Epistemic Reasoning in OWL 2 DL

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Abstract

The standard semantics employed for description logics (DLs) adheres to the open world assumption (OWA). On one hand, such an assumption facilitates a modeler to formally represent problem domains with out explicitly expressing full information, which sometimes is even not possible. This is probably one of the main reasons of founding the Web Ontology Language (OWL) based on DLs. On the other hand, OWA proves to be the main cause of the impotence of DLs of capturing non-monotonic reasoning: *where new information invalidates the previously concluded information*. Non-monotonic reasoning has several applications in real life scenarios. For example, logic programming community, parallel to the OWL community, in general focuses on non-monotonic reasoning. These short-comes of DLs lead to the quest for a formalism as suitable as DLs meanwhile capable of capturing some "kind" of non-monotonic reasoning.

Several work has been done on extending DLs with some non-monotonic feature. Epistemic extensions of DLs (called epistemic DLs sometimes) is probably one of the earliest work in this direction. Such extensions enhance expressivity and querying capabilities of these DLs by knowledge base introspection. The existing approaches in this respect are limited to less expressive DLs like $\mathcal{ALC}$: *the least expressive DL which is boolean complete* whereas in real life applications, modeling problems require expressivity beyond $\mathcal{ALC}$.

The aim of this work is to extend epistemically the most expressive DL $\mathcal{SROIQ}$ which is the foundation of OWL 2 DL. We argue that unintended effects occur when imposing the semantics traditionally employed on very expressive DLs like $\mathcal{SROIQ}$. Consequently, we identify the most expressive DL for which the current approach can still be adapted. For the epistemic extension of $\mathcal{SROIQ}$ and alike expressive DLs, we suggest a revised semantics that behaves more intuitively in these cases and coincides with the traditional semantics on less expressive DLs.

ABSTRACT

Different languages can be used for formalizing knowledge bases and queries. The use of an epistemically extended formalisms as query language has been highly advocated in literature. Motivated by several use cases of using an epistemic DL as a query language we introduce a way of answering epistemic queries to DL knowledge bases by a reduction to standard DL reasoning. Hence, we can use off-the-shelf highly optimized DL reasoners for answering epistemic queries.

Finally, to evaluate our method of answering epistemic queries, we implement a tool that utilizes the introduced technique of reducing epistemic query answering to the standard DL reasoning tasks. We perform several experiments that show the practical feasibility of the tool.

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Chapter 1
Introduction

An important task in modeling a domain of interest is *conceptualization: to identify different concepts and relationships between the concepts.*. In computer science and other relevant fields, this process is usually called *ontology modeling*. An ontology provides a formal method of describing information about a particular domain. Besides several application areas like the life sciences, automobile industry, software engineering, etc, probably the most prominent one is the Semantic Web. The Web Ontology Language (OWL) [OWL WORKING GROUP 2009] is currently the single most important formalism for realizing semantic technologies. Some other prominent languages include Common Logic [ISO/IEC 24707:2007 2007], F-Logic [KIFER et al. 1995], KIF [GENESERETH and FIKES 1992], OKBC [GROUP 1998] etc. OWL is based on description logics (DLs)–family of knowledge representation formalisms. At the very basics, DLs are fragments of first-order logic and thus employ the open-world assumption (OWA) for their semantics. Hence such formalisms are capable of modeling domains with incomplete information like the Semantic Web. Nevertheless, there are application scenarios where a close-world approach is demanded. This usually requires closing (if not completely) a certain part of the ontology in consideration. Thus a hybrid formalism[1] that allows for viewing certain part of an ontology under the close-word assumption (CWA) would be highly suitable for such scenarios. A Query language with such capability is the main objective of this work.

[1]Here by a hybrid formalism when mean combination of different formalisms e.g., description logic with a non-monotonic formalism.

The following section discusses the usual assumptions taken in the logic based formalisms regarding reasoning. In Section 1.2, we present an overview of description logics. Section 1.4 sets the goals and objectives of this work. The last Section provides some guidelines for the readers.

1.1 The Two Assumptions of the Reasoning World

One of the most basic question regarding the semantics of a given formal logic is if the semantics uses the open-world assumption or the closed world assumption. In the open-world assumption or OWA, the absence of information does not allow concluding negative information. For example, given the only fact that Taylor[2] produces a wine called Taylor's Port. There is no further information that Taylor produces other wines. Under the OWA, it is not allowed to conclude that Taylor's Port is the only wine produced by Taylor. In other words, the available information is under no condition considered to be complete. Such an assumption is very useful when modeling a domain with incomplete information. An other application scenario includes representation of knowledge (on discovery) of a problem domain. In such a scenario we start by representing knowledge at hand without making an assumption about its completeness. We add on new knowledge as we discover while still holding the conclusions entailed previously.

Semantic Web is a prime example where one requires OWA in general. The very nature of the world wide web (WWW) advocates the need for assuming the incompleteness of information. In fact, with the huge amount of dynamic data on WWW, there is no way of being sure (at least not practically) if the information available is complete at all. The OWA is really handy in describing the information on the web i.e., we utilize the available information on the web for reasoning tasks while restricting the conclusions of new information from the available information only.

[2]This information has been taken from the Wine ontology available at `http://www.w3.org/TR/owl-guide/wine.rdf`

In contrast to OWA, in the close-world assumption or CWA, the absence of information implies the negation of the information i.e., if a certain fact is not known, then it is assumed to be *false*[3]. As an example suppose it is unknown if Taylor's Port is a red wine, then we conclude that Taylor's Port is not a red wine. Note that CWA is natural[4] in the sense that we human generally employ this assumption in daily life reasoning and argumentation. A classical example is reading a timetable. On observing that there is no train leaving Karlsruhe at 14:00 to Paris stated in the timetable, we conclude that in actual there is no train for Paris at that time from Karlsruhe. Whereas, it might be the case that there is a mistake in the timetable and indeed a train is leaving at 14:00 to Paris. Databases are another example where CWA is used. Whatever is stated in a database is assumed to be complete.

An important difference between the two assumptions can be observed on the availability of new information. In the OWA, the previous consequences of a given information retains. In contrast, in CWA we may have to retract from the previous concluded information once new facts are available. That is, newly added information might invalidate the previous consequences. In the previous wine example, under CWA we concluded that Taylor's Port is not a red wine. Now suppose we come to know (via some fact or deduction) that it is indeed the case that it is a red wine, we have to retract from what we had concluded earlier. Thus we can no more say that Taylor's Port is a red wine. Hence, reasoning in CWA is defeasible.

1.2 Description Logics

Description logics (DLs) provide the formal basis for the Web Ontology Language [BAADER et al. 2007]. DLs correspond to a family of knowledge representation formalisms that can be used for specifying conceptual knowledge of an application domain in a machine-processable way. There are several related formalisms that fall into this family. Less expressive DLs include the $\mathcal{EL}$ family [BAADER et al. 2005]. As tractable formalisms, these DLs have been

[3]The truth-value of the given fact is considered to be false.

[4]Such reasoning is sometimes also called *commonsense reasoning*

used in modeling huge ontologies like SNOMED CT,GALEN etc. Meanwhile $\mathcal{SROIQ}$ is one of the most expressive DLs. However reasoning in $\mathcal{SROIQ}$ is computationally far harder than in $\mathcal{EL}$. For example checking whether certain information follows from an ontology formulated in $\mathcal{SROIQK}$ requires NExpTime [Papadimitriou 1994]. This trade-off between expressiveness and computational complexity gives rise to different DLs. Depending on the expressiveness of DLs, different OWL profiles [Motik et al. 2009] correspond to different DLs, for example, OWL 2 EL corresponds to the $\mathcal{EL}$ family. Note that reasoning tasks in DLs are decidable. In fact, DLs can be seen as decidable fragments of first-order logic (with equality) [Baader et al. 2007]. Nevertheless, we use different syntax than the traditional first-order one (see Chapter 2).

The basic syntactic notions in a description logics are of *individual names* representing individual elements in a problem domain, *atomic concepts* representing classes of which elements can be member of and *atomic roles* representing binary relations among the elements. From first-order logic perspective, concepts are unary predicates, roles are binary predicates and individual names are constants. Different description logics allow for different sets of constructs for constructing complex concepts and roles from the atomic concepts, atomic roles and individual names. For example the description logic $\mathcal{ALC}$ [Schmidt-Schauss and Smolka 1991] allows for basic boolean constructs: disjunction (union) ($\sqcup$), conjunction (intersection), negation (complement) ($\neg$) and constructs (for quantifying over the elements) namely existential quantification ($\exists$) and universal quantification ($\forall$). The complex concept

$$\mathsf{RedWine} \sqcap \exists \mathsf{madeFromFruit}.\mathsf{Fruit}$$

is an example of $\mathcal{ALC}$ concept representing all red wines which are made of some fruit.

In DLs, *concept inclusion* axioms are used to describe the subclass relationship between two concepts. We use the symbol $\sqsubseteq$ to represent this inclusion. For example, the axioms RedWine $\sqsubseteq$ Wine[5] state that every red wine is a wine i.e., the class Wine is a super-concept of the class RedWine. In other words, RedWine is a sub-concept of the class Wine. Thus every individual in class RedWine must be in class Wine as well. To state a membership relationship between an individual and a class or between two individuals, we use assertions. For example, Wine(taylor-port) states that taylor-port is member of the class Wine. Similarly, the assertion like hasMaker(taylor-port, taylor) with hasMaker a role, is used to state that the relation of hasMaker holds between taylor-port and taylor.

Like in other logical formalisms, we have the notion of theory in DLs as well. Theories in DLs are called *knowledge bases* (also called *ontologies*). A knowledge base is a set of concept inclusion axioms and assertions.

The semantics of DLs is based on the first-order logic semantics and thus adheres to the open-world assumption. We have the notion of interpretations composed of a domain. In an interpretation, a concept is interpreted by a subset (called the extension of the concept) of the domain, a role is interpreted by a binary relation (called the extension of the role) over the domain and an individual name is interpreted by a member (called the interpretation of the individual name) of the domain. Usually we are interested in specific interpretations of a knowledge base which satisfy[6] all the axioms of the knowledge base. Such interpretations are called *models*.

Like first-order logic, we have different notions of reasoning tasks in DLs: *inconsistency checking* asks for the consistency (existence of a model) of a knowledge base. *Concept satisfiability checking* is to test if a given concept is interpreted by a non-empty subset of the domain within an interpretation. *Instance checking* for a given individual name and concept verifies if the interpretation of the individual is a member of the interpretation of the concept. Another reasoning task of high interest is the *query checking* to a knowledge. The typical kind of query to a

[5] It is a usual practice in DL community to represent concepts by words starting with an uppercase letter, whereas roles and individuals are represented by words starting with a lowercase letter.

[6] satisfaction conditions are formally defined in Chapter 2

knowledge base Σ is to check if an axiom of the form $C(a)$ or $r(a, b)$ is *entailed* by Σ i.e., we check if the interpretation of a belongs to the extension of C in all the models of Σ and similarly if the interpretations of a and b occur as an order-pair in the extension of r in all the models of Σ. We denote these queries as $\Sigma \models C(a)$? and $\Sigma \models r(a, b)$?. Due to the OWA taken by the semantics in DLs, consequently the answer of a query to a knowledge base is YES if the axiom in the query is entailed by the knowledge base and NO if the negation of query is entailed. The answer if UNKNOWN, otherwise. Note that the incompleteness of knowledge does not let us conclude NO as an answer to a query in case we can not conclude YES as an answer.

Deciding the aforementioned reasoning tasks requires inventing algorithms and implementing tools (DL system) based on these algorithms. Over a period of couple of decades, such systems have evolved into highly efficient and practically feasible inference engine. Some of the early systems include KL-ONE [SCHMIDT-SCHAUSS 1989], KRYPTON [BRACHMAN et al. 1985], NIKL [KACZMAREK et al. 1986], KANDOR [PATEL-SCHNEIDER 1984], LOOM [MACGREGOR and BATES 1987] CLASSIC [BORGIDA et al. 1989], KRIS [BAADER and HOLLUNDER 1991], etc. (see Chapter 8 of [BAADER et al. 2007] for a detail discussion). These early system were not suitable for many practical application scenarios: either they were not efficient enough or lacked completeness in the sense that only correct answers could be identified by these system. Nevertheless they paved the way to the modern systems. Most of the modern DL system are some optimized implementation of tableaux algorithms. Some are specifically designed for performing reasoning tasks in less expressive DLs. For example, CEL [BAADER et al. 2006] and ELK [KAZAKOV et al. 2012] are a highly optimized reasoner for the $\mathcal{EL}$ family. For expressive DLs, some reasoners include Pellet [SIRIN et al. 2007], RacerPro [HAARSLEV et al. 2012], FacT++ [TSARKOV and HORROCKS 2006], etc.

1.3 Toward a Richer Query Language

A query to a DL knowledge base is to check if a certain DL axiom is entailed by the knowledge base. Similar to knowledge bases, we have several choices from the DLs family for formulating these axioms . We call the DL for formulating the knowledge base (query) as knowledge base (query) language. An important question here is *"Is it of interest to distinguish between the knowledge base language and the query language?"*. This question can be raised for any logic-based knowledge representation formalism. Most of the time a query answering task can be translated into some other basic reasoning tasks. Regardless of how expressive a query language is, sometimes it does not suffice. Hector J. Levesque was the first to observe this in-adequacy and argued for the need for a richer query language in knowledge representation formalisms [LEVESQUE 1984]. He advocated the idea of extending a querying language by the attribute *knows* denoted by **K** (also called *epistemic operator*, used akin to modalities in modal logics). This operator enhances the query language by *knowledge base introspection* capabilities. Reiter makes a similar argument of in-adequacy of the standard first-order language for querying in the context of databases [REITER 1992].

We now consider some examples in the context of description logics, that demonstrate the in-adequacy of a DL as a query language. Consider a DL knowledge base Σ containing the following axiom:

$$\mathsf{Wine} \sqsubseteq \mathsf{RedWine} \sqcup \mathsf{WhiteWine}, \quad \text{and}$$
$$\mathsf{Wine(taylors\text{-}port)}$$

stating that every wine is either a red wine or a white wine and that Taylor's Port is a wine. Now consider the query $\Sigma \models \mathsf{RedWine(taylors\text{-}port)}?$ asking if Taylor Port is a red wine. Due to the incompleteness of information the answer to the query is UNKNOWN i.e., we do not know if Taylor Port is a red wine or a white wine. The epistemic operator (**K**) enables us asking such introspective question. To perform an introspection of Σ, consider the query $\Sigma \models \mathbf{K}\mathsf{RedWine(taylors\text{-}port)}?$. This query asks whether it is known that Taylor's Port is a red wine. The answer is simply NO as from the given axioms all we can conclude is that Taylor's Port is a red wine or it is a white wine. Our knowledge regarding the type of wine Taylor's Port belongs to, is incomplete. Note that disjunction allows for modeling incomplete information. Now adding the axiom

$$\mathbf{K}\mathsf{Wine} \sqsubseteq \mathbf{K}\mathsf{RedWine} \sqcup \mathbf{K}\mathsf{WhiteWine}$$

to the knowledge base enforces the condition that for any thing which is known to be wine, its type (red or white) needs to be known as well. Such axioms are sometimes called as *integrity constraints* as they require certain properties to be satisfied by individuals of a problem domain.

Another construct to model incompleteness in DLs is via the existential quantification. Suppose Σ contains the axiom

$$\exists\mathsf{locatedIn.Region(taylors\text{-}port)}$$

asserting that Taylor's Port has some region as its origin. Again all we know is that there is some region as the origin of this wine. Thus the answer to query $\Sigma \models \exists\mathsf{locatedIn.Region(taylors\text{-}port)}$? is YES as so is asserted in Σ. On introspecting if it is known what is the origin of Taylor's Port, we get a negative answer i.e., the answer to the query $\Sigma \models \exists\mathbf{K}\mathsf{locatedIn}.\mathbf{K}\mathsf{Region(taylors\text{-}port)}$? is NO.
The enhanced reasoning capabilities due the operator **K** has motivated many researchers in extending DLs with this operator (cf. [DONINI et al. 1998], [DONINI et al. 1992b], [DONINI et al. 1995], [DONINI et al. 1997], [MEHDI et al. 2011], [MEHDI and RUDOLPH 2011b], etc.) Note that this is not a straight forward task as introducing **K** in DLs introduces non-monotonicity in the semantics.

1.4 Aims and Objectives

The previous section identifies the added value of enriching a query language with the epistemic operator **K** for introspecting knowledge bases. The OWA taken for the semantics of DLs allows us modeling incomplete information in a very simple manner. This makes DLs very suitable for representing knowledge on the

Web. Hence, the web ontology language (OWL) is founded with DLs as the basis. Nevertheless, sometimes it is important for decision making to get answers for queries under closed-world assumption. This has lead to investigate introspective reasoning in DLs via the introduction of the epistemic operator $\mathbf{K}$. The importance of enriched DLs, for example, has been demonstrated in ([GRIMM et al. 2006]) via several application scenarios (specially in web services discovery).

Besides significantly additional capabilities, extensions of DLs with $\mathbf{K}$ have not founded their way into the OWL specification. Classical work on such extensions focuses on extending tableaux algorithms for less expressive languages. Some recent work considers even more expressive formalisms capable of capturing the epistemic extension of DLs but focuses mainly on theoretical aspects. Thus, to the best of our knowledge, there is no reasoner capable of utilizing the $\mathbf{K}$-operator for introspection of the knowledge bases. We believe this is one of the main obstacles in adapting query languages extended with the epistemic operator in practice.

The main objective of this work is to investigate the epistemic extension of expressive DLs and to devise an inference engine that match in performance with the current state-of-the-art standard DL reasoners. The main objectives for this work are:

- extending DL $\mathcal{SROIQ}$ with the operator $\mathbf{K}$
- implementing an inference engine for reasoning in such an extension

In achieving our objective, we have identified three sub-goals:

Investigate the Traditional Approach of Extending DLs with $\mathbf{K}$

The first epistemic extension explored was for the description logic $\mathcal{ALC}$. This extension was called $\mathcal{ALCK}$ where $\mathbf{K}$ stands for the operator $\mathbf{K}$ [DONINI et al. 1992b]. The semantics employed for such an extension is based on the possible-world approach: *interpretations in DLs correspond to possibles worlds in $\mathcal{ALCK}$*. The interpretation of the operator $\mathbf{K}$ depends on all the possible worlds: claiming the “knowing” of some information requires the non-existence of a possibility of falsehood of the information. While considering a set of possible worlds for the semantics, the question is how to interpret concepts, roles and individuals across the worlds and what of sets to allow as domains of interpretations? In classical or traditional semantics a fixed countable set is assumed as the set of all interpretations under consideration. Meanwhile, the semantics also enforces the uniformity of the interpretation of individual names. These two assumptions are respectively called as *the common domain assumption* and *the rigid term assumption*. However, later we will see that when extended expressive DLs like $\mathcal{SROIQ}$ [HORROCKS et al. 2006] with the **K** operator, certain constructs restrict the domains of interpretations to finite sets only, thus contradicting the common domain assumption. Further, the rigid term assumption enforces the unique assumption[7] as well. Our first goal in this work is to identify DLs for which the traditional semantics is applicable when extended with the operator **K**. Further, we need to devise techniques for deciding the reasoning tasks in such extensions.

Invent New Semantics for Expressive DLs like $\mathcal{SROIQ}$

Since the traditional semantics is not adaptable for expressive DLs extended with the operator **K**, the next goal is to come up with a new semantics for such extensions. A comparison between the new semantics and the traditional one is to be addressed as well.

Implement Efficient Inference Tool

Probably the most obvious reason of the epistemic extensions of DLs not making their ways into OWL is the unavailability of efficient tools. Though, several highly optimized inference engines are available for deciding reasoning tasks in DLs, to the best of our knowledge, none of these engines are utilizable for epistemic extensions of DLs. Thus as the final goal of this work, we strive for inventing an efficient and practically feasible inference engine for the epistemic extensions of

[7] Under the unique name assumption (UNA) distinct individual names are interpreted by distinct elements of the domain.

expressive DLs.

Each of these sub-goals are addressed in different chapters of this work on which we put light in the next section.

1.5 Guide to the Reader

The thesis in hand is divided into four parts. The first part introduces the basic notions required as preliminaries. We also briefly discuss relevant literature in this part. The goals set in the previous section are addressed in the second and third part. These two parts in fact describe the main contribution of our work. In the final part, we conclude this thesis by first discussing related work and then presenting some final remarks along with a summary of the future work. We now outline each part:

Chapter 1. This chapter introduces the reader to the research objective of this work in an informal manner.

Part I: Foundation

Chapter 2. Formal introduction to description logics is presented in this chapter. It mainly discussed the most expressive DL called $\mathcal{SROIQ}$. Most of the DLs are just fragment of $\mathcal{SROIQ}$. After formally introducing the syntax and semantics of $\mathcal{SROIQ}$, the chapter offers an overview of different reasoning tasks in $\mathcal{SROIQ}$.

Chapter 3. Some important non-monotonic logic formalisms are the topic of this chapter. This lays foundations for understanding the non-monotonic extensions of DLs which are discussed in Chapter 4.

Chapter 4. This chapter presents a bird-eye-view of different non-monotonic extensions of DLs including epistemic extension, thus provide a thorough understanding of different research directions. Meanwhile we also discusses several recent hyrbid[8] formalisms against which we need to compare our approach.

Part II: Epistemic Queries and DLs

The main contribution of this thesis are presented in the chapters in this part.

Chapter 5. Examples and application scenarios are discussed to motivate the need of enriching query languages with the operator K.

Chapter 6. The first sub-goal set in Section 1.4 is the topic of this chapter. We identify the expressive DLs upto which the classical approach of embedding the K operator in the query language can be taken. We then devise an algorithm for answering the epistemic queries under the classical semantics. The results presented in this chapter

Chapter 7. Extending expressive DLs like $\mathcal{SROIQ}$ with K is not possible under the classical approach. This chapter presents a new semantics for such expressive DLs. The semantics is back-ward compatible in the sense that for less expressive DLs it behaves identical to the classical semantics so far the entailment of axioms from a knowledge base is concerned. A reasoning technique for deciding reasoning tasks is presented as well.

Part III: Implementation and Evaluation

This part highlights our contribution from the practical point of view i.e., we discuss implementation and evaluation of a inference engine we developed based on the techniques introduced in the previous chapters.

Chapter 8. Based on the reasoning techniques introduced in Chapter 7, a tool is implemented. The tool is based on black-box approach thus off-the-shelf reasoners can be used for performing reasoning. The practical feasibility of the tool is advocated by some tests and evaluation results.

[8] By a hybrid formalism we mean combination of two or more different formalisms.

Part IV: Conclusion

Chapter 9. In this chapter we compare our approach with some existing one. A thorough comparison with the formalism presented in [MOTIK and ROSATI 2010] is a minor contribution of this work as well.
Chapter 10. Some concluding remarks are presented in this chapter. We identify some future goals.

1.6 List of Publications

Our results presented in this work are published at several conferences. The content of Chapter 6 is based on our work presented in [MEHDI et al. 2011]. In [MEHDI and RUDOLPH 2011b] we presented a new semantics of epistemic extensions of expressive DLs which we discuss in Chapter 7. The inference tool we present in Chapter 8 is based on a method we presented in [MEHDI and RUDOLPH 2011b, MEHDI and RUDOLPH 2011a] and the optimization techniques we invented for our tool were presented in [MEHDI and WISSMANN 2013].

One final remark, a basic familiarity with propositional and first-order logic is assumed for the readers of this thesis (see [FITTING 1996] for an introduction to these logics).

Part I

Foundations

Chapter 2

Description Logic

Description Logics (DLs) are family of logic-based knowledge representation formalisms. As decidable fragments of first-order logic, different DLs differ in expressiveness and thus differ in computational complexity of the usual reasoning tasks defined for DLs like knowledge base consistency, entailment of axioms from a knowledge base etc. While light-weight DLs e.g., $\mathcal{EL}^{++}$ [Baader et al. 2008] are less expressive but common reasoning tasks in such formalisms can be solved in polynomial time. Whereas expressive DLs like $\mathcal{SROIQ}$ [Horrocks et al. 2006] retain expressiveness at the cost of higher computational complexity.

Similar to other logic formalism, DLs have well-defined syntax and semantics along with different reasoning tasks . In this chapter we formally introduce $\mathcal{SROIQ}$. Section 2.1.1 present syntax of $\mathcal{SROIQ}$ along with the notions of knowledge base and several axioms. Semantics of DLs are based on the semantics of first-order logic. Section 2.1.2 introduces semantics for $\mathcal{SROIQ}$ by presenting the notion of interpretation. Finally several reasoning tasks are introduced in Section 2.2.

2.1 Introducing Description Logic $\mathcal{SROIQ}$

2.1.1 Syntax

As for the signature of $\mathcal{SROIQ}$, we assume three finite and disjoint sets N_C, N_R and N_I of *concept names*, *role names* and *individual names* respectively.[1] Additionally, we have two special concepts, namely the top concept ($\top$) and the bottom concept ($\bot$). The set N_R is partitioned into two sets namely, $\mathbf{R_s}$ and $\mathbf{R_n}$ of *simple* and *non-simple* roles respectively. The set **R** *of* $\mathcal{SROIQ}$ *roles* is

$$\mathbf{R} := U \mid N_R \mid N_R^-$$

where U is called the *universal role*. Further, we define a function Inv on roles such that $\mathsf{Inv}(R) = R^-$ if R is a role name, $\mathsf{Inv}(R) = S$ if $R = S^-$ and $\mathsf{Inv}(U) := U$.

$\mathcal{SROIQ}$ Concepts

The set of $\mathcal{SROIQ}$ (complex) concepts description or simply concepts[2] is the smallest set satisfying the following properties:

- every concept name $A \in N_C$ is a concept,
- $\top$(top concept) and $\bot$ (bottom concept) are concepts, and
- if C, D are concepts, R is a role, S is a simple role, $a_1, \ldots, a_n$ are individual names and n a non-negative integer then following are concepts:

[1] Finiteness, in particular for N_I, is required for the further considerations. However note that the signature is not bounded and can be extended whenever this should be necessary.

[2] In the sequel, by roles and concepts we mean $\mathcal{SROIQ}$ roles and $\mathcal{SROIQ}$ concept descriptions unless stated otherwise.

$\neg C$	(negation)
$\exists S.\mathsf{Self}$	(self)
$C \sqcap D$	(conjunction)
$C \sqcup D$	(disjunction)
$\forall R.C$	(universal quantification)
$\exists R.C$	(existential quantification)
$\leq nS.C$	(at least number restriction)
$\geq nS.C$	(at most number restriction)
$\{a_1, \ldots, a_n\}$	(nominals / one-of)

$\mathcal{SROIQ}$ Knowledge Bases

To define a $\mathcal{SROIQ}$ knowledge base, first we need to define its basic components. *RBox:* A $\mathcal{SROIQ}$ RBox contains axioms specifying conditions on the roles. An $\mathcal{SROIQ}$ *RBox axiom* is either a

- *role inclusion axiom: (RIA)* $R_1 \circ \cdots \circ R_n \sqsubseteq R$ where $R_1, \ldots, R_n, R \in \mathbf{R}$ and if $n = 1$ and $R_1 \in \mathbf{R_s}$ then $R \in \mathbf{R_s}$, otherwise $R \in \mathbf{R_n}$, or a
- *role characteristic:* $\mathsf{Dis}(R, R')$ (role disjointness) with $R, R \in \mathbf{R_s}$.

Now a $\mathcal{SROIQ}$ RBox $\mathcal{R}$ is a finite set of RBox axioms such that the following condition is satisfied: There is a strict (irreflexive) total order $\prec$ on $\mathbf{R}$ such that

- $S \prec R$ iff $\mathsf{Inv}(S) \prec R$ for $R \notin \{S, \mathsf{Inv}(S)\}$and
- every RIA is of the form $R \circ R \sqsubseteq R$, $\mathsf{Inv(R)} \sqsubseteq R$, $R_1 \circ \ldots R_n \sqsubseteq R$, $R \circ R_1 \circ \cdots \circ R_n \sqsubseteq R$ or $R_1 \circ \cdots \circ R_n \circ R \sqsubseteq R$ where $R, R_1 \ldots, R_n \in \mathbf{R}$ and $R_i \prec R$ for $1 \leq i \leq n$.

These conditions ensures the decidability of the reasoning tasks [HORROCKS et al. 2006]. We usually call an RBox *regular* if it satisfies these conditions.

TBox: We first define GCIs. A $\mathcal{SROIQ}$ *general concept inclusion axiom* (GCI) is an expression of the form $C \sqsubseteq D$, where C and D are concepts. Now a $\mathcal{SROIQ}$

TBox is a finite set of $\mathcal{SROIQ}$ GCIs. TBoxes represent schema-level knowledge i.e., by TBox we represents extensional knowledge of a problem domain.

ABox: A $\mathcal{SROIQ}$ *ABox* is a finite set of $\mathcal{SROIQ}$ ABox axioms which are of one of the following types:

- $C(a)$ called *concept assertion*
- $R(a, b)$ called *role assertion*
- $a \doteq b$ called *equality assertion*
- $a \not\doteq b$ called *inequality assertion*

Having defined all these notions, we now define $\mathcal{SROIQ}$ knowledge bases.

Definition 1. *A* $\mathcal{SROIQ}$ knowledge base *is a tuple* $(\mathcal{T}, \mathcal{R}, \mathcal{A})$ *where* $\mathcal{T}$ *is a* $\mathcal{SROIQ}$ *TBox,* $\mathcal{R}$ *is a* $\mathcal{SROIQ}$ *RBox and* $\mathcal{A}$ *is a* $\mathcal{SROIQ}$ *ABox.*

We will drop the term $\mathcal{SROIQ}$ and simply refer to knowledge bases, TBoxes, ABoxes, RBoxes, concepts and roles whenever it is clear from the context. Sometimes knowledge bases are also called ontologies. We use these words interchangeably.

2.1.2 Semantic

As a fragment of first-order logic, the approach to defining the semantics of $\mathcal{SROIQ}$ is model-theoretic. Like in first-order logic, the semantics of $\mathcal{SROIQ}$ does not allow us concluding falsehood of information given that its truth is not deducible. This behavior is due to the open-world assumption taken in the $\mathcal{SROIQ}$ semantics. Though having short-comes in certain scenarios, this enables us modeling incomplete information. For example, it is easy to express the fact that "every person has parents who themselves are persons" without even specifying person along without event listing the individuals explicitly. The TBox GCI

$$\mathsf{Person} \sqsubseteq \exists \mathsf{hasParent}.\mathsf{Person}$$

formalizes this fact. Note that in formalism adhering to close-world semantics, e.g., logic program, specifying the fact would require stating all people along with their parents explicitly which simply is not possible in general.

For the semantics of $\mathcal{SROIQ}$ we have the notion of interpretations.

Definition 2. *A $\mathcal{SROIQ}$* interpretation *$\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is composed of a non-empty set $\Delta^{\mathcal{I}}$ called the* domain *of $\mathcal{I}$ and a* mapping function *$\cdot^{\mathcal{I}}$ such that:*

- *$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ for every concept name A;*
- *$R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for every $R \in N_R$;*
- *$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ for every individual name a.*

Further the universal role U is interpreted as a total relation on $\Delta^{\mathcal{I}}$ i.e., $U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The bottom concept $\bot$ and top concept $\top$ are interpreted by $\emptyset$ and $\Delta^{\mathcal{I}}$ respectively.
The mapping $.^{\mathcal{I}}$ is extended to concepts as following where C, D are $\mathcal{SROIQ}$ concepts, R is a role and S is a simple role, n is a non-negative integer and $\#M$ represents the cardinality of a set M.

$$
\begin{aligned}
(\neg C)^{\mathcal{I}} &= \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(\exists S.\textsf{\textit{Self}})^{\mathcal{I}} &= \{x \mid (x,x) \in S^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} &= C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} &= C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(\forall R.C)^{\mathcal{I}} &= \{x \in \Delta \mid \forall y.(x,y) \in R^{\mathcal{I}} \to y \in C^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} &= \{x \in \Delta \mid \exists y.(x,y) \in R^{\mathcal{I}} \land y \in C^{\mathcal{I}}\} \\
(\leq nS.C)^{\mathcal{I}} &= \{x \in \Delta \mid \#\{y \mid (x,y) \in S^{\mathcal{I}} \land y \in C^{\mathcal{I}}\} \leq n\} \\
(\geq nS.C)^{\mathcal{I}} &= \{x \in \Delta \mid \#\{y \mid (x,y) \in S^{\mathcal{I}} \land y \in C^{\mathcal{I}}\} \geq n\} \\
(\{a_1, \ldots, a_n\})^{\mathcal{I}} &= \{a_1^{\mathcal{I}}, \ldots, a_n^{\mathcal{I}}\}
\end{aligned}
$$

Similarly, the interpretation of the inverse of a role R is defined as follows:

$$(R^-)^\mathcal{I} = \{(x, y) \mid (y, x) \in R^\mathcal{I}\}$$

Having defined interpretations, we now see how to interpret axioms in a given interpretation. Given an axiom α (TBox, RBox or ABox axiom), we say an interpretation $\mathcal{I}$ *satisfies* α, written $\mathcal{I} \models \alpha$, if it satisfies the corresponding condition given in Table 2.1. Similarly $\mathcal{I}$ satisfies a TBox $\mathcal{T}$, written $\mathcal{I} \models \mathcal{T}$, if it satisfies all the axioms in $\mathcal{T}$. The satisfaction of an RBox and an ABox by an interpretation is defined analogously. We are mainly interested in interpretations that satisfy all the components of a knowledge base.

Axiom α	$\mathcal{I} \models \alpha$, if
$C \sqsubseteq D$	$C^\mathcal{I} \subseteq D^\mathcal{I}$
$R_1 \circ \cdots \circ R_n \sqsubseteq R$	$R_1^\mathcal{I} \circ \cdots \circ R_n^\mathcal{I} \subseteq R^\mathcal{I}$
Dis(S,T)	$S^\mathcal{I} \cap T^\mathcal{I} = \emptyset$
$C(a)$	$a^\mathcal{I} \in C^\mathcal{I}$
$R(a, b)$	$(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$
$a \doteq b$	$a^\mathcal{I} = a^\mathcal{I}$
$a \neq b$	$a^\mathcal{I} \neq b^\mathcal{I}$

Table 2.1: Semantics of $\mathcal{SROIQ}$ axioms

Definition 3. *A given interpretation $\mathcal{I}$ satisfies a knowledge base $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ (written $\mathcal{I} \models \Sigma$) if it satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$ in which case we call $\mathcal{I}$ a* model *of Σ.*

Having defined the semantics of $\mathcal{SROIQ}$, we next introduce several reasoning tasks based on the notion of satisfaction.

2.2 Reasoning Problems in $\mathcal{SROIQ}$

The very goal of a logic-based knowledge representation formalism is to deductively provide implicit information from the explicitly stated information. Deduction is usually understood in terms of entailment.

Definition 4. *For a given $\mathcal{SROIQ}$ knowledge base Σ and $\mathcal{SROIQ}$ axiom α we say, Σ* entails *α (written as $\Sigma \models \alpha$) if and only if for each model $\mathcal{I}$ of Σ we have that $\mathcal{I} \models \alpha$.*

Based on Definition 3 and Definition 4, we define the following reasoning tasks.

- **consistency checking**: in this task, we test if a given knowledge base is consistent i.e., if it has a model.
- **subsumption checking**: for a given knowledge base Σ the subsumption checking of a concept C by a concept D requires checking if $\Sigma \models C \sqsubseteq D$.
- **concept satisfiability checking**: for a given knowledge base Σ and concept C, the satisfiability checking requires finding a model $\mathcal{I}$ of Σ such that we have $C^{\mathcal{I}} \neq \emptyset$. On the existence of such a model, the concept C is said to be satisfiable with respect to Σ.

  Note that satisfiability problem is reducible to subsumption problem: C is unsatisfiable iff C is subsumed by $\perp$.
- **instance checking**: for a given knowledge base Σ, concept C and individual name a, this task checks if $\Sigma \models C(a)$ in which case we say a to be an instance of C.

