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Review Article

Precision Calculations in Supersymmetric Theories

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In this paper we report on the newest developments in precision calculations in supersymmetric theories. An important issue related to this topic is the construction of a regularization scheme preserving simultaneously gauge invariance and supersymmetry. In this context, we discuss in detail dimensional reduction in component field formalism as it is currently the preferred framework employed in the literature. Furthermore, we set special emphasis on the application of multi-loop calculations to the analysis of gauge coupling unification, the prediction of the lightest Higgs boson mass, and the computation of the hadronic Higgs production and decay rates in supersymmetric models. Such precise theoretical calculations up to the fourth order in perturbation theory are required in order to cope with the expected experimental accuracy on the one hand and to enable us to distinguish between the predictions of the Standard Model and those of supersymmetric theories on the other hand.

1. Introduction

Today we know that the Standard Model (SM) of particle physics [1–7], which is a renormalizable gauge theory for the group $SU(3)_C \times SU(2)_W \times U(1)$, is extremely successful at short distances of the order of 10^{-16} cm. Up to now, all experiments verify it without any conclusive hint towards new physics. On the other hand, Einstein's gravitational theory based on the same concept of gauging the symmetries gives a very good classical theory for long distances. However, the classical theory of gravity could not be quantized due to its abundant number of singularities. There seems to be a deep conflict between the classical theory of gravity and the quantum field theory. Thus, the question whether gauging is the only organizing principle or there is a deeper connection between space time and internal space symmetries arises naturally. In a long series of "no-go theorems" among which the Coleman-Mandula theorem [8] is the most important one, it was shown that the only possible symmetry group of a consistent four-dimensional quantum field theory is the direct product of the internal symmetry group and the Poincaré group. Precisely, it states that internal symmetries cannot interact nontrivially with space time symmetry. Surprisingly, there is a unique way of combining nontrivially space time and inner space symmetries, namely, supersymmetry (SUSY). It was shown by Haag et al. [9] that weakening the assumptions of the Coleman-Mandula theorem by allowing both commuting and anticommuting symmetry generators, there is a nontrivial extension of the Poincaré algebra, namely, the supersymmetry algebra. The supersymmetry generators transform bosonic particles into fermionic ones and vice versa, but the commutator of two such transformations yields a translation in space time. In case of four-dimensional space time, the algebra generated by the SUSY generators will contain the algebra of Einstein's general relativity.

The first attempts to construct physical models respecting SUSY can be traced back in the early seventies to the works by Golfand and Likhtman [10] and Volkov and Akulov [11]. However, the first known example of a renormalizable supersymmetric four-dimensional quantum field theory is the Wess-Zumino model [12]. Within SUSY it is very natural to extend the concept of space time to the concept of superspace [13]. Along with the four-dimensional Minkowski space there are also two new "anticommuting" coordinates θ_α and $\overline{\theta}_{\dot{\alpha}}$, that are labeled in Grassmann numbers rather than real numbers:

$$\begin{aligned} &\left\{\theta_\alpha, \theta_\beta\right\} = 0, \qquad \left\{\overline{\theta}_{\dot{\alpha}}, \overline{\theta}_{\dot{\beta}}\right\} = 0, \\ &\theta_\alpha^2 = 0, \qquad \overline{\theta}_{\dot{\alpha}}^2 = 0, \quad \text{with } \alpha, \beta, \dot{\alpha}, \dot{\beta} = 1, 2. \end{aligned} \tag{1}$$

The ordinary space dimensions correspond to bosonic degrees of freedom, the anticommuting dimensions to fermionic degrees of freedom. The fields are now functions of the superspace variables $(x_\mu, \theta_\alpha, \overline{\theta}_{\dot{\alpha}})$ and they are organized into supersymmetric multiplets in a natural way [13]. Expanding the multiplets in Taylor series over the Grassmannian variables, one obtains the components of the superfield as the coefficients of the expansion. They are ordinary functions of the space time coordinates and can be identified with the usual fields. Furthermore, in the superfield notation the manifestly supersymmetric Lagrangians are polynomials of the superfields. In the same way, as the ordinary action is the integral over the space time of the Lagrangian density, in the supersymmetric case the action may be expressed as an integral over the whole superspace.

As quantum field theories the supersymmetric theories are less divergent as they would be in the absence of SUSY. These properties can be traced back to the cancellation of diagrams containing bosonic or fermionic particles, as, for example, the cancellation of quadratic divergences present in the radiative corrections to the Higgs boson mass. Even more, it was shown [14–16] that there are parameters of the theory that do not get any radiative corrections; that is a very special feature in quantum field theories. The most important consequence for the particle phenomenology is the fact that in a supersymmetric theory there should be an equal number of bosons and fermions with equal masses. In other words, for every SM particle there should exist a supersymmetric partner with an equal mass. But in Nature we do not observe such a situation. An elegant solution to break SUSY in such a way that its renormalization properties remain valid (in particular the nonrenormalization theorems and the cancellation of quadratic divergences) is to introduce the so-called soft terms [17]. In this way, the mass difference between supersymmetric partners can become of the order of SUSY breaking scale. Moreover, there will also be parameters that receive only finite radiative corrections of the order of magnitude of SUSY breaking parameters. This is the case of the Higgs masses and Higgs couplings. Accordingly, the SUSY partners of the SM particles should not be very heavy in order to account for the smallness of the Higgs mass and couplings. For example, requiring for consistency of the perturbation theory that the radiative corrections to the Higgs boson mass do not exceed the mass itself gives [18]

$$\delta M_h^2 \approx g^2 M_{\mathrm{SUSY}}^2 \approx M_h^2, \tag{2}$$

where M_{SUSY} denotes the mass scale of SM superpartners. Thus, for $M_h \approx 100\,\mathrm{GeV}$ and $g \approx 10^{-1}$ one obtains $M_{\mathrm{SUSY}} \approx 1000\,\mathrm{GeV}$. This feature is one of the great achievements of supersymmetric theories, namely, the solution to the hierarchy problem in particle physics.

The very old concept of the existence of an organizing principle that allows the unification of all interactions present in Nature is nowadays embedded in the so-called Grand Unified Theories (GUT). The predictions of such theories can be even precisely tested with the help of the experiments conducted at modern particle colliders. The most prominent example concerns, for sure, the prediction of gauge coupling unification. Once the gauge couplings for the electroweak and strong interactions had been precisely measured at the Large Electron-Proton Collider (LEP) [19], we could verify this hypothesis with high precision. The amazing result of evolving the low-energy values of the gauge couplings according to the SM predictions [20–22] is that unification is excluded by more than eight standard deviations. This means that unification can be achieved only if new physics occurs between the electroweak and the Planck scales. If one considers that a supersymmetric theory describes the new physics, one obtains that unification at an energy scale of about 10^{16} GeV can be realized if the typical supersymmetric mass scale is of the order of 10^3 GeV. This observation was interpreted as first "evidence" for SUSY, especially because the supersymmetric mass scale was in the same range as that derived from the solution to the hierarchy problem.

Another virtue of SUSY is that it provides a candidate for the cold dark matter. Nowadays, it is well established that the visible matter amounts to only about 4% of the matter in the Universe. A considerable fraction of the energy is made up from the so-called dark matter. The direct evidence for the existence of dark matter is the flat rotation curves of spiral galaxies (see, e.g., [23] and references cited therein), the gravitational lensing caused by invisible gravitating matter in the sky [24, 25], and the formation of large structures like clusters of galaxies. The dark matter is classified in terms of the mass of the constituent particle(s) and its (their) typical velocity: The hot dark matter, consisting of light relativistic particles and the cold one, made of massive weakly interacting particles (WIMPs) [26]. The hot dark matter might consist of neutrinos; however, this hypothesis cannot explain galaxy formation. For the cold dark matter, there is obviously no candidate within the SM. Nevertheless, SUSY provides an excellent candidate for WIMP, namely, the neutralino as the lightest supersymmetric particle.

These three fundamental predictions of SUSY make it one of the preferred candidates for physics beyond the SM. This explains the enormous efforts devoted to searches for SUSY in particle physics experiments at accelerators, in the deep sky with the help of telescopes, and with the help of underground facilities, that last already for four decades. The exclusion bounds on the supersymmetric mass spectrum are in general model dependent. In the case of the constrained MSSM (CMSSM), the current status is as follows: if one combines the excluded regions from the direct searches at the LHC [27], the stringent lower bound on the mass of the pseudoscalar Higgs from XENON100 [28], the constraints from the relic density from WMAP [29], and those from muon anomalous magnetic moment [30], one can set a lower limit on the WIMP mass of 230 GeV and on strongly interacting supersymmetric particles of about 1300 GeV. If in addition, the mass of the lightest Higgs boson of 125 GeV in agreement with the recent measurement at the LHC [31, 32] is imposed; one can exclude strongly interacting superpartners below 2 TeV. Nevertheless, such exclusion bounds concern the gluinos and mainly the first two generation of squarks. On the other hand, for the third generations of squarks, masses of the order of few hundred GeV are still allowed.

In this context, the question whether low-energy SUSY is still a valid candidate for physics beyond the SM arises naturally. Despite the slight tension that appears in particular models, as, for example, the constrained MSSM, (the constrained MSSM model is based on the universality hypothesis and is described by a set of five free parameters defining the mass scale for the Higgs potential and the scalar and fermion masses) the supersymmetric parameter space is large enough to accommodate all the experimental data known at present. However, the main prediction of low-energy SUSY, that is, the existence of supersymmetric particles at the TeV scale, is falsifiable at the LHC at the full energy run of 14 TeV. If no supersymmetric particle will be found at the TeV scale, we have to give up the main arguments in favor of SUSY, namely, the gauge coupling unification and the solution to the hierarchy problem.

To draw such powerful conclusions, one definitely needs an accurate comparison of the experimental data with the theory predictions based on SUSY models. There are various possibilities to perform such comparisons; one of them is high precision analyses, that requires precision data both at the experimental and theoretical level. On the theory side, the observables for which precise theoretical predictions up to the next-to-next-to leading order in perturbation theory are required are the electroweak precision observables (EWPO) [33], the muon anomalous magnetic moment [34], the lightest Higgs boson mass [33], the decay rate for the rare decay of a bottom quark into a strange quark and a photon $\Gamma(\overline{B} \to X_s\gamma)$ [35], and, of course, the production and decay rates of the Higgs boson at hadron colliders [36]. Details about the various topics can be found in the excellent review articles cited above. In this paper we report on the newest developments in precision calculations within SUSY models and set special emphasis on the recent calculations at the three-loop order involving several different mass scales. The latter constitute in many cases essential ingredients for the state of the art analyses of the experimental data taken currently at the LHC.

This paper is organized as follows. In the next section we briefly review the main results concerning the renormalizability of supersymmetric theories that can be derived from their holomorphic properties. In Sections 3 and 4 we describe the regularization method based on dimensional reduction applied to nonsupersymmetric and supersymmetric theories up to the fourth order in perturbation theory. In the second part of the paper we present the phenomenological applications of such precision calculations. Namely, in Section 5 we concentrate on computation of the three-loop gauge beta functions within the SM that allows us to predict the gauge couplings at high energies with very high accuracy. Furthermore, in Section 6 we report on the gauge coupling unification within SUSY models taking into account the most precise theoretical predictions and experimental measurements. Section 7 is devoted to the computation of the lightest Higgs boson mass within SUSY models with three-loop accuracy. In Section 8, the hadronic Higgs production and decay in SUSY models are reviewed and the required computations up to the third order in perturbation theory are presented. Finally, we draw our conclusions and present our perspective on precision calculations in SUSY models in Section 9. In the Appendix A we give details about the computation of the group invariants required in multiloop calculations. Appendix B contains the main renormalization constants needed for three-loop calculations in supersymmetric quantum chromodynamics (SUSY-QCD) within the modified minimal subtraction, that has been employed for the computations reviewed in Sections 7 and 8.

2. Holomorphy and Exact Beta Functions in Supersymmetric Theories

In the last decades, enormous progress has been made in understanding the dynamics of supersymmetric gauge theories. For many models even exact renormalization group equations (RGEs) for the gauge couplings have been derived. However, the connections between the exact results and those obtained in perturbation theory are still not completely elucidated. Shifman and Vainshtein [37] were the first to propose a solution to this puzzle. They based their argumentation on the difference between the quantities involved in the exact beta functions derived within the Wilsonian renormalization approach and those adopted in the common perturbative framework. A different derivation of the exact beta functions was presented in [38], where only the Wilsonian renormalization approach was used but the authors distinguished between the holomorphic and canonical normalization of the gauge kinetic term in the bare Lagrangian.

Within the Wilsonian framework [39] any field theory is defined by the fundamental Lagrangian, the bare couplings, and the cutoff parameter. Varying the cutoff parameter and the bare couplings in a concerted way so that the low-energy physics remains fixed, one finds the dependence of the bare couplings on the cutoff parameter which is encoded in the Wilsonian renormalization group equations (WRGEs). The transition from a fundamental Lagrangian to an effective Lagrangian involves integrating out the high momentum modes of the quantum fields (i.e., degrees of freedom with momenta between some large cutoff scale Λ and some renormalization scale μ). The coefficients of the resulting operators play the role of renormalized couplings and we will call them Wilsonian effective couplings. The virtue of this approach is the lack of any infrared effects, since none of the calculations involves infrared divergences.

Let us consider as an example supersymmetric electrodynamics (SQED). The vector superfield in the Wess-Zumino gauge has the following Grassmannian expansion:

$$V\left(x,\theta,\overline{\theta}\right) = -\theta\sigma^{\mu}\overline{\theta}v_{\mu}(x) + i\theta\theta\overline{\theta\lambda}(x) - i\overline{\theta\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta\theta}D(x), \tag{3}$$

where the physical degrees of freedom correspond to the vector gauge field v_{μ} and the Majorana spinor field λ, known also as gaugino field. The field D is an auxiliary field without any physical meaning and can be eliminated with the help of equations of motion for the physical fields.

The Lagrangian of the model at an energy scale μ can be written as follows:

$$L = \frac{1}{4g^2(\mu)} \int \mathrm{d}^4x \mathrm{d}^2\theta \mathscr{W}^\alpha \mathscr{W}_\alpha + \frac{1}{4} Z(\mu) \int \mathrm{d}^4x \mathrm{d}^2\theta \left(\overline{T} e^V T + \overline{U} e^{-V} U\right), \tag{4}$$

where the superfield strength tensor is defined through the following relation:

$$\mathscr{W}_\alpha = \frac{1}{8}\overline{D}^2 D_\alpha V = i\lambda_\alpha(x) - \theta_\alpha D(x) - i\theta^\beta F_{\alpha\beta}(x) + \theta^2 \partial_{\alpha\dot{\alpha}} \overline{\lambda}^{\dot{\alpha}}(x), \tag{5}$$

with

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu. \tag{6}$$

Here D and $\overline{D}$ are the supercovariant derivatives. The superfields $T(x,\theta,\overline{\theta})$ and $U(x,\theta,\overline{\theta})$ are chiral matter superfields with charges 1 and -1, respectively. $g(\mu)$ stands for the gauge coupling and $Z(\mu)$ denotes the superfield renormalization constant.

The maximal value of μ is equal to Λ, the ultraviolet cutoff parameter. At this point the Lagrangian (4) is just the original SQED Lagrangian and the coefficients $1/g^2(\Lambda)$ and $Z(\Lambda)$ are bare parameters.

Because the momentum integrals are performed in $d = 4$ dimension and the regularization is introduced through the cutoff parameter, the Wilsonian renormalization procedure preserves SUSY. Thus, if one calculates the Wilsonian effective Lagrangian, it is manifestly supersymmetric. As a consequence, the resulting effective superpotential (the part of the Lagrangian density that does not contain any derivative) must be a holomorphic function of the couplings [14–16]. This constraint restricts the running of the Wilsonian couplings to just the one-loop order.

For example, let us assume that we integrate out the matter superfields passing to the low-energy limit of the theory. The low-energy effective coupling at the low-energy μ is given through the following relation:

$$\frac{\pi}{\alpha_W(\mu)} = \frac{\pi}{\alpha_{W,0}(\Lambda)} - 2b_0 \ln\frac{\Lambda}{\mu}, \tag{7}$$

where $\alpha_W(\mu)$ denotes the renormalized or the Wilsonian low-energy effective coupling constant and $\alpha_{W,0}(\Lambda)$ is the cutoff-dependent bare coupling constant. b_0 is the coefficient of the one-loop beta function of the underlying theory, where the beta function is defined through

$$\beta(\alpha) = \mu^2 \frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{\alpha}{\pi} = -\left(\frac{\alpha}{\pi}\right)^2 \sum_{n\geq 0}\left(\frac{\alpha}{\pi}\right)^n b_n, \quad \text{with } \alpha = \frac{g^2}{4\pi}. \tag{8}$$

Let us emphasize that (7) is exact at all orders. The two- and higher-loop RGEs involve at least $\ln(\ln(\alpha_{W,0}))$ which is a nonholomorphic function of the bare coupling and thus cannot contribute to (7). In [37], it was proved through a direct calculation using the supergraphs method that the two-loop contributions to the running of the effective coupling vanish. The generalization of this assertion to higher loops is based on the extension of the nonrenormalization theorem for F-terms in supersymmetric theories [14–16].

