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by Mher Safarian

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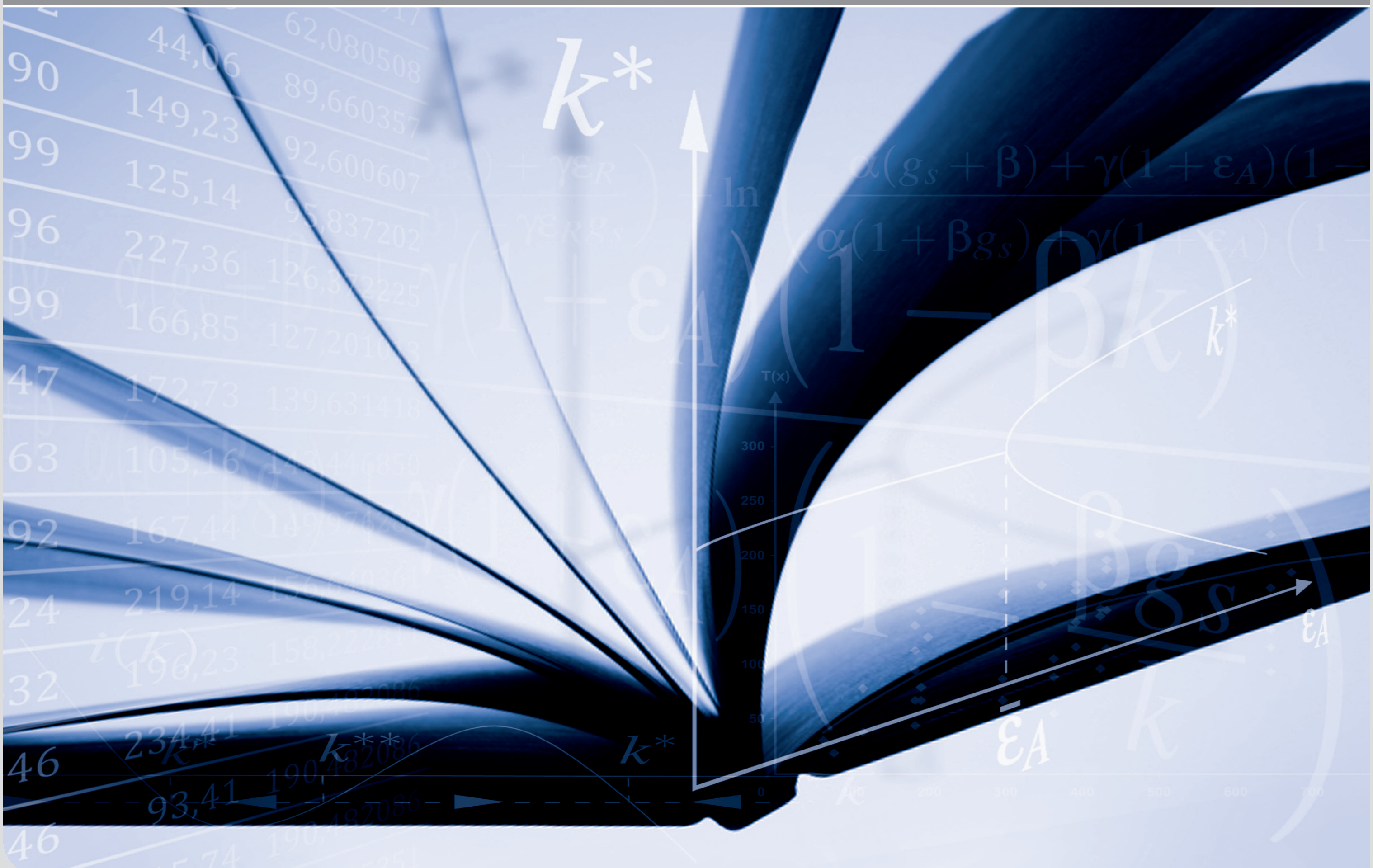

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Hedging options including transaction costs in incomplete markets

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Abstract

In this paper we study a hedging problem for European options taking into account the presence of transaction costs. In incomplete markets, i.e. markets without classical restriction, there exists a unique martingale measure. Our approach is based on the Föllmer-Schweizer-Sondermann concept of risk minimizing. In discret time Markov market model we construct a risk minimizing strategy by backwards iteration. The strategy gives a closed-form formula. A continuous time market model using martingale price process shows the existence of a risk minimizing hedging strategy.

Key words: hedging of options, incomplete markets, transaction costs, risk minimization, mean-self strategies

1 Discrete-Time Model

In this section we formulate terminology for the basic problem of taking into account transaction costs, studied in this paper. The idea is based on the approach taken by the Föllmer-Schweizer-Sondermann concept of risk minimization. A detailed description of this concept in discrete time and in the absence of transaction costs is found in the one of the best Monographs for Financial Stochastic by Föllmer/Schied [4]. An introduction to the problem of transaction costs in a complete markets is provided in the monography of Kabanov/Safarian [6].

