# Interbank Lending in a Generalized Walras Equilibrium

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aus Pforzheim

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Referent: Prof. Dr. Nicole Bäuerle
Korreferent: Prof. Dr. Günter Last

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# Contents

# 1 Introduction

In this work we will study an interbank market model, which is based on a model used in an ECB working paper by Heider and Hoerova [15]. There is no doubt about the importance of interbank markets as a way to smooth out different liquidity levels of banks without intervention of a central bank. There are various markets for different durations of the credits. We are interested in the different interest rates on secured and unsecured markets and therefore consider one fixed time horizon in this work. During the financial crisis, the spread between the various rates has increased significantly, which is why we try to model this spread and try to analyze in which way it depends on liquidity shocks for individual banks.

A version of the model we study has been discussed by Freixas and Holthausen [12]. This paper is one of a series of papers on the topic of liquidity, risk and in particular the effects on inetrbank markets ([10], [11], [12], [15], [16], [17]).

An interbank market of the type at hand has been introduced by Diamond and Dybvig [8] and Bhattacharya and Gale [2]. The interbank markets and interest rates are determined by a Walrasian auction. There are various papers, which discuss equilibria in the sense of Walras or an adapted version thereof. An introduction and discussion of the Walrasian approach is given by Katzner [21]. A general article on auctions and equilibria, as well as applications, was published by de Vries and Vohra [7].

Walras described an equilibrium and showed properties of equilibrium prices, but did not address the question of existence of such an equilibrium. Among many others, Arrow and Debreu [1] discuss the existence of an equilibrium. There are two established modifications of the classical Walras equilibrium, the Stackelberg-Walras equilibrium and the Cournot-Walras equilibrium. They are described and compared in a work by Julien [19]. Codognato and Gabsewicz [5] discuss Cournot-Walras equilibria in markets with a continuum of traders. An early example, where a continuum of traders is used, is a paper by Diamond and Dybvig [8]. This is particularly relevant, since we will assume to have a continuum of agents (i.e. banks) as well.

Our model goes als follows: We consider a continuum of identical banks maximizing expected utility of their terminal wealth in a three-period model ($t = 0, t_1, t_2$). There are households, which have future liquidity needs (in $t_1$ and $t_2$) and therefore let banks manage their funds.

In $t = 0$, banks can invest households' funds in riskless bonds and a risky investment. Those investments can only be liquidated in $t = t_2$. The remaining funds can be stored at no cost, i.e. there is a riskless interest-free asset, which can be accessed at any time. We will refer to this asset as the liquid asset. The risky investment is assumed to have a higher expected return than the bonds. It represents a combination of loans, venture capital investments and stock investments. In contrast to usual models, the $t_2$-value of the investment is determined independently for each bank. The distribution of the value is the same for all banks. Thus there is a tradeoff between liquidity and return as well as a tradeoff between expected return and risk.

In $t_1$ and $t_2$, the banks have to satisfy the households' liquidity needs. Every household can withdraw an arbitrary fraction of its funds in $t_1$. We assume that only the distribution of those withdrawals are known, but not the exact amount for every bank.

Since the bonds and the risky investment cannot be liquidated in $t_1$, the withdrawals have to be paid out of the liquid asset. Without a possibility to smooth out the different levels of liquidity demand between the banks, every bank has to hold an amount of liquid asset, which covers the highest possible withdrawal in $t_1$. This obviously harms the return in $t_2$.

Therefore, we introduce two interbank markets, a secured one and an unsecured one. The security on the secured market would typically be bonds, thus we allow an outright sale of bonds. This is easier to handle and yields equivalent results.

As in [15], the interest rates will be determined by a Walras-type auction. We will derive results for a very general case, thus we will define a generalization of the classical Walras equilibrium theory in chapter 2. This is necessary, since the most common generalizations, namely the Stackelberg-Walras equilibrium and the Cournot-Walras equilibrium, do not fit our needs perfectly. In some aspects, we need a more general formulation, in other aspects we need to consider a more special situation.

In chapter 3, we will specify the mathematical setup for our model and give an overview over the three-period market.

The core of this work will be chapter 3, where we will derive criteria for the development of an unsecured interbank market as well as a formula for the unsecured interest rate. This will be made with little to no assumptions on the distributions of $t_1$-withdrawals and return on the risky investment.

In the last chapter, we will discuss optimal portfolios in $t = 0$ by means of certain examples. Since only very few general results can be shown, the distributions of $t_1$-withdrawals and return on the risky investment will be specified.

# 2 A generalized Walras equilibrium

In this section, we will generalize the classical Walras equilibrium theory. We start from the trivial classical situation and generalize exactly the parts we consider necessary. We introduce a way to deal with non-unique demand functions and allow for situations where the buying price for a certain good exceeds its selling price. We add external cashflows to the model and introduce storage of excess capital as well as lending between agents. Finally, instead of limiting the model to a finite number of agents, we let the set of agents be arbitrary. Even uncountable sets will be possible.

## 2.1 Classical Walras equilibrium

First of all, let us recap the Walras equilibrium for a market with $n$ agents and $m$ goods. Let $n \geq 2$ be the number of agents on the market, $m \geq 2$ the number of goods on the market and $w_i^j$ agent $i$'s initial endowment of good $j$, measured in units of the respective good ($i = 1, \dots n; j = 1, \dots m$). The auction is held as follows: First, the auctioneer announces a price $m$-tuple for the goods, then the agents state their demand for each good given those prices. We denote by $d_i^j(p_1, \dots, p_m)$ the demand function of agent $i$ for good $j$ given prices $p_1, \dots, p_m$ (i=1,...n; j=1,...m). If the market clears (aggregate demand equals aggregate initial endowment for each good), then the announced price $m$-tuple is an equilibrium price $m$-tuple. Else, the auctioneer adjusts the prices (raise prices for goods where aggregate demand surpassed aggregate initial endowment; lower prices for goods where aggregate demand was lower than aggregate initial endowment) and the process is repeated. Thus by definition, an equilibrium price $m$-tuple $p_1^*, \dots, p_m^*$ satisfies

$$\sum_{i=1}^{n} d_i^j(p_1^*, \dots, p_m^*) = \sum_{i=1}^{n} w_i^j, \quad \text{for all } j = 1, \dots m.$$

## 2.2 Modifications

### 2.2.1 Multiple demand functions

Assuming that demand is uniquely determined by a given price $m$-tuple is not reasonable. From an agent's point of view, demand depends on a utility function and the initial endowment. In general, the agents' utility functions allow for multiple optimal allocations. Thus, we replace the demand functions by a set of functions, namely the optimal order sets. Let $f^i : \mathbb{R}^m \to \mathbb{R}$ be the reward function, mapping an allocation $(x_1, \ldots, x_m)$ of goods to its numerical value to the $i$-th agent. Every agent's preferences are included in this reward function. We assume, that $f^i$ is increasing in every element. The function $f^i$ may depend on the price $m$-tuple $(p_1, \ldots, p_m)$, a typical choice is $f^i(x_1, \ldots, x_m) = U^i(\sum_{j=1}^m p_j x_j)$ where $U^i$ is a utility function.

**Definition 2.1.** *Let $(p_1, \ldots, p_m)$ be fixed. The* ***optimal order set*** *of agent $i$ is the set $O^i = O^i(p_1, \ldots, p_m) \subset \mathbb{R}^m_{\geq 0}$, s.t. every $x = (x_1, \ldots, x_m) \in O^i$ satisfies*

- *budget constraint: $\sum_{j=1}^m p_j x_j = \sum_{j=1}^m p_j w_i^j$,*
- *optimality:*

$$f^i(x) = \max_{(y_1,\ldots,y_m)\in\mathbb{R}^m_{\geq 0}} f^i(y_1, \ldots, y_m).$$

By stating an optimal order set, the agent agrees with being assigned an arbitrary $x \in O^i$. It is now up to the auctioneer to assign to every agent an allocation out of his order set such that the market clears. The following definition provides all possible ways to do so.

**Definition 2.2.** *Let $(p_1, \ldots, p_m)$ be fixed. An* ***optimal market-clearing assignment*** *is a function*

$$a : \{1, \ldots, n\} \to \mathbb{R}^m_{\geq 0}, \quad i \mapsto (a_1(i), \ldots, a_m(i)),$$

*such that for every $i \in \{1, \ldots, n\}$*

$$a(i) \in O^i$$

*and the market clearing conditions hold:*

$$\sum_{i=1}^n a_j(i) = \sum_{i=1}^n w_i^j, \quad j = 1, \ldots, m.$$

*If such a function exists, we call $(p_1, \ldots, p_m)$ an* ***equilibrium price*** $m$***-tuple****.*

If there is no such function, the auctioneer adjusts the prices as in the classical model and the process is repeated.

### 2.2.2 Asymmetric markets

One can easily adjust the model to allow asymmetric markets. The classical model assumes that buying prices equal selling prices for each good. There are various situations, in which that is not the case. An example is the existence of transaction costs or delivery costs. Obviously, the buying price should always be at least as high as the selling price. Let us assume that the buying price may depend on the selling price and the amount of goods sold, and denote this relation by a function $g_j$. Thus for selling $x_j$ units of good $j$ at price $p_j$, an agent receives $p_j x_j$. For buying $x_j$ units of good $j$ at price $p_j$, an agent has to pay $g_j(p_j, x_j)x_j$. It makes sense to assume that $g_j(p_j, x_j) \geq p_j$. Note that for goods with symmetric price structure we have $g_j(p_j, x_j) \equiv p_j$. The market clearing condition and the optimality condition remain the same. We have to adjust the budget constraint to

$$\sum_{j=1}^{m} g_j(p_j, x_j)(x_j - w_i^j)^+ = \sum_{j=1}^{m} p_j(x_j - w_i^j)^-.$$

The left-hand side is the amount of money spent on the market, the right hand side is the amount of money earned, since for $x_j - w_i^j > 0$, agent $i$ is a buyer on the market for good $j$ and for $x_j - w_i^j < 0$, agent $i$ is a seller on the market for good $j$. Note that for $g_j(p_j, x_j) \equiv p_j$ $(j = 1, \ldots, m)$, the equation is the same as the old budget constraint:

$$\sum_{j=1}^{m} p_j(x_j - w_i^j)^+ = \sum_{j=1}^{m} p_j(x_j - w_i^j)^- \Longleftrightarrow \sum_{j=1}^{m} p_j(x_j - w_i^j) = 0.$$

### 2.2.3 Payment obligations, income, storage and lending

Agents on the market might have income from and payment obligations to either other agents which participate in the Walras auction or external counterparties. We aggregate all those cashflows in the (possibly negative) net income and denote it by $c_i \in \mathbb{R}$ for agent $i$. This value can be added to the endowment vector as an $(m+1)$-th element. The budget constraint of agent $i$ for given $(p_1, \ldots, p_m)$ now reads

$$\sum_{j=1}^{m} g_j(p_j, x_j)(x_j - w_i^j)^+ = \sum_{j=1}^{m} p_j(x_j - w_i^j)^- + c_i.$$

But for $c_i < -\sum_{j=1}^m p_j w_i^j$ this is not possible: Assume, the budget constraint holds, then

$$c_i = \sum_{j=1}^m g_j(p_j, x_j)(x_j - w_i^j)^+ - \sum_{j=1}^m p_j(x_j - w_i^j)^- \geq \sum_{j=1}^m p_j(x_j - w_i^j) \geq -\sum_{j=1}^m p_j w_i^j.$$

We want to allow for general net incomes and since there might be agents with spare liquidity from their cashflows, we introduce lending between agents. Generally, there are two possible ways to do so:

- Allow every agent to lend to any other agent individually. This means that we get up to $\frac{n(n-1)}{2}$ different lending markets, possibly with different interest rates.
- Have a centralized lending market, where all lenders and borrowers are perfectly diversified.

In the current setup, we can handle both options. But since we want to generalize the set of agents later on, the second option is the more sensible choice. There might be a positive probability that a borrower cannot pay back such a debt. Therefore we model the lending market between agents as an asymmetric market. We introduce two variables referring to the amount lent and borrowed on the market, $y_+$ and $y_-$. Since there might be tradeable goods or assets outside the auction, we introduce an additional variable $z$, which represents the aggregated investments in anything else except the goods $1, \ldots, m$ and the interbank market.

A rather strong assumption we have to make at this point is that the redemption rate on the lending market is known to the auctioneer when making the assignments. This means that agents state their optimal order sets depending on an assumed overall redemption rate $\hat{p}$ and on the interest rate $r$ on the lending market, thus

$$O^i = O^i(p_1, \ldots, p_m, r, \hat{p}).$$

The interpretation of this modification is the following: by stating an optimal order set, the agent agrees with being assigned an arbitrary $x \in O^i$, given prices $(p_1, \ldots, p_m)$ and interest rate $r$, if the redemption rate is $\hat{p}$. The interest rate and a possible surcharge for borrowers enter the reward functions. An element of the optimal order set now is an $(m+2)$-tuple $(x_1, \ldots, x_m, y_+, y_-)$, the budget constraint of agent $i$ becomes

$$\sum_{j=1}^m g_j(p_j, x_j)(x_j - w_i^j)^+ + y_+ + z = \sum_{j=1}^m p_j(x_j - w_i^j)^- + c_i + y_-.$$

The reward functions are modified to include $y_+$ and $y_-$. For example, the typical reward function from Section 2.2.1 becomes $f^i(x_1, \dots, x_m, y_+, y_-) = U^i(\sum_{j=1}^m p_j x_j + (1+r)\hat{p}y_+ - (1+r)y_-)$. We now assume that $f^i$ is increasing in the first $m+1$ elements and decreasing in the $(m+2)$-th element. An optimal market clearing assignment now also has two more entries $a_{m+1}$ and $a_{m+2}$. The old market clearing conditions still have to hold, additionally we require

$$\sum_{i=1}^n a_{m+1}(i) = \sum_{i=1}^n a_{m+2}(i),$$

which simply means that the market for interagent loans clears. We do not restrict external investments (except for obviously only allowing positive investments) and they have no influence on the auction, thus there is no need for another entry to the assignment vector.

An additional step is required since the auctioneer has to check for consistency: if $(p_1, \dots, p_m, r, \hat{p})$ is a market-clearing price-$(m+1)$-tuple, the auctioneer can only make the corresponding assignments, if $\hat{p}$ is actually the real redemption rate in this market. This requires knowledge of the borrowers' situation when the credits are due. Assume that there is a function $V$, s.t. the wealth of agent $i$ at maturity of the loans, before clearing, is given by

$$V(a_1(i), \dots, a_m(i), p_1, \dots, p_m, v_i),$$

where $v_i \in \mathbb{R}$ is the net income of agent $i$ at maturity. We write shortly $V(i, v_i) := V(a_1(i), \dots, a_m(i), p_1, \dots, p_m, v_i)$.

The individual redemption rate of agent $i$, $R^{i,v_i} = R^{i,v_i}(\hat{p}, r, p_1, \dots, p_m, a)$ is defined by

$$R^{i,v_i} := \begin{cases} a_{m+2}(i)(1+r), & V(i,v_i) + a_{m+1}(i)(1+r)\hat{p} \geq a_{m+2}(i)(1+r), a_{m+2}(i) > 0, \\ (V(i,v_i) + a_{m+1}(i)\hat{p})^+, & V(i,v_i) + a_{m+1}(i)(1+r)\hat{p} < a_{m+2}(i)(1+r), a_{m+2}(i) > 0, \\ 0, & a_{m+2}(i) = 0. \end{cases}$$

The first case is clear: if solvent (i.e. wealth exceeds debts), the agent pays back all his debts. Otherwise, he pays back as much as he can, i.e. the positive part of his current wealth. Obviously

$$R^{i,v_i}(\hat{p}, r, p_1, \dots, p_m, a, v_i) \leq a_3(s)(1+r).$$

Since there is full diversification in the lending market, every lender gets paid back the same fraction of its investment i.e. $\tilde{p} \cdot a_{m+1}(i)$ with $\tilde{p} \in [0,1]$. At maturity, the market in has to clear the same way as in the auction, that means aggregated redemption paid by borrowers equals aggregated redemption received by lenders:

$$\sum_{i=1}^{n} R^{i,v_i}(\hat{p}, r, p_1, \ldots, p_m, a, v_i) = \sum_{i=1}^{n} \tilde{p} a_{m+1}(i) = \tilde{p} \sum_{i=1}^{n} a_{m+2}(i).$$

This yields the following definition:

**Definition 2.3.** *For given price parameters $(p_1, \ldots, p_m)$, interest rate $r$ and (assumed) redemption rate $\hat{p}$, let $a$ be an optimal market clearing assignment. The **real redemption rate** on the lending market is defined by $\tilde{p} := 1$, if $a_{m+2} \equiv 0$ and*

$$\tilde{p} := \frac{\sum_{i=1}^{n} R^{i,v_i}(\hat{p}, r, p_1, \ldots, p_m, a, v_i)}{\sum_{i=1}^{n} a_{m+2}(i)}$$

*else. If $\hat{p} = \tilde{p}$, then $(p_1, \ldots, p_m, r)$, $a$ and $\hat{p}$ are called **consistent**.*

### 2.2.4 Arbitrary agent set

In this section we assume, that agents are identical in the sense that there is only one reward function $f$ valid for all agents. This means that two agents with the same endowment have the same optimal order set. Furthermore, we assume that the optimal assignments for two agents with the same optimal order sets coincide, i.e. for $i_1, i_2 \in \{1, \ldots, n\}$

$$O^{i_1} = O^{i_2} \Longrightarrow a(i_1) = a(i_2).$$

These assumptions are necessary for the following reformulation of the problem. Let $I = \{1, \ldots, n\}$ be the set of agents in the market and let $S$ be a state space. Agent $i$'s state is denoted by $s_i$. We assume that an agent's endowment (and therefore his demand) as well as his net income are determined by such a state, consequently there is a function $h : S \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}$ and agent $i$ has the endowment $h(s_i)$, i.e. $w_i^j = h_j(s_i)$ and $c_i = h_{m+1}(s_i)$.

