The Bloch Transform on L^p-Spaces

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Contents

Chapter 1

Introduction

An important tool in the mathematical study of light propagation in certain periodic crystals is the Bloch Transform. These physical effects are often described by a partial differential operator defined on a suitable function space. The Bloch Transform allows to represent the spectrum of such an operator as the union of spectra of a system of "reduced" operators, each of them has compact resolvent. This representation of the spectrum is called *band-gap structure* and provides a starting point for the search of band gaps. Band gaps are subintervals I of $\mathbb{R}$ such that $\sigma(A) \cap I = \emptyset$. In view of applications band-gaps are related to wavelengths of monochromatic light which can not propagate inside the crystal described by the operator under consideration. We will explain this in more detail in the next subsection.

The main focus of the present thesis is an expansion of the mathematical theory of the Bloch Transform. "Classically" it is used in a Hilbert space setting and applied to self adjoint partial differential operators A with periodic coefficients. Here the Fourier Transform and Plancherel's theorem are used to give a direct integral decomposition of A into a family of differential operators defined on a function space over the compact set $\mathbb{I}^d = [0, 1]^d$. Each of this so called *fiber operators* has a compact resolvent and therefore a discrete spectrum. Our approach interprets this decomposition in terms of Fourier multiplier operators instead of using Plancherel's theorem. This allows us to extend the reach of the Bloch Transform to non-self-adjoint periodic operators on more general spaces, i.e. vector-valued L^p-spaces. The class of operators for which similar results as in the "classical" setting are obtained covers a large family of partial differential operators with periodic coefficients.

The reinterpretation of periodic operators on the spaces $L^p(\mathbb{R}^d, E)$ as Bloch multipliers is the goal of Chapter 3, which gives a detailed framework for periodic operators and Bloch multipliers. In a first step we show how these operators are related to more general translation invariant operators on the sequence spaces $l^p(\mathbb{Z}^d, F)$. Their interpretation as Fourier multiplication operators allows for a description of periodic operators as Bloch multipliers.

In Chapter 4 we first prove a Fourier multiplier theorem for translation invariant operators on $l^p(\mathbb{Z}^d, F)$. The relation between translation invariant operators on $l^p(\mathbb{Z}^d, F)$ and periodic operators on $L^p(\mathbb{R}^d, E)$ of the previous chapter then allows for a reinterpretation as a rather general boundedness theorem for periodic operators in terms of "their" Bloch Transform.

Chapter 5 applies the theory to prove the band-gap structure for a large family of periodic and sectorial operators on $L^p(\mathbb{R}^d, E)$ whose decomposition into fiber operators on the fiber space $L^p(\mathbb{I}^d, E)$ depends analytically on the fiber parameter. In the classical case, which we introduced above, the analytic dependence is obtained by an eigenvalue expansion of the resolvent operator. Since such an expansion is not longer available in the general case we have to make this assumption. Finally we are also able to show how the functional calculus for these operators is decomposed in the same manner.

After this rather abstract theoretical part we include explicit examples of periodic, cylindrical, boundary value problems in Chapter 6.

Motivation and Background

As mentioned before, the main focus of the present thesis is the study of the Bloch Transform. Before we go into mathematical detail let us give a brief motivation, which originates in the technology of integrated chips, such as CPU's and GPU's.

Technical Motivation

In 1956 Gordon E. Moore predicted that transistor counts on integrated circuits will double approximately every two years. His prediction is known as Moore's Law and has proven to be highly accurate. This resulted in dramatic reduction of feature size of electronic devices and denser circuits. As a consequence new challenges appeared, since higher energy consumption on smaller scales cause electric interferences, a highly unpleasant effect. In recent years photonics became more and more popular as a possible replacement for the electronic technology. Besides the possible reduction of power consumption, photonic devices also promise a higher bandwidth and are not affected by electromagnetic interference. On the other hand, the realization of such devices requires a suitable implementation of optical switches and waveguides on a small scale. Fortunately is was shown that optical waveguides, guiding the light around sharp corners, are realizable [MCK+96]. The appropriate tools for such manipulations are photonic crystals.

Photonic Crystals

A photonic crystal is a certain optical nanostructure that rigs the propagation of light in a predefined way. One desirable manipulation is to prevent the propagation of light with a specific wavelength in one region, whereas the propagation is not affected in an other region. Having such material at hand one is able to build a waveguide. The effect that light of a specific wavelength is not able to propagate is achieved by a periodic dielectric modulation on the order of wavelength of light which is somewhere in between 400 and 700 nanometers. In recent years the investigation of such structures became increasingly popular both in mathematics and physics. First physical observations of theoretic nature where made in the late nineties of the previous century [Yab87, Joh87]. While the physical fabrication of these materials is still a difficult task some progress has been made [vELA+01, THB+02]. For an overview of the current state we recommend [Arg13].

The Mathematical Modeling of Photonic Crystals

As we said before, the periodic structure of a photonic crystal in on a scale of $400-700$ nanometers. Since this scale is large enough to neglect effects taking place on a atomic level we may assume a classical setting in the mathematical model of such structures[1].

The classical, macroscopic Maxwell Equations describe how electric and magnetic fields are generated and altered by each other. It is therefore not surprising that these equations are used as a starting point for a mathematical modeling of 'photonic crystals'. We will shortly outline how one can derive an eigenvalue problem from the Maxwell Equations by some suitable simplifying assumptions. The general 'macroscopic' Maxwell Equations in a spacial region Ω are given by

$$\begin{aligned} \partial_t D - \nabla \times H &= -j && \text{(Ampère's circuital law)} \\ \nabla \cdot D &= \rho && \text{(Gauss's law)} \\ \partial_t B + \nabla \times E &= 0 && \text{(Faraday's law of induction)} \\ \nabla \cdot B &= 0 && \text{(Gauss's law for magnetism)} \end{aligned} \tag{1.1}$$

Here E, B, D, H refer -in order- to the electric field, magnetic induction, electric displacement field, magnetic field density - functions that depend on time and space, giving vector fields in $\mathbb{R}^3$. The functions $j : \Omega \to \mathbb{R}^3$ and $\rho : \Omega \to \mathbb{R}$ are called the electric current density and electric charge density and equal to zero in absence of electric charges.

The material properties enter via constitutive laws which relate the electric field to the electric displacement field and the magnetic induction with the magnetic field density. In vacuum these relations are given by a linear coupling

$$\begin{aligned} D(t,x) &= \epsilon_0 E(t,x), \\ B(t,x) &= \mu_0 H(t,x) \end{aligned}$$

with the permittivity of free space ϵ_0 and the permeability of free space μ_0, both of them are real constants with values depending on the choice of units.

[1] By a classical setting we mean that the macroscopic Maxwell equations give a sufficient description of the electromagnetic phenomena, that take place on such a scale. For smaller scales the macroscopic description is inaccurate and one has to consider microscopic Maxwell Equations.

The influence of matter leads in specific models to the relations

$$D = \epsilon E = \epsilon_r \epsilon_0 E, \\ B = \mu H = \mu_r \mu_0 H \tag{1.2}$$

where the functions ϵ_r and μ_r are space but not time dependent, bounded and stay away from zero, with values in $\mathbb{R}$. They describe the properties of the material. For a discussion of these linear relations, which are a suitable approximation in various cases, we refer to any standard physics book about electrodynamics such as [Gre98, Jac75]. In general the functions ϵ_r and μ_r are also frequency dependent, a circumstance we will neglect[2].

To derive the eigenvalue problem, mentioned previously, we have to make some simplifications. The first one is, that we assume monochromatic waves. Hence all fields that arise in (1.1) are of the form $A(x,t) = e^{i\omega t}\mathbf{A}(x)$. Plugging this Ansatz into (1.1) as well as the linear constitutive laws (1.2) leads to the so called 'time-harmonic Maxwell Equations'

$$\begin{aligned} i\omega\epsilon\mathbf{E} - \nabla \times \frac{1}{\mu}\mathbf{B} &= 0, \\ \nabla \cdot (\epsilon\mathbf{E}) &= 0, \\ i\omega\mathbf{B} + \nabla \times \mathbf{E} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{1.3}$$

Note that we have already included the assumption that there are no electrical charges and currents, i.e. $\rho = j = 0$. Since both functions ϵ and μ are bounded away from zero we can eliminate the electric field $\mathbf{E}$ in (1.3) which leads to the following equations for the magnetic induction field $\mathbf{B}$

$$\begin{aligned} \nabla \times \left(\frac{1}{\epsilon}\nabla \times \frac{1}{\mu}\mathbf{B}\right) &= \omega^2\mathbf{B}, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned} \tag{1.4}$$

In the same way an elimination of the magnetic induction field $\mathbf{B}$ in (1.3) yields for the electric field $\mathbf{E}$

$$\begin{aligned} \nabla \times \left(\frac{1}{\mu}\nabla \times \mathbf{E}\right) &= \omega^2\epsilon\mathbf{E}, \\ \nabla \cdot (\epsilon\mathbf{E}) &= 0. \end{aligned} \tag{1.5}$$

These two sets of equations are already eigenvalue problems but we may simplify them even more. The assumption of a non-magnetic material, i.e. the relative permeability μ_r equals to one, transfers (1.4) and (1.5) via the identity[3] $\sqrt{\epsilon_0\mu_0} = 1/c_0$ into

$$\begin{aligned} \nabla \times \left(\frac{1}{\epsilon_r}\nabla \times \mathbf{B}\right) &= \frac{\omega^2}{c_0^2}\mathbf{B}, \\ \nabla \cdot \mathbf{B} &= 0, \end{aligned} \tag{1.6}$$

[2]Recent results concerning this situation are covered in [Sch13].
[3]c_0 denotes the speed of light in vacuum

and

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \frac{\omega^2}{c_0^2} \epsilon_r \mathbf{E}, \\ \nabla \cdot (\epsilon \mathbf{E}) &= 0. \end{aligned} \tag{1.7}$$

Now, if ϵ_r is a two-dimensional function, i.e. $\epsilon(x) = \epsilon(x_1, x_2)$ we decompose the electric- and the magnetic induction field accordingly. In particular we write

$$\mathbf{E}(x) = \begin{pmatrix} \mathbf{E}_1(x) \\ \mathbf{E}_2(x) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{E}_3(x) \end{pmatrix} = \mathbf{E}^{\mathrm{TE}}(x) + \mathbf{E}^{\mathrm{TM}}(x)$$

and

$$\mathbf{B}(x) = \begin{pmatrix} \mathbf{B}_1(x) \\ \mathbf{B}_2(x) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{B}_3(x) \end{pmatrix} = \mathbf{B}^{\mathrm{TM}}(x) + \mathbf{B}^{\mathrm{TE}}(x).$$

Here TM and TE abbreviate 'transverse magnetic' and 'transverse electric' and refer to the orientation of the oscillations of the electromagnetic field. In this 'two-dimensional' setting one speaks of TE-polarization, if the fields are in the form above where the magnetic induction is parallel and the electric field is normal to the axis of homogeneity (which is x_3 here). If the orientation of the electromagnetic field is the other way around we speak of TM-polarization.

Let us now assume, that the electromagnetic field is TM-polarized[4]. Due to the homogeneity of the material in x_3-direction it is reasonable to assume, that also the electrical field $\mathbf{E}$ and magnetic induction field $\mathbf{B}$-field depend only on the directions x_1 and x_2. In this case we can rewrite (1.7) and obtain

$$\nabla \times \left(\nabla \times \mathbf{E}^{\mathrm{TM}}\right) = \begin{pmatrix} -\partial_{x_3}\partial_{x_1}\mathbf{E}_3 \\ \partial_{x_3}\partial_{x_2}\mathbf{E}_3 \\ -\Delta \mathbf{E}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\Delta \mathbf{E}_3 \end{pmatrix} = \frac{\omega^2}{c_0^2}\epsilon_r \begin{pmatrix} 0 \\ 0 \\ \mathbf{E}_3 \end{pmatrix}.$$

It is important to note that in this special situation the constraint $\nabla \cdot (\epsilon \mathbf{E}^{\mathrm{TM}}) = 0$ is automatically fulfilled. Indeed

$$\nabla \cdot (\epsilon \mathbf{E}^{\mathrm{TM}}) = \begin{pmatrix} 0 \\ 0 \\ \partial_{x_3}\epsilon \mathbf{E}_3 \end{pmatrix} = 0.$$

Thus we finally end up with a eigenvalue problem for the scalar valued function $\mathbf{E}_3$ which we write in the form

$$-\frac{1}{\epsilon_r}\Delta \mathbf{E}_3 = \frac{\omega^2}{c_0^2}\mathbf{E}_3 \ \text{ in } \mathbb{R}^2. \tag{1.8}$$

[4]An analogous consideration is possible if one assumes TE-polarization, leading to an eigenvalue problem for the magnetic induction field.

We may interpret (1.8) in a physical manner as follows. If ω^2/c_0^2 is not in the spectrum of the operator $-\frac{1}{\epsilon_r}\Delta$, which has to be realized on a suitable space, we can not find a non-trivial function $\mathbf{E}_3$ in that space such that (1.8) is satisfied. Hence there can not be a monochromatic polarized electromagnetic wave, with frequency ω that is able to exist (propagate) inside the medium.

Finding such frequencies is desirable in applications. As mentioned before, one is interested in an optical realization of certain electrical devices. One of the fundamental tools for building electrical devices is the possibility to guide a current via a conductor on a spacial restricted area, typically some wire. Now, if one has a photonic crystal that is homogenous in one direction and a frequency ω such that ω^2/c_0 is not in the spectrum of the operator on the right hand side in (1.8), the realization of an 'optical wire' namely a waveguide works similar to the electrical case, by surrounding the path by a material where the light is not able to propagate. Note that we do not want to absorb the energy of the incoming light, but force it to stay inside the path by diffraction and refraction.

This should be enough motivation for the following task. For a given material, i.e. a given permittivity function ϵ_r, find frequencies ω such that ω^2/c_0^2 is not in the spectrum of $\frac{1}{\epsilon_r}\Delta$ realized on a suitable space.

Since for a photonic crystal the permittivity function ϵ_r is always periodic we have to solve a 'periodic' eigenvalue problem. A well developed tool for such a study is the so called Bloch Transform which we now introduce in short. We will give a more detailed introduction in Section 2.2.

The Bloch Transform

Introducing the Bloch Transform we follow the standard books [Kuc93, RS78]. Nevertheless we slightly adopt the presentation to our specific needs.

Consider a compactly supported function f defined on the real line with values in the complex numbers. Then the sum

$$[Zf](\theta, x) := \sum_{z\in\mathbb{Z}} e^{2\pi i\theta z} f(x-z), \tag{1.9}$$

is finite for all x, θ in $\mathbb{R}$. There are two immediate consequences of this definition. The function $Zf : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ is periodic with respect to the first variable (periodicity 1) and quasi-periodic with respect to the second variable (quasi-periodicity 1). This means

$$[Zf](\theta+1, x) = [Zf](\theta, x) \qquad \text{and} \qquad [Zf](\theta, x+1) = e^{2\pi i\theta}[Zf](\theta, x)$$

for all $(\theta, x) \in \mathbb{R}\times\mathbb{R}$. If we modify Z in the following way

$$[\Phi f](\theta, x) := e^{-2\pi i\theta x}[Zf](\theta, x) = e^{-2\pi i\theta x}\sum_{z\in\mathbb{Z}} e^{2\pi i\theta z} f(x-z), \tag{1.10}$$

then Φf is quasi-periodic in the first variable and periodic in the second one, each time with (quasi-)periodicity 1. Restricting the variables θ, x to an interval of length one, where we choose $[-1/2, 1/2]$ for θ and $\mathbb{I} := [0,1]$ for x, leads to one of the most important results concerning these transforms.

Theorem 1.1. *Both Z and Φ have a unitary extension to operators*

$$Z, \Phi : L^2(\mathbb{R}, \mathbb{C}) \to L^2([-1/2, 1/2] \times \mathbb{I}) \cong L^2([-1/2, 1/2], L^2(\mathbb{I}, \mathbb{C})).$$

We will give a more detailed study of the Zak- and Bloch Transform Φ in Section 2.2, where we also include a proof of the above theorem.

At this point we only mention that both the Zak- and the Bloch Transform have meaningful versions in the d-dimensional case if one replaces products by the inner product given on $\mathbb{R}^d$. For these operations a similar statement as the above theorem holds true.

A crucial step towards locating frequencies not lying in the spectrum of an operator, is the so called band-gap structure of the spectrum. We now give a short overview of the classical, well know theory. This should provide an impression why the later development is interesting. In upcoming chapters we extend the results of the next subsection to much wider generality.

Band-Gap Structure of the Spectrum - The Classical Approach

We have chosen (1.8) as our standard problem and will continue with the study of it. We just mention, that it is also possible to extend the results of this subsection to a more general class of partial differential operators.

Recall that the permittivity function ϵ_r is periodic with respect to some periodicity. Polarization lead to a 2-dimensional setting. Thus we restrict our attention to the variables x_1, x_2 and assume without loss of generality[5] that ϵ_r in (1.8) is periodic with respect to $\mathbb{Z}^2$, i.e. $\epsilon(x+z) = \epsilon(x)$ for all $(x, z) \in \mathbb{R}^2 \times \mathbb{Z}^2$. Let us write (1.8) in the form

$$-\frac{1}{\epsilon_r}\Delta u = \lambda u \quad \text{in } \mathbb{R}^2, \tag{1.11}$$

where the frequency ω is linked to λ via the relation $\lambda = \omega^2/c_0^2$. Define the L^2-realization of the eigenvalue problem (1.11) by

$$\begin{aligned} D(A) &:= H^2(\mathbb{R}^2), \\ Au &:= -\frac{1}{\epsilon_r}\Delta u. \end{aligned} \tag{1.12}$$

In the classical theory a fundamental observation is a commutator relation between Φ and a given differential operator with periodic coefficients. Let us briefly show this calculation, exemplary for the operator (1.12).

[5] We show in Section 3.1 how every other periodicity may be transformed into this special type, by a simple rescaling.

Let $f : \mathbb{R}^2 \to \mathbb{C}$ be a smooth function with compact support. Then

$$\begin{aligned}\Phi[\frac{1}{\epsilon_r}\Delta f](\theta,x) &= e^{-2\pi i x\theta}\sum_{z\in\mathbb{Z}^2} e^{2\pi i\theta z}\frac{1}{\epsilon_r(x-z)}[\Delta f](x-z)\\ &= \frac{1}{\epsilon_r(x)}e^{-2\pi i x\theta}\big[\sum_{j=1}^{2}\partial_j^2\sum_{z\in\mathbb{Z}^2} e^{2\pi i\theta z} f(\cdot - z)\big](x)\\ &= \frac{1}{\epsilon_r(x)}\big[\sum_{j=1}^{2}(\partial_j + 2\pi i\theta_j)^2 e^{-2\pi i x\theta}\sum_{z\in\mathbb{Z}^2} e^{2\pi i\theta z} f(\cdot - z)\big](x)\\ &= \frac{1}{\epsilon_r(x)}\big[\sum_{j=1}^{2}(\partial_j + 2\pi i\theta_j)^2[\Phi f](\theta,\cdot)\big](x).\end{aligned}$$

Here we have used the finiteness of the sum corresponding to z and a commutator relations[6] of the partial derivative with the exponential term.

At this point it is worth to repeat that for fixed θ the function $x \mapsto \Phi f(\theta,x)$ is periodic with period one and the previous calculation showed how the operator A defined in (1.12) turns into a family of operators which are formally given by 'shifted' versions of A, in terms of the Bloch Transform. For this observation we only needed periodicity of the coefficient function ϵ_r.

In fact since the operator A is self-adjoint one can use the theory of direct integral decompositions to deduce that A is given in terms of so called fiber operators on a fiber space. We do not go into detail here but refer to [RS78] for a rigorous discussion concerning the operator under consideration here and to [Dix81] for an abstract framework.

As a result of this theory one obtains

Theorem 1.2. *The self-adjoint operator A decomposes under the Bloch Transform into fiber operators $A(\theta)$ which are again self-adjoint and precisely given by*

$$\begin{aligned} D(A(\theta)) &:= H^2_{per}(\mathbb{I}^2),\\ A(\theta)u &:= -\frac{1}{\epsilon_r}[(\partial_1 + 2\pi i\theta_1)^2 + (\partial_2 + 2\pi i\theta_2)^2]u \ \text{ for } u \in D(A).\end{aligned}$$

Moreover it holds for $f \in D(A)$ and $\theta \in [-1/2, 1/2]^2$ that $\Phi f(\theta,\cdot) \in D(A(\theta))$ for almost all $\theta \in [-1/2, 1/2]^2$ and

$$Af = \Phi^{-1}\Big[\theta \mapsto A(\theta)[\Phi f](\theta,\cdot)\Big]. \tag{1.13}$$

Thus the eigenvalue problem (1.11) transfers into a family of eigenvalue problems on the space L^2 over the 'compact' set $\mathbb{I}^2$, where each problem is symmetric and given by

$$A(\theta)u = \lambda u, \quad u \in H^2_{\text{per}}(\mathbb{I}^2).$$

[6] For fixed θ we have $(\partial_j + 2\pi i\theta_j)e^{-2\pi i\theta x}f(\theta,x) = -2\pi i\theta_j e^{-2\pi i\theta x}f(\theta,x) + e^{-2\pi i\theta x}\partial_j f(\theta,x) + 2\pi i\theta_j e^{-2\pi i\theta x}f(\theta,x) = e^{-2\pi i\theta x}\partial_j f(\theta,x)$.
Applying this calculation a second time yields $(\partial_j + 2\pi i\theta_j)^2 e^{-2\pi i\theta x}f(\theta,x) = e^{-2\pi i\theta x}\partial_j^2 f(\theta,x)$.

The main result of the classical theory concerns the spectrum of A. It states, that $\sigma(A)$ is given by the union of the spectra of the fiber operators $A(\theta)$, i.e.

$$\sigma(A) = \bigcup_{\theta \in [-1/2, 1/2]^2} \sigma(A(\theta)), \tag{1.14}$$

which is often called band-gap structure of $\sigma(A)$. For a proof we refer to [RS78, XIII,16]. Let us briefly explain the term band-gap structure.

The Rellich-Kondrachov theorem implies that the domain of each $A(\theta)$ is compactly embedded in $L^2(\mathbb{I}^2)$ so that the spectrum of each operator $A(\theta)$ is discrete, i.e. $\sigma(A(\theta)) = (\lambda_n(\theta))_{n \in \mathbb{N}}$ with

$$\lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \leq \lambda_j(\theta) \leq \lambda_{j+1}(\theta) \leq \cdots \to \infty \quad \text{for } j \to \infty$$

and fixed $\theta \in [-1/2, 1/2]^2$.

The continuous dependence of the operator family $A(\theta)$ on the parameter θ implies continuous dependence of each 'band function' $\theta \mapsto \lambda_n(\theta)$ for fixed $n \in \mathbb{N}$ [Kat66, Ch.IV]. Self-adjointness of each $A(\theta)$ gives $\lambda_n(\theta) \in \mathbb{R}$ and compactness of $[-1/2, 1/2]^2$ implies that the image of each 'band function' is a compact interval in $\mathbb{R}$.

Let us plot some of the functions $\lambda_n(\cdot)$ schematically to give an visual impression

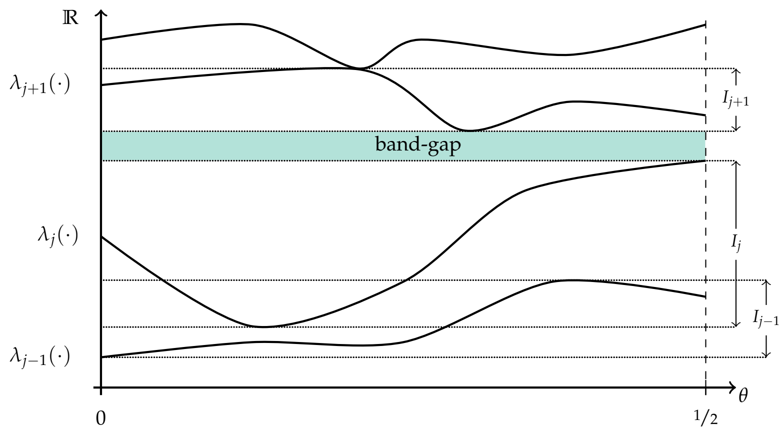

Figure 1.1: Schematic, one-dimensional visualization of band functions for the operator $-\frac{1}{\epsilon_r}\Delta$. The min-max-principle shows that the functions are even, hence we can restrict to the interval $[0, 1/2]$.

In Figure 1.1 we have already illustrated an open 'gap' in the spectrum of A, a situation which is -as mentioned before- highly pleasant for applications but not guaranteed. Starting with the band-gap structure (1.14) it is another challenging task to decide whether there are gaps or not. One possible approach to this problem is via a 'computer assisted proof' as in [HPW09]. Finally we mention, that there are also works, addressing the task of finding materials that provide gaps of specific width and a predefined location, see for example [Khr12].

A more detailed presentation of the ideas given above may be found in [DLP+11]. Similar results for a larger class of partial differential operators are contained in [Kuc93]. For further reading we recommend [Kuc93, Sca99] to mention only two examples from of the rich literature concerning this topic.

CHAPTER 2

Preliminaries

In this first part we want to fix our notations and introduce some basic, mostly well known, results which will be used frequently all through the thesis. Section 2.2 is devoted to a detailed introduction of the Bloch Transform previously mentioned in Chapter 1. In particular we give a useful decomposition of it which allows to reduce may considerations concerning the Bloch Transform to the study of Fourier Series. This observation is crucial for our treatment of Bloch multiplier in L^p-spaces. For further details, we refer to [Ama03, Gra08, Lan93, Kuc93, Con85] and the references mentioned in the specific subsections.

2.1 Basic Notations

For some integer[1] $d \geq 1$ and an arbitrary set Ω we denote by Ω^d the d- fold Cartesian product $\Omega^d := \Omega \times \cdots \times \Omega$, consisting of d-tuples $(\omega_1, \ldots, \omega_d)$ of elements in Ω (usually we will have $\Omega \in \{\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{I}, B\}$[2]).

If Ω is normed we denote by $|x| := \left(\sum_{j=1}^d |x_j|^2\right)^{1/2}$ the euclidean norm of $x \in \Omega^d$. A multi-index is a vector $\alpha \in \mathbb{N}_0^d$. Operations for two multi indices are preformed component wise. For $x \in \mathbb{R}^d$, $k \in \mathbb{N}$ and a multi-index $\alpha \in \mathbb{N}_0^d$ we have the following useful estimates.

$$|x^\alpha| \leq |x|^{|\alpha|} \qquad \text{and} \qquad |x|^k \leq \sum_{\substack{\beta \in \mathbb{N}_0^d \\ |\beta| = k}} |x^\beta|. \tag{2.1}$$

Let three sets Ω, X and Y be given, such that $\Omega \subset X$. For a given function $g : \Omega \to Y$ we define the extension (by zero) to X by $[\mathfrak{E}_X g](x) := g(x)$, for $x \in \Omega$ and $[\mathfrak{E}_X g](x) = 0$ for $x \in X \setminus \Omega$. Accordingly if $f : X \to Y$, the restriction to Ω is denoted by $[\mathfrak{R}_\Omega f](x) := f(x)$, $x \in \Omega$. Note that $\mathfrak{E}_X$ does not preserve any smoothness.

[1] d will always be an integer, greater or equal than one, which is assigned to the dimension

[2] $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z} := \mathbb{N}_0 \cup -\mathbb{N}$, $\mathbb{R}$ denotes the real numbers, $\mathbb{C}$ the complex numbers, B is the interval $[-1/2, 1/2]$ and $\mathbb{I} := [0, 1]$.

A Banach space is a normed vector space over the complex numbers $\mathbb{C}$, which is complete with respect to its norm. Following [Ama03] we consider a general setting for a multiplication. Let E_0, E_1, E_2 be Banach spaces. A mapping $\bullet : E_0 \times E_1 \to E_2$ is called multiplication, if $\bullet$ is a continuous, bi-linear map with norm less or equal to 1. Three spaces $(E_0, E_1, E_2, \bullet)$ together with a multiplication are called *multiplication triple*.

We mostly use this general multiplication in a very specific situation namely as 'operator-vector-multiplication'. In particular, if $\mathcal{B}(E_0, E_1)$ is the space of bounded linear operators from one Banach space E_0 to an other E_1, the evaluation map $\mathcal{B}(E_0, E_1) \times E_0 \to E_1$, $(T, x) \to Tx$ is a multiplication in the sense above.

Other canonical examples are scalar multiplication, composition, duality pairing and so forth.

Rapidly Decreasing Sequences

If E is a normed space we define

$$l^\infty(\mathbb{Z}^d, E) := \{\phi : \mathbb{Z}^d \to E \ : \ \|\phi\|_{l^\infty(\mathbb{Z}^d,E)} := \sup_{z\in\mathbb{Z}^d} \|\phi(z)\|_E < \infty\}$$

and for $p \in [1, \infty)$

$$l^p(\mathbb{Z}^d, E) := \{\phi : \mathbb{Z}^d \to E \ : \ \|\phi\|^p_{l^p(\mathbb{Z}^d,E)} := \sum_{z\in\mathbb{Z}^d} \|\phi(z)\|^p_E < \infty\}.$$

It is easy to see, that $\|\cdot\|_{l^p(\mathbb{Z}^d,E)}$ is a norm on $l^p(\mathbb{Z}^d, E)$. Moreover if E is a Banach space so is $\big(l^p(\mathbb{Z}^d, E), \|\cdot\|_{l^p(\mathbb{Z}^d,E)}\big)$ for every $p \in [1, \infty]$.

Definition 2.1. *Let E be a Banach space. For every $\alpha \in \mathbb{N}_0^d$ we define a mapping $p^E_\alpha : l^\infty(\mathbb{Z}^d, E) \to [0, \infty]$ by*

$$p^E_\alpha(\phi) := \sup_{z\in\mathbb{Z}^d} \|z^\alpha \phi(z)\|_E \ \text{for all}\ \phi \in l^\infty(\mathbb{Z}^d, E),$$

and set $\mathfrak{s}(\mathbb{Z}^d, E) := \{\phi \in l^\infty(\mathbb{Z}^d, E) \mid p^E_\alpha(\phi) < \infty$ for all $\alpha \in \mathbb{N}_0^d\}$.

Clearly $\mathfrak{s}(\mathbb{Z}^d, E)$ is a linear space, which is non-empty (consider $z \mapsto e^{-|z|^2}$) and $(p^E_\alpha)_{\alpha\in\mathbb{N}_0^d}$ is a family of semi-norms on $\mathfrak{s}(\mathbb{Z}^d, E)$. Denote by τ the topology on $\mathfrak{s}(\mathbb{Z}^d, E)$ that has the sets $\{f : p^E_\alpha(\phi - \psi) < \varepsilon\}$ as sub-base (here $\varepsilon > 0$ and $\psi \in \mathfrak{s}(\mathbb{Z}^d, E)$). Then the topological vector space $(\mathfrak{s}(\mathbb{Z}^d, E), \tau)$ is metrizable. Indeed a metric is given by

$$\mathbf{d}(\phi, \psi) := \sum_{\alpha\in\mathbb{N}_0} 2^{-|\alpha|} \frac{p^E_\alpha(\phi - \psi)}{1 + p^E_\alpha(\phi - \psi)}$$

and the topology τ coincides with the topology defined by $\mathbf{d}$. For details of the general theory of locally convex spaces with a countable system of semi-norms we refer to ([Con85, IV.Prop. 2.1]).

Lemma 2.2. *The metric space $s(\mathbb{Z}^d, E) := (\mathfrak{s}(\mathbb{Z}^d, E), \mathbf{d})$ is complete and convergence with respect to $\mathbf{d}$ is equivalent to convergence with respect to every semi-norm p^E_α.*

By the inequalities given in (2.1) it is easy to see that the system $(p_N)_{N\in\mathbb{N}_0}$ defined by

$$p_N(\phi) := \sup_{z\in\mathbb{Z}^d} (1+|z|)^N \|\phi(z)\|_E \quad \text{for } \phi \in s(\mathbb{Z}^d, E)$$

is equivalent to the system $(p^E_\alpha)_{\alpha\in\mathbb{N}_0^d}$. Hence we have the following convenient characterization of sequences in $s(\mathbb{Z}^d, E)$.

Lemma 2.3. *A sequence ϕ belongs to $s(\mathbb{Z}^d, E)$ if and only if, for all $N \in \mathbb{N}_0$ there is a constant $C_N > 0$ such that $\|\phi(z)\|_E \le C_N(1+|z|)^{-N}$ for all $z \in \mathbb{Z}^d$.*

Given a multiplication triple $(E_0, E_1, E_2, \bullet)$ we may define the discrete convolution for functions $\phi \in l^1(\mathbb{Z}^d, E_0)$ and $\psi \in l^1(\mathbb{Z}^d, E_1)$ by

$$\phi * \psi(j) := \sum_{z\in\mathbb{Z}^d} \phi(j-z) \bullet \psi(z) \quad \text{for every } j \in \mathbb{Z}^d.$$

The sum on the right hand side is absolute convergent. Moreover a reduction to the scalar case via triangle inequality and continuity of $\bullet$ shows, that $\phi * \psi$ is an element of $l^1(\mathbb{Z}^d, E_2)$[3].

We summarize the subsequent facts known in the scalar case for the group $\mathbb{R}^d$ (cf. [Gra08]), which transfers to the present situation under slight modifications of the proofs.

Lemma 2.4. *Consider a multiplication triple $(E_0, E_1, E_2, \bullet)$. Let $\phi \in s(\mathbb{Z}^d, E_0)$ and $\psi \in s(\mathbb{Z}^d, E_1)$.*

(i) Define $\psi \cdot \phi(j) := \psi(j) \bullet \phi(j)$ for $j \in \mathbb{Z}^d$. Then $\psi \cdot \phi \in s(\mathbb{Z}^d, E_2)$.

(ii) $\psi * \phi \in s(\mathbb{Z}^d, E_2)$,

(iii) Define $\tilde{\phi}(z) := \phi(-z)$ for $z \in \mathbb{Z}^d$, then $\tilde{\phi} \in s(\mathbb{Z}^d, E_0)$.

(iv) For $y \in \mathbb{Z}^d$ we define $\tau_y\phi(z) := \phi(z-y)$ for all $z \in \mathbb{Z}^d$. Then $\tau_y\phi \in s(\mathbb{Z}^d, E_0)$.

(v) If $T \in \mathcal{B}(E_0, E_1)$, $\phi \in s(\mathbb{Z}^d, E_0)$. Define $[\tilde{T}\phi](z) := [T\phi(z)]$ for all $z \in \mathbb{Z}^d$. Then $\tilde{T}\phi \in s(\mathbb{Z}^d, E_1)$ and $p^{E_1}_\alpha(\tilde{T}\phi) \le \|T\| p^{E_0}_\alpha(\phi)$ for all $\alpha \in \mathbb{N}_0^d$.

Remark 2.5. *An inspection of the proof of Lemma 2.4 shows, that all the operations are continuous with respect to the metric $\mathbf{d}$ on the spaces $s(\mathbb{Z}^d, E_i)$ for $(i = 0, 1, 2)$.*

[3] It is also easy to see, that Young's general inequality for convolutions transfers to this situation, i.e. $\|\phi * \psi\|_{l^r(\mathbb{Z}^d,E_2)} \le \|\phi\|_{l^p(\mathbb{Z}^d,E_0)} \|\psi\|_{l^q(\mathbb{Z}^d,E_1)}$ if $1 + 1/r = 1/p + 1/q$.

Smooth and Periodic Functions

Periodic functions will play an important role in our considerations. Hence we give a short introduction here. We focus on algebraic operations as well as the introduction of a topology that fits to our requirements. Before we go into detail, let us fix the term *periodic* in the multi-dimensional case.

For two vectors $a = (a_1, \dots, a_d)^T, b = (b_1, \dots, b_d)^T \in \mathbb{R}^d$ we denote by $a \times b$ the vector of *component wise multiplication*, i.e.

$$a \times b := (a_1 b_1, \dots, a_d b_d)^T \in \mathbb{R}^d.$$

A discrete subset $\mathcal{P} \subset \mathbb{R}^d$ is called *lattice*, if we can find positive, real numbers $p_1, \dots, p_d$ such that

$$\mathcal{P} = \{z \times (p_1, \dots, p_d),\ z \in \mathbb{Z}^d\}.$$

The vector $\mathfrak{p} := (p_1, \dots, p_d)^T$ is called *lattice vector*. Note that a lattice vector is uniquely determined by the condition that all entries of $\mathfrak{p}$ are positive. For convenience we write $1/\mathfrak{p} := (1/p_1, \dots, 1/p_d)^T$.

Definition 2.6. *Let Ω be a set and $\mathcal{P} \subset \mathbb{R}^d$ a lattice with lattice vector $\mathfrak{p} \in \mathbb{R}^d_{>0}$. A function $f : \mathbb{R}^d \to \Omega$ is called periodic with period $\mathfrak{p}$ if f satisfies the equation $f(x + p) = f(x)$ for all $x \in \mathbb{R}^d$, $p \in \mathcal{P}$.*

Now its easy to see, that we may switch between different lattices by multi-dimensional dilatation.

Lemma 2.7. *Let E be a metric space and $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^d$ be two lattices. If $f : \mathbb{R}^d \to E$ is periodic with respect to $\mathcal{P}_1$ then $g : \mathbb{R}^d \to E$ defined by $x \mapsto g(x) := f(\mathfrak{p}_1 \times 1/\mathfrak{p}_2 \times x)$ is periodic with respect to $\mathcal{P}_2$. Moreover if $f \in C^k(\mathbb{R}^d, E)$, then $g \in C^k(\mathbb{R}^d, E)$ and $\partial^\alpha g(y) = (\mathfrak{p}_1 \times 1/\mathfrak{p}_2)^\alpha [\partial^\alpha f](\mathfrak{p}_1 \times 1/\mathfrak{p}_2 \times y)$ for all $y \in \mathbb{R}^d$ and all multi-indices $|\alpha| \leq k$.*

Lemma 2.7 allows to transfer any lattice to $\mathbb{Z}^d$. Hence we call a function $f : \mathbb{R}^d \to E$ periodic, if it is periodic with respect to $\mathbb{Z}^d$. The lattice vector of $\mathbb{Z}^d$ is given by $(1, \dots, 1)$. For $k \in \mathbb{N}_0 \cup \infty$ we define[4]

$$C^k_{\text{per}}(\mathbb{R}^d, E) := \{f \in C^k(\mathbb{R}^d, E) : f \text{ is periodic}\}.$$

Since the behavior of a periodic function is uniquely determined on one cell of periodicity (lets say $B^d := [-1/2, 1/2]^d$) it is reasonable to set

$$C^k_p(B^d, E) := \{\mathfrak{R}_{B^d} f : f \in C^k_{\text{per}}(\mathbb{R}^d, E)\}.$$

Let us mention that this space is significantly smaller than $C^k(B^d, E)$. The reason is that we have beside differentiability also periodicity. Nevertheless $C^k_p(B^d, E)$ is a $\mathbb{C}$-vector space for every $k \in \mathbb{N}_0 \cup \{\infty\}$.

[4] For a definition of $C^k(\mathbb{R}^d, E)$, see Appendix A.

Remark 2.8. *For a function $f : [-1/2, 1/2)^d \to E$ define the periodic extension via*

$$[\mathfrak{E}_p f](x) := \sum_{z \in \mathbb{Z}^d} [\mathfrak{E}_{\mathbb{R}^d} f](x - z), \text{ for all } x \in \mathbb{R}^d.$$

We have two immediate consequences of this definition.

(i) $f \in C_p^k(B^d, E)$ *if and only if* $\mathfrak{E}_p f \in C_{per}^k(\mathbb{R}^d, E)$.

(ii) For $f \in C_p^k(B^d, E)$ *and* $\alpha \in \mathbb{N}_0^d$ *with* $|\alpha| \leq k$ *we have*

$$\partial^\alpha f(x) = \partial^\alpha \mathfrak{E}_p f(x) \text{ for all } x \in (-1/2, 1/2)^d.$$

Hence we define $\partial^\alpha f(x) := \partial^\alpha \mathfrak{E}_p f(x)$ *for all* $x \in B^d$.

We introduce a system of semi-norms $(\bar{p}_\alpha^E)_{\alpha \in \mathbb{N}_0^d}$ on $C_p^\infty(B^d, E)$ by

$$\bar{p}_\alpha^E(f) := \sup_{x \in B^d} \|\partial^\alpha f(x)\|_E \quad \text{for all } f \in C_p^\infty(B^d, E).$$

Periodicity of $\mathfrak{E}_p f$ combined with Remark 2.8 yields finiteness of $\bar{p}_\alpha^E(f)$ for all $\alpha \in \mathbb{N}_0^d$. Again denote by $\bar{\tau}$ the topology on $C_p^\infty(B^d, E)$ that has the sets $\{f : \bar{p}_\alpha^E(f - g) < \varepsilon\}$ as a sub-base (here $\alpha \in \mathbb{N}_0^d, g \in C_p^\infty(B^d, E)$ and $\varepsilon > 0$). Then $(C_p^\infty(B^d, E), \bar{\tau})$ is a topological vector space.

This space is locally convex and metrizable, e.g. a metric is given by

$$\bar{\mathbf{d}}(f, g) := \sum_{\alpha \in \mathbb{N}_0} 2^{-|\alpha|} \frac{\bar{p}_\alpha^E(f - g)}{1 + \bar{p}_\alpha^E(f - g)}.$$

Furthermore the topology defined by $\bar{\mathbf{d}}$ coincides with $\bar{\tau}$. For details we refer once more to [Con85, IV.Prop. 2.1], where a general approach is presented.

Lemma 2.9. *The metric space* $D(B^d, E) := (C_p^\infty(B^d, E), \bar{\mathbf{d}})$ *is complete and convergence with respect to* $\bar{\mathbf{d}}$ *is equivalent to convergence with respect to every semi-norm* $\bar{p}_\alpha^E$, $\alpha \in \mathbb{N}_0^d$.

As before we summarize some properties of $D(B^d, E)$ under algebraic operations.

Lemma 2.10. *Let* $(E_0, E_1, E_2, \bullet)$ *be a given multiplication triple,* $\phi \in D(B^d, E_0)$ *and* $\psi \in D(B^d, E_1)$.

(i) Define $[\psi \cdot \phi](\theta) := \psi(\theta) \bullet \phi(\theta)$ *for* $\theta \in B^d$. *Then* $\psi \cdot \phi \in D(B^d, E_2)$.

(ii) Define for $y \in B^d$ $\tau_y \phi := \mathfrak{R}_{B^d} \tau_y \mathfrak{E}_p \phi$. *Then* $\tau_y \phi \in D(B^d, E_0)$.

(iii) Define $[\psi * \phi](\theta) := \int_{B^d} [\tau_\theta \psi](x) \cdot \phi(x) dx$ *for* $\theta \in B^d$. *Then* $\psi * \phi$ *is an element of* $D(B^d, E_2)$ *and* $\partial^\alpha [\psi * \phi] = [\partial^\alpha \psi] * \phi = \psi * [\partial^\alpha \phi]$.

(iv) Define $\tilde{\phi}(\theta) := \phi(-\theta)$ *for* $\theta \in B^d$. *Then* $\tilde{\phi} \in D(B^d, E_0)$.

(v) *If* $T \in \mathcal{B}(E_0, E_1)$, $\phi \in D(B^d, E_0)$. *Define* $[\tilde{T}\phi](\theta) := T[\phi(\theta)]$ *for* $\theta \in B^d$. *Then* $\tilde{T}\phi \in D(B^d, E_1)$ *and* $\bar{p}_\alpha^{E_1}(\tilde{T}\phi) \leq \|T\| \bar{p}_\alpha^{E_0}(\phi)$ *for all* $\alpha \in \mathbb{N}_0^d$.

Remark 2.11. *Corresponding to the discrete case, the translation by* $y \in \mathbb{R}^d$ *of a function* f *defined on* $\mathbb{R}^d$ *is given by* $\tau_y f(x) := f(x - y)$. *As before the proof of the scalar case (cf. [Gra08]) transfers to the present situation under slight modifications and shows, that all operations in Lemma 2.10 are continuous with respect to* $\bar{\mathbf{d}}$ *on the spaces* $D(B^d, E_i)$ *for* $i = 0, 1, 2$.

Distributions

It is often desirable to extend operations defined on the spaces $D(B^d, E)$ and $s(\mathbb{Z}^d, E)$ to the whole of $L^p(\mathbb{I}^d, E)$ (or $l^p(\mathbb{Z}^d, E)$ respectively). Since this is not always possible on the level of functions we have to introduce 'generalized functions'. We do this for a general multiplication but consider first two arbitrary Banach spaces E_0 and E. Let us define

$$s'_E(\mathbb{Z}^d, E_0) := \{S : s(\mathbb{Z}^d, E_0) \to E;\ S \text{ is linear and continuous}\}$$
$$D'_E(B^d, E_0) := \{D : D(B^d, E_0) \to E;\ D \text{ is linear and continuous}\}.$$

Here continuity refers to continuity with respect to the metrics $\mathbf{d}, \bar{\mathbf{d}}$ and the norm topology in E. We also used the designation S for elements in $s'_E(\mathbb{Z}^d, E_0)$ and D for elements in $D'_E(B^d, E_0)$, which we will keep during the whole text.

On the spaces $s'_E(\mathbb{Z}^d, E_0)$ and $D'_E(B^d, E_0)$ we are always given the topology of bounded convergence. Then these spaces are Montel spaces (compare [Yos94, IV.7] and [Ama03, Ch.1.1]). Elements of this spaces are called *E-valued distributions*.

The next Lemma provides a characterization of distributions which turns out to be very useful in practice.

Lemma 2.12. *Let* E *be a Banach space and* $F_{E_0} \in \{s(\mathbb{Z}^d, E_0), D(B^d, E_0)\}$. *A linear mapping* $T : F_{E_0} \to E$ *is a distribution if and only if there is a constant* $C > 0$ *and a* $m \in \mathbb{N}_0$ *such that*

$$\|T(\varphi)\|_E \leq C \sum_{|\alpha| \leq m} \rho_\alpha^{E_0}(\varphi) \ \text{ for all } \varphi \in F_{E_0}. \tag{2.2}$$

Here $\rho_\alpha^{E_0}$ *denotes the semi-norms given on* F_{E_0}. *Moreover continuity is equivalent to sequentially continuity.*

Proof. First of all it is clear that (2.2) implies sequentially continuity. But the space F_{E_0} is a metric space so that sequentially continuity implies continuity (see [BC11]). For the converse statement recall that the sets $\{g \in F_{E_0} : \rho_\alpha^{E_0}(g) < \varepsilon\}$ where $\alpha \in \mathbb{N}_0^d$ and $\varepsilon > 0$ from a sub-base for the topology on F_{E_0}. Hence if T is continuous, we find $m \in \mathbb{N}$ and $\delta > 0$ such that

$$\text{if } \ \rho_\alpha^{E_0}(\varphi) < \delta \text{ for all } |\alpha| \leq m, \text{ then } \|T(\varphi)\|_E \leq 1.$$

Now for $\varphi \neq 0$ define $\phi := \frac{\delta}{2 \sum_{|\beta| \leq m} \rho_\beta^{E_0}(\varphi)} \varphi$. Then $\rho_\alpha^{E_0}(\phi) < \delta$ for all $|\alpha| \leq m$ which implies $\|T(\phi)\|_Y \leq 1$. Hence

$$\|T\varphi\|_E \leq \frac{2}{\delta} \sum_{|\alpha| \leq m} \rho_\alpha^{E_0}(\varphi)$$

and (2.2) holds. □

As usual we carry over operations known for functions to the level of distributions by applying them to the argument.

Lemma 2.13. *Assume we have a multiplication triple $(E_0, E_1, E_2, \bullet)$ and another Banach space E. Let $F'_{E_i,E}$ be one of the spaces $s'_E(\mathbb{Z}^d, E_i)$, $D'_E(\mathbb{I}^d, E_i)$.*
If $F'_{E_i,E} = s'_E(\mathbb{Z}^d, E_i)$ we set $F_{E_i} := s(\mathbb{Z}^d, E_i)$ and if $F'_{E_i,E} = D'_E(\mathbb{I}^d, E_i)$ we set $F_{E_i} := D(\mathbb{I}^d, E_i)$, $(i = 0, 1, 2)$. For $T \in \mathcal{B}(E_1, E_2)$, $G \in F'_{E_2,E}$, $\varphi \in F_{E_0}$, $\psi \in F_{E_1}$ and $\chi \in F_{E_2}$ define

(a) $[\varphi \cdot G](\psi) := G(\varphi \cdot \psi)$, *(b)* $[\varphi * G](\psi) := G(\tilde{\varphi} * \psi)$,

(c) $[TG](\psi) := G(\tilde{T}\psi)$, *(d)* $\tilde{G}(\chi) := G(\tilde{\chi})$,

(e) $[\tau_x G](\chi) := G(\tau_{-x}\chi)$, *here x is a element of $\mathbb{Z}^d$ or B^d according to the situation.*

Then $\varphi \cdot G, \varphi * G, TG \in F'_{E_1,E}$ and $\tilde{G}, \tau_x G \in F'_{E_2,E}$.

Proof. Follows directly by Lemma 2.12, Remark 2.5 and 2.11. □

Regular Distributions

As in the scalar case it is possible to identify certain functions as distributions. In fact the class of function for which such an identification is possible consists of more functions than the one presented here, but the smaller class is sufficient for our needs. The next Lemma follows directly from the scalar case and our results concerning vector valued functions.

Lemma 2.14. *If $\psi \in D(B^d, E)$ and $\varphi \in s(\mathbb{Z}^d, E)$. Then for every $p \in [1, \infty]$ and $\alpha \in \mathbb{N}_0^d$ we have $\|\partial^\alpha \psi\|_{L^p(B^d,E)} \leq \bar{p}_\alpha^E(\psi)$. Moreover we find a constant $C_{d,p} > 0$ and $M \in \mathbb{N}$ such that $\|(\cdot)^\alpha \varphi(\cdot)\|_{l^p(\mathbb{Z}^d,E)} \leq C_{d,p} \sum_{|\beta| \leq M} p_{\alpha+\beta}^E(\varphi)$.*

Proof. The first assertion follows by Hölders inequality, whereas for the second we have to use (2.1). □

We now assume again a given multiplication triple $(E_0, E_1, E_2, \bullet)$. For fixed $p \in [1, \infty]$, $g \in L^p(B^d, E_0)$ and $h \in l^p(\mathbb{Z}^d, E_0)$ define mappings D_g and S_h by

$$D_g : D(B^d, E_1) \to E_2 \qquad \psi \mapsto \int_{B^d} g(\theta) \bullet \psi(\theta) d\theta$$

$$S_h : s(\mathbb{Z}^d, E_1) \to E_2 \qquad \varphi \mapsto \sum_{z \in \mathbb{Z}^d} h(z) \bullet \varphi(z).$$

Lemma 2.15. *In the situation above we have $D_g \in D'_{E_2}(B^d, E_1)$ and $S_h \in s'_{E_2}(\mathbb{Z}^d, E_1)$.*

Proof. Apply Hölders inequality, Lemma 2.14 and Lemma 2.12. □

Distributions of the form D_g, S_h are called *regular*. One easily verifies that operations given for functions and distributions are consistent in the way, that taking the operation on the level of regular distributions is the same as taking the regular distribution after applying the operation.

For this reason we always identify a given function with its induced distribution, whenever we apply an operation that is not defined for the particular function.

Fourier Coefficients and Series

In the study of periodic problems a Fourier Series approach seems to be reasonable. As we will see in Section 2.2 the Bloch Transform can be expressed in terms of Fourier Series. Hence we start with a short review of Fourier- coefficients and series of both functions and distributions.

For two elements $x, y \in \mathbb{R}^d$ we use the standard notation for the inner product $x \cdot y := \sum_{i=1}^d x_i y_i$.

Definition 2.16. *Let $u \in D(B^d, E)$. We define the Fourier coefficients of u by*

$$[\mathcal{F}u](z) := \hat{u}(z) := \int_{B^d} e^{-2\pi i \theta \cdot z} u(\theta) d\theta \text{ for all } z \in \mathbb{Z}^d.$$

Since functions in $D(B^d, E)$ are integrable the definition is meaningful and we get from Hölder's inequality $\|\mathcal{F}u\|_{l^\infty(\mathbb{Z}^d, E)} \leq \|u\|_{L^1(B^d, E)}$ for all $u \in D(B^d, E)$. The latter inequality also shows, $\mathcal{F} \in \mathcal{B}(L^1(B^d, E), l^\infty(\mathbb{Z}^d, E))$.

For a sequence $g \in s(\mathbb{Z}^d, E)$ and $\theta \in B^d$ we define the inverse Transform $\mathcal{F}^{-1}$ by

$$[\mathcal{F}^{-1}g](\theta) := \check{g}(\theta) := \sum_{z \in \mathbb{Z}^d} e^{2\pi i z \cdot \theta} g(z). \tag{2.3}$$

Because sequences in $s(\mathbb{Z}^d, E)$ are absolutely summable, the series in (2.3) is uniformly convergent with respect to $\theta \in B^d$. Combined with the periodicity of the exponential function we get $\mathcal{F}^{-1}g \in C_p(B^d, E)$.
Furthermore the inequality $\|\mathcal{F}^{-1}g\|_{L^\infty(B^d, E)} \leq \|g\|_{l^1(\mathbb{Z}^d, E)}$ for all $g \in l^1(\mathbb{Z}^d, E)$ shows $\mathcal{F}^{-1} \in \mathcal{B}(l^1(\mathbb{Z}^d, E), L^\infty(B^d, E))$.

The next Lemma provides both classical and essential rules which are well known in the scalar case [Gra08, Prop. 3.1.2] and the proofs directly carry over to the vector-valued setting. Recall the notations in Lemma 2.4, 2.10 and the subsequent remarks.

Lemma 2.17. *Let $(E_0, E_1, E_2, \bullet)$ be a multiplication triple. Consider $u, v \in D(B^d, E_0)$, $w \in D(B^d, E_1)$, $f, g \in s(\mathbb{Z}^d, E_0)$, $h \in s(\mathbb{Z}^d, E_1)$, $\theta \in B^d$, $z \in \mathbb{Z}^d$ and $T \in \mathcal{B}(E_0, E_1)$ as well as $\alpha \in \mathbb{N}_0^d$. Then we have* *(a)* $\mathcal{F}\tilde{u} = \widetilde{\mathcal{F}u}$,

(b) $\mathcal{F}[\tau_\theta u](z) = e^{-2\pi i\theta\cdot z}\hat{u}(z)$,

(c) $\mathcal{F}(\tilde{T}u) = \tilde{T}\mathcal{F}(u)$,

(d) $\mathcal{F}(\theta \mapsto e^{2\pi iz\cdot\theta}u(\theta)) = \tau_z(\mathcal{F}u)$,

(e) $\mathcal{F}(D^\alpha u)(z) = (2\pi iz)^\alpha\hat{u}(z)$,

(f) $\mathcal{F}u(0) = \int_{B^d} u(\theta)d\theta$,

(g) $\mathcal{F}[u * w] = \hat{u}\cdot\hat{w}$,

(h) $\mathcal{F}(u) \in s(\mathbb{Z}^d, E_0)$,

(i) $\mathcal{F}^{-1}\tilde{g} = \widetilde{\mathcal{F}^{-1}g}$,

(j) $[\mathcal{F}^{-1}\tau_z g](\theta) = e^{2\pi iz\cdot\theta}[\mathcal{F}^{-1}g](\theta)$,

(k) $\mathcal{F}^{-1}(\tilde{T}g) = \tilde{T}\mathcal{F}^{-1}(g)$,

(l) $\mathcal{F}^{-1}(g * h) = \check{g}\cdot\check{h}$,

(m) $D^\alpha(\mathcal{F}^{-1}g) = (z \mapsto (2\pi iz)^\alpha g(z))^\vee$,

(n) $\mathcal{F}^{-1}(g) \in D(B^d, E_0)$,

(o) $\mathcal{F}^{-1}[\mathcal{F}u] = u$, $\mathcal{F}[\mathcal{F}^{-1}g] = g$,

(p) *if* $g_k \to g$ *in* $s(\mathbb{Z}^d, E_0)$ *then* $\check{g}_k \to \check{g}$ *in* $D(B^d, E_0)$,

(q) *if* $u_k \to u$ *in* $D(B^d, E_0)$ *then* $\hat{u}_k \to \hat{u}$ *in* $s(\mathbb{Z}^d, E_0)$.

The Hilbert Space Case - Plancherel's Theorem

For the moment let E be a Hilbert space. We use the notation $E = H$ to emphasis this special assumption and denote by $\langle\cdot,\cdot\rangle_H$ the given inner product. Note that $L^2(B^d, H)$ and $l^2(\mathbb{Z}^d, H)$ are Hilbert spaces as well, with the inner products

$$\langle f, g\rangle_{l^2(\mathbb{Z}^d,H)} := \sum_{z\in\mathbb{Z}^d}\langle f(z), g(z)\rangle_H,$$
$$\langle u, v\rangle_{L^2(B^d,H)} := \int_{B^d}\langle u(\theta), g(\theta)\rangle_H d\theta.$$

We want to extend the mapping $\mathcal{F} : D(B^d, H) \to s(\mathbb{Z}^d, H)$ to a bounded linear operator $L^2(\mathbb{I}^d, H) \to l^2(\mathbb{Z}^d, H)$. For this reason we state Plancherel's Theorem in the next Lemma.

Lemma 2.18. *For* $u \in D(B^d, H)$ *and* $g \in s(\mathbb{Z}^d, H)$ *we have*

(a) $\|\hat{u}\|_{l^2(\mathbb{Z}^d,H)} = \|u\|_{L^2(B^d,H)}$,

(b) $\|\check{g}\|_{L^2(B^d,H)} = \|g\|_{l^2(\mathbb{Z}^d,H)}$.

Proof. (a) Lemma 2.17 (o) gives

$$\|\hat{u}\|^2_{l^2(H)} = \langle\hat{u},\hat{u}\rangle_{l^2(H)} = \sum_{z\in\mathbb{Z}^d}\langle\hat{u}(z),\hat{u}(z)\rangle_H = \sum_{z\in\mathbb{Z}^d}\langle\hat{u}(z), \int_{B^d} e^{-2\pi i\theta\cdot z}u(\theta)d\theta\rangle_H$$
$$= \int_{B^d}\langle\sum_{z\in\mathbb{Z}^d} e^{2\pi i\theta\cdot z}\hat{u}(z), u(\theta)\rangle_H d\theta = \int_{B^d}\langle u(\theta), u(\theta)\rangle_H d\theta = \|u\|^2_{L^2(H)}.$$

Note that the inner product is continuous and because of $u \in D(B^d, H)$ and $\hat{u} \in s(\mathbb{Z}^d, H)$ we may interchange summation and integration by Proposition A.6.

(b) For $g \in s(\mathbb{Z}^d, H)$ there is a $u \in D(B^d, H)$ with $g = \hat{u}$. Hence (b) follows by (a) and Lemma 2.17 (o).

□

Denseness now allows us to extend $\mathcal{F}$ and $\mathcal{F}^{-1}$ to isometric, isomorphisms $\mathcal{F}_2 : L^2(B^d, H) \to l^2(\mathbb{Z}^d, H)$ and $\mathcal{F}_2^{-1} : l^2(\mathbb{Z}^d, H) \to L^2(B^d, H)$. Furthermore it is clear that we have $\mathcal{F}_2\mathcal{F}_2^{-1} = id_{L^2(B^d,H)}$ and $\mathcal{F}_2^{-1}\mathcal{F}_2 = id_{l^2(\mathbb{Z}^d,H)}$. In the following we will denote $\mathcal{F}_2$ and $\mathcal{F}_2^{-1}$ again by $\mathcal{F}$ and $\mathcal{F}^{-1}$ since no confusion will appear.

Remark 2.19. *The assertions (a)-(d) and (i)-(k) of Lemma 2.17 remain valid for $\mathcal{F}_2$ and $\mathcal{F}_2^{-1}$.*

Fourier Coefficients and Series of Distributions

Following the main idea we extend the definition of Fourier coefficients and Fourier series to distributions by applying the transform to the argument. In order to be consistent with the Transform defined for functions, we have to apply a reflection first. Recall that the reflection of a sequence $\varphi \in s(\mathbb{Z}^d, E)$ is defined by $\tilde{\varphi}(z) := \varphi(-z)$ for all $z \in \mathbb{Z}^d$ and accordingly, the reflection of a function $\psi \in D(B^d, E)$ was defined by $\tilde{\psi}(\theta) := \psi(-\theta)$. As always E_0 and E refer to Banach spaces.

Lemma 2.20. *Let $D \in D'_E(B^d, E_0)$ and $S \in s'_E(\mathbb{Z}^d, E_0)$ define*

(i) $[\mathcal{F}D](\phi) := D(\mathcal{F}^{-1}\tilde{\phi})$ *for* $\phi \in s(\mathbb{Z}^d, E_0)$,

(ii) $[\mathcal{F}^{-1}S](\psi) := S(\mathcal{F}\tilde{\psi})$ *for* $\psi \in D(B^d, E_0)$.

Then $\mathcal{F}D \in s'_E(\mathbb{Z}^d, E_0)$ and $\mathcal{F}^{-1}S \in D'_E(B^d, E_0)$. Moreover $\mathcal{F}\mathcal{F}^{-1} = id_{s'_E(\mathbb{Z}^d,E_0)}$ and $\mathcal{F}^{-1}\mathcal{F} = id_{D'_E(B^d,E_0)}$.

Proof. Continuity follows by Lemma 2.17 (p), (q) and Lemma 2.12. Linearity is clear and the last statement follows by Lemma 2.17 (o). □

The rules of Lemma 2.17 carry over to this situation. We only state a selection. The proof of them follows directly by definition and the corresponding statement for functions. Recall Lemma 2.13 and let $(E_0, E_1, E_2, \bullet)$ be a multiplication triple.

Lemma 2.21. *Consider $D \in D'_E(B^d, E_2)$, $S \in s'_E(\mathbb{Z}^d, E_2)$. Then for $\varphi \in s(\mathbb{Z}^d, E_0)$, $\psi \in D(B^d, E_0)$ and $T \in \mathcal{B}(E_1, E_2)$ we have*

(i) $\mathcal{F}\tilde{D} = \widetilde{\mathcal{F}D}$, $\mathcal{F}^{-1}\tilde{S} = \widetilde{\mathcal{F}^{-1}S}$,

(ii) $\mathcal{F}[\psi \cdot D] = \hat{\psi} * [\mathcal{F}D]$, $\mathcal{F}^{-1}[\varphi \cdot S] = \hat{\varphi} * [\mathcal{F}^{-1}s]$,

(iii) $\mathcal{F}[\psi * D] = \hat{\psi} \cdot [\mathcal{F}D]$, $\mathcal{F}^{-1}[\varphi * S] = \check{\varphi} \cdot [\mathcal{F}^{-1}S]$,

(iv) $\mathcal{F}[TD] = T[\mathcal{F}D]$, $\mathcal{F}^{-1}[TS] = T[\mathcal{F}^{-1}S]$,

(v) $\mathcal{F}[e^{-2\pi i z \cdot} D] = \tau_z \mathcal{F} D, \quad \mathcal{F}^{-1}[\tau_z S] = e^{-2\pi i z \cdot} \mathcal{F}^{-1} S,$

where the first equation in (i) *and* (v) *holds in* $s'_E(\mathbb{Z}^d, E_2)$ *and the second one in* $D'_E(B^d, E_2)$. *Similarly the first equation in* (ii), (iii), (iv) *hold in* $s'_E(\mathbb{Z}^d, E_2)$ *whereas the second one holds in* $D'_E(B^d, E_2)$.

2.2 The Bloch Transform and its Decomposition

Recall our discussion of the Bloch Transform in Chapter 1. We only gave a definition in the one-dimensional case and mentioned that this definition can be extended to the multi-dimensional situation. With the previous observations it is now possible to give a consistent definition for all $d \geq 1$. Moreover we will replace the scalar field $\mathbb{C}$ by an arbitrary Banach space E. Clearly we have to be careful with the previous statement concerning unitarity, which only holds if $E = H$ is a Hilbert space and $p = 2$.

In order to prepare for our later studies we introduce the Zak/Bloch Transform as a composition of operators which get defined now.

The Mapping Γ

For any subset A of $\mathbb{R}^d$ the indicator function of A is given by

$$\mathbb{1}_A(x) := \begin{cases} 1 & : \ x \in A \\ 0 & : \ \text{else.} \end{cases}$$

Recall the definitions of the restriction operator $\mathfrak{R}_{\mathbb{I}^d}$ and the (zero) extension operator $\mathfrak{E}_{\mathbb{R}^d}$.

Clearly $\mathfrak{R}_{\mathbb{I}^d}$ is an element of $\mathcal{B}(L^p(\mathbb{R}^d, E), L^p(\mathbb{I}^d, E))$ and $\mathfrak{E}_{\mathbb{R}^d}$ is contained in $\mathcal{B}(L^p(\mathbb{I}^d, E), L^p(\mathbb{R}^d, E))$ for every Banach space E and $p \in [1, \infty]$. Furthermore $\mathfrak{E}_{\mathbb{R}^d} \mathfrak{R}_{\mathbb{I}^d} g = \mathbb{1}_{\mathbb{I}^d} g$ for all $g \in L^p(\mathbb{R}^d, E)$. Next we want to define a mapping that reflects periodicity of a given function (and later on of bounded operators). Recall that we have the agreement to consider only periodicity with respect to $\mathbb{Z}^d$. Let $g : \mathbb{R}^d \to E$ be any function and $z \in \mathbb{Z}^d$. We set

$$[\Gamma g](z) := \mathfrak{R}_{\mathbb{I}^d} \tau_z g. \tag{2.4}$$

For fixed $z \in \mathbb{Z}^d$, $[\Gamma g](z)$ is a function defined on the cube $\mathbb{I}^d$ with values in E. Moreover if g is periodic $z \mapsto [\Gamma g](z)$ is constant, i.e. for any $z_1, z_2 \in \mathbb{Z}^d$ we have $\Gamma g(z_1) = \Gamma g(z_2)$.

Lemma 2.22. *For all* $p \in [1, \infty]$ *the mapping* $\Gamma : L^p(\mathbb{R}^d, E) \to l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$ *is an isometric isomorphism and its inverse is given by*

$$[\Gamma^{-1} \varphi] := \sum_{z \in \mathbb{Z}^d} \tau_{-z} [\mathfrak{E}_{\mathbb{R}^d} \varphi(z)] \quad \textit{for all} \ \ \varphi \in l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E)).$$

For later purposes we include a characterization of $s(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$ in terms of Γ. Observe[5] that $s(\mathbb{Z}^d, L^p(\mathbb{I}^d, E)) \subset l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$ is dense for $p \in [1, \infty)$. Define

$$L^p_s(\mathbb{R}^d, E) := \Gamma^{-1} s(\mathbb{Z}^d, L^p(\mathbb{I}^d, E)).$$

Since Γ^{-1} is bounded, linear and maps $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$ onto $L^p(\mathbb{R}^d, E)$, the set $L^p_s(\mathbb{R}^d, E)$ is a dense and linear subspace of $L^p(\mathbb{R}^d, E)$ for all $p \in [1, \infty)$.

Lemma 2.23. *We have for $p \in [1, \infty)$*

$$L^p_s(\mathbb{R}^d, E) = \{f \in L^p(\mathbb{R}^d, E) \ : \ \forall k \in \mathbb{N} \ , \ x \mapsto (1 + |x|)^k f(x) \in L^p(\mathbb{R}^d, E)\}.$$

Let us again emphasize, that $L^p_s(\mathbb{R}^d, E)$ is dense in $L^p(\mathbb{R}^d, E)$ and seems to be the natural space for the study of the Bloch Transform on $L^p(\mathbb{R}^d, E)$.

The Decomposition of Φ

First of all we remind of the definition of the Zak Transform in Chapter 1 which was given by

$$[Zf](\theta, x) = \sum_{z \in \mathbb{Z}} e^{2\pi i \theta z} f(x - z),$$

for $f : \mathbb{R} \to \mathbb{C}$ with compact support. We may rewrite Zf in the following way

$$Zf = \mathcal{F}^{-1} \circ \Gamma f. \tag{2.5}$$

This decomposition together with the previous discussions makes it possible, to extend the definition of Z to functions $f \in L^p_s(\mathbb{R}^d, E)$ for all $1 \leq p < \infty$ in a consistent way.

Definition 2.24. *Let $1 \leq p < \infty$ and E be a Banach space. The Zak Transform of any function $f \in L^p_s(\mathbb{R}^d, E)$ is defined by*

$$Zf := \mathcal{F}^{-1} \circ \Gamma f.$$

Thanks to Lemma 2.18 we see that in case $E = H$ is a Hilbert space we may extend Z to an isometric isomorphism

$$Z : L^2(\mathbb{R}^d, H) \to L^2(B^d, L^2(\mathbb{I}^d, H)).$$

For $p \neq 2$ and a general Banach space E we only get the following weaker statement.

Lemma 2.25. *Let E be a Banach space and $p \in [1, \infty)$. Then*

$$Z : L^p_s(\mathbb{R}^d, E) \to D(B^d, L^p(\mathbb{I}^d, E))$$

is one-to-one and onto.

[5] see Appendix A.

Note that we did not state anything about continuity in the Lemma above. The reason for this is, that the Fourier Transform does not extend in general to an bounded operator $L^p(\mathbb{I}^d, E) \to l^{p'}(\mathbb{Z}^d, E)$. Although this is true in the scalar case (for some p) it is not longer true for general Banach spaces E.

The Bloch Transform was a variant of the Zak Transform. Again we remind of the definition given in Chapter 1. For a function $f : \mathbb{R} \to \mathbb{C}$ with compact support we had

$$[\Phi f](\theta, x) = e^{-2\pi i \theta x} \sum_{z \in \mathbb{Z}} e^{2\pi i \theta z} f(x - z).$$

In order to obtain a decomposition of Φ that is consistent with the formula above we define an operator Ξ by

$$\begin{aligned} \Xi : L^p(B^d, L^p(\mathbb{I}^d, E)) &\to L^p(B^d, L^p(\mathbb{I}^d, E)) \\ f &\mapsto \left[\theta \mapsto [x \mapsto e^{-2\pi i \theta \cdot x} f(\theta, x)]\right]. \end{aligned}$$

Then Ξ is an isometric isomorphism for all $1 \le p \le \infty$ and any Banach space E. Clearly Φf is given by

$$\Phi f = \Xi \circ Z f = \Xi \circ \mathcal{F}^{-1} \circ \Gamma f. \tag{2.6}$$

Definition 2.26. *Let $1 \le p < \infty$ and E be a Banach space. The Bloch Transform of a function $f \in L^p_s(\mathbb{R}^d, E)$ is defined by*

$$\Phi f := \Xi \circ Z f = \Xi \circ \mathcal{F}^{-1} \circ \Gamma f.$$

Clearly the statement of Lemma 2.25 for the Zak Transform carries over to Φ, thanks to the fact that Ξ is an isometric isomorphism. Note that for fixed $\theta \in B^d$, $\Xi(\theta)$ is a multiplication operator on $L^p(\mathbb{I}^d, E)$, multiplying with the function $x \mapsto e^{-2\pi i \theta x}$. The advantage of Φ will become apparent in Chapter 5.

