Quo vadis, Aut($\mathfrak{n}$)?

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PREFACE

In this thesis we deepen the relations between rational homotopy theory and nilpotent Lie algebras which were first discovered by D. Sullivan and A. Malcev between 1960 and 1980. In particular, we study cohomological representations of the automorphism group of a nilpotent Lie algebra. As already pointed out by Sullivan [20] and Malcev [13], such representations play a fundamental role in the context of automorphism groups of spaces and classifications of manifolds. As revealed by the work of O. Baues and F. Grunewald [1], they are of similar importance in the context of arithmetic automorphism groups occurring in group theory. The first main result of this thesis is:

Theorem A. *Let* $\mathfrak{n}$ *be a two-step nilpotent Lie algebra over a field of characteristic zero,* $\mathrm{Aut}(\mathfrak{n})$ *the group of Lie algebra automorphisms and*

$$H^1 : \mathrm{Aut}(\mathfrak{n}) \to \mathrm{Aut}(H^1(\mathfrak{n}))$$

the group homomorphism induced by the Lie algebra cohomology functor of degree one. Then

$$\mathrm{Aut}(\mathfrak{n}) = \ker(H^1) \rtimes \mathrm{im}(H^1)\ .$$

That is, the short exact sequence of groups

$$\{1\} \longrightarrow \ker(H^1) \longrightarrow \mathrm{Aut}(\mathfrak{n}) \xrightarrow{H^1} H^1(\mathrm{Aut}(\mathfrak{n})) \longrightarrow \{1\}$$

splits.

Since the automorphism group of a nilpotent Lie algebra is a linear algebraic group, there exists a semi-direct decomposition

$$\mathrm{Aut}(\mathfrak{n}) = U \rtimes \mathrm{Aut}(\mathfrak{n})/U$$

where U is a maximal normal unipotent subgroup and $\mathrm{Aut}(\mathfrak{n})/U$ is reductive [3]. This decomposition is called the Levi–Mostow decomposition. We give a wide class of examples where the decomposition of Theorem A is significantly different from the Levi–Mostow decomposition. Our next result is:

Theorem B. *Let* $\mathfrak{n}$ *be a nilpotent Lie algebra over a field of characteristic zero with one dimensional commutator,* $\mathrm{Inn}(\mathfrak{n})$ *the group of inner automorphisms and*

$$H^* : \mathrm{Aut}(\mathfrak{n}) \to \mathrm{Aut}(H^*(\mathfrak{n}))$$

the group homomorphism induced by the Lie algebra cohomology functor $H^*(\mathfrak{n}) := \bigoplus_{k \in \mathbb{N}} H^k(\mathfrak{n})$*. Then*

$$\mathrm{Inn}(\mathfrak{n}) = \ker(H^*) \ .$$

In particular this implies that $\mathrm{Out}(\mathfrak{n}) = H^*(\mathrm{Aut}(\mathfrak{n}))$ *an thus* $\mathrm{Out}(\mathfrak{n})$ *acts faithfully on cohomology.*

Note that a nilpotent Lie algebra with one dimensional commutator is two-step nilpotent. However Theorem B does not imply in this case that $\mathrm{Out}(\mathfrak{n})$ already acts faithfully on $H^1(\mathfrak{n})$, since $\ker(H^*) = \ker(H^1)$ holds if and only the center of the Lie algebra is also one dimensional. Thus Theorem A and Theorem B may be considered as independent except for the class of generalised Heisenberg algebras.

Moreover, we contrast Theorem B with the following example:

Example C. *There exists a three-step nilpotent Lie algebra where an automorphism acts trivially on all Lie algebra cohomology groups but is not an inner one.*

At present we do not know whether Theorem B holds more generally in the two-step nilpotent case.

Furthermore, we work out a detailed dictionary between notions of Lie algebras and notions of differential graded algebras. We want to highlight the following correspondence briefly. Let $(\mathfrak{n}, [\cdot,\cdot])$ be a n-step nilpotent Lie algebra over a field $\mathbb{K}$ of characteristic zero. On the one hand, the descending central series $\mathcal{C}^k := [\mathfrak{n}, \mathcal{C}^{k-1}]$ induces a filtration

$$\mathbb{K} = (\mathcal{C}^0)^\perp \subset \cdots \subset (\mathcal{C}^k)^\perp \subset \cdots \subset (\mathcal{C}^n)^\perp = \bigwedge \mathfrak{n}^*$$

of the Koszul complex by duality with

$$(\mathcal{C}^k)^\perp := \{\omega \in \mathfrak{n}^* \mid \omega(x) = 0 \ \forall x \in \mathcal{C}^k\} \ .$$

On the other hand, one can iteratively construct the corresponding minimal model $p : \mathcal{M} \to \bigwedge \mathfrak{n}^*$. The iterative construction process induces a filtration

$$\mathbb{K} = \mathcal{M}_0 \subset \cdots \subset \mathcal{M}_k \subset \cdots \subset \mathcal{M}_m = \mathcal{M}$$

of the minimal model and since p is actually an isomorphism we get another induced filtration

$$\mathbb{K} = \mathcal{N}_0 \subset \cdots \subset \mathcal{N}_k \subset \cdots \subset \mathcal{N}_m = \bigwedge \mathfrak{n}^*$$

of the Koszul complex with $\mathcal{N}_K := p(\mathcal{M}_k)$. We prove:

Theorem D. *Let $\mathfrak{n}$ be a n-step nilpotent Lie algebra over a field $\mathbb{K}$ of characteristic zero. Then the two filtrations*

$$\mathbb{K} = (\mathcal{C}^0)^\perp \subset \cdots \subset (\mathcal{C}^k)^\perp \subset \cdots \subset (\mathcal{C}^n)^\perp = \bigwedge \mathfrak{n}^*$$
$$\mathbb{K} = \mathcal{N}_0 \subset \cdots \subset \mathcal{N}_k \subset \cdots \subset \mathcal{N}_m = \bigwedge \mathfrak{n}^*$$

of the Koszul complex coincide. In particular they have the same length n, that is $m = n$.

Morita [17] and Morgan [8] formulate a comparable statement in the context of the Malcev hull of a nilpotent group. Even if it might contain Theorem D as a special case, we give a direct algebraic proof in contrast to their proofs which utilise deeper properties of the Malcev hull or the Postnikov decomposition of spaces.

We can also apply Theorem A, B and Example C directly to the automorphism group of a connected simply connected two-step nilpotent Lie group or to the group of homotopy classes of homotopy self-equivalences of certain Eilenberg–MacLane spaces. For the sake of clarity we refer the reader to Chapter 5 for precise statements of these results.

Last but not least our results may be of use in characterising or even classifying compact Kähler manifolds, which was a motivating aspect for this thesis and will be discussed in more detail below.

Relations to rational homotopy theory

Since their discovery by Sophus Lie in the nineteenth century, Lie algebras have been studied intensively and insight about their structure has been applied successfully to many different areas of mathematics and physics. Surprisingly, little seems to be known about their automorphism group, at least with a view towards cohomological representations. Explicit examples, computations and structural results are hard to find in the literature.

Sullivans celebrated work [20] introduces a geometric interpretation of rational homotopy theory in terms of differential graded algebras. This also contains a surprisingly new view onto Lie algebras and their homomorphisms which appear as degree one generated algebras in this theory. Sullivan proves that for a nilpotent Lie algebra $\mathfrak{n}$ the kernel of the map

$$H^* : \mathrm{Aut}(\mathfrak{n}) \to \mathrm{Aut}(H^*(\mathfrak{n}))$$

is a unipotent matrix group. Furthermore he shows that $\mathrm{Inn}(\mathfrak{n}) \subset \ker(H^*)$. Theorem B and Example C shed some light on the latter result. Namely there exist non trivial nilpotent Lie algebras where $\mathrm{Inn}(\mathfrak{n}) = \ker(H^*)$ as well as $\mathrm{Inn}(\mathfrak{n}) \subsetneq \ker(H^*)$ occurs.

The first result of Sullivan mentioned above can be sharpened to the fact that even the kernel of the map

$$H^1 : \mathrm{Aut}(\mathfrak{n}) \to \mathrm{Aut}(H^1(\mathfrak{n}))$$

is a unipotent matrix group [1]. Now Theorem A and the subsequent discussion shows that the reductive part of the automorphism group is in general not the biggest normal subgroup operating faithfully on the first Lie algebra cohomology group.

Motivation and context

J. W. Morgan proves in [15] that the real one-minimal model of a smooth complex variety admits a mixed Hodge structure which is compatible with the graded algebra structure and invariant under certain automorphisms. This also implies that if a finitely generated nilpotent group is the fundamental group of a smooth complex variety, its real Malcev hull admits a mixed Hodge structure. As explained below, our interest concerns Kähler manifolds which appear as a special case of the spaces considered by Morgan. He uses rational homotopy theory combined with techniques from algebraic geometry to prove his results. In order to avoid the abstract arguments from algebraic geometry we developed the following simpler idea in order to gain more insight with respect to this special case. The real one-minimal model of a reasonable manifold M is given by a Koszul complex $\bigwedge \mathfrak{n}^*$ of a real finite dimensional nilpotent Lie algebra $\mathfrak{n}$ together with a homomorphism

$$p_1 : \bigwedge \mathfrak{n}^* \to C^*_{\mathrm{DR}}(M)$$

into the de Rham algebra such that $H^1(p_1)$ is bijective and $H^2(p_1)$ is injective. Furthermore, the Lie algebra $\mathfrak{n}$ is the real Malcev hull of $\pi_1(M)$ up to natural identification. Now a Hodge structure on a vector space can be defined to be a faithful representation of the multiplicative group of units $\mathbb{C}^*$ on that vector space. If M is a Kähler manifold, its cohomology groups

$H^k(M, \mathbb{R})$ admit a Hodge structure and thus a faithful representation of $\mathbb{C}^*$ on $H^k(M, \mathbb{R})$ for all k. So a natural question is: can we lift such a representation to a faithful representation on $\mathfrak{n}^* \cong \mathfrak{n}$ which is invariant under some reasonable automorphisms? In conclusion this motivated to study cohomological representations of the automorphism group of a nilpotent Lie algebra $\mathfrak{n}$ and in particular representations on $H^1(\mathfrak{n})$ via methods of rational homotopy theory.

My interest for Morgans work arose out of my diploma thesis [19] and the subsequent joint work with Baues [2]. There we studied the following question: When is a finitely generated virtually abelian group Π a Kähler or, more restrictively, a projective group? That is, when can it be realised as the fundamental group of a closed Kähler or compact complex projective manifold, respectively? We gave a complete answer in the following sense. A finitely generated virtually abelian group contains a lattice $\Gamma \cong \mathbb{Z}^n$ as normal subgroup of finite index. Now a finitely generated, virtually abelian group is Kähler if and only if there exists a complex structures on $\Gamma \otimes \mathbb{R}$ which is invariant under the action of the finite group Π/Γ on $\Gamma \otimes \mathbb{R}$ induced by conjugation. Furthermore, Π is Kähler if and only if it is projective.

A natural generalisation of this question is to look at finitely generated, nilpotent or even virtually nilpotent groups. The first problem one faces in order to generalise the ideas of the abelian case is that for a nilpotent group N and a field $\mathbb{K}$ the tensor product $N \otimes \mathbb{K}$ is generally not defined. Precisely this gap is bridged by virtue of rational homotopy theory and the Malcev hull. That way the work of Morgan is a generalisation of our ansatz in the abelian case. Thus the main results of this thesis may also be regarded as first step to a better understanding of two-step nilpotent Kähler groups.

Structure of this thesis

Chapter one gives an overview about rational homotopy theory or to be more precise about that part of rational homotopy theory which deals with differential graded algebras. In particular, Sullivan algebras, minimal models and the homotopy theory of differential graded algebras are introduced. Also the cohomology theory of differential graded algebras and some tools from homological algebra are presented.

Chapter two deals with properties of the automorphism group of a Sullivan algebra, especially general relations between homotopic maps and maps on cohomology. It introduces the concept of inner and outer automorphisms and it is shown that inner automorphisms are precisely those automorphisms which are homotopic to the identity. Furthermore, we discuss some algebraic aspects as for example unipotent and reductive matrix groups and the Levi–

Mostow decomposition.

Chapter three connects the theory of nilpotent Lie algebras with the theory developed in Chapter one and two. We also recall all needed notions of nilpotent Lie algebras. In particular the Koszul complex and Lie algebra cohomology groups are introduced. Furthermore, we work out a dictionary between notions of Lie algebras and notions of differential graded algebras inclusive detailed proofs. This Chapter also sets the stage for the proofs of the main results of the next chapter.

Chapter four is where we prove Theorem A and B and provide Example C. Furthermore, we discuss some differences of the decomposition provided by Theorem A and the Levi–Mostow decomposition. We also shortly repeat all needed facts from previous chapters and give an overview of the known results.

Chapter five is devoted to applications. We discuss some consequences of the results of Chapter four in the context of Lie groups and Eilenberg–MacLane spaces.

CONTENTS

CHAPTER

ONE

A SURVEY ABOUT THE CATEGORY OF DIFFERENTIAL GRADED ALGEBRAS

This chapter is intended to give an overview about rational homotopy theory and its main results. Most facts are well documented in the literature as for example in [5], I.3, [17], Chapter 1, [8]. These books are also our main references for this chapter. We decided to give proofs of those results which are not so well documented or which details of the proof are of further significance for this thesis, as for example the precise construction of the homological model in Section 1.6.1.

1.1 Differential graded algebras and basic constructions

Definition 1.1.1. A graded vector space V is a vector space which admits a decomposition

$$V = \bigoplus_{l \in \mathbb{N}_{\geq 0}} V^l \ ,$$

where V^l are vector spaces. An element $v \in V$ is called of degree $\deg(v) = k$, if $v \in V^k$. Furthermore, a graded vector space is called of homogeneous degree k, if $V = V^k$.

Definition 1.1.2. A linear map $\phi : V \to W$ between graded vector spaces is called of degree k, if $\phi(V^i) \subset W^{i+k}$ for all $i \in \mathbb{N}$.

Definition 1.1.3. A graded algebra A is a graded vector space A together with a product satisfying

$$xy \in A^{n+k} \text{ for all } x \in A^n \text{ and } y \in A^k \ .$$

Remark 1.1.4. Note that in a graded algebra $\deg(1) = 0$, that is $1 \in A^0$, and that 0 is of any degree. Moreover, the sub algebra $\mathbb{K}\cdot 1 \subset A^0$ is isomorphic to $\mathbb{K}$ as graded algebra.

Definition 1.1.5. A graded algebra A is called commutative if

$$xy = (-1)^{n\cdot k} yx \text{ for all } x \in A^n \text{ and } y \in A^k.$$

Definition 1.1.6. A homomorphism between graded algebras A and B is a linear map $f : A \to B$ of degree 0, such that

$$f(xy) = f(x)f(y) \text{ for all } x, y \in A\ .$$

Since we consider algebras with units, we also require $f(1_A) = 1_B$.

Definition 1.1.7. A linear map $\theta : A \to A$ is called a derivation of degree k, if it is a linear map of degree k and

$$\theta(xy) = \theta(x)y + (-1)^{n\cdot k} x\theta(y)$$

for all $x \in A^n$ and $y \in A$.

Remark 1.1.8. Note that for all derivations $\theta_{|\mathbb{K}} = 0$ holds, since

$$\theta(1) = \theta(1 \cdot 1) = \theta(1) \cdot 1 + 1 \cdot \theta(1) = 2\theta(1)$$

and θ is linear.

Definition 1.1.9. Let $f : A \to B$ be a linear map of degree k between graded algebras. Then

$$f_n : A^n \to B^{n+k}$$

denotes the induced linear map on the vector space A^n.

There are several standard constructions for creating a new graded algebra out of old ones, as for example

Definition 1.1.10. The graded tensor algebra $A \otimes B$ of two graded algebras is given by the following data. The grading on $A \otimes B$ is defined by

$$(A \otimes B)^n := \bigoplus_{k=0}^{n} A^k \otimes B^{n-k}.$$

The multiplication is defined by $(x \otimes y)(x' \otimes y') := (-1)^{\deg(y)\deg(x')} xx' \otimes yy'$ and the addition is the usual one of the tensorproduct of the involved (graded) vectorspaces.

Remark 1.1.11. We can view multiplication in a graded algebra $\mathcal{A}$ as a linear map $A \otimes A \to A$ of degree 0.

Further important commutative graded algebras are constructed in the following way.

Definition 1.1.12. Let $V = \bigoplus V^k$ be a graded $\mathbb{K}$-vector space. The graded tensor algebra $T(V)$ over V is the graded algebra

$$T(V) := \bigoplus_{q=0}^{\infty} T^q(V) \ ,$$

where $T^q(V) := \underbrace{V \otimes \cdots \otimes V}_{q-\text{times}}$, $T^0 V = \mathbb{K}$ and the degree of an element

$$v_1 \otimes \cdots \otimes v_n \in T^n V$$

is defined by $\sum_{k=1}^{n} \deg(v_k)$. The multiplication is defined by

$$ab := a \otimes b \ .$$

For the scalar multiplication we canonically identify $\mathbb{K} \otimes \mathbb{K}$ with $\mathbb{K}$. The unit is given by $1 \in \mathbb{K}$.

Definition 1.1.13. Let $T(V)$ be the tensor algebra over V and $I \subset T(V)$ be the ideal generated by the elements

$$x \otimes y - (-1)^{k \cdot n} y \otimes x; \ x \in V^k, y \in V^n \ .$$

Then the quotient

$$\bigwedge V := T(V)/I$$

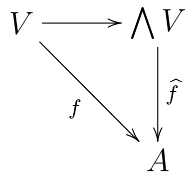
is called the free commutative graded algebra over V. The corresponding equivalence classes are denoted by $v_1 \wedge \cdots \wedge v_n$. Also note that by definition $\bigwedge^0 V = \mathbb{K}$.

A free commutative graded algebra has the following well-known properties:

Lemma 1.1.14.

i) $\bigwedge V$ *is indeed a commutative graded algebra, that is* $v \wedge w = (-1)^{n \cdot k} w \wedge v$ *for all* $v \in \bigwedge V^n$ *and* $w \in \bigwedge V^k$.

ii) If $f : V \to A$ is a linear map of degree zero into a commutative graded algebra A, then there exists a homomorphism of commutative graded algebras $\widehat{f} : \bigwedge V \to A$, such that the diagram



commutes. Similarly any linear map $\theta : V \to \bigwedge V$ of degree k extends to a derivation $\widehat{\theta} : \bigwedge V \to \bigwedge V$ of degree k. This is called the universal property of free commutative graded algebras.

iii) If V and W are graded $\mathbb{K}$-vector spaces, then

$$\theta : \bigwedge V \otimes \bigwedge W \to \bigwedge(V \oplus W)$$
$$\theta(v_1 \wedge \cdots \wedge v_n \otimes w_1 \wedge \cdots \wedge w_m) = v_1 \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge w_m$$

defines an isomorphism of commutative graded algebras.

Remark 1.1.15. Because of property ii), it is enough a homomorphism or a derivation of free commutative graded algebras $\bigwedge V \to \bigwedge V$ is uniquely defined on generators.

Definition 1.1.16. A commutative graded algebra A is called free, if there exists a graded vector space $V := \bigoplus_{k \in \mathbb{N}_{\geq 0}} V^k$, such that there exists an isomorphism

$$A \cong \bigwedge V$$

of commutative graded algebras.

Definition 1.1.17. Let A be a commutative graded algebra. A linear map

$$d : A \to A$$

of degree 1, that is $d(A^n) \subset A^{n+1}$, is called differential if

$$d^2 = d \circ d = 0$$

and

$$d(xy) = d(x)y + (-1)^n x dy$$

for all $x \in A^n$ and $y \in A$. The second condition is called Leibniz rule. In particular d is a derivation of degree 1.

Definition 1.1.18. A pair $(\mathcal{A}, d)$, where $\mathcal{A}$ is a commutative graded algebra and d is a differential, is called a differential graded algebra. We also write $\mathcal{A}$ instead of $(\mathcal{A}, d)$, suppressing the differential, which we call $d_{\mathcal{A}}$ or, if no confusion is possible, only d.

Definition 1.1.19. A homomorphism of differential graded algebras $\mathcal{A}$ and $\mathcal{B}$ is a homomorphism of graded algebras $f : \mathcal{A} \to \mathcal{B}$ that commutes with the differentials, that is $f \circ d_{\mathcal{A}} = d_{\mathcal{B}} \circ f$.

Definition 1.1.20. A differential graded algebra $\mathcal{A}$ is called free, if it is free as a commutative graded algebra as defined in 1.1.13.

Remark 1.1.21. The differential of a free differential graded algebra is determined by its images on the generators.

The universal property of free algebras is also useful to define homomorphisms of free differential graded algebras.

Lemma 1.1.22. *Let* $\left(\bigwedge V, d\right)$ *and* $\left(\bigwedge W, d'\right)$ *be free differential graded algebras and* $f : \bigwedge V \to \bigwedge W$ *be a homomorphism of graded algebras. If* $f(d(v)) = d' f(v)$ *for all* $v \in V$, *then* f *is a homomorphism of differential graded algebras.*

In other words, for a homomorphism of free commutative graded algebras, to be a homomorphism of differential graded algebras, it is sufficient to commute with the differentials on generators.

Proof. Let $f(d(v)) = d' f(v)$ for all $v, \in V$. For $v_1, v_2 \in V$ we compute

$$\begin{aligned} d' f(v_1 \wedge v_2) &= (d' f(v_1)) \wedge f(v_2)) + (-1)^{\deg(f(v_2))} f(v_1) \wedge (d' f(v_2)) \\ &= f(dv_1) \wedge f(v_2) + (-1)^{\deg(f(v_2))} f(v_1) \wedge f(dv_2)) \\ &= f(dv_1 \wedge v_2) + f((-1)^{\deg(v_2)} v_1 \wedge dv_2) \\ &= f(d(v_1 \wedge v_2)) \end{aligned}$$

□

Definition 1.1.23. Let $\mathcal{A}$ and $\mathcal{B}$ be differential graded algebras. Then the differential graded tensor algebra $\mathcal{A} \otimes \mathcal{B}$ is the tensor algebra of graded algebras with differential

$$d_{\mathcal{A}\otimes\mathcal{B}}(a \otimes b) := (d_{\mathcal{A}} a) \otimes b + (-1)^n a \otimes (d_{\mathcal{B}} b)$$

for all $a \in A^n$ and $b \in B$.

An example of a differential graded algebra is the singular cohomology of a topological space with coefficients in a field, where multiplication is given by the cup product and the differential is trivial. Another one is the de Rham complex of differential forms on a manifold, where multiplication is given by the wedge product and the differential by the exterior derivative. For details see [22] chapter 4+5. Last but not least $\mathbb{K}$ is a differential graded algebra with trivial differential.

1.2 Cohomology theory of differential graded algebras

To a differential graded algebra one can naturally associate a sequence of vector spaces called cohomology groups. Moreover, we can view a differential graded algebra as a cochain complex by ignoring the multiplicative structure. Thus we can use tools from homological algebra as for example a long exact cohomology sequence. All these facts and tools are introduced in this section.

Definition 1.2.1. Let $(\mathcal{A}, d)$ be a differential graded algebra. Then

$$Z^n(\mathcal{A}) := \ker(d_n)$$

and

$$B^n(\mathcal{A}) := \mathrm{im}(d_{n-1})$$

are vector subspaces of $\mathcal{A}^n$ since d_n and d_{n-1} are linear maps. Elements in $Z^n(\mathcal{A})$ are called cocycles of degree n and elements in $B^n(\mathcal{A})$ are called coboundaries of degree n respectively.

Since $d^2 = 0$, $B^n(\mathcal{A}) \subset Z^n(\mathcal{A})$ and thus we can define

Definition 1.2.2. Let $(\mathcal{A}, d)$ be a differential graded algebra. Then the $\mathbb{K}$-vector space

$$H^n(\mathcal{A}) := Z^n(\mathcal{A})/B^n(\mathcal{A})$$

is called the n-th cohomology group of $\mathcal{A}$.

Remark 1.2.3. $H^*(\mathcal{A}) := \bigoplus_{i\in\mathbb{N}_0} H^i(\mathcal{A})$ is a graded algebra, where the product is induced by the product of $\mathcal{A}$.

Proof. Let $x, y \in Z^n(\mathcal{A})$ be cocycles. Then

$$d(xy) = (dx)y + (-1)^{\deg(x)}x(dy) = 0\ .$$

For $x = dx'$ and $y = dy'$ coboundaries, we compute

$$d(x'(dy')) = (dx')(dy') + (-1)^{\deg(x)}x(d^2y) = xy\ .$$

Thus the product of $\mathcal{A}$ descends to a product of $H^*(\mathcal{A})$. □

Definition 1.2.4. A differential graded algebra $\mathcal{A}$ over $\mathbb{K}$ is called connected if $\mathcal{A}^0 = \mathbb{K}$.