The optimal order sets only depend on the state, thus we write $O^{s_i}$ instead of $O^i$. Here we have $S = \{z_1, \ldots, z_l\}$ with $l \leq n$ (the number of states is $n$, if every agent faces a different state). We denote by $\delta_k$ the number of agents, who face situation $z_k$, i.e. $\delta_k = |\{i \in \{1, \ldots, n\} : s_i = z_k\}|$. Obviously, $\sum_{k=1}^{l} \delta_k = n$ has to hold. Since the optimal assignment now only depends on the optimal order set, which in turn only depends on the state an agent faces, we can think of it as a function with domain $S$ instead of $I$.

The market clearing conditions can be rewritten:

$$\sum_{i=1}^{n} a_j(s_i) = \sum_{i=1}^{n} h_j(s_i) \quad \Longleftrightarrow \quad \sum_{k=1}^{l} \delta_k a_j(z_k) = \sum_{k=1}^{l} \delta_k h_j(z_k),$$

for $j = 1, \dots, m$ and

$$\sum_{i=1}^{n} a_{m+1}(i) = \sum_{i=1}^{n} a_{m+2}(i) \Longleftrightarrow \quad \sum_{k=1}^{l} \delta_k a_{m+1}(z_k) = \sum_{k=1}^{l} \delta_k a_{m+2}(z_k).$$

Now consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and the random variable $Y : \Omega \to S$ with distribution $\mathbb{P}(Y = z_k) = \frac{\delta_k}{n}$ $(k = 1, \dots, l)$. Then the market clearing conditions can again be rewritten as

$$E(a_j(Y)) = E(h_j(Y)), \quad j = 1, \dots, m,$$

$$E(a_{m+1}(Y)) = E(a_{m+2}(Y)).$$

We can proceed similarly with the net income of agent $i$ at maturity, $v_i$. Define a random variable $X$ by $\mathbb{P}(X = v_i) = \frac{|\{j \in \{1,\dots,n\} : v_j = v_i\}|}{n}$. With the setup of this section, the individual redemption rate depends on the state of an agent and the net income at maturity. Thus the definition of the real redemption rate becomes

$$\tilde{p} := \frac{E(R^{Y,X}(\hat{p}, r, p_1, \dots, p_m, a))}{E a_{m+2}(Y)}.$$

With this interpretation we can now allow for arbitrary agents sets $I$, state sets $S$ and state distributions, represented by a random variable $Y$ as well as income distributions, represented by a random variable $X$. The only necessary assumption is the existence of $E(h(Y))$.

## 2.3 The generalized Walras equilibrium with lending

Now let us sum up all the changes and list the necessary steps for determining equilibrium prices and assignments. A generalized Walras equilibrium with lending for $m$ tradeable goods is derived as follows: Let $I$ and $S$ be arbitrary sets, $Z \in \{\{0\}, \mathbb{R}_{\geq 0}\}$, $h : S \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}$ and $v : \mathbb{R}^{2m}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ functions and $f : \mathbb{R}^{m+3}_{\geq 0} \to \mathbb{R}$ a function, which is strictly decreasing in the $(m+2)$-th element and strictly increasing in every other element. For $j = 1, \dots, m$ let $g_j : \mathbb{R}^2_{>0} \to \mathbb{R}_{>0}$ be a function with $g_j(p, x) \geq p$, for all $(p, x) \in \mathbb{R}^2_{>0}$. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $Y : \Omega \to S$ and $X : \Omega \to \mathbb{R}$ be random variables, with $E(|h_j(Y)|) < \infty$ for all $j = 1, \dots, m$.

**Definition 2.4.** *Let $(p_1, \dots, p_m)$, $r$ and $\hat{p}$ be fixed. The **optimal order set** of an agent facing state $s \in S$ is the set $O^s = O^s(p_1, \dots, p_m, r, \hat{p}) \subset \mathbb{R}^{m+2}_{\geq 0}$, s.t. every $x = (x_1, \dots, x_m, y_+, y_-) \in O^s$ satisfies:*

- *budget constraint:*
$$z := \sum_{j=1}^{m} p_j(x_j - h_j(s))^- - \sum_{j=1}^{m} g_j(p_j, x_j)(x_j - h_j(s))^+ + h_{m+1}(s) + y_- - y_+ \in Z,$$
- *optimality:*
$$f(x, z) = \max_{(y_1, \dots, y_{m+3}) \in \mathbb{R}^{m+3}_{\geq 0}} f(y_1, \dots, y_{m+3}).$$

Note that the choice of $Z$ decides whether external investments are allowed ($Z = \mathbb{R}_{\geq 0}$) or not ($Z = \{0\}$). $z$ is the optimal amount invested outside the Walras auction.

**Definition 2.5.** *Let $(p_1, \dots, p_m)$, $r$ and $\hat{p}$ be fixed. An **optimal market-clearing assignment** is a function*
$$a : S \to \mathbb{R}^{m+2}_{\geq 0}, \quad s \mapsto (a_1(s), \dots, a_{m+2}(s)),$$
*such that for every $s \in S$*
$$a(s) \in O^s$$
*and the market clearing conditions hold:*
$$E(a_j(Y)) = E(h_j(Y)), \quad j = 1, \dots, m,$$
$$E(a_{m+1}(Y)) = E(a_{m+2}(Y)).$$

*If such a function exists, we call* $(p_1, \ldots, p_m, r)$ *an* ***equilibrium price*** $(m+1)$***-tuple****.* *We call* $(p_1, \ldots, p_m, r)$*,* $a$ *and* $\hat{p}$ ***consistent****, iff*

$$\hat{p} = \frac{E(R^{Y,X}(\hat{p}, r, p_1, \ldots, p_m, a))}{Ea_{m+2}(Y)},$$

*where for* $s \in S$ *and* $x \in \mathbb{R}$*,* $R^{s,x} = R^{s,x}(\hat{p}, r, p_1, \ldots, p_m, a)$ *is defined by*

$$R^{s,x} := \begin{cases} a_{m+2}(s)(1+r), & V(s,x) + a_{m+1}(s)\hat{p} \geq a_{m+2}(s)(1+r), a_{m+2}(s) > 0, \\ (V(s,x) + a_{m+1}(s)\hat{p})^+, & V(s,x) + a_{m+1}(s)\hat{p} < a_{m+2}(s)(1+r), a_{m+2}(s) > 0, \\ 0, & a_{m+2}(s) = 0, \end{cases}$$

*with* $V(s,x) := v(a_1(s), \ldots, a_m(s), p_1, \ldots, p_m, x)$*.*

# 3 The Interbank Market

## 3.1 The model

In this section we will specify the mathematical setup for our model and give an overview over the three-period market.

We assume WLOG that there is no discounting between the three dates $t = 0,\ t_1,\ t_2$ and that the set of banks is given by $[0, 1]$.

Utility is measured by a strictly increasing, strictly concave and continuous function $u$.

We normalize the problem by assuming that each bank has one unit of the households' funds under management at $t = 0$. Thus all other values are to be considered relative to the actual value of funds managed by each bank. Any fraction of those claims can be withdrawn either at $t = t_1$ or $t = t_2$ and the resulting payout is calculated as withdrawal times a factor $c_1$ or $c_2$ respectively.

The demand for liquidity on individual bank level is determined by a $[0, 1]$-valued non-constant random variable $\Lambda$ with known distribution. $\Lambda$ describes the fraction of claims, which are withdrawn in $t = t_1$, thus a bank pays out $\Lambda c_1$ in $t = t_1$ and $(1 - \Lambda)c_2$ in $t = t_2$.

The value of the risky investment is modeled by a constant $S_0$ and a random variable $S_{t_2}$, where $S_{t_2}$ is the $t_2$-value of the risky asset and for all banks the distribution of said value is the same.

The bond prices are denoted by $P_0$ in $t = 0$ and $P_2$ in $t = t_2$. The liquid asset obviously has a constant price of 1.

Since all banks are identical and face the same situation in $t_0$, we can assume WLOG, that their investment decisions in $t_0$ coincide. We denote the allocation by $(\alpha, \beta_0, \delta_0)$, which represents the fraction of funds invested in the risky investment, bonds and the liquid asset, where $\alpha, \beta_0, \delta_0 \geq 0$ and $\alpha + \beta_0 + \delta_0 = 1$. Note that $\alpha, \beta_0$ and $\delta_0$ are the

$t_0$-value of the assets held, which is equivalent to holding $\frac{\alpha}{S_0}$, $\frac{\beta_0}{P_0}$ and $\delta_0$ units.

Assuming that we find equilibrium prices $P_1$ (bond price) and $r$ (unsecured interest rate) and a consistent redemption rate $\hat{p}$, the assets and financial claims (measured in monetary value) are given by:

| $t$ | 0 | $t_1$ | $t_2$ |
|---|---|---|---|
| **cash** | $-\delta_0$ | $\delta_0$ | |
| | | $-\delta_1$ | $\delta_1$ |
| **stock** | $-\alpha$ | 0 | $\alpha\frac{S_{t_2}}{S_0}$ |
| **bond** | $-\beta_0$ | $\beta_0\frac{P_1}{P_0}$ | |
| | | $-\beta_1$ | $\beta_1\frac{P_2}{P_1}$ |
| **unsecured ib debt** | | $-\gamma_+$ | $\gamma_+\hat{p}(1+r)$ |
| | | $\gamma_-$ | $-\gamma_-(1+r)$ |
| **cash flow** | 1 | $-\Lambda c_1$ | $-(1-\Lambda)c_2$ |

The budget constraints in $t=0$ and $t=t_1$ are included in this table. In order to retrieve them, we take the sum over all entries in the corresponding column.

The terminal wealth can be calculated by taking the sum over all entries in the last column.

## 3.2 The existence of an interbank market and equilibrium prices

In this chapter we will study the banks' behaviour in $t_1$ given an arbitrary investment decision from $t_0$. Banks have to satisfy households' liquidity needs. The illiquid asset cannot be traded, bonds will be traded in a generalized Walras auction with lending. Spare liquidity can be put into the liquid asset.

As we assume that bonds can be traded, there is one good that will be auctioned, thus $m = 1$. The agent set for the Walras auction is $I = [0, 1]$, since every bank is an agent. We choose the space of all possible payouts a bank has to make to the households as the state space, i.e. $S = \text{supp}(\Lambda c_1) \subset [0, c_1]$. The state distribution is given by $Y \equiv \Lambda c_1$. We assume that there are no transaction costs on any trade of bonds, thus $g_1(p, x) \equiv p$. There is an external investment possibility, namely the liquid asset. Thus $Z = \mathbb{R}_{\geq 0}$. What remains is deriving the endowment function $h$ and the reward function $f$.

The initial endowment of bonds is the same for all banks:

$$h_1 \equiv \frac{\beta_0}{P_0}.$$

The net income is the difference between what was stored in the liquid asset and the payout to households:

$$h_2(s) = \delta_0 - s, \quad s \in S.$$

The goal is to maximize expected utility of terminal wealth (i.e. wealth in $t_2$). If there is an optimal assignment $a$ with a corresponding investment in the liquid asset $\delta_1$, then a bank facing state $s$ will have a terminal wealth of

$$V_2^s(\alpha, \beta_0, \delta_0) = \alpha \frac{S_{t_2}}{S_0} + a_1(s)P_2 + a_2(s)(1 + r)\hat{p} - a_3(s)(1 + r) + \delta_1(s) - (1 - \frac{s}{c_1})c_2.$$

Thus a possible function $v$, as introduced in section 2.3, is

$$v(a_1(s), \dots, a_3(s), P_1, x) := \alpha \frac{x}{S_0} + a_1(s)P_2 + \delta_1(s) - (1 - \frac{s}{c_1})c_2$$

and the distribution of the net income is $X \overset{d}{=} S_{t_2}$.

Before checking for optimal assignments, we have to determine optimal order sets by looking for maxima. For the optimization problem at hand, the reward function of an agent facing state $s$ for given $p_1$, $r$ and $\hat{p}$ should be

$$E\left[U(\alpha\frac{S_{t_2}}{S_0}+x_1P_2+y_+(1+r)\hat{p}-y_-(1+r)+z-(1-\frac{s}{c_1})c_2)\right].$$

But we can simplify this function in this chapter, since a bank can only influence $x_1$, $y_+$, $y_-$ and $z$. The only random part is $\frac{S_{t_2}}{S_0}$ and since expectation is strictly increasing, it can be left out of the maximization problem at hand. $u$ is also assumed to be strictly increasing, therefore neither changes the result and can be left out. Finally, $(1-\frac{s}{c_1})c_2$ is a constant and thus also ignored. Summing up, maximizing the above term in $t_1$ is equivalent to maximizing

$$f(x_1,y_+,y_-,z):=x_1P_2+y_+(1+r)\hat{p}-y_-(1+r)+z.$$

The definiton of $z$ in the budget constraint $z\in Z=\mathbb{R}_{\geq 0}$ can be written more simple as in the definition, since we have a symmetric market:

$$z=\sum_{j=1}^{m}p_j(x_j-h_j(s))^- -\sum_{j=1}^{m}g_j(p_j,x_j)(x_j-h_j(s))^+ +h_{m+1}(s)+y_- -y_+$$

$$\Leftrightarrow z=p_1(\frac{\beta_0}{P_0}-x_1)+\delta_0-s+y_- -y_+.$$

We replace the notation from chapter 2 by the notation in chapter 3.1. Thus we will write $P_1$ instead of $p_1$, $\beta_1$ instead of $x_1$, $\delta_1$ instead of $z$ (the liquid asset is the only external investment possibility) and $\gamma_+/\gamma_-$ instead of $y_+/y_-$ respectively. We define for every $s\in S$ the constraint set $C^s$ by

$$C^s:=\{(\beta_1,\gamma_+,\gamma_-)\in\mathbb{R}^3_{\geq 0}:\beta_1P_1+\gamma_+-\gamma_-\leq\beta_0\frac{P_1}{P_0}+\delta_0-s\}.$$

This is the set of all $(\beta_1,\gamma_+,\gamma_-)\in\mathbb{R}^3_{\geq 0}$, which satisfy $\delta_1\in\mathbb{R}_{\geq 0}$, i.e the budget constraint in the definition of the optimal order set.

Thus we have $X=S_{t_2}$ and for $x\in\text{supp}(S_{t_2})$ and $s\in\text{supp}(Y)$, the individual redemption rate is given by

$$R^{s,x}(\hat{p},r,P_1,a):=\begin{cases}a_3(s)(1+r), & V_2^{s,x}(\alpha,\beta_0,\delta_0)\geq 0, a_3(s)>0,\\ (V_2^{s,x}(\alpha,\beta_0,\delta_0)+a_3(s)(1+r))^+, & V_2^{s,x}(\alpha,\beta_0,\delta_0)<0, a_3(s)>0,\\ 0, & a_3(s)=0.\end{cases}$$

Recall that the real redemption rate on the unsecured interbank market is given by $\tilde{p}:=1$, if $a_3\equiv 0$ and

$$\tilde{p}:=\frac{E\left[R^{Y,S_{t_2}}(\hat{p},r,P_1,a)\right]}{E\left[a_3(Y)(1+r)\right]}=\frac{E\left[R^{Y,S_{t_2}}(\hat{p},r,P_1,a)1_{\{a_3(Y)>0\}}\right]}{E\left[a_3(Y)(1+r)1_{\{a_3(Y)>0\}}\right]}$$

else.

Now we can determine the optimal order sets. The reward function $f$ is affine and the constraint set $C^s$ is a convex polytope. Therefore, $O^s$ either is empty, contains only one element or is uncountable for a given $(P_1, r, \hat{p})$.
Write $o = (\beta_1, \gamma_+, \gamma_-)$ for an element of the order set. For $\hat{p} = 0$, define $\frac{1}{\hat{p}} - 1 := +\infty$.