2.3 Some Results from Operator Theory

For a closed operator $(A, D(A)) : E \to E$ we denote by $\rho(A)$ its resolvent set which is defined in the usual way

$$\rho(A) := \Big\{ \lambda \in \mathbb{C} \mid (\lambda - A) : D(A) \to E \text{ is bijective and } R(\lambda, A) := (\lambda - A)^{-1} \in \mathcal{B}(E) \Big\}.$$

For $\lambda \in \rho(A)$ the bounded operator $R(\lambda, A) : E \to D(A)$ is called resolvent operator and by $\sigma(A) := \mathbb{C} \setminus \rho(A)$ we denote the spectrum of A. For two elements $\lambda, \mu \in \rho(A)$ we have the well known resolvent identity [RS80, Thm.VIII.2]

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda) R(\lambda, A) R(\mu, A). \tag{2.7}$$

(2.7) shows that the resolvent operators commute. It is worth to mention that $\rho(A)$ is open and the mapping $\rho(A) \ni \lambda \mapsto R(\lambda, A)$ is bounded analytic in the sense of Definition 5.5, facts which are also deduced by (2.7).

Two closed operators $(A, D(A))$, $(B, D(B))$ are equal, if their graphs are equal. We say $A \subset B$ if $\text{graph}(A) \subset \text{graph}(B)$, i.e. $D(A) \subset D(B)$ and $Ax = Bx$ for $x \in D(A)$.

Lemma 2.27. *Let E be a Banach space and $(A, D(A))$, $(B, D(B))$ be two closed operators $E \to E$. If $\rho(A) \cap \rho(B) \neq \emptyset$ and $A \subset B$, then $A = B$.*

In the study of unbounded operators on a Banach space E it is often more convenient to deal with their resolvent operators. Then, after a few calculations, one is often faced with a family of operators satisfying (2.7) on an open subset of $\mathbb{C}$. Our next objective is, to give results which determine conditions under which such a family is the resolvent of a closed and densely defined operator. Let start with the following definition.

Definition 2.28. *Let Ω be a subset of the complex plane and $(J(\omega))_{\omega\in\Omega}$ be a family of bounded, linear operators on a Banach space E such that for all $\omega_1, \omega_2 \in \Omega$ we have*

$$J(\omega_1) - J(\omega_2) = (\omega_2 - \omega_1)J(\omega_1)J(\omega_2). \tag{2.8}$$

In this case we call the family $(J(\omega))_{\omega\in\Omega}$ pseudo resolvent on E.

The first statement in a positive direction comes with rather natural assumptions concerning the range, $\text{rg}(J(\omega)) := \{y \in E \;:\; \exists x \in E \text{ with } y = J(\omega)x\}$ and nullspace, $\ker(J(\omega)) := \{x \in E \;:\; J(\omega)x = 0\}$, of the operators $J(\omega)$. Observe that both sets $\text{rg}(J(\omega))$ and $\ker(J(\omega))$ are independent of ω by (2.8).

Theorem 2.29 ([Paz83, §1.9, Cor. 9.3])**.** *Let E be a Banach space, Ω a subset of the complex plane and $(J(\omega))_{\omega\in\Omega} \subset \mathcal{B}(E)$ a pseudo resolvent on E. Then the following assertions are equivalent.*

(i) There is a unique, densely defined closed linear operator $(A, D(A))$ on E such that $\Omega \subset \rho(A)$ and $J(\omega) = R(\omega, A)$ for $\omega \in \Omega$.

(ii) $\ker(J(\omega)) = \{0\}$ and $\overline{rg(J(\omega))}^{E} = E$ for some (or equivalently all) $\omega \in \Omega$.

In concrete situations one often gets the kernel condition from a growth estimate of J.

Theorem 2.30 ([Paz83, §1.9, Thm.9.4])**.** *Let Ω be an unbounded subset of the complex plane and $(J(\omega))_{\omega\in\Omega}$ be a pseudo resolvent on a Banach space E. If $rg(J(\omega))$ is dense in E and there is a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \Omega$ such that $|\omega_n| \overset{n\to\infty}{\longrightarrow} \infty$ and*

$$\|\omega_n J(\omega_n)\| \leq M,$$

for some $M \in \mathbb{R}$, then (ii) of Theorem 2.29 is satisfied.

It is often possible to get the range condition in Theorem 2.30 by the strong convergence $\omega_n J(\omega_n) \xrightarrow{s} \mathrm{id}_E$.

Theorem 2.31 ([Paz83, §1.9, Cor. 9.5]). *Let Ω be an unbounded subset of the complex plane and $(J(\omega))_{\omega\in\Omega}$ be a pseudo resolvent on a Banach space E. If there is a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \Omega$ such that $|\omega_n| \xrightarrow{n\to\infty} \infty$ and*

$$\lim_{n\to\infty} \omega_n J(\omega_n)x = x \text{ for all } x \in E$$

then the assertions of Theorem 2.30 are satisfied.

Later on we will see, that the second result fits perfectly into the theory of C_0-semigroups thanks to the characterization theorem of Hille and Yoshida. In the context of a general pseudo resolvent Mazur's Theorem allows to weaken the latter condition.

Theorem 2.32 ([Bre11, Ch.3.3]). *Let E be a Banach space and $(e_n)_{n\in\mathbb{N}} \subset E$ be a sequence that converges weakly to some element e, i.e. for all $e' \in E'$ we have*

$$e'(e_n) \xrightarrow{n\to\infty} e'(e) \quad \text{in } \mathbb{C}.$$

Then there exists a sequence y_n made up of convex combinations of the x_n's that converges strongly to e, i.e.

$$y_n \xrightarrow{n\to\infty} e \quad \text{in } E.$$

For applications it is often convenient to have a version of Theorem 2.31 with slightly weaker assumptions on the family $J(\omega)$. In order to proceed we state the following lemma which is well known but hard to find in the literature.

Lemma 2.33. *Let $T_n, T \in \mathcal{B}(E)$ be such that $\sup\{\|T_n\|, \|T\|\} := M < \infty$. Further assume there is a dense subset D of E with $T_n x \to Tx$ weakly for all $x \in D$. Then $T_n e \to Te$ weakly for all $e \in E$.*

Proof. Let $e \in E$ and $(x_n)_{n\in\mathbb{N}} \subset D$ with $x_n \to e$. We have for any $e' \in E'$ and $n, j \in \mathbb{N}$

$$\begin{aligned} |e'[Te - T_n e]| &\le |e'[Te - Tx_j]| + |e'[Tx_j - T_n x_j]| + |e'[T_n x_j - T_n e]| \\ &\le 2M\|e'\|_{E'}\|e - x_j\|_E + |e'[Tx_j - T_n x_j]|. \end{aligned}$$

By assumption, the last term tends to zero as $n \to \infty$ for every fixed $j \in \mathbb{N}$. Hence if $\epsilon > 0$ is given we choose $j \in \mathbb{N}$ such that $\|e - x_j\| < (2M\|e'\|_{E'})^{-1}\epsilon$ and obtain

$$\overline{\lim_{n\to\infty}} |e'[Te - T_n e]| < \epsilon,$$

i.e. $T_n e \to Te$ weakly. □

Now here is the modified version of Theorem 2.31.

Corollary 2.34. *Let Ω be an unbounded subset of the complex plane and $(J(\omega))_{\omega\in\Omega}$ be a pseudo resolvent on a Banach space E. If there is a sequence $(\omega_n)_{n\in\mathbb{N}} \subset \Omega$ such that $|\omega_n| \overset{n\to\infty}{\longrightarrow} \infty$ and a constant $M < \infty$ with $\sup_{n\in\mathbb{N}} \|\omega_n J(\omega_n)\| \leq M$ as well as a dense subset $D \subset E$ such that*

$$\omega_n J(\omega_n)x \to x \text{ for all } x \in D \text{ weakly.}$$

Then there is a unique, densely defined closed and linear operator $(A, D(A))$ on E with $\Omega \subset \rho(A)$ and $J(\omega) = R(\omega, A)$ for all $\omega \in \Omega$.

Proof. By the previous Lemma we obtain, the weak convergence $\omega_n J(\omega_n)e \to e$ for all $e \in E$. Hence by Mazur's Theorem we get for every $e \in E$ a sequence x_j of the form

$$x_j = \sum_{k=1}^{N(j)} \alpha_k^j \omega_{n_k} J(\omega_{n_k})e \tag{2.9}$$

with $\sum_{k=1}^{N(j)} |\alpha_k^j| = 1$ such that $x_j \to e$ strongly. By (2.8) both $\ker(J(\omega))$ and $\operatorname{rg}(J(\omega))$ are independent of $\omega \in \Omega$ and both sets are linear subspaces of E. In particular $x_j \in \operatorname{rg}(J(\omega))$ for all $\omega \in \Omega, j \in \mathbb{N}$ and we obtain $\overline{\operatorname{rg}(J(\omega))} = E$.

If $e \in \ker(J(\omega))$ it follows $x_j = 0$ for all $j \in \mathbb{N}$ by (2.9). Hence $e = 0$, i.e. $\ker(J(\omega)) = 0$. Finally Theorem 2.29 applies and gives the statement. □

Bounded Multiplication Operators

First let us consider the scalar valued situation first. Let (Ω, μ) be a measure space and $m : \Omega \to \mathbb{C}$ be a function. To derive measurability of the function $\omega \mapsto m(\omega)f(\omega)$ we need to assume, that both f and m are measurable. If m is bounded and measurable, hence in $L^\infty(\Omega)$, the function $\omega \mapsto m(\omega)f(\omega)$ is in $L^p(\Omega)$ for all $p \in [1, \infty]$ as long as $f \in L^p(\Omega)$. Thus, in the scalar case measurable and bounded functions are the right framework for the study of multiplication operators on $L^p(\Omega)$.

This motivates the following definition in the case of vector-valued function spaces. Let E_0, E_1 be Banach spaces. We define

$$L^\infty(\Omega, \mathcal{B}_s(E_0, E_1)) := \Big\{ m : \Omega \to \mathcal{B}(E_0, E_1); \forall e \in E_0,\ \theta \mapsto m(\theta)e \in L^\infty(\Omega, E_1) \Big\}.$$

As a consequence one obtains the subsequent assertions.

Lemma 2.35 ([Tho03, Lem. 2.2.9 - Cor. 2.2.13]). *Let $m \in L^\infty(\Omega, B_s(E_0, E_1))$ and $f : \Omega \to E_0$ be measurable. Then*

(i) $\Omega \ni \omega \mapsto m(\omega)f(\omega)$ *is measurable,*

(ii) *there is a constant $C \geq 0$ and a set Ω_0 of measure zero such that for $e \in E_0$ and $\omega \in \Omega \setminus \Omega_0$ and we have $\|m(\omega)e\|_{E_1} \leq C\|e\|_{E_0}$,*

(iii) $\Omega \ni \omega \mapsto \|m(\omega)\|_{B(E_0,E_1)}$ *is measurable,*

(iv) *if* $f \in L^p(\Omega, E_0)$ *for some* $p \in [1, \infty]$*, then* $\Omega \ni \omega \mapsto m(\omega)f(\omega)$ *is in* $L^p(\Omega, E_1)$*.*

(v) *The set* $L^\infty(\Omega, \mathcal{B}_s(E_0, E_1))$ *is a* $\mathbb{C}$*-vector space. Endowed with the (essential) supremum norm* $\|m\|_\infty := \operatorname{ess\,sup}_{\theta \in \mathbb{I}^d} \|m\|_{\mathcal{B}(E_0,E_1)}$ *it turns into a Banach space. Moreover* $L^\infty(\Omega, \mathcal{B}_s(E_0))$ *is a Banach algebra.*

(vi) $\mathcal{M}_m : L^p(\Omega, E_0) \to L^p(\Omega, E_1)$*,* $f \mapsto \mathcal{M}_m f := [\omega \mapsto m(\omega)f(\omega)]$ *defines an element of* $\mathcal{B}(L^p(\Omega, E_0), L^p(\Omega, E_1))$*.*

(vii) *The map* $L^\infty(\Omega, \mathcal{B}_s(E_0, E_1)) \to \mathcal{B}(L^p(\Omega, E_0), L^p(\Omega, E_1))$*,* $m \mapsto \mathcal{M}_m$ *is an isometric homomorphism and in case of* $E_0 = E_1$ *and isometric algebra homomorphism.*

Definition 2.36. $M \in \mathcal{B}(L^p(\Omega, E_0), L^p(\Omega, E_1))$ *is called bounded (operator-valued) multiplication operator, if there is a* $m \in L^\infty(\Omega, \mathcal{B}_s(E_0, E_1))$ *such that* $M = \mathcal{M}_m$*.*

Unbounded Multiplication Operators

The treatment of unbounded multiplication operators is more sophisticated. To avoid unnecessary complications we start with the definition. As usual E_0, E_1 are Banach spaces.

Definition 2.37. *Let* $(A, D(A)) : L^p(\Omega, E_0) \to L^p(\Omega, E_1)$ *be an unbounded linear operator. A is called a unbounded multiplication operator if there is a family* $(A(\omega), D(A(\omega)))_{\omega \in \Omega}$ *of (unbounded) linear operators* $E_0 \to E_1$ *such that*

$$\begin{aligned} D(A) = \{ f \in L^p(\Omega, E_0) : f(\omega) \in D(A(\omega)) \text{ for almost all } \omega \in \Omega \\ \text{and } \omega \mapsto A(\omega)f(\omega) \in L^p(\Omega, E_1)\}, \\ (Af)(\omega) = A(\omega)f(\omega) \text{ for all } f \in D(A) \text{ and almost all } \omega \in \Omega. \end{aligned}$$

The operators $(A(\omega), D(A(\omega))$ *are called the fiber operators of* $(A, D(A))$*.*

A useful consequence of the definition above is the following

Lemma 2.38. *Let* $(A, D(A)) : L^p(\Omega, E_0) \to L^p(\Omega, E_1)$ *be an unbounded multiplication operator with fiber operators* $(A(\omega), D(A(\omega)))_{\omega \in \Omega}$*. If* $(A(\omega), D(A(\omega)))$ *is closed for almost all* $\omega \in \Omega$*, then* $(A, D(A))$ *is closed as well.*

Proof. By assumption there is a set $\Omega_1 \subset \Omega$ of measure zero such that the operators $(A(\omega), D(A(\omega)))$ are closed for $\omega \in \Omega \setminus \Omega_1$.

Let $(f_n)_{n \in \mathbb{N}} \subset D(A)$ be a sequence such that $f_n \to f \in L^p(\Omega, E_0)$ together with $Af_n \to g \in L^p(\Omega, E_1)$. Then we may find a sub-sequence (again denoted by $(f_n)_{n \in \mathbb{N}}$) and a set $\Omega_2 \subset \Omega$ of measure zero such that

$$f_n(\omega) \to f(\omega) \text{ for all } \omega \in \Omega \setminus \Omega_2.$$

Clearly $Af_n \to g \in L^p(\Omega, E_1)$ also for this sub-sequence, so that we find a sub-sequence of this sub-sequence (again denoted by $(f_n)_{n\in\mathbb{N}}$) and an other set $\Omega_3 \subset \Omega$ of measure zero with

$$A(\omega)f_n(\omega) = (Af_n)(\omega) \to g(\omega) \text{ for all } \omega \in \Omega \setminus \Omega_3.$$

Hence we have for $\omega \in \Omega \setminus \Omega_2 \cup \Omega_3$

$$\begin{aligned} f_n(\omega) &\to f(\omega), \\ (Af_n)(\omega) = A(\omega)f_n(\omega) &\to g(\omega) \end{aligned}$$

and the closedness of $(A(\omega), D(A(\omega))$ implies for $\omega \in \Omega \setminus \Omega_1 \cup \Omega_1 \cup \Omega_3$

$$\begin{aligned} f(\omega) &\in D(A(\omega)), \\ A(\omega)f(\omega) &= g(\omega). \end{aligned}$$

Since $\Omega_1 \cup \Omega_1 \cup \Omega_3$ is of measure zero we obtain $f \in D(A)$ and $Af = g$. □

Semigroups

In the context of evolution equations the notion of semigroups is well established and gives a useful tool for their treatment. Let us give a short overview and recall the fundamental aspects.

Definition 2.39. *Let E be a Banach space. A mapping $T(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(E)$ is called strongly continuous semigroup (C_0-semigroup in short) if the following conditions are fulfilled.*

(a) $T(0) = id_E$ and $T(t+s) = T(t) \circ T(s)$ for all $t, s \geq 0$.

(b) For each $e \in E$ the map $T(\cdot)e : \mathbb{R}_{\geq 0} \to E$, $t \mapsto T(t)e$ is continuous.

Moreover we set

$$\begin{aligned} D(A) &:= \{e \in E \mid \lim_{t\searrow 0} 1/t(T(t)e - e) \text{ exists as limit in } E\}, \\ Ae &:= \lim_{t\searrow 0} 1/t(T(t)e - e) \text{ for } e \in D(A). \end{aligned}$$

The operator $(A, D(A)) : E \to E$ is called the generator of the semigroup $T(\cdot)$.

We proceed with some well known facts concerning C_0-semigroups. For proofs and more details, we refer to [EN00, Paz83].

Lemma 2.40.

(a) Let E be a Banach space and $T(\cdot) : \mathbb{R}_{\geq 0} \to E$ be a C_0-semigroup. Then there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

(b) *If A is the generator of a C_0-semigroup, then A is closed, densely defined and the semigroup generated by A is unique.*

(c) *For every C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space E with generator $(A, D(A))$ it holds*

$$T(t)e = \lim_{n\to\infty} \left[{}^{n}/_{t}R({}^{n}/_{t}, A)\right]^{n} e = \lim_{n\to\infty} \left[id_E - {}^{t}/_{n}A\right]^{-n} e, \quad e \in E$$

uniformly (in t) on compact intervals.

The next result gives a complete picture of C_0-semigroups. The proof is based on a result of Hille and Yoshida for contraction semigoups, which got extended using a rescaling argument by Feller, Miyadera and Phillips. Nevertheless we call it, as usual the Hille-Yoshida Theorem. A proof can be found in [EN00, 3.8].

Theorem 2.41. *Let $(A, D(A))$ be a linear operator on a Banach space E and $\omega \in \mathbb{R}$, $M \geq 1$. Then the following are equivalent.*

(i) *$(A, D(A))$ generates a C_0-semigroup $(T_t)_{t\geq 0}$ with*

$$\|T(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

(i) *$(A, D(A))$ is closed, densely defined and for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\lambda-\omega)^n} \text{ for all } n \in \mathbb{N}.$$

(i) *$(A, D(A))$ is closed, densely defined and for every $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega$ one has $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leq \frac{M}{(Re(\lambda)-\omega)^n} \text{ for all } n \in \mathbb{N}.$$

Multiplication Semigroups

We briefly recall some known facts about multiplication semigroups. Again [EN00] gives a nice foundation for further reading in the case of scalar valued multiplication operators. For the vector-valued setting we refer to [Th003] where we also borrowed the presented results. For this subsection let (Ω, μ) always be a σ-finite measure space and E a separable Banach space.

We begin with the definition of a multiplication semigroup.

Definition 2.42. *A C_0-semigroup $(T(t))_{t\geq 0}$ on $L^p(\Omega, E)$ is called multiplication semigroup, if for every $t \geq 0$ the operator $T(t)$ is a bounded multiplication operator, i.e. for every $t \geq 0$ there is a function $T_{(\cdot)}(t) \in L^\infty(\Omega, \mathcal{B}_s(E))$ such that for all $f \in L^p(\Omega, E)$*

$$[T(t)f](\omega) = T_{(\omega)}(t)f(\omega) \text{ for almost all } \omega \in \Omega.$$

There are various connections between multiplication semigroups and multiplications operators. We summarize some of the most important results.

Theorem 2.43 ([Tho03, Thm.2.3.15]). *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $L^p(\Omega, E)$ with growth bound $\|T(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and some $M \geq 1$, $\omega \geq 0$. Then the following statements are equivalent.*

(i) $(T(t))_{t\geq 0}$ is a multiplication semigroup such that for almost all $\theta \in \Omega$ it holds $\|T_{(\theta)}(t)\| \leq Me^{\omega t}$.

(ii) For all $\lambda \in \mathbb{C}$ with $Re(\lambda) > \omega$ we have $\lambda \in \rho(A)$ and $R(\lambda, A)$ is a bounded multiplication operator.

(iii) The operator $(A, D(A))$ is a unbounded multiplication operator with fiber operators $(A(\theta), D(A(\theta)))_{\theta\in\Omega} : E \to E$ such that for almost all $\theta \in \Omega$ and all $\lambda \in \rho(A)$ we have:

- *$R(\lambda, A) = \mathcal{M}_{R(\lambda,A(\cdot))}$ whenever $Re(\lambda) > \omega$,*
- *$(A(\theta), D(A(\theta)))$ is the generator of a C_0-semigroup $(T_{(\theta)}(t))_{t\geq 0}$ on E with $T(t) = \mathcal{M}_{T_{(\cdot)}(t)}$ for all $t \geq 0$.*

The Bounded $\mathcal{H}^\infty$-Functional Calculus

In semigroup theory one may interpret the semigroup generated by an operator A as the 'operator-valued' function e^{tA}. The $\mathcal{H}^\infty$-calculus for sectorial operators gives the right framework for such an interpretation. For the construction we follow the usual procedure as suggested in [KW04, Sect. 9], [Haa06] and [DHP03]. All the details we omit here may be found in this references. Motivated by the characterization theorem of a C_0-semigroup we define

Definition 2.44. *A closed and densely defined operator $(A, D(A))$ on a Banach space E is called pseudo-sectorial, if $(-\infty, 0) \subset \rho(A)$ and*

$$\|t(t + A)^{-1}\|_{\mathcal{B}(E)} \leq C, \tag{2.10}$$

for all $t > 0$ and some constant $C > 0$.

Note, that the function $t \mapsto (t + A)^{-1} = R(t, -A)$ is indefinitely often differentiable with $(\frac{d}{dt})^n (t + A)^{-1} = (-1)^n n!(t + A)^{-(n+1)}$. Hence we may use Taylor's expansion for vector valued functions [Lan93, XIII,§6] to obtain for any pseudo-sectorial operator

$$|(\lambda + A)^{-1}| = |\sum_{n=0}^{\infty} (-1)^n (\lambda - t)^n (t + A)^{-(n+1)}| \leq \frac{C}{t} \sum_{n=0}^{\infty} \left(\frac{C|\lambda - t|}{t}\right)^n.$$

The right-hand side of the estimate above is finite for $|{}^{\lambda}/_{t} - 1| < {}^{1}/_{C}$. Writing λ as $te^{i\phi}$ leads to

$$|e^{i\phi} - 1| = 2 \cdot \sin({}^{\phi}/_{2}) < {}^{1}/_{C}$$

and thus

$$\|\lambda(\lambda + A)^{-1}\|_{\mathcal{B}(E)} \leq \tilde{C}_\phi$$

for all $\lambda \in \mathbb{C}$ with $|\phi| = |\arg(\lambda)| < 2 \cdot \sin^{-1}(1/2C)$. If we denote for $0 \leq \omega \leq \pi$ by

$$\Sigma_\omega := \begin{cases} \{z \in \mathbb{C} : |\arg(z)| < \omega\} & : \text{ if } \omega \in (0, \pi], \\ (0, \infty) & : \text{ if } \omega = 0 \end{cases}$$

the open, symmetric sector in $\mathbb{C}$ about the positive real line with opening angle 2ω (compare Figure 2.1) the above observations yields, that not only $(-\infty, 0)$ is part of the resolvent set of a pseudo-sectorial operator, but also all $\lambda \in \mathbb{C}$ with $|\arg(\lambda)| > \pi - 2 \cdot \sin^{-1}(1/2C)$ belong to $\rho(A)$. Hence we define the *spectral angle* of a pseudo-sectorial operator A by

$$\omega_A := \inf \Big\{ \omega : \sigma(A) \subset \overline{\Sigma}_\omega, \text{ for all } \nu > \omega \text{ there is a constant } C_\nu \text{ such that} \\ \|\lambda R(\lambda, A)\| \leq C_\nu, \text{ if } \nu \leq \arg(\lambda) \leq \pi \Big\}. \quad (2.11)$$

Now let us construct a first auxiliary functional calculus for pseudo-sectorial operators. For fixed $0 < \omega < \pi$ denote by $\mathcal{H}^\infty(\Sigma_\omega)$ the commutative algebra of bounded holomorphic functions defined on Σ_ω, that is

$$\mathcal{H}^\infty(\Sigma_\omega) := \Big\{ f : \Sigma_\omega \to \mathbb{C} : f \text{ is holomorphic with } |f|_{\infty,\omega} < \infty \Big\},$$

where $|f|_{\infty,\omega} := \sup_{\eta \in \Sigma_\omega} |f(\eta)|$. Put $\rho(\eta) := \frac{\eta}{(1+\eta)^2}$ for all $\eta \in \mathbb{C} \setminus \{-1\}$ and define

$$\mathcal{H}_0^\infty(\Sigma_\omega) := \Big\{ f \in \mathcal{H}^\infty(\Sigma_\omega) : \exists C, \varepsilon > 0 \text{ s.t. } |f(\eta)| \leq C|\rho(\eta)|^\varepsilon \text{for all } \eta \in \Sigma_\omega \Big\}.$$

Assume A is a pseudo-sectorial operator on a Banach space E with spectral angle $\omega_A \in [0, \pi)$. Choose some $\varphi > \omega_A$ and $\psi \in (\omega_A, \varphi)$. Let γ be a parametrization of the boundary $\partial\Sigma_\psi$ orientated counterclockwise. Then the growth estimate $\|R(\lambda, A)\| \sim 1/|\lambda|$ on γ ensures, that the Cauchy integral

$$f(A) := \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, A) d\lambda, \quad (2.12)$$

represents a well defined element of $\mathcal{B}(E)$ for all $f \in \mathcal{H}_0^\infty(\Sigma_\varphi)$. Moreover one can show that $f(A)$ is independent of the choice $\psi \in (\omega_A, \varphi)$.
It can also be shown that formula (2.12) defines an algebra homomorphism

$$\Psi_A : \mathcal{H}_0^\infty(\Sigma_\varphi) \to \mathcal{B}(E),$$

which is often called the Dunford calculus for pseudo-sectorial operators. Even this is not a satisfying calculus it provides the basis for an approximation argument.

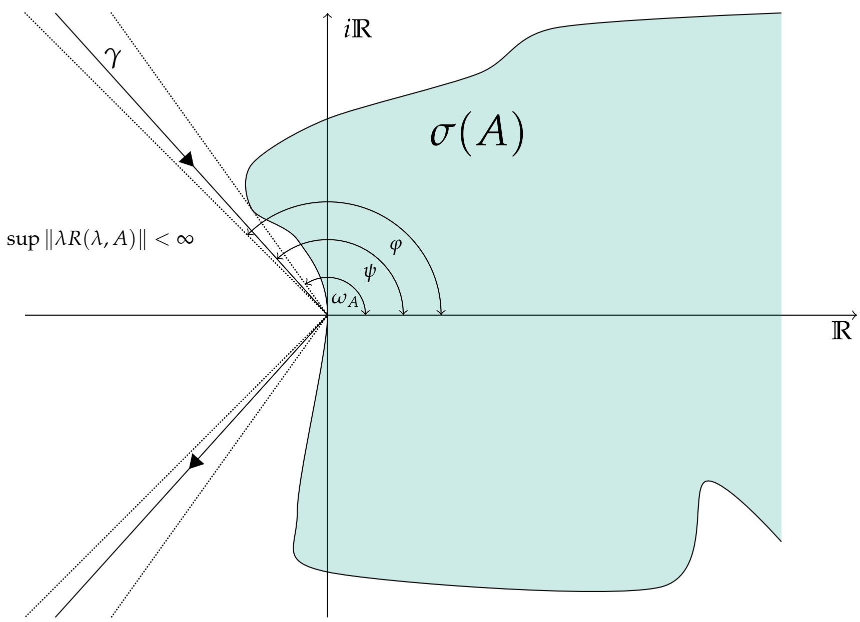

Figure 2.1: The spectrum of a sectorial operator and an integration path γ.

Lemma 2.45 ([KW04, Thm.9.2]). *Let A be a pseudo-sectorial operator on a Banach space E with angle $\omega_A \in [0, \pi)$ and $\omega_A < \psi < \varphi$. If the functions $f_n, f \in \mathcal{H}^\infty(\Sigma_\varphi)$ are uniformly bounded, and $f_n(z) \to f(z)$ for all $z \in \Sigma_\varphi$, then for all $g \in \mathcal{H}_0^\infty(\Sigma_\varphi)$ we have*

$$\lim_{n\to\infty} \Psi_A(f_n \cdot g) = \Psi_A(f \cdot g).$$

Moreover for $f \in \mathcal{H}_0^\infty(\Sigma_\varphi)$ we have the estimate

$$\|\Psi_A(f)\|_{\mathcal{B}(E)} \leq \frac{C_\varphi}{2\pi} \int_\gamma \frac{|f(\lambda)|}{|\lambda|} d\lambda,$$

where C_φ is the constant in (2.11).

In order to implement an approximation argument for more general f, we have to add more assumptions on A.

Definition 2.46. *A pseudo-sectorial operator on a Banach space E with spectral angle ω_A is called sectorial (with spectral angle ω_A) if $ker(A) = \{0\}$ and $\overline{rg(A)} = E$.*

If the space E is known to be reflexive, then one of the additional assumptions for a sectorial operator comes for free if the other is known. More precisely the following statement is shown in [KW04, Prop.15.2].

Lemma 2.47. *Let E be a reflexive Banach space and A be a pseudo-sectorial operator on a Banach space E. Then A has dense range if and only if A is injective.*

The additional assumptions for sectoriality are of technical nature and not really a loss of generality, since it can be shown that every pseudo-sectorial operator has a restriction with this additional properties, see [KW04, §15]. Nevertheless, they are needed for the following approximation procedure which extends Ψ_A to the class $\mathcal{H}^\infty_A(\Sigma_\varphi)$.

Definition 2.48. *Let A be a sectorial operator on a Banach space E and $\varphi > \omega_A$. Define*

$$\mathcal{H}^\infty_A(\Sigma_\varphi) := \Big\{ f \in \mathcal{H}^\infty(\Sigma_\varphi) : \exists (f_n)_{n\in\mathbb{N}} \subset \mathcal{H}^\infty_0(\Sigma_\varphi) \text{ with } f_n(z) \overset{n\to\infty}{\longrightarrow} f(z) \\ \text{for all } z \in \Sigma_\varphi \text{ and } \sup_{n\in\mathbb{N}} \|\!|f_n|\!\|_A < \infty \Big\},$$

where $\|\!|f_n|\!\|_A := \|f_n\|_{\mathcal{H}^\infty(\Sigma_\varphi)} + \|f_n(A)\|_{B(E)}$ denotes the 'graph norm' of Ψ_A.

Now the announced approximation works as follows. Let $\rho \in \mathcal{H}^\infty_0(\Sigma_\varphi)$ be the function $z \mapsto \frac{z}{(1+z)^2}$. Then $\rho(A) = A(1+A)^{-2}$ for any sectorial operators A and $\ker(\rho(A)) = \{0\}$ as well as $\overline{\mathrm{rg}(\rho(A))} = E$. Thus $\rho(A)$ is invertible on $\mathrm{rg}(\rho(A))$ and we obtain for $f \in \mathcal{H}^\infty_A(\Sigma_\varphi)$, $y \in \mathrm{rg}(\rho(A))$ and Lemma 2.45

$$\overline{\Psi}_A(f)y := \lim_{n\to\infty} f_n(A)\rho(A)[\rho(A)^{-1}y] = (f\cdot\rho)(A)[\rho(A)^{-1}y]$$

which may be extended to a bounded operator on E by the uniform boundedness of the f_n and denseness of $\mathrm{rg}(\rho(A))$. Lets summarize the properties of this extension, see [KW04, Thm.9.6] for a proof.

Theorem 2.49. *Let A be a sectorial operator on E and $\varphi > \omega_A$. Then the previously defined mapping $\overline{\Psi}_A : \mathcal{H}^\infty_A(\Sigma_\varphi) \to \mathcal{B}(E)$ is linear and multiplicative. Moreover if $(f_n)_{n\in\mathbb{N}} \subset \mathcal{H}^\infty_A(\Sigma_\varphi)$, $f \in \mathcal{H}^\infty(\Sigma_\varphi)$ are such that $f_n(z) \overset{n\to\infty}{\longrightarrow} f(z)$ for all $z \in \Sigma_\varphi$ and $\|\!|f|\!\|_A \le C$, then $f \in \mathcal{H}^\infty_A(\Sigma_\varphi)$ with*

$$\begin{aligned} \overline{\Psi}_A(f)e &= \lim_{n\to\infty} \overline{\Psi}_A(f_n)e \quad \text{for all } e \in E, \qquad (2.13)\\ \|\overline{\Psi}_A(f)\| &\le C. \end{aligned}$$

For $\mu \notin \overline{\Sigma}_\varphi$, $z \mapsto \tau_\mu(z) := (\mu - z)^{-1}$ belongs to $\mathcal{H}^\infty_A(\Sigma_\varphi)$ and $\overline{\Psi}_A(\tau_\mu) = R(\mu, A)$.

Of particular interest are those sectorial operators with $\mathcal{H}^\infty_A(\Sigma_\varphi) = \mathcal{H}^\infty(\Sigma_\varphi)$.

Definition 2.50. *A sectorial operator A has a bounded $\mathcal{H}^\infty$-calculus of angle $\varphi > \omega_A$, if $\mathcal{H}^\infty_A(\Sigma_\varphi) = \mathcal{H}^\infty(\Sigma_\varphi)$. In this case $\overline{\Psi}_A : \mathcal{H}^\infty(\Sigma_\varphi) \to \mathcal{B}(E)$ is a bounded algebra homomorphism with the convergence property (2.13).*

The closed graph theorem allows for a nice characterization of operators with a bounded $\mathcal{H}^\infty$-calculus.

Corollary 2.51 ([KW04, 9.11]). *A sectorial operator A has a bounded $\mathcal{H}^\infty(\Sigma_\varphi)$-calculus ($\varphi > \omega_A$) if and only if there is a constant $C > 0$ with*

$$\|\Psi_A(f)\|_{B(E)} \leq C\|f\|_{\mathcal{H}^\infty(\Sigma_\varphi)} \text{ for all } f \in \mathcal{H}_0^\infty(\Sigma_\varphi).$$

Moreover, we have in this case $|\!|\!| f |\!|\!|_A \approx \|f\|_{\mathcal{H}^\infty(\Sigma_\varphi)}$.

Remark 2.52.

(i) If A is a pseudo-sectorial operator, such that there is a $B \in \mathcal{B}(E)$ that commutes with all resolvent operators of A, i.e. $R(\lambda, A)B = BR(\lambda, A)$, then also $\Psi_A(f)$ commutes with B for all $f \in \mathcal{H}_0^\infty(\Sigma_\varphi)$ (here $\varphi > \omega_A$). This is easily deduced from the fact, that $\Phi_A(f)$ is a Bochner integral of the resolvent operators.

(ii) If A is a sectorial operator such that there is a B that commutes with the resolvent operators of A, then also $\overline{\Psi}_A(f)$ commutes with B for all $f \in \mathcal{H}_A^\infty(\Sigma_\varphi)$ where again $\varphi > \omega_A$. This follows directly form (i) and the construction of $\overline{\Psi}_A$.

(iii) It can be shown, that the $\mathcal{H}^\infty$-calculus is unique in the sense, that if Ψ_2 is an other mapping $\mathcal{H}_A^\infty(\Sigma_\varphi) \to \mathcal{B}(E)$ that satisfies the properties in Theorem 2.49, then $\Psi_2 = \overline{\Psi}_A$ on $\mathcal{H}_A^\infty(\Sigma_\varphi)$.

(iv) If a sectorial operator A has a bounded $\mathcal{H}^\infty$-calculus of angle $\varphi < \pi/2$ then for $\eta \in \mathbb{C}$ with $|arg(\eta)| < \frac{\pi}{2} - \omega_A$ the function $z \mapsto e_\eta(z) := e^{-\eta z}$ belongs to $\mathcal{H}^\infty(\Sigma_\varphi)$ for $\omega_A < \varphi < \frac{\pi}{2} - |arg(\eta)|$ and it can be shown, that $\eta \mapsto \overline{\Psi}_A(e_\eta)$ is a analytic semigroup.

For later purposes we extend the notations of (pseudo)-sectoriality to families of operators.

Definition 2.53. *Let Ω be any set and $(A(\theta), D(A(\theta)))_{\theta\in\Omega}$ be a family of operators on a Banach space E.*

(i) $(A(\theta), D(A(\theta)))_{\theta\in\Omega}$ is called uniformly pseudo-sectorial with spectral angle ω, if each operator $(A(\theta), D(A(\theta)))$ is pseudo-sectorial with spectral angle ω and the bounds in (2.11) are uniform in θ.

(ii) $(A(\theta), D(A(\theta)))_{\theta\in\Omega}$ is called uniformly sectorial of angle ω, if it is uniformly pseudo sectorial of angle ω and every operator $(A(\theta), D(A(\theta)))$ is sectorial.

(iii) If in addition (Ω, μ) is a measure space, the family $(A(\theta), D(A(\theta)))_{\theta\in\Omega}$ is called almost uniformly (pseudo)-sectorial if there is a subset $N \subset \Omega$ with $\mu(N) = 0$ such that $(A(\theta), D(A(\theta)))_{\theta\in\Omega\setminus N}$ is uniformly (pseudo)-sectorial.

It is also possible to extend the functional calculus to functions of polynomial grows at zero and infinity. But this will then lead to an unbounded operator. Since we can only handle bounded operators with the multiplier theorem that is developed in Chapter 4, we forgo the introduction to this 'extended' functional calculus. The interested reader may find a detailed description in [KW04, Haa06].

2.4 $\mathcal{R}$-bounded Sets of Operators

It was shown in [Wei01], that (beside others) the assumption of $\mathcal{R}$-boundedness makes it possible to extend the well known Mihlin Theorem in the case of scalar-valued functions to the vector valued setting. One of the main steps towards the spectral Theorem mentioned in Chapter 1 is to transfer this result to the setting of the Bloch Transform. For this reason we briefly discuss $\mathcal{R}$-bounded sets of operators. For a detailed treatment see [KW04] and [DHP03]. Beside the definition we will give workable criteria for $\mathcal{R}$-boundedness. In Chapter 4 we will show, how the assumption of $\mathcal{R}$-boundedness enters in a natural way if one begins to study vector-valued situations.

As a starting point for the definition of $\mathcal{R}$-boundedness, we follow the standard way and introduce a special family of functions -the Rademacher functions- first.

Rademacher functions

For $n \in \mathbb{N}$ define functions $r_n : [0,1] \to \{-1,1\}$ by $r_n(t) := \text{sign}(\sin(2^n \pi t))$. These functions are called Rademacher functions and form a orthogonal sequence in $L^2([0,1])$ which is not complete [WS01, Ch.7.5]. The orthogonality can visually be seen by their graphs, given the first four of them in Figure 2.2.

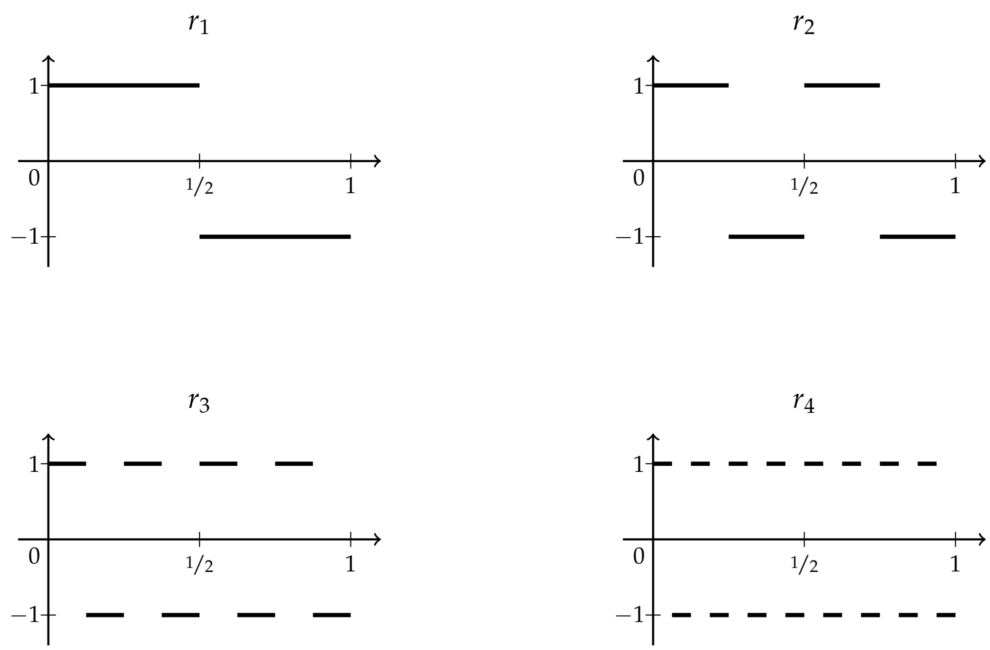

Figure 2.2: The Rademacher functions r_1, r_2, r_3 and r_4.

Denoting by λ the Lebesgue measure on $[0,1]$, it is clear that for all $n \in \mathbb{N}$ we have $\lambda(\{t \in [0,1] : r_n(t) = 1\}) = \lambda(\{t \in [0,1] : r_n(t) = -1\}) = \frac{1}{2}$. But even more is true. Consider any sequence $(\delta_n)_{n\in\mathbb{N}} \subset \{-1, 1\}$. Then for all $m \in \mathbb{N}$

$$\begin{aligned}\frac{1}{2^m} &= \lambda(\{t \in [0,1] : r_{n_1}(t) = \delta_1, r_{n_2}(t) = \delta_2, \dots, r_{n_m}(t) = \delta_m\}) \\ &= \prod_{j=1}^{m} \lambda(\{t \in [0,1] : r_{n_j}(t) = \delta_j\}).\end{aligned}$$

The first equality can be seen as follows. Without loss of generality we assume that the $n_j's$ are arranged in increasing order. Now chose the subset I_{n_1} of $[0,1]$ with $r_{n_1}(t) = \delta_j$ for $t \in I_{n_1}$. Note that I_{n_1} is a union of intervals with $\lambda(I_{n_1}) = 1/2$, which enjoys a subdivision into finer intervals by the function r_{n_2}. Denote by I_{n_2} the subset of I_{n_1} where $r_{n_2}(t) = \delta_2$. Then by construction $\lambda(I_{n_2}) = 1/4$ and $r_{n_1}(t) = \delta_1$, $r_{n_2}(t) = \delta_2$ if and only if $t \in I_{n_2}$. Repeating this m-times gives the first equality. Now the second equality is obvious.
The above observations enable us to interpret the r_n's as identically distributed, stochastically independent random variables on the probability space $([0,1], \lambda)$. For a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathbb{C}$, $m \in \mathbb{N}$ and $t \in [0,1]$ we find $(\delta_n)_{n=1}^m \subset \{-1,1\}$ with $\sum_{n=1}^m r_n(t)a_n = \sum_{n=1}^m \delta_n a_n$. Consequently every choice of signs $(\delta_n)_{n=1}^m$ occurs on a set of measure 2^{-m} and these sets are disjoint where their union is the whole interval $[0,1]$. Thus we have for all $p \in [1,\infty)$

$$\int_0^1 |\sum_{n=1}^m r_n(t)a_n|^p dt = 2^{-m} \sum_{\delta_n\in\{-1,1\}} |\sum_{n=1}^m \delta_n a_n|^p.$$

Definition 2.54. *Let E_0, E_1 be Banach spaces. A family $\tau \subset \mathcal{B}(E_0, E_1)$ is called $\mathcal{R}$-bounded, if there is a constant $C < \infty$ such that, for all $m \in \mathbb{N}$, $T_1, \dots, T_m \in \tau$ and $e_1, \dots, e_m \in E_0$, it holds*

$$\|\sum_{k=1}^m r_k T_k e_k\|_{L^2([0,1],E_1)} \le C \|\sum_{k=1}^m r_k e_k\|_{L^2([0,1],E_0)}, \tag{2.14}$$

here the r_k's are an enumeration of the Rademacher functions from above. The infimum over all constants such that (2.14) holds, is called the $\mathcal{R}$-bound of the family τ and is denoted by $\mathcal{R}_2(\tau)$.

The next Theorem states the well known Kahane's inequality as well as Kahane's contraction principle which allows for several observations in the case of scalar valued functions.

Theorem 2.55 ([Kah85]).

(a) (Kahane's inequality) For all $p \in [1,\infty)$ there is a constant $C_p < \infty$ such that for all $e_n \in E_0$ and $m \in \mathbb{N}$

$$\frac{1}{C_p}\|\sum_{n=1}^m r_n e_n\|_{L^2([0,1],E_0)} \le \|\sum_{n=1}^m r_n e_n\|_{L^p([0,1],E_0)} \le C_p \|\sum_{n=1}^m r_n e_n\|_{L^2([0,1],E_0)}$$

(b) (Kahane's contraction principle) For every $p \in [1,\infty)$ and $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ with $|a_n| \leq |b_n|$ and $m \in \mathbb{N}$ we have

$$\| \sum_{n=1}^{m} r_n a_n e_n \|_{L^p([0,1],E_0)} \leq 2 \| \sum_{n=1}^{m} r_n b_n e_n \|_{L^p([0,1],E_0)}.$$

A direct consequence of Kahane's inequality is, that the L^2-Norm in (2.14) may be replaced by any L^p-Norm for $p \in [1,\infty)$, meaning a subset $\tau \in \mathcal{B}(E_0, E_1)$ is $\mathcal{R}$-bounded if and only if for one (or all) $p \in [1,\infty)$ there is a constant C_p such that for all $m \in \mathbb{N}$, $T_1, \ldots, T_m \in \tau$ and $e_1, \ldots, e_m \in E_0$

$$\| \sum_{k=1}^{m} r_k T_k e_k \|_{L^p([0,1],E_1)} \leq C_p \| \sum_{k=1}^{m} r_k e_k \|_{L^p([0,1],E_0)}. \tag{2.15}$$

We denote by $\mathcal{R}_p(\tau)$ the infimum over all constants such that (2.15) holds and get $(C_p)^{-2}\mathcal{R}_2(\tau) \leq \mathcal{R}_p(\tau) \leq (C_p)^2\mathcal{R}_2(\tau)$ where C_p is the constant given in Theorem 2.55. This leads to equivalent descriptions of $\mathcal{R}_p$-boundedness. Hence we skip the p dependence and simply talk about $\mathcal{R}$-boundedness.

The next Lemma shows how the norm inequalities $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$, $\|T_1 \circ T_2\| \leq \|T_1\|\|T_2\|$, $\|c \cdot T\| \leq c\|T\|$ transfer to $\mathcal{R}$-boundedness.

Lemma 2.56. *Let E_0, E_1, E_2 be Banach spaces and $c > 0$. If $\tau, \sigma \subset \mathcal{B}(E_0, E_1)$ and $\gamma \subset \mathcal{B}(E_1, E_2)$ are $\mathcal{R}$-bounded, then also the sets*

$$\begin{aligned}
\tau + \sigma &:= \{T + S \;:\; T \in \tau,\ S \in \sigma\} \subset \mathcal{B}(E_0, E_1)\\
\tau \cup \sigma &:= \{T \in \mathcal{B}(E_0, E_1) \;:\; T \in \tau \text{ or } T \in \sigma\} \subset \mathcal{B}(E_0, E_1)\\
\gamma \circ \tau &:= \{G \circ T \;:\; T \in \tau,\ G \in \gamma\} \subset \mathcal{B}(E_0, E_2)\\
c \cdot \tau &:= \{c \cdot T \;:\; T \in \tau\} \subset \mathcal{B}(E_0, E_1)
\end{aligned}$$

are $\mathcal{R}$-bounded and the $\mathcal{R}$-bounds satisfy $\mathcal{R}(\tau \cup \sigma), \mathcal{R}(\tau + \sigma) \leq \mathcal{R}(\tau) + \mathcal{R}(\sigma)$, $\mathcal{R}(\gamma \circ \tau) \leq \mathcal{R}(\gamma) \cdot \mathcal{R}(\tau)$ and $\mathcal{R}(c \cdot \tau) \leq c \cdot \mathcal{R}(\tau)$.

Proof. Follows directly by definition and the inequalities given above. A detailed proof may be found in [vG06, Prop. 2.1]. □

A very important statement also deduced by Kahane's Inequality is the following (see [KW04, 2.13]).

Lemma 2.57. *Let E_0, E_1 be Banach spaces and $\tau \subset \mathcal{B}(E_0, E_1)$ be $\mathcal{R}$-bounded. Then the convex hull $co(\tau)$, the absolute convex hull*

$$absco(\tau) := \{\sum_{k=1}^{n} \lambda_k T_k :\ n \in \mathbb{N},\ T_k \in \tau,\ \lambda_k \in \mathbb{C} \text{ with } \sum_{k=1}^{n} |\lambda_k| = 1\},$$

of τ and their closures in the strong operator topology are $\mathcal{R}$-bounded with

$$\mathcal{R}(\overline{co(\tau)}^{s}) \leq \mathcal{R}(\tau) \quad \text{and} \quad \mathcal{R}(\overline{absco(\tau)}^{s}) \leq 2\mathcal{R}(\tau).$$

Now we turn to the announced workable criteria for $\mathcal{R}$-bounded sets of operators. We summarize several results given with detailed proofs in [KW04].

Lemma 2.58. *Let E_0, E_1 be Banach spaces, $\tau \subset \mathcal{B}(E_0, E_1)$ $\mathcal{R}$-bounded and (Ω, μ) a sigma finite measure space.*

(a) The set of multiplication operators $\sigma := \{\mathcal{M}_h : \|h\|_{L^\infty(\Omega,\mathbb{C})} \leq 1\}$ is a $\mathcal{R}$-bounded subset of $\mathcal{B}(L^p(\Omega, E_0))$ with $\mathcal{R}(\sigma) \leq 2$.

(b) For every strongly measurable $N : \Omega \to \mathcal{B}(E_0, E_1)$ with values in τ and every function $h \in L^1(\Omega, \mathbb{C})$ define an operator $V_{N,h} \in \mathcal{B}(E_0, E_1)$ by

$$V_{N,h}e := \int_\Omega h(\omega)N(\omega)e d\mu(\omega), \quad e \in E_0.$$

Then $\gamma := \{V_{N,h} : \|h\|_{L^1} \leq 1,\ N \text{ as above}\}$ is $\mathcal{R}$-bounded in $\mathcal{B}(E_0, E_1)$ with $\mathcal{R}(\gamma) \leq 2\mathcal{R}(\tau)$.

(c) For an operator $T \in \tau$ consider the extension $\tilde{T} : L^p(\Omega, E_0) \to L^p(\Omega, E_1)$ given as in Section 2.1. Then $\tilde{\tau} := \{\tilde{T} : T \in \tau\} \subset \mathcal{B}(L^p(\Omega, E_0), L^p(\Omega, E_1))$ is a $\mathcal{R}$-bounded set with $\mathcal{R}(\tilde{\tau}) \leq \mathcal{R}(\tau)$.

2.5 Banach Spaces of class $\mathcal{HT}$

As mentioned in the previous section there are more assumptions than $\mathcal{R}$-boundedness to obtain a Mihlin type Fourier multiplier theorem in the general setting of vector-valued functions. The proof of such a theorem was first derived by Bourgain in [Bou86] with scalar valued multiplication functions. Under the assumption, that the Hilbert Transform

$$[\mathcal{H}f](t) := \frac{1}{\pi}\text{p.v.-}\int_{\mathbb{R}} \frac{f(x-y)}{y} dy := \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy, \tag{2.16}$$

first defined for $f \in \mathcal{S}(\mathbb{R}) \otimes E$ extends to an bounded operator on $L^p(\mathbb{R}, E)$ he was able to proof a vector valued version of the well known Paley-Littlewood decomposition. This allowed him to proof the L^p-boundedness of certain singular integrals with scalar kernels in the periodic case. We will see how this assumption helps to establish the boundedness of some basic Fourier multiplication operators in Chapter 4.

Definition 2.59. *A Banach space E is called of class $\mathcal{HT}$ if the Hilbert transform given in (2.16) extends to a bounded operator on $L^p(\mathbb{R}, E)$ for some (or equivalently for all) $p \in (1, \infty)$.*

First of all it is well known, that $\mathbb{C}$ is of class $\mathcal{HT}$. If one revisits the proof of this result (see for example [Gra08, Ch.IV]) it is not surprising that there are spaces which do not have this property. We will give an example at the end of this section. But for now we give an alternative characterization.

Theorem 2.60. *For a Banach space E the following statements are equivalent.*

(i) E is of class $\mathcal{HT}$,

(ii) E is a UMD[6] space, meaning for a given probability space $(\Omega, \mathcal{A}, P)$ and some (or equivalently all) $1 < p < \infty$ there is a constant $c > 0$ such that

$$\| \sum_{k=1}^{n} \varepsilon_k (u_k - u_{k-1}) \|_{L^p(\Omega,E)} \leq c \| \sum_{k=1}^{n} (u_k - u_{k-1}) \|_{L^p(\Omega,E)},$$

for all $n \in \mathbb{N}$, $\varepsilon_k \in \{-1, 1\}$, and E-valued martingales (u_k).

Proof. (i)⇒(ii) was shown by Bourgain in [Bou83] where the converse direction (ii)⇒(i) goes back to Burkholder [Bur83]. □

Let us close this section by collecting some important results out the literature and give an example of a space that is not of class $\mathcal{HT}$.

Proposition 2.61.

(i) Every Banach space of class $\mathcal{HT}$ is reflexive [Mau75].

(ii) If E is of class $\mathcal{HT}$ so is its dual E' [Ama95, Thm 4.5.2].

(iii) If E is of class $\mathcal{HT}$ and (Ω, μ) is a σ-finite measure space, so is $L^p(\Omega, E)$ for all $1 < p < \infty$.

Statement (iii) is an simple consequence of Fubini's Theorem (A.11). Indeed we have for $f \in L^p(\mathbb{R}, L^p(\Omega, E)) \cong L^p(\mathbb{R} \times \Omega, E)$

$$\begin{aligned}
\|\mathcal{H}f\|^p_{L^p(\mathbb{R},L^p(\Omega,E))} &= \int_{\mathbb{R}} \| \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} dy \|^p_{L^p(\Omega,E)} dx \\
&= \int_{\mathbb{R}} \int_{\Omega} \| \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(x-y,\omega)}{y} dy \|^p_E dy d\omega dx \\
&= \int_{\Omega} \int_{\mathbb{R}} \| \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(x-y,\omega)}{y} dy \|^p_E dy dx d\omega \\
&\leq \|\mathcal{H}\|^p_{\mathcal{B}(L^p(\mathbb{R},E))} \int_{\Omega} \|f(\cdot,\omega)\|^p_{L^p(\mathbb{R},E)} d\omega \\
&= \|\mathcal{H}\|^p_{\mathcal{B}(L^p(\mathbb{R},E))} \|f\|^p_{L^p(\mathbb{R},L^p(\Omega,E))}.
\end{aligned}$$

One example of a Banach space that is not of class $\mathcal{HT}$ is given by $L^1(\mathbb{R}^d)$. Note that $L^1(\mathbb{R}^d)$ is not reflexive. For further reading we suggest [Bur01].

[6] *UMD* refers to 'unconditional martingale differences'

Chapter 3

Periodic Operators on $L^p(\mathbb{R}^d, E)$

In this chapter we take a closer look at periodic operators. As it turns out, these operators act as operator-valued multiplication operators under the Bloch Transform. First we consider the case of bounded periodic operators and study their behavior under the mapping Γ from (2.4). This is the first operator in the decompositions $Z = \mathcal{F}^{-1} \circ \Gamma$, $\Phi = \Xi \circ \mathcal{F}^{-1} \circ \Gamma$ of the Zak and Bloch Transform given in (2.5) and (2.6). The considerations lead to a characterization in terms of operator-valued convolutions on the sequence space $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$. Applying Fourier series turns such a convolution into a bounded multiplication operator with fiber operators acting on the space $L^p(\mathbb{I}^d, E)$. The investigation of unbounded operators is more involved as we will see in Section 3.3. However we can reduce their study to the bounded case by considering their resolvent.

3.1 Bounded Periodic Operators - Reduction to Translation Invariant Operators on Sequence Spaces

Recall the definition of a lattice in Section 2.1 and assume we are given one, denoted as usual by $\mathcal{P}$. Further let E_0, E_1 be Banach spaces and $p, q \in [1, \infty]$.

Definition 3.1. *An operator $T \in \mathcal{B}(L^p(\mathbb{R}^d, E_0), L^q(\mathbb{R}^d, E_1))$ is called periodic with respect to the lattice $\mathcal{P}$ if*

$$T\tau_p f = \tau_p T f \quad \text{for all } p \in \mathcal{P} \text{ and } f \in L^p(\mathbb{R}^d, E_0).$$

As in the case of functions it is possible to reduce all considerations to one specific lattice (where we choose again $\mathbb{Z}^d$). Let us recall our earlier definition for a multidimensional dilatation. For a function $f : \mathbb{R}^d \to E$ and $\xi \in \mathbb{R}^d$ we defined $[\delta_\xi f](x) = f(x \times \xi)$, where $x \times \xi := (x_1\xi_1, \ldots, x_d\xi_d)^T$. If ξ has only non-zero components (what is by definition the case for a lattice vector), then δ_ξ is invertible on $L^p(\mathbb{R}^d, E)$ with $\delta_\xi^{-1} = \delta_{1/\xi}$. Here $1/\xi$ is the component wise reciprocal as in Section 2.1. We have the following analogue of Lemma 2.7.

Lemma 3.2. *Let $\mathcal{P}$ be a lattice with lattice vector $\mathfrak{p}$ and $p, q \in [1, \infty]$. For a bounded operator $T : L^p(\mathbb{R}^d, E_0) \to L^q(\mathbb{R}^d, E_1)$ which is periodic with respect to $\mathcal{P}$ define*

$$\mathbb{T} := \delta_{1/\mathfrak{p}} T \delta_{\mathfrak{p}}.$$

Then $\mathbb{T} : L^p(\mathbb{R}^d, E_0) \to L^q(\mathbb{R}^d, E_1)$ is bounded and periodic with respect to $\mathbb{Z}^d$. Moreover, in the case $p = q$ the mapping $T \mapsto \mathbb{T}$ is isometric.

Proof. Let $\xi_1, \xi_2 \in \mathbb{R}^d$ where ξ_2 has only non-zero components. We have for any $f \in L^p(\mathbb{R}^d, E_0)$ the identity $\tau_{\xi_1} \delta_{\xi_2} f = \delta_{\xi_2} \tau_{1/\xi_2 \times \xi_1} f$. This yields for $z \in \mathbb{Z}^d$

$$\begin{aligned}\tau_z \mathbb{T} f &= \tau_z \delta_{1/\mathfrak{p}} T \delta_{\mathfrak{p}} f = \delta_{1/\mathfrak{p}} \tau_{\mathfrak{p} \times z} T \delta_{\mathfrak{p}} f = \delta_{1/\mathfrak{p}} T \tau_{\mathfrak{p} \times z} \delta_{\mathfrak{p}} f = \delta_{1/\mathfrak{p}} T \delta_{\mathfrak{p}} \tau_{1/\mathfrak{p} \times \mathfrak{p} \times z} f \\ &= \delta_{1/\mathfrak{p}} T \delta_{\mathfrak{p}} \tau_z f = \mathbb{T} \tau_z f.\end{aligned}$$

Hence $\mathbb{T}$ is periodic with respect to $\mathbb{Z}^d$. Linearity and boundedness of $\mathbb{T}$ follow by the same properties of its parts. Moreover we have

$$\|\mathbb{T} f\|_{L^q} = \Big(\prod_{i=1}^d \mathfrak{p}_i\Big)^{1/q} \|T \delta_{\mathfrak{p}} f\|_{L^q} \leq \Big(\prod_{i=1}^d \mathfrak{p}_i\Big)^{1/q - 1/p} \|T\| \|f\|_{L^p},$$

and by the same calculation applied to $T = \delta_{\mathfrak{p}} \mathbb{T} \delta_{1/\mathfrak{p}}$

$$\|T f\|_{L^q} \leq \Big(\prod_{i=1}^d \mathfrak{p}_i\Big)^{1/p - 1/q} \|\mathbb{T}\| \|f\|_{L^p}.$$

Hence $\|T\| = \|\mathbb{T}\|$ if $p = q$. □

As in the case of functions, we restrict ourselves to the case of bounded operators which are periodic with respect to $\mathbb{Z}^d$. With this agreement we turn our attention to the relation between periodic operators on $L^p(\mathbb{R}^d, E)$ and translation invariant operators on the sequence space $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$, mentioned previously.

Definition 3.3. *Let E_0, E_1 be Banach spaces and $p, q \in [1, \infty]$. We call an operator $\mathbf{T} \in \mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^q(\mathbb{Z}^d, E_1))$ translation invariant if*

$$\tau_z \mathbf{T} g = \mathbf{T} \tau_z g \ \text{ for all } z \in \mathbb{Z}^d \text{ and } g \in l^p(\mathbb{Z}^d, E_0).$$

Lemma 3.4. *Assume $T : L^p(\mathbb{R}^d, E_0) \to L^q(\mathbb{R}^d, E_1)$ is linear and bounded. Define for an element $g \in l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0))$*

$$\mathbf{T} g := \Gamma T \Gamma^{-1} g.$$

Then $\mathbf{T} : l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0)) \to l^q(\mathbb{Z}^d, L^q(\mathbb{I}^d, E_1))$ is linear and bounded. Further $\mathbf{T}$ is translation invariant if and only if T is periodic. Moreover, the mapping $T \mapsto \mathbf{T}$ is isometric.

Proof. Boundedness and linearity of $\mathbf{T}$ follow immediately by the corresponding properties of its parts. For $z \in \mathbb{Z}^d$, $f \in L^p(\mathbb{R}^d, E_0)$ and $g \in l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0))$ a straight forward calculation shows

$$\Gamma \tau_z f = \tau_{-z} \Gamma f \quad \text{and} \quad \tau_z \Gamma^{-1} g = \Gamma^{-1} \tau_{-z} g \tag{3.1}$$

where in both equations the translation on the left hand side is taken in $\mathbb{R}^d$ and on the right hand side in $\mathbb{Z}^d$. Now the translation invariance of $\mathbf{T}$ (respectively the periodicity of T) follows directly by the respective property of T and $\mathbf{T}$. Finally recall that Γ and Γ^{-1} are isometric mappings (Lemma 2.22), so that $\|\mathbf{T}\| = \|T\|$. □

We summarize the previous results in the following commutative diagram, which illustrates how we can transform any periodic operator into an translation invariant operator on a sequence space.

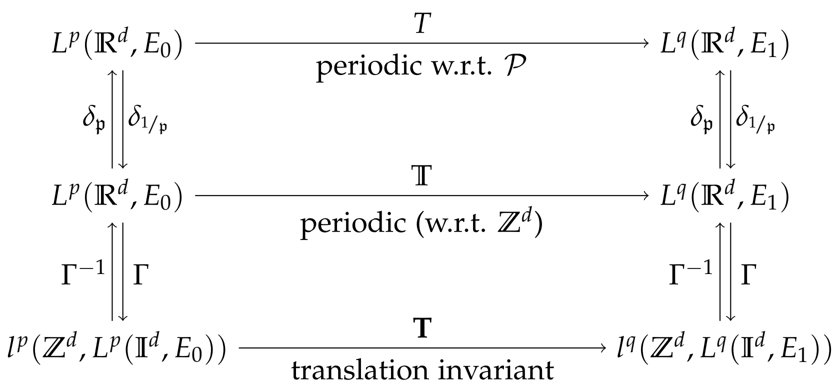

Figure 3.1: Commutative diagram for the transformation of a bounded, periodic operator on $L^p(\mathbb{R}^d, E)$ to a bounded translation invariant operator on $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$.

The explicit structure of Γ allows for a more detailed description of $\mathbf{T}$. In fact $\mathbf{T}$ is always given by convolution with a bounded, $\mathcal{B}(L^p(\mathbb{I}^d, E_0), L^q(\mathbb{I}^d, E_1))$-valued sequence.

Theorem 3.5. *Let $T : L^p(\mathbb{R}^d, E_0) \to L^q(\mathbb{R}^d, E_1)$ be linear, bounded and periodic. Then $\mathbf{T}$ is given by convolution with the sequence $\mathbf{T}(z) := \mathfrak{R}_{\mathbb{I}^d} T \tau_z \mathfrak{E}_{\mathbb{R}^d}$, $(z \in \mathbb{Z}^d)$ i.e.*

$$[\mathbf{T}\varphi](z) = \sum_{j \in \mathbb{Z}^d} \mathbf{T}(z - j)\varphi(j)$$

for all $z \in \mathbb{Z}^d$, $\varphi \in s(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0))$. Additionally we have

$$\sup_{z \in \mathbb{Z}^d} \|\mathbf{T}(z)\|_{\mathcal{B}(L^p(\mathbb{I}^d, E_0), L^q(\mathbb{I}^d, E_1))} \leq \|T\|.$$

Proof. Let $\varphi \in s(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0))$. Then the sum $\Gamma^{-1}\varphi = \sum_{j\in\mathbb{Z}^d} \tau_{-j}\mathfrak{E}_{\mathbb{R}^d}\varphi(j)$ is convergent in $L^p(\mathbb{R}^d, E_0)$. Now linearity and boundedness of the restriction, translation and T yield for every $z \in \mathbb{Z}^d$

$$[\mathbf{T}\varphi](z) = [\Gamma T\Gamma^{-1}\varphi](z) = \mathfrak{R}_{\mathbb{I}^d}\tau_z T \sum_{j\in\mathbb{Z}^d} \tau_{-j}\mathfrak{E}_{\mathbb{R}^d}\varphi(j) = \sum_{j\in\mathbb{Z}^d} \mathfrak{R}_{\mathbb{I}^d}\tau_z T\tau_{-j}\mathfrak{E}_{\mathbb{R}^d}\varphi(j).$$

Periodicity of T leads to

$$[\mathbf{T}\varphi](z) = \sum_{j\in\mathbb{Z}^d} \mathfrak{R}_{\mathbb{I}^d} T\tau_{z-j}\mathfrak{E}_{\mathbb{R}^d}\varphi(j) = \sum_{j\in\mathbb{Z}^d} \mathbf{T}(z-j)\varphi(j).$$

Note that all translations appearing in the calculations above are in $\mathbb{R}^d$. Additionally we have for any $z \in \mathbb{Z}^d$ and $f \in L^p(\mathbb{I}^d, E_0)$

$$\begin{aligned}\|\mathbf{T}(z)f\|_{L^q(\mathbb{I}^d,E_1)} &= \|\mathfrak{R}_{\mathbb{I}^d} T\tau_z\mathfrak{E}_{\mathbb{R}^d} f\|_{L^q(\mathbb{I}^d,E_1)} \le \|T\tau_z\mathfrak{E}_{\mathbb{R}^d} f\|_{L^q(\mathbb{R}^d,E_1)} \\ &\le \|T\|\|\tau_z\mathfrak{E}_{\mathbb{R}^d} f\|_{L^p(\mathbb{R}^d,E_0)} = \|T\|\|f\|_{L^p(\mathbb{I}^d,E_0)}\end{aligned}$$

and all statements are shown. □

We close this subsection by summarizing algebraic properties of periodic and translation invariant operators.

Remark 3.6.

(i) For a lattice $\mathcal{P}$ and $p, q \in [1, \infty]$ the set

$$\mathcal{B}^{p,q}_{\mathcal{P}}(E_0, E_1) := \{T \in \mathcal{B}(L^p(\mathbb{R}^d, E_0), L^q(\mathbb{R}^d, E_1)) : T \text{ is periodic w.r.t } \mathcal{P}\}$$

is a closed linear subspace of $\mathcal{B}(L^p(\mathbb{R}^d, E_0), L^q(\mathbb{R}^d, E_1))$. The mapping $T \mapsto \mathbb{T}$ sets up an isomorphism between $\mathcal{B}^{p,q}_{\mathcal{P}}(E_0, E_1)$ and $\mathcal{B}^{p,q}_{\mathbb{Z}^d}(E_0, E_1)$ that is isometric in case of $p = q$ and an algebra isomorphism if additionally $E_0 = E_1$.

(ii) The situation in Lemma 3.4 is very similar. The set $\mathcal{B}^{p,q}_{trans}(E_0, E_1)$ consisting of all $T \in \mathcal{B}(l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0)), l^q(\mathbb{Z}^d, L^q(\mathbb{I}^d, E_1)))$ that are translation invariant is a closed, linear subspace of $\mathcal{B}(l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0)), l^q(\mathbb{Z}^d, L^q(\mathbb{I}^d, E_1)))$. The mapping $T \mapsto \mathbf{T}$ sets up an isometric isomorphism between $\mathcal{B}^{p,q}_{\mathbb{Z}^d}(E_0, E_1)$ and $\mathcal{B}^{p,q}_{trans}(E_0, E_1)$ which is an algebra isomorphism if $p = q$ and $E_0 = E_1$.

The above results allow us to restrict our attention the study of translation invariant operators on sequence spaces and all results we may obtain for them have corresponding counterparts for periodic operators $L^p(\mathbb{R}^d, E_0) \to L^q(\mathbb{R}^d, E_1)$. Using the fact, that with a Banach space E also $L^p(\mathbb{I}^d, E)$ is a Banach space we may replace the space $L^p(\mathbb{I}^d, E)$ which occurred above by an other Banach space E_0 and only gain more generality. This 'generalization' will also simplify our notation. Hence we only consider linear and translation invariant operators $l^p(\mathbb{Z}^d, E_0) \to l^q(\mathbb{Z}^d, E_1)$ for the moment. At the end of the chapter we include the corresponding statements for periodic operators.