As mentioned above, one has to distinguish between the holomorphic Wilsonian gauge couplings and the physically measurable momentum-dependent effective gauge couplings present in the one-particle irreducible generating functional. Unlike the Wilsonian couplings, the physical couplings do not depend on the ultraviolet cutoff scale but on momenta of the particles involved. The dependence of the physical couplings on the overall momentum scale is governed by the Gell-Mann-Low equations [40], which have different physical meaning as the WRGEs and have different β-functions [41, 42] beyond one loop. Going from the effective Lagrangian in the Wilsonian approach to the classical effective action Γ means to integrate out all of the degrees of freedom down to zero momentum, that will generate nonholomorphic corrections. Γ is often interpreted as a sort of effective Lagrangian, but in general it does not have the form of a supersymmetric Lagrangian with holomorphic coefficients.

The connection between the Wilsonian gauge coupling α_W and a physical gauge coupling α_{ph} was derived in the so-called Novikov-Shifman-Vainshtein-Zakharov renormalization scheme (NSVZ) [37]. This scheme requires a manifestly supersymmetric regularization procedure. In addition, the definition of the physical couplings is close to that in the momentum subtraction scheme (MOM). The conversion relation reads

$$\frac{\pi}{\alpha_W(\mu)} = \frac{\pi}{\alpha_{\mathrm{ph}}(\mu)} + T(R)\ln Z(\mu), \tag{9}$$

where $Z(\mu)$ is the renormalization constant of the matter superfield and the coefficient $T(R)$ is the Dynkin index of the representation R of the matter superfield. The factor $Z(\mu)$ is related to the mass renormalization constants of the matter superfield through the nonrenormalization theorems, provided SUSY is preserved. However, in general the Z factors are not restricted by any holomorphic constraints and thus are not known analytically. They have to be computed order by order in perturbation theory. Combining (9) and (7) we get

$$\frac{\pi}{\alpha_{\mathrm{ph}}(\mu)} = \frac{\pi}{\alpha_{\mathrm{ph}}(\Lambda)} - T(R)\ln\left(\frac{Z(\mu)}{Z(\Lambda)}\right) - 2b_0\ln\frac{\Lambda}{\mu}. \tag{10}$$

Using (8) we obtain for the beta function of the physical coupling in the NSVZ scheme the following relation:

$$\beta^{\mathrm{NSVZ}}_{\mathrm{SQED}}(\alpha_{\mathrm{ph}}) = \left(\frac{\alpha_{\mathrm{ph}}}{\pi}\right)^2 \frac{1}{2} T(R)(1-\gamma), \tag{11}$$

where we have specified the value of the coefficient b_0 for the SQED case and the superfield anomalous dimension is defined through

$$\gamma = -\mu\frac{\mathrm{d}\ln Z(\mu)}{\mathrm{d}\mu}. \tag{12}$$

Because (7) is exact at all orders, also the relation between the beta function of $\alpha_{\rm ph}$ and the anomalous dimension of the matter superfields γ is valid at all orders. Let us remark, however, that this relation holds only in the NSVZ scheme. Unfortunately, it is highly nontrivial to fulfill the requirements of the NSVZ scheme in practice.

In supersymmetric nonabelian models with several matter supermultiplets, (9) becomes

$$\frac{\pi}{\alpha_W(\mu)} = \frac{\pi}{\alpha_{\rm ph}(\mu)} + \frac{1}{2}C(G)\ln\alpha_{\rm ph}(\mu) + \sum_i T(R_i)\ln Z_i(\mu), \tag{13}$$

where $C(G)$ is the quadratic Casimir operator of the adjoint representation and $T(R_i)$ is the Dynkin index of the representation R_i of the matter field i. The second term stands for the gaugino contribution, while the third one for contributions generated by the matter superfields. A simple calculation provides us with the exact relation between the gauge beta function and the anomalous dimension of the matter superfields:

$$\beta^{\rm NSVZ} = -\left(\frac{\alpha_{\rm ph}}{2\pi}\right)^2 \frac{3C(G) - 2\sum_i T(R_i)(1-\gamma_i)}{1 - C(G)\alpha_{\rm ph}/(2\pi)}. \tag{14}$$

From (14) it is easy to see that for the derivation of the L-loop beta functions in the NSVZ scheme one needs the matter anomalous dimensions γ_i at the $(L-1)$-loop order. As will be shown below this feature was intensively exploited in the literature.

In the case of SUSY-Yang-Mills theories the matter superfields are absent, so $T(R_i) = 0$, and an exact formula for the gauge coupling beta function can be derived:

$$\beta^{\rm NSVZ} = -\left(\frac{\alpha_{\rm ph}}{2\pi}\right)^2 \frac{3C(G)}{1 - C(G)\alpha_{\rm ph}/(2\pi)}. \tag{15}$$

Similar relations can also be derived for models with softly broken SUSY. The line of reasoning is as follows: the powerful supergraph method is also applicable for models with softly broken SUSY by using the "spurion" external field method [17, 43]. Perhaps, one of the most prominent example is the relation that can be established between the gaugino mass $m_{\tilde{g}}$ and the gauge beta function. In the presence of the SUSY breaking gaugino mass term, the coefficient of the gauge kinetic term in the Wilsonian action becomes

$$\left(\frac{1}{g^2}\right)_W \longrightarrow \left(\frac{1 - 2m_{\tilde{g}}^2\theta^2}{g^2}\right)_W, \tag{16}$$

where θ is the Grassmann variable.

Using the same arguments based on holomorphy, it was shown [44, 45] that a renormalization group invariant (RGI) relation for the gaugino mass can be derived within NSVZ scheme:

$$\frac{m_{\tilde{g}}\alpha}{\beta(\alpha)} = \text{RGI}. \tag{17}$$

Moreover, it was shown with the help of the spurion formalism that the renormalization constants of softly broken SUSY gauge theory can be related to the renormalization constants of the underlying exact supersymmetric model [46–48]. Even more, the connecting formulas are valid at all orders in perturbation theory. The only necessary assumption for their derivation is the existence of a gauge and SUSY invariant regularization scheme. Thus, such relations are valid only in NSVZ-like regularization schemes.

At this point, a few remarks are in order to comment on the results discussed above. The authors of [56] state that in $d = 4$ dimensions the only known regularization to conserve SUSY is the Pauli-Villars scheme for matter superfields and the higher derivative scheme for the gauge superfields. Technically this construction is rather complicated and hardly applicable to multiloop computations. In [43], an attempt was made to apply the "supersymmetric dimensional regularization" or "regularization by dimensional reduction" (DRED) [57] within the supergraph formalism. However, as pointed out by Siegel himself [58], this scheme is mathematically inconsistent in its original formulation and a consistent formulation will break supersymmetry in higher orders of perturbation theory. A similar situation occurs also for the application of DRED in component field formalism [59, 60] (A detailed analysis of this issue will be done in the next section). Thus, the exact formulas of the NSVZ scheme are not valid, in general, for calculations based on DRED since they do not involve a regularization scheme supersymmetric at all orders. For particle phenomenology, it means that the powerful predictions of (14) cannot be tested through experiments, since the beta functions are scheme dependent beyond two loops.

The breakthrough regarding this situation was obtained in [61–65], where it is stated that if the NSVZ scheme exists it can be perturbatively related to schemes based on DRED. Such arguments follow from the equivalence of different renormalization schemes in perturbation theory [66]. Precisely, the computation of the three-loop mass anomalous dimension for the chiral matter superfield in a general nonabelian supersymmetric theory and of the three-loop gauge beta function in the abelian case allowed the derivation of the three- and four-loop gauge beta function for a general supersymmetric theory. Remarkably enough, the derivation (up to a numerical coefficient) of the four-loop gauge beta function was based on a three-loop calculation and theoretical considerations about special relations valid in $N = 2$ supersymmetric theories and one-loop finite supersymmetric theories.

Let us mention at this point also the calculation of the three-loop gauge beta function for supersymmetric Yang-Mills (SYM) theories of [67]. For this calculation, DRED was employed in component field formalism rather than superfield formalism, and hence a manifestly not supersymmetric gauge was used. The computations of [64, 65, 67] coincide as a consequence of gauge invariance of the gauge beta function.

Moreover, the authors of [61, 62] noticed that the differential operators relating the beta functions for soft SUSY breaking parameters to the beta functions of the gauge and Yukawa couplings are form invariant under change of scheme

(i.e., from NSVZ to DRED scheme). Thus, similar relations for the soft SUSY breaking parameter valid to all orders of perturbation theory hold also in a DRED-like scheme (Actually, the scheme. proposed by the authors of [61, 62] is the so-called DRED′, for which beta functions of SUSY breaking parameters do not depend on the unphysical ϵ-scalar mass parameter. For more details about the DRED′ scheme see Section 3.)

In the next section we will discuss in detail the application of DRED in component field formalism and give some example of important calculations that can be done within this approach. Nevertheless, already now we want to mention the coincidence of all results obtained with DRED in component field formalism and those derived via DRED in supergraphs formalism.

3. Dimensional Reduction in the Component Field Formalism

The precision of many present or forthcoming experiments in particle physics requires inevitably higher order perturbative calculations in the SM or its extensions like the Minimal Supersymmetric Standard Model (MSSM). Regularization of the divergent loop diagrams arising in the higher order calculations is commonly performed employing Dimensional Regularization (DREG) or its variants, due to its nice feature to respect gauge invariance. Higher order calculations within the SM predominantly use DREG in its original form [68, 69], while for calculations within supersymmetric theories DRED as defined in [57] is commonly employed. It is not *a priori* known whether SUSY as a symmetry of a given Lagrangian is still a symmetry of the full quantum theory in any particular case. Nevertheless, a detailed formal renormalization program has been pursued in [70] including a proof that SUSY is not anomalous. If the regularized theory does not respect SUSY, the finite amplitude will not satisfy the Ward identities required by SUSY, giving rise to an apparent anomaly. If SUSY is not anomalous, it is possible to restore the invariance by introducing finite counterterms.

In practice, the choice of regularization scheme is of considerable significance for the extraction of physical predictions. This is the case for the NSVZ scheme we alluded in the previous section, that rarely found direct practical applicability. It rather provides important checks for results predicted within DRED. In this section we discuss in detail the application of DRED in the component field formalism and its application to practical calculations.

3.1. Framework. DRED consists of continuing the number of space dimensions from 4 to d, where d is less than 4, but keeping the dimension of all the fields fixed. In component field language, this means that the vector bosons and fermions preserve their four-dimensional character. Furthermore, it is assumed that all fields depend on d rather than 4 space time coordinates, so that the derivatives ∂_μ and momenta p_μ become d-dimensional. It is the four-dimensional nature of the fields that is supposed to restore the supersymmetric Ward-Takahashi [71, 72] or Slavnov-Taylor [73] identities, while the d-dimensional space time coordinates cure, as in DREG, the singularities of the loop integrals.

However, potential inconsistencies of DRED, arising from the use of purely four-dimensional relations between the Levi-Civita tensor and the metric tensor, have been pointed out by Siegel himself [58]. Even more, inconsistencies of DRED arising without the direct use of Levi-Civita tensors have been revealed in [60]. The authors have correlated them with the impossibility of decomposing the finite four-dimensional space into a direct sum of infinite-dimensional spaces. The solution proposed by the same authors is to introduce a formal space, called quasi-four-dimensional space (Q_4), with "noninteger valued" vector and spinor indices (thus, the two types of indices range over an infinite set of values), obeying certain algebraic identities inspired from the properties of the four-dimensional Minkowski space. The existence of such a space was demonstrated by construction [74] starting from similar arguments as those used to prove the existence of the formal d-dimensional space of DREG [75]. In this way the consistency of the calculation rules is guaranteed. By construction, Q_4 is represented as the direct sum of two infinite-dimensional spaces: Q_d which is formally d-dimensional and is identical with the one of DREG and $Q_{2\epsilon}$ which is formally $2\epsilon = 4 - d$-dimensional. (One needs to perform twice the construction of n-dimensional integrals and metric tensors for $n = d$ and $n = 2\epsilon$. The d-dimensional integral is the momentum integral in DRED, while 2ϵ integral is involved only in the definition of the 2ϵ-dimensional metric tensor.)

$$Q_4 = Q_d \oplus Q_{2\epsilon}. \tag{18}$$

According to the properties of the three formal spaces at hand Q_4, Q_d, $Q_{2\epsilon}$ one can derive the following relations for the corresponding metric tensors $g^{\mu\nu}$, $\widehat{g}^{\mu\nu}$, $\overline{g}^{\mu\nu}$ [59, 74]:

$$\begin{gathered} g^{\mu\nu} = \widehat{g}^{\mu\nu} + \overline{g}^{\mu\nu}, \qquad g^{\mu\mu} = 4, \qquad \widehat{g}^{\mu\mu} = d, \\ \overline{g}^{\mu\mu} = 2\epsilon, \qquad g^{\mu\nu}\widehat{g}^{\rho}_{\nu} = \widehat{g}^{\mu\rho}, \\ g^{\mu\nu}\overline{g}^{\rho}_{\nu} = \overline{g}^{\mu\rho}, \qquad \widehat{g}^{\mu\nu}\overline{g}^{\rho}_{\nu} = 0. \end{gathered} \tag{19}$$

Furthermore, any quasi-four-dimensional vector can be decomposed with the help of the projectors $\widehat{g}^{\mu\nu}$, $\overline{g}^{\mu\nu}$:

$$t^\mu = \widehat{t}^\mu + \overline{t}^\mu, \qquad \widehat{t}^\mu = \widehat{g}^{\mu\nu}t_\nu, \qquad \overline{t}^\mu = \overline{g}^{\mu\nu}t_\nu. \tag{20}$$

Imposing the Dirac algebra for the γ-matrices defined in Q_4

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}, \tag{21}$$

we can derive similar commutation relations for the components in Q_d and $Q_{2\epsilon}$:

$$\begin{gathered} \{\widehat{\gamma}^\mu, \widehat{\gamma}^\nu\} = 2\widehat{g}^{\mu\nu}\mathbf{1}, \qquad \{\overline{\gamma}^\mu, \overline{\gamma}^\nu\} = 2\overline{g}^{\mu\nu}\mathbf{1}, \\ \{\overline{\gamma}^\mu, \widehat{\gamma}^\nu\} = 0. \end{gathered} \tag{22}$$

These relations together with the trace condition

$$\mathrm{Tr}\,\mathbf{1} = 4 \tag{23}$$

are sufficient for computing Feynman diagrams. Equation (23) is particularly useful in supersymmetric theories, because it ensures that the numbers of degrees of freedom for fermions and bosons are equal.

For practical computations, it is useful to note that the fermion traces that contain both types of γ-matrices can be factored out as follows:

$$\begin{aligned}&\mathrm{Tr}\left(\widehat{\gamma}^{\mu_1}\cdots\widehat{\gamma}^{\mu_n}\overline{\gamma}^{\nu_1}\cdots\overline{\gamma}^{\nu_l}\right)\\&\quad=\frac{1}{4}\,\mathrm{Tr}\left(\widehat{\gamma}^{\mu_1}\cdots\widehat{\gamma}^{\mu_n}\right)\mathrm{Tr}\left(\overline{\gamma}^{\nu_1}\cdots\overline{\gamma}^{\nu_l}\right).\end{aligned}\tag{24}$$

This relation can be derived from (23), (22), and the algebra of Dirac matrices in d dimensions. Thus, the Dirac algebra can be performed separately in d and in $4-d=2\epsilon$ dimensions.

Once we introduced "noninteger valued" spinor indices, we need infinite-dimensional γ-matrices to represent the Dirac algebra. Thus, the Fierz identities valid in the genuine four-dimensional space do not hold anymore in Q_4. Their use was identified with one of the sources of DRED inconsistencies. Moreover, within Q_4 the invariance of the original Lagrangian under SUSY transformations might be broken. This feature can be directly correlated with the lack of Fierz identities that would ensure the cancellation of Lagrangian variation under SUSY transformations in the genuine four-dimensional space. However, it has been shown [60, 74] that such inconsistencies become active only in the higher orders of perturbation theory, when, for example, traces over at least ten γ-matrices and antisymmetrization over five indices are involved. Thus, DRED also breaks SUSY, but starting from higher orders of perturbation theory. This explains, why one- and even two-loop calculations of QCD corrections within DRED [76–80] based on genuine four-dimensional Dirac algebra and even Fierz rearrangement provided correct results. Even the supersymmetric character of DRED at low orders has been exploited in the context of QCD with massless quarks in [78]. However, beyond the one-loop level the distinction between $g^{\mu\nu}$ resulting from contractions of the quasi-four-dimensional vector fields and $\widehat{g}^{\mu\nu}$ resulting from momentum integrals is difficult to follow. It turned out [81] that for higher order computations it is useful to decompose the quasi-four-dimensional vector fields according to (20). As we shall see in the next section, in the case of gauge theories the d-dimensional components behave as vectors under the gauge transformations whereas the 2ϵ components as scalars, usually called ε-scalars.

