Quantified Stability Investigations for Reaction-Diffusion Systems

Zur Erlangung des akademischen Grades eines

DOKTORS DER NATURWISSENSCHAFTEN

von der Fakultät für Mathematik
des Karlsruher Instituts für Technologie (KIT)
genehmigte

DISSERTATION

von

M.Sc. Madina Omarova
aus Almaty in Kasachstan

Tag der mündlichen Prüfung: 12.06.2013
Referent: Prof. Dr. M. Plum
Korreferent: Prof. Dr. W. Reichel

Acknowledgements

Foremost, I would like to express my deep gratitude to my supervisor Prof. Dr. Michael Plum for his support and patience during my PhD studies. This thesis would not have been possible without his valuable advice and encouragement.

I am grateful to Prof. Dr. Wolfgang Reichel for introducing me to the field of pattern formation models and giving the idea for the thesis.

I would also like to thank the Carl Zeiss, Fazit, and Kronmüller Foundations for their financial support during this research.

Last but not least I am eternally thankful to my parents and my sister for their love and support during my life.

Contents

1

Introduction

1.1 Reaction-diffusion systems

The main topic of this thesis are parabolic differential equations of the form

$$\frac{\partial u_i}{\partial t} = d_i \Delta u_i + f_i(x, u_1, \ldots, u_n) \quad (i = 1, \ldots, n). \tag{1.1}$$

Systems (1.1) are called reaction-diffusion systems and have a wide range of applications in chemistry, physics, biology, ecology and geology. They serve as a typical mathematical model for many processes which are time and space dependent. Here the unknown functions $u_1, \ldots, u_n$ could, for example, represent the densities of interacting populations or the concentrations of chemical reactants. The functions $f_1, \ldots, f_n$, which are in many cases nonlinear, describe the reaction between the participants. The diffusion terms $d_i \Delta u_i$ reflect the distribution in space. In this thesis we are going to focus on models with positive diffusion coefficients d_i, $i = 1, \ldots, n$.

The processes, which are described by the systems (1.1), usually take place in some confined area (e.g. biotop). Hence, we postulate the differential equations (1.1) for all $(x, t) \in \Omega \times (0, \infty)$, where Ω is a bounded domain in $\mathbb{R}^m$. In our investigations we consider the case $m = 1$, that is we set $\Omega = (0, l)$ to be a bounded interval. In addition, we impose either Dirichlet or Neumann boundary conditions at $(0, t)$ and at (l, t) for all $t \in (0, \infty)$ and consider an initial condition at the moment $t = 0$.

The mathematical theory, which underlies reaction-diffusion processes, is a widely developed and fundamental field in the area of partial differential equations. The main focus of the investigations in this field lies on the existence of the solution, its structure, stability, local and global behaviour. For that purpose many qualitative techniques and approaches have been developed, among them semigroup methods and variational methods.

In our work we introduce an alternative approach to the field of reaction-diffusion equations. It combines the techniques of analysis with numerical computations and is known as "computer-assisted proofs". It is in the nature of computer-assisted methods that the verification of some assumptions, which are problematic to treat analytically, is left to the computer.

The main focus of our thesis lies on the investigation of the properties of stationary solutions $\bar{u}$ to (1.1). Using some particular examples, we intend to answer the questions of its existence and stability. Moreover, granted that the stationary solution is stable, we propose a method for the quantification of its domain of attraction.

The methods developed in this thesis are applied to some particular examples which we have picked from different areas of biology. We are going to consider the following models

1. the Schnakenberg model which describes the pattern formation in developmental biology;

2. a predator-prey model. This model simulates the interaction between two different species, one of which predates on the other;

3. the spruce budworm model which demonstrates the distribution of the pest insect spruce budworm and is important for pest control strategies;

4. a competition model, where the interaction of two different species, this time on the basis of competition, is observed.

In the next section we provide a brief description of the examples above, accentuating on their biological background.

1.2 Examples

1.2.1 Turing instability

The concept of Turing instability is an important concept in the field of pattern formation. It explains the property of some reaction-diffusion systems to exhibit stationary solutions which are heterogeneous in space. This heterogeneity, given that the observed solution is stable, corresponds to the final pattern. Basically, according to Turing [59], the pattern is caused by those modes of the solution which are stable without diffusion, but became unstable after diffusion is introduced into the system. This idea of Turing has been quite innovative, since before establishing this concept, diffusion had been understood by scientists only as a smoothing factor.

Let us consider a system of the form

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + F(u(x,t)), & t > 0, \quad x \in [0,l], \\ \dfrac{\partial u(0,t)}{\partial \nu} = \dfrac{\partial u(l,t)}{\partial \nu} = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l], \end{cases} \tag{1.2}$$

where $D = \operatorname{diag}(d_1, d_2)$ is a matrix with positive diagonal elements and $F : \mathbb{R}^2 \to \mathbb{R}^2$ is the nonlinear reaction term. For convenience we denote the elements of the vector F as $F = (f, g)^T$. The system above incorporates the Neumann boundary condition, to be understood as a no flux condition. If, for example, the interaction between two species is under consideration, then the no flux condition means that no single animal leaves the observed habitat. Additionally, there exists no external influence on the resulting solutions.

In the following we present a brief description of the notion of the Turing instability. For more details please refer to [19, 37].

Let u^* be a spatially homogeneous equilibrium of system (1.2), that is $F(u^*) = 0$. In particular, due to Turing, we are interested in the case where u^* is stable in the absence of diffusion, i.e. u^* is a stable solution of the ordinary differential system,

$$u_t(x,t) = F(u(x,t)), \tag{1.3}$$

associated with (1.2). Let us set $w = u - u^*$. It is easy to see that the linearisation of (1.2) about $u = u^*$ results in

$$\begin{cases} w_t(x,t) = Dw_{xx}(x,t) + J_F w(x,t), & t > 0, \quad x \in [0,l], \\ \dfrac{\partial w(0,t)}{\partial \nu} = \dfrac{\partial w(l,t)}{\partial \nu} = 0, & t \geq 0, \\ w(x,0) = w^0(x), & x \in [0,l], \end{cases} \tag{1.4}$$

where J_F is a 2×2 Jacobian matrix defined as

$$J_F = \begin{pmatrix} f_{u_1} & f_{u_2} \\ g_{u_1} & g_{u_2} \end{pmatrix}\Bigg|_{(u_1^*,u_2^*)}. \tag{1.5}$$

In the following we will establish some particular conditions for the elements of the Jacobian J_F, which will be the reason for an inhomogeneous equilibrium to bifurcate from u^* as either the width l or the diffusion coefficients d_1, d_2 are varied.

Let (ϕ_j, λ_j) denote the jth eigenpair of the second order derivative operator, defined on $[0,l]$, with no-flux boundary conditions. Thus, $(\phi_j, \lambda_j) = \left(\cos\left(\dfrac{j\pi x}{l}\right), \dfrac{j^2\pi^2}{l^2}\right)$ $j = 0, 1, 2, \ldots$. By the separation of the variables technique, we obtain the solutions of (1.4) of the form

$$w(x,t) = \sum_{j=0}^{\infty} \phi_j(x) s_j(t), \tag{1.6}$$

where $s_j(t) \in \mathbb{R}^2$. Substituting (1.6) into (1.4) and equating the coefficients of each ϕ_j we have

$$\frac{ds_j(t)}{dt} = C_j s_j(t), \tag{1.7}$$

where C_j is the matrix

$$C_j = J_F - \lambda_j D. \tag{1.8}$$

We investigate the stability of the trivial solution $w = 0$ by examining the behaviour of the eigenvalues of the matrices C_j. Suppose that each C_j has two eigenvalues with negative real part. This means that $s_j(t)$ decays to zero as $t \to \infty$ and hence the trivial solution $w = 0$ is asymptotically stable. Now if any C_j has an eigenvalue with positive real part, then $|s_j|$ can grow exponentially, thus causing w to grow as well. Hence the trivial solution $w = 0$ will be unstable to any spatial perturbations which are not orthogonal to ϕ_j. Now let the parameter l (or d_1 and d_2) be chosen so that some C_j has an eigenvalue with zero real part. Then, as l (or d_1 and d_2) is varied locally, the stability of $w = 0$ may switch. This change of the stability reflects a bifurcation of some inhomogeneous equilibrium from the trivial solution $u = u^*$ for (1.2). This conclusion follows after the application of the Lyapunov-Schmidt bifurcation method to the problem (1.2). We omit to go into many details on this approach and for more thorough description we refer to [19, 52]. Here we simply note that if C_j has an eigenvalue with zero real part, then some non-trivial equilibrium for (1.2) will bifurcate from $u = u^*$. The eigenvalues, σ, of C_j satisfy

$$\sigma^2 + (\lambda_j(d_1 + d_2) - (f_{u_1} + g_{u_2}))\sigma + h(\lambda_j) = 0, \tag{1.9}$$

with

$$h(\lambda) = \lambda^2 d_1 d_2 - (d_1 g_{u_2} + d_2 f_{u_1})\lambda + f_{u_1} g_{u_2} - f_{u_2} g_{u_1}. \tag{1.10}$$

Here and in the following all the partial derivatives are evaluated at u^*, unless stated otherwise. In case $j = 0$ the eigenvalues of C_0 satisfy

$$\sigma^2 - (f_{u_1} + g_{u_2})\sigma + f_{u_1} g_{u_2} - f_{u_2} g_{u_1} = 0. \tag{1.11}$$

Since u^* is a stable solution of (1.3) by hypothesis, the spectrum of C_0 belongs to the left half of the complex plane. Hence we have

$$f_{u_1} + g_{u_2} < 0, \tag{1.12}$$

$$f_{u_1} g_{u_2} - f_{u_2} g_{u_1} > 0. \tag{1.13}$$

The roots $\sigma_1,\ \sigma_2$ of (1.9) are given by

$$\sigma_{1,2}(\lambda) = \frac{-(\lambda(d_1 + d_2) - (f_{u_1} + g_{u_2})) \pm \sqrt{(\lambda(d_1 + d_2) - (f_{u_1} + g_{u_2}))^2 - 4h(\lambda)}}{2}.$$

From (1.12) and the positivity of $\lambda,\ d_1$, and d_2 follows that C_j will have an eigenvalue with zero real part, if $h(\lambda_j) = 0$. This is a necessary condition for the change of the stability (and bifurcation). For the fixed eigenvalue λ_j the condition $h(\lambda_j) = 0$ could be represented as a neutral stability curve in the (d_1, d_2) plane. If the parameters $d_1,\ d_2$ are varied, then we will observe the growth of small perturbations in the solution of (1.4) every time one of these neutral stability curves is crossed. For more details please refer to [19].

Now let us fix the diffusion coefficients $d_1,\ d_2$ and allow the width of the interval l (and thus the eigenvalues λ_j) to vary. Let us consider (1.10). It is easy to see, that as $\lambda \to \infty$, we have $h > 0$. Furthermore, (1.13) implies that $h(0) > 0$. Hence the function h posseses two positive real roots $\lambda_\pm$, which are then given by

$$\lambda_\pm = \frac{(d_1 g_{u_2} + d_2 f_{u_1}) \pm \sqrt{(d_1 g_{u_2} + d_2 f_{u_1})^2 - 4 d_1 d_2 (f_{u_1} g_{u_2} - f_{u_2} g_{u_1})}}{2 d_1 d_2},$$

if and only if the following two conditions are satisfied

$$d_1 g_{u_2} + d_2 f_{u_1} > 0, \tag{1.14}$$

$$\frac{(d_1 g_{u_2} + d_2 f_{u_1})^2}{4 d_1 d_2} > f_{u_1} g_{u_2} - f_{u_2} g_{u_1}. \tag{1.15}$$

Note that in the case of the equal diffusion coefficients, that is when $d_1 = d_2$, the conditions (1.12) and (1.14) contradict each other and therefore no Turing instability could be observed. Furthermore, from (1.12) and (1.14) one obtains one more necessary condition for the Turing instability: the components f_{u_1} and g_{u_2} should have opposite signs.

Going back to the matrix C_j it is easy to see that C_j will have an eigenvalue σ, which belongs to the right-hand side of the complex plane, if and only if $\lambda_j \in (\lambda_-, \lambda_+)$ (which is equivalent to $h(\lambda_j) < 0$). The interval (λ_-, λ_+) is the instability region for the eigenvalues λ_j. If $\lambda_j \in (\lambda_-, \lambda_+)$ then the trivial solution $u = u^*$ becomes unstable in the jth eigenmode. In addition, observe that from $\lambda_j = \dfrac{j^2\pi^2}{l^2}$ follows that the width l of the given interval should be large enough in order to surely incorporate some unstable modes.

Further we wish to comment that when d_1, d_2 and the kinetics parameters are varied the unstable window of eigenvalues is varied as well: it could be pushed around, shrinked or enlarged.

We finish our discussion of Turing instability by collecting the conditions (1.12) to (1.15), which one has to impose on the elements of the Jacobian matrix J_F in order to generate the spatial pattern. For more detailed analysis of the Turing instability please refer to [19, 37].

1.2.2 Schnakenberg model

The Schnakenberg model has been introduced in 1979 as the simplest model which describes the formation of pattern in developmental biology. The ideas of the morphogen prepattern theory became the main motivation for this model. This theory has been developed by Wolpert [62] in 1969 and it introduces the concept of *positional information*. According to Wolpert [62], pattern evolves as a result of the reaction of the cells to certain chemical concentrations or morphogens. Embryonic cells are able to "read out" the positional information from the existing morphogen map (prepattern) and differentiate themselves or migrate accordingly. Thus, as soon as the morphogen map is established, the pattern formation process continues automatically. The morphogen map could be seen as the result of the reaction of morphogens with each other combined with their diffusion throughout the medium.

The dimensionless Schnakenberg model, postulated on an interval $\Omega = (0, l)$, has the form

$$\begin{cases} u_{1t}(x,t) = u_{1xx}(x,t) + \gamma\left(a - u_1(x,t) + u_1^2(x,t)u_2(x,t)\right), & t > 0, \quad x \in [0,l], \\ u_{2t}(x,t) = du_{2xx}(x,t) + \gamma\left(b - u_1^2(x,t)u_2(x,t)\right), & t > 0, \quad x \in [0,l], \\ \dfrac{\partial u(0,t)}{\partial \nu} = \dfrac{\partial u(l,t)}{\partial \nu} = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l], \end{cases}$$

where $a,\ b,\ d,\ \gamma$ are some positive constants. The Schnakenberg model describes the mechanism of the reaction between different morphogens, which is called the activator-inhibitor mechanism. Here one of the morphogens u_1 represents the activator, which is autocatalytic, and the other one, u_2, is the inhibitor. The autocatalytic property of u_1 is reflected in the term $u_1^2 u_2$. For more details we refer to [37].

It is easy to see that the only spatially homogeneous equilibrium has the form

$$\begin{aligned} u_1^* &= a + b, \\ u_2^* &= \frac{b}{(a+b)^2}. \end{aligned} \tag{1.16}$$

Thus, the conditions (1.12) to (1.15) read

$$0 < b - a < (a + b)^3, \tag{1.17}$$

$$(a + b)^2 > 0, \tag{1.18}$$

$$d(b - a) > (a + b)^3, \tag{1.19}$$

$$\left(d(b - a) - (a + b)^3\right)^2 > 4d(a + b)^4. \tag{1.20}$$

The inequalities above define the (a, b, d) parameter space, which is called pattern formation or Turing space. As it was already mentioned above, condition (1.17) together with (1.19) implies $d > 1$. Therefore, in order for the pattern to emerge, the inhibitor should diffuse faster than the activator.

Now let us discuss the parameters a, b, d, satisfying conditions (1.17) to (1.20). In order to determine those parameters, one can express (1.12) to (1.15) in the terms of the parameter u_1^*. After that, by letting u_1^* to take on a range of positive values, one can calculate the corresponding ranges for a and b, for a given value of d.

From (1.16) we have

$$\begin{aligned} b &= u_1^* - a, \\ u_2^* &= \frac{u_1^* - a}{(u_1^*)^2}. \end{aligned} \tag{1.21}$$

Next, using (1.21), we express the elements of the Jacobian matrix in the terms of u_1^* and, by analysing the conditions (1.12) to (1.15), arrive at the following boundary curves

$$\begin{aligned} &a > \frac{1}{2}u_1^*\left(1 - (u_1^*)^2\right), \quad b = \frac{1}{2}u_1^*\left(1 + (u_1^*)^2\right), \\ &a < \frac{1}{2}u_1^*\left(1 - \frac{2u_1^*}{\sqrt{d}} - \frac{(u_1^*)^2}{d}\right), \quad b = \frac{1}{2}u_1^*\left(1 + \frac{2u_1^*}{\sqrt{d}} + \frac{(u_1^*)^2}{d}\right). \end{aligned} \tag{1.22}$$

For more details please refer to [37]. As one can see at $d = 1$ the curves in (1.22) contradict each other and hence the Turing space is empty. By letting d take on values greater than 1, we observe that above some crtitical value of d a Turing space starts to grow. We denote this value as d_c and obtain it from (1.22) by determining the d such that both curves give $a = 0$ at $b = 1$. We obtain $d = d_c = 3 + 2\sqrt{2}$ and at this value two inequalities in (1.22) are no longer contradictory. The Turing space lies between the two curves in (1.22). Note that only due to the relatively simple form of the Schnakenberg nonlinearity, it is possible to carry out the analysis above. For more complicated forms of the nonlinear terms, one has to apply some other methods, mostly numerical computations.

As we have mentioned above, for the spatial pattern to occur, the width l of the observed interval is important, namely, l should be sufficiently large in order to surely incorporate some unstable modes of the solution. Thus, by picking an appropriate constellation of the parameters $a,\ b,\ d$, and posing the problem in a sufficiently large interval, one achieves the desired pattern structure at the end.

In our numerical simulations we set $a = 0.1,\ b = 0.9,\ d = 10,\ \gamma = 1,\ l = 5$. The computed approximate stationary solution components ω_1 and ω_2 are shown on Figure 1.1.

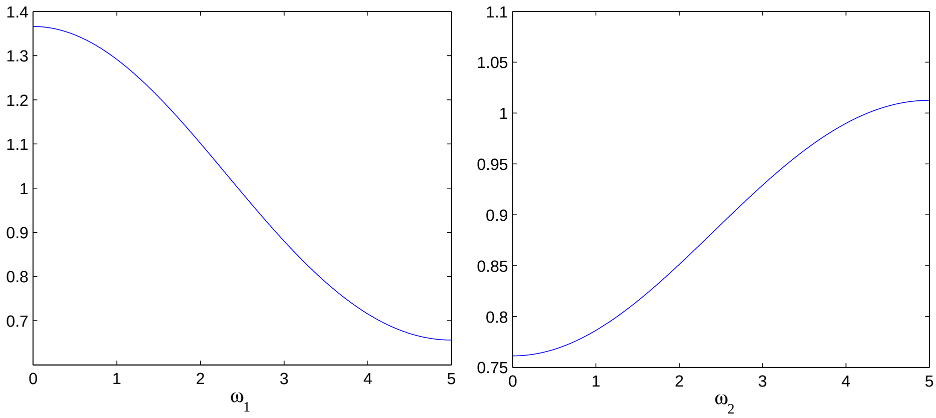

FIGURE 1.1: Approximate stationary solution of the Schnakenberg model which correspond to the parameter constellation $a = 0.1,\ b = 0.9,\ d = 10,\ \gamma = 1,\ l = 5$.

1.2.3 *Predator-prey model*

Another model, which exhibits the pattern behaviour in accordance with the Turing instability concept, is the predator-prey model. The pattern in the context of this model should be understood as the fluctuation of the densities of the predator and prey, which interact in some bounded domain. In our thesis we consider the predator-prey model, formulated on the interval $\Omega = (0, l)$.

We consider the following reaction-diffusion system

$$\begin{cases} u_{1t}(x,t) = d_1 u_{1xx}(x,t) + (h_1(u_1(x,t)) - u_2(x,t))u_1(x,t), & t > 0, \quad x \in [0,l], \\ u_{2t}(x,t) = d_2 u_{2xx}(x,t) + a u_2(x,t)(u_1(x,t) - h_2(u_2(x,t))), & t > 0, \quad x \in [0,l], \\ \dfrac{\partial u(0,t)}{\partial \nu} = \dfrac{\partial u(l,t)}{\partial \nu} = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l], \end{cases}$$

where the functions h_1 and h_2 have the form

$$h_1(s) = \varepsilon_1 \left(\gamma_1 + \gamma_2 s - s^2 \right),$$
$$h_2(s) = 1 + \varepsilon_2 s,$$

and $d_1,\ d_2,\ a,\ \varepsilon_1,\ \varepsilon_2,\ \gamma_1,\ \gamma_2$ are some positive constants.

The model above describes e.g. the interaction between two different types of plankton: phytoplankton (prey) and zooplankton (predator). It has been observed that in some cases plankton displays spatial heterogeneity, which was called patchiness. For the purpose of the investigation of that phenomenon Steele [54] in 1974 has suggested this predator-prey model. Here the $u_1,\ u_2$ components represent the densities of phytoplankton and zooplankton respectively.

Following the results of Mimura and Murray [35], we have chosen the following constellation of the parameters:

$$a = 1,\ d_1 = 0.0125,\ d_2 = 1,\ \varepsilon_1 = \frac{1}{9},\ \varepsilon_2 = \frac{2}{5},\ \gamma_1 = 35,\ \gamma_2 = 16,\ l = 1. \tag{1.23}$$

This constellation satisfies conditions (1.12) to (1.15). Note that the condition that d_1 is essentially smaller than d_2, and both d_1 and d_2 are sufficiently small is important for the generation of pattern. For more detailed analysis of the model above we refer to [35].

The result of our numerical simulations is shown on the Figure 1.2.

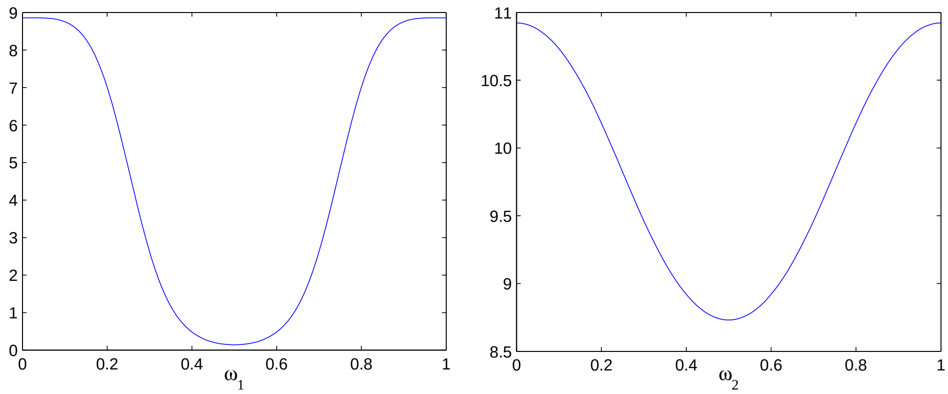

FIGURE 1.2: Approximate stationary solution of predator-prey model which correspond to the parameter constellation in (1.23).

1.2.4 Spruce budworm model

Spruce budworm (*Choristoneura fumiferana*) is a serious pest which is mostly observed in eastern Canada and northern Minnesota. This caterpillar (or moth) predates on coniferous trees and, in large numbers, is capable of damaging and killing the host. The only natural enemies of the spruce budworm are birds, which also eat other insects. Over the last century canadians have observed that every 30-40 years a sudden outbreak of the spruce budworm takes place. The outbreak may last for several years. During this time a large amount of trees are defoliated, and the forest industry, as well as the dependent communities, suffer great losses.

In 1978 Ludwig et al. [31] have proposed a model which simulates the interaction between spruce budworm and forest. Since the life-span of the tree is significantly larger than that of the spruce budworm, the forest variables were treated as constants. Thus, the model has become a single-species model. In dimensionless form, formulated on the interval $\Omega = (0, l)$, it reads

$$\begin{cases} u_t(x,t) = du_{xx}(x,t) + ru(x,t)\left(1 - \dfrac{u(x,t)}{q}\right) - \dfrac{u^2(x,t)}{1+u^2(x,t)}, & t > 0, \quad x \in [0,l], \\ u(0,t) = u(l,t) = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l]. \end{cases}$$

Here the positive parameter r is directly proportional to the linear birth rate and is inversely proportional to the intensity of predation. The positive parameter q is proportional to the carrying capacity, which is related to the density of the foliage available on the trees. Term $-\frac{u^2(x,t)}{1+u^2(x,t)}$ reflects the predation by birds and has a sigmoid character. The qualitative form of the predation term implies the existence of an approximate threshold value for the population of spruce budworm. When the population is small, the predation is moderate, when it exceeds the threshold value, the predation is " switched on".

It is easy to see that the steady state solutions are the solutions of

$$f(s) = rs\left(1 - \frac{s}{q}\right) - \frac{s^2}{1+s^2} = 0.$$

Clearly, $s = 0$, is one of these solutions. The other solutions, if they exist, satisfy

$$r\left(1 - \frac{s}{q}\right) = \frac{s}{1+s^2}. \tag{1.24}$$

The equation above can be solved explicitly or graphically. We omit the detailed discussion of the solution to (1.24) and refer to [36] for more information. Here we simply note, that there exists a domain in the r, q parameter space, where three roots of (1.24) exist. The boundary curves of that domain for some positive a are given by

$$r(a) = \frac{2a^3}{(a^2+1)^2}, \quad q(a) = \frac{2a^3}{a^2-1}. \tag{1.25}$$

One can obtain the rigorous explanation for the sudden outbreaks in the spruce budworm population by analysing the behaviour of the model as the parameters r and q change. In particular, one says that the spruce budworm model exhibits a hysterisis effect: when r and q change to some new values the system changes as well, but as r and q change back to the old values, the system does not retrace its steps in reverse. This effect is then reflected in the sudden jumps of the population levels from the smallest stable equilibrium to the largest stable equilibrium and vice versa (see [36]). The largest stable equilibrium is called an outbreak equilibrium.

Now let us comment on the spatial patterning of the spruce budworm. For that purpose we examine the trivial steady state solution $u = 0$. The linearisation of the given problem about $u = 0$ results in

$$\begin{cases} u_t(x,t) = du_{xx}(x,t) + ru(x,t), & t > 0, \quad x \in [0,l], \\ u(0,t) = u(l,t) = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l]. \end{cases} \tag{1.26}$$

Let M be a number such that $u(x,0) \leq M$, $\forall x \in (0,l)$, and let $\widehat{u}(x,t)$ be a solution of

$$\begin{cases} u_t(x,t) = du_{xx}(x,t) + ru(x,t), & t > 0, \quad x \in [0,l], \\ u(0,t) = u(l,t) = 0, & t \geq 0, \\ u(x,0) = M, & x \in [0,l]. \end{cases} \tag{1.27}$$

The Fourier analysis shows that

$$\widehat{u}(x,t) = \frac{4M}{\pi} \sum_{j=0}^{\infty} \frac{1}{2j+1} e^{\left(r - d\frac{(2j+1)^2\pi^2}{l^2}\right)t} \sin\left(\frac{(2j+1)\pi x}{l}\right). \tag{1.28}$$

Application of the comparison principle to the problems (1.26) and (1.27) results in

$$0 \leq u(x,t) \leq \widehat{u}(x,t), \quad \forall x \in (0,l), \quad t \geq 0. \tag{1.29}$$

From (1.28) we see that if $l < \pi\sqrt{\frac{d}{r}}$, then $\widehat{u}$ decays exponentially to zero as $t \to \infty$, and, due to (1.29), so is u. Therefore we conclude, that if $l < \pi\sqrt{\frac{d}{r}}$ then

$$\lim_{t\to\infty} u(x,t) = 0, \quad \forall x \in (0,l),$$

and no spatial structure occurs. Therefore, similar to the previous results, when the interval width is not large enough the pattern will not occur.

There is one more interesting relation between the size of the interval and the behaviour of the solution. Namely, it is possible to establish a correspondence between the maximum value of the solution u_m and the length l of the interval. The numerical evaluation of that correspondence, which was performed by Ludwig et. al [31], has shown that there exists a critical domain size l_0, above which the maximum population can achieve the outbreak state. In particular, if $l < l_0$, then the outbreak of the spruce budworm population is not possible. The value of l_0 could be approximately obtained by analytical means (see [37]).

In our numerical simulations we set in (1.25) $a = 1.5$, which has produced the values $r = 0.6391$ and $q = 5.4$. In addition we choose $d = 3$. In order for pattern to emerge it is essential to choose $l > \pi\sqrt{\frac{d}{r}} = 6.8068$. On the other hand, for the purpose of avoiding the sudden outbreak, we have chosen some $l < l_0$. The analytical approximation to the value l_0 has resulted in $l_0 \approx 57.5552$. The result of our numerical simulations with $l = 12$ is illustrated on Figure 1.3.

1.2.5 Competition model

The interaction of species, which are forced to coexist and have similar preferences in resources, is described by competition models. The model we are going to consider in our work is based on the interaction between grey and red squirrels in Britain. In the beginning of the 20th century North American grey squirrels have been imported

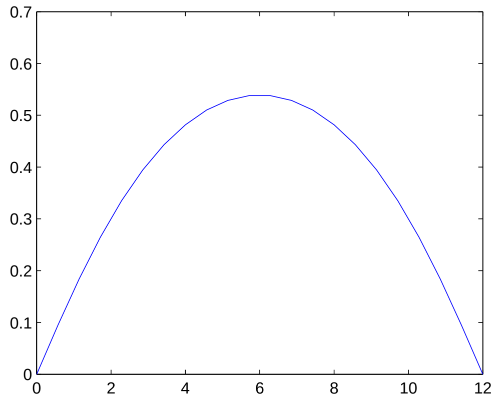

FIGURE 1.3: An approximate stationary solution of the spruce budworm model.

into various sites in Britain. They have managed to successfully spread through the country, forcing the red indigenous squirrel to drive off. Okubo et al. [39] investigated this displacement and proposed a competition model. In dimensionless form, formulated on the interval $\Omega = (0, l)$, this model reads

$$\begin{cases} u_{1t}(x,t) = u_{1xx}(x,t) + u_1(x,t)(1 - u_1(x,t) - a_{12}u_2(x,t)), & t > 0, \quad x \in [0,l], \\ u_{2t}(x,t) = du_{2xx}(x,t) + \alpha u_2(x,t)(1 - u_2(x,t) - a_{21}u_1(x,t)), & t > 0, \quad x \in [0,l], \\ \dfrac{\partial u(0,t)}{\partial \nu} = \dfrac{\partial u(l,t)}{\partial \nu} = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l]. \end{cases}$$

Here u_1, u_2 represent the densities of the grey and red squirrels respectively. The dimensionless parameter α denotes the ratio between the net birth rates of grey and red squirrels. If $\alpha > 1$, the net birth rate of the grey squirrels is higher than the net birth rate of the red squirrels. The coefficient d stands for the ratio between the diffusion coefficients. In particular, $d > 1$ implies the faster diffusion of the red squirrel. The parameters a_{12} and a_{21} measure the competitive effect of the red squirrel on the grey and vice versa.

In the absence of diffusion the above system has four homogeneous steady states, which are given by

$$\bar{u}_1 = 0, \quad \bar{u}_2 = 0; \tag{1.30}$$

$$\bar{u}_1 = 1, \quad \bar{u}_2 = 0; \tag{1.31}$$

$$\bar{u}_1 = 0, \quad \bar{u}_2 = 1; \tag{1.32}$$

$$\bar{u}_1 = \frac{1 - a_{12}}{1 - a_{12}a_{21}}, \quad \bar{u}_2 = \frac{1 - a_{21}}{1 - a_{12}a_{21}}. \tag{1.33}$$

The latter steady state exists only when $a_{12}a_{21} \neq 1$. The stability or instability, respectively, of the steady states is easy to verify with the standard methods. We obtain that the state $(0, 0)$ is unstable. For the states (1.31) to (1.33) we will distinguish between following cases

(i) $a_{12} < 1,\ a_{21} < 1,$

(ii) $a_{12} > 1,\ a_{21} > 1,$

(iii) $a_{12} < 1,\ a_{21} > 1,$

(iv) $a_{12} > 1,\ a_{21} < 1.$

Note that in cases (iii) and (iv) the steady state (1.33) does not belong to the positive quadrant and therefore is not relevant for the biological interpretation. We obtain that

(i) (1.31) and (1.32) are unstable, (1.33) is stable,

(ii) (1.31) and (1.32) are stable, (1.33) is a saddle point,

(iii) (1.31) is stable and (1.32) is unstable,

(iv) (1.31) is unstable and (1.32) is stable.

The cases (i) to (iv) are shown in Figure 1.4. In case (i), that is, when the impact of the species on each other is small, the steady state (1.33) is stable and the species coexist. Cases (ii) to (iv) illustrate the competitive exclusion principle: two different species cannot coexist and one of them eventually disappears. In case (ii) there are two stable solutions: $(1, 0)$ and $(0, 1)$. The matter of which population will ultimately win depends on the initial condition: if the initial condition starts in the area I, then the population u_2 will die out and if it starts in the area II, then the population u_1 will extinct (see [36]).

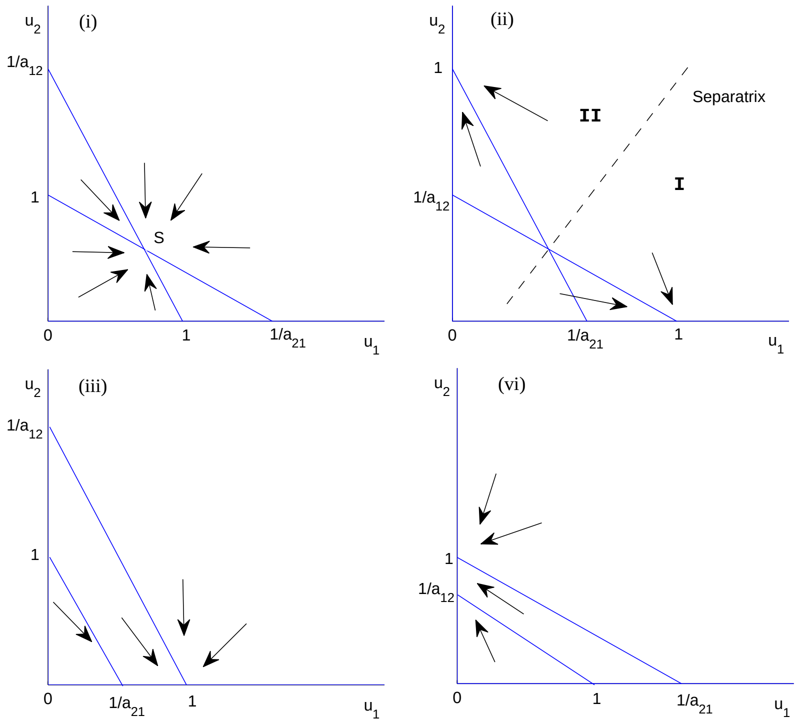

FIGURE 1.4: Schematic phase trajectories near the steady states for the competition dynamics.

For our further investigations we will be interested in the constant stationary solution which correponds to case (i) (even though it does not reflect the real interaction between grey and red squirrels). For more details on the competition models we refer to [36, 37].

1.3 Contents and scope of the thesis

As it was already mentioned above, the main subject of investigations in this thesis are stationary solutions $\bar{u}$ to problem (1.1). In particular, we are interested in the results on their existence, stability, and - in the case of the stability - in the size of their domain of attraction.

Due to the complex structure of the reaction-diffusion systems (1.1), it is usually impossible to calculate non-constant stationary solutions $\bar{u}$ in closed form. This is certainly the case with the Schnakenberg, predator-prey, and spruce budworm models. The question of existence of solutions to problems of the above type has been a subject of investigation of many scientists for many years. Concerning the examples, which are under consideration in our thesis, one can find many papers devoted to the pattern formation phenomenon, and various discussions of the numerical and analytical aspects of the models above. For example in [3] by using the homotopy analysis method, based on the fractional order differential equations, author constructs an approximate analytical solution to the Schnakenberg problem. In [51] a numerical method for the solution of the pattern formation models (and specifically for the Schnakenberg model) is proposed. In [7] one can find the examination, along with the numerical approximations, of some certain type of the travelling wave solutions of the spruce budworm model. Some numerical aspects of the modelling the spruce budworm problem are discussed in [55]. A theoretical analysis of the pattern formation, along with the computation of numerical approximations for the predator-prey model can be found in [29, 33, 35]. In [38] the pattern formation phenomenon is discussed. Most of the results about numerical investigations of these examples do not go beyond the computed approximation ω. Therefore, some rigorous quantitative results on the exact stationary solution are desirable. In our thesis we want to apply computer-assisted techniques, which can ensure the existence of a stationary solution $\bar{u}$ in some explicitly known neighbourhood of a numerical approximation ω.

In particular, we are intending to use the computer-assisted enclosure methods, which were developed for elliptic boundary value problems by Plum [5, 41, 42, 43, 45, 46], to the stationary formulation of (1.1). In the course of the implementation of the methods above, the given problem is transformed in such a way, that it becomes suitable for the application of a fixed-point theorem. As a result a constant α such that

$$\|\bar{u} - \omega\|_\infty \leq \alpha \tag{1.34}$$

is obtained. The existence of the solution is shown simultaneously. Since some of the conditions needed for the fixed-point theorem are verified numerically, this method is referred to as computer-assisted method.

Furthermore we will be concerned with stability properties of the enclosed stationary solution $\bar{u}$. For the stability investigations we consider a linearisation of problem (1.1) at $\bar{u}$, which we denote as $L_{\bar{u}}$ (rigorous definition of $L_{\bar{u}}$ will be given later). In our thesis the operator $L_{\bar{u}}$ will play an important role. In particular, by establishing the sectoriality of this operator we will be able to verify the stability of $\bar{u}$ and compute an upper bound to its domain of attraction. The notion of sectoriality has its roots in semigroup theory and defines those classes of linear operators, which have a bounded resolvent and the spectrum of which can be included into some certain sector. The sectoriality of $L_{\bar{u}}$ will be established with the help of the computer-assisted methods. Of a special help for us at this point will be the method of eigenvalue exclosure, which provides a proof for a non-existence of eigenvalues on a local basis. This method is especially useful in those cases when the operator $L_{\bar{u}}$ has complex eigenvalues. By eigenvalue exclosure, combined with some certain analytic estimations, we will be able to obtain an upper bound to the norm of the resolvent operator of $L_{\bar{u}}$ and show that its spectrum is contained in a sector, which has an opening to the right of some angle $\zeta \in \left(0, \frac{\pi}{2}\right)$ and a cusp at some real point z. The methods, which we propose in our thesis provide us with the explicit values for this constants ζ and z. Note, in particular, that if z is positive, then the stationary solution $\bar{u}$ is stable. In addition, we will be paying attention to the special case of a self-adjoint operator $L_{\bar{u}}$. Although the eigenvalue exclosure method can be applied in this case as well, one may follow more direct approach which requires less numerical effort. Namely, as opposed to the *excluding* of eigenvalues, in the self-adjoint case it is possible to compute *enclosure* intervals for eigenvalues by means of some known variational method for computing eigenvalue bounds.

Finally, while examining the stability properties of stationary solutions, we have developed approaches for the quantification of theirs domains of attraction. In our investigations we apply some results from semigroup theory. In particular, estimations of the form

$$\left\|e^{-tL_{\bar{u}}}\right\|_{\mathcal{L}(X)} \leq Ce^{-zt}, \tag{1.35}$$

are essential. As a result we obtain the estimation of the domain of attraction in the following sense: we compute some value δ_0 such that

$$\text{if} \quad \|u_0 - \bar{u}\|_\infty \leq \delta_0 \quad \text{then} \quad \lim_{t\to\infty} \|u(t) - \bar{u}\|_\infty = 0. \tag{1.36}$$

The quantification of the attraction section opens an opportunity for the investigation of the long-time behaviour of a time-depedent solution. In particular, since the system (1.1) is autonomous, one observes, that if for some fixed time $T > 0$ the solution of (1.1) is contained in the neighbourhood of $\bar{u}$ of the size δ_0, then the conclusion (1.36) is valid, and the solution converges to the stationary solution $\bar{u}$. The time T, which satisfies the above condition, can be found with the help of the computer-assisted enclosure methods for the time-dependent problems. This is the subject of further research.

The constant δ_0, which we have computed for the Schnakenberg and predator-prey models, turned out to be relatively small. The reasons for this lie in the theoretical methods, which are used for the determination of the constant C from (1.35). In this cases, the semigroup approach, which is sufficient for many qualitative purposes, has proved to be not efficient enough for explicit estimations.

When the operator $L_{\bar{u}}$ is self-adjoint, it is possible to obtain the domain of attraction, using eigenfunctions series expansion techniques. In this thesis we propose two different approaches for the quantification of the domain of attraction in case of self-adjoint $L_{\bar{u}}$. In the basis of this approaches lie explicit embedding estimations of $C[0,l] \hookrightarrow H^1(0,l)$. As a result the estimations similar to (1.36) are obtained. The computed constants δ_0 are now significantly better, compared to the cases discussed above. In the view of the improvement in the attractor's size, we have also established some certain classes of problems with non-self-adjoint linearisations, attractor of which can nevertheless be obtained by methods developed for the self-adjoint linearisations, after applying some symmetrisation technique.

Finally, before concluding this section, we wish to remark on the recent work of Cai [8]. In her work the author has considered the Schnakenberg problem, modelled on a two-dimensional domain. Similar to our results, the author was able to prove the existence and stability of some particular stationary solution, and has quantified its attractor. The corresponding value for δ_0 was quite small as well. In our thesis we investigate some other examples and extend our research to models and stationary solutions with self-adjoint linearisation. As we have already mentioned above, the results on the domain of attraction for these models are significantly better.

This thesis is organised as follows. In Chapter 2 we present some preliminary results, which we apply in the course of the thesis. In Chapter 3 we discuss computer-assisted methods for the enclosure of stationary solutions. In Chapter 4 we study the operator $L_{\bar{u}}$ and show its sectoriality. In Chapter 5 we investigate the stability properties of stationary solutions and obtain their domains of attraction. In Chapter 6 we present a brief description of variational methods for computing eigenvalue bounds and develop them in the framework of the given problems. In Chapter 7 we report on the results. The description of the corresponding numerical procedures is presented in Appendix A.

1.4 Notations

We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ the natural[1], real, and complex numbers respectively.

For $n \in \mathbb{N}$ we denote by $\mathbb{R}^n$ the n-dimensional euclidean space, endowed with the norm $|y|_2 = \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}$. $y^T = (y_1, \ldots, y_n)$ corresponds to the transpose of $y \in \mathbb{R}^n$.

For the matrix $A \in \mathbb{C}^{n\times n}$ we denote by A^* its adjoint and by $|A|_2 = \sqrt{\lambda_{\max}(A^*A)}$ its euclidean norm.

For the diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ we denote $d_{\min} := \min_{j=1,\ldots,n} d_j$ and $d_{\max} := \max_{j=1,\ldots,n} d_j$.

The identity matrix (operator) is addressed as I.

For $z \in \mathbb{R}$ and $R > 0$ we denote by $B(z, R)$ the ball with center in z and radius R. $B^C(z, R)$ is to be understood as the complement to $B(z, R)$.

We denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators between the Banach spaces X and Y. We set $\mathcal{L}(X, X) = \mathcal{L}(X)$.

The Lebesque spaces $L_p(0, l)$, $1 \le p \le \infty$ are endowed with the norms

$$\|f\|_{L_p(0,l)} = \left(\int_0^l |f(x)|^p \, dx\right)^{1/p},$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in(0,l)} |f(x)|.$$

[1] in particular, $0 \notin \mathbb{N}$

We write $\|f\|_p$ for $\|f\|_{L_p(0,l)}$. In addition, we denote by $\langle\cdot,\cdot\rangle_2$ the scalar product in $L_2(0,l)$.

The Sobolev spaces $W^{k,p}(0,l)$, where k is any positive integer and $1\le p\le\infty$, consist of all the functions $f\in L_p(0,l)$, which admit weak derivatives $D^\alpha f$ for $|\alpha|\le k$ belonging to $L_p(0,l)$. The norm on $W^{k,p}(0,l)$ is defined as

$$\|f\|_{W^{k,p}(0,l)}=\sum_{|\alpha|\le k}\|D^\alpha f\|_p\,.$$

When $p=2$, we write $H^k(0,l)$ for $W^{k,p}(0,l)$.

We denote by $C[0,l]$ the Banach space of continuous complex-valued functions on $[0,l]$, endowed with the maximum norm $\|\cdot\|_\infty$. If $k\in\mathbb{N}$, $C_k[0,l]$ is the Banach space of k-times continuously differentiable complex-valued functions on $[0,l]$, endowed with the norm

$$\|f\|_{C_k[0,l]}=\sum_{|\alpha|\le k}\|D^\alpha f\|_\infty\,.$$

Let $I\subset\mathbb{R}$ be a compact interval and X be a Banach space. We consider the functional spaces $B(I;X)$, $C(I;X)$, consisting respectively of the bounded, continuous functions $f:I\mapsto X$. $B(I;X)$ is endowed with the sup norm

$$\|f\|_{B(I;X)}=\sup_{t\in I}\|f(t)\|_X\,.$$

We set also

$$C_b(I;X)=B(I;X)\cap C(I;X),\quad \|f\|_{C(I;X)}=\|f\|_{B(I;X)}\,.$$

The Banach spaces of Hölder continuous functions $C^\alpha(I;X)$ ($\alpha\in(0,1)$), are defined by

$$C^\alpha(I;X)=\{f\in C_b(I;X):[f]_{C^\alpha(I;X)}=\sup_{t,s\in I,\,s<t}\frac{\|f(t)-f(s)\|_X}{(t-s)^\alpha}<+\infty\},$$
$$\|f\|_{C^\alpha(I;X)}=\|f\|_\infty+[f]_{C^\alpha(I;X)}.$$

We denote the corresponding spaces of $\mathbb{R}^n$-valued functions by an upper index n. For example we write $L_2^n(0,l)$ or $C^n[0,l]$. The corresponding maximum and L_2 norms have the form

$$\|f\|_\infty = \max_{i=1,\ldots,n} \operatorname*{ess\,sup}_{x\in(0,l)} |f_i(x)|,$$

$$\|f\|_2 = \left(\int_0^l f^T(x)\bar{f}(x)dx\right)^{\frac{1}{2}}.$$

2

Preliminaries

The main subject of this thesis is the reaction-diffusion system of the form

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + F(u(x,t)), & t > 0, \quad x \in [0,l], \\ B_p[u(\cdot,t)](0) = B_p[u(\cdot,t)](l) = 0, & t \geq 0, \\ u(x,0) = u_0(x), & x \in [0,l]. \end{cases} \tag{2.1}$$

In the system above $u : [0,l] \times [0,\infty) \to \mathbb{R}^n$ is the unknown, $D = \mathrm{diag}(d_1, \ldots, d_n)$, with $d_i > 0$ is the matrix of diffusion coefficients, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a given nonlinear function modelling reactions, $u_0 : [0,l] \to \mathbb{R}^n$ is a continuous function of initial conditions. The operator B_p, $(p = 0, 1)$ is the formal linear operator of boundary conditions with

$$\begin{aligned} B_0[u(\cdot,t)](x) &= u(x,t), \\ B_1[u(\cdot,t)](x) &= \left(\frac{\partial u(\cdot,t)}{\partial \nu}\right)(x). \end{aligned} \tag{2.2}$$

Throughout this work we assume that the components of the vector $F = (F_1, \dots, F_n)^T$ and the elements of its Jacobian matrix

$$F_y := \left(\frac{\partial F_i}{\partial y_j} \right)_{i,j=1,\dots,n} \tag{2.3}$$

are all continuous functions. We write $F_y(u(x,t))$, if the Jacobian is evaluated at a function $u(x,t)$.

This chapter is devoted to some preliminary results needed in the sequel. In particular, we introduce facts from the theory of analytic semigroups, theory of unbounded self-adjoint operators and derive the explicit estimations of the embedding $C^n[0,l] \hookrightarrow H_1^n(0,l)$.

2.1 Some preliminaries on analytic semigroups

The methods of semigroups provide an elegant and comprehensive approach to the field of abstract time-dependent problems. Our main concern in this subsection will be a special class of semigroups, namely analytic semigroups. Below we introduce several classical results from that field, which are going to be useful in our work. We omit the detailed description of these results. For a more thorough overview on analytic semigroups please refer to [10, 11, 21, 30, 32, 63].

2.1.1 Sectorial operators. Analytic semigroups

We start with the following

Definition 2.1. *Any real number θ satisfying $z = re^{i\theta}$ for some positive real r is called an argument of a complex number z and is an angle made by z with the positive x-axis. The particular argument of z lying in the range $-\pi < \theta \leq \pi$ is called the principal argument of z and is denoted by arg(z).*

Definition 2.2. *Let X be a Banach space. Let $T : D(T) \subset X \to X$ be a linear operator. The resolvent set $\rho(T)$ and the spectrum $\sigma(T)$ of T are defined by*

$$\rho(T) = \{\lambda \in \mathbb{C} : \exists (T - \lambda I)^{-1} \in \mathcal{L}(X)\}, \quad \sigma(T) = \mathbb{C} \setminus \rho(T).$$

The complex numbers $\lambda \in \rho(T)$ such that $T - \lambda I$ is not one-to-one are called eigenvalues. If $\lambda \in \rho(T)$, we set

$$(T - \lambda I)^{-1} = R(\lambda, T).$$

$R(\lambda, T)$ is called resolvent operator or simply resolvent.

Definition 2.3. *Let X be a Banach space. We say that a linear operator $T : D(T) \subset X \to X$ is sectorial if there are constants $a \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $M > 0$ such that*

$$\begin{cases} (i) \quad \rho(T) \supset S_{\theta,a} := \{\lambda \in \mathbb{C} : \lambda \neq a, \ |arg(\lambda - a)| < \theta\}, \\ \\ (ii) \quad \|R(\lambda, T)\|_{\mathcal{L}(X)} \leq \dfrac{M}{|\lambda - a|}, \quad \lambda \in S_{\theta,a}. \end{cases} \tag{2.4}$$

For every $t > 0$ the properties of a sectorial operator T allow us to define the operator exponential e^{tT} by means of a Dunford-Taylor integral as

$$e^{tT} := \frac{1}{2\pi i} \int_{\gamma_{r,\eta}+a} e^{t\lambda} R(\lambda, T)\, d\lambda, \quad t > 0, \tag{2.5}$$

where $r > 0$, $\eta \in \left(\frac{\pi}{2}, \theta\right)$ and $\gamma_{r,\eta}$ is the curve

$$\gamma_{r,\eta} := \{\lambda \in \mathbb{C} : |\text{arg}\lambda| = \eta, \ |\lambda| \geq r\} \cup \{\lambda \in \mathbb{C} : |\text{arg}\lambda| \leq \eta, \ |\lambda| = r\}, \tag{2.6}$$

oriented counterclockwise. Hence $\gamma_{r,\eta} + a$ is given by

$$\gamma_{r,\eta}+a = \{\lambda \in \mathbb{C} : |\text{arg}(\lambda-a)| = \eta, \ |\lambda-a| \geq r\} \cup \{\lambda \in \mathbb{C} : |\text{arg}(\lambda-a)| \leq \eta, \ |\lambda-a| = r\}.$$

Lemma 2.4. *[30, Lemma 1.3.2, p. 11] If T is a sectorial operator, the integral in* (2.5) *is well-defined, and it is independent of $r > 0$ and $\eta \in \left(\frac{\pi}{2}, \theta\right)$.*

Let us additionaly set

$$e^{0T}x = x, \; x \in X \tag{2.7}$$

and introduce the following

Definition 2.5. *Let T be a sectorial operator. The function from $[0, \infty) \to \mathcal{L}(X)$, $t \mapsto e^{tT}$ is called the analytic semigroup generated by T (in X).*

We continue with

Proposition 2.6. *[32, Proposition 2.1.1, Proposition 2.1.4]*

(i) $e^{tT}x \in D(T)$ for each $t > 0$, $x \in X$.

(ii) For every $x \in X$ and $t \geq 0$, the integral $\int_0^t e^{sT}x ds$ belongs to $D(T)$.

Below we would like to introduce one property of the analytic semigroup, which will be essential for the estimation of the domain of attraction.

Lemma 2.7. *Let T be a sectorial operator and let e^{tT} be given by* (2.5)*. Let a be the number introduced in* (2.4)*. Then the following statement holds.*

$$\left\|e^{tT}\right\|_{\mathcal{L}(X)} \leq Ce^{ta}, \quad t > 0 \tag{2.8}$$

for some positive constant C.

Proof. Let us introduce the shifted operator $\widetilde{T} := T - aE$. Then $\rho(\widetilde{T})$ contains the sector $S_{\theta,0}$ with $\theta \in \left(\frac{\pi}{2}, \pi\right)$ and $R(\lambda, \widetilde{T}) = R(\lambda + a, T)$, so that

$$\left\|\lambda R(\lambda, \widetilde{T})\right\|_{\mathcal{L}(X)} \leq M, \quad \lambda \in S_{\theta,0}. \tag{2.9}$$

From definition (2.5) follows that

$$e^{t\widetilde{T}} = e^{tT} e^{-at}. \tag{2.10}$$

Let us choose $\eta \in \left(\frac{\pi}{2}, \theta\right)$, $r > 0$ and consider

$$\left\| e^{t\widetilde{T}} \right\|_{\mathcal{L}(X)} = \left\| \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{t\lambda} R(\lambda, \widetilde{T})\, d\lambda \right\|_{\mathcal{L}(X)}.$$

Let us make a substitution $\lambda t = \xi$. Then, due to Lemma 2.4, for $t > 0$ we have

$$e^{t\widetilde{T}} = \frac{1}{2\pi i} \int_{\gamma_{rt,\eta}} e^{\xi} R\left(\frac{\xi}{t}, \widetilde{T}\right) \frac{d\xi}{t} = \frac{1}{2\pi i} \int_{\gamma_{r,\eta}} e^{\xi} R\left(\frac{\xi}{t}, \widetilde{T}\right) \frac{d\xi}{t}.$$

After obvious parametrisation of $\gamma_{r,\eta}$, we arrive at

$$\begin{aligned} e^{t\widetilde{T}} =& \frac{1}{2\pi i} \left(\int_r^{+\infty} e^{\rho e^{i\eta}} R\left(\frac{\rho e^{i\eta}}{t}, \widetilde{T}\right) \frac{e^{i\eta}}{t} d\rho - \int_r^{+\infty} e^{\rho e^{-i\eta}} R\left(\frac{\rho e^{-i\eta}}{t}, \widetilde{T}\right) \frac{e^{-i\eta}}{t} d\rho \right. \\ & \left. + \int_{-\eta}^{\eta} e^{re^{i\alpha}} R\left(\frac{re^{i\alpha}}{t}, \widetilde{T}\right) ire^{i\alpha} \frac{d\alpha}{t} \right). \end{aligned}$$

From (2.9) follows

$$\left\| e^{t\widetilde{T}} \right\|_{\mathcal{L}(X)} \le \frac{M}{2\pi} \left(2 \int_r^{+\infty} \frac{1}{\rho}\, e^{\rho \cos \eta} d\rho + \int_{-\eta}^{\eta} e^{r \cos \alpha} d\alpha \right).$$

Setting

$$C := \frac{M}{2\pi} \left(2 \int_r^{+\infty} \frac{1}{\rho}\, e^{\rho \cos \eta} d\rho + \int_{-\eta}^{\eta} e^{r \cos \alpha} d\alpha \right) \tag{2.11}$$

and taking into account (2.10), we obtain the desired estimation. $\square$

2.1.2 Cauchy problem. Mild solutions

Let us introduce the following

Definition 2.8. *Given three Banach spaces $Z \subset Y \subset X$ (with continuous embeddings) and given $\alpha \in (0,1)$, we say that Y is of class J_α between X and Z if there is $C > 0$ such that*

$$\|y\|_Y \le C \|y\|_Z^\alpha \|y\|_X^{1-\alpha}, \quad y \in Z. \tag{2.12}$$

Let $T : D(T) \subset X \to X$ be a sectorial operator. Let X_α denote a space of class J_α between X and $D(T)$. We consider the initial value problem

$$\begin{cases} u'(t) = Tu(t) + H(u(t)), & t > 0, \\ u(0) = u_0, \end{cases} \tag{2.13}$$

where $u_0 \in X_\alpha$ and $H : X_\alpha \to X$ is a continuous function. In addition for every $R > 0$ there is $K > 0$ such that

$$\|H(x) - H(y)\|_X \le K \|x - y\|_{X_\alpha}, \quad x, y \in B(0,R) \subset X_\alpha. \tag{2.14}$$

Let us introduce the following

Definition 2.9. *We say that a function u defined in an interval $I = [0,\tau)$ or $I = [0,\tau]$ is a mild solution of problem* (2.13) *if $u \in (C \setminus \{0\}; X_\alpha)$ and it satisfies*

$$u(t) = e^{tT}u_0 + \int_0^t e^{(t-s)T} H(u(s))ds, \quad t \in I. \tag{2.15}$$

Due to the embeddings $D(T) \subset X_\alpha \subset X$ it follows that $t \mapsto e^{tT}$ is analytic in $(0,+\infty)$ with values in $\mathcal{L}(X_\alpha)$. In order to avoid blowing up of $\left\|e^{tT}\right\|_{\mathcal{L}(X_\alpha)}$ as $t \to 0$ [32], we make the following assumption

$$\limsup_{t \to 0} \left\|e^{tT}\right\|_{\mathcal{L}(X_\alpha)} < +\infty. \tag{2.16}$$

Therefore $\left\|e^{tT}\right\|_{\mathcal{L}(X_\alpha)}$ is bounded on every compact interval contained in $[0,+\infty)$.

For our further investigations we will need the following

Theorem 2.10. *[30, Theorem 6.3.2, p. 91] The following statements hold.*

(a) If $u, v \in C_b((0,\tau]; X_\alpha)$ are mild solutions of (2.13) *for some $\tau \in (0,\infty)$, then $u \equiv v$.*

(b) For every $\tilde{u} \in X_\alpha$ there exist r, $\delta > 0$ such that if $\|u_0 - \tilde{u}\|_{X_\alpha} \leq r$ problem (2.13) *has a mild solution $u \in C_b((0,\delta]; X_\alpha)$. The function u belongs to $C([0,\delta]; X_\alpha)$ if and only if $u_0 \in \overline{D(T)}^{X_\alpha} :=$ closure of $D(T)$ in X_α.*

In addition, let us set

$$\begin{cases} t_{\max} = \sup\{\tau > 0 : \text{ problem (2.13) has a mild solution } u_\tau \text{ in } [0,\tau]\}, \\ u(t) = u_\tau(t), \quad \text{if} \quad t \leq \tau. \end{cases} \tag{2.17}$$

$u(t)$ is called the maximally defined solution. Due to Theorem 2.10(a), u is well defined in the interval

$$I := \bigcup \{[0,\tau] : \text{problem (2.13) has a mild solution } u_\tau \text{ in } [0,\tau]\} \tag{2.18}$$

and we have $t_{\max} = \sup I$. Of course $t_{\max}$ and I may depend on u_0. We suppress this dependency for now and write $t_{\max}$ and I unless otherwise needed.

The following lemma will be useful for the global existence result.

Lemma 2.11. *[32, Lemma 7.1.1, p.257] Let $f \in C_b((0,\tau); X)$, $\tau \in (0,\infty)$, and set*

$$\Gamma(t) = \int_0^t e^{(t-s)T} f(s) ds, \quad 0 \leq t \leq \tau.$$

If $0 < \alpha < 1$, then $\Gamma \in C^{1-\alpha}([0,\tau]; X_\alpha)$, and there is $C > 0$, not depending on τ and f, such that

$$\|\Gamma\|_{C^{1-\alpha}([0,\tau];X_\alpha)} \leq C \sup_{t \in (0,\tau)} \|f(t)\|_X \,.$$

Let us now introduce a result, concerning existence in the large of the solution of (2.13).

Theorem 2.12. *Let the function $t \mapsto \|u(t)\|_{X_\alpha}$ be bounded on I. Then $t_{\max} = \infty$. Thus the mild solution of problem* (2.13) *exists for all $t > 0$.*

Proof. Assume by contradiction that $t_{\max} < \infty$. By Theorem 2.10 there exists a mild solution $u \in C_b((0, t_{\max}); X_\alpha)$. In the following we would like to show that u can be continuously extended to $t = t_{\max}$. Mild solution u is given by

$$u(t) = e^{tT}u_0 + \int_0^t e^{(t-s)T}H(u(s))ds, \quad t \in (0, t_{\max}). \tag{2.19}$$

Since $t \mapsto \|u(t)\|_{X_\alpha}$ is bounded, then $t \mapsto H(u(t))$ is bounded and continuous with values in X in the interval $(0, t_{\max})$. By Lemma 2.11 the expression $\int_0^t e^{(t-s)T}H(u(s))ds$ belongs to $C^{1-\alpha}([0, t_{\max}]; X_\alpha)$. In addition, observe that $t \mapsto e^{tT}u_0$ is well-defined and analytic on $(0, +\infty)$. Summing up, we find that u belongs to $C^\theta((0, t_{\max}]; X_\alpha)$ with $\theta \in (0, 1)$. Thus u is a uniformly continuous function on $(0, t_{\max}]$ and

$$u(t_{\max}) = e^{t_{\max}T}u_0 + \int_0^{t_{\max}} e^{(t_{\max}-s)T}H(u(s))ds.$$

Indeed, let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $(0, t_{\max})$, which converges to $t_{\max}$. Due to the uniform continuity of u we have $\lim\limits_{n\to\infty} u(t_n) = u(t_{\max})$. In addition, due to the continuity of function H, we conclude that $\lim\limits_{t_n\to t_{\max}} H(u(t_n)) = H(u(t_{\max}))$.