3.2 Bounded Translation Invariant Operators on $l^p(\mathbb{Z}^d, E)$

As in the case of a translation invariant operator, that is obtained from a periodic operator on $L^p(\mathbb{R}^d, E)$ we will show, that every bounded translation invariant operator $l^p(\mathbb{Z}^d, E_0) \to l^q(\mathbb{Z}^d, E_1)$ is given by convolution with a distribution $S \in s'_{E_1}(\mathbb{Z}^d, E_0)$. This is in perfect correspondence with Theorem 3.5 if one considers the bounded sequence obtained there as a 'regular' distribution. Let us fix two Banach spaces E_0 and E_1 and assume $p, q \in [1, \infty]$ if not separately specified.

Theorem 3.7. *Let $T : l^p(\mathbb{Z}^d, E_0) \to l^q(\mathbb{Z}^d, E_1)$ be linear, bounded and translation invariant. Then there is some $G \in s'_{E_1}(\mathbb{Z}^d, E_0)$ such that*

$$T\varphi = \varphi * G \quad \text{for all } \varphi \in s(\mathbb{Z}^d, E_0). \tag{3.2}$$

Conversely if $T : l^p(\mathbb{Z}^d, E_0) \to l^q(\mathbb{Z}^d, E_1)$ is bounded and there is a distribution $G \in s'_{E_1}(\mathbb{Z}^d, E_0)$ such that (3.2) holds, then T is linear and translation invariant.

Proof. We start with a bounded, linear and translation invariant operator T and define a mapping $S : s(\mathbb{Z}^d, E_0) \to E_1$ by

$$S\varphi := [T\varphi](0).$$

Clearly S is linear. Moreover S is an element of $s'_{E_1}(\mathbb{Z}^d, E_0)$. Indeed we have by Lemma 2.14

$$\begin{aligned}\|S\varphi\|_{E_1} = \|[T\varphi](0)\|_{E_1} \leq \|T\varphi\|_{l^q(\mathbb{Z}^d,E_1)} &\leq \|T\|_{\mathcal{B}(l^p(\mathbb{Z}^d,E_0),l^q(\mathbb{Z}^d,E_1))}\|\varphi\|_{l^p(\mathbb{Z}^d,E_1)} \\ &\leq \|T\| C_{d,p} \sum_{|\beta|\leq M} p_\beta^{E_0}(\varphi)\end{aligned}$$

for some $M \in \mathbb{N}$, a constant $C_{d,p}$ and all $\varphi \in s(\mathbb{Z}^d, E_0)$. Hence $S \in s'_{E_1}(\mathbb{Z}^d, E_0)$ by Lemma 2.12 and the same is true for $G := \tilde{S}$. Recall, that $\mathbb{C} \times E_i \to E_i$, $(c, e) \mapsto c \cdot e$ is a multiplication for $i = 0, 1$ in the sense of Section 2.1. We get for $\varphi \in s(\mathbb{Z}^d, E_0)$ and $\psi \in s(\mathbb{Z}^d, \mathbb{C})$

$$[\varphi * G](\psi) = G(\tilde{\varphi} * \psi) = \sum_{j\in\mathbb{Z}^d} G(\widetilde{\tau_{-j}\varphi}) \cdot \psi(j).$$

Hence[1] $[\varphi * G] : \mathbb{Z}^d \to E_1$ with $[\varphi * G](z) = G(\widetilde{\tau_{-z}\varphi})$. This yields for $z \in \mathbb{Z}^d$

$$[\varphi * G](z) = \tilde{S}(\widetilde{\tau_{-z}\varphi}) = S(\tau_{-z}\varphi) = [T\tau_{-z}\varphi](0) = [\tau_{-z}T\varphi](0) = [T\varphi](z).$$

[1] At this point we want to emphasize, that if $f, g : \mathbb{Z}^d \to E$ are such that the corresponding regular distributions S_f, S_g exist and coincide on $s(\mathbb{Z}^d, \mathbb{C})$, i.e. $S_f(\phi) = S_g(\phi)$ for all $\phi \in s(\mathbb{Z}^d, \mathbb{C})$ then $f = g$. Indeed choose for $j \in \mathbb{Z}^d$ the singleton sequence $\phi_j := (\delta_{z,j})_{z\in\mathbb{Z}^d}$, where $\delta_{z,j} = 1$ if $j = z$ and $\delta_{z,j} = 0$ for $j \neq z$. Then $S_f\phi_j = \sum_{z\in\mathbb{Z}^d} f(z)\phi_j(z) = f(j)$ and $S_g\phi_j = g(j)$, which implies $f = g$.

For the reverse statement all we have to check is $\tau_z[\varphi * G] = [(\tau_z\varphi) * G]$ for all $z \in \mathbb{Z}^d$ and $\varphi \in s(\mathbb{Z}^d, E_0)$. In this case, the assertion follows by unique bounded extension and denseness. We have for any $\psi \in s(\mathbb{Z}^d, \mathbb{C})$ and $z \in \mathbb{Z}^d$

$$\tau_z[\varphi * G](\psi) = [\varphi * G](\tau_z\psi) = G(\tilde{\varphi} * \tau_z\psi) = G(\widetilde{\tau_z\varphi} * \psi) = [(\tau_z\varphi) * G](\psi)$$

and the theorem is proven. □

Remark 3.8. *The proof of the above theorem was motivated an guided by a similar proof given in the case of translation invariant operators on $L^p(\mathbb{R}^d)$ in [Gra08].*

According to our previous definitions, we define the space of all translation invariant operators on sequence spaces by

$$\mathcal{M}^{p,q}(E_0, E_1) := \{T \in \mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^q(\mathbb{Z}^d, E_1)) : T \text{ is translation invariant}\}.$$

Properties of the spaces $\mathcal{B}^{p,q}_{\text{trans}}(E_0, E_1)$ have corresponding counterparts for the space $\mathcal{M}^{p,q}(E_0, E_1)$. Indeed one easily verifies, that $\mathcal{M}^{p,q}(E_0, E_1)$ is a closed linear subspace of $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), L^q(\mathbb{Z}^d, E_1))$ and if $p = q$ and $E_0 = E_1$ it is a Banach algebra. Clearly $\mathcal{B}^{p,q}_{\text{trans}}(E_0, E_1) = \mathcal{M}^{p,q}(L^p(\mathbb{I}^d, E_0), L^q(\mathbb{I}^d, E_1))$.

Although it is easy to show, it is a surprising fact that $\mathcal{M}^{p,q}(E_0, E_1) = \{0\}$ if $1 \le q < p < \infty$. To see this observe for $r \in [1, \infty)$ and $f \in l^r(\mathbb{Z}^d, E)$

$$\lim_{\substack{z\to\infty \\ z\in\mathbb{Z}^d}} \|\tau_z f + f\|_{l^r(\mathbb{Z}^d,E)} = 2^{1/r}\|f\|_{l^r(\mathbb{Z}^d,E)},$$

which is clear for sequences with finite support and extends to all sequences by denseness. Hence if $T \in \mathcal{M}^{p,q}(E_0, E_1)$ we get

$$\begin{aligned} 2^{1/q}\|Tf\|_{l^q(\mathbb{Z}^d,E_1)} &= \lim_{z\to\infty} \|\tau_z Tf + Tf\|_{l^q(\mathbb{Z}^d,E_1)} \le \|T\| \lim_{z\to\infty} \|\tau_z f + f\|_{l^p(\mathbb{Z}^d,E_0)} \\ &= 2^{1/p}\|T\|\|f\|_{l^p(\mathbb{Z}^d,E_0)}. \end{aligned}$$

If $f \neq 0$ and $q < p$, this is only possible for $T = 0$.

Concerning duality we observe, that if E is reflexive and $p \in (1, \infty)$ then the dual of $l^p(\mathbb{Z}^d, E)$ may be identified with $l^{p'}(\mathbb{Z}^d, E')$ where $1/p + 1/p' = 1$ (c.f. Appendix A). Moreover the dual operator of a translation τ_z on $l^p(\mathbb{Z}^d, E)$ is given by τ_{-z}. We obtain

Lemma 3.9. *Let E_0, E_1 be reflexive and $p, q \in (1, \infty)$. Then*

$$\mathcal{M}^{q',p'}(E_1', E_0') = \{T' : T \in \mathcal{M}^{p,q}(E_0, E_1)\},$$

where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

Proof. Since both inclusions work the same way we only show $\supset$. For this reason let $T \in \mathcal{M}^{p,q}(E_0, E_1)$. Then by definition $T' : l^{q'}(\mathbb{Z}^d, E_1') \to l^{p'}(\mathbb{Z}^d, E_0')$ is bounded. Hence it is sufficient to show translation invariance.

For $z \in \mathbb{Z}^d$, $g \in l^{q'}(\mathbb{Z}^d, E_1')$ and $f \in l^p(\mathbb{Z}^d, E_0)$ we have

$$[\tau_z T'g](f) = [T'g](\tau_{-z}f) = g(T\tau_{-z}f) = g(\tau_{-z}Tf) = [\tau_z g](Tf) = [T'\tau_z g](f).$$

Hence $T' \in \mathcal{M}^{q',p'}(E_1', E_0')$. □

In the scalar valued case even more is true. Set $\mathcal{M}^{p,q}(\mathbb{C}) := \mathcal{M}^{p,q}(\mathbb{C},\mathbb{C})$ and $l^p(\mathbb{Z}^d) := l^p(\mathbb{Z}^d,\mathbb{C})$.

Lemma 3.10. *Let $p, q \in (1,\infty)$ and $T \in \mathcal{M}^{p,q}(\mathbb{C})$. Then T can be defined on $l^{q'}(\mathbb{Z}^d)$ with values in $l^{p'}(\mathbb{Z}^d)$, coinciding with its previous definition on the (dense) subspace $l^p(\mathbb{Z}^d) \cap l^{q'}(\mathbb{Z}^d)$ of $l^p(\mathbb{Z}^d)$, so that it maps $l^{q'}(\mathbb{Z}^d)$ to $l^{p'}(\mathbb{Z}^d)$ with norm $\|T\|_{l^{q'}(\mathbb{Z}^d)\to l^{p'}(\mathbb{Z}^d)} = \|T\|_{l^p(\mathbb{Z}^d)\to l^q(\mathbb{Z}^d)}$. This gives an isometric identification*

$$\mathcal{M}^{p,q}(\mathbb{C}) \cong \mathcal{M}^{q',p'}(\mathbb{C}).$$

Proof. First of all Theorem 3.7 yields that the operator T is given by convolution with a (scalar-valued) distribution $S \in s'_{\mathbb{C}}(\mathbb{Z}^d,\mathbb{C})$, i.e.

$$T\varphi = \varphi * S \quad \text{for all } \varphi \in s(\mathbb{Z}^d,\mathbb{C}).$$

We now show, that the adjoint T' of T is also given by convolution, this time with the distribution $\tilde{S}$. For this reason fix $\varphi, \psi \in s(\mathbb{Z}^d,\mathbb{C})$ and observe

$$\begin{aligned}\varphi(T'\psi) &= (T''\varphi)(\psi) = (T\varphi)(\psi) = (\varphi * S)(\psi) = S(\tilde{\varphi} * \psi) = S(\widetilde{\tilde{\psi} * \varphi}) \\ &= \tilde{S}(\tilde{\psi} * \varphi) = [\psi * \tilde{S}](\varphi) = \varphi(\psi * \tilde{S}).\end{aligned}$$

Because $s(\mathbb{Z}^d)$ is dense in both $l^p(\mathbb{Z}^d)$ and $l^{q'}(\mathbb{Z}^d)$, we actually obtained that T' is given by convolution with the distribution $\tilde{S}$. Now the identity

$$T'\varphi = \varphi * \tilde{S} = \widetilde{\tilde{\varphi} * S} = \widetilde{T\tilde{\varphi}} \tag{3.3}$$

yields, that T is well defined on $l^{q'}(\mathbb{Z}^d)$. But (3.3) also shows, that

$$\|T\|_{l^p(\mathbb{Z}^d)\to l^q(\mathbb{Z}^d)} = \|T'\|_{l^{q'}(\mathbb{Z}^d)\to l^{p'}(\mathbb{Z}^d)} = \|T\|_{l^{q'}(\mathbb{Z}^d)\to l^{p'}(\mathbb{Z}^d)},$$

which finishes the proof. □

Remark 3.11. *Again we were guided by a corresponding result for translation invariant operators on $L^p(\mathbb{R}^d)$, see [Gra08, Thm. 2.5.7].*

After this preparatory work we are now interested in the study of translation invariant operators on $l^p(\mathbb{Z}^d, E)$ and their relation to multiplication operators. We start with the simpler case where $p = 2$ and $E_0 = H_0$ and $E_1 = H_1$ are Hilbert spaces. Note that in this situation, $\mathcal{F} : L^2(\mathbb{I}^d, H_i) \to l^2(\mathbb{Z}^d, H_i)$ is an isometric isomorphism and so is its inverse (here $i = 0, 1$).

A Characterization of $\mathcal{M}^{2,2}(H_0, H_1)$

Over this whole subsection we assume, that H_0, H_1 are separable Hilbert spaces. Separability is an extra assumption that is needed to obtain measurability as we will see below. At the end of this subsection we include some remarks concerning this assumption.

As we already know, bounded translation invariant operators are characterized by convolution operators (see Theorem 3.7). Our aim is now to show that these convolution operators are characterized by strongly measurable bounded operator-valued functions in the case of $p = q = 2$ and Hilbert spaces H_0, H_1. Recall the notation $B = [-1/2, 1/2]$.

Theorem 3.12. *An operator $T : l^2(\mathbb{Z}^d, H_0) \to l^2(\mathbb{Z}^d, H_1)$ is in $\mathcal{M}^{2,2}(H_0, H_1)$ if and only if the inverse Fourier Transform of its convolving distribution $S \in s'_{H_1}(\mathbb{Z}^d, H_0)$ is a function $m : B^d \to \mathcal{B}(H_0, H_1)$ that belongs to $L^\infty(B^d, \mathcal{B}_s(H_0, H_1))$. In this case,*

$$Tg = \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}g \text{ for all } g \in l^2(\mathbb{Z}^d, H_0), \tag{3.4}$$

and $\|\mathcal{M}_m\|_{B(L^2(B^d,H_0),L^2(B^d,H_1))} = \operatorname{ess\,supp}_{\theta\in B^d}\|m(\theta)\|_{\mathcal{B}(H_0,H_1)} = \|T\|$.

Proof. First let us assume, that T is given as a convolution operator, convolving with a distribution $S \in s'_{H_1}(\mathbb{Z}^d, H_0)$ such that $m := \mathcal{F}^{-1}S \in L^\infty(B^d, \mathcal{B}_s(H_0, H_1))$ and $Tg = \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}g$ for all $g \in l^2(\mathbb{Z}^d, H_0)$.

Now by Lemma 2.21 (v) we obtain for $\varphi \in s(\mathbb{Z}^d, H_0)$ and $z \in \mathbb{Z}^d$

$$\begin{aligned}\tau_z T\varphi &= \tau_z\mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}\varphi = \mathcal{F}\mathcal{M}_{[\theta\mapsto e^{-2\pi i z\theta}]}\mathcal{M}_m\mathcal{F}^{-1}\varphi = \mathcal{F}\mathcal{M}_m\mathcal{M}_{[\theta\mapsto e^{-2\pi i z\theta}]}\mathcal{F}^{-1}\varphi\\ &= \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}\tau_z\varphi = T\tau_z\varphi\end{aligned}$$

an equation, that extends to $l^2(\mathbb{Z}^d, H_0)$ by denseness. Hence T is translation invariant. But Lemma 2.18 and Lemma 2.35 (vii) together yield

$$\|T\varphi\|_{l^2(\mathbb{Z}^d,H_1)} = \|\mathcal{M}_m\mathcal{F}^{-1}\varphi\|_{L^2(B^d,H_0)} = \|m\|_\infty\|\varphi\|_{l^2(\mathbb{Z}^d,H_0)}$$

for all $\varphi \in l^2(\mathbb{Z}^d, H_0)$. Thus $T \in \mathcal{M}^{2,2}(H_0, H_1)$ with $\|T\| = \|m\|_\infty$.

Conversely assume $T \in \mathcal{M}^{2,2}(H_0, H_1)$ and let $S \in s'_{H_1}(\mathbb{Z}^d, H_0)$ be the convolving distribution given by Theorem 3.7. Then $\varphi * S \in l^2(\mathbb{Z}^d, H_1)$ for all $\varphi \in s(\mathbb{Z}^d, H_0)$ and by Lemma 2.21 (iii)

$$\mathcal{F}^{-1}[\varphi * S] = [\mathcal{F}^{-1}\varphi] \cdot [\mathcal{F}^{-1}S] \in L^2(B^d, H_1). \tag{3.5}$$

Now, let $(h_n)_{n\in\mathbb{N}} \subset H_0$ be a dense subset (which we may choose thanks to our assumption of separability) and define for $z \in \mathbb{Z}^d$ and $n \in \mathbb{N}$ a sequence of functions $\varphi_n : \mathbb{Z}^d \to H_0$ by

$$\varphi_n(z) := \begin{cases} h_n & : \text{ if } z = 0 \\ 0 & : \text{ else.}\end{cases}$$

Then we obtain for $\theta \in B^d$

$$[\mathcal{F}^{-1}\varphi_n](\theta) = \mathbb{1}_{B^d}(\theta)h_n.$$

By (3.5) we find for every $n \in \mathbb{N}$ a set $\Omega_n \subset B^d$ of measure zero and a function $g_n \in L^2(B^d, H_1)$ with

$$([\mathcal{F}^{-1}\varphi_n] \cdot [\mathcal{F}^{-1}S])(\theta) = g_n(\theta) \text{ for all } \theta \in B^d \setminus \Omega_n. \tag{3.6}$$

Set $\Omega := \cup_{n\in\mathbb{N}}\Omega_n$, then $\lambda(\Omega) = 0$ and for $\theta \in B^d \setminus \Omega$, $m, n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ we have

$$([\mathcal{F}^{-1}(\lambda\varphi_n + \varphi_m)] \cdot [\mathcal{F}^{-1}S])(\theta) = \Big((\lambda \mathbb{1}_{B^d} h_n + \mathbb{1}_{B^d} h_m) \cdot [\mathcal{F}^{-1}S]\Big)(\theta) \\ = \lambda g_n(\theta) + g_m(\theta). \tag{3.7}$$

If $f \in D(B^d, \mathbb{C})$ then $f \in L^2(B^d, \mathbb{C})$ and $\mathcal{F}f \in s(\mathbb{Z}^d, \mathbb{C})$. Further, we can use Lemma 2.18, Lemma 2.20 and Lemma 2.21 (ii) to obtain for $n \in \mathbb{N}$

$$\begin{aligned} \|f \cdot g_n\|_{L^2(B^d,H_1)} &= \|(fh_n) \cdot \mathcal{F}^{-1}S\|_{L^2(B^d,H_1)} = \|\mathcal{F}[fh_n \cdot \mathcal{F}^{-1}S]\|_{l^2(\mathbb{Z}^d,H_1)} \\ &= \|[\mathcal{F}fh_n] * S\|_{l^2(\mathbb{Z}^d,H_1)} = \|T[\mathcal{F}(fh_n)]\|_{l^2(\mathbb{Z}^d,H_1)} \\ &\le \|T\|_{\mathcal{B}(l^2(\mathbb{Z}^d,H_0),l^2(\mathbb{Z}^d,H_1))} \|\mathcal{F}(fh_n)\|_{l^2(\mathbb{Z}^d,H_0)} = \|T\|\|f\|_{L^2}\|h_n\|_{H_0}. \end{aligned} \tag{3.8}$$

This inequality extends to all $f \in L^2(B^d, \mathbb{C})$ by denseness and we obtain

$$\int_{B^d} \|T\|^2 \|h_n\|_{H_0}^2 |f(\theta)|^2 - |f(\theta)|^2 \|g_n(\theta)\|_{H_1}^2 d\theta \ge 0 \tag{3.9}$$

for all $f \in L^\infty(B^d, \mathbb{C}) \hookrightarrow L^2(B^d, \mathbb{C})$ and $n \in \mathbb{N}$. But equation (3.9) implies

$$\operatorname*{ess\,sup}_{\theta\in B^d} \|g_n(\theta)\|_{H_1} \le \|T\|\|h_n\|_{H_0} \quad \text{for all } n \in \mathbb{N}. \tag{3.10}$$

Indeed fix $n \in \mathbb{N}$ and set $A := \{\theta \in B^d \ : \ \|g_n(\theta)\|_{H_1} > \|T\|\|h_n\|_{H_0}\}$.

The set A is measurable and the assumption $\lambda(A) > 0$ implies $f := \mathbb{1}_A \neq 0$, which leads to

$$\int_{B^d} \|T\|^2 \|h_n\|_{H_0}^2 |f(\theta)|^2 - |f(\theta)|^2 \|g_n(\theta)\|_{H_1}^2 d\theta \\ = \int_A \|T\|^2 \|h_n\|_{H_0}^2 - \|g_n(\theta)\|_{H_1}^2 d\theta < 0, \tag{3.11}$$

a contradiction to (3.9), whereby (3.10) is verified.

By (3.10) we find for every $n \in \mathbb{N}$ subsets $\tilde{\Omega}_n \subset B^d$ of measure zero such that the inequality

$$\|g_n(\theta)\|_{H_1} \le \|T\|\|h_n\| \quad \text{for all } \theta \in B^d \setminus \tilde{\Omega}_n \tag{3.12}$$

is satisfied. Define $\tilde{\Omega} := \cup_{n\in\mathbb{N}}\tilde{\Omega}_n \cup \Omega$ then again $\lambda(\tilde{\Omega}) = 0$ and (3.6), (3.12) hold true for all $\theta \in B^d \setminus \tilde{\Omega}$ and $n \in \mathbb{N}$.

For $\theta \in B^d \setminus \tilde{\Omega}$ define an operator $m(\theta) : \{h_n : n \in \mathbb{N}\} \to H_1$ by $h_n \mapsto g_n(\theta)$. Now (3.7) and (3.12) yield that $m(\theta)$ is well defined on

$$\operatorname{span}\{h_n : n \in \mathbb{N}\} := \{\sum_{n=1}^{m} \lambda_n h_n : m \in \mathbb{N}, \lambda_n \in \mathbb{C}\}$$

linear and bounded. Since $\operatorname{span}\{h_n : n \in \mathbb{N}\}$ is a dense and linear subspace of H_0, we can extend $m(\theta)$ to a bounded linear operator $m(\theta) : H_0 \to H_1$ for all $\theta \in B^d \setminus \tilde{\Omega}$ with $\|m(\theta)\| \leq \|T\|$. Finally setting $m(\theta) := 0$ for $\theta \in \tilde{\Omega}$ yields a function $m \in L^\infty(B^d, \mathcal{B}_s(H_0, H_1))$.

Indeed it is sufficient to show that the mapping $\theta \mapsto m(\theta)h$ is measurable for every $h \in H_0$. Hence let $h \in H_0$ be arbitrary and take an approximating sequence $(h_{n(j)})_{j\in\mathbb{N}}$ from the dense subset $(h_n)_{n\in\mathbb{N}}$. Then $m(\cdot)h_{n(j)}$ converges point wise to $m(\cdot)h$. Moreover for $\theta \in B^d \setminus \tilde{\Omega}$ we have $m(\theta)h_{n(j)} = g_{n(j)}(\theta)$, which is a measurable function. This yields measurability of $\theta \mapsto m(\theta)h$ for all $h \in H_0$.

Finally we need to show (3.4). Let us take any finite sequence $\psi \in s(\mathbb{Z}^d, H_0)$ with values in $(h_n)_{n\in\mathbb{N}}$. Then $\mathcal{F}^{-1}\psi$ is a trigonometric polynomial of the form $\theta \mapsto \sum\limits_{|j|\leq k} e^{2\pi i n(j)\theta} h_{n(j)}$ and we obtain

$$\begin{aligned}
[\mathcal{F}^{-1}\psi] \cdot [\mathcal{F}^{-1}S] &= \sum_{|j|\leq k} e^{2\pi i n(j)(\cdot)} (\mathbb{1}_{B^d} h_{n(j)}) \cdot [\mathcal{F}^{-1}S] = \sum_{|j|\leq k} e^{2\pi i n(j)(\cdot)} g_{n(j)}(\cdot) \\
&= \sum_{|j|\leq k} e^{2\pi i n(j)(\cdot)} m(\cdot) h_{n(j)} = m(\cdot) \sum_{|j|\leq k} e^{2\pi i n(j)(\cdot)} h_{n(j)} \\
&= \mathcal{M}_m[\mathcal{F}^{-1}\psi].
\end{aligned}$$

Thus we obtain

$$T\psi = \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}\psi \tag{3.13}$$

for all finite sequences ψ with values in $(h_n)_{n\in\mathbb{N}}$. Equation (3.13) extends to $l^2(\mathbb{Z}^d, H_0)$ by denseness and continuity of $\mathcal{F}^{-1}$. The equality $\|T\| = \|\mathcal{M}_m\| = \|m\|_\infty$ follows again by Lemma 2.18 and Lemma 2.35 (vii).

□

Remark 3.13.

(i) Theorem 3.12 together with Lemma 3.4 shows, that the Zak Transform gives an isometric isomorphism $\mathcal{B}^{2,2}_{\mathbb{Z}^d}(H_0, H_1) \to L^\infty(B^d, \mathcal{B}_s(H_0, H_1))$. Since the Bloch Transform is an 'isometric' variation of the Zak transform this statement holds also true for Φ.

(ii) The special situation of having Hilbert spaces was exploited through Plancherel's Theorem, which allowed several estimates during the proof.

(iii) Separability of the spaces H_0, H_1 was used to find a representative of the functions $(\mathbb{1}h_n) \cdot [\mathcal{F}^{-1}G]$. If one skips separability of H_0 one may us a version of Lebesgue differentiation theorem for vector valued functions as this was successfully done in [Miko2, Theorem 3.1.3] for the Fourier Transform on the group $\mathbb{R}$.

Let us finally collect some algebraic properties of the space $\mathcal{M}^{2,2}(H_0, H_1)$

Corollary 3.14. *If $T, S \in \mathcal{M}^{2,2}(H_0, H_1)$, then also $T + S \in \mathcal{M}^{2,2}(H_0, H_1)$ and $m_T + m_S = m_{T+S}$. If $H_0 = H_1 = H$ and $T, S \in \mathcal{M}^{2,2}(H)$, then $T \circ S \in \mathcal{M}^{2,2}(H)$ with $m_T \circ m_S = m_{T\circ S}$. The multiplication function corresponding to $T = id_{l^2(\mathbb{Z}^d,H)}$ is given by $m_{id_{l^2(\mathbb{Z}^d,H)}}(\theta) = id_H$ for almost all $\theta \in B^d$.*

Proof. The conclusions are consequences of Remark 3.6. We only show the statement for $T \circ S$. If $f \in l^2(\mathbb{Z}^d, H)$, we get

$$\mathcal{M}_{m_{T\circ S}} \mathcal{F}^{-1} f = \mathcal{F}^{-1}[T \circ S f] = \mathcal{M}_{m_T} \mathcal{F}^{-1} S f = \mathcal{M}_{m_T} \circ \mathcal{M}_{m_S} \mathcal{F}^{-1} f.$$

Hence $\mathcal{M}_{m_{T\circ S}} = \mathcal{M}_{m_T} \circ \mathcal{M}_{m_S}$. Now, the point wise almost every where equality of $m_{T\circ S}$ and $m_T \circ m_S$ follows via Lemma 2.35 (vii). The claim for the sum and the identity may be derived in the same way. □

The spaces $\mathcal{M}^{p,p}(E_0, E_1)$

In this section we extend one part of Theorem 3.12 to reflexive[2] and separable Banach spaces E_0, E_1. Since we will use an interpolation argument assuming $p = q$ seems to be appropriate, because for $p \neq q$ either $p > q$ or $p' > q'$ and thus one of the spaces $\mathcal{M}^{p,q}(E_0, E_1)$, $\mathcal{M}^{q',p'}(E_1', E_0')$ is trivial (see the discussion on page 46). We note the well known fact, that if E is separable and reflexive also its dual E' has this properties (see [Con85, Ch.V-§5]). The idea of this section is, to deduce properties of an operator in $\mathcal{M}^{p,q}(E_0, E_1)$ from its 'scalar versions'. The next lemma gives a first idea what we mean by this.

Lemma 3.15. *Let E_0, E_1 be Banach spaces. Define for $T \in \mathcal{M}^{p,p}(E_0, E_1)$, $e \in E_0$ and $\epsilon \in E_1'$ an operator T_e^ϵ by*

$$\begin{aligned} T_e^\epsilon : l^p(\mathbb{Z}^d, \mathbb{C}) &\to l^p(\mathbb{Z}^d, \mathbb{C}) \\ \varphi &\mapsto (z \mapsto \epsilon[(T(\varphi e))(z)]) \,. \end{aligned}$$

Then $T_e^\epsilon \in \mathcal{M}^{p,p}(\mathbb{C})$.

Proof. For any $\varphi \in l^p(\mathbb{Z}^d, \mathbb{C})$ the function $\varphi \cdot e$ is an element of $l^p(\mathbb{Z}^d, E_0)$ and

$$\|T_e^\epsilon \varphi\|^p_{l^p(\mathbb{Z}^d,\mathbb{C})} = \sum_{z\in\mathbb{Z}^d} |\epsilon[T(\varphi e)(z)]|^p \leq \|\epsilon\|^p_{E'} \|T\|^p_{\mathcal{B}(l^p(\mathbb{Z}^d,E))} \|\varphi\|_{l^p(\mathbb{Z}^d,\mathbb{C})} \|e\|^p_E.$$

This shows that T_e^ϵ is well defined and bounded. To show translation invariance, choose any $z \in \mathbb{Z}^d$ and observe for $m \in \mathbb{Z}^d$

$$\begin{aligned} \left[T_e^\epsilon(\tau_z \varphi)\right](m) &= \epsilon[(T\tau_z \varphi e)(m)] = \epsilon[(\tau_z T\varphi e)(m)] = \epsilon[(T\varphi e)(m - z)] \\ &= \left[\tau_z T_e^\epsilon \varphi\right](m) \end{aligned}$$

which completes the proof. □

[2] In the following chapters we will always need, that the spaces E_0, E_1 are of class $\mathcal{HT}$ so that -at least there- assuming reflexivity is not a restriction (see Section 2.5). If one skips reflexivity similar results (with weaker) properties are valid for the Fourier Transform on the group $\mathbb{R}$ [Miko2, Theorem 3.2.4] proven by arguments which might transfer to the situation given here.

The next Lemma is in fact just an application of Lemma 3.10 combined with "M. Riesz's convexity Theorem" [SW71, Ch.V-§1 Thm.1.3] and might be well known. Nevertheless we include the short proof.

Lemma 3.16. *If $T \in \mathcal{M}^{p,p}(\mathbb{C})$, then $T \in \mathcal{M}^{2,2}(\mathbb{C})$. In particular, there is a function $m : B^d \to \mathbb{C}$ with $\|m\|_\infty = \|T\|_{\mathcal{B}(l^2(\mathbb{Z}^d,\mathbb{C}))} \leq \|T\|_{\mathcal{B}(l^p(\mathbb{Z}^d,\mathbb{C}))}$ and*

$$Tf = \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}f \text{ for all } f \in l^2(\mathbb{Z}^d,\mathbb{C}) \cap l^p(\mathbb{Z}^d,\mathbb{C}).$$

Proof. By Lemma 3.10, $T \in \mathcal{M}^{p',p'}(\mathbb{C})$ with $\|T\|_{\mathcal{B}(l^{p'}(\mathbb{Z}^d,\mathbb{C}))} = \|T\|_{\mathcal{B}(l^p(\mathbb{Z}^d,\mathbb{C}))}$. Hence by M. Riesz's convexity Theorem $T \in \mathcal{M}^{2,2}(\mathbb{C})$ with

$$\|T\|_{\mathcal{B}(l^2(\mathbb{Z}^d,\mathbb{C}))} \leq \|T\|_{\mathcal{B}(l^p(\mathbb{Z}^d,\mathbb{C}))}.$$

Now, Theorem 3.12 implies the existence of a function $m \in L^\infty(B^d,\mathbb{C})$ with the stated properties. □

Remark 3.17. *The conclusions of Corollary 3.14 stay true in the case above. In particular if $S, T \in \mathcal{M}^{p,p}(\mathbb{C})$, then $T \circ S, T + S \in \mathcal{M}^{p,p}(\mathbb{C})$ and $m_{T\circ S}(\theta) = m_T(x) \cdot m_S(\theta)$, $m_{T+S}(\theta) = m_T(\theta) + m_S(\theta)$ for almost all $\theta \in B^d$.*

Before we state the main result of this subsection, let us recall that a Banach space E is reflexive if and only if the canonical (isometric) embedding

$$\begin{aligned} J &: E \to E'' \\ &e \mapsto J(e) = e'' \text{ with } [Je](e') := e'(e) \quad \text{for all } e' \in E' \end{aligned}$$

maps E onto E'' one-to-one. For later use we set

$$\begin{aligned} \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E := \Big\{ & f \in l^2(\mathbb{Z}^d,\mathbb{C}) \cap l^p(\mathbb{Z}^d,\mathbb{C}) : \\ & \exists \varphi_j \in l^2(\mathbb{Z}^d,\mathbb{C}) \cap l^p(\mathbb{Z}^d,\mathbb{C}), e_j \in E, m \in \mathbb{N} \text{ s.t. } f = \sum_{j=1}^m \varphi_j e_j \Big\}, \end{aligned} \tag{3.14}$$

for any Banach space E. Note that $\left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$ is a dense and linear subspace of $l^p(\mathbb{Z}^d, E)$ for all $p \in [1,\infty)$.

Theorem 3.18. *Let E_0, E_1 be separable and reflexive Banach spaces and $p \in (1,\infty)$. If $T \in \mathcal{M}^{p,p}(E_0, E_1)$, then there is a function $m \in L^\infty(B^d, \mathcal{B}_s(E_0, E_1))$ such that*

$$Tf = \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}f \quad \text{for all } f \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E_0. \tag{3.15}$$

Moreover $\|m\|_\infty \leq \|T\|$.

Proof. The proof is divided into several steps. At first let us reduce the situation to the scalar case. For this reason we choose countable dense subsets of E_0 and E_1', which we denote by $(e_n)_{n\in\mathbb{N}} \subset E_0$ and $(\epsilon_n)_{n\in\mathbb{N}} \subset E_1'$. By Lemma 3.15 the operators $T_{e_i}^{\epsilon_j} : l^p(\mathbb{Z}^d) \to l^p(\mathbb{Z}^d)$ defined via

$$\left[T_{e_i}^{\epsilon_j} f\right](z) := \epsilon_j\big(\left[T(fe_i)\right](z)\big) \quad \text{for all } z \in \mathbb{Z}^d$$

are elements of $\mathcal{M}^{p,p}(\mathbb{C})$ for every pair $(i,j) \in \mathbb{N}^2$. Lemma 3.16 implies the existence of functions $m_{\epsilon_j,e_i} \in L^\infty(B^d,\mathbb{C})$ with

$$T_{e_i}^{\epsilon_j} f = \mathcal{F}\mathcal{M}_{m_{\epsilon_j,e_i}}\mathcal{F}^{-1} f \text{ for all } f \in l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)$$

and $\|m_{\epsilon_j,e_i}\|_\infty \leq \|T_{e_i}^{\epsilon_j}\| \leq \|T\|\|\epsilon_j\|\|e_i\|$. Consequently we find for every pair $(i,j) \in \mathbb{N}^2$ a subset $\Omega_{ij} \subset B^d$ of measure zero such that

$$|m_{\epsilon_j,e_i}(\theta)| \leq \|T\|\|\epsilon_j\|\|e_i\| \text{ for all } \theta \in B^d \setminus \Omega_{ij}.$$

But the set $\Omega := \cup_{(i,j)\in\mathbb{N}^2}\Omega_{ij}$ is again of measure zero and we obtain

$$|m_{\epsilon_j,e_i}(\theta)| \leq \|T\|\|\epsilon_j\|\|e_i\| \text{ for all } (i,j) \in \mathbb{N}^2 \text{ and } \theta \in B^d \setminus \Omega. \tag{3.16}$$

The second step extends the above observations to the dense and linear subspaces span$\{e_n : n \in \mathbb{N}\}$ and span$\{\epsilon_n : n \in \mathbb{N}\}$. At first observe that we have for $\mu \in \mathbb{C}$, $n, m, j \in \mathbb{N}$ and $f \in l^p(\mathbb{Z}^d)$ the equalities

$$\begin{aligned} T_{e_j}^{\mu\epsilon_n+\epsilon_m} f &= \mu T_{e_j}^{\epsilon_n} f + T_{e_j}^{\epsilon_m} f, \\ T_{\mu e_n+e_m}^{\epsilon_j} f &= \mu T_{e_m}^{\epsilon_j} f + T_{e_n}^{\epsilon_j} f, \end{aligned}$$

which follow by linearity of T. This yields

$$\begin{aligned} m_{\mu\epsilon_n+\epsilon_m,e_j} &= \mu m_{\epsilon_n,e_j} + m_{\epsilon_m,e_j}, \\ m_{\epsilon_j,\mu e_n+e_m} &= \mu m_{\epsilon_j,e_n} + m_{\epsilon_j,e_m} \end{aligned} \tag{3.17}$$

in $L^\infty(B^d)$. From (3.17) and the previous discussion we obtain for $\theta \in B^d \setminus \Omega$

$$\begin{aligned} m_{\mu\epsilon_n+\epsilon_m,e_j}(\theta) &= \mu m_{\epsilon_n,e_j}(\theta) + m_{\epsilon_m,e_j}(\theta), \\ m_{\epsilon_j,\mu e_n+e_m}(\theta) &= \mu m_{\epsilon_j,e_n}(\theta) + \mu_{\epsilon_j,e_m}(\theta). \end{aligned} \tag{3.18}$$

Hence for $\theta \in B^d \setminus \Omega$ the map

$$\begin{aligned} B(\theta) : \text{span}\{\epsilon_n : n \in \mathbb{N}\} \times \text{span}\{e_n : n \in \mathbb{N}\} &\to \mathbb{C} \\ (\epsilon, e) &\mapsto m_{\epsilon,e}(\theta) \end{aligned}$$

is bilinear and continuous with

$$|B(\theta)[\epsilon, e]| \leq \|T\|\|\epsilon\|\|e\|$$

where we notice, that (3.16) extends to span$\{e_n : n \in \mathbb{N}\}$ and span$\{\epsilon_n : n \in \mathbb{N}\}$.

The third step is devoted to the extension of $B(\theta)$ to an continuous bilinear map $E_0 \times E_1' \to \mathbb{C}$. For this reason pick some arbitrary $e \in E_0$, $\epsilon \in E_1'$ and sequences $(e_k)_{k\in\mathbb{N}} \subset$ span$\{e_n : n \in \mathbb{N}\}$, $(\epsilon_l)_{l\in\mathbb{N}} \subset$ span$\{\epsilon_n : n \in \mathbb{N}\}$ with $\lim_{k\to\infty} e_k = e$ and $\lim_{l\to\infty} \epsilon_l = \epsilon$. Then we have for $\theta \in B^d \setminus \Omega$ the estimate

$$\begin{aligned} \left|B(\theta)[\epsilon_l, e_k] - B(\theta)[\epsilon_{\tilde{l}}, e_{\tilde{k}}]\right| &\leq \left|B(\theta)[\epsilon_l - \epsilon_{\tilde{l}}, e_k]\right| + \left|B(\theta)[\epsilon_{\tilde{l}}, e_k - e_{\tilde{k}}]\right| \\ &\leq \|T\|\left(\|\epsilon_l - \epsilon_{\tilde{l}}\|\|e_k\| + \|\epsilon_{\tilde{l}}\|\|e_k - e_{\tilde{k}}\|\right). \end{aligned}$$

Since $(e_k)_{k\in\mathbb{N}}$ and $(\epsilon_l)_{l\in\mathbb{N}}$ are cauchy sequences, the right hand side tends to zero and we conclude, that the limit

$$\lim_{\substack{l\to\infty \\ k\to\infty}} B(\theta)[\epsilon_l, e_k] =: \overline{B}(\theta)[\epsilon, e] \tag{3.19}$$

exists. It is clear that for $\theta \in B^d \setminus \Omega$ the mapping $\overline{B}(\theta) : E_0 \times E_1' \to \mathbb{C}$ defined by (3.19) is continuous and bilinear with $\left|\overline{B}(\theta)[\epsilon, e]\right| \le \|T\|\|\epsilon\|\|e\|$. For $\theta \in \Omega$ we define $\overline{B}(\theta) := 0$.

Now, in the fourth step we finally construct a function $m : B^d \to \mathcal{B}(E_0, E_1)$ which will serve as a candidate for our assertion. The associated operator to the continuous bilinear form $\overline{B}(\theta) : E_0 \times E_1' \to \mathbb{C}$ is given by

$$\begin{aligned} M(\theta) : E_0 &\to E_1'' \\ e &\mapsto \left[\epsilon \mapsto \overline{B}(\theta)[e, \epsilon]\right]. \end{aligned}$$

For every $\theta \in B^d$ the operator $M(\theta)$ is linear and continuity of $\overline{B}(\theta)$ implies $\|M(\theta)\| \le \|T\|$. Indeed

$$\|M(\theta)\|_{\mathcal{B}(E_0,E_1'')} = \sup_{\|e\|_{E_0}=1} \|\overline{B}(\theta)(e,\cdot)\|_{E_1''} = \sup_{\|e\|_{E_0}=1} \sup_{\|\epsilon\|_{E_1'}=1} \left|\overline{B}(\theta)(e,\epsilon)\right| \le \|T\|.$$

Recall that E_1 is reflexive. Thus if we define for $e \in E_0$ and $\theta \in B^d$

$$m(\theta)e := J^{-1}\left[M(\theta)e\right],$$

we obtain a bounded, linear operator $m(\theta) : E_0 \to E_1$. Moreover for fixed e the function $\theta \mapsto m(\theta)e$ is bounded with $\|m(\theta)e\|_\infty \le \|T\|\|e\|$.

In the last step we need to verify that $m \in L^\infty(B^d, \mathcal{B}_s(E_0, E_1))$ and (3.15) holds for this function m. We start with measurability of the mapping $\theta \mapsto m(\theta)e$ for fixed but arbitrary $e \in E_0$. Note that by assumption and Theorem A.3 it is enough to show measurability of $\theta \mapsto \epsilon[m(\theta)e]$ for all $\epsilon \in E_1'$.

We have for $e \in \text{span}\{e_n : n \in \mathbb{N}\}$ and $\epsilon \in \text{span}\{\epsilon_n : n \in \mathbb{N}\}$ by definition[3]

$$\begin{aligned} \epsilon[m(\theta)e] &= \epsilon[J^{-1}(M(\theta)[e])] = \overline{B}(\theta)[\epsilon, e] = m_{\epsilon,e}(\theta) && \text{for } \theta \in B^d \setminus \Omega, \\ \epsilon[m(\theta)e] &= 0 && \text{for } \Omega. \end{aligned} \tag{3.20}$$

which is a measurable function by the first step. Hence an approximation as in the previous step gives the measurability of

$$\theta \mapsto \epsilon[m(\theta)e] = \lim_{\substack{i\to\infty \\ k\to\infty}} \varepsilon_j[m(\theta)e_i].$$

This shows $m \in L^\infty(B^d, \mathcal{B}_s(E_0, E_1))$ with $\|m\|_\infty \le \|T\|$.

[3] It worth to mention, that for $U \in E''$ and $\epsilon \in E'$ we have $\epsilon[J^{-1}U] = U(\epsilon)$. Indeed denote $V := J^{-1}U$, then $JV \in E''$ is given by $(JV)[\epsilon] = \epsilon[V]$. Hence $U(\epsilon) = (JV)[\epsilon] = \epsilon[V] = \epsilon[J^{-1}U]$.

The last step now concerns (3.15). For a special sequence f of the form $f = g \cdot e_j$ with $g \in l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)$ and e_j from the dense subset of E, we may deduce from Lemma 3.16, (3.20), Lemma 2.17 combined with Remark 2.19 for any ϵ_n in the dense subset of E_1'

$$\begin{aligned}\epsilon_n[Tf] = \epsilon_n[Tge_j] = T_{e_j}^{\epsilon_n} g = \mathcal{F}\mathcal{M}_{m_{\epsilon_n,e_j}}\mathcal{F}^{-1}g = \mathcal{F}\epsilon_n\mathcal{M}_m e_j\mathcal{F}^{-1}g \\ = \epsilon_n[\mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}ge_j]. \qquad (3.21)\end{aligned}$$

Since $\{\epsilon_n : n \in \mathbb{N}\} \subset E_1'$ is dense (3.21) leads to

$$Tf = \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}f$$

by an application of the Hahn-Banach Theorem.

Approximating $e \in E_0$ by a sequence $(e_{n(j)})_{j\in\mathbb{N}}$ finally gives (3.15) for all functions $f = g \cdot e$ where $g \in l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)$ and $e \in E_0$. Since an arbitrary $f \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E_0$ is a finite linear combination of functions with the form above (3.15) follows by linearity of $\mathcal{F}, \mathcal{F}^{-1}$ and T. □

As before we close this subsection with the relevant algebraic properties of $\mathcal{M}^{p,p}(E_0, E_1)$, which are analogues of Corollary 3.14.

Remark 3.19. *Let E_0, E_1 be a separable and reflexive Banach spaces and $p \in (1, \infty)$.*

(i) If $T, S \in \mathcal{M}^{p,p}(E_0, E_1)$, then $T + S \in \mathcal{M}^{p,p}(E_0, E_1)$. If $E_0 = E_1 = E$ and $T, S \in \mathcal{M}^{p,p}(E)$ the also $T \circ S \in \mathcal{M}^{p,p}(E)$. We have for the corresponding multiplication functions the identities

$$m_T(\theta) \circ m_S(\theta) = m_{T\circ S}(\theta) \qquad \text{and} \qquad m_{T+S}(\theta) = m_T(\theta) + m_S(\theta)$$

for almost all $\theta \in B^d$.

(ii) The operator $id_{l^p(\mathbb{Z}^d,E)}$ is an element of $\mathcal{M}^{p,p}(E)$ with $m_{id_{l^p(\mathbb{Z}^d,E)}}(\theta) = id_E$ for almost all $\theta \in B^d$.

Proof. (i) The statement concerning the sum of operators is clear, so we only care about the composition. The translation invariance of $T \circ S$ is immediate. If we can show, that the representation

$$Sf = \mathcal{F}\mathcal{M}_{m_s}\mathcal{F}^{-1}f \qquad (3.22)$$

extends to all $f \in l^p(\mathbb{Z}^d, E)$ such that $\mathcal{F}^{-1}f \in L^1(B^d, E)$ we would obtain for $\varphi \in s(\mathbb{Z}^d, E)$

$$\mathcal{F}\mathcal{M}_{m_{T\circ S}}\mathcal{F}^{-1}\varphi = (T \circ S)(\varphi) = T\mathcal{F}\mathcal{M}_{m_s}\mathcal{F}^{-1}\varphi = \mathcal{F}\mathcal{M}_{m_T} \circ \mathcal{M}_{m_S}\mathcal{F}^{-1}\varphi$$

and the statement follows by Lemma 2.35 (vii).

So, let us assume $f \in l^p(\mathbb{Z}^d, E)$ with $\mathcal{F}^{-1}f \in L^1(B^d, E)$. Then there is a sequence $(f_n)_{n\in\mathbb{N}}$ in the dense subset $[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)] \otimes E$ with $f_n \to f$. Because of $\mathcal{F}^{-1}f_n \in L^2(B^d) \otimes E \hookrightarrow L^1(B^d, E)$ we obtain

$$\|\mathcal{F}^{-1}f_n - \mathcal{F}^{-1}f\|_{L^1(B^d,E)} \leq \|f_n - f\|_{l^\infty(\mathbb{Z}^d,E)} \leq \|f_n - f\|_{l^p(\mathbb{Z}^d,E)} \to 0.$$

Thus we find a $C > 0$ and a sub-sequence (again denoted by f_n such that $\|\mathcal{F}^{-1}f_n(\theta)\|_E \leq C$ and $\mathcal{F}^{-1}f_n(\theta) \to \mathcal{F}^{-1}f(\theta)$ for almost all $\theta \in B^d$. Now Proposition A.6 implies for any $z \in \mathbb{Z}^d$

$$\begin{aligned} Sf(z) \leftarrow Sf_n(z) &= \int_{B^d} e^{-2\pi i\theta z} m_s(\theta)\mathcal{F}^{-1}f_n(\theta)d\theta \\ &\to \int_{B^d} e^{-2\pi i\theta z} m_s(\theta)\mathcal{F}^{-1}f(\theta)d\theta. \end{aligned}$$

Thus (3.22) holds for all $f \in l^p(\mathbb{Z}^d, E)$ with $\mathcal{F}^{-1}f \in L^1(B^d, E)$.

(ii) In this case the operators $T_{e_i}^{\epsilon_j}$ defined in the proof Theorem 3.18 are given by $\epsilon_j(e_i)\mathrm{id}_{l^p(\mathbb{Z}^d)}$. The corresponding multiplication operators are $\epsilon_j(e_i)\mathbb{1}_{\mathrm{id}_E}$. Hence the functions m_{ϵ_j,e_i} are equal to $\epsilon_j(e_i)\mathbb{1}_{\mathrm{id}_E}$ almost every where. This yields that the bilinear form $\overline{B}(\theta)$ is given by $\overline{B}(\theta)[\epsilon, e] = \epsilon(e)$ for almost all $\theta \in B^d$. Hence $m(\theta)e = e$ for almost all $\theta \in B^d$ and all $e \in E$.

□

3.3 Unbounded Periodic Operators - Reduction to Translation Invariant Operators on Sequence Spaces

Before we can start with the study of unbounded operators, we have to extend the notions of periodicity and translation invariance to this situation in a way that is consisted to the previous definitions. Once this is done we will reduce our considerations to sequence spaces, as this gave a suitable simplification in the case of bounded operators.

Definition 3.20. *Let $(A, D(A)) : L^p(\mathbb{R}^d, E_0) \to L^q(\mathbb{R}^d, E_1)$ be a linear operator with domain $D(A)$. If $\mathcal{P} \subset \mathbb{R}^d$ is a lattice then A is called periodic with respect to $\mathcal{P}$ if for all $p \in \mathcal{P}$*

(a) $\tau_p D(A) = \{\tau_p f : f \in D(A)\} \subset D(A)$,

(b) $\tau_p Af = A\tau_p f$ for all $f \in D(A)$.

A linear operator $(B, D(B)) : l^p(\mathbb{Z}^d, E_0) \to l^q(\mathbb{Z}^d, E_1)$ with domain $D(B)$ is called translation invariant if for all $z \in \mathbb{Z}^d$

(a) $\tau_z D(B) = \{\tau_z f : f \in D(B)\} \subset D(B)$,

(b) $\tau_z Bf = B\tau_z f$ for all $f \in D(B)$.

It is possible to transfer all preparatory tools we derived in the bounded case, to the unbounded case. The crucial ingredient is the following Lemma whose proof is a consequence of the definitions given in Chapter 2.

Lemma 3.21. *Let E_0, E_1 be Banach spaces, $B : E_0 \to E_1$ be linear, bounded and bijective. Further let $(A, D(A)) : E_0 \to E_0$ be a linear operator. Define an operator $\mathbb{A} : E_1 \to E_1$ by*

$$\begin{aligned} D(\mathbb{A}) &:= \{x \in E_1 : \exists y \in D(A),\ By = x\}, \\ \mathbb{A}x &:= BAB^{-1}x \ \text{ for } x \in D(\mathbb{A}). \end{aligned}$$

Then we have

(a) $\mathbb{A}$ is bounded $\Leftrightarrow$ A is bounded,

(b) $\mathbb{A}$ is densely defined $\Leftrightarrow$ A is densely defined,

(c) $\mathbb{A}$ is closed $\Leftrightarrow$ A is closed,

(d) $\lambda \in \rho(\mathbb{A}) \Leftrightarrow \lambda \in \rho(A)$,

(e) $\mathbb{A}$ is the generator of a C_0-semigroup $\Leftrightarrow$ A is the generator of a C_0-semigroup,

(f) $\mathbb{A}$ is (pseudo)-sectorial of angle $\omega_A \Leftrightarrow A$ is (pseudo)-sectorial of angle ω_A,

(g) $\mathbb{A}$ has bounded $\mathcal{H}^\infty(\Sigma_\varphi)$-calculus $\Leftrightarrow$ A has bounded $\mathcal{H}^\infty(\Sigma_\varphi)$-calculus.

Recall that the dilatation δ_ξ defined in Section 3.1 is a bounded and linear operator on $L^p(\mathbb{R}^d, E_0)$ for every $p \in [1, \infty]$. δ_ξ is bounded invertible with inverse $\delta_\xi^{-1} = \delta_{1/\xi}$ if and only if all components of ξ are non-zero. We collect the analogues concerning the reduction to the lattice $\mathbb{Z}^d$ and to translation invariant operators in the next two lemmas, whose proofs are now consequences of Lemma 3.21.

Lemma 3.22. *Let $\mathcal{P} \subset \mathbb{R}^d$ be a lattice with lattice vector $\mathfrak{p}$. Assume that $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ is a linear operator that periodic with respect to $\mathcal{P}$. Define an operator $(\mathbb{A}, D(\mathbb{A})) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ by*

$$\begin{aligned} D(\mathbb{A}) &:= \{g \in L^p(\mathbb{R}^d, E) : \exists h \in D(A), g = \delta_{1/\mathfrak{p}} h\}, \\ \mathbb{A}g &:= \delta_{1/\mathfrak{p}} A \delta_{\mathfrak{p}} g. \end{aligned}$$

Then $\mathbb{A}$ is linear and periodic with respect to $\mathbb{Z}^d$. Moreover properties of A mentioned in Lemma 3.21 are equivalent to the corresponding properties of $\mathbb{A}$.

As in the case of bounded operators we restrict our attention to the lattice $\mathbb{Z}^d$ and call unbounded operators periodic if they are periodic with respect to $\mathbb{Z}^d$. The mapping Γ from Section 2.2 again relates unbounded periodic operators $L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ and unbounded translation invariant operators $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E)) \to l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$.

Lemma 3.23. *Assume $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ is a linear operator that is periodic (with respect to $\mathbb{Z}^d$). Define*

$$D(\mathbf{A}) := \Gamma D(A) = \{g \in l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E)) : \exists h \in D(A), g = \Gamma h\},$$
$$\mathbf{A}g := \Gamma A \Gamma^{-1} g.$$

Then $\mathbf{A}$ is translation invariant. Moreover properties of A mentioned in Lemma 3.21 are equivalent to the corresponding properties of $\mathbf{A}$.

Again Lemma 3.23 is reason enough to restrict our attention to unbounded translation invariant operators $l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$.

3.4 Unbounded Translation Invariant Operators on $l^p(\mathbb{Z}^d, E)$

As mentioned before, the idea for the treatment of unbounded operators is, to apply the previous results to the resolvent operators. The next Lemma allows for such an approach.

Lemma 3.24. *Let $(A, D(A)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ be a closed and translation invariant operator with $\rho(A) \neq \emptyset$. Then for $\lambda \in \rho(A)$ the resolvent operator $R(\lambda, A)$ is an element of $\mathcal{M}^{p,p}(E)$, i.e. $R(\lambda, A)$ is translation invariant.*

Proof. For $f \in D(A)$ and $z \in \mathbb{Z}^d$ we have $\tau_z(\lambda - A)f = (\lambda - A)\tau_z f$. Hence $R(\lambda, A)\tau_z(\lambda - A)f = R(\lambda, A)(\lambda - A)\tau_z f = \tau_z f = \tau_z R(\lambda, A)(\lambda - A)f$. Since $(\lambda - A)$ is bijective we obtain $R(\lambda, A)\tau_z g = \tau_z R(\lambda, A)g$ for all $g \in l^p(\mathbb{Z}^d, E)$. □

Remark 3.25. *A similar result is true for periodic operators with essentially the same proof. Let $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ be a closed and periodic operator with $\rho(A) \neq \emptyset$. Then for $\lambda \in \rho(A)$ the operator $R(\lambda, A)$ is an element of $\mathcal{B}^{p,p}_{\mathbb{Z}^d}(E)$.*

Following the procedure preformed in the bounded case we study unbounded translation invariant operators in the Hilbert space setting first. Here we will again use Plancherel's Theorem (Lemma 2.18). Since the aim is to characterize unbounded translation invariant operators in terms of multiplication operators this multiplication operator has to be unbounded as well. Our approach will involve the bounded case and therefore we have to assume that the resolvent set is nonempty. Having generators of C_0-semigroups in mind this is not a restriction as seen by Theorem 2.41.

Theorem 3.26. *Let H be a separable Hilbert space, $(A, D(A))$ a linear, closed and translation invariant operator on $l^2(\mathbb{Z}^d, H)$ such that there is an unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \rho(A)$ with $\lim_{k\to\infty} \lambda_k R(\lambda_k, A)h = h$ for all $h \in H$. Define an operator $(\mathcal{A}, D(\mathcal{A})) : L^2(B^d, H) \to L^2(B^d, H)$ by*

$$D(\mathcal{A}) := \mathcal{F}D(A) := \{g \in L^2(B^d, H) : \exists\, h \in D(A) \text{ s.t. } g = \mathcal{F}^{-1}h\},$$
$$\mathcal{A}g := \mathcal{F}^{-1}A\mathcal{F}g \ \text{ for } g \in D(\mathcal{A}).$$

Then $(\mathcal{A}, D(\mathcal{A}))$ is a closed multiplication operator, i.e there is a family of linear fiber operators $(A(\theta), D(A(\theta))) : H \to H$ which are closed for almost all $\theta \in B^d$ and

$$g(\theta) \in D(A(\theta)),$$
$$[\mathcal{A}g](\theta) = A(\theta)g(\theta)$$

for all $g \in D(\mathcal{A})$ and almost all $\theta \in B^d$. In addition there is a subset $\Omega \subset B^d$ of measure zero such that

$$\rho(A) = \rho(\mathcal{A}) \subset \bigcap_{\theta \in B^d \setminus \Omega} \rho(A(\theta)).$$

Proof. By Lemma 3.24 we have $R(\lambda, A) \in \mathcal{M}^{2,2}(H)$ for all $\lambda \in \rho(A)$. Hence Theorem 3.12 gives a function $m_\lambda \in L^\infty(B^d, \mathcal{B}_s(H))$ with

$$R(\lambda, A)g = \mathcal{F}\mathcal{M}_{m_\lambda}\mathcal{F}^{-1}g \ \text{ for all } g \in l^2(\mathbb{Z}^d, H)$$

and $\|R(\lambda, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d,H))} = \|\mathcal{M}_{m_\lambda}\|_{\mathcal{B}(L^2(\mathbb{I}^d,H))} = \|m_\lambda\|_\infty$.

We want to show that there is a set $\Omega_1 \subset B^d$ of measure zero such that for $\theta \in B^d \setminus \Omega_1$ the family $(m_\lambda(\theta))_{\lambda \in \rho(A)} \subset \mathcal{B}(H)$ is a pseudo resolvent[4].

We first observe that the resolvent equation

$$\big[R(\lambda, A) - R(\mu, A)\big]f = (\mu - \lambda)R(\lambda, A)R(\mu, A)f$$

transfers to the multiplication operators (see Corollary 3.14), i.e. we have for $\lambda, \mu \in \rho(A)$ and $f \in l^2(\mathbb{Z}^d, E)$

$$\big[\mathcal{M}_{m_\lambda} - \mathcal{M}_{m_\mu}\big]\mathcal{F}^{-1}f = (\mu - \lambda)\mathcal{M}_{m_\lambda}\mathcal{M}_{m_\mu}\mathcal{F}^{-1}f. \tag{3.23}$$

Further the properties of $\mathcal{F}$ and $\mathcal{F}^{-1}$ in this situation yield $\ker(R(\lambda, A)) = \ker(\mathcal{M}_{m_\lambda}) = 0$, and

$$L^2(B^d, H) = \overline{\mathcal{F}\mathrm{rg}(R(\lambda, A))}^{\|\cdot\|_{L^2(B^d,H)}} = \overline{\mathrm{rg}(\mathcal{M}_{m_\lambda})}^{\|\cdot\|_{L^2(B^d,H)}}.$$

Thus the family $(\mathcal{M}_{m_\lambda})_{\lambda \in \rho(A)} \subset \mathcal{B}(L^2(B^d, H))$ fulfills all assumptions of Theorem 2.29. Hence there is a unique, closed, densely defined and linear operator

$$(\mathcal{A}_1, D(\mathcal{A}_1)) : L^2(B^d, H) \to L^2(B^d, H),$$

with $\rho(A) \subset \rho(\mathcal{A}_1)$ and $R(\lambda, \mathcal{A}_1) = \mathcal{M}_{m_\lambda}$ for all $\lambda \in \rho(A)$.

We now show that this operator is the one given in the statement of the theorem.

First we show $\mathcal{A}_1 = \mathcal{A}$ followed by the proof that $\mathcal{A}_1$ is a multiplication operator with fiber operators $A(\theta)$.

Let $\lambda \in \rho(A) \subset \rho(\mathcal{A}_1)$ and $g \in D(\mathcal{A}_1)$. We find a function $f \in L^2(B^d, H)$ such that $g = R(\lambda, \mathcal{A}_1)f = \mathcal{M}_{m_\lambda}f$. This yields $\mathcal{F}g = \mathcal{F}\mathcal{M}_{m_\lambda}f = R(\lambda, A)\mathcal{F}f$,

[4] Recall this notion and the corresponding results from Section 2.3.

and hence $g = \mathcal{F}^{-1}h$ with $h := R(\lambda, A)\mathcal{F}f \in D(A)$. Thus $D(\mathcal{A}_1) \subset D(\mathcal{A})$. Moreover we have

$$\begin{aligned}\mathcal{A}_1 g &= \mathcal{A}_1 R(\lambda, \mathcal{A}_1)f = \lambda R(\lambda, \mathcal{A}_1)f - f \\ &= \lambda \mathcal{M}_{m_\lambda} f - f = \mathcal{F}^{-1}[\lambda R(\lambda, A) - \mathrm{id}_{l^2(\mathbb{Z}^d, H)}]\mathcal{F}f = \mathcal{F}^{-1}AR(\lambda, A)\mathcal{F}f \\ &= \mathcal{F}^{-1}Ah = \mathcal{F}^{-1}A\mathcal{F}g = \mathcal{A}g.\end{aligned}$$

Hence $\mathcal{A}_1 \subset \mathcal{A}$. By Lemma 3.21 (d) we have $\rho(A) = \rho(\mathcal{A})$ which yields by the above observations $\rho(\mathcal{A}_1) \cap \rho(\mathcal{A}) \neq \emptyset$ and $\mathcal{A}_1 = \mathcal{A}$ follows via Lemma 2.27.

Our next goal is to show, that $\mathcal{A}$ is an unbounded multiplication operator with fiber operators $(A(\theta), D(A(\theta))) : H \to H$.

Equation (3.23) states that for every $g \in L^2(B^d, H)$ and $\lambda, \mu \in \rho(A)$ there is a set $\Omega_{g,\lambda,\mu}$ of measure zero such that

$$[m_\lambda(\theta) - m_\mu(\theta)]g(\theta) = (\mu - \lambda)[m_\lambda(\theta) \circ m_\mu(\theta)]g(\theta) \tag{3.24}$$

for all $\theta \in B^d \setminus \Omega_{g,\lambda,\mu}$.

Thanks to the assumption that H is separable we can find a countable dense subset of H denoted by $(h_n)_{n\in\mathbb{N}}$. Define functions $g_n : B^d \to H$ by

$$g_n(\theta) := \mathbb{1}_{\mathbb{I}^d}(\theta)h_n.$$

Clearly $g_n \in L^2(B^d, H)$. Denote by $\overline{\Omega}_{\lambda,\mu} := \cup_{n\in\mathbb{N}}\Omega_{g_n,\lambda,\mu}$ the union of all sets of exception in (3.24), where g is replaced by g_n. Then $\overline{\Omega}_{\lambda,\mu}$ is of measure zero and we get for $\theta \in B^d \setminus \overline{\Omega}_{\lambda,\mu}$ and $n \in \mathbb{N}$

$$\begin{aligned}[m_\lambda(\theta) - m_\mu(\theta)]h_n &= [m_\lambda(\theta) - m_\mu(\theta)]g_n(\theta) = (\mu - \lambda)m_\lambda(\theta) \circ m_\mu(\theta)g_n(\theta) \\ &= (\mu - \lambda)m_\lambda(\theta) \circ m_\mu(\theta)h_n.\end{aligned}$$

Further there is another set $\tilde{\Omega}_{\lambda,\mu}$ of measure zero such that both, $m_\lambda(\theta)$ and $m_\mu(\theta)$ are elements of $\mathcal{B}(H)$ with

$$\|m_\lambda(\theta)\|_{\mathcal{B}(H)} \leq \|R(\lambda, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))} \quad \text{and} \quad \|m_\mu(\theta)\|_{\mathcal{B}(H)} \leq \|R(\mu, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))}$$

for all $\theta \in B^d \setminus \tilde{\Omega}_{\lambda,\mu}$ (compare Lemma 2.35 (ii)). This leads to

$$\begin{aligned}[m_\lambda(\theta) - m_\mu(\theta)]h_n &= (\mu - \lambda)m_\mu(\theta) \circ m_\lambda(\theta)h_n, \\ \|m_\lambda(\theta)\|_{\mathcal{B}(H)} &\leq \|R(\lambda, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))}, \\ \|m_\mu(\theta)\|_{\mathcal{B}(H)} &\leq \|R(\mu, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))}\end{aligned} \tag{3.25}$$

for $\theta \in B^d \setminus (\overline{\Omega}_{\lambda,\mu} \cup \tilde{\Omega}_{\lambda,\mu})$ and all h_n. The set $\Omega_{\lambda,\mu} := \overline{\Omega}_{\lambda,\mu} \cup \tilde{\Omega}_{\lambda,\mu}$ is of measure zero and we may extend the first equation above to all of H by unique bounded extension. Hence

$$[m_\lambda(\theta) - m_\mu(\theta)]h = (\mu - \lambda)m_\mu(\theta) \circ m_\lambda(\theta)h \tag{3.26}$$

for all $\theta \in B^d \setminus \Omega_{\lambda,\mu}$ and $h \in H$.

So far all considerations where for fixed $\lambda, \mu \in \rho(A)$ and the corresponding statements involved a set of exception depending on λ and μ. Within the next step we will overcome this problem.

First we consider $\lambda, \mu \in \rho(A) \cap (\mathbb{Q} + i\mathbb{Q}) =: \rho_D(A)$ and define another set of measure zero by $\Omega_1 := \cup_{\lambda,\mu \in \rho_D(A)} \Omega_{\lambda,\mu}$, where $\Omega_{\lambda,\mu}$ are the sets of exception in (3.25) and (3.26). Then Ω_1 is of measure zero and

$$\begin{aligned} \left[m_\lambda(\theta) - m_\mu(\theta)\right] h &= (\mu - \lambda) m_\mu(\theta) \circ m_\lambda(\theta) h \\ \|m_\lambda(\theta)\|_{\mathcal{B}(H)} &\leq \|R(\lambda, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))} \\ \|m_\mu(\theta)\|_{\mathcal{B}(H)} &\leq \|R(\mu, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))} \end{aligned} \tag{3.27}$$

for all $\lambda, \mu \in \rho_D(A)$, $\theta \in B^d \setminus \Omega_1$ and $h \in H$.

Now pick any $\lambda \in \rho(A) \setminus \rho_D(A)$ and any sequence $(\lambda_j)_{j \in \mathbb{N}} \subset \rho_D(A)$ with $\lambda_j \to \lambda$ as $j \to \infty$. Then, by continuity of the mapping $\lambda \mapsto R(\lambda, A)$, we clearly obtain $R(\lambda_j, A) \to R(\lambda, A)$ in $\mathcal{B}(l^2(\mathbb{Z}^d, H))$ as $j \to \infty$. Furthermore if $\theta \in B^d \setminus \Omega_1$ equations (3.27) yield

$$\begin{aligned} \|m_{\lambda_j}(\theta) - m_{\lambda_l}(\theta)\|_{\mathcal{B}(H)} &= |\lambda_j - \lambda_l| \|m_{\lambda_j}(\theta) \circ m_{\lambda_l}(\theta)\|_{\mathcal{B}(H)} \\ &\leq |\lambda_j - \lambda_l| \|R(\lambda_j, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))} \|R(\lambda_l, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d, H))} \end{aligned} \tag{3.28}$$

and the right hand side tends to zero as $j, l \to \infty$. This gives rise to the definition

$$\begin{aligned} \overline{m}_\lambda(\theta) &:= \lim_{j \to \infty} m_{\lambda_j}(\theta) && \text{for } \theta \in B^d \setminus \Omega_1, \\ \overline{m}_\lambda(\theta) &:= 0 && \text{for } \theta \in \Omega_1. \end{aligned}$$

We now claim that the function $\overline{m}_\lambda$ is in $L^\infty(B^d, \mathcal{B}_s(H))$ and the associated multiplication operator $\mathcal{M}_{\overline{m}_\lambda}$ coincides with the one obtained at the very beginning of this proof, i.e. $\mathcal{M}_{\overline{m}_\lambda} = \mathcal{M}_{m_\lambda}$ for all $\lambda \in \rho(A)$.

Clearly $\theta \mapsto \overline{m}_\lambda(\theta) h$ is measurable as a point wise almost everywhere limit of the measurable functions $\theta \mapsto m_{\lambda_j}(\theta) h$. Moreover

$$\|\overline{m}_\lambda(\theta) h\|_H \leq \|\overline{m}_\lambda(\theta) - m_{\lambda_j}(\theta)\| \|h\| + \|m_{\lambda_j}(\theta)\| \|h\| \leq C \|h\| < \infty$$

thanks to (3.28). Thus $\overline{m}_\lambda \in L^\infty(B^d, \mathcal{B}_s(H))$ and it remains to show the equality $\mathcal{M}_{\overline{m}_\lambda} = \mathcal{M}_{m_\lambda}$.