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1.1 Assumptions and definitions

A discrete-time model of financial market is built on a finite probability space $(\Omega, F = (\mathcal{F}_t), P)$ equipped with a filtration an increasing sequence of σ - algebras included in F $F_0 = \{\emptyset, \Omega\}$, $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_N = F, N < \infty$, where $|\Omega| < \infty$.

Definition 1

a) A pair $\varphi = (\xi, \eta)$ with random process $\xi = (\xi_t), t = 1, \ldots, N$, $\xi_0 = 0$ and random process $\eta = (\eta_t), t = 0, 1, \ldots, N$ is a trading strategy, if it satisfies the following properties:

ξ_t *is* F_{t-1}*- measurable (a predictable process) for* $t = 1, \ldots, N$ *and*

η_t *is* F_t*- measurable for* $t = 0, 1, 2 \ldots, N$.

The process ξ_t is the number of units of stock held at time t and η_t is the number of riskless units held at time t. The securities and the risk-free assets form the so-called portfolio.

We assume the interest rate r is constant over the entire period. So, we set r = 0 in order to simplify the notation.

b) The value process $V_t(\varphi)$ defined by

$$V_t(\varphi) = \xi_{t+1} S_t + \eta_t \text{ for } \; t = 0, 1, \ldots, N$$

then represents the value of the portfolio $V_t(\varphi)$ held at time k. The process S_t with $ES_t^2 < \infty$ is called price process and represents the discounted value of some risky asset.

c) The cost process $C_t(\varphi)$ of a strategy $\varphi = (\xi, \eta)$ is given by the equation

$$C_t(\varphi) = V_t(\varphi) - \sum_{j=1}^{t} \xi_t \Delta S_j \text{ for } t = 0, 1, \ldots, N,$$

where $\Delta S_j = S_j - S_{j-1}$ and $C_0 = V_0$. $C_t(\varphi)$ describes the cumulative costs up to time k incurred by using the trading strategy $\varphi = (\xi, \eta)$.

The cost process including transaction costs is then defined by the following formula:

$$C_t(\varphi) = V_t(\varphi) - \sum_{j=1}^{t} \xi_t \Delta S_j + TC\Big(S_j, \Big|\Delta \xi_j\Big|\Big) \text{ for } t = 0, 1, \ldots, N. \tag{1}$$

d)The process TC is called transaction cost process if

$$TC\Big(S_j, \Big|\Delta\xi_j\Big|\Big) = k\sum_{j=1}^{t} S_j\big|\xi_j - \xi_{j-1}\big|,$$

where k is the coefficient of transaction costs, k is constant. The process $TC\Big(S_j, \Big|\Delta\xi_j\Big|\Big)$ represents the cumulstive transaction costs up to time t.

In realistic situations the coefficient of transaction costs k may depend on the volume of sales. Our method could easily be generalized to cover such transaction costs. For the purpouse of readability we write T_t instead of TC.

Definition 2

a) A trading strategy $\varphi = (\xi, \eta)$ is called mean-self-financing if its cost process $C_t(\varphi)$ is a square-integrable martingale.

b) The risk process $r_t(\varphi)$ of a trading strategy is defined by $r_t(\varphi) = E\Big\{\Big(C_{t+1}(\varphi) - C_t(\varphi)\Big)^2\Big|\mathcal{F}_t\Big\}$ (see[2], [8] p.18), for $t = 0, 1, 2, \ldots, N-1$.

c) Let H be a contingent claim. A trading strategy $\varphi = (\xi, \eta)$ is called H - admissible if $V_N(\varphi) = H$ almost surely (a.s.), where N is the maturity time.

d) A trading strategy
$\varphi = \Big(\big(\xi_1^*, \eta_1^*\big), \big(\xi_2^*, \eta_2^*\big), \ldots, \big(\xi_k^*, \eta_k^*\big), \ldots, \big(\xi_N^*, \eta_N^*\big)\Big)$
is called risk-minimizing if for any trading time $t = 0, 1, \ldots, N$ and for any admissible strategy
$\varphi^k = \Big(\big(\xi_1^*, \eta_1^*\big), \big(\xi_2^*, \eta_2^*\big), \ldots, \big(\xi_k^*, \eta\big), \big(\xi, \eta_{k+1}^*\big), \ldots, \big(\xi_N^*, \eta_N^*\big)\Big)$
(i.e. $V_N\big(\varphi^k\big) = V_N\big(\varphi\big)$) the folowing inequality is valid:

$r_t\big(\varphi^k\big) - r_t(\varphi) \geq 0$ *for* $t = 0, 1, \ldots, N$.