Representing the underlying space of DRED Q_4 as a formal infinite-dimensional space renders the extension of γ_5 as subtle as in DREG. The consistent procedure proposed by 't Hooft-Veltman (HV) [69] for defining γ_5 as in four dimensions $\overline{\gamma}_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ has in the context of SUSY theories two drawbacks. On the one hand, it is the fact that the mathematically consistent treatment of γ_5 in DREG requires $d > 4$, whereas for DRED $d < 4$ is needed. However, it has been shown up to two loops [82, 83] that the Adler-Bardeen theorem [84] could still be satisfied in DRED with HV scheme, if relations like

$$\widehat{\gamma}^i\overline{\gamma}_5\widehat{\gamma}_i=(d-8)\,\overline{\gamma}_5,\tag{25}$$

which follow in $d > 4$ are assumed to hold also for $d < 4$. On the other hand, the use of a not anticommuting γ_5 leads to the breakdown of symmetries, for example, chiral symmetry of the SM or supersymmetry in case of the MSSM already at the one-loop level. These "spurious anomalies" would spoil the renormalizability and they have to be cured by introducing appropriate counterterms to restore Ward-Takahashi and Slavnov-Taylor identities order by order in perturbation theory (see [85]). This approach was successfully applied for SM predictions within DREG up to three-loops [86, 87]. However, for the MSSM it becomes much more involved due to the complexity introduced by supersymmetric conditions and it rarely has been employed in practice [83].

The implementation of γ_5 in DRED commonly used in practice is inspired by the naive scheme (NS) of DREG. Namely, it is treated rather like a formal object which is not well defined mathematically but anticommutes with all γ-matrices

$$\{\widehat{\gamma}^\mu,\widetilde{\gamma}_5\}=\{\overline{\gamma}^\mu,\widetilde{\gamma}_5\}=0,\qquad\left(\widetilde{\gamma}_5\right)^2=\mathbf{1}.\tag{26}$$

Nevertheless, one has to correct the false result that arises from (26), that the trace of γ_5 and four or more γ-matrices vanishes. Paying attention that now two types of γ-matrices occur, the additional constraints read

$$\begin{aligned}&\mathrm{Tr}\left(\Gamma^\alpha\Gamma^\beta\Gamma^\gamma\Gamma^\delta\widetilde{\gamma}_5\right)=4i\;\widetilde{\varepsilon}^{\alpha\beta\gamma\delta}+\mathcal{O}(\epsilon),\\&\text{with }\Gamma^\mu=\widehat{\gamma}^\mu\text{ or }\overline{\gamma}^\mu.\end{aligned}\tag{27}$$

The tensor $\widetilde{\varepsilon}^{\alpha\beta\gamma\delta}$ has some similarities with the four-dimensional Levi-Civita tensor: (i) it is completely antisymmetric in all indices; (ii) when contracted with a second one of its kind gives the following result:

$$\begin{aligned}&\widetilde{\varepsilon}^{\alpha\beta\gamma\delta}\widetilde{\varepsilon}_{\alpha'\beta'\gamma'\delta'}\\&\quad=\left[G^\alpha_{\alpha'}\;\;G^\beta_{\beta'}\;\;G^\gamma_{\gamma'}\;\;G^\delta_{\delta'}\right],\\&\quad G^{\mu\nu}=\widehat{g}^{\mu\nu}\text{ or }\overline{g}^{\mu\nu},\end{aligned}\tag{28}$$

depending on the nature of Dirac matrices Γ^μ in (27). Here the square brackets denote complete antisymmetrization. When taking the limit $d\to 4$, $\widetilde{\varepsilon}^{\alpha\beta\gamma\delta}$ converts into the four-dimensional Levi-Civita tensor and (27) and (28) ensure that the correct four-dimensional results are reproduced. This last constraint is needed in order to correctly compute fermion triangle diagrams containing an axial vector current, that is, to cope with the Adler-Bardeen-Jackiw anomaly [88–90].

At this point a comment on (27) is in order. When we combine it with the cyclic property of traces, it necessarily follows that other traces are not well defined in $d\neq 4$ dimensions. It turns out that there is an unavoidable ambiguity of order $\mathcal{O}(d-4)$ when fixing the trace condition in (27). Even if one does not use the cyclic property of the trace, an ambiguity in the distribution of the anomaly between the vector and the axial vector currents shows up [82]. The occurrence of the ambiguity is a characteristic of the extension of γ_5 away from $d = 4$ dimensions. 't Hooft and Veltman have pointed out

in their original paper [69] that an ambiguity related to the location of γ_5 shows up in HV scheme, too.

The use of an anticommuting γ_5 in $d \neq 4$ dimensions was applied for the first time to the evaluation of fermion traces with an even number of γ_5's in [91], and a few years later extended also to odd γ_5 fermion traces in [92]. The method (for more details see [93] and references cited therein) proved to be effective for SM calculations involving chiral fermions up to two-loop order [94–97]. The consistency of this γ_5 prescription has been verified even in three-loop QCD-electroweak calculations [98, 99]. Within DRED, it has been successfully employed in MSSM calculations at the two- and three-loop order [100–103]. However, let us mention at this point that for these calculations at most the finite parts of two-loop and the divergent parts of three-loop diagrams are required. For the calculation of finite parts of three-loop diagrams containing two fermion triangle subdiagrams, the HV scheme has to be applied as the naive scheme does not provide correct results.

Through the consistent formulation of DRED we gain a regularization scheme which proves to be supersymmetric only in the lower orders of perturbation theory. Due to the violation of Fierz identities, SUSY invariance will be broken at higher orders. The first consequence of SUSY breaking is that the all-order relations between different anomalous dimensions valid in the NSVZ scheme do not hold in DRED. However, although DRED consistently formulated is not a supersymmetric scheme at all orders, it provides so far the best option for computations within SUSY theories.

3.2. Minimal Subtraction Schemes $\overline{MS}$ and $\overline{DR}$. The common renormalization schemes used for multiloop calculations are the minimal subtraction (MS), momentum subtraction and on-shell schemes. Minimal subtraction, scheme has the advantage of involving the simplest computations, but it is nonphysical in the sense that it does not take into account mass threshold effects for heavy particles. Nevertheless, it is the main scheme used in renormalization group (RG) analyses relating the predictions of a given theory at different energy scales. The other two options are computationally much more involved but indispensable for the determination of the parameters of a theory from the quantities measured experimentally. We focus in this section on the minimal subtraction methods.

Minimal subtraction scheme with DREG as regulator [104] or the modified $\overline{\text{MS}}$ scheme [105] and its variant for DRED—the $\overline{\text{DR}}$ scheme—are in particular well suited for higher order calculations in perturbation theory. The advantage of these schemes is that all ultraviolet (UV) counterterms are polynomial both in external momenta and masses [106, 107]. This allows for setting to zero certain masses or external momenta, provided no spurious infrared divergences are introduced. This simplifies substantially the calculations of the Feynman integrals. It has been shown [108] by means of the infrared rearrangement (IRR) procedure [108–110] that the renormalization constants within the $\overline{\text{MS}}$ scheme can be reduced to the calculation of only massless propagator diagrams. This method was used for the first three-loop calculation of the QCD β-function [111], applying it to each individual diagram. But the most effective approach is its use in combination with multiplicative renormalization. This amounts in general to solve recursively the equation

$$Z_a = 1 - K_\epsilon \left[\Gamma_a\left(p^2\right) Z_a\right], \tag{29}$$

where $K_\epsilon[f(\epsilon)]$ stands for the singular part of the Laurent expansion of $f(\epsilon)$ in ϵ around $\epsilon = 0$. $\Gamma_a(p^2)$ denotes the renormalized Green function with only one external momentum p^2 kept nonzero. Z_a denotes the renormalization constant associated with the Green function Γ_a. In this case, the renormalization of Γ_a through $(l+1)$-loop order requires the renormalization of the Lagrangian parameters like couplings, masses, gauge parameters, mixing angles, and so forth. up to l-loop order. The method was successfully applied to the three-loop calculations of anomalous dimensions within $\overline{\text{MS}}$ or $\overline{\text{DR}}$ schemes [49, 86, 101–103, 112–114] using the package `MINCER` [115] written in `FORM` [116], which computes analytically massless propagator diagrams up to three loops.

Apart from that, a second method was proposed in [117], which has been used for the calculation of the three- and even four-loop anomalous dimensions of QCD [118–121] and the beta function of the quartic coupling of the Higgs boson in the SM [114, 122, 123]. It deals with the IRR by introducing an artificial mass for all propagators. Expanding in all particles masses and external momenta, one can reduce the evaluation of the Feynman integrals to massive tadpoles. The analytic evaluation of the massive tadpoles up to three-loop order can be obtained with the help of the package `MATAD` [124].

A third method was introduced for the evaluation of the renormalization constants for the quark mass [118] and the vector [125] and quark scalar current correlators [126] through four loops. It is based on global IRR properties and amounts essentially to set to zero the external momentum and let an arbitrary subset of the internal lines to be massive. After nontrivial manipulations, the four-loop integrals can be reduced to three-loop massless, two-point integrals, and one-loop massive vacuum integrals.

The three-loop accuracy for the anomalous dimensions of theories involving not only vector but also Yukawa and quartic scalar interactions (e.g., the SM [49, 114]) was achieved only very recently. Remarkably, for supersymmetric and softly broken supersymmetric theories like the MSSM the three-loop anomalous dimensions were computed long before [63, 64, 127]. Their derivations used intensively the exact relations established between the various anomalous dimensions in the NSVZ scheme (for more details see Section 2) as well as the observation that the NSVZ scheme and DRED can be perturbatively connected.

3.3. DRED Applied to Nonsupersymmetric Theories. Although DRED was originally proposed as a candidate for an invariant regularization in supersymmetric theories, it proved to be useful also in nonsupersymmetric theories. Its use in SM calculations up to three-loop orders was motivated either by the possibility to apply four-dimensional algebra and even Fierz rearrangements [77, 80] (the mathematical inconsistencies alluded to above do not occur at the two-loop level

in this calculations), or by the possibility to easily convert a nonsupersymmetric gauge theory into a SUSY-Yang-Mills theory and use nontrivial Ward identities as checks of complicated calculations [78, 98, 128]. Apart from the computational advantages, DRED applied to nonsupersymmetric theories, in particular to QCD, provides us with a powerful tool to verify its consistency up to three-loop order via the connection that can be established with DREG (DRED and DREG are also perturbatively connected). Finally, it is motivated by the MSSM, as a softly broken supersymmetric theory or by various models derived from the MSSM which feature lower symmetries (e.g., the intermediate energy theory obtained by integrating out the squarks and sleptons). DRED applied to effective field theories, such that QCD extended to include the Higgs-top Yukawa coupling, was useful for the calculation of the production rate for the Higgs boson in gluon-fusion channel within MSSM [54, 129].

In the following, we consider a nonabelian gauge theory with n_f Dirac fermions ψ_f transforming according to a representation R of the gauge group $\mathcal{G}$. For the moment we do not take into account any genuine scalar field.

The Lagrangian density (in terms of bare fields) reads

$$\begin{aligned}\mathscr{L}_B = & -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2(1-\xi)}\left(\partial^\mu W_\mu^a\right)^2 \\ & + \partial^\mu \bar{c}^a \left(\partial_\mu c^a - g f^{abc} c^b W_\mu^c\right) + i\sum_{f=1}^{n_f} \overline{\psi}_f \not{D} \psi_f,\end{aligned} \tag{30}$$

where the field strength tensor is defined through

$$\begin{aligned}F_{\mu\nu}^a &= \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g f^{abc} W_\mu^b W_\nu^c, \\ D_\mu &= \partial_\mu - ig\left(R^a\right) W_\mu^a\end{aligned} \tag{31}$$

is the covariant derivative. W_μ is the gauge field, c^a is the Fadeev-Popov-ghost field, f^{abc} are the structure constants of the gauge group $\mathcal{G}$, ξ is the gauge parameter, and g is the gauge coupling.

For the case when the theory admits a gauge invariant fermion mass term we will have $L_B \to L_B + L_B^m$, where

$$L_B^m = -m_f \overline{\psi}_f \psi_f. \tag{32}$$

DRED amounts to imposing that all field variables depend only on a subset of the total number of space time dimensions; in this case d out of 4 where $d = 4 - 2\epsilon$. We can then make the decomposition

$$W_\mu^a\left(x^j\right) = \widehat{W}_\mu^a\left(x^j\right) + \overline{W}_\mu^a\left(x^j\right), \tag{33}$$

where

$$\widehat{W}_\mu^a = \hat{g}_{\mu\nu} W^{\nu,a}, \qquad \overline{W}_\mu^a = \overline{g}_{\mu\nu} W^{\nu,a}, \qquad \hat{g}_{\mu\mu} = d. \tag{34}$$

It is then easy to show that [130]

$$L_B = L_B^d + L_B^\epsilon, \tag{35}$$

where

$$\begin{aligned}L_B^d = & -\frac{1}{4}\widehat{F}_{\mu\nu}^2 - \frac{1}{2(1-\xi)}\left(\partial^\mu \widehat{W}_\mu\right)^2 \\ & + \partial^\mu \bar{c}^a \left(\partial_\mu c^a - g f^{abc} c^b \widehat{W}_\mu^c\right) + \sum_{f=1}^{n_f} i\overline{\psi}_f \widehat{\gamma}^\mu \widehat{D}_\mu \psi_f,\end{aligned} \tag{36}$$

$$\begin{aligned}L_B^\epsilon = & \frac{1}{2}\left(\widehat{D}_\mu \overline{W}_\nu\right)^2 - \sum_{f=1}^{nf} g\overline{\psi}_f \overline{\gamma}^\mu R^a \psi_f \overline{W}_\mu^a \\ & - \frac{1}{4} g^2 f^{abc} f^{ade} \overline{W}_\mu^b \overline{W}_\nu^c \overline{W}^{d,\mu} \overline{W}^{e,\nu},\end{aligned} \tag{37}$$

where $\widehat{F}_{\mu\nu}$ and $\widehat{D}_\mu$ denote the projection of the field strength and covariant derivative given in (31) onto Q_d, obtained with the help of the operator $\hat{g}^{\mu\nu}$. Conventional dimensional regularization (DREG) amounts to using (36) and discarding (37).

Note that under the gauge transformations

$$\delta \widehat{W}_\mu^a = \partial_\mu \Lambda^a + g f^{abc} \widehat{W}_\mu^b \Lambda^c, \tag{38a}$$

$$\delta \overline{W}_\mu^a = g f^{abc} \overline{W}_\mu^b \Lambda^c, \tag{38b}$$

$$\delta \psi^\alpha = ig\left(R^a\right)^{\alpha\beta} \psi^\beta \Lambda^a \tag{38c}$$

each term in (37) is separately invariant. The $\overline{W}_\mu$ fields behave exactly like scalar fields and are hence known as ε-scalars. There is therefore no reason to expect the $\overline{\psi}\psi\overline{W}$ vertex to renormalize in the same way as the $\overline{\psi}\psi\widehat{W}$ vertex (except in the case of supersymmetric theories). The couplings associated with the $\overline{\psi}\psi\overline{W}$ vertex or with the quartic ε-scalar interaction are called *evanescent couplings*. They were first described in [131] and later independently discovered by van Damme and 't Hooft [132]. The vertices $\widehat{W}\overline{W}\,\overline{W}$ and $\widehat{W}\widehat{W}\overline{W}\,\overline{W}$, on the other hand, are renormalized in the same way as $\widehat{W}\widehat{W}\widehat{W}$, $\overline{C}C\widehat{W}$, and so forth because of the gauge invariance [133]. Thus we can conclude that $\widehat{W}$ is the gauge particle, while $\overline{W}$ acts as matter field transforming according to the adjoint representation. In order to avoid confusion, we denote in the following the gauge particles with G_μ^a and the ε-scalars with ε_μ^a:

$$\widehat{W}_\mu^a \longrightarrow G_\mu^a, \qquad \overline{W}_\mu^a \longrightarrow \varepsilon_\mu^a. \tag{39}$$

Since ε-scalars are present only on internal lines we could, in fact, choose the wave function renormalization of ε_μ and G_μ to be the same. However, such a renormalization prescription will break unitarity [132]. The crucial point is the correct renormalization of subdivergences, which requires that vertices involving ε-scalars renormalize in a different way as their gauge counterparts. Thus, to renormalize the ε-scalars one has to treat them as new fields present in the theory.