By Proposition 2.6, $u(t_{\max}) \in D(T) \subset \overline{D(T)}^{X_\alpha}$. Thus, u is a mild solution of (2.13) on $(0, t_{\max}]$. By Theorem 2.10, the problem

$$w'(t) = Tw(t) + H(w(t)), \quad t \geq t_{\max}, \quad w(t_{\max}) = u(t_{\max})$$

has a unique mild solution $w \in C([t_{\max}, t_{\max} + \delta]; X_\alpha)$ for some $\delta > 0$. Now let us introduce a function

$$\hat{u}(t) = \begin{cases} u(t), & t \in (0, t_{\max}), \\ w(t), & t \in [t_{\max}, t_{\max} + \delta]. \end{cases} \tag{2.20}$$

For $t \in (0, t_{\max})$ the function $\hat{u}$ satisfies

$$\hat{u}(t) = u(t) = e^{tT} u_0 + \int_0^t e^{(t-s)T} H(\hat{u}(s)) ds.$$

For $t \in [t_{\max}, t_{\max} + \delta]$, taking into account that $u(t_{\max})$ satisfies (2.19), we have

$$\begin{aligned} \hat{u}(t) = w(t) &= e^{(t-t_{\max})T} u(t_{\max}) + \int_{t_{\max}}^t e^{(t-s)T} H(w(s)) ds \\ &= e^{(t-t_{\max})T} \left(e^{t_{\max} T} u_0 + \int_0^{t_{\max}} e^{(t_{\max}-s)T} H(\hat{u}(s)) ds \right) \\ &\quad + \int_{t_{\max}}^t e^{(t-s)T} H(\hat{u}(s)) ds \\ &= e^{tT} u_0 + \int_0^t e^{(t-s)T} H(\hat{u}(s)) ds. \end{aligned}$$

Thus, $\hat{u} \in C_b((0, t_{\max} + \delta]; X_\alpha)$ is a unique mild solution of problem (2.13). This is a contradiction with the definition of $t_{\max}$. Therefore, $t_{\max} = \infty$. □

Remark 2.13. *The result of Theorem 2.12 is used to prove existence in the large when we have an a priori estimate on the norm of $u(t)$. We will be able to obtain this estimate later, during the estimation of the domain of attraction.*

2.2 Self-adjoint operators

All results in this section were taken from [27].

Definition 2.14. *Let H be a Hilbert space. A densely defined operator on H is a pair $(D(T), T)$, where $D(T) \subset H$ is a dense subspace of H, and $T : D(T) \to H$ is a linear map.*

Definition 2.15. *Let H be a Hilbert space. If $(D(T), T)$ is a densely defined operator on H, and $D_1 \subset D(T)$ is a subspace of $D(T)$ which is still dense in H, we call $(D_1, T|D_1)$ the restriction of the operator $(D(T), T)$ to D_1. An extension of a densely defined operator $(D(T), T)$ is a densely defined (D_1, T_1) such that $D(T) \subset D_1$ and $(D(T), T)$ is the restriction of (D_1, T_1) to $D(T)$.*

If (D_1, T_1) is the restriction of (D_2, T_2) to D_1, one may write $T_1 = T_2|D_1$ or $T_1 \subset T_2$.

Definition 2.16. *Let H be a Hilbert space and $(D(T), T)$ be densely defined on H. The graph $\Gamma(T)$ of $(D(T), T)$ is the linear subspace*

$$\Gamma(T) = \{(v, w) \in H \times H : v \in D(T) \text{ and } w = T(v)\} \tag{2.21}$$

of $H \times H$.

Definition 2.17. *The densely defined operator $(D(T), T)$ is said to be closed if $\Gamma(T)$ is closed in $H \times H$ when the latter is seen as a Hilbert space with the inner product*

$$\langle (v_1, w_1), (v_2, w_2) \rangle_{H \times H} = \langle v_1, v_2 \rangle_H + \langle w_1, w_2 \rangle_H. \tag{2.22}$$

The operator is said to be closable if there exists a closed extension of T.

Definition 2.18. *Let H be a Hilbert space and $(D(T), T)$ be densely defined on H. The domain of the adjoint $D(T^*)$ is defined to be the set of all $w \in H$ such that the linear map*

$$f_w^* : \begin{cases} D(T) \to \mathbb{C}, \\ v \mapsto \langle Tv, w \rangle \end{cases} \tag{2.23}$$

is continuous, i.e., those w such that, equivalently, f_w^* extends uniquely to linear functional $f_w^* \in H'$, or there exists a constant $C \geq 0$ with

$$|\langle Tv, w \rangle| \leq C \|v\|, \quad \forall v \in D(T). \tag{2.24}$$

The adjoint is the linear map

$$T^*: \begin{cases} D(T^*) \to H \\ w \mapsto \textit{the unique vector } T^* \ w \textit{ such that } f_w^*(v) = \langle v, T^* w \rangle, \end{cases} \tag{2.25}$$

where the existence of the vector is given by the Riesz Representation Theorem for Hilbert spaces.

Finally, we introduce

Definition 2.19. *Let H be a Hilbert space and $(D(T), T)$ be a densely deifned closable operator.*

(1) The operator $(D(T), T)$ is symmetric or Hermitian if it is closable and $T \subset T^*$, i.e., equivalently, if

$$\langle Tv, w \rangle = \langle v, Tw \rangle, \quad \forall v, w \in D(T). \tag{2.26}$$

(2) The operator $(D(T), T)$ is self-adjoint if it is closable and $T = T^*$, i.e., if it is symmetric and in addition $D(T^*) = D(T)$.

In our further investigations we will use the following perturbation result.

Lemma 2.20. *[27, Lemma 4.26, p. 66] Let H be a Hilbert space and let $(D(T), T)$ be a densely defined closable operator with the adjoint $(D(T^*), T^*)$ which is densely defined as well. Then for any $S \in \mathcal{L}(H)$ the operator $(D(T), S + T)$ is closable and its adjoint is given by $(D(T^*), S^* + T^*)$.*

2.3 Embedding estimations

Let us introduce the following two lemmata, which could be considered as an explicit version of the embedding: $H_1^n(0, l) \hookrightarrow C^n[0, l]$.

Lemma 2.21. *For all $\varphi \in H_1^n(0,l)$ the estimation*

$$\|\varphi\|_\infty^2 \le C_0 \|\varphi\|_2^2 + C_1 \|\varphi'\|_2^2 \tag{2.27}$$

holds with

$$\begin{cases} C_0 = \rho + \dfrac{1}{l}, \\ C_1 = \dfrac{1}{\rho} \end{cases} \tag{2.28}$$

for any $\rho > 0$.

Proof. For $\varphi \in H_1^n(0,l)$, $x \in [0,l]$ and $j = 1, \dots, n$ we have

$$\begin{aligned} x|\varphi_j(x)|^2 &= \int_0^x \frac{d}{dy}(y|\varphi_j(y)|^2)\, dy \\ &\le \int_0^x |\varphi_j(y)|^2\, dy + 2\int_0^l y|\varphi_j(y)||\varphi_j'(y)|\, dy, \end{aligned}$$

and likewise

$$\begin{aligned} (l-x)|\varphi_j(x)|^2 &= -\int_x^l \frac{d}{dy}\left((l-y)|\varphi_j(y)|^2\right)\, dy \\ &\le \int_x^l |\varphi_j(y)|^2\, dy + 2\int_0^l (l-y)|\varphi_j(y)||\varphi_j'(y)|\, dy. \end{aligned}$$

Adding both inequalities, we obtain for all $x \in [0,l]$ and $j = 1, \dots, n$

$$l|\varphi_j(x)|^2 \le \int_0^l |\varphi_j(y)|^2\, dy + 2l\int_0^l |\varphi_j(y)||\varphi_j'(y)|\, dy.$$

After application of Young's inequality with $\rho > 0$, we obtain

$$|\varphi_j(x)|^2 \le \left(\frac{1}{l} + \rho\right) \|\varphi_j\|_2^2 + \frac{1}{\rho}\|\varphi_j'\|_2^2.$$

Since for $a \in H_1^n(0,l)$

$$\|a_j\|_2^2 \le \sum_{i=1}^n \|a_i\|_2^2 = \|a\|_2^2 \tag{2.29}$$

holds, we obtain $\forall x \in (0,l)$ and $\forall j = 1,\dots,n$

$$|\varphi_j(x)|^2 \leq \left(\frac{1}{l}+\rho\right)\|\varphi\|_2^2 + \frac{1}{\rho}\|\varphi'\|_2^2.$$

Hence it follows

$$\|\varphi\|_\infty^2 \leq \left(\frac{1}{l}+\rho\right)\|\varphi\|_2^2 + \frac{1}{\rho}\|\varphi'\|_2^2.$$

The proof of the lemma is complete. $\square$

We will comment on the appropriate choice of the parameter ρ later.

Remark 2.22. *Note, that Lemma 2.21 holds for all $\varphi \in H_1^n(0,l)$ without any boundary conditions. As a matter of fact, when Dirichlet boundary conditions are imposed, it is possible to obtain better embedding constants.*

Lemma 2.23 (Dirichlet boundary conditions)**.** *For $\varphi \in (H_0^1(0,l))^n$ the estimate*

$$\|\varphi\|_\infty^2 \leq C_0\,\|\varphi\|_2^2 + C_1\|\varphi'\|_2^2 \tag{2.30}$$

holds with

$$\begin{cases} C_0 = 0, \\ C_1 = \dfrac{l}{4}, \end{cases} \tag{2.31}$$

and also with

$$\begin{cases} C_0 = \dfrac{\rho}{2}, \\ C_1 = \dfrac{1}{2\rho} \end{cases} \tag{2.32}$$

for any $\rho > 0$.

Proof. For $\varphi \in (H_0^1(0,l))^n$, $x \in [0,l]$ and $j = 1,\dots,n$ we have

$$|\varphi_j(x)| \leq \int_0^x |\varphi_j'(y)|dy,$$

$$|\varphi_j(x)| \leq \int_x^l |\varphi_j'(y)|dy.$$

Adding both expressions and applying Cauchy-Schwarz inequality, we obtain

$$|\varphi_j(x)|^2 \leq \frac{l}{4} \left\|\varphi_j'\right\|_2^2 .$$

Hence from (2.29) estimation (2.30) with C_0, C_1 as in (2.31) follows.

For the second estimation we proceed as follows.

$$|\varphi_j(x)|^2 = \int_0^x \frac{d}{dy}|\varphi_j(y)|^2 dy \leq 2 \int_0^x |\varphi_j(y)||\varphi_j'(y)|dy$$

and

$$|\varphi_j(x)|^2 = \int_x^l \frac{d}{dy}|\varphi_j(y)|^2 dy \leq 2 \int_x^l |\varphi_j(y)||\varphi_j'(y)|dy.$$

Addition of these inequalities and application of Young's inequality with $\rho > 0$ results in

$$\begin{aligned} |\varphi_j(x)|^2 &\leq \int_0^l |\varphi_j(y)||\varphi_j'(y)|dy \\ &\leq \frac{\rho}{2} \int_0^l |\varphi_j(y)|^2 \, dy + \frac{1}{2\rho} \int_0^l \left|\varphi_j'(y)\right|^2 dy \\ &= \frac{\rho}{2} \left\|\varphi_j\right\|_2^2 + \frac{1}{2\rho} \left\|\varphi_j'\right\|_2^2 . \end{aligned}$$

Hence from (2.29) estimation (2.30) with C_0, C_1 as in (2.32) follows. $\square$

3

Enclosure of stationary solutions

In this chapter we are going to describe a computer-assisted method which provides the existence and enclosure results for a *stationary* solution of the reaction-diffusion system of the form

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + F(u(x,t)), & t > 0, \quad x \in [0,l], \\ B_p[u(\cdot,t)](0) = B_p[u(\cdot,t)](l) = 0, & t \geq 0, \\ u(x,0) = u_0(x), & x \in [0,l]. \end{cases} \tag{3.1}$$

Let $\bar{u}$ denote a stationary solution of (3.1). System (3.1) in its stationary formulation reads

$$\begin{cases} -D\bar{u}''(x) - F(\bar{u}(x)) = 0, & x \in [0,l], \\ B_p[\bar{u}](0) = B_p[\bar{u}](l) = 0. \end{cases} \tag{3.2}$$

Let $H_2^B(0,l) := \{\varphi \in H_2^n(0,l) : B_p[\varphi](0) = B_p[\varphi](l) = 0\}$ and let $\omega \in H_2^B(0,l)$ denote a numerical approximation to $\bar{u}$. We aim at the existence and enclosure results in the following sense: we want to derive a constant α, such that a stationary solution

$\bar{u}$ satisfying

$$\|\bar{u} - \omega\|_\infty \le \alpha \tag{3.3}$$

exists. In other words we are looking for some "sufficiently small" and explicitly described neighbourhood of ω which contains $\bar{u}$. The estimation (3.3), as well as the existence of $\bar{u}$ will follow after the application of the Schauder's fixed point theorem to a suitable formulation of (3.2). In order to obtain this formulation we will have to verify some certain conditions with the help of the computer.

3.1 Some preliminary facts and notations

Let us introduce a notation

$$C_\omega(x) := -F_y(\omega(x)), \quad x \in [0, l]. \tag{3.4}$$

Throughout this chapter we make the following assumption:

(G_0) there exists a monotonically non-decreasing function $G : [0, +\infty) \to [0, +\infty)$ such that

$$\begin{cases} |F(y + \omega(x)) - F(\omega(x)) + C_\omega(x)y|_2 \le G(|y|_2), & y \in \mathbb{R}^n,\ x \in [0, l], \\ \text{with} \quad G(h) = o(h) \quad \text{as} \quad h \to 0+. \end{cases} \tag{3.5}$$

Let us introduce an operator $\mathbf{F} : C^n[0, l] \to L_2^n(0, l)$ as

$$(\mathbf{F}(\varphi))(x) := F(\varphi(x)), \quad x \in [0, l]. \tag{3.6}$$

We want to show that

Lemma 3.1. *The operator $\boldsymbol{F} : C^n[0, l] \to L_2^n(0, l)$ is Fréchet differentiable at ω.*

Proof. In order to show the assertion we introduce an operator $\mathcal{C}_\omega : C^n[0, l] \to L_2^n(0, l)$ as

$$(\mathcal{C}_\omega \varphi)(x) := C_\omega(x)\varphi(x), \quad x \in [0, l] \tag{3.7}$$

and note that for any $y \in \mathbb{R}^n$ the inequality

$$\max_{j=1,\dots,n} |y_j| \le |y|_2 \le \sqrt{n} \max_{j=1,\dots,n} |y_j| \tag{3.8}$$

holds.

By assumption (G_0) for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\varphi \in C^n[0,l]$, satisfying $|\varphi(x)|_2 \le \delta \; \forall x \in [0,l]$ follows

$$\begin{aligned} \|\mathbf{F}(\varphi+\omega) - \mathbf{F}(\omega) + \mathcal{C}_\omega\varphi\|_2 &= \sqrt{\int_0^l |F(\varphi(x)+\omega(x)) - F(\omega(x)) + C_\omega(x)\varphi(x)|_2^2 dx} \\ &\overset{(3.5)}{\le} \sqrt{\int_0^l G^2(|\varphi(x)|_2) dx} \\ &\overset{(3.5)}{\le} \varepsilon\sqrt{\int_0^l |\varphi(x)|_2^2 dx} = \varepsilon \|\varphi\|_2 \le \varepsilon\sqrt{l}\, \|\varphi\|_\infty . \end{aligned}$$

Taking into account (3.8) we obtain the following assertion: for all $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\varphi \in C^n[0,l]$, satisfying $\|\varphi\|_\infty \le \frac{\delta}{\sqrt{n}}$ we have

$$\|\mathbf{F}(\varphi+\omega) - \mathbf{F}(\omega) + \mathcal{C}_\omega\varphi\|_2 \le \varepsilon\sqrt{l}\, \|\varphi\|_\infty .$$

Thus $\mathbf{F} : C^n[0,l] \to L_2^n(0,l)$ is Fréchet differentiable at ω with Fréchet derivative given by

$$(\mathbf{F}'(\omega)[\varphi])(x) = -(\mathcal{C}_\omega\varphi)(x) = -C_\omega(x)\varphi(x) = F_y(\omega(x))\varphi(x), \quad x \in [0,l]. \tag{3.9}$$

$\square$

Next let us introduce a function $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ as

$$\mathbf{g}(\varphi,\omega) := \mathbf{F}(\varphi+\omega) - \mathbf{F}(\omega) + \mathcal{C}_\omega\varphi. \tag{3.10}$$

We will need the following result

Lemma 3.2. *Let $\alpha > 0$. For any $\varphi \in C^n[0,l]$, satisfying $\|\varphi\|_\infty \le \alpha$ we have*

$$\|\mathbf{g}(\varphi,\omega)\|_2 \le \sqrt{l}G(\alpha\sqrt{n}). \tag{3.11}$$

Proof. Let $\alpha > 0$ and let $\varphi \in C^n[0,l]$ satisfy $\|\varphi\|_\infty \le \alpha$. Then according to (3.8) we have $|\varphi(x)|_2 \le \alpha\sqrt{n}$ for all $x \in [0,l]$. From (3.5) and the monotonicity of the function G follows

$$|\mathbf{g}(\varphi(x),\omega(x))|_2 \le G(|\varphi(x)|_2) \le G(\alpha\sqrt{n}), \quad x \in [0,l].$$

Consequently,

$$\|\mathbf{g}(\varphi,\omega)\|_2 = \sqrt{\int_0^l |\mathbf{g}(\varphi(x),\omega(x))|_2^2 dx} \le \sqrt{l}G(\alpha\sqrt{n}).$$

□

Next we introduce an operator $A : D_p(A) \to L_2^n(0,l)$ as

$$D_p(A) = H_2^B(0,l) = \begin{cases} H_2^n(0,l) \cap \left(H_0^1(0,l)\right)^n, & \text{if} \quad p = 0, \\ \{\varphi \in H_2^n(0,l) : \varphi'(0) = \varphi'(l) = 0\}, & \text{if} \quad p = 1, \end{cases} \tag{3.12}$$

$$A\varphi := D\varphi''.$$

Finally, given that $\mathbf{F}$ is Fréchet differentiable at ω, we introduce a linear operator $L_\omega : D_p(L_\omega) \to L_2^n(0,l)$, which denotes the operator obtained by the linearisation of problem (3.2) at ω. Thus, L_ω is defined via

$$D_p(L_\omega) = H_2^B(0,l), \quad L_\omega\varphi := -A\varphi + \mathcal{C}_\omega\varphi. \tag{3.13}$$

3.2 Existence and enclosure theorem

At first we would like to show several preliminary results. We start with

Proposition 3.3. *The embedding* $E : H_2^n(0,l) \hookrightarrow C^n[0,l]$ *is compact.*

Proof. The assertion of the proposition follows from the Rellich-Kondrachov theorem [2, Theorem 6.2, p. 144]. □

In our next result we will be using the embedding $I \in \mathcal{L}(C^n[0,l], L_2^n(0,l))$.

Proposition 3.4. *Let* $A : H_2^B(0,l) \to L_2^n(0,l)$ *be the operator introduced in* (3.12). *Then the operator* $-A + IE : H_2^B(0,l) \to L_2^n(0,l)$ *is one-to-one and onto.*

Proof. For $v \in H_2^B(0,l)$, $r \in L_2^n(0,l)$ consider a boundary value problem of the form

$$-Av + IEv = r. \tag{3.14}$$

Note that problem (3.14) is a system of linear ordinary differential equations of second order with constant coefficients. The existence of the unique solution v follows after the application of the standard methods from the theory of ordinary differential equations (see uniqueness and existence Theorem [61, Theorem I, p. 169]).

□

Corollary 3.5. *Let* $A : H_2^B(0,l) \to L_2^n(0,l)$ *be the operator introduced in* (3.12). *Then the following implication is satisfied for every* $\xi \in \mathcal{L}(C^n[0,l], L_2^n(0,l))$*:*

$$\begin{gathered} \textit{If} \quad -A + \xi E : H_2^B(0,l) \to L_2^n(0,l) \quad \textit{is one-to-one, then it is also onto,} \\ \textit{and} \quad (-A + \xi E)^{-1} \in \mathcal{L}(L_2^n(0,l), H_2^B(0,l)). \end{gathered} \tag{3.15}$$

Proof. Consider for $v \in H_2^B(0,l)$ and $r \in L_2^n(0,l)$ the boundary value problem

$$-Av + \xi Ev = r. \tag{3.16}$$

According to Proposition 3.4 the operator $-A + IE : H_2^B(0,l) \to L_2^n(0,l)$ is one-to-one and onto. Let us introduce an operator $K : H_2^B(0,l) \to H_2^B(0,l)$ and a function $s \in H_2^B(0,l)$ as

$$K := (-A + IE)^{-1}(IE - \xi E),$$
$$s := (-A + IE)^{-1} r.$$

It is easy to see that the boundary value problem (3.16) is equivalent to

$$v = Kv + s. \tag{3.17}$$

Let us consider the operator K. Since $-A + IE$ is one-to-one and onto, the Open Mapping Theorem implies that $(-A+IE)^{-1} : L_2^n(0,l) \to H_2^B(0,l)$ is bounded. In addition, the embedding $E : H_2^n(0,l) \hookrightarrow C^n[0,l]$ is compact. Thus $K : H_2^B(0,l) \to H_2^B(0,l)$ is compact as well. By assumption $-A + \xi E$ is one-to-one. Therefore the homogeneous problem (3.16), and hence also the homogeneous problem (3.17), has only the trivial solution. An application of the Fredholm's Alternative to (3.17) results in the existence of a unique solution $v \in H_2^B(0,l)$ for every $r \in L_2^n(0,l)$. Hence $(-A + \xi E)^{-1} : L_2^n(0,l) \to H_2^B(0,l)$ exists and, due to Open Mapping Theorem, is bounded. □

Finally let us introduce the existence and enclosure theorem, which was developed by Plum.

Theorem 3.6. *Let ω be an approximate solution of a boundary value problem* (3.2)*. Let $L_\omega : H_2^B(0,l) \to L_2^n(0,l)$ be given by* (3.13)*. Suppose that positive constants δ, K are known such that*

$$\|-A\omega - \boldsymbol{F}(\omega)\|_2 \leq \delta, \tag{3.18}$$

$$\|u\|_\infty \leq K \|L_\omega u\|_2 \quad \forall u \in H_2^B(0,l). \tag{3.19}$$

In addition, let there exist a monotonically non-decreasing function $G : [0,+\infty) \to [0,+\infty)$, satisfying (3.5)*. If*

$$\delta \leq \frac{\alpha}{K} - \sqrt{l}G(\alpha\sqrt{n}) \tag{3.20}$$

holds for some $\alpha \geq 0$*, then there exists a solution* $\bar{u} \in C_2^n[0,l]$ *to problem* (3.2) *satisfying*

$$\|\bar{u} - \omega\|_\infty \leq \alpha. \tag{3.21}$$

Proof. Let us set $u = \bar{u} - \omega$ and denote

$$d[\omega] := -A\omega - \mathbf{F}(\omega). \tag{3.22}$$

Consider the boundary value problem

$$L_\omega u - \mathbf{g}(u,\omega) = -d[\omega] \quad \text{on } (0,l), \quad B_p[u](0) = B_p[u](l) = 0. \tag{3.23}$$

In the following we are going to show that a solution $u \in H_2^B(0,l)$ of (3.23) exists and satisfies $\|u\|_\infty \leq \alpha$. If this is the case then $\bar{u} := u+\omega$ is a solution of (3.2), satisfying (3.21). The required smoothness of $\bar{u}$ will eventually follow from the differential equation (3.2).

From (3.19) follows that $L_\omega = -A + \mathcal{C}_\omega$ is one-to-one on $H_2^B(0,l)$. In addition, by Lemma 3.1 the operator $\mathcal{C}_\omega$ is the Fréchet derivative of $\mathbf{F}$ and $\mathcal{C}_\omega \in \mathcal{L}(C^n[0,l], L_2^n(0,l))$. Hence the application of Corollary 3.5 to the operator L_ω results in the existence of a bounded operator $L_\omega^{-1} : L_2^n(0,l) \to H_2^B(0,l)$. Therefore, taking into account the compactness of the embedding $E : H_2^n(0,l) \hookrightarrow C^n[0,l]$, boundedness of L_ω^{-1}, and continuity of $\mathbf{g}$, we may represent (3.23) as

$$u = -L_\omega^{-1}(d[\omega] - \mathbf{g}(u,\omega)) =: Tu, \tag{3.24}$$

where $T : C^n[0,l] \to C^n[0,l]$ is a continuous and compact operator. The existence of a fixed point $u \in C^n[0,l]$ of problem (3.24) would follow from the Schauder's fixed-point theorem if we would be able to find a closed, convex, bounded set U, such that $TU \subset U$. Let us set

$$U := \{u \in C^n[0,l] : \|u\|_\infty \leq \alpha\}. \tag{3.25}$$

Note that $Tu \in H_2^B(0,l)$. Since G satisfies (3.5), by Lemma 3.2 $\mathbf{g}(u,\omega)$ satisfies (3.11). From (3.19), (3.24), (3.18), and (3.11) follows

$$\|Tu\|_\infty \leq K \|L_\omega Tu\|_2 = K \|d[\omega] - \mathbf{g}(u,\omega)\|_2 \leq K(\delta + \sqrt{l}G(\alpha\sqrt{n})).$$

The operator T maps the set U from (3.25) onto itself, if

$$K(\delta + \sqrt{l}G(\alpha\sqrt{n})) \leq \alpha,$$

which is equivalent to

$$\delta \leq \frac{\alpha}{K} - \sqrt{l}G(\alpha\sqrt{n}).$$

Hence, if the inequality above holds true, then due to Schauder's fixed point theorem a fixed point $u \in C^n[0,l]$, satisfying $\|u\|_\infty \leq \alpha$, of problem (3.24) exists. Therefore $u \in H_2^B(0,l)$ is a solution of (3.23). Consequently, $\bar{u} \in H_2^B(0,l)$ is a solution of (3.2) satisfying $\|\bar{u} - \omega\|_\infty \leq \alpha$. The smoothness of $\bar{u}$ follows from the differential equation (3.2). The proof of the theorem is complete. $\square$

In order to obtain the enclosure interval for the stationary solution $\bar{u}$, we need to

1. find a constant δ satisfying (3.18),
2. find a constant K satisfying (3.19),
3. find a monotonically non-decreasing function G satisfying (3.5).

In the next section we are going to present a method which provides us with the constant K. It is obvious, that for condition (3.20) to hold, the defect bound δ should be sufficiently small. Thus, a highly accurate numerical approximation ω is required. The accuracy of ω can be achieved with the help of the Newton algorithm. We comment on the computation of the highly accurate numerical solution ω, as well as on the computation of the corresponding defect bound δ in the Appendix A.

In Section 3.4 we discuss the computation of constant α satisfying (3.20). We report on the function G in Chapter 7.

3.3 Computation of constant K

In this section we describe the calculation of constant K satisfying (3.19). For that purpose we are going to use the estimations of the embedding $H_1^n(0,l) \hookrightarrow C^n[0,l]$, which were presented in Lemma 2.21 and Lemma 2.23 in Chapter 2. Recall that for $u \in H_1^n(0,l)$ we have

$$\|u\|_\infty^2 \leq C_0 \|u\|_2^2 + C_1 \|u'\|_2^2, \tag{3.26}$$

with constants C_0 and C_1 being chosen as in (2.28) or as in (2.31) (specifically for the Dirichlet boundary conditions).

For the computation of the constant K we consider the weak form of the eigenvalue problem for $L_\omega^* L_\omega$:

$$u \in H_2^B(0,l), \quad \langle L_\omega u, L_\omega v\rangle_2 = \lambda \left(\beta \langle u, v\rangle_2 + \langle u', v'\rangle_2\right) \quad \forall v \in H_2^B(0,l), \tag{3.27}$$

where $\beta > 0$ is a fixed constant. For simplicity in the following we are going to use the notation

$$M(u,v) := \langle L_\omega u, L_\omega v\rangle_2 ,$$
$$N(u,v) := \beta \langle u, v\rangle_2 + \langle u', v'\rangle_2 .$$

It is easy to see that the bilinear forms M and N are positive definite self-adjoint forms on the space $H_2^B(0,l)$ and N is bounded. Therefore problem (3.27) is equivalent to an eigenvalue problem for a self-adjoint operator in $H_2^B(0,l)$ and the usual spectral terms are well-defined for this problem.[2]

The variational characterization of the smallest eigenvalue λ_1 of problem (3.27) gives

$$\lambda_1 \leq \frac{\langle L_\omega u, L_\omega u\rangle_2}{\beta\langle u, u\rangle_2 + \langle u', u'\rangle_2} \quad \forall u \in H_2^B(0,l) \setminus \{0\}.$$

[2] We will discuss the spectral properties of self-adjoint operator $L_\omega^* L_\omega$ in Remark 4.11. In particular, the existence of the orthonormal basis of eigenfunctions follows from the compactness of the resolvent of the self-adjoint operator under consideration.

Hence, if we can compute some $\underline{\lambda}$ satisfying

$$0 < \underline{\lambda} \leq \lambda_1, \tag{3.28}$$

we obtain

$$(\beta \|u\|_2^2 + \|u'\|_2^2) \leq \frac{1}{\underline{\lambda}} \|L_\omega u\|_2^2 \quad \forall u \in H_2^B(0,l). \tag{3.29}$$

Next, we introduce the following

Proposition 3.7. *If* $\underline{\lambda}$ *satisfies* (3.28)*, then* (3.19) *is satisfied with*

$$K = \sqrt{\frac{1}{\underline{\lambda}} \max\left\{\frac{C_0}{\beta}, C_1\right\}}. \tag{3.30}$$

Proof. Starting with

$$\|u\|_\infty^2 \leq \frac{C_0}{\beta}\beta \|u\|_2^2 + C_1 \|u'\|_2^2$$

and using (3.29), we obtain

$$\begin{aligned}
\|u\|_\infty &\leq \sqrt{\max\left\{\frac{C_0}{\beta}, C_1\right\}}\sqrt{\beta \|u\|_2^2 + \|u'\|_2^2} \\
&\leq \sqrt{\frac{1}{\underline{\lambda}} \max\left\{\frac{C_0}{\beta}, C_1\right\}} \|L_\omega u\|_2 = K \|L_\omega u\|_2
\end{aligned}$$

Hence, the proof is complete. □

In case when Neumann boundary conditions are imposed, the constants C_0 and C_1 should be chosen as in (2.28). Thus, K becomes a function of ρ. Observe that we will have more chances to satisfy inequality (3.20), if constant K is small. Therefore, it makes sense to choose ρ such that

$$\max\left\{\frac{1}{\beta}\left(\rho + \frac{1}{l}\right), \frac{1}{\rho}\right\} \to \min. \tag{3.31}$$

The choice of the positive constant β is made in such a way that K is as small as possible. We accomplish this task by making several tests with the different values of β. Note that $\underline{\lambda}$ depends on β.

3.3.1 Alternative approach for self-adjoint L_ω

In this section we are going to operate under the assumtion that

(A) The operator L_ω is self-adjoint in $L_2^n(0,l)$ and its resolvent is compact.

The fact that this is true will be shown later in Proposition 4.10. In that case one may follow a different approach for computation of the constant K. The main advantage of this approach, compared to the method described above, is that it requires less numerical effort.

Let us assume, that the constants $K_0 > 0$ and $K_1 > 0$ satisfying the inequalities

$$\|u\|_2 \leq K_0 \|L_\omega u\|_2 , \quad u \in H_2^B(0,l), \tag{3.32}$$

$$\|u'\|_2 \leq K_1 \|L_\omega u\|_2 , \quad u \in H_2^B(0,l) \tag{3.33}$$

are known. It is easy to see that (3.26) combined with (3.32) and (3.33) yields in

$$K = \sqrt{C_0 K_0^2 + C_1 K_1^2}. \tag{3.34}$$

Let us assume at the moment, that the value of constant K_0 is known. Then we obtain the constant K_1 as it is described in the following lemma, which for $n = 1$ can be found in [42, Theorem 2, p. 44].

Lemma 3.8. *Let* (3.32) *hold with some constant* K_0*, and let* $\underline{c}$ *be defined as*

$$\underline{c} = \min_{x\in[0,l]} \lambda_{\min}(C_\omega(x)). \tag{3.35}$$

Then (3.33) *holds with*

$$K_1 := \begin{cases} \sqrt{\dfrac{1}{d_{\min}} K_0 (1 - \underline{c} K_0)}, & \textit{if} \quad \underline{c} K_0 \le \frac{1}{2}, \\ \\ \dfrac{1}{2\sqrt{\underline{c} d_{\min}}}, & \textit{otherwise.} \end{cases} \tag{3.36}$$

Proof. Let $u \in H_2^B(0,l)$, $u \not\equiv 0$. From (3.32) follows the injectivity of L_ω and therefore $L_\omega u \not\equiv 0$. We have

$$\begin{aligned} \langle L_\omega u, u \rangle_2 &= -\int_0^l u''(x)^T D \overline{u(x)} dx + \int_0^l u(x)^T C_\omega(x)^T \overline{u(x)} dx \\ &= \int_0^l u'(x)^T D \overline{u'(x)} dx + \int_0^l u(x)^T C_\omega(x)^T \overline{u(x)} dx \\ &\ge d_{\min} \|u'\|_2^2 + \underline{c} \|u\|_2^2 . \end{aligned} \tag{3.37}$$

On the other hand the application of the Cauchy-Schwarz inequality results in

$$\langle L_\omega u, u \rangle_2 \le \|u\|_2 \|L_\omega u\|_2 . \tag{3.38}$$

Combining (3.37) and (3.38), we obtain

$$\|u'\|_2^2 \le \frac{1}{d_{\min}} \left(\|u\|_2 \|L_\omega u\|_2 - \underline{c} \|u\|_2^2 \right) . \tag{3.39}$$

Let us set $\mu = \dfrac{\|u\|_2}{\|L_\omega u\|_2}$. From (3.32) follows $\mu \le K_0$. Expression (3.39) reads

$$\|u'\|_2^2 \le \frac{1}{d_{\min}} \mu (1 - \underline{c}\mu) \|L_\omega u\|_2^2 . \tag{3.40}$$

Observe that a function $f(\mu) = -\underline{c}\mu^2 + \mu$ achieves its maximum at $\mu^* = \dfrac{1}{2\underline{c}}$. Hence if $\mu^* \ge K_0$ then we set $\mu = K_0$ in (3.40). Otherwise we have

$$\|u'\|_2^2 \le \frac{1}{d_{\min}} f(\mu^*) \|L_\omega u\|_2^2 = \frac{1}{4\underline{c} d_{\min}} \|L_\omega u\|_2^2 .$$

The proof of lemma is complete. $\square$

Now let us discuss the computation of the constant K_0. We consider the eigenvalue problem of the form

$$u \in H_2^B(0,l), \quad L_\omega u = \lambda u; \quad \text{on} \quad [0,l]. \tag{3.41}$$

Since L_ω is self-adjoint in $L_2^n(0,l)$ and its resolvent is compact (according to Assumption (A)), there exists a system of eigenfunctions of (3.41), which is orthonormal and complete with respect to $\langle \cdot, \cdot \rangle_2$. The spectrum of L_ω consist only of the eigenvalues, which are real and converge to infinity. If

$$0 < \underline{\lambda} \leq \min\{|\lambda| : \lambda \text{ eigenvalue of } (3.41)\}, \tag{3.42}$$

then by the eigenfunctions expansion we obtain

$$\|u\|_2 \leq \frac{1}{\underline{\lambda}} \|L_\omega u\|_2 .$$

Thus, (3.32) is satisfied with

$$K_0 = \frac{1}{\underline{\lambda}}.$$

3.3.2 Additional comments

Note, that for the computation of the constant K a positive lower bound $\underline{\lambda}$, defined as in (3.28) or as in (3.42) respectively, should be determined.

At first, observe that the positivity of $\underline{\lambda}$ implies that the operator L_ω is one-to-one. The converse is also true. Therefore during the computation of $\underline{\lambda}$ the injectivity of L_ω, which is essential for the fixed point theorem, will be proven. In addition, let us consider the approach described in Section 3.3.1. As we will see later in Chapter 4, the eigenvalue problem (3.41) is the same eigenvalue problem, which occurs in the course of the verification of the stability of $\bar{u}$. Thus, since we are looking for stable $\bar{u}$ we are interested in cases where the eigenvalues of (3.41) are positive.

Finally let us remark that due to the self-adjointness of (3.27) and (3.41) the variational methods for computing eigenvalue bounds will be applied for computing eigenvalue bounds. These methods will be presented in Chapter 6.

3.4 Enclosure statement

Having a monotonically nondecreasing function G satisfying (3.5) at hand, we insure the enclosure inequality by proceeding as follows:

1. We approximately solve the equation

$$\delta = \frac{\tilde{\alpha}}{K} - \sqrt{l}G(\tilde{\alpha}\sqrt{n})$$

   looking for $\tilde{\alpha}$ with the help of the Newton algorithm. Thus, we set as starting value $\tilde{\alpha}_0 = 0$ and proceed as:

   - $f(\tilde{\alpha}_k) = \delta - \dfrac{\tilde{\alpha}_k}{K} + \sqrt{l}G(\tilde{\alpha}_k\sqrt{n})$,
   - $\tilde{\alpha}_{k+1} = \tilde{\alpha}_k - \dfrac{f(\tilde{\alpha}_k)}{f'(\tilde{\alpha}_k)}, \quad (k = 0, \ldots, k_0).$

   Here k_0 is the index at which the iteration should be stopped.

2. We set $\alpha := 1.01\,\tilde{\alpha}$ and check the inequality (3.20) using interval arithmetic.

4

Operator $L_{\bar{u}}$

The main objective of this chapter is an operator $L_{\bar{u}} : D_p(L_{\bar{u}}) \to L_2^n(0,l)$ $(p = 0, 1)$ defined as

$$D_p(L_{\bar{u}}) = H_2^B(0,l) = \begin{cases} H_2^n(0,l) \cap \left(H_0^1(0,l)\right)^n, & \text{if} \quad p = 0, \\ \\ \{\varphi \in H_2^n(0,l) : \varphi'(0) = \varphi'(l) = 0\}, & \text{if} \quad p = 1, \end{cases} \tag{4.1}$$

$$L_{\bar{u}}\varphi = -D\varphi'' + \mathcal{C}_{\bar{u}}\varphi,$$

where

$$(\mathcal{C}_{\bar{u}}\varphi)(x) := C_{\bar{u}}(x)\varphi(x) = -F_y(\bar{u}(x)), \quad x \in [0,l]. \tag{4.2}$$

Our main task will be to show that $-L_{\bar{u}}$ is a sectorial operator, i.e. it satisfies the conditions of Definition 2.3. For the reason of convenience let us reproduce them here. We want to show that there are constants $a \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, $M > 0$ such that

$$\begin{cases} (i) \quad \rho(-L_{\bar{u}}) \supset S_{\theta,a} := \{\lambda \in \mathbb{C} : \lambda \neq a, \ |\arg(\lambda - a)| < \theta\}, \\ \\ (ii) \quad \|R(\lambda, -L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq \dfrac{M}{|\lambda - a|}, \quad \lambda \in S_{\theta,a}. \end{cases}$$

As we will see later the sectoriality of $-L_{\bar{u}}$, and, specifically, the fact that $-L_{\bar{u}}$ generates an analytic semigroup $e^{-tL_{\bar{u}}}$, will be essential for the quantification of the domain of attraction of $\bar{u}$ and establishing the global existence of a solution of (2.1).

The required properties (i) and (ii) will be shown to some certain extent by computer assistance. We will proceed as follows. At first, by analytic estimations we will obtain properties (i) and (ii) outside some bounded domain on the complex plane. Inside this domain the investigation will be reduced to the determination of those local areas, where no eigenvalues of $L_{\bar{u}}$ can lie. We will accomplish this task by implementing a numerical method, called the eigenvalue exclosure method. In particular, we will introduce some auxiliary self-adjoint eigenvalue problem and will compute the eigenvalue bounds to the eigenvalues of this problem with the help of some known variational methods. In addition, during this process, we will be able to obtain the estimation of the reslovent as in (ii). This approach will be especially helpful in the case when the spectrum of $L_{\bar{u}}$ contains complex eigenvalues.

In the particular case of the self-adjoint $L_{\bar{u}}$, its sectoriality can be shown by making use of the spectral properties of the self-adjoint operators. We discuss this approach in Section 4.2.

4.1 Sectoriality of $-L_{\bar{u}}$

Let us introduce for $z \in \mathbb{R}$ and $\zeta \in \left(0, \frac{\pi}{2}\right)$ a sector

$$\hat{S}_{\zeta,z} := \{\lambda \in \mathbb{C} : |\arg(\lambda - z)| \leq \zeta\}. \tag{4.3}$$

In addition, recall that for $z \in \mathbb{R}$, $R > 0$ we denote $B(z, R)$ to be a ball of radius r with center in z.

As it was already mentioned above, we prove the sectoriality of $-L_{\bar{u}}$ in two steps. At first, by performing certain estimations of terms involved into the resolvent equation, we will show properties (i) and (ii) for all such λ, which belong to the complement of the sector $\hat{S}_{\zeta,z}$ and lie outside the circle $B(z,R)$. Thus in the first step the domain $\hat{S}^C_{\zeta,z} \cap B^C(z,R)$ is under consideration. In the second step we will show that properties (i) and (ii) are satisfied for all such λ, which are still in the complement of the sector $\hat{S}_{\zeta,z}$, but lie inside the circle $B(z,R)$. This will be done with the help of the eigenvalue exclosure procedure.

4.1.1 Estimation in $\hat{S}^C_{\zeta,z} \cap B^C(z,R)$

Let us assume that for $z \in \mathbb{R}$ a constant K_z is known, such that

$$\|\mathcal{C}_{\bar{u}} - z\|_{\mathcal{L}(L_2^n(0,l))} \leq K_z. \tag{4.4}$$

Note that we understand $\mathcal{C}_{\bar{u}} - z$ in the following sense

$$((\mathcal{C}_{\bar{u}} - z)\varphi)(x) := C_{\bar{u}}(x)\varphi(x) - zI\varphi(x), \quad x \in [0,l].$$

Details on the computation of K_z will be given down below.

We continue with the following

Theorem 4.1. *Let $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ be the linear operator, introduced in* (4.1)*, and let $z \in \mathbb{R}$ and $\zeta \in \left(0, \frac{\pi}{2}\right)$. Then the estimation*

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{\widetilde{M}}{|\lambda - z|}, \quad \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B^C(z,R) \tag{4.5}$$

holds true. Here

$$\widetilde{M} = \frac{R}{R\sin(\zeta) - K_z}, \tag{4.6}$$

where K_z as in (4.4) *and*

$$R > \frac{K_z}{\sin(\zeta)}. \tag{4.7}$$

Proof. Let $\lambda \in \mathbb{C}$. For $\varphi \in H_2^B(0,l)$ and $f \in L_2^n(0,l)$ consider the differential equation

$$L_{\bar{u}}\varphi - \lambda\varphi = f, \tag{4.8}$$

which is equivalent to

$$-A\varphi + (z-\lambda)\varphi + (\mathcal{C}_{\bar{u}} - z)\varphi = f. \tag{4.9}$$

Recall that the operator A is defined by (3.12). Let us consider the operator $-A+z$. Depending on conditions, imposed on the boundary, the operator $-A+z$ has the eigenvalues $\lambda^p_{j,k} \in \mathbb{R}$, $j = 1, 2, \dots, n$, which are given by

$$\lambda^p_{j,k}(-A+z) = \begin{cases} d_j \dfrac{\pi^2 k^2}{l^2} + z, & \text{if} \quad p = 0, \\ \\ d_j \dfrac{\pi^2 (k-1)^2}{l^2} + z, & \text{if} \quad p = 1, \end{cases} \qquad k \in \mathbb{N}.$$

In both cases it follows that the spectrum of $-A+z$ is included in a sector $\hat{S}_{\zeta,z}$. Therefore the operator $(-A+z-\lambda)^{-1}$ exists for all $\lambda \in \hat{S}^C_{\zeta,z}$.

After regrouping the terms in (4.9), for all $\lambda \in \hat{S}^C_{\zeta,z}$ we obtain

$$\varphi = (-A+z-\lambda)^{-1} f - (-A+z-\lambda)^{-1} (\mathcal{C}_{\bar{u}} - z)\varphi.$$

Thus, if the following condition is satisfied

$$\text{if} \quad \lambda \in \hat{S}^C_{\zeta,z} \quad \text{is such that} \quad \left\| (-A+z-\lambda)^{-1} \right\|_{\mathcal{L}(L_2^n(0,l))} < \frac{1}{K_z}, \tag{4.10}$$

then $\|\varphi\|_2$ can be estimated from above as

$$\|\varphi\|_2 \le \frac{\|(-A+z-\lambda)^{-1}\|_{\mathcal{L}(L_2^n(0,l))}}{1 - \|(-A+z-\lambda)^{-1}\|_{\mathcal{L}(L_2^n(0,l))} K_z} \; \|f\|_2 \, .$$

Consequently, for λ, which satisfy (4.10), follows that $\lambda \in \rho(L_{\bar{u}})$ and

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \le \frac{\|(-A+z-\lambda)^{-1}\|_{\mathcal{L}(L_2^n(0,l))}}{1 - \|(-A+z-\lambda)^{-1}\|_{\mathcal{L}(L_2^n(0,l))} K_z}. \tag{4.11}$$

We proceed with the estimation of $\|(-A+z-\lambda)^{-1}\|_{\mathcal{L}(L_2^n(0,l))}$. After a straightforward computation, taking into the account the imposed boundary conditions (either Dirichlet or Neumann), for $\varphi \in H_2^B(0,l)$ we obtain

$$\begin{aligned}\|-A\varphi + z\varphi - \lambda\varphi\|_2^2 =& \sum_{j=1}^{n} \left\|-d_j\varphi_j'' + z\varphi_j - \lambda\varphi_j\right\|_2^2 \\ =& \sum_{j=1}^{n} \left(\left\|-d_j\varphi_j'' + z\varphi_j - \mathrm{Re}\lambda\varphi_j\right\|_2^2 + (\mathrm{Im}\lambda)^2 \left\|\varphi_j\right\|_2^2\right). \end{aligned} \tag{4.12}$$

Let us introduce the following two domains

$$\Omega_1 := \{\lambda \in \mathbb{C} : 0 < \mathrm{Re}\lambda - z \le \cot(\zeta)|\mathrm{Im}\lambda|\}, \tag{4.13}$$

$$\Omega_2 := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \le z\}. \tag{4.14}$$

We have $\hat{S}^C_{\zeta,z} = \Omega_1 \cup \Omega_2$.

For $\lambda \in \Omega_1$ the following estimation holds

$$|\lambda - z| = \sqrt{|\mathrm{Re}\lambda - z|^2 + |\mathrm{Im}\lambda|^2} \le \frac{1}{\sin(\zeta)}|\mathrm{Im}\lambda|.$$

Hence, from (4.12) for $\lambda \in \Omega_1$ we obtain

$$\begin{aligned}\left\|-d_j\varphi_j'' + z\varphi_j - \lambda\varphi_j\right\|_2^2 \ge& (\mathrm{Im}\lambda)^2 \left\|\varphi_j\right\|_2^2 \\ \ge& \sin^2(\zeta)|\lambda - z|^2 \left\|\varphi_j\right\|_2^2. \end{aligned} \tag{4.15}$$

Let us consider $\lambda \in \Omega_2$. Since $\mathrm{Re}\lambda - z \le 0$ and taking into account the boundary conditions, we obtain

$$\begin{aligned}\left\|-d_j\varphi_j'' + z\varphi_j - \mathrm{Re}\lambda\varphi_j\right\|_2^2 =& \left\|-d_j\varphi_j''\right\|_2^2 - 2(\mathrm{Re}\lambda - z)d_j \left\|\varphi_j'\right\|_2^2 + (\mathrm{Re}\lambda - z)^2 \left\|\varphi_j\right\|_2^2 \\ \ge& (\mathrm{Re}\lambda - z)^2 \left\|\varphi_j\right\|_2^2. \end{aligned}$$

Hence, from (4.12) for $\lambda \in \Omega_2$ follows

$$\left\|-d_j\varphi_j'' + z\varphi_j - \lambda\varphi_j\right\|_2^2 \ge |\lambda - z|^2 \left\|\varphi_j\right\|_2^2. \tag{4.16}$$

Gathering together (4.15), (4.16), and (4.12), we arrive at

$$\|-A\varphi + z\varphi - \lambda\varphi\|_2 \geq \begin{cases} \sin(\zeta)\,|\lambda - z|\ \|\varphi\|_2\,, & \text{if} \quad \lambda \in \Omega_1, \\ |\lambda - z|\ \|\varphi\|_2\,, & \text{if} \quad \lambda \in \Omega_2. \end{cases}$$

Therefore we have

$$\left\|(-A + z - \lambda)^{-1}\right\|_{\mathcal{L}(L_2^n(0,l))} \leq \begin{cases} \dfrac{\frac{1}{\sin(\zeta)}}{|\lambda - z|}, & \text{if} \quad \lambda \in \Omega_1, \\ \dfrac{1}{|\lambda - z|}, & \text{if} \quad \lambda \in \Omega_2. \end{cases} \tag{4.17}$$

On the other hand, in order for the estimation (4.11) to hold, condition (4.10) should be satisfied. Combining (4.17) and (4.10) together, we obtain that (4.11) holds if

$$|\lambda - z| > \begin{cases} \dfrac{K_z}{\sin(\zeta)}, & \text{if} \quad \lambda \in \Omega_1, \\ K_z, & \text{if} \quad \lambda \in \Omega_2. \end{cases} \tag{4.18}$$

From (4.17) and (4.18) for all $\lambda \in \hat{S}^C_{\zeta,z}$ follows

$$\left\|(-A + z - \lambda)^{-1}\right\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{\frac{1}{\sin(\zeta)}}{|\lambda - z|}, \quad \text{if} \quad |\lambda - z| > \frac{K_z}{\sin(\zeta)} \tag{4.19}$$

Let us choose $R > \dfrac{K_z}{\sin(\zeta)}$. Thus for $\lambda \in \hat{S}^C_{\zeta,z} \cap B^C(z,R)$, using (4.11) and (4.19), we proceed as follows

$$\begin{aligned} \|R(\lambda, L_{\bar{u}})\|_2 &\leq \frac{\frac{1}{\sin(\zeta)}}{|\lambda - z|} \frac{1}{1 - \frac{\frac{1}{\sin(\zeta)}}{|\lambda - z|} K_z} \leq \frac{\frac{1}{\sin(\zeta)}}{|\lambda - z|} \frac{1}{1 - \frac{\frac{1}{\sin(\zeta)}}{R} K_z} \\ &= \frac{R}{R\sin(\zeta) - K_z} \frac{1}{|\lambda - z|}. \end{aligned}$$

We set $\widetilde{M} := \dfrac{R}{R\sin(\zeta) - K_z}$ and obtain the assertion of the theorem. □

As one can see, Theorem 4.1 provides the estimation of the resolvent for those $\lambda \in \hat{S}^C_{\zeta,z}$, which lie outside the circle $B(z,R)$, that is for $\lambda \in \hat{S}^C_{\zeta,z} \cap B^C(z,R)$. In addition it follows that $\rho(L_{\bar{u}}) \supset \hat{S}^C_{\zeta,z} \cap B^C(z,R)$. The domain $\hat{S}^C_{\zeta,z} \cap B(z,R)$ is considered in the next section.

4.1.2 Estimation in $\hat{S}^C_{\zeta,z} \cap B(z,R)$. Exclosure of eigenvalues

The method of eigenvalues exclosure provides a proof of a non-existence of eigenvalues on a local basis. Recall from Chapter 3 that ω denotes a numerical approximation to $\bar{u}$ and the operator $L_\omega : D_p(L_\omega) \to L_2^n(0,l)$ is given by

$$D_p(L_\omega) := H_2^B(0,l), \quad L_\omega := -A + \mathcal{C}_\omega, \tag{4.20}$$

with

$$(\mathcal{C}_\omega \varphi)(x) := C_\omega(x)\varphi(x), \quad x \in [0,l].$$

In addition, let us introduce a notation

$$|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}} := \max_{x\in[0,l]} |C_\omega(x) - C_{\bar{u}}(x)|_2. \tag{4.21}$$

We comment on the computation of $|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}}$ later.

Now we continue with the following

Theorem 4.2. *Let $L_\omega : H_2^B(0,l) \to L_2^n(0,l)$ be the linear operator introduced in (4.20). Let $\mu \in \mathbb{C}$ be some given point in the complex plane. Assume that the bottom eigenvalue of the eigenvalue problem*

$$\begin{gathered} \varphi \in H_2^B(0,l), \quad \langle (L_\omega - \mu)\varphi, (L_\omega - \mu)\psi \rangle_2 = \tilde{\kappa} \langle \varphi, \psi \rangle_2 , \\ \forall\, \psi \in H_2^B(0,l), \end{gathered} \tag{4.22}$$

which we will denote as $\tilde{\kappa}_1$, is positive. Then there exists no eigenvalue $\tilde{\lambda}$ of L_ω in the circle $B(\mu, \sqrt{\tilde{\kappa}_1})$.

Proof. The assertion of the theorem follows from Poincaré's min-max principle: for $\tilde{\kappa}_1$ the following estimate

$$\tilde{\kappa}_1 \leq \frac{\langle (L_\omega - \mu)\varphi, (L_\omega - \mu)\varphi \rangle_2}{\langle \varphi, \varphi \rangle_2} = \frac{|\tilde{\lambda} - \mu|^2 \, \|\varphi\|_2^2}{\|\varphi\|_2^2} = |\tilde{\lambda} - \mu|^2$$

holds true for any given eigenvalue $\tilde{\lambda}$ of L_ω and eigenelement $\varphi \in H_2^B(0, l)$. Hence no eigenvalue $\tilde{\lambda}$ can lie inside the circle $B(\mu, \sqrt{\tilde{\kappa}_1})$. □

Since we are interested in the eigenvalues λ of $L_{\bar{u}}$ a transition from L_ω to $L_{\bar{u}}$ should be made. For that purpose let us introduce the following

Theorem 4.3. *Let $L_{\bar{u}} : H_2^B(0, l) \to L_2^n(0, l)$ be the linear operator introduced in* (4.1)*. Let $\mu \in \mathbb{C}$ be some given point in the complex plane and $\tilde{\kappa}_1$ be a positive lower bound to the bottom eigenvalue of* (4.22)*. Then if*

$$|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{Sp} < \sqrt{\tilde{\kappa}_1}, \tag{4.23}$$

there exists no eigenvalue λ of $L_{\bar{u}}$ in the circle $B(\mu, \sqrt{\kappa_1})$ with

$$\kappa_1 := (\sqrt{\tilde{\kappa}_1} - |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{Sp})^2. \tag{4.24}$$

Proof. Let $\varphi \in H_2^B(0, l)$ be an eigenelement of L_ω. From Poincaré's min-max principle follows:

$$\begin{aligned}
\|\varphi\|_2 &\leq \frac{1}{\sqrt{\tilde{\kappa}_1}} \|(L_\omega - \mu)\varphi\|_2 = \frac{1}{\sqrt{\tilde{\kappa}_1}} \|(-A + \mathcal{C}_\omega - \mu)\varphi\|_2 \\
&= \frac{1}{\sqrt{\tilde{\kappa}_1}} \|(-A + \mathcal{C}_{\bar{u}} - \mu + \mathcal{C}_\omega - \mathcal{C}_{\bar{u}})\varphi\|_2 \\
&\leq \frac{1}{\sqrt{\tilde{\kappa}_1}} \left(\|(L_{\bar{u}} - \mu)\varphi\|_2 + \|(\mathcal{C}_\omega - \mathcal{C}_{\bar{u}})\varphi\|_2\right) \\
&\leq \frac{1}{\sqrt{\tilde{\kappa}_1}} \left(\|(L_{\bar{u}} - \mu)\varphi\|_2 + |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}} \, \|\varphi\|_2\right).
\end{aligned}$$

Consequently,

$$\left(1 - \frac{1}{\sqrt{\tilde{\kappa}_1}} |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}}\right) \|\varphi\|_2 \le \frac{1}{\sqrt{\tilde{\kappa}_1}} \|(L_{\bar{u}} - \mu)\varphi\|_2 .$$

Hence if

$$|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}} < \sqrt{\tilde{\kappa}_1}$$

holds, then

$$\|\varphi\|_2 \le \frac{1}{\sqrt{\tilde{\kappa}_1} - |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}}} \|(L_{\bar{u}} - \mu)\varphi\|_2 . \tag{4.25}$$

With (4.24) the assertion of the theorem follows from Poincaré's min-max principle.

□

Now we demonstrate how during the implementation of the eigenvalue exclosure procedure an upper bound to $\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n[0,l])}$ can be gained.

Theorem 4.4. *Let $L_{\bar{u}}$, μ, κ_1 be defined as above. Let $0 < \xi < 1$. Then for every $\lambda \in B(\mu, \xi\sqrt{\kappa_1})$ the estimation*

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \le \frac{1}{(1-\xi)\sqrt{\kappa_1}}. \tag{4.26}$$

holds.

Proof. Let $\varphi \in H_2^B(0,l)$ be an eigenelement of $L_{\bar{u}}$. Then for any constant $q \in \mathbb{C}$ we have

$$\|(L_{\bar{u}} - (\mu + q))\varphi\|_2 \ge \|(L_{\bar{u}} - \mu)\varphi\|_2 - |q| \, \|\varphi\|_2 \ge (\sqrt{\kappa_1} - |q|) \, \|\varphi\|_2 , \tag{4.27}$$

where the last inequality follows from the definition of κ_1 and Poincaré's min-max principle.

From (4.27) for $\varphi \not\equiv 0$ and $|q| < \sqrt{\kappa_1}$ follows that

$$\|(L_{\bar{u}} - (\mu + q))\varphi\|_2 > 0.$$

Therefore, if $|q| < \sqrt{\kappa_1}$, the operator $L_{\bar{u}} - (\mu + q) : H_2^B(0,l) \to L_2^n(0,l)$ is injective. Now for $\varphi \in H_2^B(0,l)$ and $f \in L_2^n(0,l)$ consider

$$(L_{\bar{u}} - (\mu + q))\varphi = f. \tag{4.28}$$

Since $L_{\bar{u}} - (\mu + q)$ is injective, the Fredholm's Alternative applied to (4.28) results in the bijectivity of $L_{\bar{u}} - (\mu + q)$. Hence for $|q| < \sqrt{\kappa_1}$ and for all $f \in L_2^n(0,l)$ from (4.27) we have

$$\left\|(L_{\bar{u}} - (\mu + q))^{-1} f\right\|_{\mathcal{L}(L_2^n(0,l))} \le \frac{1}{\sqrt{\kappa_1} - |q|} \|f\|_2 .$$

Consequently, we obtain

$$\left\|(L_{\bar{u}} - (\mu + q))^{-1}\right\|_{\mathcal{L}(L_2^n(0,l))} \le \frac{1}{\sqrt{\kappa_1} - |q|}.$$

Let us choose $|q| \le \xi\sqrt{\kappa_1}$ with $0 < \xi < 1$. Then for every $\lambda \in B(\mu, \xi\sqrt{\kappa_1})$ we obtain the resolvent estimation as

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \le \frac{1}{(1-\xi)\sqrt{\kappa_1}}.$$

$\square$

Let us briefly sum up the eigenvalue exclosure process. We choose $\mu \in \mathbb{C}$ such that we suspect that no eigenvalue $\tilde{\lambda}$ of L_ω lies near to μ. After that we compute the positive lower bound $\tilde{\kappa}_1$ to the bottom eigenvalue of the eigenvalue problem (4.22). If condition (4.23) holds, then due to Theorem 4.3 we obtain a circle $B(\mu, \sqrt{\kappa_1})$, with κ_1 as in (4.24), which does not contain eigenvalues of $L_{\bar{u}}$. In addition, due to Theorem 4.4 in the circle $B(\mu, \xi\sqrt{\kappa_1})$ a resolvent estimation (4.26) is valid.

During the implementation of the eigenvalue exclosure method we cover the bounded domain $\hat{S}^C_{\zeta,z} \cap B(z, R)$ with a finite union of circles, each of which does not contain eigenvalues of $L_{\bar{u}}$. Furthermore, we gather estimations for the resolvent as in (4.26). Hence, for some $i = 1, \ldots, K,\ K > 0$, we obtain

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq M_{\mathrm{IC}}^i := \frac{1}{(1-\xi)\sqrt{\kappa_1^i}}, \quad \forall \lambda \in \hat{S}_{\zeta,z}^C \cap B(z,R) \cap B\left(\mu_i, \xi\sqrt{\kappa_1^i}\right)$$

with

$$\hat{S}_{\zeta,z}^C \cap B(z,R) \subset \bigcup_{i=1}^{K} B\left(\mu_i, \xi\sqrt{\kappa_1^i}\right).$$

We denote the resulting vector of the resolvent estimations as M_{IC}, that is

$$M_{\mathrm{IC}} = \left(M_{\mathrm{IC}}^1, \ldots, M_{\mathrm{IC}}^K\right)^T, \tag{4.29}$$

and set

$$M_{\mathrm{IC}}^{\max} := \max_{i=1,\ldots,K} M_{\mathrm{IC}}^i. \tag{4.30}$$

Therefore,

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq M_{\mathrm{IC}}^{\max}, \quad \forall \lambda \in \hat{S}_{\zeta,z}^C \cap B(z,R). \tag{4.31}$$

Now let us comment on the choice of the constant ξ.

