbf{R}_0 = \begin{pmatrix} \mathbf{L}_0 & 0 \\ 0 & \mathbf{I}_\Omega \end{pmatrix}.$$

Here, $\mathbf{I}_\Omega$ is the identity matrix with the size of the dimension of the space $\mathbf{D}(\Omega)$.

Now, we can define the coarse operator matrix $\mathbf{S}_0$:

$$\mathbf{S}_0 = \mathbf{R}_0^T \mathbf{S} \mathbf{R}_0 = \begin{pmatrix} \mathbf{L}_0^T \mathcal{S}_\Gamma \mathbf{L}_0 & (\mathcal{B}_\Omega \mathbf{L}_0)^T \\ \mathcal{B}_\Omega \mathbf{L}_0 & \mathbf{0} \end{pmatrix}.$$

Since we do not compute the Schur-complement operator $\mathbf{S}$, the coarse matrix $\mathbf{S}_0$ cannot be computed by matrix-vector operations. We compute the matrix $\mathbf{S}_0$ by applying the Schur-complement operator to each of the columns of the matrix $\mathbf{R}_0$. Consequently to compute the matrix $\mathbf{S}_0$, we need to solve for each of the m_0^v vectors of the matrix $\mathbf{L}_0$ on each subdomain a local Dirichlet problem. This is the main cost factor in the set up for the balancing Neumann-Neumann preconditioner.

Now, we can define the coarse operator matrix for the balancing Neumann-Neumann preconditioner P_0^{bNN}:

$$P_0^{bNN} = \mathbf{R}_0 \mathbf{S}_0^{-1} \mathbf{R}_0^T.$$

Algorithm 6.3 describes the steps, when applying the coarse operator $P_0^{bNN}$to a coupling residual vector $(\mathbf{v}_\Gamma, \mathbf{v}_\Omega)$.

Algorithm 6.3 Apply coarse operator P_0^{bNN}

1: **function** $P_0^{bNN}((\mathbf{v}_\Gamma, \mathbf{v}_\Omega))$
2: Restrict: $\mathbf{v}_0 \leftarrow \mathbf{R}_0^T(\mathbf{v}_\Gamma, \mathbf{v}_\Omega)$
3: **if** exact **then**
4: Solve (preconditioned): $S_0 \mathbf{w}_0 = \mathbf{v}_0$
5: **else if** inexact **then**
6: Solve l_Γ (preconditioned) steps: $S_0 \mathbf{w}_0 = \mathbf{v}_0$
7: **end if**
8: Project: $(\mathbf{w}_\Gamma, \mathbf{w}_\Omega) \leftarrow \mathbf{R}_0 \mathbf{w}_0$
9: **return** $(\mathbf{w}_\Gamma, \mathbf{w}_\Omega)$
10: **end function**

In step 2 (or 8), we restrict (or project) the coupling vector to the coarse space (or vice versa). In step 4, we solve a coarse saddle point problem. Either with a direct solver or an iterative solver up to a given stopping tolerance. In our implementation this is a local operation. Since the coarse matrix is very small compared to the subdomain matrices, we store the coarse matrix as well as the projection matrix $\mathbf{R}_0$ on every process. The skeleton vector is also globally known. On each process, we solve the same problem independently. Consequently, the result is directly known to all processes which avoids communication.

Using these definitions, we define the balancing Neumann-Neumann preconditioner:

$$P^{bNN} = P_0^{bNN} + (I - P_0^{bNN} \mathbf{S}) P^{NN}.$$

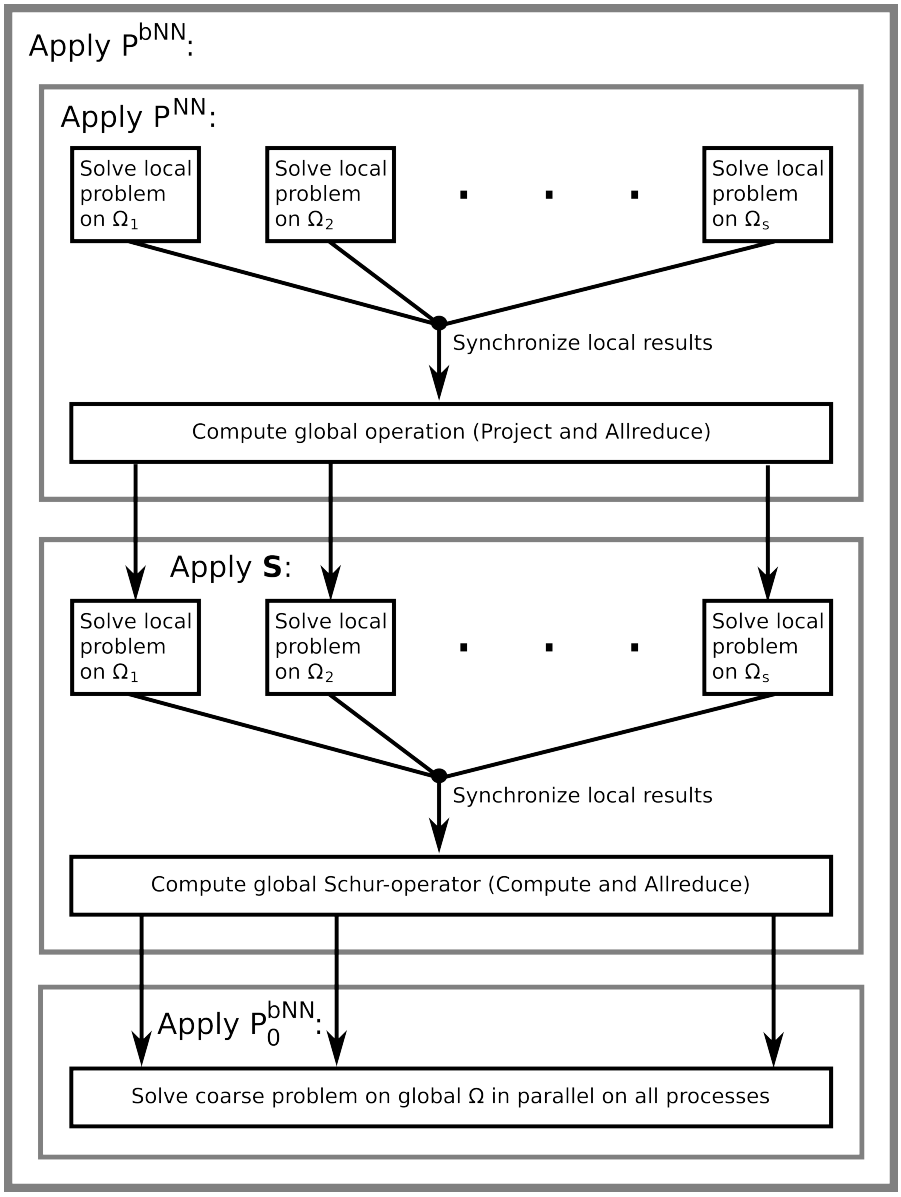


Figure 20: This figure outlines diagrammatically the application of the balancing Neumann-Neumann preconditioner. In the first step, we apply the Neumann-Neumann preconditioner, secondly, we compute the Schur-complement operator and finally, we solve a coarse problem.

We chose the non-symmetric definition of the balancing Neumann-Neumann preconditioner. This is convenient, since we solve non-symmetric model problems and use a suitable solver. The preconditioner is of hybrid type because it combines the coarse solver, which is treated multiplicative, whit the additively treated Neumann-Neumann preconditioner, see [51, Chapter 5]. Algorithm 6.4 describes the steps, when applying the preconditioner P^{bNN} to a residual vector $(\mathbf{v}_\Gamma, \mathbf{v}_\Omega)^T \in \mathbb{R}^{m_\Gamma + m_\Omega}$.

Algorithm 6.4 Apply balancing Neumann-Neumann preconditioner

1: **function** $P^{bNN}((\mathbf{v}_\Gamma, \mathbf{v}_\Omega))$
2: Apply P^{NN}: $\mathbf{v}_\Gamma^1 \leftarrow P^{NN}(\mathbf{v}_\Gamma)$ ▷ Alg. 6.2
3: Apply S: $(\mathbf{v}_\Gamma^2, \mathbf{v}_\Omega^2) \leftarrow S(\mathbf{v}_\Gamma^1, \mathbf{v}_\Omega)$ ▷ Alg. 6.1
4: Compute: $(\mathbf{v}_\Gamma^2, \mathbf{v}_\Omega^2) \leftarrow (\mathbf{v}_\Gamma, \mathbf{v}_\Omega) - (\mathbf{v}_\Gamma^2, \mathbf{v}_\Omega^2)$
5: Apply P_0^{bNN}: $(\mathbf{w}_\Gamma, \mathbf{w}_\Omega) \leftarrow P_0^{bNN}(\mathbf{v}_\Gamma^2, \mathbf{v}_\Omega^2)$ ▷ Alg 6.3
6: Compute: $\mathbf{w}_\Gamma \leftarrow \mathbf{v}_\Gamma^1 + \mathbf{w}_\Gamma$
7: **return** $(\mathbf{w}_\Gamma, \mathbf{w}_\Omega)$
8: **end function**

Applying the balancing Neumann-Neumann preconditioner can be subdivided in two categories. In category one, described by step 4 and 6, the vectors are updated. This category is a vector-vector operation, which is not very costly. The second category covers the steps where operators must be applied, see step 2, 3 and 5. Applying a operator always implies solving a saddle point problem. Either on the subdomain level as in step 2 and 3 including global communication or on the coarse level as in step 5. The application of the balancing Neumann-Neumann preconditioner is illustrated schematically in Fig. 20.

6.3 COMPARISON OF THE NEUMANN-NEUMANN AND THE BALANCING NEUMANN-NEUMANN PRECONDITIONER

When comparing the two preconditioners, we need to consider on the one hand the characteristics of the preconditioners and on the other hand the cost for the set up and its application.

Both preconditioners consider the local saddle point structure and consequently are a good approximation for the local part of the Schur-complement operator. They both exchange information with its direct neighbors. When applying the Neumann-Neumann preconditioner to many subdomains the exchange of information between two subdomains takes as least as many iterations as subdomains are in between two subdomains. Numerical experiments in the next chapter show that the number of iterations for the Neumann-Neumann preconditioner increases rapidly with the number of subdomains. The idea of the balancing Neumann-Neumann preconditioner is to improve the exchange of information by solving a global but very coarse problem. By doing

this, in every iteration information is exchanged globally. Therefore, the coarse part of the preconditioner considers the global saddle point which is completely ignored by the one level Neumann-Neumann preconditioner. By considering an inf-sup coarse implementation, we can even achieve that the balancing Neumann-Neumann preconditioner gets independent of the number of subdomains, see [45, Section 6].

Comparing the cost for the set up and in each iteration, the one level Neumann-Neumann preconditioner is much cheaper. For the set up, we only need to determine the weights for the matrices $\mathbf{D}_{\Gamma_i}$ and assemble the local subdomain matrices with natural boundary conditions. To set up the balancing Neumann-Neumann preconditioner two more steps are needed. We additionally need to set up the projection matrix $\mathbf{R}_0$ by determining the counting functions. The most costly part is the computation of the coarse matrix $\mathbf{S}_0$. The Schur-complement operator needs to be applied m_0^v times which implies solving local Dirichlet problems on each subdomain and global communication. The more subdomains the more Dirichlet problems on the one hand, but the more subdomains the smaller the Dirichlet problems on the other hand. Furthermore the global skeleton grows with the number of subdomains. Consequently, the communication gets more costly.

The cost to apply the Neumann-Neumann preconditioner in each iteration can be split into two parts: First, solving in parallel local subdomain problems with natural boundary conditions on the local skeleton and second, communicating the results globally.

One iteration of the balancing Neumann-Neumann preconditioner is more costly, additionally the Schur-complement operator is applied in each iteration. That involves solving local subdomain problems with inhomogeneous Dirichlet boundary conditions on the local skeleton and global communication between all processes to distribute the result. Additionally a coarse problem is solved in each iteration.

Summarizing, we can say that on the one hand the mathematical characteristics of the two level balancing Neumann-Neumann preconditioner improve the one level Neumann-Neumann preconditioner but on the other hand the two level balancing Neumann-Neumann preconditioner is more costly. In our numerical experiments in Chapter 7, in which we study not only the number of iterations but also the run time for the set-up of the solver and the solving time, we observe a substantial improvement of the two level balancing Neumann-Neumann preconditioner compared to the one level Neumann-Neumann preconditioner regarding the reduction of iterations and the runtime.

Algorithm 6.5 Compute global solution via Schur-complement equation

1: **function** COMPUTEGLOBALSOLUTION
2: Compute $(\mathbf{b}_\Gamma, \mathbf{b}_\Omega) \leftarrow$ RightHandSide ▷ Alg. 6.6
3: $(\mathbf{x}_\Gamma, \mathbf{x}_\Omega) \leftarrow$ SolveSkeleton$((\mathbf{x}_\Gamma^0, \mathbf{x}_\Omega^0), (\mathbf{b}_\Gamma, \mathbf{b}_\Omega))$ ▷ Alg. 6.7
4: Compute FinalSolution$(\mathbf{x}_\Gamma, \mathbf{x}_\Omega)$ ▷ Alg. 6.8
5: **end function**

6.4 SOLUTION ALGORITHM FOR THE SCHUR-COMPLEMENT EQUATION

In this section, we describe the steps how to solve the global model problems by solving the Schur-complement equation and in which step we apply the Neumann-Neumann type preconditioners.

Algorithm 6.5 gives an overview about the three main steps of the solution algorithm: Before we can solve the Schur-complement equation as described in Algorithm 6.7 (step 3), we need to compute the right hand side $\mathbf{r}$ (step 2) as outlined in more detail in Algorithm 6.6. Solving the Schur-complement equation is the main part of the solution process. The outcome of solving the Schur-complement equation is a coupling vector containing the solution for the velocity on the skeleton and global constants for the pressure on each subdomain. The coupling vector is the input to compute the final global solution on Ω (step 4) as further specified in Algorithm 6.8.

Algorithm 6.6 Compute right hand side

1: **function** RIGHTHANDSIDE
2: Given: $\mathbf{b}_i, \mathbf{b}_{\Gamma_i}, \mathbf{b}_{\Omega_i}$
3: Solve: $\mathbf{K}_{ii}\mathbf{x}_i = \mathbf{b}_i - \mathbf{K}_{iD_i}\mathbf{x}_{D_i}$
4: Compute: $\mathbf{b}_{\Gamma_i} \leftarrow \mathbf{b}_{\Gamma_i} + K_{\Gamma_i i}\mathbf{x}_i$
5: Allreduce:$(\mathbf{b}_\Gamma, \mathbf{b}_\Omega)^T \leftarrow \sum_i (\mathbf{R}_{\Gamma_i}\mathbf{b}_{\Gamma_i}, \mathbf{R}_{\Omega_i}\mathbf{b}_{\Omega_i})$
6: **return** $(\mathbf{b}_\Gamma, \mathbf{b}_\Omega)^T$
7: **end function**

Computing the right hand side $\mathbf{r}$, see Algorithm 6.6, is done analogously to applying the Schur-complement equation. We solve a local Dirichlet problem on each subdomain, see step 3. In step 4, we project the Dirichlet data to natural boundary (outflow) data by matrix-vector operations. The global coupling vector containing the right hand side is then computed via global communication as described in step 5.

We solve the Schur-complement equation via a flexible right preconditioned variant of the GMRES method [48, Chapter 9.4], see also Algorithm 6.7. The GMRES method is an appropriate choice since our Schur-complement operator is not positive definite and non-symmetric. As a preconditioner, we apply the one level Neumann-Neumann or the two level balancing Neumann-Neumann preconditioner. Both precon-

ditioners are applied in an operator form which internally solves local linear systems up to a given stopping tolerance. Therefore the number of iterations of the local linear solvers may change in each iteration of the skeleton solver. Consequently, applying the preconditioner is not a linear operation anymore. To be able to compute the final solution or the solution for the restart, the flexible variant stores in each iteration not only the basis of the Krylov-space but also the vector $\mathbf{z}^j$, see step 8 and 17. In our implementation the global skeleton vector is known to all processes. Therefore the FGMRES method 6.7 is executed on all processes at the same time. Like this every process knows the next iterate and we avoid communication between the processes. The most costly parts, which are applying the preconditioner (step 6 or 8) and applying the Schur-complement operator (step 10), are done in parallel for each subdomain.