**Theorem 3.1.** *For every $s \in S$, the order sets $O^s = O^s(P_1, r, \hat{p})$ are uniquely determined by*

$$O^s = \begin{cases} \emptyset, & \hat{p} \in [0,1], r \in (-1,0), P_1 \in (0,\infty) \textit{ or } \\ & \hat{p} \in [0,1], r \in [0,\infty), P_1 \in (0, \frac{P_2}{1+r}), \\ \{o \in C^s : \beta_1 = \gamma_+ = 0\}, & \hat{p} \in [0,1), r = 0, P_1 \in (P_2, \infty), \\ \{(0, 0, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-)\}, & \hat{p} \in [0,1), r \in (0, \frac{1}{\hat{p}} - 1), P_1 \in (P_2, \infty), \\ \left\{ o \in C^s : \begin{matrix} \gamma_+ \le (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^+, \\ \beta_1 = 0, \gamma_- = (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^- \end{matrix} \right\}, & \hat{p} \in (0,1), r = \frac{1}{\hat{p}} - 1, P_1 \in (P_2, \infty), \\ \{o \in C^s : \gamma_+ = 0\}, & \hat{p} \in [0,1), r = 0, P_1 = P_2, \\ \{o \in C^s : \gamma_+ = 0, \beta_1 P_1 - \gamma_- = \frac{P_1 \beta_0}{P_0} + \delta_0 - s\}, & \hat{p} \in [0,1), r \in (0,\infty), P_1 = \frac{P_2}{1+r}, \\ \{(\frac{\delta_0}{P_1} + \frac{\beta_0}{P_0} - \frac{s}{P_1}, 0, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-)\}, & \hat{p} \in (0,1), r \in (0, \frac{1}{\hat{p}} - 1], P_1 \in (\frac{P_2}{1+r}, P_2) \\ & \textit{or } \hat{p} = 0, r \in (0,\infty), P_1 \in (\frac{P_2}{1+r}, P_2) \textit{ or } \\ & \hat{p} \in (0,1), r > \frac{1}{\hat{p}} - 1, P_1 \in (\frac{P_2}{1+r}, \frac{P_2}{(1+r)\hat{p}}), \\ \left\{ o \in C^s : \begin{matrix} \beta_1 P_1 \le (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^+, \\ \gamma_+ = 0, \gamma_- = (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^- \end{matrix} \right\}, & \hat{p} \in [0,1), r \in (0, \frac{1}{\hat{p}} - 1), P_1 = P_2, \\ \left\{ o \in C^s : \begin{matrix} \beta_1 P_1 + \gamma_+ \le (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^+, \\ \gamma_- = (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^- \end{matrix} \right\}, & \hat{p} \in (0,1), r = \frac{1}{\hat{p}} - 1, P_1 = P_2, \\ \left\{ o \in C^s : \begin{matrix} \beta_1 P_1 + \gamma_+ = (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^+, \\ \gamma_- = (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^- \end{matrix} \right\}, & \hat{p} \in (0,1), r \in (\frac{1}{\hat{p}} - 1, \infty), P_1 = \frac{P_2}{(1+r)\hat{p}}, \\ \{(0, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^+, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-)\}, & \hat{p} \in (0,1), r \in (\frac{1}{\hat{p}} - 1, \infty), P_1 \in (\frac{P_2}{(1+r)\hat{p}}, \infty), \\ \{o \in C^s : \beta_1 = 0\}, & \hat{p} = 1, r = 0, P_1 \in (P_2, \infty), \\ \{o \in C^s : \beta_1 = 0, \gamma_+ - \gamma_- = \frac{P_1}{P_0} \beta_0 + \delta_0 - s\}, & \hat{p} = 1, r \in (0,\infty), P_1 \in (\frac{P_2}{(1+r)}, \infty), \\ \{o \in C^s : \beta_1 P_1 + \gamma_+ - \gamma_- = \frac{P_1}{P_0} \beta_0 + \delta_0 - s\}, & \hat{p} = 1, r \in (0,\infty), P_1 = \frac{P_2}{(1+r)}, \\ C^s, & \hat{p} = 1, r = 0, P_1 = P_2. \end{cases}$$

With knowledge of the order sets, we can determine optimal assignments and equilibrium prices. Recall that an optimal assignment is a function $a : S \to \mathbb{R}^3_{\ge 0}$ with $a(s) \in O^s$ for all $s \in S$ and

$$E[a_1(Y)] = \frac{\beta_0}{P_0}, \quad E[a_2(Y)] = E[a_3(Y)].$$

Write $A$ for the set of all functions from $S$ to $\mathbb{R}^3_{\geq 0}$, which (as an assignment) satisfy the market clearing conditions i.e.

$$A := \{a \in \left(\mathbb{R}^3_{\geq 0}\right)^S : E[a_1(Y)] = \frac{\beta_0}{P_0}, E[a_2(Y)] = E[a_3(Y)]\}.$$

For given $(\alpha, \beta_0, \delta_0)$, we write all possible equilibrium prices and corresponding optimal market clearing assignments as a subset of

$$B := [0,1] \times (-1, \infty) \times (0, \infty) \times A.$$

Write $b = (\hat{p}, r, P_1, a)$ for an element of $B$ and define

$$\overline{Y} := \text{ess sup } Y = \inf_{N \subset [0,1], P(N)=0} \sup_{i \in [0,1] \setminus N} Y(i).$$

**Theorem 3.2.** *For an initial allocation $(\alpha, \beta_0, \delta_0)$, the set of all possible equilibrium prices and corresponding optimal market clearing assignments is given by*

$$\begin{cases}
\emptyset, & \delta_0 < E(Y), \\
\begin{array}{c}\{b\in B: \hat{p}\in(0,1), r\in[\frac{1}{\hat{p}}-1,\infty), P_1\in[\frac{P_2}{(1+r)\hat{p}},\infty), a\in A_1\} \\ \bigcup\{b\in B: \hat{p}=1, r\in[0,\infty), P_1\in[\frac{P_2}{1+r},\infty), a\in A_2\}\end{array}, & \delta_0 = EY, \beta_0 = 0, \\
\begin{array}{c}\{b\in B: \hat{p}\in(0,1), r=\frac{1}{\hat{p}}-1, P_1\in[P_2,\infty), a\in A_3\} \\ \bigcup\{b\in B: \hat{p}=1, r=0, P_1\in[P_2,\infty), a\in A_4\}\end{array}, & \delta_0\in(EY,\overline{Y}), \beta_0=0, \\
\begin{array}{c}\{b\in B: \hat{p}\in[0,1), r\in[0,\frac{1}{\hat{p}}-1], P_1\in[P_2,\infty), a\equiv 0\} \\ \bigcup\{b\in B: \hat{p}=1, r=0, P_1\in[P_2,\infty), a\in A_4\}\end{array}, & \delta_0\in[\overline{Y},1], \beta_0=0, \\
\begin{array}{c}\{b\in B: \hat{p}\in(0,1), r\in[\frac{1}{\hat{p}}-1,\infty), P_1=\frac{P_2}{(1+r)\hat{p}}, a\in A_5\} \\ \bigcup\{b\in B: \hat{p}=1, r\in[0,\infty), P_1=\frac{P_2}{1+r}, a\in A_6\}\end{array}, & \begin{array}{c}\delta_0=EY, \beta_0>0 \\ \beta_0<\frac{P_0}{P_2}(\overline{Y}-EY),\end{array} \\
\begin{array}{c}\{b\in B: \hat{p}\in(0,1), r\in[\frac{1}{\hat{p}}-1,\infty), P_1=\frac{P_2}{(1+r)\hat{p}}, a\in A_5\} \\ \bigcup\{b\in B: \hat{p}=1, r\in[0,\infty), P_1=\frac{P_2}{1+r}, a\in A_6\}\bigcup \\ \{b\in B: \hat{p}\in[0,1), P_1\in[P_0\frac{\overline{Y}-EY}{\beta_0},P_2], r\in[\frac{P_2}{P_1}-1,\frac{P_2}{P_1\hat{p}}-1), a\in A_7\}\end{array}, & \begin{array}{c}\delta_0=EY, \beta_0\leq 1-EY, \\ \beta_0\geq\frac{P_0}{P_2}(\overline{Y}-EY),\end{array} \\
\begin{array}{c}\{b\in B: \hat{p}\in(0,1), r=\frac{1}{\hat{p}}-1, P_1=P_2, a\in A_8\} \\ \bigcup\{b\in B: \hat{p}=1, r=0, P_1=P_2, a\in A_9\}\end{array}, & \begin{array}{c}\delta_0\in(EY,\overline{Y}), \beta_0>0 \\ \beta_0<\frac{P_0}{P_2}(\overline{Y}-\delta_0),\end{array} \\
\begin{array}{c}\{b\in B: \hat{p}\in(0,1), r=\frac{1}{\hat{p}}-1, P_1=P_2, a\in A_8\} \\ \bigcup\{b\in B: \hat{p}=1, r=0, P_1=P_2, a\in A_9\} \\ \{b\in B: \hat{p}\in[0,1), r\in[0,\frac{1}{\hat{p}}-1), P_1=P_2, a\in A_{10}\}\end{array}, & \begin{array}{c}\delta_0\in(EY,1), \beta_0\leq 1-\delta_0, \\ \beta_0\geq\frac{P_0}{P_2}(\overline{Y}-\delta_0).\end{array}
\end{cases}$$

*The sets of corresponding optimal market clearing assignments above are defined as* *follows:*

$$\begin{aligned}
A_1 &:= \{a \in A : a(s) = (0, (EY - s)^+, (EY - s)^-) \ \forall s \in S\} \\
A_2 &:= \{a \in A : a_1(s) = 0, a_2(s) - a_3(s) = EY - s \ \forall s \in S\} \\
A_3 &:= \{a \in A : a_1(s) = 0, a_2(s) \leq (\delta_0 - s)^+, a_3(s) = (\delta_0 - s)^- \ \forall s \in S\} \\
A_4 &:= \{a \in A : a_1(s) = 0, a_2(s) - a_3(s) \leq \delta_0 - s \ \forall s \in S\} \\
A_5 &:= \{a \in A : a_1(s)P_1 + a_2(s) = (EY + \beta_0 \tfrac{P_1}{P_0} - s)^+, a_3(s) = (EY + \beta_0 \tfrac{P_1}{P_0} - s)^- \ \forall s \in S\} \\
A_6 &:= \{a \in A : a_1(s)P_1 + a_2(s) - a_3(s) = EY + \beta_0 \tfrac{P_1}{P_0} - s \ \forall s \in S\} \\
A_7 &:= \{a \in A : a(s) = (\tfrac{EY}{P_1} + \tfrac{\beta_0}{P_0} - \tfrac{s}{P_1}, 0, 0) \ \forall s \in S\} \\
A_8 &:= \{a \in A : a_1(s)P_2 + a_2(s) \leq (\delta_0 + \beta_0 \tfrac{P_2}{P_0} - s)^+, a_3(s) = (\delta_0 + \beta_0 \tfrac{P_2}{P_0} - s)^- \ \forall s \in S\} \\
A_9 &:= \{a \in A : a_1(s)P_2 + a_2(s) - a_3(s) \leq \delta_0 + \beta_0 \tfrac{P_2}{P_0} - s \ \forall s \in S\} \\
A_{10} &:= \{a \in A : a_2(s) = a_3(s) = 0, a_1(s)P_2 \leq \delta_0 + \beta_0 \tfrac{P_2}{P_0} - s \ \forall s \in S\}
\end{aligned}$$

The last step is to check the parameter $\hat{p}$ for consistency in the sense of definition 2.5. Theorem 3.2 implies, that there is no consistent quadruple for $\delta_0 < EY$. Thus, assume $\delta_0 \geq EY$.

We have a fix point problem. There might be multiple redemption rates that allow for consistent equilibrium prices and assignments. In this chapter, we will characterize the existence of a consistent quadruple $(\hat{p}, r, P_1, a)$.

But first we discuss a trivial case, $\delta_0 + \beta_0 \frac{P_2}{P_0} \geq \overline{Y}$. In this case, there is no need for interbank lending. All banks can pay out the households' claims by liquidating bonds and using the liquid asset. Thus $(\hat{p}, r, P_1, a) = (1, 0, P_2, a)$ is consistent for all $a \in A$ with $a_2 \equiv a_3 \equiv 0$.

For the rest of this chapter we consider the non-trivial case $\delta_0 + \beta_0 \frac{P_2}{P_0} < \overline{Y}$. If we find a consistent quadruple, then there is definitely unsecured interbank lending, since banks with $s > \delta_0 + \beta_0 \frac{P_2}{P_0}$ need to borrow in order to pay out the households' claims.

Define $\underline{S} := \frac{\text{ess inf } S_{t_2}}{S_0}$ and $\overline{S} := \frac{\text{ess sup } S_{t_2}}{S_0}$. Set $A := \{Y > \delta_0 + \beta_0 \frac{P_2}{P_0}\} = Y^{-1}\left(\left(\delta_0 + \beta_0 \frac{P_2}{P_0}, c_1\right]\right) \subset [0,1]$ and define

$$\underline{Y} := \text{ess} \inf_{A} Y = \sup_{N \subset A, P(N) = 0} \inf_{i \in A \setminus N} Y(i).$$

Note that $\underline{Y} \geq \delta_0 + \beta_0 \frac{P_2}{P_0}$.

Furthermore, define $h : [0,1] \to [0,1]$ by

$$h(0) := 0, \quad h(p) := p \frac{E\left[R^{Y, S_{t_2}}(p, \frac{1}{p} - 1, P_2, a)\right]}{E\left[(\delta_0 + \beta_0 \frac{P_2}{P_0} - Y)^-\right]}, \quad p \in (0,1],$$

with $a_3(s) = (\delta_0 + \beta_0 \frac{P_2}{P_0} - s)^-$ and $(a_1(s) + a_2(s))a_3(s) = 0$ for all $s \in S$. Thus, by the definiton of $R^{s,x}$, $h$ is well-defined.

**Lemma 3.3.** *Set $p_0 := 0$ and $p_n := h(p_{n-1}), n \in \mathbb{N}$. Then $\overline{p} := \lim_{n\to\infty} p_n \in [0,1]$ exists and depends only on $(\alpha, \beta_0, \delta_0)$. $\overline{p}$ is the largest fixpoint of $h$.*

We consider the model parameters as constant. Obviously, $\overline{p}$ depends on $P_2$ and the distributions of $S_{t_2}$ and $Y$. The lemma states, that $\overline{p}$ is independent of the choice of a particular assignment or equilibrium prices where there is ambiguity. This is important in the next theorem, since the existence of a consistent quadruple depends only on the value of $\overline{p}$.

**Theorem 3.4.** *There is a consistent quadruple $(\hat{p}, r, P_1, a)$, iff $\overline{p} > 0$ and for $\overline{p} > 0$ there is an assignment $a \in A$, s.t. $(\overline{p}, \frac{1}{\overline{p}} - 1, P_2, a)$ is consistent. A consistent quadruple $(\hat{p}, r, P_1, a)$ with $\hat{p} = 1$ exists, iff $\overline{p} = 1$, which is equivalent to*

$$\alpha \underline{S} + \beta_0 \frac{P_2}{P_0} + \delta_0 + \underline{Y}(\frac{c_2}{c_1} - 1)^+ - \overline{Y}(\frac{c_2}{c_1} - 1)^- \geq c_2.$$

The value of $\overline{p}$ also tells us whether a redemption rate of 1 is possible, which is, economically speaking, the optimal case. There might be other criteria to choose a consistent quadruple than maximizing the redemption rate. We will discuss some of those in a more specific setup in the next chapter.

Finally, we will show that we can get unique optimal assignments for given $(\hat{p}, r, P_1)$ by making only the following two assumptions:

**Assumption 3.5.** *The auctioneer assigns bonds and unsecured interbank loans according to the following rules:*

- *suppliers in the bond market may sell their minimal supply only*
- *demanders in both markets get assigned a constant fraction of their maximal demand*

*Minimal supply and maximal demand are taken over all possible optimal assignments, given by theorem 3.2.*

**Theorem 3.6.** *For equilibirum prices* $(\hat{p}, r, P_1)$*, assuming 3.5, the optimal assignment is unique and given by*

$$a_1(s) = \begin{cases} 0 & , \quad s > \delta_0 + \beta_0 \frac{P_1}{P_0} \\ \delta_0 + \beta_0 \frac{P_1}{P_0} - s & , \quad \delta_0 \leq s \leq \delta_0 + \beta_0 \frac{P_1}{P_0} \\ \beta_0 \frac{P_1}{P_0} + \frac{AS^B}{AD}(\delta_0 - s) & , \quad s < \delta_0 \end{cases}$$

$$a_2(s) = \begin{cases} 0 & , \quad s \geq \delta_0 \\ \frac{AS^U}{AD}(\delta_0 - s) & , \quad s < \delta_0 \end{cases}$$

$$a_3(s) = \begin{cases} s - \delta_0 - \beta_0 \frac{P_1}{P_0} & , \quad s > \delta_0 + \beta_0 \frac{P_1}{P_0} \\ 0 & , \quad s \leq \delta_0 + \beta_0 \frac{P_1}{P_0} \end{cases}$$

$$\delta_1(s) = \begin{cases} 0 & , \quad s \geq \delta_0 \\ (1 - \frac{AS^B + AS^U}{AD})(\delta_0 - s) & , \quad s < \delta_0 \end{cases}$$

*where*

$$AS^U = E[(Y - \delta_0 - \beta_0 \frac{P_1}{P_0}) 1_{\{Y > \delta_0 + \beta_0 \frac{P_1}{P_0}\}}],$$

$$AS^B = \beta_0 \frac{P_1}{P_0} P(Y > \delta_0 + \beta_0 \frac{P_1}{P_0}) + E[(Y - \delta_0) 1_{\{\delta_0 < Y \leq \delta_0 + \beta_0 \frac{P_1}{P_0}\}}].$$

*and*

$$AD^U = E[(\delta_0 - Y) 1_{\{Y < \delta_0\}}].$$

# 4 Optimal interest rates and initial allocation

As we have seen in the previous section, the auctioneer can choose between different interest rates, iff $\delta_0 = EY$. This choice has an impact on the expected utility of terminal wealth, in contrast to the choice of an optimal assignment. Therefore we will determine the interest rate, which maximizes expected utility of terminal wealth, given an initial allocation $(\alpha, \beta_0, \delta_0)$.

Depending on the distribution of $\Lambda$, $\sup_{P_1 \in (0,P_2]} V_2(P_1)$ might not be attained. Thus we will choose an interest rate $P_1^* \in \text{argmax} V_2(P_1)$, if possible and $P_1^* = P_2$ else ($P_1 = P_2$ is a possible equlibrium price in every case and is also the unique equilibirum price for any initial allocation with $\delta_0 > EY$). Note that $P_1^*$ depends on the initial allocation: $P_1^* = P_1^*(\alpha, \beta_0, \delta_0)$.

The banks are aware of the choice of $P_1$ a priori, i.e. they know which interest rate will result depending on their initial allocation. Thus we can look for an optimal initial allocation considering the choice of interest rate described above.

We will calculate some examples in this chapter, but first we will derive a general result, namely the optimal amount invested in the liquid asset.