Sometimes entailment of certain axioms is also called query answering. Formally,

Definition 5. *We now define the notion of* query answering. *A* query *Q to a knowledge base Σ is of the form $\Sigma \models \alpha$? where α is an axiom. In this work, we only consider concept assertions as axioms in a query*[3]. *The answer to the query Q (denoted by $ans(Q)$) is defined as follows:*

$$ans(Q) = \begin{cases} \textsf{YES} & \textit{if } \Sigma \models \alpha \\ \textsf{NO} & \textit{if } \Sigma \models \neg\alpha \\ \textsf{UNKNOWN} & \textit{otherwise} \end{cases}$$

where by $\neg\alpha$ we mean $\neg C(a)$ for $\alpha = C(a)$.

[3] This we do for the sack of simplicity. Our notions are applicable to other forms of axioms as well.

Before concluding this chapter one final remark. DLs are fragments of first-order logic. Hence, any DL axiom (knowledge base) is expressible as first-order formula (theory). Here we do not provide a description of this translation rather we refer the interested reader to [Baader et al. 2007, Krötzsch 2010].

Chapter 3

Nonmonotonic Logic Formalisms

There come several formalisms that are "non-monotonic" in the sense that they capture *defeasible inference: inferencing where conclusions are based on the available information and thus retractable in the light of new information.* To be more formal, let Θ represent a theory in some logic $\mathcal{L}$. Let $Th(\Theta)$ represents the set of all formulas that are deducible from Θ i.e.,

$$Th(\Theta) = \{\phi | \Theta \models \phi \text{ for an } \mathcal{L} \text{ formula } \phi\}$$

Non-monotonicity of a logic depends on the behavior of the consequence relation $\models$.

Definition 6. *A logic $\mathcal{L}$ is non-monotonic if and only if $Th(\Theta) \not\subseteq Th(\Theta \cup \Theta')$ for some theory Θ'.*

Had $\mathcal{L}$ be a monotonic logic like DLs, the addition of new information does not invalidate the previous drawn conclusions and thus would have been a monotonic logic i.e., $Th(\Sigma) \subseteq Th(\Sigma \cup \Sigma')$ with Σ and Σ' theories of $\mathcal{L}$. Note that $\Sigma \cup \Sigma'$ represent the addition of new information.

Defeasible inferencing or reasoning are commonly observable in daily life. Common sense reasoning is a prime example. In Section 1.1 of Chapter 1, we presented the timetable example of trains from Karlsruhe to Paris. Lets consider a daily life example where we make decisions considering certain facts by defaults. When going for grocery on weekdays, by default we assume that the market is open. Though we do not know with certain if so is the case. Nevertheless we take the decision based on the knowledge acquired due to routine. However, it is quite possible that the market might be closed due to some construction work. On knowing this, we can not further conclude that the market is open. Thus, we retract from our conclusion which we would make by default and may choose another market.

In this chapter we will present an overview of some important non-monotonic formalisms. We distinguish them in three categories:

- classical Non-Monotonic Formalisms
- logic Programs
- expressive Non-monotonic Formalisms

In the following we discuss these formalisms in detail. This will prove as preliminary in understanding non-monotonic reasoning in description logics which is the topic of Chapter 4.

3.1 Classical Non-monotonic Formalisms

Some early non-monotonic formalisms include [McDermott and Doyle 1980, McDermott 1982, Reiter 1980, Moore 1983a, McCarthy 1980] but here we limit ourselves to the following ones.

3.1.1 Default Logic

Default logic is one of the earlier approaches to capturing default reasoning. Consider the following classic example in artificial intelligence. From the daily life, we know that almost every bird flies except for penguins and some others. Given a particular bird, we want to conclude that it flies so far we do not know if its one of the exceptional birds. As described in [Reiter 1980] at first it seems that a first-order sentence representing the above rule:

$$\forall x.\mathsf{Bird}(x) \wedge \neg\mathsf{DisableBird}(x) \rightarrow \mathsf{Fly}(x)$$

where $\mathsf{DisableBird}(x)$ is a disjunction of all birds that cannot fly i.e.,

$$\mathsf{DisableBird}(x) = \mathsf{Penguin}(x) \vee \mathsf{Kiwi}(x) \vee \mathsf{Ostrich}(x) \vee \ldots$$

Now given that tweety is a bird, we still are unable to conclude that tweety can fly. The reason is that all we know is just that tweety is bird but we do not have any information whether tweety is a Penguin, an Ostrich or a Kiwi, etc. Naturally we want that by *default* every bird can fly and thus so does tweety i.e., if tweety is a bird and there is no information regarding if it is one of the disabled birds, conclude that it can fly. By the absence of information here we mean that it is consistent to assume that a bird can fly. We now introduce the notion of default rules for representing such defaults.

Definition 7. *A* default rule (or just default) *is of the form:*

$$\frac{\varphi : \psi_1, \ldots, \psi_n}{\xi}$$

where $\varphi, \psi_1, \ldots, \psi_n, \xi$ are closed first-order formulas[1]*. Further, we call*

- *the formula as the prerequisite,*
- *$\psi_1, \ldots, \psi_n$ as the justifications, and*
- *ξ as* the consequent

of the default rule.

A default rule as in the above definition is read as "If φ holds and it is consistent to assume that $\psi_1, \ldots, \psi_n$ hold, then conclude ξ". For the above bird example, the default rule thus can be represented as follows,

$$\frac{\mathsf{Bird}(x) : \mathsf{Fly}(x)}{\mathsf{Fly}(x)}$$

which is read as "If x is bird and it is consistent to assume that x can fly, then infer that x can fly".

[1]Instead of first-order logic, one can define the notion of default for any other logic.

In Default Logic, theories are thus a set of formulas and axioms along with a set of default rules. We called them as *default theories*.

Definition 8. *A default theory T is a pair (W, D) consisting of a set of formulas W and a set of defaults D.*

As for the semantics of default logic, we have already mentioned that a default $\frac{\varphi, \psi_1, \ldots, \psi_n}{\xi}$ is to be interpreted as "if φ holds, and it is consistent to assume $\psi_1, \ldots, \psi_n$, then conclude ξ" [ANTONIOU 1996]. To this end we present the notion of extensions based on a fixed-point operator defined below.

Definition 9. *Given a default theory $T = (W, D)$ with D containing defaults of the form $\varphi : \psi_1, \ldots, \psi_n/\xi$, for any set of closed formulas E, let $\Gamma(E)$ is the smallest set satisfying:*

- $W \subseteq \Gamma(E)$
- $Th(\Gamma(S)) = \Gamma(E)$
- *if* $(\varphi : \psi_1, \ldots, \psi_n/\xi) \in D$ *and* $\varphi \in \Gamma(E)$, *and* $\neg\psi_1, \ldots, \neg\psi_n \notin S$, *then* $\xi \in \Gamma(E)$

where by $Th(\Gamma(E))$ we mean the set of all closed formulas deducible from $\Gamma(E)$. We called the set E as an extension of T *if and only if* $\Gamma(E) = E$ *i.e., E is the fixed point of* Γ.

Note that we can think of an extension of a default theory as a set of acceptable beliefs [REITER 1980]. For our bird example, let $T_B = (W_B, D_B)$ be the theory representing the scenario. Here we simplify the example by assuming the set $W_B = \{\text{Bird(tweety)}, \text{Disabled(tweety)} \rightarrow \neg\text{Fly(tweety)}\}$ and $D_B = \{\text{Bird(tweety)} : \text{Fly(tweety)}/\text{Fly(tweet)}\}$. Now let E_B a set closed formulas such that

$$E_B = \{\text{Bird}(tweety), \text{Disabled(tweety)} \rightarrow \neg\text{Fly(tweety)}, \text{Fly}(tweety)\}$$

It is easy to see that E_B is an extension of T_B. Thus, by default we can conclude that tweety can fly. Now if only later we come to know that tweety is one of the disable birds i.e., if Disable(tweety) holds. Then ¬Fly(tweety) must hold as well. Consequently we get $W_B = \{\text{Bird(tweety)}, \text{Disabled(tweety)} \rightarrow \neg\text{Fly(tweety)}, \neg\text{Fly(tweety)}\}$. Hence we need to retract from our previous conclusion that Fly(tweety) and thus E_B is no more an extension. In this case, the only extension of T_B is the set

$$\{\text{Bird(tweety)}, \text{Disabled(tweety)} \rightarrow \neg\text{Fly(tweety)}, \neg\text{Fly(tweety)}\}$$

For further discussion regarding different types of default rules and theories and their characteristics we refer the interested reader to [ANTONIOU 1996].

3.1.2 Autoepistemic Logic

Autoepistemic logic is a nonmonotonic formalism invented in early 80s by Moore [MOORE 1983a]. This logic enables us of questioning our own knowledge i.e., introspective capabilities. Thus the world "autoepistemic" coined to mean *reflection upon self-knowledge*. For example, assuming that John is a regular student, consider the question "Is there a quiz next week?". Since there was no announcement about it in the lecture hall, John replies "No, else I would have known about it." Note that, we can not claim of John's knowledge to be complete. He is just making a conjecture in reply negatively. Suppose only later John comes to know via the homepage of the coarse that indeed there is a quiz next week. Now John knows about the happening of the quiz and thus the conclusion drawn earlier is not valid any more. Notice the difference between default reasoning and autoepistemic reasoning. The former deals in capturing reasoning by defaults whereas the latter is context-sensitive. Conclusions drawn by an agent in autoepistemic logic are based on the knowledge of the agent or what the agent believes. Different agents believe differently and thus draw conclusions differently. We quote Moore [MOORE 1983a]

> "...The nonmonotonicity associated with autoepistemic statements should therefore be no more puzzling than the fact that 'I am hungry' can be true when uttered by a particular speaker at a particular time, but false when uttered by a different speaker at the same time or the

> same speaker at a different time. So we might say that, whereas default reasoning is nonmonotonic because it is defeasible, autoepistemic reasoning is nonmonotonic because it is indexical."

Formally, we now define the language of autoepistemic logic.

Definition 10. Autoepistemic (AE) formulas [2] *are the smallest set satisfying the following:*

- *each closed first-order formula is an AE formula*
- *if φ is an AE formula, then so is $\mathbf{K}\varphi$*
- *if φ and ψ are AE formulas, then so are $\neg\varphi$ and $(\varphi \vee \psi)$*

An autoepistemic (AE) theory *is then just a a set of AE-formulas.*

The symbol $\mathbf{K}$ before a formula reflects knowledge/belief about it. For example, $\mathbf{K}\varphi$ can be read as "I know/believe φ". Lets make an statement about Brazilians liking football. We want to state that if I do not know whether a Brazilian likes football, then I conclude that he likes football. We formalize this by the following AE formula:

$$\mathsf{Brazilian} \wedge \neg\mathbf{K}\neg\mathsf{likesFootball} \rightarrow \mathsf{likesFootball}$$

The original semantics for autoepistemic logic is based on the notion of autoepistemic interpretations [MOORE 1983a, MOORE 1983b]. Here we rather take the possible-world approach for the semantics as introduced in [MOORE 1984]. For this we define the notion of autoepistemic interpretations which are define in terms of S5 structures. As a remainder, S5 structures are Kripke structures [KRIPKE 1971, BLACKBURN et al. 2006] where the accessibility relation is an equivalence relation.

Definition 11. *An* autoepistemic interpretations *is a S5 structure (I, R, W) where I is an first-order interpretation, W is a set of first-order interpretations and R is an accessibility relation i.e., $R := W \times W \mapsto W$. The interpretation of an AE*

[2] Again we consider first-order autoepistemic logic here, though the notions can be defined for other logics as well.

formula φ such that φ does not contain K i.e., φ is a first-order logic formula, can be defined analogous to first-order logic by considering the interpretation I only. For a given AE formula of the form $\mathsf{K}\varphi$, (I, R, W) satisfies $\mathsf{K}\varphi$ or $\mathsf{K}\varphi$ is true in (I, R, W) (written $(I, R, W) \models \mathsf{K}\varphi$) iff $J \models \varphi$ for each J such that $(I, J) \in R$.

To decide entailment of an AE theory, we are interested in certain autoepistemic interpretations defined as follows:

Definition 12. *For a given AE theory T, an autoepistemic interpretation (I, R, M) is called an* autoepistemic model *of T iff*

- *$(I, R, M) \models \varphi$ for each $\varphi \in T$, and*
- *the accessibility relation R is total i.e., every world is accessible from every world.*

We write $(I, R, M) \models T$

In [MOORE 1987] such S5 structures are called complete S5 structures. Note that for an autoepistemic model (I, R, M), since the accessibility is a total and equivalence relation, we simply write (I, M). We can further simplify the notation by representing the model (I, R, M) just by M. The justification for this simplification is that since R is total, hence if we have $(I, M) \models \varphi$ for some AE formula φ, then $(J, M) \models \varphi$ for each $J \in M$. In other words, we have

$$\exists I \in M : (I, M) \models \varphi \Leftrightarrow \forall I \in M : (I, M) \models \varphi$$

Now an autoepistemic model M of theory T is just a set of interpretations such that $(I, M) \models T$ for all $I \in M$. We will use similar notion of autoepistemic models for epistemic extension of description logics defined in the subsequent chapters.

Coming back to the example about a Brazilian liking football, consider the AE theory

$$T = \{\mathsf{Brazilian} \land \neg\mathsf{K}\mathsf{likesFootball} \rightarrow \mathsf{likesFootball}, \mathsf{Brazilian}\}$$

An autoepistemic model of T is the set M containing all the interpretations which contain the propositions Brazilian and likesFootball.

3.1.3 Circumscription

Circumscription is yet another way of formalizing nonmonotonic reasoning introduced by McCarthy [McCarthy 1980]. Unlike default logic and autoepistemic logic, in circumscription for certain predicates we minimize (circumscribe) the set of objects for which the predicates are true. For a given first-order theory T, the set of predicates which we want to circumscribed is denoted by $circ(T)$.
Lets consider the following first-order theory T containing the following formulas

$$\forall x.(\mathsf{Bird}(x) \wedge \neg\mathsf{Disable}(x) \rightarrow \mathsf{Fly}(x))$$
$$\mathsf{Bird(tweety)}$$

Under the classical first-order logic, we can not deduce $\mathsf{Fly}(tweety)$ as we can not prove $\mathsf{Disable}(tweety)$. In circumscription, we circumscribe the predicate Disable i.e., we minimize the set of all objects c for which $\mathsf{Disable}(c)$ is true. In our example, since there is no evidence about tweety being a disable bird. Thus we add the formula $\forall x.(\neg\mathsf{Disable}(x))$ to the set $circ(T)$. Now concluding that Fly(tweety) requires considering certain models of T. In this example, we consider only models I of T in which $\mathsf{Disable}^I = \emptyset$. This allows us to conclude ¬Disable(tweety) and consequently Fly(tweety). Note that later if we come to know the fact $\mathsf{Disable}(tweety)$, we eliminate all the models I of T for which the set $\mathsf{Disable}^I$ is strictly larger than $\{\mathsf{tweety}\}$. In that case, we have to retract from the conclusion Fly(tweety).

The above example explains circumscription in an informal way. A detailed description of the formalisms is given in ([Antoniou 1996, McCarthy 1980]).

3.2 Logic Programs

Rule-based formalisms are well-studied approaches to knowledge representation. Some are fragments of first-order logic like *datalog* and others are even not comparable with first-order logic. The community of rule-based formalisms is now very old and several annual conferences focus on such formalisms including ICLP, LPNMR, ILP etc. Here we focus on a non-monotonic rules only. For first-order (monotonic) rules, one can refer to [KRÖTZSCH 2010, HORROCKS et al. 2004] etc. The formalisms with non-monotonic rules usually known as normal logic programs (NLPs).

To define an NLP, the signature $\mathcal{A}$ is a countably infinite disjoint set of constants, predicate symbols and functions symbols along with a countably infinite set of variables. A *term over* $\mathcal{A}$ is defined as follows:

- all the variables and constants are term
- if $t_, \ldots, t_n$ are terms and f is function symbol of arity n then, $f(t_1, \ldots, t_n)$ is a term as well.

An *atom over* $\mathcal{A}$ is of an expression of the form $P(t_1, \ldots, t_n)$, where P is an n-ary predicate and $t_1, \ldots, t_n$ are terms. Then a *literal* is an atom A or *default* negated atom $not\ A$. By $\mathcal{H}$ we represent the set of all ground atoms and call it the *Herbrand Base* of $\mathcal{A}$.

Definition 13. *A* Normal Logic Program (NLP) *is a set of rules r of the form:*

$$H \leftarrow B_1, \ldots, B_n, not\ C_1, \ldots, not\ C_m$$

where $H, B_1, \ldots, B_n, C_1, \ldots, C_m$ are atoms and $m, n \geq 0$ are finite integers. Further, H is called the head *of r, where as $B_1, \ldots, B_n, C_1, \ldots, C_m$ are the* body atoms *of r. A* fact *is a rule with empty body i.e., a rule of the form $H \leftarrow$. We abbreviate a fact $H \leftarrow$ by H.*

As an example of NLP consider the program P containing the following rule

$$\mathsf{picnic} \leftarrow \mathsf{sunny}, not\ \mathsf{strike}$$

The negation not (usually called as the *default negation*) introduces nonmonotonicity in reasoning. It is in contrast to the classical negation $\neg$. Default negation is interpreted as "if is un-provable". Thus the above rule states that if it is a sunny day and it is not provable that there is a strike, then we go to picnic. Note that if later we come to know that there indeed is a strike, we can no more conclude picnic, thus retracting from our previous conclusion.

There is no one semantics that the logic program adheres to. Among many, the two-valued *stable model semantics* [GELFOND and LIFSCHITZ 1988] and the three-valued *well-founded semantics* [GELDER et al. 1991] are probably the most popular ones. Stable model semantics is two-valued in the sense that truth values true or false are assign to the atoms in the semantics whereas the three-valued may also assigns a third value of unknown. A recent semantics for NLPs called *the minimal hypothesis semantics* is presented [PINTO and PEREIRA 2011] which is two-valued as well but exhibits certain characteristics like commulativity, relevance, model existence etc (cf. [PINTO 2011] for thorough discussion). In this work, we mainly focus on the stable models and well-founded semantics. Note that the answer set programming [GELFOND and LIFSCHITZ 1991, LIFSCHITZ 2008] is based the stable model semantics. In the following we assume that the reader is familiar with definite logic programs and their semantics (cf [NILSSON and MALUSZYNSKI 1995, LLOYD 1987] etc.).

3.2.1 Stable Model Semantics

We first describe the notion of two-valued interpretations. A *(two-value) interpretation* I for an NLP P is a set of ground literals such that

$$I = I^+ \cup \mathcal{H} \setminus I^+$$

where $\mathcal{H}_P$ is the Herbrand base of P and $I^+ \subseteq \mathcal{H}_P$ is the a set of atoms. An atom A is true in an interpretation I iff $A \in I^+$ whereas a literal $not\ A$ is true in I iff $A \notin I^+$. We say I satisfies a rule r of the form

$$H \leftarrow B_1, \ldots, B_n, not\ C_1, \ldots, not\ C_m$$

iff H is true in I whenever $B_1, \ldots, B_n, not\ C_1, \ldots, not\ C_m$ are true in I. Finally I is a model of P iff I satisfies all the the rules in P.

The stable models of NLP are defined in terms of Gelfond-Lifschitz operator [ALFERES and PEREIRA 1996].

Definition 14. *Let I be a two-value interpretation of an NLP P. The GL-transformation of P modulo I is the program $P_{|I}$ obtained from P by deleting:*

- *each rule with a literal $not\ B$ in its body from P such that $B \in I$;*
- *all the negative literals in the bodies of the remaining rules.*

Since $P_{|I}$ is a positive or a definite program (i.e., without negative literals), it has unique minimal model that is its minimal Herbrand Model J. The GL-operator Γ is then defined as $\Gamma(I) := J$.

The fixed-point of the operator Γ preserves the modelhood of P i.e., $\Gamma(I)$ is a model of P as well. The stable models of P are defined based on $\Gamma(P)$.

Definition 15. *Given a two-valued interpretation I, we say I is a stable model of an NLP P iff $\Gamma(I) = I$. An atom A is true under the stable model semantics of P iff A belongs to all the stable models of P.*

As an example consider the following program:

$$P = \{\mathsf{picnic} \leftarrow \mathsf{sunny}, not\ \mathsf{strike}$$
$$\mathsf{sunny}\}$$

Let $M = \{\mathsf{sunny}, \mathsf{picnic}\}$. Thus, we get

$$P_M = \{\mathsf{picnic} \leftarrow \mathsf{sunny}$$
$$\mathsf{sunny}\}$$

Now the minimal (Herbrand) model of P_M is $\{\mathsf{picnic}, \mathsf{sunny}\}$ which is the same as M. Hence, M is a stable model of P. In fact, it is the only stable model of P. Thus, we can say that sunny (and similarly picnic) is true under the stable model semantics of P.

Note that there are programs with more than one stable models. E.g., the program with only rules

$$A \leftarrow not\ B$$
$$B \leftarrow not\ A$$

has two stable models namely $\{A\}$ and $\{B\}$. Meanwhile there are programs with no stable models at all. For example, the program

$$\{A \leftarrow not\ A\}$$

has no stable models.

3.2.2 Well-Founded Semantics

Again we first define the notion of three-valued interpretations. The *three-valued interpretation* I of a given NLP P is defined as a set $\mathsf{T} \cup not\ \mathsf{F}$ where T and F are disjoint subsets of $\mathcal{H}(P)$ containing atoms true respectively false in I. The truth values of the remaining atoms are undefined (the third truth value). Note that every two-valued interpretation by definition is a three-value interpretation as well. In such a case, we have $\mathsf{T} \cup \mathsf{F} = \mathcal{H}(P)$.

In well-founded semantics, models are three-valued interpretations. We can define the well-found semantics based by extending GL-operator of the stable model semantics. Like in two-valued interpretations, every positive program has three-valued least model [Przymusinska and Przymusinski 1990].

Definition 16. *Given an NLP P, let I be three-valued interpretation. The extended GL-transformation of P modulo I is the program $P_{|I}$ obtained by:*

- *deleting each rule with a literal $\mathrm{not}\ B$ in its body from P such that $B \in I$;*
- *replacing in the remaining rules of P literals of the form $\mathrm{not}\ B$ by u[3], for which B is unknown;*
- *deleting all the negative literals in the bodies of the remaining rules.*

[3] u represent the third truth value i.e., the unknown

$P_{|I}$ is a program without negative literals and thus has a unique three-valued least model J. We define $\Gamma * (I) = J$

Based on the operator $\Gamma*$, we now define partial stable models. For a given NLP P, a three-valued interpretation I of P is called partial stable model of P iff $\Gamma * (I) = I$. The well-founded semantics is then defined as following:

Definition 17. *Given an NLP P, let $\Im(P)$ represents the set of all partial stable models of P. An interpretation $I \in \Im(P)$ is called well-founded model of P iff I is the least partial stable model of P in the sense that $I \subseteq J$ for each $J \in \Im(P)$.*

To compute a well-founded model of program, we can use $\Gamma*$ iteratively starting with the empty interpretation (cf. [ALFERES and PEREIRA 1996] for details and an example.).

3.3 Expressive Formalisms

In this section, we introduce the logic of minimal knowledge and negation as failure usually abbreviated as MKNF [LIFSCHITZ 1991] which is very similar to the logic of minimal belief and negation as failure (MBNF) [LIFSCHITZ 1994]. MKNF is expressive enough to capture many of the existing nonmonotonic formalisms. We introduce MKNF for propositional logic whereas the notions can easily be extended to the first-order logic (see [MOTIK and ROSATI 2008]).

Definition 18. *Given a set of proposition letters $\{p_1, \ldots, p_n\}$, the set of (propositional) MKNF formula is the smallest set such that:*

- *every propositional letter p_i ($1 \leq i \leq n$) is an MKNF formula,*
- *if φ is an MKNF formula, so are $\neg\varphi$, $\mathbf{K}\varphi$ and $not\ \varphi$*
- *if φ and ψ are MKNF formula, so is $\varphi \vee \psi$*

Where we use $\varphi \wedge \psi$ and $\varphi \rightarrow \psi$ as an abbreviation to $\neg(\neg\varphi \vee \neg\psi)$ and $\neg\varphi \vee \psi$ respectively. An MKNF *theory is then just a set of MKNF formulas.*

For the semantics of MKNF, the possible-world approach is adapted by defining the notion of MKNF interpretation. As usual, an (propositional) *interpretation* is a set of atoms. We denote the set of all interpretations with $\mathfrak{I}$.

Definition 19. *An* MKNF interpretation *is a tuple* $(I, W^{\mathbf{K}}, W^{not})$ *such that* I *is an interpretation whereas* $W^{\mathbf{K}}$ *and* W^{not} *are set of interpretations. The truth value of an MKNF formula in* $(I, W^{\mathbf{K}}, W^{not})$ *is defined as follows:*

- *if* φ *is an atom (proposition letter),* φ *is true in* $(I, W^{\mathbf{K}}, W^{not})$ *iff* $\varphi \in I$.
- $\neg\varphi$ *is true in* $(I, W^{\mathbf{K}}, W^{not})$ *iff* φ *is not true in* $(I, W^{\mathbf{K}}, W^{not})$.
- $\varphi \vee \psi$ *is true in* $(I, W^{\mathbf{K}}, W^{not})$ *iff either* φ *is true in* $(I, W^{\mathbf{K}}, W^{not})$ *or* ψ *is true in* $(I, W^{\mathbf{K}}, W^{not})$.
- $\mathbf{K}\varphi$ *is true in* $(I, W^{\mathbf{K}}, W^{not})$ *iff* φ *is true in* $(J, W^{\mathbf{K}}, W^{not})$ *for all* $J \in W^{\mathbf{K}}$.
- *not* φ *is true in* $(I, W^{\mathbf{K}}, W^{not})$ *iff there is an interpretation* $J \in W^{not}$ *such that* φ *is not true in* $(J, W^{\mathbf{K}}, W^{not})$.

If a formula φ is true in an MKNF interpretation $(I, W^{\mathbf{K}}, W^{not})$, we write $(I, W^{\mathbf{K}}, W^{not}) \models \varphi$. An MKNF theory T is true in $(I, W^{\mathbf{K}}, W^{not})$ (written as $(I, W^{\mathbf{K}}, W^{not}) \models T$) iff for each formula $\varphi \in T$ we have $(I, W^{\mathbf{K}}, W^{not}) \models \varphi$. For a given MKNF theory T, a set of interpretations M is called *an MKNF model* of T if and only if

1. $(I, M, M) \models T$ for each $I \in M$, and
2. M is a maximal set with this property i.e., there is no set M' with $M \neq M'$ and $M \subseteq M'$ such that $(I, M', M) \models T$.

The requirement of the maximality of M introduces non-monotonicity of reasoning in MKNF. It is this requirement that enforces the minimality of knowledge. The more the worlds (interpretations) are there in M for an MKNF interpretation (I, M, N), the fewer are the formula of the form $\mathbf{K}\varphi$ that are true in (I, M, N). As an example consider the MKNF theory T given as follows

$$T = \{\mathbf{K}\text{sunny} \vee \mathbf{K}\text{cloudy}\}$$

where sunny and cloudy are just propositional letters. Since there is no MKNF formula in T with not occurring in it, we can drop the third component of the MKNF interpretations we consider. This is possible as third component is relevant only for interpreting formulas containing not . Note that there are two MKNF models of T: the first is the set of all interpretations in which sunny is true and the second is the set of all interpretations in which cloudy is true.

The prime interest of introducing the MKNF formalism is its capability in capturing most of the approaches to nonmonotonic reasoning. We elaborate it in the following.

MKNF and NLPs

A normal logic program P can be translated into an MKNF theory P_{MKNF} such that every stable model of P corresponds to an MKNF model of P_{MKNF}. Formally let r be a rule in P and suppose it has the following form:

$$A \leftarrow B_1, \ldots, B_n, not\ C_1, \ldots, not\ C_m$$

Then its translation r_{MKNF} is given as following

$$r_{\text{MKNF}} = \mathbf{K}B_1 \wedge \cdots \wedge \mathbf{K}B_n \wedge not\ C_1 \wedge \cdots \wedge not\ C_m \rightarrow A$$

Now P can be translated into P_{MKNF} as following

$$P_{\text{MKNF}} = \{r_{\text{MKNF}} \mid r \in P\}$$

We now establish correspondence between stable models of P and MKNF models of P_{MKNF}. For any set of atoms I, let $\omega(I)$ represents the set of all supersets of I i.e., $\omega(I) = \{J \mid I \subseteq I\}$. Suppose that $\mathfrak{M}(P)$ represents the set of all stable models of P, then the set of all MKNF models P_{MKNF} is given as by the set $\{\omega(M) \mid \mathfrak{M}(P)\}$. This relationship has been proved in (Theorem 1.Part B [GELFOND and LIFSCHITZ 1988]).

MKNF and Default Logic

For a given default rule r of the form

$$\frac{\varphi : \psi_1, \ldots, \psi_n}{\xi}$$

It's translation is represented by r_{MKNF} and given as the the following MKNF formula

$$r_{\text{MKNF}} = \mathbf{K}\varphi \wedge not\ \neg\psi_1 \wedge \cdots \wedge not\ \neg\psi_n \rightarrow \mathbf{K}\xi$$

Similarly, a set of default rules D can be translated into an MKNF theory D_{MKNF} given as follows:

$$D_{\text{MKNF}} = \{r_{\text{MKNF}} \mid r \in D\}$$

For a default theory (W, D) and a formula φ[4], we have that φ is in every extension of (W, D) if and only if $W \cup D_{\text{MKNF}} \models \varphi$. For proof and technical details we refer to [Gelfond and Lifschitz 1988].

The first-order version of MKNF is presented in [Motik and Rosati 2008]. It is one of the most expressive hybrid formalisms[5] that can expresses most of the approaches to nonmonotonic reasoning in OWL.

We have seen so far different approaches to defeasible reasoning. These approaches have been extended to embed non-monotonic reasoning of some kind in description logics. In the next chapter, we put light on some of the important ones.

[4]Note that earlier we defined default logic for the first-order logic, but here we consider propositional default logic only and thus φ is a propositional formula

[5]Hybrid in the sense that it can capture combination of first-order logic with logic programs.

Chapter 4

Towards Defeasible Reasoning in OWL

In this chapter we provide an overview of different approaches to nonmonotonic reasoning in OWL. We have seen different nonmonotonic logic in the previous chapter. A somewhat similar development of formalisms can be observed in OWL community when it comes to defeasible inferencing. Probably the most mature work in this respect include approaches presented in [MOTIK and ROSATI 2008] and [KNORR et al. 2008]. We discuss the former in detail in this chapter. The latter is a similar approach but not directly comparable with our work. Hence, we will briefly introduce it. In the following section we outline some of the drawbacks of using DLs and thus OWL for representing knowledge.

4.1 Short-comes of Description Logics

Historically, DLs can be seen as a successor of the framed-base systems [MINSKY 1974] and semantic networks [QUILLIAN 1968]. As the semantics of DLs is first-order based, this leaves out several features of framed-base system that are important for modeling a problem domain. In [DONINI et al. 1998], these features have been classified into different groups of which we are interested only in two:

- *query features:* there are scenarios where a somewhat enriched query language is required. Some example were presented in Section 1.3 of Chapter 1.

- *nonmonotonic features:* capabilities to captures incomplete information.

These lack of expressiveness in DLs is due to the open-word assumption in semantics. Consequently, introspection and nonmonotonic inferencing cannot be captured in standard DLs [MOTIK and ROSATI 2008] and hence DLs are impotent in

- axiomatizing integrity constraints e.g., the constraint that 'for each wine, the sugar level should be known' is impossible to be expressed in DLs,
- handling close-world reasoning i.e., to draw conclusion when certain facts can not be proved,
- reasoning by default e.g., by default birds can fly.

Besides, to acquire decidability in DLs, several syntactic restrictions are imposed. For example in DLs we are not allowed to formalize a non-tree-like-relationship. Yet another drawback, though not a serious one, is the lack of higher arity predicates symbols other than unary and binary. Sometimes, there are relationships that can be expressed more intuitively using predicates with higher. Although, such relationships can be expressed using binary predicates only via *reification* [HITZLER et al. 2009], it is a quite tedious and unintuitive task as most of the time, humans are responsible for modeling a problem domain.

The quest for a richer formalism i.e., a formalisms composing all the features of DLs and still enable to overcome the aforementioned problems, has led several researchers to looking for new or hybrid formalisms (formalisms obtained from combining different formalisms e.g., DLs with rules.) In the following section we discuss some important work in this respect.

4.2 Defeasible Reasoning in DLs

We distinguish the approaches to non-monotonic reasoning in DLs into three categories. The first group covers classical approaches to non-monotonicity in DLs. A combination of DLs with ruled-based formalisms are discussed in the second group. Finally the more expressive approaches are discussed in the third group.

Note that we did a some similar categorization of non-monotonic formalisms in Chapter 3. This indeed is intentional as we observe a similar trend in the development of embedding nonmonotonic features in DLs as in nonmonotonic logic formalisms community.

4.2.1 Classical Approaches

The approaches we present here are default and DLs, circumscription in DLs, and autoepistemic reasoning in DLs. Although we present an overview to the first two, our main focus in this Section is on autoepistemic reasoning in DLs as it is highly related to our work.

Defaults and Description Logics

One of the early approaches to nonmonotonic reasoning in DLs is the embedding of defaults in DLs presented in [BAADER and HOLLUNDER 1992]. We have already seen that pure DLs are incapable of capturing reasoning by defaults. Normally the embedding of defaults in DLs introduces undecidability of reasoning. However, [BAADER and HOLLUNDER 1992] acquires decidability by restricting the application of defaults to the *named individuals only: individual names that explicitly occur in a knowledge base*. Note that DLs are fragments of first-order logic, and a DL knowledge base Σ can be considered as a theory $(\Sigma_{FOL}, \emptyset)$ in default logic where Σ_{FOL} represents the first-order equivalent theory of Σ. However in the following we stick to the notion as per [BAADER and HOLLUNDER 1992] where only ABoxes are considered.

A *DL default theory/knowledge base* is a pair $(\mathcal{K}, \mathcal{D})$ where $\mathcal{K}$ is a DL KB and $\mathcal{D}$ is a finite set of defaults rules with concepts as their prerequisites, justifications and consequents. For example, consider the knowledge base

$$\mathcal{K} = \{\mathsf{Bird} \sqsubseteq \mathsf{FlyingBird} \sqcup \mathsf{NonFlyingBird},\ \mathsf{Bird}(\mathsf{tweety})\}$$

stating that every bird is either a flying bird or a non-flying bird and that tweety is a bird. Now the default rule that birds fly by default can be formalized as following

$$\frac{\mathsf{Bird} : \mathsf{FlyingBird}}{\mathsf{FlyingBird}}$$

Hence, for a set $\mathcal{D}$ containing the above rule, $(\mathcal{K}, \mathcal{D})$ is a DL default theory. As for semantics, we still have the notion of extensions but we do not discuss it here. A thorough discussion along with some examples can be found in [BAADER and HOLLUNDER 1992].

Circumscription

The idea of non-monotonic inferencing via circumscription of predicates can easily be applied to DLs as DLs are fragments of first-order logic. In DLs, concepts as well as roles can be circumscribed. Note that in circumscription we have preferential semantics in the sense that we prefer some models (namely, the minimal ones) with respect to the set of circumscribed predicates. Similarly, for a set $M \subseteq N_C \cup N_R$ of concepts and roles (that are to be circumscribed), we define a preference relation $<_M$ with respect to M on interpretations. For any two interpretations $\mathcal{I}$ and $\mathcal{J}$, we say $\mathcal{I} <_M \mathcal{J}$ if

- $\Delta^{\mathcal{I} = \Delta^{\mathcal{J}}}$
- $a^{\mathcal{I}} = a^{\mathcal{J}}$ for each $a \in N_I$
- for each concept $C \in M$ (role $R \in M$), we have that $C^{\mathcal{I}} \subset C^{\mathcal{J}}$ ($R^{\mathcal{I}} \subset R^{\mathcal{J}}$)[1]
- for each concept $C \in N_C \cup N_R \setminus M$ (role $R \in N_C \cup N_R \setminus M$), we have that $C^{\mathcal{I}} = C^{\mathcal{J}}$ ($R^{\mathcal{I}} = R^{\mathcal{J}}$)

Now the reasoning tasks can be defined as in DLs but considering the minimal models only. For further details see [BAADER and HOLLUNDER 1992, BONATTI et al. 2009]. A form of nonmonotonic inferencing called *local closed-word reasoning* via circumscription is presented in [KRISNADHI et al. 2011] where

[1] By $\subset$ we mean strict subset relation i.e., $C \subset D$ iff $C \subseteq D$ but $C \neq D$.

as [GRIMM and HITZLER 2009] presents a tableaux based algorithm for reasoning in circumscriptive $\mathcal{ALCO}$[2].