Algorithm 6.7 Solve Schur-complement equation via FGMRES

1: **function** SOLVESKELETON($(\mathbf{x}_\Gamma^0, \mathbf{x}_\Omega^0), (\mathbf{b}_\Gamma, \mathbf{b}_\Omega)$)
2: Compute Residual: $(\mathbf{r}_\Gamma^0, \mathbf{r}_\Omega^0) \leftarrow (\mathbf{b}_\Gamma, \mathbf{b}_\Omega) - S(\mathbf{x}_\Gamma^0, \mathbf{x}_\Omega^0)$ ▷ Alg. 6.1
3: Normalize: $(\mathbf{v}_\Gamma^1, \mathbf{v}_\Omega^1) \leftarrow \frac{1}{\|(\mathbf{r}_\Gamma^0, \mathbf{r}_\Omega^0)\|_2}(\mathbf{r}_\Gamma^0, \mathbf{r}_\Omega^0)$
4: **for** $j = 1, \ldots, m$ **do**
5: **if** Neumann-Neumann preconditioner **then**
6: Apply P^{NN}: $(\mathbf{z}_\Gamma^j, \mathbf{z}_\Omega^j) \leftarrow P^{NN}(\mathbf{v}_\Gamma^j, \mathbf{v}_\Omega^j)$ ▷ Alg. 6.2
7: **else if** balancing Neumann-Neumann preconditioner **then**
8: Apply P^{bNN}: $(\mathbf{z}_\Gamma^j, \mathbf{z}_\Omega^j) \leftarrow P^{bNN}(\mathbf{v}_\Gamma^j, \mathbf{v}_\Omega^j)$ ▷ Alg. 6.4
9: **end if**
10: Apply S: $(\mathbf{w}_\Gamma, \mathbf{w}_\Omega) \leftarrow S(\mathbf{z}_\Gamma^j, \mathbf{z}_\Omega^j)$ ▷ Alg. 6.1
11: **for** $i = 1, \ldots, j$ **do**
12: Compute: $h_{i,j} \leftarrow ((\mathbf{w}_\Gamma, \mathbf{w}_\Omega), (\mathbf{v}_\Gamma^i, \mathbf{v}_\Omega^i))_2$
13: Orthogonalize: $(\mathbf{w}_\Gamma, \mathbf{w}_\Omega) \leftarrow (\mathbf{w}_\Gamma, \mathbf{w}_\Omega) - h_{i,j}(\mathbf{v}_\Gamma^i, \mathbf{v}_\Omega^i)$
14: **end for**
15: Compute: $h_{j+1,j} \leftarrow \|(\mathbf{w}_\Gamma, \mathbf{w}_\Omega)\|_2$
16: Normalize: $(\mathbf{v}_\Gamma^{j+1}, \mathbf{v}_\Omega^{j+1}) \leftarrow \frac{1}{h_{j+1,j}}(\mathbf{w}_\Gamma, \mathbf{w}_\Omega)$
17: Define: $\mathbf{Z}_m := [\mathbf{z}_1, \ldots, \mathbf{z}_m]$
18: Define: $\mathbf{H}_m := \{h_{i,j}\}_{1 \le i \le j+1; 1 \le j \le m}$
19: **end for**
20: Compute: $y_m = \operatorname{argmin}_\mathbf{y} \|\|(\mathbf{r}_\Gamma^0, \mathbf{r}_\Omega^0)\|_2 \mathbf{e}_1 - \mathbf{H}_m \mathbf{y}\|_2$
21: Compute: $(\mathbf{x}_\Gamma^m, \mathbf{x}_\Omega^m) = (\mathbf{x}_\Gamma^0, \mathbf{x}_\Omega^0) + \mathbf{Z}_m \mathbf{y}_m$
22: **if** Stopping criterion fulfilled **then**
23: **return** $(\mathbf{x}_\Gamma, \mathbf{x}_\Omega) \leftarrow (\mathbf{x}_\Gamma^m, \mathbf{x}_\Omega^m)$
24: **else**
25: $(\mathbf{x}_\Gamma^0, \mathbf{x}_\Omega^0) \leftarrow (\mathbf{x}_\Gamma^m, \mathbf{x}_\Omega^m)$ Goto 2
26: **end if**
27: **end function**

Once we know the solution on the skeleton and the pressure constants for the subdomains, we can compute the global solution on the

Algorithm 6.8 Compute final solution

1: **function** FINALSOLUTION($\mathbf{x}_\Gamma, \mathbf{x}_\Omega$)
2: Restrict: $(\mathbf{x}_{\Gamma_i}, \mathbf{x}_{\Omega_i}) \leftarrow (\mathbf{R}_{\Gamma_i}^T \mathbf{x}_\Gamma, \mathbf{R}_{\Omega_i}^T \mathbf{x}_\Omega)$
3: Solve: $\mathbf{K}_{ii}\mathbf{x}_i = \mathbf{b}_i - \mathbf{K}_{i\Gamma}\mathbf{x}_\Gamma - \mathbf{K}_{iD_i}\mathbf{x}_{D_i}$
4: Add pressure global constant $\mathbf{x}_{\Omega_i}$ to local pressure
5: **end function**

domain Ω. Therefore, we need to solve a Dirichlet problem on each subdomain. Then on each processor the global solution restricted to the corresponding subdomain is stored.

6.5 PARALLELIZATION OF GLOBAL SYSTEM VIA DOMAIN DECOMPOSITION

Another possibility to exploit the characteristics of the domain decomposition method arises when directly solving the coupled global linear system. Here, we use the domain decomposition as a parallelization method which provides parallel matrix-vector operations. The global linear systems corresponding to the four different model problems can be written in the following general block format:

$$\mathbf{K}\mathbf{x} = \mathbf{b} \Leftrightarrow \begin{pmatrix} \mathbf{K}_{II} & \mathbf{K}_{I\Gamma} \\ \mathbf{K}_{\Gamma I} & \mathbf{K}_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{x}_I \\ \mathbf{x}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{b}_I \\ \mathbf{b}_\Gamma \end{pmatrix}, \tag{6.2}$$

where the different blocks are defined as

$$\mathbf{K}_{II} = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{ss} \end{pmatrix}, \qquad \mathbf{K}_{I\Gamma} = \begin{pmatrix} \mathbf{K}_{1\Gamma_1}\mathbf{R}_{L_1}^T \\ \vdots \\ \mathbf{K}_{1\Gamma_s}\mathbf{R}_{L_s}^T \end{pmatrix},$$

$$\mathbf{K}_{\Gamma I} = \begin{pmatrix} \mathbf{R}_{L_1}\mathbf{K}_{\Gamma_1 1} & \cdots & \mathbf{R}_{L_s}\mathbf{K}_{\Gamma_s s} \end{pmatrix}, \qquad \mathbf{K}_{\Gamma\Gamma} = \sum_{i=1}^{s} \mathbf{R}_{L_i}\mathbf{K}_{\Gamma_i\Gamma_i}\mathbf{R}_{L_i}^T.$$

Using these block matrices, we get another formulation of the general Schur-complement operator:

$$\mathbf{S} := \mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1}\mathbf{K}_{I\Gamma}. \tag{6.3}$$

Note that in the following, the subscript Γ of a vector refers to the whole coupling vector, including the skeleton and the pressure constants.

In our parallelization approach, we use as many processors as subdomains. In that way each processor corresponds to one subdomain. On processor i, we store one local subdomain matrix $\mathbf{K}_i$ (6.1) and the local subdomain vector $(\mathbf{x}_i, \mathbf{x}_{\Gamma_i})^T$. Additionally, we store on each subdomain the global skeleton vector. Like this we get a good framework for the matrix-vector operations needed to solve the global linear system in parallel with an iterative method. In each operation, we have to make sure, that the local skeleton vectors and the global skeleton vector on each subdomain are updated and contain the same data. This set-up then fits very well to the global Schur-complement preconditioner, we derive in the next subsection. Note that each of the blocks $\mathbf{K}_{ii}$ contains the local saddle point structure and each of the the blocks $\mathbf{K}_{\Gamma_i\Gamma_i}$ contains the local part of the global saddle point structure. The matrix $\mathbf{K}_{\Gamma\Gamma}$ then forms the global saddle point structure.

Note that depending on the number of subdomains, we get a different discretization for the pressure. They are equivalent on the continuous level but not the same on the discrete level. The pressure is coupled through global constants on each subdomain. Thus the more subdomains we have, the more larger gets the global pressure space. Therefore we cannot expect to solve the global linear system with the same number of iterations, when using different number of subdomains/processes, even though no preconditioner is applied.

6.6 GLOBAL SCHUR-COMPLEMENT PRECONDITIONER

Next, we derive a global preconditioner for the global linear system (6.2) by exploiting the block structure of the matrix. This leads us to a global Schur-complement preconditioner which is based on the ideas presented in [9, Chapter 5].

Note that since the local block matrices $\mathbf{K}_{ii}$ are invertible, it directly follows that the global block matrix $\mathbf{K}_{II}$ is invertible. Thus, we can decompose the global matrix as follows:

$$\mathbf{K} = \begin{pmatrix} \mathbf{I}_I & 0 \\ \mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1} & \mathbf{I}_\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{K}_{II} & 0 \\ 0 & \mathbf{S} \end{pmatrix} \begin{pmatrix} \mathbf{I}_I & \mathbf{K}_{II}^{-1}\mathbf{K}_{I\Gamma} \\ 0 & \mathbf{I}_\Gamma \end{pmatrix}$$

with the Schur-complement operator $\mathbf{S}$ as defined before (6.3) and $\mathbf{I}_I$ and $\mathbf{I}_\Gamma$ the identity matrices with dimensions of the blocks $\mathbf{K}_{II}$ and $\mathbf{K}_{\Gamma\Gamma}$, respectively.

Since $\mathbf{S}$ is the Schur-complement operator, we know that it is invertible. Then we can invert the global matrix $\mathbf{K}$:

$$\mathbf{K}^{-1} = \begin{pmatrix} \mathbf{I}_I & -\mathbf{K}_{II}^{-1}\mathbf{K}_{I\Gamma} \\ 0 & \mathbf{I}_\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{K}_{II}^{-1} & 0 \\ 0 & \mathbf{S}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{I}_I & 0 \\ -\mathbf{K}_{\Gamma I}\mathbf{K}_{II}^{-1} & \mathbf{I}_\Gamma \end{pmatrix}.$$

A good preconditioner approximates the inverse of the system matrix. This leads to the following proposal for a global preconditioner P_K :

$$P_K = \begin{pmatrix} \mathbf{I}_I & -P_{II}\mathbf{K}_{I\Gamma} \\ 0 & \mathbf{I}_\Gamma \end{pmatrix} \begin{pmatrix} P_{II} & 0 \\ 0 & P_\Gamma \end{pmatrix} \begin{pmatrix} \mathbf{I}_I & 0 \\ -\mathbf{K}_{\Gamma I}P_{II} & \mathbf{I}_\Gamma \end{pmatrix}.$$

Consequently, we need to find a good subdomain preconditioner P_{II} and a good skeleton preconditioner P_Γ.

6.6.1 *Subdomain Preconditioner P_{II}*

The subdomain preconditioner P_{II} is an approximation of the inverse of $\mathbf{K}_{II}$. Consequently, it can be interpreted as solving local Dirichlet problems on each subdomain. Thus, our proposal for the subdomain preconditioner P_{II} is to apply the a preconditioned Dirichlet solver with a fixed number l_I of iterations. Analyzing the block structure, we have two different cases when applying the subdomain preconditioner P_{II}:

1. $\mathbf{w}_I = P_{II}\mathbf{v}_I$: The subdomain preconditioner is applied directly to an "inner" vector. Applying the subdomain preconditioner in this case, means that we solve on each subdomain l_I steps of a local Dirichlet problem with a given right hand side for the inner degrees of freedom and homogeneous Dirichlet conditions on the skeleton.

2. $\mathbf{w}_I = P_{II}\mathbf{K}_{I\Gamma}\mathbf{w}_\Gamma$: This case can be interpreted as solving l_I iterations of a local Dirichlet problem on each subdomain with given inhomogeneous Dirichlet data $\mathbf{w}_\Gamma$ on the skeleton.

Algorithm 6.9 Subdomain preconditioner (Case 1)

1: **function** $P_{II}(\mathbf{v}_I)$
2: Restrict: $(\mathbf{v}_i, \mathbf{v}_{\Gamma_i}) \leftarrow (\mathbf{v}_I|_{\Omega_i}, 0)$
3: Solve l_I (preconditioned) steps: $\mathbf{K}_{ii}\mathbf{w}_i = \mathbf{v}_i$
4: Project: $(\mathbf{w}_I)|_{\Omega_i} \leftarrow \mathbf{w}_i$
5: **return** $(\mathbf{w}_I)$
6: **end function**

Algorithm 6.9 describes the first variant of the subdomain preconditioner P_{II}. In step 2 the global inner vector is restricted to the subdomain. In step 3, we apply l_I steps of a preconditioned iterative local Dirichlet solver on each subdomain. In step 4, we project the local inner vector to the global inner vector. Step 2 and 4 are trivial, since we store on each processor the vector with the entries corresponding to the subdomain.

Algorithm 6.10 describes the second variant of the subdomain preconditioner P_{II}. In step 2 we restrict the global Dirichlet data $\mathbf{w}_\Gamma$ to the

Algorithm 6.10 Subdomain preconditioner (Case 2)

1: **function** $P_{II}(\mathbf{w}_\Gamma)$
2: Restrict: $\mathbf{w}_{\Gamma_i} \leftarrow \mathbf{R}_{\Gamma_i}^T \mathbf{w}_\Gamma$
3: Solve l_I (preconditioned) steps: $\mathbf{K}_{ii}\mathbf{w}_i = \mathbf{K}_{i\Gamma}\mathbf{w}_{\Gamma_i}$
4: Project: $(\mathbf{w}_I)|_{\Omega_i} \leftarrow \mathbf{w}_i$
5: **return** $(\mathbf{w}_I)$
6: **end function**

local skeleton. Since the global vector $\mathbf{w}_\Gamma$ is known to all processors, no communication is needed. In step 3, we apply l_I steps of a local preconditioned Dirichlet solver on each processor to a subdomain problem with given Dirichlet data on the skeleton Γ. In step 4, we project the local inner result to the global vector. Analogously to the steps 2 and 4 of Algorithm 6.9, this step is trivial.

6.6.2 *Skeleton Preconditioner P_Γ*

The skeleton preconditioner P_Γ approximates the inverse of the Schur-complement operator $\mathbf{S}$. Analogously to the subdomain preconditioner, we propose to apply a preconditioned Schur-complement solver with a fixed number of iteration l_Γ, see Algorithm 6.11. Our proposal is to use the inexact variant of the skeleton solver as described in Algorithm 6.7. As a preconditioner for the Schur-complement solver, we can either apply the inexact variant of the Neumann-Neumann preconditioner or the inexact variant of the balancing Neumann-Neumann preconditioner. As described in Algorithm 6.2 and Algorithm 6.4, only a fixed number of iterations l_Γ is computed when applying the local operator P_i^{NN} and the coarse operator P_0^{bNN}. Applying the Schur-complement operator implies solving local Dirichlet problems on each subdomain. We propose to apply the inexact Schur-complement operator because it is only used as a preconditioner and thus it is not necessary to solve the Schur-complement equation exactly, which would also be too expensive. The inexact Schur-complement operator is defined as follows

$$\widetilde{\mathbf{S}} = \mathbf{K}_{\Gamma\Gamma} - \mathbf{K}_{\Gamma I} P_{II} \mathbf{K}_{I\Gamma}.$$

Instead of solving the local Dirichlet problems accurately, we apply the local subdomain preconditioner and only compute a fixed number of iterations as described in the inexact variant of Algorithm 6.1.