Remark 4.5. *Let $\xi \in (0,1)$. Observe that for $\lambda \in \hat{S}_{\zeta,z}^C \cap B(z,R) \cap B\left(\mu, \xi\sqrt{\kappa_1}\right)$ we have*

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{1}{(1-\xi)\sqrt{\kappa_1}} \leq \frac{|\mu - z| + \xi\sqrt{\kappa_1}}{|\lambda - z|} \frac{1}{(1-\xi)\sqrt{\kappa_1}}. \tag{4.32}$$

Let us set $\hat{M} := \dfrac{|\mu - z| + \xi\sqrt{\kappa_1}}{(1-\xi)\sqrt{\kappa_1}}$. *Then we obtain*

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{\hat{M}}{|\lambda - z|}, \quad \forall \lambda \in \hat{S}_{\zeta,z}^C \cap B(z,R) \cap B\left(\mu, \xi\sqrt{\kappa_1}\right).$$

Now let ξ converge to zero. Then constant $\hat{M}$ converges to $\dfrac{|\mu - z|}{\sqrt{\kappa_1}}$ *whereas the radius of the circle, in which the estimation of the resolvent is valid, converges to zero. Hence there is no optimal choice for ξ. In our computations we set $\xi = \frac{1}{2}$.*

4.1.3 Final estimation of $\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))}$. Sectoriality

Combination of (4.5) and (4.31) results in

$$\|R(\lambda, L_{\bar{u}})\|_2 \leq \begin{cases} \dfrac{\widetilde{M}}{|\lambda - z|}, & \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B^C(z, R), \\ M^{\max}_{\mathrm{IC}}, & \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B(z, R). \end{cases}$$

The estimation of the resolvent above is not exactly the "classical" estimation in the sense of the property (ii) from Definition 2.3. Nevertheless, this estimation will be quite useful for our further investigations on the domain of attraction. In particular, starting with the problem

$$L_{\bar{u}}\varphi - \lambda\varphi = f,$$

with $\varphi \in H_2^B(0, l)$ and $f \in L_2^n(0, l)$, we will use the following estimation

$$\|\varphi\|_2 \leq \begin{cases} \dfrac{\widetilde{M}}{|\lambda - z|} \|f\|_2, & \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B^C(z, R), \\ M^{\max}_{\mathrm{IC}} \|f\|_2, & \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B(z, R). \end{cases} \tag{4.33}$$

The application of (4.33) will be demonstrated in Chapter 5.

Now let us show the property (ii). We will follow the approach introduced in (4.32). Namely, for each $\lambda \in \hat{S}^C_{\zeta,z} \cap B(z, R) \cap B\left(\mu_i, \xi\sqrt{\kappa_1^i}\right)$, $i = 1, \ldots, K$ we have

$$\|R(\lambda, L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{\hat{M}_i}{|\lambda - z|},$$

with $\hat{M}_i = \dfrac{|\mu - z| + \xi\sqrt{\kappa_1^i}}{(1 - \xi)\sqrt{\kappa_1^i}}$. Now set $\hat{M}_{\max} := \max\limits_{i=1,\ldots,K} \hat{M}_i$. Thus we obtain

$$\|R(\lambda, L_{\bar{u}})\|_2 \leq \frac{\hat{M}_{\max}}{|\lambda - z|}, \quad \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B(z, R). \tag{4.34}$$

Setting $M := \max\{\widetilde{M}, \hat{M}_{\max}\}$, from (4.5) and (4.34) we have

$$\|R(\lambda, L_{\bar{u}})\|_2 \leq \frac{M}{|\lambda - z|} \quad \forall \lambda \in \hat{S}^C_{\zeta,z}. \tag{4.35}$$

In addition, by combination of the eigenvalue exclosure procedure with Theorem 4.1 we have shown that

$$\rho(L_{\bar{u}}) \supset \hat{S}^C_{\zeta,z}. \tag{4.36}$$

Now let λ be an eigenvalue of $L_{\bar{u}}$. Then $-\lambda$ is an eigenvalue of $-L_{\bar{u}}$ and from (4.36), (4.35) follows

(A) $\rho(-L_{\bar{u}}) \supset S_{\pi-\zeta,-z}$ for some $z \in \mathbb{R}$ and $\zeta \in \left(0, \frac{\pi}{2}\right)$,

(B) $\|R(\lambda, -L_{\bar{u}})\|_{\mathcal{L}(L_2^n[0,l])} \leq \dfrac{M}{|\lambda + z|}, \quad \forall \lambda \in S_{\pi-\zeta,-z}$.

Therefore the properties (i) and (ii) are satisfied and the operator $-L_{\bar{u}}$ is sectorial in $L_2^n(0, l)$.

4.1.4 Some additional remarks

To conclude this section, let us present the following remarks.

Remark 4.6. *Let us discuss the computation of* $|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{Sp}$. *Due to* (4.21) *and the definition of the euclidean norm we obtain*

$$|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{Sp} = \max_{x\in[0,l]} \sqrt{\lambda_{\max}(C_\omega(x) - C_{\bar{u}}(x))^*(C_\omega(x) - C_{\bar{u}}(x))}.$$

After some elementary transformations, the estimation of $|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{Sp}$ *can be reduced to the estimation of the expressions which contain only* $\|\omega\|_\infty$ *and* $\|\bar{u} - \omega\|_\infty$. *It is possible to obtain the estimation of* $\|\omega\|_\infty$, *since the numerical approximation* ω *is available. For the estimation of* $\|\bar{u} - \omega\|_\infty$ *we use* (3.21).

Remark 4.7. *Let us briefly discuss the calculation of* K_z. *At first, consider the following estimation*

$$\begin{aligned}
\|(\mathcal{C}_{\bar{u}} - z)\varphi\|_2 &\leq \|(\mathcal{C}_{\bar{u}} - \mathcal{C}_\omega + \mathcal{C}_\omega - z)\varphi\|_2 \\
&\leq \|(\mathcal{C}_{\bar{u}} - \mathcal{C}_\omega)\varphi\|_2 + \|(\mathcal{C}_\omega - z)\varphi\|_2 \\
&\leq \left(|\mathcal{C}_{\bar{u}} - \mathcal{C}_\omega|_{Sp} + |\mathcal{C}_\omega - z|_{Sp}\right) \|\varphi\|_2 ,
\end{aligned}$$

where

$$|\mathcal{C}_\omega - z|_{Sp} = \max_{x\in[0,l]} \sqrt{\lambda_{\max}(C_\omega(x) - z)^*(C_\omega(x) - z)}.$$

Hence, we set

$$K_z := |\mathcal{C}_{\bar{u}} - \mathcal{C}_\omega|_{Sp} + |\mathcal{C}_\omega - z|_{Sp}. \tag{4.37}$$

The term $|\mathcal{C}_\omega - z|_{Sp}$ *is given by*

$$|\mathcal{C}_\omega - z|_{Sp} = \max_{x\in[0,l]} \sqrt{\lambda_{\max}(C_\omega(x) - z)^*(C_\omega(x) - z)}$$

and can be reduced to the estimation of the expressions which contain only $\|\omega\|_\infty$.

Remark 4.8. *As one can see the estimation* (4.5) *holds for all* $z \in \mathbb{R}$ *and* $\zeta \in \left(0, \frac{\pi}{2}\right)$. *Let us make the following observations concerning the choice of this constants.*

1. *Observe from Theorem 4.1 that the constant* R *- radius of the circle, outside of which (by Theorem 4.1) no eigenvalue can lie, depends on* z. *Thus, if one chooses constant* z *"close" to the spectrum or even "in" the spectrum of* $L_{\bar{u}}$, *the radius will increase in such a way that the circle* $B(z, R)$ *will cover the "critical" regions, where the eigenvalues may lie.*

2. *In our computations we proceed as follows. We compute approximate eigenvalues of the operator* L_ω *and establish the sector* $\hat{S}_{\tilde{\zeta},\tilde{z}}$, *which contains these eigenvalues. After that we set in Theorem 4.1* $z = \tilde{z}$ *and* $\zeta = \tilde{\zeta}$. *Now, by Theorem 4.1 we conclude that no eigenvalue of* $L_{\bar{u}}$ *can lie outside* $\hat{S}^C_{\tilde{\zeta},\tilde{z}} \cap B^C(\tilde{z}, \tilde{R})$, *where* $\tilde{R} > \dfrac{K_{\tilde{z}}}{\sin(\tilde{\zeta})}$. *The domain* $\hat{S}^C_{\tilde{\zeta},\tilde{z}} \cap B(\tilde{z}, \tilde{R})$ *is now the domain where we implement the eigenvalue exclosure process. After that implementation we establish values for* z *and* ζ, *such that* $\rho(L_{\bar{u}}) \supset \hat{S}^C_{\zeta,z} \cap B(\tilde{z}, \tilde{R})$. *Now we can apply Theorem 4.1 again, this time with this new values for* z *and* ζ. *As a result we*

obtain the non-existence of eigenvalues in $\hat{S}^C_{\zeta,z} \cap B^C(z,R)$*, combined with the resolvent estimation* (4.5).

4.2 Self-adjoint $L_{\bar{u}}$

In this section we would like to discuss a particular case when the operator $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$, introduced in (4.1) and given by

$$L_{\bar{u}}\varphi = -D\varphi'' + \mathcal{C}_{\bar{u}}\varphi,$$

is self-adjoint. In that case the results from the spectral theory for self-adjoint operators can be employed. Therefore the sectoriality of the self-adjoint $L_{\bar{u}}$ can be shown by a simpler method, than the approach that has been described above. In addition, we will introduce a condition on the matrix $C_{\bar{u}}$, which is equivalent to the self-adjointness of $L_{\bar{u}}$. This condition will be helpful for the determination of those classes of problems, which can be treated by the methods we propose for the self-adjoint operators.

4.2.1 Sectoriality of $-L_{\bar{u}}$

At first we introduce

Lemma 4.9. *The operator* $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ *given by* (4.1) *is self-adjoint if and only if*

$$C^*_{\bar{u}} = C_{\bar{u}}. \tag{4.38}$$

Proof. At first observe that the operator $A : H_2^B(0,l) \to L_2^n(0,l)$ given by $A\varphi := -D\varphi''$ is self-adjoint. One can find a proof of this assertion, which is based on the application of the Friedrichs extension procedure in e.g. [57, Proposition 2.1, Proposition 2.2, p. 100-101]. Let us consider the operator $\mathcal{C}_{\bar{u}} : C^n[0,l] \to L_2^n(0,l)$ which is defined via

$$(\mathcal{C}_{\bar{u}}\varphi)(x) := C_{\bar{u}}(x)\varphi(x), \quad x \in [0,l].$$

For our further purposes we extend $\mathcal{C}_{\bar{u}}$ to the whole $L_2^n(0,l)$. Thus, $\mathcal{C}_{\bar{u}} : L_2^n(0,l) \to L_2^n(0,l)$. Observe, that (4.38) is equivalent to the symmetry of $\mathcal{C}_{\bar{u}}$. Therefore, (4.38) implies self-adjointness of $\mathcal{C}_{\bar{u}}$ and vice versa.

Now let us assume that $\mathcal{C}_{\bar{u}}^* = \mathcal{C}_{\bar{u}}$. Since $\mathcal{C}_{\bar{u}} \in \mathcal{L}(L_2^n(0,l), L_2^n(0,l))$ by Lemma 2.20 (with $-A$ instead of T and $\mathcal{C}_{\bar{u}}$ instead of S) the operator $L_{\bar{u}} = -A + \mathcal{C}_{\bar{u}}$ is closable and its adjoint $L_{\bar{u}}^*$ is given by $L_{\bar{u}}^* = -A^* + \mathcal{C}_{\bar{u}}^*$. Since $-A$ and $\mathcal{C}_{\bar{u}}$ are self-adjoint, $L_{\bar{u}}$ is self-adjoint as well with $D_p(L_{\bar{u}}^*) = D_p(L_{\bar{u}}) = H_2^B(0,l)$.

On the other hand, if $L_{\bar{u}}$ is self-adjoint then it is closable and $L_{\bar{u}} = L_{\bar{u}}^*$. Again, from the self-adjointness of $-A$ follows that $\mathcal{C}_{\bar{u}}^* = \mathcal{C}_{\bar{u}}$. □

Let us introduce the following notation

$$\lambda_1^C := \min_{x\in[0,l]} \lambda_{\min}(C_{\bar{u}}(x)).$$

Now we investigate the spectrum of $L_{\bar{u}}$. We consider an eigenvalue problem

$$L_{\bar{u}}u = \lambda u \quad (0 \le x \le l), \quad B_p[u](0) = B_p[u](l) = 0. \tag{4.39}$$

We would like to show that

Proposition 4.10. *There exists an orthonormal basis $\{\tilde{\varphi}_k\}_{k=1}^{\infty}$ of $L_2^n(0,l)$ of the eigenfunctions of $L_{\bar{u}}$ such that*

$$L_{\bar{u}}\tilde{\varphi}_k = \lambda_k\tilde{\varphi}_k, \quad k = 1, 2, \ldots. \tag{4.40}$$

The eigenvalue sequence $\lambda_k \to \infty$, as $k \to \infty$. Additionally, $\lambda_k \in \mathbb{R}$, $k \in \mathbb{N}$, and $\lambda_1 \le \lambda_2 \le \ldots$.

Proof. We start by introducing a positive constant σ such that

$$\lambda_1^C + \sigma > 0. \tag{4.41}$$

For $u \in H_2^B(0,l)$ and $r \in L_2^n(0,l)$ consider the eigenvalue problem

$$(L_{\bar{u}} + \sigma I)u = r \quad (0 \le x \le l), \quad B_p[u](0) = B_p[u](l) = 0. \tag{4.42}$$

For $u \in H_2^B(0,l),\ u \not\equiv 0$ we have

$$\begin{aligned}\langle (L_{\bar u} + \sigma I)u, u\rangle_2 &= \langle -Du'', u\rangle_2 + \langle (\mathcal{C}_{\bar u} + \sigma I)u, u\rangle_2 \\ &= \langle Du', u'\rangle_2 + \langle (\mathcal{C}_{\bar u} + \sigma I)u, u\rangle_2 \\ &\geq \underbrace{d_{\min}}_{>0} \|u'\|_2^2 + \underbrace{(\lambda_1^C + \sigma)}_{>0} \|u\|_2^2 > 0.\end{aligned}$$

Thus, the operator $L_{\bar u} + \sigma I : H_2^B(0,l) \to L_2^n(0,l)$ is injective. Application of the Fredholm's Alternative to problem (4.42) results in the bijectivity of $L_{\bar u} + \sigma I$.

Further, the operator $L_{\bar u} + \sigma I$ is bounded. Indeed, we have that

$$\begin{aligned}\|(L_{\bar u} + \sigma I)u\|_2 &= \|-Du'' + (\mathcal{C}_{\bar u} + \sigma I)u\|_2 \\ &\leq \|-Du''\|_2 + \|(\mathcal{C}_{\bar u} + \sigma I)u\|_2 \\ &\leq d_{\max} \|u\|_{H_2^n(0,l)} + \max_{x\in[0,l]} \lambda_{\max}(C_{\bar u}(x) + \sigma I) \|u\|_{H_2^n(0,l)} \\ &\leq \max\{d_{\max}, \max_{x\in[0,l]} \lambda_{\max}(C_{\bar u}(x) + \sigma I)\} \|u\|_{H_2^n(0,l)}.\end{aligned}$$

Due to the bijectivity and boundedness of $L_{\bar u} + \sigma I$, by Open Mapping Theorem we conclude that the inverse operator $(L_{\bar u} + \sigma I)^{-1} : L_2^n(0,l) \to H_2^B(0,l)$ is bounded. Further, due to Rellich-Kondrachov Theorem [2, Theorem 6.2, p. 144] the embedding $E : H_2^n(0,l) \to L_2^n(0,l)$ is compact. Therefore $E(L_{\bar u} + \sigma I)^{-1} : L_2^n(0,l) \to L_2^n(0,l)$ is compact as well.

Moreover $(L_{\bar u} + \sigma I)^{-1}$ is symmetric. Indeed, from the symmetry of $L_{\bar u}$ the symmetry of $(L_{\bar u} + \sigma I)$ follows. Further let $U, V \in L_2^n(0,l)$ and consider

$$\begin{aligned}u &:= (L_{\bar u} + \sigma I)^{-1}U \in H_2^B(0,l), \\ v &:= (L_{\bar u} + \sigma I)^{-1}V \in H_2^B(0,l).\end{aligned}$$

Then we have

$$\langle (L_{\bar u} + \sigma I)^{-1}U, V\rangle_2 = \langle u, (L_{\bar u} + \sigma I)v\rangle_2 = \langle (L_{\bar u} + \sigma I)u, v\rangle_2 = \langle U, (L_{\bar u} + \sigma I)^{-1}V\rangle_2.$$

Since $(L_{\bar{u}}+\sigma I)^{-1}$ is symmetric, then $E(L_{\bar{u}}+\sigma I)^{-1} : L_2^n(0,l) \to L_2^n(0,l)$ is a compact and symmetric operator. Moreover, $E(L_{\bar{u}}+\sigma I)^{-1}$ is self-adjoint, since it is defined on the whole space $L_2^n(0,l)$. Due to the spectral theorem for the compact and self-adjoint operators [64, Theorem 1, p. 325] there exists an orthonormal basis $(\tilde{\varphi}_k)_{k\in\mathbb{N}}$ of $L_2^n(0,l)$ of the eigenfunctions of $E(L_{\bar{u}}+\sigma I)^{-1}$. The corresponding sequence of eigenvalues $(\mu_k)_{k\in\mathbb{N}}$ is real and converges to 0. In addition, due to the injectivity of $E(L_{\bar{u}}+\sigma I)^{-1}$, we have $\mu_k \neq 0$. Hence, for all $k \in \mathbb{N}$ we have

$$E(L_{\bar{u}}+\sigma I)^{-1}\tilde{\varphi}_k = \mu_k\tilde{\varphi}_k,$$

and consequently,

$$(L_{\bar{u}}+\sigma I)\tilde{\varphi}_k = \frac{1}{\mu_k}\tilde{\varphi}_k.$$

It follows that

$$L_{\bar{u}}\tilde{\varphi}_k = \underbrace{\left(\frac{1}{\mu_k} - \sigma\right)}_{:=\lambda_k}\tilde{\varphi}_k. \tag{4.43}$$

Since $\mu_k \to 0$ for $k \to \infty$, the sequence $(\lambda_k)_{k\in\mathbb{N}}$ has no finite accumulation point. Consequently, it can be considered as monotone non-decreasing. We have shown the assertion. □

Remark 4.11. *As discussed in Chapter 3, Section 3.3 in case of the non-self-adjoint operator $L_{\bar{u}}$ one proceeds with the enclosure of the stationary solution (and in particular with the calculation of constant K) by considering eigenvalue problem* (3.27) *for the operator $L_{\bar{u}}^*L_{\bar{u}}$. Following the steps of the proof of Proposition 4.10 and taking $L_{\bar{u}}^*L_{\bar{u}}$ instead of $L_{\bar{u}}$ and $H_4^B(0,l) := \{\varphi \in H_4^n(0,l) : B_p[\varphi](0) = B_p[\varphi](l) = 0\}$ instead of $H_2^B(0,l)$ one can show the compactness of the resolvent of the self-adjoint operator $L_{\bar{u}}^*L_{\bar{u}}$ and consequently conclude the existence of the orthonormal basis of the eigenfunctions of $L_{\bar{u}}^*L_{\bar{u}}$ as well as the existence of the non-decreasing sequence of the eigenvalues of the correspondent eigenvalue problem.*

Now let us discuss the computation of the eigenvalues of $L_{\bar{u}}$. In particular, we will be interested in the lower bound to the spectrum of $L_{\bar{u}}$. We introduce the following

Theorem 4.12. *Let $L_{\bar{u}}$ and L_ω be defined as in* (4.1) *and* (3.13) *respectively. Let $\tilde{\lambda}_1$ denote a positive lower bound to the bottom eigenvalue of the eigenvalue problem*

$$\varphi \in H_2^B(0,l), \quad \langle L_\omega \varphi, \psi \rangle_2 = \tilde{\lambda} \langle \varphi, \psi \rangle_2 \,, \quad \forall \psi \in H_2^B(0,l).$$

If

$$|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{Sp} < \tilde{\lambda}_1,$$

then

$$\lambda_1 \geq \tilde{\lambda}_1 - |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{Sp}. \tag{4.44}$$

Proof. Let $\varphi \in H_2^B(0,l)$ be an eigenfunction of L_ω. From Poincaré's min-max principle follows

$$\begin{aligned}
\langle \varphi, \varphi \rangle_2 &\leq \frac{1}{\tilde{\lambda}_1} \langle L_\omega \varphi, \varphi \rangle_2 = \frac{1}{\tilde{\lambda}_1} \langle -D\varphi'' + \mathcal{C}_\omega \varphi - \mathcal{C}_{\bar{u}} \varphi + \mathcal{C}_{\bar{u}} \varphi, \varphi \rangle_2 \\
&= \frac{1}{\tilde{\lambda}_1} \left(\langle L_{\bar{u}} \varphi, \varphi \rangle_2 + \langle (\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}) \varphi, \varphi \rangle_2 \right) \\
&\leq \frac{1}{\tilde{\lambda}_1} \left(\langle L_{\bar{u}} \varphi, \varphi \rangle_2 + |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}}| \, \|\varphi\|_2^2 \right).
\end{aligned}$$

Thus, if

$$|\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}} < \tilde{\lambda}_1,$$

then

$$\langle L_{\bar{u}} \varphi, \varphi \rangle_2 \geq (\tilde{\lambda}_1 - |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}}) \, \|\varphi\|_2^2 \,.$$

The assertion of the theorem follows from Poincaré's min-max principle. ∎

Due to the self-adjointness of $L_{\bar{u}}$, a lower bound of $\tilde{\lambda}_1$ can be computed with the help of the variational methods for the computation of eigenvalue bounds. We present a detailed description of these methods in Chapter 6.

In the following we assume that we were able to compute a constant $z := \tilde{\lambda}_1 - |\mathcal{C}_\omega - \mathcal{C}_{\bar{u}}|_{\mathrm{Sp}}$ - the lower bound to the spectrum of $L_{\bar{u}}$. Thus for every eigenelement $\varphi \in H_2^B(0,l)$ we have

$$\langle L_{\bar{u}}\varphi, \varphi\rangle_2 \geq z \langle \varphi, \varphi\rangle_2 \,,$$

and hence

$$\langle -L_{\bar{u}}\varphi, \varphi\rangle_2 \leq -z \langle \varphi, \varphi\rangle_2 \,. \tag{4.45}$$

We want to show the following

Proposition 4.13. *Let operator $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ be the self-adjoint operator defined above. Then $-L_{\bar{u}}$ is a sectorial operator with an arbitrary $\theta < \pi$ and $a = -z$, where θ and a are the constants from Definition 2.3.*

Proof. By Proposition 4.10 and definition of z we have $\rho(-L_{\bar{u}}) \supset S_{\theta,-z}$ for an arbitrary $\theta < \pi$.

Let us verify condition (2.4)(ii). From (4.45) follows

$$\langle (-L_{\bar{u}} + z)\varphi, \varphi\rangle_2 \leq 0. \tag{4.46}$$

Let us denote $\tilde{L} := -L_{\bar{u}} + z : H_2^B(0,l) \to L_2^n(0,l)$. Then we have that $\rho(\tilde{L}) \supset S_{\theta,0}$. Let $\lambda \in S_{\theta,0}$. Then $\lambda = \rho e^{i\theta}$ with $\rho > 0$, $-\pi < \theta < \pi$. For $\varphi \in H_2^B(0,l)$ and $f \in L_2^n(0,l)$ we consider

$$\lambda\varphi - \tilde{L}\varphi = f. \tag{4.47}$$

Since $\lambda \in S_{\theta,0}$ problem (4.47) has a unique solution $\varphi = R(\lambda, \tilde{L})f$. Let us multiply (4.47) by $e^{\frac{-i\theta}{2}}$ and take the inner product with φ. We obtain

$$\rho e^{\frac{i\theta}{2}} \|\varphi\|_2^2 - e^{\frac{-i\theta}{2}} \langle \tilde{L}\varphi, \varphi\rangle_2 = e^{\frac{-i\theta}{2}} \langle f, \varphi\rangle_2.$$

Taking the real part of the equation above, we get

$$\rho \cos\left(\frac{\theta}{2}\right) \|\varphi\|_2^2 - \cos\left(\frac{\theta}{2}\right) \langle \tilde{L}\varphi, \varphi\rangle_2 = \mathrm{Re}\left(e^{\frac{-i\theta}{2}} \langle f, \varphi\rangle_2\right) \leq \|f\|_2 \|\varphi\|_2 \,.$$

Therefore, taking into account that $\cos\left(\dfrac{\theta}{2}\right) > 0$ and, by (4.46), $\langle \tilde{L}\varphi, \varphi\rangle_2 \leq 0$, we obtain

$$\|\varphi\|_2 \leq \frac{1}{\rho \cos\left(\frac{\theta}{2}\right)} \|f\|_2 = \frac{\dfrac{1}{\cos\left(\frac{\theta}{2}\right)}}{|\lambda|} \|f\|_2 .$$

Thus, for $\lambda \in S_{\theta,0}$ we have

$$\left\|R(\lambda, \tilde{L})\right\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{M_{L_2}}{|\lambda|}, \quad \text{with} \quad M_{L_2} := \frac{1}{\cos\left(\frac{\theta}{2}\right)}. \tag{4.48}$$

Since $R(\lambda, \tilde{L}) = R(\lambda - z, -L_{\bar{u}})$, we obtain

$$\|R(\lambda - z, -L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{M_{L_2}}{|\lambda|}, \quad \text{for} \quad \lambda \in S_{\theta,0},$$

and consequently,

$$\|R(\lambda, -L_{\bar{u}})\|_{\mathcal{L}(L_2^n(0,l))} \leq \frac{M_{L_2}}{|\lambda + z|}, \quad \text{for} \quad \lambda \in S_{\theta,-z}.$$

We have obtained the assertion. □

5

Domain of attraction

In this chapter we are going to consider a nonlinear system of parabolic differential equations of the form

$$\begin{cases} u_t(x,t) = Du_{xx}(x,t) + F(u(x,t)), & t > 0, \quad x \in [0,l], \\ B_p[u(\cdot,t)](0) = B_p[u(\cdot,t)](l) = 0, & t \geq 0, \\ u(x,0) = u_0(x), & x \in [0,l] \end{cases} \tag{5.1}$$

from the point of view of stability. In Chapter 3 we have established the existence of the stationary solution $\bar{u}$ of (5.1) and found an explicitly described neighbourhood of numerical approximation $\omega \in H_2^B(0,l)$, which contains $\bar{u}$. In Chapter 4 we investigated the operator $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ and established its sectoriality. In this chapter we show that if the lower bound to the spectrum of $L_{\bar{u}}$ is positive, then $\bar{u}$ is asymptotically stable and its domain of attraction can be quantified by the methods we are going to propose in sequel.

Throughout this chapter we make the following assumption

(G) There exists a monotonically non-decreasing function $G : [0,+\infty) \to [0,+\infty)$

such that

$$\begin{cases} |F(y+\bar{u}(x)) - F(\bar{u}(x)) + C_{\bar{u}}(x)y|_2 \le G(|y|_2), \quad y \in \mathbb{R}^n,\ x \in [0,l] \\ \\ \text{with} \quad G(h) = o(h), \quad h \to 0+\,. \end{cases} \tag{5.2}$$

Thus, from (5.2) follows that for each $\varepsilon > 0$ there exist $\delta > 0$ such that

$$|G(h)| \le \varepsilon|h|, \quad \text{for} \quad |h| < \delta. \tag{5.3}$$

For our further investigations the following remark will be important.

Remark 5.1. *Consider* (5.3). *Note that if G is known, then a function $\delta : (0, \varepsilon_0] \to (0, \infty)$ satisfying*

$$\forall \varepsilon > 0 \quad \forall |h| < \delta(\varepsilon) \quad |G(h)| \le \varepsilon|h| \tag{5.4}$$

can be computed.

5.1 Basic framework

In the following we write system (5.1) as a Cauchy problem in some suitable Banach space X. We wish to construct a proper framework, in which the existence of the mild solutions to a Cauchy formulation of (5.1) can be established. For that purpose let us recall the abstract setting to the Cauchy problem from Chapter 2. We have considered three Banach spaces $D(T) \subset X_\alpha \subset X$, where X_α was the space of class J_α between $D(T)$ and X. This is the kind of framework, which we are aiming at, while considering problem (5.1). Let us introduce

Proposition 5.2. *The space $C^n[0,l]$ is of class $J_{1/2}$ between $L_2^n(0,l)$ and $H_B^2(0,l)$.*

Proof. We have that $H_B^2(0,l) \subset C^n[0,l] \subset L_2^n(0,l)$. Hence we will prove the assertion, if we can find a positive constant C such that

$$\|\varphi\|_\infty \le C\, \|\varphi\|_{H_2}^{\frac{1}{2}}\, \|\varphi\|_2^{\frac{1}{2}}\,, \quad \varphi \in H_B^2(0,l). \tag{5.5}$$

By partial integration, taking into account the conditions, imposed on the boundary (Dirichlet or Neumann), we have

$$\|\varphi'\|_2^2 = \int_0^l |\varphi'(x)|^2 dx = \underbrace{\varphi(x)^T \overline{\varphi'(x)}|_0^l}_{=0} - \int_0^l \varphi(x)^T \overline{\varphi''(x)} dx \le \|\varphi\|_2 \|\varphi''\|_2 . \tag{5.6}$$

According to Lemma 2.21 and Lemma 2.23 the embedding estimation

$$\|\varphi\|_\infty^2 \le C_0 \|\varphi\|_2^2 + C_1 \|\varphi'\|_2^2 , \tag{5.7}$$

with C_0 and C_1 given either by (2.28) or (2.31) (for Dirichlet boundary conditions) holds. Inserting (5.6) into (5.7) we obtain

$$\begin{aligned}
\|\varphi\|_\infty^2 &\le C_0 \|\varphi\|_2^2 + C_1 \|\varphi'\|_2^2 \le C_0 \|\varphi\|_2^2 + C_1 \|\varphi\|_2 \|\varphi''\|_2 \\
&= \|\varphi\|_2 (C_0 \|\varphi\|_2 + C_1 \|\varphi''\|_2) \\
&\le \|\varphi\|_2 \sqrt{C_0^2 + C_1^2} \sqrt{\|\varphi\|_2^2 + \|\varphi''\|_2^2} \le \sqrt{C_0^2 + C_1^2} \|\varphi\|_2 \|\varphi\|_{H_2} .
\end{aligned}$$

Thus, we obtain

$$\|\varphi\|_\infty \le C \|\varphi\|_{H_2}^{\frac{1}{2}} \|\varphi\|_2^{\frac{1}{2}} , \quad \varphi \in H_B^2(0,l),$$

with $C := (C_0^2 + C_1^2)^{\frac{1}{4}}$. We have shown the assertion. $\square$

Taking into account Proposition 5.2, we make the following setting in the abstract framework from Chapter 2: $X = L_2^n(0,l)$, $X_\alpha = C^n[0,l]$, and $D(T) = D_p(L_{\bar{u}}) = H_2^B(0,l)$, where $L_{\bar{u}}$ is the operator which was introduced in (4.1). Now we write (5.1) as a Cauchy problem in $L_2^n(0,l)$. For that purpose we switch our viewpoint and consider the function $u = u(x,t)$ not as a function of x and t, but rather as a mapping of t into the space $L_2^n(0,l)$ of functions in x. Thus, we introduce

$$\mathbf{u} : [0,\infty) \to L_2^n(0,l),$$

defined by

$$\mathbf{u}(t)(x) := u(x,t) \quad \forall x \in [0,l], \quad t \ge 0.$$

Now (5.1) reads

$$\begin{cases} \mathbf{u}'(t) = A\mathbf{u}(t) + \mathbf{F}(\mathbf{u}(t)), & t > 0, \\ \mathbf{u}(0) = u_0. \end{cases} \tag{5.8}$$

At first, let us remark on the nature of $\mathbf{u}$.

Remark 5.3. *Here and in the following we regard the solutions of Cauchy problems only as mild solutions. In particular, this implies that the time-derivative* $\mathbf{u}'(t)$ *(or* $v'(t)$*, which will appear later) has only symbolic character.*

In (5.8) the operator $A : D_p(A) \to C^n[0,l]$ is the operator, which was introduced in (3.12), that is

$$D_p(A) = H_2^B(0,l) = \begin{cases} H_2^n(0,l) \cap (H_1^0(0,l))^n, & \text{if} \quad p = 0, \\ \{\varphi \in H_2^n(0,l) : \varphi'(0) = \varphi'(l) = 0\}, & \text{if} \quad p = 1, \end{cases}$$

$$A\varphi = D\varphi''$$

and the mapping

$$\mathbf{F} : C^n[0,l] \to L_2^n(0,l)$$

is the mapping, which was defined via (3.6), i.e

$$(\mathbf{F}\varphi)(x) := F(\varphi(x)), \quad \forall x \in (0,l).$$

Now let us introduce a new unknown

$$v(t) = \mathbf{u}(t) - \bar{u},$$

with

$$v_0 := \mathbf{u}(0) - \bar{u}.$$

A stationary solution $\bar{u}$ satisfies

$$A\bar{u} + \mathbf{F}(\bar{u}) = 0. \tag{5.9}$$

Substraction of (5.9) from (5.8) results in

$$\begin{cases} v'(t) = -L_{\bar{u}} v(t) + \mathbf{g}(v(t), \bar{u}), \qquad t > 0, \\ \\ v(0) = v_0, \end{cases} \tag{5.10}$$

where the operator $L_{\bar{u}} : D_p(L_{\bar{u}}) \to L_2^n(0,l)$ is the operator from (4.1). Thus,

$$D_p(L_{\bar{u}}) = H_2^B(0,l), \quad L_{\bar{u}} = -A + \mathcal{C}_{\bar{u}},$$

and a function $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ is given by

$$\mathbf{g}(\varphi, \bar{u}) := \mathbf{F}(\varphi + \bar{u}) - \mathbf{F}(\bar{u}) + \mathcal{C}_{\bar{u}} \varphi. \tag{5.11}$$

Let us list some properties of $\mathbf{g}$. At first, we have $\mathbf{g}(0, \bar{u}) = 0$. Next we show that

Lemma 5.4. *For each $\varepsilon > 0$ there exist $\delta > 0$ such that for any $\varphi \in C^n[0,l]$ satisfying $\|\varphi\|_\infty \le \dfrac{\delta}{\sqrt{n}}$, we have*

$$\|\mathbf{g}(\varphi, \bar{u})\|_\infty \le \varepsilon \sqrt{n}\, \|\varphi\|_\infty \tag{5.12}$$

and

$$\|\mathbf{g}(\varphi, \bar{u})\|_2 \le \varepsilon \|\varphi\|_2 \,. \tag{5.13}$$

Proof. At first, note that for each $y \in \mathbb{R}^n$

$$\max_{j=1,\dots,n} |y_j| \le |y|_2 \le \sqrt{n} \max_{j=1,\dots,n} |y_j| \tag{5.14}$$

holds. Due to (3.6),(5.2), and (5.11) for any $\varphi \in C^n[0,l]$ we have

$$\begin{cases} |\mathbf{g}(\varphi(x), \bar{u}(x))|_2 \le G(|\varphi(x)|_2), \quad x \in [0,l] \\ \\ \text{with} \quad G(h) = o(h), \quad \text{as} \quad h \to 0+ \,. \end{cases} \tag{5.15}$$

Thus, for each $\varepsilon > 0$ there exist $\delta > 0$ such that for any $\varphi \in C^n[0,l]$ satisfying $|\varphi(x)|_2 \leq \delta \ \ \forall x \in [0,l]$, we have

$$
\begin{aligned}
\|\mathbf{g}(\varphi, \bar{u})\|_\infty &\overset{(5.14)}{\leq} \max_{x\in[0,l]} |\mathbf{g}(\varphi(x), \bar{u}(x))|_2 \\
&\overset{(5.15)}{\leq} \max_{x\in[0,l]} G(|\varphi(x)|_2) \overset{(5.15)}{\leq} \varepsilon \max_{x\in[0,l]} |\varphi(x)|_2 \\
&\overset{(5.14)}{\leq} \varepsilon\sqrt{n} \max_{x\in[0,l]} \max_{j=1,\ldots,n} |\varphi_j(x)| = \varepsilon\sqrt{n}\, \|\varphi\|_\infty ,
\end{aligned}
\tag{5.16}
$$

and

$$
\begin{aligned}
\|\mathbf{g}(\varphi, \bar{u})\|_2 =& \sqrt{\int_0^l |\mathbf{g}(\varphi(x), \bar{u}(x))|_2^2 dx} \overset{(5.15)}{\leq} \sqrt{\int_0^l G^2(|\varphi(x)|_2) dx} \\
&\overset{(5.15)}{\leq} \varepsilon \sqrt{\int_0^l |\varphi(x)|_2^2 dx} = \varepsilon \, \|\varphi\|_2 .
\end{aligned}
\tag{5.17}
$$

Now, if $\|\varphi\|_\infty \leq \dfrac{\delta}{\sqrt{n}}$, then, due to (5.14), $|\varphi(x)|_2 \leq \delta \ \forall x \in [0,l]$. Hence the above inequalities hold for all $\varphi \in C^n[0,l]$ satisfying $\|\varphi\|_\infty \leq \dfrac{\delta}{\sqrt{n}}$. We have shown the assertion. $\square$

Next, having the local existence result in mind, let us impose one further assumption on $\mathbf{g}$. Namely, we assume that for each $R > 0$ there is $\tilde{K}(R, \bar{u}) > 0$ such that

$$
\|\mathbf{g}(x, \bar{u}) - \mathbf{g}(y, \bar{u})\|_2 \leq \tilde{K}(R, \bar{u}) \, \|x - y\|_\infty , \ \ x, y \in B(0, R) \subset C^n[0,l], \ x \neq y. \tag{5.18}
$$

In the previous section we have shown that the operator $-L_{\bar{u}}$ is sectorial in $L_2^n(0,l)$. Therefore, taking condition (5.18) also into account, we see that problem (5.10) satisfies the assumptions made in subsection 2.1.2. Hence, by Theorem 2.10 if $v_0 \in C^n[0,l]$, there exists a mild solution $v \in C_b(I, C^n[0,l])$ to problem (5.10) such that

$$
v(t) = e^{-tL_{\bar{u}}} v_0 + \int_0^t e^{-(t-s)L_{\bar{u}}} \mathbf{g}(v(s), \bar{u}) ds, \quad t \in I, \tag{5.19}
$$

where I is given by (2.18).

We have established an appropriate framework for our further investigations on the stability and the domain of attraction.

5.2 Strategy

In the following we are going to investigate the stability properties of the stationary solution $\bar{u}$. In particular, under some certain conditions we will establish its asymptotic stability and will quantify its domain of attraction. Based on the fact, that the function $\mathbf{g}(v, \bar{u})$ is in general $\mathbf{g}(v, \bar{u}) = O(\|\varphi\|_\infty)$, but not necessarily $\mathbf{g}(v, \bar{u}) = O(\|\varphi\|_2)$, we will have to consider the semigroup $e^{-tL_{\bar{u}}}$ in the space $C^n[0, l]$ (and not in the space $L_2^n(0, l)$). In particular, an upper estimation to $\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])}$ will be of importance for our stability investigations. Therefore, in the following we will be concentrating our attention on the restriction of the operator $L_{\bar{u}}$ to the space $C^n[0, l]$, which will be addressed as $\mathcal{S}$. We are going to show that $-\mathcal{S}$ is a sectorial operator in $C^n[0, l]$. Thus, it generates an analytic semigroup and an estimation of $\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])}$ can be derived. Once the esimation of $\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])}$ is at hand, we will proceed with the stability investigations by using property (5.12) of the function $\mathbf{g}$ and Gronwall's Lemma. As a result, we will establish that under some certain conditions a stationary solution $\bar{u}$ is asymptotically stable and the upper bound to its domain of attraction is computable.

When the operator $-L_{\bar{u}}$ is self-adjoint a different approach for the above stability investigations can be implemented. As a matter of fact it is possible to obtain better results for the domain of attraction, using the technique of expansion into the series of eigenfunctions of $L_{\bar{u}}$, together with the explicit version of the embedding $H_1^n(0, l) \hookrightarrow C^n[0, l]$. In the following we are going to propose two different methods for the verification of the stability and quantification of the domain of attraction in case of self-adjoint $-L_{\bar{u}}$.

Having mentioned that the embedding estimations yield better results for the domain of attraction, let us briefly discuss why they cannot be applied in the general case, when the operator $L_{\bar{u}}$ is not necessarily self-adjoint. Let $\varphi \in H_1^n(0,l)$ and consider the estimate

$$\|\varphi\|_\infty^2 \le C_0 \|\varphi\|_2^2 + C_1 \|\varphi'\|_2^2. \tag{5.20}$$

As we will see later, for the stability investigations the upper bound to $\|\varphi\|_\infty$ is required. Thus, due to (5.20) an upper bound to $\|\varphi'\|_2$ is of interest. This bound can be obtained as follows. By partial integration, we estimate

$$\langle L_{\bar{u}}\varphi, \varphi\rangle_2 \ge d_{\min}\|\varphi'\|_2^2 + \min_{x\in[0,l]} \lambda_{\min}(C_{\bar{u}}^T + \overline{C_{\bar{u}}})\|\varphi\|_2^2.$$

On the other hand, by the Cauchy-Schwarz inequality we have

$$\langle L_{\bar{u}}\varphi, \varphi\rangle_2 \le \|L_{\bar{u}}\varphi\|_2 \|\varphi\|_2. \tag{5.21}$$

Thus, if $\underline{c} := \min\limits_{x\in[0,l]} \lambda_{\min}(C_{\bar{u}}^T + \overline{C_{\bar{u}}}) \le 0$, we obtain

$$\|\varphi'\|_2^2 \le \frac{1}{d_{\min}} \left(\|L_{\bar{u}}\varphi\|_2 \|\varphi\|_2 - \underline{c}\|\varphi\|_2^2\right). \tag{5.22}$$

As one can see from (5.22) the term $\|L_{\bar{u}}\varphi\|_2$ is now under consideration. In the sequel this term gives rise to the term $\|L_{\bar{u}}e^{-tL_{\bar{u}}}\|_2$, which can be estimated only as $\|L_{\bar{u}}e^{-tL_{\bar{u}}}\|_2 \le \frac{C}{t}e^{at}$, for some constants C and a. Thus, it is not useful for the stability investigations, since for $t \to 0$, $\|\varphi\|_\infty$ will not be bounded. The reason for this result lies in estimation (5.21), which is not quite optimal for our purposes. In case of the self-adjoint $L_{\bar{u}}$ it is possible to avoid estimation (5.21) by using the eigenfunction series expansion techniques.

5.3 Results from Chapter 4

Recall from Chapter 4 that for $z \in R$ and $\zeta \in \left(0, \dfrac{\pi}{2}\right)$ the sector $\hat{S}_{\zeta,z}$ is given by

$$\hat{S}_{\zeta,z} := \{\lambda \in \mathbb{C} : |\arg(\lambda - z)| \leq \zeta\}. \tag{5.23}$$

From now on we are going to assume that by the implementation of the methods, which were presented in Chapter 4, namely, by the eigenvalue exclosure procedure and Theorem 4.1 we have shown that the operator $L_{\bar{u}}$ satisfies

(A_0) $\rho(L_{\bar{u}}) \supset \hat{S}^C_{\zeta,z}$ for some given $z \in \mathbb{R}$ and $\zeta \in \left(0, \dfrac{\pi}{2}\right)$.

In addition, estimation (4.33) holds.

5.4 Operator $\mathcal{S}$

We introduce an operator $\mathcal{S} : D_p(\mathcal{S}) \subset C^n[0,l] \to C^n[0,l]$ as

$$\begin{aligned} &D_p(\mathcal{S}) = \{\varphi \in C^n_2[0,l] : B_p[\varphi](0) = B_p[\varphi](l) = 0\}, \\ &\mathcal{S}\varphi = L_{\bar{u}}\varphi. \end{aligned} \tag{5.24}$$

Consequently, it follows

$$R(\lambda, \mathcal{S}) = R(\lambda, L_{\bar{u}})|_{C^n[0,l]}. \tag{5.25}$$

In the following we intend to show that $-\mathcal{S}$ satisfies conditions (i) and (ii), listed in Definition 2.3.

5.4.1 Resolvent set of $\mathcal{S}$

Lemma 5.5. *Let (A_0) be satisfied. Then*

$$\rho(\mathcal{S}) \supset \hat{S}^C_{\zeta,z}. \tag{5.26}$$

Proof. Let λ be an eigenvalue of $\mathcal{S}$ and let $\varphi \in D_p(\mathcal{S})$ be a corresponding eigenelement. From (5.24) follows

$$\lambda\varphi = \mathcal{S}\varphi = L_{\bar{u}}\varphi.$$

Since $D_p(\mathcal{S}) \subset D_p(L_{\bar{u}}) = H_2^B(0,l)$, then $\varphi \in D_p(L_{\bar{u}})$ and λ is an eigenvalue of $L_{\bar{u}}$ as well. Hence, the following implication holds true:

$$\sigma(\mathcal{S}) \subset \sigma(L_{\bar{u}}). \tag{5.27}$$

From (5.27) and assumption (A_0) follows that $\rho(\mathcal{S}) \supset \rho(L_{\bar{u}}) \supset \hat{S}^C_{\zeta,z}$. $\square$

5.4.2 "New" differential equation

Let $\lambda \in \hat{S}^C_{\zeta,z}$. For $\varphi \in H_2^B(0,l)$ and $f \in C^n[0,l]$ we consider a differential equation

$$L_{\bar{u}}\varphi - \lambda\varphi = f. \tag{5.28}$$

In the view of assumption (A_0) problem (5.28) has a unique solution

$$\varphi = R(\lambda, L_{\bar{u}})f.$$

Due to (5.25) we write

$$\varphi = R(\lambda, L_{\bar{u}})f = R(\lambda, \mathcal{S})f. \tag{5.29}$$

Next, let us introduce a constant diagonal matrix $C_0 = \operatorname{diag}(c_1^0, \ldots, c_n^0)$, with $c_j^0 \in \mathbb{R}$, $(j = 1, \ldots, n)$ and with the additional property

$$\min_{j=1,\ldots,n} c_j^0 = z, \tag{5.30}$$

with z from (A_0). In the sequel we will need the following notation

$$(\mathcal{C}_0\varphi)(x) := C_0\varphi(x), \quad x \in [0,l].$$

Now let us write differential equation (5.28) as

$$-A\varphi + \mathcal{C}_0\varphi - \lambda\varphi = f - (\mathcal{C}_{\bar{u}} - \mathcal{C}_0)\varphi$$

and denote

$$\tilde{f} := f - (\mathcal{C}_{\bar{u}} - \mathcal{C}_0)\varphi. \tag{5.31}$$

We want to show the following

Proposition 5.6. *For $\lambda \in \hat{S}^C_{\zeta,z}$, $\varphi \in H^B_2(0,l)$ and $\tilde{f} \in C^n[0,l]$ the differential equation*

$$-A\varphi + \mathcal{C}_0\varphi - \lambda\varphi = \tilde{f} \tag{5.32}$$

is uniquely solvable.

Proof. Observe that the inverse operator $(-A+\mathcal{C}_0-\lambda)^{-1}$ exists for all $\lambda \in \rho(-A+\mathcal{C}_0)$. Since $\lambda \in \hat{S}^C_{\zeta,z}$ then (5.32) is uniquely solvable for all λ if

$$\hat{S}^C_{\zeta,z} \subset \rho(-A + \mathcal{C}_0). \tag{5.33}$$

The spectrum of the operator $-A+\mathcal{C}_0$ with either Dirichlet or Neumann conditions, imposed on the boundary, has the form

$$\lambda^p_{j,k}(-A+\mathcal{C}_0) = \begin{cases} d_j \dfrac{\pi^2 k^2}{l^2} + c^0_j, & \text{if} \quad p = 0, \\ \\ d_j \dfrac{\pi^2 (k-1)^2}{l^2} + c^0_j, & \text{if} \quad p = 1, \end{cases} \qquad j = 1, \ldots, n, \quad k \in \mathbb{N}.$$

By (5.30) condition (5.33) is satisfied. We have shown the assertion. $\square$

In the following, in order to obtain an upper bound to $\|R(\lambda, \mathcal{S})\|_{\mathcal{L}(C^n[0,l])}$, we will shift our focus from problem (5.28) to problem (5.32). Due to the simple form of (5.32) it will be possible to write the solution of this problem using the Green's functions. Consequently we will apply the maximum norm to this solution and obtain the desired results. During this process estimation (4.33) will be applied.

5.4.3 Preliminary results

Before starting with the estimation of $\|R(\lambda,\mathcal{S})\|_{\mathcal{L}(C^n[0,l])}$, let us show several auxiliary results.

Lemma 5.7. *For given $\lambda \in \hat{S}^C_{\zeta,z}$ let $m_j = \mu_j + i\nu_j$ be a complex number such that*

$$m_j^2 = \frac{\lambda - c_j^0}{d_j} \quad (j = 1, \ldots, n). \tag{5.34}$$

Then

$$|\nu_j| \geq |m_j| \sin\left(\frac{\zeta}{2}\right). \tag{5.35}$$

Proof. From

$$m_j^2 = \left(\mu_j^2 - \nu_j^2\right) + 2\mu_j\nu_j i,$$

and (5.34) follows

$$\mathrm{Re}\lambda - c_j^0 = d_j\left(\mu_j^2 - \nu_j^2\right), \tag{5.36}$$

$$\mathrm{Im}\lambda = 2d_j\mu_j\nu_j. \tag{5.37}$$

For all $\lambda \in \hat{S}^C_{\zeta,z}$ we have

$$\mathrm{Re}\lambda - z \leq \cot(\zeta)|\mathrm{Im}\lambda|. \tag{5.38}$$

Hence, by (5.30) due to (5.38), we obtain

$$\mathrm{Re}\lambda - c_j^0 \leq \mathrm{Re}\lambda - z \leq \cot(\zeta)|\mathrm{Im}\lambda|. \tag{5.39}$$

Inserting (5.36) and (5.37) into (5.39), we arrive at

$$\mu_j^2 - \nu_j^2 \leq 2\cot(\zeta)|\mu_j\nu_j|, \quad (j = 1, \ldots, n).$$

Solving the inequality above with respect to $|\nu_j|$, we obtain

$$|\nu_j| \geq \frac{1-\cos(\zeta)}{\sin(\zeta)}|\mu_j| = \tan\left(\frac{\zeta}{2}\right)|\mu_j|.$$

Thus,

$$|m_j| = \sqrt{|\mu_j|^2 + |\nu_j|^2} \leq |\nu_j| \sqrt{1 + \cot^2\left(\frac{\zeta}{2}\right)} = |\nu_j| \frac{1}{\sin\left(\frac{\zeta}{2}\right)},$$

with $\sin\left(\frac{\zeta}{2}\right) > 0$ for $\zeta \in \left(0, \frac{\pi}{2}\right)$. Hence we have obtained the assertion of the lemma. □

Remark 5.8. *Observe that m_j from (5.34) satisfies*

$$|m_j| = \sqrt{\frac{|\lambda - c_j^0|}{d_j}} \quad (j = 1, \ldots, n). \tag{5.40}$$

Proof. We show the assertion by a straightforward computation. Starting with the system of equations

$$\begin{cases} \mathrm{Re}\lambda - c_j^0 = d_j \left(\mu_j^2 - \nu_j^2\right), \\ \\ \mathrm{Im}\lambda = 2 d_j \mu_j \nu_j, \end{cases}$$

we derive

$$\nu_j^2 = \frac{|\lambda - c_j^0| - (\mathrm{Re}\lambda - c_j^0)}{2d_j},$$

$$\mu_j^2 = \frac{|\lambda - c_j^0| + (\mathrm{Re}\lambda - c_j^0)}{2d_j}.$$

Hence the assertion follows. □

We continue with

Lemma 5.9. *Let $\lambda \in \hat{S}^C_{\zeta,z}$. Then*

$$|\lambda - c_j^0| \geq |\lambda - z| \sin(\zeta). \tag{5.41}$$

Proof. Let us recall the definitions of the sets Ω_1 and Ω_2 from (4.13) and (4.14). For convenience we reproduce them here again

$$\Omega_1 := \{\lambda \in \mathbb{C} : 0 < \mathrm{Re}\lambda - z \leq \cot(\zeta)|\mathrm{Im}\lambda|\},$$
$$\Omega_2 := \{\lambda \in \mathbb{C} : \mathrm{Re}\lambda \leq z\}.$$

For all $\lambda \in \Omega_1$, taking into account that $\sin(\zeta) > 0$ for $\zeta \in (0, \frac{\pi}{2})$, we obtain

$$|\lambda - z| = \sqrt{|\mathrm{Re}\lambda - z|^2 + |\mathrm{Im}\lambda|^2} \leq |\mathrm{Im}\lambda|\sqrt{1 + \cot(\zeta)^2} = |\mathrm{Im}\lambda|\frac{1}{\sin(\zeta)}. \tag{5.42}$$

Since $c_j^0 \in \mathbb{R}$ it follows, that

$$|\lambda - c_j^0| \geq |\mathrm{Im}\lambda|. \tag{5.43}$$

Combining (5.42) and (5.43), we obtain for all $\lambda \in \Omega_1$:

$$|\lambda - c_j^0| \geq |\lambda - z| \sin(\zeta). \tag{5.44}$$

For $\lambda \in \Omega_2$, by (5.30), we have

$$\mathrm{Re}\lambda - c_j^0 \leq \mathrm{Re}\lambda - z \leq 0,$$

and therefore $|\mathrm{Re}\lambda - c_j^0|^2 \geq |\mathrm{Re}\lambda - z|^2$. Consequently, for all $\lambda \in \Omega_2$ we obtain

$$|\lambda - c_j^0| = \sqrt{|\mathrm{Re}\lambda - c_j^0|^2 + |\mathrm{Im}\lambda|^2} \geq \sqrt{|\mathrm{Re}\lambda - z|^2 + |\mathrm{Im}\lambda|^2} = |\lambda - z|. \tag{5.45}$$

From (5.44) and (5.45) for all $\lambda \in \hat{S}^C_{z,\zeta}$ we have

$$|\lambda - c_j^0| \geq |\lambda - z| \sin(\zeta).$$

The proof of the lemma is complete. □

Finally, we will need the following

Lemma 5.10. *Let $q \in \mathbb{C}$, $q = a + bi$, $a \in \mathbb{R}$, $b \in \mathbb{R}$. Then for each $y \in \mathbb{R}$ the following inequalities hold.*

$$|\sinh(by)| \leq |\sin(qy)| \leq \cosh(by),$$

$$|\cos(qy)| \leq \cosh(by).$$

Proof. Let us start with

$$|\sin(qy)| = |\sin(ay + biy)| = |\sin(ay)\cosh(by) + i\cos(ay)\sinh(by)|$$

$$= \sqrt{\sin^2(ay)\cosh^2(by) + \cos^2(ay)\sinh^2(by)}$$

$$= \begin{cases} \sqrt{\sin^2(ay) + \sinh^2(by)} \geq |\sinh(by)|, \\ \sqrt{\cosh^2(by) - \cos^2(ay)} \leq \cosh(by). \end{cases}$$

We continue with

$$|\cos(qy)| = |\cos(ay + biy)| = |\cos(ay)\cosh(by) - i\sin(ay)\sinh(by)|$$

$$= \sqrt{\cos^2(ay)\cosh^2(by) + \sin^2(ay)\sinh^2(by)}$$

$$= \sqrt{\cosh^2(by) - \sin^2(ay)} \leq \cosh(by).$$

$\square$

5.4.4 Estimation of $\|R(\lambda, \mathcal{S})\|_{\mathcal{L}(C^n[0,l])}$

At first, for convenience, let us reproduce here estimation (4.33). For $\varphi \in H_2^B(0,l)$ and $f \in L_2^n(0,l)$ we have

$$\|\varphi\|_2 \leq \begin{cases} \dfrac{\widetilde{M}}{|\lambda - z|}\|f\|_2, & \forall\lambda \in \hat{S}^C_{\zeta,z} \cap B^C(z,R), \\ M_{\mathrm{IC}}^{\max}\|f\|_2, & \forall\lambda \in \hat{S}^C_{\zeta,z} \cap B(z,R), \end{cases}$$

where $\widetilde{M}$, $M_{\mathrm{IC}}^{\max}$ and R are given by (4.6), (4.30), and (4.7) respectively. In the sequel we will be using a notation

$$\tilde{C} := C_{\bar{u}} - C_0. \tag{5.46}$$

In addition, for $\zeta \in \left(0, \dfrac{\pi}{2}\right)$, $a, b \in \mathbb{R}$ such that $ab > 0$ and $b \neq 0$ let us introduce a function $P : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$P(a,b) := \sqrt{ab\sin(\zeta)}\tanh\left(\sqrt{\frac{a\sin(\zeta)}{b}}\sin\left(\frac{\zeta}{2}\right)l\right). \tag{5.47}$$

Now we have all necessary machinery in order to introduce the main result of this section.

Theorem 5.11. *Let $\mathcal{S}$ be the linear operator introduced in* (5.24). *Let $\widetilde{M}$, $M_{IC}^{\max}$ and R be given by* (4.6), (4.30), *and* (4.7) *respectively, and z, ζ be the constants introduced in assumption (A_0). Let us denote*

$$\widehat{M} := \frac{1}{\sin(\zeta)\sin\left(\frac{\zeta}{2}\right)},$$

$$Q := \max_{j=1,\dots,n} \frac{\sqrt{\sum_{k=1}^{n}\left\|\tilde{C}_{jk}\right\|_2^2}}{P(R, d_j)}.$$

Then the estimation

$$\|R(\lambda, \mathcal{S})\|_{\mathcal{L}(C^n[0,l])} \leq \frac{M_\infty}{|\lambda - z|} \quad \forall \lambda \in \hat{S}_{\zeta,z}^C \tag{5.48}$$

holds with

$$M_\infty = \widehat{M} + \sqrt{nl}\,Q\max\left(\widetilde{M}, RM_{IC}^{\max}\right). \tag{5.49}$$

Proof. For all $\varphi \in H_2^B(0,l)$ and $\tilde{f} \in C^n[0,l]$ we consider the boundary value problem (5.32):

$$-A\varphi + \mathcal{C}_0\varphi - \lambda\varphi = \tilde{f}.$$

Rewriting the problem above componentwise, we obtain

$$d_j\varphi_j'' + (\lambda - c_j^0)\varphi_j = -\tilde{f}_j, \quad B_p[\varphi_j](0) = B_p[\varphi_j](l) = 0.$$

For $m_j^2 := \dfrac{\lambda - c_j^0}{d_j}$ $(j = 1, \dots, n)$ the corresponding Green's function is represented by a diagonal matrix with elements $G_j^p(x, y)$ of the form[3]

$$G_j^p(x,y) = \begin{cases} \dfrac{1}{d_j m_j \sin(m_j l)} \begin{cases} \sin(m_j(l-y))\sin(m_j x), & x \le y, \\ \sin(m_j(l-x))\sin(m_j y), & x \ge y, \end{cases} & \text{if} \quad p = 0, \\ \\ \dfrac{1}{d_j m_j \sin(m_j l)} \begin{cases} \cos(m_j(l-y))\cos(m_j x), & x \le y, \\ \cos(m_j(l-x))\cos(m_j y), & x \ge y, \end{cases} & \text{if} \quad p = 1. \end{cases} \tag{5.50}$$

Hence, for all $x \in [0, l]$, taking into account (5.31), a solution of (5.32) has the form

$$\varphi(x) = \int_0^l G^p(x,y)\tilde{f}(y)dy = \int_0^l G^p(x,y)f(y)dy - \int_0^l G^p(x,y)\tilde{C}(y)\varphi(y)dy.$$

Therefore for $j = 1, \dots, n$ we obtain

$$\varphi_j(x) = \int_0^l G_j^p(x,y)f_j(y)dy - \int_0^l G_j^p(x,y)\sum_{k=1}^n \tilde{C}_{jk}(y)\varphi_k(y)dy.$$

Thus, it follows

$$|\varphi_j(x)| \le \left|\int_0^l G_j^p(x,y)f_j(y)dy\right| + \left|\int_0^l G_j^p(x,y)\sum_{k=1}^n \tilde{C}_{jk}(y)\varphi_k(y)dy\right|. \tag{5.51}$$

For the convenience of the notation let us introduce for all $j = 1, \dots, n$:

$$T_{1_j}^p(x) := \left|\int_0^l G_j^p(x,y)f_j(y)dy\right|,$$

$$T_{2_j}^p(x) := \left|\int_0^l G_j^p(x,y)\sum_{k=1}^n \tilde{C}_{jk}(y)\varphi_k(y)dy\right|.$$

Hence, from (5.51) follows

$$\begin{aligned} \|\varphi\|_\infty &\le \max_{x\in[0,l]} \max_{j=1,\dots,n} \left(T_{1_j}^p(x) + T_{2_j}^p(x)\right) \\ &\le \max_{x\in[0,l]} \max_{j=1,\dots,n} T_{1_j}^p(x) + \max_{x\in[0,l]} \max_{j=1,\dots,n} T_{2_j}^p(x). \end{aligned} \tag{5.52}$$

[3] This is a standard result from the Green's function theory. For more details see e.g. [34].

We start with the estimation of $\max\limits_{x\in[0,l]}\max\limits_{j=1,\dots,n} T^p_{1_j}(x)$.

$$
\begin{aligned}
T^p_{1_j}(x) &\le \max_{y\in[0,l]}|f_j(y)|\int_0^l \left|G^p_j(x,y)\right| dy \\
&= \max_{y\in[0,l]}|f_j(y)|\left(\int_0^x \left|G^p_j(x,y)\right| dy + \int_x^l \left|G^p_j(x,y)\right| dy\right).
\end{aligned}
$$

Due to (5.50) we obtain

$$
\begin{aligned}
T^0_{1_j}(x) \le \max_{y\in[0,l]}|f_j(y)| \; \frac{1}{|d_j m_j \sin(m_j l)|} \Bigg(&\int_0^x |\sin(m_j(l-x))\sin(m_j y)|\, dy \\
&+ \int_x^l |\sin(m_j(l-y))\sin(m_j x)|\, dy\Bigg),
\end{aligned}
$$

if $p=0$ and

$$
\begin{aligned}
T^1_{1_j}(x) \le \max_{y\in[0,l]}|f_j(y)| \; \frac{1}{|d_j m_j \sin(m_j l)|} \Bigg(&\int_0^x |\cos(m_j(l-x))\cos(m_j y)|\, dy \\
&+ \int_x^l |\cos(m_j(l-y))\cos(m_j x)|\, dy\Bigg),
\end{aligned}
$$

if $p=1$.