Autoepistemic Description Logic

The formalisms discussed now are the most relevant ones regarding our work. In fact, the epistemic extension of DLs like $\mathcal{SROIQ}$ that we present in this work, are just motivated by these early approaches: we mainly extend the notions of autoepistemic logic to expressive DLs as well as invent new reasoning techniques.

Early work on extending DLs with the epistemic operator $\mathbf{K}$ include [DONINI et al. 1992b, DONINI et al. 1992a] where the logic $\mathcal{ALCK}$ (DL $\mathcal{ALC}$ extended with the operator $\mathbf{K}$) is presented. The syntax of $\mathcal{ALCK}$ is very much similar to $\mathcal{ALC}$ except that we now allow for $\mathbf{K}$ to occur within concepts. Formally, an $\mathcal{ALCK}$ concept C is defined as follows:

$$C ::= A|\neg C|C \sqcup C|C \sqcap C|\exists R.C|\forall R.C|\mathbf{K}C$$

where A is a concept and R is an $\mathcal{ALCK}$ role defined as

$$R ::= P|\mathbf{K}R$$

with P a role name.

The notion of axioms, TBox, RBox, ABox and knowledge base can be defined similar to standard $\mathcal{ALC}$ but now considering $\mathcal{ALCK}$ concept and roles.

As for the semantics, we define the notion of $\mathcal{ALCK}$ interpretation base on the possible world approach similar to the one presented in [LIFSCHITZ 1991]. Such an approach is quite straight forward for propositional logic: each world correspond to the set of proposition that are true in that world (see Section 3.1.2). Similarly in (any fragment of) first-order logic with some modal operator (like $\mathbf{K}$ here), each world corresponds to an interpretation. However several issues need

[2] A DL that allows for nominals besides being boolean complete.

to be addressed once the interpretations are more than just set of proposition letters [FITTING and MENDELSOHN 1998]. Here two main questions are:

1. Do the domains of interpretation vary across the worlds?

2. Do the interpretation of individual names vary across the worlds?

To address these issue the following two assumptions are made in defining the semantics of $\mathcal{ALCK}$.

- All the interpretations under consideration are defined over a fixed infinite domain. This is also called as *the common domain assumption (CDA).*

- The interpretation of individual names is fixed i.e., in every interpretation an individual is mapped to the same element of the domain. This is termed as *the rigid term assumption (RTA).*

Based on these two assumptions, the notion of $\mathcal{ALCK}$ interpretations is defined as follow:

Definition 20. *An $\mathcal{ALCK}$* interpretation *is a pair $(\mathcal{I}, \mathcal{W})$ where $\mathcal{I}$ is an $\mathcal{ALC}$ interpretation and $\mathcal{W}$ is a set of $\mathcal{ALC}$ interpretations. Let Δ represent the common domain of the interpretation under consideration. Interpretation of concepts and roles are then given as following:*

$$
\begin{aligned}
\top^{\mathcal{I},\mathcal{W}} &= \Delta \\
\bot^{\mathcal{I},\mathcal{W}} &= \emptyset \\
(C \sqcap D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cap D^{\mathcal{I},\mathcal{W}} \\
(C \sqcup D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cup D^{\mathcal{I},\mathcal{W}} \\
(\neg C)^{\mathcal{I},\mathcal{W}} &= \Delta \setminus C^{\mathcal{I},\mathcal{W}} \\
(\exists R.\mathsf{Self})^{\mathcal{I},\mathcal{W}} &= \{x \mid (x,x) \in R^{\mathcal{I},\mathcal{W}}\} \\
(\exists R.C)^{\mathcal{I},\mathcal{W}} &= \{x \mid \exists y.(x,y) \in R^{\mathcal{I},\mathcal{W}} \wedge y \in C^{\mathcal{I},\mathcal{W}}\} \\
(\forall R.C)^{\mathcal{I},\mathcal{W}} &= \{x \mid \forall y.(x,y) \in R^{\mathcal{I},\mathcal{W}} \rightarrow y \in C^{\mathcal{I},\mathcal{W}}\} \\
(\boldsymbol{K}C)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J} \in \mathcal{W}} (C^{\mathcal{J},\mathcal{W}}) \\
(\boldsymbol{K}R)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J} \in \mathcal{W}} (R^{\mathcal{J},\mathcal{W}})
\end{aligned}
$$

Note that for the $\mathcal{ALCK}$ concepts of the form $\mathbf{K}C$ (and similarly for roles of the form $\mathbf{K}R$), the interpretation of $\mathbf{K}C$ is defined as the intersection of the extensions of C in all the interpretations in $\mathcal{W}$. Thus the operator $\mathbf{K}$ is interprated in the same way as the operator $\Box$ in modal logic. The intuition is that for a given $x \in \Delta$, we say x is known to be a member of the class represented by C if x is a member of the class in each possible world (interpretation).

The notion of satisfiability of axioms, TBox, RBox, ABox and knowledge base is obvious e.g., for $\mathcal{ALCK}$ concept C and D, an $\mathcal{ALCK}$ interpretation $\mathcal{I}, \mathcal{W}$ satisfies a GCI $C \sqsubseteq D$ if and only if $C^{\mathcal{I},\mathcal{W}} \subseteq D^{\mathcal{I},\mathcal{W}}$ in which case write $\mathcal{I}, \mathcal{W} \models C \sqsubseteq D$. The notion of models in $\mathcal{ALCK}$ is defined as follow:

Definition 21. *Given an $\mathcal{ALCK}$ knowledge base Σ and a set of $\mathcal{ALC}$ interpretation $\mathcal{W}$ such that*

- *$\mathcal{I}, \mathcal{W}$ satisfies all the axioms in Σ i.e., $\mathcal{I}, \mathcal{W} \models \Sigma$ for each $\mathcal{I} \in \mathcal{W}$.*

- *$\mathcal{W}$ is a maximal set with this property i.e., there is no set $\mathcal{W}'$ of $\mathcal{ALC}$ interpretation such that $\mathcal{W} \subset \mathcal{W}'$ and $\mathcal{I}, \mathcal{W}' \models \Sigma$.*

The second condition indeed induces nonmonotonic behavior in the reasoning since only maximal sets of interpretations are preferred as models. Hence, the semantics is preferential. As an important remark, note the correspondence between $\mathcal{ALCK}$ interpretations/models and the MKNF interpretations/models defined in Section 3.3.

In [DONINI et al. 1998], the notion of answer to a query is presented which we use in this work as well. Given an $\mathcal{ALCK}$ knowledge base Σ, an $\mathcal{ALCK}$ concept C and an individual name a, a query is of the form $\Sigma \models C(a)$?. The answer to a query Q of the form $\Sigma \models C(a)$? is denoted by $ans(Q)$ and is defined as following:

$$ans(Q) = \begin{cases} \textsf{YES} & \text{if } \Sigma \models C(a) \\ \textsf{NO} & \text{if } \Sigma \models \neg C(a) \\ \textsf{UNKNOWN} & \text{otherwise} \end{cases}$$

There are several scenarios where a DL enriched with the operator **K** is useful (cf. [DONINI et al. 1998, DONINI et al. 1992b]). We will come back to the usefulness of epistemic extension of DLs in Chapter 5.

4.2.2 Description Logic and Rules

When it comes to combining DLs with rules there are several choices for types of rules. Approaches to such combinations can be categorized into: monotonic rule (e.g., rules in datalog) and nonmonotonic rule approaches. Here by latter we mean a rule that allows for default negation 'not '. In this work, we focus only on the latter. For the monotonic rule approaches we refer the interested reader to [KRÖTZSCH 2010]. In the following we discuss some of the important hybrid formalisms (i.e., formalisms combining rules and description logic) that are capable of capturing defeasible reasoning of some kind. Note that a direct combination of rules (both monotonic and nonmonotonic) with DLs leads to undecidability. Different techniques have been introduced to acquire decidability by restricting the interaction of rules with DLs axioms in one way or the other.

$\mathcal{DL} + log$

In $\mathcal{DL}$+log [ROSATI 2006a], a knowledge base $\mathcal{K}$ is a tuple $(\Sigma, \mathcal{P})$ with Σ a DL knowledge base and $\mathcal{P}$ a set of rules of the form

$$H_1 \vee \cdots \vee H_l \leftarrow B_1, \ldots, B_m, not\ C_1, \ldots, not\ C_n$$

called disjunctive rules (due to the presence of disjunction in the head)[3]. Note that a normal logic program rule is a disjunctive rule with $l = 1$. Further the set of predicates is partitioned into DL and non-DL predicates. As a restriction, DL atoms (atoms constructed with a DL predicate) cannot occur in the scope of not . Note that a DL atom correspond to an assertion in DLs. Information exchanged between Σ and $\mathcal{P}$ is via these atoms.

[3]We refer to [PINTO 2011] for a details on disjunctive rules and disjunctive logic programs.

For the semantics of $\mathcal{DL}$+log, stable model approach [GELFOND and LIFSCHITZ 1991, GELFOND and LIFSCHITZ 1988] is adapted. Let Δ be a countably infinite set called *universe*. By $\mathcal{P}_\Delta$ we mean the program obtained by replacing every rule r in $\mathcal{P}$ with a set of rules by substituting variables in r with elements of Δ in all possible ways. Now let $\mathcal{I}$ be a first-order interpretation such that $\mathcal{I}_\Sigma$ and $\mathcal{I}_\mathcal{P}$ represent projection of $\mathcal{I}$ to Σ and $\mathcal{P}$ respectively. Then $\mathcal{I}$ is a model of $\mathcal{K}$ if and only if

- $\mathcal{I}_\Sigma$ is a model of Σ i.e., $\mathcal{I}_\Sigma \models \Sigma$
- $\mathcal{I}_\mathcal{P}$ is a stable model of $\mathcal{I}(\mathcal{P}_\Delta)$ where $\mathcal{I}(\mathcal{P}_\Delta)$ represents the program obtained by replacing every DL atom A in a rules of $\mathcal{P}_\Delta$ with true if $\mathcal{I} \models A$ and false if $\mathcal{I} \not\models A$.

In general $\mathcal{DL}$+log is undecidable until *week DL-safety* is ensured which requires variables in DL atom occurring as a head in some rule, to occur in a body of non-DL-atom as well. In other words, new information can be concluded from head of the rules only for the named individuals. In semantics of $\mathcal{DL}$+log, the assumption of standard name is taken which we will discuss in Section 4.2.3.

Disjunctive dl-programs

Similar to $\mathcal{DL}$+log, the formalism called *disjunctive dl-programs* is presented in [LUKASIEWICZ 2010b]. Syntactically, the difference is that now only function free atoms are considered i.e., a knowledge base in disjunctive dl-program is a pair $(\Sigma, \mathcal{P})$ where Σ is a DL knowledge base and $\mathcal{P}$ is a disjunctive program containing rules with no function symbols and no literal with classical negation.
For the semantics, an interpretation $\mathcal{I}$ is a set of ground atoms from the Herbrand base $\mathcal{H}_{\mathcal{P}_\Sigma}$of the ground program $\mathcal{P}_\Delta$ where Δ is a finite set of containing at least the named individuals. Now $\mathcal{I}$ is a *model* of Σ if the knowledge base $\Sigma \cup \mathcal{I} \cup \{\neg|A \in \mathcal{H}_{\mathcal{P}_\Sigma} \setminus \mathcal{I}\}$ is satisfiable in DL sense. We say $\mathcal{I}$ *is model of the knowledge base* $(\Sigma, \mathcal{P})$ if $\mathcal{I}$ is a model of Σ and $\mathcal{I} \models \mathcal{P}_\Delta$. Intuitively, we can think of the set of atoms $\mathcal{I}$ as the set of facts we belief satisfying $\mathcal{P}_\Delta$ and is

consistent with the information in the DL knowledge base Σ. Finally, $\mathcal{I}$ is a stable model of $(\Sigma, \mathcal{P}_\Delta)$ if it is $\mathcal{I}$ is a model of $(\Sigma, \mathcal{P}_\Delta)$ and it is minimal i.e., there is no $\mathcal{I}' \subseteq \mathcal{H}_{\mathcal{P}_\Delta}$ with $\mathcal{I}' \subset \mathcal{I}$ such that $\mathcal{I}'$ is a a model of $(\Sigma, \mathcal{P}_\Delta)$.

A nice characteristic of disjunctive dl-program is that a knowledge base $(\emptyset, \mathcal{P}) \models A$ if and only if $\mathcal{P} \models A$ i.e., if A is true in all the stable models of $\mathcal{P}$. This characteristic is usually called as *faithfulness* [MOTIK and ROSATI 2008].

dl-programs

Finally let us consider *dl-programs* presented in [EITER et al. 2008]. DL-programs combine normal generalized logic program with description logic knowledge bases. The rules of generalized logic programs are of the form

$$H \leftarrow B_1, \ldots, B_m, not\ C_1, \ldots, not\ C_n$$

where $H, B_1, \ldots, B_m, C_1, \ldots, C_n$ are atoms or classical negation ($\neg$) of atoms. In Section 3.2 of Chapter 3, we have seen normal program and defined the stable model semantics for such programs. Extending the notion of semantics to the generalized programs is straight-forward: treat $\neg A$ for an atom A as a new atom (by renaming it) and apply the same procedure of computing stable models. For details we refer to [GELFOND and LIFSCHITZ 1991, ALFERES and PEREIRA 1996].

The basic idea of dl-programs is to allow the logic program part for query the DL knowledge base. To this end, the notion of dl-queries is defined as following:

Definition 22. *A* dl query Q(t) *is one of the following [*EITER *et al. 2008]:*

- *a GCI or its negation, or*
- *a concept assertion of the form $C(t)$ or its negation $\neg C(t)$ for some concept C and a term T, or*
- *a (negative) role assertion $R(t, t')$ $(\neg R(t, t'))$ for a role R and terms t and t'*

Note terms are used instead of the individual names in the above definition. A *dl-atom* is of the form

$$DL[S_1 op_1 p_1, \ldots, S_m op_m p_m; Q](t)$$

where S_i's are DL-predicates, p_i's are non-DL-predicates, $op_i \in \{\uplus, \cup^-, \cap^-\}$ and $Q(t)$ is a dl-query. Intuitively, $\uplus$/$\cup^-$ increases/decreases S_i by the extension of p_i and $\cap^-$ limits S_i to p_i. Now a *dl-rule* is of the form

$$H \leftarrow B_1, \ldots, B_m, not\ C_1, \ldots, not\ C_n \qquad (*)$$

where besides literals dl-atoms are allowed for B_i's and C_i's. A *dl-program* is a pair $(\Sigma, \mathcal{P})$ where Σ is a DL knowledge base and $\mathcal{P}$ is a finite set of dl-rules.

For the semantics, the Herbrand base $\mathcal{H}_{\mathcal{P}}$ contains all the ground atoms formed from predicates in $\mathcal{P}$ and constants in Σ and $\mathcal{P}$. An interpretation I is a consistent subset of $\mathcal{H}_{\mathcal{P}}$. A (non dl-atom) literal A is true in I under Σ iff $A \in I$. We write $I \models_\Sigma A$. Similarly for a dl-atom

$$a = DL[S_1 op_1 p_1, \ldots, S_m op_m p_m; Q](c)$$

Now I satisfies a under Σ, written $I \models_\Sigma a$ iff $\Sigma \cup \bigcup_{i=1}^{m} A_i(I) \models Q(c)$ where

$$A_i(I) = \begin{cases} \{S_i(e) \mid p_i(e) \in I\} & \text{if } op_i = \uplus \\ \{\neg S_i(e) \mid p_i(e) \in I\} & \text{if } op_i = \cup^- \\ \{\neg S_i(e) \mid p_i(e) \notin I\} & \text{if } op_i = \cap^- \end{cases}$$

Note the satisfaction of the dl-atom a requires checking if the dl query $Q(c)$ is entailed by the DL knowledge base embedded with the additional information from the logic program part through non-dl predicates. These additional information are the ABox assertion $A_i(I)$. Now I satisfies a rule r of the form $(*)$ written $I \models_L r$ iff $I \models_\Sigma H$ whenever $I \models_\Sigma B_i$ for $1 \leq i \leq m$ and $I \not\models_\Sigma C_j$ for $1 \leq j \leq n$. Based on these notions and on the stable model semantics, the answer sets of a dl-programs are defined as below:

Definition 23. *Given a dl-program $\Sigma, \mathcal{P}$, let C represents the set of all constant occurring in $(\Sigma, \mathcal{P})$. Then for an interpretation I, the dl-program $P_{C|I}$ is obtained by*

- *deleting each rule r from $\mathcal{P}_C$ such that r contains an atom $\text{not } A$ with $I \models_\Sigma A$;*
- *deleting all the atoms of the form $\text{not } C$ from the rest of the rules.*

Now I is an answer set[4] *of $\Sigma, \mathcal{P}$ if $I \models_\Sigma r$ for each $r \in P_{C|I}$ and I is the least interpretation with this characteristic i.e., there is no interpretation I' such that $I' \subset I$ and $I' \models_L P_{C|I}$.*

In the next section we present some expressive formalisms capable of expressing most of the approaches mentioned above. One final remark, in [MEHDI et al. 2012] DLs and logic programs are integrated where the minimal hypothesis semantics [PINTO 2011] is considered for the logic program parts. However, the work is not directly comparable with our approach here.

4.2.3 Expressive Hybrid Formalisms

We now discuss some modern approaches to extending DLs with non-monotonic features. All these approaches are based somewhat on the logic of minimal knowledge and negation as failure (MKNF) [LIFSCHITZ 1991].

In Section 3.3 we discussed the the logic of minimal knowledge and negation as failure (or MKNF) which is an expressive enough to capture many of the non-monotonic formalisms. In [LIFSCHITZ 1991], Lifschitz presented the logic of *minimal knowledge and negation as failure* (MKNF). Thus MKNF is usually seen as a unifying framework for different non-monotonic logics as well as logic programs. As a fragment of first-order logic, DLs are inherently expressible by (first-order) MKNF. Consequently MKNF is a natural choice for extending DLs

[4] In [EITER et al. 2008], two notions of answer sets are defined: *weak answer sets* and *strong answer sets*. We limit ourselves to the strong answer sets only and believe that it is sufficient for understanding the basic idea of the approach.

with non-monotonic features and rules. Much work has been done in this direction including [KNORR et al. 2008, MOTIK and ROSATI 2008, MOTIK et al. 2006, DONINI et al. 2002, MOTIK and ROSATI 2006]. In the following we focus only on some relevant ones.

Description Logics of MKNF

In [DONINI et al. 2002] the DL $\mathcal{ALC}$ is extended with two modal operators **K** and **A**. This extension is called $\mathcal{ALC}_{\mathcal{NF}}$. The syntax is similar to that of $\mathcal{ALC}$ additionally considering the operator **A**.

Definition 24. *An $\mathcal{ALC}_{\mathcal{NF}}$ concept C is a given as*

$$C ::= \top|\bot|A|C \sqcap C|C \sqcup C|\neg C|\exists R.C|\forall R.C|\mathbf{K}C|\mathbf{A}C$$

where A is a concept name and R is an $\mathcal{ALC}_{\mathcal{NF}}$ role given by

$$R ::= P|\mathbf{K}R|\mathbf{A}R$$

with P a role name.

All the other notions (e.g., axioms, TBoxes , knowledge bases etc) can be analogously defined as in standard DLs. The semantics of $\mathcal{ALC}_{\mathcal{NF}}$ is similar to that of MKNF presented in Section 3.3 and hence assumes the common domain assumption and the rigid term assumption.

Definition 25. *An $\mathcal{ALC}_{\mathcal{NF}}$ interpretation is a triple $(\mathcal{I}, \mathcal{M}, \mathcal{N})$ where $\mathcal{I}$ is an ALC interpretation, and $\mathcal{M}$ and $\mathcal{N}$ are set of $\mathcal{ALC}$ interpretations.*

$\mathcal{ALC}_{\mathcal{NF}}$ concepts are interpreted similar to Definition 20 for concepts without the **A** operator. For a concept of the form $\mathbf{A}C$ we now have

$$(\mathbf{A}C)^{\mathcal{I},\mathcal{M},\mathcal{N}} = \bigcup_{\mathcal{J}\in\mathcal{N}} C^{\mathcal{J},\mathcal{M},\mathcal{N}}$$

and similarly for role $\mathbf{A}R$ we have

$$(\mathbf{A}R)^{\mathcal{I},\mathcal{M},\mathcal{N}} = \bigcup_{\mathcal{J}\in\mathcal{N}} R^{\mathcal{J},\mathcal{M},\mathcal{N}}$$

Note that the definition of interpretation of $\mathbf{K}$ and $\mathbf{A}$ are identical where M is considered in interpreting $\mathbf{K}$ and N is considered in interpreting $\mathbf{A}$. The notion of satisfiability of an axiom, a Tbox, an ABox and a knowledge base can be defined easily similar to standard DLs. Similar to $\mathcal{ALCK}$, models in $\mathcal{ALCK}_{\mathcal{NF}}$ are defined by preferring some sets of interpretations over the others.

Definition 26. *An $\mathcal{ALC}_{\mathcal{NF}}$ model of a $\mathcal{ALC}_{\mathcal{NF}}$ knowledge base Σ is a set of interpretations $\mathcal{M}$ such that*

- *$(\mathcal{I}, \mathcal{M}, \mathcal{M})$ satisfies Σ for each $\mathcal{I} \in \mathcal{M}$ and*
- *for each set of interpretations $\mathcal{M}'$, if $\mathcal{M} \subset \mathcal{M}'$ then there is an interpretation $\mathcal{J} \in \mathcal{M}'$ such that $(\mathcal{J}, \mathcal{M}', \mathcal{M})$ does not satisfy Σ i.e., we want $\mathcal{M}$ to be a maximal set.*

As demonstrated in [DONINI et al. 2002], $\mathcal{ALC}_{\mathcal{NF}}$ is expressive enough to reconstruct several features of framed-based systems which are not realizable in standard DLs.

First lets consider defaults. A default rule of the form

$$\frac{\varphi : \psi_1, \ldots, \psi_n}{\xi}$$

can be expressed by the following $\mathcal{ALC}_{NF}$ axiom

$$\mathbf{K}I \sqcap \mathbf{K}\varphi \neg\mathbf{A}\neg\psi_1 \sqcap \cdots \sqcap \neg\mathbf{A}\psi_n \sqsubseteq \mathbf{K}\xi$$

where $\mathbf{K}I$ enforces the applicability of the axiom only to named individuals to acquire the decidability of reasoning as in [BAADER and HOLLUNDER 1992].

Impotence of expressing integrity constraints is yet another short-come of standard DLs which is easily realizable in $\mathcal{ALC}_{\mathcal{NF}}$. For example, the following axiom

$$\mathbf{K}\text{Person} \sqsubseteq \mathbf{A}\text{Male} \sqcup \mathbf{A}\text{Female}$$

enforces that the gender of every known person shall be specified.

Formalism like $\mathcal{ALC}_{\mathcal{NF}}$ can be used for expressing closure of a concept or a role. In this work, we do not focus on these features.

Stable Model Based Hybrid MKNF Knowledge Base

In [MOTIK and ROSATI 2008], the notion of MKNF$^+$ knowledge bases is presented. The formalism is expressive enough to capture many of the existing approaches to non-monotonic extension of DLs and approaches to integrate DLs with rules (see [ROSATI 2006b, BAADER and HOLLUNDER 1992, CALVANESE et al. 2007, EITER et al. 2008, LUKASIEWICZ 2010a] etc). Thereby, the expressiveness of the underlying language is not restricted to some less expressive DLs rather it can be any fragment of first-order logic. We now formally define MKNF$^+$ knowledge bases.

Definition 27. *An* MKNF$^+$ KB $\mathcal{K}$ *is a pair* $(\mathcal{O}, \mathcal{P})$ *where* $\mathcal{O}$ *is a description logic KB and* $\mathcal{P}$ *a general logic program. Thus, by definition, every DL KB* $\mathcal{O}$ *is an MKNF*$^+$ *KB with empty program part.*

The semantics employed for MKNF$^+$ is the standard MKNF semantics as presented in Section 3.3 except that the standard name assumption is required to be satisfied by the interpretations under consideration. Let Δ the common universe of all the interpretations.

Definition 28. *A first-order interpretation* $\mathcal{I}$ *employs the standard name assumption if*

- *the universe* Δ *contains all the constants of a given signature and a countably infinite number of additional constants.*
- *each term is interpreted by itself i.e.,* $t^{\mathcal{I}} = t$

- $\doteq$ *is interpreted as a congruence relation*

Interpretations employing the standard name assumption preserve the satisfiability of formulas (see Proposition 3.2 of [MOTIK and ROSATI 2008]). The interpretation of formulas are defined similar to Definition 19 in Section 3.3.

A comparison between MKNF$^+$ approaches with the related one is presented in Section 7 of [MOTIK and ROSATI 2008] where it is shown how MKNF$^+$ knowledge bases can capture most of the approaches to non-monotonic reasoning in DLs. In Chapter 9 we compare it with our approach in detail.

Well-founded Model Based Hybrid MKNF Knowledge Base

Similar to the MKNF$^+$ knowledge bases, in [KNORR et al. 2008] the notion of hybrid knowledge bases based on MKNF is presented but instead of the stable model semantics, the authors employ the well-founded semantics [GELDER et al. 1991]. Hence the notion of three-valued MKNF structure is presented. A further description of the semantics is out of the scope of this work.

With this we conclude the second part of the thesis. In the next part, we present our contributions by answering the challenges we raised in Section 1.4.

Part II

Epistemic Queries and Description Logic

Chapter 5

Epistemic DLs as Query Languages

The inadequacy of standard KR languages for querying a knowledge base was initially discussed by Hector J. Levesque in [LEVESQUE 1984] where he suggests to incorporate the epistemic operator **K** into the query language. In practice one encounters many scenarios where a richer query language is highly desirable. In the context of databases, Reiter motivates the need for a more expressive language via several examples [REITER 1992]. The prime situation are the ones where some sort of disjunction or existential quantifies have been used to model incomplete information. In such cases, querying a given knowledge base usually results in an unknown answer. For example, given the information that John is either a graduate or undergraduate student, on asking if John a graduate student is, we get an unknown answer. This is so as the only information stated about John is via the disjunction and unlike non-monotonic logic, absence of information does not allow us concluding the falsehood of the information. Similarly, via existential quantification one can say there is a course which John attends. Again we don't know anything about the course. Thus no conclusion can be drawn about a concrete course that John is attending unless stated so.

A somewhat similar case we have in description logic. There are several places where we are interested in decision making when only little information is available. In the following we discuss several examples emphasizing the need of a richer language. Indeed, we argue why $\mathcal{SROIQK}$ shall be used rather than $\mathcal{SROIQ}$ to get conclusions that can help in decision making.

In [Donini et al. 1998, Donini et al. 1992b], the added value of using $\mathcal{ALCK}$ as querying language for $\mathcal{ALC}$ knowledge bases has been discussed. The authors demonstrated several kinds of queries easily realizable using $\mathcal{ALCK}$ as the query language. Such queries cannot be formulated in $\mathcal{ALC}$. In the following we consider similar examples thus advocating the use of $\mathcal{SROIQK}$ rather than $\mathcal{SROIQ}$ in many scenarios of an application domain. For these examples, we consider the wine ontology[1] which is an ontology formulated in OWL specifying different wines along with their relationships.

5.1 Incompleteness of Information

As a consequence of the open-world assumption, modeling incomplete information is one of the fundamental capabilities of OWL. One way of modeling incompleteness is via use of the existential quantification. Let Σ represent the Wine ontology. Consider the following query.

$$\Sigma \models \exists \mathsf{locatedIn}.\mathsf{Region}(\mathsf{chianti})?$$

The query asks if according to Σ there is a region as the origin of the wine chianti. In the wine ontology we have the axioms $\mathsf{Wine} \sqsubseteq \mathsf{locatedIn}.\mathsf{Region}$ and $\mathsf{Wine}(\mathsf{chianti})$. Thus the answer to this query is YES as it has been explicitly asserted regardless of if the region is known or unknown. But the following query

$$\Sigma \models \exists \mathbf{K}\mathsf{locatedIn}.\mathbf{K}\mathsf{Region}(\mathsf{chianti})?$$

asks if this region is known. On assuming that all we have in Σ about chianti is the assertion $\exists \mathsf{locatedIn}.\mathsf{Region}(\mathsf{chianti})$ or Σ entails $\exists \mathsf{locatedIn}.\mathsf{Region}(\mathsf{chianti})$, the answer to the above epistemic query is NO. Now suppose we later come to know that the origin of chianti is some Italian regions i.e., suppose there is a region r such that we have $\Sigma \models \mathsf{locatedIn}(\mathsf{chianti}, \mathsf{r})$ and $\Sigma \models \mathsf{ItalianRegion}(\mathsf{r})$. If we ask the same epistemic query again, we now get YES as an answer. This is so since

[1] http://www.w3.org/TR/owl-guide/wine.rdf

we now know about the exact origin of chianti which is the region r known to be an Italian region.

Lets consider a query dealing with disjunction. In Σ we have

$$\mathsf{Fruit} \equiv \mathsf{NonSweetFruit} \sqcup \mathsf{SweetsFruit}$$

Now given that $\mathsf{CFGrape}(\mathsf{CabernetFrancGrape})$ is a member of class Fruit, the answer to the query

$$\Sigma \models \mathsf{NonSweetFruit} \sqcup \mathsf{SweetFruit}(\mathsf{CFGrape})?$$

is YES. However as it is not known if CFGrape is a sweet or non sweet fruit, asking query

$$\Sigma \models \mathbf{K}\mathsf{NonSweetFruit} \sqcup \mathbf{K}\mathsf{SweetFruit}(\mathsf{CFGrape})?$$

leads to NO.

5.2 Local Closed-world Reasoning

Another use of **K**-operator is reasoning in a locally *closed-world*: employing close-world assumption for the concepts and classes preceded by the **K**-operator. Note that in Σ we do not have the assertion that Wine is disjoint from Region. Hence there are models of Σ where $\mathsf{Region}(chianti)$ holds. Nevertheless, **K**Region(chianti) does not hold. Thus the answer to the query

$$\Sigma \models \mathbf{K}\mathsf{Region(r)}?$$

is YES only for those individuals r for which either it is explicitly stated in Σ or provable via deduction that r is a region. Lets consider the behavior of universal quantification in a closed-world. In Σ the role producesWine relates a wine to its maker. For example, we have bancroft as a maker of bancroftChardonnay i.e., we have the assertion

$$\text{producesWine(bancroft, bancroftChardonnay)}$$

in Σ. Further Σ also states that bancroftChardonnay is a white table wine i.e., it contains the assertion

$$\text{WhiteTableWine(bancroftChardonnay)}$$

Now consider the following query

$$\Sigma \models \forall \text{producesWine.WhiteTableWine}(\text{bancroft})$$

asking if bancroft makes white table wines only. Note that there are models of Σ according to which bancroft produces only white table wine. Meanwhile there are models stating bancroft as a maker of some other kind of wines. Hence the answer to this query is simply UNKNOWN. In other words, in standard DLs we cannot restrict the universal quantification to quantify only over known relationships. This, however, is easily expressible with the **K** operator. In this case the query would be

$$\Sigma \models \forall \mathbf{K}\text{producesWine}.\mathbf{K}\text{WhiteTableWine}(\text{bancroft})$$

and thus it is easy to see that the answer to this query is YES since only one wine, namely bancroftChardonnay is stated to be produced by bancroft and we know that bancroftChardonnay is a white table wine.

From the above examples, we can observe that the epistemic operator allows us to view certain concept or properties under the closed-world assumption. Since the assumption is employed to certain concept or roles, this is usually called as the local closed-world assumption. The advantages of local closed-world reasoning has been discussed in [GRIMM et al. 2006, GRIMM and HITZLER 2007] in the context of semantic match-making of web services.

Standard DL reasoners like Racer, Pellet, FacT++, etc do not support the epistemic query answering task. We thus need to device a new algorithm for this purpose. In this work, we present a procedure for answering epistemic queries while still using the standard DL reasoners.

5.3 Integrity Constraint Checking

According to Reiter [REITER 1992], integrity constraints (ICs) are of epistemic nature in the sense that they are not statements about the world represented by a KB rather they are statements about what the KB is required to know i.e., what shall be known given the KB. Formally,

Definition 29. *An integrity constraints is a property P that a given knowledge base needs to satisfy.*

Reiter suggests formalizing such constraints as epistemic queries. Then checking the integrity constraint is just to test if the knowledge base entails the query. We now demonstrate integrity constraint checking via an example. Suppose we want to enforce that we know the degree of sugar of every wine we know. This constraint can easily be expressed by the following axiom:

$$\alpha := \mathbf{K}\mathsf{Wine} \sqsubseteq \exists \mathbf{K}\mathsf{hasSugar}\{\mathsf{Dry}\} \sqcup \exists \mathbf{K}\mathsf{hasSugar}.\{\mathsf{OffDry}\} \sqcup \exists \mathbf{K}\mathsf{hasSugar}.\{\mathsf{Sweet}\}$$

Now if $\Sigma \models \alpha$, then we are sure that for every named individual which is known to be wine, we know (i.e. it can be logically derived) its degree of sugar as well. Note that in [DONINI et al. 2002], the same constraint would be formulated as

$$\mathbf{K}\mathsf{Wine} \sqsubseteq \exists \mathbf{A}\mathsf{hasSugar}\{\mathsf{Dry}\} \sqcup \exists \mathbf{A}\mathsf{hasSugar}.\{\mathsf{OffDry}\} \sqcup \exists \mathbf{A}\mathsf{hasSugar}\text{-}\{\mathsf{Sweet}\}$$

where $\mathbf{A}$ is an operator for representing default assumptions. The difference is that in our case we can only enforce the constraint via query checking and thus can not formulate them within the knowledge base whereas in formalism presented in [DONINI et al. 2002], constraints can be formulated within the knowledge base as well.

Chapter 6

Classical Epistemic Semantics

The extension of $\mathcal{ALC}$ with the $\mathbf{K}$ operator, called $\mathcal{ALCK}$, has already been introduced in Section 4.2.1. In this chapter we extend this notion to the description logic $\mathcal{SROIQ}$. Note that $\mathcal{SROIQ}$ is one of the most expressive DL and is the foundation of OWL 2 DL. For the semantics we adopt the approach presented in [DONINI et al. 1998] and thus call it the *classical (approach to) semantics*. We will see later why we have to restrict the classical semantics to DLs up to $\mathcal{SRIQ} \setminus \mathcal{U}$, which is the DL $\mathcal{SROIQ}$ without the nominals and the universal role (see [KRÖTZSCH 2010] for naming convention in DLs).

6.1 Epistemic Extension of $\mathcal{SROIQ}$

The extension of $\mathcal{SROIQ}$ with $\mathbf{K}$ is called *autoepistemic* $\mathcal{SROIQ}$. We represent it by $\mathcal{SROIQK}$. Syntactically $\mathcal{SROIQK}$ is very much similar to $\mathcal{SROIQ}$ except that we now allow for $\mathbf{K}$ operator to occur in concepts.

Definition 30. *We first define $\mathcal{SROIQK}$ roles as follows:*

$$R := P \mid \mathbf{K}P$$

where P is a $\mathcal{SROIQ}$ role. Further we call $R = \mathbf{K}P$ a simple role if P is a simple role.

A $\mathcal{SROIQK}$ concept C is defined as

$$\begin{aligned} C ::= \quad & A \mid \neg C \mid \mathbf{K}C \mid C \sqcap C \mid C \sqcup C \mid \exists R.\textit{Self} \mid \\ & \exists R.C \mid \forall R.C \mid \leq nS.C \mid \geq nS.C \mid \{a_1, \ldots, a_n\} \end{aligned}$$

where A is a concept name, R is a $\mathcal{SROIQK}$ role, S is a simple $\mathcal{SROIQK}$ role, $a_1, \ldots, a_n$ are individual names, and n is a non-negative integer.

Note that by definition every $\mathcal{SROIQ}$ concept is a $\mathcal{SROIQK}$ concept. The notions of knowledge base, TBox, RBox, ABox and their respective axioms can be extended from $\mathcal{SROIQ}$ to $\mathcal{SROIQK}$ in the obvious way[1]. For example, the following are $\mathcal{SROIQK}$ concepts:

$$\exists \mathbf{K}\textsf{hasMaker}.\mathbf{K}\textsf{Region} \qquad \mathbf{K}\textsf{WhiteWine} \qquad \mathbf{K}\textsf{hasSugar}.\top$$

Following the approach of [DONINI et al. 1998], in the next section we now define the semantics for $\mathcal{SROIQK}$.

