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Hadronic contribution to the muon anomalous magnetic moment to next-to-next-to-leading order

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A B S T R A C T

We compute the next-to-next-to-leading order hadronic contribution to the muon anomalous magnetic moment originating from the photon vacuum polarization. The corresponding three-loop kernel functions are calculated using asymptotic expansion techniques which lead to analytic expressions. Our final result, $a_\mu^{\mathrm{had,NNLO}} = 1.24 \pm 0.01 \times 10^{-10}$, has the same order of magnitude as the current uncertainty of the leading order hadronic contribution and should thus be included in future analyses.

1. Introduction

The anomalous magnetic moment of the electron and the muon is measured with high precision and at the same time also accurately predicted including high-order quantum corrections (see, e.g., Refs. [1–4] for reviews on this topic). Notable recent achievements in this context are the five-loop QED corrections obtained in Refs. [5,6].

In the case of the muon the largest input to the uncertainty comes from hadronic contributions which to a large extent rely on experimental measurements of the cross section $\sigma(e^+e^- \to \text{hadrons})$. Several groups have performed the leading order (LO) [7–10] and next-to-leading order (NLO) [8,11–13] analysis. In this paper we compute the next-to-next-to-leading order (NNLO) hadronic corrections to the anomalous magnetic moment of the electron and the muon. We evaluate the three-loop kernels in the limit $M_\mu \ll m_\pi$ and show that four expansion terms are sufficient to obtain a precision far below the per cent level. Note that we do not consider the light-by-light contribution where the external photon couples to the hadronic loop (see, e.g., Ref. [14]) but only the contributions involving the hadronic vacuum polarizations.

In the next section we briefly mention some technical details of our calculation and discuss the NLO contribution. Section 3 contains the results of the various NNLO contributions for the muon anomalous magnetic moment and in Section 4 we apply our results to the anomalous magnetic moment of the electron. We conclude in Section 5.

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2. Technicalities and NLO result

The LO hadronic contribution to the anomalous magnetic moment of the muon (see Fig. 1) can be computed via

$$a_\mu^{(1)} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{m_\pi^2}^{\infty} \mathrm{d}s \frac{R(s)}{s} K^{(1)}(s), \tag{1}$$

where α is the fine structure constant and $R(s)$ is given by the properly normalized total hadronic cross section in electron positron collisions

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_{pt}}, \tag{2}$$

with $\sigma_{pt} = 4\pi\alpha^2/(3s)$. A convenient integral representation for $K^{(1)}(s)$ is given by

$$K^{(1)}(s) = \int_0^1 \mathrm{d}x \frac{x^2(1-x)}{x^2 + (1-x)\frac{s}{M_\mu^2}}, \tag{3}$$

analytic results can be found in Refs. [15,16].

A crucial input for the evaluation of a_μ^{had} is a compilation of the experimental data for $R(s)$ as obtained by various experiments. In our analysis we use a FORTRAN code which is provided to us by the authors of Ref. [8]. This gives us access to both the central values and the upper and lower limit of $R(s)$. However, the use of the latter leads to a vast overestimation of the final uncertainty since we have no information about the correlations of

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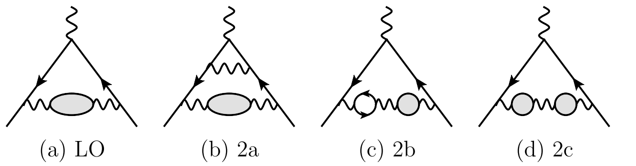

Fig. 1. LO and sample NLO Feynman diagrams contributing to a_μ^{had}.

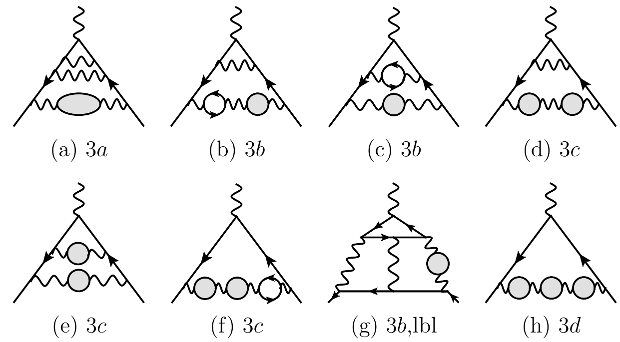

Fig. 2. Sample NNLO Feynman diagrams contributing to a_μ^{had}. The external fermions are muons and the fermions in the closed loops represent electrons.

the individual data points. Thus, we use a heuristic method and consider the difference between a_μ^{had} as obtained from the central and upper or lower limit of $R(s)$ and divide it by three which leads to realistic (and still conservative) error estimates at LO and NLO. In fact, for the energy region [0.32 GeV, 1.43 GeV] we obtain the LO contribution $608.19 \pm 3.97 \times 10^{-10}$ which is in a good agreement with $606.50 \pm 3.35 \times 10^{-10}$ from Table 5 of Ref. [8]. Note that in this paper we do not aim for an improved prediction of the LO or NLO contribution. Rather we present for the first time NNLO hadronic predictions. Obviously, for that purpose, the described prescription for the determination of the uncertainty is sufficient.