Assume that there is an optimal allocation $(\alpha, \beta_0, \delta_0)$ with $\delta_0 > EY$. Set $(\tilde{\alpha}, \tilde{\beta_0}, \tilde{\delta_0}) :=$

$(\alpha, \beta_0 + \delta_0 - EY, EY)$. Then the bond price in $t = t_1$ is given by $P_2 > P_0$ and we have

$$
\begin{aligned}
& V_2(\alpha, \beta_0, \delta_0, P_2) \\
= \; & \left(\alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} + \delta_0 - \Lambda c_1 - (1-\Lambda) c_2\right) \cdot 1_{\{\Lambda \leq \frac{1}{c_1}(\delta_0 + \beta_0 \frac{P_2}{P_0})\}} \\
& + \left(\alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0 \hat{p}} + (\delta_0 - \Lambda c_1)\frac{1}{\hat{p}} - (1-\Lambda) c_2\right) \cdot 1_{\{\Lambda > \frac{1}{c_1}(\delta_0 + \beta_0 \frac{P_2}{P_0})\}} \\
& < \left(\alpha \frac{S_{t_2}}{S_0} + \tilde{\beta}_0 \frac{P_2}{P_0} + EY - \Lambda c_1 - (1-\Lambda) c_2\right) \cdot 1_{\{\Lambda \leq \frac{1}{c_1}(\delta_0 + \beta_0 \frac{P_2}{P_0})\}} \\
& + \left(\alpha \frac{S_{t_2}}{S_0} + \tilde{\beta}_0 \frac{P_2}{P_0 \hat{p}} + (EY - \Lambda c_1)\frac{1}{\hat{p}} - (1-\Lambda) c_2\right) \cdot 1_{\{\Lambda > \frac{1}{c_1}(\delta_0 + \beta_0 \frac{P_2}{P_0})\}} \\
= \; & V_2(\tilde{\alpha}, \tilde{\beta}_0, \tilde{\delta}_0, P_2)
\end{aligned}
$$

Expected utility is strictly increasing, thus we have

$$E(U(V_2(\alpha, \beta_0, \delta_0, P_2))) < E(U(V_2(\tilde{\alpha}, \tilde{\beta}_0, \tilde{\delta}_0, P_2))) \leq E(U(V_2(\tilde{\alpha}, \tilde{\beta}_0, \tilde{\delta}_0, P_1^*))).$$

by definition of $P_1^*$. This is a contradiction to $(\alpha, \beta_0, \delta_0)$ being optimal.

Using $\delta_0 = EY$, the equality constraint on the initial allocation is equivalent to $\alpha = 1 - EY - \beta_0$. The inequality constraints on $\alpha$ and $\beta_0$ become

$$0 \leq \beta_0 \leq 1 - EY.$$

The constraint on $\delta_0$ is obviously fulfilled, since $EY > 0$. We can rewrite the terminal wealth as a function in one variable:

$$V_2(\beta_0) := V_2(1 - EY - \beta_0, \beta_0, EY, P_1^*(1 - EY - \beta_0, \beta_0, EY)).$$

The optimization problem becomes

$$\max_{\beta_0 \in [0, 1-EY]} V_2(\beta_0).$$

Where convenient, we will consider $V_2$ as a function in $\alpha$ and $\beta_0$ or only in $\alpha$.

## 4.1 Example: Binomial distributions for risky asset and liquidity demand

Let $0 \le d < u$, $p \in (0,1)$ and assume

$$P(\frac{S_{t_2}}{S_0} = u) = p = 1 - P(\frac{S_{t_2}}{S_0} = d).$$

Furthermore, let $0 < \lambda_l < \lambda_h < 1$, $\pi_h \in (0,1)$ assume

$$P(\Lambda = \lambda_h) = \pi_h = 1 - P(\Lambda = \lambda_l).$$

Set $\pi_l := 1 - \pi_h$. Recall that $Y = c_1\Lambda$. Thus $EY = c_1(\pi_h\lambda_h + \pi_l\lambda_l)$, $\overline{Y} = c_1\lambda_h$.

Let the utility function of banks be given by $U : \mathbb{R} \to \mathbb{R}$, $U(x) := -\frac{1}{\kappa}\exp(-\kappa x)$ for some $\kappa > 0$.

First, we determine optimal bond prices.

(i) $\delta_0 = c_1(\pi_h\lambda_h + \pi_l\lambda_l)$, $\beta_0 \ge \frac{P_0}{P_2}c_1\pi_l(\lambda_h - \lambda_l)$
In this case, there is no need for unsecured lending, if $P_1 \in \left[P_0\frac{c_1\pi_l(\lambda_h-\lambda_l)}{\beta_0}, P_2\right] \neq \emptyset$.
Substitute $x = \frac{1}{P_1}$. Expected utility of terminal wealth is given by

$$\begin{aligned}
E(U(V_2(x))) &= \pi_l p U\left(\alpha u + \beta_0\frac{P_2}{P_0} + \pi_h(\lambda_h - \lambda_l)c_1P_2x - (1-\lambda_l)c_2\right) \\
&\quad + \pi_l(1-p)U\left(\alpha d + \beta_0\frac{P_2}{P_0} + \pi_h(\lambda_h - \lambda_l)c_1P_2x - (1-\lambda_l)c_2\right) \\
&\quad + \pi_h p U\left(\alpha u + \beta_0\frac{P_2}{P_0} - \pi_l(\lambda_h - \lambda_l)c_1P_2x - (1-\lambda_h)c_2\right) \\
&\quad + \pi_h(1-p)U\left(\alpha d + \beta_0\frac{P_2}{P_0} - \pi_l(\lambda_h - \lambda_l)c_1P_2x - (1-\lambda_h)c_2\right)
\end{aligned}$$

Taking the derivative with respect to $x$ and evaluating at $x = \frac{c_2}{P_2c_1}$, we get

$$\begin{aligned}
&\frac{\partial E(U(V_2(x)))}{\partial x}\Big|_{x=\frac{c_2}{P_2c_1}} \\
&= \pi_l\pi_h(\lambda_h - \lambda_l)c_1P_2pU'\left(\alpha u + \beta_0\frac{P_2}{P_0} + \pi_h(\lambda_h - \lambda_l)c_2 - (1-\lambda_l)c_2\right) \\
&\quad + \pi_l\pi_h(\lambda_h - \lambda_l)c_1P_2(1-p)U'\left(\alpha d + \beta_0\frac{P_2}{P_0} + \pi_h(\lambda_h - \lambda_l)c_2 - (1-\lambda_l)c_2\right) \\
&\quad - \pi_h\pi_l(\lambda_h - \lambda_l)c_1P_2pU'\left(\alpha u + \beta_0\frac{P_2}{P_0} - \pi_l(\lambda_h - \lambda_l)c_2 - (1-\lambda_h)c_2\right) \\
&\quad - \pi_h\pi_l(\lambda_h - \lambda_l)c_1P_2(1-p)U'\left(\alpha d + \beta_0\frac{P_2}{P_0} - \pi_l(\lambda_h - \lambda_l)c_2 - (1-\lambda_h)c_2\right) \\
&= 0,
\end{aligned}$$

since

$$\pi_h(\lambda_h - \lambda_l)c_2 - (1-\lambda_l)c_2 = (\pi_h\lambda_h + \pi_l\lambda_l - 1)c_2 = -\pi_l(\lambda_h - \lambda_l)c_2 - (1-\lambda_h)c_2.$$

The second derivative with respect to $x$ is strictly negative:

$$\begin{aligned}
&\frac{\partial^2 E(U(V_2(x)))}{\partial^2 x} \\
= \quad & \pi_l(\pi_h(\lambda_h-\lambda_l)c_1P_2)^2 pU''\left(\alpha u + \beta_0\frac{P_2}{P_0} + \pi_h(\lambda_h-\lambda_l)c_2 - (1-\lambda_l)c_2\right) \\
& +\pi_l(\pi_h(\lambda_h-\lambda_l)c_1P_2)^2(1-p)U''\left(\alpha d + \beta_0\frac{P_2}{P_0} + \pi_h(\lambda_h-\lambda_l)c_2 - (1-\lambda_l)c_2\right) \\
& +\pi_h(\pi_l(\lambda_h-\lambda_l)c_1P_2)^2 pU''\left(\alpha u + \beta_0\frac{P_2}{P_0} - \pi_l(\lambda_h-\lambda_l)c_2 - (1-\lambda_h)c_2\right) \\
& +\pi_h(\pi_l(\lambda_h-\lambda_l)c_1P_2)^2(1-p)U''\left(\alpha d + \beta_0\frac{P_2}{P_0} - \pi_l(\lambda_h-\lambda_l)c_2 - (1-\lambda_h)c_2\right) \\
< \quad & 0,
\end{aligned}$$

since $U$ is strictly concave an therefore $U'' < 0$.

Thus $x = \frac{c_2}{P_2c_1}$ is the unique zero of $\frac{\partial E(U(V_2(x)))}{\partial x}$, $E(U(V_2(x)))$ is increasing for $x < \frac{c_2}{P_2c_1}$ and decreasing for $x > \frac{c_2}{P_2c_1}$. Resubstituting $P_1 = \frac{1}{x}$, we get as expected-utility-maximizing bond price $P_1 = P_2\frac{c_1}{c_2}$. This value does not necessarily lie in the interval $\left[P_0\frac{c_1\pi_l(\lambda_h-\lambda_l)}{\beta_0}, P_2\right]$, thus we get different optimal interest rates depending on the values of $c_1$ and $c_2$:

$$P_1^* = \begin{cases} P_2 & , \quad c_1 \geq c_2 \\ P_2\frac{c_1}{c_2} & , \quad c_1 < c_2 \leq \frac{P_2}{P_0}\frac{\beta_0}{\pi_l(\lambda_h-\lambda_l)} \\ P_0\frac{c_1\pi_l(\lambda_h-\lambda_l)}{\beta_0} & , \quad c_2 > \frac{P_2}{P_0}\frac{\beta_0}{\pi_l(\lambda_h-\lambda_l)} \end{cases}$$

Note that this result holds for all strictly increasing, strictly concave utility functions.

(ii) $\delta_0 = c_1(\pi_h\lambda_h + \pi_l\lambda_l)$, $\beta_0 < \frac{P_0}{P_2}c_1\pi_l(\lambda_h - \lambda_l)$

In this situation, banks with $s = \lambda_h c_1$ need to borrow on the unsecured interbank market and banks with $s = \lambda_l c_1$ can pay households' liquidity needs out of the liquid asset.

By theorem 3.4, a consistent quadruple $(\hat{p}, r, P_1, a)$ with $\hat{p} = 1$ exists, iff

$$\alpha\underline{S} + \beta_0\frac{P_2}{P_0} + \delta_0 + \underline{Y}(\frac{c_2}{c_1} - 1)^+ - \overline{Y}(\frac{c_2}{c_1} - 1)^- \geq c_2.$$

Here we have $\underline{Y} = \text{ess inf } (Y 1_{\{Y > \delta_0 + \beta_0 \frac{P_2}{P_0}\}}) = c_1 \lambda_h$ and $\underline{S} = \frac{\text{ess inf } S_{t_2}}{S_0} = d$. The condition becomes

$$\alpha d + \beta_0 \frac{P_2}{P_0} + \delta_0 + c_1 \lambda_h (\frac{c_2}{c_1} - 1)^+ - c_1 \lambda_h (\frac{c_2}{c_1} - 1)^- \geq c_2$$

$$\Longleftrightarrow \alpha d + \beta_0 \frac{P_2}{P_0} + \delta_0 - \lambda_h c_1 - (1 - \lambda_h) c_2 \geq 0$$

$$\Longleftrightarrow \alpha \left( \frac{P_2}{P_0} - d \right) \leq \frac{P_2}{P_0} - \delta_0 \left( \frac{P_2}{P_0} - 1 \right) - \lambda_h c_1 - (1 - \lambda_h) c_2.$$

Since $\frac{P_2}{P_0} > d$, this is possible, iff

$$\frac{P_2}{P_0} - \delta_0 \left( \frac{P_2}{P_0} - 1 \right) - \lambda_h c_1 - (1 - \lambda_h) c_2 > 0.$$

This also means, that for

$$\alpha \left( \frac{P_2}{P_0} - d \right) > \frac{P_2}{P_0} - \delta_0 \left( \frac{P_2}{P_0} - 1 \right) - \lambda_h c_1 - (1 - \lambda_h) c_2,$$

there is no consistent quadruple $(\hat{p}, r, P_1, a)$ with $\hat{p} = 1$ and thus only banks with $s = \lambda_h c_1$ are borrowers on the unsecured market. Therefore the redemption rate can either be $p$ or 0. In the latter case, there is no consistent quadruple. A redemption rate of $p$ is possible, if

$$\alpha \left( u - \frac{P_2}{P_0} \right) \geq -\frac{P_2}{P_0} + \delta_0 \left( \frac{P_2}{P_0} - 1 \right) + \lambda_h c_1 + (1 - \lambda_h) c_2.$$

Expected utility of terminal wealth is given by

$$\begin{aligned} E(u(V_2(x))) &= \pi_l p U \left( \alpha u + \beta_0 \frac{P_2}{P_0} + \pi_h (\lambda_h - \lambda_l) c_1 P_2 x - (1 - \lambda_l) c_2 \right) \\ &\quad + \pi_l (1 - p) U \left( \alpha d + \beta_0 \frac{P_2}{P_0} + \pi_h (\lambda_h - \lambda_l) c_1 P_2 x - (1 - \lambda_l) c_2 \right) \\ &\quad + \pi_h p U \left( \alpha u + \beta_0 \frac{P_2}{P_0 p} - \pi_l (\lambda_h - \lambda_l) c_1 \frac{P_2}{p} x - (1 - \lambda_h) c_2 \right) \\ &\quad + \pi_h (1 - p) U \left( \alpha d + \beta_0 \frac{P_2}{P_0 p} - \pi_l (\lambda_h - \lambda_l) c_1 \frac{P_2}{p} x - (1 - \lambda_h) c_2 \right) \end{aligned}$$

Taking the derivative with respect to $x$, evaluating at

$$x = \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1 - \frac{1}{p}) - (\lambda_h - \lambda_l) c_2)}{\kappa(\pi_h + \frac{1}{p} \pi_l)(\lambda_h - \lambda_l) c_1 P_2}$$

and dividing by the positive constant $\pi_l \pi_h (\lambda_h - \lambda_l) c_1 P_2 \exp(\kappa c_2)$, we get

$$
\begin{aligned}
& \frac{1}{\pi_l \pi_h (\lambda_h - \lambda_l) c_1 P_2 \exp(\kappa c_2)} \frac{\partial E(U(V_2(x)))}{\partial x}\bigg|_{x=\frac{\ln(p)-\kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p})-(\lambda_h-\lambda_l)c_2)}{\kappa(\pi_h+\frac{1}{p}\pi_l)(\lambda_h-\lambda_l)c_1 P_2}} \\
= \; & pU'\left(\alpha u + \beta_0 \frac{P_2}{P_0} + \pi_h \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(\pi_h + \frac{1}{p}\pi_l)} + \lambda_l c_2\right) \\
& +(1-p)U'\left(\alpha d + \beta_0 \frac{P_2}{P_0} + \pi_h \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(\pi_h + \frac{1}{p}\pi_l)} + \lambda_l c_2\right) \\
& -U'\left(\alpha u + \beta_0 \frac{P_2}{P_0 p} - \pi_l \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(p\pi_h + \pi_l)} + \lambda_h c_2\right) \\
& -\frac{1-p}{p}U'\left(\alpha d + \beta_0 \frac{P_2}{P_0 p} - \pi_l \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(p\pi_h + \pi_l)} + \lambda_h c_2\right) \\
= \; & 0,
\end{aligned}
$$

since

$$
pU'\left(\alpha u + \beta_0 \frac{P_2}{P_0} + \pi_h \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(\pi_h + \frac{1}{p}\pi_l)} + \lambda_l c_2\right) =
$$

$$
pU'\left(\ln(p)\frac{\pi_h}{\kappa(\pi_h+\frac{1}{p}\pi_l)} + \alpha u + \beta_0 \frac{P_2}{P_0}\left(1 - \frac{\pi_h(1-\frac{1}{p})}{\pi_h+\frac{1}{p}\pi_l}\right) + c_2\left(\frac{\pi_h(\lambda_h-\lambda_l)}{\pi_h+\frac{1}{p}\pi_l} + \lambda_l\right)\right)
$$

$$
= p^{\frac{\pi_l}{p\pi_h+\pi_l}} \exp\left(-\kappa\left(\alpha u + \beta_0 \frac{P_2}{P_0}\frac{1)}{p\pi_h + \pi_l} + c_2 \frac{\pi_h \lambda_h + \frac{1}{p}\pi_l \lambda_l)}{\pi_h + \frac{1}{p}\pi_l}\right)\right) =
$$

$$
U'\left(-\ln(p)\frac{\pi_l}{\kappa(p\pi_h + \pi_l)} + \alpha u + \beta_0 \frac{P_2}{P_0}\frac{1}{p}\left(1 + \frac{\pi_l(1-\frac{1}{p})}{\pi_h + \frac{1}{p}\pi_l}\right) + c_2\left(\frac{-\pi_l(\lambda_h - \lambda_l)}{p\pi_h + \pi_l} + \lambda_h\right)\right)
$$

$$
= U'\left(\alpha u + \beta_0 \frac{P_2}{P_0 p} - \pi_l \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(p\pi_h + \pi_l)} + \lambda_h c_2\right)
$$

and analoguosly

$$
(1-p)U'\left(\alpha d + \beta_0 \frac{P_2}{P_0} + \pi_h \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(\pi_h + \frac{1}{p}\pi_l)} + \lambda_l c_2\right) =
$$

$$
\frac{1-p}{p}U'\left(\alpha d + \beta_0 \frac{P_2}{P_0 p} - \pi_l \frac{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1-\frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}{\kappa(p\pi_h + \pi_l)} + \lambda_h c_2\right).
$$

Again, the second derivative with respect to $x$ is strictly negative:

$$\begin{aligned}
&\frac{\partial^2 E(U(V_2(x)))}{\partial^2 x} \\
= \; &\pi_l(\pi_h(\lambda_h - \lambda_l)c_1 P_2)^2 p U''\left(V_2^{u,l}(P_1)\right) \\
&+\pi_l(\pi_h(\lambda_h - \lambda_l)c_1 P_2)^2 (1-p) U''\left(V_2^{d,l}(P_1)\right) \\
&+\pi_h(\pi_l(\lambda_h - \lambda_l)c_1 P_2)^2 \frac{1}{p} U''\left(V_2^{u,h}(P_1)\right) \\
&+\pi_h(\pi_l(\lambda_h - \lambda_l)c_1 P_2)^2 \frac{1-p}{p^2} U''\left(V_2^{d,h}(P_1)\right) \\
< \; &0,
\end{aligned}$$

since $U$ is strictly concave an therefore $U'' < 0$.