For this let $g \in L^2(B^d, H)$, $\lambda \in \rho(A) \setminus \rho_D(A)$ be given and choose an approximating sequence $(\lambda_j)_{j \in \mathbb{N}} \subset \rho_D(A)$ with $\lambda_j \to \lambda$ for $j \to \infty$. We start with the inequality

$$\begin{aligned} \|(\mathcal{M}_{\overline{m}_\lambda} - \mathcal{M}_{m_\lambda}) g\|_{L^2(B^d, H)} &\leq \|(\mathcal{M}_{\overline{m}_\lambda} - \mathcal{M}_{m_{\lambda_j}}) g\|_{L^2(B^d, H)} \\ &\quad + \|(\mathcal{M}_{m_{\lambda_j}} - \mathcal{M}_{m_\lambda}) g\|_{L^2(B^d, H)}. \end{aligned} \tag{3.29}$$

The second term tends to zero by continuity of the map $\lambda \mapsto R(\lambda, A)$ and Lemma 2.18. Indeed

$$\|(\mathcal{M}_{m_{\lambda_j}} - \mathcal{M}_{m_\lambda})g\|_{L^2(B^d,H)} = |\lambda - \lambda_j| \|R(\lambda_j, A)R(\lambda, A)\mathcal{F}g\|_{l^2(\mathbb{Z}^d,H)} \overset{j\to\infty}{\longrightarrow} 0.$$

For the first term we observe, that $m_{\lambda_j} \to \overline{m}_\lambda$ point wise almost everywhere (by construction) and $\operatorname{ess\,sup}_{\theta\in B^d} \|\overline{m}_\lambda(\theta) - m_{\lambda_j}(\theta)\| < C$ by (3.28). Thus, we may apply the theorem of dominated convergence (Proposition A.6) to obtain that also the first term on the right hand side in (3.29) tends to zero as j goes to infinity. This yields $\mathcal{M}_{\overline{m}_\lambda} g = \mathcal{M}_{m_\lambda} g$ for all $g \in L^2(B^d, H)$. Finally, we define for $\lambda \in \rho_D(A)$

$$\begin{aligned} \overline{m}_\lambda(\theta) &:= m_\lambda(\theta) && \text{for } \theta \in B^d \setminus \Omega_1, \\ \overline{m}_\lambda(\theta) &:= 0 && \text{for } \theta \in \Omega_1. \end{aligned}$$

Now let us verify that for $\theta \in B^d \setminus \Omega$ the family $(\overline{m}_\lambda(\theta))_{\lambda\in\rho(A)}$ is a pseudo resolvent on H, where Ω is a set of measure zero containing Ω_1. So far we have shown that for $\theta \in B^d \setminus \Omega_1$ the resolvent equality is valid. Our aim is to apply Theorem 2.31. Hence we have to show the equality

$$\lim_{k\to\infty} \lambda_k m_{\lambda_k}(\theta)h = h \text{ for all } h \in H\ ,\ \theta \in B^d \setminus \Omega,$$

where $(\lambda_k)_{k\in\mathbb{N}}$ is the unbounded sequence in $\rho(A)$ given by the assumptions of the theorem. By assumption we have $\lim_{k\to\infty} \lambda_k R(\lambda_k, A)f = f$ for all $f \in l^2(\mathbb{Z}^d, H)$, which implies $\sup_{k\in\mathbb{N}} \lambda_k R(\lambda_k, A)f < \infty$. In result, the principle of uniform boundedness gives a finite constant $C > 0$ with

$$\sup_{k\in\mathbb{N}} \|\lambda_k R(\lambda_k, A)\| \le C. \tag{3.30}$$

Further Theorem 3.12 yields

$$\begin{aligned} \|\lambda_k R(\lambda_k, A)\|_{\mathcal{B}(l^2(\mathbb{Z}^d,H))} &= \|\lambda_k \mathcal{M}_{m_{\lambda_k}}\|_{\mathcal{B}(L^2(\mathbb{I}^d,H))} = \|\lambda_k \mathcal{M}_{\overline{m}_{\lambda_k}}\|_{\mathcal{B}(L^2(\mathbb{I}^d,H))} \\ &= \sup_{\theta\in B^d\setminus\Omega_1} \|\lambda_k \overline{m}_{\lambda_k}(\theta)\|_{\mathcal{B}(H)}. \end{aligned}$$

Hence $\|\lambda_k \overline{m}_{\lambda_k}(\theta)\|_{\mathcal{B}(H)} \le C$, for $\theta \in B^d \setminus \Omega_1$, $k \in \mathbb{N}$, where $C < \infty$ is as in (3.30). Further Lemma 2.18 shows,

$$\begin{aligned} \|\lambda_k R(\lambda_k, A)f - f\|_{l^2(\mathbb{Z}^d,H)} &= \|\mathcal{F}^{-1}[\lambda_k R(\lambda_k, A)f - f]\|_{L^2(\mathbb{I}^d,H)} \\ &= \|\lambda_k \mathcal{M}_{m_{\lambda_k}} \mathcal{F}^{-1}f - \mathcal{F}^{-1}f\|_{L^2(\mathbb{I}^d,H)} \\ &= \|\lambda_k \mathcal{M}_{\overline{m}_{\lambda_k}} \mathcal{F}^{-1}f - \mathcal{F}^{-1}f\|_{L^2(\mathbb{I}^d,H)}. \end{aligned}$$

The left hand side of the equation above converges to zero as k tends to zero. Hence there is a sub-sequence of $(\lambda_k)_{k\in\mathbb{N}}$ (again denoted by $(\lambda_k)_{k\in\mathbb{N}}$) and another set of measure zero, (depending on f and denoted by Ω_f) such that for all θ not contained in Ω_f

$$\|\overline{m}_{\lambda_k}(\theta)[\mathcal{F}^{-1}f](\theta) - [\mathcal{F}^{-1}f](\theta)\|_H \overset{k\to\infty}{\longrightarrow} 0. \tag{3.31}$$

Recall the definition of the functions g_n and denote by Ω_2 the union of all sets of exception such that (3.31) holds with $\mathcal{F}^{-1}f$ replaced by g_n.

Finally set $\Omega := \Omega_1 \cup \Omega_2$. Then Ω is of measure zero and we have for $\theta \in B^d \setminus \Omega$

$$\|\lambda_k \overline{m}_{\lambda_k}(\theta)h_n - h_n\|_H \overset{k\to\infty}{\longrightarrow} 0. \tag{3.32}$$

If $h \in H$ is arbitrary we approximate with elements of the dense subset $(h_n)_{n\in\mathbb{N}}$ which yields for $\theta \in B^d \setminus \Omega$

$$\|\lambda_k \overline{m}_{\lambda_k}(\theta)h - h\|_H \le \|\lambda_k \overline{m}_{\lambda_k}(\theta)h - \lambda_k \overline{m}_{\lambda_k}(\theta)h_{n_j}\|_H + \|\lambda_k \overline{m}_{\lambda_k}(\theta)h_{n_j} - h_{n_j}\|_H + \|h_{n_j} - h\|_H.$$

The first term on the right hand side can be estimated by $C\|h_{n_j} - h\|_H$ with C from (3.30) and the second is controlled by (3.32). Thus we obtain

$$\|\lambda_k \overline{m}_{\lambda_k}(\theta)h - h\|_H \overset{k\to\infty}{\longrightarrow} 0,$$
$$\overline{m}_\lambda(\theta) - \overline{m}_\mu(\theta) = (\mu - \lambda)\overline{m}_\mu(\theta) \circ \overline{m}_\lambda(x)$$

for all $h \in H$, $\theta \in B^d \setminus \Omega$ and $\lambda, \mu \in \rho(A)$.

Theorem 2.31 implies for fixed $\theta \in B^d \setminus \Omega$ the existence of a unique, densely defined, closed and linear operator

$$(A(\theta), D(A(\theta)) : H \to H,$$

with $\rho(A) \subset \rho(A(\theta))$ and $\overline{m}_\lambda(\theta) = R(\lambda, A(\theta))$ for all $\lambda \in \rho(A)$. For $\theta \in \Omega$ we set $D(A(\theta)) = H$, $A(\theta) := 0$, and define an operator on $L^2(B^d, H)$ by

$$\begin{aligned} D(\mathcal{A}_2) &:= \{f \in L^2(B^d, H) \mid f(\theta) \in D(\mathcal{A}_2(\theta)) \text{ for almost all } \theta \in B^d \\ &\qquad \text{and } \theta \mapsto A(\theta)f(\theta) \in L^2(B^d, H)\}, \\ [\mathcal{A}_2 f](\theta) &:= A(\theta)f(\theta) \quad \text{for } f \in D(\mathcal{A}_2). \end{aligned}$$

To finish the proof it remains to show $\mathcal{A} = \mathcal{A}_2$. Note that for any $\lambda \in \rho(\mathcal{A})$ we have by (3.29) and the discussion that followed

$$\begin{aligned} D(\mathcal{A}) = R(\lambda, \mathcal{A})L^2(B^d, H) &= \{f \in L^2(B^d, H) \mid \exists g \in L^2(B^d, H) \;:\; f = \mathcal{M}_{m_\lambda} g\} \\ &= \{f \in L^2(B^d, H) \mid \exists g \in L^2(B^d, H) \;:\; f = \mathcal{M}_{\overline{m}_\lambda} g\}. \end{aligned}$$

Now choose $f \in D(\mathcal{A})$ and let $g \in L^2(B^d, H)$ be such that $f = \mathcal{M}_{\overline{m}_\lambda} g$. Then for almost all $\theta \in B^d$ we have

$$f(\theta) = [\mathcal{M}_{\overline{m}_\lambda} g](\theta) = \overline{m}_\lambda(\theta)g(\theta) = R(\lambda, A(\theta))g(\theta),$$

which shows $f(\theta) \in D(A(\theta))$ for almost all $\theta \in B^d$. Further we have

$$\begin{aligned} A(\cdot)f(\cdot) = A(\cdot)R(\lambda, A(\cdot))g(\cdot) &= \lambda R(\lambda, A(\cdot))g(\cdot) - g(\cdot) = \lambda[\mathcal{M}_{\overline{m}_\lambda} g](\cdot) - g(\cdot) \\ &= \lambda f(\cdot) - g(\cdot) \in L^2(B^d, H). \end{aligned}$$

So far we have $D(\mathcal{A}) \subset D(\mathcal{A}_2)$ and we will show $\mathcal{A}f = \mathcal{A}_2 f$ for all $f \in D(\mathcal{A})$. Pick some $f \in D(\mathcal{A})$, $\lambda \in \rho(A)$ and $g \in L^2(B^d, H)$ with $f = R(\lambda, \mathcal{A})g$. Then we have for almost all $\theta \in B^d$

$$\begin{aligned}[\mathcal{A}f](\theta) &= [\mathcal{A}R(\lambda,\mathcal{A})g](\theta) = \lambda[R(\lambda,\mathcal{A})g - g](\theta) = \lambda[\mathcal{M}_{m_\lambda}g - g](\theta) \\ &= \lambda[\mathcal{M}_{\overline{m}_\lambda}g - g](\theta) = \lambda\overline{m}_\lambda(\theta)g(\theta) - g(\theta) = \lambda R(\lambda, A(\theta))g(\theta) - g(\theta) \\ &= A(\theta)R(\lambda, A(\theta))g(\theta) = A(\theta)f(\theta) \\ &= [\mathcal{A}_2 f](\theta).\end{aligned}$$

Hence $\mathcal{A} \subset \mathcal{A}_2$. Since $\rho(\mathcal{A}) \cap \rho(\mathcal{A}_2) \neq \emptyset$ we obtain $\mathcal{A} = \mathcal{A}_2$ by Lemma 2.27 and the proof is completed. □

In the last proof we frequently used Plancherel's Theorem to obtain several equalities. If we want to state a result according to Theorem 3.26 in the Banach space case, i.e. $p \neq 2$ and E not a Hilbert space we need adequate replacements for the calculations done there. For this reason let us collect some preparatory tools.

Lemma 3.27. *Let $p \in (1, \infty)$ and E be a reflexive Banach space. Further assume that $(A, D(A)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ is a linear, closed, densely defined and translation invariant operator. Then the adjoint operator $(A', D(A')) : l^{p'}(\mathbb{Z}^d, E') \to l^{p'}(\mathbb{Z}^d, E')$ is linear, closed, densely defined and translation invariant. Further A is bounded if and only if A' is bounded.*

Proof. The dual of closed densely defined operator is again closed. This is well known. But under the additional assumption, that E is reflexive it also follows, that the dual operator is densely defined. For both facts we refer to [Phi55]. Hence it remains to show the statement concerning translation invariance.

Assume $g \in D(A')$ and $z \in \mathbb{Z}^d$. We have to show $\tau_z g \in D(A')$ and $\tau_z A' g = A' \tau_z g$. By definition of A' we find for $g \in D(A')$ an element $\tilde{g} \in l^{p'}(\mathbb{Z}^d, E')$ such that $(A'g)[f] = g[Af] = \tilde{g}[f]$ for all $f \in D(A)$. Now

$$\begin{aligned}[\tau_z g](Af) &= \sum_{j\in\mathbb{Z}^d} g(j-z)(Af)(j) = \sum_{j\in\mathbb{Z}^d} g(j)(Af)(j+z) \\ &= \sum_{j\in\mathbb{Z}^d} g(j)(A\tau_{-z}f)(j) = \tilde{g}(\tau_{-z}f) = [\tau_z\tilde{g}](f).\end{aligned}$$

Since $\tau_z \tilde{g} \in l^{p'}(\mathbb{Z}^d, E')$ we obtain $\tau_z g \in D(A')$. But the above equation also shows $A'\tau_z g = \tau_z \tilde{g} = \tau_z A' g$. Hence $(A', D(A'))$ is translation invariant. □

Concerning the adjoint of an bounded translation invariant operator we next develop a result covering strong convergence. This may be seen as a supplementary statement for Remark 3.19. Recall that we denoted the embedding $E \to E''$ by J which is bijective in the case that E is reflexive.

Lemma 3.28. *Let E be a reflexive Banach space and $T_n, T : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ be linear and translation invariant operators with*

$$T_n \xrightarrow{s} T \qquad \text{in } l^p(\mathbb{Z}^d, E),$$
$$T_n' \xrightarrow{s} T' \qquad \text{in } l^{p'}(\mathbb{Z}^d, E').$$

If $m_{T_n}, m_T \in L^\infty(B^d, \mathcal{B}_s(E))$ denote the multiplication functions corresponding to T_n and T respectively, then there is a sub-sequence $(n_j)_{j\in\mathbb{N}}$ such that

$$m_{T_{n_j}}(\theta)e \xrightarrow{w} m_T(\theta)e \quad \text{for almost all } \theta \in B^d \text{ and } e \in E.$$

Proof. First of all we note, that the 'scalar version' of T_n and T_n' converge strongly to the 'scalar versions' of T and T' respectively. Indeed let $e \in E$ and $\epsilon \in E'$. Then we have for $f \in l^p(\mathbb{Z}^d, \mathbb{C})$ and $g \in l^{p'}(\mathbb{Z}^d, \mathbb{C})$

$$\|(T_n)_e^\epsilon f - (T)_e^\epsilon f\|_{l^p(\mathbb{Z})} \leq \|\epsilon\| \|T_n fe - Tfe\|_{l^p(\mathbb{Z}^d, E)} \to 0,$$
$$\|(T_n')_\epsilon^{(Je)} g - (T')_\epsilon^{(Je)} g\|_{l^{p'}(\mathbb{Z}^d)} \leq \|Je\| \|T_n' g\epsilon - T' g\epsilon\|_{l^{p'}(\mathbb{Z}^d, E')} \to 0.$$

Furthermore we obtain

$$\begin{aligned}
[(T_e^\epsilon)' g](f) = g(T_e^\epsilon f) &= \sum_{z\in\mathbb{Z}^d} g(z)\epsilon\big((T(f\cdot e))(z)\big) = \sum_{z\in\mathbb{Z}^d} [g(z)\cdot\epsilon]\big((T(f\cdot e))(z)\big) \\
&= [g\cdot\epsilon]\big(T(f\cdot e)\big) = \big[T'(g\cdot\epsilon)\big](f\cdot e) \\
&= \sum_{z\in\mathbb{Z}^d} \big[(T'(g\cdot\epsilon))(z)\big]\big((f\cdot e)(z)\big) = \sum_{z\in\mathbb{Z}^d} \big[(T'(g\cdot\epsilon))(z)\big](e)\cdot f(z) \\
&= \sum_{z\in\mathbb{Z}^d} \big[Je\big]\big((T'(g\cdot\epsilon))(z)\big) f(z) \\
&= \big[(T')_\epsilon^{(Je)} g\big](f).
\end{aligned}$$

Hence $(T_e^\epsilon)' = (T')_\epsilon^{(Je)}$. Recall from Lemma 3.10 that the realization of T_e^ϵ on $l^{p'}(\mathbb{Z}^d)$ was given by

$$T_e^\epsilon f = \widetilde{(T_e^\epsilon)' \tilde{f}}.$$

Thus we obtain $(T_n)_e^\epsilon \xrightarrow{s} T_e^\epsilon$ in $l^{p'}(\mathbb{Z}^d)$. Now 'Riesz's Convexity Theorem' implies $(T_n)_e^\epsilon \xrightarrow{s} T_e^\epsilon$ in $l^2(\mathbb{Z}^d)$. Since the Fourier Transform is isometric on $l^2(\mathbb{Z}^d)$ we obtain

$$\|\mathcal{M}_{m_{(T_n)_e^\epsilon}} \mathcal{F}^{-1} f - \mathcal{M}_{m_{T_e^\epsilon}} \mathcal{F}^{-1} f\|_{L^2(B^d)} = \|(T_n)_e^\epsilon f - T_e^\epsilon f\|_{l^2(\mathbb{Z}^d)} \to 0$$

for all $f \in l^2(\mathbb{Z}^d)$. Choosing in particular $f = (\delta_{0,z})_{z\in\mathbb{Z}^d}$ yields

$$\|m_{(T_n)_e^\epsilon} - m_{T_e^\epsilon}\|_{L^2(B^d)} \to 0 \ \text{ as } n \to \infty.$$

Hence we find a sub-sequence $(n_j)_{j\in\mathbb{N}}$ and a set of measure zero such that for all θ not contained in this particular set

$$|m_{(T_{n_j})_e^\epsilon}(\theta) - m_{T_e^\epsilon}(\theta)| \to 0 \ \text{ as } j \to \infty. \tag{3.33}$$

Note, that the functions $m_{(T_{n_j})^\epsilon_e}$ where related to the multiplication function $m_{T_{n_j}} \in L^\infty(B^d, \mathcal{B}_s(E))$ via the bilinear forms given in the proof of Theorem 3.18. They satisfy the equation

$$m_{(T_{n_j})^\epsilon_e}(\theta) = \epsilon[m_{T_{n_j}}(\theta)e] \quad \text{for almost all } \theta \in B^d.$$

Thus we finally obtain $m_{T_{n_j}}(\theta)e \xrightarrow{w} m_T(\theta)e$ for almost all $\theta \in B^d$ and all $e \in E$ by (3.33). □

If the operators T_n are resolvent operators the assumption regarding the adjoint may be skipped in the case of a reflexive Banach space. Precisely we have

Lemma 3.29. *Let E be a reflexive Banach space and $(A, D(A)) : E \to E$ a linear, closed and densely defined operator. If there is a unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \rho(A)$ such that $\lambda_k R(\lambda_k, A) \xrightarrow{s} id_E$ for $k \to \infty$, then also $\lambda_k R(\lambda_k, A)' \xrightarrow{s} id_{E'}$.*

Proof. Since E is reflexive the adjoint of A is a closed and densely defined operator with $\rho(A) = \rho(A')$ [Phi55]. The principle of uniform boundedness yields a constant $C > 0$ with

$$\sup_{k\in\mathbb{N}} \|\lambda_k R(\lambda_k, A)\| = \sup_{k\in\mathbb{N}} \|\lambda_k R(\lambda_k, A)'\| = \sup_{k\in\mathbb{N}} \|\lambda_k R(\lambda_k, A')\| \le C.$$

We have for every $x \in D(A')$

$$\begin{aligned} \|\lambda_k R(\lambda_k, A')x - x\|_{E'} &= \|\lambda_k R(\lambda_k, A')x - R(\lambda_k, A')(\lambda_k - A')x\|_{E'} \\ &= \|R(\lambda_k, A')A'x\|_{E'} \le \frac{C}{|\lambda_k|}\|A'x\|_{E'} \end{aligned} \tag{3.34}$$

and the right hand side goes to zero as k tends to infinity. Since $D(A')$ is dense in E' we find for any $y \in E'$ a sequence $(x_n) \subset D(A')$ with $x_n \to y$ in E'. For a given $\epsilon > 0$ choose $n \in \mathbb{N}$ with $\|y - x_n\| \le \max\{\frac{\epsilon}{3C}, \frac{\epsilon}{3}\}$. For this n we find $k_0 \in \mathbb{N}$ such that $\|\lambda_k R(\lambda_k, A')x_n - x_n\|_{E'} \le \frac{\epsilon}{3}$ for all $k \ge k_0$ and obtain

$$\begin{aligned} \|\lambda_k R(\lambda_k, A')y - y\|_{E'} \le{}& \|\lambda_k R(\lambda_k, A')(y - x_n)\|_{E'} \\ &+ \|\lambda_k R(\lambda_k, A')x_n - x_n\|_{E'} + \|x_n - y\|_{E'} < \epsilon \end{aligned}$$

for all $k \ge k_0$. Hence the lemma is proven. □

Now we have collected all ingredients to proof a version of Theorem 3.26 in the general 'non-Hilbert space' case. For a clearer notation let us use the abbreviation

$$\Delta_E := \{f : \mathbb{Z}^d \to E; \exists N \in \mathbb{N} \text{ such that } f(z) = 0 \text{ for } |z| > N\}. \tag{3.35}$$

Theorem 3.30. *Let E be a reflexive and separable Banach space. Further assume that $(A, D(A)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ is a closed, linear, densely defined and translation invariant operator with $\rho(A) \neq \emptyset$, such that there is a unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \rho(A)$ satisfying*

$$\lim_{k\to\infty} \lambda_k R(\lambda_k, A) f = f \quad \text{for all } f \in l^p(\mathbb{Z}^d, E). \tag{3.36}$$

Then there is an unbounded, closed and linear multiplication operator $(\mathcal{A}, D(\mathcal{A}))$ defined on $L^p(B^d, E)$ with a family of unbounded, linear, closed and densely defined fiber operators $(A(\theta), D(A(\theta)))_{\theta\in B^d} : E \to E$ such that

(i) $\mathcal{F}^{-1} f \in D(\mathcal{A})$ *for all* $f \in \mathbf{D}_A := R(\lambda_1, A)\Delta_E$,

(ii) $Af = \mathcal{F}\mathcal{A}\mathcal{F}^{-1} f$ *for all* $f \in \mathbf{D}_A$,

(iii) there is a subset $\Omega \subset B^d$ *of measure zero with*

$$\rho(A) \subset \bigcap_{\theta\in B^d\setminus\Omega} \rho(A(\theta)).$$

Note that the set $\mathbf{D}_A$ is a core[5] *for A.*

Proof. Lets first show, that $\mathbf{D}_A$ is a core for A. Since $R(\lambda_1, A)$ maps $l^p(\mathbb{Z}^d, E)$ boundedly onto $(D(A), \|\cdot\|_A)$ and Δ_E is dense in $l^p(\mathbb{Z}^d, E)$ the set $\mathbf{D}_A$ is clearly a core for A.

For $\lambda \in \rho(A)$ the operator $R(\lambda, A) \in B(l^p(\mathbb{Z}^d, E))$ is translation invariant by Lemma 3.24. Thus Theorem 3.18 gives a function $m_\lambda \in L^\infty(B^d, \mathcal{B}_s(E))$ and a subset $\Omega_\lambda \subset B^d$ of measure zero such that

$$\begin{aligned} R(\lambda, A) f &= \mathcal{F}\mathcal{M}_{m_\lambda}\mathcal{F}^{-1} f \quad \text{for all } f \in \Delta_E, \\ \|m_\lambda(\theta)\| &\le R(\lambda, A) \quad \text{for } \theta \in B^d \setminus \Omega_\lambda. \end{aligned} \tag{3.37}$$

Note that $\Delta_E \subset \left[l^2(\mathbb{Z}^d \cap l^p(\mathbb{Z}^d)\right] \otimes E$. Remark 3.19 together with (2.7) imply for $\lambda, \mu \in \rho(A)$

$$\mathcal{M}_{m_\lambda} - \mathcal{M}_{m_\mu} = (\mu - \lambda)\mathcal{M}_{m_\lambda} \circ \mathcal{M}_{m_\mu},$$

in particular we find for every function $f \in L^p(B^d, E)$ a set of measure zero $\Omega_{\lambda,\mu,f} \subset B^d$ depending on λ, μ and f such that

$$\begin{aligned} m_\lambda(\theta) f(\theta) - m_\mu(\theta) f(\theta) &= \left[(\mathcal{M}_{m_\lambda} - \mathcal{M}_{m_\mu}) f\right](\theta) \\ &= (\mu - \lambda)\left[\mathcal{M}_{m_\lambda} \circ \mathcal{M}_{m_\mu} f\right](\theta) \\ &= (\mu - \lambda) m_\lambda(\theta) \circ m_\mu(\theta) f(\theta), \end{aligned} \tag{3.38}$$

for all $\theta \in B^d \setminus \Omega_{\lambda,\mu,f}$.

[5] A core is a subset of $D(A)$ which is dense in $D(A)$ with respect to the graph norm.

Let us again consider the set $\rho_D(A) := \rho(A) \cap (\mathbb{Q} + i\mathbb{Q})$ and choose any dense subset $(e_n)_{n\in\mathbb{N}}$ of E which exists due to the assumptions. The functions $\varphi_n := \mathbb{1}_{B^d} e_n$ belong to $L^p(B^d, E)$ for every $p \in [1, \infty]$.

Taking the union of all exceptional sets in (3.37) and (3.38) where f is replaced by φ_n and $\mu, \lambda \in \rho_D(A)$ yields a set Ω_1 which is the countable union of sets of measure zero and hence it self of measure zero. Moreover we have for all $\lambda, \mu \in \rho_D(A)$, $n \in \mathbb{N}$ and $\theta \in B^d \setminus \Omega_1$

$$\begin{aligned} m_\lambda(\theta)e_n - m_\mu(\theta)e_n &= m_\lambda(\theta)\varphi_n(\theta) - m_\mu(\theta)\varphi_n(\theta) \\ &= (\mu - \lambda) m_\lambda(\theta) \circ m_\mu(\theta)\varphi_n(\theta) \\ &= (\mu - \lambda) m_\lambda(\theta) \circ m_\mu(\theta) e_n. \end{aligned} \tag{3.39}$$

Again Lemma 2.35 (ii) yields another subset $\Omega_2 \subset B^d$ of measure zero such that

$$\|m_\lambda(\theta)\|_{\mathcal{B}(E)} \leq \|R(\lambda, A)\|_{\mathcal{B}(l^p(\mathbb{Z}^d, E))} \tag{3.40}$$

for all $\lambda \in \rho_D(A)$ and $\theta \in B^d \setminus \Omega_2$. Set $\Omega := \Omega_1 \cup \Omega_2$. Then (3.39) and (3.40) hold true for all $\theta \in B^d \setminus \Omega$ and $\lambda, \mu \in \rho_D(A)$. Moreover we can extend (3.39) to all $e \in E$. Hence, for fixed $\theta \in B^d \setminus \Omega$ the family of bounded operators

$$\{m_\lambda(\theta) : \lambda \in \rho_D(A)\}$$

is a pseudo resolvent on E. Now we may extend this family consistently to all $\lambda \in \rho(A)$ as in the proof of Theorem 3.26. Let us choose some arbitrary $\lambda \in \rho(A) \setminus \rho_D(A)$ and a approximation sequence $(\lambda_j)_{j\in\mathbb{N}} \subset \rho_D(A)$. Then for every $\theta \in B^d \setminus \Omega$ the sequence $m_{\lambda_j}(\theta)$, $j \in \mathbb{N}$ is a cauchy sequence. Indeed we have the same estimate as in (3.28).

Hence we define again a function $\overline{m}_\lambda : B^d \to \mathcal{B}(E)$ by

$$\begin{aligned} \overline{m}_\lambda(\theta) &:= \lim_{j\to\infty} m_{\lambda_j}(\theta) \ \text{ for } \theta \in B^d \setminus \Omega, \\ \overline{m}_\lambda(\theta) &:= 0 \ \text{ for } \theta \in \Omega \end{aligned}$$

and obtain $\overline{m} \in L^\infty(B^d, \mathcal{B}_s(E))$ with $\mathcal{M}_{\overline{m}_\lambda} = \mathcal{M}_{m_\lambda}$ for all $\lambda \in \rho(A)$. In fact measurability follows along the lines in the proof of Theorem 3.26 and the estimate (3.29) may be replaced by

$$\|[\mathcal{M}_{\overline{m}_\lambda} - \mathcal{M}_{m_\lambda}]f\|_{L^p(B^d,E)} \leq \|[\overline{m}_\lambda - m_{\lambda_j}]f\|_{L^p(B^d,E)} + \|m_{\lambda_j} - m_\lambda\|_\infty \|f\|_{L^p(B^d,E)}$$

for every $f \in L^p(B^d, E)$. Again the first term goes to zero by dominated convergence where for the second term we use continuity of the mapping $\lambda \mapsto R(\lambda, A)$, i.e.

$$\|m_{\lambda_j} - m_\lambda\|_\infty = \|\mathcal{M}_{m_{\lambda_j} - m_\lambda}\| \leq \|R(\lambda_j, A) - R(\lambda, A)\| \to 0 \ \text{ as } j \to \infty.$$

Defining $\overline{m}_\lambda(\theta) := m_\lambda(\theta)$ for all $\theta \in B^d$ and $\lambda \in \rho_D(A)$ we finally obtain a family of functions $\overline{m}_\lambda \in L^\infty(B^d, \mathcal{B}_s(E))$, $\lambda \in \rho(A)$ such that for all $\theta \in B^d \setminus \Omega$ and $e \in E$ we have

$$(\overline{m}_\lambda(\theta) - \overline{m}_\mu(\theta))e = (\mu - \lambda)\overline{m}_\lambda(\theta) \circ \overline{m}_\mu(\theta)e, \tag{3.41}$$

i.e. for fixed $\theta \in B^d \setminus \Omega$ the family $\left(\overline{m}_\lambda(\theta)\right)_{\lambda \in \rho(A)}$ is a pseudo resolvent on the Banach space E. Assumption (3.36) together with Lemma 3.29 and Lemma 3.28 yield an other set of measure zero denoted by $\tilde{\Omega}$ and a sub-sequence of $(\lambda_k)_{k\in\mathbb{N}}$ again denoted by $(\lambda_k)_{k\in\mathbb{N}}$ with

$$\lambda_k \overline{m}_{\lambda_k}(\theta)e \xrightarrow{w} e \;\text{ for all } \theta \in B^d \setminus \tilde{\Omega} \text{ and } e \in E \;\text{ as } k \to \infty.$$

For fixed $\theta \in B^d \setminus (\Omega \cup \tilde{\Omega})$, Corollary 2.34 implies the existence of a unique, closed, densely defined and linear operator

$$(A(\theta), D(A(\theta))) : E \to E$$

such that $\rho(A) \subset \rho(A(\theta))$ and $R(\lambda, A(\theta)) = \overline{m}_\lambda(\theta)$. For $\theta \in \Omega \cup \tilde{\Omega}$ we define $D(A(\theta)) := E$, $A(\theta)e := 0$ for all $e \in E$. Note that $\Omega \cup \tilde{\Omega}$ has measure zero.

With this fiber operators we are now able to define an unbounded and linear multiplication operator $(\mathcal{A}, D(\mathcal{A})) : L^p(B^d, E) \to L^p(B^d, E)$ by

$$\begin{aligned} D(\mathcal{A}) := \{f \in L^p(B^d, E) : f(\theta) \in D(A(\theta)) \;\text{ for almost all } \theta \in B^d \\ \text{and } \theta \mapsto A(\theta)f(\theta) \in L^p(B^d, E)\}, \\ [\mathcal{A}f](\theta) := A(\theta)f(\theta) \;\text{ for all } f \in D(\mathcal{A}) \text{ and almost all } \theta \in B^d. \end{aligned}$$

The operator $\mathcal{A}$ is closed by Lemma 2.38. Now we claim that $\mathcal{A}$ is the the operator in the statement of the theorem. In fact (iii) is fulfilled by construction. For (i) we take any $f \in \mathbf{D}_A$. Then there is a $g \in \Delta_E$ such that $f = R(\lambda_1, A)g$, i.e. for almost all $\theta \in B^d$

$$\begin{aligned} [\mathcal{F}^{-1}f](\theta) &= \left[\mathcal{M}_{\overline{m}_{\lambda_1}} \mathcal{F}^{-1}g\right](\theta) = \overline{m}_{\lambda_1}(\theta)[\mathcal{F}^{-1}f](\theta) \\ &= R(\lambda_1, A(\theta))[\mathcal{F}^{-1}f](\theta). \end{aligned}$$

But the last expression is an element of $D(A(\theta))$. Further, we have for almost all $\theta \in B^d$

$$\begin{aligned} A(\theta)[\mathcal{F}^{-1}f](\theta) &= A(\theta)R(\lambda_1, A(\theta))[\mathcal{F}^{-1}g](\theta) \\ &= \lambda_1 R(\lambda_1, A(\theta))\mathcal{F}^{-1}g(\theta) - \mathcal{F}^{-1}g(\theta) \end{aligned}$$

and $\theta \mapsto \lambda_1 R(\lambda_1, A(\theta))\mathcal{F}^{-1}g(\theta) - \mathcal{F}^{-1}g(\theta) \in L^p(B^d, E)$, because the function $\theta \mapsto \mathcal{F}^{-1}g(\theta)$ is a trigonometric polynomial and $\theta \mapsto R(\lambda, A(\theta))$ is essentially bounded. Hence we have shown (i) and it remains to show (ii). Let again f, g be such that $f = R(\lambda_1, A)g$ with $g \in \Delta_E$. Then

$$\begin{aligned} Af &= AR(\lambda_1, A)g = \lambda_1 R(\lambda_1, A)g - g = \left[\lambda_1 R(\lambda_1, A) - \mathrm{id}_{l^p(\mathbb{Z}^d, E)}\right]g \\ &= \mathcal{F}\lambda_1 \mathcal{M}_{m_{\lambda_1}} - \mathcal{M}_{\mathrm{id}_E}\mathcal{F}^{-1}g = \mathcal{F}\mathcal{M}_{\lambda_1 m_{\lambda_1} - \mathrm{id}_E}\mathcal{F}^{-1}g \\ &= \mathcal{F}\left[\theta \mapsto \lambda_1 R(\lambda_1, A(\theta)) - \mathrm{id}_E\right]\mathcal{F}^{-1}g \\ &= \mathcal{F}\left[\theta \mapsto A(\theta)R(\lambda_1, A(\theta))\right]\mathcal{F}^{-1}g \\ &= \mathcal{F}\mathcal{A}\mathcal{M}_{m_{\lambda_1}}\mathcal{F}^{-1}g = \mathcal{F}\mathcal{A}\mathcal{F}^{-1}R(\lambda_1, A)g = \mathcal{F}\mathcal{A}\mathcal{F}^{-1}f \end{aligned}$$

and all assertions are proven. □

3.5 C_0-semigroups and the Functional Calculus

Theorem 3.30 enables us to study C_0-semigroups which are generated by translation invariant operators. We get

Corollary 3.31. *Let $(A, D(A)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ be linear, translation invariant and the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$. Then the assumption of Theorem 3.30 are satisfied. In particular the multiplication operator $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a multiplication semigroup, where the fiber semigroups $(T_\theta(t))_{t\geq 0}$ are generated by the fiber operators $(A(\theta), D(A(\theta))$. Moreover we have $\|T_\theta(t)\| \leq Me^{\omega t}$ for almost all $\theta \in B^d$ and*

$$T(t)f = \mathcal{F}\mathcal{M}_{[\theta \mapsto T_\theta(t)]}\mathcal{F}^{-1}f,$$

for all $f \in \left(l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right) \otimes E$ and $t \geq 0$.

Proof. If A is the generator of a C_0-semigroup then by Theorem 2.41 A is closed and densely defined with $(\omega, \infty) \in \rho(A)$ and

$$\|R(\lambda, A)\| \leq \frac{M}{(\lambda - \omega)} \tag{3.42}$$

for all $\lambda \in (\omega, \infty)$. As in (3.34) equation (3.42) yields $\lim_{\lambda\to\infty} \lambda R(\lambda, A)f = f$ for all $f \in l^p(\mathbb{Z}^d, E)$.

Thus all assumptions of Theorem 3.30 are verified and we obtain a multiplication operator $(\mathcal{A}, D(\mathcal{A})) : L^p(B^d, E) \to L^p(B^d, E)$ with fiber operators $(A(\theta), D(A(\theta)))_{\theta\in B^d} : E \to E$ satisfying (i)-(iii) of Theorem 3.30. Moreover the resolvent estimate (3.42) transfers to almost all resolvent operators $R(\lambda, A(\theta))$ by Theorem 3.18. Hence the fiber operators $(A(\theta), D(A(\theta)))$ are generators of C_0-semigroups $(T_\theta(t))_{t\geq 0}$ on E for almost all $\theta \in B^d$ with $\|T_\theta(t)\| \leq Me^{\omega t}$ by Theorem 2.41.

Lemma 2.40 (c) gives for every fixed $t \in \mathbb{R}_{\geq 0}$

$$T(t)f = \lim_{n\to\infty} \left[{}^n\!/\!_t R({}^n\!/\!_t, A)\right]^n f \quad \text{for every } f \in l^p(\mathbb{Z}^d, E). \tag{3.43}$$

Since $\left[R({}^n\!/\!_t, A)\right]^n$ is a translation invariant operator for every $n \in \mathbb{N}$ we obtain translation invariance of $T(t)$. Again Theorem 3.18 yields the existence of a function $m_{T(t)} \in L^\infty(B^d, \mathcal{B}_s(E))$ such that for all $f \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$

$$T(t)f = \mathcal{F}\mathcal{M}_{m_{T(t)}}\mathcal{F}^{-1}f.$$

Now, for fixed $z \in \mathbb{Z}^d$ the evaluation map $f \mapsto f(z)$ is a continuous and linear operation $l^p(\mathbb{Z}^d, E) \to E$. Hence we obtain for $f \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$ by Remark 3.19

$$\begin{aligned} \left[T(t)f\right](z) &= \left[\lim_{n\to\infty} \left({}^n\!/\!_t R({}^n\!/\!_t, A)\right)^n f\right](z) = \lim_{n\to\infty}\left[\left({}^n\!/\!_t R({}^n\!/\!_t, A)\right)^n f\right](z) \\ &= \lim_{n\to\infty}\left[\mathcal{F}\left(\theta \mapsto \left({}^n\!/\!_t R({}^n\!/\!_t, A(\theta))\right)^n [\mathcal{F}^{-1}f](\theta)\right)\right](z) \\ &= \lim_{n\to\infty} \int_{B^d} e^{-2\pi i\theta\cdot z}\left({}^n\!/\!_t R({}^n\!/\!_t, A(\theta))\right)^n [\mathcal{F}^{-1}f](\theta)d\theta. \end{aligned} \tag{3.44}$$

By (3.43) and the principle of uniform boundedness we find a constant $C > 0$ such that $\|\left({}^{n}\!/\!{}_{t}R({}^{n}\!/\!{}_{t}, A)\right)^n\| \leq C$. This implies the point wise almost every where estimate $\|\left({}^{n}\!/\!{}_{t}R({}^{n}\!/\!{}_{t}, A(\cdot))\right)^n\|_\infty \leq C$. Further the function $\mathcal{F}^{-1}f$ is in $L^2(B^d) \otimes E$ and hence integrable. We obtain, that the integrand in (3.44) is uniformly (in n) bounded by the integrable function $C\mathcal{F}^{-1}f$. But for almost all $\theta \in B^d$ we also have

$$\left({}^{n}\!/\!{}_{t}R({}^{n}\!/\!{}_{t}, A(\theta))\right)^n [\mathcal{F}^{-1}f](\theta) \overset{n\to\infty}{\longrightarrow} T_\theta(t)[\mathcal{F}^{-1}f](\theta).$$

Finally, the theorem of dominated convergence (Proposition A.6) yields

$$[T(t)f](z) = \left[\mathcal{F}\mathcal{M}_{T_{(\cdot)}(t)}\mathcal{F}^{-1}f\right](z)$$

for all $z \in \mathbb{Z}^d$ and $f \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$ and we have proven all assertions. □

Decomposition of the bounded $\mathcal{H}^\infty$-Functional Calculus

Before we go on to the study of the bounded $\mathcal{H}^\infty$-functional calculus let us give a simple consequence of Theorem 3.30 for pseudo-sectorial operators.

Corollary 3.32. *Let E be a reflexive and separable Banach space. Further assume that $(A, D(A)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ is a linear closed and translation invariant operator, such that $\nu + A$ is pseudo-sectorial of angle $\omega_A \in [0, \pi)$ for some $\nu \in \mathbb{R}$. Then the assumptions of Theorem 3.30 are satisfied. In particular the family of fiber operators $(\nu + A(\theta), D(A(\theta)))_{\theta\in B^d}$ is almost uniformly pseudo-sectorial of angle less or equal to ω_A.*

Proof. By definition $\rho(A) \neq \emptyset$ and $\rho(A)$ contains an unbounded sequence $(\lambda_k)_{k\in\mathbb{N}}$. We have

$$\begin{aligned}\|\lambda_k R(\lambda_k, A)f - f\| &= \|(\lambda_k + \nu)R(\lambda_k + \nu, \nu + A)f - f - \nu R(\lambda_k + \nu, \nu + A)f\| \\ &\leq \|(\lambda_k + \nu)R(\lambda_k + \nu, \nu + A)f - f\| + \|\nu R(\lambda_k + \nu, \nu + A)f\|.\end{aligned}$$

The same arguments as in Corollary 3.31 combined with the pseudo-sectoriality estimate for the resolvent operators show, that $\lambda_k R(\lambda_, A)f \to f$ as $k \to \infty$, for any unbounded sequence $\lambda_k \subset (-\infty, -\nu)$ whence (3.36) holds true. Theorem 3.30 yields a family of fiber operators $(A(\theta), D(A(\theta))) : E \to E$ and a set Ω of measure zero, such that

$$\rho(\nu + A) \subset \bigcap_{B^d\setminus\Omega} \rho(\nu + A(\theta)),$$

$$\sup_{\theta\in B^d\setminus\Omega} \|\lambda R(\lambda, \nu + A(\theta))\| \leq \|\lambda R(\lambda, \nu + A)\|,$$

for all $\lambda \in \rho(\nu + A)$. But the two relations above state in particular, due to the pseudo sectoriality of $\nu + A$, that

$$\bigcup_{\theta\in B^d\setminus\Omega} \sigma(\nu + A(\theta)) \subset \overline{\Sigma}_{\omega_A}$$

and for any $\nu > \omega_A$ and $\theta \in B^d \setminus \Omega$

$$\|\lambda R(\lambda, \nu + A(\theta))\| \leq \|\lambda R(\lambda, \nu + A)\| \leq C_\nu$$

if $\nu \leq |\arg(\lambda)| \leq \pi$. Thus the family $(A(\theta), D(A(\theta)))_{\theta \in B^d \setminus \Omega}$ is almost uniformly pseudo-sectorial of angle at most ω_A. □

This observation has an immediate consequence for the decomposition of the 'auxiliary' functional calculus Ψ_A for pseudo-sectorial operators (recall Remark 2.52 (i)).

Corollary 3.33. *Let E be a separable and reflexive Banach space. If the operator $(A, D(A)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ is pseudo-sectorial and translation invariant, then $\Psi_A(f) \in \mathcal{M}^{p,p}(E)$ for all $f \in \mathcal{H}_0^\infty(\Sigma_\varphi)$, where $\varphi > \omega_A$. Moreover the fiber operators corresponding to A are almost uniformly pseudo sectorial of angle at most ω_A and*

$$\Psi_A(f)g = \mathcal{F}\mathcal{M}_{\theta \mapsto \Psi_{A(\theta)}(f)}\mathcal{F}^{-1}g \tag{3.45}$$

for all $g \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$ and $f \in \mathcal{H}_0^\infty(\Sigma_\varphi)$. Additionally we have

$$\operatorname*{ess\,sup}_{\theta \in B^d} \|\Psi_{A(\theta)}(f)\|_{\mathcal{B}(E)} \leq \|\Psi_A(f)\|_{\mathcal{B}(l^p(\mathbb{Z}^d, E))}.$$

Proof. By Remark 2.52 (i) we have $\Psi_A(f) \in \mathcal{M}^{p,p}(E)$ for all $f \in \mathcal{H}_0^\infty(\Sigma_\varphi)$ and everything else except (3.45) has been shown before. For $z \in \mathbb{Z}^d$ denote by δ_z the evaluation map $l^p(\mathbb{Z}^d, E) \to E$, $\delta_z(f) := f(z)$. As we have seen before δ_z is linear and bounded. Fubini's Theorem yields for any $z \in \mathbb{Z}^d$, a path γ as in (2.12), $f \in \mathcal{H}_0^\infty(\Sigma_\varphi)$ and $g \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$

$$\begin{aligned}
\delta_z[\Psi_A(f)g] &= \frac{1}{2\pi i}\int_\gamma \delta_z[R(\lambda, A)g]f(\lambda)d\lambda \\
&= \frac{1}{2\pi i}\int_\gamma \int_{B^d} e^{-2\pi i z\theta} R(\lambda, A(\theta))[\mathcal{F}^{-1}g](\theta)d\theta f(\lambda)d\lambda \\
&= \frac{1}{2\pi i}\int_{B^d} e^{-2\pi i z\theta} \int_\gamma R(\lambda, A(\theta))f(\lambda)d\lambda[\mathcal{F}^{-1}g](\theta)d\theta \\
&= \int_{B^d} e^{-2\pi i z\theta}\Psi_{A(\theta)}(f)[\mathcal{F}^{-1}g](\theta)d\theta \\
&= \delta_z[\mathcal{F}\mathcal{M}_{\theta \mapsto \Psi_{A(\theta)}(f)}\mathcal{F}^{-1}g]
\end{aligned}$$

and (3.45) is shown. The norm estimate now follows by Theorem 3.18 and the fact, that $\theta \mapsto \Psi_{A(\theta)}(f) \in L^\infty(B^d, \mathcal{B}_s(E))$ is the multiplication function corresponding to the operator $\Psi_A(f) \in \mathcal{M}^{p,p}(E)$. □

With this preparation we are able to decompose the $\mathcal{H}^\infty$-calculus for sectorial operators. Note that it is necessary to transfer the additional assumptions of injectivity and/or dense range to the fiber operators. Again Lemma 3.28 makes this transference possible.

Theorem 3.34. *Assume E is a separable and reflexive Banach space, $p \in (1, \infty)$ and $(A, D(A)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ is sectorial of angle ω_A. If A is in addition translation invariant, then the fiber operators given by Corollary 3.33 are almost uniformly sectorial. Moreover there is a subset $\Omega \subset B^d$ of measure zero such that for any $\varphi > \omega_A$ we have*

$$\mathcal{H}^\infty_A(\Sigma_\varphi) \subset \bigcap_{\theta \in B^d \setminus \Omega} \mathcal{H}^\infty_{A(\theta)}(\Sigma_\varphi) \tag{3.46}$$

and

$$\overline{\Psi}_A(f)g = \mathcal{F}\mathcal{M}_{\theta \mapsto \overline{\Psi}_{A(\theta)}(f)}\mathcal{F}^{-1}g \tag{3.47}$$

for all $g \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$ and $f \in \mathcal{H}^\infty_A(\Sigma_\varphi)$. Additionally the property of a bounded $\mathcal{H}^\infty$-calculus of angle $\varphi > \omega_A$ for A, transfers to almost all fiber operators $(A(\theta), D(A(\theta)))$.

Proof. Since sectorial operators are pseudo-sectorial, we already know, that the family of fiber operators is almost uniformly pseudo sectorial. Since E is reflexive it is enough to show, by Lemma 2.47, that $\overline{\mathrm{rg}(A(\theta))} = E$ for almost all $\theta \in E$. The sectoriality of A implies

$$A(1/n + A)^{-1}g \to g \;\text{ for all } g \in l^p(\mathbb{Z}^d, E) \text{ as } n \to \infty.$$

Indeed the equality $A(t + A)^{-1} = \mathrm{id}_{l^p(\mathbb{Z}^d,E)} - t(t + A)^{-1}$ together with the resolvent estimate for sectorial operators shows, that the set

$$\{A(t + A)^{-1} : t > 0\} \subset \mathcal{B}(l^p(\mathbb{Z}^d, E))$$

is bounded. But for any $g \in D(A)$ it holds $A(t + A)^{-1}g = (t + A)^{-1}Ag$, so that

$$A(1/n + A)^{-1}g - g = -1/n(1/n + A)^{-1}Ag = -1/nA(1/n + A)^{-1}g \to 0$$

as n tends to infinity. By the denseness of $D(A)$ in $l^p(\mathbb{Z}^d, E)$ this extends to all $g \in l^p(\mathbb{Z}^d, E)$. Since we assumed the space E to be reflexive and $p \in (1, \infty)$ also $l^p(\mathbb{Z}^d, E)$ is reflexive and the 'orthogonality' relation $\ker(A') = \mathrm{rg}(A)^\perp$ (see [Bre11, Cor.2.18]) yield the sectoriality of the adjoint A'. Hence the previous calculation keeps valid for the adjoint operator A' and we get

$$\lim_{n\to\infty} A(1/n + A)^{-1} \xrightarrow{s} \mathrm{id}_{l^p(\mathbb{Z}^d,E)}$$
$$\lim_{n\to\infty} A'(1/n + A')^{-1} \xrightarrow{s} \mathrm{id}_{l^{p'}(\mathbb{Z}^d,E')}.$$

Now again Lemma 3.28 together with Theorem 2.32 show, that for almost all $\theta \in B^d$ the range of $A(\theta)$ is dense in E, in particular there is a set $\Omega \subset B^d$ of measure zero such that the family $(A(\theta), D(A(\theta)))_{\theta \in B^d \setminus \Omega}$ is uniformly sectorial of angle at most ω_A.

Finally assume $f \in \mathcal{H}^\infty_A(\Sigma_\varphi)$. Then there is a sequence $f_n \subset \mathcal{H}^\infty_0(\Sigma_\varphi)$ with $f_n(z) \to f(z)$ for all $z \in \Sigma_\varphi$ and $\sup_{n\in\mathbb{N}} \|f_n\|_A \le C$. Corollary 3.33 yields

$$\sup_{n\in\mathbb{N}} \operatorname*{ess\,sup}_{\theta\in B^d} \|f_n\|_{A(\theta)} \le \sup_{n\in\mathbb{N}} \|f_n\|_A \le C, \tag{3.48}$$

which shows (3.46). Finally we obtain for any $g \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes E$ and $z \in \mathbb{Z}^d$ again by boundedness of the evaluation map δ_z and Corollary 3.33

$$\begin{aligned} \delta_z[\overline{\Psi}_A(f)g] &= \delta_z[\lim_{n\to\infty} \Psi_A(f_n)g] = \lim_{n\to\infty} [\Psi_A(f_n)g](z) \\ &= \lim_{n\to\infty} \mathcal{F}\mathcal{M}_{\theta\mapsto\Psi_{A(\theta)}(f_n)}\mathcal{F}^{-1}g(z). \end{aligned}$$

The sequence of functions $\theta \mapsto e^{-2\pi i\theta z}\Psi_{A(\theta)}(f_n)[\mathcal{F}^{-1}g](\theta)$, $(n \in \mathbb{N})$ is uniformly bounded by the integrable function $\theta \mapsto C\mathcal{F}^{-1}g(\theta)$ with C from (3.48) and converges almost every where to $e^{-2\pi i\theta z}\overline{\Psi}_{A(\theta)}(f)[\mathcal{F}^{-1}g](\theta)$. Hence the theorem of dominated convergence applies and gives

$$\overline{\Psi}_A(f)g = \mathcal{F}\mathcal{M}_{\theta\mapsto\overline{\Psi}_{A(\theta)}(f)}\mathcal{F}^{-1}g.$$

If A has a bounded $\mathcal{H}^\infty$-calculus of angel φ, i.e. $\mathcal{H}^\infty_A(\Sigma_\varphi) = \mathcal{H}^\infty(\Sigma_\varphi)$, we obtain from (3.46) that almost all fiber operators have a bounded $\mathcal{H}^\infty$-calculus of angle φ and all assertions are proven. □

Remark 3.35. *We mentioned the 'extended' functional calculus in Chapter 2. A detailed inspection of the construction shows, that even this calculus is decomposable in the sense above, if the underlying operator is periodic and sectorial.*

Within the next subsection we extend the results obtained here to both the Zak and the Bloch Transform.

3.6 Back to Periodic Operators and the Bloch Transform

After the detailed study of translation invariant operators on $l^p(\mathbb{Z}^d, E)$ in the previous subsections, we will now give corresponding results for the Zak and Bloch 'decomposition' for periodic operators on $L^p(\mathbb{R}^d, E)$. We remind of the decomposition of Z and Φ given in (2.5), (2.6) as well as Figure 3.1. All we have to do is to reverse the reduction done in Lemma 3.4 and Lemma 3.23 respectively. Note that both reductions resulted in the study of translation invariant operators on a sequence space $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$. In the statements of Theorem 3.18, 3.30, 3.34 and Corollary 3.31 we always used a dense subset of this sequence space to obtain a representation of the corresponding operators in terms of multiplication operators. In order to get a unified framework we define for $p \in [1, \infty]$

$$L^p_c(\mathbb{R}^d, E) := \Big\{ f \in L^p(\mathbb{R}^d, E) : \operatorname{supp}(f) \text{ is compact} \Big\}.$$

Lemma 3.36. *Let $p \in [1, \infty)$. Then the set $L^p_c(\mathbb{R}^d, E)$ is a linear and dense subspace of $L^p(\mathbb{R}^d, E)$. The image of $L^p_c(\mathbb{R}^d, E)$ under Γ is $\Delta_{L^p(\mathbb{I}^d,E)}$*[6]*, which is a linear and dense subspace of $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$ that is contained in $\left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes L^p(\mathbb{I}^d, E)$.*

Proof. It is clear that $L^p_c(\mathbb{R}^d, E)$ is dense in $L^p(\mathbb{R}^d, E)$ because every simple function is contained. It is also obvious that $L^p_c(\mathbb{R}^d, E)$ is a linear space. Now fix any $f \in L^p_c(\mathbb{R}^d, E)$ and recall $[\Gamma f](z) = \mathfrak{R}_{\mathbb{I}^d} \tau_z f$ for all $z \in \mathbb{Z}^d$. Since $\operatorname{supp}(f)$ is compact there is a $N \in \mathbb{N}$ such that $[\Gamma f](z) \equiv 0$ for $|z| > N$, whence $\Gamma f \in \Delta_{L^p(\mathbb{I}^d,E)}$. Conversely, if $f \in \Delta_{L^p(\mathbb{I}^d,E)}$ then $\Gamma^{-1} f = \sum_{z \in \mathbb{Z}^d} \tau_{-z} \mathfrak{E}_{\mathbb{R}^d} f(z)$ is a finite sum. Since $\operatorname{supp}(\tau_{-z} \mathfrak{E}_{\mathbb{R}^d} f(z)) \subset z + \mathbb{I}^d$ we obtain $\Gamma^{-1} f \in L^p_c(\mathbb{R}^d, E)$. Writing $g \in \Delta_{L^p(\mathbb{I}^d,E)}$ as $g = \sum_{|z|<N} (\delta_{jz})_{j \in \mathbb{Z}^d} g(z)$, we see $g \in \left[l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)\right] \otimes L^p(\mathbb{I}^d, E)$. □

Concerning bounded periodic operators we have the following analogue of Theorem 3.18.

Theorem 3.37. *Let E_0, E_1 be separable and reflexive Banach spaces, $p \in [1, \infty)$. Further assume $T : L^p(\mathbb{R}^d, E_0) \to L^p(\mathbb{R}^d, E_1)$ is linear, bounded and periodic. Then there are operator valued functions $m, \overline{m} \in L^\infty(B^d, \mathcal{B}_s(E_0, E_1))$ such that*

$$Tf = Z^{-1} \mathcal{M}_m Z f = \Phi \mathcal{M}_{\overline{m}} \Phi^{-1} f \tag{3.49}$$

for all $f \in L^p_c(\mathbb{R}^d, E)$. Moreover we have $\|m\|_\infty = \|\overline{m}\|_\infty \le \|T\|$.

If $p = 2$ and E_0, E_1 are Hilbert spaces then (3.49) holds true for all $f \in L^2(\mathbb{R}^d, H_0)$ and $\|m\|_\infty = \|\overline{m}\|_\infty = \|T\|$.

Proof. As a first step we use Lemma 3.4 to reduce the situation to the study of a bounded translation invariant operator

$$\mathbf{T} : l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0)) \to l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_1)).$$

Theorem 3.18 yields a function $m \in L^\infty(B^d, \mathcal{B}_s(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1)))$ such that

$$\mathbf{T} f = \mathcal{F} \mathcal{M}_m \mathcal{F}^{-1} f \quad \text{for all } f \in \left[l^p(\mathbb{Z}^d) \cap l^2(\mathbb{Z}^d)\right] \otimes L^p(\mathbb{I}^d, E_0)$$

and $\|m\|_\infty \le \|\mathbf{T}\|$. The operator $\mathbf{T}$ is given by $\mathbf{T} f = \Gamma T \Gamma^{-1} f$. Hence the mapping properties of Γ and Γ^{-1} yield together with Lemma 3.36

$$Tf = \Gamma^{-1} \mathbf{T} \Gamma f = \Gamma^{-1} \mathcal{F} \mathcal{M}_m \mathcal{F}^{-1} \Gamma f = Z^{-1} \mathcal{M}_m Z f \tag{3.50}$$

for all $f \in L^p_c(\mathbb{R}^d, E_0)$. Concerning the Bloch Transform we note that the function

$$\overline{m}(\theta) := \Xi(\theta) m(\theta) \Xi^{-1}(\theta)$$

is an element of $L^\infty(B^d, \mathcal{B}_s(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1))$ with $\mathcal{M}_{\overline{m}} = \Xi \mathcal{M}_m \Xi^{-1}$ and $\|\overline{m}\|_\infty = \|m\|_\infty$. Thus we may write (3.50) in the form

$$Tf = \Gamma^{-1} \mathbf{T} \Gamma f = \Gamma^{-1} \mathcal{F} \mathcal{M}_m \mathcal{F}^{-1} \Gamma f = Z^{-1} \Xi^{-1} \mathcal{M}_{\overline{m}} \Xi Z f = \Phi^{-1} \mathcal{M}_{\overline{m}} \Phi f \tag{3.51}$$

[6] Recall the definition of $\Delta_E := \{f : \mathbb{Z}^d \to E; \exists N \in \mathbb{N} \text{ such that } f(z) = 0 \text{ for } |z| > N\}$.

for all $f \in L^p_c(\mathbb{R}^d, E_0)$. Finally we mention that in the case $p = 2$ and E_0, E_1 Hilbert spaces also $L^2(\mathbb{I}^d, E_i)$, $(i = 1, 2)$ is a Hilbert space so we may use Theorem 3.12 instead of Theorem 3.18 to obtain

$$Tf = \Gamma^{-1}\mathbf{T}\Gamma f = \Gamma^{-1}\mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}\Gamma f = Z^{-1}\mathcal{M}_m Z f = \Phi^{-1}\mathcal{M}_{\overline{m}}\Phi f$$

for all $f \in L^2(\mathbb{R}^d, H_0)$ and and $\|m\|_\infty = \|\overline{m}\| = \|T\|$. □

Concerning unbounded periodic operators we obtain by exactly the same arguments

Theorem 3.38. *Let $p \in (1, \infty)$ and E be a separable and reflexive Banach space. Further assume $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ is linear, closed, densely defined and periodic with $\rho(A) \neq \emptyset$. Further assume, that there is a unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \rho(A)$ with*

$$\lim_{k\to\infty} \lambda_k R(\lambda_k, A)f = f \quad \text{for all } f \in L^p(\mathbb{R}^d, E). \tag{3.52}$$

Then there is an unbounded, closed and linear multiplication operator $(\mathcal{A}, D(\mathcal{A}))$ defined on $L^p(B^d, L^p(\mathbb{I}^d, E))$ with linear, closed and densely defined fiber operators $(A(\theta), D(A(\theta)))$ defined on $L^p(\mathbb{I}^d, E)$ such that

(i) $Z^{-1}f \in D(\mathcal{A})$ for all $f \in \mathbf{D}_A := R(\lambda_1, A)L^p_c(\mathbb{R}^d, E)$,

(ii) $Af = Z^{-1}\mathcal{A}Zf$ for all $f \in \mathbf{D}_A$,

(iii) there is a subset $\Omega \subset B^d$ of measure zero with

$$\rho(A) \subset \bigcap_{\theta\in B^d\setminus\Omega} \rho(A(\theta)).$$

The set $\mathbf{D}_A$ is again a core for A. Modifying the fiber operators $(A(\theta), D(A(\theta)))$ to operators $(\tilde{A}(\theta), D(\tilde{A}(\theta)))$ by

$$D(\tilde{A}(\theta)) := \Xi(\theta)D(A(\theta))$$
$$\tilde{A}(\theta)g := \Xi(\theta) \circ A(\theta) \circ \Xi^{-1}(\theta)g \ \text{ for all } g \in D(\tilde{A}(\theta))$$

yields a family of fiber operators on $L^p(\mathbb{I}^d, E)$ such that (i)-(iii) holds true with Z replaced by Φ and $A(\theta)$ replaced by $\tilde{A}(\theta)$. Moreover, if A is (pseudo)-sectorial of angel ω both families are almost everywhere uniformly (pseudo)-sectorial of angle ω. In this case also the functional calculus decomposes according to Z and Φ. In particular for $h \in \mathcal{H}^\infty_A(\Sigma_\varphi)$, where $\varphi \in (\omega, \pi)$ we have

$$\begin{aligned} \overline{\Psi}_A(h)f &= Z^{-1}[\theta \mapsto \overline{\Psi}_{A(\theta)}(h)]Zf \\ \overline{\Psi}_A(h)f &= \Phi^{-1}[\theta \mapsto \overline{\Psi}_{\tilde{A}(\theta)}(h)]\Phi f \end{aligned} \tag{3.53}$$

for all $f \in L^p_c(\mathbb{R}^d, E)$. If A is the generator of a C_0-semigroup, so are the fiber operators $A(\theta), \tilde{A}(\theta)$ for almost all $\theta \in B^d$ and the semigroup has a decomposition according to Z and Φ respectively.
Again if $p = 2$ and E is a Hilbert space (i) and (ii) are true for all $f \in D(A)$ and (3.53) is valid for all $f \in L^2(\mathbb{R}^d, E)$.

Proof. The proof works essentially the same way as the one of Theorem 3.37. But this time we use Lemma 3.23 for the reduction to the case of unbounded translation invariant operators on the sequence space $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$. Then Theorem 3.30 gives a family of fiber operators $(A(\theta), D(A(\theta)))$ defined on $L^p(\mathbb{I}^d, E)$, that satisfies (i)-(iii) of Theorem 3.30. For sectoriality we invoke Corollary 3.32 and the decomposition of the functional calculus follows by Theorem 3.34. The statements concerning Φ are now simple consequences of the properties of Ξ and Ξ^{-1}. In the Hilbert space setting we use Theorem 3.26 instead of Theorem 3.30. □

Remark 3.39.

(i) *In the case of bounded periodic operators the statements of Remark 3.19 transfer in a natural way.*

(ii) *Concerning the assumptions (3.36) and (3.52) we note, that they where only used to derive range and kernel properties of the fiber operators that correspond to the resolvent operators, i.e. we used them to show that $rg(m_\lambda(\theta))$ is dense in E and $ker(m_\lambda(\theta)) = \{0\}$. In explicit applications it might happen, that one can derive this properties in another way. Then these assumptions may be skipped.*

Let us close this section by giving a schematic overview of the results regarding the decomposition of periodic operators.

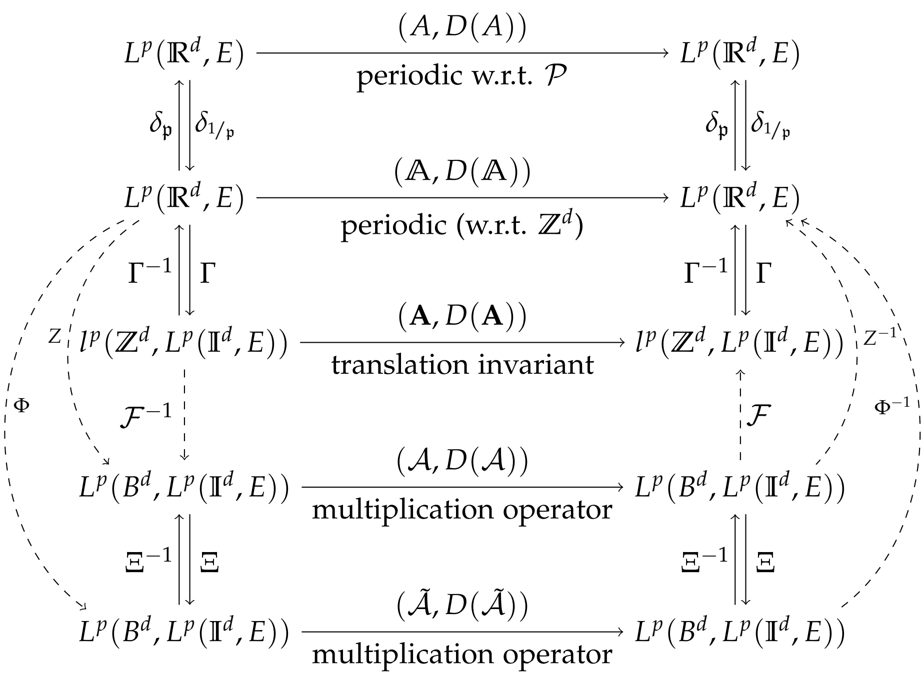

Figure 3.2: Schematic diagram of the decomposition of a periodic operator according to the Zak- and Bloch-Transform. Dashed lines refer to operations that are only defined on suitable subspaces.

Chapter 4

Bloch Multiplier Theorems

In the previous chapter we developed several decomposition results of periodic and translation invariant operators in terms of multiplication operators under the Zak-/Bloch- and Fourier Transform. We have seen that every bounded, translation invariant operator $l^p(\mathbb{Z}^d, E_0) \to l^P(\mathbb{Z}^d, E_1)$ has an associated multiplication operator in the Fourier image where the multiplication function m belongs to $L^\infty(B^d, \mathcal{B}_s(E_0, E_1))$. The Zak/Bloch Transform is an appropriate way to transfer this correspondence to periodic operators. In the Hilbert space case those operators are actually characterized in terms of boundedness of the multiplication function, thanks to Plancherel's theorem. As we will see in this chapter, this characterization fails in the non Hilbert space case dramatically (even in the scalar valued setting). Therefore we have to find suitable conditions, replacing the boundedness of the multiplication function $m : B^d \to \mathcal{B}(E_0, E_1)$, such that the corresponding multiplication operator is associated to a bounded translation invariant operator $l^p(\mathbb{Z}^d, E_0) \to l^p(\mathbb{Z}^d, E_1)$. If we have found such a condition we can easily transfer it to the case of periodic operators on $L^p(\mathbb{R}^d, E)$ by the same ideas used in the previous chapter.

Definition 4.1. *A bounded function $m : B^d \to \mathcal{B}(E_0, E_1)$ is called p-Fourier multiplication function for some $p \in (1, \infty)$, if the operator*

$$T_m \varphi := \mathcal{F} \mathcal{M}_m \mathcal{F}^{-1} \varphi,$$

first defined for $\varphi \in s(\mathbb{Z}^d, E_0)$, extends to an operator in $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$. In this case T_m is called the p-Fourier multiplier operator (corresponding to m).

According to the notation in the previous chapter we denote by $\mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ the set of all functions $m : B^d \to B(E_0, E_1)$ such that $T_m \in \mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$.

The development of results that guarantee the boundedness of operators of the form T_m has a long history. Originally these questions where studied for the continuous Fourier Transform and scalar valued functions. A classical result concerning this situation, whose proof is based on an earlier observation of Marcinkiewicz addressing the same question for periodic functions [Mar39], goes back to Mihlin [Mih56] and states, that the condition

$$|x|^{|\alpha|}|D^\alpha m(x)| \leq C \quad \text{for all } \alpha \in \mathbb{N}^d \text{ with } |\alpha|_\infty \leq 1 \text{ and } x \in \mathbb{R}^d \setminus \{0\}$$

is sufficient for m to be a p-Fourier multiplication function, for the Fourier Transform on $\mathbb{R}^d$ and all $p \in (1, \infty)$. Due to the wide applicability of this result several variants where investigated by various authors. The breakthrough to the vector-valued setting, with a scalar valued multiplication function, is due to Bourgain [Bou86] under the additional assumption that the Banach space under consideration is of class $\mathcal{HT}$. The step towards a fully vector-valued situation, i.e. an operator-valued function m requires an additional assumption, namely that of $\mathcal{R}$-boundedness. A first result into this direction was given by Weis [Wei01]. Later on several variants followed. Some of them weaken the condition on m as well as variants that change the group $\mathbb{R}^d$. Results for the one-dimensional Torus are given in [AB02, AB10]. A generalization to the case $d \geq 1$ may be found in [BK04]. For the group $\mathbb{Z}^d$ only the case $d = 1$ is known [Blu01]. We will give a multidimensional variant of Mihlin's theorem in the fully vector-valued case for the group $\mathbb{Z}^d$ in this section. Before we begin let us motivate the necessary assumptions.

4.1 Necessary Conditions for a Multiplier Theorem

This first section is devoted to find necessary conditions for a Banach space E and a function $m : B \to \mathcal{B}(E)$ so that $T_m \in \mathcal{B}(l^p(\mathbb{Z}, E))$. Before we introduce them, let us give the previously mentioned example of a bounded (relativity natural) function that is not a p-Fourier multiplication function.

A Function $m \in L^\infty(B^d)$ that is not a Fourier Multiplication Function

It is sufficient to show, that there is a function $m \in L^\infty(B^d, \mathbb{C})$ such that the linear operator

$$T_m f := \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1} f \tag{4.1}$$

first defined for $f \in s(\mathbb{Z}^d, \mathbb{C})$ does not extend to a bounded operator in on $l^p(\mathbb{Z}^d)$, for some $p \in (1, \infty)$. We do this by an application of two important results concerning multiplication functions in general.