The contribution to a_μ from the J/Ψ, $\Psi(2S)$ and $\Upsilon(nS)$ ($n = 1, \dots, 4$) resonances is obtained with the help of the narrow-width approximation as described in Ref. [13].

At NLO three different contributions are distinguished as shown in Fig. 1(b), (c) and (d). We have computed the kernels $K^{(2a)}$ and $K^{(2b)}$ using the methods of asymptotic expansion [17] and in that way confirmed the results provided in Ref. [11]. Ref. [11] also contains analytic expressions for $K^{(2c)}(s, s')$. It is, however, convenient to work with the one-dimensional integral representation which reads [11]

$$K^{(2c)}(s, s') = \int_0^1 \mathrm{d}x \frac{x^4(1-x)}{[x^2 + (1-x)\frac{s}{M_\mu^2}][x^2 + (1-x)\frac{s'}{M_\mu^2}]}. \tag{4}$$

The contributions $a_\mu^{(2a)}$ and $a_\mu^{(2b)}$ are obtained from Eq. (1) after replacing $K^{(1)}$ by either $K^{(2a)}$ or $K^{(2b)}$ and $(\alpha/\pi)^2$ by $(\alpha/\pi)^3$. $a_\mu^{(2c)}$ requires an integration over both s and s' and is obtained from

$$a_\mu^{(2c)} = \frac{1}{9}\left(\frac{\alpha}{\pi}\right)^3 \int_{m_\pi^2}^\infty \mathrm{d}s\mathrm{d}s' \frac{R(s)}{s}\frac{R(s')}{s'} K^{(2c)}(s, s'). \tag{5}$$

Our results for the three contributions read

$$\begin{aligned} a_\mu^{(2a)} &= -20.90 \times 10^{-10}, \\ a_\mu^{(2b)} &= 10.68 \times 10^{-10}, \\ a_\mu^{(2c)} &= 0.35 \times 10^{-10}, \end{aligned} \tag{6}$$

which leads to

$$a_\mu^{\text{had,NLO}} = -9.87 \pm 0.09 \times 10^{-10}, \tag{7}$$

in a good agreement with Refs. [8,13].

3. NNLO hadronic contributions to a_μ

We classify the NNLO contributions in analogy to NLO according to the number of hadronic insertions and closed electron loops. This leads to five different kernels which contain the following contributions (see Fig. 2 for sample Feynman diagrams):

- $K^{(3a)}$: one hadronic insertion; up to two additional photons to the LO Feynman diagram; contains also the contributions with one or two closed muon loops and the light-by-light-type diagram with a closed muon loop.
- $K^{(3b)}$: one hadronic insertion and one or two closed electron loops and additional photonic corrections; the external photon couples to the muon.
- $K^{(3b,\text{lbl})}$: light-by-light-type contribution with closed electron loop and one hadronic insertion; the external photon couples to the electron.
- $K^{(3c)}$: two hadronic insertions and additional photonic corrections and/or closed electron or muon loops.
- $K^{(3d)}$: three hadronic insertions.

Note that we do not consider contributions with closed tau lepton loops since they are suppressed by an additional factor M_μ^2/M_τ^2. Actually, at NLO these contributions amount to 0.01×10^{-10} and thus we anticipate that the corresponding NNLO terms are even smaller.

The calculation of $K^{(3a)}(s)$ proceeds in analogy to the corresponding one- and two-loop cases: we apply an asymptotic expansion for $\sqrt{s} \gg M_\mu$ and compute terms up to order $(M_\mu^2/s)^4$. The minimal value of $\sqrt{s}$ is given by m_π and thus the largest value of the expansion parameter is $M_\mu^2/m_\pi^2 \approx 0.6$. Note, however, that the contribution from the energy interval $[m_\pi, 2m_\pi]$ is very small such that in practice the expansion parameter is $M_\mu^2/(4m_\pi^2) \approx 0.15$ or smaller for higher energies. We observe a good convergence of the series as can be seen by considering the difference for $a_\mu^{(3a)}$ ($a_\mu^{(3b)}$) computed from $K^{(3a)}(s)$ ($K^{(3b)}(s)$) by including and neglecting the highest available term which is at the per mil level. For $K^{(3b)}$ and $K^{(3b,\text{lbl})}$ we consider in addition the limit $M_\mu \gg M_e$ and compute terms up to quartic order in M_e. Corrections of order M_e/M_μ or higher turn out to be negligibly small. In the case of $K^{(3b)}$ the leading term for $M_e \to 0$ can be obtained using renormalization group techniques (see, e.g., Ref. [18] where four-loop correction to a_μ with closed electron loops have been considered). However, a non-zero electron mass is crucial for the light-by-light-type contribution $K^{(3b,\text{lbl})}$ since the Feynman integrals are divergent in case $M_e = 0$ is chosen. Thus, a non-trivial asymptotic expansion has to be applied. The latter is realized with the help of the program `asy` [19,20].