The goal is to find a strategy that is H-admissible, risk-minimizing and mean-self-financing, including transaction costs incurred after conversion of the portfolio.

Remark 1

a) The risk-mimimizing strategies without transactions costs have been constructed in Föllmer/Schweizer [1], Föllmer/Sondermann [2], Schweizer [8].

b) In [2], [3], [7] the risk at time t is defined as follows:

$$R_t(\varphi) = E\Big\{\Big(C_N(\varphi) - C_t(\varphi)\Big)^2\Big|\mathcal{F}_t\Big\},$$

where N is maturity time and $t = 0, 1, \ldots, N$.
It is easily seen that the above formulated problem is similar to the following problem in discrete time (see also the remark in M. Schweizer [8] pp. 25-26):

$$R_t(\varphi) = E\Big(\Big(C_N(\varphi) - C_{t+1}(\varphi) + C_{t+1}(\varphi) - C_t(\varphi)\Big)^2\Big|\mathcal{F}_t\Big) = \\ = E\Big(R_{t+1}(\varphi)\Big|\mathcal{F}_t\Big) + r_t(\varphi),$$

where $r_t(\varphi) = E\Big\{\big(C_{t+1} - C_t\big)^2\big|\mathcal{F}_t\Big\}$ *with* $R_t(\varphi) \geq r_t(\varphi)$ *for any* $t = (0, 1, 2, \ldots, N)$.

The theorem of existence of a risk minimizing hedging strategy for a finite probability space can be stated as:

Theorem 1
Let $\Delta S_n = \rho_n S_{n-1}(\rho_k > -1)$ be a price process and ρ_n be a sequence of independent identically distributed random variables, such that $\rho_n \in \big(\alpha_1, \ldots, \alpha_m\big)$ with probability $\big(p_1, \ldots, p_m\big)$ und $E\rho_k = 0,\ E\rho_n^2 = \sigma^2$.
The H-admissible local risk-minimizing strategy under transaction costs is given by explicit formulas:

$$\xi_n = \Theta_n^* = \Theta_n^*\big(S_{n-1}\big) = \operatorname*{arg\,min}_{o\leq i\leq m+1} r_n\big(Z_i^*\big) = J_{N-n}\big(S_{n-1}\big), \tag{2}$$

where $Z_i^* = Z_i^*\big(S_{n-1}\big) = \operatorname*{arg\,min}_{Z_{i-1}<Z\leq Z_i} r_n\big(Z\big)$

and $\quad \eta_n = V_n(\varphi) - \xi_n S_n,\ n = \{1, \ldots, N\}$ *with*

$$V_{n-1} = E\big(V_n\big|F_{n-1}\big) + kE\big(\xi_{n+1}S_n\chi_n\big(J_{N-n}(S_{n-1})\big)\big|F_{n-1}\big) - \\ -kJ_{N-n}\big(S_{n-1}\big)E\Big\{S_n\chi_n\big(J_{N-n}(S_{n-1})\big)\big|F_{n-1}\Big\}, \tag{3}$$

$n \in \{0, 1, \ldots, N\}$, *$N$ being the maturity time.*

The risk function is given by

$$r_t\big(\xi_n\big) = E\Big\{\Big(V_n - V_{n-1} - \xi_n\Delta S_n + kS_n\big|\xi_{n+1} - \xi_n\big|\Big)^2\big|F_{n-1}\Big\}. \tag{4}$$

.

Proof. The proof of Theorem 1 is done step by step, going backwards from time N. We apply the argument of the preceding section step by step to retroactively determine our trading strategy. At time k, we would choose ξ_{k+1} and φ_k such that the conditional risk

$$E\Big((C_n - C_{n-1})^2 \big| F_{n-1}\Big) = E\bigg\{\Big(V_n - V_{n-1} - \xi_n \Delta S_n + k S_n \big|\xi_{n+1} - \xi_n\big|\Big)^2 \big| F_{n-1}\bigg\}$$

is minimized and the cost process of a strategy $\varphi = (\xi, \eta)$ including transaction costs

$$C_t(\varphi) = V_t(\varphi) - \sum_{j=1}^{t} \xi_j \Delta S_j + k \sum_{j=1}^{t} S_j \big|\xi_j - \xi_{j-1}\big|$$

is a martingale.
For the proof of the theorem we consider the following:

Step 1
We will assume without impairing the generality that $\sigma^2 = 1$.

Let $n = N$ and $\rho_n \in \big(\alpha_1, \ldots, \alpha_m\big)$ with probabilities $\big(p_1, \ldots, p_m\big)$.
According to definition 1a), we set $\xi_{N+1} = \xi_N$. This follows directly from the property that ξ_t is a predictable process.

We admit only strategies such that each V_n is square-integrable and such that the contingent claim H is produced in the end, i.e. $V_N = H$.