Note that this immediately implies

Lemma 1.2.5. *Let $\mathcal{A}$ be a connected differential graded algebra over $\mathbb{K}$. Then $H^0(\mathcal{A}) = \mathcal{A}^1 = \mathbb{K}$ and $H^1(\mathcal{A}) = Z^1(\mathcal{A})$.*

Proof. Since the differential d of $\mathcal{A}$ is a derivation of degree 1 and $\mathcal{A}$ is an algebra, we compute $d_0(1) = d_0(1 \cdot 1) = d_0(1) \cdot 1 + (-1)^0 \cdot 1 \cdot d(1) = 2d_0(1)$ and thus $d_0(1) = 0$. Now d is a linear map and thus $d_0(k) = kd_0(1) = 0$ for all $k \in \mathbb{K}$. Thus $\mathrm{im}(d_0) = 0$ and the claim follows. □

Lemma 1.2.6. *A homomorphism of differential graded algebras*

$$f : (\mathcal{A}, d) \to (\mathcal{B}, d')$$

induces a $\mathbb{K}$-linear map $H^*(f) : H^*(\mathcal{A}) \to H^*(\mathcal{B})$ by restriction.

Proof. Let $z \in Z^n(\mathcal{A})$. Then $d'f(z) = f(dz) = f(0) = 0$ and thus $f(z) \in Z^n(\mathcal{B})$. Let $b \in B^n(\mathcal{A})$ with $db' = b$. Then $f(b) = f(db) = d'f(b)$ and thus $f(b) \in B^n(\mathcal{B})$. □

Also note that the category of differential graded algebras is a subcategory of the category of cochain complexes. By a cochain complex we mean

Definition 1.2.7. A cochain complex is a graded vector space C with a linear map $d : C \to C$ of degree 1 such that $d^2 = 0$.

Definition 1.2.8. Let (C, d) and (C', d') be cochain complexes. A cochain map is a linear map $f : C \to C'$ of degree 0 (between graded vector spaces) such that

$$f \circ d = d' \circ f \; .$$

Analogues to the definition of the tensor product of differential graded algebras as described in Section 1.1, we can define the tensor product $C \otimes D$ of cochain complexes C and D and we can compute its cohomology with

Theorem 1.2.9 (Künneth theorem)**.** *Let C and D be cochain complexes (over a field). Then*

$$H^n(C \otimes D) = \bigoplus_{k=0}^{n} H^k(C) \otimes H^{n-k}(D) \; .$$

Moreover, this isomorphism is natural, i.e. the diagram

$$\begin{array}{ccc}
H^n(C \otimes D) & \longrightarrow & \bigoplus_{k=1}^{n} H^k(C) \otimes H^{n-k}(D) \\
\big\downarrow {\scriptstyle H^n(f \otimes g)} & & \big\downarrow {\scriptstyle \bigoplus_{k=0}^{n} H^k(f) \otimes H^{n-k}(g)} \\
H^n(C' \otimes D') & \longrightarrow & \bigoplus_{k=1}^{n} H^k(C') \otimes H^{n-k}(D') \; .
\end{array}$$

commutes for all chain maps $f : C \to C'$ and $g : D \to D'$.

Proof. For a proof see [16] Section 17.2 and note that we work over fields. □

Definition 1.2.10. Let $\mathcal{A}$ be a differential graded algebra. Then the differential graded algebra $\mathcal{A}[-1]$ is defined via

$$\mathcal{A}[-1]^n := \mathcal{A}^{n-1}$$
$$d_{\mathcal{A}[-1]} := d_{\mathcal{A}}$$

Definition 1.2.11 (Mapping cone)**.** Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of differential graded algebras (or more generally of cochain complexes). Then the chain complex $\mathcal{C}\text{one}(f)$ is defined by

$$\mathcal{C}\text{one}(f)^n := \mathcal{A}^n \oplus \mathcal{B}^{n-1}$$

with addition component wise and differential

$$d_{\mathcal{C}\text{one}(f)}(a, b) := (d_{\mathcal{A}}(a), -d_{\mathcal{B}}(b) + f(a)) \; . \tag{1.1}$$

Digression on the mapping cone

The definition of the mapping cone seems to be quite artificial. But it arises naturally by studying the topology of basic geometric constructions. Let X and Y be topological spaces and

$$f : X \to Y$$

a continuous map. For topological spaces

$$\text{Cone}(X) := X \times I/X \times \{0\}$$

is called the cone of X and

$$\mathrm{Cone}(f) := \mathrm{Cone}(X)\,\dot{\cup}\, Y/(x,1) \sim f(x)$$

the mapping cone of f. Here $I := [0,1]$ denotes the unit interval. So the mapping cone arises by gluing the bottom of Cone(X) onto Y along f, illustrated in the following picture: Now let

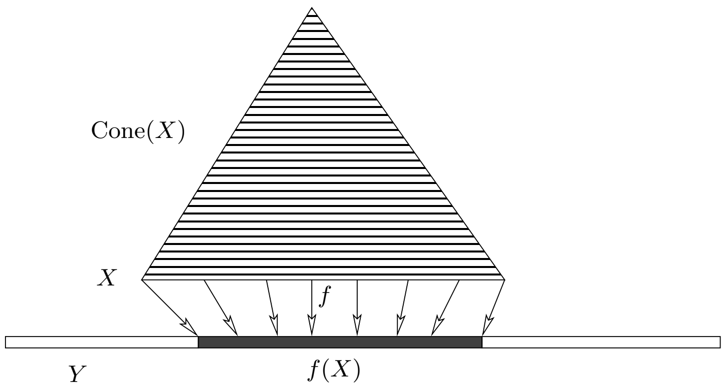


Figure 1.1: Gluing Cone(X) on Y along f

$$C^* : \{\text{topological spaces}\} \to \{\text{cochain complexes}\}$$

be the singular cochain functor. It can be shown that the cochain complexes

$$C^*(\mathrm{Cone}(f)) \cong \mathfrak{Cone}(C^*(f))$$

are chain homotopic, which makes the motivation perfect. For a discussion see [23] 1.5.

With the help of the mapping cone we can proof the existence of a long exact cohomology sequence, namely:

Proposition 1.2.12 (Mayer-Vietoris sequence)**.** *Let $f : \mathcal{S} \to \mathcal{A}$ be a homomorphism of differential graded algebras. Then there exists a long exact sequence*

$$\cdots \longrightarrow H^{n+1}(\mathfrak{Cone}(f)) \xrightarrow{H^{n+1}(\pi_1)} H^{n+1}(\mathcal{S}) \xrightarrow{H^{n+1}(f)} H^{n+1}(\mathcal{A})$$

$$\xrightarrow{H^{n+2}(-i_2)} H^{n+2}(\mathfrak{Cone}(f)) \longrightarrow \cdots ,$$

where $\pi_1 : \mathcal{C}\mathrm{one}(f) := \mathcal{S} \oplus \mathcal{A}[-1] \to \mathcal{S}$ *is the projection onto the first factor and* $-i_2 : \mathcal{A}[-1] \to \mathcal{C}\mathrm{one}(f) := \mathcal{A} \oplus \mathcal{A}[-1]$ *the inclusion. It is called Mayer-Vietoris Sequence of the mapping cone.*

Proof. The short exact sequence of chain complexes

$$1 \longrightarrow \mathcal{A}[-1] \xrightarrow{-i_2} \mathcal{C}\mathrm{one}(f) \xrightarrow{\pi_1} \mathcal{S} \longrightarrow 1$$

induces the long exact cohomology sequence

$$\cdots \longrightarrow H^{n+1}(\mathcal{C}\mathrm{one}(f)) \xrightarrow{H^{n+1}(\pi_1)} H^{n+1}(\mathcal{S}) \xrightarrow{\delta^{n+2}} H^{n+2}(\mathcal{A}[-1])$$

$$\xrightarrow{H^{n+2}(i_2)} H^{n+2}(\mathcal{C}\mathrm{one}(f)) \longrightarrow \cdots ,$$

where $\delta^{n+2} : H^{n+1}(\mathcal{S}) \to H^{n+2}(\mathcal{A}[-1])$ is the so-called connecting homomorphism. Since $H^{n+2}(\mathcal{A}[-1]) = H^{n+1}(\mathcal{A})$, the claim follows with the observation, that $\delta^{n+2} = H^{n+1}(f)$. For details see [23] 1.5. □

1.3 Hirsch extensions and Hirsch automorphisms

Given a differential graded algebra, a vector space and a linear map, one can construct a new differential graded algebra which includes the old one as a differential graded subalgebra. This new differential graded algebra is called a Hirsch extension. Moreover, if the starting algebra is free and the vector space is finite dimensional, then the Hirsch extension is free.

A homomorphism of Hirsch extensions is then defined to be a homomorphism of differential graded algebras which restrict to a homomorphism of the initial involved differential graded algebras. All these concepts will be introduced now.

Let $(\mathcal{A}, d)$ be a differential graded algebra as defined in 1.1.18, V a vector space of homogeneous degree k as defined in 1.1.1 and

$$\sigma : V \to Z^{k+1}(\mathcal{A})$$

a linear map, where $Z^{k+1}(\mathcal{A})$ are the cocycles of degree $k+1$ of $\mathcal{A}$ as defined in 1.2.1. Furthermore, let $\bigwedge V$ be the free commutative graded algebra over V as defined in 1.1.13 and $\mathcal{A} \otimes \bigwedge V$ the tensor product of graded algebras as defined in 1.1.10.

Now define a differential d_σ on $\mathcal{A} \otimes \bigwedge V$ as follows:

1. For elements $a \in \mathcal{A}$ and $v \in V$ define $d_\sigma(a \otimes 1) = da \otimes 1$ and $d_\sigma(1 \otimes v) = \sigma(v) \otimes 1$.
2. Extend d_σ via the Leibniz rule

$$\begin{aligned} d_\sigma((x_1 \otimes y_1)(x_2 \otimes y_2)) :=& (d_\sigma(x_1 \otimes y_1))(x_2 \otimes y_2) \\ &+ (-1)^{\deg(x_1 \otimes y_1)}(x_1 \otimes y_1)(d_\sigma(x_2 \otimes y_2)) \ . \end{aligned}$$

   In particular this yields

$$\begin{aligned} d_\sigma((a \otimes v) &= d_\sigma((a \otimes 1)(1 \otimes v)) \\ &= (d_\sigma(a \otimes 1))(1 \otimes v) + (-1)^{\deg(a \otimes 1)}(a \otimes 1)(d_\sigma(1 \otimes v)) \\ &= (da \otimes 1)(1 \otimes v) + (-1)^{\deg(a \otimes 1)}(a \otimes 1)(\sigma(v) \otimes 1)) \\ &= da \otimes v + (-1)^{\deg a}\sigma(v)a \otimes 1 \ . \end{aligned}$$

Definition 1.3.1. Let $(\mathcal{A}, d)$ be a differential graded algebra, V be a vector space of homogeneous degree k and $\sigma : V \to Z^{k+1}(\mathcal{A})$ a linear map. Then the differential graded algebra $(\mathcal{A} \otimes \bigwedge V, d_\sigma)$ as constructed above is called a Hirsch extension of $\mathcal{A}$ of degree k and is denoted by

$$\mathcal{A} \otimes_{d_\sigma} \bigwedge V \ .$$

Lemma 1.3.2. *Let $\mathcal{A} \otimes_{d_\sigma} \bigwedge V$ be a Hirsch extension (of arbitrary degree). Then the following holds.*

i) $\iota : (\mathcal{A}, d) \to \mathcal{A} \otimes_{d_\sigma} \bigwedge V$, $\iota(a) = a \otimes 1$ is a homomorphism of differential graded algebras.

ii) If $\mathcal{A}$ is free and V is finite dimensional, then $(\mathcal{A} \otimes \bigwedge V, d_\sigma)$ is free.

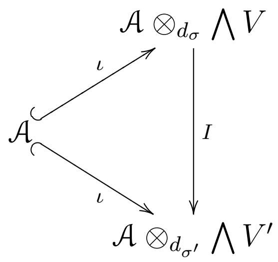
Proof. i) $d_\sigma(a \otimes 1) = da \otimes 1$ by definition.

ii) This directly follows from Lemma 1.1.14 iii). □

Definition 1.3.3. Two Hirsch extensions $\mathcal{A} \otimes_{d_\sigma} \bigwedge V$ and $\mathcal{A} \otimes_{d_{\sigma'}} \bigwedge V'_k$ are called equivalent, if there exists an isomorphism

$$I : \mathcal{A} \otimes_{d_\sigma} \bigwedge V \to \mathcal{A} \otimes_{d_{\sigma'}} \bigwedge V$$

of differential graded algebras, such that the diagram



commutes.

Definition 1.3.4. Let $(\mathcal{S}', d)$ be a differential graded algebra and

$$\mathcal{S} = \mathcal{S}' \otimes_{d_\sigma} \bigwedge V$$

a Hirsch extension. An automorphism $f \in \mathrm{Aut}(\mathcal{S})$ is called a Hirsch automorphism, if $f_{|\mathcal{S}'} \in \mathrm{Aut}(\mathcal{S}')$.

Thus for a Hirsch automorphism f we get a commutative diagram of differential graded algebras

$$\begin{array}{ccc} \mathcal{S}' & \xrightarrow{\ \iota\ } & \mathcal{S}' \otimes_{d_\sigma} \bigwedge V \\ \downarrow{\scriptstyle f_{|\mathcal{S}'}} & & \downarrow{\scriptstyle f} \\ \mathcal{S}' & \xrightarrow{\ \iota\ } & \mathcal{S}' \otimes_{d_\sigma} V \ . \end{array}$$

Conversely an automorphism of graded algebras with the latter property is also an automorphism of $\mathcal{S}$. Also note that a Hirsch equivalence is given by the special case $f_{|\mathcal{S}'} = id$.

1.4 Sullivan algebras

A Sullivan algebra is a differential graded algebra which is inductively build out of Hirsch extensions as defined in 1.3.1. One might think of it as a pendant to a sequence of fibrations of topological spaces. For a discussion see for example [8] Chapter XI. After introducing the notion of homotopy between homomorphisms of differential graded algebras in the next section, it turns out that Sullivan algebras posses special homotopical and cohomological properties. These will be demonstrated in Section 1.5.

Definition 1.4.1. Let $\mathcal{A}$ be a differential graded algebra. It is called minimal, if it is free, connected and $d\mathcal{A}^+ \subset \mathcal{A}^+ \cdot \mathcal{A}^+$, where $\mathcal{A}^+ := \bigcup_{n>0} \mathcal{A}^n$ are the elements of positive degree and $\mathcal{A}^+ \cdot \mathcal{A}^+ := \langle ab \mid a \in \mathcal{A}^+, b \in \mathcal{A}^+ \rangle$ are the decomposable elements.

Recall that being connected was defined in 1.2.4 and being free in 1.1.20.

Definition 1.4.2. A differential graded algebra $\mathcal{A}$ is called generalised nilpotent, if there exists a sequence of differential graded sub algebras

$$\mathbb{K} =: \mathcal{A}_0 \subset \cdots \subset \mathcal{A}_\alpha \subset \mathcal{A}_{\alpha+1} \subset \cdots \ ; \alpha \in \mathbb{N} \ ,$$

such that

i) $\mathcal{A} = \bigcup_{\alpha \in \mathbb{N}} \mathcal{A}_\alpha$,

ii) $\mathcal{A}_{\alpha+1}$ is a Hirsch extension of $\mathcal{A}_\alpha$ for all α, that is $\mathcal{A}_{\alpha+1} = \mathcal{A}_\alpha \otimes_{d_\sigma} \bigwedge V_\alpha$, where V_α is the homogeneous vector space of new generators.

If in addition the property

iii) for all m there exists α such that $\mathcal{A}^m \subset \mathcal{A}_\alpha$ holds, then $\mathcal{A}$ is called nilpotent. To recall the notations, this means that all elements of degree m, denoted by $\mathcal{A}^m$, are exhausted by some sub algebra $\mathcal{A}_\alpha$ of the increasing sequence of Hirsch extensions.

iv) We also call a sequence satisfying i) and ii) a Hirsch filtration of $\mathcal{A}$ and an automorphism is said to respect the Hirsch filtration, if it is an Hirsch automorphism for all Hirsch extensions. More general, if we replace the index set $\mathbb{N}$ of the sequence by any well ordered index set, we still call it a Hirsch filtration of $\mathcal{A}$.

We are now ready to define the central object of this text.

Definition 1.4.3. A Sullivan algebra is a minimal, generalised nilpotent differential graded algebra. If it is nilpotent and not only generalised nilpotent, we call it nilpotent Sullivan algebra to prevent any confusion. If it is nilpotent and finitely generated we call it a Sullivan algebra of finite type.

Remark 1.4.4. Some authors also drop the condition of being minimal in the definition of a Sullivan algebra, as for example [5] Section 12. They explicitly distinguish between minimal Sullivan algebras and Sullivan algebras.

A special case of a Sullivan algebra is the homological model of a connected differential graded algebra introduced in the next section. Also the Koszul complex of a nilpotent Lie algebra is naturally a Sullivan algebra, which will be discussed in more detail in Section 3.2.

1.5 Some aspects of a model structure

We now list and discuss some properties of the category of differential graded algebras. There are two approaches to prove them. A quite abstract one is as follows. Daniel Quillen introduced in [18] a purely categorial framework for homotopy theory. He defined so-called closed model categories and proved, that whenever a category is a model category, most properties we list here are actually true. So one only has to show that the category of differential graded algebras is indeed a closed model category, which is for example proved in [7]. The other approach is a direct one using the language of differential graded algebras only. Proofs of that kind can be found in [8], [20] and [17]. However, since the author is not well educated in abstract homotopical algebra, we outline some arguments of the second approach here.

For differential graded algebras there exists the notion of a homotopy between homomorphisms of differential graded algebras. To be more precise in some cases there are two definitions. One is analogous to the definition of a homotopy $H : X \times I \to Y$ of topological spaces. For the source a Sullivan algebra, there exists a second definition, analogous to the definition of a homotopy $H : X \to Y^I$ as a continuous map into the mapping space $Y^I := \{f : [0,1] \to Y \mid f \text{ continuous}\}$. For topological spaces the mentioned definitions coincide under some restriction on the space, as for example for CW-complexes. For a discussion see [21] Chapter 0. The situation in the case of differential graded algebras is similar. The two definitions coincide for Sullivan algebras. For convenience this fact follows just by abstract arguments as described at the beginning of this section. Throughout this thesis we use the first definition. Let

$$\langle t, dt\rangle := \langle t\rangle \oplus \langle dt\rangle$$

be the graded vector space generated by t with degree $\deg(t) = 0$ and dt with degree $\deg(dt) = 1$. Then Define

$$(t, dt) := \big(\bigwedge\langle t, dt\rangle, d\big)$$

to be the differential graded algebra with differential d defined on generators via $d(t) = dt, d(dt) = 0$ and usual extension to a derivation of degree 1 onto the whole algebra. This defines a differential, since all elements of $\bigwedge\langle t, dt\rangle$ with degree greater then 1 are zero. For simplicity reasons, we denote the multiplication in (t, dt) by plain multiplication instead of $\wedge$. For $\mathcal{B}$ a differential graded algebra, $\mathcal{B} \otimes (t, dt)$ denotes the canonical tensor product of differential graded algebras as described in Definition 1.1.23. Moreover, let

$$\epsilon_0, \epsilon_1 : (t, dt) \to \mathbb{K}$$

be the augmentation homomorphisms of differential graded algebras defined by

$$\begin{aligned} &\epsilon_0(t) = 0,\ \epsilon_0(dt) = 0 \ \text{ and} \\ &\epsilon_1(t) = 1, \epsilon_1(dt) = 0 \end{aligned}$$

on generators and natural extension. Here $\mathbb{K}$ is considered as trivial differential graded algebra which implies that this map is a homomorphism of differential graded algebras.

Definition 1.5.1. Let $f, g : \mathcal{A} \to \mathcal{B}$ be two homomorphisms of differential graded algebras. f and g are called right homotopic, if there exists a homomorphism of differential graded algebras

$$H : \mathcal{A} \to \mathcal{B} \otimes (t, dt)$$

such that $(\operatorname{id} \otimes \epsilon_0) \circ H = f$ and $(\operatorname{id} \otimes \epsilon_1) \circ H = g$ (To be precise, $\mathcal{B} \otimes \mathbb{K}$ is canonically identified with $\mathcal{B}$). Right homotopic maps are denoted by $f \sim g$.

We will also use the following notations which are adapted from the usual definition of a homotopy between continuous functions on spaces:

$$\begin{aligned} H_{|t=0} &:= (\operatorname{id} \otimes \epsilon_0) \circ H = f, \\ H_{|t=1} &:= (\operatorname{id} \otimes \epsilon_1) \circ H = g \end{aligned}$$

for a homotopy H between two homomorphism of differential graded algebras

$$f, g : \mathcal{A} \to \mathcal{B}\ .$$

Similar denote

$$\begin{aligned} \beta_{|t=0} &:= (\operatorname{id} \otimes \epsilon_0)(\beta), \\ \beta_{|t=1} &:= (\operatorname{id} \otimes \epsilon_1)(\beta) \end{aligned}$$

for $\beta \in \mathcal{B} \otimes (t, dt)$.

Lemma 1.5.2. *One can compose homotopies.*

Proof. Let

$$f, g : \mathcal{A} \to \mathcal{B} \text{ and } f', g' : \mathcal{B} \to \mathcal{C}$$

be homomorphisms of differential graded algebras and

$$H : \mathcal{A} \to \mathcal{B} \otimes (t, dt), \quad H' : \mathcal{B} \to \mathcal{C} \otimes (t', dt')$$

homotopies from f to g and f' to g' respectively. Then define a homotopy

$$(H' \circ H) : \mathcal{A} \to \mathcal{C} \otimes (t, dt)$$

from $f \circ f'$ to $g \circ g'$ in the following way.

First note that

$$(H' \otimes \mathrm{id}) \circ H : \mathcal{A} \to \mathcal{C} \otimes \big((t, dt) \otimes (t', dt'), d_{(t,dt)\otimes(t',dt')}\big) = \mathcal{C} \otimes \big(\bigwedge \langle t, dt, t', dt' \rangle, d \oplus d\big)$$

defines a homomorphism of differential graded algebras. Now

$$\big(t + t', (d \oplus d)(t + t')\big) \subset \big(\bigwedge \langle t, dt, t', dt' \rangle, d \oplus d\big)$$

is a differential graded sub algebra and define a homomorphism of graded algebras

$$\pi : \big(\bigwedge \langle t, dt, t', dt' \rangle, d \oplus d\big) \to \big(t + t', (d \oplus d)(t + t')\big)$$

on generators by

$$\begin{aligned} &\pi(t) = t + t', \quad \pi(t') = t + t' \\ &\pi(dt) = dt + dt' = (d \oplus d)(t + t'), \\ &\pi(dt') = dt + dt' = (d \oplus d)(t + t') \ . \end{aligned}$$

Since π commutes obviously with the differential on generators, it is also a homomorphism of differential graded algebras by Lemma 1.1.22. Finally

$$H' \circ H \ := \pi \circ (H' \otimes \mathrm{id}) \circ H$$

is a homotopy from $f \circ f'$ to $g \circ g'$. □

Lemma 1.5.3. *Homotopic maps induce the same map on cohomology.*

Proof. Let $f, g : \mathcal{A} \to \mathcal{B}$ be homomorphisms of differential graded algebras and $H : \mathcal{A} \to \mathcal{B} \otimes (t, dt)$ a homotopy between f and g. Since differential graded algebras are defined over a field, the corresponding chain complex is free. Moreover, note that $H^*((t, dt)) = H^0((t, dt)) = \mathbb{K}$. Thus the Künneth Theorem 1.2.9 implies that the diagram

$$\begin{array}{ccc} H^n(\mathcal{B}) \otimes \mathbb{K} & \xrightarrow{\cong} & H^n(\mathcal{B} \otimes (t, dt)) \\ \Big\downarrow{\scriptstyle \mathrm{id} \otimes H^0(\epsilon_i)} & & \Big\downarrow{\scriptstyle H^n(\mathrm{id} \otimes \epsilon_i)} \\ H^n(\mathcal{B}) \otimes \mathbb{K} & \xrightarrow{\cong} & H^n(\mathcal{B} \otimes \mathbb{K}) \end{array}$$

commutes for $i \in \{0, 1\}$. Since $H^0(\epsilon_i) = (\epsilon_i)_{|\mathbb{K}} = \mathrm{id}$, we conclude that $H^*(f) = H^*(\mathrm{id} \otimes \epsilon_0 \circ H) = H^*(\mathrm{id} \otimes \epsilon_1 \circ H) = H^*(g)$. For an alternative proof see [5], Proposition 12.8 (i). Among differential graded algebras, Sullivan algebras admit some special properties, which are presented here. □

We now discuss lifting of maps and homotopies along Hirsch extension. The main result is the Hirsch Lemma, which yields an obstruction for this lifting problem in terms of cohomology.