Thus the optimal bond price in this case is

$$P_1 = \frac{\kappa(\pi_h + \frac{1}{p}\pi_l)(\lambda_h - \lambda_l)c_1 P_2}{\ln(p) - \kappa(\beta_0 \frac{P_2}{P_0}(1 - \frac{1}{p}) - (\lambda_h - \lambda_l)c_2)}.$$

Next, we look for an optimal initial allocation. Using $\delta_0 = EY = c_1(\pi_h\lambda_h + \pi_l\lambda_l)$ and $\alpha + \beta_0 + \delta_0 = 1$, we can write the portfolio value as a function of only one variable, for example for $\lambda = \lambda_l$ and $\frac{S_{t_2}}{S_0} = u$:

$$\begin{aligned}
V_2^{\lambda_l,u}(\alpha) &= \alpha u + (1 - \alpha - c_1(\pi_h\lambda_h + \pi_l\lambda_l))\frac{P_2}{P_0} + \pi_h(\lambda_h - \lambda_l)c_1\frac{P_2}{P_1} - (1-\lambda_l)c_2 \\
&= \alpha(u - \frac{P_2}{P_0}) + (1 - c_1(\pi_h\lambda_h + \pi_l\lambda_l))\frac{P_2}{P_0} + \pi_h(\lambda_h - \lambda_l)c_1\frac{P_2}{P_1} - (1-\lambda_l)c_2.
\end{aligned}$$

The first derivative of the expectation of terminal wealth with respect to $\alpha$ is

$$\begin{aligned}
&\frac{\partial E(U(V_2(\alpha)))}{\partial \alpha} \\
= \; &\pi_l p(u - \frac{P_2}{P_0}) \exp\left(-\kappa\left(V_2^{\lambda_l,u}(\alpha)\right)\right) \\
&+\pi_l(1-p)(d - \frac{P_2}{P_0}) \exp\left(-\kappa\left(V_2^{\lambda_l,d}(\alpha)\right)\right) \\
&+\pi_h p(u - \frac{P_2}{P_0}) \exp\left(-\kappa\left(V_2^{\lambda_h,u}(\alpha)\right)\right) \\
&+\pi_h(1-p)(d - \frac{P_2}{P_0}) \exp\left(-\kappa\left(V_2^{\lambda_h,d}(\alpha)\right)\right)
\end{aligned}$$

Thus $\frac{\partial E(U(V_2(\alpha)))}{\partial \alpha} = 0$, iff

$$\begin{aligned}
&p(u - \frac{P_2}{P_0})\left(\pi_l \exp\left(-\kappa\left(V_2^{\lambda_l,u}(\alpha)\right)\right) + \pi_h \exp\left(-\kappa\left(V_2^{\lambda_h,u}(\alpha)\right)\right)\right) = \\
&(1-p)(\frac{P_2}{P_0} - d)\left(\pi_l \exp\left(-\kappa\left(V_2^{\lambda_l,d}(\alpha)\right)\right) + \pi_h \exp\left(-\kappa\left(V_2^{\lambda_h,d}(\alpha)\right)\right)\right).
\end{aligned}$$

This equation can be solved to

$$\exp\left(-\kappa\alpha(u-d)\right) = \frac{1-p}{p}\frac{\frac{P_2}{P_0}-d}{u-\frac{P_2}{P_0}}$$

$$\iff \alpha = \frac{1}{-\kappa(u-d)}\ln\left(\frac{1-p}{p}\frac{\frac{P_2}{P_0}-d}{u-\frac{P_2}{P_0}}\right).$$

Since $pu+(1-p)d > \frac{P_2}{P_0}$, we have $\frac{1-p}{p}\frac{\frac{P_2}{P_0}-d}{u-\frac{P_2}{P_0}} < 1$, thus the right-hand side is positive.

# 5 Proofs

*Proof of 3.1.* We will prove Theorem 3.1 by stating the KKT conditions for our optimization problem and then deriving the set of optimal solutions for every case. In order to get the standard KKT representation, we rewrite the constraints with $\delta_1$ as a slack variable.

The Karush Kuhn Tucker conditions

For a fixed state $s \in S$ and fixed $(P_1, r, \hat{p})$ we get as necessary conditions for the optimization problem $\max f(\beta_1, \gamma_+, \gamma_-, \delta_1)$ with $f(\beta_1, \gamma_+, \gamma_-, \delta_1) := \beta_1 P_2 + \gamma_+(1+r)\hat{p} - \gamma_-(1+r) + \delta_1$.

(1) primal feasibility

$g(\beta_1, \gamma_+, \gamma_-, \delta_1) \geq 0$ with $g : \mathbb{R}^4 \to \mathbb{R}^4$,

$$g(\beta_1, \gamma_+, \gamma_-, \delta_1) = (\beta_1, \gamma_+, \gamma_-, \delta_1)$$

$h(\beta_1, \gamma_+, \gamma_-, \delta_1) = 0$ with $h : \mathbb{R}^4 \to \mathbb{R}$,

$$h(\beta_1, \gamma_+, \gamma_-, \delta_1) = \beta_1 P_1 + \gamma_+ - \gamma_- + \delta_1 - \beta_0 \frac{P_1}{P_0} - \delta_0 + s$$

(2) dual feasibility

Let $\mu_i$, $i = 1, 2, 3, 4$ be the KKT multipliers on $\beta_1 \geq 0$, $\gamma_+ \geq 0$, $\gamma_- \geq 0$ and $\delta_1 \geq 0$ and $\nu$ the multiplier on the budget constraint. Then

$$\mu_1, \mu_2, \mu_3, \mu_4 \geq 0$$

(3) complementary slackness

$$\beta_1 \mu_1 = \gamma_+ \mu_2 = \gamma_- \mu_3 = \delta_1 \mu_4 = 0$$

(4) stationarity: $\nabla f + \sum_{i=1}^{4} \mu_i \nabla g + \nu \nabla h = 0$

$$P_2 + \mu_1 + \nu P_1 = 0$$

$$(1+r)\hat{p} + \mu_2 + \nu = 0$$

$$-(1+r) + \mu_3 - \nu = 0$$

$$1 + \mu_4 + \nu = 0$$

We can rewrite the first condition as $\frac{P_2}{P_1} + \frac{1}{P_1}\mu_1 + \nu = 0$, since $P_1 \in (0, \infty)$. Dual feasibility and complementary slackness hold for $\mu_1$ iff they hold for $\tilde{\mu}_1 := \frac{1}{P_1}\mu_1$, i.e. the first stationarity condition can be replaced by

$$\frac{P_2}{P_1} + \mu_1 + \nu = 0$$

Regularity

Since $f$ and $g$ are affine functions, a maximum point must satisfy the above conditions.

Sufficiency

Since $f$ and $g_j$ ($j = 1, \ldots, 4$) are continuously differentiable and concave and $h$ is affine, those conditions are also sufficient for optimality.

For every element of the optimal order set, there is a unique $\delta_1$, which represents the amount invested in the liquid asset. Since it can be calculated via

$$\delta_1 = \frac{P_1}{P_0}\beta_0 - \beta_1 P_1 + \delta_0 - s + \gamma_- - \gamma_+,$$

it will not be stated explicitely for every case.

We will have to consider various different cases in order to determine optimal order sets for $\hat{p}$, $r$ and $P_1$. The following charts sum up all case differentiations: For $\hat{p} \in [0,1]$, $r \in (-1,0)$, $P_1 \in (0,\infty)$, refer to case (i) and for $\hat{p} \in [0,1]$, $r \in [0,\infty)$, $P_1 \in (0, \frac{P_2}{1+r})$, refer to case (v). What remains is $\hat{p} \in [0,1]$, $r \in [0,\infty)$, $P_1 \in (\frac{P_2}{1+r}, \infty)$, which splits up into the following cases:

- For $\hat{p} = 0$:

| | $P_1 = \frac{P_2}{1+r}$ | $P_1 \in (\frac{P_2}{1+r}, P_2)$ | $P_1 = P_2$ | $P_1 \in (P_2, \infty)$ |
|---|---|---|---|---|
| $r = 0$ | - | - | (vi) | (ii) |
| $r \in (0,\infty)$ | (vii) | (viii) | (vi) | (iii) |

- For $\hat{p} \in (0,1)$:

| | $P_1 = \frac{P_2}{1+r}$ | $P_1 \in (\frac{P_2}{1+r}, P_2)$ | $P_1 = P_2$ | $P_1 \in (P_2, \infty)$ |
|---|---|---|---|---|
| $r = 0$ | - | - | (vi) | (ii) |
| $r \in (0, \frac{1}{\hat{p}} - 1)$ | (vii) | (viii) | (ix) | (iii) |
| $r = \frac{1}{\hat{p}} - 1$ | (vii) | (viii) | (x) | (iv) |

| | $P_1 = \frac{P_2}{1+r}$ | $P_1 \in (\frac{P_2}{1+r}, \frac{P_2}{(1+r)\hat{p}})$ | $P_1 = \frac{P_2}{(1+r)\hat{p}}$ | $P_1 \in (\frac{P_2}{(1+r)\hat{p}}, \infty)$ |
|---|---|---|---|---|
| $r \in (\frac{1}{\hat{p}} - 1, \infty)$ | (vii) | (xi) | (xii) | (xiii) |

- For $\hat{p} = 1$:

| | $P_1 = \frac{P_2}{1+r}$ | $P_1 \in (\frac{P_2}{1+r}, P_2)$ | $P_1 = P_2$ | $P_1 \in (P_2, \infty)$ |
|---|---|---|---|---|
| $r = 0$ | - | - | (xvi) | (xiv) |
| $r \in (0,\infty)$ | (xvii) | (xv) | (xv) | (xv) |

(i) $\hat{p} \in [0,1]$, $r \in (-1,0)$, $P_1 \in (0,\infty)$

By stationarity we have

$$\mu_3 + \mu_4 = r < 0,$$

which is a contradiction to dual feasibility. Thus there is no maximum. (For a given strategy, lending more on the unsecured interbank market and investing it into the liquid asset always yields higher return)

$$O^s = \emptyset, \quad s \in S$$

(ii) $\hat{p} \in [0,1)$, $r = 0$, $P_1 \in (P_2, \infty)$
Then by stationarity

$$\mu_1 = 1 - \frac{P_2}{P_1} + \mu_4 > \mu_4 \geq 0,$$
$$\mu_2 = 1 - (1+r)\hat{p} + \mu_4 > \mu_4 \geq 0$$

and therefore by complementary slackness $\beta_1 = \gamma_+ = 0$. Given the price parameters, the objective function now is independent of the choice of $\gamma_-$ and $\delta_1$:

$$f(0,0,\gamma_-,\delta_1) = -\gamma_- + \delta_1 = \beta_0 \frac{P_1}{P_0} + \delta_0 - s.$$

The resulting order sets are uncountable:

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 = \gamma_+ = 0\}, \quad s \in S.$$

(iii) $\hat{p} \in (0,1)$, $r \in (0, \frac{1}{\hat{p}} - 1)$, $P_1 \in (P_2, \infty)$ or $\hat{p} = 0$, $r \in (0, \infty)$, $P_1 \in (P_2, \infty)$
By stationarity,

$$\mu_1 = 1 - \frac{P_2}{P_1} + \mu_4 > \mu_4 \geq 0,$$
$$\mu_2 = 1 - (1+r)\hat{p} + \mu_4 > \mu_4 \geq 0,$$
$$\mu_3 + \mu_4 = r > 0$$

and therefore by complementary slackness $\beta_1 = \gamma_+ = 0$ and $(\gamma_- = 0 \vee \delta_1 = 0)$. The resulting order sets are singletons:

$$O^s = \{(0, 0, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-\}, \quad s \in S.$$

(iv) $\hat{p} \in (0,1)$, $r = \frac{1}{\hat{p}} - 1$, $P_1 \in (P_2, \infty)$

By stationarity

$$\mu_1 = 1 - \frac{P_2}{P_1} + \mu_4 > \mu_4 \geq 0,$$
$$\mu_3 + \mu_4 = r > 0,$$
$$\mu_3 + \mu_2 = r > 0$$

and therefore by complementary slackness $\beta_1 = 0$ and $(\gamma_- = 0 \vee \delta_1 = \gamma_+ = 0)$.

a) $\beta_0 \frac{P_1}{P_0} + \delta_0 - s > 0$

The budget constraint implies

$$\gamma_+ - \gamma_- + \delta_1 = \beta_0 \frac{P_1}{P_0} + \delta_0 - s > 0.$$

Since $\gamma_- \geq 0$, we get $\gamma_+ + \delta_1 > 0$ and thus $\gamma_- = 0$. Again, the objective function is constant and $\gamma_+$ and $\delta_1$ can be chosen arbitrarily as long as $\gamma_+ + \delta_1 = \beta_0 \frac{P_1}{P_0} + \delta_0 - s$.

b) $\beta_0 \frac{P_1}{P_0} + \delta_0 - s = 0$

$\gamma_+ - \gamma_- + \delta_1 = 0$ and the above conditions imply $\gamma_+ = \gamma_- = \delta_1 = 0$.

c) $\beta_0 \frac{P_1}{P_0} + \delta_0 - s < 0$

Since $\gamma_+ - \gamma_- + \delta_1 < 0$ we get $\gamma_+ = \delta_1 = 0$ and $\gamma_- = -\beta_0 \frac{P_1}{P_0} - \delta_0 + s$.

The optimal order sets can be summed up in the following form for $s \in S$:

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 = 0, \gamma_+ \leq (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^+, \gamma_- = (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-\}$$

(v) $\hat{p} \in [0, 1]$, $r \in [0, \infty)$, $P_1 \in (0, \frac{P_2}{1+r})$

$$\mu_1 + \mu_3 = (1 + r) - \frac{P_2}{P_1} < 0,$$

a contradiction to dual feasibility and again we have no maximum. In this case lending more on the unsecured interbank market and investing it into bonds always yields higher return.

$$O^s = \emptyset, \quad s \in S.$$

(vi) $\hat{p} \in [0, 1)$, $r = 0$, $P_1 = P_2$

$$\mu_2 = \mu_1 + 1 - \hat{p} > \mu_1 \geq 0,$$

which by complementary slackness means $\gamma_+ = 0$.

The objective function becomes $\beta_1 P_1 - \gamma_- + \delta_1 \equiv \beta_0 \frac{P_1}{P_0} + \delta_0 - s$ by primal feasibility. Thus all allocations with $\beta_1 P_1 - \gamma_- + \delta_1 = \beta_0 \frac{P_1}{P_0} + \delta_0 - s$ and $\beta_1, \gamma_- \delta_1 \geq 0$ are optimal.