Classical Semantics

Autoepistemic DLs are basically fragments of first-order modal logic [FITTING and MENDELSOHN 1998] with a single modal operator **K** (similar to the necessity operator) employing possible world semantics. In Section 4.2.1, we have seen that we need to cope with several conceptual issues in defining a uniform and intuitive semantics for extended DLs. The classical semantics take the two assumptions namely:

1. *Common Domain Assumption (CDA)*: All interpretations are defined over a fixed countably infinite domain Δ.

2. *Rigid Term Assumption (RTA)*: For all interpretations, the mapping from individual names to domain elements is fixed (it is just the identity function).

These assumptions are imposed in order to ensure that (sets of) domain elements can be referred to and dealt with uniformly in a cross-domain manner. Further, without loss of generality we assume $\Delta := N_I \cup \mathbb{N}$. We adapt the possible world

[1] When clear from the context, we sometimes call $\mathcal{SROIQK}$ concepts as epistemic concepts and similar is the case for notions like role, axioms, TBox, etc.

approach for the classical semantics of $\mathcal{SROIQK}$. To this end we define the notion of epistemic interpretations where we provide a "context" of relevant models which are inspected whenever the extension of an epistemic concept or role is to be determined.

Definition 31. *An* epistemic interpretation *for $\mathcal{SROIQK}$ is a pair $(\mathcal{I}, \mathcal{W})$ where $\mathcal{I}$ is a $\mathcal{SROIQ}$ interpretation and $\mathcal{W}$ is a set of $\mathcal{SROIQ}$ interpretations, where $\mathcal{I}$ and all of $\mathcal{W}$ have the same infinite domain Δ with $N_I \subset \Delta$. For given $\mathcal{SROIQK}$ concepts C and D, role R and simple role S, the interpretation function $\cdot^{\mathcal{I},\mathcal{W}}$ is then defined as follow:s*

$$
\begin{aligned}
a^{\mathcal{I},\mathcal{W}} &= a \quad \textit{for } a \in N_I \\
X^{\mathcal{I},\mathcal{W}} &= X^{\mathcal{I}} \;\; \textit{for } X \in N_C \cup N_R \cup \{\top, \bot\} \\
\{a_1,\ldots,a_n\}^{\mathcal{I},\mathcal{W}} &= \{a_1,\ldots,a_n\} \\
(\boldsymbol{K}C)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J}\in\mathcal{W}}(C^{\mathcal{J},\mathcal{W}}) \\
(\boldsymbol{K}R)^{\mathcal{I},\mathcal{W}} &= \textstyle\bigcap_{\mathcal{J}\in\mathcal{W}}(R^{\mathcal{J},\mathcal{W}}) \\
(C \sqcap D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cap D^{\mathcal{I},\mathcal{W}} \\
(C \sqcup D)^{\mathcal{I},\mathcal{W}} &= C^{\mathcal{I},\mathcal{W}} \cup D^{\mathcal{I},\mathcal{W}} \\
(\neg C)^{\mathcal{I},\mathcal{W}} &= \Delta \setminus C^{\mathcal{I},\mathcal{W}} \\
(\exists S.\mathsf{Self})^{\mathcal{I},\mathcal{W}} &= \{x \mid (x,x)\in R^{\mathcal{I},\mathcal{W}}\} \\
(\exists R.C)^{\mathcal{I},\mathcal{W}} &= \{x \mid \exists y.(x,y)\in R^{\mathcal{I},\mathcal{W}} \wedge y\in C^{\mathcal{I},\mathcal{W}}\} \\
(\forall R.C)^{\mathcal{I},\mathcal{W}} &= \{x \mid \forall y.(x,y)\in R^{\mathcal{I},\mathcal{W}} \rightarrow y\in C^{\mathcal{I},\mathcal{W}}\} \\
(\leqslant nS.C)^{\mathcal{I},\mathcal{W}} &= \{x \mid \#\{y\in C^{\mathcal{I},\mathcal{W}} \mid (x,y)\in R^{\mathcal{I},\mathcal{W}}\} \leq n\} \\
(\geqslant nS.C)^{\mathcal{I},\mathcal{W}} &= \{x \mid \#\{y\in C^{\mathcal{I},\mathcal{W}} \mid (x,y)\in R^{\mathcal{I},\mathcal{W}}\} \geq n\}
\end{aligned}
$$

From the above one can see that $\mathbf{K}C$ is interpreted as the set of objects that are in the extension of C under every interpretation in $\mathcal{W}$. This also makes clear why the common domain and rigid term assumption have to be imposed; otherwise the respective extension intersections would be empty. Note that the rigid term assumption implies the unique name assumption (UNA) i.e., for any epistemic interpretation $\mathcal{I} \in \mathcal{W}$ and for any two distinct individual names a and b we have that $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.

Similar to the standard DLs, the notion of satisfaction of a $\mathcal{SROIQK}$ (TBox, RBox or ABox) axiom α by an epistemic interpretation $(\mathcal{I}, \mathcal{W})$ is defined as in Table 6.1.

Axiom α	$(\mathcal{I}, \mathcal{W}) \models \alpha$, if
$C \sqsubseteq D$	$C^{\mathcal{I},\mathcal{W}} \subseteq D^{\mathcal{I},\mathcal{W}}$
$R_1 \circ \cdots \circ R_n \sqsubseteq R$	$R_1^{\mathcal{I},\mathcal{W}} \circ \cdots \circ R_n^{\mathcal{I},\mathcal{W}} \subseteq R^{\mathcal{I},\mathcal{W}}$
Dis(S,T)	$S^{\mathcal{I},\mathcal{W}} \cap T^{\mathcal{I},\mathcal{W}} = \emptyset$
$C(a)$	$a^{\mathcal{I},\mathcal{W}} \in C^{\mathcal{I},\mathcal{W}}$
$R(a,b)$	$(a^{\mathcal{I},\mathcal{W}}, b^{\mathcal{I},\mathcal{W}}) \in R^{\mathcal{I},\mathcal{W}}$
$a \doteq b$	$a^{\mathcal{I},\mathcal{W}} = b^{\mathcal{I},\mathcal{W}}$
$a \not\doteq b$	$a^{\mathcal{I},\mathcal{W}} \neq b^{\mathcal{I},\mathcal{W}}$

Table 6.1: Classical Semantics of $\mathcal{SROIQK}$ axioms

Note that we write $(\mathcal{I}, \mathcal{W}) \models \alpha$ whenever $(\mathcal{I}, \mathcal{W})$ satisfies α. Given a TBox $\mathcal{T}$ we say $(\mathcal{I}, \mathcal{W})$ satisfies $\mathcal{T}$ iff $(\mathcal{I}, \mathcal{W})$ satisfies all the axioms in $\mathcal{T}$, in which case we write $(\mathcal{I}, \mathcal{W}) \models \mathcal{T}$. The satisfaction of an RBox and an ABox in $(\mathcal{I}, \mathcal{W})$ can be defined analogously. Similarly $(\mathcal{I}, \mathcal{W})$ satisfies a knowledge base $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$, written as $(\mathcal{I}, \mathcal{W}) \models \Sigma$ iff it satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$.

Next we define the notion of modelhood of an epistemic interpretation. Like in MKNF we take a preferential semantics approach as well. In other words, not every epistemic interpretation satisfying all axioms of a knowledge base is a model. Indeed, we prefer certain interpretations over the others thus acquiring a non-monotonic reasoning behavior for $\mathcal{SROIQK}$.

Definition 32. *Given a $\mathcal{SROIQK}$ knowledge base $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$, a set $\mathcal{M}$ of interpretations is called* epistemic model *of Σ if*

- $(\mathcal{I}, \mathcal{M}) \models \Sigma$ *for each $\mathcal{I} \in \mathcal{M}$, and*
- *for each set $\mathcal{M}'$ of interpretations such that $\mathcal{M} \subset \mathcal{M}'$ there is an interpretation $\mathcal{J} \in \mathcal{M}'$ such that $(\mathcal{J}, \mathcal{M}') \not\models \Sigma$. In other words, $\mathcal{M}$ needs to be a maximal set.*

Based on the definition above we now define the entailment of axioms from a $\mathcal{SROIQK}$ knowledge base.

Definition 33. *A knowledge base* Σ epistemically entails[2] *an axiom* α *(written* $\Sigma \mid\models \alpha$*), if for every epistemic model* $\mathcal{M}$ *of* Σ*, we have that for each* $\mathcal{I} \in \mathcal{M}$*, the epistemic interpretation* $(\mathcal{I}, \mathcal{W})$ *satisfies* α*.*

Unlike the standard entailment, we use the symbol $\mid\models$ to represent the entailment relation under the classical semantics. The *epistemic entailment problem* is then to check if a given knowledge base epistemically entails an axiom.

Note that by definition every $\mathcal{SROIQ}$ knowledge base Σ is a $\mathcal{SROIQK}$ knowledge base and thus it has a unique epistemic model which is the set of all models of Σ having infinite domain and satisfying the unique name assumption.

Fact 1. *A given* $\mathcal{SROIQ}$ *knowledge base* Σ *(***K***-free) has up to renaming only one unique epistemic model. We denote this model by* $\mathcal{M}(\Sigma)$*.*

In Section 2.2, we have already seen the notion of queries in $\mathcal{SROIQ}$. A very similar notion can be defined for $\mathcal{SROIQK}$.

Definition 34. *For a given knowledge base* Σ*, an* epistemic query Q *(to* Σ*) is of the form* $\Sigma \mid\models \alpha$? *where* α *is an axiom formulated in some epistemic DL. Then the answer to an epistemic query, denoted by* $ans(Q)$*, is defined as*

$$ans(Q) = \begin{cases} \textsf{YES} & \text{if } \Sigma \mid\models \alpha \\ \textsf{NO} & \text{if } \Sigma \mid\models \neg\alpha \\ \textsf{UNDEFINED} & \text{otherwise} \end{cases}$$

Different languages can be used for formalizing a knowledge base and a query. In this work, we call the formalism used for expressing knowledge bases as the *knowledge base language* and the formalism used for expressing queries as the *query language*. Note that query answering requires entailment checking. In the next section, we present a technique of entailment checking reasoning in $\mathcal{SROIQK}$.

[2] We drop the word ”epistemically” whenever it is clear from the context.

6.2 Epistemic Query Answering under the Classical Semantics

We now present a procedure for answering epistemic queries. More specifically, we devise an approach for solving the epistemic entailment problem based on techniques for non-epistemic standard reasoning. The idea is to translate an epistemic query in to a standard query via intermediate reasoner calls. For example, suppose the epistemic concept $\mathbf{K}C$ occurs as a sub-concept within a given concept. We compute all individuals known to be in the extension of C.. In other words, we collect individuals $a_1, \ldots, a_n$ such that the given knowledge base entails $C(a_i)$ for $1 \leq i \leq n$. We then replace any instance of $\mathbf{K}C$ by the one-of concept $\{a_1, \ldots, a_n\}$. Similarly in a recursive fashion we remove every epistemic sub-concept within a given concept with an semantically equivalent $\mathbf{K}$-free concept. Consequently, we get a $\mathbf{K}$-free concept. This enables us to check whether a $\mathcal{SROIQK}$ axiom is entailed from a standard DL knowledge base, using a standard DL reasoner. An important point here is that for answering queries formalized in an epistemic extension of a DL, the underlying reasoner needs to support the one-of construct. For example, if a query is formulated in $\mathcal{ALCK}$, then we translate it in $\mathcal{ALCO}$ query. As a result we get a $\mathbf{K}$-free query and thus any standard reasoner for $\mathcal{ALCO}$ can answer $\mathcal{ALCK}$ queries.

6.3 DL $\mathcal{SRIQ}\backslash U$ as a Query Language

As already mentioned we distinguish the (epistemic) querying language from the (non-epistemic) knowledge base language similar to the strategies pursued in [Reiter 1992], [Levesque 1984] and [Calvanese et al. 2007] etc. For the classical semantics of $\mathcal{SROIQK}$, we consider the epistemic entailment problem of a $\mathcal{SROIQK}$ axiom α from a $\mathcal{SRIQ}\backslash U$ knowledge base Σ, where $\mathcal{SRIQ}\backslash U$ is defined as $\mathcal{SROIQ}$ excluding nominals and the universal role. The reason for this choice will become clear in the subsequent chapters.

The basic, rather straightforward idea to answer an epistemic query Q to a given knowledge base Σ is to disassemble the axioms in Q, query for the named individuals contained in extensions for every sub-expression preceded by $\mathbf{K}$, and use the results to rewrite Q into one that is free of $\mathbf{K}$s. But first note that as a consequence of the rigid name assumption, every $\mathcal{I} \in \mathcal{M}(\Sigma)$ satisfies the condition that different individual names are interpreted by different individuals (this condition per se is commonly referred to as the *unique name assumption*), where $\mathcal{M}(\Sigma)$ is the unique epistemic model of Σ. In order to enforce this behavior (which is not ensured by the non-epistemic standard DL semantics) we have to explicitly axiomatize this condition.

Definition 35. *Given a $\mathcal{SRIQ}\backslash U$knowledge base Σ, we denote by Σ_{UNA} the knowledge base $\Sigma \cup \{a \not\approx b \mid a, b \in N_I, a \neq b\}$.*

The following is an immediate consequence from the definition of the UNA.

Fact 2. *The set of models of Σ_{UNA} is exactly the set of those models of Σ that satisfy the unique name assumption.*

In the classical semantics, the common domain assumption constrains the domain of the interpretations to a common infinite set. However, standard DL reasoning as performed by DL reasoners adheres to a semantics that allows for both finite and infinite models. We therefore restrict the knowledge base language to $\mathcal{SRIQ}\backslash U$, since we will show that considering only infinite models does not affect results. The justification for this consideration is that for any finite interpretation we find an infinite one which satisfies the same set of axioms and hence will make up for the loss of the former. In the following we prove it formally.

Definition 36. *For any $\mathcal{SRIQ}\backslash U$ interpretation $\mathcal{I}$, the* lifting *of $\mathcal{I}$ to ω is the interpretation $\mathcal{I}_\omega$ defined as follows:*

- $\Delta^{\mathcal{I}_\omega} := \Delta^{\mathcal{I}} \times \mathbb{N}$,
- $a^{\mathcal{I}_\omega} := \langle a^{\mathcal{I}}, 0\rangle$ *for every* $a \in N_I$,
- $A^{\mathcal{I}_\omega} := \{\langle x, i\rangle \mid x \in A^{\mathcal{I}}$ *and* $i \in \mathbb{N}\}$ *for each concept name* $A \in N_C$,

- $r^{\mathcal{I}_\omega} := \{(\langle x,i\rangle, \langle x',i\rangle) \mid (x,x') \in r^{\mathcal{I}}$ *and* $i \in \mathbb{N}\}$ *for every role name* $r \in N_R$.

Note that $\mathcal{I}_\omega$ is infinite as $\Delta^{\mathcal{I}_\omega} = \Delta^{\mathcal{I}} \times \mathbb{N}$ where $\mathbb{N}$ is the set of non-negative integers. Next we prove that the extension of a concept in an interpretation $\mathcal{I}$ agrees with that in the interpretation $\mathcal{I}_\omega$ in the follow sense:

Lemma 1. *For all* $\langle x,i\rangle \in \Delta^{\mathcal{I}_\omega}$ *and all* $\mathcal{SRIQ}\backslash U$ *concepts* C *holds that* $\langle x,i\rangle \in C^{\mathcal{I}_\omega}$ *if and only if* $x \in C^{\mathcal{I}}$.

Proof. The proof is by the induction on the structure of C:

- For the atomic concept, $\top$ or $\bot$ it follows immediately from the definition of $\mathcal{I}_\omega$.

- Let $C = \neg D$ with D a concept. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (\neg D)^{\mathcal{I}}$
  $\Leftrightarrow x \notin D^{\mathcal{I}}$
  $\Leftrightarrow \langle x,i\rangle \notin D^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ (Induction)
  $\Leftrightarrow \langle x,i\rangle \in (\neg D)^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$.

- Let $C = C_1 \sqcap C_2$ where C_1 and C_2 are concepts. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (C_1 \sqcap C_2)^{\mathcal{I}}$
  $\Leftrightarrow x \in C_1^{\mathcal{I}}$ and $x \in C_2^{\mathcal{I}}$
  $\Leftrightarrow \langle x,i\rangle \in C_1^{\mathcal{I}_\omega}$ and $\langle x,i\rangle \in C_2^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ (Induction)
  $\Leftrightarrow \langle x,i\rangle \in (C_1 \sqcap C_2)^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$.

- Let $C = \exists R.D$ for a role R and concept D. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (\exists R.D)^{\mathcal{I}}$
  $\Leftrightarrow$ there is a $y \in \Delta^{\mathcal{I}}$ such that $(x,y) \in R^{\mathcal{I}}$ and $y \in D^{\mathcal{I}}$
  $\Leftrightarrow$ there is $\langle y,i\rangle \in \Delta^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ with $(\langle x,i\rangle, \langle y,i\rangle) \in R^{\mathcal{I}_\omega}$ and $\langle y,i\rangle \in D^{\mathcal{I}_\omega}$ (Definition 36 and Induction)
  $\Leftrightarrow \langle x,i\rangle \in (\exists R.D)^{\mathcal{I}_\omega}$

- Let $C = \leq nR.D$ for a role R and concept D. For any $x \in \Delta^{\mathcal{I}}$ we have that
  $x \in (\leq nR.D)^{\mathcal{I}}$

$\Leftrightarrow$ there are pair-wise distinct $y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that we haves $(x, y_l) \in R^{\mathcal{I}}$ and $y_l \in D^{\mathcal{I}}$ for $1 \leq l \leq n$
$\Leftrightarrow$ there are $\langle y_l, i\rangle \in \Delta^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$ with $(\langle x_l, i\rangle, \langle y_l, i\rangle) \in R^{\mathcal{I}_\omega}$ and $\langle y_l, i\rangle \in D^{\mathcal{I}_\omega}$ for $1 \leq l \leq n$. This steps follows from Definition 36 and Induction. Further it also follows from the definition that $\langle y_1, i\rangle, \ldots, \langle y_n, i\rangle$ are pair-wise distinct for $i \in \mathbb{N}$. Thus by semantics the final step is equivalent to $\langle x, i\rangle \in (\leq nR.D)^{\mathcal{I}_\omega}$.

- The remaining cases can be proved analogously.

□

We now prove that we can indeed drop off finite interpretations from consideration when dealing with epistemic models of a $\mathcal{SRIQ}\backslash U$ knowledge base.

Lemma 2. *Let Σ be a $\mathcal{SRIQ}\backslash U$ knowledge base. For any interpretation $\mathcal{I}$ we have that*

$$\mathcal{I} \models \Sigma \text{ if and only if } \mathcal{I}_\omega \models \Sigma.$$

Proof. First we note that from the definition of $\mathcal{I}_\omega$, for any $\mathcal{SRIQ}\backslash U$ role $R \in \mathbf{R}$ and $(\langle x, i\rangle, \langle y, i'\rangle) \in \Delta^{\mathcal{I}_\omega}$ with $i, i' \in \mathbb{N}$ we have that $(\langle x, i\rangle, \langle y, i'\rangle) \in R^{\mathcal{I}_\omega}$ if an only if $(x, y) \in R^{\mathcal{I}}$ and $i = i'$ for an interpretation $\mathcal{I}$. Now for any RIA $R_1 \circ \ldots R_n \sqsubseteq R$ we have that:
$\mathcal{I} \models R_1 \circ \ldots R_n \sqsubseteq R$
$\Leftrightarrow \mathcal{I} \models R_1^{\mathcal{I}} \circ \ldots R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$, whenever $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ then $(x_0, x_n) \in R^{\mathcal{I}}$
$\Leftrightarrow$ for any $x_0, \ldots, x_n \in \Delta^{\mathcal{I}}$ and any $j \in \mathbb{N}$, whenever $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$ for $1 \leq i \leq n$ then $(\langle x_0, j\rangle, \langle x_n, j\rangle) \in R^{\mathcal{I}_\omega}$
$\Leftrightarrow \mathcal{I}_\omega \models R_1 \circ \ldots R_n \sqsubseteq R$.
The second last equivalence holds as by definition $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ for $1 \leq i \leq n$ and any non-negative integer j implies that $(\langle x_{i-1}, j\rangle, \langle x_i, j\rangle) \in R_i^{\mathcal{I}_\omega}$. Similarly $(\langle x_{i-1}, j_{i-1}\rangle, \langle x_i, j_i\rangle) \in R_i^{\mathcal{I}_\omega}$ for $1 \leq i \leq n$ implies that $(x_{i-1}, x_i) \in R_i^{\mathcal{I}}$ and that $j_{i-1} = j_i$. The same holds for the role R.

Now consider any role characteristic $\mathsf{Dis}(S,T)$ with simple roles S and T. We prove the left to right direction only. The other direction can be proved analogously. Let $\mathcal{I} \models \mathsf{Dis}(S,T)$. It means that $(x,y) \in S^{\mathcal{I}}$ if and only if $(x,y) \notin T^{\mathcal{I}}$ for all $x,y \in \Delta^{\mathcal{I}}$. Now by Lemma 1, $(\langle x,j\rangle,\langle y,j\rangle) \in S^{\mathcal{I}_\omega}$ if and only if $(\langle x,j\rangle,\langle y,j\rangle) \notin S^{\mathcal{I}_\omega}$ for any $j \in \mathbb{N}$ and $x,y \in \Delta^{\mathcal{I}}$. This by semantic implies that $\mathcal{I}_\omega \models \mathsf{Dis}(S,T)$.
Consequently we get that $\mathcal{I}$ satisfies a role hierarchy $\mathcal{R}$ if and only if so does $\mathcal{I}_\omega$.

Now for any GCI $C \sqsubseteq D$, Lemma 1 implies that for any interpretation $\mathcal{I}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ if and only if $C^{\mathcal{I}_\omega} \subseteq D^{\mathcal{I}_\omega}$. Thus we get that for any TBox $\mathcal{T}$, $\mathcal{I} \models \mathcal{T}$ if and only if $\mathcal{I}_\omega \models \mathcal{T}$.

Finally for an ABox $\mathcal{A}$ we show that for each assertion in $\alpha \in \mathcal{A}$, $\mathcal{I} \models \alpha$ if and only if $\mathcal{I}_\omega \models \alpha$.

- α is of the form $C(a)$ for a concept C and individual name a:
  Now for an interpretation $\mathcal{I}$ it follows from the definition of $\mathcal{I}_\omega$ that $a^{\mathcal{I}_\omega} = (a^{\mathcal{I}},0)$. As we have already shown that $a^{\mathcal{I}} \in C^{\mathcal{I}}$ if and only if $(a^{\mathcal{I}},i) \in C^{\mathcal{I}_\omega}$ for $i \in \mathbb{N}$. Hence we get that $a^{\mathcal{I}} \in C^{\mathcal{I}}$ if and only if $(a^{\mathcal{I}},0) \in C^{\mathcal{I}_\omega}$.

- α is of the form $R(a,b)$ for a role R and individual names a and b:
  Let $\mathcal{I}$ be some interpretation such that $\mathcal{I} \models R(a,b)$. By semantics it is the case if and only if $(a^{\mathcal{I}},b^{\mathcal{I}}) \in R^{\mathcal{I}}$. This by Definition 36 is equivalent to $(\langle a,0\rangle,\langle b,0\rangle) \in R^{\mathcal{I}_\omega}$ and thus to $(a^{\mathcal{I}_\omega},b^{\mathcal{I}_\omega}) \in R^{\mathcal{I}_\omega}$. By semantics this is the case if and only if $\mathcal{I}_\omega \models R(a,b)$.

- α is of the form $a \doteq b$ for individual name a and b:
  Again for any interpretation $\mathcal{I}$ we have that $\mathcal{I} \models a \doteq b$ if and only if $a^{\mathcal{I}} = b^{\mathcal{I}}$ and thus equivalent to $\langle a^{\mathcal{I}},0\rangle = \langle b^{\mathcal{I}},0\rangle$. By Definition 36 this is equivalent to $a^{\mathcal{I}_\omega} = b^{\mathcal{I}_\omega}$ and thus by semantics this is the case if and only if $\mathcal{I}_\omega \models a = b$.

- For the rest of the cases, we can proceed in a similar manner.

□

Similar results can not be proved when we consider $\mathcal{SROIQ}$ simply because there are $\mathcal{SROIQ}$ knowledge bases with no infinite models at all (see Chapter 7).

6.4 Rewriting of Epistemic Queries

In [MEHDI et al. 2011], we presented an approach for rewriting axioms containing **K**s into **K**-free ones. The basic idea is that the semantic extension of a concept preceded by the operator K may contain named individuals only (except of some special cases which we describe subsequently). This allows us to collect all the named individuals into a one-of concept and replace the actual concepts with this one-of concept thus getting rid of **K**. We prove this by exploiting certain symmetries on the model set $\mathcal{M}(\Sigma)$. Intuitively, one can freely swap or permute anonymous individuals (i.e., domain elements which do not correspond to any individual name) in a model of some knowledge base without losing modelhood (see Lemma 3.3 in [DONINI et al. 1998]). To this end, we define the following:

Definition 37. *Given an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, a set Δ with $N_I \subseteq \Delta$, and a bijection $\varphi : \Delta^{\mathcal{I}} \to \Delta$ with $\varphi(a^{\mathcal{I}}) = a$ for all $a \in N_I$, the* renaming *of $\mathcal{I}$ according to φ, denoted by $\varphi(\mathcal{I})$, is defined as the interpretation $(\Delta, \cdot^{\varphi(\mathcal{I})})$ with:*

- $a^{\varphi(\mathcal{I})} = \varphi(a^{\mathcal{I}}) = a$ *for every individual name* a
- $A^{\varphi(\mathcal{I})} = \{\varphi(z) \mid z \in A^{\mathcal{I}}\}$ *for every concept name* A
- $P^{\varphi(\mathcal{I})} = \{(\varphi(z), \varphi(w)) \mid (z, w) \in P^{\mathcal{I}}\}$ *for every role name* P

We now show that renaming a model of a given knowledge base does not invalidates its modelhood.

Lemma 3. *Let Σ be a $\mathcal{SRIQ}\backslash U$ knowledge base and let $\mathcal{I}$ be a model of Σ with infinite domain. Then, every renaming $\varphi(\mathcal{I})$ of $\mathcal{I}$ satisfies $\varphi(\mathcal{I}) \in \mathcal{M}(\Sigma)$.*

Proof. By definition, the renaming satisfies the common domain and rigid term assumption. Modelhood w.r.t. Σ immediately follows from the isomorphism lemma of first-order interpretations [VAN DALEN 1989] since $\mathcal{I}$ and $\varphi(\mathcal{I})$ are isomorphic and φ is an isomorphism from $\mathcal{I}$ to $\varphi(\mathcal{I})$. □

Note that the model, obtained by renaming a model of a knowledge base, interprets every individual name a by a itself. Nevertheless, the interpretations of the anonymous individuals are not fixed. This allows us to swap every anonymous individual with another individual which serves as a counterexample for membership in some given concept D, unless the concept is equivalent to $\top$. Consequently we are able to to prove that an epistemic concept $\mathbf{K}D$ may contain merely named individuals, given that it is not universal. Now for translating $\mathbf{K}D$ into a $\mathbf{K}$-free concept, all we need to do is to replace it with the concept $\{a_1, \ldots, a_n\}$ such that $\Sigma \models D(a)$ whenever D is not universal. In case, $\Sigma_{UNA} \models \top \sqsubseteq D$, we simply replace it with $\top$ in our rewriting procedure. As a justification, it follows from the semantics that in such a case

$$\mathbf{K}D^{\mathcal{I},\mathcal{M}(\Sigma)} = \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} \top^{\mathcal{I},\mathcal{M}} = \Delta$$

We now formally prove the aforementioned relationship between the named individual and the operator $\mathbf{K}$.

Lemma 4. *Let Σ be a $\mathcal{SRIQ}\backslash U$ knowledge base. For any epistemic concept $C = \mathbf{K}D$ with D a $\mathbf{K}$-free concept such that $\Sigma_{\text{UNA}} \not\models D \equiv \top$ and $x \in \Delta$, we have that $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ iff there is an individual $a \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models D(a)$.*

Proof. Lets first prove the left to right direction. Suppose that $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$. Since $C = \mathbf{K}D$, by semantics we get that

$$x \in \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

To the contrary, suppose that there is no $a \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$ and $\Sigma_{\text{UNA}} \models D(a)$ i.e., x is an anonymous element. By assumption $\Sigma_{\text{UNA}} \not\models \top \equiv D$, this means that there is a model $\mathcal{I}'$ of Σ_{UNA} such that $D^{\mathcal{I}'} \neq \Delta^{\mathcal{I}'}$. This implies that there is a $y \in \Delta^{\mathcal{I}'}$ such that $y \notin D^{\mathcal{I}'}$. Considering $\mathcal{I}'_\omega$, we can invoke Lemma 2 to ensure $\mathcal{I}'_\omega \models \Sigma_{\text{UNA}}$. Further Lemma 1 guarantees $\langle y, i\rangle \notin D^{\mathcal{I}'_\omega}$ for $i \in \mathbb{N}$. Specifically, $\langle y, 1\rangle \notin D^{\mathcal{I}'_\omega}$ and thus by construction, $\langle y, 1\rangle$ is anonymous[3]. Let $\varphi : \Delta^{\mathcal{I}'} \times \mathbb{N} \to \Delta$ be a bijection such that $\varphi(a^{\mathcal{I}'_\omega}) = a^{\mathcal{I}}$ for all $a \in N_I$ and

[3]Note that we can consider any $i \in \mathbb{N}$ in $\langle y, 1\rangle$ instead of 1.

$\varphi(\langle y, 1\rangle) = x$. Such a bijection exists, as $|\Delta^{\mathcal{I}'} \times \mathbb{N}| = |\Delta|$ and $\mathcal{I}'_\omega$ satisfies the unique name assumption. By Lemma 3, we get that $\varphi(\mathcal{I}'_\omega) \in \mathcal{M}(\Sigma)$. By the choice of φ we get $x \notin D^{\varphi(\mathcal{I}'_\omega)}$ as $\langle y, 1\rangle \notin D^{\mathcal{I}'_\omega}$ has been renamed by x and since φ is an isomorphism. In particular,

$$x \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

which is a contradiction.

For the right to left direction, suppose there is an individual name $a \in N_I$ such that $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$ and $\Sigma_{\text{UNA}} \models D(a)$. This means that each model $\mathcal{I}$ of Σ_{UNA} is such that $a \in D^{\mathcal{I}}$. Now since $\mathcal{M}(\Sigma)$ is the set of all models This implies that for any $\mathcal{I} \in \mathcal{M}(\Sigma)$ we have that $x \in D^{\mathcal{I}}$ i.e.,

$$x \in \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J}}$$

Hence by semantics, $x \in \mathbf{K}D^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $C = \mathbf{K}D$. □

The above lemma shows that for a given concept $C = \mathbf{K}D$ with D a $\mathbf{K}$-free concept, we can replace the occurrence of $\mathbf{K}D$ in C with $\top$ if $\Sigma_{\text{UNA}} \models \top \sqsubseteq D$ and otherwise with $\{a_1, \ldots, a_n\}$ where $\Sigma_{\text{UNA}} \models D(a_i)$ and $a_i \in N_I$ for $1 \leq i \leq n$. A similar property can be proved for the roles as well. Again, we have to take care of the exceptional case of the universal role.

Claim 1. *Let Σ be a knowledge base. For the universal role U we have:*

$$\mathbf{K}U^{\mathcal{I},\mathcal{M}(\Sigma)} = U^{\mathcal{I},\mathcal{M}(\Sigma)}$$

Proof. The claim follows trivially as $U^{\mathcal{J}} = \Delta \times \Delta$ for any $\mathcal{J} \in \mathcal{M}(\Sigma)$. Specifically $U^{\mathcal{I}} = U^{\mathcal{I},\mathcal{M}(\Sigma)} = \Delta \times \Delta$. Now by semantics

$$\mathbf{K}U^{\mathcal{I},\mathcal{M}(\Sigma)} = \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} U^{\mathcal{J}} = \Delta \times \Delta$$

and thus $U^{\mathcal{I},\mathcal{M}(\Sigma)} = \mathbf{K}U^{\mathcal{I},\mathcal{M}(\Sigma)}$.

□

This allows us to simply replace a role of the form $\mathbf{K}U$ in an epistemic concept with U. Using construction from Definition 36 we can show a somewhat similar result as in Lemma 4 for roles.

Lemma 5. *Let Σ be a $\mathcal{SRIQ}\backslash U$ knowledge base. For any epistemic role $R = \mathbf{K}P$ with P a $\mathbf{K}$-free role such that $P \neq U$, and for any $x, y \in \Delta$ we have that $(x, y) \in R^{\mathcal{I},\mathcal{M}(\Sigma)}$ iff at least one of the following holds:*

1. *$x = y$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$.*
2. *there are individual names $a, b \in N_I$ with $a^{\mathcal{I},\mathcal{M}(\Sigma)} = x$ and $b^{\mathcal{I},\mathcal{M}(\Sigma)} = y$ such that $\Sigma_{\text{UNA}} \models P(a, b)$.*

Proof. For the "only-if" direction, we make the following case distinctions:

- Suppose that $x = y$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$. As $\mathcal{M}(\Sigma)$ is the epistemic model of Σ, therefore every interpretation in $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies the UNA and by Fact 2 we get that $\mathcal{J} \models \Sigma_{\text{UNA}}$. This means for every $x' \in \Delta$ and every interpretation $\mathcal{J} \in \mathcal{M}(\Sigma)$ we have that $(x', x') \in P^{\mathcal{I}}$ i.e.,

  $$(x', x') \in \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J},\mathcal{M}(\Sigma)}$$

  By semantics, therefore, $(x', x') \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ for any $x' \in \Delta$. In particular, we have that $(x, y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $x = y$.

- Suppose there are $a, b \in N_I$ with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$, $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models P(a, b)$. By assumption we have that $\Sigma_{\text{UNA}} \models P(a, b)$. Therefore, we have that $(x, y) \in P^{\mathcal{I}}$ for any interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$ as each such $\mathcal{I}$ satisfies UNA i.e.,

  $$(x, y) \in \bigcap_{\mathcal{I} \in \mathcal{M}(\Sigma)} P^{\mathcal{I},\mathcal{M}(\Sigma)}$$

  Hence by semantics, $(x, y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$.

This completes the proof of lemma in right to left direction.

Now for the "if" direction, we suppose that the first case of the lemma does not hold. Therefore, we have to show that there are a, b with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$, $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models P(a, b)$. To the contrary suppose that there are no such individual names a and b in N_I. We distinguish two cases.

- There are a and b with $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y = b^{\mathcal{I},\mathcal{M}(\Sigma)}$ but $\Sigma_{\text{UNA}} \not\models P(a, b)$. Now $\Sigma_{\text{UNA}} \not\models P(a, b)$ implies that there is a model $\mathcal{I}'$ of Σ_{UNA} with $(a^{\mathcal{I}'}, b^{\mathcal{I}'}) \notin P^{\mathcal{I}'}$. But as the interpretation of a and b does not change across the interpretation in $\mathcal{M}(\Sigma)$ (rigid term assumption), we get that $a^{\mathcal{I}'} = a^{\mathcal{I}}$ and $b^{\mathcal{I}'} = b^{\mathcal{I}}$ and thus $a^{\mathcal{I}'} = x$ and $b^{\mathcal{I}'} = y$. This means that $(x, y) \notin P^{\mathcal{I}'}$ and hence
$$(x, y) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J},\mathcal{M}(\Sigma)}$$
which is a contradiction to the assumption that $(x, y) \in R^{\mathcal{J},\mathcal{M}(\Sigma)}$.