Let $\mathcal{A} \otimes_{d_\sigma} \bigwedge V$ be a Hirsch extension of degree n of $\mathcal{A}$ and

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{g} & \mathcal{B} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle \varphi} \\ \mathcal{A} \otimes_{d_\sigma} \bigwedge V & \xrightarrow{f} & \mathcal{C} \end{array} \tag{1.2}$$

a diagram of homomorphisms of differential graded algebras, such that there exists a homotopy $H : \mathcal{A} \to \mathcal{C} \otimes (t, dt)$ from $\varphi \circ g$ to $f_{|\mathcal{A}}$. For all $v \in V$ define $\sigma'(v) \in \mathcal{C}\mathrm{one}(\varphi)^{k+1} := \mathcal{B}^{k+1} \oplus \mathcal{C}^k$ by

$$\sigma'(v) := (g(dv), f(v) + \int_0^1 H(dv)),$$

where

$$\int_0^1 : \mathcal{B} \otimes (t, dt) \to \mathcal{B}$$

is defined by $\int_0^1 b \otimes t^i = 0$ and $\int_0^1 b \otimes t^i \cdot dt = (-1)^{\deg(b)} \frac{b}{i+1}$. It is easy to see, that actually $\sigma'(v) \in Z^{k+1}(\mathcal{C}\mathrm{one}(\varphi))$ for all $v \in V$. With that we can go on to define

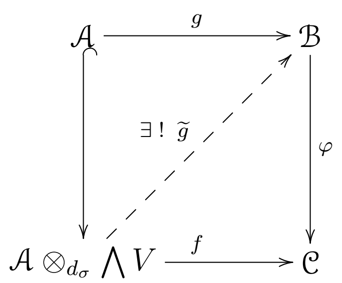
$$\sigma : V \to H^{k+1}(\mathfrak{Cone}(\varphi))$$
$$\sigma(v) := [\sigma'(v)] \; .$$

The map σ is called obstruction class for the lifting problem 1.2. The name is justified by the following

Proposition 1.5.4 (Hirsch lemma). *The obstruction class*

$$\sigma \in \mathrm{Hom}\big(V, H^{k+1}(\mathfrak{Cone}(\varphi))\big)$$

vanishes, if and only if there exists a lifting



together with a homotopy $\widetilde{H}$ from $\varphi \circ \widetilde{g}$ to f, such that $\widetilde{H}_{|\mathcal{A}} = H$. Moreover, if H is constant, that is $g \circ \varphi = f_{|\mathcal{S}}$, and φ is surjective, then there exists a lifting $\widetilde{f}'$, such that $\varphi \circ \widetilde{f}' = f$. If φ is surjective, then one can even choose $\widetilde{g}$ such that $\widetilde{H}$ is constant.

Definition 1.5.5. Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of differential graded algebras. Then f is called a quasi isomorphism, if $H^*(f) : H^*(\mathcal{A}) \to H^*(\mathcal{B})$ is an isomorphism.

Proposition 1.5.6 (Whitehead Lemma). *Let $f : \mathcal{S} \to \mathcal{S}'$ be a quasi isomorphism of Sullivan algebras, that is $H^*(f) : H^*(\mathcal{S}) \to H^*(\mathcal{S}')$ is an isomorphism. Then f is an isomorphism.*

There exists also a refinement of this fact, namely:

Proposition 1.5.7. *Let $f : \mathcal{S} \to \mathcal{S}'$ be a homomorphism between Sullivan algebras. Assume that $\mathcal{S}'$ is generated by elements of degree smaller or equal to n and that $H^k(f)$ is an isomorphism for $k \leq n$ and injective for $k = n+1$. Then f is an isomorphism.*

The proofs are as usual by induction. Since it is well covered in the literature, as for example [17], Proposition 1.46, we do not repeat a proof here.

Lemma 1.5.8. *If the source algebra is a Sullivan algebra, then being homotopic defines an equivalence relation.*

Proof. We prove consecutive all necessary properties. Proofs for definition I of a homotopy can also be found in [8], p. 125. or [5], Proposition 12.7. If one takes definition II of a homotopy, a proof can be found in [17], Proposition 1.45, or [20], Corollary 3.4. From here let $\mathcal{S}$ be a Sullivan algebra and $\mathcal{A}$ be arbitrary.
Reflexivity: Let $f : \mathcal{S} \to \mathcal{A}$ be a homomorphism of differential graded algebras and

$$\begin{aligned} \iota : \mathcal{S} &\to \mathcal{S} \otimes (t, dt) \\ \iota(s) &= s \otimes 1 \end{aligned}$$

the natural inclusion. Obviously $(f \otimes \mathrm{id}) \circ \iota : \mathcal{S} \to \mathcal{A} \otimes (t, dt)$ defines a homomorphism of differential graded algebras with $\mathrm{id} \otimes \epsilon_0 \circ (f \otimes \mathrm{id}) \circ \iota = \mathrm{id} \otimes \epsilon_1 \circ (f \otimes \mathrm{id}) \circ \iota = f$.
Symmetry: Let $H : \mathcal{S} \to \mathcal{A} \otimes (t, dt)$ be a homotopy from f to g. Now

$$i : (t, dt) \to (1 - t, d(1 - t))$$

defined on generators by $i(t) = 1 - t$ defines a homomorphism of differential graded algebras and hence $(\mathrm{id} \otimes i) \circ H$ defines a homotopy from g to f.

Transitivity: For a proof see [8], p. 125. □

Definition 1.5.9. Let S be a Sullivan algebra and $\mathcal{A}$ be an arbitrary differential graded algebra. Then

$$[\mathcal{S}, \mathcal{A}] := \big\{ f : \mathcal{A} \to \mathcal{A} \mid ; f \text{ is a homomorphism of differential graded algebras} \big\} / \sim$$

denotes the set of homomorphisms modulo homotopy.

Proposition 1.5.10. *Let S be a Sullivan algebra and $\phi : \mathcal{A} \to \mathcal{B}$ be a homomorphism of differential graded algebras. If $H^*(\phi)$ is an isomorphism, then the map*

$$\begin{aligned} \phi_* : [\mathcal{S}, \mathcal{A}] &\to [\mathcal{S}, \mathcal{B}] \\ \phi_*(f) &:= f \circ \phi \end{aligned}$$

is an isomorphism.

Proof. See [8] Theorem 10.8 or [17] Theorem 1.47. □

1.6 Minimal models

A minimal model is a way to describe a given differential graded algebra by a Sullivan algebra, similar as the Postnikov tower describes a topological space. For a discussion see for example [8] Chapter XI. It is unique only up to homotopy and there is a specific candidate, which is constructed out of cohomological data of the given algebra and which admits a canonical Hirsch filtration. We will call it the homological model. It is also not unique, but unique up to an isomorphism which respects the canonical Hirsch filtration. Our main references are [8] IX and XII, [17] 1.2, [5] and [7]. The algorithm for the explicit construction of the homological model in Section 1.6.1 is taken out of [7].

Recall that a differential graded $\mathbb{K}$-algebra $(\mathcal{A}, d)$ is called connected, if $\mathcal{A}^0 = \mathbb{K}$. Compare Definition 1.2.4.

Definition 1.6.1. Let $\mathcal{A}$ be a connected differential graded algebra. A n-minimal model for $\mathcal{A}$ is a Sullivan algebra $\mathcal{M}_n$ as defined in 1.4.3, together with a homomorphism of differential graded algebras

$$p_n : \mathcal{M}_n \to \mathcal{A} \ ,$$

such that

$$H^k(p_n) : H^k(\mathcal{M}_n) \to H^k(\mathcal{A})$$

is bijective for all $k \leq n$ and injective for $k = n + 1$. If bijectivity holds for all k, then it is just called a minimal model and is denoted by

$$p : \mathcal{M} \to \mathcal{A} \ .$$

Lemma 1.6.2. *If $\mathcal{S}$ is a Sullivan algebra generated by elements of degree smaller or equal to n, then*

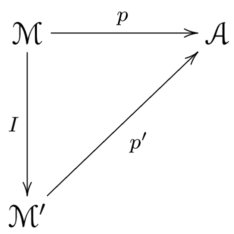
$$p_n : \mathcal{M}_n \to \mathcal{S}$$

is actually an isomorphism of differential graded algebras. In particular this means that in this situation the n-minimal model is the minimal model.

Proof. The claim is a direct consequence of the Whitehead Lemma 1.5.7. □

The existence of a minimal model will be proved in the next section and we first want to say some words about its uniqueness, namely:

Proposition 1.6.3. *Let $p : \mathcal{M} \to \mathcal{A}$ and $p' : \mathcal{M}' \to \mathcal{A}$ be minimal models of $\mathcal{A}$. Then there exists an isomorphism $I : \mathcal{M} \to \mathcal{M}'$ such that $p' \circ I \sim p$, that is, the diagram*



commutes up to homotopy. Moreover, the map I is unique up to homotopy.

Proof. According to Theorem 1.5.10 p' induces an isomorphism

$$p'_* : [\mathcal{M}, \mathcal{M}'] \to [\mathcal{M}, \mathcal{A}] ,$$

that is there exists $I \in [\mathcal{M}, \mathcal{M}']$ such that $I \circ p' \sim p$. Since $H^*(p)$ and $H^*(p')$ are isomorphisms, so is $H^*(I)$. Thus I is an isomorphism of differential graded algebras by the Whitehead Lemma 1.5.6. □

1.6.1 Existence and the homological model

Proposition 1.6.4. *Let $\mathcal{A}$ be a connected differential graded algebra. Then there exists a minimal model $p : \mathcal{M} \to \mathcal{A}$.*

Definition 1.6.5. We call the minimal model constructed via the following algorithm the homological model and the corresponding constructed Hirsch filtration the canonical Hirsch filtration.

Proof of Proposition 1.6.4. The proof is by induction.
Base: Start with

$$\mathcal{M}_0 := \mathbb{K}$$

with trivial differential together with

$$p_0 : \mathbb{K} \to \mathcal{A}$$

the natural inclusion. Clearly this is a 0-minimal model, since $H^0(\mathcal{A}) = \mathbb{K}$ by assumption and $H^1(\mathbb{K}) = 0$, which implies that $H^0(p_0)$ is bijective and that $H^1(p_0)$ is injective.

Induction step "killing the cokernel":

Suppose an n-minimal model $p_n : \mathcal{M}_n \to \mathcal{A}$ has already been constructed.

Start with $\mathcal{M}_{n,0} := \mathcal{M}_n$ and $p_{n,0} = p_n$ respectively. In particular $H^{n+1}(p_{n,0})$ is injective. Now recall the definition of

$$\operatorname{coker}(H^{n+1}(p_{n,0})) := H^{n+1}(\mathcal{A}) / \operatorname{im}(H^{n+1}(p_{n,0}))$$

and choose a sub vector space $C \subset Z^{n+1}(\mathcal{A})$, that is mapped isomorphically onto $\operatorname{coker}(H^{n+1}(p_{n,0}))$ under the compositions of maps

$$Z^{n+1}(\mathcal{A}) \to H^{n+1}(\mathcal{A}) \to H^{n+1}(\mathcal{A}) / \operatorname{im}(H^{n+1}(p_{n,0}))$$

(this is possible, since short exact sequences of vector spaces always split). Then define

$$\mathcal{M}_{n.1} := \mathcal{M}_{n,0} \otimes_{d_0} \bigwedge C ,$$

where $0 : C \to \mathcal{M}_{n,0}$ is the zero map and C is of degree $n+1$. Accordingly define $p_{n,1} := p_{n,0} \otimes id$, which obviously yields a map of differential graded algebras $p_{n,1} : \mathcal{M}_{n,1} \to \mathcal{A}$ by natural extension. Moreover, $H^{n+1}(\mathcal{M}_{n,1}) = H^{n+1}(\mathcal{A})$ and $H^{n+1}(p_{n,1})$ is bijective by construction. But $H^{n+2}(p_{n,1})$ need not be injective. To correct this error, one constructs now inductively $p_{n,k+1} : \mathcal{M}_{n,k+1} \to \mathcal{A}$ by the following algorithm, called

Induction step "killing the kernel":

Suppose $p_{n,k} : \mathcal{M}_{n,k} \to \mathcal{A}$ has already been constructed, such that $H^i(p_{n,k})$ is bijective for all $i \leq n+1$. This gives rise to the Mayer-Vietoris sequence

$$\begin{array}{c} \qquad\qquad\qquad\qquad\qquad\qquad\qquad V_{p_{n,k}} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \cdots H^{n+1}(\mathcal{M}_{n,k}) \xrightarrow{H^{n+1}(p_{n,k})} H^{n+1}(\mathcal{A}) \xrightarrow{H^{n+1}(-i_2)} H^{n+2}(\mathfrak{Cone}(p_{n,k})) \\ \xrightarrow{H^{n+2}(\pi_1)} H^{n+1}(\mathcal{M}_{n,k}) \xrightarrow{H^{n+2}(p_{n,k})} H^{n+2}(\mathcal{A}) \cdots , \end{array} \tag{1.3}$$

where $\pi_1 : \mathcal{M}_{n,k} \oplus \mathcal{A}[-1] \longrightarrow \mathcal{M}_{n,k}$ denotes the canonical projection and $i_2 : \mathcal{A}[-1] \longrightarrow \mathcal{M}_{n,k} \oplus \mathcal{A}[-1]$ the canonical inclusion. For details see Proposition 1.2.12. Define

$$V_{p_{n,k}} := H^{n+2}(\operatorname{Cone}(p_{n,k}))$$

and choose a section

$$s_{n,k} : H^{n+2}(\operatorname{Cone}(p_{n,k})) \to Z^{n+2}(\operatorname{Cone}(p_{n,k})) .$$

(As mentioned before, this is always possible, since a short exact sequence of vector spaces always splits). By Definition 1.2.11 of the mapping cone, this section is of the form $s_{n,k}([v]) = (m_v, a_v)$ with $m_v \in Z^{n+2}(\mathcal{M}_{n,k})$ and $a_v \in \mathcal{A}[-1]^{n+2} = \mathcal{A}^{n+1}$. Thus we can define maps

$$\begin{aligned} &\sigma_{s_{n,k}} : V_{p_{n,k}} \to Z^{n+2}(\mathcal{M}_{n,k}),\\ &\sigma_{s_{n,k}}([v]) := (\pi_1 \circ s_{n,k})([v]) \end{aligned}$$

and

$$\begin{aligned} &p^{s_{n,k}} : V_{p_{n,k}} \to \mathcal{A}^{n+1},\\ &p^{s_{n,k}}([v]) := (\pi_2 \circ s_{n,k})([v]) \end{aligned}$$

which yields a Hirsch extension

$$\mathcal{M}^{s_{n,k}}_{n,k+1} := \mathcal{M}_{n,k} \otimes_{d_{\sigma_{s_{n,k}}}} \bigwedge V_{p_{n,k}},$$

where $V_{p_{n,k}}$ is of degree $n+1$, together with a homomorphism of differential graded algebras

$$p^{s_k}_{n,k+1} := p_{n,k} \otimes p^{s_{n,k}} : \mathcal{M}^{s_{n,k}}_{n,k+1} \to \mathcal{A}$$

by natural extension. To prove that $p^{s_{n,k}}_{n,k+1}$ is indeed a homomorphism of differential graded algebras, it is enough to show $p^{s_{n,k}}_{n,k+1}(d_{\mathcal{M}}(v)) := p_{n,k}(m_v) = d_{\mathcal{A}}(\ p^{s_{n,k}}_{n,k+1}(v)) := d_{\mathcal{A}}(a_v)$ on the new generators, since $(p^{s_k}_{n,k+1})_{|\mathcal{M}_{n,k}} = p_{n,k}$. So let $v = (m_v, a_v) \in Z^{n+2}(\mathcal{C}\mathrm{one}(p_{n,k}))$. We have $d_{Z^{n+2}(\mathcal{C}\mathrm{one}(p_{n,k}))} v = (dm_v, m_{1,k}(m_v) - d_{\mathcal{A}} a_v) = 0$ and hence $p_{n,k}(m_v) = d_{\mathcal{A}} a_v$.

By exactness of the sequence and since $H^{n+1}(p_{n,k})$ is bijective,

$$V_{p_{n,k}} = \ker(H^{n+2}(p_{n,k}))$$

and we get a commutative diagram

$$\begin{array}{ccccc} & & H^{n+2}(\mathcal{M}_{n,k+1}) & & \\ & & \uparrow{\scriptstyle H^{n+2}(\iota)} & \searrow{\scriptstyle H^{n+2}(p_{n,k+1})} & \\ V_{p_{n,k}} & \longrightarrow & H^{n+2}(\mathcal{M}_{n,k}) & \xrightarrow{H^{n+2}(p_{n,k})} & H^{n+2}(\mathcal{A}) \end{array} , \tag{1.4}$$

where $\iota : \mathcal{M}_{n,k} \to \mathcal{M}_{n,k+1}$ denotes the canonical inclusion. So, by constructing $\mathcal{M}_{n,k+1}$, we have killed the kernel of $H^{n+2}(p_{n,k})$, but $H^{n+2}(p_{n,k+1})$ might still have one. Moreover, note that $H^{n+1}(\mathcal{M}_{n,k+1}) = \mathcal{M}_{n,k}$, since $d_{\sigma_{s_{n,k}}}$ is injective by commutativity of the diagram

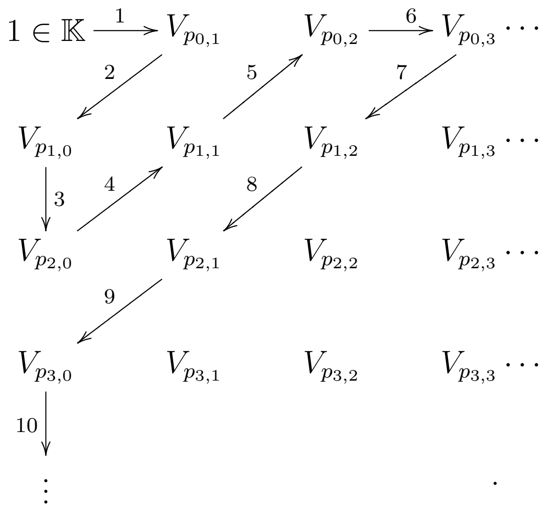
$$\begin{array}{ccc}
Z^{n+2}(\mathcal{C}\mathrm{one}(p_{n.k})) & \xrightarrow{\pi_1} & Z^{n+2}(\mathcal{M}_{n,k}) \\
{\scriptstyle s_{n,k}}\uparrow & & \downarrow{\scriptstyle H^{n+2}} \\
H^{n+2}(\mathcal{C}\mathrm{one}(p_{n.k})) & \xrightarrow{H^{n+2}(\pi_1)} & H^{n+2}(\mathcal{M}_{n,k})\ .
\end{array}$$

Hence $H^{n+1}(p_{n,k+1}) = H^{n+1}(p_{n,k})$ is bijective and we can continue the induction process. If at some point, say l, the kernel will be trivial, we can define $\mathcal{M}_{n+1} := \mathcal{M}_{n,l}$.

If this i not the case, we define the $n+1$-minimal model by the direct limit (which is the union of sets) $\mathcal{M}_{n+1} := \bigcup_k \mathcal{M}_{n,k}$ corresponding to the direct system $\iota_{n,k} : \mathcal{M}_{n,k} \to \mathcal{M}_{n,k+1}$ of natural inclusions and the canonical induced map corresponding to the maps $p_{n,k}$, which is denoted by p_{n+1}. Now $H^i(p)_{n+1}$ is bijective for all $i \leq n+1$, since, as already demonstrated, all $H^i(p)_{n,k}$ are bijective. p is injective for $k = n+1$ by commutativity of diagram (1.4) and the definition of the induced map on the direct limit. To obtain a minimal model, we take again the direct limit $\mathcal{M} := \bigcup_n \mathcal{M}_n$ corresponding to the inclusions $\mathcal{M}_n \to \mathcal{M}_{n,0} \subset \mathcal{M}_{n+1}$ and the canonical induced map p.

Some words on why this process indeed yields a Sullivan algebra and the simply connected case

The resulting differential graded algebra is free by construction. For convenience use inductively the isomorphism of Lemma 1.1.14 iii). Since the index set of the produced Hirsch filtration can be of order ω^2, it is not obvious that $\mathcal{M}$ is indeed generalised nilpotent, where the index set of the Hirsch filtration is required to be at most of order ω. Arrange all generators in the following system



and define consecutive Hirsch extensions via the Cantor diagonal scheme in the following way. Suppose we have already built up a Hirsch sequence $\mathcal{M}_0 \subset \cdots \subset \mathcal{M}_{l-1}$ of $\mathcal{M}$, such that $V_{p_{i,j}} \in \mathcal{M}_{l-1}$, where $V_{p_{i,j}}$ are the vector spaces along the arrows of the cantor scheme up to the $(l-1)$-th arrow. For the l-th arrow, let's say from $V_{p_{u,v}} \to V_{p_{n,m}}$, we would like to define $\mathcal{M}_l := \mathcal{M}_{l-1} \otimes_{d_{\sigma_{n,m}}} \bigwedge V_{p_{n,m}}$. The problem is, that this may not be a Hirsch extension of A_{l-1}. But for all generators $v_{n,m}$ of $V_{p_{n,m}}$, we get $d(v_{n,m}) \subset V_{p_{r,s}}$, where $r < n$ if $m = 1$ and $s < m$ if $m > 1$. Hence we can insert a finite Hirsch sequence from $\mathcal{M}_{l-1}$ to $\mathcal{M}_l$ and the claim follows. $\square$

CHAPTER

TWO

THE AUTOMORPHISM GROUP OF A SULLIVAN ALGEBRA

Let $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ be differential graded algebras as defined in 1.1.18. Recall that a homomorphism of differential graded algebras $f : \mathcal{A} \to \mathcal{B}$ is a linear map of degree 0, such that $f \circ d_{\mathcal{A}} = d_{\mathcal{B}} \circ f$. For details see Chapter 1.1 and in particular Definition 1.1.19.

Definition 2.0.6. Let $(\mathcal{A}, d)$ be a differential graded algebra. Then

$$\mathrm{Aut}(\mathcal{A}) := \Big\{ f \in \mathrm{GL}(\mathcal{A}) \mid d \circ f = f \circ d \Big\}$$

is called the automorphism group of $\mathcal{A}$, where $\mathrm{GL}(\mathcal{A})$ is the general linear group of the (graded) vector space $\mathcal{A}$. It is obviously a group.

Now if $\mathcal{S}$ is a Sullivan algebra, then by Proposition 1.5.8 being homotopic defines an equivalence relation $\sim$ on the automorphism group $\mathrm{Aut}(\mathcal{S})$.

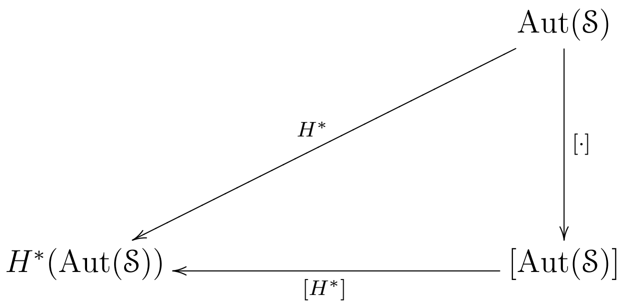
Lemma 2.0.7. *Let $\mathcal{S}$ be a Sullivan algebra. Then*

$$[\mathrm{Aut}(\mathcal{S})] := \mathrm{Aut}(\mathcal{S}) / \sim$$

is a group called the homotopy automorphism group. Moreover, there exists an induced map

$$[H^*] : [\mathrm{Aut}(\mathcal{S})] \to H^*(\mathrm{Aut}(\mathcal{S}))\ ,$$

such that the diagram of groups



commutes.