For the renormalization of the theory we distinguish two new types of couplings: a Yukawa like coupling g_e associated with the vertex $\overline{\psi}\psi\varepsilon$ and a set of p quartic couplings λ_r associated with vertices containing four ε-scalars.

The number p is given by the number of independent rank four tensors H^{abcd} which are nonvanishing when symmetrized with respect to (ab) and (cd) interchange. We address the issue of the quartic vertex renormalization in more detail in the next section.

The renormalization constants for the couplings, masses, and fields and vertices are defined as

$$\begin{gathered}
g^0 = \mu^\epsilon Z_g g, \qquad g_e^0 = \mu^\epsilon Z_e g_e, \qquad \sqrt{\lambda_r^0} = \mu^\epsilon Z_{\lambda_r}\sqrt{\lambda_r}, \\
1-\xi^0 = (1-\xi)\, Z_3, \qquad m_f^0 = m_f Z_{m_f}, \\
m_\varepsilon^0 = m_\varepsilon Z_m^\varepsilon, \qquad \psi^0 = \sqrt{Z_2}\psi, \qquad G_\mu^{0,a} = \sqrt{Z_3} G_\mu^a, \\
\varepsilon_\mu^{0,a} = \sqrt{Z_3^\varepsilon}\varepsilon_\mu^a, \qquad c^{0,a} = \sqrt{\widetilde{Z}_3} c^a, \\
\Gamma^0_{\bar\psi\psi G} = Z_1 \Gamma_{\bar\psi\psi G}, \qquad \Gamma^0_{\bar\psi\psi\varepsilon} = Z_1^\varepsilon \Gamma_{\bar\psi\psi\varepsilon}, \\
\Gamma^0_{\bar c c G} = \widetilde{Z}_1 \Gamma_{\bar c c G}, \qquad \Gamma^0_{\varepsilon\varepsilon G} = Z_1^{\varepsilon\varepsilon G}\Gamma_{\varepsilon\varepsilon G}, \\
\Gamma^0_{\varepsilon\varepsilon GG} = Z_1^{\varepsilon\varepsilon GG}\Gamma_{\varepsilon\varepsilon GG}, \qquad \Gamma^0_{GGGG} = Z_1^{4G}\Gamma_{GGGG}, \\
\Gamma^0_{\varepsilon\varepsilon\varepsilon\varepsilon} = Z_1^{4\varepsilon}\Gamma_{\varepsilon\varepsilon\varepsilon\varepsilon},
\end{gathered} \tag{40}$$

where μ is the renormalization scale and the bare quantities are marked by the superscript "0." $\Gamma xyz(w)$ stands for one-particle irreducible Green functions involving the external particles x, y, z, (w). Equation (36) takes under renormalization the usual expression in terms of renormalized parameters as in DREG scheme. The renormalized Lagrangians L^ϵ is the new term that distinguishes DRED from DREG and it is given by

$$\begin{aligned}
L^\epsilon = {} & \frac{1}{2} Z^{\varepsilon\varepsilon}\left(\partial_\mu \varepsilon_\nu^a\right)^2 + Z^{\varepsilon\varepsilon G} g f^{abc}\partial_\mu \varepsilon_\nu^a G^{b,\mu}\varepsilon^{c,\nu} \\
& + Z^{\varepsilon\varepsilon GG} g^2 f^{abc} f^{ade} G_\mu^b \varepsilon_\nu^c G^{d,\mu}\varepsilon^{e,\nu} \\
& - Z^{\psi\psi\varepsilon} g_e \overline{\psi} R^a \overline{\gamma}^\mu \psi \varepsilon_\mu^a \\
& - \frac{1}{4}\sum_{r=1}^{p} Z_{\lambda_r}\lambda_r H_r^{abcd}\varepsilon_\mu^a \varepsilon_\nu^c \varepsilon^{b,\mu}\varepsilon^{d,\nu}.
\end{aligned} \tag{41}$$

Strictly speaking, (41) should also have a mass term for the ε-scalars; but since this mass term does not affect renormalization of the couplings and fermion masses we omit it here. We discuss this issue in more detail in Section 3.3.5.

The charge renormalization constants are obtained from the Slavnov-Taylor identities. For example, if one computes the N-point Green function with external fields $\phi_1, \ldots, \phi_n$ and denotes its coupling constant by g, one obtains

$$Z_g = \frac{Z_{\phi_1\cdots\phi_N}}{\sqrt{Z_{\phi_1}\cdots Z_{\phi_N}}}, \tag{42}$$

where the Z_{ϕ_i} are the wave function renormalization constants for the ϕ_i, $Z_{\phi_1\cdots\phi_N}$ is the corresponding vertex renormalization constant, and Z_g the charge renormalization. Within the minimal subtraction scheme, one is free to choose any masses and external momenta, as long as infrared divergences are avoided. One can set all masses to zero, as well as one of the two independent external momenta in three-point functions. In this case, one arrives at three-loop integrals with one nonvanishing external momentum q which can be calculated with the help of MINCER. One can also calculate the three-point functions setting a common mass m to all particles and expanding the Feynman integrals in the limit $m^2/q^2 \ll 1$ with the help of asymptotic expansions [117]. This approach is much more tedious, but possible infrared singularities would manifest in $\ln m^2/q^2$ terms. If such terms are absent in the final expression, the limit $m \to 0$ can be taken and the result should coincide with the one obtained with the massless setup (for a comprehensive overview about the multiloop techniques within DREG see the review article [134]).

Precisely, the charge renormalization of the gauge coupling can be derived from the ghost-gauge boson, fermion-gauge boson, ε-scalar-gauge boson vertices, or the gauge boson self-interaction

$$Z_g = \frac{\widetilde{Z}_1}{\widetilde{Z}_3\sqrt{Z_3}} = \frac{Z_1}{Z_2\sqrt{Z_3}} = \frac{Z_1^{\varepsilon\varepsilon G}}{Z_3^\varepsilon\sqrt{Z_3}} = \text{etc.} \tag{43}$$

as a consequence of gauge invariance.

Similarly, for the charge renormalization constants of the evanescent couplings, the following relations hold:

$$Z_e = \frac{Z_1^\varepsilon}{Z_2\sqrt{Z_3^\varepsilon}}, \qquad Z_{\lambda_r} = \frac{Z_1^{4\varepsilon}}{\left(Z_3^\varepsilon\right)^2}. \tag{44}$$

In general, $Z_g \neq Z_e$ even at one-loop order. However, in supersymmetric theories $Z_g = Z_e$ should hold at all orders because of SUSY. This can be understood following the same line of reasoning as for the derivation of the equality of the charge renormalization constants for the interactions involving gluons and those involving gluinos.

3.3.1. The ε-Scalar Self-Couplings. Let us discuss the structure of the quartic ε-scalar couplings for an arbitrary gauge group. These interactions are invariant under the symmetry $\mathcal{G} \otimes O(2\epsilon)$, where only $\mathcal{G}$ is gauged. The number of independent quartic ε-scalar couplings is given by the number of independent rank $n = 4$ tensors H^{abcd} invariant with respect to (a, b) and (c, d) exchange, because of the $O(2\epsilon)$ invariance. It has been shown that for $\mathcal{G} = \mathrm{SU}(N)$, $\mathrm{SO}(N)$, $\mathrm{SP}(N)$ with $N \geq 4$ there are four such tensors [135]. For the case $N = 3$ only three independent tensors can be built [136], while for $N = 2$ their number reduces to two [133]. The answer to the general question concerning rank n tensors is not yet known. For the explicit construction of the set of tensors H^{abcd} we consider first the $\mathrm{SU}(N)$ group and then generalize the results for the other two groups.

A natural choice for a basis for rank $n = 4$ tensors when $N \geq 4$ is given by [137]. (An alternative way to define a basis which has the virtue of being immediately generalizable to any group [136] is in terms of traces of

products of the generators in the defining representation, thus $\mathrm{Tr}(T^aT^bT^cT^d)$, $\mathrm{Tr}(T^aT^b)\,\mathrm{Tr}(T^cT^d)$, etc.) one has

$$\begin{aligned} K_1 &= \delta^{ab}\delta^{cd}, \qquad K_4 = d^{abe}d^{cde}, \qquad K_7 = d^{abe}f^{cde}, \\ K_2 &= \delta^{ac}\delta^{bd}, \qquad K_5 = d^{ace}d^{bde}, \qquad K_8 = d^{ace}f^{bde}, \\ K_3 &= \delta^{ad}\delta^{bc}, \qquad K_6 = d^{ade}d^{bde}, \qquad K_9 = d^{ade}f^{bce}. \end{aligned} \tag{45}$$

Here d^{abc} stands for the completely symmetric rank $n = 3$ tensors. The dimension of the basis reduces to 8 in the case of SU(3). This is achieved via the relation [137, 138]

$$K_4 + K_5 + K_6 = \frac{1}{3}\left(K_1 + K_2 + K_3\right) \tag{46}$$

which is not valid for $N \geq 4$.

To describe the ε-scalar quartic interactions one needs to construct rank $n = 4$ tensors invariant with respect to exchange of pairs of indices. Thus, one has to take linear combinations of the basis tensors and symmetrize them with respect to the pair of indices (ab) and (cd). A possible choice for $N \geq 4$ is given by

$$\begin{aligned} H_1 &= \frac{1}{2}K_1, \qquad H_2 = \frac{1}{2}\left(K_2 + K_3\right), \\ H_3 &= \frac{1}{2}K_4, \qquad H_4 = \frac{1}{2}\left(K_5 + K_6\right). \end{aligned} \tag{47}$$

Note that the absence of a $d - f$ type term from (47) follows from the identity [137]

$$K_8 + K_9 = -f^{abe}d^{cde}. \tag{48}$$

However, for practical purposes a basis constructed with the help of the structure constants f^{abc} and avoiding the use of the d-tensors is more suited. For example, it would allow to explore more easily the supersymmetric case and to generalize to other groups. It is natural to consider the alternative choice [113, 131]

$$\begin{aligned} \overline{H}_1 &= \frac{1}{2}\left(f^{ace}f^{bde} + f^{ade}f^{bce}\right), \\ \overline{H}_2 &= \delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc} + \delta^{ab}\delta^{cd}, \\ \overline{H}_3 &= \frac{1}{2}\left(\delta^{ac}\delta^{bd} + \delta^{ad}\delta^{bc}\right) - \delta^{ab}\delta^{cd}, \\ \overline{H}_4 &= \frac{1}{2}\left(f^{aef}f^{bfg}f^{cgh}f^{dhe} + f^{aef}f^{bfg}f^{dgh}f^{che}\right). \end{aligned} \tag{49}$$

Let us introduce the coupling constants

$$\alpha_s = \frac{g^2}{4\pi}, \qquad \alpha_e = \frac{g_e^2}{4\pi}, \qquad u_r = \frac{\lambda_r}{4\pi}. \tag{50}$$

Then we can write the last term in (41)

$$\sum_{r=1}^{4} Z_{\lambda_r}\lambda_r H_r^{abcd} = 4\pi\sum_{r=1}^{4} Z_{u_r}u_r H_r^{abcd} = 4\pi\sum_{r=1}^{4} Z_{\eta_r}\eta_r \overline{H}_r^{abcd}, \tag{51}$$

where η_r denote the quartic ε-scalar couplings in the basis $\overline{H}^{abcd}$. The renormalization constants Z_η, Z_u, and so forth have been computed through one loop in the $\overline{\mathrm{DR}}$ scheme for a general gauge group in [131, 135] and in [113] for SU(3). The calculation performed in [113] has employed the method of [117] to introduce an artificial mass for all propagators in order to avoid spurious infrared divergences. For the calculation of the results in terms of group invariants the package `color` [139] has been used. For completeness, we reproduce here the one-loop results for the couplings η_r:

$$\begin{aligned} Z_{\eta_1} &= 1 + \frac{1}{\epsilon} \\ &\times \Bigg[-\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}C_A\frac{3}{2} + \frac{\eta_1}{\pi}C_A\frac{1}{2} + \frac{\eta_2}{\pi}C_A 2 \\ &\quad - \frac{\eta_3}{\pi}\frac{7}{2} - \frac{\eta_2}{\pi}\frac{\eta_4}{\eta_1}C_A - \frac{\eta_3}{\pi}\frac{\eta_4}{\eta_1}C_A\frac{1}{2} + \frac{\eta_4}{\pi} \\ &\quad \times \frac{C_A^4\left(-61 + 7N_A\right) + 48D_4\left(AA\right)\left(N_A - 1\right)/N_A}{36C_A^2\left(N_A - 3\right)} \\ &\quad + \frac{\alpha_e}{\pi}T_f - \frac{\alpha_e}{\pi}\frac{\alpha_e}{\eta_1} \\ &\quad \times \Bigg(\Bigg(\frac{4C_A\left(2 + N_A\right)D_4\left(RA\right)}{I_2\left(R\right)} \\ &\qquad + 5C_A^3\left(7C_A - 2C_R\right)N_A \\ &\qquad -16\left(2 + N_A\right)D_4\left(AA\right) \Bigg) \\ &\qquad \times\left(2\left(25C_A^4N_A - 12D_4\left(AA\right)\left(2+N_A\right)\right)\right)^{-1} \Bigg) \\ &\quad \times T_f \\ &\quad - \frac{\eta_4}{\pi}\frac{\eta_4}{\eta_1} \\ &\quad \times \Bigg(\left(1\right) \times \Big(54C_AN_A\left(N_A - 3\right) \\ &\qquad \times \left(25C_A^4N_A \right. \\ &\qquad \left. -12D_4\left(AA\right)\left(2 + N_A\right)\right)\Big)^{-1} \Bigg) \\ &\quad \times \Big(-144D_4(AA)^2\left(2 + N_A\right)\left(1 + 2N_A\right) \\ &\qquad + 216C_A^2D_4\left(AAA\right)N_A\left(2 + N_A\right)\left(N_A - 3\right) \\ &\qquad - 12C_A^4D_4\left(AA\right)N_A\left(-191 - 56N_A + N_A^2\right) \\ &\qquad -25C_A^8N_A^2\left(4N_A + 23\right)\Big) \Bigg], \end{aligned}$$

$$
\begin{aligned}
Z_{\eta_2} &= 1+\frac{1}{\epsilon}\\
&\times\Bigg[-\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}C_A\frac{3}{2}-\frac{\eta_1}{\pi}C_A\frac{1}{6}+\frac{\eta_3}{\pi}\frac{N_A-1}{6}\\
&\quad+\frac{\eta_4}{\pi}C_A^2\frac{13}{12}+\frac{\eta_2}{\pi}\frac{2\left(8+N_A\right)}{3}\\
&\quad+\frac{\eta_3}{\pi}\left(\frac{\eta_1}{\eta_2}C_A\frac{1}{6}+\frac{\eta_4}{\eta_2}C_A^2\frac{1}{6}-\frac{\eta_3}{\eta_2}\frac{\left(N_A-1\right)}{12}\right)\\
&\quad-\frac{\eta_4}{\pi}\frac{\eta_4}{\eta_2}\frac{2}{9}\\
&\quad\times\Bigg(\Bigg(\frac{72D_4(AA)^2}{N_A}-90C_A^2D_4\left(AAA\right)\\
&\qquad+25C_A^4D_4\left(AA\right)\Bigg)\\
&\qquad\times\left(25C_A^4N_A-12D_4\left(AA\right)\left(2+N_A\right)\right)^{-1}\Bigg)\\
&\quad+\frac{\alpha_e}{\pi}\\
&\quad\times\Bigg(T_f-\frac{\alpha_e}{\eta_2}2\\
&\qquad\times\Bigg(\Bigg(\frac{5C_A^2D_4\left(RA\right)}{I_2\left(R\right)}+\left(C_A-6C_R\right)D_4\left(AA\right)\Bigg)\\
&\qquad\times\left(25C_A^4N_A-12D_4(AA)\left(2+N_A\right)\right)^{-1}\Bigg)\\
&\qquad\times T_f\Bigg)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
Z_{\eta_3} &= 1+\frac{1}{\epsilon}\times\Bigg[-\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}C_A\frac{3}{2}+\frac{\eta_4}{\pi}C_A^2\frac{5}{12}\\
&\quad+\frac{\eta_2}{\pi}\frac{2\left(2+N_A\right)}{3}+\frac{\eta_3}{\pi}\frac{-26+5N_A}{12}\\
&\quad+\frac{\eta_4}{\pi}\frac{\eta_4}{\eta_3}\frac{7}{108}\frac{12D_4\left(AA\right)-5C_A^4N_A}{\left(N_A-3\right)N_A}\\
&\quad-\frac{\eta_2}{\pi}\left(\frac{\eta_4}{\eta_3}C_A^2\frac{2}{3}+\frac{\eta_2}{\eta_3}\frac{\left(2+N_A\right)}{3}\right)\\
&\quad+\frac{\eta_1}{\pi}\Bigg(-C_A\frac{5}{6}-\frac{\eta_2}{\eta_3}\frac{2C_A}{3}\\
&\qquad+\frac{\eta_4}{\eta_3}\frac{12D_4\left(AA\right)-5C_A^4N_A}{9C_AN_A\left(N_A-3\right)}\Bigg)\\
&\quad+\frac{\alpha_e}{\pi}T_f\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
Z_{\eta_4} &= 1+\frac{1}{\epsilon}\\
&\times\Bigg[-\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}C_A\frac{3}{2}-\frac{\eta_1}{\pi}C_A\frac{1}{4}+\frac{\eta_2}{\pi}8\\
&\quad-\frac{\eta_3}{\pi}\frac{1}{2}+\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\eta_4}\frac{3}{4}-\frac{\eta_1}{\pi}\frac{\eta_1}{\eta_4}\frac{1}{4}+\frac{\eta_4}{\pi}\\
&\quad\times\Big(-1152D_4\left(AAA\right)\left(2+N_A\right)+5C_A^2\\
&\qquad\times\left(125C_A^4N_A+4D_4(AA)(98+N_A)\right)\Big)\\
&\quad\times\left(48\left(25C_A^4N_A-12D_4\left(AA\right)\left(2+N_A\right)\right)\right)^{-1}\\
&\quad+\frac{\alpha_e}{\pi}\\
&\quad\times\Bigg(T_f+\frac{\alpha_e}{\eta_4}\\
&\qquad\times\Bigg(\Bigg(5C_A^2\left(C_A-6C_R\right)N_A\\
&\qquad+\frac{12\left(2+N_A\right)D_4\left(RA\right)}{I_2\left(R\right)}\Bigg)\\
&\qquad\times\left(25C_A^4N_A-12D_4(AA)(2+N_A)\right)^{-1}\Bigg)\\
&\quad\times T_f\Bigg)\Bigg],
\end{aligned}
\tag{52}
$$