Algorithm 6.11 Skeleton preconditioner

1: **function** $P_\Gamma(\mathbf{w}_\Gamma)$
2: Apply inexact SolveSkeleton($\mathbf{v}_\Gamma$) **return** $(\mathbf{w}_I)$
3: **end function**

6.7 ALGORITHM FOR THE GLOBAL SCHUR-COMPLEMENT PRECONDITIONER

In this subsection, we give an algorithmic description, how to apply the global Schur-complement preconditioner P_K, see Algorithm 6.12.

Algorithm 6.12 Global Schur-complement preconditioner

1: **function** $P_K(\mathbf{v}_I, \mathbf{v}_\Gamma)$
2: Apply P_{II}: $\mathbf{w}_I^1 \leftarrow P_{II}\mathbf{v}_I$ ▷ Alg. 6.9
3: Compute: $\mathbf{v}_\Gamma^1 \leftarrow \mathbf{K}_{\Gamma I}\mathbf{w}_I^1$
4: Compute: $\mathbf{v}_\Gamma^2 \leftarrow \mathbf{v}_\Gamma - \mathbf{v}_\Gamma^1$
5: Apply P_Γ: $\mathbf{w}_\Gamma \leftarrow P_\Gamma \mathbf{v}_\Gamma^2$ ▷ Alg. 6.11
6: Apply P_{II} : $\mathbf{w}_I^2 \leftarrow P_{II}\mathbf{K}_{I\Gamma}\mathbf{w}_\Gamma$ ▷ Alg. 6.10
7: Compute: $\mathbf{w}_I \leftarrow \mathbf{w}_I^1 - \mathbf{w}_I^2$
8: **return** $(\mathbf{w}_I, \mathbf{w}_\Gamma)$
9: **end function**

For a better understanding, how to apply the preconditioner, we expand the block structure of the preconditioner applied to a residual vector $P_K(\mathbf{v}_I, \mathbf{v}_\Gamma)^T$:

$$\begin{pmatrix} \mathbf{w}_I \\ \mathbf{w}_\Gamma \end{pmatrix} = \begin{pmatrix} P_{II}\mathbf{v}_I - P_{II}\mathbf{K}_{I\Gamma}P_\Gamma(-\mathbf{K}_{\Gamma I}P_{II}\mathbf{v}_I + \mathbf{v}_\Gamma) \\ P_\Gamma(-\mathbf{K}_{\Gamma I}P_{II}\mathbf{v}_I + \mathbf{v}_\Gamma) \end{pmatrix}.$$

In step 2, we apply the first variant of the subdomain preconditioner to the given residual vector $\mathbf{v}_I$. It is a local operation. The result is stored in $\mathbf{w}_I^1$ and reused in step 7. Step 2 - 4 prepare the residual vector $\mathbf{v}_\Gamma$ to which the skeleton preconditioner P_Γ is applied in step 5. The matrix-vector operation in step 3 requires global communication. It projects Dirichlet data to natural boundary data. The application of the skeleton preconditioner in step 5 also requests global communication. This step provides the final result on the global skeleton $\mathbf{w}_\Gamma$. Step 6 is again a local step, where the second variant of the subdomain preconditioner is applied. The vector-vector operation described in step 7 computes the final result for the inner degrees of freedom in form of the vector $\mathbf{w}_I$.

Analogously to the case of the interface preconditioners the global Schur-complement preconditioner is not a linear operation. Furthermore the problem is not symmetric. Consequently, an appropriate choice for an iterative solver for the global linear system is again a FGMRES-method [47, 48, Chapter 9.4].

SUMMARY

In this chapter, we derived a solution algorithm for the Schur-complement equation. Furthermore, we developed and compared a Neumann-

Neumann and a balancing Neumann-Neumann preconditioner for the interface equation. Based on the ideas of the solution algorithm for the Schur-complement equation and of these Neumann-Neumann type preconditioners, we derived a global Schur-complement preconditioner for the global linear system. In the next chapter, we study these solvers and preconditioners based on numerical experiments in the framework of high performance computing for the different model problems considered in Chapter 2-5.

7

NUMERICAL EXPERIMENTS

This chapter is dedicated to the numerical experiments. After introducing the used software and hardware, we give an overview about the set-up of the different numerical experiments. They mostly cover the model problems introduced in Chapter 2 - 5. We apply both the Schur-complement and the global approach, which we presented in the previous Chapter 6 to solve the different model problems. After verifying the correctness of the methods and their implementation by means of solving a problem for which the analytical solution is known, we analyze both methods in the context of high performance computing. Based on the number of iterations of the solvers and the runtime, we analyze effectiveness and the differences between the preconditioners and evaluate the speed up and efficiency for both methods.

7.1 HIFLOW^3-SOFTWARE

The domain decomposition methods are implemented with the parallel finite element software package HiFlow^3[3, 4]. We adapted various parts of the software to make it work in the context of non-overlapping domain decomposition methods: We developed a partitioner for the degrees of freedom, which distinguishes between the inner degrees of freedom and the degrees of freedom on the local and global skeleton. Based on this partitioning, we implemented a new linear algebra structure for the Schur-complement operator and a suitable global skeleton vector structure. Exploiting this domain decomposition structure especially with respect to the partitioning of the degrees of freedoms, we implemented a structure for the linear algebra suitable for the global approach. Furthermore, we implemented the Neumann-Neumann preconditioner, the balancing Neumann-Neumann preconditioner and the global Schur-complement preconditioner, based on the implementation for the Neumann-Neumann type preconditioners.

For the mesh partitioning we used the software tool Metis [35] to which an interface is implemented in HiFlow^3. Metis partitions the global mesh into non-overlapping subdomain meshes which coincide with our non-overlapping subdomains. The finite element triangulation is done directly on those subdomains. We assume that

Parameter	Value
Preconditioner Number	11
Preprocessing Type	0
Maximum of Multi Levels	20
Memory Factor	0.8
Pivot Threshold	2.5
Minimal Pivot	0.01

Table 1: ILU++ parameters.

each subdomain created by Metis is connected. The parallelization is achieved by mapping each subdomain to one process.

Moreover, we used the HiFlow^3-interface to the ILU++ library for the preconditioners of the local solvers. While this library provides various implementations of the incomplete LU-factorization, we use it more or less as a black box and set the parameters, as listed in Table 1. The parameters were chosen based on empirical testing. We refer to the literature for details about the meaning of the parameters [41, 42].

All numerical experiments were conducted on a cluster consisting of ten two socket nodes with a total of 120 cores. The nodes are connected via infiniband and each of the nodes provides between 48 and 192 GB of memory.

7.2 OVERVIEW AND STRUCTURE OF THE NUMERICAL EXPERIMENTS

In this section, we provide an overview about the different numerical experiments. First, we describe the two example problems which we used, then, we review shortly the two solution approaches introduced in Chapter 6 and state the parameters for the solvers.

For the numerical experiments, we defined two examples, which cover the model problems introduced in Chapter 2 - 5 equipped with mixed outflow and Dirichlet boundary conditions. In our examples, we assume distributed control. The first one is based on the Oseen equations and the second on the Navier-Stokes equations. Both examples are handled in the same way as outlined in Fig. 21. First, we solve a Stokes problem, whose solution is used as the desired state $\mathbf{u}_0$ for the optimization problem and in case of the Oseen example also for the advection $\mathbf{a}$, see Fig. 25a and Fig. 26a. In a second step, we solve either the linear Oseen or the non-linear Navier-Stokes equations without control. By setting the start solution to the solution of the Stokes problem, we begin directly with a divergence free solution. Next, we solve several optimal flow control problems assuming distributed control constrained either by the Oseen or by the Navier-Stokes equations. In each experiment, we successively reduce the regularization parameter $\alpha \in \{1.0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}\}$. For $\alpha = 1.0$, we use the solution of

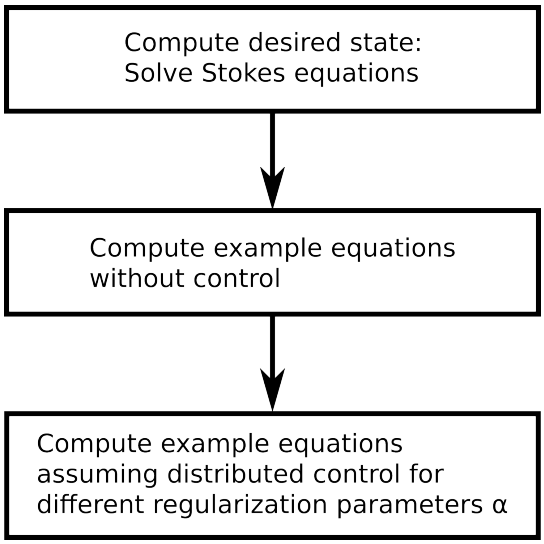
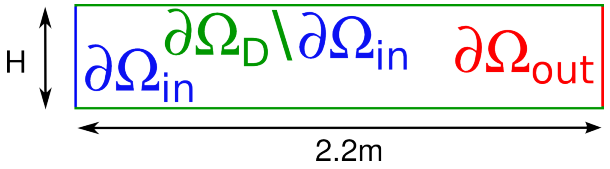
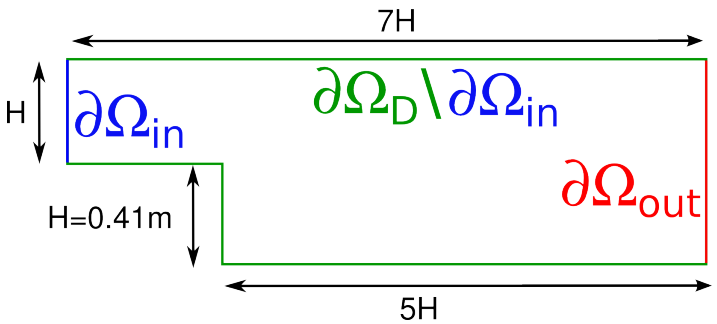


Figure 21: Overview about the program execution for both examples.

the Oseen or Navier-Stokes equation as a start solution, respectively. In this case the adjoint velocity and adjoint pressure are set to 0. In the other cases, we use the solution of the last optimal control problem as a start solution. In all cases, we start with a divergence free solution. In Chapter 2, we assume $\mathbf{a} \cdot \mathbf{n} = 0$ for the Oseen equations on the outflow boundary. By choosing the Stokes solution as given advection, this assumption does not hold. Therefore, we have to adapt the formulation of the bilinear form $a(\cdot,\cdot)$ such that it accounts for the outflow conditions analogously as we defined the trilinear form $n(\cdot,\cdot,\cdot)$ on the outflow boundary in Chapter 4.



(a) Channel geometry.



(b) Backward facing step geometry.

Figure 22: Geometries.

For both examples, we have

- $\Omega_0 = \Omega_C = \Omega$,
- and the right hand sides are set to $f = 0$ and $h = 0$.

By setting $h = 0$, we assume natural boundary conditions on the boundary $\partial\Omega_{out}$. The Dirichlet boundary Ω_D is split into two parts, see also Fig. 22: On $\partial\Omega_D \setminus \partial\Omega_{in}$, we assume $\mathbf{d} = 0$ and on $\partial\Omega_{in}$, we assume the following inflow profile:

$$\mathbf{d}(x_1, x_2) = \left(4\mathbf{u}_m x_2(1 - x_2)/H^2, \quad 0\right)^T,$$

where $\mathbf{u}_m$ is the maximum inflow speed and H is the height of the geometry at the inflow boundary. We execute the example using two different geometries: a channel and a backward facing step geometry, see Fig. 22. The channel geometry is chosen for the verification of the method and its implementation, since an analytical solution is known. For the other geometry, no analytical solution is known. We use this backward facing step geometry to study both solution methods in the context of high performance computing. Therefore, we run numerical experiments with different numbers of subdomains, which are listed in Table 2. Table 3 shows the number of degrees of

	Schur-complement Method							Global Method					
	Oseen												
NN	4	8	16	32	-	-	-	4	8	16	32	-	-
bNN	4	8	16	32	48	64	80	4	8	16	32	48	64
	Navier-Stokes												
NN	4	8	16	-	-	-	-	4	8	16	-	-	-
bNN	4	8	16	32	48	64	80	4	8	16	32	48	64

Table 2: Number of subdomains used in the numerical experiments for the backward facing step geometry.

freedom for the chosen triangulation on the global domain for each variable and the resulting total when partitioned into four subdomains. Since the pressure discretization changes depending on the number of subdomains, the total number differs. In our implementation, we do not discretize the control variable. We use the relationship defined by the optimality condition $\mathbf{c} = \frac{1}{\alpha}\mathbf{z}$ and reduce the optimality system to a system of four partial differential equations. The control $\mathbf{c}$ can be computed in a post processing step based on the solution of $\mathbf{z}$, see also [23, Chapter 6]. For both examples, we analyze the Reynolds number [38, Chapter 5.4]

$$Re := \frac{\rho V L}{\mu}.$$

	Channel		Backward facing step	
	No Control	Control	No Control	Control
Total #DoFs	187237	374474	445688	891376
Total #DoFs for $\mathbf{u}$	166172	166172	395534	395534
Total #DoFs for p	21065	21065	50154	50154
Total #DoFs for $\mathbf{z}$	-	166172	-	395534
Total #DoFs for r	-	21065	-	50154

Table 3: Number of degrees of freedom (DoFs) for the two different geometries for an example with four subdomains.

Therefore, we define the reference length $L := H = 0.41\,m$ and the reference speed $V := \frac{2}{3}\mathbf{u}_m$. In both examples, we assume a normalized density of $\rho = 1.0\,kg\,m^{-3}$. For the Oseen example, we set the maximum inflow speed to $\mathbf{u}_m = 0.2\,m/s$ and the dynamic viscosity to $\mu = 10\,kg\,m^{-1}s^{-1}$, which leads to a Reynolds number $Re = 5.5e-3$. This is reasonable, since the Oseen equations model creeping flow which is characterized amongst others by a Reynolds number tending to 0. In case of the Navier-Stokes example, we set $\mathbf{u}_m = 1.0\,m/s$ and $\mu = 0.1\,kg\,m^{-1}s^{-1}$, thus $Re = 2.73$.

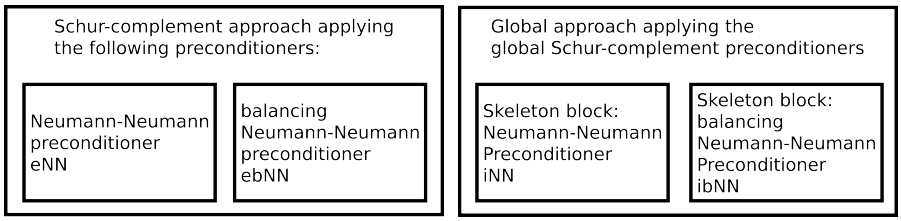


Figure 23: Method Overview.

We solve both examples applying the Schur-complement and the global approach. For the Navier-Stokes example, we apply a Newton-method as an outer loop to resolve the non-linearity, see also Alg. 4.2. In case of the Schur-complement method, we either apply the exact Neumann-Neumann preconditioner (eNN) or the exact balancing Neumann-Neumann (ebNN) preconditioner. For the global approach, we apply the global Schur-complement preconditioner using either the inexact Neumann-Neumann (iNN) or the inexact balancing Neumann-Neumann (ibNN) preconditioner for the skeleton block. An overview about the methods is given in Fig. 23. The stopping tolerances for the Schur-complement and global approach are given in Table 4. The relative and absolute tolerances are defined as follows:

$$\text{relative tolerance: } \left\|r^k\right\|_2 / \left\|r^0\right\|_2,$$
$$\text{absolute tolerance: } \left\|r^k\right\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm, k the iteration of the solver, such that r^0 denotes the start residual and r^k the residual of the kth iteration step. The stopping tolerance is given either by the relative or absolute tolerance or by the maximum number of iterations. For the Schur-complement method, we need to define stopping tolerances for the Schur-complement solver, solving the Schur-complement equation. Furthermore, we need to define stopping tolerances for the Dirichlet solver used for the local Dirichlet problems needed when applying the Schur-complement operator. The solutions of these problems form the basis of the Krylov-subspace for the skeleton equation and must therefore be solved very accurately, see also Remark 41. This Dirichlet solver is also used for the ebNN preconditioner to compute the coarse operator $\mathbf{S}_0$ and to apply the Schur-complement operator inside the ebNN preconditioner. Moreover, we define stopping tolerances for the Neumann solver used to solve the local problems with natural boundary conditions inside the eNN and ebNN preconditioners, when applying the operator P^{NN} and also for the coarse solver needed when applying the coarse operator P_0^{bNN} of the ebNN preconditioner. Since the Neumann and coarse solvers are used inside the preconditioner, the stopping tolerances are lower at least for the relative tolerance.