In the following we are going to use the results of Lemma 5.10, namely, for each $y\in[0,l]$ we estimate

$$
\begin{aligned}
&|\sin(m_j y)| \le \cosh(\nu_j y), \\
&|\cos(m_j y)| \le \cosh(\nu_j y), \\
&|m_j \sin(m_j y)| \ge |m_j| \;\; |\sinh(\nu_j y)|\,.
\end{aligned}
$$

Thus, for $p = 0, 1$ we obtain

$$
\begin{aligned}
T_{1_j}^p(x) \leq \max_{y\in[0,l]} |f_j(y)| \ \frac{1}{d_j\, |m_j|\, |\sinh(\nu_j l)|} &\left(\int_0^x \cosh(\nu_j(l-x)) \cosh(\nu_j y) dy \right. \\
&\left. + \int_x^l \cosh(\nu_j(l-y)) \cosh(\nu_j x) dy \right) \\
= \max_{y\in[0,l]} |f_j(y)| \ \frac{1}{d_j\, |m_j|\ \ |\sinh(\nu_j l)|} &\left(\frac{1}{\nu_j} \cosh(\nu_j(l-x)) \sinh(\nu_j x) \right. \\
&\left. + \frac{1}{\nu_j} \cosh(\nu_j x) \sinh(\nu_j(l-x)) \right) \\
= \max_{y\in[0,l]} |f_j(y)| \ \frac{1}{d_j\, |m_j|\ \ |\sinh(\nu_j l)|} &\ \frac{\sinh(\nu_j l)}{\nu_j} \\
= \max_{y\in[0,l]} |f_j(y)| \ \frac{1}{d_j\, |m_j|\ \ |\sinh(\nu_j l)|} &\ \frac{|\sinh(\nu_j l)|}{|\nu_j|} = \max_{y\in[0,l]} |f_j(y)| \ \frac{1}{d_j\, |m_j|\ \ |\nu_j|}.
\end{aligned}
$$

After the consecutive application of Lemma 5.7, Remark 5.8, and Lemma 5.9 we arrive at

$$
\frac{1}{d_j\, |m_j|\ \ |\nu_j|} \leq \frac{1}{d_j\, |m_j|^2 \sin\left(\frac{\zeta}{2}\right)} = \frac{1}{\left|\lambda - c_j^0\right| \sin\left(\frac{\zeta}{2}\right)} \leq \frac{1}{|\lambda - z| \sin(\zeta) \sin\left(\frac{\zeta}{2}\right)}.
$$

Thus, $\forall \lambda \in \hat{S}_{\zeta,z}^C$, $x \in [0, l]$

$$
T_{1_j}^p(x) \leq \max_{y\in[0,l]} |f_j(y)| \ \frac{1}{|\lambda - z| \sin(\zeta) \sin\left(\frac{\zeta}{2}\right)}
$$

and

$$
\begin{aligned}
\max_{j=1,\ldots,n} \max_{x\in[0,l]} T_{1_j}^p(x) \leq \max_{j=1,\ldots,n} \max_{y\in[0,l]} |f_j(y)| \ \frac{1}{|\lambda - z| \sin(\zeta) \sin\left(\frac{\zeta}{2}\right)} \\
= \frac{1}{|\lambda - z| \sin(\zeta) \sin\left(\frac{\zeta}{2}\right)} \|f\|_\infty \\
= \frac{\widehat{M}}{|\lambda - z|} \|f\|_\infty . \qquad (5.53)
\end{aligned}
$$

Now we continue with the estimation of $\max_{x\in[0,l]} \max_{j=1,\dots,n} T^p_{2_j}(x)$. Using Hölder's inequality, we obtain

$$\max_{x\in[0,l]} \max_{j=1,\dots,n} T^p_{2_j}(x) \leq \max_{x\in[0,l]} \max_{j=1,\dots,n} \left(\left\| G^p_j(x,\cdot) \right\|_\infty \sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2 \left\| \varphi_k \right\|_2 \right), \tag{5.54}$$

We continue as follows

$$\left\| G^p_j(x,\cdot) \right\|_\infty = \max \left(\max_{y\in[0,x]} \left| G^p_j(x,y) \right|, \max_{y\in[x,l]} \left| G^p_j(x,y) \right| \right).$$

Due to (5.50) and the results of Lemma 5.10, we have

$$\max_{y\in[0,x]} \left| G^p_j(x,y) \right| \leq \max_{y\in[0,x]} \frac{\cosh(\nu_j y)\cosh(\nu_j(l-x))}{d_j\,|m_j|\,|\sinh(\nu_j l)|} \leq \frac{\cosh(\nu_j x)\cosh(\nu_j(l-x))}{d_j\,|m_j|\,|\sinh(\nu_j l)|},$$

$$\max_{y\in[x,l]} \left| G^p_j(x,y) \right| \leq \max_{y\in[x,l]} \frac{\cosh(\nu_j x)\cosh(\nu_j(l-y))}{d_j\,|m_j|\,|\sinh(\nu_j l)|} \leq \frac{\cosh(\nu_j x)\cosh(\nu_j(l-x))}{d_j\,|m_j|\,|\sinh(\nu_j l)|}.$$

Therefore,

$$\max_{x\in[0,l]} \left\| G^p_j(x,\cdot) \right\|_\infty \leq \max_{x\in[0,l]} \frac{\cosh(\nu_j x)\cosh(\nu_j(l-x))}{d_j\,|m_j|\,|\sinh(\nu_j l)|}.$$

The function $h(x) = \cosh(\nu_j x)\cosh(\nu_j(l-x))$ is symmetric with respect to the axis $x = \frac{l}{2}$ and on the interval $[0,l]$ attains its maximum at $x = 0$ and $x = l$. Thus,

$$\max_{x\in[0,l]} \left\| G^p_j(x,\cdot) \right\|_\infty \leq \frac{\cosh(\nu_j l)}{d_j\,|m_j|\,|\sinh(\nu_j l)|} = \frac{1}{d_j\,|m_j|\tanh(|\nu_j|\,l)}.$$

Applying again Remark 5.8 and Lemma 5.9, we arrive at

$$\begin{aligned}
\max_{x\in[0,l]} \left\| G^p_j(x,\cdot) \right\|_\infty &\leq \frac{1}{d_j\,|m_j|\tanh\left(|m_j|\sin\left(\frac{\zeta}{2}\right) l\right)} \\
&= \frac{1}{\sqrt{d_j\left|\lambda - c^0_j\right|}\tanh\left(\sqrt{\frac{\left|\lambda - c^0_j\right|}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)} \\
&\leq \frac{1}{\sqrt{d_j|\lambda - z|\sin(\zeta)}\tanh\left(\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)}.
\end{aligned} \tag{5.55}$$

We continue with the estimation of the term $\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2 \|\varphi_k\|_2$. Applying the Cauchy-Schwarz inequality in $\mathbb{R}^n$, we obtain

$$\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2 \|\varphi_k\|_2 \leq \sqrt{\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2^2} \sqrt{\sum_{k=1}^{n} \|\varphi_k\|_2^2} = \sqrt{\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2^2} \|\varphi\|_2 . \tag{5.56}$$

From (5.54), (5.55), and (5.56) follows

$$\max_{x\in[0,l]} \max_{j=1,\dots,n} T_{2_j}^p(x)$$

$$\leq \max_{j=1,\dots,n} \frac{\sqrt{\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2^2}}{\sqrt{d_j|\lambda - z|\sin(\zeta)} \tanh\left(\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}} \sin\left(\frac{\zeta}{2}\right) l\right)} \|\varphi\|_2 . \tag{5.57}$$

For $\lambda \in \hat{S}_{\zeta,z}^C$, $|\lambda - z| \geq R$ follows

$$\max_{j=1,\dots,n} \frac{\sqrt{\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2^2}}{\sqrt{d_j|\lambda - z|\sin(\zeta)} \tanh\left(\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}} \sin\left(\frac{\zeta}{2}\right) l\right)} \|\varphi\|_2$$

$$\leq \max_{j=1,\dots,n} \frac{\sqrt{\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2^2}}{\sqrt{d_j R\sin(\zeta)} \tanh\left(\sqrt{\frac{R\sin(\zeta)}{d_j}} \sin\left(\frac{\zeta}{2}\right) l\right)} \|\varphi\|_2$$

$$= \max_{j=1,\dots,n} \frac{\sqrt{\sum_{k=1}^{n} \left\| \tilde{C}_{jk} \right\|_2^2}}{P(R, d_j)} \|\varphi\|_2$$

$$= Q \|\varphi\|_2 .$$

Thus, for $\lambda \in \hat{S}^C_{\zeta,z}$, such that $|\lambda - z| \geq R$, we have

$$\max_{x\in[0,l]} \max_{j=1,\dots,n} T^p_{2_j}(x) \leq Q \|\varphi\|_2 . \tag{5.58}$$

Let us introduce for $x \in \mathbb{R}$ a function

$$\hat{\psi}(x) := \frac{\tanh(x)}{x}. \tag{5.59}$$

It is easy to see that $\hat{\psi}(x)$ for $x > 0$ is a decreasing function. It converges to 0 as $x \to \infty$ and converges to 1 as $x \to 0$. Rewriting the expression in (5.57) in terms of function $\hat{\psi}$, we obtain

$$\max_{x\in[0,l]} \max_{j=1,\dots,n} T^p_{2_j}(x)$$

$$\leq \max_{j=1,\dots,n} \frac{\sqrt{\sum_{k=1}^{n} \left\|\tilde{C}_{jk}\right\|_2^2}}{\sqrt{d_j|\lambda - z|\sin(\zeta)}\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\hat{\psi}\left(\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)} \|\varphi\|_2$$

$$= \max_{j=1,\dots,n} \frac{\sqrt{\sum_{k=1}^{n} \left\|\tilde{C}_{jk}\right\|_2^2}}{|\lambda - z|\sin(\zeta)\sin\left(\frac{\zeta}{2}\right) l\hat{\psi}\left(\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)} \|\varphi\|_2 .$$

For $\lambda \in \hat{S}^C_{\zeta,z}$, such that $|\lambda - z| \leq R$ we have

$$\frac{1}{\hat{\psi}\left(\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)} \leq \frac{1}{\hat{\psi}\left(\sqrt{\frac{R\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)}.$$

Therefore

$$\frac{1}{|\lambda - z|\sin(\zeta)\sin\left(\frac{\zeta}{2}\right) l\hat{\psi}\left(\sqrt{\frac{|\lambda - z|\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)}$$

$$\leq \frac{R}{|\lambda - z| R\sin(\zeta)\sin\left(\frac{\zeta}{2}\right) l\hat{\psi}\left(\sqrt{\frac{R\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)}$$

$$= \frac{R}{|\lambda - z|\sqrt{Rd_j\sin(\zeta)}\tanh\left(\sqrt{\frac{R\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right) l\right)} = \frac{R}{|\lambda - z| P(R, d_j)}.$$

Thus, for $\lambda \in \hat{S}^C_{\zeta,z}$, $|\lambda - z| \leq R$, we obtain

$$\begin{aligned} \max_{x\in[0,l]}\max_{j=1,\ldots,n} T^p_{2_j}(x) &\leq \frac{R}{|\lambda - z|}\max_{j=1,\ldots,n}\frac{\sqrt{\sum_{k=1}^{n}\left\|\tilde{C}_{jk}\right\|_2^2}}{P(R, d_j)}\left\|\varphi\right\|_2 \\ &= \frac{RQ}{|\lambda - z|}\left\|\varphi\right\|_2 \end{aligned} \tag{5.60}$$

Combining (5.58) and (5.60), we have

$$\max_{x\in[0,l]}\max_{j=1,\ldots,n} T^p_{2_j}(x) \leq \begin{cases} Q\left\|\varphi\right\|_2, & \forall\lambda \in \hat{S}^C_{\zeta,z}\cap B^C(z,R), \\ \dfrac{RQ}{|\lambda - z|}\left\|\varphi\right\|_2, & \forall\lambda \in \hat{S}^C_{\zeta,z}\cap B(z,R). \end{cases} \tag{5.61}$$

Application of (4.33) to (5.61) results in

$$\max_{x\in[0,l]}\max_{j=1,\ldots,n} T^p_{2_j}(x) \leq \begin{cases} \dfrac{Q\widetilde{M}}{|\lambda - z|}\left\|f\right\|_2, & \forall\lambda \in \hat{S}^C_{\zeta,z}\cap B^C(z,R), \\ \dfrac{RQM^{\max}_{\mathrm{IC}}}{|\lambda - z|}\left\|f\right\|_2, & \forall\lambda \in \hat{S}^C_{\zeta,z}\cap B(z,R). \end{cases}$$

Since $\|f\|_2 \leq \sqrt{nl}\,\|f\|_\infty$, we continue as

$$\max_{x\in[0,l]} \max_{j=1,\dots,n} T^p_{2_j}(x)$$

$$\leq \begin{cases} \dfrac{Q\widetilde{M}\sqrt{nl}}{|\lambda - z|}\,\|f\|_\infty\,, & \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B^C(z,R), \\ \dfrac{RQM^{\max}_{\mathrm{IC}}\sqrt{nl}}{|\lambda - z|}\,\|f\|_\infty\,, & \forall \lambda \in \hat{S}^C_{\zeta,z} \cap B(z,R). \end{cases}$$

Hence for all $\lambda \in \hat{S}^C_{\zeta,z}$ we have

$$\max_{x\in[0,l]} \max_{j=1,\dots,n} T^p_{2_j}(x) \leq Q\sqrt{nl}\max\left(\widetilde{M}, RM^{\max}_{\mathrm{IC}}\right)\frac{\|f\|_\infty}{|\lambda - z|}. \tag{5.62}$$

Finally, combining (5.52) with (5.53) and (5.62), we obtain

$$\|\varphi\|_\infty \leq \left(\widehat{M} + Q\sqrt{nl}\max\left(\widetilde{M}, RM^{\max}_{\mathrm{IC}}\right)\right)\frac{\|f\|_\infty}{|\lambda - z|}.$$

From $\varphi = R(\lambda, \mathcal{S})f$ and (5.49) the assertion of the theorem follows. □

Remark 5.12. *Let j^* be chosen such that $c^0_{j^*} = \min_{j=1,\dots,n} c^0_j = z$. As we will show later for the stability of $\bar{u}$ and the consequent quantification of its domain of attraction the constant z should be positive. In the process of quantification of the domain of attraction we aim at the largest possible upper bound to this domain. For that purpose, as we will see later, the constant M_∞ from* (5.49) *should be kept as small as possible. Therefore, one can try to choose c^0_j, $j \neq j^*$ in such a way that the expression $\sum_{k=1}^{n} \left\|\tilde{C}_{jk}\right\|_2^2$ does not result in a large value. In our computation we have chosen c^0_j, $j \neq j^*$ to be the mean values of $(C_{\bar{u}}(x))_{jk}$ over $x \in [0,l]$.*

5.4.5 Sectoriality of $-\mathcal{S}$

Let λ be an eigenvalue of $\mathcal{S}$. Then $-\lambda$ is an eigenvalue of $-\mathcal{S}$. Thus, from (5.26) and (5.48) follows

(A) $\rho(-\mathcal{S}) \supset S_{\pi-\zeta,-z}$ for some given $z \in \mathbb{R}$ and $\zeta \in \left(0, \frac{\pi}{2}\right)$,

(B) $\|R(\lambda, -\mathcal{S})\|_{\mathcal{L}(C^n[0,l])} \leq \dfrac{M_\infty}{|\lambda + z|}$ $\forall \lambda \in S_{\pi-\zeta,-z}$, with the constant M_∞ from (5.49).

Therefore, the operator $-\mathcal{S}$ satisfies all of the requirements in (2.4) and is sectorial in $C^n[0, l]$. We denote $\theta := \pi - \zeta$.

5.4.6 Computation of the constant C

Since $-\mathcal{S}$ is sectorial, then it generates an analytic semigroup $e^{-t\mathcal{S}}$ and the estimation

$$\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])} \leq Ce^{-tz}, \quad t > 0, \tag{5.63}$$

holds with constant C given by

$$C := \frac{M_\infty}{2\pi}\left(2\int_r^{+\infty} \frac{1}{\rho} e^{\rho\cos\eta} d\rho + \int_{-\eta}^{\eta} e^{r\cos\alpha} d\alpha\right), \quad \eta \in \left(\frac{\pi}{2}, \theta\right), \ r > 0. \tag{5.64}$$

Let us discuss computation of C. We introduce

Definition 5.13. *The $E_1(x)$ function is defined by the integral*

$$E_1(x) = \int_x^\infty \frac{e^{-y}}{y} dy, \quad x > 0.$$

We cannot compute the value of the first integral in (5.64) directly, therefore we have to estimate it from above. Since $\cos(\eta) < 0$ for $\eta \in \left(\frac{\pi}{2}, \theta\right)$, we can represent the first integral in (5.64) as the E_1 function of the positive real argument $r|\cos(\eta)|$:

$$\int_r^{+\infty} \frac{1}{\rho} e^{\rho\cos\eta} d\rho = \int_r^{+\infty} \frac{1}{\rho} e^{-\rho|\cos\eta|} d\rho \overset{\tilde\rho = \rho|\cos\eta|}{=} \int_{r|\cos(\eta)|}^{+\infty} \frac{|\cos(\eta)|}{\tilde\rho} e^{-\tilde\rho} \frac{d\tilde\rho}{|\cos(\eta)|}$$
$$= \int_{r|\cos(\eta)|}^{+\infty} \frac{1}{\tilde\rho} e^{-\tilde\rho} d\tilde\rho = E_1(r|\cos(\eta)|),$$

which could be estimated from above by elementary functions (see [1]) as

$$E_1(r|\cos(\eta)|) < e^{-r|\cos(\eta)|}\ln\left(1+\frac{1}{r|\cos(\eta)|}\right).$$

We obtain the value for the second integral in (5.64) by means of numerical integration. In particular, we apply the trapezoidal rule. Since $\cos(\alpha)$ is an even function, we have

$$\int_{-\eta}^{\eta} e^{r\cos\alpha}d\alpha = 2\int_{0}^{\eta} e^{r\cos\alpha}d\alpha.$$

For any fixed positive r we set

$$f_r(\alpha) := e^{r\cos\alpha}. \tag{5.65}$$

Let N_q denote the number of quadrature points $\tilde{\alpha}_k = \dfrac{\eta k}{N_q}$, $k = 0, \dots, N_q - 1$. Then, according to the trapezoidal rule, we have

$$\int_{-\eta}^{\eta} e^{r\cos\alpha}d\alpha = 2(Q(f_r) + E(f_r)), \tag{5.66}$$

with

$$Q(f_r) = \frac{\eta}{N_q}\left(\frac{1}{2}f_r(\tilde{\alpha}_0) + \sum_{k=1}^{N_q-2} f_r(\tilde{\alpha}_k) + \frac{1}{2}f_r(\tilde{\alpha}_{N_q-1})\right),$$

$$|E(f_r)| \leq \frac{\eta^3}{12N_q^2}\left\|f_r''\right\|_\infty.$$

By a straightforward computation we obtain

$$f_r''(\alpha) = re^{r\cos(\alpha)}(r\sin^2(\alpha) - \cos(\alpha)),$$

and consequently,

$$\|f_r''\|_\infty \leq e^r r(r+1).$$

Choosing large enough N_q, we can make $|E(f_r)|$ sufficiently small. For the verified results we perform our computations in interval arithmetic. Let us denote

$$I(f_r) := 2(Q(f_r) + E(f_r)).$$

Hence, we arrive at the following inequality for C

$$C < \frac{M_\infty}{2\pi}\left(2e^{-r|\cos(\eta)|}\ln\left(1 + \frac{1}{r|\cos(\eta)|}\right) + I(f_r)\right). \tag{5.67}$$

Note that we may adjust the values for η and r as follows: for a fixed value of η we find a numerical approximation $\tilde{r}^*$ for the value r^* at which the expression on the right of (5.67) has its minimum. Thus, we obtain the estimation of the semigroup as

$$\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])} \leq C_\infty e^{-tz} \tag{5.68}$$

with C_∞ given by

$$C_\infty := \frac{M_\infty}{2\pi}\left(2e^{-\tilde{r}^*|\cos(\eta)|}\ln\left(1 + \frac{1}{\tilde{r}^*|\cos(\eta)|}\right) + I(f_{\tilde{r}^*})\right). \tag{5.69}$$

Now let us introduce the following remark.

Remark 5.14. *In the course of our investigations with the hope of obtaining a better result for the domain of attraction we have tried to implement a more direct approach for the computation of the constant C. Let us briefly comment on this approach.*

Let us go back to the proof of the Theorem 5.11 and repeat all the steps of the proof up until the estimation (5.58)*. Further, let us denote*

$$Q_1(|\lambda - z|) = \max_{j=1,\ldots,n} \frac{\sqrt{\sum_{k=1}^{n}\left\|\tilde{C}_{jk}\right\|_2^2}}{\sqrt{d_j|\lambda - z|\sin(\zeta)}\tanh\left(\sqrt{\frac{|\lambda-z|\sin(\zeta)}{d_j}}\sin\left(\frac{\zeta}{2}\right)l\right)}. \tag{5.70}$$

Thus, we obtain

$$\max_{x\in[0,l]}\max_{j=1,\ldots,n} T^p_{2_j}(x) \le \begin{cases} Q\left\|\varphi\right\|_2, & \forall\lambda\in\hat{S}^C_{\zeta,z}\cap B^C(z,R),\\ Q_1(|\lambda-z|)\left\|\varphi\right\|_2, & \forall\lambda\in\hat{S}^C_{\zeta,z}\cap B(z,R).\end{cases}$$

Following the same strategy as earlier, we arrive at

$$\max_{x\in[0,l]}\max_{j=1,\ldots,n} T^p_{2_j}(x) \le \begin{cases} \dfrac{Q\widetilde{M}\sqrt{nl}}{|\lambda-z|}\left\|f\right\|_\infty, & \forall\lambda\in\hat{S}^C_{\zeta,z}\cap B^C(z,R),\\ Q_1(|\lambda-z|)M^{\max}_{IC}\sqrt{nl}\left\|f\right\|_\infty, & \forall\lambda\in\hat{S}^C_{\zeta,z}\cap B(z,R).\end{cases}$$

Hence, we have

$$\left\|R(\lambda,\mathcal{S})\right\|_{\mathcal{L}(C^n[0,l])} \le \begin{cases} \left(\widehat{M}+Q\widetilde{M}\sqrt{nl}\right)\dfrac{1}{|\lambda-z|}, & \forall\lambda\in\hat{S}^C_{\zeta,z}\cap B^C(z,R),\\ \dfrac{\widehat{M}}{|\lambda-z|}+Q_1(|\lambda-z|)M^{\max}_{IC}\sqrt{nl}, & \forall\lambda\in\hat{S}^C_{\zeta,z}\cap B(z,R).\end{cases}$$

Making the transition from $\mathcal{S}$ to $-\mathcal{S}$, we arrive at

$$\begin{aligned}&\left\|R(\lambda,-\mathcal{S})\right\|_{\mathcal{L}(C^n[0,l])}\\ &\le \begin{cases} \left(\widehat{M}+Q\widetilde{M}\sqrt{nl}\right)\dfrac{1}{|\lambda+z|}, & \forall\lambda\in S_{\theta,-z}\cap B^C(-z,R),\\ \dfrac{\widehat{M}}{|\lambda+z|}+Q_1(|\lambda+z|)M^{\max}_{IC}\sqrt{nl}, & \forall\lambda\in S_{\theta,-z}\cap B(-z,R).\end{cases}\end{aligned} \tag{5.71}$$

The above estimation has enabled us to follow a more direct approch for the estimation of constant C. The general idea was it to use the "non-classical" estimation (5.71) *of the resolvent operator directly in the definition of the semigroup* (2.5) *with the hope of obtaining a better result for the constant C. Finally, we have arrived at*

$$\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])} \le \frac{1}{2\pi}F(r,t), \quad t>0,$$

with $F(r,t)$ defined as

$$F(r,t) = \begin{cases} \left(\widehat{M}+Q\widetilde{M}\sqrt{nl}\right)\left(2\int_r^\infty \frac{1}{\rho}e^{\rho\cos(\eta)}d\rho+\int_{-\eta}^{\eta} e^{r\cos(\alpha)}d\alpha\right), & \textit{if} \quad r>Rt,\\ F_1(r,t), & \textit{if} \quad r\le Rt,\end{cases}$$

and the function $F_1(r,t)$ defined as

$$F_1(r,t) := 2\widehat{M}\int_r^{Rt}\frac{1}{\rho}e^{\rho\cos(\eta)}d\rho + 2\sqrt{nl}\,Q_2\int_r^{Rt}\frac{M_{IC}^{\max}e^{\rho\cos(\eta)}}{\sqrt{\rho t\sin(\zeta)d_{\min}}\tanh\left(\sqrt{\frac{\rho\sin(\zeta)}{d_{\min}t}}\sin\left(\frac{\zeta}{2}\right)l\right)}d\rho$$

$$+ 2\left(\widehat{M} + \sqrt{nl}\,Q\widetilde{M}\right)\int_{Rt}^{\infty}\frac{1}{\rho}e^{\rho\cos(\eta)}d\rho + \widehat{M}\int_{-\eta}^{\eta}e^{r\cos(\alpha)}d\alpha$$

$$+ \sqrt{nl}\,Q_2\frac{M_{IC}^{\max}\sqrt{r}}{\sqrt{t\sin(\zeta)d_{\min}}\tanh\left(\sqrt{\frac{r\sin(\zeta)}{d_{\min}t}}\sin\left(\frac{\zeta}{2}\right)l\right)}\int_{-\eta}^{\eta}e^{r\cos(\alpha)}d\alpha,$$

where $Q_2 = \max_{j=1,\ldots,n}\sqrt{\sum_{k=1}^{n}\left\|\tilde{C}_{jk}\right\|_2^2}$. In order to obtain the value of C the estimation of $\max_t\min_r F(r,t)$ was required. This estimation did not result in a better value for the constant C.

5.5 Domain of attraction

Consider problem (5.10). Recall from Chapter 2 that $t_{\max}$ is given by

$$\begin{cases} t_{\max} = \sup\{\tau > 0 : \text{ problem (5.10) has a mild solution } v_\tau \text{ in } [0,\tau]\}, \\ v(t) = v_\tau(t), \quad \text{if} \quad t \le \tau. \end{cases}$$

v is called a maximally defined solution on the interval I given by

$$I := \bigcup\{[0,\tau] : \text{problem (5.10) has a mild solution } u_\tau \text{ in } [0,\tau]\}$$

and we have $t_{\max} = \sup I$. Note that I and $t_{\max}$ depend on v_0, i.e. $I = I(v_0)$ and $t_{\max} = t_{\max}(v_0)$.

Finally, let us introduce

$$\hat{C}_\infty := \max\{C_\infty, 1\}. \tag{5.72}$$

Thus, by (5.68), it follows

$$\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])} \le \hat{C}_\infty e^{-tz} \tag{5.73}$$

Theorem 5.15. *Let $-\mathcal{S}$ be the sectorial operator in $C^n[0,l]$ introduced in* (5.24) *and suppose that z, introduced in assumption (A_0), satisfies $z > 0$. Let $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ satisfy* (5.12)*. Then there exist δ_0, $\hat{C}_\infty > 0$, and $a < 0$ such that if $v_0 \in C^n[0,l]$, $\|v_0\|_\infty < \delta_0$, we have $t_{\max}(v_0) = \infty$ and*

$$\|v(t)\|_\infty \leq \hat{C}_\infty e^{at} \|v_0\|_\infty, \quad t \geq 0. \tag{5.74}$$

The trivial solution of (5.10) *is asymptotically stable.*

Proof. Since $-\mathcal{S}$ is a sectorial operator, it generates an analytic semigroup $e^{-t\mathcal{S}}$. In subsection 5.4.6 we have computed a positive constant C_∞, and consequently (by (5.73)), a positive constant $\hat{C}_\infty$ such that

$$\left\|e^{-t\mathcal{S}}\right\|_{\mathcal{L}(C^n[0,l])} \leq \hat{C}_\infty e^{-tz}, \quad t > 0.$$

Let $\beta > 0$ be some small constant and let us set

$$\varepsilon = \varepsilon_0 := \frac{z}{\sqrt{n}(1+\beta)\hat{C}_\infty}. \tag{5.75}$$

Due to condition (5.12) there exists $\delta(\varepsilon_0)$ with

$$\|\mathbf{g}(v(t), \bar{u})\|_\infty \leq \varepsilon_0 \sqrt{n} \|v(t)\|_\infty \text{ for } v \in C^n[0,l],\ \|v(t)\|_\infty \leq \frac{\delta(\varepsilon_0)}{\sqrt{n}} \text{ and } t \geq 0. \tag{5.76}$$

Now let v be a maximally defined solution of (5.10) on the interval $I(v_0)$, with v_0 satisfying

$$\|v_0\|_\infty < \frac{\delta(\varepsilon_0)}{\sqrt{n}\hat{C}_\infty} =: \delta_0. \tag{5.77}$$

The mild solution v of (5.10) is given by

$$v(t) = e^{-t\mathcal{S}} v_0 + \int_0^t e^{-(t-s)\mathcal{S}} \mathbf{g}(v(s), \bar{u}) ds, \quad t \in I.$$

After applying the $\|\cdot\|_\infty$ to the equation above, we obtain

$$\|v(t)\|_\infty \leq \left\|e^{-t\mathcal{S}}v_0\right\|_\infty + \int_0^t \left\|e^{-(t-s)\mathcal{S}}\mathbf{g}(v(s),\bar{u})\right\|_\infty ds, \quad t \in I.$$

Using (5.73) and (2.7), we estimate

$$\|v(t)\|_\infty \leq \hat{C}_\infty e^{-tz}\|v_0\|_\infty + \int_0^t \hat{C}_\infty e^{-(t-s)z}\|\mathbf{g}(v(s),\bar{u})\|_\infty ds, \quad t \in I.$$

Using (5.76), we obtain

$$\|v(t)\|_\infty \leq \hat{C}_\infty e^{-tz}\|v_0\|_\infty + \int_0^t \hat{C}_\infty e^{-(t-s)z}\varepsilon_0\sqrt{n}\,\|v(s)\|_\infty ds,$$

$$\text{as long as } \|v(t)\|_\infty \leq \frac{\delta(\varepsilon_0)}{\sqrt{n}}, \quad t \in I.$$

Note that since $\hat{C}_\infty \geq 1$, by (5.77) we have $\|v_0\|_\infty < \dfrac{\delta(\varepsilon_0)}{\sqrt{n}\hat{C}_\infty} \leq \dfrac{\delta(\varepsilon_0)}{\sqrt{n}}$.

Let us set $p(t) := e^{tz}\|v(t)\|_\infty$. Then the inequality above reads

$$p(t) \leq \hat{C}_\infty p(0) + \hat{C}_\infty\varepsilon_0\sqrt{n}\int_0^t p(s)ds, \quad \text{as long as } \|v(t)\|_\infty \leq \frac{\delta(\varepsilon_0)}{\sqrt{n}}, \quad t \in I. \tag{5.78}$$

Gronwall's Lemma applied to (5.78) implies

$$p(t) \leq \hat{C}_\infty e^{\hat{C}_\infty\varepsilon_0\sqrt{n}t}p(0), \quad \text{as long as } \|v(t)\|_\infty \leq \frac{\delta(\varepsilon_0)}{\sqrt{n}}, \quad t \in I,$$

and, consequently,

$$\|v(t)\|_\infty \leq \hat{C}_\infty e^{(\hat{C}_\infty\varepsilon_0\sqrt{n}-z)t}\|v_0\|_\infty, \quad \text{as long as } \|v(t)\|_\infty \leq \frac{\delta(\varepsilon_0)}{\sqrt{n}}, \quad t \in I. \tag{5.79}$$

Inserting (5.75) into (5.79), we obtain

$$\|v(t)\|_\infty \leq \hat{C}_\infty e^{at}\|v_0\|_\infty, \quad \text{as long as } \|v(t)\|_\infty \leq \frac{\delta(\varepsilon_0)}{\sqrt{n}}, \quad t \in I, \tag{5.80}$$

with $a := -\frac{\beta z}{1+\beta} < 0$.

Since $\|v_0\|_\infty < \dfrac{\delta(\varepsilon_0)}{\sqrt{n}\hat{C}_\infty}$ (due to (5.77)) from (5.80) follows

$$\|v(t)\|_\infty < \frac{\delta(\varepsilon_0)}{\sqrt{n}}, \quad \text{as long as } \|v(t)\|_\infty \le \frac{\delta(\varepsilon_0)}{\sqrt{n}}, \quad t \in I, \tag{5.81}$$

which implies (by continuity):

$$\|v(t)\|_\infty \le \hat{C}_\infty e^{at} \|v_0\|_\infty \quad \text{for all} \quad t \in I. \tag{5.82}$$

From (5.82) we see that $\|v(t)\|_\infty < \dfrac{\delta(\varepsilon_0)}{\sqrt{n}}$ for all $t \in I$. Therefore, by Theorem 2.12 $t_{\max}(v_0) = \infty$. We have obtained the global existence of the mild solution v and the estimation (5.82) holds for all $t \ge 0$. □

Remark 5.16. *As we see from the proof of Theorem 5.15 the upper bound to the domain of attraction is given by* $\delta_0 := \dfrac{\delta(\varepsilon_0)}{\sqrt{n}\hat{C}_\infty}$. *By Remark 5.1 if the function* G *is known, the value for* $\delta(\varepsilon_0)$ *can be computed from* (5.4) *by setting* $\varepsilon := \varepsilon_0 = \dfrac{z}{\sqrt{n}(1+\beta)\hat{C}_\infty}$. *Hence, having the computable constants* $\hat{C}_\infty$ *and* z *at hand, we can quantify the domain of attraction. Finally, we wish to remark that in our applications* C_∞ *is larger than* 1, *and therefore* $\hat{C}_\infty = C_\infty$.

To conclude this section let us introduce the following remark, concerning the eigenvalue exclosure method and the consequent choice of the constant z from (A_0).

Remark 5.17. *Let us consider* δ_0. *We want the upper bound to the domain of attraction to be as large as possible. Therefore, it is desirable for the constant* $\hat{C}_\infty$, *and, consequently, for the constant* M_∞ *to be as small as possible. Now let us recall the eigenvalue exclosure method presented in Chapter 4: by choosing some appropriate point* μ *in the complex plane, it is possible to obtain the result on a local* *non-existence of the eigenvalues, as it was described in Theorem 4.2. During the exclosure of eigenvalues one observes that as a parameter $\mu \in \mathbb{C}$ approaches the spectrum of the operator $L_{\bar{u}}$, the upper bounds to the norm of the resolvent operator from* (4.29)*, and consequently, the constant M_∞, increase. Thus, in order to avoid the unnecessarily large M_∞, in the course of the eigenvalue exclosure one may consider looking for some $0 < z^* < z$ (given that* (4.23) *holds) such that*

$$M_\infty(z^*) < M_\infty(z),$$

where under $M_\infty(z)$ we formally understand the value of M_∞ computed using z. In the case of the Schnakenberg and predator-prey model we have found such z^*, which has returned better results for the domain of attraction.

5.6 Self-adjoint $L_{\bar{u}}$

In this section we consider a special case of the self-adjoint operator $L_{\bar{u}}$ and propose two approaches for the quantification of the domain of attraction of $\bar{u}$. These approaches are based on the eigenfunction series expansions techniques and embedding estimations of $H_1^n(0,l) \hookrightarrow C^n[0,l]$.

5.6.1 Models with self-adjoint linearisation

Before starting with the description of the approaches mentioned above, let us comment on the classes of problems, which can be treated by these methods. At first recall from Chapter 4, Lemma 4.9 that the operator $L_{\bar{u}}$ is self-adjoint, if and only if the condition

$$C_{\bar{u}}^* = C_{\bar{u}} \tag{5.83}$$

is satisfied. It is obvious, that in case $n = 1$ and $c_{11} \in \mathbb{R}$ condition (5.83) automatically holds. Hence, a single differential equation with real values is a problem with the self-adjoint linearisation. As an example of that case we consider the spruce budworm model.

Our search for the system of the differential equations, which describes a real life situation and satisfies the requirement (5.83) directly, was not quite successful. As a matter of fact, for many reaction-diffusion systems, it is even essential for the non-diagonal elements of $C_{\bar{u}}$ to have different signs (e.g. activator-inhibitor or predator-prey mechanisms). But we were still able to specify special classes of problems, for which the methods we are going to propose will be valid. We introduce them in the following

Proposition 5.18. *Let $C_{\bar{u}} \in \mathbb{R}^{n\times n}$ be a constant matrix. If the elements of $C_{\bar{u}}$ satisfy the following conditions*

1. *For $n = 2$: the non-diagonal elements have the same sign*
2. *For $n = 3$: $\dfrac{c_{ij}}{c_{ji}} > 0$ $(i, j = 1, \dots, 3)$ and $c_{12}c_{23}c_{31} = c_{21}c_{32}c_{13}$*

then there exists a constant diagonal matrix $T \in \mathbb{R}^{n\times n}$, $n = 2, 3$, such that the matrix $\widetilde{C}_{\bar{u}} = T^{-1}C_{\bar{u}}T$ is symmetric.

Proof. After direct computation we obtain

$$\tilde{c}_{ij} = \frac{c_{ij}t_j}{t_i}, \quad i, j = 1, \dots, n,$$

where $\tilde{c}_{ij}$, t_i $(i, j = 1, \dots, n)$ denote the elements of matrices $\widetilde{C}_{\bar{u}}$ and T respectively. Therefore the matrix $\widetilde{C}_{\bar{u}}$ is symmetric if and only if

$$\frac{c_{ij}t_j}{t_i} = \frac{c_{ji}t_i}{t_j},$$

and hence, if and only if

$$\frac{t_i}{t_j} = \sqrt{\frac{c_{ij}}{c_{ji}}}, \quad i, j = 1, \dots, n \tag{5.84}$$

holds. For n = 2 it immediately follows that in order for $\frac{t_1}{t_2}$ to exist the non-diagonal elements of matrix $C_{\bar{u}}$ must have the same sign and vice versa.

Let us consider the case when n = 3. From (5.84) follows

$$\frac{t_1}{t_2} = \sqrt{\frac{c_{12}}{c_{21}}},$$

$$\frac{t_3}{t_1} = \sqrt{\frac{c_{31}}{c_{13}}},$$

$$\frac{t_2}{t_3} = \sqrt{\frac{c_{23}}{c_{32}}}.$$

Above equations imply: $\frac{c_{ij}}{c_{ji}} > 0$ $(i, j = 1, \ldots, 3)$ and

$$1 = \frac{t_1 t_3 t_2}{t_2 t_1 t_3} = \sqrt{\frac{c_{12} c_{31} c_{23}}{c_{21} c_{13} c_{32}}}. \tag{5.85}$$

Thus, from (5.84) the conditions on $C_{\bar{u}}$, which were listed in item 2 follow. Now let $\frac{c_{ij}}{c_{ji}} > 0$ $(i, j = 1, \ldots, 3)$ and

$$c_{12} c_{23} c_{31} = c_{21} c_{32} c_{13}. \tag{5.86}$$

Since $\frac{c_{ij}}{c_{ji}} > 0$ $(i, j = 1, \ldots, 3)$ from (5.86) we have

$$0 < \frac{c_{12} c_{23} c_{31}}{c_{21} c_{32} c_{13}} = 1. \tag{5.87}$$

Consequently, we obtain

$$\sqrt{\frac{c_{12} c_{23} c_{31}}{c_{21} c_{32} c_{13}}} = 1 = \frac{t_1 t_2 t_3}{t_1 t_2 t_3}. \tag{5.88}$$

Finally, from (5.88) we choose

$$\frac{t_1}{t_2} = \sqrt{\frac{c_{12}}{c_{21}}},$$

$$\frac{t_3}{t_1} = \sqrt{\frac{c_{31}}{c_{13}}},$$

$$\frac{t_2}{t_3} = \sqrt{\frac{c_{23}}{c_{32}}}.$$

Thus, the assertion of the proposition follows. $\square$

In the proposition that follows we would like to demonstrate how the transformation T can be introduced into problem (5.10) in case when the corresponding linearised operator $L_{\bar{u}}$ is non-self-adjoint. Clearly, after the transformation we will be aiming at a self-adjoint $\tilde{L}_{\bar{u}}$. Additionaly, in order to apply the methods we developed in the thesis, we need for a corresponding nonlinear part of the problem after the transformation, which we will denote as $\tilde{g}(w(t),\bar{u})$,to be "small" for "small" $w(t)$. Hence, $\tilde{g}(w(t),\bar{u})$ must satisfy some conditions, which are similar to conditions (5.12) and (5.13).

Proposition 5.19. *Let us consider Cauchy problem* (5.10)*. Suppose that $\bar{u}$ is a constant stationary solution of* (5.10)*. Let the operator $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ which is given by*

$$L_{\bar{u}}\varphi(x) = -D\varphi''(x) + C_{\bar{u}}\varphi(x), \quad x \in [0,l],$$

be non-self-adjoint with the constant matrix $C_{\bar{u}}$, satisfying conditions of Proposition 5.18. Then (5.10) *can be transformed into a Cauchy problem of the form*

$$\begin{cases} w'(t) = -\tilde{L}_{\bar{u}}w(t) + \tilde{g}(w(t),\bar{u}), & t > 0, \\ w(0) = w_0, \end{cases} \tag{5.89}$$

where the operator $\tilde{L}_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ *is self-adjoint and the nonlinearity* $\tilde{g}$: $C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ *satisfies the following conditions: for each* $\bar{\varepsilon} > 0$ *there exists* $\bar{\delta} > 0$ *such that for any* $w(t) \in C^n[0,l]$ *satisfying* $\|w(t)\|_\infty \le \bar{\delta}$ *follows*

$$\|\tilde{g}(w(t), \bar{u})\|_\infty \le \bar{\varepsilon} \|w(t)\|_\infty \tag{5.90}$$

and

$$\|\tilde{g}(w(t), \bar{u})\|_2 \le \bar{\varepsilon} \|w(t)\|_2 . \tag{5.91}$$

Proof. Let us introduce the T-transformation from Proposition 5.18 into problem (5.10). For this purpose we rewrite (5.10) as

$$v'(t) = Av(t) + \mathbf{F}(v(t) + \bar{u}) - \mathbf{F}(\bar{u}) \tag{5.92}$$

and define

$$w(t) := T^{-1}v(t). \tag{5.93}$$

Applying T^{-1} from both sides of (5.92), we obtain

$$\begin{aligned} w'(t) &= T^{-1}Av(t) + T^{-1}[\mathbf{F}(v(t) + \bar{u}) - \mathbf{F}(\bar{u})] \\ &= T^{-1}ATw(t) + T^{-1}[\mathbf{F}(Tw(t) + \bar{u}) - \mathbf{F}(\bar{u})] \\ &= Aw(t) - \widetilde{C}_{\bar{u}}w(t) + T^{-1}[\mathbf{F}(Tw(t) + \bar{u}) - \mathbf{F}(\bar{u})] + \widetilde{C}_{\bar{u}}w(t) \\ &= -\widetilde{L}_{\bar{u}}w(t) + \tilde{g}(w(t), \bar{u}), \end{aligned}$$

with

$$\widetilde{L}_{\bar{u}}w(t) := -Aw(t) + \widetilde{C}_{\bar{u}}w(t), \tag{5.94}$$

$$\tilde{g}(w(t), \bar{u}) := T^{-1}[\mathbf{F}(Tw(t) + \bar{u}) - \mathbf{F}(\bar{u})] + \widetilde{C}_{\bar{u}}w(t). \tag{5.95}$$

By Proposition 5.18 the matrix $\widetilde{C}_{\bar{u}}$ is symmetric. Therefore, by Lemma 4.9 the operator $\widetilde{L}_{\bar{u}}$ is self-adjoint.

Let us show the properties (5.90) and (5.91). Observe that

$$T\tilde{g}(w(t), \bar{u}) = \mathbf{g}(v(t), \bar{u}).$$

Thus, we obtain

$$\tilde{g}(w(t),\bar{u}) = T^{-1}\mathbf{g}(v(t),\bar{u}). \tag{5.96}$$

Recall that function $\mathbf{g}$ satisfies conditions (5.12) and (5.13). Thus, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $v(t) \in C^n[0,l]$ satisfying $\|v(t)\|_\infty \le \frac{\delta}{\sqrt{n}}$, $t \ge 0$ follows

$$\begin{aligned}\|\tilde{g}(w(t),\bar{u})\|_\infty = \left\|T^{-1}\mathbf{g}(v(t),\bar{u})\right\|_\infty &\overset{(5.12)}{\le} \varepsilon\sqrt{n}\left\|T^{-1}\right\|_\infty \|v(t)\|_\infty \\ &\overset{(5.93)}{\le} \varepsilon\sqrt{n}\left\|T^{-1}\right\|_\infty \|T\|_\infty \|w(t)\|_\infty,\end{aligned}$$

and

$$\|\tilde{g}(w(t),\bar{u})\|_2 = \left\|T^{-1}\mathbf{g}(v(t),\bar{u})\right\|_2 \overset{(5.13)}{\le} \varepsilon\left\|T^{-1}\right\|_2 \|v(t)\|_2 \overset{(5.93)}{\le} \varepsilon\left\|T^{-1}\right\|_2 \|T\|_2 \|w(t)\|_2.$$

Both inequalities above hold as long as $\|Tw(t)\|_\infty = \|v(t)\|_\infty \le \frac{\delta}{\sqrt{n}}$, $t \ge 0$. Thus, they hold for any $w(t) \in C^n[0,l]$ such that $\|w(t)\|_\infty \le \|T\|_\infty^{-1}\frac{\delta}{\sqrt{n}}$, $t \ge 0$. Now set $\bar{\delta} = \|T\|_\infty^{-1}\frac{\delta}{\sqrt{n}}$ and set $\bar{\varepsilon}$ as either $\bar{\varepsilon} = \varepsilon\sqrt{n}\,\|T^{-1}\|_\infty \|T\|_\infty$ for estimation (5.90), or $\bar{\varepsilon} = \varepsilon\,\|T^{-1}\|_2 \|T\|_2$ for estimation (5.91). We have shown the assertion. □

Thus, we were able to transform the initial value problem (5.10), formulated for the non-self-adjoint operator $L_{\bar{u}}$, into the initial value problem (5.89), which corresponds to the self-adjoint linearisation at $\bar{u}$ and contains nonlinearity $\tilde{g}$, satisfying conditions (5.90), (5.91).

As one can see the above transformation is valid only in the case, when matrix $C_{\bar{u}}$ is a *constant* matrix, that is when the *constant* stationary solution $\bar{u}$ is under consideration. Thinking about the models of biological interaction, which could serve as a good candidates for the above transformation, one can certainly point out the symbiosis and competition models. In our work we apply our results to the competition model.

Later we will comment on the possible extensions of the classes above.

5.6.2 *Preliminary results*

Before starting with the formulation of the main results of this section, we need some preliminary information.

Let $\{\tilde{\varphi}_k\}_{k=1}^{\infty}$ be an orthonormal basis of $L_2^n(0,l)$, where $\tilde{\varphi}_k$ is an eigenelement of $L_{\bar{u}}$, corresponding to the eigenvalue λ_k (see Proposition 4.10). A function $f \in L_2^n(0,l)$ could be represented as

$$f = \sum_{k=1}^{\infty} \langle f, \tilde{\varphi}_k \rangle_2 \, \tilde{\varphi}_k, \tag{5.97}$$

where the series is convergent in $L_2^n(0,l)$. In addition,

$$\|f\|_2^2 = \sum_{k=1}^{\infty} | \langle f, \tilde{\varphi}_k \rangle_2 |^2. \tag{5.98}$$

Further, let us introduce a notation

$$H_1^B(0,l) = \begin{cases} (H_0^1(0,l))^n, & \text{if} \quad p = 0, \\ \\ H_1^n(0,l), & \text{if} \quad p = 1. \end{cases} \tag{5.99}$$

Finally, recall from Chapter 4 that we use the notation λ_1^C as

$$\lambda_1^C := \min_{x \in [0,l]} \lambda_{\min}(C_{\bar{u}}(x)),$$

and choose the positive constant σ so that

$$\lambda_1^C + \sigma > 0. \tag{5.100}$$

We continue with the following

Lemma 5.20. *Let $f \in H_1^B(0,l)$. Then for any constant σ satisfying* (5.100), *the following identity holds*

$$\int_0^l \Big(f'(x)^T D \overline{f'(x)} + f(x)^T (C_{\bar{u}}(x)^T + \sigma I) \overline{f(x)} \Big) \, dx = \sum_{k=1}^{\infty} (\lambda_k + \sigma) | \langle f, \tilde{\varphi}_k \rangle_2 |^2 \tag{5.101}$$

Proof. Let $g \in H_2^B(0,l)$ and $f \in H_1^B(0,l)$. Partial integration, paying regard to the boundary conditions, yields

$$\begin{aligned}\langle (L_{\bar{u}} + \sigma I)g, f\rangle_2 &= \int_0^l \left(-g''(x)^T D\overline{f(x)} + g(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)}\right) dx \\ &= \int_0^l \left(g'(x)^T D\overline{f'(x)} + g(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)}\right) dx.\end{aligned}$$

On the other hand, using the representation (5.97), the self-adjointness of the operator $L_{\bar{u}}$, and (4.40), we obtain

$$\begin{aligned}\langle (L_{\bar{u}} + \sigma I)g, f\rangle_2 &= \sum_{k=1}^{\infty} \langle (L_{\bar{u}} + \sigma I)g, \tilde{\varphi}_k\rangle_2 \overline{\langle f, \tilde{\varphi}_k\rangle_2} \\ &= \sum_{k=1}^{\infty} \langle g, (L_{\bar{u}} + \sigma I)\tilde{\varphi}_k\rangle_2 \overline{\langle f, \tilde{\varphi}_k\rangle_2} \\ &= \sum_{k=1}^{\infty} (\lambda_k + \sigma) \langle g, \tilde{\varphi}_k\rangle_2 \overline{\langle f, \tilde{\varphi}_k\rangle_2}.\end{aligned}$$

Hence, for each $g \in H_2^B(0,l)$ and $f \in H_1^B(0,l)$ we have

$$\begin{aligned}&\int_0^l \left(g'(x)^T D\overline{f'(x)} + g(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)}\right) dx \\ &\qquad = \sum_{k=1}^{\infty} (\lambda_k + \sigma) \langle g, \tilde{\varphi}_k\rangle_2 \overline{\langle f, \tilde{\varphi}_k\rangle_2}. \qquad (5.102)\end{aligned}$$

Since $H_2^B(0,l)$ is dense in $H_1^B(0,l)$, we can choose a sequence $(f_n)_{n\in\mathbb{N}} \in H_2^B(0,l)$, which converges to $f \in H_1^B(0,l)$, as $n \to \infty$. Let us consider (5.102) with f_n instead of g. We obtain

$$\begin{aligned}&\int_0^l \left(f_n'(x)^T D\overline{f'(x)} + f_n(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)}\right) dx \\ &\qquad = \sum_{k=1}^{\infty} (\lambda_k + \sigma) \langle f_n, \tilde{\varphi}_k\rangle_2 \overline{\langle f, \tilde{\varphi}_k\rangle_2}. \qquad (5.103)\end{aligned}$$

Since $f_n \to f$ in $H_1^B(0,l)$ as $n \to \infty$, the left-hand side of (5.103) converges to $\int_0^l \Big(f'(x)^T D\overline{f'(x)} + f(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)} \Big)\, dx$ in $H_1^B(0,l)$, as $n \to \infty$. Now let us consider the right-hand side of (5.103).

At first, let us introduce for $f,\ h \in H_1^B(0,l)$ the following norm (which is equivalent to $H_1^B(0,l)$ norm) and the scalar product:

$$|||f||| := \left(\int_0^l \Big(f'(x)^T D\overline{f'(x)} + f(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)} \Big)\, dx \right)^{\frac{1}{2}},$$

$$\langle\langle f, h \rangle\rangle := \int_0^l \Big(f'(x)^T D\overline{h'(x)} + f(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{h(x)} \Big)\, dx.$$

Now let us set

$$\psi_k := \frac{1}{\sqrt{\lambda_k + \sigma}} \tilde{\varphi}_k, \quad k \in \mathbb{N}.$$

The functions $\{\psi_k\}_{k \in \mathbb{N}}$ build an orthonormal basis in $(H_1^B(0,l), \langle\langle \cdot, \cdot \rangle\rangle)$. Indeed, for all $j, k \in \mathbb{N}$ we have

$$\begin{aligned}
\langle\langle \psi_k, \psi_j \rangle\rangle &= \frac{1}{\sqrt{(\lambda_k + \sigma)(\lambda_j + \sigma)}} \int_0^l \Big(\tilde{\varphi}_k'(x)^T D\overline{\tilde{\varphi}_j'(x)} + \tilde{\varphi}_k(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{\tilde{\varphi}_j(x)} \Big)\, dx \\
&= \frac{1}{\sqrt{(\lambda_k + \sigma)(\lambda_j + \sigma)}} \langle \tilde{\varphi}_k, (L_{\bar{u}} + \sigma I)\tilde{\varphi}_j \rangle_2
\end{aligned}$$

$$= \frac{1}{\sqrt{(\lambda_k+\sigma)(\lambda_j+\sigma)}}(\lambda_j+\sigma)\langle\tilde{\varphi}_k,\tilde{\varphi}_j\rangle_2$$

$$= \sqrt{\frac{(\lambda_j+\sigma)}{(\lambda_k+\sigma)}}\delta_{kj} = \delta_{kj}. \tag{5.104}$$

Further, the set $\{\psi_k\}_{k\in\mathbb{N}}$ is complete in $(H_1^B(0,l),\langle\langle\cdot,\cdot\rangle\rangle)$. Let $v \in H_1^B(0,l)$, such that $v\perp^{\langle\langle\cdot,\cdot\rangle\rangle}\psi_j$, $\forall j$. Then we have

$$\begin{aligned}0 =&\langle\langle v,\psi_j\rangle\rangle = \int_0^l \left(v'(x)^T D\overline{\psi_j'(x)} + v(x)^T(C_{\bar{u}}(x)^T+\sigma I)\overline{\psi_j(x)}\right)dx\\ &= \langle v,(L_{\bar{u}}+\sigma I)\psi_j\rangle_2 = (\lambda_j+\sigma)\langle v,\psi_j\rangle_2 = \sqrt{\lambda_j+\sigma}\langle v,\tilde{\varphi}_j\rangle_2.\end{aligned}$$

Since the set $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}}$ is complete in $L_2^n(0,l)$, it follows that $v\equiv 0$. Therefore, $\{\psi_k\}_{k\in\mathbb{N}}$ is the orthonormal basis of $(H_1^B(0,l),\langle\langle\cdot,\cdot\rangle\rangle)$.

Thus, for each $v \in H_1^B(0,l)$ we have

$$\begin{aligned}|||v|||^2 = \sum_{k=1}^{\infty}|\langle\langle v,\psi_k\rangle\rangle|^2 &= \sum_{k=1}^{\infty}\left|\int_0^l\left(v'(x)^T D\overline{\psi_k'(x)} + v(x)^T(C_{\bar{u}}(x)^T+\sigma I)\overline{\psi_k(x)}\right)dx\right|^2\\ &= \sum_{k=1}^{\infty}|\langle v,(L_{\bar{u}}+\sigma I)\psi_k\rangle_2|^2 = \sum_{k=1}^{\infty}\left|\sqrt{\lambda_k+\sigma}\langle v,\tilde{\varphi}_k\rangle_2\right|^2\\ &= \sum_{k=1}^{\infty}(\lambda_k+\sigma)\left|\langle v,\tilde{\varphi}_k\rangle_2\right|^2\end{aligned} \tag{5.105}$$

Now let us go back to (5.103). By the Cauchy-Schwarz inequality and (5.105) we estimate the following difference

$$\begin{aligned}&\left|\sum_{k=1}^{\infty}(\lambda_k+\sigma)(\langle f_n,\tilde{\varphi}_k\rangle_2 - \langle f,\tilde{\varphi}_k\rangle_2)\overline{\langle f,\tilde{\varphi}_k\rangle_2}\right|\\ &\qquad\le \sqrt{\sum_{k=1}^{\infty}(\lambda_k+\sigma)|\langle f_n-f,\tilde{\varphi}_k\rangle_2|^2}\sqrt{\sum_{k=1}^{\infty}(\lambda_k+\sigma)|\langle f,\tilde{\varphi}_k\rangle_2|^2}\\ &\qquad= |||f_n-f|||\;|||f|||.\end{aligned} \tag{5.106}$$

As $n \to \infty$ the difference $f_n - f \to 0$ in $H_1^B(0,l)$, and therefore, due to the norm equivalence, the right-hand side of (5.106) converges to zero. Thus, the right-hand side of (5.103) converges to $\sum_{k=1}^{\infty}(\lambda_k + \sigma)|\langle f, \tilde{\varphi}_k\rangle_2|^2$, as $n \to \infty$. Therefore, for $f \in H_1^B(0,l)$, we obtain

$$\int_0^l \left(f'(x)^T D\overline{f'(x)} + f(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)}\right) dx = \sum_{k=1}^{\infty}(\lambda_k + \sigma)|\langle f, \tilde{\varphi}_k\rangle_2|^2.$$

The proof of lemma is complete. □

For simplicity, we denote

$$q(f) := \left(\int_0^l \left(f'(x)^T D\overline{f'(x)} + f(x)^T (C_{\bar{u}}(x)^T + \sigma I)\overline{f(x)}\right) dx\right)^{\frac{1}{2}}. \tag{5.107}$$

Thus, by Lemma 5.20 we have

$$q(f) = \left(\sum_{k=1}^{\infty}(\lambda_k + \sigma)|\langle f, \tilde{\varphi}_k\rangle_2|^2\right)^{\frac{1}{2}}. \tag{5.108}$$

For our further investigations the connection between $\|\varphi\|_\infty$ and $q(\varphi)$ will be important. We continue with the following

Lemma 5.21. *Let $\varphi \in H_1^B(0,l)$. The inequality*

$$\|\varphi\|_\infty \le \sqrt{\frac{C_1}{d_{\min}}}q(\varphi) \tag{5.109}$$

is satisfied with:

1.

$$C_1 = \frac{1}{\rho}, \quad \rho = \frac{1}{2l}\left(\sqrt{1 + \frac{4l^2(\lambda_1^C + \sigma)}{d_{\min}}} - 1\right). \tag{5.110}$$

2. If $\varphi \in (H_0^1(0,l))^n$, then

$$C_1 = \begin{cases} \dfrac{l}{4}, & \text{if} \quad d_{\min} \geq \dfrac{l^2}{4}(\lambda_1^C + \sigma), \\ \dfrac{\sqrt{d_{\min}}}{2\sqrt{\lambda_1^C + \sigma}}, & \text{otherwise.} \end{cases} \tag{5.111}$$

Proof. At first, let us consider estimation (2.27) (or (2.30)) with C_0, C_1 chosen as in (2.28) (or as in (2.32)). It follows

$$\begin{aligned} d_{\min} \|\varphi\|_\infty^2 &\leq d_{\min} C_0 \|\varphi\|_2^2 + d_{\min} C_1 \|\varphi'\|_2^2 \\ &= d_{\min} C_0 \|\varphi\|_2^2 + d_{\min} C_1 \|\varphi'\|_2^2 \\ &\quad + C_1 \left(\lambda_1^C + \sigma\right) \|\varphi\|_2^2 - C_1 \left(\lambda_1^C + \sigma\right) \|\varphi\|_2^2 . \end{aligned}$$

Thus if

$$d_{\min} C_0 - C_1 \left(\lambda_1^C + \sigma\right) = 0, \tag{5.112}$$

then

$$d_{\min} \|\varphi\|_\infty^2 \leq d_{\min} C_1 \|\varphi'\|_2^2 + C_1 \left(\lambda_1^C + \sigma\right) \|\varphi\|_2^2 \leq C_1 q^2(\varphi).$$

Let us rewrite condition (5.112) with C_0, C_1 chosen as in (2.28). We obtain

$$d_{\min} \left(\rho + \frac{1}{l}\right) - \frac{1}{\rho}\left(\lambda_1^C + \sigma\right) = \frac{d_{\min}\rho^2 + \frac{1}{l} d_{\min}\rho - \left(\lambda_1^C + \sigma\right)}{\rho} = 0.$$

Solving the equation above with respect to ρ, we obtain

$$\rho = \frac{1}{2l}\left(\sqrt{1 + \frac{4l^2(\lambda_1^C + \sigma)}{d_{\min}}} - 1\right).$$

Hence (5.109) holds with C_1 as in (5.110). In order to obtain (5.111) we proceed the similar way. Let us rewrite condition (5.112) with C_0, C_1 as in (2.32). We obtain

$$d_{\min}\frac{\rho}{2} - \frac{1}{2\rho}(\lambda_1^C + \sigma) = 0.$$

Hence

$$\rho = \sqrt{\frac{\lambda_1^C + \sigma}{d_{\min}}}$$

and (5.109) is satisfied with C_1 as in the second case of (5.111). Now let us consider (2.30) with C_0, C_1 as in (2.31). It follows

$$d_{\min} \|\varphi\|_\infty^2 \le d_{\min} C_1 \|\varphi'\|_2^2 \le C_1 q^2(\varphi).$$

Hence in that case (5.109) holds with $C_1 = \frac{l}{4}$. We obtain the estimation (5.111) by distinguishing between the choices (2.31) and (2.32) with respect to the smaller value of $\sqrt{\frac{C_1}{d_{\min}}}$. $\square$

5.6.3 Computation of the constant C_{L_2}

In Chapter 4, Proposition 4.13 we have shown that the self-adjoint operator $-L_{\bar{u}}$ is sectorial in $L_2^n(0,l)$. Therefore it generates an analytic semigroup $e^{-tL_{\bar{u}}}$ and there exist $C > 0$ such that

$$\left\|e^{-tL_{\bar{u}}}\right\|_{\mathcal{L}(L_2^n(0,l))} \le Ce^{-tz},$$

where z is the constant from assumption (A_0). Here we would like to briefly comment on the computation of the constant C. As matter of fact, C will be computed exaclty the same way as it was described in the subsection 5.4.6 for the general $L_{\bar{u}}$. In particular, we use (5.69). The only difference now is that in (5.69) instead of the constant M_∞ we use the constant M_{L_2}, which was introduced in (4.48). Thus, we obtain the following estimation

$$\left\|e^{-tL_{\bar{u}}}\right\|_2 \le C_{L_2} e^{-tz}, \tag{5.113}$$

with

$$C_{L_2} := \frac{M_{L_2}}{2\pi} \left(2e^{-\tilde{r}^*|\cos(\eta)|} \ln\left(1 + \frac{1}{\tilde{r}^*|\cos(\eta)|}\right) + I(f_{\tilde{r}^*})\right). \tag{5.114}$$

5.6.4 First result on the domain of attraction

Let us consider problem (5.10). As earlier, $t_{\max}(v_0)$ and the time interval $I(v_0)$ are given by (2.17) and (2.18) respectively. At first let us introduce one preliminary result.