with the group invariants defined in Appendix A and the abbreviation $T_f = I_2(R)n_f$, where n_f denotes the number of active fermions. Let us notice at this point the presence of negative power of couplings in the expressions of the renormalization constants. This results in beta functions that are not proportional to the coupling itself. This feature is specific to scalar couplings and it implies that, even if we set such a coupling to zero at a given scale, it will receive nonvanishing radiative corrections due to the other couplings present in the theory.

The above results have been computed using an SU(N) gauge group. However, they are parametrized in terms of group invariants. Thus they are also valid for other physically interesting groups like SO(N) and SP(N). The explicit values of the group invariants for the three groups can be found in Appendix A.

In the case of SU(3) group, the invariant $\overline{H}_4$ becomes a linear combination of $\overline{H}_i$, $i = 1, 2, 3$, because of relation (46). The same is also true for the coupling η_4 that can be expressed in terms of the other three couplings. Thus in this case one can ignore η_4.

Actually, the one- and two-loop renormalization constants for totally symmetric quartic scalar couplings with scalars in an arbitrary representation have been known for

long time [140]. However, these results cannot be directly applied to ε-scalar self-interactions, due to their particular symmetry with respect to exchange between pairs of indices.

3.3.2. Three-Loop Renormalization Constants for a Non-supersymmetric Theory. In this section we report on the explicit computation of the charge Z_g, Z_e and mass Z_m^q, Z_m^ϵ renormalization constants to three-loop order within $\overline{\text{DR}}$ scheme. This requires the calculation of divergent parts of logarithmically divergent integrals. One can exploit the fact that such contributions are independent of the masses and external momenta. Precisely, one sets all internal masses to zero and keeps only one external momentum different from zero and then solve recursively (29). In practice, use of the automated programs `QGRAF` [141], `q2e` and `exp` [142, 143] and `MINCER` are essential due to the large number of diagrams that occur.

The analytical form of $Z_g^{\overline{\text{DR}}}$ up to two-loop order is identical to the corresponding result in the $\overline{\text{MS}}$ scheme. This has been shown by an explicit calculation for the first time in [81] and is a consequence of the minimal renormalization. The three- and four-loop results for a general theory have been derived in [113, 135, 144]. For completeness we present in the following the three-loop results:

$$\begin{aligned}
Z_g^{\overline{\text{DR}}} &= 1 + \frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\frac{1}{\epsilon}\left(-\frac{11}{24}C_A + \frac{1}{6}T_f\right) + \left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^2 \\
&\times\left[\frac{1}{\epsilon^2}\left(\frac{121}{384}C_A^2 - \frac{11}{48}C_AT_f + \frac{1}{24}T_f^2\right)\right. \\
&\left.+\frac{1}{\epsilon}\left(-\frac{17}{96}C_A^2 + \frac{5}{48}C_AT_f + \frac{1}{16}C_RT_f\right)\right] \\
&+\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^3 \\
&\times\left[\frac{1}{\epsilon^3}\left(\frac{-6655}{27648}C_A^3 + \frac{605}{2304}C_A^2T_f\right.\right. \\
&\left.-\frac{55}{576}C_AT_f^2 + \frac{5}{432}T_f^3\right) \\
&+\frac{1}{\epsilon^2}\left(\frac{2057}{6912}C_A^3 - \frac{979}{3456}C_A^2T_f + \frac{11}{288}C_RT_f^2\right. \\
&\left.-\frac{121}{1152}C_AC_RT_f + \frac{55}{864}C_AT_f^2\right) \\
&+\frac{1}{\epsilon}\left(-\frac{3115}{20736}C_A^3 + \frac{1439}{10368}C_A^2T_f\right. \\
&\left.\left.+\frac{193}{3456}C_AC_RT_f - \frac{79}{5184}C_AT_f^2\right.\right. \\
&\left.\left.-\frac{1}{192}C_R^2T_f - \frac{11}{864}C_RT_f^2\right)\right] \\
&+\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^2\frac{\alpha_e}{\pi}\frac{1}{\epsilon}\left(\frac{1}{32}C_R^2T_f\right) + \frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\left(\frac{\alpha_e}{\pi}\right)^2\frac{1}{\epsilon} \\
&\times\left(\frac{1}{96}C_AC_RT_f - \frac{1}{48}C_R^2T_f - \frac{1}{96}C_RT_f^2\right).
\end{aligned} \tag{53}$$

The one-loop result for Z_e can be found in [133]. For the particular case of QCD, that is, $\mathcal{G} = \text{SU}(3)$ and $\eta_4 = 0$, the two-, three-, and four-loop results have been computed in [113, 144]. The two-, three-, and four-loop results for a general theory have been derived in [135]. Because of the complexity of the results, we reproduce below only the two-loop contributions that are, however, enough for most of the practical applications:

$$\begin{aligned}
Z_e &= 1 + \frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\frac{1}{\epsilon}\left(-\frac{3}{4}C_R\right) \\
&+\frac{\alpha_e}{\pi}\frac{1}{\epsilon}\left(-\frac{1}{4}C_A + \frac{1}{2}C_R + \frac{1}{4}T_f\right) + \left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^2 \\
&\times\left[\frac{1}{\epsilon^2}\left(\frac{11}{32}C_AC_R + \frac{9}{32}C_R^2 - \frac{1}{8}C_RT_f\right)\right. \\
&+\frac{1}{\epsilon}\left(\frac{7}{256}C_A^2 - \frac{55}{192}C_AC_R\right. \\
&\left.\left.-\frac{3}{64}C_R^2 - \frac{1}{32}C_AT_f + \frac{5}{48}C_RT_f\right)\right] \\
&+\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\frac{\alpha_e}{\pi} \\
&\times\left[\frac{1}{\epsilon^2}\left(\frac{3}{8}C_AC_R - \frac{3}{4}C_R^2 - \frac{3}{8}C_RT_f\right)\right. \\
&\left.+\frac{1}{\epsilon}\left(\frac{3}{32}C_A^2 - \frac{5}{8}C_AC_R + \frac{11}{16}C_R^2 + \frac{5}{32}C_RT_f\right)\right] \\
&+\left(\frac{\alpha_e}{\pi}\right)^2\left[\frac{1}{\epsilon^2}\left(\frac{3}{32}C_A^2 - \frac{3}{8}C_AC_R + \frac{3}{8}C_R^2\right.\right. \\
&\left.-\frac{3}{16}C_AT_f + \frac{3}{8}C_RT_f + \frac{3}{32}T_f^2\right) \\
&+\frac{1}{\epsilon}\left(-\frac{3}{32}C_A^2 + \frac{5}{16}C_AC_R - \frac{1}{4}C_R^2\right. \\
&\left.\left.+\frac{3}{32}C_AT_f - \frac{3}{16}C_RT_f\right)\right] \\
&+\frac{\alpha_e}{\pi}\frac{1}{\epsilon}\left[\frac{\eta_1}{\pi}\left(\frac{1}{32}C_A^2\right) + \frac{\eta_2}{\pi}\left(\frac{1}{16}C_A - \frac{3}{8}C_R\right)\right. \\
&+\frac{\eta_3}{\pi}\left(-\frac{1}{16}C_A\right) \\
&\left.+\frac{\eta_4}{\pi}\left(\frac{1}{192}C_A^3 - \frac{1}{8}D_4(RA)I_2(R)\right)\right] \\
&+\left(\frac{\eta_1}{\pi}\right)^2\frac{1}{\epsilon}\left(-\frac{3}{256}C_A^2\right) + \left(\frac{\eta_2}{\pi}\right)^2\frac{1}{\epsilon}\left(\frac{3}{32}(N_A+2)\right)
\end{aligned}$$

$$+\frac{\eta_1}{\pi}\frac{\eta_3}{\pi}\frac{1}{\epsilon}\left(\frac{3}{64}C_A\right)+\left(\frac{\eta_3}{\pi}\right)^2\frac{1}{\epsilon}\left(-\frac{3}{128}\left(N_A-1\right)\right)$$
$$+\frac{\eta_1}{\pi}\frac{\eta_4}{\pi}\frac{1}{\epsilon}\left(-\frac{1}{256}C_A^3\right)+\frac{\eta_2}{\pi}\frac{\eta_4}{\pi}\frac{1}{\epsilon}\left(\frac{5}{32}C_A^2\right)$$
$$+\frac{\eta_3}{\pi}\frac{\eta_4}{\pi}\frac{1}{\epsilon}\left(\frac{1}{128}C_A^2\right)$$
$$+\left(\frac{\eta_4}{\pi}\right)^2\frac{1}{\epsilon}\left(-\frac{1}{3072}C_A^4+\frac{1}{32}D_4\left(AA\right)\right). \tag{54}$$

The group invariants $C_A, C_R, I_2(R), D_4(XY)$ occurring in the above equations are defined in Appendix A and we used the abbreviation $T_f = I_2(R)n_f$.

There is also an indirect way to derive the three-loop gauge beta function in the $\overline{\text{DR}}$ scheme starting from the knowledge of the three-loop gauge beta function in the $\overline{\text{MS}}$ scheme and the fact that the gauge couplings defined in the two schemes can be perturbatively related to each other. This method will be discussed in more detail in the next section. Let us mention, however, that using the expression for the three-loop gauge beta function in the $\overline{\text{MS}}$ scheme $\beta_s^{\overline{\text{MS}}}$ and the two-loop conversion relation of α_s given in (57) one obtains exactly the same results for $\beta_s^{\overline{\text{DR}}}$ as given in (53). This is a powerful consistency check for the calculation reviewed in this section. It is interesting to mention that the equality of the two results can be obtain only if one keeps $\alpha_s^{\overline{\text{DR}}} \neq \alpha_e$ during the calculation and renormalize them differently. The identification of $\alpha_s^{\overline{\text{DR}}}$ and α_e leads to inconsistent results. In case of $\beta_s^{\overline{\text{DR}}}$ the error is a finite, gauge parameter independent term [128]. For quark mass renormalization, this identification (precisely the identification of the renormalization constants for the two couplings) generates much more severe problems. Namely, the renormalization constant for the quark mass $Z_m^{\overline{\text{DR}}}$ will contain nonlocal terms at three-loop order and the mass anomalous dimension will erroneously become divergent at this loop order.

The renormalization constant for the fermion masses $Z_m^{\overline{\text{DR}}}$ has been computed in [113] to three- and in [135, 144] even to four-loop order. Whereas in [113, 135, 144] only the anomalous dimensions were given we want to present the explicit three-loop result for the renormalization constant, that reads

$$Z_m^{\overline{\text{DR}}} = 1+\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\frac{1}{\epsilon}\left(-\frac{3}{4}C_R\right)+\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^2$$
$$\times\left[\frac{1}{\epsilon^2}\left(\frac{11}{32}C_AC_R+\frac{9}{32}C_R^2-\frac{1}{8}C_RT_f\right)\right.$$
$$\left.+\frac{1}{\epsilon}\left(-\frac{91}{192}C_AC_R-\frac{3}{64}C_R^2+\frac{5}{48}C_RT_f\right)\right]$$
$$+\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\frac{\alpha_e}{\pi}\left(\frac{3}{16}\frac{1}{\epsilon}C_R^2\right)$$
$$+\left(\frac{\alpha_e}{\pi}\right)^2\frac{1}{\epsilon}\left(\frac{1}{16}C_AC_R-\frac{1}{8}C_R^2-\frac{1}{16}C_RT_f\right)$$
$$+\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^3$$
$$\times\left[\frac{1}{\epsilon^3}\left(-\frac{121}{576}C_A^2C_R-\frac{33}{128}C_AC_R^2-\frac{9}{128}C_R^3\right.\right.$$
$$\left.+\frac{11}{72}C_AC_RT_f+\frac{3}{32}C_R^2T_f-\frac{1}{36}C_RT_f^2\right)$$
$$+\frac{1}{\epsilon^2}\left(\frac{1613}{3456}C_A^2C_R+\frac{295}{768}C_AC_R^2+\frac{9}{256}C_R^3\right.$$
$$\left.-\frac{59}{216}C_AC_RT_f-\frac{29}{192}C_R^2T_f+\frac{5}{216}C_RT_f^2\right)$$
$$+\frac{1}{\epsilon}\left(-\frac{10255}{20736}C_A^2C_R+\frac{133}{768}C_AC_R^2-\frac{43}{128}C_R^3\right.$$
$$+\left(\frac{281}{2592}+\frac{1}{4}\zeta_3\right)C_AC_RT_f$$
$$\left.\left.+\left(\frac{23}{96}-\frac{1}{4}\zeta_3\right)C_R^2T_f+\frac{35}{1296}C_RT_f^2\right)\right]$$
$$+\left(\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\right)^2\frac{\alpha_e}{\pi}$$
$$\times\left[\frac{1}{\epsilon^2}\left(-\frac{11}{192}C_AC_R^2-\frac{15}{64}C_R^3+\frac{1}{48}C_R^2T_f\right)\right.$$
$$+\frac{1}{\epsilon}\left(\frac{5}{256}C_A^2C_R+\frac{7}{32}C_AC_R^2\right.$$
$$\left.\left.+\frac{9}{64}C_R^3-\frac{3}{32}C_R^2T_f\right)\right]$$
$$+\frac{\alpha_s^{\overline{\text{DR}}}}{\pi}\left(\frac{\alpha_e}{\pi}\right)^2$$
$$\times\left[\frac{1}{\epsilon^2}\left(-\frac{9}{64}C_AC_R^2+\frac{9}{32}C_R^3+\frac{9}{64}C_R^2T_f\right)\right.$$
$$+\frac{1}{\epsilon}\left(-\frac{1}{64}C_A^2C_R+\frac{7}{32}C_AC_R^2-\frac{3}{8}C_R^3\right.$$
$$\left.\left.-\frac{1}{64}C_AC_RT_f-\frac{1}{8}C_R^2T_f\right)\right]$$
$$+\left(\frac{\alpha_e}{\pi}\right)^3$$
$$\times\left[\frac{1}{\epsilon^2}\left(-\frac{1}{48}C_A^2C_R+\frac{1}{12}C_AC_R^2-\frac{1}{12}C_R^3\right.\right.$$
$$\left.+\frac{1}{24}C_AC_RT_f-\frac{1}{12}C_R^2T_f-\frac{1}{48}C_RT_f^2\right)$$
$$+\frac{1}{\epsilon}\left(\frac{1}{32}C_A^2C_R-\frac{1}{8}C_AC_R^2+\frac{1}{8}C_R^3\right.$$
$$\left.\left.-\frac{1}{24}C_AC_RT_f+\frac{5}{48}C_R^2T_f+\frac{1}{96}C_RT_f^2\right)\right]$$
$$+\left(\frac{\alpha_e}{\pi}\right)^2\frac{1}{\epsilon}$$

$$\times\left[\frac{\eta_1}{\pi}\left(-\frac{1}{96}C_A^2C_R\right)+\frac{\eta_2}{\pi}\left(-\frac{1}{48}C_AC_R+\frac{1}{8}C_R^2\right)\right.$$
$$+\frac{\eta_3}{\pi}\left(\frac{1}{48}C_AC_R\right)$$
$$\left.+\frac{\eta_4}{\pi}\left(-\frac{1}{576}C_A^3C_R+\frac{1}{12}C_RD_4\left(RA\right)\right)\right]$$
$$+\frac{\alpha_e}{\pi}\frac{1}{\epsilon}$$
$$\times\left[\left(\frac{\eta_1}{\pi}\right)^2\left(\frac{1}{256}C_A^2C_R\right)\right.$$
$$+\left(\frac{\eta_2}{\pi}\right)^2\left(-\frac{1}{32}C_R\left(N_A+2\right)\right)$$
$$+\left(\frac{\eta_3}{\pi}\right)^2\left(\frac{1}{128}C_R\left(N_A-1\right)\right)$$
$$+\left(\frac{\eta_4}{\pi}\right)^2\left(\frac{1}{9216}C_A^4C_R-\frac{1}{96}C_RD_4\left(AA\right)\right)$$
$$+\frac{\eta_1}{\pi}\frac{\eta_3}{\pi}\left(-\frac{1}{64}C_AC_R\right)+\frac{\eta_1}{\pi}\frac{\eta_4}{\pi}\left(\frac{1}{768}C_A^3C_R\right)$$
$$\left.+\frac{\eta_2}{\pi}\frac{\eta_4}{\pi}\left(-\frac{5}{96}C_A^2C_R\right)+\frac{\eta_3}{\pi}\frac{\eta_4}{\pi}\left(-\frac{1}{384}C_A^2C_R\right)\right], \tag{55}$$

where $\zeta(3)$ is Riemann's zeta function with $\zeta(3) = 1.20206\ldots$.