For the global method, the stopping tolerance for the global solver is defined analogously to the stopping tolerances of the Schur-complement solver of the Schur-complement approach. The Dirichlet solver 1 is used to compute the coarse operator $\mathbf{S}_0$ for the ibNN preconditioner applied for the skeleton block P_Γ. For the other linear solvers, we use the inexact variants and therefore only define the maximum number of iterations corresponding to l_I and l_Γ. We only solve one iteration of the inexact Schur-complement solver, but for the Dirichlet solver 2, Neumann solver and coarse solver used for the preconditioner of the Schur-complement solver, we compute each four iterations. For the inner block of the preconditioner P_I, we solve again only one iteration. This is an appropriate choice having in mind the trade-off between the cost to apply the preconditioner in each iteration and the efficiency of the preconditioner. When analyzing the results for the global approach, we will observe later that the parameters may also depend on the problem type. For the non-linear solver needed for the Navier-Stokes example, we chose the same stopping tolerances for the Schur-complement and the global method.

For all solvers, except the ones used inside the global Schur-complement preconditioner, we used a restart variant [48] for which the restart parameter is also given in the Table 4.

		rel. tol.	abs. tol.	max. #iter.	restart
		Schur-complement approach			
Schur-complement solver		1e-8	1e-12	1000	500
Dirichlet solver		1e-14	1e-16	1000	100
Neumann solver		1e-8	1e-16	1000	100
Coarse solver		1e-8	1e-10	1000	10
Non-linear solver		1e-6	1e-10	10	
		Global approach			
Global solver		1e-8	1e-12	10000	500
P_Γ	Schur-complement solver	-	-	1	
	Dirichlet solver 1	1e-14	1e-16	1000	100
	Dirichlet solver 2	-	-	4	
	Neumann solver	-	-	4	
	Coarse solver	-	-	4	
P_{II}	Dirichlet solver	-	-	1	
Non-linear solver		1e-6	1e-10	10	

Table 4: The table shows the stopping criteria and restart parameters for the different solvers. rel. tol. abbreviates relative tolerance, abs. tol. absolute tolerance and max. #iter. the maximum number of iterations.

7.3 VERIFICATION OF BOTH METHODS AND THEIR IMPLEMENTATION

To verify the both methods and their implementation in HiFlow[3], we solve the Oseen and Navier-Stokes example in a channel domain, for which the exact analytical solution is given by:

$$
\begin{aligned}
u_1(x_1, x_2) &= \frac{4\mathbf{u}_m}{H^2} x_2(H - x_2), \\
u_2(x_1, x_2) &= 0, \\
p(x_1, x_2) &= \mu \left(\frac{-8\mathbf{u}_m}{H^2} x_1 + \frac{17.6\mathbf{u}_m}{H^2} \right), \\
z_1(x_1, x_2) &= 0, \\
z_2(x_1, x_2) &= 0, \\
r(x_1, x_2) &= 0, \\
c_1(x_1, x_2) &= 0, \\
c_2(x_1, x_2) &= 0.
\end{aligned}
$$

The solution of the velocity u_1 and u_2 for the Oseen and for the Navier-Stokes equations matches the desired state, which arises as

			No Control		Control $\alpha = 1.0$			
			H^1-error of $\mathbf{u}$	L^2-error of p	H^1-error of $\mathbf{u}$	L^2-error of p	H^1-error of $\mathbf{z}$	L^2-error of r
Oseen	Sc.	eNN	1.2e-09	1.1e-10	1.2e-09	1.9e-10	1.4e-12	2.9e-11
		ebNN	8.8e-10	3.1e-11	1.1e-09	6.0e-11	6.6e-13	1.9e-11
	Gl.	iNN	1.2e-09	8.2e-12	1.2e-09	8.2e-12	9.1e-15	2.6e-14
		ibNN	1.3e-09	8.3e-12	1.3e-09	6.9e-12	2.5e-15	5.6e-14
Navier-St.	Sc.	eNN	4.9e-09	7.7e-11	4.9e-09	7.7e-11	-	-
		ebNN	4.2e-09	1.2e-11	4.2e-09	1.2e-11	-	-
	Gl.	iNN	3.8e-09	1.1e-09	3.8e-09	1.1e-09	-	-
		ibNN	4.2e-09	2.8e-11	4.2e-09	2.8e-11	-	-

Table 5: The table shows the global errors between finite element approximation of the analytical solution and computed finite element solution in the channel domain for both examples solved with Schur-complement (Sc.) and the global (Gl.) approach.

the solution of the Stokes equations. Thus, our objective is directly fulfilled and we do not need to control the systems. Hence, the adjoint variables and the control are 0. Due to this fact, we only run the experiment for the regularization parameter $\alpha = 1.0$.

For both methods, the experiments were conducted using the corresponding preconditioners, see Fig 23. Since the exact solution is a Poiseuille profile, which can be exactly represented by the used Taylor-Hood finite elements, we expect a very good approximation. Due to this fact, we also used stricter stopping tolerances than for the numerical experiments in the backward facing step geometry. We reduced the relative tolerance for the global and Schur-complement solver to $1.0e-12$ and the absolute tolerance to $1.0e-14$. The H^1-errors for the velocity and adjoint velocity and the L^2-errors for the pressure and adjoint pressure are shown in Table 5. The left part of the table shows the result for the case without control, and the right part the case with distributed control. In the upper part, we listed the errors for the Oseen example and in the lower part for the Navier-Stokes example. Since the absolute error of the start residual for the Newton method already fulfilled the stopping criterion, we did not explicitly compute the error for the adjoint variable for the Navier-Stokes equations. For the Oseen example the stopping criterion was not fulfilled for the start residual, such that we could compute all errors. The numerical experiments for the NN-type preconditioners were run with four subdomains and for the bNN-type preconditioner with 16 subdomains. Based on the errors, we can evaluate the suitability of the proposed methods.

7.4 SIMULATION RESULTS IN BACKWARD FACING STEP GEOMETRY

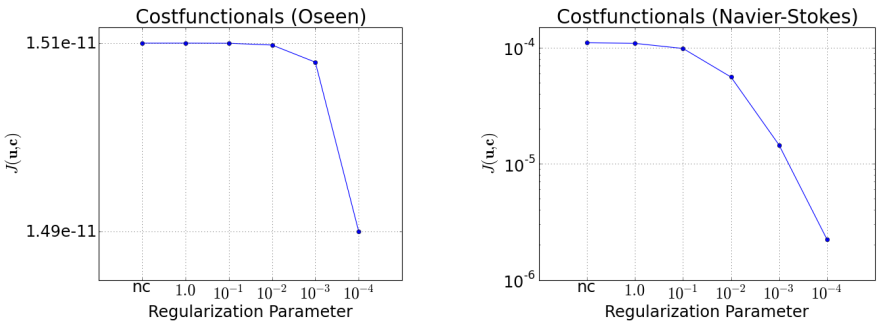


Figure 24: Values of costfunctionals for Oseen and Navier-Stokes examples for different regularization parameters. nc abbreviates the case without control.

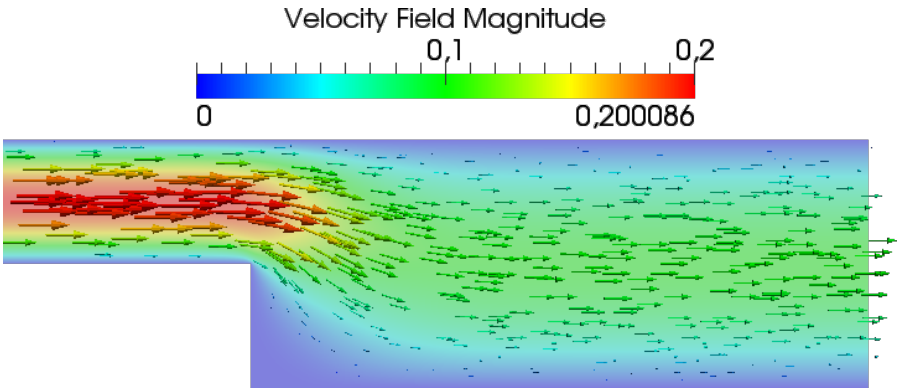
Before we study both methods in the context of high performance computing, we look at some simulation results in the backward facing step geometry. Fig. 24 shows the values of the costfunctional for the Oseen and the Navier-Stokes example:

$$J(\mathbf{u}, \mathbf{c}) = \frac{1}{2} \|\mathbf{u} + \mathbf{u}_D - \mathbf{u}_0\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|\mathbf{c}\|^2_{L^2(\Omega)} .$$

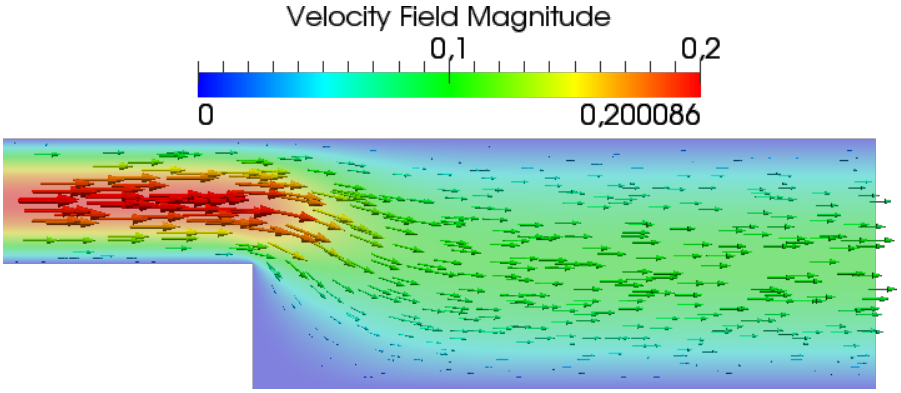
For the Oseen example, the values of the costfunctional have an order of magnitude of $1.0e-11$ and thus are very small even without applying a distributed control. Looking at the simulation results shown in Fig. 25, there is no difference visible to the eye between the desired state, the solution of the Oseen equation without control and the solution of the Oseen equation. Nevertheless, we observe that the control field shown in Fig. 25d develops three eddies. Since the order of magnitude of the control filed is three times lower than of the velocity field the influence is relatively small. The situation changes for the Navier-Stokes example. The values of the costfunctional are still very small but between five and seven magnitudes larger than for the Oseen example, see Fig. 24. Comparing the velocity field of the desired state, the solution without control and with control in Fig. 26, almost no difference can be seen, analogously to the Oseen example. But in this case, the control field develops two eddies of the same magnitude as the velocity field. A smaller one with less influence before the step and a bigger one directly behind the step.

7.5 ANALYSIS OF THE PRECONDITIONERS

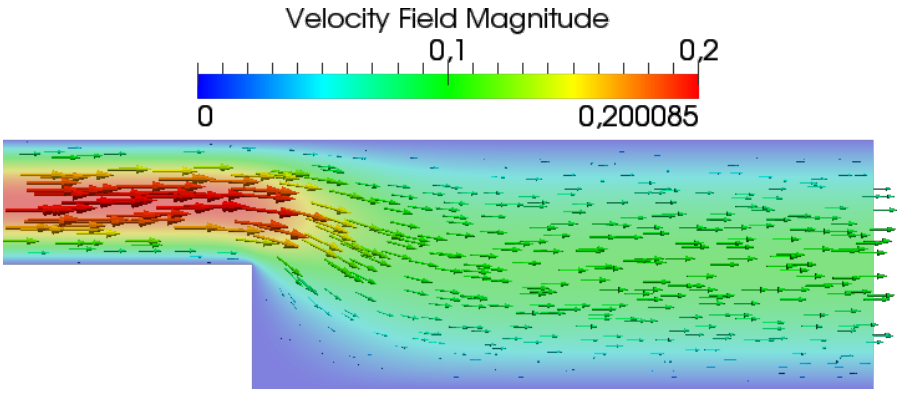
In this section, we analyze the exact and inexact Neumann-Neumann type preconditioners in the framework of high performance computing. Therefore, we look at the numbers of iterations for the Schur-



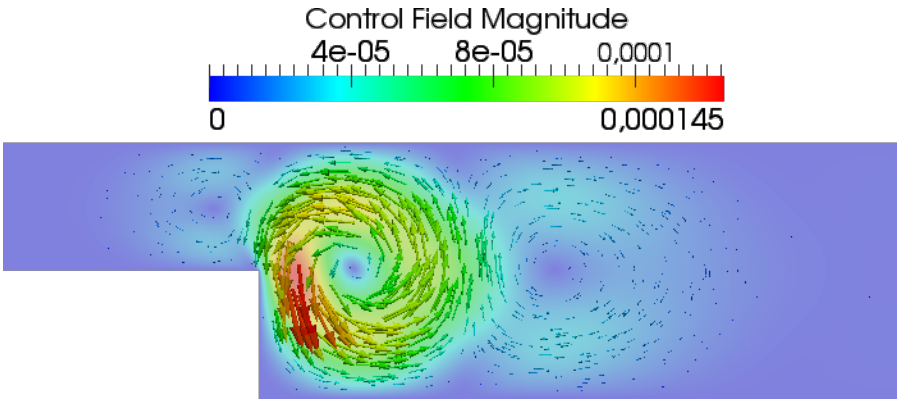
(a) Desired State (Computed solution of the Stokes equations).



(b) Velocity Field of the Oseen example without control.

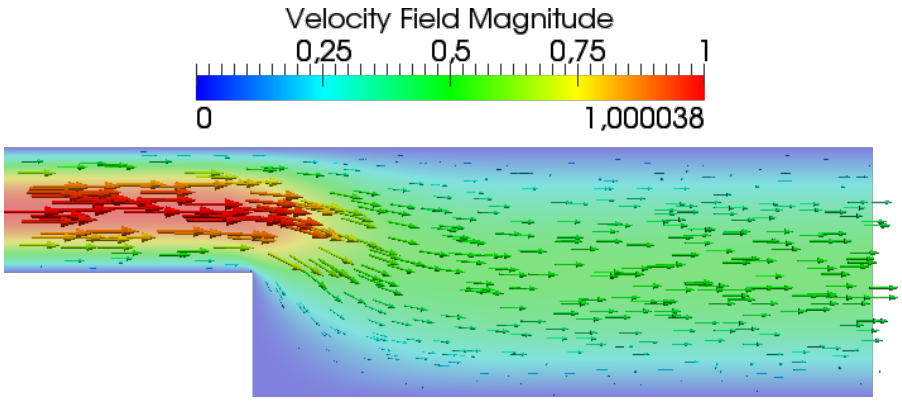


(c) Velocity Field of the Oseen example with control at $\alpha = 1.0e-4$.

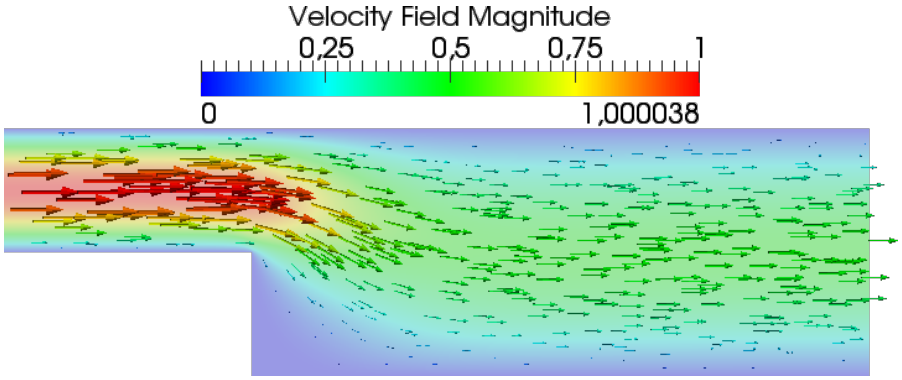


(d) Control Field of the Oseen example at $\alpha = 1.0e-4$.