For the computation of $K^{(3c)}(s, s')$ we use asymptotic expansions in the limits $s \gg s' \gg M_\mu^2$, $s \approx s' \gg M_\mu^2$ and $s' \gg s \gg M_\mu^2$ and construct an interpolating function by combining the results from the individual limits. This procedure can be tested in the case of $K^{(2c)}(s, s')$ where a comparison to the exact result is possible. In Fig. 3(a) we show $K^{(2c)}(s, s')$ for $\sqrt{s} = 1$ GeV as a function

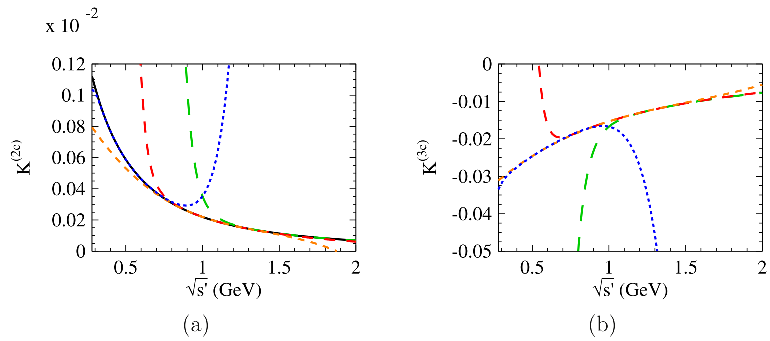

Fig. 3. (a) Comparison of exact result (solid, black) for $K^{(2c)}(s,s')$ and the various approximations for $s \gg s'$ (blue, dotted), $s \approx s'$ (orange and red, short and medium dashed) and $s \ll s'$ (green long dashed) for $\sqrt{s} = 1$ GeV as a function of $\sqrt{s'}$. (b) Approximations for $K^{(3c)}(s,s')$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of $\sqrt{s'}$.[1] (For larger values of $\sqrt{s}$ the convergence properties are even better.) One observes that for each value of $\sqrt{s'}$ there is perfect agreement between the exact result (solid line) and at least one of the approximations (dotted and dashed lines). Furthermore, the final results for $a_\mu^{(2c)}$ computed from the exact and approximated kernels differ by less than 1%.

Fig. 3(b) shows the corresponding results for $K^{(3c)}(s,s')$. For each value of s' we have at least two approximations which agree with each other. Thus it is evident that a function can be defined which agrees piecewise with one of the approximations.

For the kernel of the triple-hadronic insertion, $K^{(3d)}(s,s',s'')$, we derive a one-dimensional integral representation which is given by

$$K^{(3d)}(s,s',s'') = \int_0^1 dx \frac{x^6(1-x)}{[x^2+(1-x)\frac{s}{M_\mu^2}][x^2+(1-x)\frac{s'}{M_\mu^2}][x^2+(1-x)\frac{s''}{M_\mu^2}]}. \tag{8}$$

We refrain from listing explicit results for the NNLO kernels but provide the results in computer-readable form on the web page [23].

For the computation of $a_\mu^{(3a)}$, $a_\mu^{(3b)}$ and $a_\mu^{(3b,\mathrm{lbl})}$ one inserts the corresponding kernel in Eq. (1) and replaces $(\alpha/\pi)^2$ by $(\alpha/\pi)^4$. Furthermore, $a_\mu^{(3c)}$ is obtained from Eq. (5) with $K^{(2c)}$ replaced by $K^{(3c)}$ and $(\alpha/\pi)^3$ by $(\alpha/\pi)^4$ and the three-fold hadronic insertion is calculated from

$$a_\mu^{(3d)} = \frac{1}{27}\left(\frac{\alpha}{\pi}\right)^4 \int_{m_\pi^2}^{\infty} ds\,ds'\,ds'' \frac{R(s)}{s}\frac{R(s')}{s'}\frac{R(s'')}{s''} K^{(3d)}(s,s',s''). \tag{9}$$