Proof. We can compose homotopies , see Lemma 1.5.2, and this composition is associative. If $\sigma \in \mathrm{Aut}(\mathcal{S})$ is homotopic to the identity, then $H^*(\sigma) = \mathrm{id}$ by Lemma 1.5.3 and thus H^* induces a homomorphism of groups

$$[H^*] : [\mathrm{Aut}(\mathcal{S})] \to H^*(\mathrm{Aut}(\mathcal{S})) .$$

□

2.1 Inner and outer automorphisms

Sullivan introduced the notion of inner automorphisms. Our main reference is the original paper of Sullivan [20] Section 6 and [9]. Also in [6] some of these results are mentioned and proven while proving other statements, but are not explicitly stated. A good reference is also [9]. We decided to outline Sullivan arguments here again.

Throughout this section $(\mathcal{S}, d)$ is a **Sullivan algebra of finite type** as defined in 1.4.3. The central definitions are

Definition 2.1.1. Let $i : \mathcal{S} \to \mathcal{S}$ be a derivation of degree -1. Then

- $\mathcal{I}\mathrm{nn}(\mathcal{S}) := \{d \circ i + i \circ d \mid i \text{ derivation of degree } -1\}$ are called inner derivations of $\mathcal{S}$.
- $\exp(d\circ i+i\circ d) := \sum_{k=0}^{\infty} \frac{1}{k!}(d\circ i+i\circ d)^k = \mathrm{id} + (d\circ i+i\circ d) + \frac{1}{2}(d\circ i+i\circ d)^2 + \cdots .$
- $\mathrm{Inn}(\mathcal{S}) := \{\exp(I) \mid I \in \mathcal{I}\mathrm{nn}(\mathcal{S})\}$ are called inner automorphisms.
- $\mathrm{Out}(\mathcal{S}) := \mathrm{Aut}(\mathcal{S})/\mathrm{Inn}(\mathcal{S})$ are called outer automorphisms.

We have to prove that these definitions indeed make sense. First of all note that $\mathcal{D}\mathrm{er}(\mathcal{S}) := \{\theta : \mathcal{S} \to \mathcal{S} \mid \theta \text{ is a derivation}\}$ forms a Lie algebra under the bracket $[X, Y] := X \circ Y - Y \circ X$. Now the inner derivations form a Lie sub algebra of $\mathcal{D}\mathrm{er}(\mathcal{S})$ since $[d, d \circ j + j \circ d] = 0$ by the property $d^2 = 0$, which implies

$$[i \circ d + d \circ i, j \circ d + d \circ j] = d \underbrace{[i, j \circ d + d \circ j]}_{\in \mathrm{inn}(\mathcal{S})} + \underbrace{[i, j \circ d + d \circ j]}_{\in \mathrm{inn}(\mathcal{S})} d \,.$$

Moreover, an inner derivation obviously commutes with the differential d.

Because $\mathcal{S}$ is of finite type, for all n there exists k such that $\bigwedge^k \mathcal{S}^n = 0$. Now d increases the monomial weight, i preserves it and thus by the previous observation $\exp(di + id)$ is nilpotent in each degree and hence well defined. Nilpotent means as usual $\exp(d \circ i + i \circ d)^k = 0$ for some $k \in \mathbb{N}$. Furthermore, inner automorphisms are closed under composition by virtue of the Baker-Campbell-Hausdorff formula and since inner derivations commute with d, so inner automorphism do. Altogether this implies that $\mathrm{Inn}(\mathcal{S}) \subset \mathrm{Aut}(\mathcal{S})$ is a subgroup.

Now we can turn to Sullivans main results. The first one is

Proposition 2.1.2. *Inner automorphisms are homotopic trivial, that is*

$$\exp(d \circ i + i \circ d) \sim \mathrm{id}$$

for all derivations i of degree -1.

Proof. Proofs can be found in [20] or [9] Theorem 11.22. We sketch the latter one. Define a derivation j of degree -1 on $\mathcal{S} \otimes \bigwedge(t, dt)$ on generators by $j(t) := 0$, $j(dt) := 0$ and $j(s) := ti(s)$ for all $s \in \mathcal{S}$. Then define a homomorphism of differential graded algebras $H : \mathcal{S} \to \mathcal{S} \otimes \bigwedge(t, dt)$ by

$$H(s) := \exp(d' \circ j + j \circ d')(s \otimes 1)$$

where d' is the differential in $\mathcal{S} \otimes \bigwedge(t, dt)$. Then $\pi_0 \circ H = \mathrm{id}$ and $\pi_1 \circ H = \exp(d \circ i + i \circ d)$. □

Proposition 2.1.3. *Let $\sigma \in \mathrm{Aut}(\mathcal{S})$ be homotopic trivial. Then*

$$\sigma = \exp(d \circ i + i \circ d) \,,$$

where i is some derivation of degree -1.

Before we can prove it, we have to introduce some more notations. An automorphism σ is called **unipotent** if $(\sigma - \mathrm{id})$ is nilpotent (degree wise). Furthermore, the cocycles modulo the decomposables of $\mathcal{S}$, in symbols

$$H^*_{\mathrm{spherical}}(\mathcal{S}) := Z^*(\mathcal{S})/\mathcal{S}^+ \cdot \mathcal{S}^+ ,$$

is called the **spherical homology** of $\mathcal{S}$. Obviously $H^*_{\mathrm{spherical}}$ is a functor.

Lemma 2.1.4. *An automorphism $\sigma \in \mathrm{Aut}(\mathcal{S})$ is unipotent if and only if $H^*_{spherical}(\sigma)$ is unipotent.*

Proof. By definition $H^*_{\mathrm{spherical}}(\sigma)$ is unipotent if σ is. Now let $H^*_{\mathrm{spherical}}(\sigma)$ be unipotent. Furthermore, let $\mathcal{S}_k$ be the Hirsch filtration of $\mathcal{S}$ and suppose that σ is unipotent on generators of $\mathcal{S}_n$ up to some n. If x is a new generator in $\mathcal{S}_{n+1}$, then $d(x) \in \mathcal{S}_n$. Now $(\sigma - \mathrm{id})$ is a homomorphism of differential graded algebras and by assumption there exists a $k \in \mathbb{N}$ such that $d((\sigma - \mathrm{id})^k(x)) = (\sigma - \mathrm{id})^k(d(x)) = 0$, which implies that $(\sigma - \mathrm{id})^k(x) \in Z^*(\mathcal{S})$. But on cocycles σ is unipotent by assumption and by recalling that a homomorphism of differential graded algebras is the identity on $\mathcal{S}_0 := \mathbb{K}$, the claim follows by induction. $\square$

Now note that since $\mathcal{S}$ is a Sullivan algebra, coboundaries are decomposable and hence $\ker(H^*) \subset \ker(H^*_{\mathrm{spherical}})$. In particular this directly implies

Corollary 2.1.5. *Let $H^*(f) = \mathrm{id}$. Then f is unipotent.*

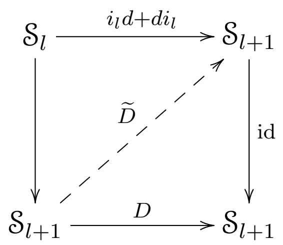
Moreover, we can write any unipotent automorphism $\sigma \in \mathrm{Aut}(\mathcal{S})$ as $\sigma = exp(D)$, where $D = \log(\sigma)$ is a derivation of degree 0 commuting with d. For convenience we use the usual formula

$$\log(\sigma) = \log(\mathrm{id} + X) = X - \frac{1}{2}X^2 + \cdots ,$$

where X is a nilpotent automorphism.

We are now ready for

Proof of Proposition 2.1.3. Let $\sigma \in \mathrm{Aut}(\mathcal{S})$ be homotopic trivial. As already mentioned above σ is unipotent and we can write $\sigma = \exp D$ with D a derivation of degree 0 commuting with d. We have to show that $D = id + di$ for some $i \in \mathcal{D}\mathrm{er}(\mathcal{S})$ of degree -1. The proof is by induction again. Suppose $D = i_l d + d i_l$ on $\mathcal{S}_l$. By Proposition 1.5.4 this yields a commuting diagram of the form



since id is surjective and $H^n(\mathcal{C}\mathrm{one}(\mathrm{id})) = 0$ for all $n \in \mathbb{N}$. The last statement is a direct consequence of the long exact cohomology sequence 1.2.12. Moreover, on a new generator $v \in \mathcal{S}_{l+1}$ we can choose $\widetilde{D}(v) = di_{l+1}(v) + i_l d(v)$ where i_{l+1} is of degree -1 and the claim follows. □

Summing up we get

Theorem 2.1.6. *Let $\mathcal{S}$ be a nilpotent Sullivan algebra of finite type, then* $\mathrm{Inn}(\mathcal{S}) := \{\sigma \in \mathrm{Aut}(\mathcal{S}) \mid \sigma \sim \mathrm{id}\}$ *and* $\mathrm{Out}(\mathcal{S}) \cong [\mathrm{Aut}(\mathcal{S})]$ *accordingly.*

2.2 Algebraic structure

The automorphism group of a Sullivan algebra has an additional algebraic structure, namely it is a linear algebraic group. For such groups there exists a decomposition into a semi-direct product consisting of a reductive part and a unipotent one. We will briefly discuss these results and also combine them with the results of the last section about inner and outer automorphisms.

Our main references for this section are [20] Section 6 and the appendix about algebraic groups subsequent to it, [3] IV.11 and [4].

By a linear algebraic group we mean

Definition 2.2.1. Let V be a finite dimensional $\mathbb{K}$-vector space. A linear algebraic group is a Zariski closed subgroup of $\mathrm{GL}(V)$.

As usual a subset of $\mathrm{GL}(V)$ is called Zariski closed, if, after choosing a basis for V, it is the zero set of a set of polynomial equations. Now the first observation is

Lemma 2.2.2. *Let $\mathcal{S}$ be a Sullivan algebra of finite type. Then the groups* $\mathrm{Aut}(\mathcal{S})$ *and* $\mathrm{Out}(\mathcal{S}) := \mathrm{Aut}(\mathcal{S})/\mathrm{Inn}(\mathcal{S})$ *are linear algebraic groups.*

Proof. Let $\mathcal{S}$ be a Sullivan algebra of finite type. Since $\mathcal{S}$ is finitely generated, $\mathcal{S} = \bigwedge V$ where V is a finite dimensional (graded) vector space. Since an automorphism of a free algebra is determined on generators, for details see the Remark after Lemma 1.1.14, we get an embedding $\mathrm{Aut}(\mathcal{S}) \subset \mathrm{GL}(V)$. Now all automorphisms φ of the latter type fulfil a set of equations defined by $d \circ \varphi - \varphi \circ d = 0$ and thus the automorphism group is the zero set of a set of polynomial equations. □

As already mentioned, for linear algebraic groups there exists a semi-direct decomposition into a unipotent part and a reductive complement. This decomposition is called the Levi–Mostow decomposition which will be discussed in the rest of this section. But first we should explain what precisely is meant by a semi-direct product.

2.2.1 Semi-direct products and splittings of short exact sequences of groups

Definition 2.2.3. Let $(G', \circ)$ and $(G'', \cdot)$ be groups and $\varphi : G'' \to \mathrm{Aut}(G')$ a homomorphism. Then the semi-direct product $G' \rtimes_\varphi G''$ is the cartesian product $G' \times G''$ as a set together with the multiplication given by

$$(g_1', g_1'') \star (g_2', g_2'') := (g_1' \circ \varphi(g_1'')(g_2'), g_1'' \cdot g_2'') \ .$$

Usually the homomorphism φ is suppressed within the notation and we only write $G' \rtimes G''$ for a semi-direct product.

Definition 2.2.4. Let

$$1 \longrightarrow G' \xrightarrow{f} G \xrightarrow{g} G'' \longrightarrow 1 \tag{2.1}$$

be a short exact sequence of groups. This means that f is injective, g is surjective and $\ker(g) = \mathrm{im}(f)$. One says that the sequence (2.1) splits, if there exists a homomorphism of groups $s : G'' \to G$, such that $f \circ s = \mathrm{id}_{G''}$. The homomorphism s is also called a splitting.

Lemma 2.2.5. *Let G be a group. Then $G = G' \rtimes G''$ if and only if there exists a short exact sequences of groups*

$$1 \longrightarrow G' \xrightarrow{f} G \xrightarrow{g} G'' \longrightarrow 1$$

which splits.

Proof. For a proof see [14] Chapter XII Scetion 2. □

2.2.2 Levi–Mostow decomposition

Definition 2.2.6. A linear algebraic group G is called unipotent, if $g - \mathrm{id}$ is nilpotent for all $g \in G$. Nilpotent means that there exists $n \in \mathbb{N}$ such that $(g - \mathrm{id})^n = 0$.

Corollary 2.2.7. *Let G be a linear algebraic group and $R \subset G$ be a reductive group and $U \subset G$ be a unipotent normal subgroup. Then $R \cap U = \{\mathrm{id}\}$.*

Lemma 2.2.8. *Let G be a linear algebraic group. Then there exists a maximal normal unipotent subgroup, called the maximal unipotent radical $R_u(G)$.*

Proof. For a proof see [3] 11.22. □

Definition 2.2.9. A linear algebraic group G is called reductive, if

$$\mathrm{R}_u(G) = \{1\} .$$

Theorem 2.2.10 (Levi–Mostow decomposition)**.** *Let G be a linear algebraic group over a field of characteristic 0 and $R_u(G)$ its maximal unipotent radical. Then $G/R_u(G)$ is reductive and*

$$G = R_u(G) \rtimes G/R_u(G) .$$

Proof. For details see [3] Definition 11.22 or [20]. □

Corollary 2.2.11 (Levi–Mostow decomposition)**.** *Let $\mathcal{S}$ be a Sullivan algebra of finite type. Then there exists a unipotent, normal subgroup $U \subset \mathrm{Aut}(\mathcal{S})$ such that $\mathrm{Aut}(\mathcal{S})/U$ is reductive and*

$$\mathrm{Aut}(\mathcal{S}) = U \rtimes \mathrm{Aut}(\mathcal{S})/U .$$

CHAPTER

THREE

NILPOTENT LIE ALGEBRAS, DUAL LIE ALGEBRAS AND AUTOMORPHISMS

A special kind of a Sullivan algebra of finite type, as defined in 1.4.3, is given by the Koszul complex of a nilpotent Lie algebra. Differential graded algebras of that type are also called dual Lie algebras. In particular a dual Lie algebra admits a Hirsch filtration which is induced by the descending central series of the Lie algebra. On the other hand we can construct the homological model of a dual Lie algebra, which also induces a Hirsch filtration. This chapter is intended to explain these terms and to show that the Hirsch extensions mentioned above coincide. Moreover, it introduces the concept of inner automorphisms of a Lie algebra and explains their connection to Sullivans concept of inner automorphisms on Sullivan algebras as defined at the beginning of Section 2.1. The term Sullivan algebra of finite type was already explained in Section 1.4, Definition 1.4.3.

3.1 Nilpotent Lie algebras

Definition 3.1.1. A Lie algebra is a pair $\big(\mathfrak{g}, [\cdot,\cdot]\big)$, where $\mathfrak{g}$ is a finite dimensional $\mathbb{K}$-vector space and

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is an alternating, bilinear map satisfying the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Alternating means $[X,Y] = -[Y,X]$ for all $X, Y \in \mathfrak{g}$.

We also just write $\mathfrak{g}$ for a Lie algebra and $[\cdot,\cdot]_{\mathfrak{g}}$ then denotes the corresponding Lie bracket or, if no confusion is possible, just $[\cdot,\cdot]$.

Definition 3.1.2. A Lie algebra homomorphism is a $\mathbb{K}$-linear map

$$f : (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}) \to (\mathfrak{h}, [\cdot,\cdot]_{\mathfrak{h}})$$

commuting with the Lie brackets, that is $f([X,Y]_{\mathfrak{g}}) = [f(X), f(Y)]_{\mathfrak{h}}$.

Accordingly one defines

Definition 3.1.3.

$$\mathrm{Aut}(\mathfrak{g}) := \Big\{ f \in \mathrm{GL}(\mathfrak{g}) \mid f[X,Y] = [f(X), f(Y)] \;\forall\, X, Y \in \mathfrak{g} \Big\}$$

is called the automorphism group of $\mathfrak{g}$, where $\mathrm{GL}(\mathfrak{g})$ is the general linear group of the vector space $\mathfrak{g}$.

Definition 3.1.4. Let $\mathfrak{g}$ be a Lie algebra. Then

$$\mathcal{Z}(\mathfrak{g}) := \{ X \in \mathfrak{g} \mid [X,Y] = 0 \;\forall\, Y \in \mathfrak{g} \}$$

is called the center of $\mathfrak{g}$.

Definition 3.1.5. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. Then

$$\mathcal{C}^k(\mathfrak{g}) := [\mathfrak{g}, \mathcal{C}^{k-1}(\mathfrak{g})],\ \mathcal{C}^0(\mathfrak{g}) := \mathfrak{g}$$

is called the descending central series associated to $\mathfrak{g}$.

Definition 3.1.6. A Lie algebra $(\mathfrak{n}, [\cdot,\cdot])$ is called nilpotent of step $n \in \mathbb{N}$, if $\mathcal{C}^n(\mathfrak{n}) = 0$ and $\mathcal{C}^k(\mathfrak{n}) \neq 0$ for all $k < n$. In the special case $n = 1$, which means $\mathcal{C}^1(\mathfrak{n}) = [\mathfrak{n}, \mathfrak{n}] = 0$, it is called an abelian Lie algebra.

Since for a nilpotent Lie algebra $\mathfrak{n}$ we always have

$$[\mathcal{C}^{k-1}(\mathfrak{n}), \mathcal{C}^{k-1}(\mathfrak{n})] \subset [\mathfrak{n}, \mathcal{C}^{k-1}(\mathfrak{n})] = \mathcal{C}^k(\mathfrak{n})\ ,$$

one can also define the quotient Lie algebras

$$\mathfrak{a}_k := \mathcal{C}^{k-1}(\mathfrak{n})/\mathcal{C}^k(\mathfrak{n}) \text{ and } \mathfrak{n}_k := \mathfrak{n}/\mathcal{C}^k(\mathfrak{n})\ ,$$

where the Lie brackets are just given by restriction. In particular $\mathfrak{a}_k$ is abelian with respect to this Lie bracket. The induced Lie bracket on $\mathfrak{n}_k$ is denoted by $[\cdot,\cdot]_k$. This leads to abelian, central extension of Lie algebras

$$0 \longrightarrow \mathfrak{a}_k \longrightarrow \mathfrak{n}_k \longrightarrow \mathfrak{n}_{k-1} \longrightarrow 0$$

for all $k \leq n$, where the maps are the obvious ones. Abelian and central refers to the fact that $\mathfrak{a}_k$ is abelian with respect to the induced Lie bracket and that the image of $\mathfrak{a}_k$ lies in the center of $\mathfrak{n}_k$.

Lemma 3.1.7. *There exists an isomorphism of Lie algebras*

$$(\mathfrak{n}_k, [\cdot,\cdot]_k) = (\mathfrak{n}_{k-1} \oplus \mathfrak{a}_k,\ [\cdot,\cdot]_{\varphi_{k-1}})\ .$$

where $[(X,A),(Y,B)]_{\varphi_{k-1}} := ([X,Y]_{k-1}, \varphi(X,Y))$ *and* $\varphi_{k-1} \in Z^2(\mathfrak{n}_{k-1}, 0, \mathfrak{a}_k)$ *is a cocycle in the Lie-algebra cohomology of* $\mathfrak{n}_{k-1}$ *with trivial action on* $\mathfrak{a}_k$*. This means by definition that* $\varphi_{k-1} : \mathfrak{n}_{k-1} \times \mathfrak{n}_{k-1} \to a_k$ *is an alternating, bilinear map such that*

$$\varphi_{k-1}(X_i, [X_j, X_k]) + \varphi_{k-1}(X_j, [X_k, X_i]) + \varphi_{k-1}(X_k, [X_i, X_j]) = 0\ .$$

3.2 Koszul complex, dual Lie algebra and Lie algebra cohomology

If V is a $\mathbb{K}$-vector space, then as usual

$$V^* := \{\omega : V \to \mathbb{K} \mid \omega \text{ linear }\}$$

denotes the dual vector space. Moreover, if $\{v_1, \cdots, v_n\}$ is a basis of V, then the dual basis $\{v_1^*, \cdots, v_n^*\}$ of V^* is characterised by $v_i^*(v_j) = \delta_{i,j}$.

A linear map $f : V \to W$ induces a linear map $f^* : W^* \to V^*$ defined by $(f^*\omega)(v) := \omega(f(v))$ for $\omega \in W^*$ and $v \in V$. Obviously $(f \circ g)^* = g^* \circ f^*$ for all linear maps $f : V \to W$ and $g : W \to U$.

Definition 3.2.1. Let $(\mathfrak{n}, [\cdot,\cdot])$ be a nilpotent Lie algebra, $\mathfrak{n}^*$ be the dual vector space and $\{x_i\}_{i\in I}$ be the dual basis to a basis $\{X_i\}_{i\in I}$ of $\mathfrak{n}$. Then the **Koszul complex** of $\mathfrak{n}$ is the differential graded algebra

$$(\mathcal{N}, d) := \left(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\right),$$

where $\bigwedge \mathfrak{n}^*$ is the free graded algebra over $\mathfrak{n}^*$, as defined in 1.1.13. The differential $d_{[\cdot,\cdot]}$ is defined on a basis by

$$\left(d_{[\cdot,\cdot]}x_i\right)(X_j, X_k) := x_i([X_j, X_k]) \tag{3.1}$$

and continuation via the Leibniz rule

$$d_{[\cdot,\cdot]}(\omega \wedge \eta) = d_{[\cdot,\cdot]}\omega \wedge \eta + (-1)^{\deg(\omega)}\omega \wedge d_{[\cdot,\cdot]}\eta$$

for $\omega, \eta \in \bigwedge \mathfrak{n}^*$. Indeed $d^2 = 0$ since

$$\begin{aligned} d^2x_i(X_j, X_k, X_l) &= \frac{1}{3}\Big(d^2x_i(X_j, X_k, X_l) + d^2x_i(X_k, X_l, X_j) + d^2x_i(X_l, X_j, X_k)\Big) \\ &= \frac{1}{3}x_i\Big([X_j, [X_k, X_l] + [X_k, [X_l, X_j] + [X_l, [X_j, X_k]\Big) \quad (3.2) \\ &= 0 \text{ by the Jacobi identity.} \end{aligned}$$

Thus $(\mathcal{N}, d)$ is a differential graded algebra generated by elements of degree one.

Since the Koszul complex is a differential graded algebra, one can take cohomology of it, which leads to

Definition 3.2.2. Let $(n, [\cdot, \cdot])$ be a nilpotent Lie algebra and

$$(\mathcal{N}, d) := \big(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\big)$$

the corresponding Koszul complex. Then

$$H^n(\mathfrak{n}) := H^n(\mathcal{N})$$

is called the n-th Lie algebra cohomology group of $\mathfrak{n}$, where the right hand side denotes the cohomology group of a differential graded algebra as defined in 1.2.2. Furthermore, H^n is a contravariant functor for all n. For a homomorphism of Lie algebras

$$f : \mathfrak{n} \to \mathfrak{m};,$$

the linear map

$$H^n(f) : H^n(\mathfrak{n}) \to H^n(\mathfrak{m})$$

is defined in the following way. By definition f is a linear map. Thus we get the dual map $f^* : \mathfrak{m}^* \to \mathfrak{n}^*$. By Lemma 1.1.14, we get an induced map $\widehat{f^*} : \bigwedge \mathfrak{m}^* \to \mathfrak{n}^*$ of free graded algebra. By the definition of the differential, it is easy to see that this map is a homomorphism of differential graded algebras. Thus it induces a map $H^*(\widehat{f^*})$ on cohomology by Lemma 1.2.6. One then defines $H^n(f) := H^n(\widehat{f^*})$. Clearly $H^n(f \circ g) = H^n(g) \circ H^n(f)$.

In what follows we will frequently use some well known facts from linear algebra concerning dual vector spaces.

Definition 3.2.3. Let V be a vector space and $U \subset V$ be a sub vector space. Then

$$U^{\perp_V} := \{\omega \in V^* \mid \omega(u) = 0 \;\forall\, u \in U\}$$

is called the annihilator of U in V^*. If no confusion is possible concerning the vector space V , we also drop the index and just write $U^{\perp}$.

To get used to the previous definitions, observe the following

Lemma 3.2.4. *Let $\mathfrak{n}$ be a nilpotent Lie algebra and $\mathcal{N}$ the corresponding Koszul complex. Then*

$$H^1(\mathfrak{n}) = Z^1(\mathcal{N}) = \mathcal{C}^1(\mathfrak{n})^{\perp_{\mathfrak{n}}} .$$

Proof. By definition the Koszul complex is a free algebra and thus connected, that is $\mathcal{N}^0 = \mathbb{K}$. Compare Definition 1.1.13. So by Lemma 1.2.5 $H^1(\mathfrak{n}) = Z^1(\mathcal{N})$. That $Z^1(\mathcal{N}) = \mathcal{C}^1(\mathfrak{n})^{\perp_\mathfrak{n}}$ follows directly by definition of the differential.