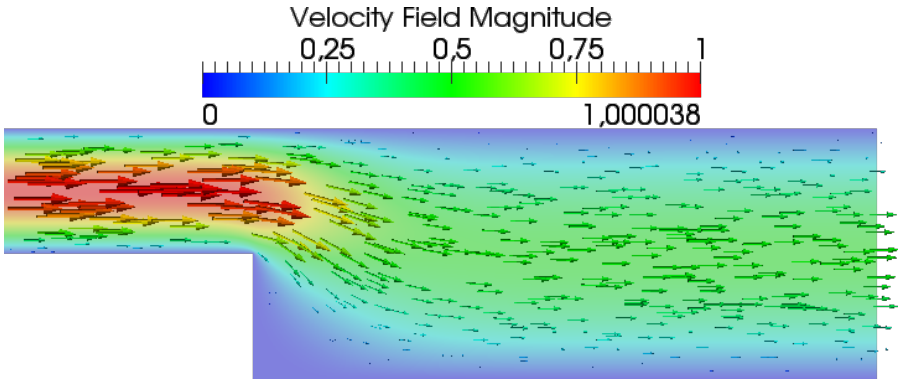
Figure 25: Results of the flow simulations for the Oseen example.



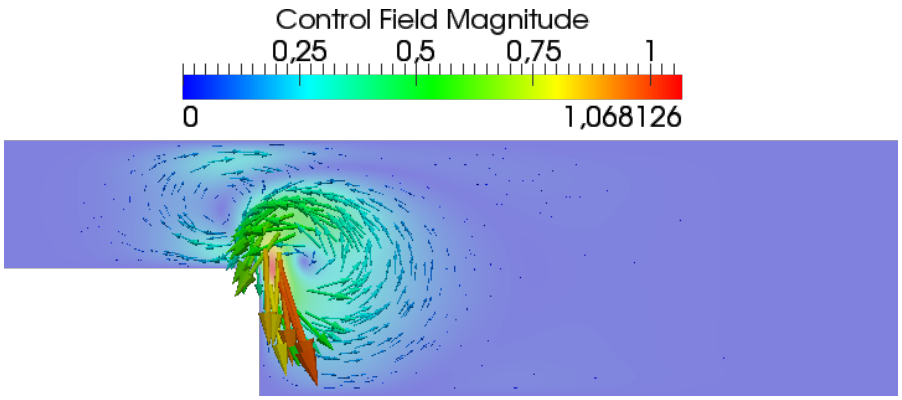
(a) Desired State (Computed solution of the Stokes equations).



(b) Velocity Field of Navier-Stokes solution without control.



(c) Velocity Field of Navier-Stokes example with control at $\alpha = 1.0e-4$.



(d) Control Field of the Navier-Stokes example at $\alpha = 1.0e-4$.

Figure 26: Results of the flow simulations for Navier-Stokes example.

Number of Newton iterations applying Schur-complement solver (Navier-Stokes)						
α	no control	1.0	$1.0e-1$	$1.0e-2$	$1.0e-3$	$1.0e-4$
s	with preconditioner P^{NN}					
4,8	2	7	5	5	4	3
16	2	6	4	4	3	2
s	with preconditioner P^{bNN}					
4, 48, 64, 80	2	6	4	4	3	2
8, 16, 32	2	5	4	4	3	2

Table 6: The table shows the mean values of the number of FGMRES iterations of the skeleton solver per Newton step applying the two Neumann-Neumann-type preconditioners to solve the Navier-Stokes example.

Number of Newton iterations applying global solver (Navier-Stokes)						
α	no control	1.0	$1.0e-1$	$1.0e-2$	$1.0e-3$	$1.0e-4$
#iter	2	6	5	5	4	3*

Table 7: The table shows the number of Newton iterations when applying the global solver. (*) The method did not terminate when applying the inexact balancing Neumann-Neumann preconditioner for the skeleton block for 32, 48 and 64 subdomains for the control case with $\alpha = 1.0e-4$.

complement and global solver and for the non-linear Navier-Stokes example also at the number of Newton steps. Furthermore, we analyze the time per iteration and compare the set-up time of the preconditioners with the solving time.

For the different numerical experiments, Tables 8 and 10 list the number of iterations of the Schur-complement solver for the Oseen and Navier-Stokes example, while Tables 9 and 11 specify the number of iterations for the global solver. For the non-linear Navier-Stokes example, the tables list the mean values with respect to the Newton steps, which we record in Tables 6 and 7 for the Schur-complement and global approach, respectively. Analyzing the number of Newton iterations, we note that for the global solver the number of Newton steps only depends on whether we solve a problem without control or in case of applying distributed control on the regularization parameter α. We expected this result due to the fact that the Newton method is applied as an outer loop. In all cases the Newton method is stopped with an absolute residual in the order of magnitude of 10^{-10}. The relative residual was reduced at least by five orders of magnitude. Applying the Schur-complement solver as an inner loop to compute

the correction, we observe a slightly different situation. Depending on the preconditioner and the number of subdomains, the number of Newton steps differs for some cases by one iteration taking the number of Newton-iterations of the global approach as a reference. The absolute residual also does not reach the same order of magnitude as in the case when applying the global solver. But in all cases the residual is reduced by an order of magnitude between three and four which indicates that the method converged. To assure that we reached the minimal residual, we applied an inexact variant of the Newton method [13], in case that the residual was not reduced anymore at a step size of 1.0, we reduce the step size successively. We stopped the Newton method, in case that even at a step size of 0.1 the residual could not be reduced anymore. Even thought, we did not reach the same order of magnitude for the absolute residual, the values for the costfunctionals are the same. Therefore, we conclude that we resolved the problem sufficiently.

	#Iter of Schur-complement solver for Oseen example					
α	no control	1.0	$1.0e-1$	$1.0e-2$	$1.0e-3$	$1.0e-4$
s	eNN					
4	41	65	42	48	60	63
8	78	141	107	114	130	141
16	218	284	204	214	234	290
32	999	478	491	396	414	1000(*)
s	ebNN					
4	20	27	16	18	21	25
8	32	49	29	35	43	50
16	51	73	44	52	69	88
32	141	124	120	119	121	138
48	435	187	184	175	180	210
64	486	213	207	205	211	253
80	1000(*)	265	258	257	264	302

Table 8: The table shows the number of FGMRES iterations of the Schur-complement solver applying the two Neumann-Neumann-type preconditioners to solve the Oseen example. (*)The maximum of iterations was set to 1000.

Next, we analyze the number of iterations of the Schur-complement and global solver for both examples see Tables 8 - 11. As expected, with an increasing number of subdomains, the number of iterations for the eNN and iNN preconditioners increases much stronger compared to the increase of iterations for the ebNN and ibNN preconditioners. Only based on this observation, we can conclude that both variants of the NN-type preconditioners are not suitable for high performance

	#Iter of global solver for Oseen example					
α	no control	1.0	$1.0e-1$	$1.0e-2$	$1.0e-3$	$1.0e-4$
s	iNN					
4	399	239	196	216	251	384
8	1321	353	276	283	359	10000(*)
16	8501	1001	296	451	710	10000(*)
32	10000(*)	1962	803	1501	10000(*)	10000(*)
s	ibNN					
4	274	132	102	100	135	249
8	361	151	124	139	142	360
16	419	133	110	130	168	380
32	758	133	109	133	303	1501
48	933	208	131	250	501	10000(*)
64	1340	220	158	423	2944	10000(*)

Table 9: The table shows the number of FGMRES iterations of the global solver applying the two variants of the global Schur-complement preconditioner to solve the Oseen example.(*) The maximum of iterations is set to 10000.

computing. When looking at the total runtime in Tab. 12, we also see that starting from 16 subdomains the bNN-type preconditioners are quicker than the NN-type preconditioners. This is another record that the NN-type preconditioners are not suitable for high performance computing. Graphically this is also depicted for the Navier-Stokes example without control in the upper part of Fig 33. Analyzing the number of iterations for the bNN-type preconditioners, we note that the number of iterations increases for an increasing number of subdomains but much slower as in the case when applying the variant of the NN-type preconditioners. This also coincides with our theoretical considerations. Only for inf-sup stable coarse finite elements, we can expect that the number of iterations is independent of the number of subdomains at least for the Schur-complement approach. Furthermore, we observe that looking at the ebNN variant the number of iterations is more or less stable for a specific number of subdomains while the number of iterations for the ibNN preconditioner increases along with decreasing the regularization parameter. From that point of view the ebNN preconditioner seems to be more robust.

In the next step, we look at the time per iteration for both approaches applied to the Oseen and the Navier-Stokes example, as it is shown in Fig. 27 and 28. Sub-figures on the left side correspond to the Schur-complement approach and on the right side to the global approach. Exemplary, we show the case without control for each method in the upper sub-figures while the lower sub-figures display the case apply-

Mean value of #iter of Schur-complement solver per Newton step (Navier-Stokes)						
α	no control	1.0	$1.0e-1$	$1.0e-2$	$1.0e-3$	$1.0e-4$
s	eNN					
4	29	38	43	50	57	63
8	55	89	94	102	109	118
16	107	193	201	217	236	255
s	ebNN					
4	18	33	34	38	35	32
8	33	61	61	66	58	56
16	50	91	95	111	103	87
32	73	147	149	167	177	151
48	97	161	165	180	216	190
64	121	225	218	238	287	246
80	143	246	252	262	334	275

Table 10: The table shows the mean values of the number of FGMRES iterations of the skeleton solver per Newton step applying the two Neumann-Neumann-type preconditioners to solve the Navier-Stokes example.

ing distributed control for the regularization parameter $\alpha = 1.0e-3$. In each sub-figure, we compare the time per iteration of the NN and bNN-type preconditioners. The time per iteration was computed by dividing the solving time of the Schur-complement or global solver by the total number of iterations. In case of the non-linear Navier-Stokes example, we used the mean values with respect to the Newton-steps for the solving time and the number of iterations. Therefore, the time includes all steps executed in each iteration of the solver, especially the costly steps of applying the preconditioner and the Schur-complement or global operator depending on the approach, (see also steps 6, 8 and 10 of Alg. 6.7). As expected, we observe that both variants of the bNN preconditioner consume more time per iteration than the variants of the NN preconditioner. This results from the fact that for the bNN type we additionally need to apply the Schur-complement and coarse operator, see also Fig. 20. Moreover, we see that for an increasing number of subdomains, the time per iterations decreases proportional to the subdomain size which is shown in Fig. 31. It indicates that the smaller the problem the faster we can solve it. Furthermore, we observe that one iteration of the Schur-complement solver is between six and ten times slower than an iteration of the global solver. To be able to conclude, which approach is more reasonable in the context of high performance computing, we compare the total time for the Schur-complement approach with the total time for the global approach in

Mean value of #iter of global solver per Newton step (Navier-Stokes)						
α	no control	1.0	$1.0e-1$	$1.0e-2$	$1.0e-3$	$1.0e-4$
s	iNN					
4	151	112	122	211	162	137
8	218	174	187	311	261	194
16	345	325	369	9629	1398	983
s	ibNN					
4	120	55	68	150	190	272
8	138	69	76	155	439	361
16	132	85	119	243	3493	600
32	143	122	176	359	1422	-
48	157	174	368	733	920	-
64	181	204	493	4021	1256	-

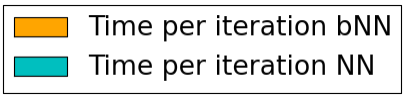
Table 11: The table shows the mean values of the number of FGMRES iterations of the global solver per Newton step applying the two Neumann-Neumann-type preconditioners to solve the Navier-Stokes example.

Total solving time for Navier-Stokes example without control				
s		4	8	16
Sc.	eNN	1631.5	1022.8	761.1
	ebNN	2094.1	1231.1	613.1
Gl.	iNN	1422.0	829.1	663.0
	ibNN	1616.7	934.0	481.3

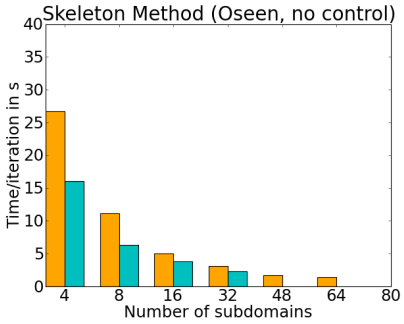
Table 12: Comparing the total time depending on preconditioner type and method to solve Navier-Stokes example without control.

Fig. 29. For values larger than one the global approach is faster. For most cases, the global approach is faster even though, more iterations are needed. From that point of view, the global approach features a greater potential in high performance computing.

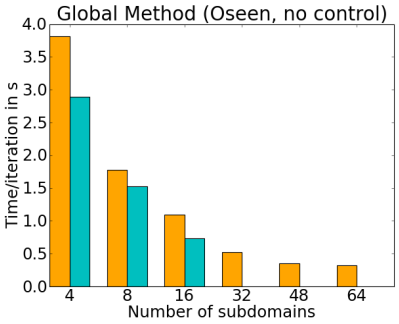
Another important aspect for the analysis of the preconditioners in the framework of high performance computing is their set-up time. Therefore, in Fig. 32 and 33, we compare the set-up time for the bNN-type preconditioners with the solving time for both examples solved with the two approaches applying the preconditioners. The figures and sub-figures are arranged analogously to Fig. 27 and 28. In the upper part of Fig 28, we additionally compare the NN-type preconditioner and the bNN-type preconditioners. As expected, we observe that the set-up time for the NN preconditioners is smaller than for the bNN



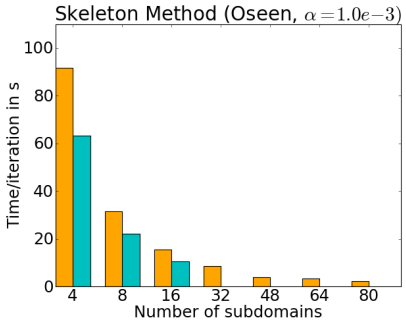
(a) Legend for Fig. 27 and Fig. 28.



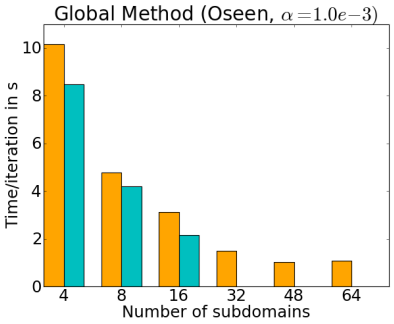
(b) Oseen example without control.



(c) Oseen example with distributed control for $\alpha = 1.0e-3$.



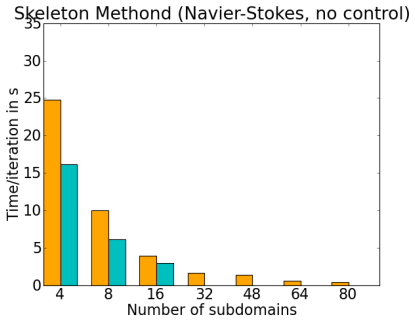
(d) Oseen example without control.



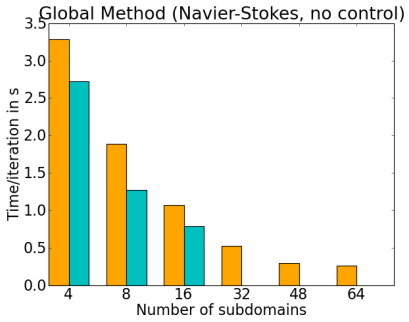
(e) Oseen example with distributed control for $\alpha = 1.0e-3$.

Figure 27: The figure shows the time per iteration for the two solving approaches applied to the Oseen example exemplarily for the non-control and the control case for $\alpha = 1.0e-3$.