Again, the consistency of the above results can be proved using the indirect method alluded above. To derive the three-loop quark mass anomalous dimension in the $\overline{\text{DR}}$ scheme $\gamma_m^{\overline{\text{DR}}}$, one needs the three-loop result for $\gamma_m^{\overline{\text{MS}}}$ and the two-loop conversion relation for the quark mass as given in (58). Full agreement has been found between the two methods [113], that provides a further consistency check of the calculation.

3.3.3. The General Four-Loop Order Results in the $\overline{DR}$ Scheme. The direct way to compute the renormalization constants in minimal subtraction schemes as $\overline{\text{MS}}$ or $\overline{\text{DR}}$ requires the calculation of divergent parts of logarithmically divergent integrals. Up to three loops there are well established methods and automated programs exist to perform such calculations (see, e.g., [115, 124]). Also at four-loop order a similar approach is applicable. Nevertheless, it is technically much more involved [118, 119, 121, 145, 146]. There is, however, an indirect method discussed in [113, 128] to derive the renormalization constants in the $\overline{\text{DR}}$ scheme starting from their $\overline{\text{MS}}$ expressions. It relies on the perturbative relation that can be established between the couplings and masses defined in the two schemes and takes into account that the four-loop results in the $\overline{\text{MS}}$ scheme are known [118, 119, 121]. For example, to derive the beta function for the gauge coupling to four-loop order in $\overline{\text{DR}}$ scheme one needs the relation between the gauge couplings defined in the $\overline{\text{MS}}$ and $\overline{\text{DR}}$ schemes up to three-loop order. The latter can be determined using the following arguments.

To compute the relations between running parameters defined in two different renormalization schemes, one has to relate them to physical observables which cannot depend on the choice of scheme. For example, the relationship between the strong coupling constant defined in the $\overline{\text{MS}}$ and $\overline{\text{DR}}$ schemes can be obtained from the S-matrix amplitude of a physical process involving the gauge coupling computed in the two schemes. However, beyond one loop the computation of the physical amplitudes becomes very much involved and requires the computation of multiloop and multiscale on-shell Feynman integrals that is a highly nontrivial task. Nevertheless, one can avoid the use of on-shell kinematics introducing a physical renormalization scheme defined through convenient kinematics, for which the renormalization constants can be computed applying the "large-momentum" or the "hard-mass" procedures. Up to three loops, there are well established methods (for details see previous sections) to compute the divergent as well as finite pieces of the Feynman integrals and automated programs exist to perform such calculations. Once the renormalization constants in the physical renormalization scheme are determined, one uses the constraint that the *effective* gauge coupling constant defined in such a scheme is unique and thus independent of the regularization procedure. Furthermore, one relates the running gauge couplings defined in the two regularization schemes through the following relations:

$$\alpha_s^{\text{ph}} = \left(z_s^{\text{ph},X}\right)^2\alpha_s^X, \qquad z_s^{\text{ph},X} = \frac{Z_s^X}{Z_s^{\text{ph},X}}, \quad X \in \left\{\overline{\text{MS}}, \overline{\text{DR}}\right\}$$
$$\Longrightarrow \alpha_s^{\overline{\text{DR}}} = \left(\frac{Z_s^{\text{ph},\overline{\text{DR}}}}{Z_s^{\text{ph},\overline{\text{MS}}}}\frac{Z_s^{\overline{\text{MS}}}}{Z_s^{\overline{\text{DR}}}}\right)^2\alpha_s^{\overline{\text{MS}}}, \tag{56}$$

where $Z_s^{\overline{\text{MS}}/\overline{\text{DR}}}$ are the charge renormalization constants using minimal subtraction in DREG/DRED, as defined above. Note that the various Z_s in (56) depend on differently renormalized α_s, so that the equations have to be used iteratively at higher orders of perturbation theory. Working out these considerations for the gauge coupling and for the fermion mass up to the three-loop order, one obtains

$$\alpha_s^{\overline{\text{DR}}} = \alpha_s^{\overline{\text{MS}}}\left[1+\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\frac{1}{12}C_A+\left(\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\right)^2\frac{11}{72}C_A^2\right.$$
$$\left.-\frac{\alpha_s^{\overline{\text{MS}}}}{\pi}\frac{\alpha_e}{\pi}\frac{1}{8}C_RT_f+\delta_\alpha^{(3)}\right], \tag{57}$$

$$m^{\overline{\mathrm{DR}}} = m^{\overline{\mathrm{MS}}}\left[1 - \frac{\alpha_e}{\pi}\frac{1}{4}C_R + \left(\frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\right)^2 \frac{11}{192}C_A C_R - \frac{\alpha_s^{\overline{\mathrm{MS}}}}{\pi}\frac{\alpha_e}{\pi}\frac{1}{32}C_R\left(3C_A + 8C_R\right) + \left(\frac{\alpha_e}{\pi}\right)^2 \frac{1}{32}\left[3C_R + T_f\right] + \delta_m^{(3)}\right], \tag{58}$$

where we have suppressed the explicit dependence on the renormalization scale μ. $\delta_\alpha^{(3)}$ and $\delta_m^{(3)}$ denote the three-loop terms and they are obtained from the finite parts of three-loop diagrams (see [113] for details). They read [131, 135]

$$\begin{aligned}
\pi^3\delta_\alpha^{(3)} = {} & \frac{1}{96}\alpha_s^{\overline{\mathrm{MS}}}\alpha_e^2 T_f \\
& \times \left[2C_A^2 - 3C_A C_R + 2C_R^2 - C_A T_f + 7C_R T_f\right] \\
& - \frac{1}{192}\left(\alpha_s^{\overline{\mathrm{MS}}}\right)^2 \alpha_e T_f \left(5C_A^2 + 60C_A C_R + 6C_R^2\right) \\
& + \frac{1}{9216}\alpha_s^{\overline{\mathrm{MS}}} \\
& \times \left(36C_A^3\eta_1^2 - 576C_A\eta_2^2 - 144C_A^2\eta_1\eta_3 - 72C_A\eta_3^2 \right. \\
& \quad + 12C_A^4\eta_1\eta_4 - 480C_A^3\eta_2\eta_4 - 24C_A^3\eta_3\eta_4 \\
& \quad \left. + C_A^5\eta 4^2 - 288C_A N_A\eta_2^2 + 72C_A N_A\eta_3^2\right) \\
& - \frac{1}{96N_A}\alpha_s^{\overline{\mathrm{MS}}}\eta_4^2 C_A D_4(AA) \\
& + \frac{1}{48}\left(\alpha_s^{\overline{\mathrm{MS}}}\right)^2 \eta_4 D_4(AA) + \frac{1}{4608}\left(\alpha_s^{\overline{\mathrm{MS}}}\right)^2 \\
& \times \left(-6C_A^3\eta_1 + 240C_A^2\eta_2 + 12C_A^2\eta_3 - C_A^4\eta_4\right) \\
& + \frac{1}{10368}\left(\alpha_s^{\overline{\mathrm{MS}}}\right)^3 \\
& \times \left[3049C_A^3 - 416C_A^2 T_f - 138C_A C_R T_f\right], \\
\pi^3\delta_m^{(3)} = {} & -\frac{1}{384}\alpha_e^3 C_R \\
& \times \left[-10C_A^2 + 14C_A C_R + 27C_R^2 - 7C_A T_f \right. \\
& \quad + 39C_R T_f - 10I_2(R)^2 T_f^2 + 12C_A^2\zeta_3 \\
& \quad \left. -36C_A C_R\zeta_3 + 24C_R^2\zeta_3\right] - \alpha_e^2 C_R \\
& \times \left(\frac{1}{322}\left[\left(6C_R - C_A\right)\eta_2\right] \right. \\
& \quad + \frac{1}{16I_2(R)N_A}D_4(RA)\eta_4 + \frac{1}{384}\alpha_s^{\overline{\mathrm{MS}}} \\
& \quad \times \left[47C_A^2 + 10C_R^2 - 3C_A T_f - 19C_R T_f \right. \\
& \quad\quad - 165C_A C_R + 144C_R^2\zeta_3 - 48C_A T_f\zeta_3 \\
& \quad\quad \left. \left. +48C_R T_f\zeta_3 + 72C_A^2\zeta_3 - 216C_A C_R\zeta_3\right]\right) \\
& + \alpha_e C_R \\
& \times \left(\frac{1}{12288}\right. \\
& \quad \times \left[-36C_A^2 C_R\eta_1^2 + 1728C_R\eta_2^2 \right. \\
& \quad\quad + 144C_A C_R\eta_1\eta_3 + 72C_R\eta 3^2 \\
& \quad\quad - 12C_A^3 C_R\eta_1\eta_4 + 1440C_A^2 C_R\eta_2\eta_4 \\
& \quad\quad + 24C_A^2 C_R\eta_3\eta_4 - C_A^4 C_R\eta 4^2 \\
& \quad\quad \left. +864C_R N_A\eta_2^2 - 72C_R N_A\eta 3^2\right] \\
& \quad + \frac{1}{3072}\left(\alpha_s^{\overline{\mathrm{MS}}}\right)^2 \\
& \quad \times \left[2880C_R^2\zeta_3 - 168C_A T_f - 1544C_A C_R \right. \\
& \quad\quad - 52C_R^2 - 128C_R T_f + 1440C_A^2\zeta_3 \\
& \quad\quad \left. \left. -4320C_A C_R\zeta_3 - 79C_A^2\right]\right) \\
& + \frac{1}{20736}\left(\alpha_s^{\overline{\mathrm{MS}}}\right)^3 C_R C_A\left[4354C_A + 135C_R + 304T_f\right] \\
& + \frac{3}{128N_A}D_4(AA)\eta_4^2.
\end{aligned} \tag{59}$$

Inserting (57) and (58) into the definition of the beta function for the gauge coupling equation (8) and the mass anomalous dimension equation (12), one can show that

$$\begin{aligned}
\beta_s^{\overline{\mathrm{DR}}} &= \mu^2\frac{\mathrm{d}}{\mathrm{d}\mu^2}\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi} \\
&= \beta_s^{\overline{\mathrm{MS}}}\frac{\partial\alpha_s^{\overline{\mathrm{DR}}}}{\partial\alpha_s^{\overline{\mathrm{MS}}}} + \beta_e\frac{\partial\alpha_s^{\overline{\mathrm{DR}}}}{\partial\alpha_e} + \sum_r\beta_{\eta_r}\frac{\partial\alpha_s^{\overline{\mathrm{DR}}}}{\partial\eta_r}, \\
\gamma_m^{\overline{\mathrm{DR}}} &= \frac{\mu^2}{m^{\overline{\mathrm{DR}}}}\frac{\mathrm{d}}{\mathrm{d}\mu^2}m^{\overline{\mathrm{DR}}} \\
&= \gamma_m^{\overline{\mathrm{MS}}}\frac{\partial\ln m^{\overline{\mathrm{DR}}}}{\partial\ln m^{\overline{\mathrm{MS}}}} + \frac{\pi\beta_s^{\overline{\mathrm{MS}}}}{m^{\overline{\mathrm{DR}}}}\frac{\partial m^{\overline{\mathrm{DR}}}}{\partial\alpha_s^{\overline{\mathrm{MS}}}} + \frac{\pi\beta_e}{m^{\overline{\mathrm{DR}}}}\frac{\partial m^{\overline{\mathrm{DR}}}}{\partial\alpha_e} \\
&\quad + \sum_r\frac{\pi\beta_{\eta_r}}{m^{\overline{\mathrm{DR}}}}\frac{\partial m^{\overline{\mathrm{DR}}}}{\partial\eta_r},
\end{aligned} \tag{60}$$

where the first equality is due to the definition of $\beta_s^{\overline{\mathrm{DR}}}$ and $\gamma_m^{\overline{\mathrm{DR}}}$ and the second one is a consequence of the chain rule. Let us

briefly discuss the order in perturbation theory up to which the individual building blocks are needed. Of course, the $\overline{\text{MS}}$ quantities are needed to four-loop order; they can be found in [118–121]. The dependence of $\alpha_s^{\overline{\text{DR}}}$ and $m^{\overline{\text{DR}}}$ on α_e starts at two- and one-loop order [113], respectively. Thus, β_e is needed up to the three-loop level (cf. (60)). On the other hand, both $\alpha_s^{\overline{\text{DR}}}$ and $m^{\overline{\text{DR}}}$ depend on η_r starting from three loops and consequently only the one-loop term of β_{η_r} enters in (60). The $\overline{\text{DR}}$ four-loop results were derived for QCD in [113] and for a general theory in [131, 135]. The explicit four-loop results are too lengthy to be presented in this review and we refer to the original papers for the explicit results. We discuss however their supersymmetric limit in the next section.

3.3.4. The Four-Loop Supersymmetric Case. An important check of the complicated formulas derived in the previous sections can be obtained by converting them to a supersymmetric Yang-Mills theory. For this case, one has to replace the fermions by the supersymmetric partner of the gauge bosons, the so-called gauginos. Technically, this amounts to setting the fermions in the adjoint representation of the gauge group. In addition, closed fermion loops have to be multiplied by an extra factor 1/2 in order to take into account the Majorana character of the gauginos. Explicitly, for the derivation of the three- and four-loop results one needs the replacements

$$\begin{aligned} C_R &\longrightarrow C_A, \\ I_2(R) &\longrightarrow C_A, \\ n_f &\longrightarrow \frac{1}{2}, \\ D_4(RR) &\longrightarrow D_4(AA), \\ D_4(RA) &\longrightarrow D_4(AA), \\ D_4(RAA) &\longrightarrow D_4(AAA). \end{aligned} \tag{61}$$

Furthermore, SUSY requires that the gauge coupling α_s equals the evanescent coupling α_e to all orders of perturbation theory, and therefore, the β functions are also equal $\beta_e^{\text{SYM}} = \beta_s^{\text{SYM}}$. Moreover, SUSY also requires that the ε-scalar quartic interaction containing the structure constants is equal to the gauge coupling to all orders of perturbation theory. In this case, the other three quartic couplings have to vanish, so that the decomposition equation (33) holds to all orders of perturbation theory. Indeed, using (52) one can easily derive the corresponding one-loop beta functions for supersymmetric theories and obtains

$$\beta_{\eta_1}^{\text{SYM}} = \beta_e^{\text{SYM}} = \beta_s^{\text{SYM}}, \qquad \beta_{\eta_2}^{\text{SYM}} = \beta_{\eta_3}^{\text{SYM}} = \beta_{\eta_4}^{\text{SYM}} = 0, \tag{62}$$

when the SUSY restrictions

$$\eta_1 = \alpha_3 = \alpha_s, \qquad \eta_2 = \eta_3 = \eta_4 = 0 \tag{63}$$

are imposed. It is also interesting to notice that the terms in the renormalization constants equations (52) that contain negative power of couplings cancel out in the SUSY limit, so that the limit $\eta_2 = \eta_3 = \eta_4 \to 0$ can be computed trivially. Thus, if relations (63) are imposed at the tree level, they will not be spoiled by the renormalization at the one-loop order. Checks of this statement at two- and three-loop orders are available so far only for the evanescent coupling α_e [113, 144].