□

A bit more advanced is the following well known fact from linear algebra.

Lemma 3.2.5. *Let V be a finite dimensional vector space and $W \subset U \subset V$ sub vector spaces. Then we get the following isomorphisms:*

i) $(V/U)^* \cong U^{\perp_V}$.

ii) $(W^{\perp_V}/U^{\perp_V}) \cong (U/W)^*$.

The combination of these two isomorphisms yields

iii) $W^{\perp_U} \cong (W^{\perp_V}/U^{\perp_V}) \cong (U/W)^*$.

Applying this lemma to nilpotent Lie algebras leads to

Proposition 3.2.6. *The Koszul complex $\big(\mathcal{N}, d\big) := \big(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\big)$ of a nilpotent Lie algebra $(\mathfrak{n}, [\cdot,\cdot])$ is a Sullivan algebra of finite type.*

Proof. The series of Lie algebras $\big(\mathfrak{n}_k, d_{[\cdot,\cdot]_k}\big) := \big(\mathfrak{n}/\mathcal{C}^k(\mathfrak{n}), d_{[\cdot,\cdot]_k}\big)$, dualises to a sequence of differential graded algebras $\big(\bigwedge \mathfrak{n}_k^*, d_{[\cdot,\cdot]_k}\big)$. Using isomorphism i) of Lemma 3.2.5 we get

$$\big(\bigwedge \mathfrak{n}_k^*, d_{[\cdot,\cdot]_k}\big) = \big(\bigwedge \mathcal{C}^k(\mathfrak{n})^\perp, d_{[\cdot,\cdot]}\big) .$$

Thus by defining $\big(\mathcal{C}^k(\mathcal{N}), d\big) := \big(\bigwedge \mathcal{C}^k(\mathfrak{n})^\perp, d_{[\cdot,\cdot]}\big)$,

$$\{0\} \subset \cdots \subset \mathcal{C}^{k-1}(\mathcal{N}) \subset \mathcal{C}^k(\mathcal{N}) \subset \cdots \subset \mathcal{C}^n(\mathcal{N}) = \mathcal{N}$$

yields a filtration of the Koszul complex $\mathcal{N}$. As already pointed out in Lemma 3.1.7, we have an isomorphism of Lie algebras

$$(\mathfrak{n}_k, [\cdot,\cdot]_k) = (\mathfrak{n}_{k-1} \oplus \mathfrak{a}_k, [\cdot,\cdot]_{\varphi_{k-1}}) ,$$

where $\mathfrak{a}_k := \mathcal{C}^{k-1}(\mathfrak{n})/\mathcal{C}^k(\mathfrak{n})$ and $\varphi_{k-1} : \mathfrak{n}_{k-1} \times \mathfrak{n}_{k-1} \to a_k$ is an an alternating, bilinear map with

$$\varphi_{k-1}(X_i, [X_j, X_l]) + \varphi_{k-1}(X_j, [X_l, X_i]) + \varphi_{k-1}(X_l, [X_i, X_j]) = 0 . \qquad (3.3)$$

This map dualises to a map $\varphi^*_{k-1} : \mathfrak{a}_k^* \to \bigwedge^2 \mathfrak{n}^*_{k-1}$. For $\omega \in \mathfrak{a_t}^*$ and $X_i, X_j, X_l \in \mathfrak{n}$ we compute

$$
\begin{aligned}
(d\varphi^*_{k-1}(\omega))(X_i, X_j, X_l) &= \frac{1}{3}\Big((d\varphi^*_{k-1}(\omega))(X_i, X_j, X_l) + (d\varphi^*_{k-1}(\omega))(X_j, X_l, X_i) \\
&+ (d\varphi^*_{k-1}(\omega))(X_l, X_i, X_j) \Big) \\
&= \frac{1}{3}\Big((\varphi^*_{k-1}(\omega))(X_i, [X_j, X_l]) + (\varphi^*_{k-1}(\omega))(X_j, [X_l, X_i]) \\
&+ (\varphi^*_{k-1}(\omega))(X_l, [X_i, X_j]) \Big) \\
&= \frac{1}{3}\Big(\omega(\varphi_{k-1}(X_i, [X_j, X_l])) + \omega(\varphi_{k-1}(X_j, [X_l, X_i])) \\
&+ \omega(\varphi_{k-1}((X_l, [X_i, X_j])) \Big) \\
&\overset{\text{by (3.3)}}{=} \omega(0) = 0
\end{aligned}
$$

and thus φ^*_{k-1} actually defines a map

$$\varphi^*_{k-1} : \mathfrak{a}_k^* \to Z^2(\mathcal{N}^*_{k-1}) \ .$$

By Lemma 1.1.14 iii), we have an isomorphism of algebras

$$
\begin{aligned}
\theta : \bigwedge \mathfrak{n}^*_{k-1} \otimes \bigwedge \mathfrak{a}^*_k &\to \bigwedge (\mathfrak{n}^*_{k-1} \oplus \mathfrak{a}^*_k) \\
\theta(\omega_1 \otimes \omega_2) &:= \omega_1 \wedge \omega_2 \ .
\end{aligned}
$$

If we show that

$$\theta : \bigwedge \mathfrak{n}^*_{k-1} \otimes_{d_{\varphi^*_{k-1}}} \bigwedge \mathfrak{a}^*_k \to \Big(\bigwedge (\mathfrak{n}^*_{k-1} \oplus \mathfrak{a}^*_k), d_{[\cdot,\cdot]_{\varphi^*_{k-1}}} \Big)$$

is actually an isomorphism of differential graded algebras, then the Koszul complex $\mathcal{N}$ of $\mathfrak{n}$ is indeed a Sullivan algebra. Note that by definition

$$d_{[\cdot,\cdot]_{\varphi^*_{k-1}}} = \widehat{d_{[\cdot,\cdot]_k} \oplus \varphi^*_{k-1}} \ ,$$

where $\widehat{d_{[\cdot,\cdot]_k} \oplus \varphi^*_{k-1}}$ is the natural extension given by Lemma1.1.14. In order to avoid notational overload, we call all differentials d and the involved Lie bracket by $[\cdot,\cdot]$. For $X, Y \in \mathfrak{n}_{k-1}, A, B \in \mathfrak{a}_k$ we compute on generators

$$\begin{aligned}\big(d\theta(1\otimes\omega)\big)(X+A, Y+B) &= d(1\wedge\omega)(X+A,Y+B)\\ &= -(1\wedge\omega)([X+A,Y+B])\\ &= -(1\wedge\omega)([X,Y]+\varphi(X,Y))\\ &= -1\wedge\varphi^*(\omega)(X,Y)\\ &= -\varphi^*\omega(X,Y)\end{aligned}$$

and on the other hand

$$\begin{aligned}\big(\theta d(1\otimes\omega)\big)(X+A, Y+B) &= -(1\otimes d(\omega))(X+A,Y+B))\\ &= -\theta(\varphi^*\omega\otimes 1)(X+A,Y+B)\\ &= -(\varphi^*\omega\wedge 1)(X+A,Y+B)\\ &= \varphi^*(X,Y)\wedge 1\\ &= \varphi^*(X,Y)\ .\end{aligned}$$

Thus the claim follows by 1.1.22 since for generators of the form $n\otimes 1$ it is obviously true. □

This formalism also works in the opposite direction and we end up with

Proposition 3.2.7. *The functor*

$$\left\{\textit{Nilpotent Lie algebras}\right\} \longrightarrow \left\{\begin{array}{l}\textit{Sullivan algebras of finite type}\\ \textit{generated by elements of degree one}\end{array}\right\}$$

$$(\mathfrak{n},[\cdot,\cdot]) \longrightarrow \left(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\right)$$

is an equivalence of categories.

Proof. First the object level. If we start with a nilpotent Lie algebra $(\mathfrak{n},[\cdot,\cdot])$ then Proposition 3.2.6 shows, that the corresponding Koszul complex

$$\mathcal{N} := \left(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\right)$$

is a Sullivan algebra of finite type generated by elements of degree one. If we start with a Sullivan algebra of finite type $\mathcal{S}$ generated by elements of degree one, then $\mathcal{S} = \bigwedge W$, where $\deg W = 1$ and thus $\mathfrak{n} := W^*$ is a finite dimensional vector space over $\mathbb{K}$. Since on a free Algebra the differential is determined on generators, we can define a Lie bracket on $\mathfrak{n}$ by reading definition (3.1) backwards.

Now the map level. With respect to above definitions, a homomorphism of Lie algebras dualises to a homomorphism of the corresponding Koszul complex via natural extension and vice versa, since a homomorphism of a free differential graded algebras is determined on generators. □

This equivalence of categories also suggests the following

Definition 3.2.8. A dual Lie algebra is a Sullivan algebra of finite type generated by elements of degree one.

3.3 Inner and outer automorphisms

We define the notion of inner and outer automorphism of a Lie algebra. Moreover, we show that these definitions and Sullivans definitions of inner and outer automorphisms on the dual Lie algebra as defined in Section 2.1 are consistent. All statements about Lie algebras in this section are quite standard and can be found for example in [10].

Throughout this section $\mathfrak{g}$ denotes a Lie algebra and $[\cdot,\cdot]_{\mathfrak{g}}$ the corresponding Lie bracket.

Definition 3.3.1. A linear map $D : \mathfrak{g} \to \mathfrak{g}$ is called a derivation of $\mathfrak{g}$, if

$$D([X,Y]_{\mathfrak{g}}) = [D(X),Y]_{\mathfrak{g}} + [X,D(Y)]_{\mathfrak{g}}$$

for all $X, Y \in \mathfrak{g}$. Moreover, the set of derivations $\mathfrak{Der}(\mathfrak{g})$ forms a Lie algebra with the Lie bracket defined by the usual matrix commutator

$$[D,E]_{\mathfrak{Der}(\mathfrak{g})} := DE - ED\ .$$

Now for a fixed element $Z \in \mathfrak{g}$

$$\mathrm{ad}_Z(X) := [Z,X]_{\mathfrak{g}}$$

defines a linear map. Clearly ad_Z is a derivation for all $Z \in \mathfrak{g}$ and thus we get a linear map

$$\begin{aligned} \mathrm{ad} : \mathfrak{g} &\to \mathfrak{Der}(\mathfrak{g}) \\ Z &\mapsto \mathrm{ad}_Z\ , \end{aligned}$$

called the adjoint representation of $\mathfrak{g}$.

Lemma 3.3.2. *The image* $\mathcal{I}nn(\mathfrak{g}) := \mathrm{im}(\mathrm{ad}(\mathfrak{g}))$ *is a sub algebra of* $\mathcal{D}\mathrm{er}(\mathfrak{g})$ *called inner derivations.*

It is a well known fact that the automorphism group of a Lie algebra is a linear Lie group and that its Lie algebra is precisely $\mathcal{D}\mathrm{er}(\mathfrak{g})$. Furthermore, the matrix exponential yields a map

$$\begin{aligned}\exp : \mathcal{D}\mathrm{er}(\mathfrak{g}) &\to \mathrm{Aut}(\mathfrak{g})\\ \exp(D) &:= \sum_{k=0}^{\infty} \frac{1}{k!} D^k\end{aligned}$$

and the restriction $\exp_{|\mathcal{I}\mathrm{nn}(\mathfrak{g})} : \mathcal{I}\mathrm{nn}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g})$ is injective. The inverse map is given by

$$\log(A) := \sum_k (-1)^{k+1} \frac{(A-I)^k}{k} \ .$$

Definition 3.3.3. The image

$$\mathrm{Inn}(\mathfrak{g}) := \mathrm{im}(\exp_{|\mathcal{I}\mathrm{nn}(\mathfrak{g})})$$

is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$ called inner automorphisms. Accordingly

$$\mathrm{Out}(\mathfrak{g}) := \mathrm{Aut}(\mathfrak{g})/\mathrm{Inn}(\mathfrak{g})$$

is called outer automorphism group.

Now for the Sullivan algebra of finite type $(\mathcal{N}, d) := (\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]})$ we defined in Section 2.1 inner automorphisms in the following way. Take any derivation i of degree -1 on $\mathcal{N}$. Then $d \circ i + i \circ d$ is obviously a derivation of degree 0. Since $\mathcal{N}$ is generated by elements of degree one, $d \circ i(\omega) = 0$ for any one form ω since $i(\omega) \in \mathbb{K}$ and $d_{|\mathbb{K}} = 0$. Thus $i \circ d + d \circ i = i \circ d$. Accordingly an inner automorphism of $\mathcal{N}$ is defined by $\exp(i \circ d)$.

Lemma 3.3.4. *Let* $(\mathfrak{n}, [\cdot,\cdot])$ *be a nilpotent Lie algebra,* $\left(\bigwedge \mathfrak{n}^*, d\right)$ *the corresponding Koszul complex and* i *be a derivation of degree* -1 *on* $\bigwedge \mathfrak{n}^*$*. Then there exists* $Z \in \mathfrak{n}$*, such that*

$$(\mathrm{ad}_Z)^* = i \circ d \ .$$

Proof. Since i is a derivation of degree -1, a derivation is determined by its image on generators and the Koszul complex is generated by elements of degree one, we can regard i as linear map $i : \mathfrak{g}^* \to \mathbb{K}$, that is $i \in \mathfrak{g}^{**}$. Let

$$\begin{aligned}\Phi &: \mathfrak{n} \to \mathfrak{n}^{**}\\ \Phi(Z)(\omega) &:= \omega(Z)\end{aligned}$$

be the canonical isomorphism of a vector space with it's bidual vector space. Then $i = \Phi(Z)$ for some $Z \in \mathfrak{n}$. Furthermore, $d\omega = \sum_k \omega'_k \wedge \omega_k''$ where ω'_k and ω''_k are one forms for all k. We compute

$$\begin{aligned}\big(\mathrm{ad}_Z^*\omega\big)(X) &= \omega\big(\mathrm{ad}_Z(X)\big) = \omega([Z,X]) = d\omega(Z,X)\\ &= \Big(\sum_k \omega'_k \wedge \omega''_k\Big)(Z,X)\\ &= \sum_k \Big(\omega'_K(Z)\cdot\omega''_k(X) - \omega'_k(X)\cdot\omega''_k(Z)\Big)\end{aligned}$$

and

$$\begin{aligned}\big(i(d\omega)\big)(X) &= (i(\sum_k \omega'_k \wedge \omega''_k))(X) = (\sum_k i(\omega'_k \wedge \omega''_k))(X)\\ &= (\sum_k \underbrace{i(\omega'_k)}_{\in\mathbb{K}} \wedge \omega''_k - \omega'_k \wedge \underbrace{i(\omega''_k)}_{\in\mathbb{K}})(X)\\ &= (\sum_k i(\omega'_k)\cdot\omega''_k - \omega'_k \cdot i(\omega''_k))(X)\\ &= (\sum_k \Phi(Z)(\omega'_k)\cdot\omega''_k - \omega'_k\cdot\Phi(Z)(\omega''_k))(X)\\ &= (\sum_k \omega'_k(Z)\cdot\omega''_k - \omega'_k\cdot\omega''_k(Z))(X)\\ &= \sum_k \omega'_K(Z)\cdot\omega''_k(X) - \omega'_k(X)\cdot\omega''_k(Z)\ .\end{aligned}$$

Thus the claim follows. □

The last lemma directly implies

Proposition 3.3.5. *The functor*

$$\big(\mathfrak{n},[\cdot,\cdot]\big) \to \big(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\big)$$

from Proposition 3.2.7 yields an isomorphism between the group of inner automorphisms $\mathrm{Inn}\big((\mathfrak{n},[\cdot,\cdot])\big)$ *of the Lie algebra and the group of inner automorphisms* $\mathrm{Inn}\big(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\big)$ *of the Sullivan algebra. In particular*

$$\mathrm{Out}\big((\mathfrak{n},[\cdot,\cdot])\big) = \mathrm{Out}\Big(\big(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\big)\Big)\ .$$

3.4 Canonical filtrations on dual Lie algebras

As we have seen in Section 3.2, the Koszul complex of a nilpotent Lie algebra is a Sullivan algebra of finite type in a canonical way, where the Hirsch filtration was induced by the descending central series of the corresponding Lie algebra. On the other hand one can construct the homological model of the Koszul complex, which also induces a filtration on the Koszul complex in a canonical way. We first briefly recall both filtrations and show afterwards, that they are equal.

Throughout this section $\big(\mathfrak{n}, [\cdot,\cdot]\big)$ denotes a nilpotent Lie algebra and $(\mathcal{N}, d) := \big(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\big)$ the corresponding Koszul complex.

3.4.1 The homological model of a dual Lie algebra in detail

The homological model

$$p : \mathcal{M} \to \mathcal{N} ,$$

as introduced in Definition 1.6.1, is constructed in the following way.

The first step

The algorithm starts by choosing the differential graded algebra

$$\mathcal{M}_{1,1} := \bigwedge H^1(\mathcal{N})$$

with the zero map as differential. Since $\mathcal{N}$ is free, it is connected and thus $H^1(\mathcal{N}) = Z^1(\mathcal{N})$ by Lemma 1.2.5 and

$$\mathcal{M}_{1,1} = \bigwedge Z^1(\mathcal{N}) .$$

Since $\mathcal{M}_{1,1}$ is a free graded algebra generated by elements of degree one, the natural inclusion $\iota : Z^1(\mathcal{N}) \hookrightarrow \mathcal{N}^1$ induces a map $\widehat{\iota} : \mathcal{M}_{1,1} \to \mathcal{N}$ by Lemma 1.1.14. One defines

$$p_{1,1} := \widehat{\iota} .$$

Clearly

$$p_{1,1} : (\mathcal{M}_{1,1}, 0) \to (\mathcal{N}, d)$$

is a homomorphism of differential graded algebras. Moreover,

$$H^1(\mathcal{M}_{1,1}) = H^1\big(\bigwedge Z^1(\mathcal{N})\big) = Z^1(\mathcal{N})$$

and thus $H^1(p_{1,1}) = \iota$ is an isomorphism. For the sake of completeness we have $H^0(\mathcal{M}_{1,1}) = \mathbb{K}$ since $\mathcal{M}_{1,1}$ is a free algebra over a vector space, compare Definition 1.1.13 , it is connected and thus $H^0(\mathcal{N}) = \mathbb{K}$ by Lemma 1.2.5. Hence $H^0(p_{1,1}) = \mathrm{id}$ is an isomorphism too.

Note that if we would start with the zero minimal model $\mathcal{M}_{0,0} := \mathbb{K}$, this first construction step corresponds to a "Killing the cokernel step" in the algorithm described in 1.6.1. Also note that $H^2(p_{1,1})$ is clearly not injective if the differential of $\mathcal{N}$ is non trivial.

The second step

The previous map $p_{1,1} : \mathcal{M}_{1,1} \to \mathcal{N}$ induces the mapping cone

$$\mathcal{C}\mathrm{one}(p_{1,1}) := \mathcal{M}_{1,1} \oplus \mathcal{N}[-1]$$

as defined in 1.2.11. Then choose a section

$$s_{1,1} : H^2(\mathcal{C}\mathrm{one}(p_{1,1})) \to Z^2(\mathcal{C}\mathrm{one}(p_{1,1})) \ . \tag{3.4}$$

Section means $H^2(s_{1,1}) = \mathrm{id}_{H^2(\mathcal{C}\mathrm{one}(p_{1,1}))}$. Note that this is indeed always possible, since every short exact sequence of vector spaces splits and in particular the sequence

$$0 \longrightarrow B^2(\mathcal{C}\mathrm{one}(p_{1,1})) \longrightarrow Z^2(\mathcal{C}\mathrm{one}(p_{1,1})) \longrightarrow H^2(\mathcal{C}\mathrm{one}(p_{1,1})) \longrightarrow 0 \ .$$

By the definition of the differential of the mapping cone we clearly have

$$Z^2(\mathcal{C}\mathrm{one}(p_{1,1})) \subset Z^2(\mathcal{M}_{1,1}) \oplus \mathcal{N}^1$$

and thus the canonical projections induce maps

$$\pi_1 : Z^2(\mathcal{C}\mathrm{one}(p_{1,1})) \to Z^2(\mathcal{M}_{1,1}) \text{ and}$$
$$\pi_2 : Z^2(\mathcal{C}\mathrm{one}(p_{1,1})) \to \mathcal{N}^1$$

by restriction. Then one defines

$$\sigma_{s_{1,1}} = \pi_1 \circ s_{1,1}$$

and, as described at the beginning of Section 1.3.1, we get the corresponding Hirsch extension

$$\mathcal{M}_{1,2} := \mathcal{M}_{1,1} \otimes_{d_{\sigma_{s_{1,1}}}} \bigwedge H^2(\mathcal{C}\mathrm{one}(p_{1,1})) \ .$$

To be precise $H^2(\mathcal{C}\mathrm{one}(p_{1,1}))$ is considered as vector space of homogeneous degree one, i.e. the above Hirsch extension is of degree one.

The homomorphism of differential graded algebras

$$p_{1,2} : \mathcal{M}_{1,2} \to \mathcal{N}$$

is defined by

$$p_{1,2} := p_{1,1} \otimes \widehat{\pi_2 \circ s_{1,1}} \ .$$

The map $\widehat{\pi_2 \circ s_{1,1}}$ denotes the map induced by the map $\pi_2 \circ s_{1,1}$ on the free graded algebra $\bigwedge H^2(\mathcal{C}\mathrm{one}(p_{1,1}))$ as described in 1.1.14.

For Section 4.2 another form of this model is quite useful, namely

Lemma 3.4.1. *There exists an equivalence of Hirsch extensions*

$$\bigwedge Z^1(\mathcal{N}) \otimes_{d_{\sigma_{s_{1,1}}}} \bigwedge H^2(\mathcal{C}\mathrm{one}(p_{1,1})) = \bigwedge Z^1(\mathcal{N}) \otimes_{d_\iota} \bigwedge \ker(H^2(p_{1,1})) \ ,$$

where $\iota : \ker(H^2(p_{1,1})) \to Z^2(\mathcal{M}_{1,1})$ *is the natural inclusion and* $\sigma_{s_{1,1}}$ *as defined above.*

Proof. The map $p_{1,1} : \mathcal{M}_{1,1} \to \mathcal{N}$ induces the long exact sequence

$$\cdots \longrightarrow H^1(\mathcal{M}_{1,1}) \xrightarrow{H^1(p_{1,1})} H^1(\mathcal{N}) \longrightarrow$$

$$H^2(\mathcal{C}\mathrm{one}(p_{1,1})) \xrightarrow{H^2(\pi_1)} H^2(\mathcal{M}_{1,1}) \xrightarrow{H^2(p_{1,1})} H^2(\mathcal{N}) \longrightarrow \cdots \ .$$

As proved in "the first step", the map $H^1(p_{1,1})$ is bijective. Thus, by exactness of the sequence, $H^2(\pi_1)$ is injective and thus (again by exactness)

$$H^2(\mathcal{C}\mathrm{one}(p_{1,1})) = \ker(H^2(p_{1,1})) \ .$$

Since the differential on $\mathcal{M}_{1,1}$ is trivial, $H^2(\mathcal{M}_{1,1}) = Z^2(\mathcal{M}_{1,1})$ and since $s_{1,1}$ is a section, we get a commutative diagram of the form

$$\begin{array}{ccccccc}
\cdots \longrightarrow H^2(\mathcal{C}\mathrm{one}(p_{1,1})) & \xrightarrow{H^2(\pi_1)} & H^2(\mathcal{M}_{1,1}) & \xrightarrow{H^2(p_{1,1})} & H^2(\mathcal{N}) \longrightarrow \cdots \\
\downarrow s_{1,1} & & \updownarrow \mathrm{id} & & \\
Z^2(\mathcal{C}\mathrm{one}(p_{1,1})) & \xrightarrow{\pi_1} & Z^2(\mathcal{M}_{1,1}) & &
\end{array}$$

Thus we have

$$\iota \circ H^2(\pi_1) = \pi_1 \circ s_{1,1}$$

and hence $\mathrm{id} \otimes H^2(\pi_1)$ is the claimed Hirsch equivalence. □

The induction step

Except of the indices and some minor differences, the induction step is a copy of the second step. Suppose $p_{1,k} : \mathcal{M}_{1,k} \to \mathcal{N}$ has already been constructed. As before we get the corresponding mapping cone

$$\mathcal{C}\mathrm{one}(p_{1,k}) := \mathcal{M}_{1,k} \oplus \mathcal{N}[-1] \ .$$

Then also choose a section

$$s_{1,k} : H^2(\mathcal{C}\mathrm{one}(p_{1,k})) \to Z^2(\mathcal{C}\mathrm{one}(p_{1,k})) \ .$$

Since $Z^2(\mathcal{C}\mathrm{one}(p_{1,k})) \subset Z^2(\mathcal{M}_{1,k}) \oplus \mathcal{N}^1$, the canonical projections induce maps

$$\begin{aligned} &\pi_1 : Z^2(\mathcal{C}\mathrm{one}(p_{1,k})) \to Z^2(\mathcal{M}_{1,k}) \text{ and} \\ &\pi_2 : Z^2(\mathcal{C}\mathrm{one}(p_{1,k})) \to \mathcal{N}^1 \end{aligned}$$

by restriction. Then one defines $\sigma_{s_{1,k}} := \pi_1 \circ s_{1,k}$ and, as described at the beginning of Section 1.3.1, we get a corresponding Hirsch extension

$$\mathcal{M}_{1,k+1} := \mathcal{M}_{1,k} \otimes_{d_{\sigma_{s_{1,k}}}} \bigwedge H^2(\mathcal{C}\mathrm{one}(p_{1,k}))$$

of degree one.