Proposition 5.22. *Let $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ be the sectorial operator introduced in* (4.1)*, and suppose that constant z, introduced in assumption (A_0), satisfies $z > 0$. Let function $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ satisfy* (5.13)*. Then there exist δ_1, $C_{L_2} > 0$, and $a < 0$ such that if $v(t) \in C^n[0,l]$, $\|v(t)\|_\infty \le \delta_1$, $t \in I(v_0)$, then we have*

$$\|v(t)\|_2 \le C_{L_2} e^{at} \|v_0\|_2 , \quad t \in I(v_0).$$

Proof. Let β be some small positive constant and let us set

$$\varepsilon = \varepsilon_1 := \frac{z}{(1+\beta)C_{L_2}}, \tag{5.115}$$

with C_{L_2} from (5.113).

According to condition (5.13) there exists $\delta(\varepsilon_1)$ with

$$\|\mathbf{g}(v(t),\bar{u})\|_2 \le \varepsilon_1 \|v(t)\|_2$$
$$\text{for } v \in C^n[0,l],\ \|v(t)\|_\infty \le \frac{\delta(\varepsilon_1)}{\sqrt{n}} =: \delta_1 \text{ and } t \ge 0. \tag{5.116}$$

The mild solution v of (5.10) is given by

$$v(t) = e^{-tL_{\bar{u}}} v_0 + \int_0^t e^{-(t-s)L_{\bar{u}}} \mathbf{g}(v(s),\bar{u}) ds, \quad t \in I.$$

Applying the $\|\cdot\|_2$ to the equation above, we obtain

$$\|v(t)\|_2 \le \left\|e^{-tL_{\bar{u}}} v_0\right\|_2 + \int_0^t \left\|e^{-(t-s)L_{\bar{u}}} \mathbf{g}(v(s),\bar{u})\right\|_2 ds, \quad t \in I.$$

Using (5.113) and (2.7), we have

$$\|v(t)\|_2 \le C_{L_2} e^{-tz} \|v_0\|_2 + \int_0^t C_{L_2} e^{-(t-s)z} \|\mathbf{g}(v(s), \bar{u})\|_2 \, ds, \quad t \in I.$$

Using (5.116), we obtain

$$\|v(t)\|_2 \le C_{L_2} e^{-tz} \|v_0\|_2 + \int_0^t C_{L_2} e^{-(t-s)z} \varepsilon_1 \|v(s)\|_2 \, ds,$$

$$\text{as long as} \quad \|v(t)\|_\infty \le \delta_1 \quad t \in I.$$

Let us set $p(t) := e^{tz} \|v(t)\|_2$. Then the inequality above reads

$$p(t) \le C_{L_2} p(0) + C_{L_2} \varepsilon_1 \int_0^t p(s) ds, \quad \text{as long as } \|v(t)\|_\infty \le \delta_1, \quad t \in I. \tag{5.117}$$

Gronwall's inequality applied to (5.117) results in

$$p(t) \le C_{L_2} e^{C_{L_2} \varepsilon_1 t} p(0), \quad \text{as long as } \|v(t)\|_\infty \le \delta_1, \quad t \in I,$$

and consequently,

$$\|v(t)\|_2 \le C_{L_2} e^{(C_{L_2} \varepsilon_1 - z)t} \|v_0\|_2, \quad \text{as long as } \|v(t)\|_\infty \le \delta_1, \quad t \in I. \tag{5.118}$$

Inserting (5.115) into (5.118), we obtain

$$\|v(t)\|_2 \le C_{L_2} e^{at} \|v_0\|_2, \quad \text{as long as } \|v(t)\|_\infty \le \delta_1, \quad t \in I,$$

with $a = -\frac{\beta z}{1+\beta} < 0$. We have shown the assertion. $\square$

Remark 5.23. *Observe that the result of Proposition 5.22 is valid for any (not necessarily self-adjoint) operator, which is sectorial in $L_2^n(0,l)$.*

For our following investigations we need to impose one further assumption on function $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$. From now on we are going to assume that

(H_1^B) the function $\mathbf{g}$ is continuous in t with values in $H_1^B(0,l)$, namely $\mathbf{g} \in C(I; H_1^B(0,l))$.

Now we introduce the first result on the domain of attraction

Theorem 5.24. *Let $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ be the sectorial operator introduced in* (4.1). *Let $L_{\bar{u}}$ be self-adjoint and suppose that constant z, introduced in assumption* (A_0), *satisfies $z > 0$. Let function $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ satisfy* (5.13). *Then there exist P, $\delta_2 > 0$ and $a < 0$ such that if $v_0 \in H_1^B(0,l)$, $q(v_0) + P \|v_0\|_2 < \delta_2$, we have $t_{\max}(v_0) = \infty$ and*

$$\|v(t)\|_\infty \leq K e^{at}(q(v_0) + P\|v_0\|_2), \quad t \geq 0, \tag{5.119}$$

where $v(t)$ is a solution of (5.10). *The trivial solution of* (5.10) *is asymptotically stable.*

Proof. Let $\beta > 0$ be some small constant and let us set

$$\varepsilon = \varepsilon_1 := \frac{z}{(1+\beta)C_{L_2}}, \tag{5.120}$$

with C_{L_2} from (5.113).

Due to condition (5.13) there exist $\delta(\varepsilon_1)$ with

$$\begin{aligned} &\|\mathbf{g}(v(t), \bar{u})\|_2 \leq \varepsilon_1 \|v(t)\|_2 \\ &\text{for } v \in C^n[0,l], \ \|v(t)\|_\infty \leq \frac{\delta(\varepsilon_1)}{\sqrt{n}} =: \delta_1 \text{ and } t \geq 0. \end{aligned} \tag{5.121}$$

Let σ be a positive constant satisfying (5.100). Let us set

$$P := \sqrt{\frac{2\pi}{e}}\sqrt{z+\sigma}(1+\beta). \tag{5.122}$$

Now let v be a maximally defined mild solution of (5.10) on the interval $I(v_0)$, satisfying

$$q(v_0) + P\|v_0\|_2 < \frac{\delta_1\sqrt{d_{\min}}}{\sqrt{C_1}} =: \delta_2, \tag{5.123}$$

where q is from (5.107) and C_1 is the embedding constant from either (5.110) or (5.111). Note that from Lemma 5.21 and (5.123) follows

$$\|v_0\|_\infty < \delta_1. \tag{5.124}$$

The mild solution of (5.10) is given by

$$v(t) = e^{-tL_{\bar{u}}}v_0 + \int_0^t e^{-(t-s)L_{\bar{u}}}\mathbf{g}(v(s), \bar{u})ds, \quad t \in I. \tag{5.125}$$

Let us take the inner product with the eigenfunction $\tilde{\varphi}_k$ of the operator $L_{\bar{u}}$. Due to the self-adjointness of $L_{\bar{u}}$ and by Fubini's theorem, we obtain

$$\begin{aligned}
\langle v(t), \tilde{\varphi}_k\rangle_2 &= \left\langle e^{-tL_{\bar{u}}}v_0, \tilde{\varphi}_k\right\rangle_2 + \int_0^t \left\langle e^{-(t-s)L_{\bar{u}}}\mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k\right\rangle_2 ds \\
&= \left\langle v_0, e^{-tL_{\bar{u}}}\tilde{\varphi}_k\right\rangle_2 + \int_0^t \left\langle \mathbf{g}(v(s), \bar{u}), e^{-(t-s)L_{\bar{u}}}\tilde{\varphi}_k\right\rangle_2 ds \\
&= e^{-\lambda_k t}\langle v_0, \tilde{\varphi}_k\rangle_2 + \int_0^t e^{-\lambda_k(t-s)}\langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k\rangle_2 ds, \quad t \in I.
\end{aligned}$$

From the definition of z follows $0 < z \leq \lambda_k,\ k \in \mathbb{N}$. Now let us multiply the above equation with $\sqrt{\lambda_k + \sigma}$, multiply the result with its adjoint, summ it for all k, take the square root and apply Minkowski's inequality. All these operations yield

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty}(\lambda_k + \sigma)|\langle v(t), \tilde{\varphi}_k\rangle_2|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty}(\lambda_k + \sigma)e^{-2\lambda_k t}|\langle v_0, \tilde{\varphi}_k\rangle_2|^2\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k=1}^{\infty}(\lambda_k + \sigma)\left|\left(\int_0^t e^{-\lambda_k(t-s)}\langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k\rangle_2 ds\right)\right|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.126}$$

Considering the last term of the inequality above, let us introduce a sequence

$$f(s) = (f_k(s))_{k\in\mathbb{N}} \in l^2$$

with elements $f_k : I \to \mathbb{R}$ defined as

$$f_k(s) := \sqrt{\lambda_k + \sigma}\, e^{-\lambda_k(t-s)} \left\langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k \right\rangle_2 . \tag{5.127}$$

Let us at the moment make the following assumption

$(\star)$ f(s) is a Bochner integrable function.

If $(\star)$ is satisfied, then one can apply Bochner's theorem (see Theorem 5.36 later) in order to obtain

$$\begin{aligned}
\left\| \left(\int_0^t f_k(s) ds \right)_{k\in\mathbb{N}} \right\|_{l_2} &= \left(\sum_{k=1}^{\infty} (\lambda_k + \sigma) \left| \left(\int_0^t e^{-\lambda_k(t-s)} \left\langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k \right\rangle_2 ds \right) \right|^2 \right)^{\frac{1}{2}} \\
&\leq \int_0^t \left\| (f_k(s))_{k\in\mathbb{N}} \right\|_{l_2} ds \\
&= \int_0^t \left(\sum_{k=1}^{\infty} (\lambda_k + \sigma) e^{-2\lambda_k(t-s)} \left| \left\langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k \right\rangle_2 \right|^2 \right)^{\frac{1}{2}} ds.
\end{aligned} \tag{5.128}$$

Later in this subsection we present the notion of Bochner integrability and show that assumption $(\star)$ is satisfied.

Inserting (5.128) into (5.126), we obtain

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty} (\lambda_k + \sigma) \left| \left\langle v(t), \tilde{\varphi}_k \right\rangle_2 \right|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty} (\lambda_k + \sigma) e^{-2\lambda_k t} \left| \left\langle v_0, \tilde{\varphi}_k \right\rangle_2 \right|^2 \right)^{\frac{1}{2}} \\
&\quad + \int_0^t \left(\sum_{k=1}^{\infty} (\lambda_k + \sigma) e^{-2\lambda_k(t-s)} \left| \left\langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k \right\rangle_2 \right|^2 \right)^{\frac{1}{2}} ds.
\end{aligned} \tag{5.129}$$

Now let α be some given constant chosen in an interval

$$\alpha \in \left(\frac{2|a| + \sigma - \lambda_1}{\sigma + \lambda_1}, 1 \right), \tag{5.130}$$

where a is the constant used in Proposition 5.22, that is $a = -\frac{\beta z}{1+\beta}$. Let us represent $-2\lambda_k(t-s)$ as

$$-2\lambda_k(t-s) = -(1-\alpha)(\lambda_k+\sigma)(t-s) - (1+\alpha)(\lambda_k+\sigma)(t-s) + 2\sigma(t-s),$$

and denote

$$\phi(\lambda_k+\sigma) := (\lambda_k+\sigma)e^{-(1-\alpha)(\lambda_k+\sigma)(t-s)}.$$

Function $\phi(\lambda_k+\sigma)$ has its maximum at $(\lambda_k+\sigma)^* = \frac{1}{(1-\alpha)(t-s)}$. Thus, we have

$$\phi(\lambda_k+\sigma) \leq \phi((\lambda_k+\sigma)^*) = \frac{e^{-1}}{(1-\alpha)(t-s)}. \tag{5.131}$$

Hence, using (5.131), we estimate the third term in (5.129) as

$$\int_0^t \left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)e^{-2\lambda_k(t-s)}|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} ds$$

$$= \int_0^t \left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)e^{-(1-\alpha)(\lambda_k+\sigma)(t-s)-(1+\alpha)(\lambda_k+\sigma)(t-s)+2\sigma(t-s)}|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} ds$$

$$= \int_0^t \left(\sum_{k=1}^{\infty}\phi(\lambda_k+\sigma)e^{-(1+\alpha)(\lambda_k+\sigma)(t-s)+2\sigma(t-s)}|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} ds$$

$$\leq \int_0^t \left(\sum_{k=1}^{\infty}\frac{e^{-1}}{(1-\alpha)(t-s)}\, e^{-(1+\alpha)(\lambda_k+\sigma)(t-s)+2\sigma(t-s)}|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} ds$$

$$\leq \frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\int_0^t \frac{1}{\sqrt{t-s}}\, e^{\frac{1}{2}(-(1+\alpha)(\lambda_1+\sigma)(t-s)+2\sigma(t-s))}\left(\sum_{k=1}^{\infty}|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} ds,$$

where in the last estimation we have used $e^{-\lambda_k t} \leq e^{-\lambda_1 t}$.

Further, taking into account (5.98) and (5.121), we obtain

$$\frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\int_0^t \frac{1}{\sqrt{t-s}}\, e^{\frac{1}{2}(-(1+\alpha)(\lambda_1+\sigma)(t-s)+2\sigma(t-s))}\left(\sum_{k=1}^{\infty} |\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} ds$$

$$=\frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\int_0^t \frac{1}{\sqrt{t-s}}\, e^{\frac{\sigma(1-\alpha)-\lambda_1(1+\alpha)}{2}(t-s)}\left\|\mathbf{g}(v(s),\bar{u})\right\|_2 ds$$

$$\le\frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\int_0^t \frac{1}{\sqrt{t-s}}\, e^{\frac{\sigma(1-\alpha)-\lambda_1(1+\alpha)}{2}(t-s)}\varepsilon_1\left\|v(s)\right\|_2\, ds,$$

$$\text{as long as}\quad \|v(t)\|_\infty\le\delta_1,\quad t\in I. \tag{5.132}$$

Let us define $m := -\frac{\sigma(1-\alpha)-\lambda_1(1+\alpha)}{2}$. Note that from (5.130) follows that $m > 0$. Hence the last integral in (5.132) reads

$$\int_0^t \frac{1}{\sqrt{t-s}}\, e^{\frac{\sigma(1-\alpha)-\lambda_1(1+\alpha)}{2}(t-s)}\left\|v(s)\right\|_2\, ds = \int_0^t \frac{e^{-m(t-s)}}{\sqrt{t-s}}\left\|v(s)\right\|_2\, ds.$$

Combining the result above with Proposition 5.22, we continue as follows

$$\frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\varepsilon_1\int_0^t \frac{1}{\sqrt{t-s}}\, e^{-m(t-s)}\left\|v(s)\right\|_2\, ds$$

$$\le\frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\varepsilon_1 C_{L_2}\int_0^t \frac{1}{\sqrt{t-s}}\, e^{-m(t-s)}e^{as}\left\|v_0\right\|_2\, ds$$

$$\le\frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\varepsilon_1 C_{L_2}e^{-mt}\left\|v_0\right\|_2\int_0^t \frac{1}{\sqrt{t-s}}\, e^{(m+a)s}\, ds,$$

$$\text{as long as}\quad \|v(t)\|_\infty\le\delta_1,\quad t\in I. \tag{5.133}$$

Let us consider $\int_0^t \frac{1}{\sqrt{t-s}}\, e^{(m+a)s}\, ds$. After introducing the substitution $\nu=\sqrt{t-s}$, we obtain

$$\int_0^t \frac{e^{(m+a)s}}{\sqrt{t-s}}\, ds = 2e^{(m+a)t}\int_0^{\sqrt{t}} e^{-(m+a)\nu^2}\, d\nu.$$

From (5.130) follows $m+a>0$. Indeed, since $a=-\frac{\beta z}{1+\beta}<0$, we have

$$m+a = m-|a| = \frac{1}{2}(\lambda_1-\sigma+\alpha(\lambda_1+\sigma))-|a| \overset{(5.130)}{>} 0.$$

Therefore, we continue with estimation

$$2\int_0^{\sqrt{t}} e^{-(m+a)\nu^2}\, d\nu \leq 2\int_0^{\infty} e^{-(m+a)\nu^2}\, d\nu = \sqrt{\frac{\pi}{m+a}}. \tag{5.134}$$

Combining all of the above, we arrive at

$$\int_0^t \left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)e^{-2\lambda_k(t-s)}\left|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2\right|^2\right)^{\frac{1}{2}} ds$$
$$\leq \frac{e^{-\frac{1}{2}}}{\sqrt{1-\alpha}}\varepsilon_1 C_{L_2}\left\|v_0\right\|_2\sqrt{\frac{\pi}{m+a}}e^{at},$$
$$\text{as long as}\quad \|v(t)\|_\infty \leq \delta_1, \quad t\in I.$$

Let us define

$$B(\alpha) = \left(\frac{1}{(1-\alpha)(m+a)}\right)^{\frac{1}{2}}$$

and find $B(\alpha^*) = \min_{\alpha\in\left(\frac{2|a|+\sigma-\lambda_1}{\sigma+\lambda_1},1\right)} B(\alpha)$. Some technical computations result in $\alpha^* = \frac{\sigma+|a|}{\sigma+\lambda_1}$. Hence,

$$B(\alpha^*) = \frac{\sqrt{2(\lambda_1+\sigma)}}{\lambda_1-|a|}.$$

It follows that

$$\int_0^t \left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)e^{-2\lambda_k(t-s)}\left|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2\right|^2\right)^{\frac{1}{2}} ds \leq \sqrt{\frac{2\pi}{e}}\frac{\sqrt{\lambda_1+\sigma}}{\lambda_1-|a|}\varepsilon_1 C_{L_2}e^{at}\left\|v_0\right\|_2,$$
$$\text{as long as}\quad \|v(t)\|_\infty \leq \delta_1, \quad t\in I. \tag{5.135}$$

Inserting (5.135) into (5.129) and taking into account the fact that $|a| < z \leq \lambda_k,\ k\in$

$\mathbb{N}$, we obtain

$$\left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)|\left\langle v(t),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} \le \left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)e^{-2\lambda_k t}|\left\langle v_0,\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} + \sqrt{\frac{2\pi}{e}}\frac{\sqrt{\lambda_1+\sigma}}{\lambda_1-|a|}\varepsilon_1 C_{L_2}e^{at}\left\|v_0\right\|_2$$

$$\le e^{at}\left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)|\left\langle v_0,\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}} + Pe^{at}\left\|v_0\right\|_2,$$

$$\text{as long as} \quad \left\|v(t)\right\|_\infty \le \delta_1, \quad t\in I,$$

with $P=\sqrt{\frac{2\pi}{e}}\sqrt{z+\sigma}(1+\beta)$ (recall also that ε_1 satisfies (5.120)).

According to (5.108) the inequality above reads

$$q(v(t)) \le e^{at}(q(v_0)+P\left\|v_0\right\|_2), \quad \text{as long as} \quad \left\|v(t)\right\|_\infty \le \delta_1, \quad t\in I. \tag{5.136}$$

Note that by (5.124) we have $\left\|v_0\right\|_\infty < \delta_1$. By Lemma 5.21 we obtain

$$\left\|v(t)\right\|_\infty \le \sqrt{\frac{C_1}{d_{\min}}}q(v(t)) \le \sqrt{\frac{C_1}{d_{\min}}}e^{at}\left(q(v_0)+P\left\|v_0\right\|_2\right),$$

$$\text{as long as} \quad \left\|v(t)\right\|_\infty \le \delta_1, \quad t\in I. \tag{5.137}$$

Since $q(v_0)+P\left\|v_0\right\|_2 < \frac{\delta_1\sqrt{d_{\min}}}{\sqrt{C_1}}$ (by (5.123)), from (5.137) follows

$$\left\|v(t)\right\|_\infty < \delta_1, \quad \text{as long as} \quad \left\|v(t)\right\|_\infty \le \delta_1,$$

which implies (by continuity):

$$\left\|v(t)\right\|_\infty \le \sqrt{\frac{C_1}{d_{\min}}}e^{at}\left(q(v_0)+P\left\|v_0\right\|_2\right) \quad \forall t\in I. \tag{5.138}$$

From (5.138) we see that the mapping $t\mapsto\left\|v(t)\right\|_\infty$ is bounded on I. By Theorem 2.12 the mild solution v exists in the large and the estimation (5.138) holds for all $t\ge 0$. We have proven the assertion. □

Remark 5.25. *As one can see from the proof of Theorem 5.24 the upper bound to the domain of attraction is given by* $\delta_2 := \dfrac{\delta_1\sqrt{d_{\min}}}{\sqrt{C_1}} = \dfrac{\delta(\varepsilon_1)\sqrt{d_{\min}}}{\sqrt{C_1 n}}$. *If the monotonically* *non-decreasing function $G : [0,+\infty) \to [0,+\infty)$ is khown, then the constant $\delta(\varepsilon_1)$ can be computed from (5.4) by setting $\varepsilon = \varepsilon_1 := \dfrac{z}{C_{L_2}(1+\beta)}$. Hence, having the computable constants C_{L_2} and z and the monotonically non-decreasing function G at hand, we can quantify the domain of attraction.*

5.6.5 Second result on the domain of attraction

In order for the second approach to work we need to impose some further assumptions on $\mathbf{g}$.

Additional requirements on $\mathbf{g}$

Recall that function $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ is given by

$$\mathbf{g}(\varphi, \bar{u}) = \mathbf{F}(\varphi + \bar{u}) - \mathbf{F}(\bar{u}) + \mathcal{C}_{\bar{u}}\varphi$$

and satisfies assumptions (G), (H_1^B) and (5.18). At first, observe, that from assumption (G) follows that

(G_1) There exists a monotonically non-decreasing function $G_1 : [0,+\infty) \to [0,+\infty)$ such that

$$\begin{cases} |\mathbf{g}(\varphi(x), \bar{u}(x))|_2^2 \le G_1(|\varphi(x)|_2) \text{ for all } x \in [0,l], \\[2ex] \text{with } G_1(h) = o(h^3), \quad \text{as} \quad h \to 0+. \end{cases}$$

Now we list all the additional requirements we are going to impose on $\mathbf{g}$:

(G_2) There exist a monotonically non-decreasing function $G_2 : [0,+\infty) \to [0,+\infty)$ such that

$$\begin{cases} |\mathbf{g}_\varphi(\varphi(x), \bar{u}(x))|_2^2 \le G_2(|\varphi(x)|_2) \text{ for all } x \in [0,l], \\[2ex] \text{with } G_2(h) = o(h), \quad \text{as} \quad h \to 0+. \end{cases}$$

(G_3) There exist a monotonically non-decreasing function $G_3 : [0, +\infty) \to [0, +\infty)$ such that

$$\begin{cases} |\mathbf{g}_{\bar{u}}(\varphi(x), \bar{u}(x))|_2^2 \leq G_3(|\varphi(x)|_2) \text{ for all } x \in [0, l], \\ \\ \text{with } G_3(h) = o(h^3), \quad \text{as} \quad h \to 0+. \end{cases}$$

From (G1), (G2), and (G3) follows that for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ there exist $\delta > 0$ such that

$$\forall |h| < \delta \quad |G_1(h)| \leq \varepsilon_1 |h|^3, \tag{5.139}$$

$$\forall |h| < \delta \quad |G_2(h)| \leq \varepsilon_2 |h|, \tag{5.140}$$

$$\forall |h| < \delta \quad |G_3(h)| \leq \varepsilon_3 |h|^3. \tag{5.141}$$

As in the case with the general operator $L_{\bar{u}}$, in order to compute an upper bound to the domain of attraction we will need a result, which is similar to Remark 5.1. We introduce the following

Remark 5.26. *Let $a_1,\ a_2,\ a_3 > 0$ be given. If the functions $G_1,\ G_2,\ G_3$ are known, then a function $\delta : [0, \hat{\varepsilon}_0) \to (0, \infty)$ satisfying*

$$\forall \hat{\varepsilon} > 0 \quad \forall |h| < \delta(\hat{\varepsilon}) \quad a_1|G_1(h)| + a_2|G_2(h)||h|^2 + a_3|G_3(h)| \leq \hat{\varepsilon}|h|^3 \tag{5.142}$$

can be computed.

Now let us briefly comment on our intentions in the following. Observe, that property (5.12) (or (5.13)) implies that function $\mathbf{g}(\varphi, \bar{u})$ is "small" for "small" φ. In the following we are going to establish the similar result. Namely, we will show that after the application of q to $\mathbf{g}(\varphi, \bar{u})$ and φ, given that $\mathbf{g}(\varphi, \bar{u})$ satisfies (H_1^B), (G_1), (G_2), and (G_3), the property above does not change. Thus, $q(\mathbf{g}(\varphi, \bar{u}))$ will also be "small" for "small" φ .

We continue with

Lemma 5.27. *[22] A Hermitian positive definite matrix $B \in \mathbb{C}^{n\times n}$ has a unique Hermitian positive definite square root.*

Observe that since the positive constant σ satisfies (5.100), the matrix $C_{\bar{u}} + \sigma I$ is positive definite. Let us introduce

Lemma 5.28. *Let the assumptions (G_1), (G_2), and (G_3) be satisfied. Then for each $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 > 0$ there exist $\delta > 0$ such that for any $\varphi \in C_1^n[0,l]$ satisfying $\|\varphi\|_\infty \le \frac{\delta}{\sqrt{n}}$ we have*

$$\mathbf{g}(\varphi(x),\bar{u}(x))^T(C_{\bar{u}}(x)^T+\sigma I)\overline{\mathbf{g}(\varphi(x),\bar{u}(x))}$$
$$\le \varepsilon_1\hat{C}(x)\delta\ \varphi(x)^T(C_{\bar{u}}(x)^T+\sigma I)\ \overline{\varphi(x)}, \tag{5.143}$$

$$(\mathbf{g}_\varphi(\varphi(x),\bar{u}(x))\varphi'(x))^T D\overline{\mathbf{g}_\varphi(\varphi(x),\bar{u}(x))\varphi'(x)} \le \varepsilon_2\frac{d_{\max}}{d_{\min}}\delta\ (\varphi'(x))^T D\ \overline{\varphi'(x)}, \tag{5.144}$$

$$(\mathbf{g}_{\bar{u}}(\varphi(x),\bar{u}(x))\bar{u}'(x))^T D\overline{\mathbf{g}_{\bar{u}}(\varphi(x),\bar{u}(x))\bar{u}'(x)} \le \varepsilon_3 d_{\max}\delta\ |\varphi(x)|_2^2|\bar{u}'(x)|_2^2, \tag{5.145}$$

for all $x \in [0,l]$, where $\hat{C}(x) := \dfrac{\lambda_{\max}(C_{\bar{u}}(x)+\sigma I)}{\lambda_{\min}(C_{\bar{u}}(x)+\sigma I)}$.

Proof. In the following for simplicity we write $\mathbf{g}$, $\mathbf{g}_\varphi$, $\mathbf{g}_{\bar{u}}$ instead of $\mathbf{g}(\varphi(x),\bar{u}(x))$, $\mathbf{g}_\varphi(\varphi(x),\bar{u}(x))$, $\mathbf{g}_{\bar{u}}(\varphi(x),\bar{u}(x))$ respectively. In addition, we drop letter x and address matrix $C_{\bar{u}}(x)^T + \sigma I$ as C.

Since C is a positive definite and Hermitian matrix, due to Lemma 5.27, it has a Hermitian positive definite square root $C^{\frac{1}{2}}$. Then by (G_1) and, consequently, by (5.139) for each $\varepsilon_1 > 0$ there exists $\delta > 0$ such that

$$\mathbf{g}^T C\overline{\mathbf{g}} = \mathbf{g}^T C^{\frac{1}{2}}\ C^{\frac{1}{2}}\overline{\mathbf{g}} = \left(\left(C^{\frac{1}{2}}\right)^T\mathbf{g}\right)^T\overline{\left(\left(C^{\frac{1}{2}}\right)^T\mathbf{g}\right)}$$
$$= \left|\left(C^{\frac{1}{2}}\right)^T\mathbf{g}\right|_2^2 \le \left|\left(C^{\frac{1}{2}}\right)^T\right|_2^2|\mathbf{g}|_2^2 \overset{(G_1)}{\le} \left|\left(C^{\frac{1}{2}}\right)^T\right|_2^2 G_1(|\varphi|_2)$$
$$\overset{(5.139)}{\le} \left|\left(C^{\frac{1}{2}}\right)^T\right|_2^2 \varepsilon_1|\varphi|_2^2\delta, \quad \text{if} \quad |\varphi|_2 \le \delta.$$

Now let us introduce

$$z = \left(C^{\frac{1}{2}}\right)^T \varphi.$$

Then

$$|z|_2^2 = \varphi^T C\,\overline{\varphi}, \tag{5.146}$$

and

$$\varphi = \left(C^{-\frac{1}{2}}\right)^T z. \tag{5.147}$$

Using (5.146) and (5.147), we continue as follows: for each $\varepsilon_1 > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}\mathbf{g}^T C \overline{\mathbf{g}} &\le \left|\left(C^{\frac{1}{2}}\right)^T\right|_2^2 \varepsilon_1 |\varphi|_2^2 \delta = \left|\left(C^{\frac{1}{2}}\right)^T\right|_2^2 \varepsilon_1 \left|\left(C^{-\frac{1}{2}}\right)^T z\right|_2^2 \delta \\ &\le \left|\left(C^{\frac{1}{2}}\right)^T\right|_2^2 \left|\left(C^{-\frac{1}{2}}\right)^T\right|_2^2 \varepsilon_1 \delta\ \varphi^T C\,\overline{\varphi}, \quad \text{if} \quad |\varphi|_2 \le \delta. \end{aligned} \tag{5.148}$$

Using again the fact that C is a positive definite and Hermitian matrix, we obtain

$$\left|\left(C^{\frac{1}{2}}\right)^T\right|_2^2 = \lambda_{\max}(C),$$

$$\left|\left(C^{-\frac{1}{2}}\right)^T\right|_2^2 = \lambda_{\max}\left(C^{-1}\right) = \frac{1}{\lambda_{\min}(C)}.$$

Setting in (5.148)

$$\hat{C}(x) := \frac{\lambda_{\max}\left(C_{\bar{u}}(x)^T + \sigma I\right)}{\lambda_{\min}\left(C_{\bar{u}}(x)^T + \sigma I\right)} = \frac{\lambda_{\max}\left(C_{\bar{u}}(x) + \sigma I\right)}{\lambda_{\min}\left(C_{\bar{u}}(x) + \sigma I\right)}, \tag{5.149}$$

we obtain estimation (5.143).

Let us continue with the estimation (5.144). This time we set

$$z = D^{\frac{1}{2}} \varphi'.$$

Hence

$$|z|^2 = (\varphi')^T D \overline{\varphi'} \tag{5.150}$$

and

$$\varphi' = D^{-\frac{1}{2}} z. \tag{5.151}$$

We continue as follows: by (G_2) and, consequently, by (5.140) for each $\varepsilon_2 > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
(\mathbf{g}_\varphi \varphi')^T D \overline{(\mathbf{g}_\varphi \varphi')} &= (\varphi')^T \mathbf{g}_\varphi^T D \overline{\mathbf{g}_\varphi} \overline{\varphi'} \overset{(5.151)}{=} z^T \left(D^{-\frac{1}{2}} \mathbf{g}_\varphi^T D^{\frac{1}{2}} \right) \left(D^{\frac{1}{2}} \overline{\mathbf{g}_\varphi} D^{-\frac{1}{2}} \right) \overline{z} \\
&= \left(D^{\frac{1}{2}} \mathbf{g}_\varphi D^{-\frac{1}{2}} z \right)^T \overline{\left(D^{\frac{1}{2}} \mathbf{g}_\varphi D^{-\frac{1}{2}} z \right)} = \left| D^{\frac{1}{2}} \mathbf{g}_\varphi D^{-\frac{1}{2}} z \right|_2^2 \\
&\le \left| D^{\frac{1}{2}} \right|_2^2 \left| D^{-\frac{1}{2}} \right|_2^2 |\mathbf{g}_\varphi|_2^2 |z|_2^2 \overset{(G_2)}{\le} \left| D^{\frac{1}{2}} \right|_2^2 \left| D^{-\frac{1}{2}} \right|_2^2 G_2(|\varphi|_2) |z|_2^2 \\
&\overset{(5.140)}{\le} \left| D^{\frac{1}{2}} \right|_2^2 \left| D^{-\frac{1}{2}} \right|_2^2 \varepsilon_2 \delta |z|_2^2 \overset{(5.150)}{=} \left| D^{\frac{1}{2}} \right|_2^2 \left| D^{-\frac{1}{2}} \right|_2^2 \varepsilon_2 \delta \, (\varphi')^T D \overline{\varphi'} \\
&= \varepsilon_2 \frac{d_{\max}}{d_{\min}} \delta \, (\varphi')^T D \overline{\varphi'}, \quad \text{if} \quad |\varphi|_2 \le \delta.
\end{aligned}$$

We have obtained (5.144).

In order to obtain (5.145), we proceed as follows. By (G_3) and consequently, by (5.141) for each $\varepsilon_3 > 0$ there exists $\delta > 0$ such that

$$\begin{aligned}
(\mathbf{g}_{\bar{u}} \bar{u}')^T D \overline{(\mathbf{g}_{\bar{u}} \bar{u}')} &= (\bar{u}')^T \mathbf{g}_{\bar{u}}^T D^{\frac{1}{2}} D^{\frac{1}{2}} \overline{(\mathbf{g}_{\bar{u}} \bar{u}')} = \left(D^{\frac{1}{2}} \mathbf{g}_{\bar{u}} \bar{u}' \right)^T \overline{\left(D^{\frac{1}{2}} \mathbf{g}_{\bar{u}} \bar{u}' \right)} = \left| D^{\frac{1}{2}} \mathbf{g}_{\bar{u}} \bar{u}' \right|_2^2 \\
&\le \left| D^{\frac{1}{2}} \right|_2^2 |\mathbf{g}_{\bar{u}}|_2^2 |\bar{u}'|_2^2 \overset{(G_3)}{\le} \left| D^{\frac{1}{2}} \right|_2^2 G_3(|\varphi|_2) |\bar{u}'|_2^2 \overset{(5.141)}{\le} \left| D^{\frac{1}{2}} \right|_2^2 \varepsilon_3 |\varphi|_2^2 \delta |\bar{u}'|_2^2 \\
&= \varepsilon_3 d_{\max} \delta |\varphi|_2^2 |\bar{u}'|_2^2, \quad \text{if} \quad |\varphi|_2 \le \delta.
\end{aligned}$$

Note that if $\|\varphi\|_\infty \le \frac{\delta}{\sqrt{n}}$ then, due to (5.14), $|\varphi(x)|_2 \le \delta$ for all $x \in [0, l]$. Hence all the results above hold for $\varphi \in C_1^n[0, l]$ satisfying $\|\varphi\|_\infty \le \frac{\delta}{\sqrt{n}}$. The proof of lemma is complete. $\square$

Now let us introduce several notations, which we will use in the sequel. We denote the upper bound to $\|\bar{u}'\|_2^2$ as $\bar{U}_x$. Computation of $\bar{U}_x$ is not difficult. Since $\bar{u}$ is a stationary solution of (5.1), it follows

$$d_{\min} \|\bar{u}'\|_2^2 \leq \langle D\bar{u}', \bar{u}'\rangle_2 \leq \|\mathbf{F}(\bar{u})\|_2 \|\bar{u}\|_2 \leq (\|\mathbf{F}(\bar{u}) - \mathbf{F}(\omega)\|_2 + \|\mathbf{F}(\omega)\|_2)(\|\bar{u} - \omega\|_2 + \|\omega\|_2)$$

where the terms $\|\mathbf{F}(\bar{u}) - \mathbf{F}(\omega)\|_2$ and $\|\bar{u} - \omega\|_2$ in the estimation above can be obtained with the help of (3.21). Hence for some positive computable $\hat{K}$ we have

$$(\|\mathbf{F}(\bar{u}) - \mathbf{F}(\omega)\|_2 + \|\mathbf{F}(\omega)\|_2)(\|\bar{u} - \omega\|_2 + \|\omega\|_2) \leq \hat{K}$$

and set

$$\|\bar{u}'\|_2^2 \leq \bar{U}_x := \frac{1}{d_{\min}} \hat{K}. \tag{5.152}$$

In addition we denote

$$\eta := \max_{x \in [0,l]} \hat{C}(x), \tag{5.153}$$

Now we introduce

Lemma 5.29. *Let the assumptions (G_1), (G_2), and (G_3) be satisfied. Then for any $\tilde{\varepsilon} > 0$ there exists $\delta > 0$ such that for any $\varphi \in H_1^B(0,l)$, $\|\varphi\|_\infty \leq \dfrac{\delta}{\sqrt{n}}$ we have*

$$q(\mathbf{g}(\varphi, \bar{u})) \leq \tilde{\varepsilon} q(\varphi). \tag{5.154}$$

Proof. According to (5.107) we have (note that $\mathbf{g}(\varphi, \bar{u}) \in H_1^B(0,l)$)

$$\begin{aligned} q^2(\mathbf{g}(\varphi, \bar{u})) = &\int_0^l \mathbf{g}_x(\varphi(x), \bar{u}(x))^T D \overline{\mathbf{g}_x(\varphi(x), \bar{u}(x))} dx \\ &+ \int_0^l \mathbf{g}(\varphi(x), \bar{u}(x))^T (C_{\bar{u}}(x)^T + \sigma I)\, \overline{\mathbf{g}(\varphi(x), \bar{u}(x))}\, dx. \end{aligned} \tag{5.155}$$

Let us consider the expression $\mathbf{g}_x(\varphi(x), \bar{u}(x))$. We have

$$\mathbf{g}_x(\varphi(x), \bar{u}(x)) = \mathbf{g}_\varphi(\varphi(x), \bar{u}(x))\varphi'(x) + \mathbf{g}_{\bar{u}}(\varphi(x), \bar{u}(x))\bar{u}'(x).$$

Each component $(\mathbf{g}_x)_j$, $(j = 1, \dots, n)$ of the vector $\mathbf{g}_x(\varphi(x), \bar{u}(x))$ can be estimated with the help of Young's inequality with $\xi > 0$ as

$$
\begin{aligned}
((\mathbf{g}_x)_j)^2 &= \left(\sum_{i=1}^{n}\left(\frac{\partial \mathbf{g}_j}{\partial \varphi_i}(\varphi')_i + \frac{\partial \mathbf{g}_j}{\partial \bar{u}_i}(\bar{u}')_i\right)\right)^2 = \left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \varphi_i}(\varphi')_i + \sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \bar{u}_i}(\bar{u}')_i\right)^2 \\
&= \left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \varphi_i}(\varphi')_i\right)^2 + 2\left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \varphi_i}(\varphi')_i\right)\left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \bar{u}_i}(\bar{u}')_i\right) + \left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \bar{u}_i}(\bar{u}')_i\right)^2 \\
&\le (1+\xi)\left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \varphi_i}(\varphi')_i\right)^2 + \left(1 + \frac{1}{\xi}\right)\left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \bar{u}_i}(\bar{u}')_i\right)^2,
\end{aligned}
$$

where $\dfrac{\partial \mathbf{g}_j}{\partial \varphi_i}$ and $\dfrac{\partial \mathbf{g}_j}{\partial \bar{u}_i}$ denote the elements of matrices $\mathbf{g}_\varphi(\varphi(x), \bar{u}(x))$ and $\mathbf{g}_{\bar{u}}(\varphi(x), \bar{u}(x))$ and $(\varphi')_i$, $(\bar{u}')_i$ are the elements of vectors $\varphi'(x)$, $\bar{u}'(x)$ respectively.

Further, let us consider the term $\mathbf{g}_x(\varphi(x), \bar{u}(x))^T D\overline{\mathbf{g}_x(\varphi(x), \bar{u}(x))}$. From the estimation above follows

$$
\begin{aligned}
&\mathbf{g}_x(\varphi(x), \bar{u}(x))^T D\overline{\mathbf{g}_x(\varphi(x), \bar{u}(x))} \\
&\quad = \sum_{j=1}^{n} d_j((\mathbf{g}_x)_j)^2 \\
&\quad \le \ (1+\xi)\sum_{j=1}^{n}\left(d_j\left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \varphi_i}(\varphi')_i\right)^2\right) + \left(1+\frac{1}{\xi}\right)\sum_{j=1}^{n}\left(d_j\left(\sum_{i=1}^{n}\frac{\partial \mathbf{g}_j}{\partial \bar{u}_i}(\bar{u}')_i\right)^2\right) \\
&\quad = (1+\xi)(\mathbf{g}_\varphi(\varphi(x), \bar{u}(x))\varphi'(x))^T D\overline{(\mathbf{g}_\varphi(\varphi(x), \bar{u}(x))\varphi'(x))} \\
&\qquad + \left(1+\frac{1}{\xi}\right)(\mathbf{g}_{\bar{u}}(\varphi(x), \bar{u}(x))\bar{u}'(x))^T D\overline{(\mathbf{g}_{\bar{u}}(\varphi(x), \bar{u}(x))\bar{u}'(x))}. \qquad (5.156)
\end{aligned}
$$

Since the assumptions (G_1),(G_2), and (G_3) are satisfied, by Lemma 5.28 the estimations (5.143), (5.144), and (5.145) hold. Thus, combining (5.144) and (5.145) with

(5.156), we obtain that for each $\varepsilon_2,\ \varepsilon_3 > 0$ there exists $\delta > 0$ such that

$$\int_0^l \mathbf{g}_x(\varphi(x), \bar{u}(x))^T D\overline{\mathbf{g}_x(\varphi(x), \bar{u}(x))}dx$$

$$\leq (1+\xi)\varepsilon_2 \frac{d_{\max}}{d_{\min}}\delta \int_0^l \varphi'(x)^T D\overline{\varphi'(x)}dx + \left(1+\frac{1}{\xi}\right)\varepsilon_3 d_{\max}\delta \int_0^l |\bar{u}'(x)|^2|\varphi(x)|^2 dx,$$

$$\text{if} \quad \|\varphi\|_\infty \leq \frac{\delta}{\sqrt{n}}. \tag{5.157}$$

Further, by (5.143) we have: for each $\varepsilon_1 > 0$ there exists $\delta > 0$ such that

$$\int_0^l \mathbf{g}(\varphi(x), \bar{u}(x))^T (C_{\bar{u}}(x)^T + \sigma I)\, \overline{\mathbf{g}(\varphi(x), \bar{u}(x))}dx$$

$$\leq \varepsilon_1\delta \int_0^l \hat{C}(x)\varphi(x)^T (C_{\bar{u}}(x)^T + \sigma I)\, \overline{\varphi(x)}dx$$

$$\leq \varepsilon_1\delta\eta \int_0^l \varphi(x)^T (C_{\bar{u}}(x)^T + \sigma I)\, \overline{\varphi(x)}dx, \quad \text{if} \quad \|\varphi\|_\infty \leq \frac{\delta}{\sqrt{n}}, \tag{5.158}$$

with η from (5.153). Combining (5.155), (5.157), and (5.158), we obtain: for each $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} q^2(\mathbf{g}(\varphi, \bar{u})) &\leq (1+\xi)\varepsilon_2 \frac{d_{\max}}{d_{\min}}\delta \int_0^l \varphi'(x)^T D\overline{\varphi'(x)}dx \\ &\quad + \left(1+\frac{1}{\xi}\right)\varepsilon_3 d_{\max}\delta \int_0^l |\bar{u}'(x)|^2|\varphi(x)|^2 dx \\ &\quad + \varepsilon_1\delta\eta \int_0^l \varphi(x)^T (C_{\bar{u}}(x)^T + \sigma I)\, \overline{\varphi(x)}dx \\ &\leq \max\left\{(1+\xi)\varepsilon_2 \frac{d_{\max}}{d_{\min}}\delta,\ \varepsilon_1\delta\eta\right\}\, q^2(\varphi) \\ &\quad + \left(1+\frac{1}{\xi}\right)\varepsilon_3 d_{\max}\delta \int_0^l |\bar{u}'(x)|^2|\varphi(x)|^2 dx, \\ &\quad \text{if} \quad \|\varphi\|_\infty \leq \frac{\delta}{\sqrt{n}}. \end{aligned} \tag{5.159}$$

Using (5.152) and (5.109), we estimate the last term in (5.159) as follows

$$\int_0^l |\bar{u}'(x)|^2 |\varphi(x)|^2 dx \le \bar{U}_x \, \|\varphi\|_\infty^2 \le \bar{U}_x \frac{C_1}{d_{\min}} q^2(\varphi).$$

Inserting the last estimation into (5.159), we obtain: for each $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 > 0$ there exists $\delta > 0$ such that

$$q^2(\mathbf{g}(\varphi, \bar{u})) \le \max\left\{(1+\xi)\varepsilon_2 \frac{d_{\max}}{d_{\min}},\ \varepsilon_1 \eta,\ \left(1 + \frac{1}{\xi}\right) \varepsilon_3 d_{\max}\, \bar{U}_x \frac{C_1}{d_{\min}}\right\} \delta q^2(\varphi),$$

$$\text{if} \quad \|\varphi\|_\infty \le \frac{\delta}{\sqrt{n}} \tag{5.160}$$

Choose $\xi = \dfrac{\varepsilon_3}{\varepsilon_2} \bar{U}_x C_1$, then

$$(1+\xi)\varepsilon_2 \frac{d_{\max}}{d_{\min}} = \left(1 + \frac{1}{\xi}\right) \varepsilon_3 d_{\max} \bar{U}_x \frac{C_1}{d_{\min}} = \frac{d_{\max}}{d_{\min}} (\varepsilon_2 + \varepsilon_3 \bar{U}_x C_1).$$

Thus (5.160) reads: for each $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3 > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} q^2(\mathbf{g}(\varphi, \bar{u})) &\le \max\left\{\varepsilon_1 \eta,\ \frac{d_{\max}}{d_{\min}} (\varepsilon_2 + \varepsilon_3 \bar{U}_x C_1)\right\} \delta q^2(\varphi) \\ &\le \left(\eta \varepsilon_1 + \frac{d_{\max}}{d_{\min}} \varepsilon_2 + \frac{d_{\max}}{d_{\min}} \bar{U}_x C_1 \varepsilon_3\right) \delta q^2(\varphi), \quad \text{if} \quad \|\varphi\|_\infty \le \frac{\delta}{\sqrt{n}}. \end{aligned}$$

With $\tilde{\varepsilon}^2 := \left(\eta\varepsilon_1 + \frac{d_{\max}}{d_{\min}}\varepsilon_2 + \frac{d_{\max}}{d_{\min}} \bar{U}_x C_1 \varepsilon_3\right) \delta$ we obtain the assertion of the lemma. □

Domain of attraction

Now we are ready to formulate

Theorem 5.30. *Let $L_{\bar{u}} : H_2^B(0,l) \to L_2^n(0,l)$ be the sectorial operator introduced in (4.1). Let $L_{\bar{u}}$ be self-adjoint and suppose that constant z, introduced in assumption (A_0), satisfies $z > 0$. Let function $\mathbf{g} : C^n[0,l] \times C^n[0,l] \to L_2^n(0,l)$ satisfy (5.154). Then there exist $\delta_3,\ K > 0$ and $a < 0$ such that if $v_0 \in H_1^B(0,l)$, $q(v_0) < \delta_3$ then $t_{\max}(v_0) = \infty$ and*

$$\|v(t)\|_\infty \le K e^{at} q(v_0), \quad t \ge 0, \tag{5.161}$$

where $v(t)$ is a solution of (5.10)*. The trivial solution of* (5.10) *is asymptotically stable.*

Proof. Let $\beta > 0$ be some small constant. We set

$$\tilde{\varepsilon} = \tilde{\varepsilon}_0 := \frac{z}{1+\beta}. \tag{5.162}$$

By property (5.154) there exists $\delta(\tilde{\varepsilon}_0) > 0$ such that

$$q(\mathbf{g}(v(t), \bar{u})) \leq \tilde{\varepsilon}_0 q(v(t)), \quad \text{if} \quad \|v(t)\|_\infty \leq \frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}}, \ \forall t \geq 0. \tag{5.163}$$

Now let v be a maximally defined mild solution of (5.10) on the interval $I(v_0)$, with v_0 satisfying

$$q(v_0) < \frac{\delta(\tilde{\varepsilon}_0)\sqrt{d_{\min}}}{\sqrt{C_1 n}} := \delta_3, \tag{5.164}$$

where q is from (5.107) and C_1 is the embedding constant from either (5.110) or (5.111). Note that from Lemma 5.21 and (5.164) follows

$$\|v_0\|_\infty < \frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}}. \tag{5.165}$$

The mild solution v to (5.10) is given by

$$v(t) = e^{-tL_{\bar{u}}} v_0 + \int_0^t e^{-(t-s)L_{\bar{u}}} \mathbf{g}(v(s), \bar{u}) ds, \quad t \in I$$

Now let us repeat the steps of the proof of Theorem 5.24 up until (5.129). Thus, we arrive at

$$\left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)|\left\langle v(t),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)e^{-2\lambda_k t}|\left\langle v_0,\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}}$$

$$+\int_0^t\left(\sum_{k=1}^{\infty}(\lambda_k+\sigma)e^{-2\lambda_k(t-s)}|\left\langle \mathbf{g}(v(s),\bar{u}),\tilde{\varphi}_k\right\rangle_2|^2\right)^{\frac{1}{2}}ds,\quad t\in I. \tag{5.166}$$

Since $\lambda_k \geq z > 0$ and using (5.108), we obtain

$$q(v(t))\leq e^{-zt}q(v_0)+\int_0^t e^{-z(t-s)}q(\mathbf{g}(v(s),\bar{u}))ds,\quad t\in I.$$

Using (5.163), we have

$$q(v(t))\leq e^{-zt}q(v_0)+\tilde{\varepsilon}_0\int_0^t e^{-z(t-s)}q(v(s))ds,$$

$$\text{as long as}\quad \|v(t)\|_\infty\leq\frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}},\quad t\in I. \tag{5.167}$$

Note that by (5.165) we have $\|v_0\|_\infty < \frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}}$.

Gronwall's lemma applied to (5.167) yields

$$q(v(t))\leq e^{-(z-\tilde{\varepsilon}_0)t}q(v_0),\quad\text{as long as}\quad \|v(t)\|_\infty\leq\frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}},\quad t\in I.$$

From (5.109) follows

$$\|v(t)\|_\infty\leq\sqrt{\frac{C_1}{d_{\min}}q(v(t))}\leq\sqrt{\frac{C_1}{d_{\min}}e^{-(z-\tilde{\varepsilon}_0)t}q(v_0)},$$

$$\text{as long as}\quad \|v(t)\|_\infty\leq\frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}},\quad t\in I. \tag{5.168}$$

Inserting (5.162) into (5.168), we obtain

$$\|v(t)\|_\infty \le \sqrt{\frac{C_1}{d_{\min}}} e^{at} q(v_0), \quad \text{as long as} \quad \|v(t)\|_\infty \le \frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}}, \quad t \in I, \tag{5.169}$$

with $a = -\frac{\beta z}{1+\beta} < 0$.

Now since $q(v_0) < \dfrac{\delta(\tilde{\varepsilon}_0)\sqrt{d_{\min}}}{\sqrt{C_1 n}}$ (by (5.164)) from (5.169) follows

$$\|v(t)\|_\infty < \frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}}, \quad \text{as long as} \quad \|v(t)\|_\infty \le \frac{\delta(\tilde{\varepsilon}_0)}{\sqrt{n}} \quad t \in I,$$

which implies (by continuity):

$$\|v(t)\|_\infty \le K e^{at} q(v_0) \quad \forall t \in I, \tag{5.170}$$

with $K := \sqrt{\dfrac{C_1}{d_{\min}}}$. We see from (5.170) that the mapping $t \mapsto \|v(t)\|_\infty$ is bounded on I. By Theorem 2.12 we obtain the existence of the mild solution v in the large and the estimation (5.170) holds for all $t \ge 0$. □

Remark 5.31. *As one can see from the proof of Theorem 5.30 the upper bound to the domain of attraction is given by* $\delta_3 = \dfrac{\delta(\tilde{\varepsilon}_0)\sqrt{d_{\min}}}{\sqrt{C_1 n}}$. *Recalling Remark 5.26, we set* $a_1 = \eta$, $a_2 = \dfrac{d_{\max}}{d_{\min}}$, $a_3 = \dfrac{d_{\max}}{d_{\min}} \bar{U}_x C_1$, $\hat{\varepsilon}|h| = \tilde{\varepsilon}_0^2$ *and compute* $\delta(\tilde{\varepsilon}_0)$ *from* (5.142).

5.6.6 Proof of assumption ($\star$)

Now let us show that assumption ($\star$) holds. For that purpose we will need some results from measure theory. At first, let us introduce

Definition 5.32. *Let* $(S, \mathcal{B}, m)$ *be a measure space, and* f *a mapping defined on* S *with values in a Banach space* X. f *is called weakly measurable if, for any* $\phi \in X^*$, *the numerical function $\phi(f(s)) = \langle f(s), \phi\rangle$ of s is measurable. f is said to be finitely-valued if it is constant $\neq 0$ on each of a finite number of disjoint measurable sets B_j with $m(B_j) < \infty$ and $f(s) = 0$ on $S - \bigcup_j B_j$. f is said to be strongly measurable if there exists a sequence of finitely-valued functions convergent (in the norm of X) to $f(s)$ m-a.e. on S.*

Definition 5.33. *f is said to be separably-valued if its range $\{f(s); s \in S\}$ is separable. It is m-almost-separably-valued if there exists a measurable set B_0 of m-measure zero such that $\{f(s); s \in S - B_0\}$ is separable.*

Definition 5.34. *A function f defined on a measure space $(S, \mathcal{B}, m)$ with values in a Banach space X is said to be Bochner integrable, if there exists a sequence of finitely-valued functions f_n which converges to f m-a.e. in such a way that*

$$\lim_{n\to\infty} \int_S \|f(s) - f_n(s)\|_X \; m(ds) = 0.$$

Finally, we will need the following two theorems

Theorem 5.35. *[64, Pettis Theorem, p. 131] f is strongly measurable if and only if it is weakly measurable and m-almost separably valued.*

Theorem 5.36. *[12, Theorem 8, p. 650] A strongly measurable function $f : [0, t] \to X$ is Bochner integrable if and only if $s \mapsto \|f(s)\|_X$ is integrable. In this case*

$$\left\| \int_0^t f(s)\, ds \right\|_X \leq \int_0^t \|f(s)\|_X \; ds.$$

Now we introduce

Proposition 5.37. *Let $\mathbf{g} \in C(I, H_1^B(0, l))$. Let $f(s) = (f_k(s))_{k\in\mathbb{N}}$ with $f_k(s)$ given by* (5.127). *Then*

$$\left\| \int_0^t f(s)\, ds \right\|_{l_2} \leq \int_0^t \|f(s)\|_{l_2} \; ds.$$

Proof. According to Theorem 5.36 if we can show that

(1) $f(s)$ is a strongly measurable function

(2) $\int_0^t \|f(s)\|_{l_2}\, ds < \infty$

then the assertion of the proposition would follow.

Let us start with assumption (1). Due to Theorem 5.35 if f is weakly measurable and m-almost separably valued, then it is strongly measurable. The m-almost separability immediately follows from the fact that $X = l_2$, which is a separable space. Let us show that the first requirement of Theorem 5.35 holds as well. Recall that every $\phi \in (l_2)^*$ is of the form

$$\phi : (l_2)^* \to \mathbb{R} : (x_k)_{k\in\mathbb{N}} \mapsto \sum_{k=1}^{\infty} x_k z_k$$

for some $z = (z_k)_{k\in\mathbb{N}} \in l_2$. Then for all $s \in [0,t]$ we have

$$\phi(f(s)) = \sum_{k=1}^{\infty} f_k(s) z_k.$$

Since $(f_k)_{k\in\mathbb{N}} \subset C[0,t]$, then $\phi(f)$ is measurable. Hence the first requirement of Theorem 5.35 is satisfied. Thus f is strongly measurable.

We show assumption (2) as follows. Due to the positivity of the spectrum of $L_{\bar{u}}$ we have

$$\begin{aligned}\|f(s)\|_{l_2} &= \left(\sum_{k=1}^{\infty} (\lambda_k + \sigma) e^{-2\lambda_k (t-s)} \left|\langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k \rangle_2\right|^2\right)^{\frac{1}{2}} \\ &\leq \left(\sum_{k=1}^{\infty} (\lambda_k + \sigma) \left| \langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k \rangle_2 \right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Since $\mathbf{g} \in C(I; H_1^B(0,l))$ by Lemma 5.20 we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty} (\lambda_k + \sigma) \left| \langle \mathbf{g}(v(s), \bar{u}), \tilde{\varphi}_k \rangle_2 \right|^2 \\
&= \int_0^l \mathbf{g}_x(v(x,s), \bar{u}(x))^T D \overline{\mathbf{g}_x(v(x,s), \bar{u}(x))} dx \\
&\quad + \int_0^l \mathbf{g}(v(x,s), \bar{u}(x))^T (C_{\bar{u}}(x)^T + \sigma I) \, \overline{\mathbf{g}(v(x,s), \bar{u}(x))} \, dx \\
&\leq d_{\max} \|\mathbf{g}_x(v(s), \bar{u})\|_2^2 + \max_{x \in [0,l]} \lambda_{\max}(C_{\bar{u}}(x)^T + \sigma I) \, \|\mathbf{g}(v(s), \bar{u})\|_2^2 \\
&\leq \underbrace{\max\{d_{\max}, \max_{x \in [0,l]} \lambda_{\max}(C_{\bar{u}}(x)^T + \sigma I)\}}_{=:\tilde{K}} \|\mathbf{g}(v(s), \bar{u})\|_{H_1^n(0,l)}^2 ,
\end{aligned}$$

and therefore

$$\int_0^t \|f(s)\|_{l_2} \, ds \leq \sqrt{\tilde{K}} \int_0^t \|\mathbf{g}(v(s), \bar{u})\|_{H_1^n(0,l)} \, ds.$$

Since $\mathbf{g}$ is continuous on I, its integral over $[0,t]$ is finite and we have

$$\int_0^t \|f(s)\|_{l_2} \, ds < \infty.$$

Hence the requirements of Theorem 5.36 are satisfied.

The proof of proposition is complete. $\square$

5.6.7 Outlook

In this subsection we would like to discuss whether it is possible to extend the classes presented in Proposition 5.18.

We start with the general case, when the transformation matrix $T \in \mathbb{R}^{n \times n}$ is *not diagonal*. The matrix $C_{\bar{u}} \in \mathbb{R}^{n \times n}$ is still required to be constant at the moment. After the transformation we arrive at the linearised operator of the form

$$\widetilde{L}_{\bar{u}} w(t) = -T^{-1} D T w_{xx}(t) + T^{-1} C_{\bar{u}} T w(t).$$

Our goal is to find conditions on the elements of the matrix $C_{\bar{u}}$ which provide the symmetry of $\widetilde{D} = T^{-1}DT$ and $\widetilde{C}_{\bar{u}} = T^{-1}C_{\bar{u}}T$. In the following let us denote the elements of the inverse matrix T^{-1} as t_{ij}^{-1}, $i, j = 1, \ldots, n$. A straightforward computation shows that the symmetry requirement for $\widetilde{D}$ and $\widetilde{C}_{\bar{u}}$ results in the following system of nonlinear equations

$$\begin{cases} \sum_{i=1}^{n} d_i \left(t_{ik}t_{li}^{-1} - t_{il}t_{ki}^{-1}\right) = 0, \\ \sum_{i,j=1}^{n} c_{ji} \left(t_{ik}t_{lj}^{-1} - t_{il}t_{kj}^{-1}\right) = 0, \end{cases} \qquad k, l = 1, \ldots, n. \tag{5.171}$$

It is easy to see that in case $n = 2$ system (5.171) reduces to the system

$$\begin{cases} t_{11}t_{21} + t_{12}t_{22} = 0, \\ \dfrac{t_{11}^2 + t_{12}^2}{t_{21}^2 + t_{22}^2} = \dfrac{c_{12}}{c_{21}}, \end{cases}$$

which results in the following elements of the matrix T:

$$\begin{cases} t_{11} = \mp\sqrt{\dfrac{c_{12}}{c_{21}}}t_{22}, \\ t_{12} = \pm\sqrt{\dfrac{c_{12}}{c_{21}}}t_{21}. \end{cases}$$

Hence, for $n = 2$, the conditions on the elements of matrix $C_{\bar{u}}$ remain the same as it was stated in Proposition 5.18. In general case of $n > 3$ the situation becomes more complicated, since it is hard to find an analytical solution of system (5.171). Nevertheless, for some fixed values of c_{ij} and d_i it is possible to solve the nonlinear system above with the help of the verified Newton computation (combined with interval arithmetic). As a result one obtains an interval-valued transformation matrix T and, consequently, an interval-valued matrix $\widetilde{C}_{\bar{u}}$. In that case for the further investigations it would be advisable to use the midpoints of the elements of matrix $\widetilde{C}_{\bar{u}}$. In addition, one would have to introduce a perturbation argument to this matrix.