Applying the substitutions given in (61) and (63) one can obtain the four-loop results for the gauge beta function β_s^{SYM} [113, 135] and compare it with the expression derived in [147]

$$\begin{aligned} \beta_s^{\text{SYM}} = &-\left(\frac{\alpha_s}{\pi}\right)^2 \left[\frac{3}{4}C_A + \frac{3}{8}C_A^2\frac{\alpha_s}{\pi} + \frac{21}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2 \right. \\ &\left. + \frac{51}{128}C_A^4\left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}\left(\alpha_s^6\right). \end{aligned} \tag{64}$$

The method employed in [147] to obtain the four-loop result was very indirect, in particular relying on the existence of the NSVZ formula for β_s^{SYM} [148, 149] (for more details see Section 2). It is therefore a remarkable check on both calculations that indeed precise agreement was obtained.

Turning now to the case of softly broken supersymmetry, there exists an exact result relating β_s and γ_m [44, 61] within the NSVZ scheme:

$$\gamma_m^{\text{SYM}} = \pi\alpha_s \frac{\mathrm{d}}{\mathrm{d}\alpha_s}\left[\frac{\beta_s^{\text{SYM}}}{\alpha_s}\right], \tag{65}$$

that nevertheless holds in $\overline{\text{DR}}$ scheme too. Hence, it follows that

$$\begin{aligned} \gamma_m^{\text{SYM}} = &-\left(\frac{\alpha_s}{\pi}\right)\left[\frac{3}{4}C_A + \frac{3}{4}C_A^2\frac{\alpha_s}{\pi} + \frac{63}{64}C_A^3\left(\frac{\alpha_s}{\pi}\right)^2 \right. \\ &\left. + \frac{51}{32}C_A^4\left(\frac{\alpha_s}{\pi}\right)^3\right] + \mathcal{O}\left(\alpha_s^5\right). \end{aligned} \tag{66}$$

Inserting (61) in (60), one can easily reproduce (66).

The invariant $D_4(AAA)$ does not occur in either calculation, and the dependence on $D_4(AA)$, N_A, ζ_3, ζ_4, and ζ_5 all cancels although they appear in individual terms. It is tempting to speculate that this absence of higher order invariants and transcendental numbers (other than π) is related to the existence of the NSVZ scheme, in which the gauge β-function for any simple gauge group is given (in the supersymmetric case without matter fields) by the expression in (15), which is manifestly free of transcendental numbers to all orders. It is natural to conjecture that the same property holds in the DRED scheme, too.

3.3.5. ε-Scalar Mass. Although there is in general no tree-level term in the Lagrangian for the mass of the ε-scalars there are loop-induced contributions to it that require the introduction of the corresponding counterterm. Let us introduce first the renormalization constant for the ε-scalar mass:

$$\left(m_\varepsilon^0\right)^2 = Z_{m_\varepsilon} m_\varepsilon^2. \tag{67}$$

The relevant Feynman diagrams contributing to the ε-scalar propagator show quadratic divergences and therefore one needs to consider only contributions from massive particles.

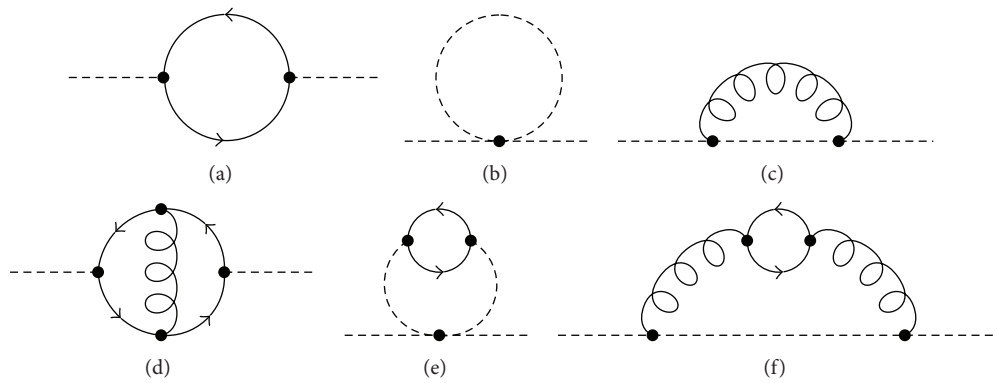

Figure 1: One- and two-loop Feynman diagrams contributing to the ε-scalar propagator. Dashed lines denote ε-scalars, curly lines denote the gauge bosons, and solid lines denote massive fermions with mass M_f.

Thus, in this case, only diagrams involving massive fermions have to be taken into account, since they are the only particles allowed by the gauge invariance to have nonzero masses. Sample diagrams are shown in Figure 1.

It is advantageous in practice to renormalize m_ε on shell and require that the renormalized mass is zero to each order in perturbation theory. In this scheme the ε-scalar mass completely decouples from the physical observables.

At the one-loop order there is only one relevant diagram (cf. Figure 1(a)) which has to be evaluated for vanishing external momentum. A closer look at the two-loop diagrams shows that they develop infrared divergences in the limit $m_\varepsilon \to 0$ (cf., e.g., Figure 1(e)). They can be regulated by introducing a small but nonvanishing mass for the ε-scalars. After the subsequent application of an asymptotic expansion [150] in the limit $q^2 = m_\varepsilon^2 \ll M_f^2$ the infrared divergences manifest themselves as $\ln(m_\varepsilon)$ terms. Furthermore, one-loop diagrams like the ones in Figures 1(b) and 1(c) do not vanish anymore and have to be taken into account as well. Although they are proportional to m_ε^2, after renormalization they induce two-loop contributions which are proportional to M_f^2, partly multiplied by $\ln(m_\varepsilon)$ terms. It is interesting to note that in the sum of the genuine two-loop diagrams and the counterterm contributions the limit $m_\varepsilon \to 0$ can be taken which demonstrates the infrared finiteness of the on-shell mass of the ε-scalar. The two-loop renormalization constant within QCD has been computed in [151]. It is given by

$$
\begin{aligned}
\frac{m_\varepsilon^2}{M_f^2}\left(Z_{m_\varepsilon}^{\mathrm{OS}}-1\right) = & -\frac{\alpha_e}{\pi} n_h I_2(R) \\
& \times\left[\frac{2}{\epsilon}+2+2L_\mu\right. \\
& \left.+\epsilon\left(2+\frac{1}{6}\pi^2+2L_\mu+L_\mu^2\right)\right] \\
& -\left(\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right)^2 n_h I_2(R)\left(\frac{3}{4}\frac{1}{\epsilon}+\frac{1}{4}+\frac{3}{2}L_\mu\right)C_A \\
& +\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\frac{\alpha_e}{\pi} n_h I_2(R) \\
& \times\left\{\frac{1}{\epsilon^2}\left(\frac{3}{8}C_A+\frac{3}{2}C_R\right)\right. \\
& +\frac{1}{\epsilon}\left[\frac{7}{8}C_A+\frac{3}{2}C_R+\left(\frac{3}{4}C_A+\frac{3}{2}C_R\right)L_\mu\right] \\
& +\left(\frac{15}{8}+\frac{1}{16}\pi^2\right)C_A+\left(\frac{3}{2}+\frac{1}{8}\pi^2\right)C_R \\
& +\left(\frac{7}{4}C_A+\frac{3}{2}C_R\right)L_\mu \\
& \left.+\left(\frac{3}{4}C_A+\frac{3}{4}C_R\right)L_\mu^2\right\} \\
& +\left(\frac{\alpha_e}{\pi}\right)^2 n_h I_2(R) \\
& \times\left\{\frac{1}{\epsilon^2}\left(\frac{1}{4}C_A-\frac{1}{2}C_R-\frac{1}{2}T_f\right)\right. \\
& +\frac{1}{\epsilon}\left[\frac{1}{2}C_R-\frac{1}{2}\left(1+L_\mu\right)T_f\right]-\frac{1}{2}C_A \\
& +\frac{5}{2}C_R-\left(\frac{1}{2}+\frac{1}{24}\pi^2\right)T_f \\
& -\left(\frac{1}{2}C_A-2C_R+\frac{1}{2}T_f\right)L_\mu \\
& \left.-\left(\frac{1}{4}C_A-\frac{1}{2}C_R+\frac{1}{4}T_f\right)L_\mu^2\right\} \\
& +\frac{\alpha_e}{\pi}\frac{\eta_1}{\pi} n_h \\
& \times\left[\frac{3}{16}\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\frac{3}{16}+\frac{3}{8}L_\mu\right)\right. \\
& \left.+\frac{3}{16}+\frac{1}{32}\pi^2+\frac{3}{8}L_\mu+\frac{3}{8}L_\mu^2\right] \\
& -\frac{\alpha_e}{\pi}\frac{\eta_2}{\pi} n_h
\end{aligned}
$$

$$\times\left[\frac{5}{4}\frac{1}{\epsilon^2}+\left(5+\frac{5}{2}L_\mu\right)\frac{1}{\epsilon}+\frac{25}{2}+\frac{5}{24}\pi^2+10L_\mu+\frac{5}{2}L_\mu^2\right]$$
$$-\frac{\alpha_e}{\pi}\frac{\eta_3}{\pi}n_h\times\left[\frac{7}{16}\frac{1}{\epsilon^2}+\left(\frac{7}{16}+\frac{7}{8}L_\mu\right)\frac{1}{\epsilon}+\frac{7}{16}+\frac{7}{96}\pi^2+\frac{7}{8}L_\mu+\frac{7}{8}L_\mu^2\right], \tag{68}$$

where $L_\mu = \ln(\mu^2/M_f^2)$, $T_f = n_f I_2(R)$, where n_f and n_h denote the number of fermions and heavy fermions, respectively. The overall factor n_h in front of the one- and two-loop corrections shows that the renormalization of m_ε is only influenced by those diagrams which contain a closed heavy fermion loop.

It is also possible to renormalize m_ε so that $m_\varepsilon^{\mathrm{OS}} \neq 0$ or adopt the $\overline{\mathrm{DR}}$ scheme for it. In the latter case, the physical observables will depend on m_ε. In order to get rid of this unphysical dependence, one has to introduce additional finite shifts in the renormalization constants of the physical parameters. This new renormalization scheme is called $\overline{\mathrm{DR}}'$ and it will be discussed in more detail in the next section. Nevertheless, in context of QCD, the $\overline{\mathrm{DR}}'$ has rarely been used [152].

4. Dimensional Reduction Applied to SUSY-QCD at Three Loops

All the appealing features of supersymmetric theories have to be confirmed by an accurate comparison with the experimental data like those measured in collider experiments [31, 32, 153]. Such an ambitious task requires precision data as well as precision calculations. But, precise predictions for observables implies computations of higher order radiative corrections. Thus, it necessarily rises the question of constructing regularization and renormalization schemes that are gauge and SUSY invariant. As discussed in the previous sections, DRED scheme was proposed as a solution, although it could violate SUSY at higher orders of perturbation theory. Currently, it is believed that DRED preserves SUSY at three-loop order as was explicitly checked in [101–103] and that it breaks SUSY at four-loop order, taking into consideration formal arguments [60, 74]. Nevertheless, renormalization by combining DRED with minimal subtraction (the $\overline{\mathrm{DR}}$ scheme) or the on-shell scheme has become the preferred schemes in higher order supersymmetric calculations [100, 129, 154–156].

4.1. Renormalization of the Gauge Coupling and Fermion Masses at Three Loops. As was already reviewed in Section 2, for supersymmetric gauge theories one can devise a particular renormalization scheme, the so-called NSVZ scheme [157], where an all-order relation between the gauge β function and the anomalous dimension of the chiral supermultiplet is valid. So, in the absence of the matter supermultiplet, that is, for SUSY-Yang-Mills theory, the β function is known to all orders in the coupling constant. Applying the same method based on the connection between the holomorphic and the NSVZ schemes to softly broken SUSY gauge theory, the authors of [44] derived the renormalization group equation governing the running of the gaugino and sfermion masses as functions of the gauge and Yukawa coupling β functions, valid to all orders in perturbation theory. Actually, all these calculations received important phenomenological applications only after the authors of [64] found the three-loop conversion formula between the NSVZ and $\overline{\mathrm{DR}}$. This allowed the derivation of three-loop order beta functions for the parameters of the MSSM in the $\overline{\mathrm{DR}}$ scheme [158, 159].

The goal of this section is to report on another confirmation of the results for the anomalous dimensions of SUSY-QCD parameters, that is based on a direct calculation of relevant three-loop Feynman diagrams implementing the DRED approach in the component field formalism. The agreement of the two independent and conceptually completely different calculations is a very important check of the two methods on the one side, and on the other side it establishes the DRED as a consistent framework for computations of radiative corrections in supersymmetric theories.

The renormalized Lagrangian of a supersymmetric theory will obey SUSY constraints, only if the decomposition of (33) holds at all orders of perturbation theory. Therefore, the renormalized gluon and ε-scalar coupling constants must be equal, that is, their β functions must be the same. An all-order proof of this statement is currently not available. However, it was explicitly shown [103] that the coupling constant arising from the vertices $gc\bar{c}$, ggg, $gq\bar{q}$, and $\tilde{g}q\bar{q}$ and that from the vertices $q\bar{q}\epsilon$, $\tilde{g}\tilde{g}\epsilon$, and $g\epsilon\epsilon$ are equal through three loops.

Even more, in order to renormalize the quartic ε-scalar vertex, one has to take into account all possible color structures for it and attribute to each one a separate coupling constant (for details see Sections 3.3.1 and 3.3.4). For SUSY-QCD, it has been explicitly checked [135] that at the one-loop order only the β function associated with the usual color structure of the four-gluon interaction, that is, $f_{abe}f_{cde}$ (f_{abe} denotes the structure constants of the gauge group), does not vanish and it equals the one loop gauge β function. Thus, through one-loop, one can identify the coupling constant of the corresponding ε-scalar quartic interaction with the strong coupling constant and set to zero the other three quartic couplings. This order of accuracy is sufficient for the results discussed here, as the ε-scalar quartic interactions contribute to the anomalous dimensions starting from the two-loop order. A similar observation was made also in the previous section when the SUSY-Yang-Mills theory was discussed at four-loop accuracy. All these tests confirm the consistency of DRED with SUSY at next-to-next-to-next-to-leading order (NNNLO) of perturbation theory.

For the calculation of renormalization constants within supersymmetric theories one can apply the same methods as the ones discussed in Section 3.3.2 in the context of non-SUSY theories. Let us however mention at this point a technical subtlety related to the implementation of γ_5 matrix.

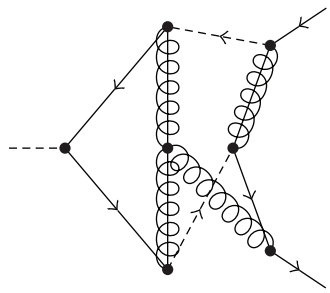

Figure 2: Sample diagram for the three-loop $q\bar{q}\epsilon$ vertex where a nonvanishing trace with a single γ_5-matrix occurs. Solid lines are quarks, dashed lines are squarks, slashed springy lines are gluinos, and the external dashed line depicts an ε-scalar. The arrows on the lines denote the charge flow.

Traces with a single γ_5 and at least four γ-matrices do not contribute to any of the two-point functions (for a detailed discussion about this aspect see [160]). They do contribute for some of the three-point functions though, in particular the $q\bar{q}\bar{g}$, the $\bar{g}\bar{g}\epsilon$, and the $q\bar{q}\epsilon$ vertex. An example diagram for the latter vertex is shown in Figure 2. Such diagrams contribute (among others) a color factor $d_R^{abcd}d_A^{abcd}$ (for the notation, see Appendix A), but they cancel against the same factors from other sources in the final result for the renormalization constants and the β functions. Precisely, the naive scheme for the implementation of the γ_5 gives rise to incorrect results. One has to supplement it with the relations given in (27) and (28). The first equation takes into account the contributions arising in triangle diagrams containing Dirac fermions, whereas the second one generalizes the contraction properties of the pseudo-Levi-Civita tensors defined away from $d = 4$ dimensions.