- Assume at least one of x, y is anonymous. W.l.o.g. let x be anonymous, the other case follows by symmetry. Let $\mathcal{I}$ be a model of Σ_{UNA}. Considering $\mathcal{I}_\omega$, we again have $\mathcal{I}_\omega \models \Sigma_{\text{UNA}}$ by Lemma 2. By construction, $\langle x, 1\rangle$ is anonymous and $(\langle x, 1\rangle, \langle y, 0\rangle) \notin P^{\mathcal{I}_\omega}$. Let $\varphi : \Delta^{\mathcal{I}} \times \mathbb{N} \rightarrow \Delta$ be a bijection such that $\varphi(\langle x, 1\rangle) = x$ and $\varphi(\langle y, 0\rangle) = y$. Such a φ exists, since $|\Delta^{\mathcal{I}} \times \mathbb{N}| = |\Delta|$ and $\mathcal{I}_\omega$ satisfies the unique name assumption. By Lemma 3, we get that $\varphi(\mathcal{I}_\omega) \in \mathcal{M}(\Sigma)$. Moreover $(\varphi(\langle x, 1\rangle), \varphi(\langle y, 0\rangle)) = (x, y) \notin P^{\varphi(\mathcal{I}_\omega)}$ and thus
$$(x, y) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J},\mathcal{M}(\Sigma)}$$
which again is a contradiction.

Now we suppose that the first case does not hold. We have to show that $x = y$ and $\Sigma \models \top \sqsubseteq \exists P.\mathsf{Self}$. Again we assume to its contrary and make the following case distinctions:

- $x \neq y$:
  Now either both of x and y are named individuals but $\Sigma \not\models P(a, b)$ or at least one of them is anonymous. We can generate contradiction as above.

- $x = y$ and but $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists P.\mathsf{Self}$:
  We have to distinguish the following two cases.

  – suppose that x is a named individual i.e., there is $a \in N_I$ with $a^{\mathcal{I}} = x$. Now as $\Sigma_{\text{UNA}} \not\models P(a, a)$, this leads to contradiction as shown above.

  – suppose that x is anonymous. Since every $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, it follows from Fact 2 that $\mathcal{J} \models \Sigma_{\text{UNA}}$ for every $\mathcal{J} \in \mathcal{M}(\Sigma)$. This along with the fact that $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists P.\mathsf{Self}$ implies that there is some $\mathcal{I}' \in \mathcal{M}(\Sigma)$ such that $(u, u) \notin P^{\mathcal{I}'}$ for some $u \in \Delta$. We define a bijection $\varphi : \Delta \to \Delta$ such that $\varphi(u) = x$. By Lemma 3, we get that $\varphi(\mathcal{I}') \in \mathcal{M}(\Sigma)$. Moreover $(\varphi(u), \varphi(u)) \notin P^{\varphi(\mathcal{I}')}$. Particularly,

  $$(\varphi(u), \varphi(u)) = (x, x) \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} P^{\mathcal{J}, \mathcal{M}(\Sigma)}$$

  and therefore, by semantics, $(x, y) \notin \mathbf{K}P^{\mathcal{I}, \mathcal{M}(\Sigma)}$ which is a contradiction.

Hence we have shown that the lemma must hold in left to right direction as well.

□

Using the above lemma, we can define a translation procedure that maps (complex) epistemic concept expressions to non-epistemic ones which are equivalent in all models of Σ.

Definition 38. *Let Σ be a $\mathcal{SRIQ}\backslash U$ knowledge base. Further let P be a role and R and S be non-epistemic roles with each different from the universal role U. We define the function Φ_Σ mapping $\mathcal{SROIQK}$ concept expressions to $\mathcal{SROIQ}$ concept expressions as given in Figure 6.1 where we let $\{\} = \emptyset = \bot$[4].*

[4] W.l.o.g. we assume that in the definition of Φ_Σ, $n \geq 1$.

$$
\Phi_\Sigma : \begin{cases}
C & \mapsto C \quad \text{if } C \text{ is an atomic or one-of concept, } \top \text{ or } \bot; \\
\mathbf{K}D & \mapsto \begin{cases} \top & \text{if } \Sigma_{\text{UNA}} \models \Phi_\Sigma(D) \equiv \top \\ \{a \in N_I \mid \Sigma_{\text{UNA}} \models \Phi_\Sigma(D)(a)\} & \text{otherwise} \end{cases} \\
\exists \mathbf{K}S.\mathsf{Self} & \mapsto \begin{cases} \exists S.\mathsf{Self} & \text{if } \Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \{a \in N_I \mid \Sigma_{\text{UNA}} \models S(a,a)\} & \text{otherwise} \end{cases} \\
C_1 \sqcap C_2 & \mapsto \Phi_\Sigma(C_1) \sqcap \Phi_\Sigma(C_2) \\
C_1 \sqcup C_2 & \mapsto \Phi_\Sigma(C_1) \sqcup \Phi_\Sigma(C_2) \\
\neg C & \mapsto \neg \Phi_\Sigma(C) \\
\exists R.D & \mapsto \exists R.\Phi_\Sigma(D) \\
\exists \mathbf{K}P.D & \mapsto \bigsqcup_{a \in N_I} \{a\} \sqcap \exists P.(\{b \in N_I \mid \Sigma_{\text{UNA}} \models P(a,b)\} \sqcap \Phi_\Sigma(D)) \\
& \qquad \sqcup \begin{cases} \Phi_\Sigma(D) & \text{if } \Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \\
\forall R.D & \mapsto \forall R.\Phi_\Sigma(D) \\
\forall \mathbf{K}P.D & \mapsto \neg \Phi_\Sigma(\exists \mathbf{K}P.\neg D) \\
\geqslant nS.D & \mapsto \geqslant nS.\Phi_\Sigma(D) \\
\geqslant n\mathbf{K}S.D & \mapsto \begin{cases} \bigsqcup_{a \in N_I} \{a\} \sqcap \geqslant nS.(\{b \in N_I \mid \Sigma_{\text{UNA}} \models S(a,b)\} \sqcap \Phi_\Sigma(D)) & \text{if } n>1 \\ \Phi_\Sigma(\exists \mathbf{K}S.D) & \text{otherwise} \end{cases} \\
\leqslant nS.D & \mapsto \leqslant nS.\Phi_\Sigma(D) \\
\leqslant n\mathbf{K}S.D & \mapsto \neg \Phi_\Sigma(\geqslant (n+1)\mathbf{K}S.D) \\
\Xi \mathbf{K}U.D & \mapsto \Xi U.\Phi_\Sigma(D) \quad \text{for } \Xi \in \{\forall, \exists, \geqslant n, \leqslant n\}
\end{cases}
$$

Figure 6.1: Translation Function Φ_Σ

In the following , we prove the correctness of this method. Note that our method can be thought of as a reduction from epistemic queries to non-epistemic queries through some computation, i.e., we perform several reasoning tasks (in this case checking some entailments and instance retrieval) in order to answer the original query. The important point is that our translation depends on the set of individuals occurring in the knowledge base. For an epistemic concept C which is not equivalent to $\top$, one can deal with $\mathbf{K}$ by considering individuals occurring in the knowledge base only (see Definition 38). We now prove that the extension of a $\mathcal{SROIQK}$ concept and that of its $\mathbf{K}$-free equivalent concept, obtained using the translation function Φ_Σ, agree under $\mathcal{M}(\Sigma)$.

Lemma 6. *Let Σ be a $\mathcal{SRIQ}\backslash U$ knowledge base, let x be an element of Δ, and let C be a $\mathcal{SROIQK}$ concept. Then for any interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$, we have that $C^{\mathcal{I},\mathcal{M}(\Sigma)} = (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$.*

Proof. It suffices to show that

$$x \in C^{\mathcal{I},\mathcal{M}(\Sigma)} \text{ iff } x \in \Phi_\Sigma(C)^{\mathcal{I},\mathcal{M}(\Sigma)}$$

for any $x \in \Delta$. For the proof, we use induction on the structure of the C. For the base case where C is an atomic concept and the cases where $C = \top$ or $C = \bot$, the translation function Φ_Σ has no effect i.e., the concept C is mapped to itself. Thus the lemma follows immediately. Similar is the case of one-of concepts. For the cases, where $C = C_1 \sqcap C_2$, $C = C_1 \sqcup C_2$ or $C = \neg D$, it follows from the standard semantics and induction hypothesis. We now focus on the rest of the cases in the following.

i. $C = \mathbf{K}D$ and $\Sigma_{\text{UNA}} \not\models D \equiv \top$:
By Lemma 4, $x \in (\mathbf{K}D)^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if there is an $a \in N_I$ such that $x = a^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models D(a)$. This is equivalent to

$$x \in \{a \in N_I \mid \Sigma_{\text{UNA}} \models D(a)\}^{\mathcal{I},\mathcal{M}(\Sigma)}$$

and hence, by definition of Φ_Σ, equivalent to $x \in [\Phi_\Sigma(\mathbf{K}D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

ii. $C = \mathbf{K}D$ and $\Sigma_{\text{UNA}} \models D \equiv \top$:
Note that it trivially holds that if $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ then $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $\Phi_\Sigma(C) = \top$. Hence we just prove that whenever $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$ then $x \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ also. Now suppose that $x \in (\Phi_\Sigma(C))^{\mathcal{I},\mathcal{M}(\Sigma)}$. To contrary, suppose $x \notin C^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, by definition, we get that

$$x \notin \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} D^{\mathcal{J},\mathcal{M}(\Sigma)}$$

which means that there is an interpretation $\mathcal{I}' \in \mathcal{M}(\Sigma)$ such that $x \notin D^{\mathcal{I}'}$. Since $\mathcal{M}(\Sigma)$ is the epistemic model of Σ, hence $\mathcal{I}' \in \mathcal{M}(\Sigma)$ respects the unique name assumption and therefore, $\mathcal{I}' \models \Sigma_{\text{UNA}}$ with $D^{\mathcal{I}'} \neq \Delta$ as $x \notin D^{\mathcal{I}'}$. Hence $\Sigma_{\text{UNA}} \not\models D \equiv \top$, which is a contradiction.

iii. $C = \exists \mathsf{K}S.\mathsf{Self}$:
For the "if" condition we distinguish two cases. First, we suppose that

$\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$. Therefore, by definition $\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists S.\mathsf{Self}$. Now $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$ along with the semantics implies that for each $\mathcal{J} \in \mathcal{M}(\Sigma)$, we have that $(x,x) \in S^{\mathcal{J},\mathcal{M}(\Sigma)}$. In particular, $(x,x) \in S^{\mathcal{I},\mathcal{M}(\Sigma)}$. Again by semantics therefore $x \in [\exists S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and hence $x \in [\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
Second, suppose that $\Sigma_{\text{UNA}} \not\models \top \sqsubseteq \exists S.\mathsf{Self}$. We need to show that $x \in \{c \in N_I \mid \Sigma_{\text{UNA}} \models S(c,c)\}$. As $x \in [\exists \mathbf{K}S.\mathsf{Self}]$ implies that $(x,x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$, by Lemma 5 there is $a \in N_I$ such that $a^\mathcal{I} = x$ and $\Sigma_{\text{UNA}} \models S(a,a)$ i.e., $a \in \{c \in N_I \mid \Sigma_{\text{UNA}} \models S(c,c)\}$ which immediately implies that $x = a^\mathcal{I} \in [\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\mathcal{I},\mathcal{M}(\Sigma)}$ as per definition of Φ_Σ.

Now for the "only-if" condition again we have to consider two cases as per definition of Φ_Σ. First suppose that $\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists S.\mathsf{Self}$. Hence from the definition of Φ_Σ it must be the case that $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$. Now as each model in $\mathcal{M}(\Sigma)$ satisfies UNA, by Fact 2, we have that $\mathcal{J} \models \Sigma_{\text{UNA}}$ and hence $\mathcal{J} \models \top \sqsubseteq \exists S.\mathsf{Self}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$ i.e., for every $u \in \Delta$, we have that $(u,u) \in S^{\mathcal{J},\mathcal{M}(\Sigma)}$. In other words, for every $u \in \Delta$, we have that $(u,u) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. In particular, we have that $x \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore by semantics, $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Now for the second case assume that $\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \{c \in N_I \mid \Sigma_{\text{UNA}} \models S(c,c)\}$. Since $x \in [\Phi_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\mathcal{I},\mathcal{M}(\Sigma)}$, it implies there is $a \in N_I$ with $a^\mathcal{I} = x$ and $\Sigma_{\text{UNA}} \models S(a,a)$ which, by Lemma 5, implies that $(x,x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore, we get that $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

iv. $C = \exists P.D$ and P is a simple role:
By semantics, $x \in (\exists P.D)^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if there is $y \in \Delta$ such that $(x,y) \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $y \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence it is equivalent to $x \in (\Phi_\Sigma(KD))^{\mathcal{I},\mathcal{M}(\Sigma)}$.

v. $C = \exists \mathbf{K}P.D$:
First let $x \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$. this implies that there is some $y \in \Delta$ with $(x,y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ such that $y \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $y \in$

$[\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 5, $(x, y) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that at least one of the following should hold.

- There are $a, b \in N_I$ with $a^\mathcal{I} = x$ and $b^\mathcal{I} = y$ such that $\Sigma_{\text{UNA}} \models P(a, b)$: Consequently we have that $y = b^\mathcal{I} \in [\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a, c)\} \sqcap \Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Now as $\mathcal{M}(\Sigma)$ is an epistemic model, every interpretation in $\mathcal{M}(\Sigma)$ satisfies the UNA, and hence by Fact 2, for every $\mathcal{J} \in \mathcal{M}(\Sigma)$ we have that $\mathcal{J} \models \Sigma_{\text{UNA}}$. This along with $\Sigma_{\text{UNA}} \models P(a, b)$ implies that $(a^\mathcal{I}, b^\mathcal{I}) = (x, y) \in P^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore, $x \in [\exists P.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a, c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Hence, $x = a^\mathcal{I} \in [\{a\} \sqcap \{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a, c)\} \sqcap \Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, which by definition of Φ_Σ implies that $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
- $x = y$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$: As $y \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore it immediately follows from the definition that $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Now suppose that $x \in [\Phi_\Sigma(\exists \mathbf{K}P.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. This means that at least one of the following should hold.

- there is an $a \in N_I$ such that $a^\mathcal{I} = x$ and $a^\mathcal{I} \in [\exists P.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a, c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Consequently there is some $b \in N_I$ such that $b^\mathcal{I} \in [[\{c \in N_I \mid \Sigma_{\text{UNA}} \models P(a, c)\} \sqcap \Phi_\Sigma(D)]]^{\mathcal{I},\mathcal{M}(\Sigma)}$ i.e., $\Sigma_{\text{UNA}} \models P(a, b)$ and $b^\mathcal{I} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $b^\mathcal{I} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 5, $\Sigma_{\text{UNA}} \models P(a, b)$ implies that $(a^\mathcal{I}, b^\mathcal{I}) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. Therefore we get that $x = a^\mathcal{I} \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
- $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists P.\mathsf{Self}$:
  Note that each $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, $\mathcal{J} \models \Sigma_{\text{UNA}}$. This implies that $\mathcal{J} \models \top \sqsubseteq \exists P.\mathsf{Self}$. In other words, for every $u \in \Delta$, we have that $(u, u) \in P^\mathcal{J}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$ and therefore, by semantics, we get that $(u, u) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. In particular, $(x, x) \in \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)}$. Now as $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, we get that $x \in [\exists \mathbf{K}P.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

vi. $C = \geqslant n\mathbf{K}S.D$:
We first prove the left to right direction. Suppose that $n = 1$. Then, $x \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ means that $x \in [\exists \mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$. Earlier we showed that

this is the case iff $x \in [\Phi_\Sigma(\exists \mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and therefore by definition, $x \in [\Phi_\Sigma(\geqslant 1\mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Now suppose that $n > 1$. From $x \in [\geqslant n\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ we get that there are $y_1, \ldots, y_m$ with $m \geq n$ such that $(x, y_i) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $y_i \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. By induction, $y_i \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. By Lemma 5, we have $a, b_1, \ldots, b_n \in N_I$ such that $a^\mathcal{I} = x$, $b_i^\mathcal{I} = y_i$ and $\Sigma_{\text{UNA}} \models S(a, b_i)$ for each $i \leq m$. Now as $m \geq n$ and $b_i^\mathcal{I} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for $i \leq m$, it follows from the semantics that $x = a^\mathcal{I}$ belongs to

$$[\geqslant nS.(\{c \in N_I \mid \Sigma_{\text{UNA}} \models S(a, c)\} \sqcap \Phi_\Sigma(D))]^{\mathcal{I},\mathcal{M}(\Sigma)}$$

Hence, using definition of Φ_Σ, we obtain that $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ as $x \in \{a\}^{\mathcal{I},\mathcal{M}(\Sigma)}$.

Suppose that $n > 1$. Therefore,

$$\Phi_\Sigma(\geqslant n\mathbf{K}S.D) = \bigsqcup_{c \in N_I} \{c\} \sqcap \geqslant nS.(\{c' \in N_I \mid \Sigma_{\text{UNA}} \models S(c, c')\} \sqcap \Phi_\Sigma(D))$$

Now $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}S.D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that there are $a, b_1, \ldots, b_m \in N_I$, for $m \geq n$, such $a^\mathcal{I} = x$, $\Sigma_{\text{UNA}} \models S(a, b_i)$ and $b_i^\mathcal{I} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. Since each $\mathcal{J} \in \mathcal{M}(\Sigma)$ satisfies UNA, therefore, $\mathcal{J} \models \Sigma_{\text{UNA}}$ and hence we get that $(a^\mathcal{J}, b_i^\mathcal{J}) \in S^\mathcal{J}$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$. Hence, It follows from the semantics that $(a^\mathcal{I}, b_i^\mathcal{I}) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ for each $i \leq m$. Now as $m \geq n$ and $b_i^\mathcal{I} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$(by induction), we get that $x \in [\geqslant n\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
Now assume that $n = 1$. Hence, $\Phi_\Sigma(\geqslant n\mathbf{K}S.D) = \Phi_\Sigma(\exists \mathbf{K}S.D)$. Now for $x \in [\exists \mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$ at least one of the following holds:

- there is $a, b \in N_I$ with $a^\mathcal{I} = x$ such that $\Sigma_{\text{UNA}} \models S(a, b)$ and $b^\mathcal{I} \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$, therefore by induction, $b^\mathcal{I} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$. By Lemma 5, we get that $(a^\mathcal{I}, b^\mathcal{I}) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$ which along with $b^\mathcal{I} \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ implies that $x = a^\mathcal{I} \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.
- $x \in [\Phi_\Sigma(D)]^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $\Sigma_{\text{UNA}} \models \top \sqsubseteq \exists S.\mathsf{Self}$. By Lemma 5, we get that $(x, x) \in \mathbf{K}S^{\mathcal{I},\mathcal{M}(\Sigma)}$. By induction we have that $x \in D^{\mathcal{I},\mathcal{M}(\Sigma)}$ which immediately implies that $x \in [\geqslant 1\mathbf{K}S.D]^{\mathcal{I},\mathcal{M}(\Sigma)}$.

vii. The rest of the cases can be proved in a similar fashion.

□

Thus we have shown that the extension of a $\mathcal{SROIQK}$ concept and that of its translation using Φ_Σ are equal. With this we have proved the correctness of our translation procedure. Using this function, we can reduce the problem of entailment of $\mathcal{SROIQK}$ axioms to that of $\mathcal{SROIQ}$ axioms in $\mathcal{SRIQ}\backslash U$ knowledge bases. Formally,

Theorem 1. *For a $\mathcal{SRIQ}\backslash U$ knowledge base Σ, $\mathcal{SROIQK}$ concepts C and D and an individual a we have that*

1. $\Sigma \Vdash C(a)$ *if and only f* $\Sigma_{\mathrm{UNA}} \models \Phi_\Sigma(C)(a)$.
2. $\Sigma \Vdash C \sqsubseteq D$ *if and only if* $\Sigma_{\mathrm{UNA}} \models \Phi_\Sigma(C) \sqsubseteq \Phi_\Sigma(D)$.

Proof. 1. For the first case, we see that $\Sigma \Vdash C(a)$ is equivalent to $a^{\mathcal{I},\mathcal{M}(\Sigma)} \in C^{\mathcal{I},\mathcal{M}(\Sigma)}$ which by Lemma 6 is the case if and only if

$$a^{\mathcal{I},\mathcal{M}(\Sigma)} \in \Phi_\Sigma(C)^{\mathcal{I},\mathcal{M}(\Sigma)}$$

for all $\mathcal{I} \in \mathcal{M}(\Sigma)$. This is equivalent to $a^{\mathcal{I}} \in \Phi_\Sigma(C)^{\mathcal{I}}$ as $\Phi_\Sigma(C)$ does not contain any **K**s. Thus by semantics, this is equivalent to $\mathcal{I} \models \Phi_\Sigma(C)(a)$ for all $\mathcal{I} \in \mathcal{M}(\Sigma)$. Consequently, Fact 2 and Lemma 2 immediately imply that this is the case if and only if $\Sigma_{\mathrm{UNA}} \models \Phi_\Sigma(C)(a)$.

2. The second case can be proved analogously.

□

Theorem 1 shows that the epistemic entailment problem of epistemic axioms is reducible into the standard entailment problem. Consequently, epistemic queries answering can be translated in to the standard query answering.

Corollary 1. *For a given $\mathcal{SRIQ}\backslash U$ knowledge base Σ and an epistemic query Q, let Q' be the (non-epistemic) query obtained by translating axioms in Q into* **K***-free using Φ_Σ . Then we have*

$$ans(Q) = ans(Q')$$

i.e. answers of Q and Q' coincide when posed to Σ.

It can be seen from the definition of Φ_Σ that deciding epistemic entailment along those lines may require deciding many classical entailment problems and hence involves many calls to a standard DL reasoner. Nevertheless, Lemma 6 shows that the translation preserves the semantics of the axiom being translated. Hence we can use standard DL (OWL) reasoners in order to answer epistemic queries. Note that the number of reasoner calls is bounded by the number of **K**s occurring in the query. The advantage in doing so is that the standard DL reasoners have been developed over period of several years thus implement several optimization techniques. Thus we do not need to implement reasoners capable of epistemic reasoning from the scratch.

We have considered $\mathcal{SRIQ}\backslash U$ knowledge bases and introduced a procedure for answering epistemic query when put to such knowledge bases. The restriction is due to the fact that Lemma 2 can only be proved upto $\mathcal{SRIQ}\backslash U$. In the next chapter, we come back to this issue in detail.

Chapter 7

Revising the Semantics

In the previous chapter we presented a method for translating an epistemic concept into a **K**-free one via intermediate reasoner calls. This is achieved by translating an epistemic axiom into a non-epistemic one using the translation function given in Definition 38. Though $\mathcal{SROIQK}$ was our query language, we restricted the knowledge base language to $\mathcal{SRIQ}\backslash U$. In the following, we explain the reason behind this restriction formally as first identified in [MEHDI and RUDOLPH 2011b, MEHDI et al. 2011].

7.1 Problems with the Classical Semantics

Following the intuition that led to the introduction of the **K** operator as an extension of **K**-free standard DL reasoning, a rather intuitive basic requirement to an epistemic DL semantics is arguably the one provided in the following definition.

Definition 39. *For a given DL $\mathcal{L}$, an epistemic DL semantics represented by an entailment relation $\mid\!\approx$ is called $\mathcal{L}$-*backward-compatible *if it coincides with the (non-epistemic) standard semantics (represented by $\models$) on non-epistemic axioms, i.e. for an $\mathcal{L}$ knowledge base Σ and an $\mathcal{L}$ axiom α both of which do not contain* **K***, we have*

$$\Sigma \mid\!\approx \alpha \textit{ iff } \Sigma \models \alpha$$

*Moreover, $\mid\!\approx$ is called $\mathcal{L}$-*UNA-backward-compatible*, $\Sigma \mid\!\approx \alpha$ if and only if $\Sigma \models \alpha$ under the unique name assumption.*

In the classical semantics of epistemic extension of DLs, the common domain assumption requires the following:

- all the interpretations considered in an epistemic interpretation share the same fixed domain, and
- the domain is infinite.

However, there is no prima facie reason, why the domain that is described by a knowledge base should not be finite, yet finite models are excluded from the consideration entirely. We showed that this does not pose problems up to $\mathcal{SRIQ}\backslash U$ due to Lemma 2: every finite model of a knowledge base can be extended to an infinite one such that the two models cannot be distinguished by means of the underlying logic. This allows to directly establish the following result.

Corollary 2. *$|\!\models$ is $\mathcal{SRIQ}\backslash U$-UNA-backward-compatible.*

As a consequence, the restriction to infinite models imposed by the common domain assumption turns out to be insignificant in case of $\mathcal{SRIQ}\backslash U$. However, this situation changes drastically once nominals or the universal role are involved.

Fact 3. *There are $\mathcal{SROIQ}$ knowledge bases with only finite models.*

It is easy to see that a knowledge base Σ containing the axiom $\top \sqsubseteq \{a, b, c\}$ or $\top \sqsubseteq {\leqslant}3U.\top$ has only models with at most three elements. Consequently, in both cases we have that Σ is unsatisfiable w.r.t. the classical epistemic semantics. Thus by *ex falso quodlibet* we, e.g., obtain $\Sigma \mathrel{|\!\models} \top \sqsubseteq \bot$ whereas we clearly have $\Sigma \not\models \top \sqsubseteq \bot$ even under the UNA. So we conclude that $|\!\models$ is not UNA-backward-compatible for any description logic that features nominals or both number restrictions and the universal role; in particular, it is not $\mathcal{SROIQ}$-UNA-backward-compatible.

Corollary 3. *$|\!\models$ is **not** $\mathcal{SROIQ}$-UNA-backward-compatible.*

Note that the imposition of UNA in the classical semantics may be a deliberate decision. Nevertheless, we believe the non-UNA-backward-compatibility is mainly due to the side effect of the classical semantics crafted for and probed against less expressive description logics. This contradicts the intuition behind the

K operator. Meanwhile, the rigidity of terms across the world (interpretations) required by the classical semantics makes no sense once the common domain assumption is not taken. This motivates us for the quest for a more appropriate, "domain-flexible" epistemic semantics.

In the following we introduce a new semantics which overcomes the problems of classical semantics when employed for an expressive DL like $\mathcal{SROIQ}$.

7.2 Extended Epistemic Semantics

For a flexible semantics, we need to relinquish the common domain assumption and the rigid term assumption. In other words, the domains we consider in the possible worlds should be allowed to have arbitrary (yet non-empty) size and be composed of arbitrary elements. Besides, the semantics should allow for the individual names to refer to different elements in different possible worlds. In other words, elements referred to by different individual names may coincide in some worlds but not in others. Now the very question is how to interpret the epistemic operator across different interpretations. We have to find a substitute for the common domain and rigid term assumptions as otherwise every epistemic role or concept would have the empty set as extension. We solve the problem by introducing one abstraction layer that assigns *abstract individual names* to domain elements. These abstract individual names are elements from $N_I \cup \mathbb{N}$ and hence common to all interpretations, thus they can serve as the "common ground" for different interpretations with different domains. We require that every domain element is associated with at least one abstract name, however, we also allow for several abstract names to refer to the same domain element (thus allowing for the possibility of finite domains). This intuition leads to the definition of extended interpretations.

Definition 40. *An* extended $\mathcal{SROIQ}$ interpretation $\tilde{\mathcal{I}}$ *is a triple* $(\Delta^{\tilde{\mathcal{I}}}, \cdot^{\tilde{\mathcal{I}}}, \varphi_{\tilde{\mathcal{I}}})$ *such that*

- $(\Delta^{\tilde{\mathcal{I}}}, \cdot^{\tilde{\mathcal{I}}})$ *is a standard DL interpretation,*
- $\varphi_{\tilde{\mathcal{I}}} : N_I \cup \mathbb{N} \twoheadrightarrow \Delta^{\tilde{\mathcal{I}}}$ *is a surjective function from* $N_I \cup \mathbb{N}$ *onto* $\Delta^{\tilde{\mathcal{I}}}$*, such that for all* $a \in N_I$ *we have that* $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}}$.

The definition of $\varphi_{\tilde{\mathcal{I}}}$ can be extended to subsets of $N_I \cup \mathbb{N}$. For a set S,

$$\varphi_{\tilde{\mathcal{I}}}(S) := \{\varphi_{\tilde{\mathcal{I}}}(t) \mid t \in S\}$$

Similarly we extend $\varphi_{\tilde{\mathcal{I}}}$ to ordered pairs and set of ordered pairs on $N_I \cup \mathbb{N}$ as follows:

- $\varphi_{\tilde{\mathcal{I}}}((s,t)) := (\varphi_{\tilde{\mathcal{I}}}(s), \varphi_{\tilde{\mathcal{I}}}(t))$ for ordered pairs $(s,t) \in (N_I \cup \mathbb{N})^2$.
- $\varphi_{\tilde{\mathcal{I}}}(T) := \{\varphi_{\tilde{\mathcal{I}}}((s,t)) \mid (s,t) \in T\}$ for sets $T \subseteq (N_I \cup \mathbb{N})^2$.

We also define the inverse $\varphi_{\tilde{\mathcal{I}}}^{-1}$ of the mapping $\varphi_{\tilde{\mathcal{I}}}$ for an extended interpretation $\tilde{\mathcal{I}}$ as follows:

- $\varphi_{\tilde{\mathcal{I}}}^{-1}(x) := \{t \in N_I \cup \mathbb{N} \mid \varphi_{\tilde{\mathcal{I}}}(t) = x\}$ for every $x \in \Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}(E) := \bigcup_{x \in E} \varphi_{\tilde{\mathcal{I}}}^{-1}(x)$ for $E \subseteq \Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}((x,y)) := \{(s,t) \mid \varphi_{\tilde{\mathcal{I}}}((s,t)) = (x,y)\}$ for ordered pairs $(x,y) \in \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\tilde{\mathcal{I}}}$.
- $\varphi_{\tilde{\mathcal{I}}}^{-1}(H) := \bigcup_{(x,y) \in H} \varphi_{\tilde{\mathcal{I}}}^{-1}((x,y))$ for any $H \subseteq \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\tilde{\mathcal{I}}}$.

Note that for any extended interpretation $\tilde{\mathcal{I}}$, we have that $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}}$ for any $a \in N_I$. This means that $\varphi_{\tilde{\mathcal{I}}}$ guarantees that each individual name a is the designator (abstract name) of the interpretation of a under $\tilde{\mathcal{I}}$. For the rest of the elements of $\Delta^{\tilde{\mathcal{I}}}$, we use elements of $\mathbb{N}$ as their designators.

Based on the notion of extended interpretation we define a new semantics for $\mathcal{SROIQK}$.

Definition 41. *An* extended epistemic interpretation *for $\mathcal{SROIQK}$ is a pair $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$, where $\tilde{\mathcal{I}}$ is an extended $\mathcal{SROIQ}$ interpretation and $\tilde{\mathcal{W}}$ is a set of extended $\mathcal{SROIQ}$ interpretations. Similar to epistemic interpretations, for $\mathcal{SROIQK}$ concepts C and D, role R and simple role S, we define an extended interpretation function $\cdot^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}$ as follows:*

$$
\begin{aligned}
a^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= a^{\tilde{\mathcal{I}}} \quad \textit{for } a \in N_I \\
X^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= X^{\tilde{\mathcal{I}}} \;\; \textit{for } X \in N_C \cup N_R \cup \{\top, \bot\} \\
\{a_1,\ldots,a_n\}^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{a_1^{\tilde{\mathcal{I}}},\ldots,a_n^{\tilde{\mathcal{I}}}\} \\
(C \sqcap D)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \cap D^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \\
(C \sqcup D)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \cup D^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \\
(\neg C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \Delta^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \setminus C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \\
(\exists S.\mathsf{Self})^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{x \mid \; (x,x) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \\
(\forall R.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{x \mid \forall y.(x,y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \rightarrow y \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \\
(\exists R.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{x \mid \exists y.(x,y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \wedge y \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \\
(\leqslant nS.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{x \mid \#\{y{\in}C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} | (x,y){\in}R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \leq n\} \\
(\geqslant nS.C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \{x \mid \#\{y{\in}C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} | (x,y){\in}R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\} \geq n\} \\
(\mathbf{K}C)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \varphi_{\tilde{\mathcal{I}}}\left(\textstyle\bigcap_{\tilde{\mathcal{J}}\in\tilde{\mathcal{W}}} \varphi_{\tilde{\mathcal{I}}}^{-1}\left(C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\right)\right) \\
(\mathbf{K}R)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} &= \varphi_{\tilde{\mathcal{I}}}\left(\textstyle\bigcap_{\tilde{\mathcal{J}}\in\tilde{\mathcal{W}}} \varphi_{\tilde{\mathcal{I}}}^{-1}\left(R^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}\right)\right)
\end{aligned}
$$

Again, we set $[(\mathbf{K}R)^-]^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} := (\mathbf{K}R^-)^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}$ for an epistemic role $(\mathbf{K}R)^-$. We now elaborate our semantics by interpreting (step by step) a concept KC in $\tilde{\mathcal{I}}, \tilde{\mathcal{W}}$ where C is $\mathbf{K}$-free concept:

1. for every interpretation $\tilde{\mathcal{J}} \in \tilde{\mathcal{W}}$, take the extension of C under $\tilde{\mathcal{J}}$ which is just a subset of $\Delta^{\tilde{\mathcal{J}}}$. (see Figure 7.1)

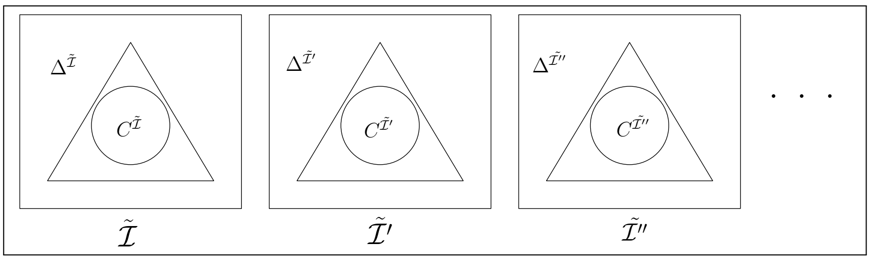

Figure 7.1: Step 1 of interpreting C in $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$

2. intersect all the subsets of $N_I \cup \mathbb{N}$ which are mapped to $C^{\tilde{\mathcal{J}}}$ by $\varphi_{\tilde{\mathcal{J}}}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{W}}$ (see Figure 7.2).

3. $KC^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}$ is then the subset of $\Delta^{\tilde{\mathcal{I}}}$ which is the image of this intersect under $\varphi_{\tilde{\mathcal{I}}}$ (see Figure 7.3) i.e.,

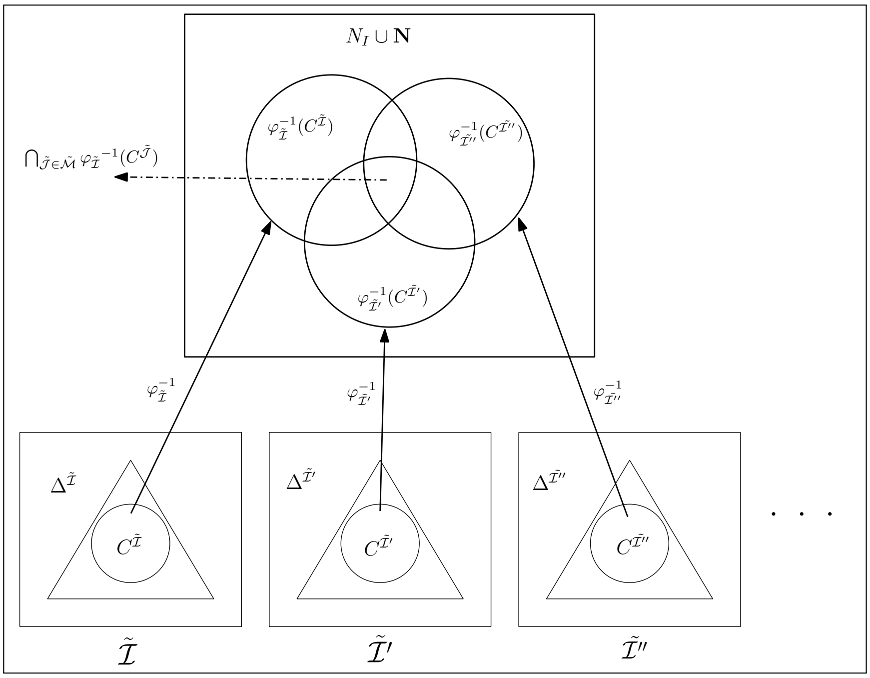

Figure 7.2: Step 2 of interpreting C in $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$

$$\mathbf{K}C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} = \varphi_{\tilde{\mathcal{I}}}(\bigcap \tilde{\mathcal{J}} \in \tilde{\mathcal{M}} \varphi_{\tilde{\mathcal{J}}}^{-1}(C^{\tilde{\mathcal{J}}}))$$

Then φ of the current world (interpretation) is applied to get the extension of the concept (role) in the current world.