If one chooses the portfolio so that

$$V_{N-1} = E\big(H(S_N)\big|F_{N-1}\big) = \sum_{i=1}^{m} H\big((1+\alpha_i)S_{N-1}\big)p_i = H_1(S_{N-1}),$$

then the cost process C_t is a martingale.

For the next steps, we agree again to use simplified notations:

$$H_0(U) := H(U), \qquad H_1(U) := \sum_{i=1}^{m} H_0\big((1+\alpha_i)U\big)p_i.$$

In this notation, we need to solve the following minimization problem:

$$E\bigg\{\Big(H_0(S_N) - H_1(S_{N-1}) - \xi_N \Delta S_N\Big)^2 \Big| F_{N-1}\bigg\} \longrightarrow min.$$

The risk is minimized by choosing

$$\begin{aligned}
\xi_N \quad &= \frac{E\Big\{\Big(H_0(S_N) - H_1(S_{N-1})\Big)(\Delta S_N)\Big|F_{N-1}\Big\}}{E\Big\{(\Delta S_N)^2\Big|F_{N-1}\Big\}} = \\
&= \frac{E\Big\{\Big(H_0(S_N) - H_1(S_{N-1})\Big)\rho_N\Big|F_{N-1}\Big\}}{S_{N-1}} = \\
&= \frac{\sum_{i=1}^m \Big(H_0\big((1+\alpha_i)S_N\big) - H_1(S_{N-1})\Big)\alpha_i p_i}{S_{N-1}} = J_0(S_{N-1}), \text{ i.e.}
\end{aligned} \tag{5}$$

$$J_0(U) = \frac{1}{U}\sum_{i=1}^m \alpha_i\big(H_0(\beta_i U) - H_1(U)\big)p_i \quad \text{ or } \quad \xi_N = J_0(S_{N-1}).$$

Remark 2

We assume that there are no transaction costs at the time N the option is exercised. This assumption is founded in economics, because at the time of exercise, no reallocation of the portfolio takes place.

Step 2

Let $n = N-1$. For the risk function taking into account transaction costs, the following minimization problem is to solve:

$$E\Big\{\Big(V_{N-1} - V_{N-2} - \xi_{N-1}\Delta S_{N-1} + k\big|\xi_N - \xi_{N-1}\big|S_{N-1}\Big)^2\Big|F_{N-2}\Big\} \longrightarrow min. \tag{6}$$

The portfolio will be selected in the same way as in the step 1 so that the cost process is a martingale, i.e.

$$\begin{aligned}
V_{N-2} \;&=\; E\Big(V_{N-1}\Big|F_{N-2}\Big) + kE\Big(\big|\xi_N - \xi_{N-1}\big|S_{N-1}\Big|F_{N-2}\Big) = \\
&=\; E\Big(V_{N-1}\Big|F_{N-2}\Big) + kE\Big\{(\xi_N - \xi_{N-1})S_{N-1}\chi^+_{N-1}\Big|F_{N-2}\Big\} + \\
&+\; kE\Big\{(\xi_{N-1} - \xi_N)S_{N-1}\chi^-_{N-1}\Big|F_{N-2}\Big\} = E\Big(V_{N-1}\Big|F_{N-2}\Big) + \\
&+\; kE\Big(\xi_N S_{N-1}\chi^+_{N-1}\Big|F_{N-2}\Big) - k\xi_{N-1}E\Big(S_{N-1}\chi^+_{N-1}\Big|F_{N-2}\Big) + \\
&+\; k\xi_{N-1}E\Big(S_{N-1}\chi^-_{N-1}\Big|F_{N-2}\Big) - kE\Big(\xi_N S_{N-1}\chi^-_{N-1}\Big|F_{N-2}\Big) = \\
&=\; E\Big(V_{N-1}\Big|F_{N-2}\Big) + kE\Big\{\xi_N S_{N-1}\big(\chi^+_{N-1} - \chi^-_{N-1}\big)\Big|F_{N-2}\Big\} - \\
&-\; k\xi_{N-1}E\Big\{S_{N-1}\big(\chi^+_{N-1} - \chi^-_{N-1}\big)\Big|F_{N-2}\Big\},
\end{aligned}$$

where $\chi^+_{N-1} = \chi(\xi_N \geq \xi_{N-1})$ and $\chi^-_{N-1} = \chi(\xi_N < \xi_{N-1})$ or

$$\chi_N(Z) = \chi(\xi_N \geq Z) - \chi(\xi_N \leq Z).$$

It follows that

$$\begin{aligned} V_{N-2} &= E\Big(V_{N-1}\big|F_{N-2}\Big) + kE\Big\{\xi_N S_{N-1}\chi_N(\xi_{N-1})\big|F_{N-2}\Big\} - \\ &- k\xi_{N-1}E\Big\{S_{N-1}\chi_N(\xi_{N-1})\big|F_{N-2}\Big\}. \end{aligned} \tag{7}$$