We use an example given by Fefferman in [Fef70] for the Fourier Transform on the group $\mathbb{R}^d$. He was able to show, that the characteristic function of the unit ball does not define a bounded Fourier multiplication operator on $L^p(\mathbb{R}^d)$ if $d \geq 2$[1] and $p \notin (\frac{2d}{d+1}, \frac{2d}{d-1})$. Then about one year later Fefferman extended this example to all $p \in (1, \infty)$ ([Fef71]), i.e. the characteristic function of the unit sphere does not define a bounded Fourier multiplication operator for any $2 \neq p \in (1, \infty)$ if $d \geq 2$.

[1]If $d = 1$ the ball is just an interval which is in fact a multiplication function. Compare Theorem 4.7.

If we want to use this example in our situation we need a relationship between multiplication functions on $\mathbb{R}^d$ and multiplication functions on B^d. Such a connection is set up by a generalization of 'de Leeuw's multiplier restriction theorem' ([dL65]) to the multidimensional case. Precisely we will use the following result.

Theorem 4.2 ([Jod70, Thm.2.1]). *If $m : \mathbb{R}^d \to \mathbb{C}$ is periodic with respect to $\mathbb{Z}^d$, then m defines a bounded Fourier multiplication operator in $L^p(\mathbb{R}^d)$ if and only if the restriction of m to the cube $[-1/2, 1/2]^d$ defines a bounded Fourier multiplication operator on $l^p(\mathbb{Z}^d)$.*

Theorem 4.2 may be used in the following way. Assume that $\varphi : B^d \to \mathbb{C}$ defines a bounded multiplication operator on $l^p(\mathbb{Z}^d, E)$. Then the periodic extension $\mathfrak{E}_p m$ of m to the whole of $\mathbb{R}^d$ would define a bounded Fourier multiplication operator for the continuous Fourier Transform.

Now take in particular $\varphi := \mathbb{1}_{\{x \in B^d : |x| \leq 1/3\}}$ and assume φ defines a bounded Fourier multiplication operator on $l^p(\mathbb{Z}^d)$ for some $2 \neq p \in (1, \infty)$. The periodic extension of m, $\mathfrak{E}_p m$ is a sum of translates of φ (see Figure 4.1). The boundedness of T_φ on $l^p(\mathbb{Z}^d)$ would now imply the boundedness of $T_{\mathfrak{E}_p m}$ on $L^p(\mathbb{R}^d)$ for the continuous Fourier transform via Theorem 4.2. Since $T_{\mathbb{1}_{[-1/2,1/2]^d}}$ is bounded in $L^p(\mathbb{R}^d)$ and $\mathbb{1}_{[-1/2,1/2]^d} \cdot \varphi = \varphi$ this would imply the boundedness of T_φ on $L^p(\mathbb{R}^d)$ which is false by the discussion above if $d \geq 2$ and $p \neq 2$. Hence φ is not a multiplication function for $l^p(\mathbb{Z}^d)$ if $d \geq 2$ and $p \neq 2$.

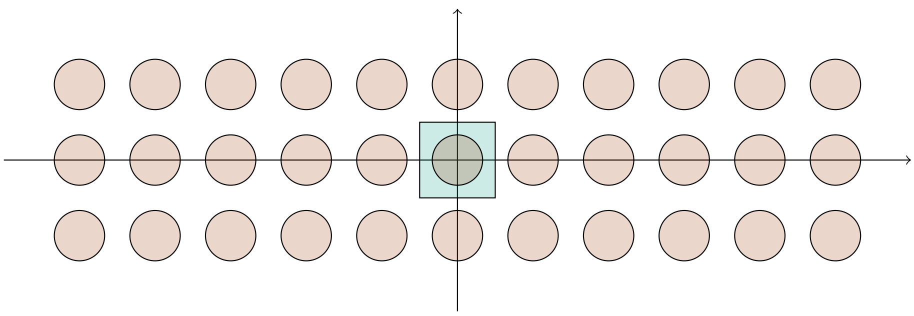

Figure 4.1: Periodic extension of the characteristic function $\mathbb{1}_{\{x \in B^2 : |x| \leq 1/3\}}$ to $\mathbb{R}^2$

Now that we know being bounded is not enough for a function to define a bounded Fourier multiplication operator we are searching for suitable necessary conditions.

Necessity of $\mathcal{R}$-boundedness

Assume we are given two arbitrary Banach spaces E_0, E_1 and an operator valued function $m \in L^\infty(B^d, \mathcal{B}(E_0, E_1))$ such that T_m defined as in (4.1) extends to a bounded operator $l^p(\mathbb{Z}^d, E_0) \to l^p(\mathbb{Z}^d, E_1)$. We will show that this already implies the $\mathcal{R}$-boundedness of the set

$$\{m(\theta) : \theta \in L(m)\} \tag{4.2}$$

where $L(m)$ denotes the sets of Lebesgue points of m. For this reason, consider for $n \geq 2$ the functions $\psi_n := \mathbb{1}_{[-n^{-1},n^{-1}]^d}$ and define

$$f_n(\theta) := \lambda([-n^{-1}, n^{-1}]^d)^{-1}\psi_n(\theta) \text{ for } \theta \in B^d.$$

Then we obtain[2] for any $\theta_0 \in B^d$ by translation invariance of λ,

$$\begin{aligned}(f_n * m)(\theta_0) &= \int_{B^d} f_n(\theta_0 - \theta)m(\theta)d\theta \\ &= \frac{1}{\lambda(\theta_0 + [-n^{-1}, n^{-1}])} \int_{\theta_0+[-n^{-1},n^{-1}]} m(\theta)d\theta.\end{aligned} \tag{4.3}$$

The vector valued version of Lebesgue's differentiation Theorem now yields

$$f_n * m(\theta_0) \overset{n\to\infty}{\longrightarrow} m(\theta_0), \tag{4.4}$$

if θ_0 is a Lebesgue point of m. For the computation of the Fourier coefficients of ψ_n we observe that we have for any $z \in \mathbb{Z}^d$

$$[\mathcal{F}\psi_n](z) = \int_{B^d} e^{-2\pi i\theta\cdot z}\psi_n(\theta)d\theta = \prod_{j=1}^{d} \int_{-n^{-1}}^{n^{-1}} e^{-2\pi i z_j y}dy.$$

If $z_j = 0$ the last integral equals $2n^{-1}$, while for $z_j \neq 0$ we obtain by Euler's Formula and the symmetry of the sin-function

$$\left|\int_{-n^{-1}}^{n^{-1}} e^{-2\pi i z_j y}dy\right| = \left|\int_{-n^{-1}}^{n^{-1}} \cos(2\pi z_j y)dy\right| = \left|\frac{1}{\pi z_j}\sin(2\pi z_j n^{-1})\right|.$$

Thus we can estimate for any $q > 1$

$$\begin{aligned}\|\mathcal{F}\psi_n\|_{l^q(\mathbb{Z}^d)}^q &= \sum_{z\in\mathbb{Z}^d} |\mathcal{F}\psi_n(z)|^q = \sum_{z\in\mathbb{Z}^d} \prod_{j=1}^{d} \left|\int_{-n^{-1}}^{n^{-1}} e^{-2\pi i z_j y}dy\right|^q \\ &\leq 2^d\left(\sum_{k\in\mathbb{N}_0} \left|\int_{-n^{-1}}^{n^{-1}} e^{-2\pi i k y}dy\right|^q\right)^d \\ &\leq 2^d\left((2n^{-1})^q + \sum_{k\in\mathbb{N}} (\pi k)^{-q}|\sin(2\pi k n^{-1})|^q\right)^d \\ &\leq 2^d\left((2n^{-1})^q + \sum_{k=1}^{n} (\pi k)^{-q}|\sin(2\pi k n^{-1})|^q + \sum_{k=n+1}^{\infty} (\pi k)^{-q}\right)^d.\end{aligned}$$

[2] Recall the convention for convolutions and translations on B^d given in Section 2.1.

Since $(\pi k n^{-1})^{-q} |\sin(2\pi k n^{-1})|^q \leq C$ we obtain, using the boundedness of the last sum by some constant depending only on q,

$$\|\mathcal{F}\psi_n\|_{l^q(\mathbb{Z}^d)} \leq c(q,d)\big(n \cdot n^{-q} + n \cdot n^{-q}\big)^{\frac{d}{q}} = c(q,d) n^{d(1/q-1)}. \tag{4.5}$$

Now define $\phi_n := (2n)^d \mathcal{F}\psi_n$, which is an element of $l^p(\mathbb{Z}^d, \mathbb{C})$ for every $p > 1$ by (4.5), and write $f_n = \psi_n[\mathcal{F}^{-1}\phi_n]$. Then for any $e \in E_0$ we obtain

$$\begin{aligned}
(m * f_n)(\theta_0)e &= \int_{B^d} m(\theta)(\tau_{\theta_0} f_n)(\theta) e d\theta = \int_{B^d} m(\theta)(\tau_{\theta_0}\psi_n)(\theta)(\tau_{\theta_0}[\mathcal{F}^{-1}\phi_n e])(\theta) d\theta \\
&= \int_{B^d} \big(\mathcal{F} T_m (e^{2\pi i \theta_0 (\cdot)} \phi_n e)\big)(\theta)(\tau_{\theta_0}\psi_n)(\theta) d\theta \\
&= \sum_{z \in \mathbb{Z}^d} \big[e^{2\pi i \theta_0(\cdot)} T_m e^{2\pi i \theta_0 (\cdot)}\big](\phi_n e)(z) \widetilde{\mathcal{F}\psi_n}(z).
\end{aligned}$$

The assumption $T_m \in \mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$ together with Lemma 2.58 (a) shows, that the set

$$\{g T_m h : g, h \in l^\infty(\mathbb{Z}^d) \text{ with } \|g\|_\infty, \|h\|_\infty \leq 1\}$$

is $\mathcal{R}$-bounded with $\mathcal{R}$-bound less or equal to $4\|T_m\|$. Now we can estimate for all $k \in \mathbb{N}$, $e_1, \ldots, e_k \in E_0$ and Lebesgue points $\theta_1, \ldots, \theta_k \in \mathcal{B}^d$ using (4.4), Fatou's Lemma (A.7) and Hölder's inequality

$$\begin{aligned}
&\int_0^1 \| \sum_{j=1}^k r_j(u) m(\theta_j) e_j \|_{E_1}^p du \\
&\leq \varliminf_{n\to\infty} \int_0^1 \| \sum_{j=1}^k r_j(u) \sum_{z\in\mathbb{Z}^d} [e^{2\pi i\theta_0(\cdot)} T_m e^{2\pi i \theta_0(\cdot)}(\phi_n e_j)](z) \widetilde{\mathcal{F}\psi_n}(z) \|_{E_1}^p du \\
&\leq \varliminf_{n\to\infty} \int_0^1 \Big(\sum_{z\in\mathbb{Z}^d} \| \sum_{j=1}^k r_j(u) [e^{2\pi i\theta_0(\cdot)} T_m e^{2\pi i \theta_0(\cdot)}(\phi_n e_j)](z) \|_{E_1} |\widetilde{\mathcal{F}\psi_n}(z)|_{\mathbb{C}} \Big)^p du \\
&\leq \varliminf_{n\to\infty} \int_0^1 \| \sum_{j=1}^k r_j(u) [e^{2\pi i\theta_0(\cdot)} T_m e^{2\pi i \theta_0(\cdot)}](\phi_n e_j) \|_{l^p(\mathbb{Z}^d, E_1)}^p \|\widetilde{\mathcal{F}\psi_n}\|_{l^{p'}(\mathbb{Z}^d)}^p du \\
&= \varliminf_{n\to\infty} \| \sum_{j=1}^k r_j(\bullet) [e^{2\pi i\theta_0(\cdot)} T_m e^{2\pi i \theta_0(\cdot)}](\phi_n e_j) \|_{L^p([0,1], l^p(\mathbb{Z}^d, E_1))}^p \|\widetilde{\mathcal{F}\psi_n}\|_{l^{p'}(\mathbb{Z}^d)}^p \\
&\leq 4\|T_m\| \varliminf_{n\to\infty} \| \sum_{j=1}^k r_j(\bullet)(\phi_n e_j) \|_{L^p([0,1], l^p(\mathbb{Z}^d, E_0))}^p \|\widetilde{\mathcal{F}\psi_n}\|_{l^{p'}(\mathbb{Z}^d)}^p \\
&= 4\|T_m\| \varliminf_{n\to\infty} \| \sum_{j=1}^k r_j(\bullet) e_j \|_{L^p([0,1], E_0)}^p \|\phi_n\|_{l^p(\mathbb{Z}^d)}^p \|\widetilde{\mathcal{F}\psi_n}\|_{l^{p'}(\mathbb{Z}^d)}^p \\
&= 4\|T_m\| \| \sum_{j=1}^k r_j(\bullet) e_j \|_{L^p([0,1], E_0)}^p \varliminf_{n\to\infty} \|\phi_n\|_{l^p(\mathbb{Z}^d)}^p \|\widetilde{\mathcal{F}\psi_n}\|_{l^{p'}(\mathbb{Z}^d)}^p
\end{aligned}$$

Recall (4.5) and the definition of ϕ_n, which implies

$$\|\phi_n\|_{l^p(\mathbb{Z}^d)}\|\widetilde{\mathcal{F}\psi_n}\|_{l^{p'}(\mathbb{Z}^d)} \leq c(d,p,p')(2n)^d n^{d(1/p-1)} n^{d(1/p'-1)} = c(d,p,p')2^d.$$

All together now gives that the set in (4.2) is $\mathcal{R}$-bounded with $\mathcal{R}$-bound less or equal to $4\|T_m\|$.

Remark 4.3. *Necessity of $\mathcal{R}$-boundedness was first shown for the Fourier Transform on the group $\mathbb{R}^d$ in [CP01]. At this time it was already known, that the $\mathcal{R}$-boundedness of the set*

$$\{tm'(t), m(t) : t \in \mathbb{R} \setminus \{0\}\}.$$

for an operator-valued function m, together with the property of E to be of class $\mathcal{HT}$ is sufficient to obtain a bounded operator $T_m \in \mathcal{B}(L^p(\mathbb{R}, E))$, again for the Fourier Transform on the group $\mathbb{R}^d$. Necessity of $\mathcal{R}$-boundedness for more general groups was proven in [Blu01] where we also found the idea for the calculation above.

Class $\mathcal{HT}$ is the Right Framework

As we have already seen in an earlier chapter, Fourier multiplication operators are closely related to convolution operators on a sequence space. One of the most basic (non trivial) convolution operator on $l^p(\mathbb{Z})$ is the discrete Hilbert Transform, denoted by $\mathcal{H}_D$, which is given by convolution with the sequence

$$h(z) := \begin{cases} 1/z & : z \neq 0 \\ 0 & : z = 0. \end{cases} \tag{4.6}$$

It is known for a long time, that convolution with h yields a bounded operation $l^p(\mathbb{Z}) \to l^p(\mathbb{Z})$ for all $p \in (1, \infty)$ (see [HLP34, §9.2]). The multiplication function corresponding to h is of a very simple form i.e.

$$m_h(\theta) = 2\pi i\left(\frac{1}{2}\operatorname{sign}(\theta) - \theta\right). \tag{4.7}$$

Hence if we want the function m_h to be covered by the multiplier Theorem, not only in the case of scalar- but also in the case of vector-valued functions, this would lead to the boundedness of the discrete Hilbert Transform on $l^p(\mathbb{Z}, E)$ for all $p \in (1, \infty)$, which then implies the boundedness of the Fourier multiplier operator corresponding to characteristic functions of sub intervals of B (see Section 4.2).

Fortunately it was shown in [BGM86] that the discrete Hilbert Transform is bounded on $l^p(\mathbb{Z}, E)$ if and only if the continuous Hilbert Transform from Section 2.5 is bounded on $L^p(\mathbb{R}, E)$ for some (or equivalently all) $p \in (1, \infty)$, i.e. if and only if E is of class $\mathcal{HT}$.

Closely related to the boundedness of the Hilbert Transform is the derivation of the so called Paley-Littlewood decomposition of $L^p(B^d, E)$ which will be a crucial ingredient for proofing the multiplier theorem. Originally class $\mathcal{HT}$

was asked by Bourgain, to obtain the boundedness of certain singular integrals on the circle, which enabled him to show the Paley-Littlewood decomposition in this case. Later the Paley-Littlewood theorem was transferred to the group $\mathbb{R}$ [Zim89, Wei01] and $\mathbb{Z}$ [Blu01]. We will use the latter result to extend the Paley-Littlewood theorem to $\mathbb{Z}^d$ in Section 4.3.

4.2 Fourier Multiplication Operators (Generalities)

We start with a general study of Fourier multiplication operators, where we first examine some standard modifications of multipliers in $\mathcal{M}_p(E_0, E_1)$. Afterward we show, that class $\mathcal{HT}$ of the Banach spaces implies that characteristic functions of sub intervals of B are p-Fourier multiplication functions. It turns out that the set of all operators corresponding to characteristic functions of a sub interval is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}, E))$ for all $p \in (1, \infty)$. This observation extends to operators corresponding to indicator functions of cubes in the case $d > 1$. But for now let us first start with a closer study of $\mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$.

Lemma 4.4. *Let E_0, E_1, E_2 be Banach spaces, $1 < p < \infty$ and $S \subset B^d$ be measurable.*

(a) ***Algebra of multipliers***
Assume $m_1, m_2 \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ and $c_1, c_2 \in \mathbb{C}$. Then the functions $c_1 m_1$ and $c_1 m_1 + c_2 m_2$ belong to $\mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ with

$$T_{c_1 m_1} = c_1 T_{m_1}$$
$$T_{c_1 m_1 + c_2 m_2} = c_1 T_{m_1} + c_2 T_{m_2}.$$

Furthermore if $m_1 \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ and $m_2 \in \mathcal{M}_p(\mathbb{Z}^d, E_1, E_2)$ then $m_2 \circ m_1$ belongs to $\mathcal{M}_p(\mathbb{Z}^d, E_0, E_2)$ with $T_{m_2 \circ m_1} = T_{m_2} \circ T_{m_1}$.

(b) ***Averaging multiplier functions (I)***
Let $\{m(s, \cdot) \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1) : s \in S\}$ be a family of Fourier multiplication functions satisfying the conditions

(i) $\|T_{m(s,\cdot)}\|_{\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))} \leq C$ *for all* $s \in S$,

(ii) $m \in L^\infty(S \times B^d, \mathcal{B}(E_0, E_1))$.

For $h \in L^1(S, \mathbb{C})$ define $m_h : B^d \to \mathcal{B}(E_0, E_1)$ by

$$m_h(\theta) := \int_S m(s, \theta) h(s) ds \text{ for almost all } \theta \in B^d.$$

Then $m_h \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ with $\|T_{m_h}\|_{\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))} \leq C \|h\|_{L^1(S, \mathbb{C})}$ and T_{m_h} is given by

$$T_{m_h} f = \int_S (T_{m(s,\cdot)} f) h(s) ds, \quad \text{for all } f \in l^p(\mathbb{Z}^d, E_0),$$

where the right hand side integral is a Bochner integral in $l^p(\mathbb{Z}^d, E_1)$.

(c) ***Averaging multiplier functions (II)***
Let $\gamma \subset \mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d E_1))$ be $\mathcal{R}$-bounded and define

$$S_\gamma := \Big\{ m \in L^\infty(S \times B^d, \mathcal{B}(E_0, E_1)) : \forall\, s \in S,\ m(s, \cdot) \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1) \\ \text{and } T_{m(s,\cdot)} \in \gamma \Big\}.$$

Then the set $\tilde{\gamma} := \{T_{m_h} : m \in S_\gamma,\ \|h\|_{L^1(S,\mathbb{C})} \le 1\}$ is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$ with $\mathcal{R}(\tilde{\gamma}) \le 2 \cdot \mathcal{R}(\gamma)$.

Proof. (a) The first statements follow directly from the definition of the space $\mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ and linearity of the Fourier Transform. For the last one, we pick a $f \in \Delta_{E_0}$. Then

$$T_{m_1} f = \mathcal{F}\mathcal{M}_{m_1}\mathcal{F}^{-1} f$$

and $\mathcal{M}_{m_1}\mathcal{F}^{-1} f \in L^1(B^d, E_1)$. Thus the operations

$$\begin{aligned}\mathcal{F}\mathcal{M}_{m_2}\mathcal{F}^{-1} T_{m_1} f &= \mathcal{F}\mathcal{M}_{m_2}\mathcal{F}^{-1}\mathcal{F}\mathcal{M}_{m_1}\mathcal{F}^{-1} f = \mathcal{F}\mathcal{M}_{m_2}\mathcal{M}_{m_1}\mathcal{F}^{-1} f \\ &= \mathcal{F}\mathcal{M}_{m_2 \circ m_1}\mathcal{F}^{-1} f\end{aligned}$$

are well defined. Since the left hand side extends to the bounded operator $T_{m_2} \circ T_{m_1} : l^p(\mathbb{Z}^d, E_0) \to l^p(\mathbb{Z}^d, E_2)$, so does the right hand side and we obtain $m_2 \circ m_1 \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_2)$ with $T_{m_2} \circ T_{m_1} = T_{m_2 \circ m_1}$

(b) By assumption (bii) we have

$$\operatorname*{ess\,sup}_{\theta \in B^d} \|m_h(\theta)\|_{\mathcal{B}(E_0,E_1)} \le \sup_{\theta \in B^d} \int_S \|m(s,\theta)\|_{E_1} |h(s)| ds \le C \|h\|_{L^1(S,\mathbb{C})}.$$

Further the function $\theta \mapsto m_h(\theta)$ is measurable as a point wise almost everywhere limit of measurable functions. Hence $m_h \in L^\infty(B^d, \mathcal{B}(E_0, E_1))$. Now for fixed $f \in s(\mathbb{Z}^d, E_0)$, $s \mapsto m(s, \cdot)(\mathcal{F}^{-1} f)(\cdot) h(s)$ is measurable with values in $L^1(B^d, E_1)$ and because of

$$\int_S \|m(s,\cdot)(\mathcal{F}^{-1} f)(\cdot) h(s)\|_{L^1(B^d,E_1)} ds \le C \int_S \|\mathcal{F}^{-1} f\|_{L^1(B^d,E_1)} |h(s)| ds < \infty,$$

the function

$$s \mapsto m(s,\cdot)(\mathcal{F}^{-1} f)(\cdot) h(s)$$

is integrable by Theorem A.5. But $\mathcal{F} \in \mathcal{B}(L^1(B^d, E_1), l^\infty(\mathbb{Z}^d, E_1))$ which yields, using Theorem A.8,

$$\begin{aligned}T_{m_h} f = \mathcal{F}\Big[\int_S m(s,\cdot)(\mathcal{F}^{-1} f)(\cdot) h(s) ds\Big] &= \int_S \mathcal{F}[m(s,\cdot)(\mathcal{F}^{-1} f)(\cdot)] h(s) ds \\ &= \int_S (T_{m(s,\cdot)} f) h(s) ds,\end{aligned} \tag{4.8}$$

where the first integral is in $L^1(B^d, E_1)$ and the latter two are in $l^\infty(\mathbb{Z}^d, E_1)$. Thus as an integrable function, $s \mapsto (T_{m(s,\cdot)}f)h(s)$ is measurable with values in $l^\infty(\mathbb{Z}^d, E_1)$. Since for fixed $s \in S$ the operator $T_{m(s,\cdot)}$ belongs to $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$, we obtain that the function $s \mapsto (T_{m(s,\cdot)}f)h(s)$ is even measurable with values in $l^p(\mathbb{Z}^d, E_1)$. Now (bi) implies

$$\begin{aligned}\int_S \|(T_{m(s,\cdot)}f)h(s)\|_{l^p(\mathbb{Z}^d,E_1)}ds &\le C\int_S \|f\|_{l^p(\mathbb{Z}^d,E_0)}|h(s)|ds \\ &\le C\|f\|_{l^p(\mathbb{Z}^d,E_0)}\|h\|_{L^1(S,\mathbb{C})}.\end{aligned}$$

So once again, by Theorem A.5, the function $s \mapsto (T_{m(s,\cdot)}f)h(s)$ is integrable with values in $l^p(\mathbb{Z}^d, E_1)$. Hence (4.8) is in fact an equality in $l^p(\mathbb{Z}^d, E_1)$ i.e.

$$T_{m_h}f = \int_S (T_{m(s,\cdot)}f)h(s)ds, \tag{4.9}$$

as an integral in $l^p(\mathbb{Z}^d, E_1)$. Clearly $f \mapsto T_{m_h}f$ is linear and because of

$$\begin{aligned}\|T_{m_h}f\|_{l^p(\mathbb{Z}^d,E_1)} &\le \int_S \|T_{m(s,\cdot)}f\|_{l^p(\mathbb{Z}^d,E_1)}|h(s)|ds \\ &\le \sup_{s\in S}\|T_{m(s,\cdot)}\|_{\mathcal{B}(l^p(\mathbb{Z}^d,E_0),l^p(\mathbb{Z}^d,E_1))}\|f\|_{l^p(\mathbb{Z}^d,E_0)}\|h\|_{L^1(S,\mathbb{C})},\end{aligned}$$

T_{m_h} has a continuous extension to an operator in $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$ with norm

$$\|T_{m_h}\|_{\mathcal{B}(l^p(E_0),l^p(E_1))} \le \sup_{s\in S}\|T_{m(\cdot,s)}\|_{\mathcal{B}(l^p(E_0),l^p(E_1))}\|h\|_{L^1(S,\mathbb{C})}.$$

But this means $m_h \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ and we have proven (b).

(c) The assumptions on m are stronger than in (b), hence all calculations above remain valid. In particular (4.9) holds. By the additional assumption, that for fixed $s \in S$ the operator $T_{m(\cdot,s)}$ is an element of an $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$, the claim follows by Lemma 2.58 (b).

□

Remark 4.5. *The Lemma above is a modification of statements given in [KW04] adapted to the given situation, where the essential arguments are the same. We also have the following extension of part* (a)*. Consider a bounded operator $F \in \mathcal{B}(E_1, E_2)$ and $m \in \mathcal{M}_p(E_0, E_1)$. Then $\tilde{F} \circ m \in \mathcal{M}_p(E_1, E_2)$ with $T_{\tilde{F}\circ m} = \tilde{F} \circ T_m$, where $\tilde{F}$ denotes the extension of F to the corresponding function spaces.*

Intervals as Fourier Multiplication Operators

A very useful consequence of the assumption that E is a Banach space of class $\mathcal{HT}$ is, that the indicator function of any sub interval $J \subset B$ is a Fourier multiplication function. To show this we consider a symmetric interval $[-a, a]$ with

$0 < a < 1/2$ first. Let us take a sequence $f : \mathbb{Z}^d \to E$ with finite support, i.e. $f(z) = 0$ for $|z| > N$ and some $N \in \mathbb{N}$. Then for any $z \in \mathbb{Z}$ we get

$$\mathcal{F}\mathcal{M}_{\mathbb{1}_{[-a,a]}}\mathcal{F}^{-1}f(z) = \int_{-a}^{a} e^{-2\pi iz\theta} \sum_{j=-N}^{N} e^{2\pi i\theta j} f(j) d\theta = \sum_{j=-N}^{N} f(j) \int_{-a}^{a} e^{2\pi i(j-z)\theta} d\theta.$$

An evaluation of the last integral yields

$$\left[T_{\mathbb{1}_{[-a,a]}} f\right](z) = \frac{1}{2\pi i}\left[T_{-a}\mathcal{H}_D T_a f\right](z) - \frac{1}{2\pi i}\left[T_a \mathcal{H}_D T_{-a} f\right](z) + 2a \cdot f(z)$$

where $T_\eta \in \mathcal{B}(l^p(\mathbb{Z}, E))$ is given by $[T_\eta f](z) = e^{2\pi iz\eta} f(z)$. Hence the boundedness of $\mathcal{H}_D$ implies the boundedness of $T_{\mathbb{1}_{[-a,a]}}$ with $\|T_{\mathbb{1}_{[-a,a]}}\| \le C(\|\mathcal{H}_D\| + 1)$. An arbitrary interval $J \subset B$ is the translate[3] of a symmetric interval of the form above, i.e. $\mathbb{1}_J = \tau_{\theta_0}\mathbb{1}_{[-a,a]}$ for appropriate $\tau_0, a \in B$. This yields by Lemma 2.17 (b) that $T_{\mathbb{1}_J} = T_{-\theta_0} T_{\mathbb{1}_{[-a,a]}}$ and we obtain from the previous discussion $\|T_{\mathbb{1}_J}\| \le C(\|\mathcal{H}_D\| + 1)$ for all intervals $J \subset B$. These observations are motivating once more, that class $\mathcal{HT}$ is the right assumption for E.

For the set of Fourier multiplier operators corresponding to indicator functions of intervals, we can even obtain a stronger result than (uniform) boundedness, i.e. $\mathcal{R}$-boundedness. Such a statement was first observed by Bourgain [Bou86, Lem. 7] for the circle group. Applying vector-valued transference principles allow the passage to the group $\mathbb{R}$ and $\mathbb{Z}$ [BG94, Lem. 3.5]. Since the direct calculation is not too difficult and gives a general idea how to proof $\mathcal{R}$-boundedness of certain sets, we include it here. In fact it is a direct consequence of the observations above and the following Lemma, which is a special case of Lemma 2.58 (a)

Lemma 4.6. *The set of multiplication operators $\sigma := \{\mathcal{M}_h : \|h\|_{l^\infty(\mathbb{Z},\mathbb{C})} \le 1\}$ is an $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E))$ with $\mathcal{R}_p(\sigma) \le 2$.*

Proof. Let $h_j \in l^\infty(\mathbb{Z}, \mathbb{C})$ with $\|h_j\|_{l^\infty} \le 1$ and $f_j \in l^p(\mathbb{Z}^d, E)$. Then Kahane's contraction principle (Theorem 2.55 (b)) and absolute convergence yields

$$\begin{aligned}\|\sum_{j=1}^{k} r_j \mathcal{M}_{h_j} f_j\|^p_{L^p([0,1],l^p(\mathbb{Z}^d,E))} &= \int_0^1 \sum_{z\in\mathbb{Z}^d} \|\sum_{j=1}^{k} r_j(t) h_j(z) f_j(z)\|_E^p dt \\ &= \sum_{z\in\mathbb{Z}^d} \int_0^1 \|\sum_{j=1}^{k} r_j(t) h_j(z) f_j(z)\|_E^p dt \le 2^p \|\sum_{j=1}^{k} r_j f_j\|^p_{L^p([0,1],l^p(\mathbb{Z}^d,E))}.\end{aligned}$$

□

Since we have already shown in the discussion before this Lemma, how to write the Fourier multiplication operator corresponding to the indicator function of an arbitrary interval, as a composition of multiplication operators corresponding to symmetric intervals and the Hilbert transform, we now get as a consequence the following theorem.

[3] Recall the convention for translates on B from Section 2.1.

Theorem 4.7. *Let $1 < p < \infty$ and E be a Banach space of class $\mathcal{HT}$. Then the set*

$$\gamma_J := \{T_{\mathbb{1}_J} \ : \ J \subset B \text{ is an interval}\}$$

is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}, E))$.

Proof. As before, we find for any interval $J \subset B$ suitable θ_0 and $0 < a \leq 1/2$ with

$$\begin{aligned} T_{\mathbb{1}_J} &= T_{-\theta_0} \circ T_{\mathbb{1}_{[-a,a]}} \\ &= T_{-\theta_0} \circ \big[\frac{1}{2\pi i} T_{-a} \circ \mathcal{H}_D \circ T_a - \frac{1}{2\pi i} T_a \circ \mathcal{H}_D \circ T_{-a} + 2a \cdot \mathrm{id}_{l^p(\mathbb{Z},E)}\big]. \end{aligned}$$

Hence $T_{\mathbb{1}_J} \in \sigma \circ \big(\sigma \circ \{\mathcal{H}_D\} \circ \sigma + \sigma\big)$, which is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}, E))$ by Lemma 4.6 and Lemma 2.56. □

It is not too hard to extend the properties of intervals in B to cubes in B^d for $d \geq 2$. Basically its a component wise application of the results for $d = 1$.

Cubes as Fourier Multiplication Operators

The next Lemma shows how one can pass from a result concerning ($\mathcal{R}$-) boundedness of Fourier multiplication operators in the one dimensional case to a corresponding component wise result in the multi-dimensional case.

Lemma 4.8. *Let E_0, E_1 be Banach spaces, $d \in \mathbb{N} \setminus \{1\}$ and $j \in \{1, \dots, d\}$ be fixed. For a function $m : B \to \mathcal{B}(E_0, E_1)$ define $\tilde{m} : B^d \to \mathcal{B}(E_0, E_1)$ by $\tilde{m}(\theta) := m(\theta_j)$.*

(a) If $m \in \mathcal{M}_p(\mathbb{Z}, E_0, E_1)$, then $\tilde{m} \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ with

$$(T_{\tilde{m}} f)(z) = [(T_m f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d)](z_j),$$

for all $z \in \mathbb{Z}^d$ and $\|T_{\tilde{m}}\|_{\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))} \leq \|T_m\|_{\mathcal{B}(l^p(\mathbb{Z}, E_0), l^p(\mathbb{Z}, E_1))}$.

(b) If $\mathcal{M} \subset \mathcal{M}_p(\mathbb{Z}, E_0, E_1)$ is such that $\gamma := \{T_m : m \in \mathcal{M}\}$ is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}, E_0), l^p(\mathbb{Z}, E_1))$, then $\{T_{\tilde{m}} : m \in \mathcal{M}\}$ is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$ with

$$\mathcal{R}_p(\{T_{\tilde{m}} : m \in \mathcal{M}\}) \leq \mathcal{R}_p(\{T_m : m \in \mathcal{M}\}). \tag{4.10}$$

Proof. We will use again single valued sequences denoted by $\delta_{\tilde{z},z}$, where $\delta_{\tilde{z},z} = 1$ if $z = \tilde{z}$ and $\delta_{\tilde{z},z} = 0$ if $z \neq \tilde{z}$ for all $z \in \mathbb{Z}^d$.

(a) First let f be a function of the form $f(z) = e \cdot \delta_{\tilde{z},z}$ with fixed $\tilde{z} \in \mathbb{Z}^d$ and $e \in E$. For $\theta \in B^d$ denote by $\check{\theta}_j$ the vector $(\theta_1, \dots, \theta_{j-1}, \theta_{j+1}, \dots, \theta_d) \in B^{d-1}$

and accordingly $\check{z}_j$ for $z \in \mathbb{Z}^d$. Then we have for arbitrary $z \in \mathbb{Z}^d$

$$
\begin{aligned}
(T_{\tilde{m}} f)(z) &= \int_{B^d} e^{-2\pi i (z-\tilde{z})\cdot\theta} \tilde{m}(\theta) e d\theta \\
&= \int_{B^{d-1}} e^{-2\pi i (\check{z}_j - \check{\tilde{z}}_j)\cdot\check{\theta}_j} d\check{\theta}_j \int_{-1/2}^{1/2} e^{-2\pi i z_j \theta_j} m(\theta_j) e^{2\pi i \tilde{z}_j \theta_j} e d\theta_j \\
&= \int_{-1/2}^{1/2} e^{-2\pi i z_j \theta_j} m(\theta_j) e^{2\pi i \check{z}_j \check{\theta}_j} \delta_{\check{\tilde{z}}_j, \check{z}_j} e d\theta_j \\
&= (T_m f(z_1, \dots, z_{j-1}, \cdot, z_{j+1} \dots, z_d))(z_j).
\end{aligned} \tag{4.11}
$$

Hence

$$
\begin{aligned}
\|T_{\tilde{m}} f\|^p_{l^p(\mathbb{Z}^d, E_1)} &= \sum_{\check{z}_j \in \mathbb{Z}^{d-1}} \|T_m f(z_1, \dots, z_{j-1}, \cdot, z_{j+1} \dots, z_d)\|^p_{l^p(\mathbb{Z}, E_1)} \\
&\le \|T_m\|^p_{\mathcal{B}(l^p(\mathbb{Z}, E_0), l^p(\mathbb{Z}, E_1))} \|f\|^p_{l^p(\mathbb{Z}^d, E_0)}.
\end{aligned} \tag{4.12}
$$

Now every finite sequence is a finite linear combination of functions of the type above so that (4.11) and (4.12) hold for those functions too. The claim now follows by denseness and bounded extension.

(b) Let $l \in \mathbb{N}$, $m_1, \dots, m_l \in \mathcal{M}$ and $f_1, \dots, f_l \in l^p(\mathbb{Z}^d, E_0)$. Using the same notation as in part (a) we get

$$
\begin{aligned}
&\int_0^1 \|\sum_{k=1}^{l} r_k(t) T_{\tilde{m}_k} f_k\|^p_{l^p(\mathbb{Z}^d, E_1)} dt \\
&= \int_0^1 \sum_{\check{z}_j \in \mathbb{Z}^{d-1}} \|\sum_{k=1}^{l} r_k(t) (T_{m_k} f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d))(\cdot)\|^p_{l^p(\mathbb{Z}, E_1)} dt \\
&= \sum_{\check{z}_j \in \mathbb{Z}^{d-1}} \int_0^1 \|\sum_{k=1}^{l} r_k(t) (T_{m_k} f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d))(\cdot)\|^p_{l^p(\mathbb{Z}, E_1)} dt \\
&\le \mathcal{R}_p^p(\gamma) \sum_{\check{z}_j \in \mathbb{Z}^{d-1}} \int_0^1 \|\sum_{k=1}^{l} r_k(t) f(z_1, \dots, z_{j-1}, \cdot, z_{j+1}, \dots, z_d)(\cdot)\|^p_{l^p(\mathbb{Z}, E_0)} dt \\
&= \mathcal{R}_p^p(\gamma) \int_0^1 \|\sum_{k=1}^{l} r_k(t) f_k\|^p_{l^p(\mathbb{Z}^d, E_0)} dt
\end{aligned}
$$

which shows (4.10). Note that the interchange of integration and summation as well as the splitting of the sum is justified by absolute convergence.

□

Based on this extension we can pass to the multi-dimensional analogue of Theorem 4.7.

Theorem 4.9. *Let E be a Banach space of class $\mathcal{HT}$ and $1 < p < \infty$. Let $\mathcal{Q}$ be the set of all cubes in B^d. Then $\{T_{\mathbb{1}_Q} : Q \in \mathcal{Q}\}$ is an $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E))$.*

Proof. For any $j \in \{1, \ldots, d\}$ consider $\mathcal{Q}_j$ to be the set of cubes having the form $Q_j = B \times \cdots \times B \times J \times B \times \cdots \times B$ where $J \subset B$ is an interval, the product is d-fold and J in the j-th position. An arbitrary cube $Q \subset B^d$ may be written as $Q = \cap_{j=1}^d Q_j$ with suitable Q_j of the form above (compare Figure 4.2). By Theorem 4.7 and Lemma 4.8 (b) the sets $\{T_{\mathbb{1}_{Q_j}} \mid Q_j \in \mathcal{Q}_j\}$ are $\mathcal{R}$-bounded subsets of $\mathcal{B}(l^p(\mathbb{Z}^d, E))$ with

$$\mathcal{R}_p(\{T_{\mathbb{1}_{Q_j}} : Q_j \in \mathcal{Q}_j\}) \leq \mathcal{R}_p(\gamma_J).$$

Now, for every cube $Q \subset B^d$ it holds $\mathbb{1}_Q(\theta) = \prod_{j=1}^d \mathbb{1}_{Q_j}(\theta_j)$ for all $\theta \in B^d$ and appropriately chosen Q_j. Applying Lemma 2.56 shows that $\{T_{\mathbb{1}_Q} : Q \in \mathcal{Q}\}$ is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E))$ with $\mathcal{R}_p(\{T_{\mathbb{1}_Q} : Q \in \mathcal{Q}\}) \leq \mathcal{R}_p(\gamma_J)^d$. $\square$

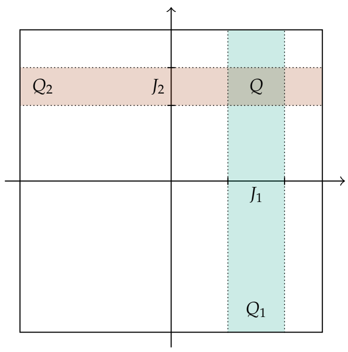

Figure 4.2: Component wise decomposition of a multidimensional cube.

4.3 Paley Littlewood Theory

Our next goal is to establish the so called Paley Littlewood estimate (4.15) for a special decomposition of $B^d \setminus \{0\}$, which we will define in a moment. A decomposition of any set Ω is a family of disjoint subsets, so that the union of all this subsets is Ω.

We first treat the one-dimensional case and lift properties to the multi-dimensional setting using the ideas from Section 4.2.

The One-Dimensional Case

A Paley Littlewood estimate (4.15) in the vector valued setting for $d = 1$ was first obtained in [BG94] and later used in a modified version in [Blu01]. In both cases a special decomposition of the one-dimensional Torus was under consideration. Let us introduce the one used in [BG94] (compare Figure 4.3).

For $j \in \mathbb{Z}$ we define 'dyadic points' by

$$t_j := \begin{cases} -\frac{1}{2} + \frac{1}{2^{j+1}} & : j \leq 0 \\ \frac{1}{2} - \frac{1}{2^{j+1}} & : j > 0 \end{cases}$$

and set $\Delta_j := \{\theta \in B : t_j \leq \theta < t_{j+1}\}$. Note that the family $(\Delta_j)_{j\in\mathbb{Z}}$ is a decomposition of $B \setminus \{\pm 1/2\}$. Since we consider B as a representation of the one-dimensional Torus the points -1/2 and 1/2 are identified.

The Paley Littlewood estimate for the decomposition $(\Delta_j)_{j\in\mathbb{Z}}$ is obtained via a transference argument form a corresponding result for the circle group, that was first proven by Bourgain [Bou86], using sophisticated arguments.

Theorem 4.10 ([BG94, Thm 3.6]). *Let $1 < p < \infty$ and E be a Banach space of class $\mathcal{HT}$. Define $P_j := T_{\mathbb{1}_{\Delta_j}}$. Then for every $f \in l^p(\mathbb{Z}, E)$ the series $\sum_{j\in\mathbb{Z}} P_j f$ converges unconditionally to f in $l^p(\mathbb{Z}^d, E)$. Moreover there is a constant $C_{p,E}$ depending only on E and p such that*

$$C_{p,E}^{-1} \|f\|_{l^p(\mathbb{Z},E)} \leq \| \sum_{j=-\infty}^{\infty} \epsilon_j P_j f \|_{l^p(\mathbb{Z},E)} \leq C_{p,E} \|f\|_{l^p(\mathbb{Z},E)},$$

for all $f \in l^p(\mathbb{Z}, E)$ and all choices $\epsilon_j \in \{-1, 1\}$.

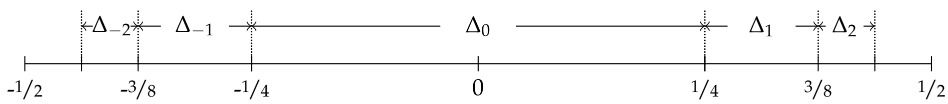

Figure 4.3: The decompostion $(\Delta_j)_{j\in\mathbb{Z}}$ of $B \setminus \{\pm 1/2\}$.

Remark 4.11. *$(\Delta_j)_{j\in\mathbb{Z}}$ is a 'dyadic' decomposition of $B \setminus \{\pm 1/2\}$ where the points $-1/2, 1/2$ are identified. If we want to transfer the situation to a 'dyadic' decomposition of $B \setminus \{a\}$ for some $a \in B$ we only have to introduce a shift.*

Let $a \in B$ and consider the sets $\Delta_j^a := a + \Delta_j$ for $j \in \mathbb{Z}$ where the translation in taken with respect to B, i.e. $\Delta_j^a \subset B$. For the indicator functions of Δ_j^a we obtain $\mathbb{1}_{\Delta_j^a} = \tau_a \mathbb{1}_{\Delta_j}$. Hence we get

$$T_{\mathbb{1}_{\Delta_j^a}} f = T_{\tau_a \mathbb{1}_{\Delta_j}} f = T_{-a} \circ T_{\mathbb{1}_{\Delta_j}} \circ T_a f,$$

where again $T_a f(z) := e^{2\pi i a z} f(z)$. Since both T_{-a} and T_a are scalar-valued multiplication operators, multiplying by a bounded sequence we obtain via Lemma 4.6

Corollary 4.12. *Let $1 < p < \infty$ and E be a Banach space of class $\mathcal{HT}$. Consider for some $a \in B$ the family of operators $P_j^a := T_{\mathbb{1}_{\Delta_j^a}}$. Then for all $f \in l^p(\mathbb{Z}^d, E)$ the series $\sum_{j\in\mathbb{Z}} P_j^a f$ converges unconditionally to f in $l^p(\mathbb{Z}, E)$ and*

$$C_{p,E}^{-1}\|f\|_{l^p(\mathbb{Z},E)} \le \|\sum_{j=-\infty}^{\infty} \epsilon_j P_j^a f\|_{l^p(\mathbb{Z},E)} \le C_{p,E}\|f\|_{l^p(\mathbb{Z},E)},$$

for all $f \in l^p(\mathbb{Z}, E)$ and all choices $\epsilon_j \in \{-1, 1\}$ with the same constant $C_{p,E}$ as in Theorem 4.10.

For the rest of this section the decomposition $(\Delta_j^{1/2})_{j\in\mathbb{Z}}$ of $B \setminus \{0\}$ is of particular interest for us. So we introduce the simplified notation $\Lambda_j := \Delta_j^{1/2}$ and $V_j := T_{\mathbb{1}_{\Lambda_j}}$ for $j \in \mathbb{Z}$. The statement of Corollary 4.12 holds true for $(\Lambda_j)_{j\in\mathbb{Z}}$. In particular, for any Banach space E which is of class $\mathcal{HT}$ and every $p \in (1, \infty)$, there is a constant $C_{p,E}$ such that

$$C_{p,E}^{-1}\|f\|_{l^p(\mathbb{Z},E)} \le \|\sum_{j=-\infty}^{\infty} \epsilon_j V_j f\|_{l^p(\mathbb{Z},E)} \le C_{p,E}\|f\|_{l^p(\mathbb{Z},E)}, \tag{4.13}$$

for all $f \in l^p(\mathbb{Z}^d, E)$ and all choices $\epsilon_j \in \{-1, 1\}$. Moreover $\sum_{j\in\mathbb{Z}} V_j f$ converges unconditionally to f in $l^p(\mathbb{Z}, E)$. Hence we may rearrange the sum and obtain for $\tilde{V}_j := V_j + V_{-j}$ if $j \neq 0$ and $\tilde{V}_0 := V_0$, that $\sum_{j\in\mathbb{N}_0} \tilde{V}_j f$ converges unconditional to f in $l^p(\mathbb{Z}, E)$ and

$$C_{p,E}^{-1}\|f\|_{l^p(\mathbb{Z},E)} \le \|\sum_{j\in\mathbb{N}_0} \epsilon_j \tilde{V}_j f\|_{l^p(\mathbb{Z},E)} \le C_{p,E}\|f\|_{l^p(\mathbb{Z},E)}, \tag{4.14}$$

for all $f \in l^p(\mathbb{Z}^d, E)$ and all choices $\epsilon_j \in \{-1, 1\}$, with the same constant as in (4.13). The decomposition $(V_j)_{j\in\mathbb{Z}}$ is illustrated below.

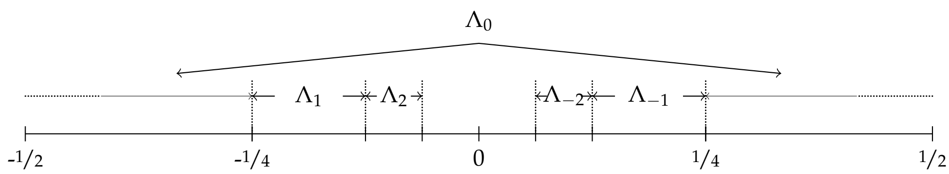

Figure 4.4: The decomposition $(\Lambda_j)_{j\in\mathbb{Z}}$ of $B \setminus \{0\}$.

Finally we note the identity $\tilde{V}_j = T_{\mathbb{1}_{\Lambda_j \cup \Lambda_{-j}}}$ for all $j \in \mathbb{N}_0$.

The Multi-Dimensional Case

The above considerations together with Lemma 4.8 allow us to extend (4.13) to the multidimensional case. As a first step we need the multi-dimensional analogue of the decomposition $(\Lambda_j)_{j\in\mathbb{Z}}$. We use the following notation. For $0 \le R \le 1/2$ we define cubes B_R by

$$B_R := \{\theta \in B^d : |\theta_j| \le R, j = 1 \dots d\}.$$

The multi-dimensional analogue is given by

$$\Lambda_j^d := B_{2^{-j-1}} \setminus B_{2^{-j-2}}$$

for $j \in \mathbb{N}_0$. Before we proceed, we mention that if $D \subset l^p(\mathbb{Z})$ is dense, then the set

$$\Big\{f \in l^p(\mathbb{Z}^d, E) : f = \sum_{j=1}^m f_j(z),\ f_j(z) = \prod_{k=1}^d g_{jk}(z_k)e_{jk},\ g_{jk} \in D,\ e_{jk} \in E\Big\}$$

is dense in $l^p(\mathbb{Z}^d, E)$. Thus we have by Corollary 4.12

Lemma 4.13. *The set $s_0(\mathbb{Z}^d, E) := \{f \in s(\mathbb{Z}^d, E) : 0 \notin supp(\mathcal{F}^{-1}f)\}$ is dense in $l^p(\mathbb{Z}^d, E)$ for all $p \in (1, \infty)$.*

This observation now allows for a simple proof of a general characterization of the Paley Littlewood estimate (4.15) for $l^p(\mathbb{Z}^d, E)$ in the case of a reflexive Banach space E.

Theorem 4.14. *Let E be a reflexive Banach space and $1 < p < \infty$. Further assume that $\mathcal{D} := \{\mathcal{D}_n\}_{n\in\mathbb{N}_0}$ is a decomposition of $B^d \setminus \{0\}$ that satisfies the following geometric conditions*

(i) if $D_n \in \mathcal{D}$ then $-D_n = \{-\theta : \theta \in D_n\} \in \mathcal{D}$,

(ii) if $0 < R < 1/2$ then there is $l \in \mathbb{N}$ such that $B^d \setminus B_R \subset \bigcup_{k=1}^{l} D_{n_k}$ for some $D_{n_1}, \dots, D_{n_l} \in \mathcal{D}$,

(iii) $P_n := T_{\mathbb{1}_{D_n}} \in \mathcal{B}(l^p(\mathbb{Z}^d, E))$ for every $n \in \mathbb{N}_0$.

For such a decomposition, the following statements are equivalent:

*(a) **(unconditional convergence)***
There is a constant $C_{E,p} > 0$ such that for all $f \in l^p(\mathbb{Z}^d, E)$

$$\sup_{\substack{\varepsilon_n\in\mathbb{C}\\|\varepsilon_n|\le1}} \|\sum_{n\in\mathbb{N}_0} \varepsilon_n P_n f\|_{l^p(\mathbb{Z}^d,E)} \le C_{E,p}\|f\|_{l^p(\mathbb{Z}^d,E)}.$$

(b) For some (or equivalently for all) choices $q_1, q_2 \in [1, \infty)$, there is a constant $C_{E,p,q_1,q_2} > 0$ such that

$$\|\sum_{n\in\mathbb{N}_0} r_n P_n f\|_{L^{q_1}([0,1],l^p(\mathbb{Z}^d,E))} \le C_{E,p,q_1,q_2}\|f\|_{l^p(\mathbb{Z}^d,E)},$$

$$\|\sum_{n\in\mathbb{N}_0} r_n P_n' g\|_{L^{q_2}([0,1],l^{p'}(\mathbb{Z}^d,E'))} \le C_{E,p,q_1,q_2}\|g\|_{l^{p'}(\mathbb{Z}^d,E')},$$

for all $f \in l^p(\mathbb{Z}^d, E)$ and $g \in l^{p'}(\mathbb{Z}^d, E')$. Here P_n' is the adjoint operator of P_n, E' the dual of E and r_n is any enumeration of the Rademacher functions.

(c) (**Paley Littlewood estimate for** $l^p(\mathbb{Z}^d, E)$)
For some (or equivalently for all) $q \in [1, \infty)$ *there is a constant* $C_{E,p,q} > 0$ *such that*

$$C_{E,p,q}^{-1} \|f\|_{l^p(\mathbb{Z}^d,E)} \le \| \sum_{n \in \mathbb{N}_0} r_n P_n f \|_{L^q([0,1], l^p(\mathbb{Z}^d,E))} \le C_{E,p,q} \|f\|_{l^p(\mathbb{Z}^d,E)}, \qquad (4.15)$$

for all $f \in l^p(\mathbb{Z}^d, E)$.

Proof. If an inequality in (a)-(c) hold for each $f \in s_0(\mathbb{Z}^d, E)$, then it holds for each $f \in l^p(\mathbb{Z}^d, E)$ (see [KW04, 2.1]). The geometric assumptions imply, that for $f \in s_0(\mathbb{Z}^d, E)$ each sum in (a)-(c) is in fact finite. Moreover each P_n is a projection, i.e. $P_n^2 = P_n$ and the P_n's are orthogonal by which we mean $P_n P_m = 0$ if $n \neq m$.
(a)⇒(b): We have for $f \in s_0(\mathbb{Z}^d, E)$ by (a):

$$\| \sum_{n \in \mathbb{N}_0} r_n P_n f \|_{L^{q_1}([0,1], l^p(\mathbb{Z}^d,E))} \le \| \sum_{n \in \mathbb{N}_0} r_n P_n f \|_{L^\infty([0,1], l^p(\mathbb{Z}^d,E))} \le C_{E,p} \|f\|_{l^p(\mathbb{Z}^d,E)}.$$

For $g \in l^{p'}(\mathbb{Z}^d, E')$ we obtain form (a), because of $\|T\| = \|T'\|$, $P_n' = T_{\mathbb{1}_{-D_n}}$ and $(\sum_{n=0}^{M} \epsilon_n P_n)' = \sum_{n=0}^{M} \epsilon_n P_n'$ that

$$\sup_{\substack{\varepsilon_n \in \mathbb{C} \\ |\varepsilon_n| \le 1}} \| \sum_{n \in \mathbb{N}_0} \varepsilon_n P_n' g \|_{l^{p'}(\mathbb{Z}^d,E')} \le C_{E,p} \|g\|_{l^{p'}(\mathbb{Z}^d,E')}.$$

Thus the same calculation as before shows for $g \in s_0(\mathbb{Z}^d, E')$

$$\| \sum_{n \in \mathbb{N}_0} r_n P_n' g \|_{L^{q_2}([0,1], l^{p'}(\mathbb{Z}^d,E'))} \le C_{E,p} \|g\|_{l^{p'}(\mathbb{Z}^d,E')}.$$

(b)⇒(c): The inequality on the right hand side is part of (b). For the lower inequality in (c) we take $f \in s_0(\mathbb{Z}^d, E)$, $g \in s_0(\mathbb{Z}^d, E')$ and obtain by the properties of the decomposition D_n

$$f = \sum_{n \in \mathbb{N}_0} P_n f \qquad \text{and} \qquad g = \sum_{n \in \mathbb{N}_0} P_n' g.$$

Hence for any fixed $f \in s_0(\mathbb{Z}^d, E)$ and $g \in s_0(\mathbb{Z}^d, E')$

$$\begin{aligned} \langle f, g \rangle &= \langle \sum_{n \in \mathbb{N}_0} P_n f, \sum_{m \in \mathbb{N}_0} P_m' g \rangle = \sum_{n \in \mathbb{N}_0} \langle P_n f, P_n' g \rangle = \sum_{n \in \mathbb{N}_0} \int_0^1 r_n^2(t) \langle P_n f, P_n' g \rangle dt \\ &= \int_0^1 \langle \sum_{n \in \mathbb{N}_0} r_n(t) P_n f, \sum_{m \in \mathbb{N}_0} r_m(t) P_m' g \rangle dt, \end{aligned}$$

where we used orthogonality of the r_n's and P_n's. Applying Hölder's inequality yields

$$|\langle f, g \rangle| \le \| \sum_{n \in \mathbb{N}_0} r_n(\cdot) P_n f \|_{L^q([0,1], l^p(\mathbb{Z}^d,E)))} \| \sum_{n \in \mathbb{N}_0} r_n(\cdot) P_n' g \|_{L^{q'}([0,1], l^{p'}(\mathbb{Z}^d,E')))}.$$

Together with the second inequality in (b), this leads to[4]

$$\|f\|_{l^p(\mathbb{Z}^d,E)} = \sup_{\substack{g\in s_0(\mathbb{Z}^d,E') \\ \|g\|_{l^{p'}(\mathbb{Z}^d,E')\leq 1}}} |\langle f,g\rangle| \leq C_{E,p,q'} \| \sum_{n\in\mathbb{N}_0} r_n(\cdot)P_n f\|_{L^q([0,1],l^p(\mathbb{Z}^d,E)))}$$

which shows the lower inequality.
(c)⇒(a): Let $f \in s_0(\mathbb{Z}^d, E)$ and $(\epsilon_n)_{n\in\mathbb{N}} \subset \mathbb{C}$ with $|\epsilon_n| \leq 1$. We obtain again by orthogonality of the r_n's and P_n's as well as Kahane' contraction principle (Theorem 2.55)

$$\begin{aligned}\| \sum_{n\in\mathbb{N}_0} \epsilon_n P_n f\|_{l^p(\mathbb{Z}^d,E)} &\leq C_{E,p}\| \sum_{m\in\mathbb{N}_0} r_m P_m \sum_{n\in\mathbb{N}_0} \epsilon_n P_n f\|_{L^p([0,1],l^p(\mathbb{Z}^d,E))} \\ &= C_{E,p}\| \sum_{n\in\mathbb{N}_0} \epsilon_n r_n P_n f\|_{L^p([0,1],l^p(\mathbb{Z}^d,E))} \\ &\leq 2C_{E,p}\| \sum_{n\in\mathbb{N}_0} r_n P_n f\|_{L^p([0,1],l^p(\mathbb{Z}^d,E))} \\ &\leq 2C_{E,p}^2 \|f\|_{l^p(\mathbb{Z}^d,E)}.\end{aligned}$$

Taking the supremum as in the statement of (a) yields the assertion. □

We mention that the decomposition $(\Lambda_j^d)_{n\in\mathbb{N}_0}$ satisfies the geometric assumptions of Theorem 4.14. Further each Λ_j^d can be decomposed into a disjoint union of cubes. For $j \in \mathbb{N}_0$ and $k \in \{1,\dots,d\}$ we define

$$\begin{aligned}\Lambda_{j,k} &:= \{\theta \in B^d : 2^{-j-2} < |\theta_k| \leq 2^{-j-1}\}, \\ B_{j,k} &:= \{\theta \in B^d : |\theta_k| \leq 2^{-j-1}\}\end{aligned}$$

then,

$$\Lambda_j^d = \bigcup_{k=1}^{d} [B_{j+1,1} \cap \cdots \cap B_{j+1,k-1} \cap \Lambda_{j,k} \cap B_{j,k+1} \cap \cdots \cap B_{j,d}], \tag{4.16}$$

and each set of the union on the right hand side is a cube as illustrated below.

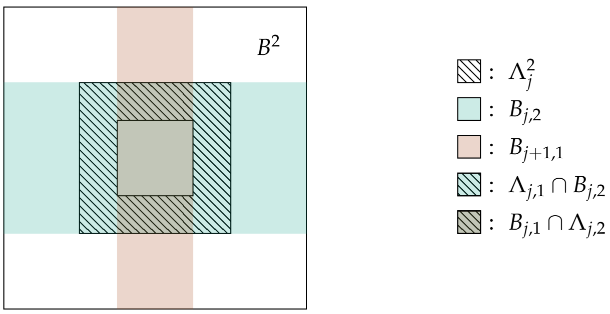

Figure 4.5: The decomposition of Λ_j into the disjoint union (4.16).

[4] Note that $s_0(\mathbb{Z}^d, E')$ is as a dense subspace of $l^{p'}(\mathbb{Z}^d, E')$ which is norming for $l^p(\mathbb{Z}^d, E)$.

According to (4.16) the indicator function of any set Λ_j^d can be written in the form

$$\mathbb{1}_{\Lambda_j^d} = \sum_{k=1}^{d} \mathbb{1}_{B_{j+1,1}} \cdots \mathbb{1}_{B_{j+1,k-1}} \cdot \mathbb{1}_{\Lambda_{j,k}} \cdot \mathbb{1}_{B_{j,k+1}} \cdots \mathbb{1}_{B_{j,d}}. \tag{4.17}$$

With this decomposition for the sets Λ_j^d we can proof the Paley Littlewood estimate for the decomposition $(\Lambda_j^d)_{j\in\mathbb{N}_0}$ of B^d, by a reduction to the one dimensional case. Note, that each set in the intersection of (4.16) is a multidimensional variant of a set covered by (4.13) and Lemma 4.8.

Theorem 4.15. *Let E be a Banach space of class $\mathcal{HT}$ and $1 < p < \infty$. For every $q \in [1,\infty)$ there is a constant $C_{p,q,E}$ such that*

$$C_{E,p,q}^{-1}\|f\|_{l^p(\mathbb{Z}^d,E)} \leq \Big\| \sum_{j\in\mathbb{N}_0} r_j T_{\mathbb{1}_{\Lambda_j^d}} f \Big\|_{L^q([0,1],l^p(\mathbb{Z}^d,E))} \leq C_{E,p,q}\|f\|_{l^p(\mathbb{Z}^d,E)}, \tag{4.18}$$

for all $f \in l^p(\mathbb{Z}^d, E)$.

Proof. By Lemma 4.4 (a) we have $T_{\mathbb{1}_{\Lambda_j^d}} = T_{\mathbb{1}_{B_{2^{-j-1}}}} - T_{\mathbb{1}_{B_{2^{-j-2}}}}$. Hence Theorem 4.9 yields $T_{\mathbb{1}_{\Lambda_j^d}} \in \mathcal{B}(l^p(\mathbb{Z}^d, E))$ for every $j \in \mathbb{N}_0$. A combination of (4.17) and Lemma 4.4 (a) gives

$$T_{\mathbb{1}_{\Lambda_j^d}} = \sum_{k=1}^{d} T_{\mathbb{1}_{B_{j+1,1}}} \circ \cdots \circ T_{\mathbb{1}_{B_{j+1,k-1}}} \circ T_{\mathbb{1}_{\Lambda_{j,k}}} \circ T_{\mathbb{1}_{B_{j,k+1}}} \circ \cdots \circ T_{\mathbb{1}_{B_{j,d}}} = \sum_{k=1}^{d} R_{j,k} \circ T_{\Lambda_{j,k}}$$

with $R_{j,k} = T_{\mathbb{1}_{B_{j+1,1}}} \circ \cdots \circ T_{\mathbb{1}_{B_{j+1,k-1}}} \circ T_{\mathbb{1}_{B_{j,k+1}}} \circ \cdots \circ T_{\mathbb{1}_{B_{j,d}}}$ (here we used the identity $T_{\mathbb{1}_A} \circ T_{\mathbb{1}_B} = T_{\mathbb{1}_B} \circ T_{\mathbb{1}_A}$, which holds true for all subsets $A, B \subset B^d$ for which the operators $T_{\mathbb{1}_A}, T_{\mathbb{1}_B}$ exist).

Since $T_{\mathbb{1}_{\Lambda_{j,k}}}$ is the multi-dimensional version of $\tilde{V}_j$ applied in direction z_k, we obtain with the notation of Lemma 4.8 from (4.13)

$$\begin{aligned} \Big\| \sum_{j\in\mathbb{N}_0} \epsilon_j T_{\mathbb{1}_{\Lambda_{j,k}}} f \Big\|_{l^p(\mathbb{Z}^d,E)}^p &= \sum_{\check{z}_k\in\mathbb{Z}^{d-1}} \Big\| \sum_{j\in\mathbb{N}_0} \epsilon_j [\tilde{V}_j f(z_1,\ldots,z_{k-1},\cdot,z_{k+1},\ldots,z_d)] \Big\|_{l^p(\mathbb{Z},E)}^p \\ &\leq C_{p,E}^p \|f\|_{l^p(\mathbb{Z}^d,E)}^p, \end{aligned} \tag{4.19}$$

for all $f \in l^p(\mathbb{Z}^d, E)$ and every choice $\epsilon_j \in \{-1, 1\}$. Furthermore, Theorem 4.9 together with Lemma 2.56 shows, that $\{R_{j,k} : j \in \mathbb{N}_0, k \in \{1,\ldots,d\}\}$ is a $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E))$. Now for any sequence $f \in s_0(\mathbb{Z}^d, E)$ we obtain by (4.19)

$$\begin{aligned}
\int_0^1 \| \sum_{j\in\mathbb{N}_0} r_j(t) T_{\mathbb{1}_{\Lambda_j^d}} f\|_{l^p(\mathbb{Z}^d,E)} dt &= \int_0^1 \| \sum_{j\in\mathbb{N}_0} r_j(t) \big(\sum_{k=1}^d R_{j,k}\circ T_{\mathbb{1}_{\Lambda_{j,k}}} \big) f\|_{l^p(\mathbb{Z}^d,E)} dt \\
&\le \sum_{k=1}^d \int_0^1 \| \sum_{j\in\mathbb{N}_0} r_j(t) R_{j,k}\circ T_{\mathbb{1}_{\Lambda_{n,k}}} f\|_{l^p(\mathbb{Z}^d,E)} dt \\
&\le \mathcal{R}_1\big(\{R_{j,k} : j\in\mathbb{N}_0, k=1,\dots,d\}\big) \sum_{k=1}^d \int_0^1 \| \sum_{n\in\mathbb{N}} r_j(t) T_{\mathbb{1}_{\Lambda_{j,k}}} f\|_{l^p(\mathbb{Z}^d,E)} dt \\
&\le \mathcal{R}_1\big(\{R_{j,k}\}_{j,k}\big) \sum_{k=1}^d \sup_{t\in[0,1]} \| \sum_{j\in\mathbb{N}_0} r_j(t) T_{\mathbb{T}_{\Lambda_{j,k}}} f\|_{l^p(\mathbb{Z}^d,E)} \\
&\le \mathcal{R}_1\big(\{R_{j,k}\}_{j,k}\big) \sum_{k=1}^d \sup_{\epsilon_j\in\{-1,1\}} \| \sum_{j\in\mathbb{N}_0} \epsilon_j T_{\mathbb{1}_{\Lambda_{j,k}}} f\|_{l^p(\mathbb{Z}^d,E)} \\
&\le d\cdot \mathcal{R}_1\big(\{R_{j,k}\}_{j,k}\big)\cdot C_{p,E}\|f\|_{l^p(\mathbb{Z}^d,E)}.
\end{aligned}$$

Denseness of $s_0(\mathbb{Z}^d,E)$ in $l^p(\mathbb{Z}^d,E)$ implies that the first inequality of Theorem 4.14 (b) is satisfied. For the second one we recall from Section 2.5 that with E also E' is of class $\mathcal{HT}$. Further the adjoint of $T_{\mathbb{1}_{\Lambda_j^d}}$ is $T_{\mathbb{1}_{\Lambda_j^d}}$ and the dual space of $l^p(\mathbb{Z}^d,E)$ may be identified with $l^{p'}(\mathbb{Z}^d,E')$. Thus we can preform the same calculation as above in the dual setting, changing only the constant $C_{p,E}$ to $C_{p',E'}$ which shows the validity of Theorem 4.14 (b). □

4.4 Multiplier Theorems for the Fourier Transform

After this preparations we are able to give sufficient conditions for a function $m\in L^\infty(B^d,\mathcal{B}(E_0,E_1))$ to be contained in $\mathcal{M}_p(\mathbb{Z}^d,E_0,E_1)$. As in the scalar valued case, for the Fourier Transform on $\mathbb{R}^d$, it seems to be impossible to find an characterization of $\mathcal{M}_p(\mathbb{Z}^d,E_0,E_1)$. We start with a typical Stečkin's type multiplier theorem. Within the proof we use the fundamental theorem of calculus, which involves boundary values. For this reason we introduce the following notation.

Let $0\neq\alpha\le(1,\dots,1)$ be a multi index in $\mathbb{N}_0^d$ and $(i_j)_{j=1,\dots,m}$ be an enumeration of the non-zero components of α, i.e. $\alpha_{i_j}\neq 0$. For $k\in\{1,\dots,d\}$ and $s=(s_1,\dots,s_d)\in B^d$ define an element $s_\alpha\in B^d$ by

$$(s_\alpha)_k := \begin{cases} s_k & : \ k=i_j \text{ for some } j\in 1,\dots,m \\ -1/2 & : \ \text{else.} \end{cases}$$

For any integrable function $g: B^d\to E$ and $t\in B^d$ we define

$$\alpha\!\!-\!\!\!\int_{-1/2}^{t} g(s_\alpha)ds_\alpha := \int_{-1/2}^{t_{i_1}}\cdots\int_{-1/2}^{t_{i_m}} g(s_\alpha)ds_{i_m}\cdots ds_{i_1},$$

in particular we integrate the function only with respect to those variables, that correspond to an index j for which $\alpha_j \neq 0$. All the other components get fixed at the value $-1/2$. Consequently if $\alpha = 0$ we set $s_0 := (-1/2, \dots, -1/2)$ and

$$0\!\!\!-\!\!\!\!\int_{-1/2}^{t} g(s_0)ds_0 := g(-1/2, \dots, -1/2).$$

With this notation we have

Lemma 4.16.

(i) If $g \in C^d(B^d, E)$ and $t \in B^d$, then

$$g(t) = \sum_{\alpha \le (1,\dots,1)} \alpha\!\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^\alpha g(s_\alpha)ds_\alpha.$$

(ii) If $g \in C^d(B^d, E)$, $t \in B^d$ and $\alpha \in \mathbb{N}_0^d$ with $\alpha \le (1, \dots .1)$ fixed. Then

$$\alpha\!\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^\alpha g(s_\alpha)ds_\alpha = \int_{B^d} Q_{s,\alpha}(t)\partial^\alpha g(s_\alpha)ds,$$

where $Q_{s,\alpha}(t) = \prod_{k=1}^{d} \mathbb{1}_{[(s_\alpha)_k, 1/2]}(t_k)$.