Now let us consider the case of a *non-constant* stationary solution $\bar{u}$ and *diagonal* matrix T. Notice that for the transformation T to work, T should be a constant matrix. It is easy to see that when the stationary solution is not constant, then condition (5.84) reads

$$\frac{t_i}{t_j} = \sqrt{\frac{c_{ij}(x)}{c_{ji}(x)}}, \quad \forall x \in [0, l], \quad \forall i, j = 1, \ldots, n. \tag{5.172}$$

Hence the transformation T is valid, if:

1. For $n = 2$: $c_{ij}(x) \neq 0$, $\forall x \in [0, l]$, $(i, j = 1, 2)$ and $\dfrac{c_{12}(x)}{c_{21}(x)} = B$, with B being some positive constant.

2. For $n = 3$: $c_{ij}(x) \neq 0$, $\forall x \in [0, l]$, $(i, j = 1, 2, 3)$ and there exist positive constants B_1, B_2, B_3 such that for all $x \in [0, l]$

$$\frac{c_{12}(x)}{c_{21}(x)} = B_1,$$

$$\frac{c_{31}(x)}{c_{13}(x)} = B_2,$$

$$\frac{c_{23}(x)}{c_{32}(x)} = B_3,$$

with $B_1 B_2 B_3 = 1$.

In case $n = 2$ the conditions above hold, for example, for the nonlinearity of the form

$$F(u_1, u_2) = \begin{pmatrix} f_1(u_1) + b_1 u_1^q u_2^r + c_1 \\ \\ f_2(u_2) + b_2 u_1^{q+1} u_2^{r-1} + c_2 \end{pmatrix},$$

with $b_1, b_2, r, q > 0$, and C^1-functions f_1 and f_2.

To conclude, we wish to remark that the results from this section could be extended to a domain $\Omega \in R^m$ with $m = 2, 3$. In that case the "explicit" embedding estimations from Lemma 2.21 (or Lemma 2.23) should be changed appropriately. One can find this estimations for the case of a single equation in [42, Corollary 1, p.42].

6

Bounds for eigenvalues

Our main concern in this chapter will be a computation of verified bounds to N smallest (with suitable $N \in \mathbb{N}$) eigenvalues of some given eigenvalue problem. For self-adjoint operators there are several approaches which aim at this purpose. Our choice will be to use the Rayleigh-Ritz method for the computation of upper bounds of eigenvalues, the right-definite Lehmann method and Lehmann-Goerisch method for the computation of lower bounds, and the homotopy method, which will provide us with some necessary a priori information. The main focus of this chapter is on the application of these variational methods to particular eigenvalue problems, which arise in the course of our investigations. Below we introduce a list of these problems. For simplicity, we drop the index ω in the notation of the operator L_ω. We consider:

1. Eigenvalue problem for computing the constant K:

$$u \in H_2^B(0,l), \quad \langle Lu, Lv\rangle_2 = \lambda\left(\beta \langle u,v\rangle_2 + \langle u', v'\rangle_2\right) \text{ for all } v \in H_2^B(0,l).$$

2. Eigenvalue problem which is under consideration during the eigenvalue exclosure procedure:

$$u \in H_2^B(0,l),\ \mu \in \mathbb{C}, \quad \langle (L-\mu)u, (L-\mu)v\rangle_2 = \lambda \langle u,v\rangle_2 \text{ for all } v \in H_2^B(0,l).$$

3. Eigenvalue problem which is associated with the self-adjoint L:

$$u \in H_2^B(0,l), \quad Lu = \lambda u. \tag{6.1}$$

The eigenvalue problem above is under consideration for two purposes: computation of the constant K and verification of the stability of a stationary solution.

As one can see, the first two types of eigenvalue problems can be united into one generic eigenvalue problem of the form

$$\begin{gathered} u \in H_2^B(0,l), \quad \langle (L-\nu)u, (L-\nu)v \rangle_2 = \lambda \left(\beta_1 \langle u, v \rangle_2 + \beta_2 \langle u', v' \rangle_2 \right) \\ \text{for all } v \in H_2^B(0,l) \end{gathered} \tag{6.2}$$

The positive constants β_1 and β_2 and $\nu \in \mathbb{C}$ are chosen as follows.

$$(\beta_1, \beta_2) = \begin{cases} (\beta, 1), & \text{for eigenvalue problem of type one,} \\ (1, 0), & \text{for eigenvalue problem of type two,} \end{cases}$$

$$\nu = \begin{cases} 0, & \text{for eigenvalue problem of type one,} \\ \mu, & \text{for eigenvalue problem of type two.} \end{cases}$$

The main reason for this unification is to simplify the description of the application of the variational methods to the problems above. This way we do not have to discuss each problem separately, but merely change the settings for the parameters β_1, β_2 and ν. The eigenvalue problem of type three will be handled separetely.

This chapter is organised as follows: in the first section we present a general outline of the variational methods. In the succeeding two sections we discuss the application of these methods to eigenvalue problems (6.2) and (6.1).

6.1 Variational methods for computing eigenvalue bounds

Let $\mathcal{M}$ be a positive definite hermitian sesquilinear form defined on $D(\mathcal{M}) = H$, where H is an infinite dimensional separable Hilbert space. Let $\mathcal{N}$ be a bounded positive definite hermitian sesquilinear form on H. We consider the following eigenvalue problem

$$\mathcal{M}(u,v) = \lambda \mathcal{N}(u,v) \quad \text{for all } v \in H. \tag{6.3}$$

There exists a sequence of eigenvalues $\lambda_k \in \mathbb{R}$, $\forall k \in \mathbb{N}$ such that

$$\lambda_1 \leq \lambda_2 \leq \ldots,$$

with $\lambda_k \to \infty$ as $k \to \infty$.

Below we introduce a sequence of theorems which we are going to apply in order to compute upper and lower bounds of the first N eigenvalues of (6.3). One can immediately see that the eigenvalue problem (6.2) is of type (6.3). The eigenvalue problem (6.1) can be seen as a special case of (6.3) as well, since in practice one would consider

$$\begin{aligned} \mathcal{M}(u,v) &= \langle Lu, v\rangle_2 = \langle Du', v'\rangle_2 + \langle Cu, v\rangle_2 \\ \mathcal{N}(u,v) &= \langle u, v\rangle_2, \end{aligned} \tag{6.4}$$

with $H = D_p(L) = H_2^B(0,l)$. We will avoid the detailed discussion of above theorems and for more thorough overview we refer to, e.g., [44]. We start with the Rayleigh-Ritz method for computing upper bounds of eigenvalues. This method is based on the Poincaré min-max-principle and could be found in [49, Theorem 40.1 and Remarks 40.1, 40.2, 39.10].

Theorem 6.1 (Rayleigh-Ritz)**.** *Let $\tilde{u}_1, \ldots, \tilde{u}_N \in H$ be linearly independent (approximative eigenelements). Define the symmetric matrices*

$$A_0 = (\mathcal{M}(\tilde{u}_i, \tilde{u}_j))_{i,j=1\ldots N}, \tag{6.5}$$

$$A_1 = (\mathcal{N}(\tilde{u}_i, \tilde{u}_j))_{i,j=1\ldots N}. \tag{6.6}$$

Let $\Lambda_1 \leq \cdots \leq \Lambda_N$ be the eigenvalues of

$$A_0 x = \Lambda A_1 x. \tag{6.7}$$

Then the following inequalities hold:

$$\lambda_k \leq \Lambda_k \quad \textit{for} \quad k = 1, \dots, N. \tag{6.8}$$

By means of the Rayleigh-Ritz method we can compute approximations for the eigenvalues and upper bounds to them. Since our aim is to obtain *verified* upper bounds, we should perform the Rayleigh-Ritz method twice. During the first computation the size of the matrix eigenvalue problem (6.7) is set to M such that $M > N$, and the trial functions $\tilde{u}_1, \dots, \tilde{u}_M \in H$ are chosen to be some suitable ansatz functions $\tilde{\varphi}_1, \dots, \tilde{\varphi}_M \in H$. After the eigenvalue problem (6.7) is solved by means of some known numerical method (e.g. Cholesky method, QR-algorithm) the values $\Lambda_1, \dots, \Lambda_M$ and the eigenvectors $x^{(1)}, \dots, x^{(M)} \in \mathbb{R}^M$, satisfying $(x^{(k)})^T A_1 \overline{x^{(l)}} = \delta_{kl}$, $(x^{(k)})^T A_0 \overline{x^{(l)}} = \Lambda_k \delta_{kl}$ are obtained. The values $\Lambda_1, \dots, \Lambda_N$ should be the good approximations to the eigenvalues $\lambda_1, \dots, \lambda_N$ and the elements

$$\tilde{u}_i^{\text{new}} := \sum_{j=1}^{M} x_j^{(i)} \tilde{u}_j, \tag{6.9}$$

approximate the eigenelements. After that we repeat the step above setting $M = N$ this time and taking $\tilde{u}_i^{\text{new}},\ i = 1, \dots, N$ as trial functions. In order to avoid rounding errors we perform this second computation using interval arithmetic.

In the following for simplicity we write $\tilde{u}_i$ instead of $\tilde{u}_i^{\text{new}}$.

Next, for computing the lower bounds to the eigenvalues we introduce the right-definite Lehmann method for problem (6.4) and the Lehmann-Goerisch method for problem (6.3). More details on these methods one can find in [15, 16, 17, 18, 28].

Theorem 6.2 (right-definite Lehmann method)**.** *Let $\tilde{u}_1, \dots, \tilde{u}_N$ be linearly independent functions (approximative eigenelements) in H. And let $\rho \in \mathbb{R}$ exists such* *that:* $\Lambda_N < \rho \le \lambda_{N+1}$*. Define the matrices*

$$A_0 = \left(\langle L\tilde{u}_i, \tilde{u}_j\rangle_2\right)_{i,j=1\ldots N},$$

$$A_1 = \left(\langle \tilde{u}_i, \tilde{u}_j\rangle_2\right)_{i,j=1\ldots N},$$

$$A_2 = \left(\langle L\tilde{u}_i, L\tilde{u}_j\rangle_2\right)_{i,j=1\ldots N}.$$

Let $\tau_1 \le \tau_2 \le \cdots \le \tau_N < 0$ *denote the negative eigenvalues of*

$$(A_0 - \rho A_1)x = \tau\,(A_2 - 2\rho A_0 + \rho^2 A_1)x. \tag{6.10}$$

Then

$$\lambda_k \ge \rho + \frac{1}{\tau_{N+1-k}} \quad (k = 1, \ldots, N). \tag{6.11}$$

Let us point out that the eigenvalue bounds derived by the means of the right-definite Lehmann method are optimal in the sense that no better bounds can be computed based on the knowledge of $\tilde{u}_1, \ldots, \tilde{u}_N$ and $L\tilde{u}_1, \ldots, L\tilde{u}_N$.

We continue with lower bounds for problem (6.3). Let us introduce

Theorem 6.3 (Lehmann-Goerisch)**.** *Let* $\tilde{u}_1, \ldots, \tilde{u}_N$ *be linearly independent functions (approximative eigenelements) in* H*. And let* $\rho \in \mathbb{R}$ *exists such that:* $\Lambda_N < \rho \le \lambda_{N+1}$*.*

Let $\mathcal{X}_G$ *be some vector space,* b *some positive definite symmetric bilinear form on* $\mathcal{X}_G$*, and* $T : H \to \mathcal{X}_G$ *be some linear operator such that*

$$b(T\varphi, T\psi) = \mathcal{M}(\varphi, \psi) \quad \textit{for all} \quad \varphi, \psi \in H. \tag{6.12}$$

Let $w^{(1)}, \ldots, w^{(N)} \in \mathcal{X}_G$ *satisfy*

$$b(T\varphi, w^{(i)}) = \mathcal{N}(\varphi, \tilde{u}_i), \quad (i = 1, \ldots, N) \quad \textit{for all} \quad \varphi \in H. \tag{6.13}$$

Form the third matrix $A_2 := (b(w^{(i)}, w^{(j)}))_{i,j=1,\ldots,N}$ *and let* $\tau_1 \le \cdots \le \tau_N < 0$ *be the eigenvalues of the matrix eigenvalue problem*

$$(A_0 - \rho A_1)x = \tau\,(A_0 - 2\rho A_1 + \rho^2 A_2)x, \tag{6.14}$$

where A_0 and A_1 are given by (6.5) *and* (6.6). *Then,*

$$\lambda_k \geq \rho - \frac{\rho}{1-\tau_{N+1-k}} \quad (k = 1, \ldots, N). \tag{6.15}$$

We would like to make a following remark concerning the Lehmann-Goerisch method.

Remark 6.4. *The method of Lehmann-Goerisch is in fact an "improved" version of the left-definite Lehmann method, earlier proposed by Lehmann. In the left-definite Lehmann method the condition (6.13) is replaced by the condition:*

$$\mathcal{M}(\varphi, \hat{w}^{(i)}) = \mathcal{N}(\varphi, \tilde{u}_i), \quad (i = 1, \ldots, N) \quad \textit{for all} \quad \varphi \in H \tag{6.16}$$

and the matrix A_2 is replaced by $\hat{A}_2 = \mathcal{M}(\hat{w}^{(i)}, \hat{w}^{(j)})_{i,j=1,\ldots,N}$. For condition (6.16) the explicite knowledge of $\hat{w}^{(1)}, \ldots, \hat{w}^{(N)}$ is required. Since it is not always possible to compute the values $\hat{w}^{(1)}, \ldots, \hat{w}^{(N)}$ explicitly, the practical implementation of the left-definite Lehmann method is rather difficult. However it is possible to overcome this problem by introducing the "$\mathcal{X}_G bT$-concept" and replacing (6.16) by (6.13), as it was done by Goerisch. The lower bounds of the Lehmann-Goerisch method are worse (respectively not better) than the bounds of left-definite Lehmann method, but they are computable.

It is intuitively clear that in order to make Lehmann-Goerisch's bounds 'close' to left-definite Lehmann methods bounds one should construct the matrices A_2 and $\hat{A}_2$ to be 'similar' to each other, i.e.:

$$b(w^{(i)}, w^{(j)}) \approx \mathcal{M}(\hat{w}^{(i)}, \hat{w}^{(j)}) \tag{6.17}$$

From the view of (6.12) follows: $w^{(i)} \approx T\hat{w}^{(i)}$, $(i = 1, \ldots, N)$. So, if $\tilde{u}_i$, $(i = 1, \ldots, N)$ are good eigenelements approximations and $\tilde{\lambda}_i$, $(i = 1, \ldots, N)$ are eigenvalue approximations, then (6.16) would provide: $\hat{w}^{(i)} \approx \dfrac{1}{\tilde{\lambda}_i}\tilde{u}_i$, $(i = 1, \ldots, N)$, i.e.

$w^{(i)}$ could be chosen as:

$$w^{(i)} \approx \frac{1}{\tilde{\lambda}_i} T\tilde{u}_i, \quad (i = 1, \ldots, N), \tag{6.18}$$

taking condition (6.13) also into account.

Notice that in both cases (right-definite Lehmann method and Lehmann-Goerisch method) an *a priori* information about the lower bound ρ to the eigenvalue λ_{N+1} is required. In order to satisfy this requirement we apply the method of homotopy which is based on the comparison principle. The main idea of the homotopy method is it to find a sequence of eigenvalue problems $(EP)_s$, $s \in [0, 1]$ which satisfies:

(H1) the eigenvalues $(\lambda_k^0)_{k\in\mathbb{N}}$ of $(EP)_0$ are computable in closed form (or at least lower bounds to them are available),

(H2) $\lambda_k^0 \le \lambda_k^t \le \lambda_k^s \le \lambda_k^1$ for all k, $0 \le t \le s \le 1$,

(H3) the eigenvalues $(\lambda_k^1)_{k\in\mathbb{N}}$ are the eigenvalues of the given problem.

Starting at the known values of $(\lambda_k^0)_{k\in\mathbb{N}}$ we go forward on s. Making usage of the monotonicity property (H2), we transfer the known information about the sequence $(\lambda_k^0)_{k\in\mathbb{N}}$ onto the sequence $(\lambda_k^{s_1})_{k\in\mathbb{N}}$, $s_1 > 0$. Having the information for the sequence $(\lambda_k^{s_1})_{k\in\mathbb{N}}$, we repeat the step above, taking this time s_1 instead of 0 and some $s_2 > s_1$ instead of s_1. The algorithm continues until we arrive at the given problem. The information which is transferred in the course of homotopy in our case is the value for the lower bound ρ. We will give a detailed description of this process later.

If the spectrum of the given eigenvalue problem does not contain any clusters, then the implementation of the homotopy process can be simplified, due to the following

Corollary 6.5. *[6, Corollary 1, p. 76] Let $\mathcal{X}_G$, b, T be defined as above. Let $\tilde{u} \in H$, $\tilde{u} \neq 0$ and $w \in \mathcal{X}_G$ such that* (6.13) *holds (with $w, \tilde{u}$ instead of $w^{(i)}, \tilde{u}_i$). Moreover,* *let $\rho > 0$ be chosen such that there are at most finitely many eigenvalues of* (6.3) *below ρ, and*

$$\frac{\mathcal{M}(\tilde{u},\tilde{u})}{\mathcal{N}(\tilde{u},\tilde{u})} < \rho. \tag{6.19}$$

Then, there is an eigenvalue λ of problem (6.3) *satisfying*

$$\frac{\rho\mathcal{N}(\tilde{u},\tilde{u}) - \mathcal{M}(\tilde{u},\tilde{u})}{\rho b(w,w) - \mathcal{N}(\tilde{u},\tilde{u})} \leq \lambda < \rho. \tag{6.20}$$

We conclude this section by presenting an outline of the homotopy method.

Let us begin with the general case, which includes the possibility of the clustered eigenvalues. Suppose that

$$u \in H, \quad \mathcal{M}_s(u,v) = \lambda^{(s)}\mathcal{N}(u,v), \quad \text{for all} \quad v \in H, \quad s \in [0,1] \tag{6.21}$$

is an eigenvalue sequence $(EP)_s$, satisfying hypothesis (H1), (H2), and (H3). We will be establishing this sequence later (see Section 6.2 and Section 6.3).

(1) We start with choosing $N_h > N$, where N denotes the number of eigenvalues we wish to estimate, such that:

$$\lambda^{(0)}_{N_h+1} > \lambda^{(1)}_{N}. \tag{6.22}$$

This choice is always possible due to the fact that $\lambda^{(0)}_{N_h} \to \infty$ as $N_h \to \infty$. Moreover, we assume that $\lambda^{(0)}_{N_h+1}$ and $\lambda^{(0)}_{N_h}$ are sufficiently separated from each other. Condition (6.22) is easy to check since a lower bound $\underline{\lambda}^{(0)}_{N_h+1}$ to $\lambda^{(0)}_{N_h+1}$ is available and the upper bound $\Lambda^{(1)}_N$ to $\lambda^{(1)}_N$ can be computed. Thus (6.22) is satisfied if:

$$\underline{\lambda}^{(0)}_{N_h+1} > \Lambda^{(1)}_{N}. \tag{6.23}$$

Since $(\lambda^{(0)}_k)_{k\in\mathbb{N}}$ are explicitly known, $\underline{\lambda}^{(0)}_{N_h+1}$ is easy to obtain. The upper bound to $\lambda^{(1)}_N$ is calculated with the Rayleigh-Ritz method.

(2) In the next step we are looking for such s_1 that:

$$\lambda_{N_h}^{(s_1)} < \lambda_{N_h+1}^{(0)}. \tag{6.24}$$

We satisfy (6.24) using the same strategy as in the step (1), namely by testing:

$$\Lambda_{N_h}^{(s_1)} < \underline{\lambda}_{N_h+1}^{(0)}. \tag{6.25}$$

Here $\Lambda_{N_h}^{(s_1)}$ denotes the upper bound to the eigenvalue $\lambda_{N_h}^{(s_1)}$, which one can obtain by means of the Rayleigh-Ritz method. Observe that s_1 exists due to (H2) and the fact that $\lambda_{N_h+1}^{(0)}$ and $\lambda_{N_h}^{(0)}$ are sufficiently separated from each other. We choose s_1 so, that it is close to $\sup\{s \in [0,1] : \Lambda_{N_h}^{(s_1)} < \underline{\lambda}_{N_h+1}^{(0)}\}$, i.e. the lower bound to $\lambda_{N_h+1}^{(0)}$ and the upper bound to $\lambda_{N_h}^{(s_1)}$ should almost 'hit' each other. One can use the bisection method for the determination of s_1.

Since (6.25) is satisfied, we proceed further with the calculation of a lower bound $\underline{\lambda}_{N_h}^{(s_1)}$ to $\lambda_{N_h}^{(s_1)}$ using the right-definite Lehmann method (or the Lehmann-Goerisch) method. As it was already mentioned above, in order to use the right-definite Lehmann (or the Lehmann-Goerisch) method we first need *a priori* information about the lower bound ρ of the eigenvalue $\lambda_{N_h+1}^{(s_1)}$. Since the sequence $(\lambda_k^{(s)})_{k\in\mathbb{N}}$ is increasing in s and with regards to (6.25), the most suitable choice for ρ which is available, is

$$\rho := \underline{\lambda}_{N_h+1}^{(0)}. \tag{6.26}$$

Having ρ, we can easily compute the lower bounds to $\lambda_k^{(s_1)}$ $(k = 1, \dots, N_h)$.

(3) The next step we perform the same way as above, this time looking for $s_2 > s_1$, such that:

$$\lambda_{N_h-1}^{(s_2)} < \lambda_{N_h}^{(s_1)}. \tag{6.27}$$

At first we consider the case when $\lambda^{(s_1)}_{N_h-1}$ and $\lambda^{(s_1)}_{N_h}$ are sufficiently separated from each other. Condition (6.27) holds true, if:

$$\Lambda^{(s_2)}_{N_h-1} < \underline{\lambda}^{(s_1)}_{N_h}, \tag{6.28}$$

where $\Lambda^{(s_2)}_{N_h-1}$ is calculated using the Rayleigh-Ritz method and $\underline{\lambda}^{(s_1)}_{N_h}$ is known from the previous step. We choose s_2 so, that it is close to $\sup\{s \in (s_1, 1] : \Lambda^{(s_2)}_{N_h-1} < \underline{\lambda}^{(s_1)}_{N_h}\}$. Next, we set $\rho := \underline{\lambda}^{(s_1)}_{N_h}$ and perform the computation of the lower bounds to $\lambda^{(s_2)}_k$ $(k = 1, \ldots, N_h - 1)$.

Now suppose that the eigenvalues $\lambda^{(s_1)}_{N_h-1}$ and $\lambda^{(s_1)}_{N_h}$ belong to a cluster $\lambda^{(s_1)}_{N_h-K_1}, \ldots \lambda^{(s_1)}_{N_h}$. In that case we choose $s_2 \in (s_1, 1]$ so that it is close to $\sup\{s \in (s_1, 1] : \lambda^{(s_2)}_{N_h-K_1-1} < \underline{\lambda}^{(s_1)}_{N_h-K_1}\}$ (observe that $\lambda^{(s_1)}_{N_h-K_1-1}$ and $\lambda^{(s_1)}_{N_h-K_1}$ are sufficiently separated from each other). After that we set $\rho := \underline{\lambda}^{(s_1)}_{N_h-K_1}$ and proceed with the right-definite Lehmann (or the Lehmann-Goerisch) method as usual, computing lower bounds to $\lambda^{(s_2)}_k$, $(k = 1, \ldots, N_h - K - 1)$.

The algorithm continues as described in steps (2) and (3) until $s = 1$ or there are no eigenvalues left any more. In the latter case, we should start the algorithm from the beginning, taking some $\tilde{N}_h > N_h$ this time. Generally the starting value of N_h should be large enough for us to expect that at $s = 1$ we will arrive at N eigenvalues.

Now let us consider the case when the given eigenvalue problem has no clustered eigenvalues. In that case during the homotopy Corollary 6.5 will be used.

We consider the sequence of eigenvalue problems (6.21). We repeat the first step of the homotopy as it was described for the general case. We proceed with the second step as follows:

(2) we look for some $s_1 > 0$ such that

$$\frac{\mathcal{M}_{s_1}\left(\tilde{u}_{N_h}^{(s_1)}, \tilde{u}_{N_h}^{(s_1)}\right)}{\mathcal{N}\left(\tilde{u}_{N_h}^{(s_1)}, \tilde{u}_{N_h}^{(s_1)}\right)} < \underline{\lambda}_{N_h+1}^{(0)}. \tag{6.29}$$

In particular, we choose s_1 close to $\sup\{s \in [0,1] : (6.29) \text{ holds }\}$. Due to Corollary 6.5 we obtain that there exists an eigenvalue $\lambda^{(s_1)}$ such that

$$\rho_1 \leq \lambda^{(s_1)} < \underline{\lambda}_{N_h+1}^{(0)}, \tag{6.30}$$

where ρ_1 denotes the lower bound of the interval in (6.20). Observe that due to the monotonicity condition (H2) we may expect that at most N_h eigenvalues lie in the interval $(0, \underline{\lambda}_{N_h+1}^{(0)})$ and therefore at most $N_h - 1$ eigenvalues belong to $(0, \rho_1)$. Thus, assuming that $\lambda_{N_h+1}^{(s_1)}$ and $\lambda_{N_h}^{(s_1)}$ are sufficiently separated from each other, we conclude that $\lambda^{(s_1)} = \lambda_{N_h}^{(s_1)}$.[4]

(3) Next, we look for $s_2 > s_1$ such that

$$\frac{\mathcal{M}_{s_2}\left(\tilde{u}_{N_h-1}^{(s_2)}, \tilde{u}_{N_h-1}^{(s_2)}\right)}{\mathcal{N}\left(\tilde{u}_{N_h-1}^{(s_2)}, \tilde{u}_{N_h-1}^{(s_2)}\right)} < \rho_1. \tag{6.31}$$

Following the same strategy as in the step (2) we conclude the existence of an eigenvalue $\lambda_{N_h-1}^{(s_2)}$ in the interval $[\rho_2, \rho_1)$, with ρ_2 being the lower bound from (6.20).

The algorithm continues until $s = 1$ or there are no eigenvalues left (in which case we have to start the homotopy from the beginning, choosing this time some $\tilde{N}_h > N_h$). Let $\hat{\rho}$ denote the lower bound of the interval in (6.20), which we have at hand when

[4] This conclusion could be verified by the means of the Rayleigh-Ritz computation of $\Lambda_{N_h}^{(s_1)}$. We choose to omit the implementation of this step due to the fact that the final Rayleigh-Ritz computation at $s = 1$ will either show the conclusion above a posteriori or show that the homotopy was not successful.

we start the enclosure of $\hat{n} \geq N$ eigenvalues at $s = 1$. At that point we have to perform the verified Rayleigh-Ritz computation in order to make sure that $\hat{\rho} \geq \Lambda_{\hat{n}}^{(1)}$, which should be the case if our previous assumptions were correct. After that we compute a lower bounds to $\lambda_k^{(1)}$ $(k = 1, \ldots, \hat{n})$ as it was described in Theorem 6.2 (or Theorem 6.3).

6.2 Variational eigenvalue bounds for problem (6.2)

6.2.1 Sequence of eigenvalue problems for the homotopy

We consider the eigenvalue problem of the form (6.2). Recall that the linear operator L is given by

$$Lu := -Du'' + \mathcal{C}u, \quad u \in H_2^B(0, l).$$

Here, $D = \mathrm{diag}(d_1, \ldots, d_n)$ and

$$(\mathcal{C}u)(x) := C(x)u(x), \quad x \in [0, l],$$

where $C(x)$ is a $n \times n$ differentiable matrix. After a straightforward computation we obtain

$$\begin{aligned}
\langle (L-\nu)u, (L-\nu)u \rangle_2 &= \langle -Du'' + \mathcal{C}u - \nu u, -Du'' + \mathcal{C}u - \nu u \rangle_2 \\
&= \langle -Du'', -Du'' \rangle_2 + \langle \mathcal{C}u, -Du'' \rangle_2 + \langle -Du'', \mathcal{C}u \rangle_2 \\
&\quad + \langle \mathcal{C}u, \mathcal{C}u \rangle_2 - \nu \langle u, -Du'' \rangle_2 - \bar{\nu} \langle -Du'', u \rangle_2 \\
&\quad - \nu \langle u, \mathcal{C}u \rangle_2 - \bar{\nu} \langle \mathcal{C}u, u \rangle_2 + |\nu|^2 \langle u, u \rangle_2 . \qquad (6.32)
\end{aligned}$$

Let us consider the second and the third terms of the right-hand side of (6.32). After partial integration, taking into account the boundary conditions (Dirichlet or

Neumann), we obtain:

$$\begin{aligned}\langle \mathcal{C}u, -Du''\rangle_2 &= -\int_0^l u(x)^T C(x)^T D\overline{u''(x)}\, dx \\ &= -u(x)^T C(x)^T D\overline{u'(x)}\Big|_0^l + \int_0^l \left(u(x)^T C(x)^T\right)' D\overline{u'(x)}\, dx \\ &= \int_0^l (u'(x))^T C(x)^T D\overline{u'(x)}\, dx + \int_0^l u(x)^T \left(C'(x)\right)^T D\overline{u'(x)}\, dx.\end{aligned}$$

Likewise

$$\langle -Du'', \mathcal{C}u\rangle_2 = \int_0^l (u'(x))^T D\overline{C(x)}\;\overline{u'(x)}\, dx + \int_0^l (u'(x))^T D\overline{C'(x)}\;\overline{u(x)}\, dx.$$

Adding both expressions yields

$$\begin{aligned}\langle \mathcal{C}u, -Du''\rangle_2 + \langle -Du'', \mathcal{C}u\rangle_2 = &\int_0^l (u'(x))^T \left(C(x)^T D + D\overline{C(x)}\right)\overline{u'(x)}\, dx \\ &+ \int_0^l u(x)^T \left(C'(x)\right)^T D\overline{u'(x)}\, dx \\ &+ \int_0^l (u'(x))^T D\overline{C'(x)}\;\overline{u(x)}\, dx. \qquad (6.33)\end{aligned}$$

Since $C(x)^T D + D\overline{C(x)}$ is a Hermitian matrix, we estimate the first term of (6.33) as

$$\int_0^l (u'(x))^T \left(C(x)^T D + D\overline{C(x)}\right)\overline{u'(x)}\, dx \geq \zeta \|u'\|_2^2,$$

with $\zeta := \min_x \lambda_{\min}\left(C(x)^T D + D\overline{C(x)}\right)$.

Let us set

$$B(x) := \left(C'(x)\right)^T D$$

and denote

$$\xi := \sqrt{\max_{x\in[0,l]} \lambda_{\max}\left((B(x))^* B(x)\right)} = |B|_{\mathrm{Sp}}. \qquad (6.34)$$

Observe that

$$\lambda\left((B(x))^* B(x)\right) = \lambda\left(B(x)\left(B(x)\right)^*\right), \quad \forall x \in [0, l]. \tag{6.35}$$

We proceed as follows.

$$\begin{aligned}
\left|\int_0^l u(x)^T \left(C'(x)\right)^T D\overline{u'(x)}\, dx\right| &\le \int_0^l \left|u(x)^T \left(C'(x)\right)^T D\overline{u'(x)}\right|\, dx \\
&\le \max_{x\in[0,l]} \sqrt{\lambda_{\max}\left((B(x))^* B(x)\right)} \int_0^l \left|u(x)^T\overline{u'(x)}\right|\, dx \\
&\le \xi \|u\|_2 \|u'\|_2.
\end{aligned} \tag{6.36}$$

In the last estimation the Cauchy-Schwarz inequality was used. Similarly, taking into account (6.35), we obtain

$$\left|\int_0^l (u'(x))^T D\overline{C'(x)}\;\overline{u(x)}\, dx\right| \le \xi \|u\|_2 \|u'\|_2. \tag{6.37}$$

Combination of (6.36) and (6.37) results in

$$\int_0^l u(x)^T \left(C'(x)\right)^T D\overline{u'(x)}\, dx + \int_0^l (u'(x))^T D\overline{C'(x)}\;\overline{u(x)}\, dx \ge -2\xi \|u\|_2 \|u'\|_2.$$

Further, due to Young's inequality with $\rho > 0$, we arrive at the following estimation of (6.33)

$$\langle \mathcal{C}u, -Du''\rangle_2 + \langle -Du'', \mathcal{C}u\rangle_2 \ge (\zeta - \xi\rho)\|u'\|_2^2 - \frac{\xi}{\rho}\|u\|_2^2. \tag{6.38}$$

Integration by parts for the terms five and six of the right-hand side of (6.32) yields

$$\begin{aligned}
-\nu \left\langle u, -Du''\right\rangle_2 - \bar{\nu}\left\langle -Du'', u\right\rangle_2 &= -2\mathrm{Re}(\nu) \int_0^l \left(u'(x)\right)^T D\overline{u'(x)}\, dx \\
&\ge -2\mathrm{Re}(\nu)\, \tilde{d}\, \|u'\|_2^2,
\end{aligned} \tag{6.39}$$

with

$$\tilde{d} = \begin{cases} d_{\max} & \text{if } \mathrm{Re}(\nu) > 0, \\ d_{\min} & \text{otherwise.} \end{cases}$$

We continue with the estimation of the seventh and eighth terms of the right-hand side of (6.32) as

$$-\nu \langle u, \mathcal{C}u\rangle_2 - \bar{\nu} \langle \mathcal{C}u, u\rangle_2 = -\int_0^l u(x)^T \left(\nu\overline{C(x)} + \bar{\nu}C(x)^T\right) \overline{u(x)}\, dx$$
$$\geq -\varepsilon_\nu \|u\|_2^2, \tag{6.40}$$

with $\varepsilon_\nu := \max_x \lambda_{\max} \left[\nu\overline{C(x)} + \bar{\nu}C(x)^T\right]$.

Finally, the fourth term of the right-hand side of (6.32) is estimated as:

$$\langle \mathcal{C}u, \mathcal{C}u\rangle_2 \geq \eta \langle u, u\rangle_2, \tag{6.41}$$

where $\eta := \min_x \lambda_{\min} \left(C(x)^T\overline{C(x)}\right)$.

Combining (6.38), (6.39), (6.40), and (6.41) with (6.32), we obtain

$$\langle (L-\nu)u, (L-\nu)u\rangle_2 \geq \langle -Du'', -Du''\rangle_2 + \left(\zeta - 2\mathrm{Re}(\nu)\tilde{d} - \xi\rho\right) \langle u', u'\rangle_2$$
$$+ \left(\eta - \frac{\xi}{\rho} - \varepsilon_\nu + |\nu|^2\right) \langle u, u\rangle_2 .$$

Finally, denoting

$$P_1 := \zeta - 2\mathrm{Re}(\nu)\tilde{d} - \xi\rho,$$
$$P_2 := \eta - \frac{\xi}{\rho} - \varepsilon_\nu + |\nu|^2,$$

we arrive at

$$\langle (L-\nu)u, (L-\nu)u\rangle_2 \geq \langle -Du'', -Du''\rangle_2 + P_1 \langle u', u'\rangle_2 + P_2 \langle u, u\rangle_2 . \tag{6.42}$$

Let us introduce a sequence of eigenvalue problems $(EP)_s,\ s \in [0, 1]$ as

$$(EP)_s:\ u \in H_2^B(0, l),$$
$$(1-s) \langle -Du'', -Dv''\rangle_2 + P_1(1-s) \langle u', v'\rangle_2 + P_2(1-s) \langle u, v\rangle_2$$
$$+ s \langle (L-\nu)u, (L-\nu)v\rangle_2 = \lambda^{(s)} \left[\beta_1 \langle u, v\rangle_2 + \beta_2 \langle u', v'\rangle_2\right],$$
$$\text{for all} \quad v \in H_2^B(0, l), \quad s \in [0, 1].$$

For the reasons which will be explained later we would like to perform a spectral shift $\sigma > 0$. Therefore, we consider the following sequence of eigenvalue problems

$$(\boldsymbol{EP})_s : \; u \in H_2^B(0,l),$$

$$\begin{aligned}&(1-s)\left\langle -Du'', -Dv''\right\rangle_2 + (P_1(1-s)+\sigma\beta_2)\left\langle u', v'\right\rangle_2 + (P_2(1-s)+\sigma\beta_1)\left\langle u, v\right\rangle_2\\ &\qquad + s\left\langle (L-\nu)u, (L-\nu)v\right\rangle_2 = \boldsymbol{\lambda}^{(s)}\left[\beta_1\left\langle u, v\right\rangle_2 + \beta_2\left\langle u', v'\right\rangle_2\right],\\ &\qquad\qquad \text{for all} \quad v \in H_2^B(0,l), \quad s \in [0,1],\end{aligned} \tag{6.43}$$

with $\boldsymbol{\lambda}^{(s)} = \lambda^{(s)} + \sigma$.

It is easy to see, that the problem $(\boldsymbol{EP})_1$ is the given problem (6.2) (with the shift σ). Thus the requirement (H3) is satisfied.

The base problem $(\boldsymbol{EP})_0$ is given by

$$\begin{aligned}&u \in H_2^B(0,l), \quad \left\langle -Du'', -Dv''\right\rangle_2 + (P_1+\sigma\beta_2)\left\langle u', v'\right\rangle_2 + (P_2+\sigma\beta_1)\left\langle u, v\right\rangle_2\\ &\qquad = \boldsymbol{\lambda}^{(0)}[\beta_1\left\langle u, v\right\rangle_2 + \beta_2\left\langle u', v'\right\rangle_2],\\ &\qquad\qquad \text{for all} \quad v \in H_2^B(0,l),\end{aligned} \tag{6.44}$$

In the next subsection we will show, that its eigenvalues can be computed in closed form.

Now we would like to show that for each fixed $k \in \mathbb{N}$ the eigenvalues $\boldsymbol{\lambda}_k^{(s)}$ satisfy property (H2), namely that they are monotonically non-decreasing with respect to s. Let us define for $u, v \in H_2^B(0,l)$ and $s \in [0,1]$ the bilinear form $\mathcal{B}_s[u,v]$ as

$$\begin{aligned}\mathcal{B}_s[u,v] :=&(1-s)\left\langle -Du'', -Dv''\right\rangle_2 + (P_1(1-s)+\sigma\beta_2)\left\langle u', v'\right\rangle_2\\ &+ (P_2(1-s)+\sigma\beta_1)\left\langle u, v\right\rangle_2 + s\left\langle (L-\nu)u, (L-\nu)v\right\rangle_2,\end{aligned}$$

and the inner product $< \cdot, \cdot >$ as

$$< u, v > := \beta_1\left\langle u, v\right\rangle_2 + \beta_2\left\langle u', v'\right\rangle_2.$$

Let us consider for all $u \in H_2^B(0,l),\ u \neq 0$ the function

$$f(s,u) := \frac{\mathcal{B}_s[u,u]}{< u, u >}.$$

After differentiating $f(s,u)$ with respect to s, we obtain

$$f_s(s,u) = \frac{\|(L-\nu)u\|_2^2 - \|Du''\|_2^2 - P\|u'\|_2^2 - P\,\|u\|_2^2}{\beta_1\,\|u\|_2^2 + \beta_2\|u'\|_2^2}.$$

From (6.42) follows $f_s(s,u) \geq 0$. Therefore, the function $f(s,u)$ is a monotonically non-decreasing function with respect to s and Poincaré's min-max principle implies that for each fixed $k \in \mathbb{N}$, $\boldsymbol{\lambda}_k^{(s)}$ is monotonically non-decreasing with respect to s.

6.2.2 Eigenvalues of the base problem

In this subsection we are going to show, that the eigenvalue problem of the form (6.44) satisfies requirement (H1). Since in our examples (Schnakenberg and predator-prey model), eigenvalue problem of type (6.2) is postulated with the Neumann conditions on the boundary, we would like to restrict our following investigations to case $p = 1$. For convenience we introduce a notation

$$H_2^N(0,l) = \{u \in H_2^n(0,l) : u'(0) = u'(l) = 0\}. \tag{6.45}$$

Eigenvalue problem (6.44), taken without σ-shift, has the form

$$u \in H_2^N(0,l),\ \langle -Du'', -Dv''\rangle_2 + P_1\,\langle u', v'\rangle_2 + P_2\,\langle u, v\rangle_2 = \lambda^{(0)}\,[\beta_1\,\langle u, v\rangle_2 + \beta_2\,\langle u', v'\rangle_2]\,,$$
$$\text{for all} \quad v \in H_2^N(0,l). \tag{6.46}$$

After partial integration of (6.46) we obtain

$$u \in H_2^N(0,l), \quad \left\langle D^2\,u^{iv}, v\right\rangle_2 - P_1\,\langle u'', v\rangle_2 + P_2\,\langle u, v\rangle_2 = \lambda^{(0)}[\beta_1\,\langle u, v\rangle_2 - \beta_2\,\langle u'', v\rangle_2],$$
$$\text{for all} \quad v \in H_2^N(0,l), \tag{6.47}$$

with the additional condition on the boundary

$$u'''(0) = u'''(l) = 0.$$

Let us denote

$$\mathcal{X} := \{u \in H_2^N(0,l) : u'''(0) = u'''(l) = 0\}. \tag{6.48}$$

Then, since $H_2^N(0,l)$ is dense in $L_2^n(0,l)$, from (6.47) we obtain

$$u \in \mathcal{X}, \quad D^2\, u^{iv} + \left(\lambda^{(0)}\beta_2 - P_1\right) u'' + \left(P_2 - \lambda^{(0)}\beta_1\right) u = 0.$$

Let us rewrite the expression above componentwise:

$$u_j \in \mathcal{X}, \quad d_j^2 u_j^{iv} + \left(\lambda_j^{(0)}\beta_2 - P_1\right) u_j'' + \left(P_2 - \lambda_j^{(0)}\beta_1\right) u_j = 0, \quad \forall j = 1, \dots, n. \tag{6.49}$$

In the following, without loss of generality, we omit writing indices j and (0). Let us consider the characteristic polynom, corresponding to (6.49)

$$d^2\xi^4 + (\lambda\beta_2 - P_1)\xi^2 + (P_2 - \lambda\beta_1) = 0. \tag{6.50}$$

Let us, for simplicity, denote

$$K_1 := \lambda\beta_2 - P_1, \tag{6.51}$$

$$K_2 := P_2 - \lambda\beta_1. \tag{6.52}$$

Setting $\xi^2 = t$ in (6.50) we obtain the quadratic equation

$$d^2t^2 + K_1t + K_2 = 0, \tag{6.53}$$

solutions of which are given by

$$t_{1,2} = \frac{-K_1 \pm \sqrt{K_1^2 - 4K_2d^2}}{2d^2}. \tag{6.54}$$

Thus, the roots of (6.50) are

$$\xi_{1,2} = \pm\sqrt{t_1}, \quad \xi_{3,4} = \pm\sqrt{t_2}. \tag{6.55}$$

Let us consider the case when $t_1 \neq t_2$. From (6.54) we have

$$K_1^2 \neq 4K_2d^2. \tag{6.56}$$

Inserting (6.51) and (6.52) into (6.56) we obtain

$$\lambda^2\beta_2^2 - 2\lambda(\beta_2P_1 - 2d^2\beta_1) + (P_1^2 - 4d^2P_2) \neq 0.$$

Hence condition (6.56) holds if and only if

$$\mathcal{D} = d^2\beta_1^2 + \beta_2^2 P_2 - P_1\beta_1\beta_2 < 0. \tag{6.57}$$

Now let us consider a system of the form

$$\{e^{\sqrt{t_1}x}, e^{-\sqrt{t_1}x}, e^{\sqrt{t_2}x}, e^{-\sqrt{t_2}x}\}. \tag{6.58}$$

Computation of the Wronskian to the set in (6.58) yields $4\sqrt{t_1t_2}(t_1 - t_2)$. Thus if

$$4\sqrt{t_1t_2}(t_1 - t_2) \neq 0, \tag{6.59}$$

then the functions in (6.58) are linearly independent and (6.58) is a fundamental system. In addition, observe that since $t_1 \neq t_2$ from (6.59) follows $t_1 \neq 0$, $t_2 \neq 0$. A general solution now reads

$$u(x) = C_1e^{\sqrt{t_1}x} + C_2e^{-\sqrt{t_1}x} + C_3e^{\sqrt{t_2}x} + C_4e^{-\sqrt{t_2}x}, \quad x \in [0, l].$$

Inserting the boundary conditions, we obtain

$$AC = 0, \tag{6.60}$$

where

$$A = \begin{pmatrix} \sqrt{t_1} & -\sqrt{t_1} & \sqrt{t_2} & -\sqrt{t_2} \\ \sqrt{t_1}e^{\sqrt{t_1}l} & -\sqrt{t_1}e^{-\sqrt{t_1}l} & \sqrt{t_2}e^{\sqrt{t_2}l} & -\sqrt{t_2}e^{-\sqrt{t_2}l} \\ t_1\sqrt{t_1} & -t_1\sqrt{t_1} & t_2\sqrt{t_2} & -t_2\sqrt{t_2} \\ t_1\sqrt{t_1}e^{\sqrt{t_1}l} & -t_1\sqrt{t_1}e^{-\sqrt{t_1}l} & t_2\sqrt{t_2}e^{\sqrt{t_2}l} & -t_2\sqrt{t_2}e^{-\sqrt{t_2}l} \end{pmatrix},$$

and C is the vector given by $C = (C_1, C_2, C_3, C_4)^T$. By a straightforward computation we obtain the following solutions to system (6.60):

$$\begin{aligned} &\text{either} \quad C_1 = C_2, \quad C_3 = C_4 = 0 \quad \text{and} \quad \sinh(\sqrt{t_1}l) = 0, \\ &\text{or} \quad C_1 = C_2 = 0, \quad C_3 = C_4 \quad \text{and} \quad \sinh(\sqrt{t_2}l) = 0. \end{aligned} \tag{6.61}$$

Therefore we have: either

$$\sqrt{t_1} = \frac{i\pi k}{l}, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{6.62}$$

or

$$\sqrt{t_2} = \frac{i\pi k}{l}, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{6.63}$$

Inserting (6.62) and (6.63) into (6.54), we obtain that either

$$\sqrt{K_1^2 - 4K_2 d^2} = -2d^2 \frac{\pi^2 k^2}{l^2} + K_1, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{6.64}$$

or

$$\sqrt{K_1^2 - 4K_2 d^2} = 2d^2 \frac{\pi^2 k^2}{l^2} - K_1, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{6.65}$$

Equation (6.64) has solutions if and only if $-2d^2 \frac{\pi^2 k^2}{l^2} + K_1 > 0$. Inserting (6.51) and (6.52) into (6.64), we obtain

$$\lambda_k = \frac{d^2 \frac{\pi^4 k^4}{l^4} + P_1 \frac{\pi^2 k^2}{l^2} + P_2}{\beta_2 \frac{\pi^2 k^2}{l^2} + \beta_1}, \quad \text{if } -2d^2 \frac{\pi^2 k^2}{l^2} + K_1 > 0, \quad k \in \mathbb{Z} \setminus \{0\}.$$

On the other hand, equation (6.65) has solutions if and only if $2d^2 \frac{\pi^2 k^2}{l^2} - K_1 > 0$. Inserting (6.51) and (6.52) into (6.65), we obtain

$$\lambda_k = \frac{d^2 \frac{\pi^4 k^4}{l^4} + P_1 \frac{\pi^2 k^2}{l^2} + P_2}{\beta_2 \frac{\pi^2 k^2}{l^2} + \beta_1}, \quad \text{if } 2d^2 \frac{\pi^2 k^2}{l^2} - K_1 > 0 \quad k \in \mathbb{Z} \setminus \{0\}.$$

Thus, combining both results together, we have

$$\lambda_k = \frac{d^2 \frac{\pi^4 k^4}{l^4} + P_1 \frac{\pi^2 k^2}{l^2} + P_2}{\beta_2 \frac{\pi^2 k^2}{l^2} + \beta_1}, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{6.66}$$

In view of (6.61) the eigenfunctions corresponding to the eigenvalues (6.66) are given by

$$\psi_k(x) = e^{\frac{i\pi k}{l} x}, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{6.67}$$

Clearly, set in (6.67) is orthogonal and complete in $L_2(0,l)$. Therefore, in case $t_1 \neq t_2$ we have found all eigenvalues.

Now let us consider the case when $t_1 = t_2$. From (6.54) we see that $t_1 = t_2$ is equivalent to

$$K_1^2 = 4K_2 d^2. \tag{6.68}$$

Since $K_1 \in \mathbb{R}$ then $K_2 \geq 0$. Consequently, from (6.52),

$$\lambda \leq \frac{P_2}{\beta_1}. \tag{6.69}$$

A fundamental system of solutions has the form

$$\{e^{\sqrt{t_1}x},\ xe^{\sqrt{t_1}x},\ e^{-\sqrt{t_1}x},\ xe^{-\sqrt{t_1}x}\}. \tag{6.70}$$

If $t_1 \neq 0$, then the functions in the set above are linearly independent. This condition follows from the computation of Wronskian. A general solution reads

$$u(x) = C_1 e^{\sqrt{t_1}x} + C_2 x e^{\sqrt{t_1}x} + C_3 e^{-\sqrt{t_1}x} + C_4 x e^{-\sqrt{t_1}x}, \quad x \in [0,l].$$

Inserting the boundary conditions, we arrive at the system

$$AC = 0, \tag{6.71}$$

where

$$A = \begin{pmatrix} \sqrt{t_1} & 1 & -\sqrt{t_1} & 1 \\ \sqrt{t_1}e^{\sqrt{t_1}l} & (1+\sqrt{t_1}l)e^{\sqrt{t_1}l} & -\sqrt{t_1}e^{-\sqrt{t_1}l} & (1-\sqrt{t_1}l)e^{-\sqrt{t_1}l} \\ t_1\sqrt{t_1} & 3t_1 & -t_1\sqrt{t_1} & 3t_1 \\ t_1\sqrt{t_1}e^{\sqrt{t_1}l} & (3+\sqrt{t_1}l)t_1 e^{\sqrt{t_1}l} & -t_1\sqrt{t_1}e^{-\sqrt{t_1}l} & (3-\sqrt{t_1}l)t_1 e^{-\sqrt{t_1}l} \end{pmatrix}$$

and $C = (C_1, C_2, C_3, C_4)^T$. By a straightforward computation we obtain the following solution to system (6.71)

$$C_1 = C_3, \quad C_2 = C_4 = 0, \quad \text{and} \quad \sinh(\sqrt{t_1}l) = 0. \tag{6.72}$$

Hence, we have

$$\sinh(\sqrt{t_1}l) = 0,$$

and consequently,

$$\sqrt{t_1} = \frac{i\pi k}{l}, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{6.73}$$

Inserting (6.54) into (6.73), and solving the resulting equation for λ, we obtain

$$\lambda_k = \frac{2d^2 \frac{\pi^2 k^2}{l^2} + P_1}{\beta_2}, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{6.74}$$

We see from (6.74) that λ_k exists only if $\beta_2 \neq 0$. By (6.69) λ_k should satisfy

$$\frac{2d^2 \frac{\pi^2 k^2}{l^2} + P_1}{\beta_2} \leq \frac{P_2}{\beta_1}, \quad k \in \mathbb{Z} \setminus \{0\},$$

which is equivalent to

$$2d^2 \frac{\pi^2 k^2}{l^2} \beta_1 \leq \beta_2 P_2 - \beta_1 P_1, \quad k \in \mathbb{Z} \setminus \{0\}. \tag{6.75}$$

From (6.75) follows that

$$\beta_2 P_2 - \beta_1 P_1 \geq 0. \tag{6.76}$$

Hence, if $\beta_2 \neq 0$ and the condition above is satisfied, then solution (6.74) exists. Now let us consider the case when $\beta_2 = 0$. Then, by (6.51), we have $K_1 = -P_1$. Then, inserting (6.54) into (6.73), we obtain $P_1 = -2d^2 \frac{\pi^2 k^2}{l^2}$, $k \in \mathbb{Z} \setminus \{0\}$, which is a contradiction, since P_1 is a fixed number.

Due to (6.72) the functions of the form (6.67) are the basis for the eigenspace, corresponding to eigenvalue (6.74). As earlier it follows that we have found all eigenspaces.

Now let us consider condition (6.57). One can see that (6.76) and (6.57) complement each other. Therefore, gathering everything together, we obtain

$$\lambda_{j,k}^{(0)} = \begin{cases} \dfrac{2d_j^2 \dfrac{\pi^2 k^2}{l^2} + P_1}{\beta_2}, \quad k \in \mathbb{Z} \setminus \{0\}, & \text{if} \quad \beta_2 \neq 0, \quad \beta_2 P_2 - \beta_1 P_1 \geq 0, \\ \\ \dfrac{d_j^2 \dfrac{\pi^4 k^4}{l^4} + P_1 \dfrac{\pi^2 k^2}{l^2} + P_2}{\beta_2 \dfrac{\pi^2 k^2}{l^2} + \beta_1}, \quad k \in \mathbb{Z} \setminus \{0\}, & \text{otherwise.} \end{cases}$$

Consequently,

$$\lambda_{j,k}^{(0)} = \begin{cases} \dfrac{2d_j^2 \dfrac{\pi^2 k^2}{l^2} + P_1}{\beta_2}, \quad k \in \mathbb{N}, & \text{if} \quad \beta_2 \neq 0, \quad \beta_2 P_2 - \beta_1 P_1 \geq 0, \\ \\ \dfrac{d_j^2 \dfrac{\pi^4 k^4}{l^4} + P_1 \dfrac{\pi^2 k^2}{l^2} + P_2}{\beta_2 \dfrac{\pi^2 k^2}{l^2} + \beta_1}, \quad k \in \mathbb{N}, & \text{otherwise.} \end{cases} \tag{6.77}$$

Thus, eigenvalue problem (6.44) satisfies requirement (H1).

6.2.3 Lehmann-Goerisch method

In this subsection we are going to obtain the terms $\mathcal{X}_G$, b, T from Theorem 6.3. Let us consider the eigenvalue problem (6.43) and set

$$\begin{aligned} \mathcal{M}_s(u,v) :=& (1-s) \langle -Du'', -Dv'' \rangle_2 + (P_1(1-s) + \sigma\beta_2) \langle u', v' \rangle_2 \\ & + (P_2(1-s) + \sigma\beta_1) \langle u, v \rangle_2 + s \langle (L-\nu)u, (L-\nu)v \rangle_2 , \\ \mathcal{N}(u,v) :=& \beta_1 \langle u, v \rangle_2 + \beta_2 \langle u', v' \rangle_2 . \end{aligned}$$

As one can see the bilinear form $\mathcal{M}_s(u,v)$ is Hermitian. Moreover, for $\sigma > 0$ large enough, it is also positive definite. $\mathcal{N}(u,v)$ is positive definite and Hermitian as well. We proceed with introducing the "$\mathcal{X}_G$bT"-terms, which are required for the Lehmann-Goerisch method.

Define the vector space $\mathcal{X}_G$ as

$$\mathcal{X}_G := L_2^n(0,l) \times L_2^n(0,l) \times L_2^n(0,l) \times L_2^n(0,l), \tag{6.78}$$

the positive definite Hermitian bilinear form b^s as

$$b^s\left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}, \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \\ \tilde{w}_4 \end{pmatrix}\right) = s\langle w_1, \tilde{w}_1\rangle_2 + (1-s)\langle w_2, \tilde{w}_2\rangle_2 + (P_1(1-s) + \sigma\beta_2)\langle w_3, \tilde{w}_3\rangle_2$$

$$+ (P_2(1-s) + \sigma\beta_1)\langle w_4, \tilde{w}_4\rangle_2 \tag{6.79}$$

and the linear operator $T\colon H_2^B(0,l) \to \mathcal{X}_G$ as

$$Tu := \begin{pmatrix} (L-\nu)u \\ -Du'' \\ u' \\ u \end{pmatrix}. \tag{6.80}$$

Due to (6.79) and (6.80) condition (6.12) holds automatically. So, it is left to find such $w^{(1)}, \dots w^{(N)} \in \mathcal{X}_G$ that

$$b^s(T\varphi, w^{(i)}) = \mathcal{N}(\varphi, \tilde{u}_i), \quad (i = 1, \dots, N) \quad \text{for all} \quad \varphi \in H_2^B(0,l). \tag{6.81}$$

Let us rewrite condition (6.81) (taking $\varphi = u$) in our case

$$s\left\langle (L-\nu)u, w_1^{(i)}\right\rangle_2 + (1-s)\left\langle -Du'', w_2^{(i)}\right\rangle_2 + (P_1(1-s)+\sigma\beta_2)\left\langle u', w_3^{(i)}\right\rangle_2$$

$$+ (P_2(1-s)+\sigma\beta_1)\left\langle u, w_4^{(i)}\right\rangle_2 = \beta_1\langle u, \tilde{u}_i\rangle_2 + \beta_2\langle u', \tilde{u}_i'\rangle_2$$

$$\text{for all } u \in H_2^B(0,l),\ s \in [0,1],\ (i = 1, \dots, N). \tag{6.82}$$

W.l.o.g. we omit writing index i in the future.

Let us choose $w_1, w_2 \in H_2^n(0,l)$, $w_3 \in H_1^n(0,l)$, and $w_4 \in L_2^n(0,l)$. Recall that we consider two different types of eigenvalue problems: with either Dirichlet or Neumann boundary conditions. Hence, the integration of (6.82) by parts yields

$$s\left\langle u,(L-\nu)^{*}w_1\right\rangle_2+(1-s)\left\langle u,-Dw_2''\right\rangle_2-(P_1(1-s)+\sigma\beta_2)\left\langle u,w_3'\right\rangle_2+$$

$$+(P_2(1-s)+\sigma\beta_1)\left\langle u,w_4\right\rangle_2=\beta_1\left\langle u,\tilde{u}\right\rangle_2-\beta_2\left\langle u,\tilde{u}''\right\rangle_2$$

$$\text{for all } u\in H_2^B(0,l),\ \ s\in[0,1], \tag{6.83}$$

combined with the condition on the boundary. In case $p=0$, we have

$$sw_1(0)+(1-s)w_2(0)=sw_1(l)+(1-s)w_2(l)=0, \tag{6.84}$$

and in case $p=1$ the additional boundary condition reads

$$\begin{aligned} sDw_1'(0)+&(1-s)Dw_2'(0)+(P_1(1-s)+\sigma\beta_2)w_3(0)\\ &=sDw_1'(l)+(1-s)Dw_2'(l)+(P_1(1-s)+\sigma\beta_2)w_3(l)=0. \end{aligned} \tag{6.85}$$

Since $H_2^B(0,l)$ is dense in $L_2^n(0,l)$, we obtain

$$s(L-\nu)^{*}w_1-(1-s)Dw_2''-(P_1(1-s)+\sigma\beta_2)w_3'+(P_2(1-s)+\sigma\beta_1)w_4=\beta_1\tilde{u}-\beta_2\tilde{u}'' \tag{6.86}$$

Let $\tilde{\boldsymbol{\lambda}}^{(s)}$ be a good numerical approximation to $\boldsymbol{\lambda}^{(s)}$, and $\tilde{u}$ a corresponding approximative eigenelement. Now we have to make a choice for the vector w. According to Remark 6.4, for the purpose of obtaining good bounds, we propose the following setting

$$w\approx\frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}}T(\tilde{u})=\frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}}\begin{pmatrix}(L-\nu)\tilde{u}\\ -D\tilde{u}''\\ \tilde{u}'\\ \tilde{u}\end{pmatrix}.$$

We choose the elements (w_1,w_2) as the approximations to $\left(\frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}}(L-\nu)\tilde{u},-\frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}}D\tilde{u}''\right)$ in the following space

$$\begin{aligned}\hat{X}:=\Bigg\{&\begin{pmatrix}v_1\\ v_2\end{pmatrix}\in H_2^n(0,l)\times H_2^n(0,l):\text{such that}\\ &\text{condition (6.84) } (p=0)\ \text{ or condition (6.85) } (p=1) \text{ holds for all } v_1,\ v_2\Bigg\}.\end{aligned}$$

Further we set

$$w_3 := \frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}} \tilde{u}',$$

and satisfy (6.86), by solving (6.86) with respect to w_4.

In our applications (Schnakenberg model with Neumann boundary conditions) the approximative elements $\tilde{u}$ are such, that condition (6.85) is automatically satisfied. Hence in that particular case we set:

$$w_1 := \frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}} (L - \nu)\tilde{u}, \tag{6.87}$$

$$w_2 := -\frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}} D\tilde{u}'', \tag{6.88}$$

$$w_3 := \frac{1}{\tilde{\boldsymbol{\lambda}}^{(s)}} \tilde{u}'. \tag{6.89}$$

So, it is left to find w_4, such that (6.86) holds, i.e.

$$\begin{aligned} w_4 = \frac{1}{P_2(1-s) + \sigma\beta_1} \Big\{ & \beta_1 \tilde{u} - \beta_2 \tilde{u}'' - s(L-\nu)^* w_1 + (1-s) D w_2'' \\ & + (P_1(1-s) + \sigma\beta_2) w_3' \Big\}. \end{aligned} \tag{6.90}$$

The expression above demonstrates again the necessity of σ-shift: if we set $\sigma = 0$, then for $s = 1$ we obtain zero in the denominator.