For the individual NNLO contributions we obtain the results

$$\begin{aligned}
a_\mu^{(3a)} &= 0.80 \times 10^{-10},\\
a_\mu^{(3b)} &= -0.41 \times 10^{-10},\\
a_\mu^{(3b,\mathrm{lbl})} &= 0.91 \times 10^{-10},\\
a_\mu^{(3c)} &= -0.06 \times 10^{-10},\\
a_\mu^{(3d)} &= 0.0005 \times 10^{-10},
\end{aligned} \tag{10}$$

[1] Note that there are two curves for the region $s \approx s'$ which correspond to the expansion parameters $1-\sqrt{s}/\sqrt{s'}$ and $1-\sqrt{s'}/\sqrt{s}$, see also Refs. [21,22].

which leads to

$$a_\mu^{\mathrm{had,NNLO}} = 1.24 \pm 0.01 \times 10^{-10}. \tag{11}$$

Our result is of the same order of magnitude as the uncertainty of the LO hadronic contribution. For example, in Ref. [8] an uncertainty of 3.72×10^{-10} is quoted due to the statistical and systematic errors of the experimental data. Furthermore, $a_\mu^{\mathrm{had,NNLO}}$ in Eq. (11) is also of the same order of magnitude as the experimental uncertainty anticipated for future experiments measuring a_μ (see, e.g., Ref. [24]). Thus, the NNLO hadronic corrections should be included in the comparison with the experimental result for a_μ.

4. NNLO hadronic contributions to a_e

In this section we apply our results to the electron anomalous magnetic moment, a_e. At LO and for $K^{(2a)}$ this means that the lepton mass has to be interpreted as M_e. $K^{(2b)}$ is absent and we have checked that $K^{(2c)}$ gives a negligible contribution (see also Ref. [11]). The situation is analogous at NNLO where we only remain with $K^{(3a)}$.

At LO and NLO our results for a_e read $a_e^{\mathrm{had,LO}} = a_e^{(1)} = 1.877 \times 10^{-12}$ and $a_e^{\mathrm{had,NLO}} = a_e^{(2a)} = -0.2246 \times 10^{-12}$ which is consistent with the recent analysis of Ref. [25] where the values $a_e^{\mathrm{had,LO}} = 1.866 \pm 0.011 \times 10^{-12}$ and $a_e^{\mathrm{had,NLO}} = -0.2234 \pm 0.0014 \times 10^{-12}$ have been obtained. At NNLO we get the result[2]

$$a_e^{\mathrm{had,NNLO}} = a_e^{(3a)} = 0.028 \pm 0.001 \times 10^{-12}, \tag{12}$$

which is almost three times larger than the uncertainty of $a_e^{\mathrm{had,LO}}$ quoted in Ref. [25]. It is furthermore of the same order of magnitude as the hadronic light-by-light contribution which amounts to $a_e^{\mathrm{had,lbl}} = 0.035 \pm 0.010 \times 10^{-12}$ [5]. Note that currently both the uncertainty in the theory prediction for a_e and the difference between theory and experiment is of order 1×10^{-12} [5] which is about a factor 40 larger than the result given in Eq. (12).

5. Conclusions

We have computed the NNLO hadronic vacuum polarization corrections to the anomalous magnetic moment of the muon. Five different contributions can be distinguished which are discussed individually. The numerically largest contribution comes from the light-by-light-type diagram with a closed electron loop followed by

[2] We neglect the contribution from $K^{(3c)}$ since it is about a factor 100 smaller than the one from $K^{(3a)}$. Similarly heavy-lepton contributions proportional to M_e^2/M_μ^2 are not taken into account.

the photonic corrections and the contribution containing a closed electron two-point function. Multiple hadronic insertions only lead to numerical results which are much smaller. The main result of this paper is given in Eq. (11).

In Ref. [6] the theory prediction $a_\mu^{\text{th}} = 116\,591\,840(59) \times 10^{-11}$ has been compared to the experimental result [26,27] $a_\mu^{\text{exp}} = 116\,592\,089(63) \times 10^{-11}$ which leads to a deviation of 2.9σ. After adding our result in Eq. (11) to a_μ^{th} this reduces to 2.7σ.

As a by-product we have also evaluated the NNLO hadronic corrections to a_e. Our result is larger than the uncertainty at LO and of the same order as the hadronic light-by-light contribution. However, it is significantly smaller than both the uncertainty from the fine structure constant and the experimental uncertainty for a_e, see the discussion in Ref. [5].

6. Note added

During the refereeing process the paper [30] appeared on the arXiv. In that paper the NLO hadronic light-by-light contribution, which is of the same perturbative order as the corrections considered in our paper, has been estimated to $a_\mu^{\text{lbl-had,NLO}} = 0.3 \pm 0.2 \times 10^{-10}$.

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