Taking that into account, we have now to solve this minimizing problem:

$$E\bigg\{\Big(C_{N-1} - C_{N-2}\Big)^2\big|F_{N-2}\bigg\} = E\bigg\{\Big(V_{N-1} - V_{N-2} - \xi_{N-2}\Delta S_{N-2} +$$
$$+ k\xi_N S_{N-1}\chi_N(\xi_{N-1}) - k\xi_{N-1}S_{N-1}\chi(\xi_{N-1})\Big)^2\big|F_{N-2}\bigg\} =$$
$$= E\bigg\{\bigg[\Big(V_{N-1} - E(V_{N-1}\big|F_{N-2})\Big) + k\xi_N S_{N-1}\chi_N(\xi_{N-2}) -$$
$$- kE\xi_N S_{N-1}\chi_N(\xi_{N-1}) - \xi_{N-1}\Big(\Delta S_{N-1} + kS_{N-1}\chi_N(\xi_{N-1}) -$$
$$- kE\{S_{N-1}\chi_N(\xi_{N-1})\big|F_{N-2}\}\Big)\bigg]^2\big|F_{N-2}\bigg\} \longrightarrow min.$$

After change the notations we get:

$$r_N(z) = E\bigg\{\Big(l_N(S_{N-1}, S_{N-2}, Z) - Zh_N(S_{N-1}, S_{N-2}, Z)\Big)^2\big|F_{N-2}\bigg\},$$

where

$$\begin{aligned} &l_N := H_1(S_{N-1}) - E(V_{N-1}\big|F_{N-2}) + kJ_0(S_{N-1})S_{N-1}\chi_N(Z) - \\ &- kEJ_0(S_{N-1})S_{N-1}\chi_N(Z), \\ &h_N := \Delta S_{N-1} + kS_{N-1}\chi_N(Z) - kE\{S_{N-1}\chi_N(Z)\big|F_{N-2}\} \end{aligned}$$

and also

$$\chi_N(Z) = \begin{cases} 1; & \xi_N \geq Z \\ -1; & \xi_N < Z. \end{cases}$$

For $\Delta S_N = \rho_N S_{N-1},\ \beta_k = 1 + \alpha_k$ we have

$$\xi_N = J_0\big(S_{N-1}\big) \in \Big\{ J_0\big(\beta_0 S_{N-2}\big), \ldots, J_0\big(\beta_k S_{N-2}\big) \Big\}.$$

Now, the following cases are considered below, in order to derive the desired strategy.

Here we investigate the behavior of the risk function r. For this, the following cases are considered to derive the desired strategy. In addition, the end intervals may coincide with the origin orwith the infinity. It means:

$$\begin{aligned}
Z_0 = &\qquad\quad -\infty; \\
Z_1 = &\quad Z_1\big(S_{N-2}\big) = \min_{1\le i\le m} J_0\big(\beta_i S_{N-2}\big); \\
Z_2 = &\quad Z_2\big(S_{N-2}\big) = \min_{2\le i\le m} J_0\big(\beta_i S_{N-2}\big); \\
\ldots & \\
Z_m = &\quad Z_m\big(S_{N-2}\big) = \max_{1\le i\le m} J_0\big(\beta_i S_{N-2}\big); \\
Z_{m+1} &= +\infty.
\end{aligned}$$

For $Z \in \big(Z_{i-1}; Z_i\big]$, i.e., $Z_{i-1} < Z \le Z_i$ we denote with

$$\chi_N\big(Z\big) := \chi\big(\xi_N \ge Z\big) - \chi\big(\xi_N < Z\big) = \chi\big(\xi_N \ge Z_i\big) - \chi\big(\xi_N \le Z_{i-1}\big) = \chi_N$$

so that l_N, h_N can be written as

$$\begin{aligned}
& l_N = l_{N,i} = H_1\big(S_{N-1}\big) - E\big(V_{N-1}\big|F_{N-2}\big) + kJ_0\big(S_{N-1}\big)S_{N-1}\chi_{N,i} - \\
& - kEJ_0\big(S_{N-1}\big)S_{N-1}\chi_{N,i}, \\
& h_N = h_{N,i} = \Delta S_{N-1} + kS_{N-1}\chi_{N,i} - kE\big\{S_{N-1}\chi_{N,i}\big|F_{N-2}\big\},\ \text{we deduce} \\
& r_N\big(Z\big) = r_{N,i}\big(Z\big) = E\Big\{\big(l_{N,i} - Zh_{N,i}\big)^2\big|F_{N-2}\Big\},\ \text{which} \\
& Z_i^* = Z_i^*\big(S_{N-2}\big) = \underset{Z_{i-1}<Z\le Z_i}{\arg\min}\ r_n\big(Z\big)
\end{aligned}$$