Combining (6.90) with (6.87) to (6.89), we obtain by a straightforward calculation

$$w_{4_i}^s = \frac{1}{P_2(1-s) + \sigma\beta_1} \left(M_{0_i}^s \tilde{u}_i^{iv} + M_{1_i}^s \tilde{u}_i'' + M_{2_i}^s \tilde{u}_i' + M_{3_i}^s \tilde{u}_i \right), \tag{6.91}$$

where

$$M_{0_i}^s := -\frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}} D^2,$$

$$M_{1_i}^s := \frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}} s\left((C-\nu E)^* D + D(C-\nu E)\right) + \frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}} (P_1(1-s)+\sigma\beta_2)E - \beta_2 E,$$

$$M_{2_i}^s := \frac{2}{\tilde{\boldsymbol{\lambda}}_i^{(s)}} sDC',$$

$$M_{3_i}^s := \beta_1 E + \frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}} sDC'' - \frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}} s\left[(C-\nu E)^*(C-\nu E)\right].$$

The next step in the implemetation of the Lehmann-Goerisch method is the construction of matrix $A_2^s := (b^s(w^{(i)}, w^{(j)}))_{i,j=1,\ldots,N}$. Combining (6.79) with (6.87) to (6.89) and (6.91), we obtain

$$\begin{aligned} A_{2_{ij}}^s :=& s\frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}\tilde{\boldsymbol{\lambda}}_j^{(s)}} W_{1_{ij}} + (1-s)\frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}\tilde{\boldsymbol{\lambda}}_j^{(s)}} W_{2_{ij}} + (P_1(1-s)+\sigma\beta_2)\frac{1}{\tilde{\boldsymbol{\lambda}}_i^{(s)}\tilde{\boldsymbol{\lambda}}_j^{(s)}} W_{3_{ij}} \\ &+ (P_2(1-s)+\sigma\beta_1) W_{4_{ij}}^s, \quad s\in[0,1],\ (i,j=1,\ldots,N), \end{aligned} \tag{6.92}$$

where

$$W_{1_{ij}} := \langle (L-\nu)\tilde{u}_i, (L-\nu)\tilde{u}_j\rangle_2$$

$$W_{2_{ij}} := \left\langle -D\tilde{u}_i'', -D\tilde{u}_j''\right\rangle_2$$

$$W_{3_{ij}} := \left\langle \tilde{u}_i', \tilde{u}_j'\right\rangle_2$$

$$\begin{aligned} W_{4_{ij}}^s := \frac{1}{(P_2(1-s)+\sigma\beta_1)^2}\Big\langle & M_{0_i}^s\tilde{u}_i^{iv} + M_{1_i}^s\tilde{u}_i'' + M_{2_i}^s\tilde{u}_i' + M_{3_i}^s\tilde{u}_i, \\ & M_{0_j}^s\tilde{u}_j^{iv} + M_{1_j}^s\tilde{u}_j'' + M_{2_j}^s\tilde{u}_j' + M_{3_j}^s\tilde{u}_j\Big\rangle_2. \end{aligned}$$

6.3 Variational eigenvalue bounds for problem (6.1)

In this subsection we consider eigenvalue problem (6.1). This time L is a self-adjoint operator in $L_2^n(0,l)$. We introduce the following

Lemma 6.6. *Let* $\underline{c}$ *be defined as*

$$\underline{c} := \min_x \lambda_{\min}(C(x)). \tag{6.93}$$

Then the sequence of eigenvalue problems $(EP)_s,\ s \in [0,1]$*:*

$$u \in H_2^B(0,l), \quad L_s u = -Du'' + ((1-s)\underline{c} + sC)u = \lambda^{(s)} u \tag{6.94}$$

satisfies conditions (H1),(H2), and (H3).

Proof. It is clear, that condition (H3) holds.

Let us consider condition (H1). Setting $s = 0$, we obtain the eigenvalue problem

$$u \in H_2^B(0,l), \quad -Du'' + \underline{c}u = \lambda^{(0)} u.$$

Hence, $\lambda^{(0)}$ are given by

$$\lambda_{j,k}^{(0)} = d_j \chi_k^2 + \underline{c}, \quad k \in \mathbb{N},\ j = 1, \ldots, n, \tag{6.95}$$

where

$$\chi_k^2 = \begin{cases} \dfrac{\pi^2 k^2}{l^2}, & \text{if} \quad p = 0, \\ \dfrac{\pi^2 (k-1)^2}{l^2}, & \text{if} \quad p = 1, \end{cases} \qquad k \in \mathbb{N}.$$

Therefore condition (H1) is satisfied.

We continue with condition (H3). For $u \in H_2^B(0,l)$ consider

$$f(s,u) := \frac{\langle L_s u, u\rangle_2}{\langle u, u\rangle_2} = \frac{\langle -Du'', u\rangle_2 + \underline{c}\,\langle u, u\rangle_2 + s\,\langle (C - \underline{c})u, u\rangle_2}{\langle u, u\rangle_2}.$$

Differentiating the expression above with respect to s, we obtain

$$f_s(s,u) = \frac{\langle (C - \underline{c})u, u\rangle_2}{\langle u, u\rangle_2}. \tag{6.96}$$

It is easy to see that, due to the choice of $\underline{c}$, $f_s(s,u) \geq 0$. Hence, the condition (H2) follows from the Poincaré's min-max principle. $\square$

In order to obtain the two-sided bounds for the eigenvalues of (6.1) we perform the homotopy method as it was described above, applying Theorem 6.1 and Theorem 6.2 to problem (6.94).

7

Results

In this chapter we are going to report on the results of the methods introduced in the thesis. We apply these methods to the Schnakenberg, predator-prey, spruce budworm and competition models.

7.1 Schnakenberg model

Recall that the dimensionless Schnakenberg model, postulated on an interval $\Omega = (0, l)$, has the form

$$
\begin{cases}
u_{1t}(x,t) = u_{1xx}(x,t) + \gamma\left(a - u_1(x,t) + u_1^2(x,t)u_2(x,t)\right), & t > 0, \quad x \in [0,l], \\
u_{2t}(x,t) = du_{2xx}(x,t) + \gamma\left(b - u_1^2(x,t)u_2(x,t)\right), & t > 0, \quad x \in [0,l], \\
\dfrac{\partial u(0,t)}{\partial \nu} = \dfrac{\partial u(l,t)}{\partial \nu} = 0, & t \geq 0, \\
u(x,0) = u^0(x), & x \in [0,l].
\end{cases}
$$

In our computations we set for all $x \in [0, l]$

$$u_1^0(x) = u_1^* + \sum_{i=1}^{20} \cos\left((i-1)\frac{\pi x}{l}\right),$$

$$u_2^0(x) = u_2^* + \sum_{i=1}^{20} \cos\left((i-1)\frac{\pi x}{l}\right),$$

where

$$u_1^* = a + b, \tag{7.1}$$

$$u_2^* = \frac{b}{(a+b)^2}. \tag{7.2}$$

In addition, we choose the following constants of the pattern formation mode: $a = 0.1,\ b = 0.9,\ \gamma = 1,\ d = 10$ and $l = 5$.

7.1.1 The function G

Let $u, v \in \mathbb{R}^n$. By a straightforward computation we obtain

$$g(u, v) = \gamma \begin{pmatrix} q \\ -q \end{pmatrix}, \quad q := u_1^2 u_2 + 2u_1 u_2 v_1 + u_1^2 v_2.$$

Thus, we have

$$\begin{aligned} |g(u,v)|_2 &\le \sqrt{2}\gamma \left(|u_1^2 u_2| + 2|u_1 u_2||v_1| + |u_1^2||v_2|\right) \\ &\le \sqrt{2}\gamma \left(\frac{2}{3\sqrt{3}}|u|_2 + |v_1| + |v_2|\right) |u|_2^2. \end{aligned}$$

Therefore we set

$$G(h) = \sqrt{2}\gamma \left(\frac{2}{3\sqrt{3}} h + \|v_1\|_\infty + \|v_2\|_\infty\right) h^2. \tag{7.3}$$

Note that for the enclosure of a stationary solution we take $v \equiv \omega$. During the stability investigation we set $v \equiv \bar{u}$ and use estimation (3.3).

7.1.2 Enclosure of the stationary solution

Recall that for the enclosure of the stationary solution the values for constants δ and K satisfying (3.18), (3.19) are required. The function G is given by (7.3). Our numerical simulations (which were performed using the interval arithmetic package INTLAB[50]) has resulted in $\delta = 0.^{1093}_{0886} \cdot 10^{-5}$. More details on that computation (as well as on the computation of the highly accurate numerical approximation ω) one can find in Appendix A.

Recall from Chapter 3, (3.30) that for computation of constant K we need to compute $\underline{\lambda}$, which is a a positive lower bound to the first eigenvalue of problem (3.27), and choose (by trial) a positive constant β so that K is as small as possible. In order to find $\underline{\lambda}$ we consider a shifted eigenvalue problem (6.43) and implement the variational methods, described in Chapter 6. In particular, we implement the homotopy algorithm.

We have started homotopy with 11 eigenvalues. At $s = 1$ we have arrived with 5 eigenvalues. In Table 7.1 the lower bounds for the eigenvalues arising in the course of homotopy are presented. Here $\underline{\boldsymbol{\lambda}}_n^{(s)}$ denotes the lower bound of the nth eigenvalue of the shifted eigenvalue problem at the moment s. This bound has been computed using interval arithmetic. Note that at the moment $s = 0$ the value for $\underline{\boldsymbol{\lambda}}_n^{(0)}$ is known. During the homotopy we have performed the Rayleigh-Ritz (by Theorem 6.1) and Lehmann-Goerisch (by Theorem 6.3) computations to find the bounds for eigenvalues. As a result we have obtained the verified lower bound to the first eigenvalue, which is given by (note, that the shift parameter $\sigma = 49.2592$) $\underline{\lambda}_1^{(1)} = 0.0053$. After setting $\underline{\lambda} = \underline{\lambda}_1^{(1)}$ and $\beta = 0.5$ in (3.30) we have arrived at $K = 18.501^5_4$.

Having computed the values for K and δ, we continued with the implementation of the Newton algorithm for the determination of the value α as it was described in Section 3.4. As a result we have obtained $\alpha = 0.^{2541}_{2056} \cdot 10^{(-4)}$

Table 7.1: Lower bounds for eigenvalues in homotopy $\underline{\boldsymbol{\lambda}}_n^{(s)}$

n \ s	0	0.5859	0.6695	0.8364	0.9637	0.9739	1
11	56.7842	-	-	-	-	-	-
10	49.1950	56.7122	-	-	-	-	-
9	42.3611	54.1490	56.6958	-	-	-	-
8	36.2623	48.9366	51.5077	56.6295	-	-	-
7	30.8614	44.4635	47.0737	52.2707	56.5443	-	-
6	26.0837	40.6785	43.3575	48.6866	56.2229	56.5420	-
5	21.7623	37.4497	40.2644	47.6987	52.7354	53.0621	53.8903
4	18.2439	34.3653	37.4971	45.8536	50.0935	50.4356	51.3025
3	17.4968	30.4691	36.0373	43.6911	48.3697	48.7464	49.7261
2	11.9154	29.2870	33.6674	41.6204	47.5759	48.0888	49.6866
1	1.0000	23.8261	29.0595	39.4937	47.4399	48.0477	49.2645

7.1.3 Stability. Domain of attraction

For the discussion that follows let us remind that we use the notation

$$v(t) = u(t) - \bar{u}, \quad t \geq 0,$$

where $\bar{u}$ is the stationary solution, the existence of which (including the error bound) we have already established.

By Theorem 5.15 there exist δ_0, $\hat{C}_\infty > 0$, such that if $v_0 \in C^n[0,l]$, $\|v_0\|_\infty < \delta_0$, we have $t_{\max}(v_0) = \infty$ and

$$\|v(t)\|_\infty \to 0, \quad \forall t \geq 0.$$

Here δ_0, $\hat{C}_\infty$ are given by

$$\delta_0 = \frac{\delta(\varepsilon_0)}{\sqrt{2}\hat{C}_\infty},$$

$$\hat{C}_\infty := \frac{M_\infty}{2\pi}\left(2e^{-\tilde{r}^*|\cos(\eta)|}\ln\left(1+\frac{1}{\tilde{r}^*|\cos(\eta)|}\right)+I(f_{\tilde{r}^*})\right).$$

At first we comment on the computation of the constant $\hat{C}_\infty$. The major difficulty here is to compute the constant M_∞. We have accomplished this task with the help of Theorem 4.1, exclosure of eigenvalues, and Theorem 5.11. For the application of Theorem 4.1 the values for constants z and ζ were required. Following the strategy described in Remark 4.8, we have computed the approximate eigenvalues of the operator L_ω. This eigenvalues were included in the sector $\hat{S}_{\tilde{\zeta},\tilde{z}}$, where $\tilde{z} = 0.0865$ and $\tilde{\zeta} = 1.4487$. Thus, in Theorem 4.1 we set $z = \tilde{z}$ and $\zeta = \tilde{\zeta}$. Using the enclosure constant α, we have obtained $|\mathcal{C}_{\bar{u}} - \mathcal{C}_\omega|_{\mathrm{Sp}} \leq 0.^{1976}_{1599} \cdot 10^{(-3)}$. In addition, using (4.37), we have computed $K_{\tilde{z}} = 5.35^{90}_{89}$. Thus, by Theorem 4.1 we have determined $\tilde{R} = 6$ - the radius of the circle, outside of which (excluding the sector $\hat{S}_{\tilde{\zeta},\tilde{z}}$) no eigenvalues could lie. In the next step we have performed the eigenvalue exclosure in the area $\hat{S}^C_{\tilde{\zeta},\tilde{z}} \cap B(\tilde{z}, \tilde{R})$. Figure 7.1 illustrates the process of the eigenvalues exclosure, where the parameter μ was chosen in the area $\tilde{\Omega}_\mu = \{\mu \in \hat{S}^C_{\tilde{\zeta},\tilde{z}} \cap B(\tilde{z}, \tilde{R}) : \mathrm{Im}(\mu) \geq 0\}$. It is clear that the eigenvalues of $L_{\bar{u}}$ lie symmetric with the respect to the real axes. Here

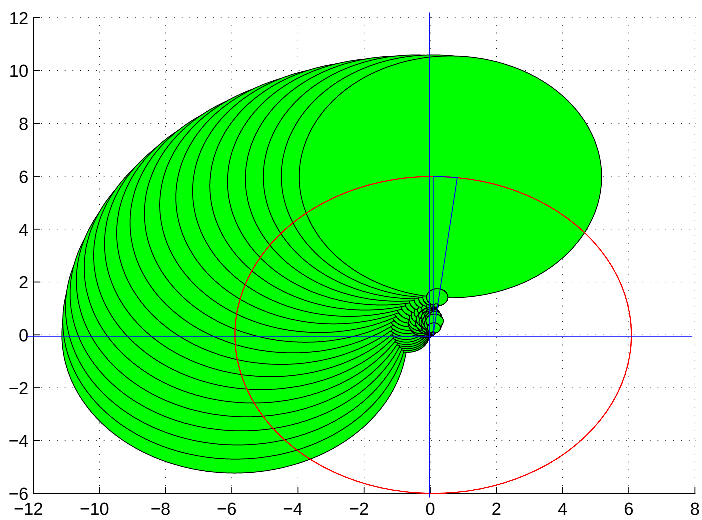

FIGURE 7.1: Eigenvalues exclosure for $\mu \in \tilde{\Omega}_\mu$

the green circles correspond to the area, where no eigenvalues exist. On Figure 7.2 and Figure 7.3 one may observe the enlarged picture of the "critical" regions, where the eigenvalues may exist. As it was already mentioned before, the implementation of the eigenvalues exclosure is possible only as long as the condition (4.23) holds true. In our numerical simulations we were able to exclude the eigenvalues of $L_{\bar{u}}$ in the left-hand side of the complex plane, therefore proving the stability of the stationary solution $\bar{u}$.

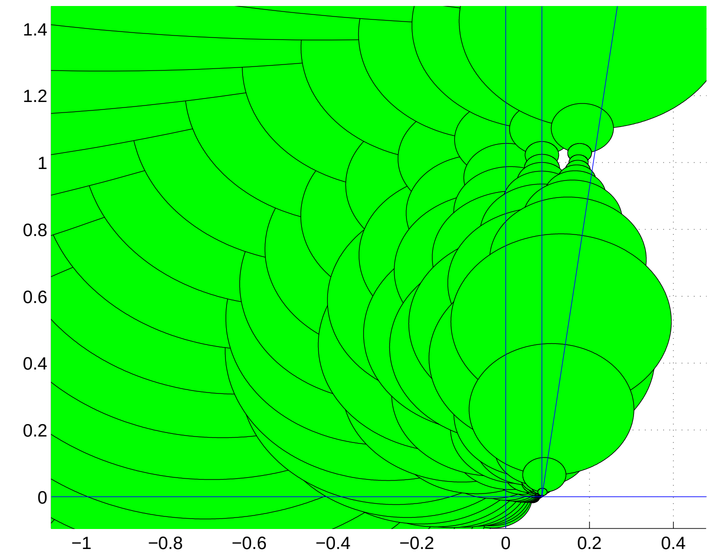

FIGURE 7.2: Eigenvalues exclosure for $\mu \in \tilde{\Omega}_\mu$

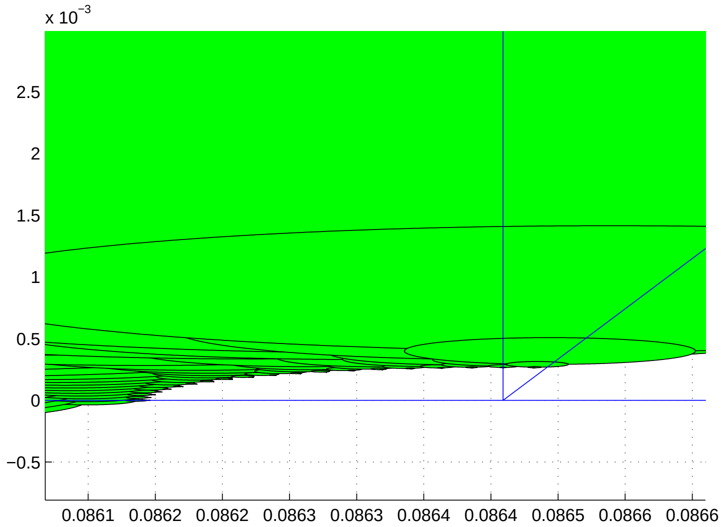

FIGURE 7.3: Eigenvalues exclosure for $\mu \in \tilde{\Omega}_\mu$

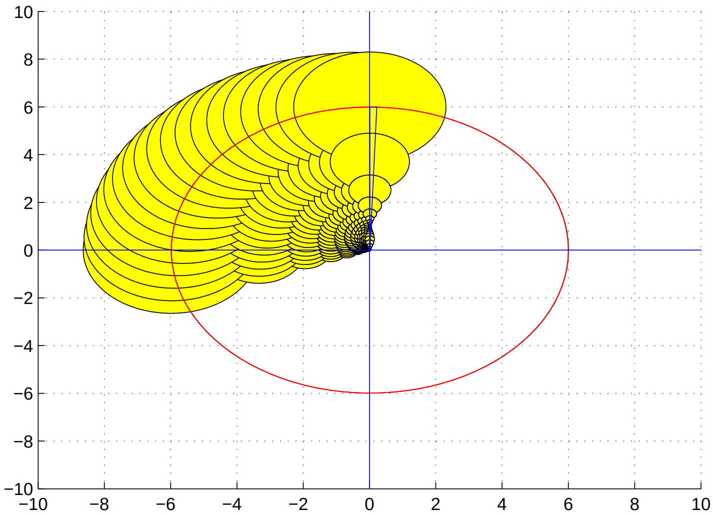

FIGURE 7.4: Eigenvalues exclosure for $\mu \in \Omega_\mu$

In the next step we have chosen $z = 0.01$ and $\zeta = 1.5359$ (see Remark 5.17). The application of Theorem 4.1 has resulted in $K_z = 5.384_7^9$, $R = 6$, and $\widetilde{M} = 9.81_{09}^{18}$. Further, we continued with the eigenvalue exclosure in the area $\hat{S}_{\zeta,z}^C \cap B(z, R)$, collecting this time the estimations for the resolvent of $L_{\bar{u}}$ as in (4.26). Figure 7.4 illustrates the second eigenvalue exclosure process in $\Omega_\mu = \{\mu \in \hat{S}_{\zeta,z}^C \cap B(z, R) : \mathrm{Im}(\mu) \geq 0\}$. Here the yellow circles correspond to the area, where the estimation for the resolvent has been conducted. After the implementation of the eigenvalues exclosure we have obtained $M_{\mathrm{IC}}^{\max} = 32.^{9221}_{8950}$. Finally we have used (5.49), (5.69) respectively in order to compute

$$M_\infty = 8.^{1111}_{1042} \cdot 10^2,$$

$$\hat{C}_\infty = 1.77_{42}^{58} \cdot 10^3, \quad \text{with} \quad r = 0.6237$$

We have obtained the value for $\delta(\varepsilon_0)$ from (5.4) by setting

$$\varepsilon_0 = \frac{z}{(1+\beta)\hat{C}_\infty\sqrt{2}} = 0.398_1^6 \cdot 10^{(-5)} \quad \text{with} \quad \beta = 10^{-15},$$

and using (7.3). Finally, we computed

$$\delta(\varepsilon_0) = 0.118_1^2 \cdot 10^{(-5)},$$

$$\delta_0 = \frac{\delta(\varepsilon_0)}{\hat{C}_\infty \sqrt{2}} = 0.470_4^9 \cdot 10^{(-8)}.$$

Therefore we have obtained the following result

$$\text{if} \quad \|u_0 - \bar{u}\|_\infty \leq 0.470_4^9 \cdot 10^{(-8)} \quad \Rightarrow \quad \lim_{t\to\infty} \|u(t) - \bar{u}\|_\infty = 0.$$

As one can see the upper bound to the domain of attraction is quite small. As we have already mentioned earlier, the reason for this are the theoretical semigroup estimations for computing constant C.

7.2 Predator-prey model

Recall from Chapter 1 that the predator-prey model has the form

$$\begin{cases} u_{1t}(x,t) = d_1 u_{1xx}(x,t) + (h_1(u_1(x,t)) - u_2(x,t))u_1(x,t), & t > 0, \quad x \in [0,l], \\ u_{2t}(x,t) = d_2 u_{2xx}(x,t) + a u_2(x,t)(u_1(x,t) - h_2(u_2(x,t))), & t > 0, \quad x \in [0,l], \\ \dfrac{\partial u(0,t)}{\partial \nu} = \dfrac{\partial u(l,t)}{\partial \nu} = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l], \end{cases}$$

where the functions h_1 and h_2 are given by

$$h_1(s) = \varepsilon_1 \left(\gamma_1 + \gamma_2 s - s^2\right),$$

$$h_2(s) = 1 + \varepsilon_2 s.$$

In our computations we have chosen the following pattern generating constellation of the parameters:

$$a = 1,\ d_1 = 0.0125,\ d_2 = 1,\ \varepsilon_1 = \frac{1}{9},\ \varepsilon_2 = \frac{2}{5},\ \gamma_1 = 35,\ \gamma_2 = 16,\ l = 1.$$

The initial conditions were set to

$$u_1^0(x) = 5 + \sum_{i=1}^{20} \cos\left((i-1)\frac{\pi x}{l}\right),$$

$$u_2^0(x) = 10 + \sum_{i=1}^{20} \cos\left((i-1)\frac{\pi x}{l}\right),$$

for all $x \in [0, l]$.

Before starting with the further description let us point out that the computations of the predator-prey model were implemented only on a partly verified basis. Namely, the results presented in the following were obtained without the computation of the lower bounds to the eigenvalues (only the upper bounds to the eigenvalues were computed). Additionaly, the interval arithmetic methods were not applied.

7.2.1 The function G

Let $u, v \in \mathbb{R}^n$. By a straightforward computation we obtain

$$g(u,v) = \begin{pmatrix} \varepsilon_1\gamma_2 u_1^2 - \varepsilon_1 u_1^3 - 3\varepsilon_1 u_1^2 v_1 - u_1 u_2 \\ a u_1 u_2 - a\varepsilon_2 u_2^2 \end{pmatrix}.$$

Estimation of the euclidean norm of $g(u,v)$ results in

$$\begin{aligned} |g(u,v)|_2 &= \sqrt{\left(\varepsilon_1\gamma_2 u_1^2 - \varepsilon_1 u_1^3 - 3\varepsilon_1 u_1^2 v_1 - u_1 u_2\right)^2 + a^2\left(u_1 u_2 - \varepsilon_2 u_2^2\right)^2} \\ &\le \varepsilon_1\gamma_2 |u_1|^2 + \varepsilon_1 |u_1|^3 + 3\varepsilon_1 |u_1|^2 |v_1| + |u_1 u_2| + a|u_1 u_2| + a\varepsilon_2 |u_2|^2 \\ &\le \varepsilon_1\gamma_2 |u|_2^2 + \varepsilon_1 |u|_2^3 + 3\varepsilon_1 |u|_2^2 |v_1| + \frac{1}{2}(1+a)|u|_2^2 + a\varepsilon_2 |u|_2^2. \end{aligned}$$

Thus, we set

$$G(h) = \left(\varepsilon_1\gamma_2 + \varepsilon_1 h + 3\varepsilon_1 \|v_1\|_\infty + \frac{1}{2}(1+a) + a\varepsilon_2\right) h^2. \tag{7.4}$$

7.2.2 Enclosure of the stationary solution

As earlier we require constants δ and K, satisfying (3.18) and (3.19).

Our computations has resulted in the following bound for defect: $\delta = 2.4314 \cdot 10^{(-6)}$. We give more details on the computation of δ and a highly accurate numerical approximation ω in Appendix A.

For compuation of K we use again (3.30). The Rayleigh-Ritz computation for $\underline{\lambda}$, with β chosen as 100 in (3.27), has resulted in value $\underline{\lambda} = 9.5671 \cdot 10^{-4}$. Inserting this value into (3.30) we obtain $K = 10.7228$. We set $v \equiv \omega$ in (7.4) and by Newton method described in Section 3.4 derive $\alpha = 2.6432 \cdot 10^{(-5)}$. Hence in the α-neighbourhood of the numerical approximation ω a stationary solution $\bar{u}$ exists.

7.2.3 Stability. Domain of attraction

Here we proceed the same way as it was described for the Schnakenberg model.

The computation of the approximate eigenvalues of the operator L_ω has resulted in $\tilde{z} = 0.5092$ and $\tilde{\zeta} = 1.2206$. Based on the enclosure constant α, the value for $|\mathcal{C}_{\bar{u}} - \mathcal{C}_\omega|_{\mathrm{Sp}}$ was computed and was bounded by $4.0786 \cdot 10^{(-4)}$. Hence, by Remark 4.8 we determine $\tilde{R} = 151$ and perform the eigenvalue exclosure process in the area $S^C_{\tilde{\zeta},\tilde{z}} \cap B(\tilde{z}, \tilde{R})$.

A first implementation of the eigenvalue exclosure process has resulted in the non-existence of eigenvalues of $L_{\bar{u}}$ in the left-hand side of the complex plane. Hence the stationary solution $\bar{u}$ is stable. In the next step we have chosen $z = 0.1$ and $\zeta = 1.4835$. Theorem 4.1, applied to this new values of z and ζ, has resulted in $R = 145$ and $\widetilde{M} = 11.9230$. After the implementation of the eigenvalue exclosure in the area $S^C_{\zeta,z} \cap B(z, R)$, which this time was combined with the estimation of the resolvent norm, we have obtained $M^{\max}_{\mathrm{IC}} = 5.5003$. We use again (5.49), (5.69), (5.75), (7.4), (5.77) respectively in order to compute

$$M_\infty = 4.8986 \cdot 10^3,$$

$$\hat{C}_\infty = 8.219 \cdot 10^3, \quad \text{with} \quad r = 0.634,$$

$$\varepsilon_0 = \frac{z}{(1+\beta)\hat{C}_\infty\sqrt{2}} = 8.6034 \cdot 10^{(-6)}, \quad \text{with} \quad \beta = 10^{-15},$$

$$\delta(\varepsilon_0) = 1.2811 \cdot 10^{(-6)}$$

$$\delta_0 = \frac{\delta(\varepsilon_0)}{\hat{C}_\infty\sqrt{2}} = 1.1022 \cdot 10^{(-10)}$$

As one can see, for the same reason as in the case of the Schnakenberg model, the result on the domain of attraction is quite small.

7.3 Spruce budworm model

In dimensionless form, formulated on the interval $\Omega = (0, l)$, the spruce budworm model reads

$$\begin{cases} u_t(x,t) = du_{xx}(x,t) + ru(x,t)\left(1 - \dfrac{u(x,t)}{q}\right) - \dfrac{u^2(x,t)}{1+u^2(x,t)}, & t > 0, \quad x \in [0,l], \\ u(0,t) = u(l,t) = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l]. \end{cases}$$

For our computations we set $r = 0.6391$, $q = 5.4$, $d = 3$, and $l = 12$. In addition we have

$$u^0(x) = \sin\left(\frac{\pi x}{l}\right) + \sin\left(\frac{2\pi x}{l}\right), \quad x \in [0,l].$$

7.3.1 *The function G*

Let $u, v \in \mathbb{R}$. By a straightforward computation we obtain

$$g(u,v) = \left(-\frac{r}{q} + \frac{3v^2 + 2vu - 1}{(1+(u+v)^2)\,(1+v^2)^2}\right) u^2. \tag{7.5}$$

Therefore we estimate

$$|g(u,v)| \leq \left(\frac{r}{q} + 3|v|^2 + 2|v||u| + 1\right)|u|^2 \tag{7.6}$$

and set

$$G(h) = \left(\frac{r}{q} + 3\left\|v\right\|_\infty^2 + 2\left\|v\right\|_\infty h + 1\right)h^2. \tag{7.7}$$

7.3.2 Enclosure of a stationary solution. Computation of the first eigenvalue

As in the cases with the Schnakenberg and predator-prey models, we need to determine the values of δ and K. Our numerical computations of the upper bound to the defect has resulted in $\delta = 0.871_6^8 \cdot 10^{(-3)}$. For detailed description of the computation of δ and a highly accurate ω please refer to Appendix A.

In the case of the spruce budworm model the linear operator $L_{\bar{u}}$ is self-adjoint. Hence we proceed as it was described in subsection 3.3.1. Recall that in order to obtain the value for the constant K, the constant K_0, which has to satisfy (3.32) and is given by

$$K_0 = \frac{1}{\underline{\lambda}}, \tag{7.8}$$

with $\underline{\lambda}$ defined as in (3.42), should be computed. In Table 7.2 we present the lower bounds for the eigenvalues obtained in the course of the corresponding homotopy process. At $s = 0$ we have started with 3 eigenvalues. At $s = 1$ we have arrived with 2 eigenvalues. We have computed the upper and lower bounds for these eigenvalues, using Theorem 6.1 and Theorem 6.2. As one can see, due to the simple implementation of the homotopy (only one step), we have actually found the lower bound to the first eigenvalue by means of comparison problems.

Inserting the value $\underline{\lambda} = \underline{\lambda}_1^{(1)}$ into (7.8) we have obtained $K_0 = 3.443_8^9$. By Lemma 3.8 we have computed $K_1 = 1.916_8^9$. Next, inserting the embedding con-

Table 7.2: Lower bounds for eigenvalues in homotopy

n	$\underline{\lambda}_n^{(0)}$	$\underline{\lambda}_n^{(1)}$
1	-0.4334	0.2904
2	0.1834	0.8347
3	1.2115	—

stants C_0 and C_1, corresponding to the Dirichlet boundary conditions, that is $C_0 = 0$, $C_1 = \frac{l}{4}$, we obtain K from (3.34) as $K = 3.320_1^2$.

Due to (7.7) and the above values of δ and K, the enclosure inequality (3.20) was satisfied with $\alpha = 0.003_1^2$.

Having the value for α, we have estimated $|\mathcal{C}_{\bar{u}} - \mathcal{C}_{\omega}|_{\mathrm{Sp}} \leq 0.013_6^7$. Thus, due to (4.44), the first eigenvalue of $L_{\bar{u}}$ was bounded by

$$\lambda_1 \geq \underline{\lambda}_1^{(1)} - |\mathcal{C}_{\bar{u}} - \mathcal{C}_{\omega}|_{\mathrm{Sp}} = 0.276_7^8 =: z. \tag{7.9}$$

7.3.3 Estimation of the attractor

Recall that in Chapter 5, in case of the self-adjoint $L_{\bar{u}}$, we have presented two approaches for the quantification of the domain of attraction. We start with the description of the first approach. By Theorem 5.24, there exist constants P, $\delta_2 > 0$ such that if $v_0 \in H_1^B(0,l)$ satisfies $q(v_0) + P\,\|v_0\|_2 < \delta_2$, then we have $t_{\max}(v_0) = \infty$ and

$$\lim_{t\to\infty} \|v(t)\|_\infty = 0.$$

Here δ_2, P are given by

$$\delta_2 = \delta(\varepsilon_1)\sqrt{\frac{d}{C_1}}, \tag{7.10}$$

$$P := \sqrt{\frac{2\pi}{e}}\sqrt{z+\sigma}(1+\beta). \tag{7.11}$$

We compute $\delta(\varepsilon_1)$ using (5.4). At first we set

$$\varepsilon := \varepsilon_1 = \frac{z}{(1+\beta)C_{L_2}}, \tag{7.12}$$

with z from (7.9). The constant C_{L_2} is computed as it was described in subsection 5.6.3. This computation has resulted in $C_{L_2} = 1.25^{71}_{70}$. The constant β in (7.12) should be small and positive. We set $\beta = 10^{-15}$. Therefore, we obtain $\varepsilon_1 = 0.220^2_1$. Having ε_1 and function G from (7.7) at hand, we obtain $\delta(\varepsilon_1) = 0.097^4_3$.

Now we are ready to compute δ_2, which is the upper bound to the domain of attraction. At first, let us comment on the embedding constant C_1. Since the Dirichlet conditions are imposed on the boundary, by Lemma 5.21 we use C_1 as it is given in (5.111). Thus, we consider the following values for C_1

$$C_1 = \begin{cases} \dfrac{l}{4}, & \text{if} \quad d \geq \dfrac{l^2}{4}(\sigma - r), \\ \dfrac{\sqrt{d}}{2\sqrt{\sigma - r}}, & \text{otherwise}, \end{cases} \tag{7.13}$$

with $\sigma > r$. Hence, for a given parameter constellation we distinguish between the following values for σ:

(A) $r < \sigma \leq \dfrac{4d}{l^2} + r$, which has resulted in $\sigma \in (0.6391, 0.7224]$,

(B) $\sigma > \dfrac{4d}{l^2} + r$, which has resulted in $\sigma > 0.7224$.

Cases (A) and (B) correspond to the first and second lines in (7.13) respectively. In Table 7.3 we present the results on the domain of attraction after the implementation of the first approach.

Table 7.3: Domain of attraction computed by the first approach

σ		δ_2
Case (A)		0.097^4_3
Case (B)	10	0.31^{70}_{69}
	100	0.572^2_1
	1000	1.018^9_8

Thus, for example, for $\sigma = 10$ from Table 7.3 we see that

$$\text{if} \quad q(u_0 - \bar{u}) + P\left\|u_0 - \bar{u}\right\|_2 < 0.31_{69}^{70}, \quad \text{then} \quad \lim_{t\to\infty}\left\|u(t) - \bar{u}\right\|_\infty = 0. \tag{7.14}$$

Notice that as σ grows, the upper bound to the domain of attraction grows as well. On the other hand, the terms on the left-hand side in the estimation of the domain of attraction, that is the constant P and the term $C_{\bar{u}} + \sigma I$, which is present in q, grow as well.

Now let us continue with the description of the second approach. By Theorem 5.30 there exist $\delta_3 > 0$ such that if $v_0 \in H_1^B(0,l)$, $q(v_0) < \delta_3$ then $t_{\max}(v_0) = \infty$ and

$$\lim_{t\to\infty}\left\|v(t)\right\|_\infty = 0. \tag{7.15}$$

Here δ_3 is given by

$$\delta_3 = \frac{\delta(\tilde{\varepsilon}_0)\sqrt{d_{\min}}}{\sqrt{C_1 n}}. \tag{7.16}$$

We compute δ_3 as it was described in Remark 5.31. At first, we set

$$\tilde{\varepsilon}_0 := \frac{z}{1+\beta}, \tag{7.17}$$

with z from (7.9) and $\beta = 10^{-15}$. We obtain $\tilde{\varepsilon}_0 = 0.276_7^8$. After that we determine the mononically non-decreasing functions G_1, G_2, and G_3 from (G1), (G2), and (G3). Starting with (7.5), after lengthy computations, we have obtained

$$G_1(h) = \frac{h^4}{q^2}\sum_{n=1}^{5} C_{6-n}h^{(n-1)}, \tag{7.18}$$

$$G_2(h) = \frac{h^2}{q^2}\sum_{n=1}^{9} K_{10-n}h^{(n-1)}, \tag{7.19}$$

$$G_3(h) = h^4\sum_{n=1}^{7} P_{8-n}h^{(n-1)}, \tag{7.20}$$

where the vectors $C = (C_1, \ldots, C_5)^T$, $K = (K_1, \ldots, K_9)^T$ and $P = (P_1, \ldots, P_7)^T$ are in fact various expressions depending on $\|\bar{u}\|_\infty$. After the application of (3.3) with $\alpha = 0.003^2_1$, we obtain

$$C = \left(1.328^4_3,\ 17.690^5_4,\ 76.629^4_3,\ 162.76^{12}_{09},\ 136.7^{102}_{100}\right)^T,$$

$$K = 10^3 \cdot \left(0.005^4_3,\ 0.05^{41}_{40},\ 0.248^9_8,\ 0.718^5_4,\ 1.476^7_6,\ 2.204^8_7,\ 2.348^7_6,\ 1.723^2_1,\ 0.685^2_1\right)^T,$$

$$P = \left(16.464^9_8,\ 83.38^{22}_{20},\ 187.01^{93}_{89},\ 269.984^9_4,\ 262.12^{86}_{80},\ 157.65^{62}_{59},\ 61.684^3_1\right)^T,$$

As soon as the functions G_1, G_2, and G_3 are established we compute $\delta(\tilde{\varepsilon}_0)$ from (5.142) by setting $a_1 = \eta$, $a_2 = d$, $a_3 = d\bar{U}_x C_1$, and $\hat{\varepsilon}|h| = \tilde{\varepsilon}_0^2$. In our computations we have to distinguish between the different cases for the constant C_1 again. In Table 7.4 we present the results on the domain of attraction after the implementation of the second method. From Table 7.4 one immediately sees that the upper bounds

Table 7.4: Domain of attraction computed by the second approach

σ		δ_3
Case (A)	0.65	0.010^2_1
	0.72	0.017^5_4
Case (B)	10	0.029^1_0
	100	0.030^3_2
	1000	0.030^6_5

to the domain of attraction are now smaller then the upper bounds, computed by the first approach. Consider, for example, $\sigma = 10$. We have

$$\text{if} \quad q(u_0 - \bar{u}) \le 0.029^1_0 \quad \text{then} \quad \lim_{t\to\infty} \|u(t) - \bar{u}\|_\infty = 0. \tag{7.21}$$

Observe that the results to the doamin of attraction are smaller in comparison to the results obtained by the first approach. This is due to the presence of the term $P\,\|u_0 - \bar{u}\|_2$ in the first inequality of (7.14).

7.4 Competition model

The competition model reads

$$\begin{cases} u_{1t}(x,t) = u_{1xx}(x,t) + u_1(x,t)(1 - u_1(x,t) - a_{12}u_2(x,t)), & t > 0, \quad x \in [0,l], \\ u_{2t}(x,t) = du_{2xx}(x,t) + \alpha u_2(x,t)(1 - u_2(x,t) - a_{21}u_1(x,t)), & t > 0, \quad x \in [0,l], \\ \dfrac{\partial u(0,t)}{\partial \nu} = \dfrac{\partial u(l,t)}{\partial \nu} = 0, & t \geq 0, \\ u(x,0) = u^0(x), & x \in [0,l], \end{cases}$$

where $\alpha,\ a_{12},\ a_{21}$ are some positive constants. The competition model is a model with the non-self-adjoint operator $L_{\bar{u}}$. On its example we would like to demonstrate the application of the T-transformation, discussed in Lemma 5.18. Hence, we consider the following constant stationary solution

$$\bar{u}_1 = \frac{1 - a_{12}}{1 - a_{12}a_{21}}, \quad \bar{u}_2 = \frac{1 - a_{21}}{1 - a_{12}a_{21}}. \tag{7.22}$$

Before starting with the quantification of the domain of attraction of the solution above, let us introduce the following

Lemma 7.1. *Let $C_{\bar{u}} \in \mathbb{R}^{n\times n}$ be a constant symmetric matrix. Let the operator $L_{\bar{u}} : D_1(L_{\bar{u}}) \to L_2^n(0,l)$ be given by*

$$\begin{aligned} &D_1(L_{\bar{u}}) = \{\varphi \in H_2^n(0,l) : \varphi'(0) = \varphi'(l) = 0\}, \\ &L_{\bar{u}}\varphi = -A\varphi + \mathcal{C}_{\bar{u}}\varphi, \quad (\mathcal{C}_{\bar{u}}\varphi)(x) := C_{\bar{u}}\varphi(x),\ x \in [0,l], \\ &D(A) = D_1(L_{\bar{u}}), \quad A\varphi = D\varphi''. \end{aligned} \tag{7.23}$$

Let $\lambda_1(L_{\bar{u}})$ and $\lambda_1(C_{\bar{u}})$ denote the smallest eigenvalue of the operator $L_{\bar{u}}$ and the matrix $C_{\bar{u}}$ respectively. Then we have

$$\lambda_1(L_{\bar{u}}) \geq \lambda_1(C_{\bar{u}}). \tag{7.24}$$

Proof. Let φ_1^L be the eigenfunction, corresponding to the first eigenvalue of $L_{\bar{u}}$ and φ_1^A be the eigenfunction, corresponding to the first eigenvalue of the operator $-A$. Due to the self-adjointness of $L_{\bar{u}}$ we have

$$\langle L_{\bar{u}}\varphi_1^L, \varphi_1^A\rangle_2 = \langle \varphi_1^L, L_{\bar{u}}\varphi_1^A\rangle_2 = \langle \varphi_1^L, -A\varphi_1^A + \mathcal{C}_{\bar{u}}\varphi_1^A\rangle_2$$

On the other hand,

$$-A\varphi_1^A = \lambda_1(-A)\varphi_1^A = 0,$$

since the first eigenvalue of the second order derivative operator, defined on $(0, l)$, with Neumann boundary conditions is zero.

Thus, taking into account the self-adjointness of $\mathcal{C}_{\bar{u}}$, we obtain

$$\langle L_{\bar{u}}\varphi_1^L, \varphi_1^A\rangle_2 = \langle \varphi_1^L, \mathcal{C}_{\bar{u}}\varphi_1^A\rangle_2 = \langle \mathcal{C}_{\bar{u}}\varphi_1^L, \varphi_1^A\rangle_2.$$

Consequently, we have

$$L_{\bar{u}}\varphi_1^L = \mathcal{C}_{\bar{u}}\varphi_1^L.$$

Therefore, we obtain

$$\lambda_1(L_{\bar{u}})\langle \varphi_1^L, \varphi_1^L\rangle_2 = \langle \mathcal{C}_{\bar{u}}\varphi_1^L, \varphi_1^L\rangle_2 \geq \lambda_{\min}(C_{\bar{u}})\langle \varphi_1^L, \varphi_1^L\rangle_2 = \lambda_1(C_{\bar{u}})\langle \varphi_1^L, \varphi_1^L\rangle_2.$$

We have obtained the assertion. □

Now let us return to the competition model. By a straightforward computation we obtain its Jacobian matrix as

$$C_u = \begin{pmatrix} -1 + 2u_1 + a_{12}u_2 & a_{12}u_1 \\ \alpha a_{21}u_2 & \alpha(-1 + 2u_2 + a_{21}u_1) \end{pmatrix}$$

Evaluating C_u at the constant stationary solution $\bar{u} = \left(\frac{1 - a_{12}}{1 - a_{12}a_{21}}, \frac{1 - a_{21}}{1 - a_{12}a_{21}}\right)^T$, we obtain

$$C_{\bar{u}} = \frac{1}{1 - a_{12}a_{21}} \begin{pmatrix} 1 - a_{12} & a_{12}(1 - a_{12}) \\ \alpha a_{21}(1 - a_{21}) & \alpha(1 - a_{21}) \end{pmatrix}.$$

As one can see, if $\alpha \neq 1$ and $a_{12} \neq a_{21}$, the matrix $C_{\bar{u}}$ is not symmetric. Following the Proposition 5.18 we choose $a_{12},\ a_{21}$ such that

$$\frac{a_{12}(1-a_{12})}{\alpha a_{21}(1-a_{21})} > 0$$

holds. Due to the positivity of a_{12} and a_{21} the condition above is satisfied if either $a_{12} < 1,\ a_{21} < 1$ or $a_{12} > 1,\ a_{21} > 1$. In the following we will be investigating the case when $a_{12} < 1,\ a_{21} < 1$.

Now let us introduce a T-transformation into the original problem. For that purpose we set

$$t_1 = \sqrt{c_{12}} = \sqrt{\frac{a_{12}(1-a_{12})}{1-a_{12}a_{21}}},$$

$$t_2 = \sqrt{c_{21}} = \sqrt{\frac{\alpha a_{21}(1-a_{21})}{1-a_{12}a_{21}}}.$$

A straightforward computation results in

$$\widetilde{C}_{\bar{u}} = T^{-1}C_{\bar{u}}T = \begin{pmatrix} \dfrac{1-a_{12}}{1-a_{12}a_{21}} & \dfrac{(\alpha a_{12}a_{21}(1-a_{12})(1-a_{21}))^{\frac{1}{2}}}{1-a_{12}a_{21}} \\ \dfrac{(\alpha a_{12}a_{21}(1-a_{12})(1-a_{21}))^{\frac{1}{2}}}{1-a_{12}a_{21}} & \dfrac{\alpha(1-a_{21})}{1-a_{12}a_{21}} \end{pmatrix}$$

with

$$\lambda_1(\widetilde{C}_{\bar{u}}) = \frac{1}{2(1-a_{12}a_{21})}\Big(1-a_{12}+\alpha(1-a_{21}) - \big((1-a_{12}+\alpha(1-a_{21}))^2 - 4\alpha(1-a_{12}a_{21})(1-a_{12})(1-a_{21})\big)^{\frac{1}{2}}\Big). \tag{7.25}$$

Due to Lemma 7.1 we have $\lambda_1(\tilde{L}_{\bar{u}}) \geq \lambda_1(\widetilde{C}_{\bar{u}})$. From $a_{12} < 1,\ a_{21} < 1$ follows that $\lambda_1(\widetilde{C}_{\bar{u}}) > 0$. Therefore the stationary solution (7.22) is stable in the presence of diffusion.

Now we set $w(t) = T^{-1}v(t)$. Further, using (5.96), by a straightforward computation, we obtain

$$\tilde{g}(w,\bar{u}) = -\frac{1}{\sqrt{1-a_{12}a_{21}}}\begin{pmatrix} \sqrt{a_{12}(1-a_{12})}w_1^2 + a_{12}\sqrt{\alpha a_{21}(1-a_{21})}w_1w_2 \\ \alpha\left(\sqrt{\alpha a_{21}(1-a_{21})}w_2^2 + a_{21}\sqrt{a_{12}(1-a_{12})}w_1w_2\right) \end{pmatrix}.$$

Thus, after the application of T-transformation, we arrive at the following problem

$$\begin{cases} w'(t) = -\tilde{L}_{\bar{u}}w(t) + \tilde{g}(w(t),\bar{u}), & t > 0, \\ w(0) = w_0, \end{cases} \tag{7.26}$$

where the operator $\tilde{L}_{\bar{u}} : D_1(L_{\bar{u}}) \to L_2^n(0,l)$ is given by

$$\tilde{L}_{\bar{u}}\varphi = -D\varphi'' + \widetilde{C}_{\bar{u}}\varphi \tag{7.27}$$

and is self-adjoint. Now we can apply Theorem 5.24 and Theorem 5.30 to problem (7.26).

Starting with the nonlinearity $\tilde{g}(w,\bar{u})$ we have computed functions G, G_1, and G_2 as follows

$$\begin{aligned} G(h) &= Kh^2, \\ G_1(h) &= K_1h^4, \\ G_2(h) &= K_2h^2, \end{aligned}$$

with

$$\begin{aligned} K := \frac{1}{\sqrt{1-a_{12}a_{21}}}\Big(&\max\left(\sqrt{a_{12}(1-a_{12})}, \alpha\sqrt{\alpha a_{21}(1-a_{21})}\right) \\ &+ \frac{1}{2}\sqrt{\alpha a_{12}a_{21}}\left(\sqrt{a_{12}(1-a_{12})} + \sqrt{\alpha a_{21}(1-a_{21})}\right)\Big), \end{aligned}$$

$$K_1 := \frac{1}{1-a_{12}a_{21}}\Bigg(\frac{1}{2}\max\big(a_{12}(1-a_{12}), \alpha^3 a_{21}(1-a_{21})\big)$$

$$+\frac{3\sqrt{3}}{8}\sqrt{\alpha a_{12}a_{21}(1-a_{12})(1-a_{21})}\,\big(a_{12}+\alpha^2 a_{21}\big)$$

$$+\frac{1}{4}\alpha a_{12}a_{21}(a_{12}(1-a_{21})+\alpha a_{21}(1-a_{12}))\Bigg),$$

$$K_2 := \frac{1}{1-a_{12}a_{21}}\Big(4\max\big(a_{12}(1-a_{12}), \alpha^3 a_{21}(1-a_{21})\big)$$

$$+2\sqrt{\alpha a_{12}a_{21}(1-a_{12})(1-a_{21})}\,\big(a_{12}+\alpha^2 a_{21}\big)$$

$$+\alpha a_{12}a_{21}\left(a_{12}(1-a_{21})+\alpha a_{21}(1-a_{12})\right)\Big).$$

By Theorem 5.24 there exist $P,\ \delta_2 > 0$ such that if $w_0 \in H_2^B(0,l)$, $q(w_0)+P\,\|w_0\|_2 < \delta_2$, we have $t_{\max}(w_0)=\infty$ and

$$\lim_{t\to\infty}\|w(t)\|_\infty = 0.$$

As earlier the constants $\delta_2,\ P$ are given by

$$\delta_2 = \delta(\varepsilon_1)\sqrt{\frac{d}{2C_1}}, \tag{7.28}$$

$$P := \sqrt{\frac{2\pi}{e}}\sqrt{z+\sigma}(1+\beta). \tag{7.29}$$

We compute $\delta(\varepsilon_1)$ using again (5.4). We set

$$\varepsilon := \varepsilon_1 = \frac{z}{(1+\beta)C_{L_2}}, \tag{7.30}$$

with $z = \lambda_1(\tilde{C}_{\bar{u}})$ from (7.25) and C_{L_2} computed as above. Since the Neumann conditions are imposed on the boundary, we have chosen C_1 as in (5.110), that is

$$C_1 = \frac{1}{\rho}, \quad \rho = \frac{1}{2l}\left(\sqrt{1+\frac{4l^2(z+\sigma)}{\min(1,d)}}-1\right). \tag{7.31}$$

Note that $\lambda_1^C = z$ in (5.110). Due to the positivity of z, we set $\sigma = 0$. Thus, having $\delta(\varepsilon_1)$ and C_1 at hand we compute δ_2 in (7.28).

In Table 7.5 we introduce the results on the domain of attraction for the different constellation of the parameters $(\alpha,\ d,\ l,\ a_{12},\ a_{21})$.

Table 7.5: Domain of attraction computed by the first approach

$\alpha,\ d,\ l$	a_{12}	a_{21}	$\bar{u}_1$	$\bar{u}_2$	δ_2
1, 1, 1	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.096^6_5
2, 1, 1	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.074^4_3
1, 2, 1	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.096^6_5
1, 1, 2	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.116^9_8

Let us take, e.g., the parameter constellation $(\alpha,\ d,\ l,\ a_{12},\ a_{21}) = \left(1,\ 2,\ 1, \frac{1}{3}, \frac{2}{3}\right)$. By Theorem 5.24 we obtain

$$\text{if} \quad q(T^{-1}(u_0 - \bar{u})) + P \left\|T^{-1}(u_0 - \bar{u})\right\|_2 < 0.096^6_5,$$

$$\text{then} \quad t_{\max}(u_0) = \infty \quad \text{and} \quad \lim_{t\to\infty} \|(u(t) - \bar{u})\|_\infty = 0,$$

where $\bar{u} = (0.8571, 0.4286)^T$.

We proceed with the second approach as follows. By Theorem 5.30 there exists $\delta_3 > 0$ such that if $w_0 \in H_1^B(0,l)$ with $q(w_0) < \delta_3$ then $t_{\max}(w_0) = \infty$ and

$$\|w(t)\|_\infty \to 0.$$

We start by setting

$$\tilde{\varepsilon}_0 = \frac{z}{1+\beta}, \tag{7.32}$$

with z from (7.25) and $\beta = 10^{-15}$. After that, following Remark 5.31, we set in

(5.142):

$$a_1 = \eta = \frac{\lambda_{\max}(\tilde{C}_{\bar{u}})}{\lambda_{\min}(\tilde{C}_{\bar{u}})},$$

$$a_2 = \frac{d_{\max}}{d_{\min}},$$

$$\hat{\varepsilon}|h| = \tilde{\varepsilon}_0^2 := \left(\frac{z}{1+\beta}\right)^2.$$

Inserting the functions G_1 and G_2 into (5.142) we obtain $\delta(\tilde{\varepsilon}_0)$ as

$$\delta(\tilde{\varepsilon}_0) = \frac{z}{(1+\beta)\sqrt{a_1 K_1 + a_2 K_2}}, \tag{7.33}$$

and finally compute δ_3, which is given by

$$\delta_3 = \frac{\delta(\tilde{\varepsilon}_0)\sqrt{d_{\min}}}{\sqrt{2C_1}}. \tag{7.34}$$

In Table 7.6 we present the results on the domain of attraction, computed by the second approach. We used the same constellations of the parameters $(\alpha,\ d,\ l,\ a_{12},\ a_{21})$, as in the case with the first approach. Let us again consider the constellation of the

Table 7.6: Domain of attraction computed by the second approach

$\alpha,\ d,\ l$	a_{12}	a_{21}	$\bar{u}_1$	$\bar{u}_2$	δ_3
1, 1, 1	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.054^{8}_{7}
2, 1, 1	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.044^{1}_{0}
1, 2, 1	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.044^{4}_{3}
1, 1, 2	$\frac{1}{3}$	$\frac{2}{3}$	0.8571	0.4286	0.066^{3}_{2}

parameters $(\alpha,\ d,\ l,\ a_{12},\ a_{21}) = \left(1,\ 2,\ 1, \frac{1}{3}, \frac{2}{3}\right)$. By Theorem 5.30 we have

$$\text{if} \quad u_0 \in H_1^B(0,l), \quad q(T^{-1}(u_0 - \bar{u})) \le 0.044^{4}_{3}$$

$$\text{then} \quad t_{\max}(u_0) = \infty \quad \text{and} \quad \|u(t) - \bar{u}\|_\infty \to 0,$$

where $\bar{u} = (0.8571, 0.4286)^T$.

Finally, let us briefly comment on the change of the upper bounds to the domain of attraction with respect to the change in the parameters. In both tables, we have examined the case, where the competitive effect of the grey squirrels is higher than the competitive effect of the red squirrels. By changing the parameters of the model for a fixed stationary solution, we have observed the changes in the domain of attraction. As one can see the growth in l corresponds to the growth of both δ_2 and δ_3, whereas the growth in α causes the decade in both δ_2 and δ_3. When the parameter d increases δ_2 remains the same, while δ_3 decreases, which is due to the presence of d in $\delta(\tilde{\varepsilon}_0)$.

Appendix A

Numerical treatment

In the present chapter we comment on the numerical computations which were carried out in the course of this thesis. Before starting with the actual description of the numerical procedures, we would like to comment on the notations in this chapter. Although we have to present the numerical algorithms for three different models, for simplicity reasons we will use more general notations for the terms under consideration. For example, we use the notation $\mathbf{N}$ for the number of the ansatz functions in general, although this number is different for each model. When the distinguishing between different problems is essential, we will comment on the corresponding differences. In that case the k-index in the notation is used, with $k = 1, 2, 3$ for Schnakenberg, predator-prey and spruce budworm models respectively.

A.1 Ansatz space

We start with the definition of the ansatz functions for the problems above. Taking into account the specifics of the given models, we introduce for all $x \in [0, l]$

$$\phi_j(x) = \begin{cases} \tilde{\varphi}_j(x) & \text{for} \quad k = 1, 2, \\ \sin\left(j\pi \dfrac{x}{l}\right), & \text{for} \quad k = 3 \end{cases} \qquad j = 1, \ldots, \mathbf{N},$$

with

$$\begin{aligned} \tilde{\varphi}_{2i-1}(x) &= \begin{pmatrix} \cos\left((i-1)\frac{\pi x}{l}\right) \\ 0 \end{pmatrix}, \\ \tilde{\varphi}_{2i}(x) &= \begin{pmatrix} 0 \\ \cos\left((i-1)\frac{\pi x}{l}\right) \end{pmatrix}, \quad i = 1, \ldots, \mathbf{M} + 1, \end{aligned} \tag{A.1}$$

where $\mathbf{M},\ \mathbf{N}$ are some positive constants and $\mathbf{N} = 2(\mathbf{M} + 1)$ in the case of the Schnakenberg and predator-prey problems. In addition, let us denote

$$\varphi_i(x) = \cos\left((i-1)\frac{\pi x}{l}\right), \quad i = 1, \ldots, \mathbf{M} + 1.$$

Thus, we perform all computations in the following ansatz space

$$V = \operatorname{span}\left\{\, \phi_j(x),\ j = 1, \ldots, \mathbf{N}\right\}. \tag{A.2}$$

It follows that the numerical approximation $\omega(x)$ is represented in the form

$$\omega(x) = \sum_{j=1}^{\mathbf{N}} \alpha_j \phi_j(x) \tag{A.3}$$

with α_j being appropriate Fourier coefficients. In addition, let us introduce for $k = 1, 2$ the following representation of the components $\omega_1(x),\ \omega_2(x)$:

$$\omega_1(x) = \sum_{i=1}^{\mathbf{M}+1} \alpha_{2i-1} \varphi_i(x), \tag{A.4}$$

$$\omega_2(x) = \sum_{i=1}^{\mathbf{M}+1} \alpha_{2i} \varphi_i(x). \tag{A.5}$$

A.2 Newton algorithm

In order to obtain a sufficiently small bound δ from (3.18) we need to compute a highly accurate numerical solution ω. We are going to carry out this step using Newton's algorithm. In the following, in order to avoid misunderstanding, we denote the defect of numerical solution ω as $\mathbf{d}[\omega]$.

Starting with some rough numerical approximation $\omega^{(0)}$ we proceed with Newton's algorithm as follows:

$$\begin{array}{l} \bullet \quad Lv^{(n)} = -\mathbf{d}[\omega^{(n)}] \\ \bullet \quad \omega^{(n+1)} := \omega^{(n)} + v^{(n)} \end{array} \qquad (n = 0, 1, \ldots, n_0),\ v^{(n)} \in H_2^B(0, l). \tag{A.6}$$

Here $Lv^{(n)}$ and $\mathbf{d}[\omega^{(n)}]$ are given by

$$Lv^{(n)} = -D\left(v^{(n)}\right)'' + C_{\omega^{(n)}} v^{(n)}, \tag{A.7}$$

$$\mathbf{d}[\omega^{(n)}] = -D\omega^{(n)} - F(\omega^{(n)}). \tag{A.8}$$

We terminate the iteration at some index n_0. There exist two possible reasons for this: either the Fourier coefficients of the solution

$$v^{(n_0)}(x) = \sum_{j=1}^{\mathbf{N}} \beta_j^{(n_0)} \phi_j(x) \tag{A.9}$$

satisfy

$$\left|\beta_j^{(n_0)}\right| < \varepsilon, \tag{A.10}$$

where ε is some given tolerance, or some maximal iteration number has been reached.

When $k = 1, 2$ in order to solve (A.6) we use a Galerkin method. In case $k = 3$ we find the solution of (A.6) with the help of the collocation procedure.