The semantics of TBox, RBox, and ABox axioms follows exactly that for the classical semantics. Formally, we say an extended epistemic interpretation $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$ satisfies a GCI $C \sqsubseteq D$ for some $\mathcal{SROIQK}$ concepts C and D, if and only if $C^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}} \subseteq D^{\tilde{\mathcal{I}},\tilde{\mathcal{W}}}$. We write $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}}) \models C \sqsubseteq D$. Similarly we say $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$ satisfies a TBox $\mathcal{T}$, written $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}}) \models \mathcal{T}$ iff it satisfies all its axioms. The satisfaction of other axioms, RBox, ABox and knowledge base in $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$ can be defined analogously. For the notion of models, like in classical semantics, we prefer certain extended epistemic interpretations over the others i.e., our semantics is preferential.

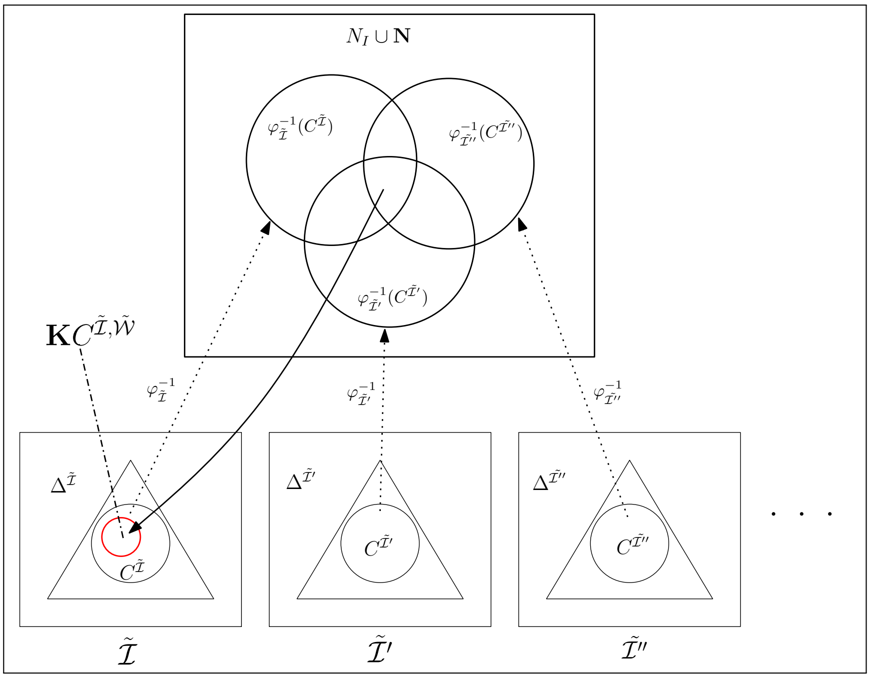

Figure 7.3: Step 3 of interpreting C in $(\tilde{\mathcal{I}}, \tilde{\mathcal{W}})$

Definition 42. *A set $\tilde{\mathcal{M}}$ of extended interpretations is called extended epistemic model of a given $\mathcal{SROIQK}$ knowledge base $\Sigma = (\mathcal{T}, \mathcal{R}, \mathcal{A})$ iff*

- *$(\tilde{\mathcal{I}}, \tilde{\mathcal{M}})$ satisfies Σ i.e, it satisfies $\mathcal{T}$, $\mathcal{R}$ and $\mathcal{A}$ for each $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$, and*
- *$\tilde{\mathcal{M}}$ is maximal in the sense that for every set of extended interpretations $\tilde{\mathcal{M}}'$ such that $\tilde{\mathcal{M}} \subset \tilde{\mathcal{M}}'$, there is some $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}'$ such that $\tilde{\mathcal{J}}, \tilde{\mathcal{M}}$ does not satisfy Σ.*

We say a $\mathcal{SROIQK}$ knowledge base Σ is satisfiable *(under the extended semantics) if it has an extended epistemic model.*

Based on the above definition, we now define the notion of entailment in this new semantics.

Definition 43. *For a given $\mathcal{SROIQK}$ knowledge base Σ and a $\mathcal{SROIQK}$ axiom α, we say Σ* entails *α under the extended semantics, written $\Sigma \mid\mid\!\!\models_{\text{e}} \alpha$, if for every* *extended epistemic model $\tilde{\mathcal{M}}$ of Σ, we have that for every $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$, the extended epistemic interpretation $(\tilde{\mathcal{I}}, \tilde{\mathcal{M}})$ satisfies α.*

Note that we now use the symbol $\mid\!\models_{\mathrm{e}}$ to represent the entailment relation in extended epistemic semantics. Further, as the semantics is preferential, $\mid\!\models_{\mathrm{e}}$ has a non-monotonic behavior.

Next we show that when considering non-epistemic axioms, the notions of satisfaction under the extended semantics and under the standard semantics coincide. More precisely, given a standard interpretation $\mathcal{I}$, by $\mathcal{E}(\mathcal{I})$ we mean the set of all extended interpretation $\tilde{\mathcal{I}}$ such that $\Delta^{\tilde{\mathcal{I}}} = \Delta^{\mathcal{I}}$ and the mappings $\cdot^{\mathcal{I}}$ and $\cdot^{\tilde{\mathcal{I}}}$ are identical. In other words, $\mathcal{E}(\mathcal{I})$ represents all the extended interpretations obtained by adding some mapping $\varphi_{\tilde{\mathcal{I}}} : N_I \cup \mathbb{N} \rightarrow \Delta^{\mathcal{I}}$ satisfying $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}}$ for each $a \in N_I$. Now as in case of standard epistemic semantics, a **K**-free knowledge base Σ has a unique extended epistemic model. We denote it with $\tilde{\mathcal{M}}(\Sigma)$ which is the set of all extended epistemic models obtained by extending extended interpretations in $\mathcal{M}(\Sigma)$. Note that $\mathcal{M}(\Sigma)$ is the set of all standard models of Σ. Formally,

Fact 4. *For any $\mathcal{SROIQ}$ knowledge base Σ, we have that*

$$\tilde{\mathcal{M}}(\Sigma) = \{\tilde{\mathcal{I}} \mid \tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I}) \textit{ for each } \mathcal{I} \textit{ with } \mathcal{I} \models \mathcal{M}(\Sigma)\}$$

We abbreviate this by writing $\tilde{\mathcal{M}}(\Sigma) = \mathcal{E}(\mathcal{M}(\Sigma))$.

Further the following relation holds between an interpretation $\mathcal{I}$ and $\mathcal{E}(\mathcal{I})$.

Lemma 7. *Let C be a non-epistemic concept, R a non-epistemic role and $\mathcal{I}$ a standard interpretation. Then*

- *for any $x \in \Delta^{\mathcal{I}}$ and for each $\tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I})$, we have that*

$$x \in C^{\mathcal{I}} \textit{iff } x \in C^{\tilde{\mathcal{I}}}$$

- *for any $(x, y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and for each $\tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I})$, we have that*

$$(x, y) \in R^{\mathcal{I}} \textit{iff } (x, y) \in R^{\tilde{\mathcal{I}}}$$

Proof. Note that as C contains no occurrence of **K**, the mapping $\varphi_{\tilde{\mathcal{I}}}$ play no role in interpreting C under $\tilde{\mathcal{I}}$. Hence, by simple induction, the proof follows simply from the definition of the extended interpretation. Similar is the case of roles. □

Since the interpretation of any axiom depends on the interpretation of concept names and role names occurring in it and since the interpretation of a knowledge base, depends on the interpretation of its axioms, as a consequence of the above lemma, we get

Corollary 4. *For any non-epistemic axiom α and a standard interpretation $\mathcal{I}$, we have that*

$$\mathcal{I} \models \alpha \textit{ iff } \tilde{\mathcal{I}} \models \alpha \textit{ for each } \tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I}).$$

Consequently we get

Corollary 5. *For a standard (non-epistemic) knowledge base Σ and a standard interpretation $\mathcal{I}$, we have that*

$$\mathcal{I} \models \Sigma \textit{ iff } \tilde{\mathcal{I}} \models \Sigma \textit{ for each } \tilde{\mathcal{I}} \in \mathcal{E}(\mathcal{I}).$$

Now it is easy to see that the (non-epistemic) consequences of a standard knowledge under the extended epistemic semantics and under the standard semantics coincide.

Corollary 6. *For a given non-epistemic knowledge base Σ and a non-epistemic axiom α, we have*

$$\Sigma \models \alpha \textit{ if and only if } \Sigma \mid\!\models_{\mathrm{e}} \alpha$$

Note that unlike the classical semantics, our proposed semantics does not enforce the unique name assumption either. Hence, this new semantics is more compatible with standard inference engines. Moreover, with this new semantics, the problem that arises when allowing for nominals and universal roles in the knowledge base language, is avoided, thus making it a more suitable and appropriate choice for **K**-extensions of expressive description logics, like $\mathcal{SROIQ}$.
It is now easy to see that the newly established semantics has the desired compatibility property as a direct consequence of Corollary 6.

Theorem 2. *$\Vdash_{\mathrm{e}}$ is $\mathcal{SROIQ}$-backward-compatible.*

Consequently, this new semantics is more adequate for very expressive DLs such as $\mathcal{SROIQ}$. Yet, as will be shown later, it is also generic in the sense that for $\mathcal{SRIQ}\backslash U$ knowledge bases it behaves similar to the (classical) epistemic interpretation introduced earlier. With this new semantics, we avoid the aforementioned problems arising from nominals and the universal role in the language of a knowledge base. In the following section, we extend the translation procedure, presented in Definition 38, such that it allows for $\mathcal{SROIQ}$ as the knowledge base language.

7.3 Reasoning in Extended Semantics

In Chapter 6.2, we presented a procedure for translating epistemic axioms into equivalent non-epistemic ones. We now extend the definition of Φ_Σ (see Definition 38) such that it also handles a richer knowledge base language like, $\mathcal{SROIQ}$.

7.3.1 Deciding Entailment of Extended Epistemic Axioms

To devise a translation procedure similar to Definition 38 for expressive DLs, the idea is similar i.e., we replace certain epistemic concepts with $\mathbf{K}$-free ones obtained via several intermediate reasoning steps. Nevertheless, there are some changes for handling the additional modeling constructs (e.g., the universal role and nominals) in $\mathcal{SROIQ}$. But first we notice that a similar argument can be made as in Lemma 4 in case of extended epistemic semantics i.e., the extension of an epistemic concept $\mathbf{K}D$ may contain only named individuals. Formally,

Lemma 8. *Let Σ be a $\mathcal{SROIQ}$-knowledge base and $C = \mathbf{K}D$ an epistemic concept with $\Sigma \not\models D \equiv \top$. For an extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$ and $x \in \Delta^{\tilde{\mathcal{I}}}$, we have that $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff x is named such that there is an individual name $a \in N_I$ with $x = a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $\Sigma \models D(a)$.*

Proof. For the left to right direction, suppose $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. It means that

$$x \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})\Big)$$

but suppose that there is no $a \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $\Sigma \not\models D(a)$.

Since $\Sigma \not\models D \equiv \top$, there is a model $\mathcal{I}'$ of Σ such that $\Delta^{\mathcal{I}'} \neq D^{\mathcal{I}'}$. In other words, there is a $y \in \Delta^{\mathcal{I}'}$ with $y \notin D^{\mathcal{I}'}$. By Lemma 7, $y \notin D^{\tilde{\mathcal{I}}'}$ for each $\tilde{\mathcal{I}}' \in \mathcal{E}(\mathcal{I}')$. As by definition, $\mathcal{E}(\mathcal{I}')$ contains all the extended interpretation obtained from $\mathcal{I}'$ by augmenting any mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\mathcal{I}'}$, therefore, there is an extended interpretation $\tilde{\mathcal{J}}' \in \mathcal{E}(\mathcal{I}')$ such that $\varphi_{\tilde{\mathcal{J}}'}^{-1}(y) = \varphi_{\tilde{\mathcal{I}}}^{-1}(x)$ as $\varphi_{\tilde{\mathcal{I}}}$ and $\varphi_{\tilde{\mathcal{J}}'}$ share the same domain, namely $N_I \cup \mathbb{N}$. Since $\mathcal{I}' \models \Sigma$ (as $\mathcal{I}' \in \mathcal{M}(\Sigma)$), by corollary 4 we get that $\tilde{\mathcal{J}}' \mathrel{|\!\!\models_{\mathrm{e}}} \Sigma$ and therefore $\tilde{\mathcal{J}}' \in \tilde{\mathcal{M}}(\Sigma)$. Now $y \notin D^{\tilde{\mathcal{J}}'}$

$$
\begin{array}{lll}
\Rightarrow & \varphi_{\tilde{\mathcal{J}}'}^{-1}(y) \not\subseteq \varphi_{\tilde{\mathcal{J}}'}^{-1}(D^{\tilde{\mathcal{J}}'}) & \\
\Rightarrow & \varphi_{\tilde{\mathcal{J}}'}^{-1}(y) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}) & \text{as } \tilde{\mathcal{J}}' \in \tilde{\mathcal{M}}(\Sigma) \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}) & \text{as } \varphi_{\tilde{\mathcal{I}}}^{-1}(x) = \varphi_{\tilde{\mathcal{J}}'}^{-1}(y) \\
\Rightarrow & x \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})\Big) &
\end{array}
$$

which is a contradiction.

Now for the right to left direction, suppose there is $a \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $\Sigma \models D(a)$. Corollary 4 along with the fact that both Σ and D are non-epistemic implies that $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} D(a)$. This implies that for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ we have that $a^{\tilde{\mathcal{J}}} \in D^{\tilde{\mathcal{J}}}$, which by definition of $\varphi^{\tilde{\mathcal{J}}}$ implies that $\varphi_{\tilde{\mathcal{J}}}^{-1}(a^{\tilde{\mathcal{J}}}) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$. Now as $a^{\tilde{\mathcal{J}}} = \varphi_{\tilde{\mathcal{J}}}(a)$ for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get that $\varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(a)) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$. This implies that $a \in \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$ for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. In other words,

$$a \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}})$$

By definition of $\varphi_{\tilde{\mathcal{I}}}$,

$$\varphi_{\tilde{\mathcal{I}}}(a) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}))$$

Now as, $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}}$, therefore,

$$a^{\tilde{\mathcal{I}}} \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}))$$

and hence

$$x \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}}}))$$

as $a^{\tilde{\mathcal{I}}} = a^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = x$. By semantics of $\mathbf{K}$, we get that $x \in C^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$.

□

A similar property can be proved for the roles as well. But again, here we have to be careful about the exceptional case of the roles equivalent to the universal role. The idea is that the extension of a role $\mathbf{K}R$, with R equivalent to the universal role U, and that of the role R, under the extended semantics, coincides. To see this we prove the following:

Claim 2. *Let Σ be a non-epistemic knowledge base. For any non-epistemic role R with $\Sigma \models R \equiv U$ and for each extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$, we have $\mathbf{K}R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$.*

Proof. Note that by definition

$$\mathbf{K}R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(R^{\tilde{\mathcal{J}}})\Big) \qquad (*)$$

Since $\Sigma \models R \equiv U$, $R^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for each $\mathcal{I} \in \mathcal{M}(\Sigma)$. Now as both R and Σ are $\mathbf{K}$-free, by Fact 4 we get $\tilde{\mathcal{M}}(\Sigma) = \mathcal{E}(\mathcal{M}(\Sigma))$ and therefore, it follows from Lemma 7 that $R^{\tilde{\mathcal{J}}} = \Delta^{\tilde{\mathcal{J}}} \times \Delta^{\tilde{\mathcal{I}}}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. This together with $(*)$ implies that

$$\mathbf{K}R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = \varphi_{\tilde{\mathcal{I}}}((N_I \cup \mathbb{N}) \times (N_I \cup \mathbb{N}))$$

as $N_I \cup \mathbb{N}$ is the domain of $\varphi^{\tilde{\mathcal{J}}}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. Now since $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{I}}}$, we get that $\mathbf{K}R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\tilde{\mathcal{I}}}$ and hence $\mathbf{K}R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}} \times \Delta^{\tilde{\mathcal{I}}}$.

□

Results similar to Lemma 5 can be proved for roles of the form $\mathbf{K}R$. Like in classical semantics here we have that

- two elements are "known‘" to be related via R in Σ if so is stated in Σ via some role assertion in which case both elements are named,

- every elements is “known” to be related to itself via R whenever Σ enforce so via some axiom.

Additionally in our semantics, via axiomatization in Σ we can have:

- an element x is “known” to be related to some other element via R, whenever Σ enforces x to be related to all the elements of the domain,
- whenever Σ enforces every element to be related to an element x via R, then the relationship is “’known’.

Note that the additional cases we have here are due to the additional constructs we have in $\mathcal{SROIQ}$. Further, in the last two cases, the element x need to be named in order to automatize the restrictions. In the following we formally prove the above characteristic of $\mathbf{K}$ in $\mathcal{SROIQ}$.

Lemma 9. *Let Σ be a $\mathcal{SROIQ}$ knowledge base. Let $R = \mathbf{K}P$ be an epistemic role such that $\Sigma \not\models P \equiv U$. For any extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$ and any $x, y \in \Delta^{\tilde{\mathcal{I}}}$, we have that $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ if and only if at least one of the following holds:*

(a) there are individual names $a, b \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$, $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$ and $\Sigma \models P(a, b)$.

(b) there is an individual name $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models \top \sqsubseteq \exists P^-.\{a\}$.

(c) there is an individual name $b \in N_I$ with $b^{\tilde{\mathcal{I}}} = y$ and $\Sigma \models \top \sqsubseteq \exists P.\{b\}$.

(d) $x = y$ and $\Sigma \models \exists\top \sqsubseteq P.\mathsf{Self}$.

Proof. For the left to right direction, suppose that $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ but neither of (a), (b), (c) or (d) holds. We distinguish the following cases:

(1) There are $a, b \in N_I$ with $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$. Regardless of whether $x \neq y$ or $\Sigma \not\models \top \sqsubseteq \exists P.\mathsf{Self}$, since (a) does not hold, we have that $\Sigma \not\models P(a, b)$. It means that there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$ but $\mathcal{I} \not\models P(a, b)$ i.e., $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin P^{\mathcal{I}}$. Define an extended interpretation $\tilde{\mathcal{K}}$ as follows:

- $\Delta^{\tilde{\mathcal{K}}} = \Delta^{\mathcal{I}}$,
- $\cdot^{\tilde{\mathcal{K}}} = \cdot^{\mathcal{I}}$,
- $\varphi_{\tilde{\mathcal{K}}}(c) = c^{\tilde{\mathcal{K}}}$ for each $c \in N_I$.

By definition, $\tilde{\mathcal{K}} \in \mathcal{E}(\mathcal{I})$. As $\mathcal{I} \models \Sigma$, by Corollary 4, we get that $\tilde{\mathcal{I}} \mid\!\models_{\mathrm{e}} \Sigma$ and by Lemma 7, we get that $(a^{\tilde{\mathcal{K}}}, b^{\tilde{\mathcal{K}}}) \notin P^{\tilde{\mathcal{K}}}$. Now by definition of $\varphi^{\tilde{\mathcal{K}}}$ we have that $\varphi^{\tilde{\mathcal{K}}}(c) = c^{\tilde{\mathcal{K}}}$ for each $c \in N_I$ and therefore,

$$\varphi_{\tilde{\mathcal{K}}}^{-1}(\varphi^{\tilde{\mathcal{K}}}(a)) \times \varphi_{\tilde{\mathcal{K}}}^{-1}(\varphi^{\tilde{\mathcal{K}}}(b)) \not\subseteq \varphi_{\tilde{\mathcal{K}}}^{-1}(P^{\tilde{\mathcal{K}}})$$

which means that $(a, b) \notin \varphi_{\tilde{\mathcal{K}}}^{-1}(P^{\tilde{\mathcal{K}}})$. Since $\tilde{\mathcal{K}} \in \tilde{\mathcal{M}}(\Sigma)$, we get

$$(a, b) \notin \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

and thus

$$(\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}(b)) \notin \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})))$$

follows from the definition of $\varphi_{\tilde{\mathcal{I}}}$. But we have $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x$ and $\varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{I}}} = y$, therefore

$$(x, y) \notin \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})))$$

and therefore by semantics of K, we get that $(x, y) \notin \mathsf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $\mathsf{K}P = R$, which is a contradiction.

(2) y is anonymous and there is $a \in N_I$ such that $a^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = x$. As (d) does not hold, therefore, either $x \neq y$ or $\Sigma \not\models \top \sqsubseteq \exists P.\mathsf{Self}$. In any of the cases, by the assumption it follows from (b) in particular that $\Sigma \not\models \top \sqsubseteq \exists P^{-}.\{a\}$. Therefore, there is an interpretation $\mathcal{I}$ with $\mathcal{I} \models \Sigma$ but $\mathcal{I} \not\models \top \sqsubseteq \exists P^{-}.\{a\}$ i.e., there is a $u \in \Delta^{\mathcal{I}}$ such that $u \notin [\exists P^{-}.\{a\}]^{\mathcal{I}}$ which implies that $(a^{\mathcal{I}}, u) \notin P^{\mathcal{I}}$. By the definition of $\mathcal{E}(\mathcal{I})$, there is an extended interpretation $\tilde{\mathcal{I}}' \in \mathcal{E}(\mathcal{I})$ such that $\varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(y)$. Again this is the case as both $\varphi_{\tilde{\mathcal{I}}}$ and $\varphi_{\tilde{\mathcal{I}}'}$ share the

common domain $N_I \cup \mathbb{N}$. Since $(a^{\mathcal{I}}, u) \notin P^{\mathcal{I}}$ and $\tilde{\mathcal{I}}' \in \mathcal{E}(\mathcal{I})$, by Lemma 7, therefore, $(a^{\tilde{\mathcal{I}}'}, u) \notin P^{\tilde{\mathcal{I}}'}$ which implies

$$
\begin{array}{lll}
& \varphi_{\tilde{\mathcal{I}}'}^{-1}(a^{\tilde{\mathcal{I}}'}) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}'}^{-1}(\varphi^{\tilde{\mathcal{I}}'}(a)) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \text{as } a^{\tilde{\mathcal{I}}'} = \varphi^{\tilde{\mathcal{I}}'}(a) \\
\Rightarrow & \{a\} \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \text{as } a^{\tilde{\mathcal{I}}'} = \varphi^{\tilde{\mathcal{I}}'}(a) \\
\Rightarrow & \{a\} \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \tilde{\mathcal{I}}' \in \tilde{\mathcal{M}}(\Sigma) \\
\Rightarrow & \{a\} \times \varphi_{\tilde{\mathcal{I}}}^{-1}(y) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(y) \\
\Rightarrow & (\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}(\varphi_{\tilde{\mathcal{I}}}^{-}(y))) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \text{by Def. of } \varphi_{\tilde{\mathcal{I}}} \\
\Rightarrow & (\varphi_{\tilde{\mathcal{I}}}(a), y) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \\
\Rightarrow & (x, y) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \text{as } \varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x \\
\Rightarrow & (x, y) \notin \mathbf{K}P^{\mathcal{I},\mathcal{M}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} & \text{by semantics}
\end{array}
$$

which is a contradiction.

(3) x is anonymous and there is $b \in N_I$ with $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$. Again as (d) does not hold, either $x \neq y$ or $\Sigma \not\models \top \sqsubseteq \exists P.\mathsf{Self}$. In either of the cases, it follows from (c) particularly that $\Sigma \not\models \top \sqsubseteq \exists P.\{b\}$. In other words, there is an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \Sigma$ but $\mathcal{I} \not\models \top \sqsubseteq \exists P.\{b\}$ i.e., there is a $u \in \Delta^{\mathcal{I}}$ such that $u \notin [\exists P.\{b\}]^{\mathcal{I}}$. Consequently, $(u, b^{\mathcal{I}}) \notin P^{\mathcal{I}}$. Again by the definition of $\mathcal{E}(\mathcal{I})$, there is an extended interpretation $\tilde{\mathcal{I}}'$ such that $\varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(x)$. Now by Lemma 7, $(u, b^{\mathcal{I}}) \notin P^{\mathcal{I}}$ implies

$$
\begin{array}{lll}
& (u, b^{\tilde{\mathcal{I}}'}) \notin P^{\tilde{\mathcal{I}}'} & \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(\varphi_{\tilde{\mathcal{I}}'}(b)) \not\subseteq \varphi_{\tilde{\mathcal{I}}'}^{-1}(P^{\tilde{\mathcal{I}}'}) & \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) \times \varphi_{\tilde{\mathcal{I}}'}^{-1}(\varphi_{\tilde{\mathcal{I}}'}(b)) \not\subseteq \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \tilde{\mathcal{I}}' \in \tilde{\mathcal{M}}(\Sigma) \\
\Rightarrow & \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \times \{b\} \notin \bigcap_{\tilde{\mathcal{J}} \subseteq \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}) & \text{as } \varphi_{\tilde{\mathcal{I}}'}^{-1}(u) = \varphi_{\tilde{\mathcal{I}}}^{-1}(x) \\
\Rightarrow & (\varphi_{\tilde{\mathcal{I}}}(\varphi_{\tilde{\mathcal{I}}}^{-1}(x)), \varphi_{\tilde{\mathcal{I}}}(b)) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \\
\Rightarrow & (x, y) \notin \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big) & \text{as } \varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{I}}} = y \\
\Rightarrow & (x, y) \notin \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} & \text{by semantics}
\end{array}
$$

which is a contradiction.

We have shown that at least one of (a), (b), (c) or (d) should hold given that $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}}$. For right to left direction, suppose either (a), (b), (c) or (d) holds. We have to show then $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Thus we make the following case distinction:

(1) There are $a, b \in N_I$ such that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $b^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = y$ and $\Sigma \models P(a, b)$:
Since both P and Σ contain no occurrence of $\mathbf{K}$, by Corollary 6 we get that $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} P(a, b)$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, $(a^{\tilde{\mathcal{J}}}, b^{\tilde{\mathcal{J}}}) \in P^{\tilde{\mathcal{J}}}$ which implies that

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(a^{\tilde{\mathcal{J}}}) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(b^{\tilde{\mathcal{J}}}) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. As $c^{\tilde{\mathcal{J}}} = \varphi_{\tilde{\mathcal{J}}}(c)$ for $c \in N_I$ and $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, therefore,

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(a)) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(b))) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ i.e.,

$$(a, b) \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

which implies that

$$(\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}(b)) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})))$$

by using the definition of $\varphi_{\tilde{\mathcal{I}}}$. Thus

$$(x, y) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))) = \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$$

as $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x$ and $\varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{I}}} = y$. Hence, $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

(2) There is an individual $a \in N_I$ with $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = x$ and $\Sigma \models \top \sqsubseteq \exists P^-.\{a\}$:
By Corollary 6 and the fact that $\Sigma \models \top \sqsubseteq \exists P^-.\{a\}$, we get $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} \top \sqsubseteq$

$\exists P^{-}.\{a\}$ as neither Σ nor $\top \sqsubseteq \exists P^{-}.\{a\}$ contains any occurrence of **K**. Hence, for each extended interpretation $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get that $\tilde{\mathcal{J}} \models \top \sqsubseteq \exists P^{-}.\{a\}$, i.e., every $u \in \Delta^{\tilde{\mathcal{J}}}$ is such that $u \in [\exists P^{-}.\{a\}]^{\tilde{\mathcal{J}}}$. This means that for every $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$, we have that $(a^{\tilde{\mathcal{J}}}, u) \in P^{\tilde{\mathcal{J}}}$. Now, using the definition of $\varphi_{\tilde{\mathcal{J}}}^{-1}$, for $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get that

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(a^{\tilde{\mathcal{J}}}) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(u) = \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi^{\tilde{\mathcal{J}}}(a)) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(u) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $u \in \Delta^{\tilde{\mathcal{I}}}$ and $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. Note that $a \in \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(a))$. Further since $\varphi_{\tilde{\mathcal{J}}}$ has domain $N_I \cup \mathbb{N}$ and $u \in \Delta^{\tilde{\mathcal{J}}}$ is arbitrary, for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get $(a, t) \in \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$ and thus

$$(a, t) \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for each $t \in (N_I \cup \mathbb{N})$. Using the definition of $\varphi_{\tilde{\mathcal{I}}}$, therefore,

$$(\varphi_{\tilde{\mathcal{I}}}(a), \varphi_{\tilde{\mathcal{I}}}^{-1}(t)) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

and since $\varphi_{\tilde{\mathcal{I}}}(a) = a^{\tilde{\mathcal{I}}} = x$ and $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping with range $\Delta^{\tilde{\mathcal{I}}}$, therefore we get that

$$(x, v) \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big)$$

for every $v \in \Delta^{\tilde{\mathcal{I}}}$. In particular,

$$(x, y) \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})\Big)$$

which, by semantics implies that $(x, y) \in \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

(3) There is $b \in N_I$ with $b^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} = y$ and $\Sigma \models \top \sqsubseteq \exists P.\{b\}$:
By Corollary 6, we get that $\Sigma \Vdash_{\mathrm{e}} \top \sqsubseteq \exists P.\{b\}$ as both Σ and $\top \sqsubseteq \exists P.\{b\}$ are **K**-free. In other words, for every $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we have that $(u, b^{\tilde{\mathcal{J}}}) \in$

$P^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)}$ for any $u \in \Delta^{\tilde{\mathcal{J}}}$. Now as $\varphi^{\tilde{\mathcal{J}}}(b) = b^{\tilde{\mathcal{J}}}$, using the definition of $\varphi_{\tilde{\mathcal{J}}}^{-1}$ we get that

$$\varphi_{\tilde{\mathcal{J}}}^{-1}(u) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi^{\tilde{\mathcal{J}}}(b)) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$$

for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$. Again as $b \in \varphi_{\tilde{\mathcal{J}}}^{-1}(\varphi_{\tilde{\mathcal{J}}}(b))$ and $\varphi_{\tilde{\mathcal{J}}}$ has domain $N_I \cup \mathbb{N}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we get $(t, b) \in \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$ and therefore,

$$(t, b) \in \varphi_{\tilde{\mathcal{J}}}^{-1} \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} (P^{\tilde{\mathcal{J}}})$$

for each $t \in (N_I \cup \mathbb{N})$. Using definition of $\varphi_{\tilde{\mathcal{I}}}$ we get

$$(\varphi_{\tilde{\mathcal{I}}}(t), \varphi_{\tilde{\mathcal{I}}}(b)) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

for any $t \in (N_I \cup \mathbb{N})$. Since $\varphi_{\tilde{\mathcal{I}}}(b) = b^{\tilde{\mathcal{J}}} = y$ and $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping with range $\Delta^{\tilde{\mathcal{I}}}$, we get that

$$(v, y) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

for any $v \in \Delta^{\tilde{\mathcal{I}}}$. In particular,

$$(x, y) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}}))$$

which by semantics implies that $(x, y) \in R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

(4) $x = y$ and $\Sigma \models \top \sqsubseteq \exists P.\mathsf{Self}$:
As both Σ and the axiom $\top \sqsubseteq \exists P.\mathsf{Self}$ are $\mathbf{K}$-free, Corollary 6 implies that $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} \top \sqsubseteq \exists P.\mathsf{Self}$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$, we have that $(u, u) \in P^{\tilde{\mathcal{J}}}$ and by definition of $\varphi_{\tilde{\mathcal{J}}}^{-1}$, therefore, $\varphi_{\tilde{\mathcal{J}}}^{-1}(u) \times \varphi_{\tilde{\mathcal{J}}}^{-1}(u) \subseteq \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$. But as $\varphi_{\tilde{\mathcal{J}}}$, for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, is a mapping with domain $N_I \cup \mathbb{N}$, we get $(t, t) \in \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}}})$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $t \in (N_I \cup \mathbb{N})$. In other words,

$$(t, t) \in \bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(P^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)})$$

for each $t \in (N_I \cup \mathbb{N})$. Now using the definition of $\varphi_{\tilde{\mathcal{I}}}$, we get that

$$(\varphi_{\tilde{\mathcal{I}}}(t), \varphi_{\tilde{\mathcal{I}}}(t)) \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)} P^{\tilde{\mathcal{J}}, \tilde{\mathcal{M}}(\Sigma)}) = \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$$

for each $t \in (N_I \cup \mathbb{N})$. As $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{I}}}$, therefore, $(v, v) \in \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ for every $v \in \Delta^{\tilde{\mathcal{I}}}$. In particular, $(x, x) \in \mathbf{K}P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ and therefore $(x, x) \in R^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $R = \mathbf{K}P$.

□

Having proved Lemma 8 and Lemma 9 we now extend the translation procedure Φ_Σ of Definition 38 in a way that it maps (complex) epistemic concept expressions to non-epistemic ones which are equivalent in all models of the given $\mathcal{SROIQ}$ knowledge base Σ. We represent this extension by $\tilde{\Phi}_\Sigma$.

Definition 44. *Let Σ be a $\mathcal{SROIQ}$ knowledge base Σ. Further let P be a role, and R and S be non-epistemic roles all different from U. We define a function $\tilde{\Phi}_\Sigma$ mapping $\mathcal{SROIQK}$ concept expressions to $\mathcal{SROIQ}$ concept expressions as in Figure 7.4 where we let $\{\} = \emptyset = \bot$.*

We now prove that the method based on the translation function $\tilde{\Phi}_\Sigma$ as in Definition 44 is indeed correct. For the formal proof of the correctness of this procedure, there are several points worth mentioning. Firstly, note that under the new semantics, we do not enforce the UNA, hence, no need to axiomatize it in the knowledge base explicitly. Secondly, we allow for both finite and infinite models of the knowledge base. In fact, a property similar to Lemma 2 can not be proved for $\mathcal{SROIQ}$ knowledge bases, i.e., it is not guaranteed that for all finite models of a $\mathcal{SROIQ}$ knowledge base there exist infinite ones with the same behavior. Hence a completely different approach is taken in proving the correctness of the extended Φ_Σ. More notably, in the current epistemic semantics, Lemma 3 allows us to exchange the role of any two anonymous individuals in a model without compromising modelhood. This holds in the extended semantics case as well, but does not suffice for showing formal correctness of Φ_Σ. Instead, we use the definition of $\varphi_{\tilde{\mathcal{I}}}$ for an extended interpretation $\tilde{\mathcal{I}}$. We now show that the extension of a $\mathcal{SROIQK}$ concept

$$
\tilde{\Phi}_\Sigma : \begin{cases}
C & \mapsto C \quad \text{if } C \text{ is an atomic or one-of concept, } \top \text{ or } \bot; \\
\mathbf{K}D & \mapsto \begin{cases} \top & \text{if } \Sigma \models \tilde{\Phi}_\Sigma(D) \equiv \top \\ \{a \in N_I \mid \Sigma \models \tilde{\Phi}_\Sigma(D)(a)\} & \text{otherwise} \end{cases} \\
\exists \mathbf{K}S.\mathsf{Self} & \mapsto \begin{cases} \exists S.\mathsf{Self} & \text{if } \Sigma \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \{a \in N_I \mid \Sigma \models S(a,a)\} & \text{otherwise} \end{cases} \\
C_1 \sqcap C_2 & \mapsto \tilde{\Phi}_\Sigma(C_1) \sqcap \tilde{\Phi}_\Sigma(C_2) \\
C_1 \sqcup C_2 & \mapsto \tilde{\Phi}_\Sigma(C_1) \sqcup \tilde{\Phi}_\Sigma(C_2) \\
\neg C & \mapsto \neg\tilde{\Phi}_\Sigma(C) \\
\exists R.D & \mapsto \exists R.\tilde{\Phi}_\Sigma(D) \\
\exists \mathbf{K}P.D & \mapsto \begin{cases} \bigsqcup_{a \in N_I} \{a\} \sqcap \exists P.(\{b \in N_I \mid \Sigma \models P(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\}) \\ \sqcup \exists P.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \\ \sqcup \{a \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^{-}\{a\}\} \sqcap \exists P.\tilde{\Phi}_\Sigma(D) \\ \sqcup \begin{cases} \tilde{\Phi}_\Sigma(D) & \text{if } \Sigma \models \top \sqsubseteq \exists P.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \end{cases} \\
\forall R.D & \mapsto \forall R.\tilde{\Phi}_\Sigma(D) \\
\forall \mathbf{K}P.D & \mapsto \neg\tilde{\Phi}_\Sigma(\exists \mathbf{K}P.\neg D) \\
\geqslant nS.D & \mapsto \geqslant nS.\tilde{\Phi}_\Sigma(D) \\
\geqslant n\mathbf{K}S.D & \mapsto \begin{cases} \bigsqcup_{a \in N_I} \{a\} \sqcap \geqslant nS.(\{b \in N_I \mid \Sigma \models S(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\}) \\ \sqcup \{a \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^{-}.\{a\}\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D) \\ \sqcup \geqslant nS.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \\ \sqcup \begin{cases} \geqslant (n-1)S.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \sqcap \\ \tilde{\Phi}_\Sigma(D) \sqcap \neg\{a \mid a \in N_I\} & \text{if } \Sigma \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \end{cases} \\
\leqslant nS.D & \mapsto \leqslant nS.\tilde{\Phi}_\Sigma(D) \\
\leqslant n\mathbf{K}S.D & \mapsto \neg\tilde{\Phi}_\Sigma(\geqslant(n{+}1)\mathbf{K}S.D) \\
\Xi\mathbf{K}R.D & \mapsto \Xi R.\tilde{\Phi}_\Sigma(D) \quad \text{for } \Xi \in \{\forall, \exists, \geqslant n, \leqslant n\} \text{ and } \Sigma \models R \equiv U
\end{cases}
$$

Figure 7.4: Translation function $\tilde{\Phi}_\Sigma$

and the extension of $\mathcal{SROIQ}$ concept obtained using the translation function $\tilde{\Phi}_\Sigma$, agree under each extended interpretation in $\tilde{\mathcal{M}}(\Sigma)$.