The homomorphism of differential graded algebras

$$p_{1,k+1} : \mathcal{M}_{1,k+1} \to \mathcal{N}$$

is defined by

$$p_{1,k+1} := p_{1,k} \otimes \widehat{\pi_2 \circ s_{1,k}} \ ,$$

where the map $\widehat{\pi_2 \circ s_{1,k}}$ denotes the map induced by the map $\pi_2 \circ s_{1,k}$ on the free graded algebra $\bigwedge H^2(\mathcal{C}\mathrm{one}(p_{1,k}))$ as described in Lemma 1.1.14.

We can also derive a different form of the k-th step similar as in the second step, but the differential is not as easy as in Lemma 3.4.1. Look at the long exact cohomology sequence

$$\cdots \longrightarrow H^1(\mathcal{M}_{1,k}) \xrightarrow{H^1(p_{1,k})} H^1(\mathcal{N}) \longrightarrow$$

$$H^2(\mathcal{C}\mathrm{one}(p_{1,k})) \xrightarrow{H^2(\pi_1)} H^2(\mathcal{M}_{1,k}) \xrightarrow{H^2(p_{1,k})} H^2(\mathcal{N}) \longrightarrow \cdots \tag{3.5}$$

Now a careful investigation of the differential of the Hirsch extension yields

Lemma 3.4.2. *Let $\mathfrak{n}$ be a nilpotent Lie algebra, $\mathcal{N} := \bigwedge \mathfrak{n}^*$ be the corresponding dual Lie algebra and $p_{1,k} : \mathcal{M}_{1,k} \to \mathcal{N}$ the k-th filtration step of the homological model. Then*

$$H^1(\mathcal{M}_{1,k}) = H^1(\mathcal{M}_{1,1}) = Z^1(\mathcal{N}) \ .$$

In particular $H^1(p_{1,k}) : H^1(\mathcal{M}_{1,k}) \to H^1(\mathcal{N})$ is an isomorphism.

Proof. The differential on $\mathcal{M}_{1,1}$ is the zero map and thus $H^1(\mathcal{M}_{1,1}) = \mathcal{M}^1_{1,1} = Z^1(\mathcal{N})$ are cocycles of degree one. Thus it is enough to show that the map $\sigma_{s_{1,k}} := \pi_1 \circ s_{1,k}$ which induces the differential on the Hirsch extension $\mathcal{M}_{1,k} \otimes_{d_{\sigma_{s_{1,k}}}} \bigwedge H^2(\mathcal{C}\mathrm{one}(p_{1,k})$ is injective for all $k > 0$ since then one adds no cocycles of degree one by constructing the next step. But this is true by commutativity of the diagram

$$\begin{array}{ccc}
Z^2(\mathcal{C}\mathrm{one}(p_{1.k})) & \xrightarrow{\ \pi_1\ } & Z^2(\mathcal{M}_{1,k}) \\
\uparrow{\scriptstyle s_{1,k}} & & \downarrow{\scriptstyle H^2} \\
H^2(\mathcal{C}\mathrm{one}(p_{1.k})) & \xrightarrow{\ H^2(\pi_1)\ } & H^2(\mathcal{M}_{1,k}) \ .
\end{array}$$

□

With the help of the last lemma and by exactness of the sequence (3.5), we get

$$H^2(\mathcal{C}\mathrm{one}(p_{1,k})) = \ker(H^2(p_{1,k})) \tag{3.6}$$

for all k. Thus by defining $\sigma_k := \pi_1 \circ s_{1,k} \circ H^2(\pi_2)^{-1}_{|\ker(H^2(p_{1,k}))}$ we get an equivalence of Hirsch extensions

$$\mathcal{M}_{1,k} \otimes_{d_{\sigma_{s_{1,k}}}} \bigwedge H^2(\mathcal{C}\mathrm{one}(p_{1,k})) = \mathcal{M}_{1,k} \otimes_{d_{\sigma_k}} \bigwedge \ker H^2((p_{1,k})) \ . \tag{3.7}$$

Note that it is not obvious that $H^2(\mathcal{C}\mathrm{one}(p_{1,n}))$ will be trivial for some $n \in \mathbb{N}$. This is equivalent to the injectivity of the map $H^2(p_{1,n})$ by (3.6). If the latter is true, then $p_{1,n} : \mathcal{M}_{1,n} \to \mathcal{N}$ would be a 1-minimal model by definition and since $\mathcal{N}$ is generated by elements of degree one, $p_{1,n}$ would already be an isomorphism by Lemma 1.6.2 and in particular $p_{1,n} : \mathcal{M}_{1,n} \to \mathcal{N}$ would already be a minimal model. That such an n exists will become clear after Section 3.4.4 and in particular will be proved in 3.4.6.

3.4.2 The filtration induced by the homological model

By construction, the maps $p_{1,k} : \mathcal{M}_{1,k} \to \mathcal{N}$ are injective. Thus

$$\mathcal{N}_{1,k} := p_{1,k}(\mathcal{M}_{1,k})$$

is a sub differential graded algebra isomorphic to $\mathcal{M}_{1,k}$ for all k. Moreover, in that way $\mathcal{N}_{1,k}$ is a Hirsch extension of $\mathcal{N}_{1,k-1}$ and thus

$$0 \subset \mathcal{N}_{1,1} \subset \cdots \subset \mathcal{N}_{1,k} \subset \cdots \subset \mathcal{N}$$

is a Hirsch filtration of $\mathcal{N}$. As already mentioned above, It is not clear that this filtration is indeed finite, i.e that there exists $n \in \mathbb{N}$ such that $\mathcal{N}_{1,n} = \mathcal{N}$. However this is proved in 3.4.6.

3.4.3 The filtration induced by the descending central series

The filtration of the Koszul complex $\mathcal{N}$ induced by the descending central series is defined by

$$\{0\} \subset \cdots \subset \mathcal{C}^{k-1}(\mathcal{N}) \subset \mathcal{C}^k(\mathcal{N}) \subset \cdots \subset \mathcal{C}^n(\mathcal{N}) = \mathcal{N} ,$$

where $(\mathcal{C}^k(\mathcal{N}), d) := \left(\bigwedge \mathcal{C}^k(\mathfrak{n})^\perp, d_{[\cdot,\cdot]}\right)$ and $\mathcal{C}^k(\mathfrak{n})^\perp$ is defined by 3.1.5 and 3.2.3.

Furthermore, one has an abelian, central extension of Lie algebras

$$0 \longrightarrow \mathfrak{a}_k \longrightarrow \mathfrak{n}_k \longrightarrow \mathfrak{n}_{k-1} \longrightarrow 0$$

for all $k \leq n$, where

$$\mathfrak{a}_k := \mathcal{C}^{k-1}(\mathfrak{n})/\mathcal{C}^k(\mathfrak{n}) \text{ and } \mathfrak{n}_k := \mathfrak{n}/\mathcal{C}^k(\mathfrak{n}) ,$$

with Lie brackets induced from the Lie bracket of $\mathfrak{n}$. This induced Lie bracket on $\mathfrak{n}_k$ is denoted by $[\cdot,\cdot]_k$ and the induced Lie bracket on $\mathfrak{a}_k$ is trivial.

Then we get isomorphisms of Hirsch extensions

$$\bigwedge \mathfrak{n}^*_{k-1} \otimes_{d_{\varphi^*_{k-1}}} \bigwedge \mathfrak{a}^*_k = \left(\bigwedge(\mathfrak{n}^*_{k-1} \oplus \mathfrak{a}^*_k), d_{[\cdot,\cdot]_{\varphi^*_{k-1}}}\right) ,$$

where $\varphi^*_{k-1} : \mathfrak{a}^*_k \to Z^2(\mathfrak{n}_{k-1})$, $d_{[\cdot,\cdot]_{\varphi^*_{k-1}}} = d_{\widehat{[\cdot,\cdot]_k \oplus \varphi^*_{k-1}}}$ and $d_{\widehat{[\cdot,\cdot]_k \oplus \varphi^*_{k-1}}}$ is the natural extension onto the free graded algebra as given by Lemma 1.1.14. Moreover, $\left(\mathcal{C}^k(\mathcal{N}), d\right) = \left(\bigwedge(\mathfrak{n}^*_{k-1} \oplus \mathfrak{a}^*_k), d_{[\cdot,\cdot]_{\varphi^*_{k-1}}}\right)$. For a proof see the proof of Proposition 3.2.6.

3.4.4 Comparison of both filtrations

Now a natural question is, if the two Hirsch filtrations presented above coincide. The answer is positive. However the author couldn't find a complete and satisfyingly detailed proof in the literature. Our proof is based on ideas given in [17] but doesn't use the Malcev hull and is quite more detailed.

Proposition 3.4.3. *The two Hirsch filtrations*

$$\{0\} \subset \cdots \subset \mathcal{C}^{k-1}(\mathcal{N}) \subset \mathcal{C}^{k}(\mathcal{N}) \subset \cdots \subset \mathcal{C}^{n}(\mathcal{N}) = \mathcal{N} \text{ and}$$
$$\{0\} \subset \cdots \subset \mathcal{N}_{1,k-1} \subset \mathcal{N}_{1,k} \subset \cdots \subset \mathcal{N}$$

as defined above coincide.

Proof. The proof is by induction.

Base: Since $H^1(\mathcal{N}) = Z^1(\mathcal{N}) = [\mathfrak{n}, \mathfrak{n}]^{\perp} = \mathcal{C}^1(\mathcal{N})$ by Lemma 3.2.4, we get an induced isomorphisms of graded algebras

$$\mathcal{N}_{1,1} = \bigwedge Z^1(\mathcal{N}) = \bigwedge [\mathfrak{g}, \mathfrak{g}]^{\perp} = \bigwedge \mathcal{C}^1(\mathcal{N})$$

by Lemma 1.1.14. Since all involved algebras have zero differential, it is also an isomorphism of differential graded algebras.

Induction step:
Suppose $\mathcal{C}^{k-1}(\mathcal{N}) = \mathcal{N}_{1,k-1}$. Since

$$\mathcal{C}^{k}(\mathcal{N}) = \mathcal{C}^{k-1}(\mathcal{N}) \otimes \bigwedge \mathfrak{a}_k^*$$

where $\mathfrak{a}_k^* := (\mathcal{C}^{k-1}(\mathfrak{n})/\mathcal{C}^{k}(\mathfrak{n}))^*$ and

$$\mathcal{N}_{1,k} = \mathcal{N}_{1,k-1} \otimes \bigwedge \ker(H^2(p_{1,k-1})),$$

it is enough to show that

$$\mathfrak{a}_k^* = \ker(H^2(p_{1,k-1})) \ .$$

This isomorphism is established with the help of the five term exact cohomology sequence corresponding to an ideal of a Lie algebra. Our main reference for this is [12]. Of particular interest is [12], Theorem 6, in the basic form

Theorem 3.4.4. *Let $\mathfrak{n}$ be a Lie algebra, $C \subset \mathfrak{n}$ an ideal and $\mathbb{K}$ a field (with trivial $\mathfrak{n}$ action). Then there exists an exact sequence*

$$\{0\} \longrightarrow H^1(\mathfrak{n}/C, \mathbb{K}) \longrightarrow H^1(\mathfrak{n}, \mathbb{K}) \longrightarrow H^1(C, \mathbb{K})^{\mathfrak{n}} \longrightarrow$$

$$H^2(\mathfrak{n}/C, \mathbb{K}) \longrightarrow H^2(\mathfrak{n}, \mathbb{K}) .$$

If we set $C := \mathcal{C}^{k-1}(\mathfrak{n})$. Then $\mathfrak{n}/C = \mathfrak{n}/\mathcal{C}^{k-1}(\mathfrak{n}) = \mathfrak{n}_{k-1}$ and we get the exact sequence

$$\{0\} \longrightarrow H^1(\mathfrak{n}_{k-1}, \mathbb{K}) \longrightarrow H^1(\mathfrak{n}, \mathbb{K}) \longrightarrow H^1(\mathcal{C}^{k-1}(\mathfrak{n}), \mathbb{K})^{\mathfrak{n}} \longrightarrow \tag{3.8}$$

$$H^2(\mathfrak{n}_{k-1}, \mathbb{K}) \longrightarrow H^2(\mathfrak{n}, \mathbb{K}) .$$

Furthermore, $H^1(\mathfrak{n}_{k-1}, \mathbb{K}) = H^1(\mathcal{C}^{k-1}(\mathcal{N}))$ and thus

$$H^1(\mathcal{C}^{k-1}(\mathcal{N})) = H^1(\mathcal{N}_{1,k}) = H^1(\mathcal{M}_{1,k}) = H^1(\mathcal{M}_{1,1}) = H^1(\mathcal{N})$$

by induction hypothesis, definition and Lemma 3.4.2. Thus, by exactness of the sequence (3.8), we get

$$H^1(\mathcal{C}^{k-1}(\mathfrak{n}), \mathbb{K})^{\mathfrak{n}} = \ker(H^2(\mathfrak{n}_{k-1}, \mathbb{K}) \to H^2(\mathfrak{n}, \mathbb{K})) .$$

By definition of this map, it is easy to see that

$$\ker(H^2(\mathfrak{n}_{k-1}, \mathbb{K}) \to H^2(\mathfrak{n}, \mathbb{K})) = \ker(H^2(p_{1,1})),$$

where $p_{1,1} : \mathcal{M}_{1,1} \to \mathcal{N}$ is the first step of the homological model. We compute

$$\begin{aligned} H^1(\mathcal{C}^{k-1}(\mathfrak{n}), \mathbb{K}) &:= H^1(\mathcal{C}^{k-1}(\mathcal{N})) = Z^1(\mathcal{C}^{k-1}(\mathcal{N})) \\ &= [\mathcal{C}^{k-1}(\mathfrak{n}), \mathcal{C}^{k-1}(\mathfrak{n})]^{\perp_{\mathcal{C}^{k-1}(\mathfrak{n})}} \subset \mathcal{C}^{k-1}(\mathfrak{n})^* . \end{aligned}$$

Now let $\omega \in H^1(\mathcal{C}^{k-1}(\mathfrak{n}), \mathbb{K})$. Since the action of $Z \in \mathfrak{n}$ on $H^1(\mathcal{C}^{k-1}(\mathfrak{n}), \mathbb{K})$ is defined by $Z * \omega(X) := \omega[Z, X]$, we compute

$$\begin{aligned} H^1(\mathcal{C}^{k-1}(\mathfrak{n}), \mathbb{K})^{\mathfrak{n}} &= ([\mathcal{C}^{k-1}(\mathfrak{n}), \mathcal{C}^{k-1}(\mathfrak{n})]^{\perp_{\mathcal{C}^{k-1}(\mathfrak{n})}})^{\mathfrak{n}} \\ &= [\mathcal{C}^{k-1}(\mathfrak{n}), \mathcal{C}^{k-1}(\mathfrak{n})]^{\perp_{\mathcal{C}^{k-1}(\mathfrak{n})}}) \cap \mathcal{C}^{k}(\mathfrak{n})^{\perp_{\mathcal{C}^{k-1}(\mathfrak{n})}} \\ &= \mathcal{C}^{k}(\mathfrak{n})^{\perp_{\mathcal{C}^{k-1}(\mathfrak{n})}} \\ &\overset{\text{Lemma 3.2.5}}{=} \mathcal{C}^{k}(\mathfrak{n})^{\perp_{\mathfrak{n}}} / \mathcal{C}^{k-1}(\mathfrak{n})^{\perp_{\mathfrak{n}}} \\ &\overset{\text{Lemma 3.2.5}}{=} (\mathcal{C}^{k-1}(\mathfrak{n})/\mathcal{C}^{k}(\mathfrak{n}))^* \\ &= \mathfrak{a}_k^* . \end{aligned}$$

□

An elementary proof of Proposition 3.4.3 in the two-step nilpotent case

Suppose that $\mathfrak{n}$ is two-step nilpotent, that is $\mathcal{C}^2(\mathfrak{n}) = 0$. We can give a self-contained proof of Proposition 3.4.3 in that case without the help of Theorem 3.4.4 , which is a quite technical result in [12].

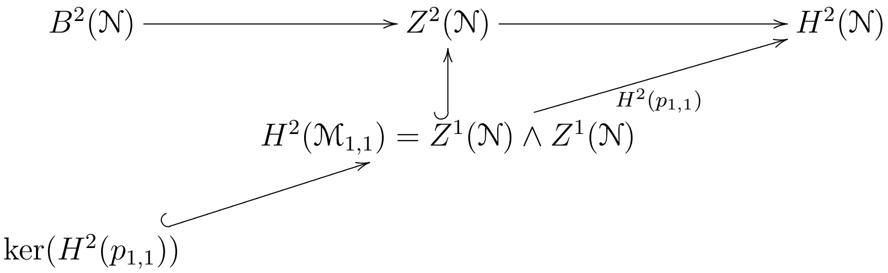
We have the canonical isomorphisms

$$\mathcal{N}_{1,2} = \bigwedge Z^1(\mathcal{N}) \otimes \bigwedge \ker(H^2(p_{1,1}))$$

and

$$\mathcal{C}^2(\mathcal{N}) = \bigwedge Z^1(\mathcal{N}) \otimes \bigwedge [\mathfrak{n}, \mathfrak{n}]^* \ .$$

Thus it is enough to show that $\ker(H^2(p_{1,1})) = [\mathfrak{n}, \mathfrak{n}]^*$. From the commutative diagram



we deduce that $ker(H^2(p_{1,1})) = B^2(\mathcal{N})$. By definition the kernel of the map $d : \mathcal{N}^1 \to B^2(\mathcal{N})$ is $Z^1(\mathcal{N}) = \mathcal{C}^1(\mathfrak{n})^\perp$ and hence

$$\ker H^2(p_{1,1}) \cong B^2(\mathcal{N}) \cong \mathcal{N}^1/\mathcal{C}^1(\mathfrak{n})^\perp \cong \mathcal{C}^1(\mathfrak{n})^* = ([\mathfrak{n}, \mathfrak{n}])^* \ .$$

Corollary 3.4.5. *For all k there exist isomorphism of Hirsch extensions*

$$\begin{aligned}\left(\bigwedge(\mathfrak{n}^*_{k-1} \oplus \mathfrak{a}^*_k), d_{[\cdot,\cdot]_{\varphi^*_{k-1}}}\right) &= \bigwedge \mathfrak{n}^*_{k-1} \otimes_{d_{\varphi^*_{k-1}}} \bigwedge \mathfrak{a}^*_k \\ &= \mathcal{M}_{1,k-1} \otimes_{d_{\sigma_{s_{1,k}}}} \bigwedge H^2(\mathcal{C}one(p_{1,k-1})) \\ &= \mathcal{M}_{1,k-1} \otimes_{d_{\sigma_k}} \bigwedge \ker H^2((p_{1,k-1})) \ .\end{aligned}$$

Proof. A direct consequence of Proposition 3.4.3. □

Corollary 3.4.6. *Let $(\mathfrak{n}, [\cdot,\cdot])$ be a n-step nilpotent Lie algebra and $(\mathcal{N}, d) := (\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]})$ the corresponding Koszul complex. Then*

$$p_{1,n} : \mathcal{M}_{1,n} \to N$$

is an isomorphism. In particular it is the minimal model.

Proof. A direct consequence of Proposition 3.4.3. □

CHAPTER

FOUR

THE AUTOMORPHISM GROUP OF A TWO-STEP NILPOTENT LIE ALGEBRA AND COHOMOLOGICAL REPRESENTATIONS

We are going to study the structure of the automorphism group of a two-step nilpotent Lie algebra. In particular we are interested in cohomological representations of this group. We start with a survey about the few known results in the general nilpotent case concerning these questions and also recall all the facts that have already been established in the preceding chapters.

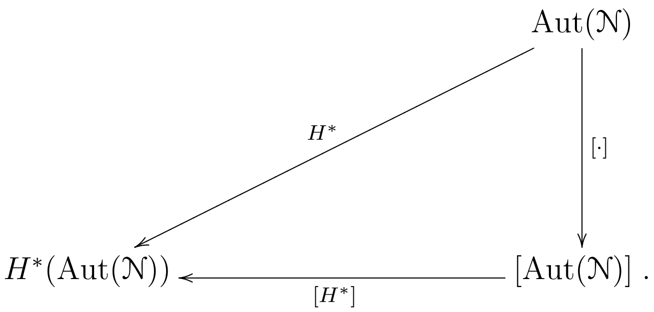
4.0.5 Known results

Let $(\mathfrak{n}, [\cdot,\cdot])$ be a nilpotent Lie algebra as defined at the beginning of Section 3.1 and $(\mathcal{N}, d) := \left(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\right)$ the corresponding Koszul complex as described at the beginning of Section 3.2. This Koszul complex can also be characterised more abstractly as a differential graded algebra, which is free and generated by elements of degree one. Moreover, it admits a series of sub differential graded algebras which are induced by the descending central series of the Lie algebra. This filtration is called a Hirsch filtration since each consecutive step is a Hirsch extension of the previous step. We also call differential graded algebras of that type dual Lie algebras since there exists an equivalence of categories between dual Lie algebras and nilpotent Lie algebras as demonstrated in Proposition 3.2.7. In particular there exists an isomorphism of groups

$$\mathrm{Aut}(\mathfrak{n}) = \mathrm{Aut}(\mathcal{N})$$

between the automorphism group of a Lie algebra as defined in 3.1.3 and the automorphism group of the corresponding dual Lie algebra as defined in

2.0.6. So in order to study the automorphism groups of Lie algebras, we can also study the automorphism groups of dual Lie algebras only. Since a dual Lie algebra is a Sullivan algebra, we get the associated automorphism group modulo homotopy $[\mathrm{Aut}(\mathcal{N})]$ as demonstrated in Lemma 2.0.7. By the same lemma we get a commutative diagram of groups



By Proposition 2.1.6 $[\mathrm{Aut}(\mathcal{N})] = \mathrm{Aut}(\mathcal{N})/\mathrm{Inn}(\mathcal{N})$, where $\mathrm{Inn}(\mathcal{N})$ are the inner automorphisms as defined in Section 2.1. In particular this implies $\mathrm{Inn}(\mathcal{N}) \subset \ker(H^k)$ for all $k \in \mathbb{N}$. By Proposition 2.1.5 the kernel of H^* is unipotent. According to O. Baues and F. Grunewald, there exists the following specialisation.