Newton-Galerkin method. Schnakenberg and predator-prey models. We start by multiplying the above system with the ansatz function $\tilde{\varphi}_j$, and taking the scalar product in $L_2^n(0,l)$. We obtain

$$\left.\begin{array}{l} \bullet \quad \left\langle Lv^{(n)}, \tilde{\varphi}_j \right\rangle_2 = \left\langle -\mathbf{d}\left[\omega^{(n)}\right], \tilde{\varphi}_j \right\rangle_2, \; j = 1, \dots, \mathbf{N} \\ \bullet \quad \omega^{(n+1)} := \omega^{(n)} + v^{(n)} \end{array}\right. \quad (n = 0, 1, \dots, n_0), \; v^{(n)} \in H_2^B(0,l).$$

Taking into account (A.7), (A.8), and (A.9), we write the Newton-Galerkin step as:

$$\left.\begin{array}{l} \bullet \quad \left(M_1 + M_2^{(n)}\right) \beta^{(n)} = M_3^{(n)} \\ \bullet \quad \alpha^{(n+1)} := \alpha^{(n)} + \beta^{(n)} \end{array}\right. \quad (n = 0, 1, \dots, n_0). \tag{A.11}$$

The matrices $M_1, M_2^{(n)}$, and the vector $M_3^{(n)}$ are defined as follows:

$$M_{1_{ij}} := -\left\langle D\tilde{\varphi}_i'', \tilde{\varphi}_j \right\rangle_2,$$

$$M_{2_{ij}}^{(n)} := \left\langle C_{\omega^{(n)}} \tilde{\varphi}_i, \tilde{\varphi}_j \right\rangle_2,$$

$$M_{3_j}^{(n)} := \left\langle -\mathbf{d}\left[\omega^{(n)}\right], \tilde{\varphi}_j \right\rangle_2,$$

for $i, j = 1, \dots, \mathbf{N}$. The matrix M_1 can be calculated explicitly and has the form

$$M_{1_{ij}} = \begin{cases} d_2 \, \dfrac{\pi^2}{2l} \left(\dfrac{i}{2} - 1\right)^2, & i = j \text{ and } i \text{ is even}, \\ d_1 \, \dfrac{\pi^2}{2l} \left(\dfrac{i-1}{2}\right)^2 & i = j \text{ and } i \text{ is odd}, \\ 0, & \text{otherwise}. \end{cases} \tag{A.12}$$

The matrix $M_2^{(n)}$ and the vector $M_3^{(n)}$ read:

$$\begin{aligned} M_{2_{ij}}^{(n)} = \int_0^l \Big(& f_1^n(x)\, \tilde{\varphi}_{i,1}(x)\, \tilde{\varphi}_{j,1}(x) + f_2^n(x)\, \tilde{\varphi}_{i,2}(x)\, \tilde{\varphi}_{j,1}(x) \\ & + f_3^n(x)\, \tilde{\varphi}_{i,1}(x)\, \tilde{\varphi}_{j,2}(x) + f_4^n(x)\, \tilde{\varphi}_{i,2}(x)\, \tilde{\varphi}_{j,2}(x) \Big)\, dx, \end{aligned} \tag{A.13}$$

$$\begin{aligned} M_{3_j}^{(n)} = \int_0^l \Big(& h_1^n(x)\, \tilde{\varphi}_{j,1}(x) + h_2^n(x)\, \tilde{\varphi}_{j,2}(x) \\ & + h_3^n(x)\, \tilde{\varphi}_{j,1}(x) + h_4^n(x)\, \tilde{\varphi}_{j,2}(x) \Big)\, dx, \end{aligned} \tag{A.14}$$

where $\tilde{\varphi}_{i,m}(x),\ m=1,2$ denotes either the first or the second component of $\tilde{\varphi}_i$ given by (A.1), and the functions $f_p^n(x),\ h_p^n(x),\ p=1,\dots,4$ are given by

$$f_1^n(x) := \gamma\left(1-2\omega_1^{(n)}(x)\omega_2^{(n)}(x)\right), \tag{A.15}$$

$$f_2^n(x) := -\gamma\left(\omega_1^{(n)}(x)\right)^2, \tag{A.16}$$

$$f_3^n(x) := 2\gamma\omega_1^{(n)}(x)\omega_2^{(n)}(x), \tag{A.17}$$

$$f_4^n(x) := \gamma\left(\omega_1^{(n)}(x)\right)^2, \tag{A.18}$$

$$h_1^n(x) := \left(\omega_1^{(n)}(x)\right)'', \tag{A.19}$$

$$h_2^n(x) := d\left(\omega_2^{(n)}(x)\right)'', \tag{A.20}$$

$$h_3^n(x) := \gamma\left(a-\omega_1^{(n)}(x)+\left(\omega_1^{(n)}(x)\right)^2\omega_2^{(n)}(x)\right), \tag{A.21}$$

$$h_4^n(x) := \gamma\left(b-\left(\omega_1^{(n)}(x)\right)^2\omega_2^{(n)}(x)\right), \tag{A.22}$$

in case of the Schnakenberg nonlinearity, and by

$$f_1^n(x) := \left(-\varepsilon_1\gamma_1-2\varepsilon_1\gamma_2\omega_1^{(n)}(x)+3\varepsilon_1\left(\omega_1^{(n)}(x)\right)^2+\omega_2^{(n)}(x)\right), \tag{A.23}$$

$$f_2^n(x) := \omega_1^{(n)}(x), \tag{A.24}$$

$$f_3^n(x) := -a\omega_2^{(n)}(x), \tag{A.25}$$

$$f_4^n(x) := a\left(2\varepsilon_2\omega_2^{(n)}(x)-\omega_1^{(n)}(x)+1\right), \tag{A.26}$$

$$h_1^n(x) := d\left(\omega_1^{(n)}(x)\right)'', \tag{A.27}$$

$$h_2^n(x) := \left(\omega_2^{(n)}(x)\right)'', \tag{A.28}$$

$$h_3^n(x) := \Bigg(\varepsilon_1\gamma_1\omega_1^{(n)}(x) + \varepsilon_1\gamma_2\left(\omega_1^{(n)}(x)\right)^2$$

$$- \varepsilon_1\left(\omega_1^{(n)}(x)\right)^3 - \omega_1^{(n)}(x)\omega_2^{(n)}(x)\Bigg), \tag{A.29}$$

$$h_4^n(x) := a\left(\omega_1^{(n)}(x)\omega_2^{(n)}(x) - \varepsilon_2\left(\omega_2^{(n)}(x)\right)^2 - \omega_2^{(n)}(x)\right), \tag{A.30}$$

when the predator-prey model is under consideration.

It is easy to see that due to (A.1), (A.4) and (A.5) the computation of the elements of $M_2^{(n)}$ and $M_3^{(n)}$ can be reduced to the computation of terms of the form

$$\sum_{i_1=1}^{\mathbf{M}+1}\tilde{\alpha}_{m(i_1)}\int_0^l \varphi_{i_1}(x)\varphi_i(x)\varphi_j(x)dx,$$

$$\sum_{i_1=1}^{\mathbf{M}+1}\sum_{i_2=1}^{\mathbf{M}+1}\tilde{\alpha}_{m(i_1)}\tilde{\alpha}_{m(i_2)}\int_0^l \varphi_{i_1}(x)\varphi_{i_2}(x)\varphi_i(x)\varphi_j(x)dx,$$

$$\sum_{i_1=1}^{\mathbf{M}+1}\tilde{\alpha}_{m(i_1)}\int_0^l \varphi_{i_1}(x)\varphi_j(x)dx,$$

$$\sum_{i_1=1}^{\mathbf{M}+1}\sum_{i_2=1}^{\mathbf{M}+1}\tilde{\alpha}_{m(i_1)}\tilde{\alpha}_{m(i_2)}\int_0^l \varphi_{i_1}(x)\varphi_{i_2}(x)\varphi_j(x)dx,$$

$$\sum_{i_1=1}^{\mathbf{M}+1}\sum_{i_2=1}^{\mathbf{M}+1}\sum_{i_3=1}^{\mathbf{M}+1}\tilde{\alpha}_{m(i_1)}\tilde{\alpha}_{m(i_2)}\tilde{\alpha}_{m(i_3)}\int_0^l \varphi_{i_1}(x)\varphi_{i_2}(x)\varphi_{i_3}(x)\varphi_j(x)dx,$$

for $i, j = 1, \ldots, \mathbf{N}$. In the integrals above, the index $m(i)$ is given by either $m(i) = 2i$ or $m(i) = 2i - 1$, depending on the term, and $\tilde{\alpha}_{m(i)}$ is given by either $\tilde{\alpha}_{m(i)} = \alpha_{m(i)}$ or $\tilde{\alpha}_{m(i)} = (i-1)^2\alpha_{m(i)}$ (when the second derivative is under consideration).

It is possible to compute the integrals above in closed form with the help of the formula (A.78), which we present later in this appendix.

Thus, going back to (A.11) again, we approximately solve the corresponding system with the help of Gauss algorithm. We continue the computation, until one of the termination conditions is satisfied.

Newton collocation method. Spruce budworm model. We consider the Newton step (A.6). We evaluate the function $v^{(n)}$ in the collocation points x_m as follows

$$v^{(n)}(x_m) = \sum_{j=1}^{\mathbf{N}} \beta_j^{(n)} \sin\left(j\pi \frac{x_m}{l}\right), \quad m = 1, \dots \mathbf{N}. \tag{A.31}$$

In particular, we choose the collocation points as

$$x_m = \frac{m}{\mathbf{N}+1} l, \quad m = 1, \dots, \mathbf{N}.$$

Evaluating the term $\mathbf{d}\left[\omega^{(n)}\right]$ and the term $Lv^{(n)}$ in x_m, we obtain

$$\begin{aligned}
&\left(\mathbf{d}\left[\omega^{(n)}\right]\right)(x_m) =: \mathbf{d}_m \\
&= -\,d\left(\omega^{(n)}\right)''(x_m) - r\omega^{(n)}(x_m)\left(1 - \frac{\omega^{(n)}(x_m)}{q}\right) + \frac{\left(\omega^{(n)}(x_m)\right)^2}{1 + \left(\omega^{(n)}(x_m)\right)^2} \\
&\left(L[v^{(n)}]\right)(x_m) \\
&= -\,d\left(v^{(n)}\right)''(x_m) + \left(-r + 2\frac{r}{q}\omega^{(n)}(x_m) + \frac{2\omega^{(n)}(x_m)}{\left(1 + \left(\omega^{(n)}(x_m)\right)^2\right)^2}\right) v^{(n)}(x_m)
\end{aligned}$$

Let us denote the matrices S^1, $K^{(n)}$, S^2 as follows

$$\begin{aligned}
S^1_{mj} &:= \sin\left(j\pi \frac{m}{\mathbf{N}+1}\right), \\
K^{(n)}_{mj} &:= \begin{cases} -r + 2\dfrac{r}{q}\omega^{(n)}(x_m) + \dfrac{2\omega^{(n)}(x_m)}{\left(1 + \left(\omega^{(n)}(x_m)\right)^2\right)^2}, & \text{if} \quad m = j, \\ 0, & \text{otherwise}, \end{cases} \\
S^2_{mj} &:= \begin{cases} dj^2\,\dfrac{\pi^2}{l^2} & \text{if} \quad m = j, \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}$$

for all $m, j = 1, \ldots, \mathbf{N}$. Then Newton step (A.6) reads

$$(S^1 S^2 + K^{(n)} S^1)\beta^{(n)} = -\mathbf{d}, \tag{A.32}$$

where $\beta^{(n)} = (\beta_1^{(n)}, \ldots, \beta_{\mathbf{N}}^{(n)})^T$ and $\mathbf{d} = (\mathbf{d}_1, \ldots, \mathbf{d}_{\mathbf{N}})^T$. We find $\beta^{(n)}$ from (A.32) with the help of Gauss algorithm. Using (A.31) again, we can construct $v^{(n)}$ and eventually obtain $\omega^{(n+1)}$. The computation continues, until one of the termination conditions is satisfied.

A.3 Calculation of the upper bound for defect

Our aim in this subsection is to find δ which satisfies (3.18). Having computed a highly accurate solution ω^k, $k = 1, 2, 3$, we may hope, that the upper bound for the defect is sufficiently small in order to satisfy (3.20).

Let us at first derive the expression $\left\|\mathbf{d}\left[\omega^{(k)}\right]\right\|_2^2$. After straightforward calculations we obtain

$$\begin{aligned}\left\|\mathbf{d}\left[\omega^{(1)}\right]\right\|_2^2 = &\int_0^l \left(-\left(\omega_1^{(1)}\right)'' - \gamma\left(a - \omega_1^{(1)} + \left(\omega_1^{(1)}\right)^2 \omega_2^{(1)}\right)\right)^2 dx \\ &+ \int_0^l \left(-d\left(\omega_2^{(1)}\right)'' - \gamma\left(b - \left(\omega_1^{(1)}\right)^2 \omega_2^{(1)}\right)\right)^2 dx,\end{aligned} \tag{A.33}$$

$$\begin{aligned}\left\|\mathbf{d}\left[\omega^{(2)}\right]\right\|_2^2 = &\int_0^l \left(-d_1\left(\omega_1^{(2)}\right)'' + \omega_1^{(2)}\left(\omega_2^{(2)} - \varepsilon_1\left(\gamma_1 + \gamma_2\omega_1^{(2)} - \left(\omega_1^{(2)}\right)^2\right)\right)\right)^2 dx \\ &+ \int_0^l \left(-d_2\left(\omega_2^{(2)}\right)'' + a\omega_2^{(2)}\left(1 + \varepsilon_2\omega_2^{(2)} - \omega_1^{(2)}\right)\right)^2 dx,\end{aligned} \tag{A.34}$$

$$\left\|\mathbf{d}\left[\omega^{(3)}\right]\right\|_2^2 = \int_0^l \left(-d\left(\omega^{(3)}\right)'' - r\omega^{(3)} + \frac{r}{q}\left(\omega^{(3)}\right)^2 + \frac{\left(\omega^{(3)}\right)^2}{1 + \left(\omega^{(3)}\right)^2}\right)^2 dx. \tag{A.35}$$

Inserting (A.4) and (A.5) into (A.33) and (A.34) one can see that the computation of $\left\|\mathbf{d}\left[\omega^{(k)}\right]\right\|_2^2$ $k = 1, 2$ could be reduced to the computation of the integrals of the form

$$\sum_{i_1=1}^{\mathbf{M}+1} \cdots \sum_{i_p=1}^{\mathbf{M}+1} \tilde{\alpha}_{m(i_1)} \dots \tilde{\alpha}_{m(i_p)} \int_0^l \varphi_{i_1}(x) \dots \varphi_{i_p}(x)\, dx, \tag{A.36}$$

where $\tilde{\alpha}_{m(i)}$ and $m(i)$ are defined as in the previous section and $p = 1, \dots, 6$. For the computation of the expression (A.36) in closed form we apply again (A.78).

We compute some of the terms of $\left\| \mathbf{d}\left[\omega^{(3)}\right] \right\|_2^2$ explicitly, using the sinus summation theorem and the orthogonal property of $\sin\left(i\pi \frac{x}{l}\right)$. The rest terms are handled with numerical integration methods. In particular, we approximate these terms with the trapezoidal rule and bound the quadrature error rigorously. We comment on this approach later, in section A.7

In (3.18) the safe bounds for defect are required. Hence, in order to pay regard to rounding errors, we implement all calculations in interval arithmetic, using the interval package INTLAB [50].

A.4 Rayleigh-Ritz Method

Recall that we need to find bounds for the eigenvalues of the problems (6.1) and (6.2). In this subsection we comment on the appilcation of Theorem 6.1 in both cases. The eigenvalue problems of the type (6.1) occur, when we consider the spruce budworm model. For the Schnakenberg and predator-prey model the eigenvalue problems of the type (6.2) are under consideration.

Problem of the form (6.2). Recall that we consider a sequence of eigenvalue problems of the form

$$u \in H_2^B(0,l),\ \mathcal{M}_s(u,v) = \boldsymbol{\lambda}^{(s)} \mathcal{N}(u,v) \quad \text{for all } v \in H_2^B(0,l),\ s \in [0,1], \tag{A.37}$$

where

$$\mathcal{M}_s(u,v) := (1-s)\langle -Du'', -Dv''\rangle_2 + (P_1(1-s)+\sigma\beta_2)\langle u', v'\rangle_2$$
$$+ (P_2(1-s)+\sigma\beta_1)\langle u, v\rangle_2 + s\langle (L-\mu)u, (L-\mu)v\rangle_2,$$
$$\mathcal{N}(u,v) := \beta_1\langle u, v\rangle_2 + \beta_2\langle u', v'\rangle_2,$$

where the constants $\beta_1,\ \beta_2,\ P,\ \sigma,\ \mu$ depend on the model.

For some linearly independent $\tilde{u}_1, \ldots, \tilde{u}_N \in H_2^B(0,l)$ we define

$$A_0^s := (\mathcal{M}_s(\tilde{u}_i, \tilde{u}_j))_{i,j=1,\ldots,N},$$
$$A_1 := (\mathcal{N}(\tilde{u}_i, \tilde{u}_j))_{i,j=1,\ldots,N}.$$

Therefore we can approximate eigenvalue problem (A.37) in the form required for Theorem 6.1, namely

$$A_0^s x = \tilde{\boldsymbol{\lambda}}^s A_1 x. \tag{A.38}$$

In the first step, in order to obtain the required approximate eigenpairs, we take as linearly independent trial functions the ansatz functions $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_M$ and construct $M \times M$ matrices A_0^s and A_1, where $M > N$. The matrices A_0^s and A_1 read

$$\begin{aligned} A_0^s &= (1-s)D_\Delta + (P(1-s)+\sigma\beta_2)G + (P(1-s)+\sigma\beta_1)U + s\tilde{L} \\ A_1 &= \beta_1 U + \beta_2 G, \end{aligned}$$

with

$$D_{\Delta ij} := \langle -D\tilde{\varphi}_i'', -D\tilde{\varphi}_j''\rangle_2$$
$$= \begin{cases} (d_2)^2 \frac{\pi^4}{2l^3}(\frac{i}{2}-1)^4, & i=j \text{ and } i \text{ is even}, \\ (d_1)^2 \frac{\pi^4}{2l^3}\left(\frac{i-1}{2}\right)^4, & i=j \text{ and } i \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{A.39}$$

$$G_{ij} := \langle \tilde{\varphi}_i', \tilde{\varphi}_j' \rangle_2 = \begin{cases} \frac{\pi^2}{2l}(\frac{i}{2}-1)^2, & i=j \text{ and } i \text{ is even}, \\ \\ \frac{\pi^2}{2l}\left(\frac{i-1}{2}\right)^2, & i=j \text{ and } i \text{ is odd}, \\ \\ 0, & \text{otherwise}, \end{cases} \tag{A.40}$$

$$U_{ij} := \begin{cases} l, & i=j=1, \\ \frac{l}{2}, & i=j>1, \\ 0, & \text{otherwise}. \end{cases} \tag{A.41}$$

The matrix $\tilde{L}$ reads

$$\tilde{L} = (\langle (L-\mu)\tilde{\varphi}_i, (L-\mu)\tilde{\varphi}_j \rangle_2)_{i,j=1,\ldots,M} = L_1 - \mu\,(L_2)^* - \bar{\mu} L_2 + |\mu|^2 U, \tag{A.42}$$

with

$$\begin{aligned} L_1 &= (\langle L\tilde{\varphi}_i, L\tilde{\varphi}_j \rangle_2)_{i,j=1,\ldots,M}, \\ L_2 &= (\langle L\tilde{\varphi}_i, \tilde{\varphi}_j \rangle_2)_{i,j=1,\ldots,M}. \end{aligned}$$

By a straightforward calculation we obtain

$$L_2 = M_1 + M_2,$$

where M_1 and M_2 are given by (A.12) and (A.13) respectively (taken without index n). The matrix L_1 takes the form

$$L_1 = D_\Delta + M_4 + (M_4)^* + M_5,$$

where the matrices M_4 and M_5 are defined as

$$M_4 := \left(\langle C\tilde{\varphi}_i, -D\tilde{\varphi}_j'' \rangle_2\right)_{i,j=1,\ldots,M},$$

$$M_5 := (\langle C\tilde{\varphi}_i,\ C\tilde{\varphi}_j \rangle_2)_{i,j=1,\ldots,M}.$$

Note that the function $\tilde{\varphi}_j''$ has the form

$$\tilde{\varphi}_j'' = \begin{cases} -\frac{\pi^2}{l^2}\left(\frac{j}{2}-1\right)^2\tilde{\varphi}_j & \text{if} \quad j \quad \text{is even}, \\ -\frac{\pi^2}{l^2}\left(\frac{j-1}{2}\right)^2\tilde{\varphi}_j & \text{if} \quad j \quad \text{is odd}. \end{cases}$$

A straightforward calculation for the elements of the matrices M_4 and M_5 yields

$$M_{4ij} = -\int_0^l \Big(d_1 f_1(x)\tilde{\varphi}_{i,1}(x)\tilde{\varphi}''_{j,1}(x) + d_1 f_2(x)\tilde{\varphi}_{i,2}(x)\tilde{\varphi}''_{j,1}(x) + d_2 f_3(x)\tilde{\varphi}_{i,1}(x)\tilde{\varphi}''_{j,2}(x) + d_2 f_4(x)\tilde{\varphi}_{i,2}(x)\tilde{\varphi}''_{j,2}(x)\Big) dx,$$

$$M_{5ij} = \int_0^l \Big(\left(f_1^2(x) + f_3^2(x)\right)\tilde{\varphi}_{i,1}(x)\tilde{\varphi}_{j,1}(x) + \left(f_1(x)f_2(x) + f_3(x)f_4(x)\right)\tilde{\varphi}_{i,2}(x)\tilde{\varphi}_{j,1}(x) + \left(f_1(x)f_2(x) + f_3(x)f_4(x)\right)\tilde{\varphi}_{i,1}(x)\tilde{\varphi}_{j,2}(x) + \left(f_2^2(x) + f_4^2(x)\right)\tilde{\varphi}_{i,2}(x)\tilde{\varphi}_{j,2}(x)\Big) dx,$$

where the functions $f_i(x),\ i = 1, \ldots, 4$ are defined as in (A.15)-(A.18) for $k = 1$ and as in (A.23) -(A.26) for $k = 2$. As before, the computation of the elements of the matrices M_4 and M_5 can be reduced to the computation of the integrals of the form

$$\sum_{i_1=1}^{\mathbf{M}+1} \cdots \sum_{i_p=1}^{\mathbf{M}+1} \tilde{\alpha}_{m(i_1)} \ldots \tilde{\alpha}_{m(i_p)} \int_0^l \varphi_{i_1}(x) \ldots \varphi_{i_p}(x)\varphi_i(x)\varphi_j(x)\, dx, \quad (i, j = 1, \ldots, M), \tag{A.43}$$

where $p = 1, \ldots, 4$. We derive the values for integrals above with the help of (A.78).

After the matrices A_0^s and A_1 are constructed, we compuite the eigenvalues and eigenvectors of the problem (A.38). As a result of our computation we obtain approximations to eigenvalues $\tilde{\boldsymbol{\lambda}}_1^s, \ldots, \tilde{\boldsymbol{\lambda}}_M^s$ and the eigenvectors $x^{(1)}, \ldots, x^{(M)}$. The required approximate eigenelements are formed as

$$\tilde{u}_i = \sum_{j=1}^{M} x_j^{(i)} \tilde{\varphi}_j, \quad i = 1, \ldots, M. \tag{A.44}$$

In the next step we construct the $N \times N$ (with $N < M$) matrices $\tilde{A}_0^s$ and $\tilde{A}_1$, taking as the trial functions in A_0^s and A_1 the eigenelements from (A.44). The elements of

$\tilde{A}_0^s$ and $\tilde{A}_1$ have the form

$$\tilde{A_0}_{ij}^{s} = \sum_{h=1}^{M}\sum_{t=1}^{M} x_h^{(i)} x_t^{(j)} {A_0}_{ht}^{s},$$

$$\tilde{A}_{1ij} = \sum_{h=1}^{M}\sum_{t=1}^{M} x_h^{(i)} x_t^{(j)} A_{1ht}.$$

This step is implemented using the interval arithmetic. In particular, for the evaluation of the expressions in (A.43) we use the interval package C-XSC [26]. Thus, we consider the eigenvalue problem

$$\tilde{A_0}^{s} x = \tilde{\boldsymbol{\lambda}}^s \tilde{A}_1 x, \tag{A.45}$$

where the matrices $\tilde{A}_0^s$ and $\tilde{A}_1$ are the matrices with interval entries. In case, when the dimension of (A.45) is small ($n = 1, 2$) the enclosure for its eigenvalues can be obtained rather directly. When $n > 2$ we use the following

Lemma A.1. *[23] Let $\mathcal{A}, \mathcal{B} \subset \mathbb{C}^{N\times N}$ be Hermitian matrices with interval entries, and with $\boldsymbol{B}$ positive definite for all $\boldsymbol{B} \in \mathcal{B}$. For some fixed Hermitian $\boldsymbol{A}_0 \in \mathcal{A}, \boldsymbol{B}_0 \in \mathcal{B}$ let $(\tilde{\lambda}_k, \tilde{\boldsymbol{x}}_k)$ $(k = 1, \dots, N)$ denote approximate eigenpairs of $\boldsymbol{A}_0\boldsymbol{x} = \lambda\boldsymbol{B}_0\boldsymbol{x}$, with $\tilde{\boldsymbol{x}}_m^* \boldsymbol{B}_0 \tilde{\boldsymbol{x}}_n \approx \delta_{m,n}$.*

Suppose that, for some $r_0, r_1 > 0$,

$$\|\boldsymbol{X}^*\boldsymbol{A}\boldsymbol{X} - \boldsymbol{X}^*\boldsymbol{B}\boldsymbol{X}\boldsymbol{\Lambda}\|_\infty \le r_0, \quad \|\boldsymbol{X}^*\boldsymbol{A}\boldsymbol{X} - \boldsymbol{E}\|_\infty \le r_1, \quad \boldsymbol{A} \in \mathcal{A}, \boldsymbol{B} \in \mathcal{B},$$

where $\boldsymbol{X} = (\tilde{\boldsymbol{x}}_1, \dots, \tilde{\boldsymbol{x}}_N)$, $\boldsymbol{\Lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_N)$. If $r_1 < 1$, we have for all $\boldsymbol{A} \in \mathcal{A}, \boldsymbol{B} \in \mathcal{B}$ and all eigenvalues λ of $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{B}\boldsymbol{x}$

$$\lambda \in \cup_{n=1}^{N} B(\tilde{\lambda}_n, r), \quad \textit{where } r = \frac{r_0}{1 - r_1}, \; \textit{and} \quad B(\lambda, r) = \{z \in \mathbb{C} : |z - \lambda| \le r\}.$$

Moreover, each connected component of this union contains as many eigenvalues as midpoints $\tilde{\lambda}_i$

After the application of Lemma A.1 we obtain the enclosures $\tilde{\boldsymbol{\lambda}}_i^s \in \Lambda_i^s$ ($i = 1, \ldots, N$) for the first N eigenvalues of the problem (A.38). By Theorem 6.1 we obtain

$$\boldsymbol{\lambda}_i^s \leq \tilde{\boldsymbol{\lambda}}_i^s \leq \sup(\Lambda_i^s), \quad (i = 1, \ldots, N).$$

Problem of the form (6.1). *Spruce budworm model.* We consider a sequence of eigenvalue problems of the form

$$u \in H_2^B(0,l),\ \langle L_s u, v\rangle_2 = \lambda^{(s)} \langle u, v\rangle_2 \quad \text{for all } v \in H_2^B(0,l),\ s \in [0,1], \tag{A.46}$$

where the operator L_s is given by

$$L_s u = -du'' + ((1-s)\underline{c} + sc(x))u$$

with the function $c(x)$ defined as

$$c(x) = -r + 2\frac{r}{q}\omega(x) + \frac{2\omega(x)}{(1+\omega^2(x))^2}$$

and $\underline{c}$ denoting its lower bound. Let us denote

$$A_0^s := (\langle L_s \tilde{u}_i, \tilde{u}_j\rangle_2)_{i,j=1,\ldots,M},$$

$$A_1 := (\langle \tilde{u}_i, \tilde{u}_j\rangle_2)_{i,j=1,\ldots,M}.$$

As in the case where $k = 1, 2$ we take as linearly independent trial functions $\tilde{u}_i$ ($i = 1, \ldots, M$) the ansatz function $\phi_i = \sin(i\pi\frac{x}{l})$ ($i = 1, \ldots M$) and consider the matrix eigenvalue problem of the form

$$A_0^s x = \tilde{\lambda}^s A_1 x.$$

A straightforward computation results in the following expressions for A_0^s and A_1

$$A_0^s = D_\Delta + ((1-s)\underline{c} - sr)S + 2sM_6,$$

$$A_1 = S,$$

where

$$D_{\Delta ij} := \langle -d\phi_i'', \phi_j \rangle_2 = \begin{cases} di^2 \dfrac{\pi}{2l}, & \text{if} \quad i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{A.47}$$

$$S_{ij} := \langle \phi_i, \phi_j \rangle_2 = \begin{cases} \dfrac{l}{2}, & \text{if} \quad i = j, \\ 0, & \text{otherwise,} \end{cases} \tag{A.48}$$

and matrix M_6 is given by

$$M_6 = \left(\left\langle \left(\frac{r}{q}\omega(x) + \frac{\omega(x)}{(1+\omega^2(x))^2} \right) \phi_i, \phi_j \right\rangle_2 \right)_{i,j=1,\dots,M}. \tag{A.49}$$

We compute the elements of matrix M_6 with the help of the trapezoidal rule. We comment on this computation in Section A.7.

Repeating the same steps as in the case of the problem (6.2), we find the approximate eigenpairs $(\tilde{\lambda}_i^s, \tilde{u}_i)$ with

$$\tilde{u}_i = \sum_{j=1}^{M} x_j^{(i)} \phi_j, \quad i = 1, \dots, M \tag{A.50}$$

and construct the "new" $N \times N$ interval matrices $\tilde{A}_0^s$ and $\tilde{A}_1$. For the verified solution of the eigenvalue problem

$$\tilde{A}_0^s x = \tilde{\lambda}^s \tilde{A}_1 x$$

we use Lemma A.1. Finally we obtain $\lambda_i^s \leq \tilde{\lambda}_i^s \leq \sup(\Lambda_i^s), \quad (i = 1, \dots, N)$, with Λ_i^s being the enclosure intervals for $\tilde{\lambda}_i^s$.

A.5 Calculation of the matrix A_2

This section is devoted to the computation of the matrix A_2^s for the Temple-Lehmann (problem (6.1)) and Lehmann-Goerisch (problem (6.2)) methods.

Problem of the form (6.2). In this paragraph we are going to treat the Schnakenberg and predator-prey models. Recall from Chapter 6 that the matrix A_2^s has the form

$$A_{2_{ij}}^s := s\,\frac{1}{\tilde{\boldsymbol{\lambda}}_i^s\tilde{\boldsymbol{\lambda}}_j^s}W_{1_{ij}} + (1-s)\,\frac{1}{\tilde{\boldsymbol{\lambda}}_i^s\tilde{\boldsymbol{\lambda}}_j^s}W_{2_{ij}} + (P_1(1-s)+\sigma\beta_2)\,\frac{1}{\tilde{\boldsymbol{\lambda}}_i^s\tilde{\boldsymbol{\lambda}}_j^s}W_{3_{ij}} + \\ + (P_2(1-s)+\sigma\beta_1)W_{4_{ij}}^s, \quad i,j=1,\ldots,N, \tag{A.51}$$

where $\tilde{\boldsymbol{\lambda}}_i^s,\ i=1,\ldots,N$ are the approximate eigenvalues. The matrices $W_1,\ W_2,\ W_3,\ W_4$ are defined as

$$W_1 := (\langle (L-\nu)\tilde{u}_i, (L-\nu)\tilde{u}_j\rangle_2)_{i,j=1,\ldots,N} \tag{A.52}$$

$$W_2 := \left(\langle -D\tilde{u}_i'', -D\tilde{u}_j''\rangle_2\right)_{i,j=1,\ldots,N} \tag{A.53}$$

$$W_3 := \left(\langle \tilde{u}_i', \tilde{u}_j'\rangle_2\right)_{i,j=1,\ldots,N} \tag{A.54}$$

$$W_4^s := \frac{1}{(P_2(1-s)+\sigma\beta_1)^2}\left(\langle \mathcal{H}(\tilde{u}_i), \mathcal{H}(\tilde{u}_j)\rangle_2\right)_{i,j=1,\ldots,N}, \tag{A.55}$$

where the expression $\mathcal{H}(\tilde{u}_i)$ has the form:

$$\mathcal{H}(\tilde{u}_i) = M_{0_i}^s \tilde{u}_i^{iv} + M_{1_i}^s \tilde{u}_i'' + M_{2_i}^s \tilde{u}_i' + M_{3_i}^s \tilde{u}_i. \tag{A.56}$$

In (A.52) to (A.56) $\tilde{u}_i$ is chosen as in (A.44). Hence the elements of matrices W_1, W_2, and W_3 can be represented as

$$W_{1_{ij}} = \sum_{h=1}^{M}\sum_{t=1}^{M} x_h^{(i)} x_t^{(j)} \tilde{L}_{ht}$$

$$W_{2_{ij}} = \sum_{h=1}^{M}\sum_{t=1}^{M} x_h^{(i)} x_t^{(j)} D_{\Delta ht}$$

$$W_{3_{ij}} = \sum_{h=1}^{M}\sum_{t=1}^{M} x_h^{(i)} x_t^{(j)} G_{ht}$$

with $\tilde{L}$, D_Δ and G as in (A.42), (A.39), and (A.40) respectively. Next we continue with the computation of the matrix W_4^s. After a straightforward calculation we represent W_4^s as

$$W_4^s = \frac{1}{(P_2(1-s)+\sigma\beta_1)^2}\sum_{m=1}^{16} E^m,$$

where

$$E_{ij}^1 := \int_0^l \tilde{u}_i^{iv^T} M_{0_i}^T \overline{M}_{0_j} \bar{\tilde{u}}_j^{iv}\, dx, \tag{A.57}$$

$$E_{ij}^2 := \int_0^l \tilde{u}_i^{iv^T} M_{0_i}^T \overline{M}_{1_j} \bar{\tilde{u}}_j''\, dx, \tag{A.58}$$

$$E_{ij}^3 := \int_0^l \tilde{u}_i^{iv^T} M_{0_i}^T \overline{M}_{2_j} \bar{\tilde{u}}_j'\, dx, \tag{A.59}$$

$$E_{ij}^4 := \int_0^l \tilde{u}_i^{iv^T} M_{0_i}^T \overline{M}_{3_j} \bar{\tilde{u}}_j\, dx, \tag{A.60}$$

$$E_{ij}^6 := \int_0^l \tilde{u}_i^{''^T} M_{1_i}^T \overline{M}_{1_j} \bar{\tilde{u}}_j''\, dx, \tag{A.61}$$

$$E_{ij}^7 := \int_0^l \tilde{u}_i^{''^T} M_{1_i}^T \overline{M}_{2_j} \bar{\tilde{u}}_j'\, dx, \tag{A.62}$$

$$E_{ij}^8 := \int_0^l \tilde{u}_i^{''^T} M_{1_i}^T \overline{M}_{3_j} \bar{\tilde{u}}_j\, dx, \tag{A.63}$$

$$E_{ij}^{11} := \int_0^l \tilde{u}_i^{'^T} M_{2_i}^T \overline{M}_{2_j} \bar{\tilde{u}}_j'\, dx, \tag{A.64}$$

$$E_{ij}^{12} := \int_0^l \tilde{u}_i^{'^T} M_{2_i}^T \overline{M}_{3_j} \bar{\tilde{u}}_j\, dx, \tag{A.65}$$

$$E_{ij}^{16} := \int_0^l \tilde{u}_i^T M_{3_i}^T \overline{M}_{3_j} \bar{\tilde{u}}_j\, dx, \tag{A.66}$$

and

$$E^{5} = \left(E^{2}\right)^{*},$$

$$E^{9} = \left(E^{3}\right)^{*},$$

$$E^{10} = \left(E^{7}\right)^{*},$$

$$E^{13} = \left(E^{4}\right)^{*},$$

$$E^{14} = \left(E^{8}\right)^{*},$$

$$E^{15} = \left(E^{12}\right)^{*}.$$

In the case of the Schnakenberg model, after straightforward computation we obtain, for the matrices in (A.56),

$$M^{s}_{0_i} = -\frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} \begin{pmatrix} 1 & 0 \\ 0 & d^{2} \end{pmatrix}, \tag{A.67}$$

$$M^{s}_{1_i} = \begin{pmatrix} -\beta_1 + \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} F^{s}_{1} & \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} s\gamma F_2 \\ \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} s\gamma F_2 & -\beta_1 + \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} F^{s}_{3} \end{pmatrix}, \tag{A.68}$$

$$M^{s}_{2_i} = \frac{4}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} \gamma s \begin{pmatrix} -F_4 & -F_5 \\ dF_4 & dF_5 \end{pmatrix}, \tag{A.69}$$

$$M^{s}_{3_i} = \begin{pmatrix} \beta_2 - \frac{2}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} \gamma s F_6 - \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} s F_8 & -\frac{2}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} \gamma s F_7 - \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} s F_9 \\ \frac{2}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} \gamma s d F_6 - \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} s \bar{F}_9 & \beta_2 + \frac{2}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} \gamma s d F_7 - \frac{1}{\tilde{\boldsymbol{\lambda}}^{s}_{i}} s F_{10} \end{pmatrix}, \tag{A.70}$$

where

$$F^{s}_{1} := 2s\left(\gamma(1 - 2\omega_1\omega_2) - \mathrm{Re}(\nu)\right) + P(1-s) + \sigma\beta_1,$$

$$F_2 := -\omega_1^{2} + 2d\omega_1\omega_2,$$

$$F^{s}_{3} := 2sd\left(\gamma\omega_1^{2} - Re(\nu)\right) + P(1-s) + \sigma\beta_1,$$

$$F_4 := \omega_1'\omega_2 + \omega_1\omega_2',$$

$$F_5 := \omega_1\omega_1',$$

$$F_6 := \omega_1''\omega_2 + 2\omega_1'\omega_2' + \omega_1\omega_2'',$$

$$F_7 := (\omega_1')^2 + \omega_1\omega_1'',$$

$$F_8 := \gamma^2\left(1 - 4\omega_1\omega_2 + 8\omega_1^2\omega_2^2\right) - 2\gamma Re(\nu)(1 - 2\omega_1\omega_2) + |\nu|^2,$$

$$F_9 := -\gamma^2\omega_1^2 + 4\gamma^2\omega_1^3\omega_2 + \gamma\bar{\nu}\omega_1^2 - 2\mu\gamma\omega_1\omega_2,$$

$$F_{10} := 2\gamma^2\omega_1^4 - 2Re(\nu)\gamma\omega_1^2 + |\nu|^2.$$

In the case of the predator-prey nonlinearity, we obtain

$$M_{0_i}^s = -\frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}\begin{pmatrix} d^2 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.71}$$

$$M_{1_i}^s = \begin{pmatrix} -\beta_1 + \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}H_1^s & \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sH_2 \\ \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sH_2 & -\beta_1 + \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}H_3^s \end{pmatrix}, \tag{A.72}$$

$$M_{2_i}^s = \frac{2}{\tilde{\boldsymbol{\lambda}}_i^s}s\begin{pmatrix} H_4 & d\omega_1' \\ -aw_2' & H_5 \end{pmatrix}, \tag{A.73}$$

$$M_{3_i}^s = \begin{pmatrix} \beta_2 + \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sdH_6 - \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sH_8 & \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sd\omega_1'' - \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sH_9 \\ -\frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sa\omega_2'' - \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}s\bar{H}_9 & \beta_2 + \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sH_7 - \frac{1}{\tilde{\boldsymbol{\lambda}}_i^s}sH_{10} \end{pmatrix}, \tag{A.74}$$

where

$$H_1^s := 2ds\left(-\varepsilon_1\gamma_1 - 2\varepsilon_1\gamma_2\omega_1 + 3\varepsilon_1\omega_1^2 + \omega_2 - \mathrm{Re}(\nu)\right) + P(1-s) + \sigma\beta_1,$$

$$H_2 := -a\omega_2 + d\omega_1,$$

$$H_3^s := 2s\left(a\left(2\varepsilon_2\omega_2 - \omega_1 + 1\right) - \mathrm{Re}(\nu)\right) + P(1-s) + \sigma\beta_1,$$

$$H_4 := d\left(-2\varepsilon_1\gamma_2\omega_1' + 6\varepsilon_1\omega_1\omega_1' + \omega_2'\right),$$

$$H_5 := a\left(2\varepsilon_2\omega_2' - \omega_1'\right),$$

$$H_6 := -2\varepsilon_1\gamma_2\omega_1'' + 6\varepsilon_1\left((\omega_1')^2 + \omega_1\omega_1''\right) + \omega_2'',$$

$$H_7 := a\left(2\varepsilon_2\omega_2'' - \omega_1''\right),$$

$$H_8 := \left|-\varepsilon_1\gamma_1 - 2\varepsilon_1\gamma_2\omega_1 + 3\varepsilon_1\omega_1^2 + \omega_2 - \nu\right|^2 + a^2\omega_2^2,$$

$$H_9 := \omega_1\left(-\varepsilon_1\gamma_1 - 2\varepsilon_1\gamma_2\omega_1 + 3\varepsilon_1\omega_1^2 + \omega_2 - \bar{\nu}\right) - a\omega_2\left(a\left(2\varepsilon_2\omega_2 - \omega_1 + 1\right) - \nu\right)$$

$$H_{10} := \omega_1^2 + |a(2\varepsilon_2\omega_2 - \omega_1 + 1) - \nu|^2 .$$

Next, combining (A.57)-(A.66) with (A.67)-(A.70) or (A.71)-(A.74), we obtain expressions for $E^m,\ m = 1, \ldots, 16$ as a functions of ω_1, ω_2, and their derivatives up to second order, and of both components of $\tilde{u}_i$, and their derivatives up to fourth order. Representing $\omega_1,\ \omega_2,\ \tilde{u}_i$ using (A.4),(A.5) and (A.44) respectively, we can rewrite each $E^m,\ m = 1, \ldots, 16$ as a combination of integrals of the form

$$\sum_{i_1=1}^{\mathbf{M}+1} \cdots \sum_{i_p=1}^{\mathbf{M}+1} \sum_{h=1}^{M} \sum_{t=1}^{M} \tilde{\alpha}_{m(i_1)} \ldots \tilde{\alpha}_{m(i_p)} \beta^{(i)}_{m(h)} \beta^{(j)}_{m(t)} \int_0^l \tilde{\phi}_{i_1}(x) \ldots \tilde{\phi}_{i_p}(x) \tilde{\phi}_h(x) \tilde{\phi}_t(x)\, dx,$$

where $p = 1, \ldots, 8$, $m(i)$ is defined as earlier, and $\tilde{\alpha}_{m(i)}$, $\beta^{(i)}_{m(h)}$ and ϕ_i are given by

$$\tilde{\alpha}_{m(i)} = \begin{cases} \alpha_{m(i)}, \\ (i-1)\alpha_{m(i)}, \\ (i-1)^2\alpha_{m(i)}, \end{cases} \qquad \beta^{(i)}_{m(h)} = \begin{cases} x^{(i)}_{m(h)}, \\ (h-1)x^{(i)}_{m(h)}, \\ (h-1)^2 x^{(i)}_{m(h)}, \end{cases} \qquad \tilde{\phi}_i = \begin{cases} \varphi_i, \\ \varphi_i'. \end{cases}$$

The choice of $\tilde{\alpha}_{m(i)}$, $\beta^{(i)}_{m(h)}$ and $\tilde{\phi}_i$ depends on the term under consideration. The integrals above can be computed in closed form with the help of the formula (A.78). For the verified computation of this integrals we use the interval package C-XSC [26]. Hence, we compute matrix W_4^s and consequently matrix A_2^s in closed form.

Problems of form (6.1) We consider

$$A_{2\,ij}^{\,s} = \langle L_s\tilde{u}_i, L_s\tilde{u}_j\rangle_2,$$

for all $i, j = 1, \ldots, N$. Due to (A.50) elements of A_2^s can be represented as

$$A_{2\,ij}^{\,s} = \sum_{h=1}^{M}\sum_{t=1}^{M} x_h^{(i)} x_t^{(j)} \langle L_s\phi_h, L_s\phi_t\rangle_2.$$

Let us define the following matrices:

$$K_{1ht} := \langle -d\phi_h'', -d\phi_t'' \rangle_2 = \begin{cases} d^2h^2t^2\dfrac{\pi^4}{2l^3}, & \text{if} \quad h = t, \\ 0, & \text{otherwise,} \end{cases}$$

$$\widetilde{D}_{ht} := \begin{cases} dh^2\dfrac{\pi^2}{l^2}, & \text{if} \quad h = t, \\ 0, & \text{otherwise,} \end{cases}$$

$$M_{7ht} := \left\langle \left(\frac{r}{q}\omega(x) + \frac{\omega(x)}{(1+\omega^2(x))^2}\right)\phi_h, \left(\frac{r}{q}\omega(x) + \frac{\omega(x)}{(1+\omega^2(x))^2}\right)\phi_t \right\rangle_2 .$$

Then, by a straightforward computation, we obtain

$$(\langle L_s\phi_h, L_s\phi_t\rangle_2)_{h,t=1,\dots,M} = K_1 + 2K_2 + K_3,$$

where K_2 and K_3 are given by

$$K_2 = ((1-s)\underline{c} - rs)D_\Delta + 2s\widetilde{D}M_6,$$

$$K_3 = ((1-s)\underline{c} - rs)^2 S + 4s((1-s)\underline{c} - rs)M_6 + 4s^2 M_7,$$

with S and M_6 as in (A.48) and (A.49) respectively. We compute the elements of matrix M_7 with the help of the trapezoidal rule with verified quadrature error bound (see Section A.7).

Final remarks In order to find the lower bounds to eigenvalues for problem (6.1) we consider the following eigenvalue problem

$$(\tilde{A}_0^s - \rho\tilde{A}_1)x = \tau\,(A_2^s - 2\rho\tilde{A}_0^s + \rho^2\tilde{A}_1)x. \tag{A.75}$$

In case of problem (6.2) we consider

$$(\tilde{A}_0^s - \rho\tilde{A}_1)x = \kappa\,(\tilde{A}_0^s - 2\rho\tilde{A}_1 + \rho^2 A_2^s)x. \tag{A.76}$$

The matrices in the problems above are the matrices with the interval entries. Therefore, we find the enclosure intervals for their eigenvalues by application of Lemma A.1. In case of (A.75) we compute the enclosures $\tau_i \in \mathcal{T}_i$ and by Theorem 6.2 estimate

$$\lambda_i \geq \rho + \frac{1}{\tau_{N+1-i}} \geq \rho + \frac{1}{\sup \mathcal{T}_{N+1-i}} \quad (i = 1, \ldots, N).$$

In case of (A.76) by Theorem 6.3 we have

$$\lambda_i \geq \rho - \frac{\rho}{1 - \kappa_{N+1-i}} \geq \rho - \frac{\rho}{1 - \sup \mathcal{K}_{N+1-i}} \quad (i = 1, \ldots, N),$$

with $\kappa_i \in \mathcal{K}_i$ - the enclosure intervals computed by Lemma A.1.

A.6 Integral computation formula

In the course of our numerical computations we have to evaluate expressions of the form

$$\sum_{k,m=1}^{\hat{M}} \sum_{i_1,i_2,\ldots,i_p=1}^{\hat{N}} \alpha_k \beta_m \gamma_{i_1} \gamma_{i_2} \ldots \gamma_{i_p} \int_0^l \varphi_k(x) \varphi_m(x) \varphi_{i_1}(x) \varphi_{i_2}(x) \ldots \varphi_{i_p}(x) dx, \quad \text{(A.77)}$$

where $\varphi_k(x) = \cos((k-1)\frac{\pi x}{l})$ and p runs from 1 to 8. Due to the orthogonality property of the functions $\varphi_k(x)$, the integral above can be computed in a closed form. Direct numerical computation of (A.77) could last long, when the expression above has a high order complexity. Observe, for example, $p = 8$ and $\hat{M} = \hat{N}$. Then (A.77) would have $O(\hat{N}^{10})$ complexity. Thus, in order to reduce the computation time, we proceed as follows

$$\sum_{k,m=1}^{\hat{M}} \sum_{i_1,i_2,\ldots,i_p=1}^{\hat{N}} \alpha_k\beta_m\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_p}\int_0^l \varphi_k(x)\varphi_m(x)\varphi_{i_1}(x)\varphi_{i_2}(x)\ldots\varphi_{i_p}(x)dx$$

$$=\frac{l}{2^{p+1}}\sum_{\tau,\sigma_1,\sigma_2,\ldots,\sigma_p\in\{-1,1\}}\sum_{k,m=1}^{\hat{M}}\sum_{i_1,i_2,\ldots,i_p=1}^{\hat{N}}\alpha_k\beta_m\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_p}\delta_{(k-1)+\tau(m-1)+\sum_{\mu=1}^{p}\sigma_\mu(i_\mu-1)}$$

$$=\frac{l}{2^{p+1}}\sum_{\tau,\sigma_1,\sigma_2,\ldots,\sigma_p\in\{-1,1\}}\underbrace{\sum_{I=-(\hat{M}-1)}^{2\hat{M}-2}\ \sum_{J=-h(\hat{N}-1)}^{h(\hat{N}-1)}\ \sum_{K=-(p-h)(\hat{N}-1)}^{(p-h)(\hat{N}-1)}}_{\text{with } I+J+K=0}\left(\sum_{\substack{k,m=1\\(k-1)+\tau(m-1)=I}}^{\hat{M}}\alpha_k\beta_m\right)\cdot$$

$$\left(\sum_{\substack{i_1,\ldots,i_h=1\\\sum_{\mu=1}^{h}\sigma_\mu(i_\mu-1)=J}}^{\hat{N}}\gamma_{i_1}\cdots\gamma_{i_h}\right)\left(\sum_{\substack{i_{h+1},\ldots,i_p=1\\\sum_{\mu=h+1}^{p}\sigma_\mu(i_\mu-1)=K}}^{\hat{N}}\gamma_{i_{h+1}}\cdots\gamma_{i_p}\right), \quad \text{(A.78)}$$

where h runs from 1 to $p-1$. As one can see, the complexity of (A.77) is now sufficiently reduced. Observe, for example, $p = 8$, $h = 4$, $\hat{M} = \hat{N}$. Then the resulting expression would have $O(\hat{N}^5)$ complexity. Notice that, with the purpose of achieving fast computation, the value of h could be adjusted accordingly.

Thus, the intergral (A.77) is computed in a closed form and within a reasonable amount of time.

A.7 Trapezoidal rule

As we have seen earlier, while considering the spruce budworm model, we approximate some of the arising integrals with the help of the trapezoidal rule. In general, we have

$$\int_0^l f(\omega(x),\phi_i(x),\phi_j(x))dx = Q(f)+E(f), \quad \text{(A.79)}$$

where under the expression $f(\omega(x),\phi_i(x),\phi_j(x))$ we understand the type of the expressions, which we obtain while computing some parts of the defect and the elements of the matrices M_6 and M_7. In particular, f depends on the numerical approximation ω and in case of matrices M_6 and M_7 also on the ansatz functions $\phi_i(x)$. Let N_q denote the number of quadrature points $x_k = \dfrac{lk}{N_q}$, $k = 0, \ldots, N_q - 1$. Then, according to trapezoidal rule, we have

$$Q(f) = \frac{l}{N_q}\left(\frac{1}{2}f(x_0) + \sum_{k=1}^{N_q-2} f(x_k) + \frac{1}{2}f(x_{N_q-1})\right), \tag{A.80}$$

$$|E(f)| \leq \frac{l^3}{12N_q^2}\left\|f''\right\|_\infty . \tag{A.81}$$

Let us comment on the computation of the term $\|f''\|_\infty$. Estimation of $\left\|f''\right\|_\infty$ reduces to the estimation of the maximum norm of the terms containing different combinations of ω, the derivatives of ω up to fourth order, and the derivatives of ϕ_i up to second order. Using the Taylor expansion, we obtain the following bounds

$$\left\|\omega^{(j)}\right\|_\infty \leq \max_{k=0,\ldots,M} |\omega^{(j)}(\xi_k)| + \frac{1}{2M}\left\|\omega^{(j+1)}\right\|_\infty, \quad j = 1, \ldots, 4. \tag{A.82}$$

where M is an arbitrary number, $\xi_k = \frac{lk}{M}$, $k = 0, \ldots, M$. Note, that we choose M to be large, in order to keep the term $\|\omega^v\|_\infty$ small. Since the numerical solution is given by (A.3), we have

$$\omega^v(x) = \frac{\pi^5}{l^5}\sum_{j=1}^{N} \alpha_j j^5 \cos\left(j\frac{\pi x}{l}\right), \quad x \in [0, l]. \tag{A.83}$$

Since $|\cos\left(j\frac{\pi x}{l}\right)| \leq 1$ we obtain the following bound for $\|\omega^v\|_\infty$

$$\|\omega^v\|_\infty \leq \frac{\pi^5}{l^5}\sum_{j=1}^{N} |\alpha_j| j^5. \tag{A.84}$$

Finally, N_q in (A.81) should be chosen large enough in order to keep $|E(f)|$ small. The verified computations were performed using the interval package INTLAB[50].

Bibliography

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth Dover printing, tenth GPO printing edition, 1964.

[2] R. A. Adams. *Sobolev Spaces*. Academic Press, New York, 1978.

[3] A. Arafa, S. Rida, and H. Mohamed. Approximate analytical solutions of schnakenberg systems by homotopy analysis method. *Applied Mathematical Modelling*, 36(10):4789 – 4796, 2012.

[4] W. Arendt, C. J. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems*. Birkhäuser, Basel, 2001.

[5] B.Breuer, M.Plum, and P.J.McKenna. Multiple solutions for a semilinear boundary value problem: a computational multiplicity proof. *Journal of Differential Equations*, 195:243–269, 2003.

[6] B. Breuer, J. Horak, P. J. McKenna, and M. Plum. A computer-assited existence and multiplicity proof for travelling waves in a nonlinearly supported beam. *Journal of Differential Equations*, 224:60 – 97, 2006.

[7] L. Buřič, A. Klíč, and L. Purmová. Canard solutions and travelling waves in the spruce budworm population model. *Applied Mathematics and Computation*, 183(2):1039 – 1051, 2006.

[8] S. Cai. *A computer-assisted proof for the pattern formation on reaction-diffusion systems*. PhD thesis, Graduate School of Mathematics, Kyushu University, 2012.

[9] D.Gilbarg and N.S.Trudinger. *Elliptic Partial Equations of Second Order*. Springer Verlag, Berlin, 1977.

[10] K.-J. Engel and R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations*. Springer, New York, 2005.

[11] K.-J. Engel and R. Nagel. *A Short Course on Operator Semigroups*. Springer, 2005.

[12] L. C. Evans. *Partial Differential Equations.* American Mathematical Society Providence, Rhode Island, 1998.

[13] A. Friedman. *Partial Differential Equations.* Holt, Rinehart and Winston, Inc., 1969.

[14] M. S. Gockenbach. *Partial Differential Equations: Analytical and Numerical Methods.* SIAM, 2002.

[15] F. Goerisch. Ein Stufenverfahren zur Berechnung von Eigenwertschranken. In: Albrecht, J., Collatz, L., Velte, W., Wunderlich, W.(eds): Numerical Treatment of Eigenvalue Problems, 4: 104-114, ISNM83, Birkhäuser, Basel, 1987.

[16] F. Goerisch and J. Albrecht. Eine einheitliche Herleitung von Einschließungssätzen für Eigenwerte. In: Albrecht, J., Collatz, L., Velte, W. (eds): Numerical Treatment of Eigenvalue Problems, 3: 55-88, ISNM69, Birkhäuser, Basel, 1984.

[17] F. Goerisch and J. Albrecht. The convergence of a new method for calculating lower bounds to eigenvalues. Equadiff 6 (Brno, 1985): 303-308, Lecture Notes in Mathematics 1192, Springer, Berlin New York, 1986.

[18] F. Goerisch and H. Haunhorst. Eigenwertschranken für Eigenwertaufgaben mit partiellen Differentialgleichungen. *ZAMM*, 65:129 – 135, 1985.

[19] P. Grindrod. *Patterns and waves. The Theory and Applications of Reaction-Diffusion Equations.* Clarendon Press, Oxford, 1991.

[20] C. Großmann and H.-G. Roos. *Numerik partieller Differentialgleichungen.* Teubner, Stuttgart, 1992.

[21] D. Henry. *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag, Berlin, 1981.

[22] N. J. Higham. *Functions of Matrices. Theory and Computation.* Siam, 2008.

[23] V. Hoang, M. Plum, and C. Wieners. A computer-assisted proof for photonic band gaps. *Z. angew. Math. Phys.*, 60:1035–1052, 2009.

[24] T. Kato. *Perturbation Theory for linear Operators.* Springer Verlag, Berlin, 1966.

[25] T. Kazuaki. *Analytic Semigroups and Semilinear Initial Boundary Value Problems.* Cambridge University Press, 1995.

[26] R. Klatte, U. Kulisch, A. Wiethoff, C. Lawo, and M. Rausch. *C-XSC. A C++ Class Library for Extended Scientific Computing.* Springer, 1993.

[27] E. Kowalski. Spectral theory in Hilbert spaces. Lecture notes. ETH Zürich, 2009.

[28] N. J. Lehmann. Optimale Eigenwerteinschließungen. *Numerische Mathematik*, 5:246–272, 1963.

[29] P.-P. Liu and Z. Jin. Pattern formation of a predator-prey model. *Nonlinear Analysis: Hybrid Systems*, 3(3):177 – 183, 2009.

[30] L. Lorenzi, A. Lunardi, G. Metafune, and D. Pallara. Analytic Semigroups and Reaction-Diffusion Problems. Internet Seminar, 2005.

[31] D. Ludwig, D. Aronson, and H. Weinberger. Spatial patterning of the spruce budworm. *Journal of Mathematical Biology*, 8:217 – 258, 1979.

[32] A. Lunardi. *Analytic Semigroups and Optimal Regularity in Parabolic Problems*. Birkhäuser, Basel, 1995.

[33] J. F. McLaughlin and J. Roughgarden. Pattern and stability in predator-prey communities: How diffusion in spatially variable environments affects the lotka-volterra model. *Theoretical Population Biology*, 40(2):148 – 172, 1991.

[34] Y. A. Melnikov. *Influence Functions and Matrices*. Marcel Dekker, Inc., 1999.

[35] M. Mimura and J. Murray. On a diffusive prey-predator model which exhibits patchiness. *Journal of Theoretical Biology*, 75(3):249 – 262, 1978.

[36] J. D. Murray. *Mathematical Biology I. An Introduction*, volume 1. Springer, third edition, 2002.

[37] J. D. Murray. *Mathematical Biology II. Spatial Models and Biomedical Applications*, volume 2. Springer, third edition, 2002.

[38] M. G. Neubert, H. Caswell, and J. D. Murray. Transient dynamics and pattern formation: reactivity is necessary for Turing instabilities. *Mathematical Biosciences*, 175:1–11, 2002.

[39] A. Okubo, P. Maini, M. Williamson, and J. Murray. On the spatial spread of the grey squirrel in britain. *Proceedings of the Royal Society of London. B. Biological Sciences*, 238(1291):113–125, 1989.

[40] M. Plum. Eigenvalue inclusions for second-order ordinary differential operators by a numerical homotopy method. *Journal of Applied Mathematics and Physic (ZAMP)*, 41:205–226, 1990.

[41] M. Plum. Computer-Assisted Existence Proofs for Two-Point Boundary Value Problems. *Computing*, 46:19 – 34, 1991.

[42] M. Plum. Explicit H2-estimates and pointwise bounds for solutions of second-order elliptic boundary value problems. *Journal of Mathematical Analysis and Applications*, 165(1):36 – 61, 1992.

[43] M. Plum. Enclosure methods with existence proof for elliptic differential equations. *International Series of Numerical Mathematics*, 123:347 – 368, 1997.

[44] M. Plum. Guaranteed Numerical Bounds for Eigenvalues. *Spectral Theory and Computational Methods of Sturm-Liouville Problems (eds.D.Hinton, P.W.Schaefer)*, pages 313 – 332, 1997.

[45] M. Plum. Computer-assisted Enclosure Methods for Elliptic Differential Equations. *Linear Algebra and its Applications*, 324:147 – 187, 2001.

[46] M. Plum. Safe Numerical Error Bounds for Solutions of Nonlinear Elliptic Boundary Value Problems. In G. Alefeld, J. Rohn, S. M. Rump, and T. Yamamoto, editors, *Symbolic Algebraic Methods and Verification Methods*, pages 195–207. Springer, 2001.

[47] R.Courant and D.Hilbert. *Methods of mathematical physics*, volume 1. Interscience publishers, Inc., New York, 1953.

[48] R. Redheffer. *Differential Equations: Theory and Applications.* Jones and Bartlett Publishers, Boston, 1991.

[49] K. Rektorys. *Variational Methods in Mathematics.* Reidel Publ. Co., Dordrecht, second edition, 1980.

[50] S. Rump. INTLAB - INTerval LABoratory. In T. Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. http://www.ti3.tu-harburg.de/rump/.

[51] F. Shakeri and M. Dehghan. The finite volume spectral element method to solve turing models in the biological pattern formation. *Comput. Math. Appl.*, 62(12):4322 – 4336, 2011.

[52] J. Smoller. *Shock Waves and Reaction-Diffusion Equations.* Springer-Verlag, New York, 1983.

[53] I. Stakgold. *Green's Functions and Boundary Value Problems.* John Wiley and Sons, New York, 1979.

[54] J. H. Steele. The quantitative ecology of marine phytoplankton. *Biological Reviews*, 34(2):129–158, 1959.

[55] B. R. Sturtevant, E. J. Gustafson, W. Li, and H. S. He. Modeling biological disturbances in landis: a module description and demonstration using spruce budworm. *Ecological Modelling*, 180(1):153 – 174, 2004.

[56] G.-Q. Sun, J. Zhang, L.-P. Song, Z. Jin, and B.-L. Li. Pattern formation of a spatial predator-prey system. *Applied Mathematics and Computation*, 218(22):11151 – 11162, 2012.

[57] M. E. Taylor. *Partial Differential Equations*, volume 2. Springer-Verlag, New York, 1996.

[58] R. Temam. *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*. Springer Verlag, New York, 1988.

[59] A. Turing. The chemical basis of morphogenesis. *Philosophical Transactions of the Royal Society of London B.*, 237(641):37 – 72, 1952.

[60] I. I. Vrabie. *Compactness Methods for Nonlinear Evolutions*. Longman Scientific and Technical, 1987.

[61] W. Walter. *Gewöhnliche Differentialgleichungen*. Springer-Lehrbuch, 1993.

[62] L. Wolpert. Positional information and the spatial pattern of cellular differentiation. *Journal of Theoretical Biology*, 25:1 – 47, 1969.

[63] A. Yagi. *Abstract Parabolic Evolution Equations and their Applications*. Springer-Verlag, Berlin, 2010.

[64] K. Yosida. *Functional Analysis*. Springer-Verlag, Berlin, sixth edition, 1980.

[65] S. Zimmermann and U.Mertins. Variational bounds to eigenvalues of self-adjoint eigenvalue problems with arbitrary spectrum. *J. Anal. Applic.*, 14:327 – 345, 1995.