and finally we obtain that

$$\Theta^*_{N-1} = \Theta^*_{N-1}\big(S_{N-2}\big) = \underset{o\le i\le m+1}{\arg\min}\ r_n\big(Z_i^*\big) = J_1\big(S_{N-2}\big). \tag{8}$$

is the solution of (6).
So the required strategy is given by $\xi_{N-1} = J_1\big(S_{N-2}\big)$ and

$$V_{N-2} = E\big(V_{N-1}\big|F_{N-2}\big) + kE\Bigg(\xi_N S_{N-1}\chi_N\Big(J_1\big(S_{N-2}\big)\Big)\big|F_{N-2}\Bigg)-$$
$$-kJ_1\Big(S_{N-2}\Big)E\Big\{S_{N-1}\chi_N J_1\Big(S_{N-2}\Big)\big|F_{N-2}\Big\}.$$

k-th step can be shown in the same way as in the step 2.
Thus, theorem 1 is proved.

□

The previous result can be extended to the case when the price process is a markovian.

Theorem 2
Assume that $S = \big(S_n\big) \quad n = \{1, \ldots, N\}$ is a markovian (or markov-prozess) with respect to given filtration and let H be a contingent claim.
Then an H-admissiblel risk minimizing strategy $\varphi = (\xi, \eta)$ under transaction costs satisfies the relations

$$\xi_n = \Theta_n^* = \Theta_n^*\big(S_{n-1}\big) = \underset{o \le i \le m+1}{\arg\min}\ r_n\big(Z_i^*\big) = J_{N-n}\big(S_{n-1}\big), \tag{9}$$

where $Z_i^* = Z_i^*\big(S_{n-1}\big) = \underset{Z_{i-1} < Z \le Z_i}{\arg\min}\ r_n\big(Z\big)$

and $\quad \eta_n = V_n(\varphi) - \xi_{n+1}S_n,\ n = \{1, \ldots, N\}$ *with*

$$V_{n-1} = E\big(V_n\big|F_{n-1}\big) + kE\big(\big|\xi_{n+1} - \xi_n\big|S_n\big|F_{n-1}\big) - \xi_n E\big(\Delta S_n\big|F_{n-1}\big) \tag{10}$$

$n \in \{0, 1, \ldots, N\}, N$ *being a maturity time.*
The risk function is given by

$$r_t\big(\xi_n\big) = E\Big\{\big(V_n - V_{n-1} - \xi_n\Delta S_n + kS_n\big|\xi_{n+1} - \xi_n\big|\big)^2\big|F_{n-1}\Big\}. \tag{11}$$

Proof. The proof is very similar to that of Theorem 1. The only difference is that we apply the general form of portfolio. □

Remark 3
It can happen in the following two theorems that $r_i(\mu_i) = r_j(\mu_j), \quad \mu_i \neq \mu_j$ where $i, j = 0, 1, \ldots, N$ and $i \neq j$. In this case, one can choose any of $r_i(\mu_i)$.

The problem of finding a risk-mimimizing strategy including transaction costs for a finite probability space has also been solved.

Finally, it should be noted that the above method can easily be generalized to American options, but with the difference that in the American-style options, the problem of optimal stopping occurs (see *Safarian* [7]). In this context, the well-known Bellman principle of backward induction is applied. One must also consider the risk-minimizing strategy including transaction costs, in the case where the price process takes infinitely many values. This can be more rigorously (see *Lamberton/Pham/Schweizer* [5]), shown by analogy.

2 Risk-Minimization under transaction costs (continuous time model)

In this section we consider generalization of the fundamental theorem of Föllmer-Sondermann in the presence of linear transaction costs. In this case, the price process is a square integrable martingale. The hedging strategy can be constructed using the Kunita-Watanabe projection technique.

2.1 Formulation of the problem

Consider the probability space $(\Omega, F, P, ((\mathcal{F}_t)_{t\geq 0})$, with a filtration $(\mathcal{F}_t)_{t\geq 0}$ and an increasing family of σ - algebras included in F.

Let $S = (S)_{t\geq 0}$ be a square-integrable semimartingale.

Definition 3
a) A pair $\varphi = (\xi, \eta)$ is a trading strategy, if it satisfies the following properties:

ξ_t *is* $\mathcal{F}_{t+1}$*-measurable,* $\quad 0 \leq t \leq T$ *and*

η_t *is* $\mathcal{F}_t$*-measurable,* $\quad 0 \leq t \leq T.$

b)A value process at time t is given by

$$V_t(\varphi) = \xi_t S_t + \eta_t.$$

c) The random process $G = g_t(\varphi)$, $0 \leq t \leq T$ is called linear transaction cost process at the time t and is given by

$$g_t(\varphi) = \int_0^t g_u du = k \int_0^t f_u S_u du, \tag{12}$$

where S_t is the stock price at time t and f_t is a number of selling or buying stock at time t and constant k is the coefficient of transaction costs.

d) The cumulative cost $C_t(\varphi)$ at time t in the presence of transaction costs can be represented in the following way

$$C_t(\varphi) = C_t(\varphi) - \int_0^t \xi_u dS_u + \int_0^t g_u du.$$

Note that both processes are well-defined, right-continuous and square-integrable.