Proposition 4.0.7 ([1], Prop 13.2)**.** *Let*

$$H^1 : \mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(H^1(\mathfrak{g}))$$

be the Lie algebra cohomology functor as defined in 3.2.2. Then $\ker(H^1)$ *is unipotent.*

Proof. The proof is by contradiction. Let $(\mathcal{N}, d)$ be the dual Lie algebra and

$$\{0\} \subset \mathcal{C}^1(\mathcal{N}) \subset \cdots \subset \mathcal{C}^k(\mathcal{N}) \subset \cdots \subset \mathcal{C}^n(\mathcal{N}) = \mathcal{N}$$

the standard filtration as described in Section 3.4.3, where $\mathcal{C}^k(\mathcal{N}) := \mathcal{C}^k(\mathfrak{n})^\perp$. Furthermore, let $\phi \in \mathrm{Aut}(\mathcal{N})$ with $H^1(\phi) = \mathrm{id}$. Since a Lie algebra homomorphisms respects the descending central series, we get an induced map

$$\phi_{|\mathcal{C}^k(\mathcal{N})} : \mathcal{C}^k(\mathcal{N}) \to \mathcal{C}^k(\mathcal{N})$$

by restriction. Moreover, $\phi_{|\mathcal{C}^1(\mathcal{N})} = \mathrm{id}$ by assumption, since $H^1(\mathcal{N}) = \mathcal{C}^1(\mathcal{N}) = [\mathfrak{n}, \mathfrak{n}]^\perp$. Now choose a basis $\{x_1, \cdots, x_l\}$ of $\mathcal{C}^k(\mathcal{N})$ and complete it to a basis

$$\{x_1, \cdots, x_l, x_{l+1}, \cdots, x_m\}$$

of $\mathcal{C}^{k+1}(\mathcal{N})$. Suppose that $\phi_{|\mathcal{C}^k(\mathcal{N})} = \mathrm{id}$ and that the vector space $\langle x_{l+1}, \cdots, x_m\rangle$ is invariant under $\phi_{|\mathcal{C}^{k+1}(\mathcal{N})}$. Then for $x \neq 0 \in \langle x_{l+1}, \cdots, x_m\rangle$ we compute

$$d(x - \phi(x)) = dx - \phi(\underbrace{dx}_{\in \mathcal{C}^k(\mathcal{N})}) = dx - dx = 0 \ .$$

Since $x \notin Z^1(\mathcal{N}) = \mathcal{C}^1(\mathcal{N})$ for all $k > 1$, it follows that $\phi(x) = x$. Thus $\phi_{|\mathcal{C}^{k+1}(\mathcal{N})} = \mathrm{id}$. So by induction ϕ is trivial. Thus if ϕ is not an upper triangular matrix with respect to the inductive chosen basis, then it is already the identity. Hence the claim follows, since upper triangular matrices correspond to unipotent automorphisms. □

A direct consequence is

Corollary 4.0.8. *Let $R \subset \mathrm{Aut}(\mathfrak{n})$ be a reductive subgroup. Then the representation $H^1 : R \to \mathrm{Aut}(H^1(\mathfrak{n}))$ is faithful.*

Proof. Since R is reductive and $\ker(H^1)$ is a unipotent normal subgroup by Proposition 4.0.7, $R \cap \ker(H^1) = \{\mathrm{id}\}$ by Corollary 2.2.7. Thus the restriction of H^1 onto R is injective. □

Moreover, we can decompose the automorphism group into a reductive part and a unipotent one, that is

Corollary 4.0.9 (Levi–Mostow decomposition)**.** *Let $\mathfrak{n}$ be a nilpotent Lie algebra. Then there exists a decomposition*

$$\mathrm{Aut}(\mathfrak{n}) = U \rtimes \mathrm{Aut}(\mathfrak{n})/U \ ,$$

where $U \subset \mathrm{Aut}(\mathfrak{n})$ is a maximal, normal unipotent subgroup. In particular $\mathrm{Aut}(\mathfrak{n})/U$ is a reductive group.

So this rises the question, if the reductive part is already the biggest normal subgroup, which acts faithful on the first cohomology group. As we will see later on in this chapter, this is not true in general.

4.1 A basic characterisation of automorphisms in the two-step nilpotent case

For the rest of this section $(\mathfrak{n}, [\cdot,\cdot])$ denotes a two-step nilpotent Lie algebra and $(\mathcal{N}, d)$ the corresponding dual Lie algebra or Koszul complex. We give an informal description of the automorphism group in that case.

First of all $\mathrm{Aut}(\mathfrak{n}) = \mathrm{Aut}(\mathcal{N})$ by Proposition 3.2.7. Furthermore, by Corollary 3.4.5 and Corollary 3.4.6, we have an isomorphism of differential graded algebras

$$(\mathcal{N}, d) = \left(\bigwedge\left([\mathfrak{n},\mathfrak{n}]^{\perp} \oplus [\mathfrak{n},\mathfrak{n}]^{*}\right), \widehat{0 \oplus \varphi_2^{*}}\right) =: (\mathcal{N}_2, d')$$

where $\varphi_2^* : [\mathfrak{n},\mathfrak{n}]^* \to \bigwedge^2 [\mathfrak{n},\mathfrak{n}]^{\perp}$ and $\widehat{0 \oplus \varphi_2^*}$ is the natural extension onto the free graded algebra as given by Lemma 1.1.14. Thus $\mathrm{Aut}(\mathcal{N}) = \mathrm{Aut}(\mathcal{N}_2)$. Now choose a basis

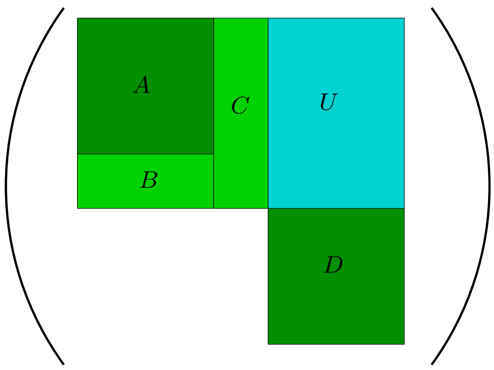
$$\{z_1, \cdots, z_k, z_{k+1}, \cdots, z_n, v_1, \cdots, v_l\}$$

of $\mathcal{N}_2^1$, such that only $\{z_{k+1}^*, \cdots, z_n^*\}$ lie in the center of $\mathfrak{n}$. Then, by definition of the differential, $d'v \in \bigwedge^2 \langle z_1, \cdots, z_k\rangle$ for all $v \in [\mathfrak{n},\mathfrak{n}]^*$ and $dz = 0$ for all $z \in [\mathfrak{n},\mathfrak{n}]^{\perp}$.

Lemma 4.1.1. *With respect to the chosen basis*

$$\{z_1, \cdots, z_k, z_{k+1}, \cdots, z_n, v_1, \cdots, v_l\}$$

as above, an automorphism $\Phi \in \mathrm{Aut}(\mathcal{N}_2)$ *is schematically represented by a block matrix of the form*

$M :=$ 

,

where A and D are invertible matrices. Moreover, If A is the identity matrix, so is D, which is indicated by their equal colouring. The block consisting of the matrices A, B and C is an invertible matrix and U is arbitrary.

Furthermore, the subgroup of $\mathrm{Aut}(\mathcal{N})$ *consisting of the matrices M with $U = 0$ is the biggest subgroup that operates faithful on* $H^1(\mathcal{N})$.

Proof. Let $\Phi \in \mathrm{Aut}(\mathcal{N}_2)$ be an automorphism. Since the Koszul complex is a free graded algebra, any automorphism is already defined by a matrix $M \in$

$\mathrm{GL}(\mathcal{N}_2^1)$ after choosing a basis by Lemma 1.1.15. Since Φ is an isomorphism of differential graded algebras, it maps cocycles to cocycles, which explains the 0 under the block consisting of the matrices A,B and C. This last block and the matrix D have to be invertible matrices, since M is invertible.

Furthermore, the matrices A and D are linked by the equation

$$\Phi \circ d' = d' \circ \Phi \ .$$

This means in particular that

$$d'D = \widehat{A} d' \ , \tag{4.1}$$

where $\widehat{A}$ is the natural extension induced by A.

This implies that A must be invertible by the following argumentation. Since $d_{|\langle\{v_1,\cdots,v_l\}\rangle}$ is injective, $\widehat{A}$ bust be an isomorphism of the subspace $\bigwedge^2\langle\{z_1,\cdots z_k\}\rangle$ by equation (4.1). Assume there exists $z_p \wedge z_q \neq 0 \in \bigwedge^2\langle\{z_1,\cdots,z_k\}\rangle$ with $Az_p = 0$ and or $Az_q = 0$. Then $Az_p \wedge Az_q = \widehat{A}(z_n \wedge z_m)$ and thus $z_n \wedge z_m = 0$ since $\widehat{A}$ is an isomorphism, a contradiction. Moreover, if A is the identity matrix, so is $\widehat{A}$ and thus D by equation (4.1).

Now the matrix U is clearly arbitrary, since $dz = 0$ for all

$$z \in \langle\{z_1,\cdots,z_k,z_{k+1},\cdots,z_n\}\rangle \ .$$

$\square$

So in order to understand which automorphisms act trivially on cohomology it is enough to look at automorphisms represented by matrices as given in Figure 4.1 where U is an arbitrary matrix.

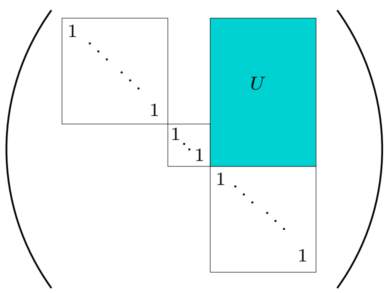
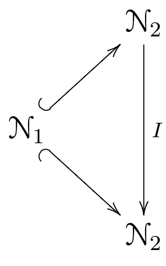


Figure 4.1: *Automorphisms acting trivially on $H^1(\mathfrak{n})$*

We can give a further characterisation of automorphisms of that kind. Let I be a Hirsch equivalence of $\mathcal{N}_2$ which means by definition that the diagram



commutes. Thus with respect to the basis as chosen above I is represented by a matrix as in figure 4.1, since it acts trivially on $Z^1(\mathcal{N}_2)$ by definition. With respect to the dual basis we get a Lie algebra homomorphism

$$I^* : \mathfrak{n} \to \mathfrak{n}$$

represented by the transpose matrix as given in Figure 4.1. So I^* defines an isomorphisms of Lie algebra extensions which means that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [\mathfrak{n},\mathfrak{n}] & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{n}/[\mathfrak{n},\mathfrak{n}] & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow I^* & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & [\mathfrak{n},\mathfrak{n}] & \longrightarrow & \mathfrak{n} & \longrightarrow & \mathfrak{n}/[\mathfrak{n},\mathfrak{n}] & \longrightarrow & 0 \end{array}$$

commutes. On the other hand such an isomorphism of Lie algebra extensions obviously dualises to a Hirsch equivalence of the corresponding Koszul complex.

4.2 Lifting cohomological representations

We now carefully investigate the interaction between automorphisms of a two-step nilpotent Lie algebra and their induced maps on the first Lie algebra cohomology group. In particular, we derive a decomposition of the automorphism group in that way. This decomposition turns out to be different from the Levi–Mostow decomposition 4.0.9 which will be discussed later on in Section 4.4.

Let $\big(\mathfrak{n}, [\cdot,\cdot]\big)$ be a two-step nilpotent Lie algebra and

$$(\mathcal{N}, d) := \big(\bigwedge \mathfrak{n}^*, d_{[\cdot,\cdot]}\big)$$

the corresponding Koszul complex or dual Lie algebra. Furthermore, let

$$H^1 : \mathrm{Aut}(\mathcal{N}) \to \mathrm{Aut}\big(H^1(\mathcal{N})\big)$$

be the cohomology functor of degree one as defined in 1.2.6 and

$$\mathcal{O}^1_{\mathcal{N}} := H^1\big(\mathrm{Aut}(\mathcal{N})\big)$$

be the image of $\mathrm{Aut}(\mathcal{N})$ under H^1 in $\mathrm{Aut}\big(H^1(\mathcal{N})\big)$. Then the following holds.

Theorem 4.2.1.

$$\mathrm{Aut}(\mathcal{N}) = \ker(H^1) \rtimes \mathcal{O}^1_{\mathcal{N}} \,.$$

Proof. The idea of the proof is to construct consecutively a section

$$\{1\} \longrightarrow \ker(H^1) \hookrightarrow \operatorname{Aut}(\mathcal{M}_{1,2}) \xrightarrow{\;H^1\;} \mathcal{O}^1_{\mathcal{N}} \longrightarrow \{1\} \qquad \mathcal{O}^1_{\mathcal{N}} \overset{\mathcal{M}_{1,2}(\cdot)}{\dashrightarrow} \operatorname{Aut}(\mathcal{M}_{1,2}) \tag{4.2}$$

from cohomology to the homological model

$$p_{1,2} : \mathcal{M}_{1,2} \to \mathcal{N} \,.$$

As demonstrated in Section 2.2.1, such a splitting induces the claimed decomposition. The homological model and it's filtration are presented in detail in Section 3.4. We also take the explicit form 3.4.1, which is shortly given by the following data:

- $\mathcal{M}_{1,1} := \bigwedge Z^1(\mathcal{N})$ with zero differential.
- A homomorphism of differential graded algebras

  $$p_{1,1} : \mathcal{M}_{1,1} \to \mathcal{N}$$

  given by $p_{1,1} := \widehat{\iota}$, where $\widehat{\iota}$ is the natural extension of the inclusion

  $$\iota : Z^1(\mathcal{N}) \to \mathcal{N}^1$$

  onto the free graded algebra $\bigwedge Z^1(\mathcal{N})$ as given by Lemma 1.1.14.
- $\mathcal{M}_{1,2} := \mathcal{M}_{1,1} \otimes_{d_\iota} \bigwedge \ker\big(H^2(p_{1,1})\big)$ is a Hirsch extension of degree one where ι is the inclusion as above.
- A homomorphism of differential graded algebras

  $$p_{1,2} : \mathcal{M}_{1,2} \to \mathcal{N}$$

  whose concrete definition is not of interest for the moment. Of importance is only the fact that this map is actually an isomorphism of differential graded algebras by Corollary 3.4.6.

Since $p_{1,2} : \mathcal{M}_{1,2} \to \mathcal{N}$ is an isomorphism of differential graded algebras, we get an isomorphism of groups

$$\operatorname{Aut}(\mathcal{M}_{1,2}) = \operatorname{Aut}(\mathcal{N})$$

by conjugation with $p_{1,2}$. Hence it is enough to prove the claim for the group $\operatorname{Aut}(\mathcal{M}_{1,2})$. Now let

$$f^* \in \mathcal{O}^1_{\mathcal{N}} \text{ and } f \in \operatorname{Aut}(\mathcal{N}) \text{ with } H^1(f) = f^*$$

be a preimage. Since $H^1(\mathcal{N}) = \mathcal{M}_{1,1}^1 = Z^1(\mathcal{N})$, we can regard f^* as a linear map

$$f^* : \mathcal{M}_{1,1}^1 \to \mathcal{M}_{1,1}^1 \ .$$

By Lemma 1.1.22 and since the differential on $\mathcal{M}_{1,1}$ is trivial we get an induced homomorphism of differential graded algebras

$$\widehat{f^*} : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1} \ .$$

Moreover, the diagram of differential graded algebras

$$\begin{array}{ccc} \mathcal{M}_{1,1} = \bigwedge Z^1(\mathcal{N}) & \xrightarrow{p_{1,1}=\widehat{\iota}} & \mathcal{N} \\ \Big\downarrow \widehat{f^*} & & \Big\downarrow f \\ \mathcal{M}_{1,1} = \bigwedge Z^1(\mathcal{N}) & \xrightarrow{p_{1,1}=\widehat{\iota}} & \mathcal{N} \end{array} \tag{4.3}$$

commutes since f is a homomorphism of differential graded algebras and thus maps cocycles to cocycles. Now the map

$$\begin{aligned} &\mathcal{M}_{1,1}(\cdot) : \mathcal{O}_{\mathcal{N}}^1 \to \mathrm{Aut}(\mathcal{M}_{1,1}), \\ &\mathcal{M}_{1,1}(f^*) := \widehat{f^*} \end{aligned}$$

is obviously a homomorphism of groups with $H^1(\mathcal{M}_{1,1}(f^*)) = f^*$ for all $f^* \in \mathcal{O}_{\mathcal{N}}^1$. Let us turn to the second filtration step.

Since Diagram (4.3) commutes, $H^2\big(\mathcal{M}_{1,1}(f^*)\big)$ defines a map

$$H^2\big(\mathcal{M}_{1,1}(f^*)\big)_{|\ker} : \ker H^2(p_{1,1}) \to \ker H^2(p_{1,1})$$

by restriction. Again by Lemma 1.1.14, we get an induced map

$$H^2\big(\widehat{\mathcal{M}_{1,1}(f^*)}\big)_{|\ker} : \bigwedge \ker H^2(p_{1,1}) \to \bigwedge \ker H^2(p_{1,1})$$

and thus an isomorphism of graded algebras

$$\mathcal{M}_{1,1}(f^*) \otimes H^2\big(\widehat{\mathcal{M}_{1,1}(f^*)}\big)_{|\ker} : \mathcal{M}_{1,2} \to \mathcal{M}_{1,2} \ .$$

By definition of the differential on $\mathcal{M}_{1,2}$ and by Lemma 1.1.22, this map is also obviously a Hirsch automorphism for all $f^* \in \mathcal{O}_{\mathcal{N}}^1$ and thus in particular an isomorphism of differential graded algebras. For the definition of a Hirsch automorphism please see Definition 1.3.4. Now define

$$\begin{aligned} &\mathcal{M}_{1,2}(\cdot) : \mathcal{O}_{\mathcal{N}}^1 \to \mathrm{Aut}(\mathcal{M}_{1,2}), \\ &\mathcal{M}_{1,2}(f^*) := \mathcal{M}_{1,1}(f^*) \otimes H^2\big(\widehat{\mathcal{M}_{1,1}(f^*)}\big)_{|\ker} \ . \end{aligned}$$

Now $\mathcal{M}_{1,1}(\cdot) : \mathcal{O}^1_{\mathcal{N}} \to \mathrm{Aut}(\mathcal{M}_{1,1})$ is a homomorphism of groups as explained before. Furthermore, H^2 and restriction of maps are clearly functorial and thus $\mathcal{M}_{1,2}(\cdot)$ is a homomorphism of groups. Since $H^1(\mathcal{M}_{1,2}) = H^1(\mathcal{M}_{1,1})$ by Lemma 3.4.2, we have

$$H^1\big(\mathcal{M}_{1,2}(f^*)\big) = f^*$$

and thus we get a splitting of (4.2).

□

A direct consequence is

Corollary 4.2.2 (H^1-decomposition)**.** *Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra and*

$$H^1 : \mathrm{Aut}(\mathfrak{n}) \to \mathrm{Aut}\big(H^1(\mathfrak{n})\big)$$

be the Lie algebra cohomology functor in degree 1 as defined in 3.2.2. Then

$$\mathrm{Aut}(\mathfrak{n}) = \ker(H^1) \rtimes \mathrm{im}(H^1) \ .$$

Proof. By the equivalence of categories 3.2.7 we have $\mathrm{Aut}(\mathcal{N}) = \mathrm{Aut}(\mathfrak{n})$ and the claim directly follows by the definition of the Lie algebra cohomology 3.2.2 and the last theorem. □

4.3 Nilpotent Lie algebras with one dimensional commutator

Let $\mathfrak{n}$ be a nilpotent Lie algebra with $\dim([\mathfrak{n}, \mathfrak{n}]) = 1$. In particular $\mathfrak{n}$ is a two-step nilpotent Lie algebra. Then, according to Section 4.1, the candidates of automorphisms of the corresponding Koszul complex acting trivially on cohomology are given schematically by matrices as shown in Figure 4.2.

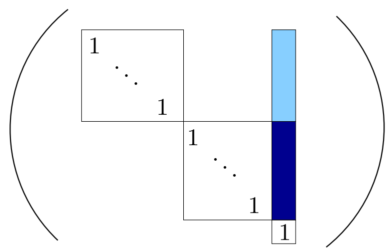


Figure 4.2: *Candidates of automorphisms acting trivially on all cohomology groups*

It is easy to see that the lighter blue ones precisely represent the inner automorphisms and thus the others are outer automorphisms. According to Section 2.1, the inner automorphisms act trivially on cohomology.

Proposition 4.3.1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra with one dimensional commutator, then*

$$\ker(H^*) = \mathrm{Inn}(\mathfrak{n}) \ .$$

Before we can prove this, we compute the automorphism group and after wards we need a small lemma about coboundaries.

Lemma 4.3.2. *Let $\mathfrak{n}$ be a nilpotent Lie algebra with one dimensional commutator and $\mathcal{N}$ the corresponding Koszul complex. Then we can choose a basis, such that*

- $\operatorname{Aut}(\mathfrak{n}) = \left\{ M^{(A,a,\lambda_A)} \mid A \in CSp(\omega),\ a \in \mathbb{K}^n \right\}$, *where*

  $$M^{(A,a,\lambda_A)} := \begin{pmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ a & \lambda_A \end{pmatrix} \text{ and}$$

  $$CSp(\omega) := \Big\{ f \in GL(W_n) \mid \omega(f(v), f(w)) = \lambda_f \omega(v,w) \text{ for some } \lambda_f \in \mathbb{K} \text{ depending on } f \Big\}$$

  is the conformal symplectic group of a specific, possibly degenerated, alternating form ω depending on the Lie bracket.

- $\operatorname{Inn}(\mathfrak{n}) = \left\{ M^{(E_n,(a_1,\cdots a_k,0,\cdots,0),1)} \mid a_1, \cdots, a_k \in \mathbb{K} \right\}$

With respect to the dual basis we get

- $\operatorname{Aut}(\mathcal{N}) = \left\{ M_{(A,a,\lambda_A)} \mid A \in CSp(\omega), a \in \mathbb{K}^n \right\}$ *with*

  $$M_{(A,a,\lambda_A)} := \begin{pmatrix} A & a \\ 0\ \ 0 & \lambda_A \end{pmatrix}.$$

- $\operatorname{Inn}(\mathcal{N}) = \{ M_{(E_n,(a_1,\cdots,a_k,0,\cdots,0),1)} \mid a_1, \cdots, a_k \in \mathbb{K} \}$.

Proof. Let $\mathfrak{n} = W_n \oplus [\mathfrak{n},\mathfrak{n}]$ with corresponding bases $\{X_1, \cdots, X_n\}$ and $\{V_1\}$. Then we can define a possibly degenerated alternating form

$$\omega : W_n \times W_n \to \mathbb{K} \tag{4.4}$$

on a basis by

$$[X_i, X_j] = \omega(X_i, X_j) V_1 .$$

Since an automorphism $\varphi \in \mathrm{Aut}(\mathfrak{n})$ respects the Lie bracket, this implies $\varphi_{|W_n} \in \mathrm{CSp}(\omega)$. If we order the basis such that $\{X_{k+1}, \cdots, X_n\}$ are in the center of $\mathfrak{n}$ as in Section 4.1 and define

$$M^{(A,a,\lambda)} := \begin{pmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ a & \lambda_A \end{pmatrix},$$

we get

$$\mathrm{Aut}(\mathfrak{n}) = \left\{ M^{(A,a,\lambda_A)} \mid A \in \mathrm{CSp}(\omega),\ a \in \mathbb{K}^n \right\}.$$

Moreover, we compute that

$$M^{(E_n,(a_1,\cdots a_k,0,\cdots,0),1)} := \begin{pmatrix} \multicolumn{2}{c}{E_n} & \begin{matrix} 0 \\ 0 \end{matrix} \\ a_1 \cdots a_k & 0 \cdots 0 & 1 \end{pmatrix}$$

are inner automorphisms and that

$$M^{(E_n(0,\cdots,0,a_{k+1},\cdots,a_n),1)} := \begin{pmatrix} \multicolumn{2}{c}{E_n} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \cdots 0 & a_{k+1} \cdots a_n & 1 \end{pmatrix}$$

are outer automorphisms.