Proof. (i) Let us start with the case $d = 1$. Then by Theorem A.10

$$g(t) - g(-1/2) = \int_{-1/2}^{t} g'(s)ds.$$

Hence $g(t) = \sum_{\alpha \in \{0,1\}} \alpha\!-\!\!\int_{-1/2}^{t} \partial^\alpha g(s_\alpha)ds_\alpha$. Denote by $\beta_d := (1, \dots, 1)$ the d-dimensional multi index with all components equal to 1 and assume the statement is true for $d - 1$. Then we have for $t \in B^d$

$$g(t_1, \dots, t_{d-1}, t_d) - g(t_1, \dots, t_{d-1}, -1/2) = \int_{-1/2}^{t_d} \partial_d g(t_1, \dots, t_{d-1}, s_d)ds_d$$

or equivalently by Fubini's Theorem (A.11) and Theorem A.9

$$\begin{aligned} g(t) &= \int_{-1/2}^{t_d} \partial_d g(t_1, \dots, t_{d-1}, s_d)ds_d + \sum_{\alpha \le \beta_{d-1}} \alpha\!\!\!-\!\!\!\!\int_{-1/2}^{(t_1,\dots,t_{d-1})} \partial^\alpha g(s_\alpha, -1/2)ds_\alpha \\ &= \sum_{\alpha \le \beta_{d-1}} \alpha\!\!\!-\!\!\!\!\int_{-1/2}^{(t_1,\dots,t_{d-1})} \int_{-1/2}^{t_d} \partial_d \partial^\alpha g(s_\alpha, s_d)ds_d ds_\alpha \\ &\qquad + \sum_{\alpha \le \beta_{d-1}} \alpha\!\!\!-\!\!\!\!\int_{-1/2}^{(t_1,\dots,t_{d-1})} \partial^\alpha g(s_\alpha, -1/2)ds_\alpha \\ &= \sum_{\alpha \le \beta_{d-1}} (\alpha, 1)\!\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^{(\alpha,1)} g(s_{(\alpha,1)})ds_{(\alpha,1)} \\ &\qquad + \sum_{\alpha \le \beta_{d-1}} (\alpha, 0)\!\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^{(\alpha,0)} g(s_{(\alpha,0)})ds_{(\alpha,0)} \\ &= \sum_{\alpha \le \beta_d} \alpha\!\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^\alpha g(s_\alpha)ds_\alpha. \end{aligned}$$

This finishes the induction.

(ii) We observe $\mathbb{1}_{[-1/2,t]}(s) = \mathbb{1}_{[s,1/2]}(t)$ for all $t, s \in [-1/2, 1/2]$. Let us again treat the case $d = 1$ first. If $\alpha = 0$ we have by definition

$$\alpha\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^\alpha g(s_\alpha) ds_\alpha = g(-1/2) = \int_{-1/2}^{1/2} g(-1/2) ds$$
$$= \int_{-1/2}^{1/2} \mathbb{1}_{[-1/2,1/2]}(s) g(-1/2) ds = \int_{-1/2}^{1/2} Q_{s,0}(t) \partial^0 g(s_0) ds.$$

If $\alpha = 1$ we obtain

$$\alpha\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^\alpha g(s_\alpha) ds_\alpha = \int_{-1/2}^{t} g'(s) ds = \int_{-1/2}^{1/2} \mathbb{1}_{[-1/2,t]}(s) g'(s) ds$$
$$= \int_{-1/2}^{1/2} \mathbb{1}_{[s,1/2]}(t) \partial^1 g(s_1) ds = \int_{-1/2}^{1/2} Q_{s,1}(t) \partial^1 m(s_1) ds.$$

Hence the statement is true for $d = 1$. Assume again, that it is also true for $d - 1$ where $d \geq 2$. Then we get for any $\alpha \leq \beta_{d-1}$

$$(\alpha, 0)\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^{(\alpha,0)} g(s_{(\alpha,0)}) ds_{(\alpha,0)} = \alpha\!\!-\!\!\!\!\int_{(-1/2,\dots,-1/2)}^{(t_1,\dots,t_{d-1})} \partial^\alpha g(s_\alpha, -1/2) ds_\alpha$$
$$= \int_{B^{d-1}} Q_{s,\alpha}(t_1, \dots, t_{d-1}) \partial^\alpha g(s_\alpha, -1/2) ds$$
$$= \int_{B^{d-1}} Q_{s,\alpha}(t_1, \dots, t_{d-1}) \int_{-1/2}^{1/2} \mathbb{1}_{[-1/2,1/2]}(t_d) \partial^\alpha g(s_\alpha, -1/2) dt_d ds$$
$$= \int_{B^d} Q_{s,(\alpha,0)}(t) \partial^{(\alpha,0)} g(s_{(\alpha,0)}) ds$$

and

$$(\alpha, 1)\!\!-\!\!\!\!\int_{-1/2}^{t} \partial^{(\alpha,1)} g(s_{(\alpha,1)}) ds_{(\alpha,1)} = \int_{-1/2}^{t_d} \alpha\!\!-\!\!\!\!\int_{(-1/2,\dots,-1/2)}^{(t_1,\dots,t_{d-1})} \partial_d \partial^\alpha g(s_\alpha, y) ds_\alpha dy$$
$$= \int_{-1/2}^{t_d} \int_{B^{d-1}} Q_{s,\alpha}(t_1, \dots, t_{d-1}) \partial_d \partial^\alpha g(s_\alpha, y) ds dy$$
$$= \int_{B^{d-1}} Q_{s,\alpha}(t_1, \dots, t_{d-1}) \int_{-1/2}^{1/2} \mathbb{1}_{[-1/2,t_d]}(y) \partial_d \partial^\alpha g(s_\alpha, y) dy ds$$
$$= \int_{B^{d-1}} Q_{s,\alpha}(t_1, \dots, t_{d-1}) \int_{-1/2}^{1/2} \mathbb{1}_{[y,1/2]}(t_d) \partial_d \partial^\alpha g(s_\alpha, y) dy ds$$
$$= \int_{B^d} Q_{s,(\alpha,1)}(t) \partial^{(\alpha,1)} g(s_{(\alpha,1)}) ds$$

and we have shown (ii).

□

Now (i) and (ii) from the Lemma above, allow for a representation of $g(t)$ for sufficiently smooth g in terms of

$$g(t) = \sum_{\alpha \leq (1,\dots,1)} \int_{B^d} Q_{s,\alpha}(t) \partial^\alpha m(s_\alpha) ds, \tag{4.20}$$

for all $t \in B^d$, which might be seen as a sum of weighted averages of the derivatives of g.

Theorem 4.17. *Let E_0, E_1 Banach spaces of class $\mathcal{HT}$ and $p \in (1,\infty)$.*

(i) If $m \in C^d(B^d, \mathcal{B}(E_0,E_1))$ then $m \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$.

(ii) Let $\tau \subset \mathcal{B}(E_0,E_1)$ be $\mathcal{R}$-bounded. Define for any constant $c>0$

$$\begin{aligned}\mathfrak{S}_c(\tau) := \{ m \in C^d(B^d, \mathcal{B}(E_0,E_1)) : \forall \alpha \in \mathbb{N}_0^d \text{ with } \alpha \le (1,\dots,1) \\ \exists h_\alpha \in L^1(B^d,\mathbb{C}) \text{ with } \|h_\alpha\|_{L^1} \le c \text{ and a } \tau\text{-valued function} \\ n_\alpha \in L^\infty(B^d, \mathcal{B}(E_0,E_1)) \text{ s.t. } \forall \theta \in B^d,\ \partial^\alpha m(\theta) = h_\alpha(\theta) n_\alpha(\theta)\}.\end{aligned}$$

Then $\gamma_{c,\tau} := \{T_m : m \in \mathfrak{S}_c(\tau)\}$ is $\mathcal{R}$-bounded in $\mathcal{B}(l^p(\mathbb{Z}^d,E_0), l^p(\mathbb{Z}^d,E_1))$ with $\mathcal{R}_p(\gamma_{c,\tau}) \le cC_{E_0,p,d}\mathcal{R}_p(\tau)$.

Proof. (i) Recall (4.20) and observe for $t \in B^d$

$$m(t) = \sum_{\alpha \le (1,\dots,1)} \int_{B^d} Q_{s,\alpha}(t) m_\alpha(s) ds,$$

where $Q_{s,\alpha}(t) = \prod_{k=1}^d \mathbb{1}_{[(s_\alpha)_k,1]}(t_k)$ and $m_\alpha(s) = \partial^\alpha m(s_\alpha)$. For fixed $s \in B^d$ we have by assumption, that $m_\alpha(s)$ is an element of $\mathcal{B}(E_0,E_1)$. Hence by Theorem 4.9 and Remark 4.5

$$T_{Q_{s,\alpha}\cdot m_\alpha(s)} = \widetilde{m_\alpha(s)} \cdot T_{Q_{s,\alpha}},$$

where $\widetilde{m_\alpha(s)}$ denotes the extension of $m_\alpha(s) \in \mathcal{B}(E_0,E_1)$ to a bounded and linear operator $l^p(\mathbb{Z}^d,E_0) \to l^p(\mathbb{Z}^d,E_1)$ given by $[\widetilde{m_\alpha(s)} f](z) := m_\alpha(s)f(z)$ for all $z \in \mathbb{Z}^d$. Moreover, for fixed $s \in B^d$ the function $t \mapsto Q_{s,\alpha}(t)$ is the indicator function of a cube. Thus we have by Theorem 4.9

$$\|T_{Q_{s,\alpha}\cdot m_\alpha(s)}\|_{\mathcal{B}(l^p(E_0),l^p(E_1))} \le \mathcal{R}_p(\gamma_I)^d \|m_\alpha(s)\|_{\mathcal{B}(E_0,E_1)},$$

where $\mathcal{R}_p(\gamma_I)$ is the $\mathcal{R}$-bound of the set of multiplier operators corresponding to indicator functions of a interval in the case $d=1$. The continuity of the mapping $s \mapsto m_\alpha(s)$ combined with compactness of B^d yields $\|T_{Q_{s,\alpha}\cdot m_\alpha(s)}\|_{\mathcal{B}(l^p(E_0),l^p(E_1))} \le C < \infty$ for all $s \in B^d$. Since $Q_{s,\alpha}(t)$ is bounded by 1 for all possible choices of $s,t \in B^d$ we also obtain

$$\|Q_{s,\alpha}(t)m_\alpha(s)\|_{\mathcal{B}(E_0,E_1)} \le C.$$

Hence Lemma 4.4 (b) applies and yields

$$g_\alpha(t) := \int_{B^d} Q_{s,\alpha}(t)\cdot m_\alpha(s)ds \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1). \tag{4.21}$$

Since m is a finite sum of of Fourier multiplication functions g_α, Lemma 4.4 (a) finishes the first part.

(ii) Let $m \in \mathfrak{S}_c(\tau)$ with $\partial^\alpha m = h_\alpha \cdot n_\alpha$ for $\alpha \leq (1,\dots,1)$. As before

$$m(t) = \sum_{\alpha \leq (1,\dots,1)} \int_{B^d} h_\alpha(s_\alpha) n_{s,\alpha}(t) ds \tag{4.22}$$

this time with $n_{s,\alpha}(t) := Q_{s,\alpha}(t) \cdot n_\alpha(s_\alpha)$ for $s,t \in B^d$. For fixed $s \in B^d$ and $\alpha \in \mathbb{N}_0^d$ with $\alpha \leq (1,\dots,1)$ we have $n_\alpha(s_\alpha) \in \mathcal{B}(E_0, E_1)$. The same argument as before shows that $n_{s,\alpha}$ is an element of $\mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$. But by assumption $n_\alpha(s_\alpha) \in \tau$, so that $T_{n_{s,\alpha}} = T_{Q_{s,\alpha}} \cdot \widetilde{n_\alpha(s_\alpha)}$ is an element of

$$\eta := \{T_{\mathbb{1}_Q \cdot B} : Q \subset B^d \text{ is a cube, } B \in \tau\} = \{T_{\mathbb{1}_Q} : Q \subset B^d \text{ is a cube}\} \circ \tilde{\tau}$$

which is an $\mathcal{R}$-bounded subset of $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$ by Theorem 4.9, Lemma 2.58 (c) and Lemma 2.56. Hence $n_{s,\alpha} \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ and $T_{n_{s,\alpha}} \in \eta$ where η is $\mathcal{R}$-bounded. The additional assumption $\|h_\alpha\|_{L^1} \leq c$ implies via Lemma 4.4 (c) that every integral on the right hand side in (4.22) defines a bounded Fourier multiplication operator. Moreover this operator is contained is the $\mathcal{R}$-bounded subset $c \cdot \tilde{\eta}$, where $\tilde{\eta}$ given in Lemma 4.4 (c). By (4.22) the Fourier multiplication operator corresponding to m is given by a finite sum of operators contained in $c \cdot \tilde{\eta}$. Thus m is a Fourier multiplication functions by Lemma 4.4 (a). Moreover $T_m \in C_1\text{absco}(c\tilde{\eta})$ for some $C_1 > 0$ and all $m \in \mathfrak{S}_C(\tau)$. But $C_1\text{absco}(c\tilde{\eta})$ is $\mathcal{R}$-bounded by Lemma 4.4 (a) and Lemma 2.57.

□

Remark 4.18. *Since functions $m \in C^d(B^d, \mathcal{B}(E_0, E_1))$ are of bounded variation part (i) of Theorem 4.17 may be seen as a special version of Stečkin Multiplier Theorem, see [EG77, Thm. 6.2.5].*

After this preparatory work we are now in the position to give a multiplier result, that allows the function m to have some oscillations at point 0. Let us mention that there is nothing special about the point 0. If the oscillation occurs at $a \in B^d$ we apply the shift τ_{-a} to reduce the situation to oscillations at the point 0. We have seen before how shifts on a multiplication function behave and do not affect the norm of the associated Fourier multiplication operator.

Theorem 4.19. *Assume E_0, E_1 are Banach spaces of class $\mathcal{HT}$. If we have a function $m \in C^d(B^d \setminus \{0\}, \mathcal{B}(E_0, E_1))$ such that the set*

$$\tau := \left\{|\theta|^{|\alpha|}\partial^\alpha m(\theta) : \theta \in B^d \setminus \{0\}, \alpha \leq (1,\dots,1)\right\}$$

is a $\mathcal{R}$-bounded subset of $\mathcal{B}(E_0, E_1)$, then $m \in \mathcal{M}_p(\mathbb{Z}^d, E_0, E_1)$ for all $p \in (1, \infty)$ with

$$\|T_m\|_{B(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))} \leq C\mathcal{R}_p(\tau),$$

for some constant C depending on p, d, E_0 and E_1.

Proof. Recall the decomposition of $B^d \setminus \{0\}$ in terms of $(\Lambda_j^d)_{j\in\mathbb{N}_0}$ given in (4.16). Now choose a functions $\varphi_0, \varphi_1 \in C^\infty_{\text{per}}(B^d)$ with the properties

$$\varphi_0 \equiv 1 \text{ on } \Lambda_0^d, \; \text{supp}\varphi_0 \subset \Lambda_0^d \cup \Lambda_1^d$$
$$\varphi_1 \equiv 1 \text{ on } \Lambda_1^d, \; \text{supp}\varphi_1 \subset B_{3/8} \setminus B_{3/32}$$

and define for $j \geq 2$ and $\theta \in B^d$, $\varphi_j(\theta) := \varphi_1(2^{j-1}\theta)$. Then $\varphi_j \equiv 1$ on Λ_j^d and $\text{supp}\varphi_j \subset \Lambda_{j-1}^d \cup \Lambda_j^d \cup \Lambda_{j+1}^d$.
Consider for $j \in \mathbb{N}_0$ the functions $m_j := \varphi_j m \in C^d(B^d, \mathcal{B}(E_0, E_1))$. Then we have for every $\alpha \in \mathbb{N}_0^d$ with $\alpha \leq (1, \ldots, 1)$ and all $\theta \in B^d$, because of $0 \notin \text{supp}\varphi_j$

$$\begin{aligned}
\partial^\alpha m_j(\theta) &= \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} (\partial^{\alpha-\gamma}\varphi_j)(\theta)\partial^\gamma m(\theta) \\
&= \frac{1}{|\theta|^{|\alpha|}} \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} |\theta|^{|\alpha|-|\gamma|}(\partial^{\alpha-\gamma}\varphi_j)(\theta)|\theta|^{|\gamma|}\partial^\gamma m(\theta) \\
&= h_{j,\alpha}(\theta) \sum_{\gamma \leq \alpha} n_{j,\alpha,\gamma}(\theta),
\end{aligned}$$

where $n_{j,\alpha,\gamma}(\theta) := \binom{\alpha}{\gamma}|\theta|^{|\alpha|-|\gamma|}(\partial^{\alpha-\gamma}\varphi_j)(\theta)|\theta|^{|\gamma|}\partial^\gamma m(\theta)$ and

$$h_{j,\alpha}(\theta) := \frac{1}{|\theta|^{|\alpha|}} \cdot \begin{cases} \mathbb{1}_{\Lambda_0^d \cup \Lambda_1^d}(\theta) & : j = 0 \\ \mathbb{1}_{\Lambda_{j-1}^d \cup \Lambda_j^d \cup \Lambda_{j+1}^d}(\theta) & : j \geq 1. \end{cases}$$

We want to apply Theorem 4.17 (ii) to obtain that the set $\{T_{m_j} : j \in \mathbb{N}_0\}$ is $\mathcal{R}$-bounded in $\mathcal{B}(l^p(\mathbb{Z}^d, E_0), l^p(\mathbb{Z}^d, E_1))$. Hence we have to show that the scalar valued functions $h_{j,\alpha}$ are uniformly bounded in $L^1(B^d)$, i.e. there is a constant $C > 0$ independent of α and j such that $\|h_{j,\alpha}\|_{L^1(B^d)} \leq C$. In addition we need to find a $\mathcal{R}$-bounded set $\sigma \subset \mathcal{B}(E_0, E_1)$ such that $n_{j,\alpha} := \sum_{\gamma\leq\alpha} n_{j,\alpha,\gamma} \in L^\infty(B^d, \mathcal{B}(E_0, E_1))$ has values in σ, uniformly in $j \in \mathbb{N}_0$ and $\alpha \leq (1, \ldots, 1)$. We have for any $\alpha \leq (1, \ldots, 1) \in \mathbb{N}_0^d$

$$\int_{B^d} |h_{0,\alpha}(\theta)| d\theta = \int_{\Lambda_0^d \cup \Lambda_1^d} \frac{1}{|\theta|^{|\alpha|}} d\theta \leq \text{vol}(\Lambda_0^d \cup \Lambda_1^d) 8^{|\alpha|} \leq 8^d(1 - (1/8)^d) \leq 2^{3d}$$

and for $j \geq 1$

$$\begin{aligned}
\int_{B^d} |h_{j,\alpha}(\theta)| d\theta &= \int\limits_{\Lambda_{j-1}^d \cup \Lambda_j^d \cup \Lambda_{j+1}^d} \frac{1}{|\theta|^{|\alpha|}} d\theta \leq \text{vol}(\Lambda_{j-1}^d \cup \Lambda_j^d \cup \Lambda_{j+1}^d) 2^{(j+3)|\alpha|} \\
&= (2^{-jd} - 2^{(-j-3)d}) 2^{(j+3)d} \leq 2^{3d}.
\end{aligned}$$

Thus $\|h_{j,\alpha}\|_{L^1(B^d)} \leq 2^{3d}$ for all $j \in \mathbb{N}_0$ and $\alpha \leq (1, \ldots, 1)$. Concerning the functions $n_{j,\alpha,\gamma}$ we recall $\gamma \leq \alpha$ and observe $\sup_{\theta\in B^d} |\theta|^{|\alpha|-|\gamma|}\partial^{\alpha-\gamma}\varphi_0(\theta) \leq C(\varphi_0, \alpha, \gamma)$, while for $j \geq 1$

$$\sup_{\theta \in B^d} |\theta|^{|\alpha|-|\gamma|} \partial^{\alpha-\gamma}\varphi_j(\theta) = \sup_{\theta \in B^d} |\theta|^{|\alpha|-|\gamma|} 2^{(j-1)|\alpha-\gamma|} (\partial^{\alpha-\gamma}\varphi_1)(2^{j-1}\theta)$$
$$\leq C(\varphi_1,\alpha,\gamma)(d2^{-j})^{|\alpha|-|\gamma|} 2^{(j-1)|\alpha-\gamma|} = C(\varphi_1,\alpha,\gamma,d).$$

Setting $C(d,\varphi_0,\varphi_1) := \max_{\substack{\alpha \leq \beta_d \\ \gamma \leq \alpha}} \{\binom{\alpha}{\gamma} C(\varphi_1,\alpha,\gamma,d), \binom{\alpha}{\gamma} C(\varphi_0,\alpha,\gamma)\}$, where again $\beta_d = (1,\dots,1) \in \mathbb{N}_0^d$, yields

$$n_{j,\alpha,\gamma}(\theta) \in C(d,\varphi_0,\varphi_1) \cdot \mathrm{absco}(\tau) \tag{4.23}$$

and the right hand side of (4.23) is a $\mathcal{R}$-bounded subset of $\mathcal{B}(E_0,E_1)$ by assumption and Lemma 2.56. Hence there is another constant $C_1 > 0$ such that $n_{j,\alpha}(\theta) \in C_1 C(d,\varphi_0,\varphi_1) \cdot \mathrm{absco}(\tau)$ which also shows boundedness of $n_{j,\alpha}$. Measurability follows from continuity.
So far we have shown all requirements of Theorem 4.17 (ii) and obtain that the set $\{T_{m_j} : j \in \mathbb{N}_0\}$ is $\mathcal{R}$-bounded in $\mathcal{B}(l^p(\mathbb{Z}^d,E_0), l^p(\mathbb{Z}^d,E_1))$.
For $f \in s_0(\mathbb{Z}^d,E_0)$ the expression $T_m f$ is a well defined element of $l^\infty(\mathbb{Z}^d,E_0)$ and we have by construction for any $j \in \mathbb{N}_0$ the identity

$$T_{m_j} T_{\mathbb{1}_{\Lambda_j^d}} f = T_{\varphi_j \cdot m \cdot \mathbb{1}_{\Lambda_j^d}} f = T_{m \cdot \mathbb{1}_{\Lambda_j^d}} f = T_{\mathbb{1}_{\Lambda_j^d}} T_m f.$$

Finally we use Theorem 4.15 to obtain for $f \in s_0(\mathbb{Z}^d,E_0)$

$$\|T_m f\|_{l^p(\mathbb{Z}^d,E_1)} \leq C_{E_1,p} \| \sum_{j \in \mathbb{N}_0} r_j T_{\mathbb{1}_{\Lambda_j^d}} T_m f\|_{L^p([0,1],l^p(\mathbb{Z}^d,E_1))}$$
$$= C_{E_1,p} \| \sum_{j \in \mathbb{N}_0} r_j T_{m_j} T_{\mathbb{1}_{\Lambda_j^d}} f\|_{L^p([0,1],l^p(\mathbb{Z}^d,E_1))}$$
$$\leq C_{E_1,p} \mathcal{R}_p(\{T_{m_j} : j \in \mathbb{N}_0\}) \| \sum_{j \in \mathbb{N}_0} r_j T_{\mathbb{1}_{\Lambda_j^d}} f\|_{L^p([0,1],l^p(\mathbb{Z}^d,E_0))}$$
$$\leq C_{E_1,p} \mathcal{R}_p(\{T_{m_j} : j \in \mathbb{N}_0\}) C_{E_0,p} \|f\|_{l^p(\mathbb{Z}^d,E_0)}.$$

Thus denseness of $s_0(\mathbb{Z}^d,E_0)$ in $l^p(\mathbb{Z}^d,E_0)$ gives the result. □

Concerning Fourier multiplication operators we make the following observation which is a simple consequence of Lemma 2.17.

Remark 4.20. *If $m : B^d \to \mathcal{B}(E_0,E_1)$ is a Fourier multiplication function, then the associated (bounded) Fourier multiplication operator $T_m : l^p(\mathbb{Z}^d,E_0) \to l^p(\mathbb{Z}^d,E_1)$ is translation invariant.*

4.5 Multiplier Theorems for Zak and Bloch Transform

In view of the decompositions (2.5) and (2.6) it is easy to formulate all results concerning Fourier multiplication functions from Section 4.4 for both the Zak and the Bloch Transform. Let us recall from Section 2.5 that if E is of class $\mathcal{HT}$ so is $L^p(\mathbb{I}^d,E)$ for $p \in (1,\infty)$. The transition for the Zak Transform is very simple and we obtain

Theorem 4.21. *Let E_0, E_1 be Banach spaces, $1 < p < \infty$ and assume that we are given a bounded and measurable function $m : B^d \to \mathcal{B}(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1))$ that is a Fourier multiplication function, i.e.*

$$T_m f := \mathcal{F}\mathcal{M}_m\mathcal{F}^{-1} f$$

extends to a bounded operator $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0)) \to l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_1))$. Then m is a Zak multiplication function, i.e.

$$Z_m f := Z^{-1}\mathcal{M}_m Z f,$$

first defined for $f \in L^p_s(\mathbb{R}^d, E_0)$, extends to a bounded linear and periodic operator $L^p(\mathbb{R}^d, E_0) \to L^p(\mathbb{R}^d, E_1)$.

Proof. Because of $Zf = \mathcal{F}^{-1}\Gamma f$ for all $f \in L^p_s(\mathbb{R}^d, E_0)$ and $\Gamma f \in s(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0))$ we obtain

$$Z_m f = \Gamma^{-1}\mathcal{F}\mathcal{M}_m\mathcal{F}^{-1}\Gamma f = \Gamma^{-1} T_m \Gamma f.$$

But both Γ and Γ^{-1} are bounded linear operators. Hence boundedness of T_m implies the one of Z_m. For periodicity of Z_m we recall the equations (3.1) as well as the translation invariance of T_m. Thus we obtain for any $z \in \mathbb{Z}^d$

$$Z_m \tau_z f = \Gamma^{-1} T_m \tau_{-z} \Gamma f = \Gamma^{-1} \tau_{-z} T_m \Gamma f = \tau_z \Gamma^{-1} T_m \Gamma f = \tau_z Z_m f.$$

□

Since the Bloch Transform involves operations on the spaces $L^p(\mathbb{I}^d, E_0)$ and $L^p(\mathbb{I}^d, E_1)$ the transition is not as easy as it was for the Zak Transform. Nevertheless the sufficient condition given in Theorem 4.17 (i) and Theorem 4.19 have natural counterparts.

Theorem 4.22. *Let E_0, E_1 be Banach spaces of class $\mathcal{HT}$ and $1 < p < \infty$.*

(i) If $m \in C^d(B^d, \mathcal{B}(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1))$, then $B_m f := \Phi^{-1}\mathcal{M}_m\Phi f$ first defined for $f \in L^p_s(\mathbb{R}^d, E_0)$ extends to a bounded, linear and periodic operator

$$B_m : L^p(\mathbb{R}^d, E_0) \to L^p(\mathbb{R}^d, E_1).$$

(ii) If $m \in C^d(B^d \setminus \{0\}, \mathcal{B}(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1))$ is such that

$$\tau := \{|\theta|^{|\alpha|}\partial^\alpha m(\theta) : \theta \in B^d, \alpha \leq (1, \ldots, 1)\}$$

is a $\mathcal{R}$-bounded subset of $\mathcal{B}(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1))$, then m is a Bloch multiplication function, i.e. the operator

$$B_m f := \Phi^{-1}\mathcal{M}_m\Phi f$$

extends to a bounded, linear and periodic operator $B_m : L^p(\mathbb{R}^d, E_0) \to L^p(\mathbb{R}^d, E_1)$ with $\|B_m\| \leq C\mathcal{R}_p(\tau)$.

Proof. (i) If $m \in C^d(B^d, \mathcal{B}(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1)))$, so is $\Xi^{-1}m\Xi$ by the definition of Ξ. Hence Theorem 4.17 (i) implies

$$\Xi^{-1}m\Xi \in \mathcal{M}_p(L^p(\mathbb{I}^d, E_0), L^p(\mathbb{I}^d, E_1)).$$

Now Theorem 4.21 applies to the function $\Xi^{-1}m\Xi$, i.e.

$$Z_{\Xi^{-1}m\Xi} \in \mathcal{B}(l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0)), l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_1))).$$

Thus $B_m = \Gamma^{-1}Z_{\Xi^{-1}m\Xi}\Gamma \in \mathcal{B}(L^p(\mathbb{R}^d, E_0), L^p(\mathbb{R}^d, E_1))$ by boundedness of Γ and Γ^{-1}.

(ii) Recall the definition of Ξ given in Section 2.2. For fixed $\theta \in B^d$, $\Xi(\theta)$ is the (bounded invertible) multiplication operator on the space $L^p(\mathbb{I}^d, E)$, multiplying with the function $x \mapsto e^{-2\pi i\theta\cdot x}$. If we are able to show that the function $\theta \mapsto \Xi^{-1}(\theta)m(\theta)\Xi(\theta)$ is a Fourier multiplication function, then Theorem 4.21 together with (2.6) gives the statement. In order to apply Theorem 4.19 to the function $\theta \mapsto \Xi^{-1}(\theta)m(\theta)\Xi(\theta)$ we note first that both Ξ and Ξ^{-1} are elements of $C^\infty(B^d, \mathcal{B}(L^p(\mathbb{I}^d, E)))$ for any Banach space E. For each $\alpha \in \mathbb{N}_0^d$ the derivatives are given by

$$\partial^\alpha \Xi(\theta) = \mathcal{M}_{x\mapsto(-2\pi ix)^\alpha e^{-2\pi ix\theta}}$$
$$\partial^\alpha \Xi^{-1}(\theta) = \mathcal{M}_{x\mapsto(2\pi ix)^\alpha e^{2\pi ix\theta}},$$

i.e. $\partial^\alpha\Xi(\theta)$ and $\partial^\alpha\Xi^{-1}(\theta)$ are multiplication operators on $L^p(\mathbb{I}^d, E)$, multiplying with scalar functions that are bounded by $(2\pi)^{|\alpha|}$. Hence we have by Lemma 2.58 (a) that the sets

$$\kappa_1 := \left\{ |\theta|^{|\beta|}\partial^\beta\Xi(\theta) : \theta \in B^d, \beta \le (1,\dots,1) \right\},$$
$$\kappa_2 := \left\{ |\theta|^{|\beta|}\partial^\beta\Xi^{-1}(\theta) : \theta \in B^d, \beta \le (1,\dots,1) \right\}$$

are $\mathcal{R}$-bounded subsets of $\mathcal{B}(L^p(\mathbb{I}^d, E))$, for any Banach space E, with $\mathcal{R}_p(\kappa_i) \le 2(2\pi)^d$.

For any $\alpha \le (1,\dots,1)$ and $\theta \in B^d \setminus \{0\}$ we have

$$\begin{aligned} &|\theta|^{|\alpha|}[\partial^\alpha\Xi^{-1}m\Xi](\theta) \\ &= \sum_{\gamma_1\le\alpha}\sum_{\gamma_2\le\alpha-\gamma_1}\binom{\alpha}{\gamma_1}\binom{\alpha-\gamma_1}{\gamma_2}|\theta|^{|\gamma_1|}\left[\partial^{\gamma_1}\Xi^{-1}\right](\theta)\circ|\theta|^{|\gamma_2|}\left[\partial^{\gamma_2}m\right](\theta)\circ \\ &\qquad\qquad |\theta|^{|\alpha-\gamma_1-\gamma_2|}\left[\partial^{\alpha-\gamma_1-\gamma_2}\Xi\right](\theta) \\ &\in \sum_{\gamma_1\le\alpha}\sum_{\gamma_2\le\alpha-\gamma_1}\binom{\alpha}{\gamma_1}\binom{\alpha-\gamma_1}{\gamma_2}\kappa_2\circ\tau\circ\kappa_1. \end{aligned}$$

But the last set in the equation above is $\mathcal{R}$-bounded by the assumption of the theorem and Lemma 2.56. Hence $\theta \mapsto \Xi^{-1}(\theta)m(\theta)\Xi(\theta)$ satisfies the assumptions of Theorem 4.19 and $T_{\Xi^{-1}m\Xi}$ extends to a bounded translation invariant operator $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_0)) \to l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E_1))$. Because of

$$B_m = \Gamma^{-1}\mathcal{F}\Xi^{-1}\mathcal{M}_m\Xi\mathcal{F}^{-1}\Gamma = Z^{-1}T_{\Xi^{-1}m\Xi}Z$$

the statement follows from Theorem 4.21.

□

Chapter 5

Applications

This chapter is devoted to applications of the multiplier theorems 4.19 and 4.21 of the previous chapter. We will show how they can be used to obtain the band gap structure of the spectrum of periodic operators on $L^p(\mathbb{R}^d, E)$. Once this is done, we proceed by reassembling a given family of unbounded operators defined on a Banach space E and parametrized by $\theta \in B^d$ to an unbounded translation on $l^p(\mathbb{Z}^d, E)$. This will then allow to consider a family of unbounded operators defined on $L^p(\mathbb{I}^d, E)$ and reassemble them to an unbounded periodic operator on $L^p(\mathbb{R}^d, E)$, by use of the Bloch Transform. The same techniques used for this procedure enables for a according reassembling of the functional calculus. At the end of this chapter we will see that even more properties of the fiber operators carry over, such as p-independence of the spectrum and stability of C_0-semigroups.

Clearly all this can not be done without further assumptions. In the complete abstract setting it is convenient to assume analyticity of the family of fiber operators, a notion we will introduce now. We are guided by [Kat66], where the same notion is used for one complex variable.

5.1 Analytic Families of Operators Depending on Several Variables

We generalize several known results of analytic families of bounded and unbounded operators depending on a parameter $z \in \mathbb{C}$ to the case, where the parameter now is taken from $\mathbb{C}^d$ for some $d \in \mathbb{N}$. A detailed treatment of the theory for $d = 1$ is given in [Kat66]. For multidimensional parameters we use Hartogs approach that characterizes analyticity by partial analyticity for scalar valued functions [Har06]. A similar approach is used in [Sca99]. This characterization allows to develop a multidimensional version of Cauchy's Integral Formula (see [KK83, Prop.1.3]), which makes it possible to extend the known theory to the case $d > 1$. As a first step we recall the definition of analyticity of scalar valued functions in the case of a multidimensional domain of definition. One treats vector-valued functions by first applying a functional, which reduces the consideration to the well known scalar case. The vector valued statement then follows by the Hahn Banach theorem.

Analyticity in the case of several variables

Definition 5.1. *Let $d \in \mathbb{N}$ and $\mathcal{D} \subset \mathbb{C}^d$ be an open set. For $\zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{C}^d$ and $\varepsilon^d := (\varepsilon_1, \ldots, \varepsilon_d)$ we call the set*

$$P_{\varepsilon^d}(\zeta) := \{z \in \mathbb{C}^d : |z_j - \zeta_j| < \varepsilon_j \ , \ j = 1, \ldots, d\}$$

a polycylinder around ζ. A function $f : \mathcal{D} \to \mathbb{C}$ is called analytic at the point $\zeta_0 \in \mathcal{D}$ if there is a polycylinder $P_{\varepsilon^d}(\zeta_0) \subset \mathcal{D}$ and a sequence $(a_\alpha)_{\alpha \in \mathbb{N}_0^d} \in \mathbb{C}$ such that the series

$$P(z) := \sum_\alpha a_\alpha (z - \zeta_0)^\alpha = \sum_\alpha a_\alpha \prod_{j=1}^d (z_j - \zeta_{0,j})^{\alpha_j}$$

converges absolutely for all $z \in P_{\varepsilon^d}(\zeta_0)$ and represents f. That is $f(z) = P(z)$ for all $z \in P_{\varepsilon^d}(\zeta_0)$. A function $f : \mathcal{D} \to \mathbb{C}$ is called analytic if it is analytic at every point $\zeta_0 \in \mathcal{D}$.

We required the power series in the definition to be absolute convergent so that we do not have to care about the order of summation. In the case of one complex variable the absolute convergence follows on a slightly smaller polycylinder.

Next we extend this definition to functions having values in a Banach space E. As in the case of one parameter the following notions of analyticity turn out to be equivalent.

Definition 5.2. *Let E be a Banach space, $\mathcal{D} \subset \mathbb{C}^d$ a domain and $f : \mathcal{D} \to E$.*

(i) The function f is called weakly analytic at the point $\zeta_0 \in \mathcal{D}$, if for all $e' \in E'$ the scalar valued function $\zeta \mapsto e'[f(z)] \in \mathbb{C}$ is analytic at ζ_0.

(ii) The function f is called strongly analytic at the point ζ_0, if there is a polycylinder $P_{\varepsilon^d}(\zeta_0) \subset \mathcal{D}$ and a sequence $(a_\alpha^{\zeta_0})_{\alpha \in \mathbb{N}_0^d} \subset E$ such that for all $z \in P_{\varepsilon^d}(\zeta_0)$ the series

$$P(z) := \sum_\alpha a_\alpha^{\zeta_0} (z - \zeta_0)^\alpha$$

converges absolutely and represents f, i.e. $P(z) = f(z)$ for $z \in P_{\varepsilon^d}(\zeta_0)$.

The function f is called weakly/strongly analytic in $\mathcal{D}$ if the corresponding property holds for all $\zeta_0 \in \mathcal{D}$.

It is clear, right from the definition, that every strongly analytic function is weakly analytic. But the converse is also true as we will show in a moment. Obviously, strongly analytic functions are continuous.

Cauchy's Integral Formula for Several Complex Variables

Cauchy's integral formula known for scalar-valued functions depending on one complex variable directly extends to functions depending on several complex variables see [BM48,KK83].

Theorem 5.3. *Let $\mathcal{D} \subset \mathbb{C}^d$ be a domain and $f : \mathcal{D} \to \mathbb{C}$ be analytic. Consider for some $\eta \in \mathcal{D}$ and $\rho = (\rho_1, \ldots, \rho_d) \in \mathbb{R}^d_{>0}$ the set $T_\rho(\eta) := \{z \in \mathbb{C}^d : |z_j - \eta_j| = \rho_j\}$. If $\rho \in \mathbb{R}^d_{>0}$ and $\eta \in \mathcal{D}$ are such that $T_\rho(\eta) \subset \mathcal{D}$, then*

$$\begin{aligned} f(z) &= \frac{1}{(2\pi i)^d} \int_{T_\rho(\eta)} \frac{f(\zeta)}{(\zeta - z)^{(1,\ldots,1)}} d\zeta \\ &= \frac{1}{(2\pi i)^d} \int_{|z_d - \eta_d| = \rho_d} \cdots \int_{|z_1 - \eta_1| = \rho_1} \frac{f(\zeta_1, \ldots, \zeta_d)}{(\zeta_1 - z_1) \cdots (\zeta_d - z_d)} d\zeta_1 \cdots d\zeta_d \end{aligned}$$

for all z in the interior of $T_\rho(\eta)$. Moreover f is complex differentiable of any order with

$$\begin{aligned} \partial_z^\alpha f(z) &= \frac{\alpha!}{(2\pi i)^d} \int_{T_\rho(\eta)} \frac{f(\zeta)}{(\zeta - z)^{(1,\ldots,1)+\alpha}} d\zeta \\ &= \frac{\alpha!}{(2\pi i)^d} \int_{|z_d - \eta_d| = \rho_d} \cdots \int_{|z_1 - \eta_1| = \rho_1} \frac{f(\zeta_1, \ldots, \zeta_d)}{(\zeta_1 - z_1)^{\alpha_1+1} \cdots (\zeta_d - z_d)^{\alpha_d+1}} d\zeta_1 \cdots d\zeta_d \end{aligned}$$

for all z in the interior of $T_\rho(\eta)$.

The formulas in the theorem above directly extend to vector valued analytic functions. Indeed let $f : \mathcal{D} \to E$ be analytic. Then for every $e' \in E'$ the function $F : \mathcal{D} \to \mathbb{C}, z \mapsto F(z) := e'[f(z)]$ is analytic and an application of Hahn Banach Theorem shows,

$$\begin{aligned} \partial_z^\alpha f(z) &= \frac{\alpha!}{(2\pi i)^d} \int_{T_\rho(\eta)} \frac{f(\zeta)}{(\zeta - z)^{(1,\ldots,1)+\alpha}} d\zeta \\ &= \frac{\alpha!}{(2\pi i)^d} \int_{|z_d - \eta_d| = \rho_d} \cdots \int_{|z_1 - \eta_1| = \rho_1} \frac{f(\zeta_1, \ldots, \zeta_d)}{(\zeta_1 - z_1)^{\alpha_1+1} \cdots (\zeta_d - z_d)^{\alpha_d+1}} d\zeta_1 \cdots d\zeta_d \end{aligned} \tag{5.1}$$

where ρ, z and η are as before. Note that the integral on the right-hand side is now a Bochner integral in E.

With this formulas at hand it is now possible to show the equivalence of strong and weak analyticity.

Lemma 5.4. *A functions $f : \mathcal{D} \to E$ is weakly analytic if and only if f is strongly analytic.*

Proof. It is clear, that strong analyticity implies the weak one. For the converse direction we argue similar as in the case of one complex variable. We see, from Cauchy's integral formula given above, that f is strongly differentiable. Applying a multi dimensional Taylor Series for vector valued functions ([Lang93, XIII, §6]) gives the result as in [KK83, Ch.0 §4, Cor. 4.8]. □

Analyticity of Families of Bounded Operators

In the previous chapters the multiplication functions under consideration have always been $\mathcal{B}(E_0, E_1)$-valued. Analyticity of these functions will be investigated now.

Definition 5.5. *Consider two Banach spaces E_1, E_2 a domain $\mathcal{D} \subset \mathbb{C}^d$ and a function $f : \mathcal{D} \to \mathcal{B}(E_1, E_2)$. We call f*

(i) weakly analytic in $\mathcal{D}$ if for all $e \in E_1$ and $e' \in E_2$ the function

$$\mathcal{D} \ni \zeta \mapsto e'\left[f(\zeta)e\right] \in \mathbb{C}$$

is analytic.

(ii) strongly analytic in $\mathcal{D}$ if for all $e \in E_1$ the function

$$\mathcal{D} \ni \zeta \mapsto f(\zeta)e \in E_2$$

is analytic.

(iii) uniformly analytic if it is strongly analytic in the sense of Definition 5.2 where E is replaced by $\mathcal{B}(E_1, E_2)$.

An immediate consequence of this definition, Lemma 5.4 and Hartogs characterization of analyticity in terms of partial analyticity is the following (compare [Kat66, III §3.1]).

Corollary 5.6. *For a function $f : \mathcal{D} \to \mathcal{B}(E_1, E_2)$ the notion of strong, weak and uniform analyticity are equivalent.*

Analyticity of Families of Unbounded Operators

Since we are not only interested in the investigation of bounded operators we need a corresponding notion of analyticity for functions with values in the set of unbounded operators. This is done via a reduction to the bounded case. In order to distinguish these two concepts we call analytic families of bounded operators 'bounded analytic families'.

For two Banach spaces E_0, E_1 denote by $\mathcal{C}(E_0, E_1)$ the set of all closed operators $(A, D(A)) : E_0 \to E_1$. As usual we set $\mathcal{C}(E) := \mathcal{C}(E, E)$.

Definition 5.7. *Let $\mathcal{D} \subset \mathbb{C}^d$ be a domain and $F : \mathcal{D} \to \mathcal{C}(E_0, E_1)$. The function F is called analytic at the point $\zeta_0 \in \mathcal{D}$, if there is a polycylinder $P_{\varepsilon^d}(\zeta_0) \subset \mathcal{D}$ and a Banach space Z as well as bounded analytic families*

$$P_{\varepsilon^d}(\zeta_0) \ni z \mapsto U(z) \in \mathcal{B}(Z, E_0) \qquad P_{\varepsilon^d}(\zeta_0) \ni z \mapsto V(z) \in \mathcal{B}(Z, E_1)$$

such that $U(z)$ maps Z onto $D(F(z))$ one-to-one and $F(z)U(z) = V(z)$, for all $z \in P_{\varepsilon^d}(\zeta_0)$. F is called analytic in $\mathcal{D}$ if it is analytic at every point $\zeta_0 \in \mathcal{D}$.

Since every bounded operator is also closed one may ask if, in the case of functions with values in the bounded operators, the two definitions given so far coincide. This is indeed the case.

Assume that $T : \mathcal{D} \to \mathcal{B}(E_0, E_1)$ is analytic at some point $\zeta_0 \in \mathcal{D}$ in the sense of Definition 5.7. Since $D(T(z)) = E_0$ for z in the polycylinder of analyticity, $U(z)$ maps Z onto E_0 one to one and hence the open mapping theorem implies $U(z)^{-1} \in \mathcal{B}(E_0, Z)$. Now a Neumann series argument gives the 'bounded analyticity' of $z \mapsto U(z)^{-1}$ for z in a possibly smaller polycylinder. Hence $z \mapsto F(z) = V(z) \circ U(z)^{-1}$ is bounded analytic in this polycylinder[1]. For the converse direction just choose $Z = E_0$, $U(z) := \mathrm{id}_{E_0}$ and $V(z) := F(z)$.

One of the most important results concerning unbounded analytic families of operators is the following theorem. It links analyticity of a family of closed operators to analyticity of their resolvent operators. The proof is based on the corresponding result for one variable given in [Kat66, Ch.VII §2-Thm1.3].

Theorem 5.8. *Let E be a Banach spaces $\mathcal{D} \subset \mathbb{C}^d$ a domain, $F : \mathcal{D} \to \mathcal{C}(E)$, $\zeta_0 \in \mathcal{D}$ and $\lambda \in \rho(F(\zeta_0))$. F is analytic at the point ζ_0 if and only if there is a polycylinder $P_{\varepsilon^d}(\zeta_0) \subset \mathcal{D}$ such that $\lambda \in \rho(F(z))$ for all $z \in P_{\varepsilon^d}(\zeta_0)$ and the mapping*

$$P_{\varepsilon^d}(\zeta_0) \ni z \mapsto R(\lambda, F(z))$$

is bounded analytic. Moreover the function $(\lambda, z) \mapsto R(\lambda, F(z))$ is bounded analytic on the set of all λ, z such that $\lambda \in \rho(F(\zeta_0))$ and z in a polycylinder depending on λ.

Proof. First suppose that F is analytic at the point ζ_0 and let $P_{\varepsilon^d}(\zeta_0)$, $U(z)$ and $V(z)$ be as in Definition 5.7. For $\lambda \in \rho(F(\zeta_0))$ and $z \in P_{\varepsilon^d}(\zeta_0)$ we have

$$(\lambda - F(z))U(z) = \lambda U(z) - V(z)$$

and $\lambda U(\zeta_0) - V(\zeta_0) = (\lambda - F(\zeta_0))U(\zeta_0)$ maps Z onto E one to one. Hence the inverse $[\lambda U(\zeta_0) - V(\zeta_0)]^{-1}$ exists and belongs to $\mathcal{B}(E, Z)$, by the open mapping theorem. A Neumann Series argument now implies the existence of a polycylinder $P_{\varepsilon_1^d}(\zeta_0)$ such that $P_{\varepsilon_1^d}(\zeta_0) \ni z \mapsto [\lambda U(z) - V(z)]^{-1}$ is bounded analytic. Hence $R(\lambda, F(z)) = U(z)[\lambda U(z) - V(z)]^{-1}$ is analytic on $P_{\varepsilon_1^d}(\zeta_0)$.

Conversely assume $z \mapsto R(\lambda, F(z))$ is bounded analytic at the point $\zeta_0 \in \mathcal{D}$. Let $P_{\varepsilon_1^d}(\zeta_0)$ be the polycylinder given in the definition of analyticity and set $Z := E$, $U(z) := R(\lambda, F(z))$ and $V(z) := \lambda U(z) - \mathrm{id}_E$ for $z \in P_{\varepsilon_1^d}(\zeta_0)$. Then U and V satisfy the properties of Definition 5.7 and moreover

$$F(z)U(z) = F(z)R(\lambda, F(z)) = \lambda R(\lambda, F(z)) - \mathrm{id}_E = V(z)$$

for $z \in P_{\varepsilon_1^d}(\zeta_0)$. Hence $z \mapsto F(z)$ is analytic at the point ζ_0. The analyticity of $(\lambda, z) \mapsto R(\lambda, F(z))$ follows with the exact same arguments as in [Kat66, Ch.IV §3-Thm3.12]. □

[1] The composition of two 'bounded analytic' functions is again bounded analytic. This fact may be derived from the scalar valued situation via Corollary 5.6.

The assumptions on the multiplication functions in the previous chapter always contained some $\mathcal{R}$-boundedness condition. Bounded analytic families of operators satisfy this assumption -at least- on compact subsets. Even if we do not use this property later on, we decided to include it for sake of completeness.

Bounded Analyticity and $\mathcal{R}$-boundedness

In view of the previous results and the multiplier theorems of Chapter 4 it would be helpful and desirable if bounded analytic operator families are $\mathcal{R}$-bounded. This is not true in this generality but one obtains

Theorem 5.9. *Let E_0, E_1 be Banach spaces, $\mathcal{D} \subset \mathbb{C}^d$ a domain and $T : \mathcal{D} \to \mathcal{B}(E_0, E_1)$ bounded analytic. If $K \subset \mathcal{D}$ is compact then the set $\{T(z) \ : \ z \in K\}$ is $\mathcal{R}$-bounded in $\mathcal{B}(E_0, E_1)$.*

Proof. First we note that by definition we find for every $\zeta_0 \in \mathcal{D}$ a polycylinder $P_{\varepsilon^d}(\zeta_0)$ and coefficients $a_\alpha^{\zeta_0}$ such that

$$T(z) = \sum_\alpha a_\alpha^{\zeta_0}(z - \zeta_0)^\alpha, \text{ for } z \in P_{\varepsilon^d}(\zeta_0), \tag{5.2}$$

where the power series is absolutely convergent. Since $P_{\varepsilon^d}(\zeta_0)$ is open we find a radius $r > 0$ such that $B_r := \{z \in \mathbb{C}^d \ : \ |z - \zeta_0| < r\} \subset P_{\varepsilon^d}(\zeta_0)$ and

$$\rho := \sum_\alpha \|a_\alpha^{\zeta_0}\|_{\mathcal{B}(E_0,E_1)} r^{|\alpha|} < \infty.$$

To see this choose $z = \zeta_0 + (r, \ldots, r)^T$ which is an element of $P_{\varepsilon^d}(\zeta_0)$ and apply the absolute convergence of (5.2). Applying Theorem 2.55 yields for $z_j \in B_r$ and $e_j \in E_0$

$$\begin{aligned}
\|\sum_{j=1}^m r_j(\cdot)T(z_j)e_j\|_{L^p([0,1],E_1)} &= \|\sum_{j=1}^m r_j(\cdot)\sum_\alpha a_\alpha^{\zeta_0}(z_j - \zeta_0)^\alpha e_j\|_{L^p([0,1],E_1)} \\
&\leq \sum_\alpha \|a_\alpha^{\zeta_0}\|_{\mathcal{B}(E_0,E_1)} \|\sum_{j=1}^m r_j(\cdot)(z_j - \zeta_0)^\alpha e_j\|_{L^p([0,1],E_0)} \\
&\leq 2\sum_\alpha \|a_\alpha^{\zeta_0}\|_{\mathcal{B}(E_0,E_1)} \|\sum_{j=1}^m r_j(\cdot) r^{|\alpha|} e_j\|_{L^p([0,1],E_0)} \\
&= 2\rho \|\sum_{j=1}^m r_j(\cdot)e_j\|_{L^p([0,1],E_0)}.
\end{aligned}$$

Hence the set $\{T(z) \ : \ z \in B_r\}$ is $\mathcal{R}$-bounded. Covering the compact set K with finitely may such sets gives the result via Lemma 2.56. ☐

5.2 Band Gap Structure of The Spectrum of Periodic Operators

For this section lets assume we are given a reflexive and separable Banach space E of class $\mathcal{HT}$ and a closed operator $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$, $p \in (1, \infty)$ which has the properties

(A.i) A is periodic ,

(A.ii) the resolvent set $\rho(A)$ contains a unbounded sequence $(\lambda_k)_{k\in\mathbb{N}}$ such that for all $f \in L^p(\mathbb{R}^d, E)$ we have $\lim_{k\to\infty} \lambda_k R(\lambda_k, A)f = f$.

In case (A.i) and (A.ii) are satisfied, we know by Theorem 3.38 that there is a family of closed operators $(A(\theta), D(A(\theta))) : L^p(\mathbb{I}^d, E) \to L^p(\mathbb{I}^d, E)$ with

$$Af = \Phi^{-1} A(\cdot)\Phi f \quad \text{for all } f \in \mathbf{D}_A$$

as well as a set $\Omega \subset B^d$ of measure zero such that

$$\rho(A) \subset \bigcap_{\theta\in B^d\setminus\Omega} \rho(A(\theta)).$$

By (A.ii) and the principle of uniform boundedness, we find a constant $M > 0$ with $\sup_{k\in\mathbb{N}} \|\lambda_k R(\lambda_k, A)\| \le M$. Without loss of generality, we may assume, that the unbounded sequence $(\lambda_k)_{k\in\mathbb{N}_0}$ is arranged in increasing order (with respect to the modulus). Then there is a $k_0 \in \mathbb{N}$ such that $\|R(\lambda_k, A))\| < 1$ for $k \ge k_0$.

Lemma 5.10. *If we have in addition to (A.i) and (A.ii) that the multiplication function $m_{\lambda_\nu} : B^d \to \mathcal{B}(L^p(\mathbb{I}^d, E))$ corresponding to $R(\lambda_\nu, A)$ is continuous for some $\nu \ge k_0$. Then m_μ is continuous for every $\mu \in \rho(A)$. In particular we obtain*

$$\rho(A) \subset \bigcap_{\theta\in B^d} \rho(A(\theta)).$$

Proof. Recall the construction of the fiber operators $(A(\theta), D(A(\theta)))$ given in the proof of Theorem 3.30. From the resolvent identity (2.7) we deduced

$$m_\lambda - m_\mu = (\mu - \lambda) m_\lambda \circ m_\mu \tag{5.3}$$

as equality in $L^\infty(B^d, \mathcal{B}(L^p(\mathbb{I}^d, E))$ for all $\lambda, \mu \in \rho(A)$. Substituting $\lambda = \lambda_\nu$ and rewriting (5.3) yields for any $\mu \in \rho(A)$

$$m_{\lambda_\nu} = (\mu - \lambda_\nu)\left[m_{\lambda_\nu} + \mathrm{id}_{L^p(\mathbb{I}^d,E)}\right] \circ m_\mu.$$

In particular we have for almost all $\theta \in B^d$

$$m_{\lambda_\nu}(\theta) = (\mu - \lambda_\nu)\left[m_{\lambda_\nu}(\theta) + \mathrm{id}_{L^p(\mathbb{I}^d,E)}\right] \circ m_\mu(\theta).$$

Now continuity together with $\|R(\lambda_\nu, A)\| < 1$ implies, that $[m_{\lambda_\nu}(\theta) + \mathrm{id}_{L^p(\mathbb{I}^d,E)}]$ is invertible for all $\theta \in B^d$. Since inversion is a continuous operation we obtain the continuity of m_μ via

$$m_\mu(\theta) = (\mu - \lambda_\nu)^{-1}[m_{\lambda_\nu}(\theta) + \mathrm{id}_E]^{-1} \circ m_{\lambda_\nu}(\theta).$$

For the statement concerning the resolvent sets we observe that (5.3) turns, by continuity, into the point wise equation

$$m_\lambda(\theta) - m_\mu(\theta) = (\mu - \lambda) m_\lambda(\theta) \circ m_\mu(\theta). \tag{5.4}$$

A review of the proof of (3.36) and Lemma 3.29 shows that $m_\lambda(\theta)$ is the resolvent operator of a uniquely defined closed and linear operator

$$(A(\theta), D(A(\theta))) : L^p(\mathbb{I}^d, E) \to L^p(\mathbb{I}^d, E)$$

for all $\theta \in B^d$ and $\lambda \in \rho(A)$. Moreover we obtain $\lambda \in \rho(A(\theta))$ for all $\theta \in B^d$ and $\lambda \in \rho(A)$ and the Lemma is proven. □

Complementing the assumptions (A.i), (A.ii) by some additional regularity of the family of fiber operators we are now able to proof the band gap structure of the spectrum for periodic operators defined on $L^p(\mathbb{R}^d, E)$, mentioned in Chapter 1.

Theorem 5.11. *Assume, that (A.i), (A.ii) hold true and there is a $\nu \geq k_0$ such that $m_{\lambda_\nu} \in C(B^d, \mathcal{B}(L^p(\mathbb{I}^d, E)))$. Further assume that one of the following additional assumptions posed on the fiber operators is satisfied.*

(i) The domain $D(A(\theta))$ is independent of θ and $\theta \mapsto A(\theta)$ is an element of $C^d(B^d, \mathcal{B}(D, L^p(\mathbb{I}^d, E)))$, where $D = D(A(\theta))$ is equipped with one of the (equivalent[2]) graph norms $\|\cdot\|_{A(\theta)}$.

(ii) There is a open set $\mathcal{D} \subset \mathbb{C}^d$ with $B^d \subset \mathcal{D}$ as well as a unbounded analytic family $[(\tilde{A}(\theta), D(\tilde{A}(\theta)))]_{\theta \in \mathcal{D}} \subset \mathcal{C}(L^p(\mathbb{I}^d, E))$ such that $\tilde{A}(\theta) = A(\theta)$ for $\theta \in B^d$.

Then we have the identities

$$\bigcap_{\theta \in B^d} \rho(A(\theta)) = \rho(A) \qquad \text{resp.} \qquad \bigcup_{\theta \in B^d} \sigma(A(\theta)) = \sigma(A). \tag{5.5}$$

Proof. First of all it is clear, that both identities in (5.5) are equivalent. Moreover the inclusion $\rho(A) \subset \cap_{\theta \in B^d} \rho(A(\theta))$ in (5.5) follows by the assumptions and Lemma 5.10.

Now assume $\lambda \in \cap_{\theta \in B^d} \rho(A(\theta))$ and (i) is satisfied. Then $\theta \mapsto R(\lambda, A(\theta))$ is an element of $C^d(B^d, \mathcal{B}(L^p(\mathbb{I}^d, E)))$ by a Neumann Series argument. Hence Theorem 4.22 (i) implies, that the operator

$$B_\lambda := \Phi^{-1} \mathcal{M}_{\theta \mapsto R(\lambda, A(\theta))} \Phi$$

[2] Equivalence of the graph norms follows by the closed graph theorem.

defines a bounded, linear and periodic operator on $L^p(\mathbb{R}^d, E)$. Next we show, that B_λ is a left and right inverse of $\lambda - A$, i.e. $\lambda \in \rho(A)$ and $B_\lambda = R(\lambda, A)$. For this reason fix any $\mu \in \rho(A)$. Then the assumptions of the theorem imply $\mu \in \rho(A(\theta))$ for all $\theta \in B^d$. Thus the resolvent identity (2.7) yields

$$\begin{aligned} R(\lambda, A(\theta)) &= (\mu - \lambda)R(\mu, A(\theta))\left[R(\lambda, A(\theta)) + \mathrm{id}_{L^p(\mathbb{I}^d,E)}\right] \\ R(\mu, A(\theta)) &= (\lambda - \mu)R(\lambda, A(\theta))\left[R(\mu, A(\theta)) + \mathrm{id}_{L^p(\mathbb{I}^d,E)}\right] \end{aligned}$$

for all $\theta \in B^d$. But $R(\mu, A(\theta)) = m_\mu(\theta)$ and we obtain by the corresponding version of Lemma 4.4 for the Bloch Transform

$$\begin{aligned} B_\lambda &= (\mu - \lambda)R(\mu, A)\left[B_\lambda + \mathrm{id}_{L^p(\mathbb{R}^d,E)}\right] \\ R(\mu, A) &= (\lambda - \mu)B_\lambda\left[R(\mu, A) + \mathrm{id}_{L^p(\mathbb{R}^d,E)}\right]. \end{aligned}$$

This shows, $\mathrm{rg}(B_\lambda) = \mathrm{rg}(R(\mu, A)) = D(A)$.

Recall that $\mathbf{D}_A = R(\lambda_1, A)L^p_c(\mathbb{R}^d, E)$, i.e. every $g \in \mathbf{D}_A$ is of the form $g = R(\lambda_1, A)f$ for some $f \in L^p_c(\mathbb{R}^d, E)$. Thus we have for every $g \in \mathbf{D}_A$

$$A(\theta)\mathcal{F}^{-1}g(\theta) = A(\theta)m_{\lambda_1}(\theta)\mathcal{F}^{-1}f(\theta) = \lambda_1 R(\lambda_1, A(\theta))\mathcal{F}^{-1}g(\theta) - \mathcal{F}^{-1}g(\theta),$$

which shows that the function $\theta \mapsto A(\theta)\mathcal{F}^{-1}g(\theta)$ belongs to $L^1(B^d, L^p(\mathbb{I}^d, E))$. Therefore the following calculation is meaningful and we obtain

$$B_\lambda(\lambda - A)g = \Phi^{-1}[\theta \mapsto R(\lambda, A(\theta))]\Phi^{-1}\Phi[\theta \mapsto \lambda - A(\theta)]\Phi g = g.$$

But we also have by a exactly the same argument

$$\begin{aligned} g &= \Phi^{-1}\left[\theta \mapsto R(\lambda, A(\theta))(\lambda - A(\theta))\right]\Phi g \\ &= \Phi^{-1}\left[\theta \mapsto (\lambda - A(\theta))R(\lambda, A(\theta))\right]\Phi g \\ &= (\lambda - A)B_\lambda g. \end{aligned}$$

Since $\mathbf{D}_A \subset L^p(\mathbb{R}^d, E)$ is dense and a core for $D(A)$ we end up with $\lambda \in \rho(A)$ and $B_\lambda = R(\lambda, A)$.

Now assume $\lambda \in \cap_{\theta \in B^d}\rho(A(\theta))$ and (ii) holds true. Then we find by Theorem 5.8, for every $\theta \in B^d$ a polycylinder $P_{\epsilon^d_\theta}(\theta) \subset \mathbb{C}^d$ such that the mapping $P_{\epsilon^d_\theta}(\theta) \ni \eta \mapsto R(\lambda, A(\eta))$ is bounded analytic. Denote by U the union of all the polycylinder $P_{\epsilon^d_\theta}(\theta)$, where $\theta \in B^d$. Then U is a open subset of $\mathbb{C}^d$ which covers B^d and $U \ni \theta \mapsto R(\lambda, A(\theta))$ is bounded analytic. In particular the mapping $B^d \ni \theta \mapsto R(\lambda, A(\theta))$ is an element of $C^d(B^d, \mathcal{B}(L^p(\mathbb{I}^d, E)))$. Once more Theorem 4.22 gives that the operator

$$B_\lambda = \Phi^{-1}\mathcal{M}_{[\theta \mapsto R(\lambda, A(\theta)])}\Phi$$

extends to an element of $\mathcal{B}(L^p(\mathbb{R}^d, E))$. With the same calculation as before we obtain $\lambda \in \rho(A)$ and $B_\lambda = R(\lambda, A)$. □

Discussion

We have seen, that under suitable additional assumptions on the multiplication functions corresponding to the resolvent operator of a unbounded periodic operator $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ we obtain the same spectral statement (1.14), which was so far only known in the Hilbert space case for a very special type of operators, i.e. symmetric partial differential operators with periodic coefficients. The additional assumption we made here, are also used in the 'classical' case, but in this special situation they are obtained by an eigenvalue expansion of the resolvent operators.

5.3 Reassembling Unbounded Operators and the Functional Calculus

In Chapter 3 we showed how a given closed and unbounded periodic operator $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ decomposes under the Bloch Transform into a family (parametrized by $\theta \in B^d$) of unbounded and closed fiber operators $(A(\theta), D(A(\theta))) : L^p(\mathbb{I}^d, E) \to L^p(\mathbb{I}^d, E)$. Now we pose the reverse question, i.e. we ask whether it is also possible to reassemble a given family of closed operators $(A(\theta), D(A(\theta))) : L^p(\mathbb{I}^d, E) \to L^p(\mathbb{I}^d, E)$ to a closed and periodic operator $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$. Of course we have to make some assumptions.

As before we first restrict ourselves to the Fourier multipliers and then extend this results in a second step to the Zak and Bloch Transform with arguments similar to those in Chapter 3. We remind once more that this first step implies that we work again with translation invariant operators on $l^p(\mathbb{Z}^d, E)$. For this section let E be a reflexive Banach space.

Theorem 5.12. *Let $\mathcal{D} \subset \mathbb{C}^d$ be a domain with $B^d \subset \mathcal{D}$. Further assume, that $[A(\theta), D(A(\theta))]_{\theta\in\mathcal{D}} : E \to E$ is a analytic family of unbounded and closed operators which is uniformly pseudo sectorial, i.e. there is a $\omega \in [0, \pi)$ such that $\sigma(A(\theta)) \subset \overline{\Sigma}_\omega$ for all $\theta \in \mathcal{D}$ and for any $\phi > \omega$ there is a constant C_ϕ with*

$$\sup_{\theta\in\mathcal{D}} \|\lambda R(\lambda, A(\theta))\| \le C_\phi \ \text{ for all } \lambda \in \mathbb{C} \setminus \overline{\Sigma}_\phi. \tag{5.6}$$

Then for every $p \in (1, \infty)$ there is a unbounded, closed and translation invariant operator $(A^p, D(A^p)) : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ which is in addition pseudo sectorial of angle ω and

$$A^p f = \mathcal{F}[\theta \mapsto A(\theta)]\mathcal{F}^{-1} f \ \text{ for all } f \in \mathbf{D}_{A^p} \tag{5.7}$$

$$\rho(A^p) = \cap_{\theta\in B^d} \rho(A(\theta)). \tag{5.8}$$

Here we use $-1 \in \rho(A^p)$ and set $\mathbf{D}_{A^p} := R(-1, A^p)[l^2(\mathbb{Z}^d, \mathbb{C}) \cap l^p(\mathbb{Z}^d, \mathbb{C})] \otimes E$.

Proof. The analyticity of $[A(\theta), D(A(\theta))]_{\theta\in\mathcal{D}}$ together with Theorem 5.8 implies for $\lambda \in \cap_{\theta\in B^d} \rho(A(\theta))$ that the mapping $\theta \mapsto R(\lambda, A(\theta))$ is bounded analytic on a open set $U(\lambda)$ containing B^d. Indeed we find for every $\theta \in B^d$ a open neighborhood $U_\theta(\lambda) \subset \mathbb{C}^d$ such that $\theta \mapsto R(\lambda, A(\theta))$ is bounded analytic on $U_\theta(\lambda)$. Setting $U(\lambda) := \cup_{\theta \in B^d} U_\theta(\lambda)$ shows the claim. Because $U(\lambda)$ is open and $B^d \subset \mathcal{D}$ is compact there is a $\rho = \rho(\lambda) \in \mathbb{R}^{2d}_{>0}$ with

$$\tilde{T}_{\rho(\lambda)}(0) := \{z \in \mathbb{C} : |\mathrm{Re}(z_j)| = \rho(\lambda)_j, |\mathrm{Im}(z_j)| = \rho(\lambda)_{d+j}, j = 1, \dots, d\} \subset U(\lambda)$$

and B^d is in the interior of $\tilde{T}_{\rho(\lambda)}(0)$. Now Cauchy's Integral Formula (5.1) applies even for the set $\tilde{T}_{\rho(\lambda)}(0)$, giving for any $\alpha \in \mathbb{N}_0^d$ and θ in the interior of $\tilde{T}_{\rho(\lambda)}(0)$

$$\partial_\theta^\alpha R(\lambda, A(\theta)) = \frac{\alpha!}{(2\pi i)^d} \int_{\tilde{T}_{\rho(\lambda)}(0)} \frac{1}{(\zeta - \theta)^{(1,\dots,1)+\alpha}} R(\lambda, A(\zeta)) d\zeta. \tag{5.9}$$

From (5.9) we deduce differentiability of $B^d \ni \theta \mapsto R(\lambda, A(\theta))$ of any order for every $\lambda \in \cap_{\theta \in B^d} \rho(A(\theta))$. Hence we can apply Theorem 4.17 (i) to obtain that $T_\lambda := T_{[\theta \mapsto R(\lambda, A(\theta))]}$ extends to a bounded and translation invariant operator T_λ^p on $l^p(\mathbb{Z}^d, E)$ for all $p \in (1, \infty)$. Moreover if $\lambda \in \cap_{\theta \in \mathcal{D}} \rho(A(\theta))$, the mapping $\mathcal{D} \ni \theta \mapsto R(\lambda, A(\theta))$ is bounded analytic and we find a $\rho \in \mathbb{R}^{2d}_{>0}$ independent of λ with $\tilde{T}_\rho(0) \subset \mathcal{D}$ and B^d in the interior of $\tilde{T}_\rho(0)$. In particular we have for $\lambda \in \cap_{\theta \in \mathcal{D}} \rho(A(\theta))$

$$\partial_\theta^\alpha R(\lambda, A(\theta)) = \frac{\alpha!}{(2\pi i)^d} \int_{\tilde{T}_{\rho}(0)} \frac{1}{(\zeta - \theta)^{(1,\dots,1)+\alpha}} R(\lambda, A(\zeta)) d\zeta.$$

The uniform pseudo sectoriality of the family $[A(\theta), D(A(\theta))]_{\theta \in \mathcal{D}}$ shows that, for any $\phi > \omega$ and θ in the interior of $\tilde{T}_\rho(0)$, there is a constant C_ϕ such that for all $\lambda \in \mathbb{C} \setminus \overline{\Sigma}_\phi \subset \cap_{\theta \in \mathcal{D}} \rho(A(\theta))$ the estimate

$$\|\partial_\theta^\alpha R(\lambda, A(\theta))\|_{B(E)} \le C_{\rho,\alpha} \frac{C_\phi}{|\lambda|} \tag{5.10}$$

holds true. An inspection of the proof of Theorem 4.17 (i) shows for those $\lambda \in \mathbb{C} \setminus \overline{\Sigma}_\phi$ that (5.10) transfers to the Fourier multiplier operator, i.e. we have

$$\|T_\lambda^p\|_{\mathcal{B}(l^p(\mathbb{Z}^d, E))} \le C_{\rho,p,d} \frac{C_\phi}{|\lambda|}. \tag{5.11}$$

Independent of these estimations we obtain by Lemma 4.4 (a) together with (2.7) for any $\mu, \lambda \in \cap_{\theta \in B^d} \rho(A(\theta))$

$$T_\lambda^p - T_\mu^p = (\mu - \lambda) T_\lambda^p \circ T_\mu^p \tag{5.12}$$

i.e. $(T_\lambda^p)_{\lambda \in \cap_{\theta \in B^d} \rho(A(\theta))}$ is a pseudo resolvent on $l^p(\mathbb{Z}^d, E)$ for every $p \in (1, \infty)$.