Lemma 10. *Let Σ be a $\mathcal{SROIQ}$-knowledge base and C be a $\mathcal{SROIQK}$ concept. Then for any extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$, we have that $C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = [\tilde{\Phi}_\Sigma(C)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.*

Proof. For the proof we use induction on the structure of C and show that for each $x \in \Delta^{\tilde{\mathcal{I}}}$, we have that $x \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ if and only if $x \in (\tilde{\Phi}_\Sigma(C))^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. For the base case; C is atomic or one-of concept, and the cases where $C = \top$ and $C = \bot$, the lemma follows immediately from the definition of $\tilde{\Phi}_\Sigma$. For the cases, where $C = C_1 \sqcap C_2$, $C = C_1 \sqcup C_2$ or $C = \neg D$, it follows from the induction hypothesis. The non-trivial cases are considered in the following.

(i) $C = \mathbf{K}D$ and $\Sigma \not\models D \equiv \top$:
By Lemma 9, $x \in \mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff there is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models D(a)$. This is equivalent to $x \in \{a \in N_I \mid \Sigma \models D(a)\}^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and hence, by definition of $\tilde{\Phi}_\Sigma$, to $x \in [\tilde{\Phi}_\Sigma(\mathbf{K}D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(ii) $C = \mathbf{K}D$ and $\Sigma \models D \equiv \top$:
By Corollary 6 we get hat $\Sigma \mathrel{|\!\!\models_{\mathrm{e}}} D \equiv \top$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ we have that $D^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}}$. Consequently, we get

$$\varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}} \varphi_{\tilde{\mathcal{J}}}^{-1}(D^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)})\Big) = \varphi_{\tilde{\mathcal{I}}}(N_I \cup \mathbb{N})$$

as $\varphi_{\tilde{\mathcal{I}}}^{-1}$ is a surjective mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\tilde{\mathcal{J}}}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$. Now $\varphi_{\tilde{\mathcal{I}}}(N_I \cup \mathbb{N})$ yields $\Delta^{\tilde{\mathcal{I}}}$ as it is a surjective mapping to $\Delta^{\tilde{\mathcal{I}}}$. Hence we get that $\mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = \Delta^{\tilde{\mathcal{I}}} = \top^{\tilde{\mathcal{I}}}$, which by definition of $\tilde{\Phi}_\Sigma$ yields that $\mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = [\tilde{\Phi}_\Sigma(\mathbf{K}D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Consequently, $x \in \mathbf{K}D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff $x \in \tilde{\Phi}_\Sigma(\mathbf{K}D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(iii) $C = \exists \mathbf{K}S.\mathsf{Self}$:
First to prove the "if" part, suppose that $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$. By semantics, $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that for each $\tilde{\mathcal{J}} \models \tilde{\mathcal{M}}(\Sigma)$ we have that $(x, x) \in S^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)}$. In particular, $(x, x) \in S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ i.e., $x \in [\exists S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

Therefore, $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as by definition $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists S.\mathsf{Self}$.

Suppose that it is not the case that $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$. By semantics, $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $(x, x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By Lemma 9, there is $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models S(a, a)$. Hence, $a \in \{c \in N_I \mid \Sigma \models S(c, c)\}$, which implies that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as by definition $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \{c \in N_I \mid \Sigma \models S(c, c)\}$.

Now we prove the "only-if" part. Suppose that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Based on the definition of $\tilde{\Phi}_\Sigma$ we distinguish the following cases.

- $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \exists S.\mathsf{Self}$:
  Like in (4) of Lemma 9, we can show that $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.
- $\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self}) = \{c \in N_I \mid \Sigma \models S(c, c)\}$:
  $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.\mathsf{Self})]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there some $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models S(a, a)$. By Lemma 9, it means that $(a^{\tilde{\mathcal{I}}}, a^{\tilde{\mathcal{I}}}) = (x, x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and hence $x \in [\exists \mathbf{K}S.\mathsf{Self}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(iv) $C = \exists P.D$ and P is a non-epistemic role:
By the semantics, $x \in [\exists P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ iff there is a $y \in \Delta^{\tilde{\mathcal{I}}}$ with $(x, y) \in P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore by induction, $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Hence, it is equivalent to $x \in [\exists P.\tilde{\Phi}\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which, by definition, is the case if and only if $x \in [\tilde{\Phi}_\Sigma(\exists P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(v) $C = \exists \mathbf{K}P.D$:
$\underline{x \in (\exists \mathbf{K}P.D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \text{ implies } x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}P.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}}$:
By semantics, $x \in (\exists \mathbf{K}P.D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ means that there is a $y \in \Delta^{\tilde{\mathcal{I}}}$ with $(x, y) \in (\mathbf{K}P)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ hence, by induction $y \in \tilde{\Phi}\Sigma(D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By Lemma 9, $(x, y) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}}$ implies that at least one of the following should hold:

- There are $a, b \in N_I$ with $a^{\tilde{\mathcal{I}}} = x, b^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models P(a, b)$. This means that $b \in \{c \in N_I \mid \Sigma \models P(a, c)\}$, therefore,

$$b^{\tilde{\mathcal{I}}} \in [\{c \in N_I \mid \Sigma \models P(a, c)\}]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$$

Now as $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$, therefore,

$$x \in [\{a\} \sqcap \exists P.(\{c \in N_I \mid \Sigma \models P(a, c)\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$$

which by definition of $\tilde{\Phi}_\Sigma$ implies that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K} P.D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$

- There is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $\Sigma \models \top \sqsubset \exists P^-.\{a\}$. This means that $a \in \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^- \{c\}\}$, therefore,

$$a^{\tilde{\mathcal{I}}} \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^- \{c\}\}]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$$

Now as Σ is $\mathbf{K}$-free, by Corollary 6 $\Sigma \models \top \sqsubseteq \exists P^-.\{a\}$ implies that $\Sigma \mid\!\models_{\mathrm{e}} \top \sqsubseteq \exists P^-.\{a\}$ i.e., $(a^{\tilde{\mathcal{J}}}, u) \in P^{\tilde{\mathcal{J}}, \tilde{\mathcal{M}}(\Sigma)}$ for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{J}}}$. In particular, $(a^{\tilde{\mathcal{I}}}, y) = (x, y) \in P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$. Consequently, we get that $x \in [\exists P.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$. This along with $(*)$ implies that $x = a^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K} P.D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

- There is a $b \in N_I$ with $b^{\tilde{\mathcal{I}}} = y$ and $\Sigma \models \top \sqsubseteq \exists P.\{b\}$ i.e., $b \in \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\}$ which implies that

$$b^{\tilde{\mathcal{I}}} \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\}]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)} \qquad (*)$$

Since Σ is $\mathbf{K}$-free we get from Corollary 6 that $\Sigma \mid\!\models_{\mathrm{e}} \top \sqsubseteq \exists P.\{b\}$ i.e., for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$ and $u \in \Delta^{\tilde{\mathcal{I}}}$ we have that $(u, b^{\tilde{\mathcal{J}}}) \in P^{\tilde{\mathcal{J}}, \tilde{\mathcal{M}}(\Sigma)}$. In particular, $(x, b^{\tilde{\mathcal{I}}}) = (x, y) \in P^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$, which along with $(*)$ and the fact that $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ immediately implies that $x \in [\exists \mathbf{K} P.D]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

- $x = y$ and $\Sigma \models \top \sqsubseteq \exists P.\mathsf{Self}$. As $\tilde{\Phi}_\Sigma(\exists \mathbf{K} P.D) = \tilde{\Phi}_\Sigma(D)$, therefore, we get that $x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K} P.D)]^{\tilde{\mathcal{I}}, \tilde{\mathcal{M}}(\Sigma)}$ as $x = y \in [\tilde{\Phi}_\Sigma(D)]^{\mathcal{I}, \mathcal{M}(\Sigma)}$.

$x \in [\tilde{\Phi}_\Sigma(\exists \mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $x \in [\exists \mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$:

According to the definition of $\tilde{\Phi}_\Sigma$, we make the following case distinction.

- There is an $a \in N_I$ such that $a^{\tilde{\mathcal{I}}} = x$ and $x \in [\exists P.(\{c \in N_I \mid \Sigma \models P(a,c)\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ i.e., there is some $b \in N_I$ such that $b^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $\Sigma \models P(a,b)$. This, by Lemma 9, implies that $(a^{\tilde{\mathcal{I}}}, b^{\tilde{\mathcal{I}}}) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Hence, we get that $x = a^{\tilde{\mathcal{I}}} \in [\exists \mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $b^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.
- $x \in [\exists P.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which implies that there is some $b \in N_I$ such that $\Sigma \models \top \sqsubseteq \exists P.\{b\}$ and $b^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. It follows from Lemma 9, that $(x, b^{\tilde{\mathcal{I}}}) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which immediately implies that $x \in [\mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $b^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.
- There is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models \top \sqsubseteq \exists P^-.\{a\}$ and $x \in [\exists P.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. It means that there is a $y \in \Delta^{\tilde{\mathcal{I}}}$ such that $y \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By Lemma 9, we get that $(x,y) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and hence, by semantics, $x \in [\exists \mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.
- $x \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore by induction $x \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Since it is the case that $\Sigma \models \exists P.\mathsf{Self}$, Lemma 9 implies that $(x,x) \in \mathbf{K}P^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and therefore, $x \in [\exists \mathbf{K}P.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(vi) $C = \geqslant n\mathbf{K}S.D$

$x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$

By semantics, $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there are pair-wise distinct $y_1, \ldots, y_m \in \Delta^{\tilde{\mathcal{I}}}$ with $m \geq n$ such that $(x, y_i) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $y_i \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore by induction, $y_i \in \tilde{\Phi}_\Sigma(D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By Lemma 9, this implies that at least one of the following should hold:

- There are $a, b_1, \ldots, b_m$ with $a^{\tilde{\mathcal{I}}} = x$ and $b_i^{\tilde{\mathcal{I}}} = y_i$ such that $\Sigma \models S(a, b_i)$ for $i \leq m$ i.e., $b_i \in \{c \in N_I \mid \Sigma \models (a,c)\}$. Since Σ is $\mathbf{K}$-free, it follows from Corollary 6 that $\Sigma \mathrel{|\!\!\models_{e}} S(a,b_i)$ for $i \leq m$. This implies that $(a^{\tilde{\mathcal{I}}}, b_i^{\tilde{\mathcal{I}}}) = (x, y_i) \in S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for each $i \leq m$. As $m \geq n$ and $b_i \in \{c \in N_I \mid \Sigma \models S(a,c)\}$ with $y_i = b_i^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore, we get that $a^{\tilde{\mathcal{I}}} \in [\geqslant nS.(\{c \in N_I \mid \Sigma \models S(a,c)\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which immediately implies that $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $x \in \{a\}^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

- There is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models \top \sqsubseteq \exists S^-.\{a\}$. This implies that

$$a \in \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\}\} \qquad (*)$$

Now by Corollary 6, $\Sigma \models \top \sqsubseteq \exists S^-.\{a\}$ implies that $\Sigma \mathrel{|\!\!\models_{e}} \top \sqsubseteq \exists S^-.\{a\}$. It means that for any $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, we have that $(a^{\tilde{\mathcal{J}}}, v) \in S^{\tilde{\mathcal{J}}}$ for arbitrary $v \in \Delta^{\tilde{\mathcal{J}}}$. In particular, $(x, y_i) \in S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $a^{\tilde{\mathcal{I}}} = x$ and $y_i \in \Delta^{\tilde{\mathcal{I}}}$ for $i \leq m$. Now since $y_i \in \tilde{\Phi}_\Sigma(D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $m \geq n$, it follows from the semantics that $x \in [\geqslant nS.\Phi_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. This along with $(*)$ implies that $x \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and therefore, by definition of $\tilde{\Phi}_\Sigma$, we get that $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]$.

- There are $b_1, \ldots, b_m \in N_I$ with $b_i^{\tilde{\mathcal{I}}} = y_i$ and $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ for $i \leq m$. This means that

$$\{b_1, \ldots, b_m\} \subseteq \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{c\}\} \qquad (*)$$

Now by Corollary 6, $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ implies that $\Sigma \mathrel{|\!\!\models_{e}} \top \sqsubseteq \exists S.\{b_i\}$ for $i \leq m$. Therefore, we get that for each $\tilde{\mathcal{J}} \in \tilde{\mathcal{M}}(\Sigma)$, $(u, b_i^{\tilde{\mathcal{J}}}) \in S^{\tilde{\mathcal{J}},\tilde{\mathcal{M}}(\Sigma)}$ for arbitrary $u \in \Delta^{\tilde{\mathcal{I}}}$ and $i \leq m$. In particular, we have that for each $i \leq m$, $(x, b_i^{\tilde{\mathcal{I}}}) \in S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Now as $b_i^{\tilde{\mathcal{I}}} = y_i \in \tilde{\Phi}_\Sigma(D)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$, consequently, it follows from $(*)$, that

$$x \in [\geqslant nS.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{c\}\} \sqcap \Phi_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$$

as $m \geq n$ and therefore, $x \in [\Phi_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

- There is a $y \in \{y_1, \ldots, y_m\}$ such that $x = y$ and $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$. Hence we have that

$$x \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \qquad (*)$$

Note that if x is named, we can proceed as the first two cases of the proof. Here hence, we assume that x is unnamed i.e.,

$$x \in [\neg\{a \mid a \in N_I\}]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \qquad (**)$$

Suppose that $x \notin [\geqslant(n-1)S.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. This means that there are $b_1, \ldots, b_k \in N_I$ with $k < (n-1)$such that $b_i \in [\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Note that $\Sigma \models \top \sqsubseteq \exists S.\mathsf{Self}$, by Corollary 6 implies that $\Sigma \mathrel{|\!\!\models_{e}} \top \sqsubseteq \exists S.\mathsf{Self}$. Hence, we get that $(x, x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which along with the fact that $x = y \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that $x[\geqslant 1S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. By the assumption here we get that there are distinct $z_1, \ldots, z_{m'} \in \Delta^{\tilde{\mathcal{I}}}$ with $(x, z_i) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $z_i \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m'$ and m' is at most $(n-1)$. Which is a contradiction as $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Therefore, it must be the case that $x \in [\geqslant(n-1)S.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which along with $(*)$ and $(**)$ implies that $x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as per definition of $\tilde{\Phi}_\Sigma$.

$x \in [\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$

According to the definition of Φ_Σ, at least one of the following is the case:

- There are $a, b_1, \ldots, b_m \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ such that $\Sigma \models S(a, b_i)$ and $b_i^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By Lemma 9, therefore, $(x, b_i^{\tilde{\mathcal{I}}}) = (a^{\tilde{\mathcal{I}}}, b_i^{\tilde{\mathcal{I}}}) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and by induction $b_i^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. This immediately implies that $x = a^{\tilde{\mathcal{I}}} \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $m \geq n$.
- $\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D) = \{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\}\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D)$:
$x \in [\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S^-.\{c\}\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there is an $a \in N_I$ with $a^{\tilde{\mathcal{I}}} = x$ and $a^{\tilde{\mathcal{I}}} \in [\geqslant nS.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ i.e., there are pair-wise disjoint $y_1, \ldots, y_m \in \Delta^{\tilde{\mathcal{I}}}$ with $m \geq n$ such that

$y_i \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By induction, therefore, for $i \leq m$ we have that $y_i \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Now $\Sigma \models \top \sqsubseteq \exists S^-.\{a\}$ implies that $(x, y) \in S^{\tilde{\mathcal{I}}}$ as $a^{\tilde{\mathcal{I}}} = x$ and therefore, by Lemma 9, $(x, y) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for each $y \in \Delta^{\tilde{\mathcal{I}}}$. In particular, $(x, y_i) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. As $y_i \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$ and $m \geq n$, consequently we get that $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

- $\tilde{\Phi}_\Sigma(\geqslant n\mathbf{K}S.D) = \geqslant nS.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{c\}\} \sqcap \tilde{\Phi}_\Sigma(D))$:
$x \in [\geqslant nS.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S\cdot\{c\}\} \sqcap \Phi_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ implies that there are distinct $b_1, \ldots, b_m \in N_I$ with $m \geq n$, such that $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ and $b_i^{\tilde{\mathcal{I}}} \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, therefore by induction $b_i^{\tilde{\mathcal{I}}} \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$. By Lemma 9, $\Sigma \models \top \sqsubseteq \exists S.\{b_i\}$ implies that $(x, b_i^{\tilde{\mathcal{I}}}) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for $i \leq m$ which immediately yields that $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

- x is anonymous with $x \in [\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ and $x \in [\geqslant(n-1)S.(\{c \in N_I \mid \Sigma \models \top \sqsubseteq \exists S.\{c\}\} \sqcap \tilde{\Phi}_\Sigma(D))]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$, which, as already proved in the previous case, implies

$$x \in [\geqslant(n-1)\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \qquad (*)$$

Now by Lemma 9, we have that $(x, x) \in \mathbf{K}S^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. This along with $(*)$ implies that $x \in [\geqslant n\mathbf{K}S.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ as $x \in D^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ by induction.

(vii) $C = \Xi\mathbf{K}R.D$ for $\Xi \in \{\forall, \exists, \geqslant n, \leqslant n\}$ and $\Sigma \models R \equiv U$:
By Claim 2, we have that $\mathbf{K}R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = R^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$. Hence, by induction, it follows immediately that $x \in [\Xi\mathbf{K}R.D]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ if and only if $x \in [\Xi R.\tilde{\Phi}_\Sigma(D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} = [\tilde{\Phi}_\Sigma(\Xi\mathbf{K}R.D)]^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$.

(viii) the rest of the cases can be proved analogously.

□

The translation function $\tilde{\Phi}_\Sigma$ in Definition 44 enables us to reduce the problem of entailment in $\mathcal{SROIQK}$ axioms by $\mathcal{SROIQ}$ knowledge bases to the problem of entailment of $\mathcal{SROIQ}$ axioms. Formally,

Theorem 3. *For a $\mathcal{SROIQ}$ knowledge base Σ, $\mathcal{SROIQK}$ concepts C, D, and an individual a, the following hold:*

1 $\Sigma \mid\!\models_{\mathrm{e}} C(a)$ *if and only if* $\Sigma \models \tilde{\Phi}_\Sigma(C)(a)$.

2 $\Sigma \mid\!\models_{\mathrm{e}} C \sqsubseteq D$ *if and only if* $\Sigma \models \tilde{\Phi}_\Sigma(C) \sqsubseteq \tilde{\Phi}_\Sigma(D)$.

Proof. For the first case, note that $\Sigma \mid\!\models_{\mathrm{e}} C(a)$ is equivalent to $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \in C^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ which by Lemma 10 implies that $a^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)} \in \tilde{\Phi}_\Sigma(C)^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}(\Sigma)}$ for all $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}(\Sigma)$. Since Σ and $\tilde{\Phi}_\Sigma(C)$ are **K**-free, by Fact 4 we get $\tilde{\mathcal{M}} = \mathcal{E}(\{\mathcal{I} \mid \mathcal{I} \models \Sigma\})$ and therefore by Lemma 7 we get the equivalent statement that $a^{\mathcal{I}} \in \tilde{\Phi}_\Sigma(C)^{\mathcal{I}}$ for each model $\mathcal{I}$ of Σ and therefore, $\Sigma \models C(a)$. We can prove the second case with similar arguments. □

Now as the answer of an epistemic query to a $\mathcal{SROIQ}$ knowledge base can be determined by checking the entailment of the axioms in the query from the knowledge base. Consequently, epistemic queries answering can be translated in to standard query answering.

Corollary 7. *For a given $\mathcal{SROIQ}$ knowledge base Σ and an epistemic query Q, let Q' be the (non-epistemic) query obtained by translating axioms in Q into* **K***-free using Φ_Σ . Then we have*

$$ans(Q) = ans(Q')$$

i.e. answers of Q and Q' coincide when posed to Σ.

Thus any tool implementing Definition 44 can be used to answer epistemic queries to $\mathcal{SROIQ}$ knowledge bases. Note that the definition of Φ_Σ and $\tilde{\Phi}_\Sigma$ coincide for a $\mathcal{SRIQ}\backslash U$ knowledge base Σ i.e.,

Lemma 11. *Let Σ be a $\mathcal{SRIQ}\backslash U$knowledge base. Then for any epistemic concept C formulated in $\mathcal{SROIQK}$ we have $\Phi_\Sigma(C) = \tilde{\Phi}_\Sigma(C)$.*

Proof. We prove the lemma by induction on the structure of C. For the base case with C an atomic concept, $C = \top$ or $C = \bot$, by Definition 38 and Definition 44 we have that $\Phi_\Sigma(C) = \tilde{\Phi}_\Sigma(C) = C$. The remaining cases are as following

- $C = \neg D$ for some epistemic concept D:
  By definition of Φ_Σ we have $\Phi_\Sigma(C) = \neg\Phi_\Sigma(D)$ and thus by induction $\Phi_\Sigma(C) = \neg\tilde{\Phi}_\Sigma(C)$ and thus by definition of $\tilde{\Phi_\Sigma}(C)$ we get that $\Phi_\Sigma(C) = \tilde{\Phi}_\Sigma(C)$.

- $C = C_1 \sqcup C_2$ for epistemic concepts C_1 and C_2:
  Again from definition of Φ_Σ and induction it follows that $\Phi_\Sigma(C) = \Phi_\Sigma(C_1) \sqcup \Phi_\Sigma(C_2) = \tilde{\Phi_\Sigma}(C_1) \sqcup \tilde{\Phi_\Sigma}(C_2)$ and thus by definition of $\tilde{\Phi_\Sigma}$ we get that $\Phi_\Sigma(C) = \tilde{\Phi_\Sigma}(C)$.

- $C = \exists R.D$ for a non epistemic role R and an epistemic concept D:

  Since R is **K**-free, by definition of Φ_Σ and induction we get that $\Phi_\Sigma(C) = \exists R.\Phi_\Sigma(D)$ and thus by induction, $\Phi_\Sigma(C) = \exists R.\tilde{\Phi_\Sigma}(D)$ and therefore by definition of $\tilde{\Phi_\Sigma}$, $\Phi_\Sigma(C) = \tilde{\Phi_\Sigma}(C)$.

- $C = \exists \mathbf{K}R.D$ for a non epistemic role R and an epistemic concept D:
  Note that the equality of $\Phi_\Sigma(C)$ to

$$\bigsqcup_{a \in N_I} a \sqcap \exists R.(\{b_1 \ldots b_n\}) \sqcap \Phi_\Sigma(D) \qquad (1)$$

  follows from the definition of Φ_Σ where $\{b_1 \ldots b_n\} = \{b \in N_I \mid \Sigma_{UNA} \models R(a,b)\}$. Now consider the case of $\tilde{\Phi_\Sigma}(\exists \mathbf{K}R.D)$ in Definition 44. Since we consider $\mathcal{SRIQ}\backslash U$ only, $\tilde{\Phi_\Sigma}(\exists \mathbf{K}R.D)$ is translated into

$$\bigsqcup_{a \in N_I} a \sqcap \exists R.(\{b_1 \ldots b_n\}) \sqcap \tilde{\Phi_\Sigma}(D) \qquad (2)$$

  where $\{b_1 \ldots b_n\} = \{b \in N_I | \Sigma \models P(a,b)\}$. The remaining disjuncts are not considered as they correspond to one-of or SELF constructs which we do not allow in $\mathcal{SRIQ}\backslash U$. Hence, we established the equality of (1) and (2) as by induction $\Phi_\Sigma(D) = \tilde{\Phi_\Sigma}(D)$ and thus $\Phi_\Sigma(C) = \tilde{\Phi_\Sigma}(C)$.

- $C = \geq nS.D$ for a simple non-epistemic role S and an epistemic concept D: By definition of Φ_Σ and induction we get that $\Phi_\Sigma(C) = \geq nS.\Phi_\Sigma(D) = \geq nS.\tilde{\Phi_\Sigma}(D)$ and thus $\Phi_\Sigma(C) = \tilde{\Phi_\Sigma}(C)$ follows from the definition of $\tilde{\Phi_\Sigma}$.

- $C = \geq nS.D$ for a simple non-epistemic role S and an epistemic concept D: Suppose that $n > 1$. Thus from the definition of Φ_Σ we get the equality of $\Phi_\Sigma(C)$ to

$$\bigsqcup_{a \in N_I} \{a\} \sqcap \geq nS.(\{B_1, \ldots, b_n\}) \sqcap \Phi_\Sigma(D) \qquad (3)$$

where $\{b_1, \ldots, b_n\} = \{b_\in N_I | \Sigma_{UNA} \models \Phi_\Sigma(D)\}$. Similarly from the definition of $\tilde{\Phi}_\Sigma$ we get the equality of $\tilde{\Phi}_\Sigma(C)$ to

$$\bigsqcup_{a \in N_I} \{a\} \sqcap \geq nS.(\{B_1, \ldots, b_n\}) \sqcap \tilde{\Phi}_\Sigma(D) \qquad (4)$$

where $\{b_1, \ldots, b_n\} = \{b_\in N_I | \Sigma_{UNA} \models \tilde{\Phi}_\Sigma(D)\}$. Again the remaining disjuncts are equivalent to $\perp$ as we do not allow for one-of and SELF constructs. By induction we immediately get the equality of (3) and (4) and thus $\Phi_\Sigma(C) = \tilde{\Phi}_\Sigma(C)$.

- $C = \exists \mathbf{K} U.D$ for an epistemic concept D: The equality $\Phi_\Sigma(C) = \tilde{\Phi}_\Sigma(C)$ follows immediately from induction hypothesis and definitions of Φ_Σ and $\tilde{\Phi}_\Sigma$.

The rest of the cases can be proved analogously. □

The lemma above, Theorem 1 and Theorem 3 immediately yields the following results.

Corollary 8. *For a given $\mathcal{SRIQ} \backslash U$ knowledge base Σ, $\mathcal{SROIQK}$ concepts C, D and an individual name a, we have that*

1. *$\Sigma \mid\models_{e} C(a)$ under the unique name assumption if and only if $\Sigma \mid\models C(a)$, similarly*

2. *$\Sigma \mid\models_{e} C \sqsubseteq D$ under the unique name assumption if and only if $\Sigma \mid\models C \sqsubseteq D$*

With this we conclude the chapter. In the next chapter we discuss issues related to the implementation of the function $\tilde{\Phi}_{\Sigma}$.

Part III

Implementation and Evaluation

Chapter 8
Implementation

In this chapter we present *EQuIKa*[1], a system realizing the translation procedure presented in Section 7.3.1. Note that a naïve implementation of the translation function presented in Definition 44 is not necessarily feasible in practice, in particular when dealing with knowledge bases containing relatively large number of individuals. We thus present several optimization rules. These rules, in general, speed up the computation time of EQuIKa in retrieving instances of an epistemic concept.

8.1 Optimization

The basic idea is to check the structure of a given epistemic concept and reduce either the number of $\mathbf{K}$'s occurring in the concept or the number of calls to the core reasoner during the translation of the concept such that the correctness of the answers is preserved. These intuition is realized by the optimization rules. In the following we discuss these rules and their correctness. Later in Section 8.3, we see that implementing these rules in EQuIKa indeed improves its performance in terms of computation time.

The rules are as follows:

- ***Rule 1 (Nominals):*** For individual names $a_1, \ldots, a_n$ we have

$$\hat{\Phi}_\Sigma(\mathbf{K}\{a_1, \ldots, a_n\}) \mapsto \{a_1, \ldots, a_n\}$$

[1]EQuIKa is a acronym for *E*pistemic *Qu*erying *I*nterface *Ka*rlsruhe)

This rule allows us to avoid instance retrieval for the concept $\{a_1, \ldots, a_n\}$ thus reduces the number of intermediate reasoner calls by 1. The correctness of Rule 1 is given as follows:

Proof. For the left to right direction, let $x \in \mathbf{K}\{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$. By semantics, therefore,

$$x \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(\{a_1, \ldots, a_n\}^{\mathcal{J}}))$$

i.e., $x \in \varphi_{\tilde{\mathcal{I}}}(\varphi_{\tilde{\mathcal{J}}}^{-1}(\{a_1, \ldots, a_n\}^{\mathcal{J}}))$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$. In particular, $x \in \varphi_{\tilde{\mathcal{I}}}(\varphi_{\tilde{\mathcal{I}}}^{-1}(\{a_1, \ldots, a_n\}^{\mathcal{I}}))$. Since $\varphi_{\tilde{\mathcal{I}}}$ is a surjective mapping, we get that $x \in \{a_1, \ldots, a_n\}^{\mathcal{I}} = \{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$.

For the right to left direction, let $x \in \{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$ and suppose that $x \notin \mathbf{K}\{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$. Note that by definition of $\varphi_{\mathcal{J}}$, we have that $a_i \in {\varphi_{\mathcal{J}}}^{-1}(a_i^{\mathcal{J}})$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$ and $1 \leq i \leq n$. It means that $\{a_1, \ldots, a_n\} \subseteq {\varphi_{\mathcal{J}}}^{-1}(\{a_1, \ldots, a_n\}^{\mathcal{J}})$ for each $\mathcal{J} \in \mathcal{M}(\Sigma)$ i.e.,

$$\{a_1, \ldots, a_n\} \subseteq \bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} {\varphi_{\mathcal{J}}}^{-1}(\{a_1, \ldots, a_n\}^{\mathcal{J}})$$

and thus by applying $\varphi_{\mathcal{I}}$ we get

$$\varphi_{\mathcal{I}}(\{a_1, \ldots, a_n\}) \subseteq \varphi_{\mathcal{I}}(\bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} {\varphi_{\mathcal{J}}}^{-1}(\{a_1, \ldots, a_n\}^{\mathcal{J}}))$$

which in other words means that $\varphi_{\mathcal{I}}(\{a_1, \ldots, a_n\}) \subseteq \mathbf{K}\{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$. By assumption, since $x \notin \mathbf{K}\{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$, consequently we get that $x \notin \varphi_{\mathcal{I}}(\{a_1, \ldots, a_n\}) = \{\varphi_{\tilde{\mathcal{I}}}(a_1), \ldots, \varphi_{\tilde{\mathcal{I}}}(a_n)\}$. By definition of $\varphi_{\tilde{\mathcal{I}}}$ we have that $\varphi_{\mathcal{I}}(a) = a^{\mathcal{I}}$ for each $a \in N_I$. Therefore we get that $x \notin \{{a_1}^{\mathcal{I}}, \ldots, {a_n}^{\mathcal{I}}\} = \{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$, which is a contradiction. Hence, $x \in \mathbf{K}\{a_1, \ldots, a_n\}^{\mathcal{I},\mathcal{M}(\Sigma)}$ must hold. □

- ***Rule 2 (Conjunction):*** Let $C_1, \ldots, C_n$ be concepts where each C_i is either an epistemic concept of the form $\mathbf{K}D$ for some concept D or a one-of concept for $1 \leq i \leq n$. By C'_i we denote the concept obtained from D_i by dropping $\mathbf{K}$ or $C'_i = C_i$ when D_i is a one-of concept. Then,

$$\hat{\Phi}_{\Sigma}(C_1 \sqcap, \ldots, \sqcap C_n) \mapsto \mathbf{K}(C'_1 \sqcap, \ldots, \sqcap C'_n)$$

This rule reduces the number of intermediate reasoner call by n. Thus it boost ups EQuIKa's performance to a great magnitude. We now present the proof of its correctness:

Proof. For a one-of concept $\{a_1, \ldots, a_k\}$, it follows from the proof of Rule 1 that $\{a_1, \ldots, a_k\}$ is equivalent to $\mathbf{K}\{a_1, \ldots, a_k\}$. Hence, we assume that each concept in $C_1 \sqcap \cdots \sqcap C_n$ is of the form $\mathbf{K}D$ for some concept D. Now we have that

$$x \in [\mathbf{K}D_1 \sqcap \ldots \mathbf{K}D_n]^{\mathcal{I},\mathcal{M}(\Sigma)}$$

if and only if by semantics of conjunction

$$x \in (\mathbf{K}D_1^{\mathcal{I},\mathcal{M}(\Sigma)} \cap \cdots \cap \mathbf{K}D_n^{\mathcal{I},\mathcal{M}(\Sigma)})$$

which again by semantics is equivalent to

$$x \in \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D_1^{\mathcal{J}})) \cap \cdots \cap \varphi_{\tilde{\mathcal{I}}}(\bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}(D_n^{\mathcal{J}}))$$

and thus from the definition of $\varphi_{\tilde{\mathcal{I}}}$ we establish its equivalence to

$$x \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} (\varphi_{\tilde{\mathcal{J}}}^{-1}(D_1^{\mathcal{J}}) \cap \cdots \cap \varphi_{\tilde{\mathcal{J}}}^{-1}(D_n^{\mathcal{J}}))\Big)$$

and therefore to

$$x \in \varphi_{\tilde{\mathcal{I}}}\Big(\bigcap_{\mathcal{J} \in \mathcal{M}(\Sigma)} \varphi_{\tilde{\mathcal{J}}}^{-1}((D_1 \sqcap \cdots \sqcap D_n)^{\mathcal{J}})\Big)$$

which by semantics of $\mathbf{K}$ is the case if and only if $x \in [\mathbf{K}(D_1 \sqcap \cdots \sqcap D_n)]^{\mathcal{I},\mathcal{M}(\Sigma)}$. $\square$

Note that a similar rule for disjunction does not work. For example, an element x in the extension of the concept

$$\mathbf{K}C_1 \sqcup \cdots \sqcup \mathbf{K}C_n$$

implies the membership of x in the extension of

$$\mathbf{K}(C_1 \sqcup \cdots \sqcup C_n)$$

whereas this does not hold in the other direction.

- ***Rule 3 (Existential Quantification):***
  Consider the case $\exists\mathbf{K}R.D$ in Definition 44 as given below:

$$\exists\mathbf{K}P.D \mapsto \begin{cases} \bigsqcup_{a\in N_I}\{a\} \sqcap \exists P.(\{b \in N_I \mid \Sigma \models P(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\}) \\ \sqcup \exists P.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \\ \sqcup \{a \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^-\{a\}\} \sqcap \exists P.\tilde{\Phi}_\Sigma(D) \\ \sqcup \begin{cases} \tilde{\Phi}_\Sigma(D) & \text{if } \Sigma \models \top \sqsubseteq \exists P.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \end{cases}$$

Suppose $\mathsf{Inst}(D)$ returns the set of instances of a concept D and let $\mathsf{OUT}_P = \mathsf{Inst}(\exists P.\tilde{\Phi}(D))$ and $\mathsf{IN}_P = \mathsf{Inst}(\exists P^-.\top)$. Using these abbreviation, Rule 3 can be given as following:

$$\hat{\Phi}_\Sigma(\exists\mathbf{K}P.D) \mapsto \begin{cases} \bigsqcup_{a\in\mathsf{OUT}_P}\{a\} \sqcap \exists P.(\{b \in \mathsf{IN}_P \mid \Sigma \models P(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\}) \\ \sqcup \exists P.(\{b \in \mathsf{IN}_P \mid \Sigma \models \top \sqsubseteq \exists P.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \\ \sqcup \{a \in \mathsf{OUT}_P \mid \Sigma \models \top \sqsubseteq \exists P^-\{a\}\} \sqcap \exists P.\tilde{\Phi}_\Sigma(D) \\ \sqcup \begin{cases} \tilde{\Phi}_\Sigma(D) & \text{if } \Sigma \models \top \sqsubseteq \exists P.\mathsf{Self} \\ \bot & \text{otherwise} \end{cases} \end{cases}$$

The basic idea is that in the first disjunct of the case $\exists\mathbf{K}R.D$ in Definition 44, instead of checking the two conditions i.e., checking for each $b \in N_I$ if

$\Sigma \models P(a,b)$ and if $b \in \tilde{\Phi}_\Sigma(D)$, we simply compute all the instances of the concept $\exists P^-.\top$ which are individual names for which we are sure that some element is related to them via the property P. Consequently we have to consider lesser number of individuals in the first disjunct of the optimization rule. Further, we reuse these instances in second disjunct as well instead of checking if for each $b \in N_I$ we have that $\Sigma \models \exists P.\{b\}$. A somewhat similar argument can me made for the third disjunct.