The aim is to construct an H-admissible mean self-financing strategy in the presence of transaction costs:

1) $E\big((C_T - C_t)\big|\mathcal{F}_t\big) = 0$

2) $R_t(\varphi) = E\big((C_T - C_t)^2\big|\mathcal{F}_t\big) \longrightarrow min$, such that $V_T = H$ *a.s.*

Lemma 1. *Let $\varphi = (\xi, \eta)$ be a trading strategy with a risk function $R_t(\varphi)$ and $t \in [0; T]$. Then there exists a trading strategy $\varphi^* = (\xi^*, \eta^*)$ satisfying*

a) $V_T(\varphi^*) = V_T(\varphi)$ a.s.

b) $C_t(\varphi^*) = E\big(C_T(\varphi^*)\big|\mathcal{F}_t\big)$ a.s. for all $t \in [0; T]$.

c) $R_t(\varphi^*) \le R_t(\varphi)$ a.s. for all $t \in [0; T]$.

Proof. a)By setting $\xi_t^* = \xi_t$ and

$$\eta_t = E\Bigg(\Bigg(V_t(\varphi) - \int_0^T \xi_u dS_u + \int_0^T g_u du\Bigg)\Bigg|\mathcal{F}_t\Bigg) + \int_0^t \xi_u du - k\int_0^t g_u du - \xi_t S_t$$

we obtain this relation

$$V_t(\varphi^*) = E\Bigg\{\Bigg(V_t(\varphi) - \int_0^T \xi_u dS_u + \int_0^T g_u du\Bigg)\Bigg|\mathcal{F}_t\Bigg\} + \int_0^t \xi_u du - \int_0^t g_u du$$

for the value process.

It implies that $V_t(\varphi^*) = V_T(\varphi)$.

b) The proof of b) follow directly from definition of cost process, i.e.

$$\begin{aligned}
C_t(\varphi^*) =& V_t(\varphi^*) - \int\limits_0^t \xi_u dS_u + \int\limits_0^t g_u du = \\
=& E\Big\{\Big(V_t(\varphi) - \int\limits_0^T \xi_u dS_u + \int\limits_0^T g_u du\Big)\Big|\mathcal{F}_t\Big\} = \\
=& E\big(C_T(\varphi^*)\big|\mathcal{F}_t\big) = E\big(C_T(\varphi)\big|\mathcal{F}_t\big).
\end{aligned}$$

c) For the risk function $R_t(\varphi)$ we have that

$$\begin{aligned}
R_t(\varphi) =& E\Big(\big(C_T(\varphi) - C_t(\varphi)\big)^2\big|\mathcal{F}_t\Big) = \\
=& E\Big(\big(C_T(\varphi^*) - C_t(\varphi^*)\big)^2\big|\mathcal{F}_t\Big) + \\
&+ E\Big(\big(C_T(\varphi^*) - C_t(\varphi^*)\big)\big(C_T(\varphi^*) - C_t(\varphi^*)\big)\big|\mathcal{F}_t\Big) + \\
&+ \big(C_t(\varphi^*) - C_t(\varphi)\big)^2 = R_t(\varphi^*) + \big(C_t(\varphi^*) - C_t(\varphi)\big)^2.
\end{aligned}$$

It results that

$$R_t(\varphi^*) < R_t(\varphi).$$

Thus, the lemma is proved. □

2.2 Theorem of Föllmer-Sondermann including transaction costs

Now we consider the special case where the price process S is a square-integrable martingale. We show how the fundamental Theorem of Föllmer-Sondermann can be generalized including transaction costs.

Let $S = (S_t)_{t\in[0;T]}$ is a square-integrable martingale, i.e. $E\big(S_{t+1}\big|\mathcal{F}_t\big) = S_t, \quad 0 \le t \le T.$

Let $\varphi = (\xi, \eta)$ be an H-admissible trading strategy. If φ is mean-self-financing, then the value process V_t $(0 \le t \le T)$ is a martingale, hence of the form

$$V_t = \overline{V_t}(\varphi) := E\big(H\big|\mathcal{F}_t\big). \tag{13}$$

Remark 4

For every contingent claim H the process $\overline{V_t}(\varphi)$ is called forecast process (see [8] Definition II.2). The process $\overline{V_t}(\varphi)$ is a right-continous square-integrable martingale.