Now let us show that the family T_λ^p fulfills the conditions of Corollary 2.34. The assumption of the theorem imply, that there is a unbounded sequence

$(\lambda_k)_{k\in\mathbb{N}}$ (contained in $\mathbb{C}\setminus\overline{\Sigma}_\phi$, for some $\phi\in(\omega,\pi)$) such that for all $\theta\in\mathcal{D}$ and $e\in E$ we have

$$\lim_{k\to\infty}\lambda_k R(\lambda_k, A(\theta))e = e. \tag{5.13}$$

This observation clearly remains true if we restrict the considerations to $\theta\in B^d$. Let us mention the following equality which holds true for every $k\in\mathbb{N}$ and $p\in(1,\infty)$

$$T^p_{[\theta\mapsto\lambda_k R(\lambda_k,A(\theta))]} = \lambda_k T^p_{\lambda_k}.$$

Now, if we fix some $f\in s(\mathbb{Z}^d,E)$ and $k\in\mathbb{N}$, it is clear by the regularity of $R(\lambda_k,A(\theta))$ in the parameter θ that the function $\theta\mapsto\lambda_k R(\lambda_k,A(\theta))[\mathcal{F}^{-1}f](\theta)$ belongs to $L^1(B^d,E)$. Moreover we get from (5.13)

$$\lim_{k\to\infty}\lambda_k R(\lambda_k,A(\theta))[\mathcal{F}^{-1}f](\theta)\to[\mathcal{F}^{-1}f](\theta)$$

for almost all $\theta\in B^d$ and form (5.10)

$$\|\lambda_k R(\lambda_k,A(\theta))\mathcal{F}^{-1}f(\theta)\|_E\le C\|\mathcal{F}^{-1}f(\theta)\|_E.$$

Since $\|\mathcal{F}^{-1}f(\cdot)\|_E$ is integrable over B^d we obtain by the theorem of dominated convergence (Proposition A.6)

$$\lim_{k\to\infty}\lambda_k R(\lambda_k,A(\cdot))[\mathcal{F}^{-1}f](\cdot)\to\mathcal{F}^{-1}f(\cdot)$$

in $L^1(B^d,E)$, which now shows because of $\mathcal{F}\in\mathcal{B}(L^1(B^d,E),l^\infty(\mathbb{Z}^d,E))$

$$\lambda_k T^p_{\lambda_k}f\to f \ \text{ in } l^\infty(\mathbb{Z}^d,E) \tag{5.14}$$

for any $p\in(1,\infty)$. But by (5.11) we also obtain that the sequence $(\lambda_k T^p_{\lambda_k}f)_{k\in\mathbb{N}}$ is bounded for any $f\in l^p(\mathbb{Z}^d,E)$. Note that $l^p(\mathbb{Z}^d,E)$ is reflexive for $p\in(1,\infty)$ so that we find a weakly convergent sub sequence. If $f\in s(\mathbb{Z}^d,E)$ the weak limit of this sub sequence has to be f thanks to (5.14). Since we can preform this argument for any sub sequence we finally obtain

$$\lambda_k T^p_{\lambda_k}f\to f \ \text{ weakly in } l^p(\mathbb{Z}^d,E) \ \text{ for any } f\in s(\mathbb{Z}^d,E) \text{ as } k\to\infty.$$

So far we have shown, that all assumptions of Corollary 2.34 are satisfied. Thus we obtain that there is a unique, closed and densely defined operator $(A^p,D(A^p))$ on $l^p(\mathbb{Z}^d,E)$ with the properties

$$\mathbb{C}\setminus\overline{\Sigma}_\phi\subset\bigcap_{\theta\in\mathcal{D}}\rho(A(\theta))\subset\bigcap_{\theta\in B^d}\rho(A(\theta))\subset\rho(A^p) \ \text{ for all } \phi>\omega,$$
$$T^p_\lambda = R(\lambda,A^p) \ \text{ for } \lambda\in\bigcap_{\theta\in B^d}\rho(A(\theta))$$

and for every $\phi > \omega$ there is a constant $C < \infty$ (depending on p, d and ϕ) such that for all $\lambda \in \mathbb{C} \setminus \overline{\Sigma}_\phi$

$$\|\lambda R(\lambda, A^p)\|_{\mathcal{B}(l^p(\mathbb{Z}^d,E))} \leq C.$$

In particular A^p is pseudo sectorial of angle ω.

Finally we show (5.7) and (5.8). For this reason let $f \in \mathbf{D}_{A^p}$. Then there is a function $g \in [l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)] \otimes E$ with $f = R(-1, A^p)g$ and we obtain

$$\begin{aligned} A^p f &= A^p R(-1, A^p)g = -R(-1, A^p)g - g \\ &= \mathcal{F}[\theta \mapsto -R(-1, A(\theta)) - \mathrm{id}_E]\mathcal{F}^{-1}g \\ &= \mathcal{F}[\theta \mapsto A(\theta)R(-1, A(\theta))]\mathcal{F}^{-1}g \\ &= \mathcal{F}[\theta \mapsto A(\theta)]\mathcal{F}^{-1}R(-1, A^p)g \\ &= \mathcal{F}[\theta \mapsto A(\theta)]\mathcal{F}^{-1}f. \end{aligned}$$

In order to get (5.8) we observe that $(A^p, D(A^p))$ satisfies the conditions of Theorem 5.11, i.e. we have

$$\rho(A^p) = \bigcap_{\theta \in B^d} \rho(A(\theta))$$

for all $p \in (1, \infty)$ and the theorem is proven.

□

Remark 5.13. *Under the hypothesis of Theorem 5.12 we hev shown that the spectrum of A^p is independent of $p \in (1, \infty)$.*

As an immediate consequence of the theorem above we are able to reassemble the auxiliary functional calculus from Chapter 2.

Corollary 5.14. *With the same hypothesis as in Theorem 5.12 and $\phi > \omega$, we have for any $p \in (1, \infty)$ and $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$*

$$B^d \ni \theta \mapsto f(A(\theta)) \in \mathcal{M}_p(\mathbb{Z}^d, E)$$

with $f(A^p) = T^p_{[\theta \mapsto f(A(\theta))]}$.

Proof. For fixed $\theta \in B^d$ and $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$ we have the integral representation of $f(A(\theta))$ given by

$$f(A(\theta)) = \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, A(\theta)) d\lambda$$

where γ is a path as in (2.12). Estimate (5.10) shows that also the function $\lambda \mapsto f(\lambda)\partial_\theta^\alpha R(\lambda, A(\theta))$ is integrable along γ for every $\alpha \in \mathbb{N}_0^d$. Hence by Theorem A.9 the function $\theta \mapsto f(A(\theta))$ is differentiable with

$$\partial_\theta^\alpha f(A(\theta)) = \frac{1}{2\pi i} \int_\gamma f(\lambda) \partial_\theta^\alpha R(\lambda, A(\theta)) d\lambda.$$

By Theorem 4.17 (i) the function $B^d \ni \theta \mapsto f(A(\theta))$ is contained in $\mathcal{M}_p(\mathbb{Z}^d, E)$. Theorem 5.12 implies pseudo sectoriality of A^p for every $p \in (1, \infty)$ with angle ω. Hence

$$f(A^p) = \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, A^p) d\lambda$$

is a well defined element of $\mathcal{B}(l^p(\mathbb{Z}^d, E))$. The only thing that is left to show is $f(A^p) = T^p_{[\theta \mapsto f(A(\theta))]}$. For this reason let us pick any finite sequence $g : \mathbb{Z}^d \to E$ and recall that the evaluation map $\delta_z : l^p(\mathbb{Z}^d, E) \to E$ is bounded an linear for every $z \in \mathbb{Z}^d$. We obtain for any $z \in \mathbb{Z}^d$ by Fubini's Theorem

$$\begin{aligned}
\delta_z[f(A_p)g] &= \delta_z\left[\frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, A_p) g d\lambda\right] = \frac{1}{2\pi i} \int_\gamma f(\lambda) \delta_z [R(\lambda, A_p) g] d\lambda \\
&= \frac{1}{2\pi i} \int_\gamma f(\lambda) \int_{B^d} e^{-2\pi i z\theta} R(\lambda, A(\theta)) [\mathcal{F}^{-1} g](\theta) d\theta d\lambda \\
&= \int_{B^d} e^{-2\pi i z\theta} \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, A(\theta)) d\lambda [\mathcal{F}^{-1} g](\theta) d\theta \\
&= \delta_z \left[\mathcal{F} \mathcal{M}_{\theta \mapsto f(A(\theta))} \mathcal{F}^{-1} g\right] = \delta_z [T^p_{[\theta \mapsto f(A(\theta))]} g],
\end{aligned}$$

which shows, by boundedness of the operators on the left and right hand side in the equation above, as desired $f(A^p) = T^p_{[\theta \mapsto f(A(\theta))]}$. □

Bounded $\mathcal{H}^\infty$-Functional Calculus

We are now concerned with the question if an operator A^p, that is reassembled from a given family of fiber operators which have a uniformly bounded $\mathcal{H}^\infty(\Sigma_\phi)$ functional calculus, attains this property too. We start with the following observation as a first step.

Lemma 5.15. *Let the assumptions of Theorem 5.12 be satisfied and assume in addition that the operators $A(\theta)$ are sectorial of angle ω, i.e. they are injective and have dense range. Then for every $p \in (1, \infty)$ the operator A^p is sectorial.*

Proof. By Lemma 2.47 is is enough to show, $\overline{\text{rg}(A^p)} = l^p(\mathbb{Z}^d, E)$. As in the proof of Theorem 3.34 sectoriality of the $A(\theta)$ imply for all $\theta \in \mathcal{D}$ and $e \in E$

$$A(\theta)[1/n + A(\theta)]^{-1} e \to e \quad \text{as } n \to \infty.$$

Now we follow the arguments in the proof of Theorem 5.12 to show that $\text{rg}(A^p)$ is a dense subset of $l^p(\mathbb{Z}^d, E)$. For this reason we observe that we have for all $\theta \in \mathcal{D}$ the equaltiy

$$A(\theta)[1/n + A(\theta)]^{-1} = \text{id}_E - 1/n[1/n + A(\theta)]^{-1}$$

and the later expression defines a bounded analytic family on $\mathcal{D}$ for every $n \in \mathbb{N}$. Again we obtain as in the proof of Theorem 5.12 the estimate

$$\|\partial_\theta^\alpha A(\theta)[1/n + A(\theta)]^{-1}\| \leq C \quad \text{for all } n \in \mathbb{N}$$

by means of Cauchy's Integral formula. Hence the sequence of operators $T^p_{[\theta\mapsto A(\theta)(1/n+A(\theta))^{-1}]}$, $(n\in\mathbb{N})$ is uniformly bounded in $\mathcal{B}(l^p(\mathbb{Z}^d,E))$. It follows by algebraic properties that

$$T^p_{[\theta\mapsto A(\theta)(1/n+A(\theta))^{-1}]} = A^p(1/n+A^p)^{-1} = A^pT^p_{[\theta\mapsto(1/n+A(\theta))^{-1}]}.$$

Applying Mazur's Theorem as in the proof of Theorem 5.12 yields that for every $f\in l^p(\mathbb{Z}^d,E)$ there is a sequence build up of convex combinations of the $A^pT^p_{[\theta\mapsto(1/n+A(\theta))^{-1}]}f$ that converges strongly to f in $l^p(\mathbb{Z}^d,E)$. But each such convex combination is contained in $\mathrm{rg}(A^p)$ by linearity of A^p, i.e. $\mathrm{rg}(A^p)$ is dense in $l^p(\mathbb{Z}^d,E)$. □

In order to pass to a statement similar to the result of Corollary 5.14 for a bounded $\mathcal{H}^\infty$-functional calculus we will make use of the following characterization which is well known (see for example [DV05, Thm.4.7]).

Proposition 5.16. *Let A be a sectorial operator on a Banach space E and $\phi>\omega_A$. Consider the functions $\rho_n\in\mathcal{H}^\infty_0(\Sigma_\phi)$ defined by*

$$\rho_n(z) := \frac{n^2z}{(1+nz)(n+z)}.$$

Then A has a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus if and only if for every function $f\in\mathcal{H}^\infty(\Sigma_\phi)$ there is a constant C such that

$$\sup_{n\in\mathbb{N}}\|\Psi_A(\rho_nf)\|\le C<\infty. \tag{5.15}$$

In this case $\overline{\Psi}_A(f)e=\lim_{n\to\infty}\Psi_A(\rho_nf)e$ for all $e\in E$.

In the next theorem we will deal again with a analytic family of operators $A(\theta)$ that is defined on a open subset $\mathcal{D}$ of $\mathbb{C}^d$ and has in addition a uniformly bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus. In order to ensure that the family of limit operators $\overline{\Psi}_{A(\theta)}(f)$ defines a Fourier multiplier function we will need the following statement concerning convergence of multiplication functions.

Lemma 5.17. *Let $m_n\in\mathcal{M}_p(\mathbb{Z}^d,E)$ for each $n\in\mathbb{N}$. Further assume that we are given a measurable function $m:B^d\to\mathcal{B}(E)$ such that for each $\theta\in B^d$, $n\in\mathbb{N}$ and $e\in E$ we have*

$$m(\theta)e=\lim_{n\to\infty}m_n(\theta)e,\qquad \sup_{n\in\mathbb{N}}\|m_n(\theta)\|_{B(E)}\le C_1,\qquad \sup_{n\in\mathbb{N}}\|T_{m_n}\|_{\mathcal{B}(l^p(\mathbb{Z}^d,E)}\le C_2.$$

Then $m\in\mathcal{M}_p(\mathbb{Z}^d,E)$ with $\|T_m\|_{\mathcal{B}(l^p(\mathbb{Z}^d,E))}\le C_2$.

Proof. Let $f\in s(\mathbb{Z}^d,E)$. Then by the assumptions and Lebesgue's Theorem $m_n\mathcal{F}^{-1}f\to m\mathcal{F}^{-1}f$ in $L^1(B^d,E)$, which implies as before $T_{m_n}f\to T_mf$ in $l^\infty(\mathbb{Z}^d,E)$. Further more we have

$$\|T_mf\|^p_{\mathcal{B}(l^p(\mathbb{Z}^d,E))}=\sum_{z\in\mathbb{Z}^d}\|T_mf(z)\|^p_E\le\varliminf_{n\to\infty}\sum_{z\in\mathbb{Z}^d}\|T_{m_n}f(z)\|^p_E\le C_2^p\|f\|^p_{l^p(\mathbb{Z}^d,E)}.$$

by Fatou's lemma (Proposition A.7). □

With this two preparatory statements Proposition 5.16 and Lemma 5.17 we are now able to reassemble a uniformly bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus.

Theorem 5.18. *Let $\mathcal{D} \subset \mathbb{C}^d$ be open with $B^d \subset \mathcal{D}$. Further assume, we are given a family of operators $(A(\theta), D(A(\theta))) : E \to E$ defined on $\mathcal{D}$ that is analytic and uniformly sectorial of angle ω. If there is a $\phi > \omega$ such that all operators $(A(\theta), D(A(\theta)))$ have a uniformly bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus, i.e. the constant in (5.15) is independent of $\theta \in \mathcal{D}$, then for every $p \in (1, \infty)$ the operator A^p defined in Theorem 5.12 has a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus. Moreover we have for any $f \in \mathcal{H}^\infty(\Sigma_\phi)$ and $g \in [l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)] \otimes E$*

$$[\overline{\Psi}_{A^p}(f)]g = \mathcal{F}[\theta \mapsto \overline{\Psi}_{A(\theta)}(f)]\mathcal{F}^{-1}g,$$

i.e. $\overline{\Psi}_{A^p}(f) = T^p_{[\theta \mapsto \overline{\Psi}_{A(\theta)}(f)]}$.

Proof. First of all we note that the assumptions on the family $A(\theta)$ are stronger than in Theorem 5.12. Hence the conclusions obtained there remain true. Moreover by Lemma 5.15 the operator $A^p : l^p(\mathbb{Z}^d, E) \to l^p(\mathbb{Z}^d, E)$ is sectorial for every $p \in (1, \infty)$. Now let us fix some $f \in \mathcal{H}^\infty(\Sigma_\phi)$ and $\theta \in \mathcal{D}$. By assumption and Proposition 5.16 we have

$$\sup_{n \in \mathbb{N}} \|\Psi_{A(\theta)}(\rho_n f)\|_{\mathcal{B}(E)} \leq C \tag{5.16}$$

$$\Psi_{A(\theta)}(f)e = \lim_{n \to \infty} \Psi_{A(\theta)}(\rho_n f)e \quad \text{for every } e \in E, \tag{5.17}$$

for every $\theta \in B^d$, where the constant in (5.16) is uniform. Since $\rho_n f \in \mathcal{H}^\infty_0(\Sigma_\phi)$ we get form the representation of $\Psi_{A(\theta)}(\rho_n f)$ as a Cauchy integral that the function $\theta \mapsto \Psi_{A(\theta)}(\rho_n f)$ is analytic on $\mathcal{D}$ for every $n \in \mathbb{N}$. Furthermore we find a $\rho \in \mathbb{R}^{2d}_{>0}$ such that $\tilde{T}_\rho(0) \subset \mathcal{D}$ and B^d lies in the interior of $\tilde{T}_\rho(0)$. This yields for all $\theta \in B^d$ by Cauchy's integral formula

$$\partial^\alpha_\theta \Psi_{A(\theta)}(\rho_n f) = \frac{\alpha!}{(2\pi i)^d} \int_{\tilde{T}_\rho(0)} \frac{1}{(\theta - \zeta)^{\alpha + (1,\ldots,1)}} \Psi_{A(\zeta)}(\rho_n f) d\zeta.$$

In particular we find a constant $C_{\rho,d,\alpha}$ such that for all $\theta \in B^d$ the estimate

$$\sup_{\theta \in B^d} \|\partial^\alpha_\theta \Psi_{A(\theta)}(\rho_n f)\|_{\mathcal{B}(E)} \leq C_{\rho,d,\alpha} \sup_{\zeta \in \mathcal{D}} \|\Psi_{A(\zeta)}(\rho_n f)\|_{\mathcal{B}(E)} \leq C_{1,\alpha,\rho}$$

holds true. The very same arguments as in the proof of Theorem 5.12 applied to the family $\Psi_{A(\theta)}(\rho_n f)$ show

$$\sup_{n \in \mathbb{N}} \|T^p_{[\theta \mapsto \Psi_{A(\theta)}(\rho_n f)]}\|_{\mathcal{B}(l^p(\mathbb{Z}^d, E))} \leq C_{p,d}. \tag{5.18}$$

By Corollary 5.14 we have $T^p_{[\theta \mapsto \Psi_{A(\theta)}(\rho_n f)]} = \Psi_{A^p}(\rho_n f)$, i.e. A^p has a bounded $\mathcal{H}^\infty(\Sigma_\phi)$-functional calculus and

$$\overline{\Psi}_{A^p}(f)g = \lim_{n \to \infty} \Psi_{A^p}(\rho_n f)g \quad \text{for all } g \in l^p(\mathbb{Z}^d, E).$$

Finally (5.16), (5.17) and (5.18) together with Lemma 5.17 show that the function

$$B^d \ni \theta \mapsto \overline{\Psi}_{A(\theta)}(f)$$

is an element of $\mathcal{M}_p(\mathbb{Z}^d, E)$ for all $p \in (1, \infty)$ and

$$\overline{\Psi}_{A^p}(f)g = T_{[\theta \mapsto \overline{\Psi}_{A(\theta)}(f)]}g \quad \text{for all } g \in [l^2(\mathbb{Z}^d) \cap l^p(\mathbb{Z}^d)] \otimes E$$

follows in the same fashion as in the proof of Corollary 5.14.

□

Corresponding Results for the Zak and Bloch Transform

Once more we remind the reader of the decompositions (2.5) and (2.6) of the Zak and Bloch Transform in terms the of Fourier Transform. They where given by

$$Z := \mathcal{F}^{-1} \circ \Gamma \qquad \text{and} \qquad \Phi := \Xi \circ Z \tag{5.19}$$

where both operations Z and Φ where defined on $L^p_s(\mathbb{R}^d, E)$ with values in $D(B^d, L^p(\mathbb{I}^d, E))$. Since both mappings $\Gamma : L^p(\mathbb{R}^d, E) \to l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$ and $\Xi \in \mathcal{B}(L^p(B^d, L^p(\mathbb{I}^d, E)))$ are isometric and invertible we can easily pass from the previous results for the Fourier Transform to similar ones for Z and Φ. We summarize them in the next theorem, whose proof is an immediate consequences of the previous discussion and the properties of Γ and Ξ given in Chapter 2.

Theorem 5.19. *Assume we are given a family of operators $[A(\theta), D(A(\theta))]_{\theta \in \mathcal{D}}$ that fulfills the assumptions of Theorem 5.12 where $E = L^p(\mathbb{I}^d, F)$ for some fixed $p \in (1, \infty)$ and a reflexive Banach space F. Then then the operator $\mathbb{A}^p : L^p(\mathbb{R}^d, F) \to L^p(\mathbb{R}^d, F)$ defined by*

$$\begin{aligned} D(\mathbb{A}^p) &:= \{f \in L^p(\mathbb{R}^d, F) : \Gamma f \in D(A^p)\} \\ \mathbb{A}^p f &:= \Gamma^{-1} A^p \Gamma f \quad \text{for } f \in D(\mathbb{A}^p) \end{aligned}$$

is periodic and pseudo sectorial of angle $\omega_{\mathbb{A}} \leq \omega$ with

$$\begin{aligned} \mathbb{A}^p f &= Z^{-1}[\theta \mapsto A(\theta)]Zf \quad \text{for all } \; f \in \Gamma^{-1}\mathbf{D}_{A^p}, \\ \rho(\mathbb{A}^p) &= \bigcap_{\theta \in B^d} \rho(A(\theta)). \end{aligned}$$

Modifying the fiber operators $(A(\theta), D(A(\theta)))$ to operators $(\tilde{A}(\theta), D(\tilde{A}(\theta)))$ by

$$\begin{aligned} D(\tilde{A}(\theta)) &:= \Xi(\theta) D(A(\theta)) \\ \tilde{A}(\theta)g &:= \Xi(\theta) \circ A(\theta) \circ \Xi^{-1}(\theta)g \quad \text{for all } g \in D(\tilde{A}(\theta)) \end{aligned}$$

yields another family of analytic operators that is uniformly sectorial of angle ω *with*

$$\mathbb{A}^p f = \Phi^{-1}[\theta \mapsto \tilde{A}(\theta)]\Phi f \text{ for all } f \in \Gamma^{-1}\mathbf{D}_{\tilde{A}^p}$$
$$\rho(\mathbb{A}^p) = \bigcap_{\theta \in B^d} \rho(\tilde{A}(\theta)) = \bigcap_{\theta \in B^d} \rho(A(\theta)).$$

In both cases we denote by A^p *and* $\tilde{A}^p$ *the unique, closed, densely defined and translation invariant operator on* $l^p(\mathbb{Z}^d, L^p(\mathbb{I}^d, E))$ *reassembled from the family* $A(\theta)$ *or* $\tilde{A}(\theta)$ *respectively. Moreover if* $f \in \mathcal{H}_0^\infty(\Sigma_\phi)$ *for some* $\phi > \omega$ *then,*

$$\Psi_{\mathbb{A}^p}(f) = Z_{[\theta \mapsto \Phi_{A(\theta)}(f)]} \tag{5.20}$$
$$\Psi_{\mathbb{A}^p}(f) = B_{[\theta \mapsto \Phi_{\tilde{A}(\theta)}(f)]}. \tag{5.21}$$

Further more, if the family $[A(\theta), D(A(\theta))]_{\theta \in \mathcal{D}}$ *is uniformly sectorial of angle* ω *so is the family* $[\tilde{A}(\theta), D(\tilde{A}(\theta))]_{\theta \in \mathcal{D}}$ *and we obtain that also the operator* $\mathbb{A}^p$ *is sectorial of angle* $\omega_{\mathbb{A}} \leq \omega$.

If in addition $[A(\theta), D(A(\theta))]_{\theta \in \mathcal{D}}$ *has a uniformly bounded* $\mathcal{H}^\infty(\Sigma_\phi)$*-calculus for some* $\phi > \omega$ *so does the family* $[\tilde{A}(\theta), D(\tilde{A}(\theta))]_{\theta \in \mathcal{D}}$ *as well as the operator* $\mathbb{A}^p$. *In this case* (5.20) *and* (5.21) *extend to all* $f \in \mathcal{H}^\infty(\Sigma_\phi)$. *In particular the functional calculus has a similar decomposition as the operator* $\mathbb{A}^p$ *in terms of Zak and Bloch Transform.*

In Remark 5.13 we obtained the q-independence of the spectrum of the operator A^q (and $\tilde{A}^q$). However this property may not hold for the $\mathbb{A}^p$. Note that in the present situation it is not even possible, to define an periodic operator on $L^q(\mathbb{R}^d, F)$ for $q \neq p$.

Nevertheless if the fiber operators are properly defined on the whole scale $L^q(\mathbb{I}^d, E)$ for $q \in (1, \infty)$ and have a spectrum independent of q we are able to define an periodic operator $\mathbb{A}_q$ on $L^q(\mathbb{R}^d, F)$ for every $q \in (1, \infty)$ with $\sigma(\mathbb{A}_q) = \sigma(\mathbb{A}_p)$ for all $p, q \in (1, \infty)$. In the next section we will give some sufficient condition for this situation to occur.

5.4 p-independence the Spectrum of Periodic Operators

We have seen in Theorem 5.12 that the spectrum of an unbounded translation invariant operator defined on $l^p(\mathbb{Z}^d, E)$ that is assembled from a family of fiber operators is the union of the spectra of the fiber operators and therefore independent of p. To assure this property we had to assume some regularity in the parameter θ. If we want to transfer such a result to periodic operators on $L^p(\mathbb{R}^d, E)$ we have to assume even more, because in this case the target space of each fiber operator is also depending on p, more precisely the fiber operators act on $L^p(\mathbb{I}^d, E)$. The goal of this section is, to complement the assumptions of Theorem 5.12 to obtain a similar result for periodic operators. For simplicity let us restrict to the case $E = \mathbb{C}$ (see remarks on the end of this section).

In a first step we have to make sure that the fiber operators are properly defined on the whole scale $1 < p < \infty$. We borrow the following construction and definitions from [Aut83]. For a general discussion about interpolation space we refer to [BL76].

Unbounded Operators on Interpolation Spaces

Let B_0, B_1 be complex Banach spaces that are both continuously embedded into a topological Hausdorff space V. As usual we define $\Delta(B_0, B_1) := B_0 \cap B_1$ and $\Sigma(B_0, B_1) := B_0 + B_1$. Equipped with the norms

$$\|b\|_{\Delta(B_0,B_1)} := \max\{\|b\|_{B_0}, \|b\|_{B_1}\},$$
$$\|b\|_{\Sigma(B_0,B_1)} := \inf\{\|b_0\|_{B_0} + \|b_1\|_{B_1} : b = b_0 + b_1, b_0 \in B_0, b_1 \in B_1\},$$

these spaces are again Banach spaces which are continuously embedded in V. Let

$$(T_0, D(T_0)) : B_0 \to B_0$$
$$(T_1, D(T_1)) : B_1 \to B_1$$

be two linear operators such that $T_0 b = T_1 b$ for all $b \in D(T_0) \cap D(T_1)$. Define a operator $(T, D(T)) : \Sigma(B_0, B_1) \to \Sigma(B_0, B_1)$ by

$$D(T) := D(T_0) + D(T_1)$$
$$Tx := T_0 b_0 + T_1 b_1 \quad \text{for } x = b_0 + b_1 \in D(T).$$

Thanks to $T_0 b = T_1 b$ for all $b \in D(T_0) \cap D(T_1)$ the operator T is well defined. On an interpolation space B of (B_0, B_1) we define T_B as the part of T on B by

$$D(T_B) := \{x \in D(T) \cap B : Tx \in B\}$$
$$T_B x := Tx \quad \text{for } x \in D(T_B).$$

Because of $\Delta(B_0, B_1) \subset B$, we have $D(T_0) \cap D(T_1) \subset D(T_B)$. To see this assume $x \in D(T_0) \cap D(T_1)$. Then $x \in D(T_0) + D(T_1)$ and $x \in \Delta(B_0, B_1) \subset B$, in particular $x \in D(T) \cap B$. But $Tx = T_1 x = T_2 x$ and hence $Tx \in \Delta(B_0, B_1) \subset B$.

Definition 5.20. *We say that T satisfies the condition (R) for $\lambda \in \mathbb{C}$ if*

(i) $\lambda \in \rho(T_0) \cap \rho(T_1)$,

(ii) $R(\lambda, T_0)x = R(\lambda, T_1)x$ *for all* $x \in \Delta(B_0, B_1)$,

i.e. the operators $R(\lambda, T_0)$ and $R(\lambda, T_1)$ are consistent on $\Delta(B_0, B_1)$. T is said to satisfy condition (R) if T satisfies the condition (R) for all $\lambda \in \rho(T_0) \cap \rho(T_1)$.

Let us consider the complex interpolation method and note that the spaces $L^p(\Omega)$ are an interpolation scale for $p \in (1, \infty)$, i.e. $L^r(\Omega) = [L^p(\Omega), L^q(\Omega)]_\vartheta$ for $\frac{1}{r} = \frac{1-\vartheta}{p} + \frac{\vartheta}{q}$ with $\vartheta \in (0, 1)$. We will use

Theorem 5.21 ([Aut83, Thm. 2.7]). *Consider any measure space (Ω, Σ, μ) and let $1 < p < q < \infty$. Assume we have two linear operators $(T_p, D(T_p)) : L^p(\mu) \to L^p(\mu)$, $(T_q, D(T_q)) : L^q(\mu) \to L^q(\mu)$ with*

$$T_p g = T_q g \quad \textit{for all } g \in D(T_p) \cap D(T_q).$$

If there is a $\lambda_0 \in \rho(T_p) \cap \rho(T_q)$ such that $R(\lambda, T_p)$ or $R(\lambda, T_q)$ is compact and condition (R) is satisfied, then $\lambda \in \rho(T_r)$ for every $r \in (p, q)$ and $R(\lambda_0, T_r)$ is compact. Moreover $\sigma(T_r) = \sigma(T_p)$.

With this tool in hand and the observation of the previous chapter we are able to obtain p-independence of the spectrum of reassembled operators.

Corollary 5.22. *Let $\mathcal{D} \subset \mathbb{C}^d$ be a domain with $B^d \subset D$. Further assume, that for every $p \in (1, \infty)$ we are given a analytic family $(A_p(\theta), D(A_p(\theta)))_{\theta \in \mathcal{D}}$ of uniformly pseudo sectorial operators on $L^p(\mathbb{I}^d, \mathbb{C})$ such that, for each $\theta \in \mathcal{D}$ and all pairs $p, q \in (1, \infty)$ we have*

(i) $A_p(\theta)g = A_q(\theta)g$ for all $g \in D(A_p(\theta)) \cap D(A_q(\theta))$,

(ii) there is $\lambda \in \rho(A_p(\theta)) \cap \rho(A_q(\theta))$ such that $R(\lambda, A_p(\theta))$ is compact and condition (R) is satisfied.

Then we have for the operators $(\mathbb{A}^p, D(\mathbb{A}^p))) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ given through Theorem 5.19

$$\sigma(\mathbb{A}^p) = \sigma(\mathbb{A}^q) \text{ for all } p, q \in (1, \infty).$$

Proof. First of all each operator $\mathbb{A}^q$ is well defined and periodic. Moreover we have for each $p \in (1, \infty)$

$$\rho(\mathbb{A}^p) = \bigcap_{\theta \in B^d} \rho(A_p(\theta)).$$

Now let us pick arbitrary $1 < p < q < \infty$. By (i), (ii) and Theorem 5.21 we obtain for each $r \in (p, q)$ and $\theta \in \mathcal{D}$

$$\rho(A_r(\theta)) = \rho(A_q(\theta)).$$

Thus

$$\rho(\mathbb{A}^r) = \bigcap_{\theta \in B^d} \rho(A_r(\theta)) = \bigcap_{\theta \in B^d} \rho(A_q(\theta)) = \rho(\mathbb{A}^q).$$

Since $p, q \in (1, \infty)$ and $r \in (p, q)$ were chosen arbitrary the statement follows. $\square$

The following statement is now a immediate consequence of the one above.

Corollary 5.23. *Let $(A, D(A)) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ be a uniformly pseudo sectorial, periodic operator that is properly defined for every $p \in (1, \infty)$. Assume that the following conditions are fulfilled.*

(i) There is a open subset $\mathcal{D} \subset \mathbb{C}^d$ as well as analytic families $(\tilde{A}_p, D(\tilde{A}_p))$ defined on $\mathcal{D}$ that satisfy the assumptions of Corollary 5.22.

(ii) For every $p \in (1, \infty)$ we have $\tilde{A}_p(\theta) = A_p(\theta)$, for all $\theta \in B^d$, where the family $(A_p(\theta), D(A_p(\theta))$ is the one related to the operator $(A, D(A))$ on $L^p(\mathbb{R}^d)$.

Then the spectrum of A is independent of p.

Remark 5.24.

(i) *We have restricted our considerations in this section to the case $E = \mathbb{C}$ only because in this case Theorem 5.21 is available in the literature. Similar results where obtained in [Gün08] for abstract interpolation spaces. It seems that these results are applicable in the general case where E is a Banach space.*

(ii) *The compactness of the resolvent operator $R(\lambda, A_p(\theta))$ is in most applications a consequence of the Rellich-Kondrachov theorem.*

5.5 Stability of Periodic C_0-Semigroups on $L^p(\mathbb{R}^d, E)$

The asymptotic behavior of semigroups is of particular interest in many applications such as abstract evolution equations. In this section we will show, how this property may be obtained for a periodic generator of a semigroup on $L^p(\mathbb{R}^d, E)$ by only posing assumptions on the fiber operators. Let us start with some standard definitions.

Definition 5.25. *A C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space E is called uniformly exponentially stable, if there is some $\epsilon > 0$ such that*

$$\lim_{t\to\infty} e^{\epsilon t} \|T(t)\|_{\mathcal{B}(E)} = 0.$$

The value $\omega_0 := \inf\{\omega \in \mathbb{R} : \exists M_\omega \geq 1 \text{ such that } \|T(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0\}$ is called the growth bound of the semigroup $(T(t))_{t\geq 0}$.

Therefore a C_0-semigroup is uniformly exponentially stable if and only if $\omega_0 < 0$. Since the growth bound is often hard to calculate it is desirable to get an alternative description. A classical approach is to relate ω_0 to the spectral bound $s(A)$ of the generator A of $(T(t))_{t\geq 0}$. One always has the estimate $s(A) \leq \omega_0$. In order to turn this into an equality for a given semigroup some type of a spectral mapping theorem is needed, i.e. a relation of the form

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} := \{e^{\lambda t} : \lambda \in \sigma(A)\}.$$

In this context the following result is useful, see [EN00, Ch.V, Lem. 1.9].

Theorem 5.26. *If for a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space E and its generator A the spectral mapping theorem*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \tag{SMT}$$

holds, then $s(A) = \omega_0$.

In particular if (SMT) is satisfied the semigroup is uniformly exponential stable if and only if $s(A) < 0$. This condition is more convenient. So one is faced with the question under which condition (SMT) is satisfied. Of course this has been under investigation for a long time. We collect some sufficient conditions for (SMT) to hold. They can be found in [EN00, Ch.III,Cor.3.12].

Lemma 5.27. (SMT) *is fulfilled for the following classes of C_0-semigroups:*

(i) eventually compact semigroups

(ii) eventually norm continuous semigroups

(iii) eventually differentiable semigroups

(iv) analytic semigroups.

Now let us turn over to a more specific situation that fits into our theory of the previous chapter.

Assume we have a periodic operator $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ that is the generator of a C_0-semigroup. In this case we have shown in Chapter 3 that both, the operator A as well as the semigroup $T(t)$ have a decomposition into fiber operators acting on $L^p(\mathbb{I}^d, E)$. We will now show how one can derive (SMT) for A and $T(t)$ under suitable assumptions on the families $(T_\theta(t))_{\theta \in B^d}$ and $(A(\theta), D(A(\theta)))_{\theta \in B^d}$.

Corollary 5.28. *Let $(A, D(A)) : L^p(\mathbb{R}^d, E) \to L^p(\mathbb{R}^d, E)$ be a periodic operator that is in addition the generator of a C_0-semigroup. Further assume that the following conditions for the fiber operators corresponding to A and $T(t)$ are fulfilled*

(i) $\rho(A) = \cap_{\theta \in B^d} \rho(A(\theta))$,

(ii) $\rho(T(t)) = \cap_{\theta \in B^d} \rho(T_\theta(t))$ *for every* $t \in \mathbb{R}_{\geq 0}$,

(iii) for every $\theta \in B^d$ it holds

$$\sigma(T_\theta(t)) \setminus \{0\} = e^{t\sigma(A(\theta))} \quad \text{for all } t \in \mathbb{R}_{\geq 0}.$$

Then we have

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \quad \text{for all } t \in \mathbb{R}_{\geq 0},$$

in particular $(T(t))_{t \geq 0}$ is uniformly exponentially stable if and only if $s(A) < 0$.

Proof. We simply calculate using (ii), (iii) and (i) in that order

$$\sigma(T(t)) \setminus \{0\} = \bigcup_{\theta \in B^d} \sigma(T_\theta(t)) \setminus \{0\} = \bigcup_{\theta \in B^d} e^{t\sigma(A(\theta))} = e^{t \bigcup_{\theta \in B^d} \sigma(A(\theta))} = e^{t\sigma(A)}.$$

□

We have developed several sufficient conditions in the previous subsections under which (i) is fulfilled. In some special cases they already imply (ii).

Lemma 5.29. *If there is an analytic family $(\tilde{A}(\theta), D(\tilde{A}(\theta))) : L^p(\mathbb{I}^d, E) \to L^p(\mathbb{I}^d, E)$ defined on an open subset $\mathcal{D} \subset \mathbb{C}^d$ with $B^d \subset \mathcal{D}$ such that*

- *each $\tilde{A}(\theta)$ is the generator of a C_0-semigroup on $L^p(\mathbb{I}^d, E)$,*

- $\tilde{A}(\theta) = A(\theta)$ *where* $(A(\theta), D(A(\theta)))$ *is the family of fiber operators corresponding to A from Corollary 5.28.*

Then (i) *and* (ii) *of Corollary 5.28 hold true.*

Proof. The validity of (i) follows from Theorem 5.11. In order to proof (ii) we show that for each $t \geq 0$ the family $(T_\theta(t))_{\theta \in \tilde{\mathcal{D}}}$ is analytic on a suitable set $B^d \subset \tilde{\mathcal{D}} \subset \mathcal{D}$. The idea is to apply Vitali's theorem [HP57, 3.14] to the sequence of functions

$$\theta \mapsto \epsilon\big[\big({}^{n}/_{t}R({}^{n}/_{t}, A(\theta))\big)^n e\big], \quad (n \in \mathbb{N})$$

for any choice $(\epsilon, e) \in E' \times E$. Note that by Lemma 2.40 (c)

$$\big({}^{n}/_{t}R({}^{n}/_{t}, A(\theta))\big)^n e \overset{n\to\infty}{\longrightarrow} T_\theta(t)e$$

for every $e \in E$. Thus $\epsilon\big[\big({}^{n}/_{t}R({}^{n}/_{t}, A(\theta))\big)^n e\big] \to \epsilon[T_\theta(t)e]$ as $n \to \infty$ for every $(\epsilon, e) \in E' \times E$. But for fixed $n \in \mathbb{N}$ the function

$$\mathcal{D} \ni \theta \mapsto \epsilon\big[\big({}^{n}/_{t}R({}^{n}/_{t}, A(\theta))\big)^n e\big]$$

is analytic by Theorem 5.8 and Lemma 2.40. Since $\mathcal{D}$ is open and B^d is compact, there is a $\rho \in \mathbb{R}^{2d}_{>0}$ such that

$$\tilde{\mathcal{D}} := \big\{z \in \mathcal{D} : |\mathrm{Re}(z_j)| < \rho_j,\ |\mathrm{Im}(z_j)| < \rho_{j+d} \ \ j = 1, \ldots, d\big\}$$

satisfies $B^d \subset \tilde{\mathcal{D}} \subset \mathcal{D}$. Hence we can apply Vital's theorem [HP57, 3.14] component wise and obtain, that for each fixed $j \in \{1, \ldots, d\}$ and corresponding $\check{\theta}_j := (\theta_1, \ldots, \theta_{j-1}, \theta_{j+1}, \ldots, \theta_d)$ the function

$$\theta_j \mapsto \epsilon[T_\theta(t)e]$$

is analytic in $\tilde{\mathcal{D}}_j := \{\theta_j : \exists \check{\theta}_j \text{ s.t. } (\theta_1, \ldots, \theta_{j-1}, \theta_j, \theta_{j+1}, \ldots, \theta_d) \in \mathcal{D}\}$. Thanks to the characterization of analyticity in terms of partial analyticity by Hartogs [Har06] this implies the analyticity of

$$\tilde{\mathcal{D}} \ni \theta \mapsto \epsilon[T_\theta(t)e]$$

for any choice $(\epsilon, e) \in E' \times E$. Now Corollary 5.6 shows the analyticity of $\tilde{\mathcal{D}} \ni \theta \mapsto T_\theta(t)$ for every $t \geq 0$. Thus we obtain with the same arguments as in the proof of Theorem 5.11 in the case of unbounded operators, that

$$\rho(T(t)) = \bigcap_{\theta \in B^d} \rho(T_\theta(t))$$

for all $t \geq 0$. □

Chapter 6

A Focus on Partial Differential Operators with Periodic Coefficients

In this last chapter we turn our attention to concrete examples. We will study partial differential operators on a cylindrical domains of the form $\mathbb{R}^{d_1} \times V$ where $V \subset \mathbb{R}^{d_2}$ is a domain satisfying some smoothness conditions. An illustration of those domains is given below.

The basic ideas for proofing sectoriality of the operators under consideration are taken from [Nau12], but we need to adjust some assumptions for our specific needs (compare the discussion on page 137).

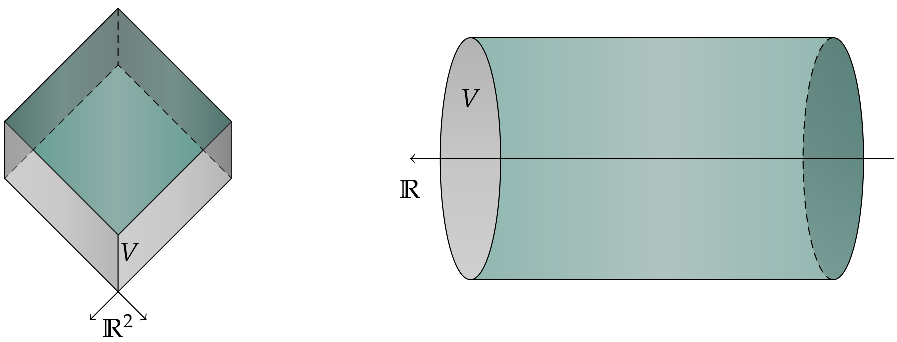

Figure 6.1: A simplified illustration of cylindrical domains. In the right picture V is an intervall and in the left one a circle. Gray surfaces are extended to infinity. The boundary operators act on the green surfaces.

In optics domains as shown in Figure 6.1 are used to model slab waveguides (left picture) or fibers (right picture). Considering, for exmaple, the wave equation in these domains and insert a time harmonic Ansatz, as we have done in Chapter 1 for the Maxwell equations, leads to a second order partial differential operator in such domains, of form we will discuss in this section. In physics boundary operators are usually given by Dirichlet-, Neumann- or Robin boundary conditions. Since the method we use allows for higher order operators as well as more general boundary conditions, these type of operators are covered. Let us start by recalling some well known results for elliptic boundary value problems.

6.1 Elliptic Boundary Value Problems

A domain $V \subset \mathbb{R}^d$ is called a standard domain, if V is given as the whole space $\mathbb{R}^d$, the half space $\mathbb{R}^d_+$ or as a domain with a compact boundary, that is as a bounded domain or the complement of a bounded domain. Now let us fix some Banach space E, integers $d, m \in \mathbb{N}$ and a standard domain $V \subset \mathbb{R}^d$ of class C^{2m} (see [AF03, 4.10] for a definition). We consider a partial differential operator

$$A(x,D)u = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha u,$$

where $\alpha \in \mathbb{N}_0^d$ and $a_\alpha : V \to \mathcal{B}(E)$. Further let the boundary operators be given by

$$B_j(x,D)u := \sum_{|\beta|\leq m_j} b_{j,\beta}(x)\mathfrak{R}_{\partial V}(D^\beta u),$$

where $m_j < 2m$, $\beta \in \mathbb{N}_0^d$, and $b_{j,\beta} : \partial V \to B(E)$ for $j = 1, \dots, m$. Consider the boundary value problem (A, B) given by

$$\begin{aligned} \lambda u + A(x,D)u &= f \ \text{ in } V \\ B_j(x,D)u &= 0 \ \text{ on } \partial V \quad (j = 1, \dots, m). \end{aligned} \tag{6.1}$$

The $L^p(V,E)$-realization of (A, B) is defined by

$$\begin{aligned} D(A) &:= \{u \in W^{2m,p}(V,E) : B_j(\cdot,D)u = 0 \ \text{ for } j = 1,\dots,m\}, \\ Au &:= A(\cdot,D)u \ \text{ for } u \in D(A). \end{aligned}$$

Definition 6.1.

(i) The differential operator $A(x,D)$ is called parameter-elliptic in V, if the principle part of its symbol $A^\#(x,\xi) := \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$ satisfies

$$\sigma(A^\#(x,\xi)) \subset \Sigma_\phi \text{ for all } x \in \overline{V},\ \xi \in \mathbb{R}^d \text{ with } |\xi| = 1 \text{ and some } \phi \in [0,\pi). \tag{6.2}$$

We call $\varphi := \inf\{\phi \in [0,\pi) : (6.2) \text{ holds}\}$ the angle of parameter ellipticity of the operator $A(x,D)$.

(ii) The boundary value problem (A, B) given by (6.1) is called parameter elliptic in V with angle of parameter ellipticity $\varphi \in [0,\pi)$, if $A(\cdot,D)$ is parameter elliptic in V with angle of parameter ellipticity φ and for each $\phi > \varphi$ the Lopatinskii-Shapiro[1] condition holds. In order to indicate that φ is the angle of parameter ellipticity of the boundary value problem (A, B), we write $\varphi_{(A,B)}$.

[1]See [DHP03] for an introduction to this topic.

Employing a finite localization procedure the following result is proven in [DHP03].

Proposition 6.2. *Let $V \subset \mathbb{R}^d$ be a C^{2m}-standard domain, $1 < p < \infty$ and E be a Banach space of class $\mathcal{HT}$. Further assume that the boundary value problem (A, B) is parameter elliptic and the coefficients of $A(x, D)$ and $B(x, D)$ satisfy*

$$\left.\begin{aligned} &a_\alpha \in C(\overline{V}, \mathcal{B}(E)) \text{ and } a_\alpha(\infty) := \lim_{|x|\to\infty} a_\alpha(x) \text{ exists for } |\alpha| = 2m \\ &a_\alpha \in [L^\infty + L^{r_k}](V, \mathcal{B}(E)),\ r_k \geq p, \frac{2m-k}{d} > \frac{1}{r_k} \text{ for } |\alpha| = k < 2m \\ &b_{j,\beta} \in C^{2m-m_j}(\partial V, \mathcal{B}(E)) \text{ for all } j = 1, \ldots, m \text{ and } |\beta| \leq m_j \end{aligned}\right\} \tag{6.3}$$

Then for each $\phi > \varphi_{(A,B)}$ there is some $\delta = \delta(\phi) \geq 0$ such that $A + \delta$ is $\mathcal{R}$-sectorial in $L^p(V, E)$ with angle less or equal to ϕ. Moreover one has

$$\mathcal{R}(\{\lambda^{1-\frac{|\gamma|}{2m}} D^\gamma (\lambda + A + \delta)^{-1} : \lambda \in \Sigma_{\pi-\phi},\ 0 \leq |\gamma| \leq 2m\}) < \infty.$$

6.2 Cylindrical Boundary Value Problems

Let $d_1, d_2 \in \mathbb{N}$ and $V \subset \mathbb{R}^{d_2}$ be a standard domain and set $\Omega := \mathbb{R}^{d_1} \times V$. According to the structure of Ω we write $x \in \Omega$ in the form $x = (x^1, x^2)$ where $x^1 \in \mathbb{R}^{d_1}$ and $x_2 \in V$ and $\alpha = (\alpha^1, \alpha^2)$ for $\alpha \in \mathbb{N}_0^{d_1+d_2}$.

We will consider a special class of boundary value problems

$$\begin{aligned} \lambda u + A(x, D)u &= f \quad \text{in } \Omega \\ B(x, D)u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{6.4}$$

Definition 6.3. *The boundary value problem (6.4) is called cylindrical if the operator $A(x, D)$ is represented as*

$$A(x, D) = A_1(x^1, D) + A_2(x^2, D)$$

with

$$A_i(x, D)u := \sum_{|\alpha^i| \leq 2m_i} a^i_{\alpha^i}(x^i) D^{\alpha^i}_{x^i} u$$

and if the boundary operator is given as

$$B(x, D) = \{B_{2,j}(x^2, D) : j = 1, \ldots, m\},$$

where

$$B_{2,j}(x^2, D)u := \sum_{|\beta^2| \leq m_{2,j}} b^2_{j,\beta^2}(x^2) \mathfrak{R}_{\partial V}(D^{\beta^2}_{x^2} u) \ \text{ for } \ j = 1, \ldots, m_2 \ \text{ where } m_{2,j} < 2m_2.$$

By definition each cylindrical boundary value problem induces a boundary value problem (A_2, B_2) on the cross section V. In view of Proposition 6.2 it is reasonable to assume that V is a C^{2m_2}- standard domain. We will denote the $L^p(V, E)$-realization of this induced boundary value problem by

$$\begin{aligned} D(A_2) &:= \{u \in W^{2m_2}(V, E) : B_{2,j}u = 0 \text{ for } j = 1, \ldots, m_2\}, \\ A_2 u &:= A_2(\cdot, D)u \quad \text{for } u \in D(A_2). \end{aligned}$$

Definition 6.4. *A cylindrical boundary value problem is called parameter elliptic in the cylindrical domain* $\Omega = \mathbb{R}^{d_1} \times V$ *if*

(i) the coefficients of $A_1(x^1, D)$ *are* $\mathbb{C}$*-valued and* $A_1(x^1, D)$ *is parameter elliptic in* $\mathbb{R}^{d_1}$ *with angle* $\varphi_{A_1} \in [0, \pi)$*,*

(ii) the boundary value problem (A_2, B_2) *on the cross section* V *is parameter elliptic with angle* $\varphi_{(A_2,B_2)} \in [0, \pi)$*,*

(iii) $\varphi_{A_1} + \varphi_{(A_2,B_2)} < \pi$*.*

We call $\varphi_{(A,B)} := \max\{\varphi_{A_1}, \varphi_{(A_2,B_2)}\}$ *the angle of parameter ellipticity of the cylindrical boundary value problem* (A, B)*.*

We define the $L^p(\Omega, E)$-realization of the boundary value problem (6.4) by

$$\begin{aligned} D(A) &:= L^p(\mathbb{R}^{d_1}, D(A_2)) \cap \bigcap_{\frac{l_1}{2m_1} + \frac{l_2}{2m_2} \le 1} W^{l_1,p}(\mathbb{R}^{d_1}, W^{l_2,p}(V, E)), \\ Au &:= A(x, D)u \quad \text{for } u \in D(A). \end{aligned}$$

In [Nau12] the following result is proven by a finite localization procedure.

Proposition 6.5. *Let* $1 < p < \infty$*,* E *be a Banach space of class* $\mathcal{HT}$*,* $V \subset \mathbb{R}^{d_2}$ *be a* C^{2m_2}*-standard domain. Further assume that the boundary value problem (6.4) on the cylindrical domain* $\Omega := \mathbb{R}^{d_1} \times V$

(i) is cylindrical,

(ii) is parameter elliptic in Ω *of angle* $\varphi_{(A,B)} \in [0, \pi)$*,*

(iii) the coefficients of $A(x, D)$ *and* $B_2(x^2, D)$ *satisfy*

$$\left.\begin{aligned} &a^1_{\alpha^1} \in C(\mathbb{R}^{d_1}, \mathbb{C}) \text{ and } a^1_{\alpha^1}(\infty) := \lim_{|x^1| \to \infty} a_{\alpha^1}(x^1) \text{ exists for all } |\alpha^1| = 2m_1 \\ &a^2_{\alpha^2} \in C(\overline{V}, E) \text{ and } a^2_{\alpha^2}(\infty) := \lim_{|x^2| \to \infty} a_{\alpha^2}(x^2) \text{ exists for all } |\alpha^2| = 2m_2 \\ &a^1_{\alpha^1} \in [L^\infty + L^{r_k}](\mathbb{R}^{d_1}, \mathbb{C}),\ r_k \ge p,\ \frac{2m_1 - k}{d_1} > \frac{1}{r_k} \text{ for } |\alpha^1| = k < 2m_1 \\ &a^2_{\alpha^2} \in [L^\infty + L^{r_k}](\mathbb{R}^{d_2}, \mathbb{C}),\ r_k \ge p,\ \frac{2m_2 - k}{d_2} > \frac{1}{r_k} \text{ for } |\alpha^2| = k < 2m_2 \\ &b^2_{j,\beta^2} \in C^{2m_2 - m_{2,j}}(\partial V, \mathcal{B}(E)) \text{ for } j = 1, \ldots, m_2 \text{ and } |\beta^2| \le m_{2,j} \end{aligned}\right\} \quad (6.5)$$

Then for each $\phi > \varphi_{(A,B)}$ there is some $\delta = \delta(\phi) \geq 0$ such that $A + \delta$ is $\mathcal{R}$-sectorial in $L^p(\Omega, E)$ of angle less or equal to ϕ. Moreover it holds

$$\mathcal{R}(\{\lambda^\rho D^\alpha(\lambda + A + \delta)^{-1} : \lambda \in \Sigma_{\pi-\phi}, \rho \in [0,1], \alpha \in \mathbb{N}_0^{d_1+d_2}, 0 \leq \rho + \frac{|\alpha^1|}{2m_2} + \frac{|\alpha^2|}{2m_2} \leq 1\}) < \infty. \quad (6.6)$$

Discussion:
Even through the above result is of interest on its own, it is does not fit our purposes. Note that the assumptions on the top order coefficients in (6.5) exclude any periodicity. Nevertheless the limit behavior is imposed in order to preform a finite localization procedure. If we are able to replace this by a uniform localization we may ignore this limit behavior and include periodic coefficients. We will show below how this may be done. Let us mention that the assumptions on the coefficients of the induced boundary value problem (A_2, B_2) on the cross section V are such that they satisfy the assertions of Proposition 6.2. The proof of the proposition above follows the idea to lift the properties of the induced boundary value problem (A_2, B_2) on the cross section V, to the cylindrical domain. This is possible thanks to the cylindrical structure of the operator $A(x, D)$. Our aim is to allow periodicity in x_1 direction. Hence there is no need to adjust anything concerning the boundary value problem (A_2, B_2).

6.3 Cylindrical Boundary Value Problems with Bounded and Uniformly Continuous Coefficients

As mentioned before, we aim for a slightly different statement as the one in Proposition 6.5. In order to set up a uniform localization we have to take a closer look at the proof of this result. Luckily most of the crucial estimates remain valid. First of all we have to adjust Definition 6.4. Let us assume, that the operator $A_1(x^1, D)$ is (M, ω_0) elliptic, i.e. that

$$\sum_{|\alpha^1|=2m_1} \|a^1_{\alpha^1}\|_{L^\infty} \leq M, \quad \sigma(A_1^\#(x^1, \xi^1)) \subset \overline{\Sigma}_{\omega_0} \setminus \{0\}, \quad |(A_1^\#(x^1, \xi^1))^{-1}| \leq M \quad (6.7)$$

for all $x^1 \in \mathbb{R}^{d_1}$ and $|\xi^1| = 1$.

Definition 6.6. *A cylindrical boundary value problem is called (M, ω_0) parameter elliptic in the cylindrical domain $\Omega = \mathbb{R}^{d_1} \times V$ if*

(i) the coefficients of $A_1(x^1, D)$ are $\mathbb{C}$-valued and $A_1(x^1, D)$ is (M, ω_0) elliptic in $\mathbb{R}^{d_1}$,

(ii) the boundary value problem (A_2, B_2) on the cross section V is parameter elliptic with angle $\varphi_{(A_2,B_2)} \in [0, \pi)$,

(iii) $\omega_0 + \varphi_{(A_2,B_2)} < \pi$.

We call $\varphi^M_{(A,B)} := \max\{\omega_0, \varphi_{(A_2,B_2)}\}$ *the angle of* (M,ω_0) *parameter ellipticity of the cylindrical boundary value problem* (A,B).

We will prove the following variant of Proposition 6.5

Theorem 6.7. *Let* $1 < p < \infty$, E *be a Banach space of class* $\mathcal{HT}$ *enjoying property* (α)[2], $V \subset \mathbb{R}^{d_2}$ *be a* C^{2m_2}*-standard domain. Further assume we have* $M \geq 0$ *and* $\omega_0 \in [0,\pi)$ *given and that the boundary value problem* (6.4) *defined on the cylindrical domain* $\Omega := \mathbb{R}^{d_1} \times V$

(i) is cylindrical

(ii) is (M,ω_0) *parameter elliptic in* Ω *of angle* $\varphi^M_{(A,B)} \in [0,\pi)$,

(iii) the coefficients of $A(x,D)$ *and* $B_2(x^2,D)$ *satisfy*

$$\left.\begin{array}{l} a^1_{\alpha^1} \in BUC(\mathbb{R}^{d_1},\mathbb{C}) \text{ for all } |\alpha^1| = 2m_1 \\ a^2_{\alpha^2} \in C(\overline{V},E) \text{ and } a^2_{\alpha^2}(\infty) := \lim\limits_{|x^2|\to\infty} a_{\alpha^2}(x^2) \text{ exists for all } |\alpha^2| = 2m_2 \\ a^1_{\alpha^1} \in L^\infty(\mathbb{R}^{d_1},\mathbb{C}) \text{ for } |\alpha^1| < 2m_1 \\ a^2_{\alpha^2} \in [L^\infty + L^{r_k}](\mathbb{R}^{d_2},\mathbb{C}),\ r_\mu \geq p,\ \dfrac{2m_2-k}{d_2} > \dfrac{1}{r_k} \text{ for } |\alpha^2| = k < 2m_2 \\ b^2_{j,\beta^2} \in C^{2m_2-m_{2,j}}(\partial V, \mathcal{B}(E)) \text{ for } j = 1,\dots,m_2 \text{ and } |\beta^2| \leq m_{2,j} \end{array}\right\} \quad (6.8)$$

Then for each $\phi > \varphi^M_{(A,B)}$ *there is some* $\delta = \delta(\phi) \geq 0$ *such that* $A+\delta$ *is* $\mathcal{R}$*-sectorial in* $L^p(\Omega,E)$ *of angle less or equal to* ϕ*. In particular we have*

$$\mathcal{R}(\{\lambda(\lambda + A + \delta)^{-1} : \lambda \in \Sigma_{\pi-\phi}) < \infty.$$

As usual the proof is divided into three parts. In the first step we consider a homogenous differential operator $A_1(x^1,D) = \sum_{|\alpha^1|=2m_1} a^1_{\alpha^1} D^{\alpha^1}$ with constant coefficients. Then a perturbation allows for small perturbations and finally a localization procedure yields the general case. In each of the first two steps we are able to use corresponding estimates given in [Nau12]. Finally we preform a uniform localization as in [KW04, §6]. The argument is along the lines of the very similar proof given there. Essentially we have to replace the space $\mathbb{C}$ by the Banach space $L^p(V,E)$ because of the additional variable x^2. We note that the argument carries over due to the cylindrical structure of our problem.

Constant Coefficients of $A_1(x^1,D)$

Let $\phi > \varphi^M_{(A,B)}$, $\lambda \in \Sigma_{\pi-\phi}$ and $\delta_2 = \delta_2(\phi) \geq 0$ given by Proposition 6.2 for the boundary value problem (A_2,B_2) on the cross section V.

[2] The assumption of property (α) is of technical nature. We will not go into detail here and refer to [KW04, 4.9] for a discussion and examples. There it is also shown, that closed subspaces of $L^p(\Omega,\mathbb{C})$ have this property if Ω is a σ-finite measure space.

If $u \in \mathcal{S}(\mathbb{R}^{d_1}, D(A_2))$ we obtain by applying partial Fourier Transform (on the group $\mathbb{R}^{d_1}$) with respect to x^1 to the equation $f = (\lambda + A_1(D) + A_2 + \delta_2)u$,

$$(\lambda + A_1(\cdot) + A_2 + \delta_2)\mathcal{F}u = \mathcal{F}f.$$

Hence we formally have

$$u = \mathcal{F}^{-1} m_\lambda \mathcal{F} f,$$

where $m_\lambda(\xi) = (\lambda + A_1(\xi) + A_2 + \delta_2)^{-1}$ for $\xi \in \mathbb{R}^{d_1}$. Note that $m_\lambda(\xi)$ is well defined if $-(\lambda + A_1(\xi)) \in \rho(A_2 + \delta_2)$, which is the case due to our specific choice $\lambda \in \Sigma_{\pi-\phi}$ and $A_1(\xi) \subset \overline{\Sigma}_{\omega_0}$. The aim is to apply a Fourier multiplier result for the Fourier Transform on $\mathbb{R}^d$, which is very similar to the one we obtained in Chapter 4 for the discrete case. In particular we will use the following result (cf. [KW04, 5.2]).

Theorem 6.8. [3] *Let E be a Banach space of class $\mathcal{HT}$ enjoying property (α) and $1 < p < \infty$ and let $\tau \subset \mathcal{B}(E)$ be $\mathcal{R}$-bounded. If $m \in C^d(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(E))$ is such that*

$$\{\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbb{R}^d \setminus \{0\}, \alpha \leq (1, \ldots, 1)\} \subset \tau, \tag{6.9}$$

then the set

$$\sigma := \{T_m : m \text{ satisfies } (6.9)\} \subset \mathcal{B}(L^p(\mathbb{R}^d, E))$$

is $\mathcal{R}$-bounded with $\mathcal{R}_p(\sigma) \leq C\mathcal{R}_p(\tau)$.

So far we are in perfect correspondence with the proof given in [Nau12, §8] for the so called 'model problem'. But now the first difference occurs.

Denote by K the set of all tuples $(a^1_{\alpha^1})_{|\alpha^1|=2m_1} \subset \mathbb{C}$ satisfying (6.7). It is shown in [KW04] that K is compact. We are now able to proof

Theorem 6.9. *For each $\phi > \varphi^M_{(A,B)}$ and $\delta_2 = \delta_2(\phi) \geq 0$ as in Proposition 6.2 we have that $A + \delta_2$ is $(\mathcal{R})$-sectorial in $L^p(\mathbb{R}^{d_1}, L^p(V, E))$ with angle less or equal than ϕ. Moreover*

$$\mathcal{R}\Big(\Big\{\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})} D^\alpha(\lambda + A + \delta_2)^{-1} : \lambda \in \Sigma_{\pi-\phi},$$
$$\alpha \in \mathbb{N}_0^{d_1+d_2},\ 0 \leq \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \leq 1,\ (a^1_{\alpha^1})_{|\alpha^1|=2m_1} \subset K\Big\}\Big) < \infty. \tag{6.10}$$

Proof. Everything except (6.10) follows with exactly the same arguments as in [Nau12, Prop.8.11]. The same sophisticated representation of

$$\xi^\alpha D^\alpha(\lambda + A_1(\xi) + A_2 + \delta_2)^{-1}$$

[3] As in the case of Fourier series T_m denotes the unique bounded extension to $L^p(\mathbb{R}^d, E)$ of the operator $\mathcal{S}(\mathbb{R}^d, E) \ni f \mapsto \mathcal{F}^{-1}[\xi \mapsto m(\xi)\mathcal{F}f(\xi)] \in \mathcal{S}'(\mathbb{R}^d, E)$. Note that in contrast to the previous chapters $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^d$.

used in their proof (cf. [Nau12, Lem.6.1], [NS11, Lem.4.4]) in combination with Kahane's contraction principle shows that, in order to proof (6.10), it is sufficient that both symbols

$$\kappa_1(\lambda,\xi,(a^1_{\alpha^1})) := \frac{\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})}\xi^{\alpha^1}}{(\lambda+A_1(\xi))^{1-\frac{|\alpha^2|}{2m_2}}},$$
$$\kappa_2(\lambda,\xi,(a^1_{\alpha^1})) := \frac{\xi^\gamma D^\gamma A_1(\xi)}{\lambda+A_1(\xi)}$$

are uniformly bounded for $(\lambda,\xi,(a^1_{\alpha^1})) \in \Sigma_{\pi-\phi}\times\mathbb{R}^{d_1}\times K$, where $\gamma \le (1,\dots,1)$. For fixed $(a^1_{\alpha^1})$, the uniform boundedness in the variables λ,ξ is already contained [Nau12] and proven by homogeneity arguments. Now for fixed α^1 and γ both mappings

$$(\lambda,\xi,(a^1_{\alpha^1})) \mapsto \kappa_1(\lambda,\xi,(a^1_{\alpha^1}))$$
$$(\lambda,\xi,(a^1_{\alpha^1})) \mapsto \frac{\xi^\gamma D^\gamma A_1(\xi)}{\lambda+A_1(\xi)}$$

are continuous. Thus compactness of K also yields uniform boundedness in $(a^1_{\alpha^1}) \subset K$. Hence (6.10) follows from Theorem 6.8 and Proposition 6.2. □

Slightly Varying Coefficients

Now we proceed by studying (M,ω_0) parameter elliptic cylindrical boundary value problems with bounded and measurable coefficients for A_1, which are close to systems with constant coefficients. As usual this is done by a perturbation argument. We will use the following result, proven in [Nau12, Lem 8.12] by a Neumann series argument.

Lemma 6.10. *Let R be a linear operator on $L^p(\mathbb{R}^{d_1},L^p(V,E))$ such that the inclusion $D(A)\subset D(R)$ holds, and let δ_2 be given as in Theorem 6.9. Assume that there are $\eta>0$ and $\delta>\delta_2$ such that*

$$\|Ru\|_{L^p(\mathbb{R}^{d_1},L^p(V,E))} \le \eta\|(A+\delta)u\|_{L^p(\mathbb{R}^{d_1},L^p(V,E))} \quad \textit{for all } u\in D(A).$$

Then $A+R+\tilde\delta$ is $(\mathcal{R})$-sectorial in $L^p(\mathbb{R}^{d_1},L^p(V,E))$ with angle less or equal to the angle of $\mathcal{R}$-sectoriality of $A+\delta_2$. Moreover, for every $\phi>\varphi_{(A,B)}$ we have

$$\mathcal{R}\Big(\Big\{\lambda^\rho D^\alpha(\lambda+A+R+\delta)^{-1} : \lambda\in\Sigma_{\pi-\phi},$$
$$\alpha\in\mathbb{N}_0^{d_1+d_2}, \rho\in[0,1],\ 0\le\rho+\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2}\le 1\Big\}\Big) \le K<\infty, \quad (6.11)$$

whenever $\eta<\mathcal{R}(\{(A+\delta)(\lambda+A+\delta)^{-1}\})^{-1}$. Here the constant K depends only on M,ω_0,p and A_2.

With this perturbation result we are now able to lift the statement of Theorem 6.9 to operators A_1 with slightly varying coefficients.

Corollary 6.11. *Let $R(x^1, D) := \sum_{|\alpha^1|=2m_1} r_{\alpha^1}(x^1)D^{\alpha^1}$ be be a differential operator such that $\sum_{|\alpha^1|=2m_1} \|r_{\alpha^1}\|_{L^\infty} < \eta$ is satisfied. Set*

$$A^{va}(x, D) := A_1(D) + R(x^1, D) + A_2(x^2, D)$$

and denote the $L^p(\mathbb{R}^{d_1}, L^p(V, E))$-realization by

$$\begin{aligned} D(A^{va}) &:= D(A), \\ A^{va}u &:= A^{va}(x, D)u \text{ for } u \in D(A^{va}). \end{aligned}$$

Then there is a $\delta > 0$ such that $A^{va} + \delta$ is $(\mathcal{R})$-sectorial on $L^p(\mathbb{R}^{d_1}, L^p(V, E))$ with angle less or equal than the angle of $\mathcal{R}$-sectoriality of $A + \delta_2$, provided η is sufficiently small. In this case we have for any $\phi > \varphi_{(A,B)}$

$$\mathcal{R}\Big(\Big\{\lambda^\rho D^\alpha(\lambda + A^{va} + \delta)^{-1} : \\ \lambda \in \Sigma_{\pi-\phi},\ \alpha \in \mathbb{N}_0^{d_1+d_2},\ \rho \in [0,1],\ 0 \le \rho + \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \le 1\Big\}\Big) < \tilde{M},$$

where $\tilde{M}$ again depends only on M, ω_0, p and A_2.

Proof. Following along the lines of the proof given [Nau12, Cro. 8.13] we obtain from Theorem 6.9

$$\|D^{\alpha^1}(A + \delta)^{-1}\| \le C \text{ for all } |\alpha^1| = 2m_1,\ \delta > \delta_2.$$

This implies

$$\begin{aligned} \|Ru\|_{L^p(\mathbb{R}^{d_1}, L^p(V,E))} &\le \sum_{|\alpha^1|=2m_1} \|r_{\alpha^1}\|_{L^\infty} \|D^{\alpha^1}(A + \delta)^{-1}(A + \delta)u\|_{L^p(\mathbb{R}^{d_1}, L^p(V,E))} \\ &\le C\eta \|(A + \delta)u\|_{L^p(\mathbb{R}^{d_1}, L^p(V,E))}. \end{aligned}$$

Thus if $\eta < 1/C\mathcal{R}(\{(A + \delta)(\lambda + A + \delta)^{-1}\})$ Lemma 6.10 applies and finishes the proof. □

The Uniform Localization Procedure

Before we set up the localization scheme let us give some crucial estimates for the treatment of 'lower order' terms. Again a proof may be found in [Nau12, Lem 8.14].