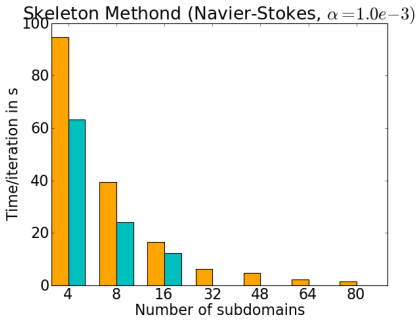
preconditioners. Furthermore, the set-up times are independent of the approach, since the same steps need to be realized. For the NN and bNN preconditioners, the set-up time includes the time for the set-up of the local ILU++ preconditioners for the local Dirichlet and Neumann solvers. In the set-up time of the bNN preconditioners additionally enters the time for the set-up of the ILU++ preconditioner for the coarse solver and the computation of the coarse operator $\mathbf{S}_0$. For its computation in the non-control case, we have to solve as many local problems as subdomains, in the control case, the coarse space is twice as large, which means that we need to solve twice as many local subdomain problems. Even though we have to solve more local problems for the set-up as the subdomain number increases, the set-up time is divided in half along with doubling the number of subdomains, which is graphically shown in Fig. 30. In sub-figure 30a, we compare the set-up times for the Oseen example and in sub-figure 30b for the Navier-Stokes example. This evolution for the set-up time can be explained by the following two observations: First, we observe that the



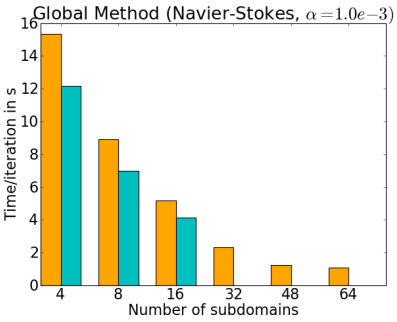
(a) Navier-Stokes example without control.



(b) Navier-Stokes example with distributed control for $\alpha = 1.0e-3$.



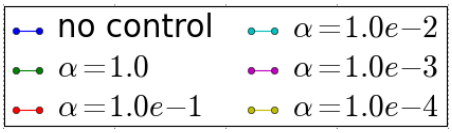
(c) Navier-Stokes example without control.



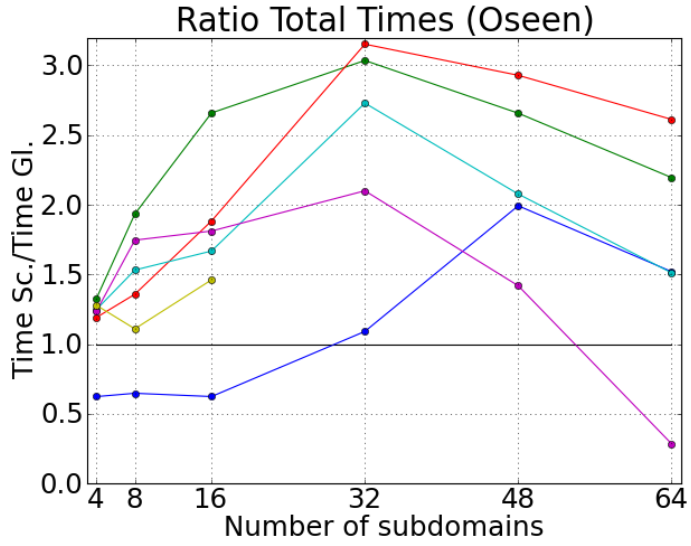
(d) Navier-Stokes example with distributed control for $\alpha = 1.0e-3$.

Figure 28: The figure shows the time per iteration for the two solving approaches applied to the Navier-Stokes example exemplarily for the non-control and the control case for $\alpha = 1.0e-3$.

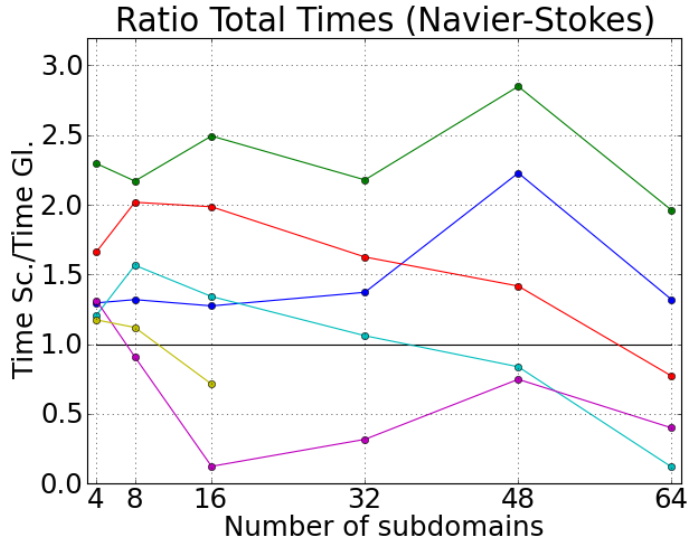
set-up time for the Neumann-Neumann preconditioner also reduces anti-proportionally to the increasing number of subdomains, therefore we conclude that the set-up time for the local ILU++ preconditioners reduces for smaller problems. Since this does not explain the whole reduction, we furthermore conclude that even thought we have to solve more local problems, the size of the problem is the crucial part. The smaller the problem the quicker a local problem can be solved.



(a) Legend for Figures 29, 30, 34 and 36.

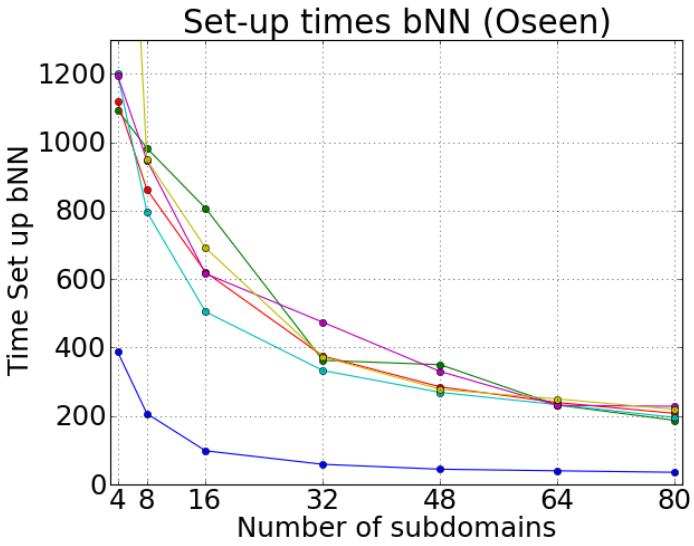


(b) Oseen example.

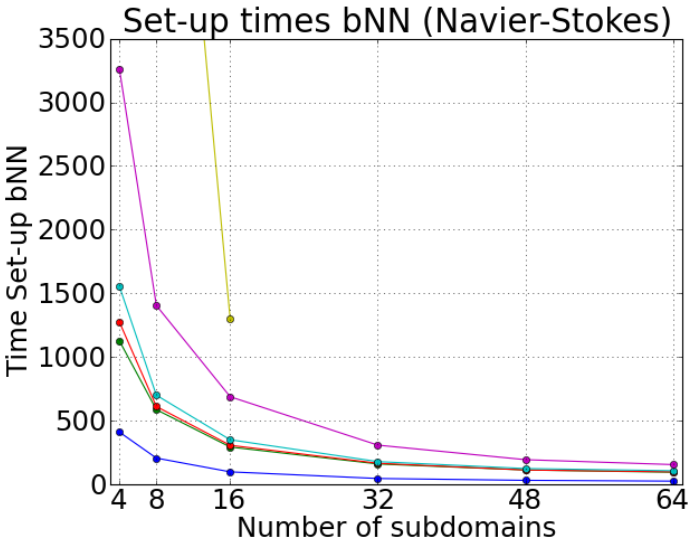


(c) Navier-Stokes example.

Figure 29: For different numbers of subdomains, this figure shows the ratio between the total time of the Schur-complement solver and the total time of the global solver for both examples applying the bNN-type preconditioner.



(a) Oseen example.



(b) Navier-Stokes example.

Figure 30: For different numbers of subdomains, this figure shows the set-up time for the bNN preconditioner for both examples. The legend is shown in Fig. 29a.

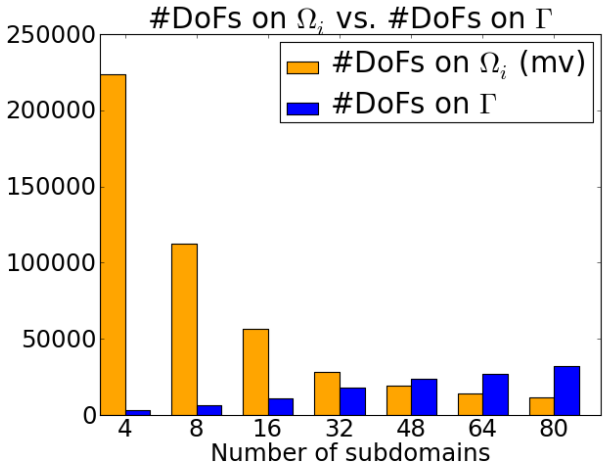
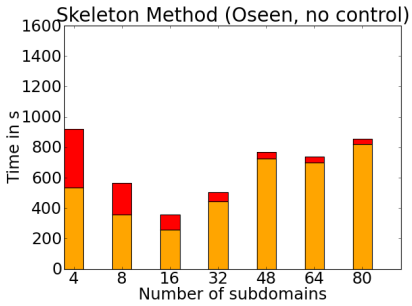
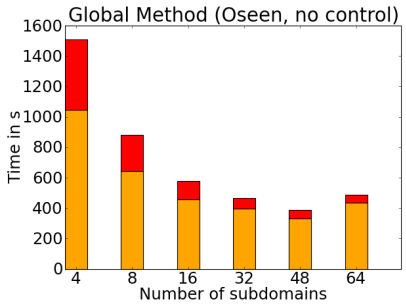


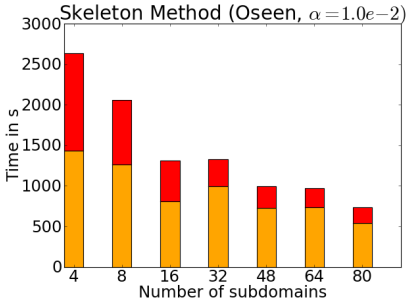
Figure 31: For different numbers of subdomains, this figure compares the mean value (mv) of the number of degrees of freedom (DoFs) on one subdomain with the number of DoFs on the global skeleton.



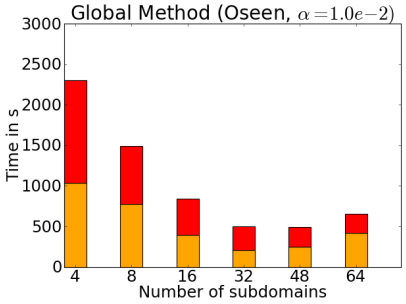
(a) Oseen example without control.



(b) Oseen example without control.

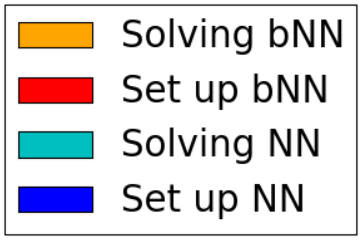


(c) Oseen example with distributed control for $\alpha = 1.0e - 2$.

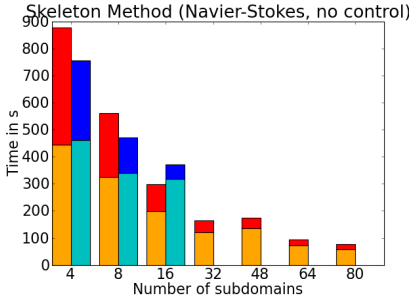


(d) Oseen example with distributed control for $\alpha = 1.0e - 2$

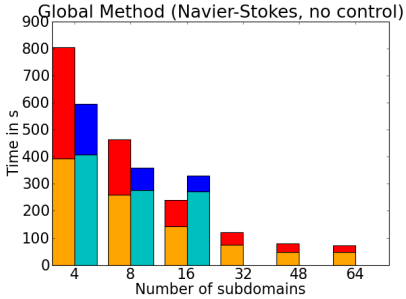
Figure 32: The figure compares the solving time with the set-up time for the two approaches with the two corresponding preconditioners for the Oseen example exemplarily for the non-control and the control case for $\alpha = 1.0e - 2$. The legend is given in Fig. 33a.



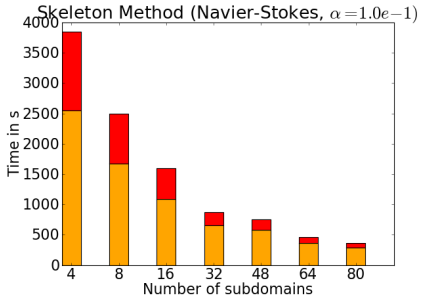
(a) Legend for Fig. 32 and Fig. 33.



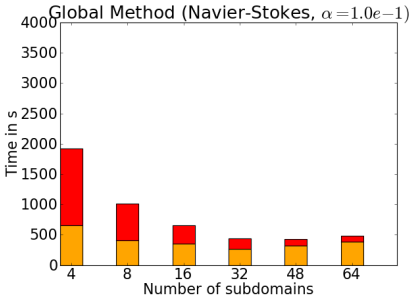
(b) Navier-Stokes example without control.



(c) Navier-Stokes example without control.



(d) Navier-Stokes example with distributed control for $\alpha = 1.0e-1$.



(e) Navier-Stokes example with distributed control for $\alpha = 1.0e-1$.

Figure 33: The figure compares the solving time with the set-up time for the two approaches with the two corresponding preconditioners for the Navier-Stokes example exemplarily for the non-control and the control case for $\alpha = 1.0e-1$.

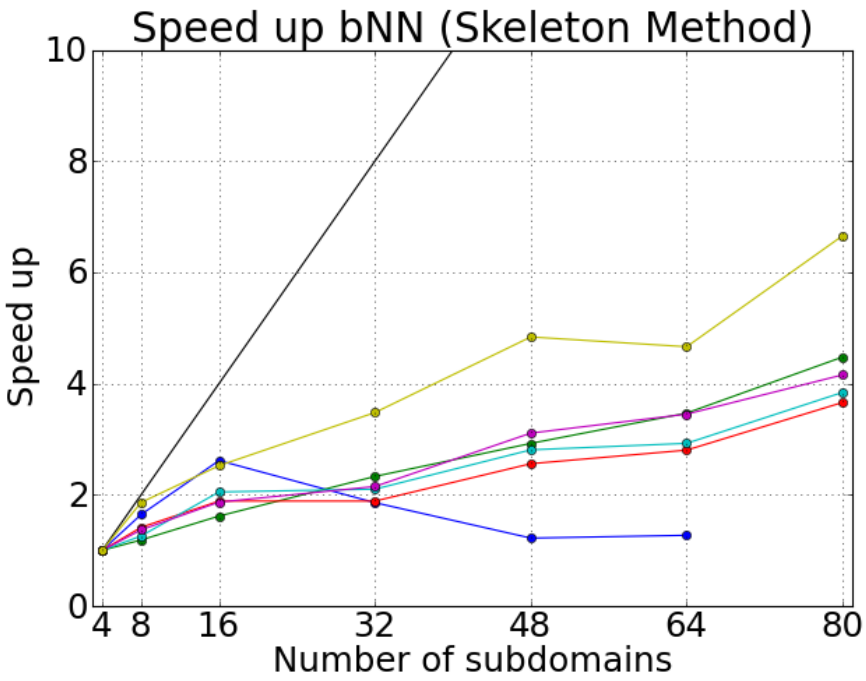
7.6 SCALABILITY STUDY

In this subsection, we study the scalability of both methods applying the bNN-type preconditioner by evaluating their speed up and efficiency. Since both methods are intrinsically parallel, we redefine the speed up and efficiency with respect to the numerical experiments with four subdomains:

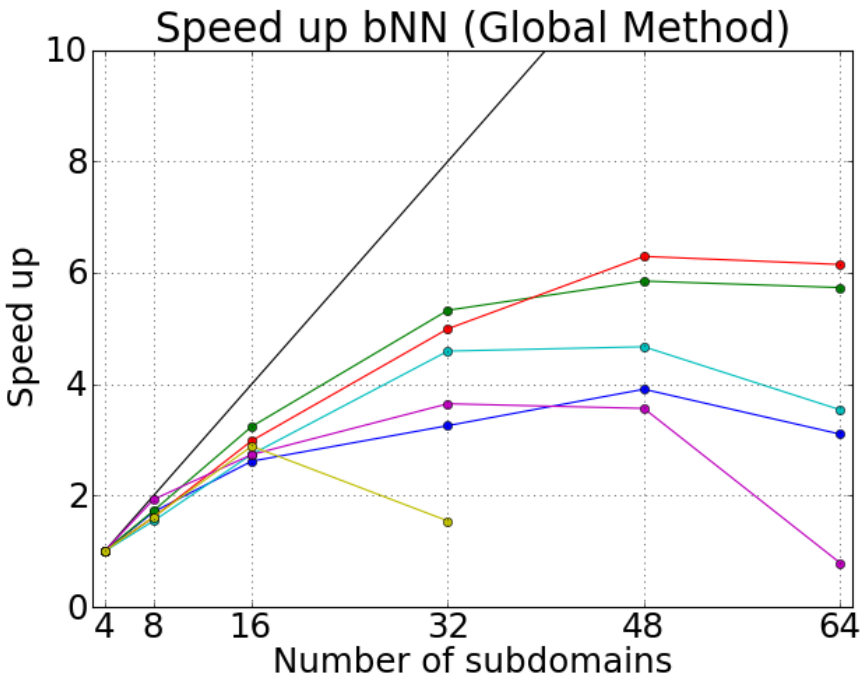
$$\text{Speed up:} \quad S_p = \frac{T_4}{T_p},$$
$$\text{Efficiency:} \quad E_p = \frac{S_p}{(p/4)},$$

where T_p denotes the total solving time using p processes which in our case coincides with the number of subdomains. We do not analyze the NN-type preconditioners, since we already concluded in the previous subsection that they are not suitable for high performance computing. Fig. 34 and 35 depict the speed up, and Fig. 36 and 37 the efficiency, for the Oseen and Navier-Stokes example, respectively. The figures are arranged analogously: The upper part shows the results for the Schur-complement method, the lower part those for the global method. We cannot expect a linear speed up and an efficiency of 1 for several reasons: Firstly, the cost for the communication increases with the number of subdomains. Secondly, as observed in the previous subsection for both approaches, the number of iterations increases with the number of subdomains. In each iteration global communication is needed three times: Two times for applying the Schur-complement operator and once for applying the NN-type preconditioner inside the bNN-type preconditioner. Taking together, these effects have a negative impact on the speed up and efficiency. Despite this face, we observe a speed up for the Schur-complement approach, which is even almost linear for the Navier-Stokes example. The better than linear speed up for the distributed control case for $\alpha = 1.0e-4$ results from the fact, that we need above the ordinary time to solve that specific problem on four subdomains. For that specific case, we observe that during the application of solving the local Dirichlet problems the Schur-complement operator, the maximal number of iterations (1000) is reached several times while for the cases with more subdomains much fewer iterations are needed. For the global solver, we note a speed up greater than 1 with up to 48 subdomains. Here, we directly see the impact that the number of iterations is not stable with respect to the number of subdomains but grows as the regularization parameter is reduced.

Looking at the efficiency of the Schur-complement approach, we obtain very good results for the Navier-Stokes example. For the Oseen example the efficiency levels out at about 32 subdomains. As

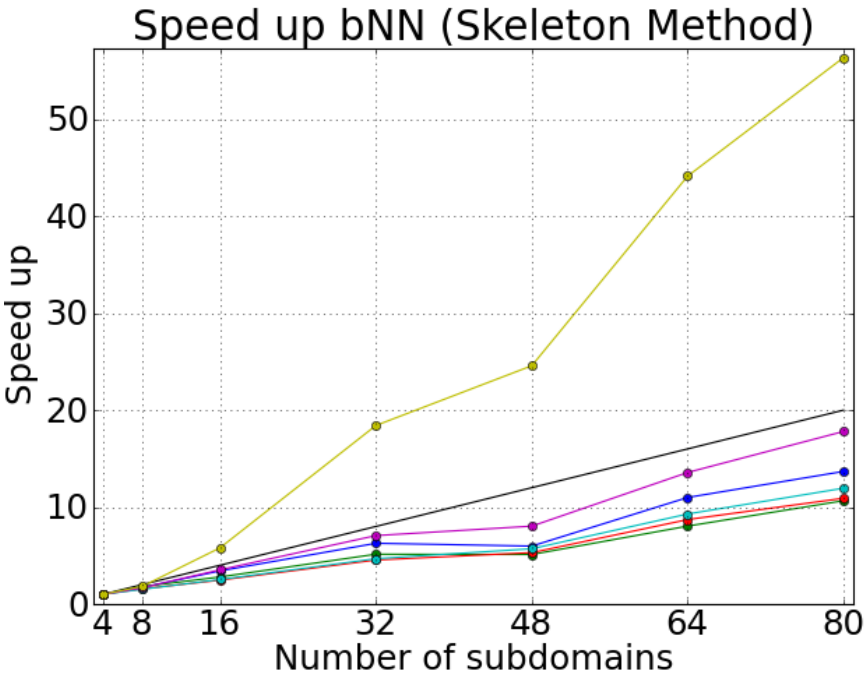


(a) Schur-complement approach.

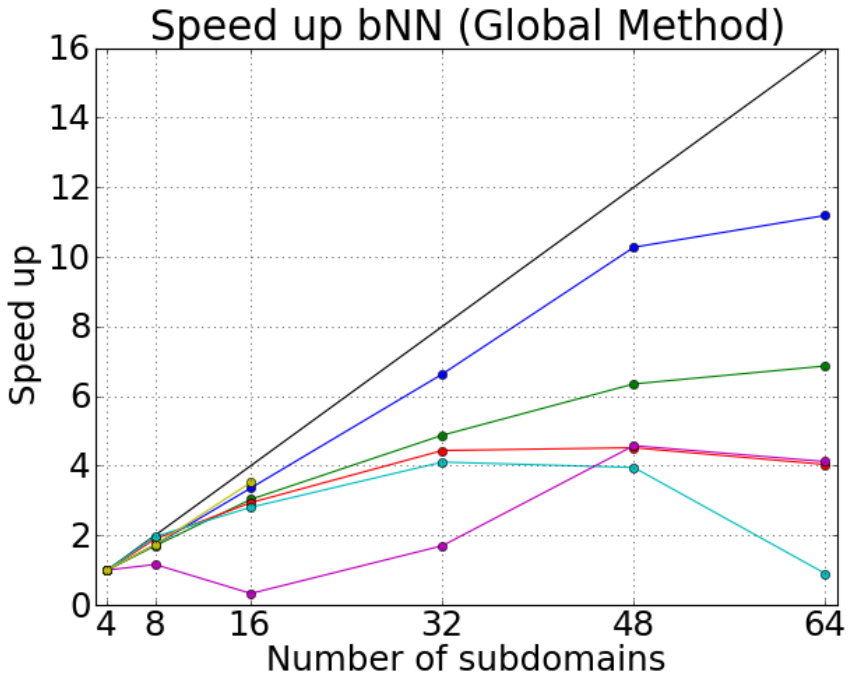


(b) Global approach.

Figure 34: The figure shows the speed up for the two approaches solving the Oseen example with the corresponding bNN type preconditioners. The legend is shown in Fig. 29a.

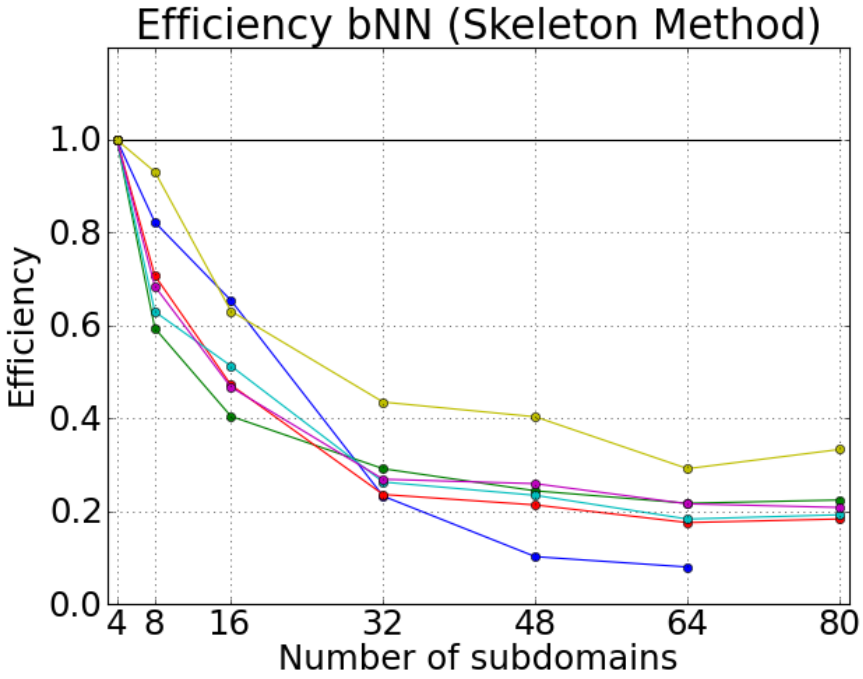


(a) Schur-complement approach.

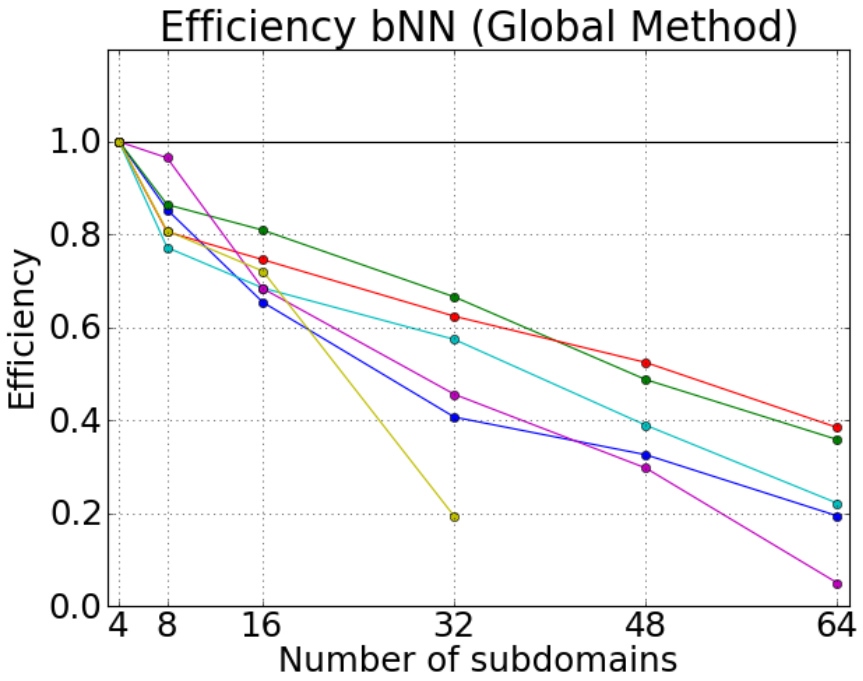


(b) Global approach.

Figure 35: The figure shows the speed up for the two approaches solving the Navier-Stokes example with the corresponding bNN type preconditioners. The legend is shown in Fig. 29a

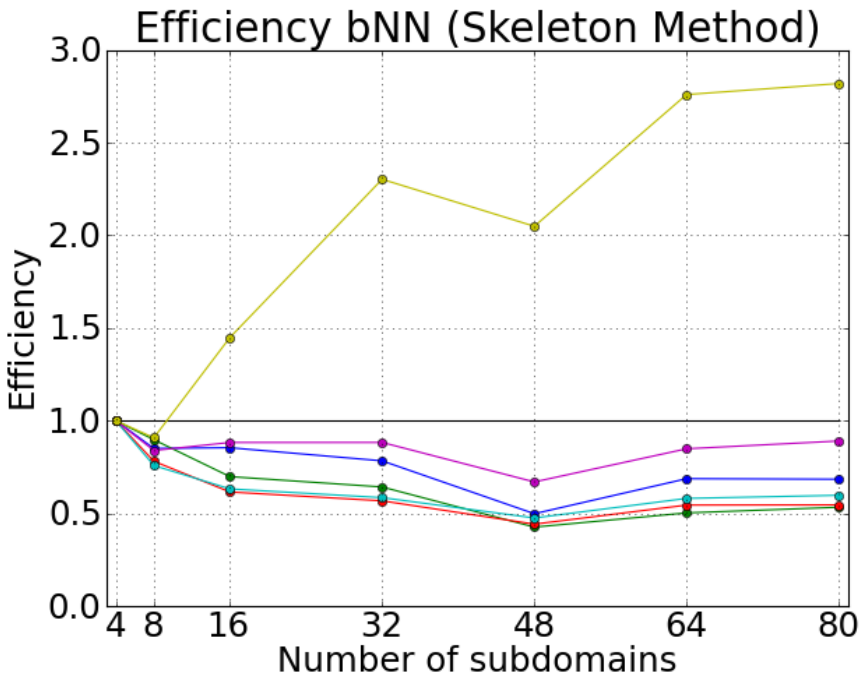


(a) Schur-complement approach.

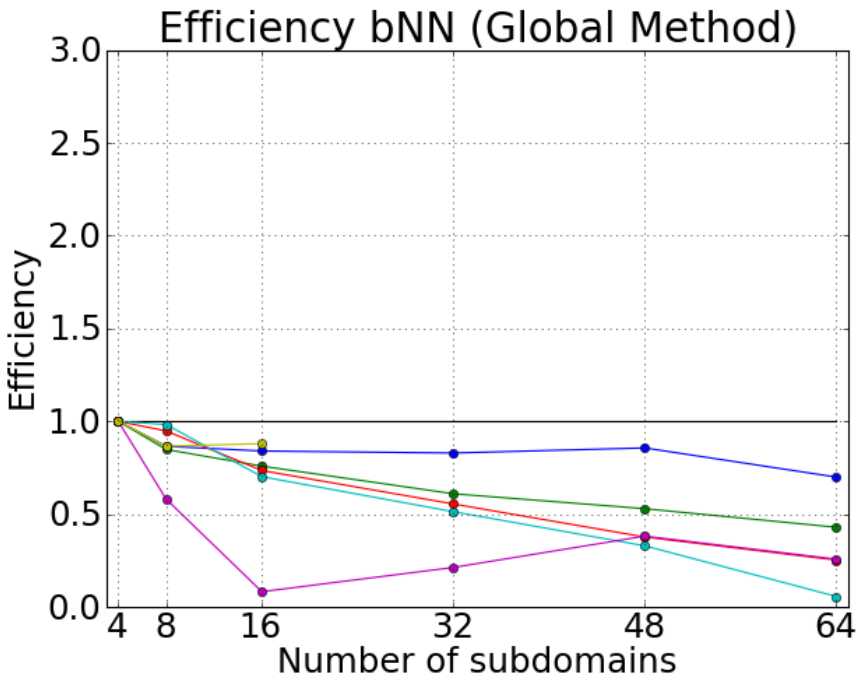


(b) Global approach.

Figure 36: The figure shows the efficiency for the two approaches each applying the bNN type preconditioners for the Oseen example. The legend is shown in Fig. 29a.



(a) Schur-complement approach.



(b) Global approach.

Figure 37: The figure shows the efficiency for the two approaches each applying the bNN type preconditioners for the Navier-Stokes examples. The legend is shown in Fig. 29a.

expected after analyzing the speed up, for the global approach the efficiency decreases with the number of subdomains. For the control case of $\alpha = 1.0e-2$, the efficiency decreases due to the fact that this specific problem requires above the ordinary number of iterations. Regarding the global solver, we assume that there exist more suitable parameters for the global Schur-complement preconditioner. Nevertheless, we conclude that both methods show a great potential in high performance computing, in part due to the fact that there exist several possibilities to improve both approaches. One option is the implementation of an inf-sup stable coarse space [21, 45]. In case of the global Schur-complement preconditioner a detailed study of the parameters for the preconditioner seems promising.

SUMMARY

In this chapter, we studied the solvers derived in the previous chapters by means of numerical experiments. To verify the method and its implementation, we solved a problem for which the analytical solution is known. Then, we investigated the effectiveness of the preconditioners based on the number of iterations. Based on the set-up time and runtime for the different preconditioners, we compared the different preconditioners. Furthermore, we studied their speed up and efficiency.