Note that for $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$ we get

$$\gamma_- = \underbrace{\beta_1 P_1}_{\geq 0} - \delta_0 - \beta_0 \frac{P_1}{P_0} + \underbrace{s}_{> \delta_0 + \beta_0 \frac{P_1}{P_0}} > 0.$$

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \gamma_+ = 0\}, \quad s \in S.$$

(vii) $\hat{p} \in [0, 1)$, $r \in (0, \infty)$, $P_1 = \frac{P_2}{1+r}$

$$\mu_2 = \mu_1 + (1 + r) - (1 + r)\hat{p} > \mu_1 \geq 0,$$
$$\mu_4 = \mu_1 + (1 + r) - 1 > \mu_1 \geq 0,$$

which by complementary slackness means $\gamma_+ = \delta_1 = 0$. The objective function becomes $(\beta_1 P_1 - \gamma_-)(1 + r) \equiv (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)(1 + r)$ by primal feasibility. Thus all allocations with $\beta_1 P_1 - \gamma_- = \beta_0 \frac{P_1}{P_0} + \delta_0 - s$ and $\beta_1, \gamma_- \geq 0$ are optimal. Again, for $s > \beta_0 \frac{P_1}{P_0} + \delta_0$ we get

$$\gamma_- = \underbrace{\beta_1 P_1}_{\geq 0} - \delta_0 - \beta_0 \frac{P_1}{P_0} + \underbrace{s}_{> \delta_0 + \beta_0 \frac{P_1}{P_0}} > 0.$$

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \gamma_+ = 0, \beta_1 P_1 - \gamma_- = \frac{P_1}{P_0} \beta_0 + \delta_0 - s\}, \quad s \in S.$$

(viii) $\hat{p} \in (0, 1)$, $r \in (0, \frac{1}{\hat{p}} - 1]$, $P_1 \in (\frac{P_2}{1+r}, P_2)$ or $\hat{p} = 0$, $r \in (0, \infty)$, $P_1 \in (\frac{P_2}{1+r}, P_2)$

$$\mu_2 = \mu_1 + \frac{P_2}{P_1} - (1 + r)\hat{p} > \mu_1 \geq 0,$$
$$\mu_4 = \mu_1 + \frac{P_2}{P_1} - 1 > \mu_1 \geq 0 \text{ and}$$
$$\mu_1 + \mu_3 = (1 + r) - \frac{P_2}{P_1} > 0,$$

which by complementary slackness means

$$\gamma_+ = \delta_1 = 0 \quad \wedge \quad (\beta_1 = 0 \quad \vee \quad \gamma_- = 0.)$$

a) $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\gamma_- = \underbrace{\beta_1 P_1}_{\geq 0} - \delta_0 - \beta_0 \frac{P_1}{P_0} + \underbrace{s}_{> \delta_0 + \beta_0 \frac{P_1}{P_0}} > 0,$$

thus $\beta_1 = 0$.

b) $s = \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\gamma_- = \beta_1 P_1,$$

which means $\gamma_- = \beta_1 = 0$.

c) $s < \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\beta_1 P_1 = \underbrace{\gamma_-}_{\geq 0} + \delta_0 + \beta_0 \frac{P_1}{P_0} - \underbrace{s}_{<\delta_0 + \beta_0 \frac{P_1}{P_0}} > 0$$

and therefore $\gamma_- = 0$, $\beta_1 P_1 = \delta_0 + \beta_0 \frac{P_1}{P_0} - s$.

$$O^s = \{((\frac{\beta_0}{P_0} + \frac{\delta_0}{P_0} - \frac{s}{P_0})^+, 0, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-)\}, \quad s \in S.$$

(ix) $\hat{p} \in (0,1)$, $r \in (0, \frac{1}{\hat{p}} - 1)$, $P_1 = P_2$ or $\hat{p} = 0$, $r \in (0, \infty)$, $P_1 = P_2$

$$\mu_2 = \mu_4 + 1 - (1+r)\hat{p} > \mu_4 \geq 0,$$

$$\mu_1 + \mu_3 = r > 0 \text{ and}$$

$$\mu_4 + \mu_3 = r > 0,$$

which by complementary slackness means

$$\gamma_+ = 0 \quad \wedge \quad (\beta_1 = \delta_1 = 0 \quad \vee \quad \gamma_- = 0).$$

a) $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\gamma_- = \underbrace{\beta_1 P_1 + \delta_1}_{\geq 0} - \delta_0 - \beta_0 \frac{P_1}{P_0} + \underbrace{s}_{>\delta_0 + \beta_0 \frac{P_1}{P_0}} > 0,$$

thus $\beta_1 = \delta_1 = 0$ and $\gamma_- = -\delta_0 - \beta_0 \frac{P_1}{P_0} + s$.

b) $s = \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\gamma_- = \beta_1 P_1 + \delta_1,$$

which means $\gamma_- = \beta_1 = \delta_1 = 0$.

c) $s < \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\beta_1 P_1 + \delta_1 = \underbrace{\gamma_-}_{\geq 0} + \delta_0 + \beta_0 \frac{P_1}{P_0} - \underbrace{s}_{<\delta_0 + \beta_0 \frac{P_1}{P_0}} > 0$$

and therefore $\gamma_- = 0$. The objective function becomes $\beta_1 P_1 + \delta_1 \equiv \delta_0 + \beta_0 \frac{P_1}{P_0} - s$, thus all allocations with $\beta_1 P_1 + \delta_1 = \delta_0 + \beta_0 \frac{P_1}{P_0} - s$ and $\beta_1, \delta_1 \geq 0$ are optimal.

for all $s \in S$

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 P_1 \leq (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^+, \gamma_+ = 0, \gamma_- = (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^-\}.$$

(x) $\hat{p} \in (0,1)$, $r = \frac{1}{\hat{p}} - 1$, $P_1 = P_2$

$$\mu_3 + \mu_1 = r > 0,$$

$$\mu_3 + \mu_2 = r > 0 \text{ and}$$

$$\mu_3 + \mu_4 = r > 0,$$

which by complementary slackness means

$$\gamma_- = 0 \quad \vee \quad \beta_1 = \gamma_+ = \delta_1 = 0.$$

a) $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$

As in (vi), by primal feasibility we get

$$\gamma_- > 0,$$

thus

$$\beta_1 = \gamma_+ = \delta_1 = 0,$$

which means

$$\gamma_- = -\delta_0 - \beta_0 \frac{P_1}{P_0} + s.$$

b) $s = \delta_0 + \beta_0 \frac{P_1}{P_0}$

Primal feasibility implies

$$\beta_1 P_1 + \gamma_+ + \delta_1 = \gamma_-,$$

thus

$$\beta_1 = \gamma_+ = \delta_1 = \gamma_- = 0.$$

c) $s < \delta_0 + \beta_0 \frac{P_1}{P_0}$

Here the budget constraint implies

$$\beta_1 P_1 + \gamma_+ + \delta_1 = \underbrace{\gamma_-}_{\geq 0} + \delta_0 + \beta_0 \frac{P_1}{P_0} - \underbrace{s}_{< \delta_0 + \beta_0 \frac{P_1}{P_0}} > 0.$$

Thus we get $\gamma_- = 0$. The objective function becomes

$$\beta_1 P_1 + \gamma_+ + \delta_1 \equiv \delta_0 + \beta_0 \frac{P_1}{P_0} - s,$$

independent of all four variables. Thus every allocation with

$$\beta_1, \gamma_+, \delta_1 \geq 0, \quad \gamma_- = 0$$

and

$$\beta_1 P_1 + \gamma_+ + \delta_1 = \delta_0 + \beta_0 \frac{P_1}{P_0} - s$$

is optimal.

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 P_1 + \gamma_+ \leq (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^+, \gamma_- = (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^-\}.$$

(xi) $\hat{p} \in (0,1)$, $r \in (\frac{1}{\hat{p}} - 1, \infty)$, $P_1 \in (\frac{P_2}{1+r}, \frac{P_2}{(1+r)\hat{p}})$

$$\mu_2 = \mu_1 + \frac{P_2}{P_1} - (1+r)\hat{p} > \mu_1 \geq 0,$$

$$\mu_4 = \mu_1 + \frac{P_2}{P_1} - 1 > \mu_1 \geq 0 \text{ and}$$

$$\mu_1 + \mu_3 = (1+r) - \frac{P_2}{P_1} > 0,$$

which by complementary slackness means

$$\gamma_+ = \delta_1 = 0 \quad \wedge \quad (\beta_1 = 0 \quad \vee \quad \gamma_- = 0).$$

This is the same situation as in case (viii).

$$O^s = \{((\frac{\beta_0}{P_0} + \frac{\delta_0}{P_0} - \frac{s}{P_0})^+, 0, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-)\}, \quad s \in S.$$

(xii) $\hat{p} \in (0,1)$, $r \in (\frac{1}{\hat{p}} - 1, \infty)$, $P_1 = \frac{P_2}{(1+r)\hat{p}}$

$$\mu_4 = \mu_1 + \frac{P_2}{P_1} - 1 > \mu_1 \geq 0,$$

$$\mu_3 + \mu_1 = (1+r) - (1+r)\hat{p} > 0 \text{ and}$$

$$\mu_3 + \mu_2 = (1+r) - (1+r)\hat{p} > 0,$$

which by complementary slackness means

$$\delta_1 = 0 \quad \wedge \quad (\gamma_- = 0 \quad \vee \quad \beta_1 = \gamma_+ = 0).$$

Analogously to case (viii) this yields the following optimal allocations:

a) $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\beta_1 = \gamma_+ = \delta_1 = 0,$$
$$\gamma_- = -\delta_0 - \beta_0 \frac{P_1}{P_0} + s.$$

b) $s = \delta_0 + \beta_0 \frac{P_1}{P_0}$

$$\beta_1 = \gamma_+ = \delta_1 = \gamma_- = 0.$$

c) $s < \delta_0 + \beta_0 \frac{P_1}{P_0}$
Every allocation with

$$\beta_1, \gamma_+ \geq 0, \quad \gamma_- = \delta_1 = 0$$

and

$$\beta_1 P_1 + \gamma_+ = \delta_0 + \beta_0 \frac{P_1}{P_0} - s.$$

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 P_1 + \gamma_+ = (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^+, \gamma_- = (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^-\}.$$

(xiii) $\hat{p} \in (0, 1)$, $r \in (\frac{1}{\hat{p}} - 1, \infty)$, $P_1 \in (\frac{P_2}{(1+r)\hat{p}}, \infty)$

$$\mu_1 = \mu_2 + (1+r)\hat{p} - \frac{P_2}{P_1} > \mu_2 \geq 0,$$
$$\mu_4 = \mu_2 + (1+r)\hat{p} - 1 > \mu_2 \geq 0,$$
$$\mu_2 + \mu_3 = (1+r)(1-\hat{p}) > 0,$$

therefore $\beta_1 = \delta_1 = 0$ and $(\gamma_+ = 0 \vee \gamma_- = 0)$.

$$O^s = \{(0, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^+, (\beta_0 \frac{P_1}{P_0} + \delta_0 - s)^-)\}, \quad s \in S.$$

(xiv) $\hat{p} = 1$, $r = 0$, $P_1 \in (P_2, \infty)$

$$\mu_1 = \mu_2 + (1+r) - \frac{P_2}{P_1} > \mu_2 \geq 0,$$

therefore $\beta_1 = 0$. The objective function becomes constant, thus

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 = 0\}, \quad s \in S.$$

(xv) $\hat{p} = 1$, $r \in (0, \infty)$, $P_1 \in (\frac{P_2}{(1+r)}, \infty)$

$$\mu_1 = \mu_2 + (1+r) - \frac{P_2}{P_1} > \mu_2 \geq 0,$$
$$\mu_4 = \mu_2 + (1+r) - 1 > \mu_2 \geq 0,$$

therefore $\beta_1 = \delta_1 = 0$. Again, the objective function becomes constant, thus

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 = 0, \gamma_+ - \gamma_- = \frac{P_1}{P_0}\beta_0 + \delta_0 - s\}, \quad s \in S.$$

(xvi) $\hat{p} = 1$, $r = 0$, $P_1 = P_2$

The objective function directly becomes constant, thus $O^s = C^s$ for all $s \in S$.

(xvii) $\hat{p} = 1$, $r \in (0, \infty)$, $P_1 = \frac{P_2}{(1+r)}$

$$\mu_4 = \mu_1 + (1+r) - 1 > \mu_2 \geq 0,$$

therefore $\delta_1 = 0$. The objective function becomes constant, thus

$$O^s = \{(\beta_1, \gamma_+, \gamma_-) \in C^s : \beta_1 P_1 + \gamma_+ - \gamma_- = \frac{P_1}{P_0}\beta_0 + \delta_0 - s\}, \quad s \in S.$$

□

*Proof of 3.2.* Now we prove Theorem 3.2 by looking for optimal assignments. By definiton, $(r, P_1)$ is an equilibirum price pair iff there is an optimal assignment. Since we now have to check the market clearing conditions, the initial endowment is important and we have to make case differentiations accordingly. All equations containing the liquidity demand $s$ are to be considered $P^Y$-a.s..

First, let $\delta_0 < E(Y)$ and assume there is an optimal assignment. Thus by definiton $a(s) \in O^s \subset C^s$, in particular $a_1(s)P_1 + a_2(s) - a_3(s) \leq \beta_0 \frac{P_1}{P_0} + \delta_0 - s$. We have

$$E[a_1(Y)P_1 + a_2(Y) - a_3(Y)] \leq E[\beta_0 \frac{P_1}{P_0} + \delta_0 - Y] = \beta_0 \frac{P_1}{P_0} + \delta_0 - E[Y] < \beta_0 \frac{P_1}{P_0},$$

in contradiction to

$$E[a_1(Y)P_1 + a_2(Y) - a_3(Y)] = \beta_0 \frac{P_1}{P_0},$$

which can be derived directly by applying the market clearing conditions $E[a_2(Y)] = E[a_3(Y)]$ and $E[a_1(Y)] = E[h_1(Y)] = \frac{\beta_0}{P_0}$. Thus there is neither interbank lending nor trading of bonds, thus banks with $\delta_0 < s$ fail. The remaining banks can only put their spare liquidity in the liquid asset. This implies

$$(\beta_1^s, \gamma_+^s, \gamma_-^s, \delta_1^s) = (\frac{\beta_0}{P_0}, 0, 0, \delta_0 - s).$$

The terminal wealth of a bank facing liquidity demand $s \leq \delta_0$ is

$$V_2(\alpha, \beta_0, \delta_0) = \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} + \delta_0 - s - (1 - \frac{s}{c_1})c_2.$$

Now let $\delta_0 = E(Y)$ and $\beta_0 > 0$, consider the cases (i) to (xvi) from above. We do not have to check each of the seventeen cases from the proof of 3.1 individually. We can group them in five categories:

- Cases (i) and (v)

  Since $O^s = \emptyset$ for all $s \in S$, there is no optimal assignment.

- Cases (ii), (iii), (iv), (xiii), (xiv) and (xv)

  For all $s \in S$ and for all $(\beta_1, \gamma_+, \gamma_-) \in O^s$ we have $\beta_1 = 0$. Assume there is an optimal assignment, then $a(s) \in O^s$ for every $s \in S$ implies

  $$E[a_1(Y)] = 0,$$

  which is a contradiction to $\beta_0 > 0$ and the market clearing condition $E[a_1(Y)] = \frac{\beta_0}{P_0}$. Again, there is no optimal assignment.

- Cases (vi), (vii), (viii), (ix) and (xi)

  This means $\hat{p} \in [0,1)$, $r \in [0,\infty)$ and $P_1 \in [\frac{P_2}{1+r}, P_2] \cap [\frac{P_2}{1+r}, \frac{P_2}{(1+r)\hat{p}})$ (where for $\hat{p} = 0$, $\frac{P_2}{(1+r)\hat{p}} = +\infty$).

  For all $s \in S$ and for all $(\beta_1, \gamma_+, \gamma_-) \in O^s$ we have $\gamma_+ = 0$. Also, for $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$ we have $\gamma_- > 0$.

  Assume $P\left(Y > \delta_0 + \beta_0 \frac{P_1}{P_0}\right) > 0$ and let $a$ be an optimal assignment. Then

$$E[a_2(Y)] = 0 \quad \text{and} \quad E[a_3(Y)] > 0,$$

  a contradiction to the market clearing condition $E[a_2(Y)] = E[a_3(Y)]$. This means that in order to find an optimal assignment in these cases we have to assume $Y \leq \delta_0 + \beta_0 \frac{P_1}{P_0}$ P-a.s..

  In cases (viii) and (xi), all order sets $O^s$ consist of only one element

$$O^s = \{(\frac{\delta_0}{P_1} + \frac{\beta_0}{P_0} - \frac{s}{P_1}, 0, 0)\}$$

  and assigning exactly those amounts accords with both market clearing conditions:

$$E[a_1(Y)] = \frac{\delta_0}{P_1} + \frac{\beta_0}{P_0} - \frac{E(Y)}{P_1} = \frac{\beta_0}{P_0},$$

$$E[a_2(Y)] = 0 = E[a_3(Y)].$$

  We get

$$a(s) = (\frac{\delta_0}{P_1} + \frac{\beta_0}{P_0} - \frac{s}{P_1}, 0, 0). \tag{5.1}$$

  In cases (vi), (vii) and (ix), the market clearing condition demands

$$E[a_2(Y)] = E[a_3(Y)] = 0,$$

  thus by $a_2(s) \geq 0$ we get $a_2 \equiv 0$. In case (vii), this results directly in the unique optimal assignment (5.1).

  In cases (vi), we apply the budget constraint:

$$a_1(s)P_1 = a_1(s)P_1 + a_2(s) - a_3(s) \leq \beta_0 \frac{P_1}{P_0} + \delta_0 - s,$$

  which also holds in case (ix). By the market clearing conditions we get

$$E[a_1(Y)P_1] = E[a_1(Y)P_1 + a_2(Y) - a_3(Y)] = \beta_0 \frac{P_1}{P_0} = E[\beta_0 \frac{P_1}{P_0} + \delta_0 - Y],$$

thus $a_1(s)P_1 = \beta_0\frac{P_1}{P_0}+\delta_0-s$ for all $s \in S$. Therefore, the unique optimal assignment is again given by (5.1)

There is no trading on the unsecured interbank market. Therefore $\hat{p}$ and $r$ can be chosen arbitrarily (they do not matter anymore regarding optimization). Also, $\hat{p}$ doesn't have to be checked for consistency. What remains is two conditions on $P_1$: $P_1 \in (0, P_2]$ and ess sup $Y \leq \delta_0 + \beta_0\frac{P_1}{P_0}$, which means

$$P_1 \in \left[P_0\frac{\text{ess sup } Y - E(Y)}{\beta_0}, P_2\right].$$

This is possible, iff $\beta_0 \geq (\text{ess sup } Y - E(Y))\frac{P_0}{P_2} > 0$.

The investment in the liquid asset is given by

$$\delta_1 = \frac{P_1}{P_0}\beta_0 - a_1(s)P_1 + \delta_0 - s + a_3(s) - a_2(s) = 0$$

for all $s \in S$.