Note that without the optimization, the number of calls to the core reasoner required for translating a concept of the form $\exists \mathbf{K} P.D$ is at least $(|N_I| \times |N_I| + 2|N_I| + 1)$ plus the number of calls needed to translate D. With the optimization Rule 3, such a translation reduces the number of calls when $|\mathsf{Inst}(\exists P.\tilde{\Phi}(D))| < |N_I|$ or $|\mathsf{Inst}(\exists P^-.\top)| < N_I$. Now to prove the correctness,

Proof. We introduce the following abbreviations,

- $C_1 = \bigsqcup_{a \in N_I} \{a\} \sqcap \exists P.(\{b \in N_I \mid \Sigma \models P(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\})$
- $C_1' = \bigsqcup_{a \in \mathsf{OUT}_P} \{a\} \sqcap \exists P.(\{b \in \mathsf{IN}_P \mid \Sigma \models P(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\})$
- $C_2 = \exists P.(\{b \in N_I \mid \Sigma \models \top \sqsubseteq \exists P.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))$
- $C_2' = \exists P.(\{b \in \mathsf{IN}_P \mid \Sigma \models \top \sqsubseteq \exists P.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D))$
- $C_3 = \{a \in N_I \mid \Sigma \models \top \sqsubseteq \exists P^-\{a\}\} \sqcap \exists P.\tilde{\Phi}_\Sigma(D)$
- $C_3' = \{a \in \mathsf{OUT}_P \mid \Sigma \models \top \sqsubseteq \exists P^-\{a\}\} \sqcap \exists P.\tilde{\Phi}_\Sigma(D)$

We first show that for an extended interpretation $\mathcal{I} \in \mathcal{M}(\Sigma)$ and $x \in \Delta^\mathcal{I}$, $x \in C_1^{\mathcal{I},\mathcal{M}(\Sigma)}$ iff $x \in C_1'^{\mathcal{I},\mathcal{M}(\Sigma)}$. For this note that if $b' \in \mathsf{Inst}(\{b \in N_I | \Sigma \models P(a',b)\})$ for individuals a' and b' then it immediately follows that $b' \in \mathsf{Inst}(\{b \in \mathsf{IN}_P | \Sigma \models (a',b)\})$ and vice versa. Now for $x \in \Delta^\mathcal{I}$, $x \in C_1^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if there is an individual $a' \in N_I$ such that $x = a'^{\mathcal{I},\mathcal{M}(\Sigma)}$ and $x \in \exists P.(\{b \in N_I \mid \Sigma \models P(a',b) \sqcap \tilde{\Phi}_\Sigma(D)\})^{\mathcal{I},\mathcal{M}(\Sigma)}$. This is the case if and only if $x \in \exists P.(\{b \in \mathsf{IN}_P \mid \Sigma \models P(a',b) \sqcap \tilde{\Phi}_\Sigma(D)\})^{\mathcal{I},\mathcal{M}(\Sigma)}$ which is equivalent to $x \in C_1'^{\mathcal{I},\mathcal{M}(\Sigma)}$.

In the same way, we can prove that $x \in C_2^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if $x \in C_2'^{\mathcal{I},\mathcal{M}(\Sigma)}$. Similarly, $x \in C_3^{\mathcal{I},\mathcal{M}(\Sigma)}$ if and only if $x \in C_3'^{\mathcal{I},\tilde{\mathcal{M}}(\Sigma)}$. Consequently, this establishes the proof of the correctness of Rule 3. □

- ***Rule 4 (Number Restriction):***
  Similarly, Rule 4 handles the case of Definition 44 where the considered concept is of the form $\geq n\mathbf{K}S.D$. The rule is given as follows:

$$\begin{cases} \bigsqcup_{a\in \mathsf{OUT}_{P,n}}\{a\} \sqcap \geqslant nS.(\{b \in \mathsf{IN}_{P,n} \mid \Sigma \models S(a,b) \sqcap \tilde{\Phi}_\Sigma(D)\}) \\ \sqcup \{a \in \mathsf{OUT}_{P,n} \mid \Sigma \models \top \sqsubseteq \exists S^-.\{a\}\} \sqcap \geqslant nS.\tilde{\Phi}_\Sigma(D) \\ \sqcup \geqslant nS.(\{b \in \mathsf{IN}_{P,n} \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \\ \sqcup \begin{cases} \geqslant (n-1)S.(\{b \in \mathsf{IN}_{P,n-1} \mid \Sigma \models \top \sqsubseteq \exists S.\{b\}\} \sqcap \tilde{\Phi}_\Sigma(D)) \sqcap \\ \tilde{\Phi}_\Sigma(D) \sqcap \neg\{a \mid a \in N_I\} \quad \text{if } \Sigma \models \top \sqsubseteq \exists S.\mathsf{Self} \\ \bot \quad \text{otherwise} \end{cases} \end{cases}$$

  where $\mathsf{OUT}_{P,n} = \geq nP.\tilde{\Phi}(D)$ and $\mathsf{IN}_{P,n} = \geq n.P^-.\top$.

The proof of correctness is similar to that of Rule 3.

These rules suffice in the sense that for most of the remaining constructs, we use their dual which corresponds to one of the rules discussed above. Later we will compare the performance of two implementations of EQuIKa: one implementing $\tilde{\Phi}_\Sigma$ without while the other implementing with the optimization rules.

8.2 Implementation

The EQuIKa system is implemented on top of the OWL-API.[2] It can be used as an API as well as within Protégé using an epistemic query tab. In the following we address several issues involved in implementing EQuIKa.

- The standard OWL-API does not suppose the epistemic constructs. Now in order to embed **K** in the syntax of queries, we extended several classes of the API. For this, note that the **K**-operator syntactically behaves similar to the

[2] http://owlapi.sourceforge.net/

```
<rdf:Description rdf:about="bancroftChardonnay">
  <rdf:type>
     <owl:Class>
        <owl:knownClass rdf:resource="Wine" />
     </owl:Class>
  </rdf:type>
</rdf:Description>
```

Figure 8.1: RDF represention of the epistemic operator for concepts

complement construct ($\neg$) for concepts and like the *inverse role construct* for roles. We therefore followed the same implementation patterns. For example, as a possible representation of the assertion

$$\mathbf{K}\mathsf{Wine}(\mathsf{bancroftChardonnay})$$

we can use the equivalent RDF[3] representation as given in figure 8.1.

Similarly, the RDF representation of the concept assertion

$$\forall \mathbf{K}\mathsf{producesWine}.\mathbf{K}\mathsf{WhiteTableWine}(\mathsf{bancroft})$$

is given in Figure 8.2.

- For parsing we created an EpistemicSyntaxParser based on the

  ManchesterOWLSyntaxOntologyParser

  The **K** operator is expressed by the token `knownClass` for concepts and by the token `knownProperty` for the roles. We have already desmonstrated the use of these new constructs in Figure 8.1 and Figure 8.2.

[3] `http://www.w3.org/RDF/`

```
<rdf:Description rdf:about="bancroft">
 <rdf:type>
  <owl:Class>
   <owl:Restriction>
    <owl:knownProperty rdf:resource="producesWine" />
     <owl:allValuesFrom>
      <owl:knownClass rdf:resource="WhiteTableWine" />
     </owl:allValuesFrom>
   </owl:Restriction>
  <owl:Class>
 </rdf:type>
</rdf:Description>
```

Figure 8.2: RDF represention of the epistemic operator for roles

- We implemented the translation function in a recursive fashion. For this, we implemented a visitor pattern by extending the

  OWLClassExpressionVisitor

  class in order to handle the epistemic operator.
- In order to support epistemic querying within the Protégé editor, we implemented an additional tab based on the DL Query tab. Figure 8.3 shows a snapshot of epistemic querying in Protégé.

An overview of the overall implementation of EQuIKa is given by the class diagram in Figure 8.4. Note that the new types OWLObjectEpistemicConcept and OWLObjectEpistemicRole are derived from the respective standard types OWLBooleanClassExpression and OWLObjectPropertyExpression to fit the design of the OWL-API. As our translation method depends on intermediate calls to a standard reasoner, the class EQuIKaReasoner implements the OWLReasoner interface. Now EQuIKa translates an epistemic concept into a **K**-free one in a recursive fashion using the class Translator that implements the OWLClassExpressionVisitor. Further, since Protégé can utilize any reasoner that implements the OWLReasoner interface, this enables Protégé to use the EQuIkaReasoner to

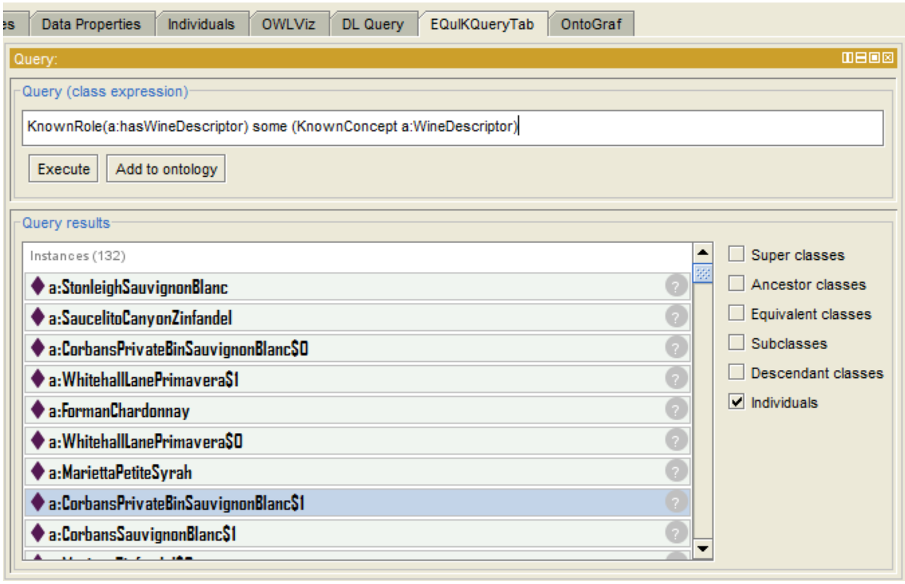

Figure 8.3: Epistemic Querying in Protégé

answer epistemic queries. Last but not least, EQuIKa has been shared on google-code for testing purposes.[4] The Protégé plugin is provided as jar file that can be installed via the Protégé 4.1 plugin folder.

In the next section we present results of some experiments we performed in order to assess the overall performance of EQuIKa in practice.

8.3 Evaluation

For the purpose of evaluation, we performed several experiments with the following setup:

- We used an IBM Thinkpad T60 dual core, 2 GHz each core and running with Windows 7 (32-bit) as the operating system. A total of 2 GB memory was available.

[4] `http://code.google.com/p/epistemicdl/`

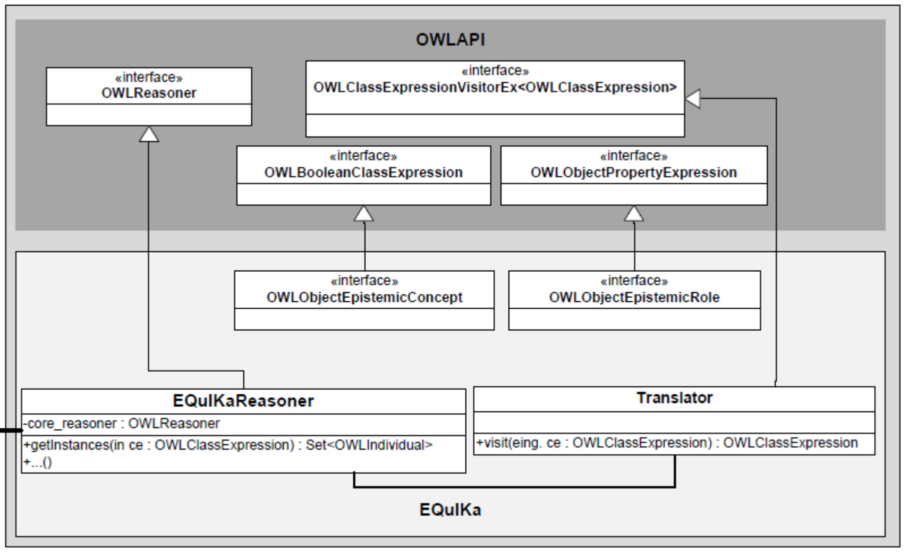

Figure 8.4: The *EQuIKa*-system extending the OWL-API

- For benchmark tests, we used a populated version of the Wine ontology[5], that contains 483 individuals. Further, to test different constructs, this version of the Wine ontology uses most of the $\mathcal{SROIQ}$/OWL 2 DL constructs. This ontology can be downloaded along with EQuIKa.

- To evaluate the performance of EQuIKa, we constructed several epistemic concepts and translated them into **K**-free ones.

  These concepts are given in Table 8.1 where $r_1, \ldots, r_{108}$ are individuals representing wine regions in the ontologies. Note that for the testing purpose we were not interested in the actual meaning of these concepts. In fact, these concepts are arbitrarily constructed using different constructs. This allows us to study the effect of different constructs in concept on the performance of EQuIKa.

[5]The version of the Wine ontology that we used in our experiments can be downloaded at `http://code.google.com/p/epistemicdl/`.

EC_1	$\exists$**K**hasWineDescriptor.**K**WineDescriptor
EC_2	$\exists$**K**hasWineDescriptor.**K**WineDescriptor $\sqcap$ $\exists$**K**madeFromFruit.**K**WineGrape
EC_3	**K**RoseWine
EC_4	**K**RoseWine $\sqcap$ **K**WhiteWine
EC_5	**K**RoseWine $\sqcap$ **K**WhiteWine $\sqcap$ $\{r_1, \ldots, r_{108}\}$
EC_6	**K**Wine $\sqcap$ $\neg\exists$**K**hasSugar.$\{Dry\}$ $\sqcap$ $\neg\exists$**K**hasSugar.$\{OffDry\}$ $\sqcap$ $\neg\exists$**K**hasSugar.$\{Sweet\}$

Table 8.1: Concepts used for instance retrieval experiments.

A better way of evaluating EQuIKa is to compare it with its alike tools. But to the best of our knowledge, EQuIKa is the only reasoner of its nature for epistemic query answering. Unfortunately, there is no other existing reasoner with these capabilities against which we could compare EQuIKa's performance. To give an impression about the runtime behavior, we performed two kind of experiments. As a measure, we considered the time required to translate the epistemic concepts (given in Table 8.1) to **K**-free equivalent ones and the instance retrieval time of the translated concept. Note that we arbitrary constructed these concepts using constructs that are potentially complex cases of the translation function presented in Defintion 44. However we did not use all the constructs in $\mathcal{SROIQ}$ simply because the wine ontology itself does not use all the constructs in $\mathcal{SROIQ}$.

Concept	EQuIKa-N			EQuIKa-O		
	T_{trans}	T_{inst}	$\#_{inst}$	T_{trans}	T_{inst}	$\#_{inst}$
EC_1	4	192.7	132	21	97.8	132
EC_2	9	198.9	3	3	37.5	3
EC_3	110	110.1	3	26	26.5	3
EC_4	203	211.7	0	122	122.1	0
EC_5	206	400.6	0	121	121.9	0
EC_6	13	–	–	0.5	487.3	119

Table 8.2: Time (in seconds) required for instance retrieval.

Experiment 1

In the first series of experiments, we evaluated the benefit of the optimization rules introduced in the previous section. We implemented two versions of EQuIKa; a naive one called EQuIKa-N implementing the translation function of Definition 44 as is and an optimized one called EQuIKa-O where the optimization rules were used. The corresponding results are shown in Table 8.2 where $\mathsf{T}_{\mathsf{trans}}$, $\mathsf{T}_{\mathsf{inst}}$ and $\#_{\mathsf{inst}}$ represent the translation time (in seconds), instance retrieval time (in seconds) and the number of instances respectively where the times are measured in seconds. Note that the optimized EQuIKa-O outperformed the naive EQuIKa-N. A time comparison in seconds between both versions of EQuIKa is presented in Figure 8.5. For each concept, the first (brown) bar represents the total time taken by EQuIKa-N and the second (blue) bar represents the total time taken by EQuIKa-O. In both bars, the dark colored part represented the time taken for the translation and the light colored part represents the instance retrieval time by the corresponding versions of EQuIKa. It can be seen from the chart that the optimized EQuIKa is one or several orders of magnitude faster than the non-optimized one. In particular for concept $EC6$, EQuIKa-N didn't responded for almost an hour and we stopped it, whereas EQuIKa-O translated $EC6$ and retrieved its instances in few seconds. Hence, beside being correct, the optimized rules introduced are of high importance toward the practical feasibility of EQuIKa.

Experiment II

In the second series of experiments, we evaluated the computation time of EQuIKa-O in general to provide an impression of how the cost of epistemic querying relates to standard reasoning tasks. For this purpose, we consider non-epistemic concepts $C1, \ldots, C_6$ where each C_i is obtained by removing all the occurrences of the operator $\mathbf{K}$ from EC_i for $1 \leq i \leq 6$. Table 8.3 shows the results of our experiments. Note that the number of instances retrieved for the concepts C_i and EC_i needs not to coincide as the concepts are semantically non-equivalent in general. Further, the instance retrieval time for EC_i's in general is great that the instance retrieval time for C_i's. The reason is that EQuIKa-O replaces sub-concepts of EC_i's by $\mathsf{one\text{-}concepts}$ thus resulting into more complex concepts. However, it can be seen that even when comparing to the $\mathbf{K}$-free counter part of the epistemic concepts, the computation time of EQuIKa-O is roughly in

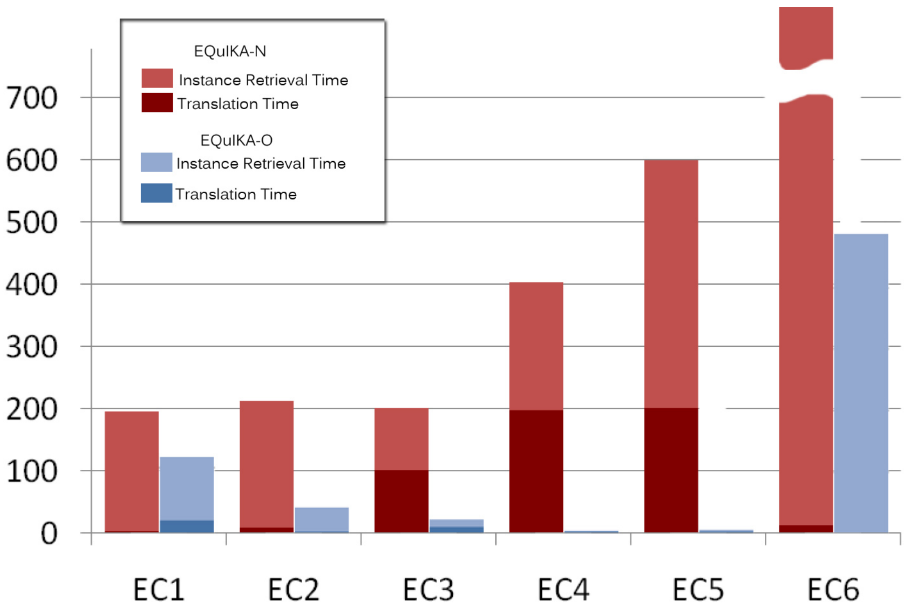

Figure 8.5: Computation time (in seconds) comparision of EQuIKa-N and EQuIKa-O.

Concept	T_{inst}	$\#_{inst}$	Concept	T_{trans}	T_{inst}	$\#_{inst}$
C_1	2.18	159	EC_1	20	95.7	132
C_2	41.9	159	EC_2	3	36.5	3
C_3	10.7	3	EC_3	10	10.8	3
C_4	2.68	0	EC_4	2	2.9	0
C_5	0.2	0	EC_5	2	2.9	0
C_6	61.1	80	EC_6	0.5	487.3	119

Table 8.3: Time (in seconds) comparison between epistemic vs. standard instance retrieval.

the same order of magnitude.

Observing the results of these experiments, we believe EQuIKa can be deployed for the real life applications. Nevertheless, finding out applications where local-closed word reasoning and thus introspective capabilities are demanded is yet a major task in itself. We leave it as one of our future goals.

Part IV

Conclusions

Chapter 9
Related Work

In this chapter we compare our approach with several related approaches which we divide into $1)$ *classical approaches*, where the standard semantics for epistemic DLs is adopted and $2)$ *modern approaches*, where new semantics have been devised for different extensions of DLs.

9.1 Classical Approaches

Donini et al. present one of the earliest approaches to extending DLs with an epistemic operator where the DL $\mathcal{ALC}$ is endowed with the operator **K** and is called $\mathcal{ALCK}$ [DONINI et al. 1992b]. We have seen that this operator is used to formalize the notion of what is known in the knowledge base and resembles the necessity modality in modal logics [BRAÜNER and GHILARDI 2006]. The semantics adapted for $\mathcal{ALCK}$ is based on the standard possible-world approach [BRAÜNER and GHILARDI 2006] where the accessibility relation is a total and equivalence relation. To cope with problems like domain of quantification and cross-domain interpretation of constants (individual names), the aforementioned assumptions regarding the rigidity of the constants and the commonality of the domains are adopted. In Chapter 7, we have seen that such an approach is suitable for DLs up to $\mathcal{SRIQ}\backslash U$, however, the classical semantics fail to handle DLs that allow for nominals or universal role together with number restriction. In this work, we have taken a similar direction. Compare to our approach,

- firstly we have devised a procedure that translated epistemic axioms into non-epistemic ones. Thus in our approach all one needs to implement is the translation functions we presented in Definition 44. Thus our approach allows us using state-of-the-art available reasoners. Note that implementing a complete new reasoner requires quite much time.

- secondly, in our approach we considered $\mathcal{SROIQ}$ as the knowledge base language unlike the classical approaches where we need to restrict the knowledge base language to $\mathcal{SRIQ}\backslash U$.

The description logic of minimal knowledge and negation as failure (DL-MKNF) is presented in [DONINI et al. 1997]. In such a formalims, a standard DL is augmented with two modal operators: **K** for representing knowledge and **A** for representing default assumptions. The semantics for DL-MKNF is similar to the semantics employed for $\mathcal{ALCK}$ where the operator **A** is interpreted similar to the **K** operator. Nevertheless, when defining the notion of models, the semantics requires the maximality of the set of interpretations considered for the operator **K** only. Restricting DL-MKNF to **A**-free queries, we compare it with our work. The semantics employed for DL-MKNF imposes the two assumptions, namely, the common domain assumption and the rigid term assumption. Due to these assumptions, the semantics suffers from the same problems as the classical semantics for epistemic extensions of DLs for expressive knowledge base languages like $\mathcal{SROIQ}$.

9.1.1 Modern Approaches

We now discuss some modern approaches to extending DLs with non-monotonic features, which avoid the semantics problems. In Section 4.2.3, we presented an overview of MKNF$^+$ knowledge bases [MOTIK and ROSATI 2008]. The formalism is expressive enough to capture many of the existing approaches to non-monotonic extension of DLs and to integrate DLs with rules (see [ROSATI 2006b, BAADER and HOLLUNDER 1992, CALVANESE et al. 2007, EITER et al. 2008, LUKASIEWICZ 2010a] etc). Thereby, the expressiveness of the underlying language is not restricted to some less expressive DL rather it can be any fragment of first-order logic. As a recall, an *MKNF$^+$ KB* $\mathcal{K}$ is a pair $(\mathcal{O}, \mathcal{P})$ where $\mathcal{O}$ is a description logic KB and $\mathcal{P}$ a general logic program. Thus, by definition, a DL KB $\mathcal{O}$ is an MKNF$^+$ KB with empty program part. The semantics employed for MKNF$^+$ is the standard MKNF semantics [LIFSCHITZ 1991] except that the standard name assumption is required to be satisfied by the interpretations. Such a semantics is capable of handling the problems mentioned in Chapter 7. Nevertheless, there is a subtle difference between MKNF$^+$ semantics and our semantics. The basic difference is how both semantics treat the equality relation $\doteq$. In our approach we exploit $\doteq$ locally i.e., given a world, our semantics respects the $\doteq$ relation between individuals. This is not the case in MKNF$^+$. To see this, we present an example. Suppose we have an knowledge base Σ containing the following axioms:

$$\begin{array}{ll} \top \sqsubseteq \{a, b, c\} & \{a\} \sqsubseteq \{b, c\} \\ b \neq c & A(a) \end{array}$$

We can see that Σ is consistent. Similar to epistemic description logics, Σ has a unique MKNF$^+$ model which is the set of all first-order models of Σ. We denote this model by $\mathcal{M}_{\mathsf{MKNF}}(\Sigma)$. Note that although $a \doteq b$ is not asserted in Σ, there are first-order models in $\mathcal{M}_{\mathsf{MKNF}}(\Sigma)$ where a is interpreted congruent to b. Let $I \in \mathcal{M}_{\mathsf{MKNF}}(\Sigma)$ be one such model. Of course there are models where this is not the case. Note that in $(I, \mathcal{M}_{\mathsf{MKNF}}(\Sigma))$ the following holds:

$$\neg \mathbf{K} A(b), \qquad \mathbf{K} A(a) \qquad a \doteq b$$

which is not intuitive as under I, $a \doteq b$ are congruent, and $\mathbf{K}A(a)$ holds as it is asserted, nevertheless since there are interpretations where a is not congruent to b, $A(b)$ does not hold in all interpretations, i.e., $\neg\mathbf{K}A(b)$ holds.

Now considering the extended model of $\tilde{\mathcal{M}}(\Sigma)$ of Σ. Again there is an extended interpretation $\tilde{\mathcal{I}}$ in $\tilde{\mathcal{M}}$ such that $\tilde{\mathcal{I}} \models a \doteq b$. Since $\mathbf{K}A(a)$ holds in each extended interpretation in Σ (so is asserted in Σ), we have that $\tilde{\mathcal{I}}, \tilde{\mathcal{M}} \models \mathbf{K}A(a)$ i.e., $\tilde{\mathcal{I}}(a) \in \mathbf{K}A^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}}$. But then we have $a^{\tilde{\mathcal{I}}} = b^{\tilde{\mathcal{I}}}$. Thus $b^{\tilde{\mathcal{I}}} \in \mathbf{K}A^{\tilde{\mathcal{I}},\tilde{\mathcal{M}}}$. Hence, $\tilde{\mathcal{I}}, \tilde{\mathcal{M}} \models \mathbf{K}A(b)$. Besides the difference we just mentioned, the entailment under both semantics do not coincide. As an example, we note that Σ entails the axiom

$$\top \sqsubseteq \exists U.(\{b, c\} \sqcap \mathbf{K}A)$$

under the extended epistemic semantics i.e.,

$$\Sigma \mid\!\models_{\text{e}} \top \sqsubseteq \exists U.(\{b, c\} \sqcap \mathbf{K}A).$$

However, this axiom is not entailed by Σ under the MKNF$^+$ semantics. The reason is as follows: Σ enforces the equality of a to exactly one of b or c since $\{\{a\} \sqsubseteq \{b, c\}, b \not\doteq c\} \subseteq \Sigma$. Further each element is connected to both b and c via the universal role U. Earlier we have seen that in a model $\tilde{\mathcal{M}}$ of Σ, for every extended interpretation $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$ we have either $\tilde{\mathcal{I}} \models a \doteq b$ or $\tilde{\mathcal{I}} \models a \doteq c$ and that $\tilde{\mathcal{I}}, \tilde{\mathcal{M}} \models \mathbf{K}A(a)$. Thus for every $\tilde{\mathcal{I}} \in \tilde{\mathcal{M}}$ we have that either $\tilde{\mathcal{I}}, \tilde{\mathcal{M}} \models \mathbf{K}A(b)$ or $\tilde{\mathcal{I}}, \tilde{\mathcal{M}} \models \mathbf{K}A(c)$. Thus $\tilde{\mathcal{I}}, \tilde{\mathcal{M}} \models \top \sqsubseteq \exists U(\{b, c\} \sqcap \mathbf{K}A)$. It can be checked that this is not the case for MKNF$^+$.

We believe that our treatment of equality in local worlds (interpretations) is more intuitive in the sense that given a world, equal individuals possesses the same characteristics: the characteristics of some $a, b \in N_I$ such that $a \doteq b$ holds (in a local worlds), are the same in that world. Hence, if some facts are known for a, the same facts are known for b as well.

Besides the aforementioned semantical differences, one of the main advantages of our approach over MKNF$^+$ is the ability to translate epistemic queries to non-epistemic ones. It is this procedure that allows us to use off-the-shelf highly optimized standard reasoners for answering epistemic queries. Thus, the tedious task of implementing a new reasoner with additional capabilities is avoided.

Chapter 10

Conclusion and Outlook

As a main objective of this work, we have presented an approach of answering queries formulated in epistemic extension of $\mathcal{SROIQ}$ that we called $\mathcal{SROIQK}$. To this end, we conclude by summing up the accomplishments resulted in addressing issues raised in Section 1.4. Finally, we set some goals as for future work directions.

10.1 Addressed Issues

We enumerate the results of our work with respect to the goals we set in Section 1.4.

10.1.1 Epistemic DLs under Classical Semantics

In Chapter 6 we have considered $\mathcal{SROIQK}$ as a query language to $\mathcal{SRIQ}\backslash U$ knowledge bases. The classical semantics imposes the common domain assumption and the rigid term assumption. The former requires all the interpretations under consideration to share a common infinite domain, whereas the latter requires the uniformity of interpretations of individual names across different interpretation. We think of these restrictions as the main reason of in-applicability of the classical semantics for the case of $\mathcal{SROIQK}$ knowledge bases. This also leads to the non-UNA-backward compatibility of $\mathcal{SROIQK}$.

We then presented a method for translating $\mathcal{SROIQK}$ queries into standard DL queries. This is in contrast to the classical methods (mainly Tableaux based) for answering epistemic queries. Further, we have seen that we had to enforce the unique name assumption explicitly in the knowledge base under consideration to acquire correctness of our method.

10.1.2 Extended Epistemic Semantics

We discussed the main reasons behind the in-applicability of the classical epistemic semantics to $\mathcal{SROIQK}$ by showing examples of knowledge bases which exhibit finite models only. We then presented a new semantics that we called *extended epistemic semantics* based on the notion of extended epistemic models. The idea is that we introduce an abstraction layer that assigns abstract individual names to domain elements. The abstraction layer (which in our case was the set $N_I \cup \mathbb{N}$) serves as a common ground for the different interpretations with different domains. Now interpreting an epistemic concept C requires interpreting it locally, then taking the abstractions of extensions of C in all the interpretations. As these abstractions are from a common set $N_I \cup \mathbb{N}$, we can simply take all the common elements in these abstractions. This is very similar to the intersecting of extensions in the classical semantics. The extension of C in a given interpretation is simply all those elements which have members of the common abstraction as their designator. This is better understood by Figure 7.1, 7.2 and 7.3.

We have shown that the new proposed semantics is backward compatible with the standard semantics as well as behaves similar to the classical semantics when the knowledge base language is restricted upto $\mathcal{SRIQ} \setminus U$. Further, our proposed semantics does not enforces the unique name assumption either. Later we presented a method of translating epistemic axioms into non-epistemic ones. Subsequently, we showed that answering an epistemic query can be performed via standard entailment checking.

10.1.3 EQuIKa: Epistemic Query Answering Tool

We provided an implementation of the method we introduced in this work for answering epistemic queries. EQuIKa takes a black-box approach by utilizing off-the-shelf standard DL reasoners for answering epistemic queries. Consequently, we avoided implementing full-fledged reasoners from the scratch with epistemic reasoning capabilities.

In order for EQuIka to be practically feasible, we invented several optimization rules that we proved to be correct. Later we showed through experiments that these rule indeed improved the computation time performance of EQuIKa to a great extent. Finally, our experiments showed that EQuIKa performs almost similar to the standard reasoners. Thus, we provided an epistemic query answering tool that is feasible in practice.

Besides the aforementioned accomplishment, we compared our work with many existing approaches. The approach of MKNF^+ knowledge bases, presented [Motik and Rosati 2008], is usually seen as one of the most mature work as it captures many of the existing approaches. We showed that our treatment of $\doteq$ is intuitive in the sense that given a particular interpretation (world) our semantics respects the $\doteq$ relation between individuals. Through example, we have seen that this is not the case in the MKNF^+ approach. Further, we also showed that both approaches do not agree on entailment of axioms in general.

10.2 Future Work

There are several goals as the future directions of our work. In the following we consider some of the important ones.

10.2.1 Embedding $\mathbf{K}$ into Knowledge Bases

For this work, we considered answering of epistemic queries posed to $\mathcal{SROIQ}$ knowledge bases. The nice property of a $\mathbf{K}$-free knowledge base is that it has a unique (extended) epistemic model. This actually allows us to translate epistemic queries to non-epistemic ones. However, this property is lost once we allow $\mathcal{SROIQK}$ as the knowledge base language, i.e., if we allow $\mathbf{K}$ within the knowledge base. As an example consider the following $\mathcal{SROIQK}$ knowledge base

$$\Sigma = \{\top \sqsubseteq \mathbf{K}C_1 \sqcup \mathbf{K}C_2\}$$

which has $\mathcal{E}(\mathcal{M}_1)$ and $\mathcal{E}(\mathcal{M}_2)$ as its only extended epistemic models where, $\mathcal{M}_1 = \{\mathcal{I}|\mathcal{I} \models \top \sqsubseteq C_1\}$ and $\mathcal{M}_2 = \{\mathcal{I}|\mathcal{I} \models \top \sqsubseteq C_2\}$. As mentioned in Section 7.2, for a set of interpretations $\mathcal{M}$ by $\mathcal{E}(\mathcal{M})$ we mean the set of extended interpretations obtained by extending every interpretation $\mathcal{I}$ in $\mathcal{M}$ with all possible surjective mapping from $N_I \cup \mathbb{N}$ to $\Delta^{\mathcal{I}}$. The first model corresponds to those extended interpretations where C_1 is interpreted as the $\top$ concept whereas the second model corresponds to those extended interpretations where C_2 is interpreted as the $\top$ concept. As a result, the translation procedure we introduced in this work does not work anymore, simply because it is based on the standard entailment of axioms from a knowledge base thus we consider all the models of the knowledge base. Whereas in this case we have a split of the interpretations into two sets namely, $\mathcal{E}(\mathcal{M}_1)$ and $\mathcal{E}(\mathcal{M}_1)$.
We will thus investigate if our method of translating epistemic queries into standard ones, can be extended to the cases where we do allow for the **K** to occur within the knowledge base as well. Note that as argued in [Donini et al. 2002], we then would require the operator **A** as well.

10.2.2 Embedding K in OWL Syntax

In Section 8.2 we presented two examples demonstrating a possible way of representing the epistemic operator **K** in OWL syntax. The current syntax does not support such non-standard features. Thus it is highly desirable to extend OWL syntax with constructs like the epistemic operator. Though a simple task, extending OWL with non-standard constructs requires motivating the whole community to acquire consensus.

10.2.3 EQuIka in Real Life

We tested and evaluated EQuIKa on a populated version of the Wine ontology. But how does it perform in real life applications is an open question. Note that we first need to find some real life application where one requires epistemic query answering. Only then, we can evaluate EQuIka in practice. To accomplish this task, we have to introduce the notion of closed-world reasoning in an easy-to-understand way to the practitioners (of Semantic Technologies) and advocate its advantages.

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