Lemma 6.12. *Let $1 < p < \infty$, $\beta \in \mathbb{N}_0^{d_1}$, $|\beta| = \mu < 2m_1$. Let $b \in L^\infty(\mathbb{R}^{d_1})$ and A^{var} be the operator defined in Corollary 6.11 and assume $\phi > \varphi_{(A,B)}$.*

(i) For every $\epsilon > 0$ there is a constant $C(\epsilon) > 0$ such that we have for all functions $u \in W^{2m_1,p}(\mathbb{R}^{d_1}, L^p(V,E))$

$$\|bD^\beta u\|_{L^p(\mathbb{R}^{d_1},L^p(V,E)))} \leq \epsilon \|u\|_{W^{2m_1,p}(\mathbb{R}^{d_1},L^p(V,E))} + C(\epsilon)\|u\|_{L^p(\mathbb{R}^{d_1},L^p(V,E)))}.$$

(ii) For every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$\mathcal{R}(\{bD^\beta(\lambda + A^{va} + \delta)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) \leq \epsilon.$$

Now let us fix some M, ω_0 and a boundary value problem as in Theorem 6.7. According to the BUC assumption on the top order coefficients of $A_1(x^1, D)$ we find for any $\epsilon > 0$ a $\delta > 0$ such that

$$\sum_{|\alpha^1|=2m_1} |a^1_{\alpha^1}(x^1) - a^1_{\alpha^1}(y^1)| < \epsilon, \text{ if } |x^1 - y^1| < \delta.$$

For a given $\epsilon > 0$ (specified later) we fix $r \in (0, \delta)$ as well as a C^∞-function $\varphi : \mathbb{R}^{d_1} \to \mathbb{R}$ with $0 \leq \varphi \leq 1$, $\operatorname{supp}(\varphi) \subset Q \subset (-r,r)^{d_1}$ such that

$$\sum_{l \in r\mathbb{Z}^{d_1}} \varphi_l^2(x) = 1 \text{ for all } x \in \mathbb{R}^{d_1},$$

where $\varphi_l(x) := \varphi(x - l)$. Further let $Q_l := Q + l$ for $l \in r\mathbb{Z}^d =: \Pi$. Additionally we choose a smooth function $\psi : \mathbb{R}^{d_1} \to \mathbb{R}$ with $\operatorname{supp}(\psi) \subset Q$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $\operatorname{supp}\varphi$ and define as before $\psi_l(x) := \psi(x - l)$ for $x \in \mathbb{R}^{d_1}$ and $l \in \Pi$.

For any $l \in \Pi$ define coefficients $a^1_{\alpha^1,l} : \mathbb{R}^{d_1} \to \mathbb{C}$ by

$$a^1_{\alpha^1,l}(x^1) := \begin{cases} a^1_{\alpha^1}(x^1) & : x^1 \in Q_l \\ a^1_{\alpha^1}(l) & : \text{else.} \end{cases}$$

We define 'local operators' by

$$\begin{aligned} D(A_l) &:= D(A), \\ A_l u(x) &:= \sum_{|\alpha^1|=2m_1} a^1_{\alpha^1,l}(x^1) D^{\alpha^1} u(x) + A_2(x^2, D)u(x). \end{aligned}$$

Choosing ϵ small enough we see, that each A_l is a small perturbation of an (M, ω_0) parameter elliptic cylindrical boundary value problem. In particular we find by Corollary 6.11 for every $\phi > \varphi_{(A,B)}$ some $\delta = \delta(\phi) > 0$ such that for all $l \in \Pi$ the operator $A_l + \delta$ is $\mathcal{R}$-sectorial of angle less or equal to ϕ. Moreover we have for any $\phi > \varphi_{(A,B)}$

$$\sup_{l \in \Pi} \mathcal{R}(\{\lambda(\lambda + A_l + \delta)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) \leq K$$

where K only depends on M, ω_0, p and A_2.

Let us write $A(x, D) = A^\#(x, D) + A_1^{\text{low}}(x^1, D)$ where

$$\begin{aligned} A^\#(x, D) &:= A_1^\#(x^1, D) + A_2(x^2, D) \\ A_1^{\text{low}}(x^1, D) &:= A(x, D) - A^\#(x, D) \end{aligned}$$

Note that $A_1^{\text{low}}(x^1, D)$ is a differential operator of order less or equal to $2m_1 - 1$, which acts only on $\mathbb{R}^{d_1}$. We have for $u \in D(A)$ with $\operatorname{supp}(u) \subset Q_l$

$$A^\#(x, D)u(x) = A_l u(x).$$

Let us define the space on which the the family of 'localized operators' act, by

$$\mathbb{X}_p := l^p(\Pi, L^p(\mathbb{R}^{d_1}, L^p(V, E)),$$

and denote elements of $\mathbb{X}_p$ by $(u_l)_{l\in\Pi}$ or in short (u_l). On $\mathbb{X}_p$ we define an operator $\mathbb{A}$ by

$$\begin{aligned} D(\mathbb{A}) &:= l^p(\Pi, D(A)) \\ \mathbb{A}(u_l) &:= (A_l u_l). \end{aligned}$$

Again let $\phi > \varphi_{(A,B)}$ and $\delta = \delta(\phi) > 0$ be such that for each $l \in \Pi$ the local operators $A_l + \delta$ are uniformly $\mathcal{R}$-sectorial with angle less or equal to ϕ. Since $\mathbb{A}$ is a diagonal operator so is $\mathbb{A} + \delta$ and the uniform sectoriality of $A_l + \delta$ implies sectoriality of $\mathbb{A} + \delta$ with angle less or equal to ϕ. But as shown in [KW04, p.149] even more is true. The operator $\mathbb{A} + \delta$ is $\mathcal{R}$-sectorial with angle less or equal to ϕ.
The operator of 'localization' J is defined by

$$\begin{aligned} J : L^p(\mathbb{R}^{d_1}, L^p(V, E)) &\to \mathbb{X}_p \\ u &\mapsto (u_l) := (\varphi_l u). \end{aligned}$$

Clearly J is injective, linear and continuous. Further J maps $D(A)$ onto $D(\mathbb{A})$. The operation of patching together is denoted by P and defined via

$$\begin{aligned} P : \mathbb{X}_p &\to L^p(\mathbb{R}^{d_1}, L^p(V, E)) \\ (u_l) &\mapsto \sum_{l\in\Pi} \varphi_l u_l. \end{aligned}$$

Now it is time to relate the operator A on $L^p(\mathbb{R}^{d_1}, L^p(V, E))$ to $\mathbb{A}$ on $\mathbb{X}_p$. For this reason we give formulas for $JA - \mathbb{A}J$ and $AP - P\mathbb{A}$ as they are obtained in [KW04]. Before we proceed we introduce the following notation. For $k \in \Pi$ denote by $k \bowtie l := \{l \in \Pi : Q_l \cap Q_k \neq \emptyset\}$. Now the same calculation as in [KW04, page 150] shows the relations

$$\begin{aligned} JA &= (\mathbb{A} + \mathbb{B})J \\ AP &= P(\mathbb{A} + \mathbb{D}) \end{aligned} \tag{6.12}$$

where the operators $\mathbb{B}$ and $\mathbb{D}$ are given by

$$\begin{aligned} \mathbb{B}(u_l) &:= \big(A_1^{\text{low}} \psi_l u_l + \sum_{k\bowtie l} [\varphi_l A - A\varphi_l]\varphi_k u_k\big)_l \\ \mathbb{D}(u_l) &:= \big(A_1^{\text{low}} u_l + \sum_{k\bowtie l} \varphi_l [A\varphi_k - \varphi_k A] u_k\big)_l, \end{aligned}$$

for $(u_l) \in D(\mathbb{A})$. Considering both operators $\mathbb{B}$ and $\mathbb{D}$ as infinitely expanded matrix operators we see that each component of this operators is a partial differential operator which only acts on $\mathbb{R}^{d_1}$ of order less or equal to $2m_1 - 1$ with coefficients in $L^\infty(\mathbb{R}^{d_1})$. Moreover the number of non zero entries in the j-th row is bounded by $\#k \bowtie j$ which is and integer independent of j. These observations allow us to apply Lemma 6.12 in each component. Hence we find for any $\epsilon > 0$ and constant $C(\epsilon) > 0$ and a $\delta = \delta(\epsilon) > 0$ such that

$$\|\mathbb{B}(u_l)\| + \|\mathbb{D}(u_l)\| \leq \epsilon\|(\delta + \mathbb{A})u\|_{\mathbb{X}_p} + C(\epsilon)\|u\|_{\mathbb{X}_p}.$$

This estimate allows for an application of the following perturbation result for $\mathcal{R}$-sectorial operators, see [KW04, Cor.6.7]

Lemma 6.13. *Let A be an $\mathcal{R}$-sectorial operator in a Banach space X with angle ω_A and $\phi > \omega_A$. Further let B be a linear operator satisfying $D(A) \subset D(B)$ and*

$$\|Bx\| \leq a\|Ax\| + b\|x\| \quad \text{for } x \in D(A)$$

for some $a, b \geq 0$. If a is small enough then there is a $\delta > 0$ such that $A + B + \delta$ is $\mathcal{R}$-sectorial with angle less or equal to ϕ.

As a result we obtain that there is a $\delta > 0$ such that both operators

$$\mathbb{A} + \mathbb{B} + \delta \qquad \text{and} \qquad \mathbb{A} + \mathbb{D} + \delta$$

are $\mathcal{R}$-sectorial in $\mathbb{X}_p$ with angle less or equal to $\varphi_{(A,B)}$. Now let $\phi > \varphi_{(A,B)}$ and pick any $\lambda \in \Sigma_{\pi-\phi}$, $u \in D(A)$ as well as $f \in L^p(\mathbb{R}^{d_1}, L^p(V,E))$ such that $(\lambda + A + \delta)u = f$ is satisfied. Then (6.12) shows

$$u = PJu = P(\lambda + (\mathbb{A} + \mathbb{B} + \delta))^{-1}Jf$$

as well as for any $f \in L^p(\mathbb{R}^{d_1}, L^p(V,E))$

$$\begin{aligned} f = PJf &= P(\lambda + (\mathbb{A} + \mathbb{D} + \delta))(\lambda + (\mathbb{A} + \mathbb{D} + \delta))^{-1}Jf \\ &= (\lambda + A + \delta)P(\lambda + (\mathbb{A} + \mathbb{D} + \delta))^{-1}Jf. \end{aligned}$$

In particular $\lambda + \mathbb{A} + \delta$ is bijective. Hence we have $-\lambda \in \rho(A + \delta)$ and

$$\lambda(\lambda + A + \delta)^{-1} = P\lambda(\lambda + \mathbb{A} + \mathbb{B})^{-1}J$$

This yields

$$\mathcal{R}(\{\lambda(\lambda + A + \delta)^{-1} : \lambda \in \Sigma_{\pi-\phi}\}) < \infty,$$

by boundedness of J and P. Thus we have proven Theorem 6.7.

6.4 Waveguide Type Boundary Value Problems

Now that we have Theorem 6.7 at hand, we are able to consider some special classes of cylindrical boundary value problems.

Definition 6.14. *A cylindrical (M, ω_0) parameter elliptic boundary value problem is said to be of wave guide type, if the coefficients of $A(x, D)$ satisfy*

$$\left.\begin{aligned} &a^1_{\alpha^1} \in C(\mathbb{R}^{d_1}, \mathbb{C}) \text{ for } |\alpha^1| = 2m_1 \text{ and } a^1_{\alpha^1} \text{ periodic w.r.t. } \mathbb{Z}^{d_1} \\ &a^2_{\alpha^2} \in C(\overline{V}, E) \text{ and } a^2_{\alpha^2}(\infty) := \lim_{|x^2|\to\infty} a_{\alpha^2}(x^2) \text{ exists for all } |\alpha^2| = 2m_2 \\ &a^1_{\alpha^1} \in L^\infty(\mathbb{R}^{d_1}, \mathbb{C}) \text{ for } |\alpha^1| < 2m_1 \text{ and } a^1_{\alpha^1} \text{ periodic w.r.t. } \mathbb{Z}^{d_1} \\ &a^2_{\alpha^2} \in [L^\infty + L^{r_k}](\mathbb{R}^{d_2}, \mathbb{C}),\ r_k \geq p,\ \frac{2m_2 - k}{d_2} > \frac{1}{r_k} \text{ for } |\alpha^2| = k < 2m_2 \\ &b^2_{j,\beta^2} \in C^{2m_2 - m_{2,j}}(\partial V, \mathcal{B}(E)) \text{ for } j = 1, \ldots, m_2 \text{ and } |\beta^2| \leq m_{2,j} \end{aligned}\right\} \quad (6.13)$$

Recall that we could have used any other lattice of periodicity. Then we would arrive at (6.13) after a rescaling as in Chapter 3. The constraints (6.13) on the coefficients are covered by Theorem 6.7.

In particular let $A^{\text{per}}(x, D)$ be $L^p(\mathbb{R}^{d_1}, L^p(V, E))$-realization of a wave guide type boundary value problem given by

$$\begin{aligned} D(A^{\text{per}}) &:= L^p(\mathbb{R}^{d_1}, D(A_2)) \cap \bigcap_{\frac{l_1}{2m_1} + \frac{l_2}{2m_2} \leq 1} W^{l_1,p}(\mathbb{R}^{d_1}, W^{l_2,p}(V, E)) \\ A^{\text{per}} u(x) &:= A^{\text{per}}(x, D) u(x) \quad \text{for } u \in D(A^{\text{per}}). \end{aligned} \quad (6.14)$$

Then for every $\phi > \varphi_{(A,B)}$, there is some $\delta = \delta(\phi) > 0$ such that

$$\mathcal{R}(\{\lambda(\lambda + A^{\text{per}} + \delta)^{-1} : \lambda \in \Sigma_{\pi - \varphi}\}) < \infty.$$

In particular the operator $A^{\text{per}} + \delta$ is sectorial with angle less or equal to ϕ. Note that the domain of A^{per} is invariant under translations τ^1_z with respect to $\mathbb{R}^{d_1}$, where $z \in \mathbb{Z}^{d_1}$. Moreover we have for $u \in D(A^{\text{per}})$ and $z \in \mathbb{Z}^{d_1}$

$$\begin{aligned} \tau^1_z A^{\text{per}} u(x) &= \tau^1_z A_1(x^1, D) u(x) + \tau^1_z A_2(x^2, D) u(x) \\ &= \sum_{|\alpha^1| = 2m_1} a^1_{\alpha^1}(x - z) D^{\alpha^1} u(x^1 - z, x_2) + \sum_{|\alpha^2 = 2m_2|} a^2_{\alpha_2}(x^2) D^{\alpha^2} u(x_1 - z, x_2) \\ &= A^{\text{per}}(\tau^1_z u)(x) \end{aligned}$$

by the periodicity assumptions on the coefficients a_{α^1}. Clearly A^{per} is closed and densely defined. Thus we can apply Theorem 3.38 to obtain a family $(A(\theta), D(A(\theta)))_{\theta \in B^{d_1}}$ of closed and densely defined operators on the 'fiber space' $L^p(\mathbb{I}^{d_1}, L^p(V, E))$ such that

$$A^{\text{per}} u = \Phi^{-1}[\theta \mapsto A(\theta)] \Phi u \quad \text{for } u \in R(\lambda, A^{\text{per}}) L^p_c(\mathbb{R}^{d_1}, L^p(V, E)),$$

where $\lambda \in \rho(A^{\text{per}})$ is arbitrary. Recall that Φ only operates in the first variable x^1 in which we required periodicity of the coefficients.

Since the operator A^{per} is given by a concrete expression we are able to calculate the fiber operators explicitly.

For this purposes let us pick $u \in C_c^\infty(\mathbb{R}^{d_1}, D(A_2))$, which is a core for A^{per}. We have with a similar calculation as we preformed in Chapter 1 for $\theta \in B^{d_1}$, $x^1 \in \mathbb{I}^{d_1}$ and $x^2 \in V$

$$
\begin{aligned}
[\Phi A^{\text{per}} u](\theta, x^1, x^2) &= [\Phi A_1(\cdot^1, D^{\alpha^1})u](\theta, x^1, x^2) + [\Phi A_2(\cdot^2, D^{\alpha^2})u](\theta, x^1, x^2) \\
&= e^{-2\pi i x^1 \theta} \sum_{z \in \mathbb{Z}^{d_1}} e^{2\pi i \theta z} \sum_{|\alpha^1| \le 2m_1} a^1_{\alpha^1}(x^1 - z) D^{\alpha^1} u(x^1 - z, x^2) \\
&\qquad + A_2(x^2, D^{\alpha^2})[\Phi u(\cdot, x^2)](x^1, \theta) \\
&= \sum_{|\alpha^1| \le 2m_1} a^1_{\alpha^1}(x^1) e^{-2\pi i x^1 \theta} D^{\alpha^1} [\sum_{z \in \mathbb{Z}^{d_1}} e^{2\pi i \theta z} \tau^1_z u(\cdot, x^2)](x^1) \\
&\qquad + A_2(x^2, D^{\alpha^2})[\Phi u(\cdot, x^2)](x^1, \theta) \\
&= \sum_{|\alpha^1| \le 2m_1} a^1_{\alpha^1}(x^1)(D + 2\pi i \theta)^{\alpha^1} e^{-2\pi i x^1 \theta} [\sum_{z \in \mathbb{Z}^{d_1}} e^{2\pi i \theta z} \tau^1_z u(\cdot, x^2)](x^1) \\
&\qquad + A_2(x^2, D^{\alpha^2})[\Phi u(\cdot, x^2)](x^1, \theta) \\
&= A_1(x^1, D + 2\pi i \theta)[\Phi u(\cdot, x^2)](\theta, x^1) + A_2(x^2, D)[\Phi u(\cdot, x^2)](x^1, \theta).
\end{aligned}
$$

Moreover, for each $x^2 \in V$, $\theta \in B^{d_1}$ the function $\mathbb{I}^{d_1} \ni x^1 \mapsto [\Phi u(\cdot, x^2)](\theta, x^1)$ satisfies periodic boundary conditions on $\mathbb{I}^d$. Hence we are led to the study of a cylindrical boundary value problem on the set $\mathbb{I}^d \times V$ with periodic boundary conditions with respect to the first variable.

Cylindrical Boundary Value Problems with Periodic Boundary Conditions

Very similar to the discussion above we will now consider 'periodic boundary value problems' on the cylindrical domain $\Omega := \mathbb{I}^{d_1} \times V$, where $V \subset \mathbb{R}^{d_1}$ is again a sufficiently smooth standard domain. More precisely we consider a boundary value problem of the form

$$
\begin{aligned}
\lambda u + A_1(x^1, D)u + A_2(x^2, D)u &= f \ \text{ in } \Omega \\
B_j(x^2, D)u &= 0 \ \text{ on } \mathbb{I}^d \times \partial V, \quad j = 1, \dots, m_2 \\
D^{\beta^1} u|_{x^1_j = 0} - D^{\beta^1} u|_{x^1_j = 1} &= 0 \ \text{ for } j = 1, \dots, d_1 \text{ and } |\beta^1| < m_1.
\end{aligned}
\tag{6.15}
$$

Problems of this type are also considered in [Nau12], see also [DN11]. Under suitable assumptions on the operators A_1 and A_2 the authors are able to show $\mathcal{R}$-sectoriality of the L^p-realization of the boundary value problem (6.15). Let us outline the result. Again we assume that A_2 is of the same type as in the previous chapter, i.e. Proposition 6.2 holds true for the $L^p(V, E)$ realization.

Denote the $L^p(\Omega, E)$-realization of (6.15) by

$$\begin{aligned} D(A) &:= L^p(\mathbb{I}^{d_1}, D(A_2)) \cap \bigcap_{\frac{l_1}{2m_1}+\frac{l_2}{2m_2}\le 1} W^{l_1,p}_{\text{per}}(\mathbb{I}^{d_1}, W^{l_2,p}(V,E)), \\ Au(x) &= A_1(x^1, D)u(x^1,x^2) + A_2(x^2, D)u(x^1,x^2) \quad \text{for } u \in D(A). \end{aligned} \tag{6.16}$$

Further assume that the following conditions for the coefficients of A_1 are satisfied.

$$\left. \begin{aligned} &a^1_{\alpha^1} \in C_{\text{per}}(\mathbb{I}^d, \mathbb{C}) \;\; \text{for } |\alpha^1| = 2m_1 \\ &a^1_{\alpha^1} \in L^{r_k}(\mathbb{I}^d, \mathbb{C}),\; r_k \ge p, \frac{2m_2-k}{d_1} > \frac{1}{r_k} \;\; \text{for } |\alpha^1| = k < 2m_1 \end{aligned} \right\} \tag{6.17}$$

The following result is obtained as a part of the proof in [Nau12, Thm. 8.10]

Proposition 6.15. *Let $1 < p < \infty$ and $\Omega = \mathbb{I}^{d_1} \times V$ where $V \subset \mathbb{R}^{d_2}$ is a C^{2m_2}-standard domain. Further we assume that the boundary value problem given by (6.15) on the cylindrical domain Ω*

(i) is cylindrical

(ii) the coefficients of A_1 satisfy (6.17) and the ones of A_2 (6.3),

(iii) is cylindrical parameter elliptic in $\overline{\Omega}$ of angle $\varphi_{(A,B)} \in [0, \pi)$.

Then for each $\phi > \varphi_{(A,B)}$ there is a $\delta = \delta(\phi)$ such that the $L^p(\Omega, E)$ realization of (6.15) given by (6.16) is $\mathcal{R}$-sectorial with angle less or equal to ϕ. Moreover we have

$$\mathcal{R}\Big(\Big\{\lambda^{1-(\frac{|\alpha^1|}{2m_1}+\frac{|\alpha^2|}{2m_2})} D^\alpha (\lambda + A + \delta)^{-1} : \\ \lambda \in \Sigma_{\pi-\phi},\; \alpha \in \mathbb{N}_0^{d_1+d_2},\; 0 \le \frac{|\alpha^1|}{2m_1} + \frac{|\alpha^2|}{2m_2} \le 1\Big\}\Big) < \infty.$$

After this short intermezzo we note that applying partial Bloch transform to the L^p realization of a waveguide type boundary value problem on the cylindrical domain $\mathbb{R}^{d_1} \times V$, results in a family of cylindrical boundary value problems with periodic boundary conditions on the cylindrical domain $\mathbb{I}^d \times V$. More precisely we will show in the next theorem, that fiber decomposition of the the L^p-realization A^{per} from (6.14) is given by the family

$$\begin{aligned} D(A(\theta)) &:= L^p(\mathbb{I}^{d_1}, D(A_2)) \cap \bigcap_{\frac{l_1}{2m_1}+\frac{l_2}{2m_2}\le 1} W^{l_1,p}_{\text{per}}(\mathbb{I}^{d_1}, W^{l_2,p}(V,E)), \\ A(\theta)u(x^1,x^2) &:= A_1(x^1, 2\pi i\theta + D)u(x^1,x^2) + A_2(x^2, D)u(x^1,x^2) \quad (u \in D(A)). \end{aligned} \tag{6.18}$$

of L^p-realizations of periodic boundary value problems on the domain $\mathbb{I}^{d_1} \times V$. The assumptions (6.13) on the coefficients of A^{per} imply the validity of (6.17) and also (M, ω_0) parameter ellipticity carries over to the family $A(\theta)$ due to the fact, that the principle part of $A_1(x^1, 2\pi i\theta + D)$ is independent of θ and equals $A_1^\#(x^1, \xi)$. We have

Theorem 6.16. *Let $1 < p < \infty$, E be a Banach space of class $\mathcal{HT}$ enjoying property (α). Assume $V \subset \mathbb{R}^{d_2}$ is a C^{2m_2}-standard domain and we are given a waveguide type boundary value problem on the cylindrical domain $\Omega := \mathbb{R}^{d_1} \times V$ such that*

(i) the coefficients of A_1 and A_2 satisfy (6.13),

(ii) it is (M, ω_0) parameter elliptic on Ω of angle $\varphi^M_{(A,B)} \in [0, \pi)$.

Then the fiber decomposition of the $L^p(\Omega)$-Realization A^{per} given through (6.14) is given by the operators (6.18). In particular we have

$$A^{per} g = \Phi^{-1}[\theta \mapsto A(\theta)]\Phi g \text{ for all } g \in \mathbf{D}_{A^{per}} := R(\lambda, A^{per}) L^p_c(\mathbb{R}^{d_1}, L^p(V, E))$$

where $\lambda \in \rho(A^{per})$ is arbitrary. Moreover it holds

$$\rho(A^{per}) = \bigcap_{\theta \in B^d} \rho(A(\theta)).$$

Proof. First of all we obtain from the assumptions and Theorem 6.9 the existence of an unbounded sequence $(\lambda_k)_{k\in\mathbb{N}} \in \rho(A)$ such that

$$\lambda_k R(\lambda_k, A^{\text{per}}) f \to f \quad \text{for } k \to \infty$$

for all $f \in L^p(\mathbb{R}^{d_1}, L^p(V, E))$. The periodicity of A^{per} with respect to the variable x^1 has been shown before. Thus we can apply Theorem 3.38 to obtain a family of closed and densely defined fiber operators $(A(\theta), D(A(\theta))$ defined on $L^p(\mathbb{I}^d, L^p(V, E))$ such that

$$A^{\text{per}} u = \Phi^{-1}[\theta \mapsto A(\theta)]\Phi u \quad \text{for } u \in \mathbf{D}_A$$

and

$$\rho(A^{\text{per}}) \subset \bigcap_{\theta \in B^d \setminus \Omega} \rho(A(\theta))$$

where $\Omega \subset B^d$ is a set of measure zero. We also note that for every $\phi > \varphi^M_{(A,B)}$ we find a $\delta = \delta(\phi) \geq 0$ such that $A^{\text{per}} + \delta$ is sectorial in $L^p(\mathbb{R}^{d_1}, L^p(V, E))$ with angle less or equal to ϕ (cf.Theorem 6.7). But if we pick $u \in C^\infty_c(\mathbb{R}^{d_1}, D(A_2))$, then $u \in D(A^{\text{per}})$ and the same calculation as on page 146 shows for $\theta \in B^d$ and $(x^1, x^2) \in \mathbb{I}^d \times V$

$$\begin{aligned} [\Phi A^{\text{per}} u](\theta, x^1, x2) &= \big(A_1(x^1, D + 2\pi i\theta) + A_2(x^2, D)\big)[\Phi u(\cdot, x^2)](\theta, x^1) \\ &=: \mathbf{A}(\theta)[\Phi u(\cdot, x^2)](\theta, x^1). \end{aligned}$$

For fixed θ we denote by $\tilde{A}(\theta)$ the $L^p(\mathbb{I}^{d_1}, L^p(V, E)))$-realization of $\mathbf{A}(\theta)$ given through (6.18). By the representation of $\mathbf{A}(\theta)$ we see that the coefficients of A_1 satisfy (6.17) and the coefficients of A_2 remain unchanged and still fulfill (6.3). Since the principle part of $\mathbf{A}_1(\theta) = A_1^\#(x_1, D)$ is independent of θ also (M, ω_0) parameter ellipticity is preserved. Hence we can apply Proposition 6.15 to find for every $\phi > \varphi^M_{(A,B)}$ a $\delta_1(\phi) \geq 0$ such that for all $\theta \in B^d$ the operator $\tilde{A}(\theta) + \delta_2$ is sectorial of angle less or equal to ϕ. Now let us fix $\phi > \varphi^M_{(A,B)}$ and accordingly $\delta > 0$ such that, $A^{\text{per}} + \delta$ and $\tilde{A}(\theta) + \delta$ are sectorial of angle less or equal to ϕ.

Since $D(\tilde{A}(\theta))$ is independent of θ and $\theta \mapsto \tilde{A}(\theta) \in C^\infty(B^d, \mathbf{D})$, where $D = D(\tilde{A}(\theta))$ is equipped with one (of the equivalent) graph norms $\|\cdot\|_{D(\tilde{A}(\theta))}$, we can apply Theorem 4.22 to $\theta \mapsto (\lambda + \tilde{A}(\theta) + \delta)^{-1}$ for $\lambda \in \Sigma_{\pi-\phi}$ and obtain, that

$$T_{\lambda,\delta} := \Phi^{-1}[\theta \mapsto (\lambda + \tilde{A}(\theta) + \delta)^{-1}]\Phi$$

defines a bounded and periodic operator on $L^p(\mathbb{R}^{d_1}, L^p(V, E))$. For u in the dense subset $C_c^\infty(\mathbb{R}^{d_1}, D(A_2))$ we have the identity

$$T_{\lambda,\delta}(\lambda + A^{\text{per}} + \delta)u = u.$$

Indeed $(\lambda + A^{\text{per}} + \delta)u \in L_c^p(\mathbb{R}^{d_1}, L^p(V, E))$. Hence

$$\begin{aligned} T_{\lambda,\delta}(\lambda + A^{\text{per}} + \delta)u &= \Phi^{-1}[\theta \mapsto (\lambda + \tilde{A}(\theta) + \delta)^{-1}]\Phi(\lambda + A^{\text{per}} + \delta)u \\ &= \Phi^{-1}[\theta \mapsto (\lambda + \tilde{A}(\theta) + \delta)^{-1}(\lambda + \tilde{A}(\theta) + \delta)\Phi u \\ &= u. \end{aligned}$$

Writing $u = (\lambda + A^{\text{per}} + \delta)^{-1} f$ with some $f \in (\lambda + A^{\text{per}} + \delta)C_c^\infty(\mathbb{R}^{d_1}, D(A_2))$ yields

$$T_{\lambda,\delta} f = (\lambda + A^{\text{per}} + \delta)^{-1} f,$$

which extends by denseness to all $f \in L^p(\mathbb{R}^{d_1}, L^p(V, E))$. In particular we have $T_{\lambda,\delta} = (\lambda + A^{per} + \delta)^{-1}$ for all $\lambda \in \Sigma_{\pi-\phi}$. But this implies by the construction of the fiber operators in Section 3.3, that

$$\lambda + \delta + A(\theta) = \lambda + \delta + \tilde{A}(\theta)$$

for almost all $\theta \in B^d$. In particular we have

$$\begin{aligned} D(A(\theta)) &= D(\tilde{A}(\theta)), \\ A(\theta)u &= \tilde{A}(\theta)u \quad \text{for all} \quad u \in D(A(\theta)) \end{aligned}$$

for almost all $\theta \in B^d$. Thus we obtain

$$A^{\text{per}} u = \Phi^{-1}[\theta \mapsto \tilde{A}(\theta)]\Phi u \quad \text{for all } u \in \mathbf{D}_{A^{\text{per}}}.$$

The uniform sectoriality of $\tilde{A}(\theta) + \delta$ together with the continuous dependence on the parameter θ shows that the assumptions of Theorem 5.11 are satisfied. Hence also

$$\rho(A^{\text{per}}) = \bigcap_{\theta \in B^d} \rho(\tilde{A}(\theta))$$

follows and the theorem is proven. □

Let us mention that at this point the advantage of the Bloch Transform gets visible. We obtained θ dependence of the fiber operators only in the differential expression, but the domains are constant. This simplified the arguments in order to show that $T_{\lambda,\delta}$ is a bounded operator on $L^p(\mathbb{R}^d, E)$. For the Zak Transform the dependency is the other way around i.e. the differential expression is independent of θ but the domain is not. In this case one needs to show analyticity of the fiber operators, which is a lots harder than continuity.

Appendix **A**

More about Vector-Valued Functions

A.1 Smooth Functions

Consider a function $f : \mathbb{R}^d \to E$, where E is an arbitrary Banach space. We will say that f is differentiable at the point $x_0 \in \mathbb{R}^d$ if there is a continuous and linear map $\lambda : \mathbb{R}^d \to E$ and a map ψ defined in a neighborhood of zero with values in E, such that

$$\lim_{\mathbb{R}^d \ni h \to 0} \psi(h) = 0 \text{ in } E$$

and

$$f(x_0 + h) = f(x_0) + \lambda(h) + |h|\psi(h).$$

It is not hard to show that the continuous linear map λ is uniquely determined by f and x_0. λ is called the *derivative* of f at the point x_0 and denoted by $f'(x_0)$. We note that $f'(x_0) \in \mathcal{B}(\mathbb{R}^d, E)$. If f is differentiable at every point $x_0 \in \mathbb{R}^d$ we will say that f is differentiable.

If in addition $x \mapsto f'(x) \in \mathcal{B}(\mathbb{R}^d, E)$ is continuous, f is called continuous differentiable. The set of all $f : \mathbb{R}^d \to E$ which are continuous differentiable is denoted by $C^1(\mathbb{R}^d, E)$. For the second derivative we observe that $\mathcal{B}(\mathbb{R}^d, E)$ is a Banach space, so that we may define differentiability of f' in the same fashion as for f. The derivative of f' is denoted by $f'' = f^{(2)}$. Now for fixed $x_0 \in \mathbb{R}^d$ the value of $f^{(2)}(x_0)$ is an element of $\mathcal{B}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d, E)))$. For higher order derivatives we define as usual

$$f^{(k)}(x_0) := (f^{(k-1)})'(x_0)$$

and call a function k-times continuous differentiable, if the derivative $f^{(l)}(x_0)$ exists for every $x_0 \in \mathbb{R}^d$ and $0 \leq l \leq k$ and the mappings

$$\mathbb{R}^d \ni x \mapsto f^{(l)}(x) \in \mathcal{B}(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d, \ldots, \mathcal{B}(\mathbb{R}^d, E)))$$

are continuous for every $0 \leq l \leq k$. The set of all k-times continuous differentiable functions is denoted by $C^k(\mathbb{R}^d, E)$. A function $f : \mathbb{R}^d \to E$ is called smooth if

$$f \in \bigcap_{k \in \mathbb{N}_0} C^k(\mathbb{R}^d, E) =: C^\infty(\mathbb{R}^d, E).$$

Since $\mathbb{R}^d$ is a product space we may also introduce directional derivatives. Fix $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in \mathbb{R}^{d-1}$ and consider for $f : \mathbb{R}^d \to E$ the function

$$t \mapsto f(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_d).$$

If this map is differentiable in the above sense at the point x_j, we call its derivative the partial derivative of f in j-th direction and denote it by $\partial_j f(x)$ for $x = (x_1, \ldots, x_d)^T$. If $\partial_j f(x)$ exists at the point x then

$$\partial_j f(x) = \lambda_{j,x} : \mathbb{R} \to E$$

is a unique continuous and linear map such that

$$f(x_1, \ldots, x_{j-1}, x_j + h, x_{j+1}, \ldots, x_d) - f(x_1, \ldots, x_d) = \lambda_{j,x}(h) + o(h)$$

for small enough $h \in \mathbb{R}$. By definition $\lambda_{j,x} \in \mathcal{B}(\mathbb{R}, E)$. But $\mathcal{B}(\mathbb{R}, E)$ may be identified with E via the isometric mapping $T \mapsto T(1)$. We have

Lemma A.1. *$f : \mathbb{R}^d \to E$ is in $C^l(\mathbb{R}^d, E)$ if and only if each partial derivative $\partial_j f : \mathbb{R}^d \to E$ is in $C^{l-1}(\mathbb{R}^d, E)$.*

A.2 The Bochner Integral

For a extensive introduction to the integration of vector valued functions we refer to standard text books like [DS58,DU77,Lan93]. Especially for the Bochner integral the original article by Bochner [Boc33] is a nice to read source.

Measurable Functions

Let (Ω, Σ, μ) be a σ-finite, positive, measure space and E a Banach space. We say that any property (P) holds true for μ-almost all $\omega \in \Omega$ if there is a set $\tilde{\Omega} \in \Sigma$ with $\mu(\tilde{\Omega}) = 0$ and (P) is valid for all $\omega \in \Omega \setminus \tilde{\Omega}$. It is always clear form the context which measure is under consideration. Hence we ignore the μ and say (P) holds for almost all $\omega \in \Omega$, or almost everywhere.

For vector valued functions there are basically two concepts of measurability, strong and weak measurability.

Definition A.2.

(i) A function $f : \Omega \to E$ is called simple, if there exist $e_1, \ldots, e_m \in E$ and $\Omega_1, \ldots, \Omega_m \in \Sigma$ with $\mu(\Omega_j) < \infty$ for $j = 1, \ldots, m$ such that $f = \sum_{j=1}^m e_j \mathbb{1}_{\Omega_j}$.

(ii) *$f : \Omega \to E$ is called (strongly) measurable, if there is a sequence f_n of simple functions, with $\lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_E = 0$ for almost all $\omega \in \Omega$.*

(iii) *$f : \Omega \to E$ is called weakly measurable, if for each $e' \in E'$ the scalar-valued functions $\omega \mapsto e'[f(\omega)]$ is measurable.*

If we say a function is measurable we always mean strongly measurable. A fundamental characterization of measurable functions is given by Pettis Theorem.

Theorem A.3. *A function $f : \Omega \to E$ is measurable if and only if*

(i) *f is almost separable valued, i.e. there is a set $\tilde{\Omega} \in \Sigma$ with $\mu(\tilde{\Omega}) = 0$ such that $f(\Omega \setminus \tilde{\Omega})$ is a separable subset of E.*

(ii) *f is weakly measurable.*

An immediate consequence for separable Banach spaces E is that strong and weak measurability coincide.

The Bochner Integral

The Bochner Integral is an abstraction of the Lebesgue integral. Some authors call it 'Dunford Schwartz integral'. The integral of any simple function is defined in the obvious way, i.e.

$$\int_\Omega f(\omega) d\mu = \sum_{j=1}^{m} \mu(A_j) e_j.$$

Definition A.4. *A measurable function $f : \Omega \to E$ is called Bochner integrable, if there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions such that*

$$\lim_{n\to\infty} \int_\Omega \|f_n(\omega) - f(\omega)\|_E d\mu = 0.$$

In this case the limit $\int_{\tilde{\Omega}} f(\omega) d\mu := \lim_{n\to\infty} \int_{\tilde{\Omega}} f_n(\omega) d\mu$ exists in E for any $\tilde{\Omega} \in \Sigma$ and is independent of the sequence f_n. Moreover

$$\| \int_{\tilde{\Omega}} f(\omega) d\mu \|_E \leq \int_{\tilde{\Omega}} \|f(\omega)\|_E d\mu. \tag{A.1}$$

Bochner gave a concise characterization of integrable functions, known as 'Bochner Theorem'.

Theorem A.5. *A measurable function $f : \Omega \to E$ is Bochner integrable, if and only if $\int_\Omega \|f(\omega)\|_E d\mu < \infty$.*

The set of all Bochner integrable functions on the measure space (Ω, Σ, μ) with values in E is denoted by $\mathcal{L}^1(\Omega, E)$. Mostly all the basic properties known, for the Lebesgue integral transfer to the vector valued setting. But there are also things, where one has to be more careful, like the theorem of Radon-Nikodym for example. Let us collect the most important results.

Proposition A.6. *Let $f_n : \Omega \to E$ be a sequence of Bochner integrable functions. If* $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ *exists for almost all $\omega \in \Omega$ and there is a real valued, Lebesgue integrable function g on Ω such that $\|f_n(\omega)\|_E \leq g(\omega)$ for almost all $\omega \in \Omega$, then f is Bochner integrable and*

$$\lim_{n\to\infty} \int_{\tilde{\Omega}} f_n(\omega) d\mu = \int_{\tilde{\Omega}} f(\mu) d\mu$$

for all $\tilde{\Omega} \in \Sigma$.

Proposition A.7. *Let $f : \Omega \to E$ be measurable and $f_n \in \mathcal{L}^1(\Omega, E)$ be sequence with $\varliminf_{n\to\infty} \int_\Omega \|f_n(\omega)\| d\omega < \infty$ and $f_n(\omega) \overset{n\to\infty}{\longrightarrow} f(\omega)$ for almost all $\omega \in \Omega$. Then $f \in \mathcal{L}^1(\Omega, E)$ and*

$$\int_\Omega \|f(\omega)\| d\mu = \int_\Omega \varliminf_{n\to\infty} \|f_n(\omega)\| d\mu \leq \varliminf_{n\to\infty} \int_\Omega \|f_n(\omega)\| d\mu.$$

Theorem A.8. *Let $(A, D(A)) : E_0 \to E_1$ be a linear and closed operator. Assume that $f \in \mathcal{L}^1(\Omega, E_0)$ is such that $f(\omega) \in D(A)$ for almost all $\omega \in \Omega$ and $\omega \mapsto Af(\omega) \in \mathcal{L}^1(\Omega, E_1)$, then $\int_{\tilde{\Omega}} f(\omega) d\mu \in \mathcal{D}(A)$ for all $\tilde{\Omega} \in \Sigma$ and*

$$A\Big(\int_{\tilde{\Omega}} f(\omega) d\mu\Big) = \int_{\tilde{\Omega}} Af(\omega) d\mu.$$

A useful consequence of the theorem above is the possibility to to interchange integration and differentiation.

Theorem A.9. *Let $U \subset \mathbb{R}$ be open. If $f : \Omega \times U \to E$ is such that for all $u \in U$ the function $\omega \mapsto f(\omega, u)$ is an element of $\mathcal{L}^1(\Omega, E)$ and for almost all $\omega \in \Omega$ the function $u \mapsto f(\omega, u)$is differentiable with $\omega \mapsto \frac{d}{du} f(\omega) \in \mathcal{L}^1(\Omega, E)$. Then $u \mapsto \int_{\tilde{\Omega}} f(\omega, u) d\mu(\omega)$ is differentiable for all $\tilde{\Omega} \in \Sigma$ and*

$$\frac{d}{du} \int_{\tilde{\Omega}} f(\omega, u) d\mu(\omega) = \int_{\tilde{\Omega}} \frac{d}{du} f(\omega, u) d\mu(\omega).$$

There is also a version of the fundamental Theorem of calculus. Recall how we identified the derivative of an E valued function with an E-valued function at the end of Section A.1.

Theorem A.10. *Let $f : [a, b] \to E$ be of class C^1. Then for any $t \in [a, b]$*

$$f(t) - f(a) = \int_a^t f'(s) ds$$

Here integration is with respect to the Lebesgue measure.

For two measure spaces $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ we consider the product measure space $(\Omega_1 \times \Omega_2, \Sigma(\Omega_1 \times \Omega_2), \mu_1 \otimes \mu_2)$. Here $\Sigma(\Omega_1 \times \Omega_2)$ denotes the sigma algebra on $\Omega_1 \times \Omega_2$ that is generated by sets of the form $U_1 \times U_2$, where $U_i \in \Sigma(\Omega_i)$. $\mu_1 \otimes \mu_2$ denoted the product measure, i.e. the unique[1] measure on $\Sigma(\Omega_1 \times \Omega_2)$ with $\mu_1 \otimes \mu_2(U_1 \times U_2) = \mu_1(U_1)\mu_2(U_2)$ for all $U_i \in \Sigma(\Omega_i)$.

[1] uniqueness is a consequence of the assumption of σ-finite measure spaces (see [DS58, III.11].

Theorem A.11. *Assume two given σ-finite, positive measure spaces $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$. Let $f : \Omega_1 \times \Omega_2 \to E$ be $\mu_1 \otimes \mu_2$-measurable. Assume that for almost all $\omega_1 \in \Omega_1$ the map $\omega_2 \mapsto f(\omega_1, \omega_2)$ is in $\mathcal{L}^1(\Omega_2, E)$ and*

$$\omega_1 \mapsto \int_{\Omega_2} \|f(\omega_1, \omega_2)\|_E d\mu_2$$

is an element of $\mathcal{L}^1(\Omega_1, \mathbb{R})$. Then $f \in \mathcal{L}^1(\Omega_1 \times \Omega_2, E)$ and

$$\int_{\Omega_1} \int_{\Omega_2} f(\omega_1, \omega_2) d\mu_2 d\mu_1 = \int_{\Omega_2} \int_{\Omega_1} f(\omega_1, \omega_2) d\mu_1 d\mu_2 = \int_{\Omega_1 \times \Omega_2} f(\omega) d(\mu_1 \otimes \mu_2).$$

Bochner-Lebesgue Spaces

The definition of the vector-valued L^p spaces works in the same fashion as in the scalar case. Denote by N the set

$$N := \{\tilde{\Omega} \in \Sigma : \mu(\tilde{\Omega}) = 0\}.$$

Definition A.12. *Let E ba a Banach space and (Ω, Σ, μ) a σ-infinite, positive measure space.*

(i) For $p \in [1, \infty)$ denote by $\mathcal{L}^p(\Omega, E)$ the set of functions $f : \Omega \to E$ such that $\mathfrak{R}_{\Omega \setminus \tilde{\Omega}}(f)$ is measurable for some $\tilde{\Omega} \in N$ and $\omega \mapsto \|\mathfrak{R}_{\Omega \setminus \tilde{\Omega}}(f)(\omega)\|_E^p$ is an element of $\mathcal{L}^1(\Omega \setminus \tilde{\Omega}, \mathbb{R})$. We set

$$\|f\|_{\mathcal{L}^p(\Omega,E)} := \Big(\int_{\Omega \setminus \tilde{\Omega}} \|f(\omega)\|_E^p d\mu \Big)^{1/p}.$$

(ii) $\mathcal{L}^\infty(\Omega, E)$ denotes the set of all functions $f : \Omega \to E$ such that $\mathfrak{R}_{\Omega \setminus \tilde{\Omega}}(f)$ is measurable for some $\tilde{\Omega} \in N$ and

$$\|f\|_{\mathcal{L}^\infty(\Omega,E)} := \inf \big\{ c \in [0, \infty] : \mu(\{\omega : \|f(\omega)\|_E > c\}) = 0 \big\} < \infty.$$

All frequently used norm inequalities that are known for the Lebesgue integral transfer to the vector-valued setting without essential changes. We obtain by an application of triangle inequality and (A.1)

Proposition A.13. *Let $f_1, f_2 \in \mathcal{L}^p(\Omega, E)$ and $g \in \mathcal{L}^{p'}(\Omega, E')$ and $h \in \mathcal{L}^{p'}(\Omega, \mathbb{C})$, where $p, p' \in [1, \infty]$. Then we have*

(i) ***(Minkowski's Inequality)***

$$\|f_1 + f_2\|_{\mathcal{L}^p(\Omega,E)} \le \|f_1\|_{\mathcal{L}^p(\Omega,E)} + \|f_2\|_{\mathcal{L}^p(\Omega,E)}.$$

(ii) ***(Hölder's Inequality)*** *If $\frac{1}{p} + \frac{1}{p'} = 1$ (we use the usual convention $\frac{1}{\infty} = 0$). Then*

$$\|hf_1\|_{\mathcal{L}^1(\Omega,E)} \le \|h\|_{\mathcal{L}^{p'}(\Omega,\mathbb{C})} \|f_1\|_{\mathcal{L}^p(\Omega,E)}$$
$$\|gf_1\|_{\mathcal{L}^1(\Omega,\mathbb{C})} \le \|g\|_{\mathcal{L}^{p'}(\Omega,E')} \|f_1\|_{\mathcal{L}^p(\Omega,E)}.$$

(iii) ***(Young's Convolution Inequality)*** *Let $(\Omega, \Sigma, \mu) = (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \lambda)$ where λ is the Lebesgue measure and $\mathcal{B}(\mathbb{R}^d)$ the Borel sigma algebra over $\mathbb{R}^d$. Assume $r \in [1, \infty]$ with $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{p'}$. Then the integrals*

$$h * f_1(x) := \int_{\mathbb{R}^d} h(x-y) f(y) d\lambda(y),$$
$$g * f_1(x) := \int_{\mathbb{R}^d} g(x-y) f(y) d\lambda(y)$$

exist for almost all $x \in \mathbb{R}^d$. Moreover $h * f_1 \in \mathcal{L}^r(\mathbb{R}^d, E)$ and $g * f_1 \in \mathcal{L}^r(\mathbb{R}^d, \mathbb{C})$ with

$$\|h * f_1\|_{\mathcal{L}^r(\mathbb{R}^d,E)} \leq \|h\|_{\mathcal{L}^{p'}(\Omega,\mathbb{C})} \|f_1\|_{\mathcal{L}^p(\Omega,E)},$$
$$\|g * f_1\|_{\mathcal{L}^r(\mathbb{R}^d,\mathbb{C})} \leq \|g\|_{\mathcal{L}^{p'}(\Omega,E')} \|f_1\|_{\mathcal{L}^p(\Omega,E)}.$$

For a general σ-finite measure space (Ω, Σ, μ) we consider the factor space

$$L^p(\Omega, E) := \mathcal{L}^p(\Omega, E) / \{f : \Omega \to E : f(\omega) = 0 \text{ for almost all } \omega \in \Omega\}.$$

$L^p(\Omega, E)$ equipped with the norm

$$\|[f]\|_{\mathcal{L}^p(\Omega,E)} := \|f\|_{\mathcal{L}^p(\Omega,E)}$$

is a Banach space. For convenience we write again $f \in L^p(\Omega, E)$ instead of $[f]$.

A special case occurs if we take $\Omega = \mathbb{Z}^d$, with sigma algebra $\Sigma(\mathbb{Z}^d)$ generated by the singleton sets $\{z\}$ and the counting measure μ. In this case the only set in $\Sigma(\mathbb{Z}^d)$ of measure zero is the empty set. Thus

$$\mathcal{L}^p(\mathbb{Z}^d, E) = L^p(\mathbb{Z}^d, E) =: l^p(\mathbb{Z}^d, E).$$

Lemma A.14. *Let $p \in [1, \infty]$. Then $f \in l^p(\mathbb{Z}^d, E)$ if and only if*

$$\|f\|_{l^p(\mathbb{Z}^d,E)} := \begin{cases} \left(\sum_{z \in \mathbb{Z}^d} \|f(z)\|_E^p \right)^{\frac{1}{p}} & : \ p \in [1, \infty) \\ \sup_{z \in \mathbb{Z}^d} \|f(z)\|_E & : \ p = \infty \end{cases}$$

is finite. Simple functions on $\mathbb{Z}^d$ are sequences with finite support. i.e. these sequences are dense in $l^p(\mathbb{Z}^d, E)$ for $p \in [1, \infty)$.

Dense subsets of $L^p(\Omega, E)$

An immediate consequence of the definition and Proposition A.6 is, that simple functions are dense in $L^p(\Omega, E)$ if $p \in [1, \infty)$. From this we obtain, that if E is separable so is $L^p(\Omega, E)$ for $p \in [1, \infty)$. Step functions on $\mathbb{Z}^d$ with values in E are sequences with finite support. Hence $s(\mathbb{Z}^d, E)$ is dense in $l^p(\mathbb{Z}^d, E)$.

Now let us consider the spaces $L^p(B^d, E)$[2]. For $N \in \mathbb{N}_0$ recall the one-dimensional Dirichlet- and Féjer kernels given by

$$D_N(\theta) := \sum_{|z| \leq N} e^{2\pi i z \theta} \qquad\qquad F_N(\theta) := \frac{1}{N+1} \sum_{j=0}^{N} D_N(\theta).$$

The multi-dimensional analogue is given by

$$F_{N,d}(\theta_1, \dots, \theta_d) := \prod_{j=1}^{d} F_N(\theta_j).$$

Functions of the form $\theta \mapsto \sum_{|z| \leq m} a(z) e^{2\pi i \theta z}$ where $a(z) \in E$ are called trigonometric polynomial. The following result is well known in the case of scalar valued functions and the proof copies verbatim to the vector valued setting.

Proposition A.15. *Let $1 \leq p < \infty$ and $f \in L^p(B^d, E)$. Then $F_{d,N} * f$ is a trigonometric polynomial for each $N \in \mathbb{N}_0$ and*

$$\|F_{N,d} * f - f\|_{L^p(B^d,E)} \to 0 \ \text{ as } N \to \infty.$$

In particular trigonometric polynomials are dense in $L^p(B^d, E)$.

Note that trigonometric polynomials are of class C^∞ and periodic. Hence $C^\infty_{\text{per}}(B^d, E)$ is dense in $L^p(B^d, E)$ for all $p \in [1, \infty)$.

Duality

Let us close this subsection by considering duality. It is well known, in the case of scalar valued functions, that $[L^p(\Omega, \mathbb{C})]' \cong L^{p'}(\Omega, \mathbb{C})$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 < p < \infty$. For vector-valued functions this is not longer true for all Banach spaces E. Nevertheless if is known, that

$$[L^p(\Omega, E)]' \cong L^{p'}(\Omega, E') \tag{A.2}$$

if either (Ω, Σ, μ) is decomposable and E' is separable [Din67, §13,5] or (Ω, Σ, μ) is σ-finite and E' has the so called Randon-Nikodym[3] property with respect to μ [DU77, §4]. However if E is reflexive and the measure space is σ-finite it is shown in [Edw65, 8.20.4], that (A.2) holds true. In particular we have for $1 < p < \infty$ and a reflexive Banach space E

$$\begin{aligned} [L^p(\mathbb{R}^d, E)]' &\cong L^{p'}(\mathbb{R}^d, E') \\ [L^p(\mathbb{I}^d, E)]' &\cong L^{p'}(\mathbb{I}^d, E') \\ [l^p(\mathbb{Z}^d, E)]' &\cong l^{p'}(\mathbb{Z}^d, E') \end{aligned} \tag{A.3}$$

and all the spaces in (A.3) are themselves again reflexive.

[2] recall that $B := [-1/2, 1/2]$

[3] For more details about such spaces we refer again to [DU77].

Vector-valued Sobolev spaces

Analogously to Section 2.1 we can define for an open set $\Omega \subset \mathbb{R}^d$ the space of E-valued test functions by

$$\mathcal{D}(\Omega, E) := \{\varphi \in C^\infty(\Omega, E) \ : \ \mathrm{supp}\varphi \subset \Omega \text{ is compact}\}.$$

Then we set $\mathcal{D}'(\Omega, E) := \mathcal{B}(\mathcal{D}(\Omega, \mathbb{C}), E)$ and call element of $\mathcal{D}'(\Omega, E)$ E-valued distributions. It is clear that the set of E-valued distributions is a linear space and endowed with the topology of uniform convergence on bounded subsets of $\mathcal{D}(\Omega, \mathbb{C})$ this space is a locally convex space (see [Ama95] for more details).

The distributional derivative of an element $T \in \mathcal{D}'(\Omega, E)$ is again by

$$[\partial^\alpha T](\varphi) := (-1)^{|\alpha|} T(\partial^\alpha \varphi) \ \text{ ,for all } \varphi \in \mathcal{D}(\Omega, \mathbb{C}).$$

Hölders inequality shows that for $u \in L^p(\Omega, E)$ the element T_u defined by

$$T_u(\varphi) := \int_\Omega u(x)\varphi(x)dx \ \text{ for } \varphi \in \mathcal{D}(\Omega, \mathbb{C}),$$

belongs to $\mathcal{D}'(\Omega, E)$. For $k \in \mathbb{N}$ and $1 \le p \le \infty$ we define the vector valued Sobolev spaces $\mathcal{W}^{k,p}(\Omega, E)$ to be the subspace of $L^p(\Omega, E)$ of functions u for which all distributional derivatives $\partial^\alpha T_u$ up to order k belong to $L^p(\Omega, E)$. More precisely u is an element of $\mathcal{W}^{k,p}(\Omega, E)$ if and only if for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \le k$ there is a $g_\alpha \in L^p(\Omega, E)$ such that $\partial^\alpha T_u = T_{g_\alpha}$. In this situation we shortly write $\partial^\alpha u = g_\alpha$. On $\mathcal{W}^{k,p}(\Omega, E)$ we introduce the norm

$$\|u\|_{W^{k,p}(\Omega,E)} := \begin{cases} \Big(\sum\limits_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \|\partial u\|^p_{L^p(\Omega,E)} \Big)^{1/p} & : \ 1 \le p < \infty \\ \max\limits_{\substack{\alpha \in \mathbb{N}_0^d \\ |\alpha| \le k}} \|\partial^\alpha u\|_{L^\infty(\Omega,E)} & : \ p = \infty \end{cases}$$

Proposition A.16. *$W^{k,p}(\Omega, E) := (\mathcal{W}^{k,p}(\Omega, E), \|\cdot\|_{W^{k,p}(\Omega,E)})$ is a Banach space.*

For the special case $\Omega = (0,1)^d$ we further introduce the periodic Sobolev spaces via traces. Theorem A.10 remains true for functions in $W^{1,p}(\mathbb{I}^d, E)$. Hence we have for $k \ge 1$

$$W^{k,p}((0,1)^d, E) \hookrightarrow L^p((0,1)^{d-1}, W^{k,p}((0,1), E)) \hookrightarrow L^p((0,1)^{d-1}, C^{k-1}([0,1], E))$$

Now we define $W^{0,p}_{\mathrm{per}}(\mathbb{I}^d, E) := L^p(\mathbb{I}^d, E)$ and if $k \ge 1$

$$W^{k,p}_{\mathrm{per}}(\mathbb{I}^d, E) := \{u \in W^{k,p}((0,1)^d, E) \ : \ \partial_j^m u|_{x_j=0} = \partial_j^m u|_{x_j=1} \\ \text{for } j = 1, \dots, d \text{ and } 0 \le m < k\}.$$

Bibliography

[AB02] Wolfgang Arendt and Shangquan Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.*, 240(2):311–343, 2002. [Cited on page 80]

[AB10] Wolfgang Arendt and Shangquan Bu. Fourier series in Banach spaces and maximal regularity. In *Vector measures, integration and related topics*, volume 201 of *Oper. Theory Adv. Appl.*, pages 21–39. Birkhäuser Verlag, Basel, 2010. [Cited on page 80]

[AF03] Robert A. Adams and John J. F. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, second edition, 2003. [Cited on page 134]

[Ama95] Herbert Amann. *Linear and quasilinear parabolic problems. Vol. 1: Abstract linear theory.* Monographs in Mathematics. 89. Basel: Birkhäuser. xxxv, 335 p. , 1995. [Cited on pages 39 and 158]

[Ama03] Herbert Amann. *(Unpublished manuscript) Vector-Valued Distributions and Fourier Multipliers.* http://user.math.uzh.ch/amann/files/distributions.pdf, 2003. [Cited on pages 11, 12 and 16]

[Arg13] Alexander Argyros. Microstructures in polymer fibres for optical fibres, thz waveguides, and fibre-based metamaterials. *ISRN Optics*, 2013. [Cited on page 3]

[Aut83] Jürgen Auterhoff. Interpolationseigenschaften des Spektrums linearer Operatoren auf L^p-Räumen. *Math. Z.*, 184(3):397–406, 1983. [Cited on pages 126 and 127]

[BC11] Heinz H. Bauschke and Patrick L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011. With a foreword by Hédy Attouch. [Cited on page 16]

[BG94] Earl Berkson and T.A. Gillespie. Spectral decompositions and harmonic analysis on UMD spaces. *Stud. Math.*, 112(1):13–49, 1994. [Cited on pages 88, 91 and 92]

[BGM86] Earl Berkson, T. A. Gillespie, and Paul S. Muhly. Abstract spectral decompositions guaranteed by the Hilbert transform. *Proc. London Math. Soc. (3)*, 53(3):489–517, 1986. [Cited on page 84]

[BK04] Shangquan Bu and Jin-Myong Kim. Operator-valued Fourier multiplier theorems on L_p-spaces on $\mathbb{T}^d$. *Arch. Math. (Basel)*, 82(5):404–414, 2004. [Cited on page 80]

[BL76] Jöran Bergh and Jörgen Löfström. *Interpolation spaces. An introduction*. Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223. [Cited on page 126]

[Blu01] Sönke Blunck. Maximal regularity of discrete and continuous time evolution equations. *Studia Math.*, 146(2):157–176, 2001. [Cited on pages 80, 84, 85 and 91]

[BM48] S. Bochner and W.T. Martin. *Several Complex Variables*. Princeon Mathematical Series. Princeton Univ. Press, 1948. [Cited on page 111]

[Boc33] S. Bochner. Integration von funktionen, deren werte die elemente eines vektorraumes sind. *Fundamenta Mathematicae*, 20(1):262–176, 1933. [Cited on page 152]

[Bou83] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. Mat.*, 21:163–168, 1983. [Cited on page 39]

[Bou86] Jean Bourgain. Vector-valued singular integrals and the H^1-BMO duality. In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 1–19. Dekker, New York, 1986. [Cited on pages 38, 80, 88 and 92]

[Bre11] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. [Cited on pages 25 and 73]

[Bur83] D. L. Burkholder. A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 270–286. Wadsworth, Belmont, CA, 1983. [Cited on page 39]

[Bur01] Donald L. Burkholder. Martingales and singular integrals in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 233–269. North-Holland, Amsterdam, 2001. [Cited on page 39]

[Con85] John B. Conway. *A course in functional analysis*, volume 96 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1985. [Cited on pages 11, 12, 15 and 51]

[CP01] Philippe Clément and Jan Prüss. An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 67–87. Dekker, New York, 2001. [Cited on page 84]

[DHP03] Robert Denk, Matthias Hieber, and Jan Prüss. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003. [Cited on pages 30, 35, 134 and 135]

[Din67] N. Dinculeanu. *Vector measures*. International Series of Monographs in Pure and Applied Mathematics, Vol. 95. Pergamon Press, Oxford, 1967. [Cited on page 157]

[Dix81] Jacques Dixmier. *von Neumann algebras*, volume 27 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1981. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett. [Cited on page 8]

[dL65] Karel de Leeuw. On L_p multipliers. *Ann. of Math. (2)*, 81:364–379, 1965. [Cited on page 81]

[DLP+11] W. Dörfler, A. Lechleiter, M. Plum, G. Schneider, and C. Wieners. *Photonic Crystals: Mathematical Analysis and Numerical Approximation*. Oberwolfach Seminars, 42. Birkhauser Verlag GmbH, 2011. [Cited on page 10]

[DN11] Robert Denk and Tobias Nau. Discrete fourier multipliers and cylindrical boundary value problems. *Konstanzer Schriften in Mathematik*, 279, 2011. [Cited on page 146]

[DS58] Nelson Dunford and Jacob T. Schwartz. *Linear Operators. I. General Theory*. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York, 1958. [Cited on pages 152 and 154]

[DU77] J. Diestel and J.J. Uhl. *Vector Measures*. Mathematical surveys and monographs. American Mathematical Society, 1977. [Cited on pages 152 and 157]

[DV05] Giovanni Dore and Alberto Venni. H^∞ functional calculus for sectorial and bisectorial operators. *Studia Math.*, 166(3):221–241, 2005. [Cited on page 123]

[Edw65] R. E. Edwards. *Functional analysis. Theory and applications*. Holt, Rinehart and Winston, New York, 1965. [Cited on page 157]

[EG77] R.E. Edwards and G.I. Gaudry. *Littlewood-Paley and multiplier theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 1977. [Cited on page 102]

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. [Cited on pages 28, 29 and 129]

[Fef70] Charles Fefferman. Inequalities for strongly singular convolution operators. *Acta Math.*, 124:9–36, 1970. [Cited on page 80]

[Fef71] Charles Fefferman. The multiplier problem for the ball. *Ann. of Math. (2)*, 94:330–336, 1971. [Cited on page 80]

[Gra08] Loukas Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008. [Cited on pages 11, 13, 16, 18, 38, 46 and 47]

[Gre98] Walter Greiner. *Classical*. Classical Theoretical Physics Series. Springer Verlag, 1998. [Cited on page 4]

[Gün08] Kerstin Günther. *Unbounded Linear Operators on Interpolation Spaces*. Dissertation, Technische Universität Berlin, 2008. [Cited on page 129]

[Haa06] Markus Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006. [Cited on pages 30 and 34]

[Har06] Fritz Hartogs. Zur Theorie der analytischen Funktionen mehrerer unabhängiger Veränderlichen, insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten. *Math. Ann.*, 62(1):1–88, 1906. [Cited on pages 109 and 131]

[HLP34] G.H. Hardy, J.E. Littlewood, and G. Pólya. *Inequalities*. Cambridge Mathematical Library. Cambridge University Press, 1934. [Cited on page 84]

[HP57] Einar Hille and Ralph S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed. [Cited on page 131]

[HPW09] Vu Hoang, Michael Plum, and Christian Wieners. A computer-assisted proof for photonic band gaps. *Z. Angew. Math. Phys.*, 60(6):1035–1052, 2009. [Cited on page 9]

[Jac75] J.D. Jackson. *Classical electrodynamics*. Wiley, 1975. [Cited on page 4]

[Jod70] Max Jodeit, Jr. Restrictions and extensions of Fourier multipliers. *Studia Math.*, 34:215–226, 1970. [Cited on page 81]

[Joh87] Sajeev John. Strong localization of photons in certain disordered dielectric superlattices. *Phys. Rev. Lett.*, 58:2486–2489, Jun 1987. [Cited on page 3]

[Kah85] Jean-Pierre Kahane. *Some random series of functions*, volume 5 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1985. [Cited on page 36]

[Kat66] Tosio Kato. *Perturbation theory for linear operators*. Die Grundlehren der mathematischen Wissenschaften, Band 132. Springer-Verlag New York, Inc., New York, 1966. [Cited on pages 9, 109, 112 and 113]

[Khr12] Andrii Khrabustovskyi. Periodic Riemannian manifold with preassigned gaps in spectrum of Laplace-Beltrami operator. *J. Differential Equations*, 252(3):2339–2369, 2012. [Cited on page 9]

[KK83] Ludger Kaup and Burchard Kaup. *Holomorphic functions of several variables*, volume 3 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1983. An introduction to the fundamental theory, With the assistance of Gottfried Barthel, Translated from the German by Michael Bridgland. [Cited on pages 109 and 111]

[Kuc93] Peter Kuchment. *Floquet theory for partial differential equations*, volume 60 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 1993. [Cited on pages 6, 10 and 11]

[KW04] Peer C. Kunstmann and Lutz Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004. [Cited on pages 30, 32, 33, 34, 35, 37, 38, 87, 95, 138, 139, 143 and 144]

[Lan93] Serge Lang. *Real and functional analysis*, volume 142 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 1993. [Cited on pages 11, 30, 111 and 152]

[Mar39] J. Marcinkiewicz. Sur les multiplicateurs des sÃľries de fourier. *Studia Mathematica*, 8(1):78–91, 1939. [Cited on page 79]

[Mau75] B. Maurey. Système de Haar (suite et fin). Sémin. Maurey-Schwartz 1974-1975, Espaces L^p, Appl. radonif., Géom. Espaces de Banach, Exposé No.2, 13 p. (1975)., 1975. [Cited on page 39]

[MCK+96] Attila Mekis, J. C. Chen, I. Kurland, Shanhui Fan, Pierre R. Villeneuve, and J. D. Joannopoulos. High transmission through sharp bends in photonic crystal waveguides. *Phys. Rev. Lett.*, 77:3787–3790, Oct 1996. [Cited on page 2]

[Mih56] S. G. Mihlin. On the multipliers of Fourier integrals. *Dokl. Akad. Nauk SSSR (N.S.)*, 109:701–703, 1956. [Cited on page 80]

[Mik02] Kalle Mikael Mikkola. *Infinite-dimensional linear systems, optimal control and algebraic Riccati equations*. ProQuest LLC, Ann Arbor, MI, 2002. Thesis (D.Sc.(Tech.))–Teknillinen Korkeakoulu (Helsinki) (Finland). [Cited on pages 50 and 51]

[Nau12] Tobias Nau. *L^p-theory of cylindrical boundary value problems*. Springer Spektrum, Wiesbaden, 2012. An operator-valued Fourier multiplier and functional calculus approach, Dissertation, University of Konstanz, Konstanz, 2012. [Cited on pages 133, 136, 138, 139, 140, 141, 146 and 147]

[NS11] Tobias Nau and Jürgen Saal. $\mathcal{R}$-sectoriality of cylindrical boundary value problems. In *Parabolic problems*, volume 80 of *Progr. Nonlinear Differential Equations Appl.*, pages 479–505. Birkhäuser/Springer Basel AG, Basel, 2011. [Cited on page 140]

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983. [Cited on pages 24, 25 and 28]

[Phi55] R. S. Phillips. The adjoint semi-group. *Pacific J. Math.*, 5:269–283, 1955. [Cited on pages 64 and 66]

[RS78] Michael Reed and Barry Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978. [Cited on pages 6, 8 and 9]

[RS80] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. Functional analysis. [Cited on page 23]

[Sca99] Bruno Scarpellini. *Stability, instability, and direct integrals*, volume 402 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 1999. [Cited on pages 10 and 109]

[Sch13] Philipp Schmalkoke. *On the spectral Properties of Dispersive Photonic Crystals*. KIT, Diss., 2013. [Cited on page 4]

[SW71] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton University Press, Princeton, N.J., 1971. Princeton Mathematical Series, No. 32. [Cited on page 52]

[THB+02] Burak Temelkuran, Shandon D. Hart, Gilles Benoit, John D. Joannopoulos, and Yoel Fink. Wavelength-scalable hollow optical fibres with large photonic bandgaps for co2 laser transmission. *Nature*, 420:650–653, 2002. [Cited on page 3]

[Tho03] Sonja Thomaschewski. *Form methods for autonomous and nonautonomous cauchy problems*. Ulm, Univ., Diss., 2003. [Cited on pages 26, 29 and 30]

[vELA+01] Martijn A. van Eijkelenborg, Maryanne C. J. Large, Alexander Argyros, Joseph Zagari, Steven Manos, Nader A. Issa, Ian Bassett, Simon Fleming, Ross C. McPhedran, C. Martijn de Sterke, and Nicolae A.P. Nicorovici. Microstructured polymer optical fibre. *Optics Express*, 9(7):319–327, 2001. [Cited on page 3]

[vG06] Onno van Gaans. On R-boundedness of unions of sets of operators. In *Partial differential equations and functional analysis*, volume 168 of *Oper. Theory Adv. Appl.*, pages 97–111. Birkhäuser, Basel, 2006. [Cited on page 37]

[Wei01] Lutz Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.*, 319(4):735–758, 2001. [Cited on pages 35, 80 and 85]

[WS01] Gilbert G. Walter and Xiaoping Shen. *Wavelets and other orthogonal systems*. Studies in Advanced Mathematics. Chapman & Hall/CRC, Boca Raton, FL, second edition, 2001. [Cited on page 35]

[Yab87] Eli Yablonovitch. Inhibited spontaneous emission in solid-state physics and electronics. *Physical review letters*, 1987. [Cited on page 3]

[Yos94] Kosaku Yosida. *Functional analysis. Repr. of the 6th ed.* Berlin: Springer-Verlag. xiv , 1994. [Cited on page 16]

[Zim89] Frank Zimmermann. On vector-valued Fourier multiplier theorems. *Studia Math.*, 93(3):201–222, 1989. [Cited on page 85]