Thus, the terminal wealth of a banking facing liquidity demand $s = \lambda c_1$ is

$$\begin{aligned} V_2(\alpha, \beta_0, \delta_0) \qquad &= \alpha\frac{S_{t_2}}{S_0} + (\frac{\delta_0}{P_1} + \frac{\beta_0}{P_0} - \frac{s}{P_1})P_2 - (1 - \frac{s}{c_1})c_2 \\ &= \alpha\frac{S_{t_2}}{S_0} + \beta_0\frac{P_2}{P_0} + \delta_0\frac{P_2}{P_1} - s\frac{P_2}{P_1} - (1 - \frac{s}{c_1})c_2 \\ &= \alpha\frac{S_{t_2}}{S_0} + \beta_0\frac{P_2}{P_0} + (E(Y) - s)\frac{P_2}{P_1} - (1 - \frac{s}{c_1})c_2 \\ &\left(= \alpha\frac{S_{t_2}}{S_0} + \beta_0\frac{P_2}{P_0} + (E(\Lambda) - \lambda)c_1\frac{P_2}{P_1} - (1 - \lambda)c_2\right) \end{aligned}$$

- Cases (x) and (xii)

This means $\hat{p} \in (0, 1)$, $r \in [\frac{1}{\hat{p}} - 1, \infty)$ and $P_1 = \frac{P_2}{(1+r)\hat{p}}$ or $\hat{p} = 1$, $r \in (0, \infty)$ and $P_1 = \frac{P_2}{1+r}$. For $s > \delta_0 + \beta_0\frac{P_1}{P_0}$ we directly get

$$a_1(s) = a_2(s) = 0, \quad a_3(s) = s - \delta_0 - \beta_0\frac{P_1}{P_0}.$$

In case (x), we have $a_1(s)P_1 + a_2(s) \leq \delta_0 + \beta_0\frac{P_1}{P_0} - s$ and by the market clearing conditions

$$E[a_1(Y)P_1 + a_2(Y)] = E[a_1(Y)P_1 + a_2(Y) - a_3(Y)] + E[a_3(Y)] = \beta_0\frac{P_1}{P_0} + E[a_3(Y)]$$

$$= \beta_0\frac{P_1}{P_0} + E[(Y - \delta_0 - \beta_0\frac{P_1}{P_0})1_{\{Y > \delta_0 + \beta_0\frac{P_1}{P_0}\}}] \overset{EY = \delta_0}{=} E[(\delta_0 + \beta_0\frac{P_1}{P_0} - Y)1_{\{Y \leq \delta_0 + \beta_0\frac{P_1}{P_0}\}}].$$

Since

$$E[a_1(Y)P_1 + a_2(Y)] = E[(a_1(Y)P_1 + a_2(Y))1_{\{Y \le \delta_0 + \beta_0 \frac{P_1}{P_0}\}}],$$

this implies $a_1(s)P_1 + a_2(s) = \delta_0 + \beta_0 \frac{P_1}{P_0} - s$ for all $s \le \delta_0 + \beta_0 \frac{P_1}{P_0}$. This equation also holds for case (xii). In both cases, an optimal assignment is characterized by

$$a_1(s)P_1 + a_2(s) = (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^+, a_3(s) = (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)^-, E[a_1(Y)] = \frac{\beta_0}{P_0}.$$

The investment in the liquid asset is given by

$$\delta_1 = \frac{P_1}{P_0}\beta_0 - a_1(s)P_1 + \delta_0 - s + a_3(s) - a_2(s) = 0$$

for all $s \in S$.

Note that if $P\left(Y > \delta_0 + \beta_0 \frac{P_1}{P_0}\right) = 0$, there is no unsecured lending and the results are as in cases (vi), (vii), (viii), (ix) and (xi). In particular, the optimal assignment is unique. Else, we get multiple solutions. Applying $P_1 = \frac{P_2}{(1+r)\hat{p}}$ ( $\iff$ $(1+r)\hat{p} = \frac{P_2}{P_1}$) , we get for the terminal wealth of a banking facing liquidity demand $s$:

$$\begin{aligned} V_2(\alpha, \beta_0, \delta_0) &= \alpha \frac{S_{t_2}}{S_0} + a_1(s)P_2 + a_2(s)(1+r)\hat{p} - a_3(s)(1+r) - (1 - \frac{s}{c_1})c_2 \\ &= \alpha \frac{S_{t_2}}{S_0} + (a_1(s)P_1 + a_2(s))\frac{P_2}{P_1} - a_3(s)\frac{P_2}{P_1\hat{p}} - (1 - \frac{s}{c_1})c_2. \end{aligned}$$

For a bank facing liquidity demand $s \le \delta_0 + \beta_0 \frac{P_1}{P_0}$ this results in

$$\begin{aligned} V_2(\alpha, \beta_0, \delta_0) &= \alpha \frac{S_{t_2}}{S_0} + (\delta_0 + \beta_0 \frac{P_1}{P_0} - s)\frac{P_2}{P_1} - (1 - \frac{s}{c_1})c_2 \\ &= \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} + (E(Y) - s)\frac{P_2}{P_1} - (1 - \frac{s}{c_1})c_2 \end{aligned}$$

And for $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$ we get

$$\begin{aligned} V_2(\alpha, \beta_0, \delta_0) &= \alpha \frac{S_{t_2}}{S_0} - (s - \delta_0 - \beta_0 \frac{P_1}{P_0})\frac{P_2}{P_1\hat{p}} - (1 - \frac{s}{c_1})c_2 \\ &= \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0\hat{p}} + (E(Y) - s)\frac{P_2}{P_1\hat{p}} - (1 - \frac{s}{c_1})c_2 \end{aligned}$$

- Cases (xvi) and (xvii)

This means $\hat{p} = 1$, $r \in [0, \infty)$ and $P_1 = \frac{P_2}{1+r}$. In case (xvi), we get

$$E[\frac{P_1}{P_0}\beta_0 + \delta_0 - Y - a_1(Y)P_1 + a_3(Y) - a_2(Y)] = 0$$

by market clearing and $\delta_0 = E(Y)$. Together with $a(s) \in O^s$, this implies

$$\frac{P_1}{P_0}\beta_0 + \delta_0 - s = a_1(s)P_1 + a_2(s) - a_3(s). \qquad (*)$$

The same holds in case (xvii). In particular, $\delta_1 = 0$. All assignments with $E[a_1(Y)] = \frac{\beta_0}{P_0}$, $E[a_2(Y)] = E[a_3(Y)]$ and $(*)$ are optimal. For a bank facing liquidity demand $s$, this results in

$$V_2(\alpha, \beta_0, \delta_0) \quad = \quad \alpha\frac{S_{t_2}}{S_0} + \beta_0\frac{P_2}{P_0} + (\delta_0 - s)(1+r) - (1 - \frac{s}{c_1})c_2.$$

Next, let $\delta_0 > E(Y)$ and $\beta_0 > 0$. Since many of the arguments are the same as in the correspondent cases for $\delta_0 = E(Y)$, we can keep the proofs shorter. As above, we can show that in the cases (i), (ii), (iii), (iv), (v), (xiii), (xiv) and (xv) there is no optimal assignment.

- Cases (vii), (viii), (xi), (xii) and (xvii)

  Assume, there is an optimal assignment. Then $a(s) \in O^s$, thus

  $$E[a_1(Y)P_1 + a_2(Y) - a_3(Y)] = E[\beta_0\frac{P_1}{P_0} + \delta_0 - Y] = \beta_0\frac{P_1}{P_0} + \delta_0 - E[Y] > \beta_0\frac{P_1}{P_0}.$$

  But the market clearing conditions imply

  $$E[a_1(Y)P_1 + a_2(Y) - a_3(Y)] = E[a_1(Y)P_1] = \beta_0\frac{P_1}{P_0},$$

  a contradiction.

- Cases (vi) and (ix)

  This means $\hat{p} \in [0,1)$, $r \in [0, \frac{1}{\hat{p}} - 1)$ and $P_1 = P_2$ (where for $\hat{p} = 0$, $\frac{1}{\hat{p}} - 1 = +\infty$). For $P\left(Y > \delta_0 + \beta_0\frac{P_1}{P_0}\right) > 0$ there is no optimal assignment, following the same arguments as above. Again we get, $a_2(s) = a_3(s) = 0$ for all $s \in S$. In case (vi), the budget constraint implies $a_1(s)P_1 \leq \delta_0 + \beta_0\frac{P_1}{P_0} - s$, which also holds in case (ix). The resulting optimal assignments are characterized by

  $$a_2(s) = a_3(s) = 0, \quad a_1(s) \leq \frac{\delta_0}{P_1} + \frac{\beta_0}{P_0} - \frac{s}{P_1}, \quad E[a_1(Y)] = \frac{\beta_0}{P_0}.$$

  The investment in the liquid asset is given by

  $$\delta_1 = \frac{P_1}{P_0}\beta_0 - a_1(s)P_1 + \delta_0 - s + a_3(s) - a_2(s) = \frac{P_1}{P_0}\beta_0 + \delta_0 - a_1(s)P_1 - s.$$

There is no trading on the unsecured interbank market, $\hat{p}$ and $r$ can be chosen arbitrarily. The resulting terminal wealth of a banking facing liquidity demand $s$ is

$$V_2(\alpha, \beta_0, \delta_0) \quad = \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} + \delta_0 - s - (1 - \frac{s}{c_1})c_2.$$

- Case (x)

  This means $\hat{p} \in (0,1)$, $r = \frac{1}{\hat{p}} - 1$ and $P_1 = P_2$. Optimal assignments are characterized by

$$a_3(s) = 0, \quad a_1(s)P_2 + a_2(s) \le \delta_0 + \beta_0 \frac{P_2}{P_0} - s, \quad \text{for } s \le \delta_0 + \beta_0 \frac{P_2}{P_0},$$
$$a_1(s) = a_2(s) = 0, \quad a_3(s) = s - \delta_0 - \beta_0 \frac{P_2}{P_0}, \quad \text{for } s > \delta_0 + \beta_0 \frac{P_2}{P_0},$$
$$E[a_1(Y)] = \frac{\beta_0}{P_0}, \quad E[a_2(Y)] = E[a_3(Y)].$$

  The investment in the liquid asset is given by

$$\delta_1 = \begin{cases} \beta_0 \frac{P_2}{P_0} + \delta_0 - a_1(s)P_2 - a_2(s) - s, & s \le \delta_0 + \beta_0 \frac{P_2}{P_0}, \\ 0, & s > \delta_0 + \beta_0 \frac{P_2}{P_0}. \end{cases}$$

  The terminal wealth of a banking facing liquidity demand $s \le \delta_0 + \beta_0 \frac{P_2}{P_0}$ is

$$V_2(\alpha, \beta_0, \delta_0) \quad = \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} + \delta_0 - s - (1 - \frac{s}{c_1})c_2,$$

  and a banking facing liquidity demand $s > \delta_0 + \beta_0 \frac{P_2}{P_0}$ has terminal wealth

$$V_2(\alpha, \beta_0, \delta_0) \quad = \alpha \frac{S_{t_2}}{S_0} + \left( \beta_0 \frac{P_2}{P_0} + \delta_0 - s \right) \frac{1}{\hat{p}} - (1 - \frac{s}{c_1})c_2.$$

- Case (xvi)

  This means $\hat{p} = 1$, $r = 0$ and $P_1 = P_2$. All assignments with $E[a_1(Y)] = \frac{\beta_0}{P_0}$, $E[a_2(Y)] = E[a_3(Y)]$ and $a(s) \in O^s$ for all $s \in S$ are optimal assignments. The terminal wealth, given liquidity demand $s$, is

$$V_2(\alpha, \beta_0, \delta_0) \quad = \quad \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} + \delta_0 - s - (1 - \frac{s}{c_1})c_2.$$

Technically, for $\beta_0 = 0$ there is only lending and 0 goods on the market. But the above results also hold (they were derived for $\beta_0 \ge 0$), i.e. instead of deriving optimal order sets on a market without bond trade, we formally allow trade of bonds but demand $E[a_1(Y)] = 0$. Thus we assume $\delta_0 = E(Y)$ and $\beta_0 = 0$. There is no optimal assignment in the cases (i) and (v).

- Cases (vii), (viii) and (xi)

  Assume, there is an optimal assignment $a$. Then for all $s \in S$ $a_2(s) = 0$. The second market clearing condition $E[a_3(Y)] = E[a_2(Y)] = 0$ implies $a_3 \equiv 0$. By the first market clearing condition $E[a_1(Y)] = 0$ we have $a_1 \equiv 0$. But then, in all three cases, the condition $a(s) \in O^s$ for all $s \in S$ is violated.

- Cases (ii), (iii), (vi) and (ix)

  Again, $a_2(s) = 0$ for all $s \in S$, thus $a_3 \equiv 0$. This can hold only if $Y \le \delta_0 + \beta_0 \frac{P_1}{P_0} = \delta_0$ P-a.s.. This is a contradiction to $\delta_0 = E(Y)$, since $Y$ is assumed not to be constant.

- Cases (iv), (x), (xii) and (xiii)

  This means $\hat{p} \in (0,1)$, $r \in [\frac{1}{\hat{p}} - 1, \infty)$ and $P_1 \in [\frac{P_2}{(1+r)\hat{p}}, \infty)$. In case (x) and (xii), we derive $a_1 \equiv 0$ with the first market clearing condition. In case (iv) and (x) we now have $a_2(s) \le (\delta_0 - s)^+$ for all $s \in S$. The second market clearing condition implies

  $$E[a_2(Y)] = E[a_3(Y)] = E[(\delta_0 - Y)^-] = E[(\delta_0 - Y)^+].$$

  The last equation holds, because $E[\delta_0 - Y] = \delta_0 - E(Y) = 0$. This concludes $a_2(s) = (\delta_0 - s)^+$ for all $s \in S$. This also holds in cases (xii) and (xiii), thus we can characterize an optimal assignment by

  $$a_1(s) = 0, \quad a_2(s) = (\delta_0 - s)^+ \quad \text{and} \quad a_3(s) = (\delta_0 - s)^- \quad \text{for all } s \in S.$$

  We get $\delta_1 = 0$ for all $s \in S$. Applying $1 = \alpha + \beta_0 + \delta_0 = \alpha + E(Y)$ yields

  $$V_2(\alpha, \beta_0, \delta_0) = \begin{cases} (1 - E(Y))\frac{S_{t_2}}{S_0} + (E(Y) - s)(1+r)\hat{p} - (1 - \frac{s}{c_1})c_2, & s \le E(Y) \\ (1 - E(Y))\frac{S_{t_2}}{S_0} + (E(Y) - s)(1+r) - (1 - \frac{s}{c_1})c_2, & s > E(Y) \end{cases}$$

- Cases (xiv), (xv), (xvi) and (xvii)

  This means $\hat{p} = 1$, $r \in [0, \infty)$ and $P_1 \in [\frac{P_2}{1+r}, \infty)$. We can show $a_1 \equiv 0$ using the first market clearing condition. In case (xiv) and (xvi) we use $a(s) \in O^s \subset C^s$ to show $a_2(s) - a_3(s) \le \delta_0 - s$. Applying

  $$E[a_2(Y) - a_3(Y)] = 0 = E[\delta_0 - Y]$$

  yields $a_2(s) - a_3(s) = \delta_0 - s$. The same holds for the cases (xv) and (xvii). An optimal assignment is characterized by

  $$a_1(s) = 0 \quad \text{and} \quad a_2(s) - a_3(s) = E(Y) - s, \quad \text{for all } s \in S.$$

Terminal wealth is

$$V_2(\alpha, \beta_0, \delta_0) \quad = \quad \alpha \frac{S_{t_2}}{S_0} + \beta_0 \frac{P_2}{P_0} + (\delta_0 - s)(1+r) - (1 - \frac{s}{c_1})c_2.$$

Finally, let $\delta_0 > E(Y)$ and $\beta_0 = 0$. The cases (i), (v), (vii), (viii) and (xi) can be treated as for $\delta_0 = E(Y)$ and $\beta_0 = 0$, where we showed that there is no optimal assignment.

- Cases (ii), (iii), (vi) and (ix)

  This means $\hat{p} \in [0,1)$, $r \in [0, \frac{1}{\hat{p}} - 1)$ and $P_1 \in [P_2, \infty)$. For an optimal assignment we have $a_2 \equiv 0$, thus $a_3 \equiv 0$. This can hold only if ess sup $Y \leq \delta_0$. In this case the optimal assignment is given by $a \equiv (0,0,0)$ and the investment in the liquid asset is $\delta_1 = \delta_0 - s$. This yields terminal wealth of

$$V_2(\alpha, \beta_0, \delta_0) \quad = \quad \alpha \frac{S_{t_2}}{S_0} + \delta_0 - s - (1 - \frac{s}{c_1})c_2.$$

- Cases (xii), (xiii), (xv) and (xvii)

  By the market clearing conditions, we get $0 = E[a_1(Y) + a_2(Y) - a_3(Y)]$, which by $a(s) \in O^s$ turns into

$$0 = E[\delta_0 - Y] = \delta_0 - E(Y) > 0,$$

  a contradiction.