Now we want to give a direct construction of the optimal hedging strategy in the presence of transaction costs.

The process of transaction costs at time t is given by

$$g_t := kf_tS_t, \quad 0 \leq t \leq T,$$

wheref_t is a nonanticipating random process.

From the martingale property of S, we can now use the fact, that H can be rewritten as

$$H = EH + \int_0^T \mu_u^H dS_u + L_T^H,$$

where μ^H is a predictable process and $L_T^H,\ 0 \leq t \leq T$ is a martingale which is orthogonal to S_t.

For any H-admissible trading strategy $\varphi = (\xi, \eta)$, the processes $\overline{V_t}(\varphi)$ and η_t under transaction costs are given by

$$\overline{V_t}(\varphi) = E\Big\{H + \int_t^T g_u du \big| \mathcal{F}_t\Big\} \tag{14}$$

and

$$\eta_t = \overline{V_t}(\varphi) - \xi_t S_t. \tag{15}$$

The cost process $C_t(\varphi)$ at time t in the presence of transaction costs is given by

$$\begin{aligned} C_t \ &= E\Big(\Big(H + \int_t^T g_u du\Big) \big| \mathcal{F}_t\Big) - \int_0^t \xi_u dS_u + \int_0^t g_u du = \\ &= E\Big(\Big(H + \int_0^T g_u du\Big) \big| \mathcal{F}_t\Big) - \int_0^t \xi_u dS_u. \end{aligned} \tag{16}$$

(16) together with Definition 3 yields

$$\begin{aligned}
C_T - C_t \quad &= H + \int_0^T g_u du + \int_0^T \xi_u dS_u - E\big(H\big|\mathcal{F}_t\big) - \\
&- E\Big(\int_t^T g_u du\big|\mathcal{F}_t\Big) + \int_0^t \xi_u dS_u = \\
&= \int_t^T (\mu_u^H - \xi_u) dS_u + k\int_t^T f_u S_u du - E\Big(k\int_t^T f_u S_u du\big|\mathcal{F}_t\Big) + L_T^H + L_t^H.
\end{aligned} \tag{17}$$

From now on, we set

$$\int_t^T f_u S_u du = J(t)S_t - \int_t^T J(u) dS_u, \text{where} \quad J(t) = \int_0^t f_u du. \tag{18}$$

Using these notations, we have

$$\int_t^T g_u du - E\Big(\int_t^T g_u du\big|\mathcal{F}_t\Big) = J(T)S_T - E\big(J(T)S_T\big|\mathcal{F}_t\big) - \int_t^T J(u) dS_u$$

and applying Kunita-Watanabe decomposition we see that

$$J(T)S_T = E\big(J(T)S_T\big) + \int_0^T \nu_u dS_u + L_T^*. \tag{19}$$

Taking equality (19) into account we obtain

$$\int_t^T g_u du - E\Big(\int_t^T g_u du\big|\mathcal{F}_t\Big) = k\Big(\int_t^T \nu_u dS_u + L_T^* - L_t^* - \int_t^T J(u) dS_u\Big).$$

We assume without impairing the generality that $k = 1$.
This implies, that

$$C_T - C_t = \int_t^T \big(\mu_u^H - \xi_u + \nu_u - J(u)\big) dS_u + L_T^* - L_t^* + L_T - L_t$$

which yields

$$E\big((C_T-C_t)^2\big|\mathcal{F}_t\big) = E\Big\{\int\limits_t^T \big(\mu_u^H-\xi_u+\nu_u-J(u)\big)^2 d\langle S\rangle_u\Big|\mathcal{F}_t\Big\}+E\Big\{(L_T^*-L_t^*+L_T-L_t)^2\big|\mathcal{F}_t\Big\}.$$

This allows us to conclude that

$$\xi_n = \mu_n^H + \nu_n - J(n) \tag{20}$$

is the optimal hedging strategy.
We have just proved the following theorem:

Theorem 3
Assume that $S = (S)_{t\geq 0}$, for all $t \in [0;T]$ is a square-integrable martingale. Then for every contingent claim $H \in L^2(P)$ there exists a unique H-admissible risk-minimizing strategy $\varphi = (\xi, \eta)$ under linear transaction costs (12) *and it is given by formulas* (15) *and* (20).

3 Conclusion

In contrast to the complete market, in the incomplete there is no unique martingale measure and a general claim is not necessarily a stochastic integral of the price process. A perfect hedge is no longer possible. From an economic point of view, this means that such a claim will have intrinsic risk. The problem is to construct strategies including transaction costs that minimize risk. In this context, it was shown that a unique risk-minimizing strategy exists. In the continuous market model, it can be constructed using the Kunita-Watanabe projection technique in the space M^2 of square-integrable martingales. In the discrete time market model, the strategy is given a closed-form formula, which facilitates practical applicability of the method.

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