If we turn to the dual Lie algebra $\mathcal{N} := \bigwedge \mathfrak{n}^*$ with dual basis

$$\{x_1, \cdots, x_k, x_{k+1} \cdots, x_n, v_1\},$$

then the automorphism group of $\mathfrak{n}$ dualises to

$$\mathrm{Aut}(\mathcal{N}) = \left\{ M_{(A,a,\lambda_A)} \mid A \in \mathrm{CSp}(\omega), a \in \mathbb{K}^n \right\}.$$

□

Lemma 4.3.3. *Let $w \in \bigwedge(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])^*$ and $v \in \bigwedge[\mathfrak{n},\mathfrak{n}]^*$. Then $w \wedge v$ is not a coboundary.*

Proof. We can write every element in $\mathcal{N}$ as $z \wedge u$ with $z \in \bigwedge Z^1(\mathcal{N})$ and $u \in \bigwedge[\mathfrak{n},\mathfrak{n}]^*$. Suppose $d(z \wedge u) = w \wedge v$. But $d(z \wedge u) = -z \wedge du$ and since $du \in \bigwedge(\mathfrak{n}/[\mathfrak{n},\mathfrak{n}])^*$ the claim follows by contradiction. □

Now we are ready for the

Proof of Proposition 4.3.1. First we choose a basis

$$\{x_1, \cdots, x_k, x_{k+1}, \cdots, x_n, v_1\},$$

of $\mathfrak{n}^*$ as in Lemma 4.3.2. Then $x_1, \cdots, x_n$ are one cocycles and

$$dv_1 = \sum_{i \in I} \sum_{j \in J_i} a_{ij} x_i \wedge x_j$$

with $a_{ij} \in \mathbb{K}$, $I \subset \{1, \cdots, k\}$ and $J_i \subset \{1, \cdots, k\} \setminus \{1, \cdots, i\}$ depends on i. Since ω induces a symplectic form on $\mathfrak{n}/\mathcal{Z}(\mathfrak{n}) = \langle x_1, \cdots, x_k \rangle$, we can choose a basis such that this expression simplifies to

$$dv_1 = \sum_{0<i<k, i \text{ odd}} x_i \wedge x_{i+1} \ .$$

For automorphisms of the form $M_{(A,a,\lambda_A)} := \begin{pmatrix} A & a \\ 0 & \lambda_A \end{pmatrix}$, we see that $H^1(M_{(A,a,\lambda_A)}) = A$ and thus the kernel of the action of the automorphism group on $H^1(\mathcal{N})$ is given by inner automorphisms of the form

$$M_{(E_n,(a_1,\cdots a_k,0,\cdots,0),1)} := \begin{pmatrix} & a_1 \\ & \vdots \\ & a_k \\ E_n & 0 \\ & \vdots \\ & 0 \\ 0 \cdots\cdots 0 & 1 \end{pmatrix}$$

and outer automorphisms of the form

$$M_{(E_n,(0,\cdots,0,a_{k+1},\cdots,a_n),1)} := \begin{pmatrix} & 0 \\ & \vdots \\ & 0 \\ E_n & a_{k+1} \\ & \vdots \\ & a_n \\ 0 \cdots\cdots 0 & 1 \end{pmatrix} .$$

Now $z := v_1 \wedge \bigwedge_{0<i<k, i \text{ odd}} x_i$ is a cocycle since

$$dz = d\Big(v_1 \wedge \bigwedge_{0<i<k, i \text{ odd}} x_i\Big) = \Big(\sum_{0<i<k, i \text{ odd}} x_i \wedge x_{i+1}\Big) \wedge \bigwedge_{0<i<k, i \text{ odd}} x_i = 0 \ .$$

Moreover $z \neq 0$ nor is it a coboundary by Lemma 4.3.3.

The action of $M_{(E_n,(0,\cdots,0,a_{k+1},\cdots,a_n),1)}$ on z is given by

$$M_{(E_n,(0,\cdots,0,a_{k+1},\cdots,a_n),1)} z = z + \sum_{l=k+1}^{n} \left(a_l x_l \wedge \bigwedge_{0<i<k, i \text{ odd}} x_i \right) .$$

Since

$$dv_1 = \sum_{0<i<k, i \text{ odd}} x_i \wedge x_{i+1} ,$$

the element

$$\left(\sum_{l=k+1}^{n} a_l x_l \right) \wedge \bigwedge_{0<i<k, i \text{ odd}} x_i$$

cannot be a coboundary and thus the action of $M_{(E_n,(0,\cdots,0,a_{k+1},\cdots,a_n),1)}$ on $H^*(\mathcal{N})$ is not trivial. □

4.4 Comparison between the Levi–Mostow decomposition and the H^1-decomposition

We now show that the H^1-decomposition 4.2.2 of the automorphism group of a two-step nilpotent Lie algebra is in general not equal to the Levi–Mostow decomposition 4.0.9.

Since we claim that they are not equal in general, it is enough to give an example where they are indeed different. We look at the class of examples already introduced in the previous Section 4.3, namely Lie algebras with one dimensional commutator. So let $\big(\mathfrak{n}, [\cdot,\cdot]\big)$ be a Lie algebra with $\dim([\mathfrak{n},\mathfrak{n}]) = 1$. Furthermore, suppose that $\dim(\mathcal{Z}(\mathfrak{n})) > 1$, that is, there are elements in the center which are not in the commutator. Thus $\mathfrak{n} = W_n \oplus [\mathfrak{n},\mathfrak{n}]$ and let $\{X_1, \cdots, X_n\}$ and $\{X_{n+1}\}$ be corresponding bases. Recall that we get a possibly degenerated alternating form

$$\omega : W_n \times W_n \to \mathbb{K}$$

as introduced in 4.4. Moreover, we have seen that (with respect to the chosen basis)

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & \lambda_A \end{pmatrix} \mid A \in \mathrm{CSp}(\omega) \right\}$$

is precisely the subgroup of $\mathrm{Aut}(\mathfrak{n})$ which operates faithful on $H^1(\mathfrak{n})$ and thus is the one summand of the H^1-decomposition. Now let $Z \in W$ with

$\omega(Z, Z) = 0$. Such an element exists by condition $\dim(\mathcal{Z}(\mathfrak{n})) > 1$. Since $\omega(AZ, AZ) = \lambda_A \omega(Z, Z) = 0$ we conclude that the vector subspace

$$W_0 := \big\{ Z \in W \mid \omega(Z, Z) = 0 \big\}$$

is A invariant. Thus we can choose a basis of W, such that

$$A = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix} .$$

The subgroup of matrices of the form $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is normal in $\mathrm{CSp}(\omega)$, since $A^{-1} = \begin{pmatrix} (A')^{-1} & * \\ 0 & (A'')^{-1} \end{pmatrix}$. Moreover, this subgroup is clearly unipotent. Thus both summands of the H^1-decomposition

$$\ker(H^1(\cdot)) \rtimes \operatorname{im}(H^1(\cdot))$$

contain a unipotent normal subgroup and are thus both not reductive. In conclusion this decomposition cannot be the Levi–Mostow decomposition, since one summand of this decomposition is reductive.

4.5 Computational example - The Heisenberg algebra

Let $\mathfrak{h}_3$ be the Heisenberg algebra defined on a basis X_1, X_2, X_3 by $[X_1, X_2] = X_3$ and $(\mathcal{H}_3, d) := \big(\bigwedge \mathfrak{h}_3^*, d_{[\cdot,\cdot]}\big)$ be the corresponding dual Lie algebra or Koszul complex. Denote the dual basis of $\mathfrak{h}_3^*$ by x_1, x_2, x_3, that is $x_i(X_j) = \delta_{i,j}$. To prevent confusion with the Lie bracket we denote the linear span of elements by angle brackets, for example $\langle x_1, x_2 \rangle$ denotes the linear span of $x_1,\ x_2$ in $\mathcal{H}_3$.

By definition we have

$$dx_k(X_i, X_j) := x_k([X_i, X_j]) = \begin{cases} x_k(X_3) & i = 1, j = 2 \\ 0 & \text{else} \end{cases}$$

and hence $dx_1 = dx_2 = 0$. Moreover,

$$\begin{aligned} (x_1 \wedge x_2)(X_i, X_j) &= x_1(X_i) \cdot x_2(X_j) - x_1(X_j) \cdot x_2(X_i) \\ &= \begin{cases} 1 & i = 1, j = 2 \\ 0 & \text{else} \end{cases} \end{aligned}$$

and thus $dx_3 = x_1 \wedge x_2$. With that we easily compute

$$H^1(\mathcal{H}_3) = Z^1(\mathcal{A}) = \langle x_1, x_2 \rangle,$$
$$H^2(\mathcal{H}_3) = \langle x_1 \wedge x_2, x_1 \wedge x_3, x_2 \wedge x_3 \rangle / \langle x_1 \wedge x_2 \rangle \text{ and}$$
$$H^3(\mathcal{H}_3) = \langle x_1 \wedge x_2 \wedge x_3 \rangle.$$

According to Section 3.4.1, the minimal model is constructed in the following way:

- $\mathcal{M}_{1,1} = \bigwedge Z^1(\mathcal{H}_3) = \bigwedge \langle x_1, x_2 \rangle$ and $m_{1,1} = \widehat{\iota}$ the natural extension of the inclusion $\iota \langle x_1, x_2 \rangle \to \mathcal{H}_3$.
- $H^2(\mathcal{M}_{1,1}) = \langle x_1 \wedge x_2 \rangle$ and $m_{1,1}(x_1 \wedge x_2) = x_1 \wedge x_2 = dx_3 = 0 \in H^2(\mathcal{H}_3)$. Hence $\ker(H^2(m_{1,1})) = \langle x_1 \wedge x_2 \rangle$ and $H^2(\mathcal{C}\mathrm{one}(m_{1,1})) = \langle (x_1 \wedge x_2, x_3) \rangle$. Since $H^2(\mathcal{C}\mathrm{one}(m_{1,1})) = Z^2(\mathcal{C}\mathrm{one}(m_{1,1}))$ we can choose $s_1 = id$. We get $\mathcal{M}_{1,2} := \bigwedge \langle x_1, x_2 \rangle \otimes_{d_{\pi_1}} \bigwedge \langle (x_1 \wedge x_2, x_3) \rangle$ and $m_{1,2} = id \otimes \pi_2$.

Now a Lie algebra automorphism ϕ has to fulfil the equation $[\phi X, \phi Y] = \phi[X, Y]$. That way we easily compute

$$\mathrm{Aut}(\mathfrak{h}_3) = \left\{ \begin{pmatrix} \multicolumn{2}{c}{A} & 0 \\ & & 0 \\ u & v & det(A) \end{pmatrix} \;\middle|\; A \in GL_2(\mathbb{R});\ u, v \in \mathbb{R} \right\} \text{ and}$$
$$\mathrm{Inn}(\mathfrak{h}_3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix} \;\middle|\; u, v \in \mathbb{R} \right\}.$$

We now explicitly compute the section given in proof of Theorem 4.2.1. Therefore let $B := \begin{pmatrix} \multicolumn{2}{c}{A} & u \\ & & v \\ 0 & 0 & det(A) \end{pmatrix} \in Aut(\mathcal{H}_3)$. We get:

- $H^1(B) = A$ since $H^1(\mathcal{A}) = (x_1, x_2)$. Hence $\mathcal{M}_{1,1}(H^1(B)) = A$.
- $\mathcal{M}_{1,1}(H^1(B))_{|\ker H^2(m_{1,1}))} = det(A)$, since $\ker H^2(m_{1,1}) = x_1 \wedge x_2$. Hence $\mathcal{M}_{1,2}(H^1(B)) = A \otimes det(A)$.

4.6 An example of a 3-step nilpotent Lie algebra where an outer automorphism acts trivially on cohomology

We have seen in Section 4.3 that in the case of Lie algebras with one dimensional center the inner automorphisms are precisely those which act trivial on all the cohomology groups. This is in general not the case as this example will show.

Let $\mathfrak{n}_4$ be the Lie algebra given by a basis X_1, X_2, X_3, X_4 with Lie bracket

$$[X_1, X_2] = X_3, [X_1, X_3] = X_4$$

and zero else.

Computation of the automorphism group

As an algebra, $\mathfrak{n}_4$ is generated by X_1 and X_2. So any automorphism has to be of the form

$$A' = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ b_{11} & b_{12} & c_1 & 0 \\ b_{21} & b_{22} & c_2 & c_3 \end{pmatrix} ; \quad det(A') \neq 0 ,$$

since it has to respect the Lie bracket and thus the descending central series of $\mathfrak{n}_4$, which explains the zeros. To respect the Lie bracket means

$$[A'X_i, A'X_j] = A'[X_i, X_j], \quad \forall i, j \in \{1, 2, 3, 4\}.$$

In particular for $i = 2, j = 3$ we get

$$\begin{aligned} &[A'X_2, A'X_3] = A'[X_2, X_3] \\ \Leftrightarrow &[a_{12}X_1 + a_{22}X_2 + b_{12}X_3 + a_{22}X_4, c_1X_3 + c_2X_4] = 0 \\ \Leftrightarrow &[a_{12}X_1, c_1X_3] = 0 \\ \Leftrightarrow &a_{12}c_1X_4 = 0 \\ \Leftrightarrow &a_{12} = 0 \vee c_1 = 0. \end{aligned}$$

Since $det(A') \neq 0$ we conclude $a_{12} = 0$. Continuing in this fashion we get for $i = 1$ and $j = 2$

$$\begin{aligned} &[A'X_1, A'X_2] = A'[X_1, X_2] \\ \Leftrightarrow &[a_{11}X_1 + a_{21}X_2 + b_{11}X_3 + b_{21}X_4, a_{22}X_2 + b_{12}X_3 + b_{22}X_4 = A'X_3 \\ \Leftrightarrow &a_{11}a_{22}[X_1, X_2] + a_{11}b_{12}[X_1, X_3] = c_1X_3 + c_2X_4 \\ \Leftrightarrow &a_{11}a_{22}X_3 + a_{11}b_{12}X_4 = c_1X_3 + c_2X_4 \end{aligned}$$

from which we can conclude $c_1 = a_{11}a_{22}$ and $c_2 = a_{11}b_{12}$. Thus for $i = 1$ and $j = 3$

$$\begin{aligned} &[a_{11}X_1 + a_{21}X_2 + b_{11}X_3 + b_{21}X_4, det(A)X_3 + a_{11}b_{12}X_4] \\ &= a_{11}^2a_{22}[X_1, X_3] = a_{11}^2a_{22}X_4 , \end{aligned}$$

which implies $c_3 = a_{11}^2 a_{22}$. In conclusion, the automorphism group is given by

$$\mathrm{Aut}(\mathfrak{n}) = \left\{ A' = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ b_{11} & b_{12} & a_{11}a_{22} & 0 \\ b_{21} & b_{22} & a_{11}b_{12} & a_{11}^2 a_{22} \end{pmatrix} ; \quad det(A') \neq 0 \right\} .$$

Inner derivations and inner automorphisms

First we compute the inner derivations with respect to the basis $\{X_1, X_2, X_3, X_4\}$. We get

$$\mathrm{ad}_{X_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{ad}_{X_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\mathrm{ad}_{X_3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}_{X_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence the inner derivations are of the form $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & 0 & * & 0 \end{pmatrix}$.

Cohomology of $\mathfrak{n}_4$

Let $(\bigwedge \mathcal{N}_4^*, d)$ be the Koszul complex of $\mathfrak{n}_4$ and $\{x_1, x_2, x_3, x_4\}$ be the dual basis of $\{X_1, X_2, X_3, X_4\}$. With that we get

$$d(x_i)(X_j, X_k) := x_i([X_j, X_k]) = \begin{cases} x_i(X_3), & j = 1, k = 2 \\ x_i(X_4), & j = 1, k = 3 \\ 0 \quad \text{else} \end{cases}$$

which implies $dx_1 = dx_2 = 0$, $d(x_3) = x_1 \wedge x_2$ and $d(x_4) = x_1 \wedge x_3$. Thus we compute

$H^1(\mathfrak{n}_4) = \langle x_1, x_2 \rangle$,
$H^2(\mathfrak{n}_4) = \langle x_1 \wedge x_4, x_2 \wedge x_3 \rangle / \langle x_1 \wedge x_2, x_1 \wedge x_3 \rangle$,
$H^3(\mathfrak{n}_4) = \langle x_1 \wedge x_3 \wedge x_4, x_2 \wedge x_3 \wedge x_4 \rangle / \langle x_1 \wedge x_2 \wedge x_3, x_1 \wedge x_2 \wedge x_4 \rangle$ and
$H^4(\mathfrak{n}_4) = \langle x_1 \wedge x_2 \wedge x_3 \wedge x_4 \rangle$.

An outer automorphism which acts trivial on cohomology

The automorphism group of $\mathcal{N}_4$ is defined dual to the automorphisms of $\mathfrak{n}_4$ with respect to the basis $\{x_1, x_2, x_3, x_4\}$. Since we know the dual inner derivations,

$$O := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

clearly represents an outer automorphism of $\mathcal{N}_4$ but $H^*(O) = \mathrm{id}$.

CHAPTER
FIVE

APPLICATIONS

We now apply our results of the last chapter to the fields of geometry and topology where automorphisms of nilpotent Lie algebras naturally appear and play a fundamental role.

5.1 The automorphism group of a connected simply connected two-step nilpotent Lie group

The first application is given in the context of connected simply connected nilpotent Lie groups. A good reference for all the following standard facts about Lie groups is [10]. As usual a Lie group is a group G which admits the structure of a smooth manifold such that the maps $(g,h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth for all $g,h \in G$. Simply connected means that the fundamental group is trivial, that is $\pi_1(G) = 0$. The tangent space T_eG at the unit element $e \in g$ forms a Lie algebra over $\mathbb{R}$ denoted by $\mathfrak{g}$. The Lie group G is called nilpotent of step n, if $\mathfrak{g}$ is a nilpotent Lie algebra of step n. One defines

$$\mathrm{Aut}(G) := \Big\{ f : G \to G \ \mid f(gh) = f(g)f(h)\ \forall g,h \in G \\ \text{and } f \text{ is a diffeomorphism} \Big\}.$$

An automorphism of the form $g \mapsto hgh^{-1}$ is called an inner automorphism and we denote by $\mathrm{Inn}(G)$ the group of inner automorphisms. In the simply connected case we have isomorphisms $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathfrak{g})$ such that $\mathrm{Inn}(G) \cong \mathrm{Inn}(\mathfrak{g})$ where $\mathrm{Aut}(\mathfrak{g})$ and $\mathrm{Inn}(\mathfrak{g})$ were defined in Section 3.3.

Moreover, we can compute the de Rham cohomology of G via the cohomology of the Lie algebra $\mathfrak{g}$ and vice versa, that is there exists an isomorphism

$$H^k(C^*_{\mathrm{DR}}) \cong H^k(\mathfrak{g})$$

where C^*_{DR} is the complex of smooth differential forms on G as a manifold. With all these identifications Corollary 4.2.2 may also be stated as:

Corollary 5.1.1. *Let N be a two-step nilpotent, connected, simply connected Lie group and $\mathfrak{n}$ the corresponding Lie algebra. Then*

$$\mathrm{Aut}(N) := \ker(H^1_{(DR)}) \rtimes \mathrm{im}(H^1_{(DR)}) \ .$$

Furthermore, Corollary 4.3.1 may be stated as:

Corollary 5.1.2. *Let N be a connected, simply connected Lie group such that the commutator $[\mathfrak{n},\mathfrak{n}]$ of the corresponding Lie algebra $\mathfrak{n}$ is one dimensional. Then*

$$Inn(N) = \ker(H^*_{(DR)}).$$

5.2 The group of homotopy self equivalences of certain Eilenberg–MacLane spaces

Our results can also be used in the context of homotopy self-equivalences of certain topological spaces. A homotopy self-equivalence of a space X is a continuous map $h : X \to X$ such that there exists a continuous map $g : X \to X$ with $f \circ g \sim \mathrm{id}_X$ and $g \circ f \sim \mathrm{id}_X$, where $\sim$ means homotopic in the usual topological sense. One defines

$$\mathcal{E}(X) := \Big\{ h : X \to X \mid h \text{ is a homotopy equivalence} \Big\} / \sim$$

as the group of homotopy classes of homotopy self-equivalences. It is indeed a group. A good reference for the following facts are [11], Chapter 5,7, [17], Section 1.4 and [5]. Let N be a finitely generated nilpotent group, that is there exists a $k \in \mathbb{N}$ such that $N_k := [N, N_{k-1}]$ is zero, where

$$[N, N_{k-1}] := \{n^{-1}m^{-1}nm \mid n \in N, m \in N_{k-1}\}$$

and $N_0 := N$. Furthermore, $K(N,1)$ denotes a connected topological space, such that $\pi_1(K(N,1)) = N$ and $\pi_k(K(N,1)) = 0$ for all $k > 1$. Such spaces are called Eilenberg–MacLane spaces and they always exist for such groups.

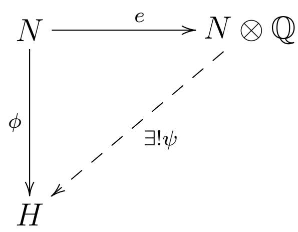
On the one hand, one can construct the Malcev hull of the group which is usually denoted by $N \otimes \mathbb{Q}$. This Malcev hull is characterised by the following properties.

i) $N \otimes \mathbb{Q}$ is a $\mathbb{Q}$-local group, which means by definition that the map

$$N \otimes \mathbb{Q} \to N \otimes \mathbb{Q}$$
$$n \mapsto n^k$$

is an isomorphism for all k.

ii) There exists a natural homomorphism of groups $e : N \to N \otimes \mathbb{Q}$ with the following universal property. If H is another $\mathbb{Q}$-local nilpotent group and $\phi : N \to H$ is a homomorphism of groups, then there exists a unique homomorphism of groups $\psi : N \otimes \mathbb{Q} \to H$ such that the diagram



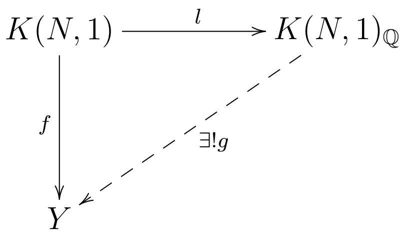
commutes.

iii) $N \otimes \mathbb{Q}$ is nilpotent of the same step as N.

On the other hand one can construct the rationalisation of $K(N, 1)$ which is denoted by $K(N, 1)_{\mathbb{Q}}$ and is characterised by the following properties.

i) $\pi_k(K(N, 1)_{\mathbb{Q}}) = 0$ for all $k > 1$ and $\pi_1(K(N, 1)_{\mathbb{Q}}) = N \otimes \mathbb{Q}$.

ii) There exists a continuous map $l : K(N, 1) \to K(N, 1)_{\mathbb{Q}}$ with the following universal property. Let Y be a space such that $\pi_k(Y)$ are $\mathbb{Q}$-vector spaces for $k > 1$ and $\pi_1(Y)$ is a $\mathbb{Q}$-local group. Then for any continuous map $f : K(N, 1) \to Y$ there exists a continuous map $g : K(N, 1)_{\mathbb{Q}} \to Y$ such that the diagram



commutes.

In particular $K(N,1)_{\mathbb{Q}} = K(N \otimes \mathbb{Q}, 1)$. Furthermore, one can show that $H^k(K(N,1)_{\mathbb{Q}}, \mathbb{Z})$ are $\mathbb{Q}$-vector spaces and that $H^k(l)$ induces an isomorphism

$$H^k(K(N,1),\mathbb{Z}) \otimes \mathbb{Q} = H^k(K(N,1)_{\mathbb{Q}}, \mathbb{Z})$$

for all k. But there are even more relations between these two objects given in terms of the minimal model. One can associate to the space $K(N,1)$ the differential graded $\mathbb{Q}$-algebra $\mathcal{A}_{\mathbb{Q}}(K(N,1))$ of piecewise linear $\mathbb{Q}$-forms. Then the minimal model

$$p : \mathcal{N} \to \mathcal{A}_{\mathbb{Q}}(K(N,1))$$

is a dual Lie algebra over $\mathbb{Q}$ and there exists an isomorphism of groups

$$[\mathrm{Aut}(\mathcal{N})] = \mathcal{E}(K(N,1)_{\mathbb{Q}}) \ .$$

Now let us denote by $\mathfrak{n}$ the nilpotent Lie algebra over $\mathbb{Q}$ corresponding to the minimal model. One can also construct a $\mathbb{Q}$-local group out of this Lie algebra, which we call the corresponding Malcev group and is denoted by $N_{\mathbb{Q}}$. Malcev proved that if N is a nilpotent group, then $N_{\mathbb{Q}} = N \otimes \mathbb{Q}$. This is also called the Malcev correspondence.

Thus with Corollary 4.3.1 we can construct spaces whose homotopy type is determined by the fundamental group and whose homotopy self-equivalences are quite understandable, namely

Corollary 5.2.1. *Let* $(\mathfrak{n}, [\cdot,\cdot])$ *be a Lie algebra over* $\mathbb{Q}$ *with* $\dim([\mathfrak{n},\mathfrak{n}]) = 1$ *and* $N_{\mathbb{Q}}$ *the corresponding Malcev group. Then* $\mathcal{E}(K(N_{\mathbb{Q}},1)) = H^*(\mathrm{Aut}(\mathfrak{n})$. *In particular two homotopy self-equivalences* $f, g : K(N_{\mathbb{Q}},1) \to K(N_{\mathbb{Q}},1)$ *are homotopic, if and only if* $H^*(f) = h^*(g)$.

In contrast to that, we can construct a space with three step nilpotent fundamental group, where a homotopy self-equivalence is not homotopic to the identity but operates trivial on cohomology.

Corollary 5.2.2. *Let* $\mathfrak{n}_4$ *be the Lie algebra over* $\mathbb{Q}$ *as defined in Section 4.6 and* $(N_4)_{\mathbb{Q}}$ *the corresponding Malcev group. Then there exists a homotopy self-equivalence*

$$o : K((N_4)_{\mathbb{Q}},1) \to K((N_4)_{\mathbb{Q}},1)$$

which is not homotopic to the identity but operates trivial on cohomology, that is $H^*(o) = \mathrm{id}$.

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