- Cases (iv) and (x)

  This means $\hat{p} \in (0,1)$, $r = \frac{1}{\hat{p}} - 1$ and $P_1 \in [P_2, \infty)$. $a_1 \equiv 0$ is clear. For $s > \delta_0$ we get $a_2(s) = 0$, $a_3(s) = \delta_0 - s$ and therefore $\delta_1 = 0$. For $s \leq \delta_0$, $a_3(s) = 0$. Thus $a_2(s) \leq \delta_0 - s$ and $\delta_1 = \delta_0 - s - a_2(s)$. $a$ is an optimal assignment iff $E[a_2(Y)] = E[Y 1_{\{Y > \delta_0\}}] - \delta_0 P(Y > \delta_0)$ and for all $s \in S$

$$a_1(s) = 0, \quad 0 \leq a_2(s) \leq (\delta_0 - s)^+, \quad a_3(s) = (\delta_0 - s)^-.$$

  Terminal wealth is

$$V_2(\alpha, \beta_0, \delta_0) = \begin{cases} \alpha \frac{S_{t_2}}{S_0} + \delta_0 - s - (1 - \frac{s}{c_1})c_2, & s \leq \delta_0 \\ \alpha \frac{S_{t_2}}{S_0} + (\delta_0 - s)\frac{1}{\hat{p}} - (1 - \frac{s}{c_1})c_2, & s > \delta_0 \end{cases}$$

- Cases (xiv) and (xvi)

  This means $\hat{p} = 1$, $r = 0$ and $P_1 \in [P_2, \infty)$. $a$ is an optimal assignment iff $E[a_2(Y)] = E[a_3(Y)]$ and for all $s \in S$

$$a_1(s) = 0, \quad a_2(s) \geq 0, a_3(s) \geq 0, \quad a_2(s) - a_3(s) \leq \delta_0 - s.$$

  Furthermore, $\delta_1 = \delta_0 - s + a_3(s) - a_2(s)$ and terminal wealth is

$$V_2(\alpha, \beta_0, \delta_0) \quad = \quad \alpha \frac{S_{t_2}}{S_0} + \delta_0 - s - (1 - \frac{s}{c_1})c_2.$$

□

*Proof of 3.3.* For $a_3(s)=(\delta_0+\beta_0\frac{P_2}{P_0}-s)^-$ and $(a_1(s)+a_2(s))a_3(s)=0$ for all $s\in S$, we get

$$R^{s,x}(p,\frac{1}{p}-1,P_2,a)=\begin{cases}(s-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}, & V_2^{s,x}(\alpha,\beta_0,\delta_0)\geq 0, s>\delta_0+\beta_0\frac{P_2}{P_0},\\ (\alpha\frac{x}{S_0}-(1-\frac{s}{c_1})c_2)^+, & V_2^{s,x}(\alpha,\beta_0,\delta_0)<0, s>\delta_0+\beta_0\frac{P_2}{P_0},\\ 0, & s\leq\delta_0+\beta_0\frac{P_2}{P_0}.\end{cases}$$

with $V_2^{s,x}(\alpha,\beta_0,\delta_0)=\alpha\frac{x}{S_0}-(s-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}-(1-\frac{s}{c_1})c_2$ for $s>\delta_0+\beta_0\frac{P_2}{P_0}$.
Thus

$$\begin{aligned}
& E\left[R^{Y,S_{t_2}}(p,\frac{1}{p}-1,P_2,a)\right]\\
= & \frac{1}{p}E\left[\left(Y-\delta_0-\beta_0\frac{P_2}{P_0}\right)1_{\{\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\geq(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p},Y>\delta_0+\beta_0\frac{P_2}{P_0}\}}\right]\\
& +E\left[\left(\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\right)1_{\{0\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p},Y>\delta_0+\beta_0\frac{P_2}{P_0}\}}\right]\\
= & \frac{1}{p}E\left[\left(Y-\delta_0-\beta_0\frac{P_2}{P_0}\right)1_{\{\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\geq(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}>0\}}\right]\\
& +E\left[\left(\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\right)1_{\{0\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}\}}\right].
\end{aligned}$$

We will show that $h$ is monotonely increasing on $[0,1]$. By definition, $h(p)\geq 0=h(0)$ for all $p\in[0,1]$. Let $0<p<q\leq 1$, then

$$\begin{aligned}
& h(p)E\left[(\delta_0+\beta_0\frac{P_2}{P_0}-Y)^-\right]\\
= & pE\left[R^{Y,S_{t_2}}(p,\frac{1}{p}-1,P_2,a)\right]\\
= & E\left[\left(Y-\delta_0-\beta_0\frac{P_2}{P_0}\right)1_{\{\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\geq(Y-\delta_0+\beta_0\frac{P_2}{P_0})\frac{1}{p}>0\}}\right]\\
& +pE\left[\left(\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\right)1_{\{0\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}\}}\right]\\
= & E\left[\left(Y-\delta_0-\beta_0\frac{P_2}{P_0}\right)1_{\{\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\geq(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}>0\}}\right]\\
& +pE\left[\left(\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\right)1_{\{0\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{q}\}}\right]\\
& +pE\left[\left(\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\right)1_{\{0<(Y-\delta_0+\beta_0\frac{P_2}{P_0})\frac{1}{q}\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}\}}\right]\\
\leq & E\left[\left(Y-\delta_0-\beta_0\frac{P_2}{P_0}\right)1_{\{\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\geq(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}>0\}}\right]\\
& +qE\left[\left(\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\right)1_{\{0\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{q}\}}\right]\\
& +pE\left[\frac{1}{p}\left(Y-\delta_0+\beta_0\frac{P_2}{P_0}\right)1_{\{0<(Y-\delta_0+\beta_0\frac{P_2}{P_0})\frac{1}{q}\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{p}\}}\right]
\end{aligned}$$

$$
\begin{aligned}
&= E\left[\left(Y-\delta_0-\beta_0\frac{P_2}{P_0}\right)1_{\{\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\geq(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{q}>0\}}\right] \\
&\quad +qE\left[\left(\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2\right)1_{\{0\leq\alpha\frac{S_{t_2}}{S_0}-(1-\frac{Y}{c_1})c_2<(Y-\delta_0-\beta_0\frac{P_2}{P_0})\frac{1}{q}\}}\right] \\
&= qE\left[R^{Y,S_{t_2}}(q,\frac{1}{q}-1,P_2,a)\right] \\
&= h(q)E\left[(\delta_0-\beta_0\frac{P_2}{P_0}-Y)^-\right].
\end{aligned}
$$

Define a $[0,1]$-sequence $(p_n)_{n\in\mathbb{N}_0}$ by

$$p_0 := 1, \quad p_n := h^n(1), \qquad n\in\mathbb{N}.$$

Then obviously for $n\in\mathbb{N}_0$ we have $p_{n+1}=h(p_n)$ and the sequence $(p_n)_{n\in\mathbb{N}_0}$ is monotone, since $p_1\leq 1=p_0$ and $p_n\leq p_{n-1}$ implies $p_{n+1}=h(p_n)\leq h(p_{n-1})=p_n$. Since $(p_n)_{n\in\mathbb{N}_0}$ is bounded, the monotone convergence theorem for sequences of real numbers implies that the sequence is convergent. We define

$$\overline{p} := \lim_{n\to\infty} p_n = \lim_{n\to\infty} h^n(1).$$

By the representation of $h(p)$ we derived above, it is clear that $\overline{p}$ only depends on $(\alpha,\beta_0,\delta_0)$.

Using this representation again and applying that the sequence $(p_n)_{n\in\mathbb{N}_0}$ is monotonely decreasing, monotone convergence (Beppo Levi) implies $h(\overline{p})=h(\lim_{n\to\infty}p_n)=\lim_{n\to\infty}h(p_n)=\overline{p}$.

Finally, assume, there is $\tilde{p}>\overline{p}$ with $h(\tilde{p})=\tilde{p}$. Then $p_0=1$ and $p_n\downarrow\overline{p}$ imply that there are $k,m\in\mathbb{N}$ with $m>k$ and $p_k>\tilde{p}>p_m$. Then $p_m=h^{m-k}(p_k)\geq h^{m-k}(\tilde{p})=\tilde{p}$ by monotonicity of $h$, which is a contradiction. Therefore, $\overline{p}$ is the largest fixpoint of $h$: □

*Proof of 3.4.* First assume, that $\overline{p} > 0$. Then by theorem 3.2, $r = \frac{1}{\overline{p}} - 1$ and $P_1 = P_2$ are equilibrium prices. A corresponding market clearing assignment is (not necessarily uniquely) given by $a \in A$, $a_3(s) = (\delta_0 + \beta_0 \frac{P_2}{P_0} - s)^-$ and $a_1(s) + a_2(s) \leq (\delta_0 + \beta_0 \frac{P_2}{P_0} - s)^+$ for all $s \in S$. In this market, the real redemption rate (see definition 2.5) is given by

$$\tilde{p} = h(\overline{p}) = \overline{p}.$$

Thus, $(\overline{p}, \frac{1}{\overline{p}} - 1, P_2, a)$ is a consistent quadruple.

Now, let $(\hat{p}, r, P_1, a)$ be a consistent quadruple. Remember that we only consider the case $\delta_0 \geq EY$ and $\delta_0 + \beta_0 \frac{P_2}{P_0} < \overline{Y}$. Therefore, $\hat{p} > 0$.

- $\hat{p} = 1$

  This means
  $$\frac{E\left[R^{Y,S_{t_2}}(1, r, P_1, a)\right]}{E\left[a_3(Y)(1+r)\right]} = 1.$$
  Since $0 \leq R^{s,x}(1, r, P_1, a) \leq a_3(s)(1+r)$, this implies

  $$R^{Y,S_{t_2}}(1, r, P_1, a) = a_3(Y)(1+r).$$

  By definition of $R^{s,x}$, this implies for all $x \in \text{supp}(S_{t_2})$

  $$\alpha \frac{x}{S_0} + a_1(s)P_2 + a_2(s)(1+r) - a_3(s)(1+r) + \delta_1(s) - (1 - \frac{s}{c_1})c_2 \geq 0 \text{ for all } s : a_3(s) > 0.$$

  In particular,

  $$\alpha \frac{x}{S_0} + a_1(s)P_2 + a_2(s)(1+r) - a_3(s)(1+r) + \delta_1(s) - (1 - \frac{s}{c_1})c_2 \geq 0 \text{ for } s > \delta_0 + \beta_0 \frac{P_1}{P_0},$$

  since all assignments satisfy $a_3(s) > 0$ for $s > \delta_0 + \beta_0 \frac{P_1}{P_0}$ (see the definition of $A_1, \ldots, A_6$, $A_8$ and $A_9$ in theorem 3.2).

  Looking at possible equilibrium prices, also given by theorem 3.2, we have for $\beta_0 > 0$
  $$r \geq 0 \text{ and } P_1 = \frac{P_2}{1+r}.$$
  This implies for all $\beta_0 \geq 0$

  $$\beta_0 \frac{P_1}{P_0} = \beta_0 \frac{P_2}{P_0(1+r)} \leq \beta_0 \frac{P_2}{P_0}$$

  and
  $$a_1(s)P_2 = a_1(s)P_1 \frac{P_2}{P_1} = a_1(s)P_1(1+r),$$

since $a_1 \equiv 0$ for $\beta_0 = 0$ (see $A_1, \ldots, A_4$).

Applying $r \geq 0$ and $\delta_1(s) = \frac{P_1}{P_0}\beta_0 - a_1(s)P_1 + \delta_0 - s - a_2(s) + a_3(s)$, (see proof of 3.1), we get

$$a_1(s)P_2 + a_2(s)(1+r) - a_3(s)(1+r) + \delta_1(s) \leq (\delta_0 + \beta_0\frac{P_1}{P_0} - s)(1+r).$$

This yields

$$\alpha\frac{x}{S_0} + (\delta_0 + \beta_0\frac{P_1}{P_0} - s)(1+r) - (1 - \frac{s}{c_1})c_2 \geq 0 \text{ for } s > \delta_0 + \beta_0\frac{P_2}{P_0}$$

and thus

$$\alpha\frac{x}{S_0} + (\delta_0 + \beta_0\frac{P_2}{P_0} - s) - (1 - \frac{s}{c_1})c_2 \geq 0 \text{ for } s > \delta_0 + \beta_0\frac{P_2}{P_0}.$$

Thus, by definition of $h$, we have $h(1) = 1$ and thus $\overline{p} = 1$.

- $0 < \hat{p} < 1$

  For the assignments, we only have to consider $A_1$, $A_3$, $A_5$ and $A_8$. Thus $a_3$ is uniquely determined by

  $$a_3(s) = (\delta_0 + \beta_0\frac{P_1}{P_0} - s)^- \text{ for all } s \in S$$

  and $a_1(s) = a_2(s) = 0$ for $s > \delta_0 + \beta_0\frac{P_1}{P_0}$. Assume $\overline{p} = 0$. Thus $R^{Y,S_{t_2}}(p, \frac{1}{p} - 1, P_2, \overline{a}) \equiv 0$, where $\overline{a}$ satisfies $\overline{a}_3(s) = (\delta_0 + \beta_0\frac{P_2}{P_0} - s)^-$ and $(\overline{a}_1(s) + \overline{a}_2(s))\overline{a}_3(s) = 0$ for all $s \in S$. This is equivalent to

  $$(\alpha\frac{S_{t_2}}{S_0} - (1 - \frac{Y}{c_1})c_2)1_{Y > \delta_0 + \beta_0\frac{P_2}{P_0}} \leq 0, \quad \mathbb{P}\text{-a.s.}$$

  Since we assumed $\delta_0 + \beta_0\frac{P_2}{P_0} < \overline{Y}$, this implies

  $$\alpha\overline{S} - (1 - \frac{\overline{Y}}{c_1})c_2) \leq 0.$$

  Therefore,

  $$\alpha\frac{S_{t_2}}{S_0} - (1 - \frac{Y}{c_1})c_2 \leq 0, \quad \mathbb{P}\text{-a.s.}$$

  ans thus

  $$R^{Y,S_{t_2}}(\hat{p}, r, P_1, a) \overset{\mathbb{P}\text{-a.s.}}{=} 0.$$

  But then

  $$\hat{p} = \frac{E\left[R^{Y,S_{t_2}}(\hat{p}, r, P_1, a)\right]}{E\left[a_3(Y)(1+r)\right]} = 0,$$

  a contradiction.

□

# Bibliography


[1] Kenneth J. Arrow and Gerard Debreu. Existence of an equilibrium for a competitive economy. *Econometrica*, 22, 1954.

[2] Sudipto Bhattacharya and Douglas Gale. Preference shocks, liquidity, and central bank policy. 1985.

[3] Florian Heider Bruno Biais and Marie Hoerova. Risk-sharing or risk-taking? counterparty risk, incentives and margins. *ECB Working Paper Series*, 1413, 2012.

[4] John Bryant and Neil Wallace. Open-market operations in a model of regulated, insured intermediaries. *The Journal of Political Economy*, pages 146–173, 1980.

[5] Giulio Codognato and Jean J. Gabszewicz. Cournot-walras equilibria in markets with a continuum of traders. *Finance and Stochastics*, 8:311–341, 2004.

[6] Hans G. Daellenbach and Stephen H Archer. The optimal bank liquidity: a multi-period stochastic model. *Journal of Financial and Quantitative Analysis*, 4(03):329–343, 1969.

[7] Sven De Vries and Rakesh V. Vohra. Combinatorial auctions: A survey. *INFORMS Journal on computing*, 15(3):284–309, 2003.

[8] Douglas W. Diamond and Philip H. Dybvig. Bank runs, deposit insurance, and liquidity. *The Journal of Political Economy*, 91:401–419, 1983.

[9] Douglas W Diamond and Raghuram G Rajan. Liquidity shortages and banking crises. *The Journal of Finance*, 60(2):615–647, 2005.

[10] Jens Eisenschmidt and Cornelia Holthausen. The minimum liquidity deficit and the maturity structure of central banks' open market operations. *ECB Working Paper Series*, 1282, 2010.

[11] Marie Hoerova Florian Heider and Cornelia Holthausen. Liquidity hoarding and interbank market spreads. the role of counterparty risk. *ECB Working Paper Series*, 1126, 2009.

[12] Xavier Freixas and Cornelia Holthausen. Interbank market integration under assymetric information. *ECB Working Paper Series*, 2001.

[13] Xavier Freixas, Bruno M. Parigi, and Jean-Charles Rochet. Systemic risk, interbank relations, and liquidity provision by the central bank. *Journal of money, credit and banking*, pages 611–638, 2000.

[14] Nicolae Gârleanu. Portfolio choice and pricing in illiquid markets. *Journal of Economic Theory*, 144(2):532–564, 2009.

[15] Florian Heider and Marie Hoerova. Interbank lending, credit risk premia and collateral. *ECB Working Paper Series*, 1107, 2009.

[16] Florian Heider and Marie Hoerova. Clearing, counterparty risk and aggregate risk. *ECB Working Paper Series*, 1481, 2012.

[17] Florian Heider and Roman Inderst. Loan prospecting. *ECB Working Paper Series*, 1439, 2012.

[18] Robert Jarrow and Philip Protter. Liquidity risk and risk measure computation. *Review of Futures Markets*, 11(1):27–39, 2005.

[19] Ludovic A. Julien. Stackelberg-walras and cournot-walras equilibria in mixed markets: A comparison. *Theoretical Economics Letters*, 2:69–74, 2012.

[20] William Karush. Minima of functions of several variables with inequalities as side constraints. Master's thesis, Dept. of Mathematics, Univ. of Chicago, 1939.

[21] Donald W Katzner. *An introduction to the economic theory of market behavior: microeconomics from a Walrasian perspective.* 2006.

[22] H.W. Kuhn and A.W. Tucker. Nonlinear programming. *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, pages 481–492, 1950.

[23] Robert E Lucas. Liquidity and interest rates. *Journal of economic theory*, 50(2):237–264, 1990.

[24] Michael Manove, A. Jorge Padilla, and Marco Pagano. Collateral versus project screening: A model of lazy banks. *RAND Journal of Economics*, pages 726–744, 2001.

[25] Robert A. Jarrow Umut Çetin and Philip Protter. Liquidity risk and arbitrage pricing theory. *Finance and Stochastics*, 8:311–341, 2004.

[26] Dimitri Vayanos and Tan Wang. Search and endogenous concentration of liquidity in asset markets. *Journal of Economic Theory*, 136(1):66–104, 2007.

[27] Jan Werner. Arbitrage and the existence of competitive equilibrium. *Econometrica*, 55(6):1403–1418, 1987.