8

CONCLUSION

In this work, we deal with optimization problems constrained by a system of partial differential equations modeling the physical dynamics of flow. Determining an optimal control for a flow problem is usually numerically challenging because it relies on solving a fully coupled optimality system. This system involves the numerical treatment of the flow problem, which is generally itself difficult to solve. To tackle the complexity of such optimal flow control problems, we develop efficient parallel numerical solvers and preconditioners based on domain decomposition methods (DDMs), which exploit the computational power provided in high performance computing.

SUMMARY OF CONTRIBUTIONS

A first challenge in deriving a non-overlapping DDM for flow and optimal flow control problems is to obtain local problems featuring the same saddle point structure as the global problem. In this work, we show that by deriving the DDM on the continuous level, we obtain local problems on the subdomains which display the same saddle point structure. Thus also for optimal flow control problems, the two main ideas of a non-overlapping DDM hold: On the one hand known and well-established methods developed for the global problems can be reused to solve the local independent problems, and on the other hand the methods can be applied in parallel due to independence of the local problems. Furthermore, we investigate in detail the differences between inhomogeneous Dirichlet boundary conditions and mixed outflow and Dirichlet conditions in the context of non-overlapping DDMs. We apply two DDMs to several different flow models and show that these methods can be used to solve optimal flow control problems, even for the non-linear case with the full Navier-Stokes equations as constraints.

For discretization of these problems, we employ an appropriate finite element method and develop efficient parallel solvers for the corresponding linear systems. Using the algebraic formulation, we derive two solution methods: one for the decoupled Schur-complement equation, and one for the globally coupled domain decomposition formulation. We extend the existing one- and two-level Neumann-

Neumann type preconditioners for flow problems to the problem of optimal flow control. Based on these ideas, a preconditioner for the global formulation is derived, which exploits the block structure of the domain decomposition matrix for efficient evaluation.

We analyze in detail several aspects of the preconditioners by means of numerical experiments. We investigate the runtime of the various steps in the computation, the effectiveness of the preconditioners in terms of the number of iterations, as well the scalability in terms of speed up and efficiency. As expected, we observe a trade-off between the cost for applying a preconditioner and the effectiveness with respect to the number of iterations. Although the Neumann-Neumann type preconditioners are less expensive with respect to set-up cost and cost per iterations, only the more costly balancing Neumann-Neumann type preconditioners enable us to solve these problems with a large number of subdomains. In particular the results obtained for the balancing Neumann-Neumann type preconditioners establish the great potential of using DDMs in the context of optimal flow control. Comparing both methods, the global approach based on a fully coupled formulation is in almost all cases faster than the Schur-complement approach. But on the other hand, the Schur-complement approach does not depend on parameters, which makes it more robust, and we obtain better results with respect to the speed up and efficiency. Summarizing, we demonstrate that both solvers applying the balancing Neumann-Neumann type preconditioners make it possible to tackle complex applications on high performance computing architectures.

We conclude that non-overlapping DDMs can also be applied to non-linear flow control problems constrained by the full Navier-Stokes equations. Furthermore, by studying the speed up and efficiency of the parallel solvers and preconditioners developed in this work for flow and optimal flow control problems, we demonstrate how to exploit the computational power available nowadays in high performance computing in a scalable and efficient manner for such sophisticated problems. We also conclude that the global exchange of information realized by the balancing Neumann-Neumann type is crucial to provide effective, scalable and efficient preconditioners.

OUTLOOK

By means of numerical experiments, we demonstrate that the DDMs that we have developed converge, but from the numerical analysis point of view, a proof of convergence for the DDM on the continuous level in the context of optimal flow control problems is still an open question. Another open and interesting issue is a theoretical estimation of the condition numbers for the preconditioners derived in this work.

From the implementational and numerical perspective, we point out some promising possibilities to improve the methods with respect

to the properties requested in high performance computing. We expect to achieve a significant benefit by implementing inf-sup stable finite elements for the coarse space used for the balancing Neumann-Neumann preconditioners. Another aspect is whether it is possible to reduce the cost to apply the exact balancing Neumann-Neumann preconditioner by actively controlling the stopping tolerances of the local solvers inside the preconditioner. Considering the global solver applied to the fully coupled linear system, an interesting question is to estimate a relation between the parameters for the inexact balancing Neumann-Neumann preconditioner and the problem, on the one hand with respect to the model problem and on the other hand with respect to the different regularization parameters.

In this work, we present and analyze parallel and scalable methods for high performance computing. The next step is to extend these methods for future exascale hardware architectures. The parallel methods need to be prepared for a dramatic increase in the number of cores which in the context of a DDM leads to a corresponding significant increase of subdomains. Thus, the interface becomes substantial larger and accordingly the cost for communication increases. Therefore, one main challenge is the reduction of global communication. One idea is to extend the two-level Neumann-Neumann type preconditioners to hierarchical Neumann-Neumann type preconditioners.

BIBLIOGRAPHY

[1] Y. Achdou, P. Le Tallec, F. Nataf, and M. Vidrascu. A domain decomposition preconditioner for an advection-diffusion problem. *Computer Methods in Applied Mechanics and Engineering*, 184(2-4): 145 – 170, 2000.

[2] R. A. Adams and J. J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and applied mathematics*. Academic Press, Amsterdam, 2. ed., repr. edition, 2007.

[3] H. Anzt, W. Augustin, M. Baumann, H. Bockelmann, T. Gengenbach, T. Hahn, V. Heuveline, E. Ketelaer, D. Lukarski, A. Otzen, S. Ritterbusch, B. Rocker, S. Ronnas, M. Schick, C. Subramanian, J.-P. Weiss, and F. Wilhelm. HiFlow3 – A Flexible and Hardware-Aware Parallel Finite Element Package. EMCL Preprint Series, 2010.

[4] H. Anzt, W. Augustin, M. Baumann, T. Gengenbach, T. Hahn, A. Helfrich-Schkarbanenko, V. Heuveline, E. Ketelaer, D. Lukarski, A. Nestler, S. Ritterbusch, S. Ronnas, M. Schick, M. Schmidtobreick, C. Subramanian, J.-P. Weiss, F. Wilhelm, and M. Wlotzka. HiFlow3: A Hardware-Aware Parallel Finite Element Package. In *Tools for High Performance Computing 2011*, pages 139–151. Springer Berlin Heidelberg, 2012.

[5] D. Barz, H. Bockelmann, and V. Heuveline. Optimization of an electrokinetic mixer for microfluidic applications. *Biomicrofluidics*, 6, 2012.

[6] R. Becker, V. Heuveline, and R. Rannacher. An Optimal Control Approach to Adaptivity in Computational Fluid Mechanics. *International Journal for Numerical Methods in Fluids*, 40(1-2):105–120, 2002.

[7] G. Biros and O. Ghattas. Parallel Lagrange–Newton–Krylov–Schur Methods for PDE-Constrained Optimization. Part I: The Krylov–Schur Solver. *SIAM Journal on Scientific Computing*, 27(2): 687–713, 2005.

[8] G. Biros and O. Ghattas. Parallel Lagrange–Newton–Krylov–Schur Methods for PDE-Constrained Optimization. Part II: The Lagrange–Newton Solver and Its Application to Optimal Control of Steady Viscous Flows. *SIAM Journal on Scientific Computing*, 27, 2005.

[9] H. Bockelmann. *High Performance Computing Based Methods for Simulation and Optimisation of Flow Problems*. PhD thesis, Karlsruhe Institute of Technology, 2010.

[10] D. Braess. *Finite Elemente : Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie*. Springer, 2007.

[11] R. Buchty, V. Heuveline, W. Karl, and J.-P. Weiss. A Survey on Hardware-aware and Heterogeneous Computing on Multicore Processors and Accelerators. *Concurrency and Computation: Practice and Experience*, 24(7):663–675, 2012.

[12] C. Calgaro and J. Laminie. On the Domain Decomposition Method for the Generalized Stokes Problem with Continuous Pressure. *Numerical Methods for Partial Differential Equations*, 16(1): 84–106, 2000.

[13] P. Deuflhard. *Newton Methods for Nonlinear Problems : Affine Invariance and Adaptive Algorithms*. Springer series in computational mathematics ; 35. Springer, 2006.

[14] J. Donea and A. Huerta. *Finite Element Methods for Flow Problems*. Wiley, 2004.

[15] A. Ern and J.-L. Guermond. *Theory and Practice of Finite Elements*, volume 159 of *Applied mathematical sciences*. Springer, 2004.

[16] A. V. Fursikov. *Optimal Control of Distributed Systems : Theory and Applications*, volume 187 of *Translations of mathematical monographs*. American Mathematical Society, Providence, RI, 2000.

[17] A. V. Fursikov, M. D. Gunzburger, and L. S. Hou. Boundary Value Problems and Optimal Boundary Control for the Navier–Stokes System: the Two-Dimensional Case. *SIAM J. Control Optim.*, 36 (3):852–894, 1998.

[18] G.P. Galdi. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations : Steady-State Problems*. Springer Monographs in Mathematics Springer Link : Bücher. Springer New York, 2011.

[19] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations : theory and algorithms*. Springer series in computational mathematics ; 5. Springer, 1986.

[20] R. Glowinski, J. L. Lions, and J. He. *Exact and Epproximate Controllability for Distributed Parameter Systems : a Numerical Approach*, volume 117 of *Encyclopedia of mathematics and its applications*. Cambridge Univ. Press, 2008.

[21] P. Goldfeld, L. F. Pavarino, and O. B. Widlund. Balancing Neumann-Neumann preconditioners for mixed approximations

of heterogeneous problems in linear elasticity. *Numerische Mathematik*, 95:283–324, 2003.

[22] M. D. Gunzburger. *Finite Element Methods for Viscous Incompressible Flows : a Guide to Theory, Practice, and Algorithms*. Computer science and scientific computing. Academic Press, 1989.

[23] M. D. Gunzburger. *Perspectives in Flow Control and Optimization*. Advances in design and control ; 5. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2003.

[24] M.D. Gunzburger, L. Hou, and T.P. Svobodny. Analysis and finite element approximation of optimal control problems for the stationary Navier–Stokes equations with distributed and Neumann controls. *Mathematics of Computation*, 57(195):123–152, 1991.

[25] M. Heinkenschloss and M. Herty. Distributed solution of optimal control problems governed by parabolic equations. In *Robust Optimization-Directed Design*, volume 81 of *Nonconvex Optimization and Its Applications*, pages 73–94. Springer US, 2006.

[26] M. Heinkenschloss and H. Nguyen. Balancing Neumann-Neumann Methods for Elliptic Optimal Control Problems. In *Domain Decomposition Methods in Science and Engineering*, volume 40 of *Lecture Notes in Computational Science and Engineering*, pages 589–596. Springer Berlin Heidelberg, 2005.

[27] M. Heinkenschloss and H. Nguyen. Neumann–Neumann Domain Decomposition Preconditioners for Linear-Quadratic Elliptic Optimal Control Problems. *SIAM Journal on Scientific Computing*, 28(3):1001–1028, 2006.

[28] M. Heinkenschloss and D. Ridzal. Integration of Sequential Quadratic Programming and Domain Decomposition Methods for Nonlinear Optimal Control Problems. In *Domain Decomposition Methods in Science and Engineering XVII*, volume 60 of *Lecture Notes in Computational Science and Engineering*, pages 69–80. Springer Verlag, 2008.

[29] V. Heuveline and F. Schieweck. On the Inf-Sup Condition for Higher Order Mixed FEM on Meshes with Hanging Nodes. *Mathematical Modelling and Numerical Analysis M2AN*, 41(1):1–20, 2007.

[30] J. G. Heywood, R. Rannacher, and S. Turek. Artificial boundaries and flux and pressure conditions for the incompressible Navier-Stokes equations. *International Journal for Numerical Methods in Fluids*, 22(5):325–352, 1996.

[31] M. Hinze. Optimal and instantaneous control of the instationary Navier-Stokes equations. Habilitation thesis, Fachbereich Mathematik,Universität Berlin, 2000.

[32] M. Hinze and K. Kunisch. Second order methods for boundary control of the instationary Navier-Stokes system. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 84(3):171–187, 2004.

[33] M. Hinze, R. Pinnau, M. Ulbrich, and S. Ulbrich. *Optimization with PDE Constraints*. Mathematical Modelling: Theory and Applications ; 23. Springer Netherlands, 2009.

[34] F. Jarre and J. Stoer. *Optimierung*. Springer-Lehrbuch. Springer, Berlin, 2004.

[35] G. Karypis and V. Kumar. A Fast and Highly Quality Multilevel Scheme for Partitioning Irregular Graphs. *SIAM Journal on Scientific Computing*, 20(1):359–392, 1999.

[36] K. Kunisch and B. Vexler. Optimal vortex reduction for instationary flows based on translation invariant cost functionals. *SIAM Journal on Control and Optimization*, 46(4):1368–1397, 2007.

[37] K. Kunisch and B. Vexler. On the Choice of the Cost Functional for Optimal Vortex Reduction for Instationary Flows. In *Active Flow Control*, volume 95 of *Notes on Numerical Fluid Mechanics and Multidisciplinary Design*, pages 339–352. Springer Berlin / Heidelberg, 2007.

[38] W. J. Layton. *Introduction to the Numerical Analysis of Incompressible Viscous Flows*. Computational Science & Engineering Series; 6. Society for Industrial and Applied Mathematics, 2008.

[39] J. L. Lions. *Optimal Control of Systems Governed by Partial Differential Equations*, volume 170 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*. Springer, 1971.

[40] G. Matthies and F. Schieweck. A Multigrid Method for Incompressible Flow Problems Using Quasi Divergence Free Functions. *SIAM Journal on Scientific Computing*, 28(1):141–171, 2006.

[41] J. Mayer. A multilevel Crout ILU preconditioner with pivoting and row permutation. *Numerical Linear Algebra with Applications*, 14(10):771–789, 2007.

[42] J. Mayer. Symmetric Permutations for I-matrices to Delay and Avoid Small Pivots During Factorization. *SIAM Journal on Scientific Computing*, 30(2):982–996, 2008.

[43] F. Nataf, H. Xiang, and V. Dolean. A two level domain decomposition preconditioner based on local Dirichlet-to-Neumann maps. *Comptes Rendus Mathematique*, 348(21-22):1163 – 1167, 2010.

[44] H. Nguyen. *Domain Decomposition Methods for Linear-Quadratic Elliptic Optimal Control Problems*. PhD thesis, Rice University, 2004.

[45] L. Pavarino and O. Widlund. Balancing Neumann-Neumann methods for incompressible Stokes equations. *Communications on Pure and Applied Mathematics*, 55(3):302–335, 2002.

[46] A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Numerical mathematics and scientific computation. Clarendon Press, 2005.

[47] Y. Saad. A Flexible Inner-Outer Preconditioned GMRES Algorithm. *SIAM Journal on Scientific Computing*, 14(2):461–469, 1993.

[48] Y. Saad. *Iterative Methods for Sparse Linear Systems*. SIAM, Society for Industrial and Applied Mathematics, 2003.

[49] Y. Saad. Multilevel ILU with Reorderings for Diagonal Dominance. *SIAM Journal on Scientific Computing*, 27(3):1032–1057, 2005.

[50] Y. Saad and M. H. Schultz. GMRES: A Generalized Minimal Residual Algorithm for Solving Nonsymmetric Linear Systems. *SIAM Journal on Scientific and Statistical Computing*, 7(3):856–869, 1986.

[51] B. F. Smith, P. E. Bjørstad, and W. D. Gropp. *Domain Decomposition : Parallel Multilevel Methods for Elliptic Partial Differential Equations*. Cambridge Univ. Press, 2004.

[52] H. Sohr. *The Navier-Stokes Equations : An Elementary Functional Analytic Approach*. Birkhäuser advanced texts. Birkhäuser, 2001.

[53] V.A. Solonnikov. Stokes and Navier-Stokes equations in domains with noncompact boundaries. *College de France Seminar*, 4:240–349, 1983.

[54] C. Taylor and P. Hood. A numerical solution of the Navier-Stokes equations using the finite element technique. *Computers & Fluids*, 1(1):73 – 100, 1973.

[55] R. Temam. *Navier-Stokes Equations : Theory and Numerical Analysis*. AMS Chelsea Publ., 2001.

[56] A. Toselli and O. Widlund. *Domain Decomposition Methods - Algorithms and Theory*. Springer series in computational mathematics ; 34. Springer, Berlin, 2005.