The results for the three-loop renormalization constants of the gauge coupling constant α_s are very compact and are given by

$$\begin{aligned}
Z_s &= 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left[-3C_A + 2T_f\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \\
&\times \left\{\frac{1}{\epsilon^2}\left[9C_A^2 - 12C_AT_f + 4T_f^2\right]\right. \\
&\quad \left. + \frac{1}{\epsilon}\left[-3C_A^2 + 2C_AT_f + 4C_RT_f\right]\right\} + \left(\frac{\alpha_s}{4\pi}\right)^3 \\
&\times \left\{\frac{1}{\epsilon^3}\left[-27C_A^3 + 54C_A^2T_f - 36C_AT_f^2 + 8T_f^3\right]\right. \\
&\quad + \frac{7}{3\epsilon^2}\left[9C_A^3 - 12C_A^2T_f - 12C_AC_RT_f\right. \\
&\qquad \left. + 4C_AT_f^2 + 8C_RT_f^2\right] + \frac{1}{3\epsilon} \\
&\quad \times \left[-21C_A^3 + 20C_A^2T_f + 52C_AC_RT_f\right. \\
&\qquad \left.\left. -16C_R^2T_f - 4C_AT_f^2 - 24C_RT_f^2\right]\right\},
\end{aligned} \tag{69}$$

where we have introduced the notation $T_f = I_2(R)n_f$, with n_f being the number of active fermions of the theory, and the invariants C_R, C_A, $I_2(R)$ are explicitly given in the Appendix A.

The case $T_f = 0$ corresponds to SUSY-Yang-Mills theory that has been treated in detail in Sections 2 and 3.3.4. Full agreement has been found between the two methods up to three-loop order.

The three-loop renormalization constants for the gluino mass read

$$\begin{aligned}
Z_{m_{\tilde{g}}} &= 1 + \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}\left[-3C_A + 2T_f\right] + \left(\frac{\alpha_s}{4\pi}\right)^2 \\
&\times \left\{\frac{1}{\epsilon^2}\left[9C_A^2 - 12C_AT_f + 4T_f^2\right]\right. \\
&\quad \left. + \frac{2}{\epsilon}\left[-3C_A^2 + 2C_AT_f + 4C_RT_f\right]\right\} + \left(\frac{\alpha_s}{4\pi}\right)^3 \\
&\times \left\{\frac{1}{\epsilon^3}\left[-27C_A^3 + 54C_A^2T_f - 36C_AT_f^2 + 8T_f^3\right]\right. \\
&\quad + \frac{4}{\epsilon^2}\left(9C_A^3 - 12C_A^2T_f - 12C_AC_RT_f\right. \\
&\qquad \left. + 4C_AT_f^2 + 8C_RT_f^2\right) + \frac{1}{\epsilon} \\
&\quad \times \left(-21C_A^3 + 20C_A^2T_f + 52C_AC_RT_f\right. \\
&\qquad \left.\left. -16C_R^2T_f - 4C_AT_f^2 - 24C_RT_f^2\right)\right\}.
\end{aligned} \tag{70}$$

The $\overline{\text{DR}}$ quark mass renormalization constant is also independent of any mass parameter and is given by the following formula:

$$\begin{aligned}
Z_{m_q} &= 1 - \frac{\alpha_s}{4\pi}\frac{1}{\epsilon}2C_R + \left(\frac{\alpha_s}{4\pi}\right)^2 \\
&\times \left\{\frac{1}{\epsilon^2}\left[3C_AC_R + 2C_R^2 - 2C_RT_f\right]\right. \\
&\quad \left. + \frac{1}{\epsilon}\left[-3C_AC_R + 2C_R^2 + 2C_RT_f\right]\right\} \\
&+ \left(\frac{\alpha_s}{4\pi}\right)^3 \\
&\times \left\{\frac{1}{\epsilon^3}\left[-6C_A^2C_R - 6C_AC_R^2 - \frac{4}{3}C_R^3\right.\right. \\
&\quad \left. + \left(8C_AC_R + 4C_R^2\right)T_f - \frac{8}{3}C_RT_f^2\right] \\
&\quad \times \frac{1}{\epsilon^2}\left[10C_A^2C_R + 2C_AC_R^2 - 4C_R^3\right. \\
&\qquad \left. + \left(-\frac{32}{3}C_AC_R - \frac{20}{3}C_R^2\right)T_f + \frac{8}{3}C_RT_f^2\right]
\end{aligned}$$

$$\times \frac{1}{\epsilon}\left[-4C_A^2C_R + 4C_AC_R^2 - \frac{16}{3}C_R^3 + \frac{8}{3}C_RT_f^2 + T_f\left(C_R^2\left(\frac{32}{3} - 16\zeta(3)\right) + C_AC_R\left(-\frac{4}{3} + 16\zeta(3)\right)\right)\right\}, \quad (71)$$

where $\zeta(3)$ is Riemann's zeta function with $\zeta(3) = 1.20206\ldots$. The results of ((69), (71), (70)) are in agreement with [63, 64]. Using (69) and (70), it is an easy exercise to confirm the relation derived in [61] between the anomalous dimension of gluino mass and the gauge β function that holds also in DRED. This result is similar to the NSVZ relation given in (17) and holds to all orders in perturbation theory. It reads

$$\gamma_n^{\tilde{g}} = (n+1)\,\beta_n, \quad (72)$$

where n denotes the number of loops.

4.2. Renormalization of the Squark Sector at Three Loops. In this section, we report on the renormalization of the squark sector of SUSY-QCD up to three-loop order within the $\overline{\text{DR}}$ scheme in the component field approach [161]. These results are on the one side important for the phenomenological analyses aiming to predict the squark masses at the TeV scale with an accuracy of the order of $\mathcal{O}$ (50 GeV), that is required by the precision achieved in the current experimental searches at the LHC. On the other side, they have also genuine theoretical significance, since they provide an independent confirmation of the three-loop results obtained with the help of the NSVZ scheme [47, 48, 158, 159].

The calculations presented in this section are performed in the framework of SUSY-QCD with $n_q = 5$ massless quarks and a massive top quark (m_t). The scalar superpartners of the latter has two mass eigenstates ($m_{\tilde{t}_1}$ and $m_{\tilde{t}_2}$) which may have different masses and thus a nonvanishing mixing angle occurs. The superpartners of the n_q light quarks are assumed to have degenerate masses ($m_{\tilde{q}}$) and vanishing mixing angles. A generalization to a nondegenerate spectrum is possible in a straightforward way from the formalism for the top squark sector which is discussed in detail in the following.

Unless stated otherwise all parameters in the following derivation are $\overline{\text{DR}}$ quantities which depend on the renormalization scale μ. For the sake of compactness the latter is omitted. Bare quantities are marked by a superscript "(0)". To define the framework, we start from the bare Lagrangian containing the kinetic energy and the mass terms for the top squarks:

$$\mathscr{L}_{\tilde{t}}^{(0)} = \frac{1}{2}\partial_\mu(\tilde{t}_L^*, \tilde{t}_R^*)^{(0)}\partial^\mu\begin{pmatrix}\tilde{t}_L\\ \tilde{t}_R\end{pmatrix}^{(0)} - \frac{1}{2}(\tilde{t}_L^*, \tilde{t}_R^*)^{(0)}\left(\mathscr{M}_{\tilde{t}}^2\right)^{(0)}\begin{pmatrix}\tilde{t}_L\\ \tilde{t}_R\end{pmatrix}^{(0)}, \quad (73)$$

where $\tilde{t}_L$ and $\tilde{t}_R$ denote the interaction eigenstates. The top squark mass matrix is given by

$$\mathscr{M}_{\tilde{t}}^2 = \begin{pmatrix} m_t^2 + M_Z^2\left(\frac{1}{2} - \frac{2}{3}\sin^2\vartheta_W\right)\cos 2\beta + M_{\widetilde{Q}}^2 & m_t\left(A_t - \mu_{\text{SUSY}}\cot\beta\right) \\ m_t\left(A_t - \mu_{\text{SUSY}}\cot\beta\right) & m_t^2 + \frac{2}{3}M_Z^2\sin^2\vartheta_W\cos 2\beta + M_{\widetilde{U}}^2 \end{pmatrix} \equiv \begin{pmatrix} m_{\tilde{t}_L}^2 & m_tX_t \\ m_tX_t & m_{\tilde{t}_R}^2 \end{pmatrix} \quad (74)$$

with $X_t = A_t - \mu_{\text{SUSY}}\cot\beta$. A_t is the soft SUSY breaking trilinear coupling, and $M_{\widetilde{U}}$ and $M_{\widetilde{Q}}$ are the soft SUSY breaking masses.

The top squark mass eigenstates are related to the interaction eigenstates through the unitary transformation

$$\begin{pmatrix}\tilde{t}_1\\ \tilde{t}_2\end{pmatrix}^{(0)} = \mathscr{R}_{\tilde{t}}^{(0)\dagger}\begin{pmatrix}\tilde{t}_L\\ \tilde{t}_R\end{pmatrix}^{(0)}. \quad (75)$$

The unitary matrix $\mathscr{R}_{\tilde{t}}$ is defined through the diagonalization relation for the mass matrix $\mathscr{M}_{\tilde{t}}^2$

$$\begin{pmatrix} m_{\tilde{t}_1}^2 & 0 \\ 0 & m_{\tilde{t}_2}^2 \end{pmatrix} = R_{\tilde{t}}^\dagger\mathscr{M}_{\tilde{t}}^2R_{\tilde{t}}. \quad (76)$$

The eigenvalues are the masses of the eigenstates $\tilde{t}_1$ and $\tilde{t}_2$. They read

$$m_{\tilde{t}_{1,2}}^2 = \frac{1}{2}\left[m_{\tilde{t}_L}^2 + m_{\tilde{t}_R}^2 \mp \sqrt{\left(m_{\tilde{t}_L}^2 - m_{\tilde{t}_R}^2\right)^2 + 4m_t^2X_t^2}\right]. \quad (77)$$

The unitary transformation can be parameterized by the mixing angle

$$R_{\tilde{t}} = \begin{pmatrix}\cos\theta_t & -\sin\theta_t\\ \sin\theta_t & \cos\theta_t\end{pmatrix}, \quad (78)$$

with

$$\sin(2\theta_t) = \frac{2m_t\left(A_t - \mu_{\text{SUSY}}\cot\beta\right)}{m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2}. \quad (79)$$

The renormalization constants connected to the top squark are extracted from the top squark propagator. At tree level it is

a diagonal 2×2 matrix which receives non-diagonal entries at loop-level. In order to be able to write down the renormalized top squark propagator we define the renormalization constants as follows. The wave function renormalization constant is introduced through the relation

$$\begin{pmatrix}\tilde{t}_1\\ \tilde{t}_2\end{pmatrix}^{(0)} = \mathcal{Z}_{\tilde{t}}^{1/2}\begin{pmatrix}\tilde{t}_1\\ \tilde{t}_2\end{pmatrix}, \quad \text{with } \mathcal{Z}_{\tilde{t}}^{1/2} = \begin{pmatrix}Z_{11}^{1/2} & Z_{12}^{1/2}\\ Z_{21}^{1/2} & Z_{22}^{1/2}\end{pmatrix}, \tag{80}$$

where it holds $\mathcal{Z}_{\tilde{t}}^{1/2} = \mathbf{I} + \mathcal{O}(\alpha_s)$.

In case of SUSY-QCD, the matrix $\mathcal{Z}_{\tilde{t}}^{1/2}$ has a particularly symmetric form. This can be derived from the observation that the left- and right-handed components of the top squark fields have the same renormalization constant for their wave functions within SUSY-QCD

$$\begin{pmatrix}\tilde{t}_L\\ \tilde{t}_R\end{pmatrix}^{(0)} = \widetilde{Z}_2^{1/2}\begin{pmatrix}\tilde{t}_L\\ \tilde{t}_R\end{pmatrix}. \tag{81}$$

Furthermore, if we introduce the renormalization constant for the mixing angle via

$$\theta_t^{(0)} = \theta_t + \delta\theta_t \tag{82}$$

and make use of (75), we obtain

$$\mathcal{Z}_{\tilde{t}}^{1/2} = \widetilde{Z}_2^{1/2}\begin{pmatrix}\cos\delta\theta_t & \sin\delta\theta_t\\ -\sin\delta\theta_t & \cos\delta\theta_t\end{pmatrix}. \tag{83}$$

When supersymmetric electroweak (SUSY-EW) corrections are taken into account, (81) becomes

$$\begin{pmatrix}\tilde{t}_L\\ \tilde{t}_R\end{pmatrix}^{(0)} = \begin{pmatrix}\widetilde{Z}_L^{1/2} & 0\\ 0 & \widetilde{Z}_R^{1/2}\end{pmatrix}\begin{pmatrix}\tilde{t}_L\\ \tilde{t}_R\end{pmatrix}. \tag{84}$$

This assignment takes into account supersymmetric constraints [162, 163] and is sufficient to absorb all divergences. As a consequence also the matrix $\mathcal{Z}_{\tilde{t}}^{1/2}$ has a more complicated structure and additional renormalization conditions are required.

Furthermore, the mass matrix equation (74) has to be renormalized. It can be parameterized as follows:

$$\begin{pmatrix}\left(m_{\tilde{t}_1}^{(0)}\right)^2 & 0\\ 0 & \left(m_{\tilde{t}_2}^{(0)}\right)^2\end{pmatrix} \longrightarrow \begin{pmatrix}m_{11}^2 Z_{m_{11}} & m_{12}^2 Z_{m_{12}}\\ m_{21}^2 Z_{m_{21}} & m_{22}^2 Z_{m_{22}}\end{pmatrix} \equiv \mathcal{M}, \tag{85}$$

where we require that the off-diagonal elements in the renormalized mass matrix vanish. This ensures that the renormalized fields are the true mass eigenstates. As a consequence, the counterterm $\delta\theta_t$ takes care of the divergences in the self-energy contribution where a $\tilde{t}_1$ transforms into a $\tilde{t}_2$ or vice versa. This can be seen in the explicit formulae given below. The diagonal elements of (85) can be identified with the renormalization constants of the masses:

$$\left(m_{\tilde{t}_i}^{(0)}\right)^2 = m_{ii}^2 Z_{m_{ii}} = m_{\tilde{t}_i}^2 Z_{m_{\tilde{t}_i}}. \tag{86}$$

In order to formulate the renormalization conditions it is convenient to consider the renormalized inverse top squark propagator given by

$$i\mathcal{S}^{-1}\left(p^2\right) = p^2\left(\mathcal{Z}_{\tilde{t}}^{1/2}\right)^\dagger \mathcal{Z}_{\tilde{t}}^{1/2} - \left(\mathcal{Z}_{\tilde{t}}^{1/2}\right)^\dagger\left[\mathcal{M} - \Sigma\left(p^2\right)\right]\mathcal{Z}_{\tilde{t}}^{1/2}, \tag{87}$$

where

$$\Sigma\left(p^2\right) = \begin{pmatrix}\Sigma_{11}\left(p^2\right) & \Sigma_{12}\left(p^2\right)\\ \Sigma_{21}\left(p^2\right) & \Sigma_{22}\left(p^2\right)\end{pmatrix}, \tag{88}$$

stands for the matrix of the squark self-energies in the mass eigenstate basis.

In the $\overline{\text{DR}}$ scheme the renormalization conditions read

$$\mathcal{S}_{ij}^{-1}\left(p^2\right)\Big|_{\text{pp}} = 0, \tag{89}$$

where "pp" stands for the "pole part."

In order to obtain explicit formulae for the evaluation of the renormalization constants it is convenient to define perturbative expansions of the quantities entering equation (89). Up to three-loop order we have

$$\begin{aligned}
Z_k &= 1 + \left(\frac{\alpha_s}{\pi}\right)\delta Z_k^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\delta Z_k^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3\delta Z_k^{(3)} + \mathcal{O}\left(\alpha_s^4\right),\\
\delta\theta_t &= \left(\frac{\alpha_s}{\pi}\right)\delta\theta_t^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\delta\theta_t^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3\delta\theta_t^{(3)} + \mathcal{O}\left(\alpha_s^4\right),\\
\Sigma_{ij} &= \left(\frac{\alpha_s}{\pi}\right)\Sigma_{ij}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2\Sigma_{ij}^{(2)} + \left(\frac{\alpha_s}{\pi}\right)^3\Sigma_{ij}^{(3)} + \mathcal{O}\left(\alpha_s^4\right),
\end{aligned} \tag{90}$$

where $i, j \in \{1, 2\}$, and $k \in \{2, m_{\tilde{t}_1}, m_{\tilde{t}_2}\}$. Inserting these equations into (87) one can solve (89) iteratively order by order in α_s. At one-loop order one gets

$$\begin{aligned}
\left\{\Sigma_{ii}^{(1)} - m_{\tilde{t}_i}^2\left(\delta\widetilde{Z}_2^{(1)} + \delta Z_{m_{\tilde{t}_i}}^{(1)}\right) + p^2\delta\widetilde{Z}_2^{(1)}\right\}\Big|_{\text{pp}} &= 0, \quad i = 1, 2,\\
\left\{\Sigma_{12}^{(1)} - \delta\theta_t^{(1)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)\right\}\Big|_{\text{pp}} &= 0.
\end{aligned} \tag{91}$$

The terms proportional to p^2 in the first equation of (91) are used to compute the wave function renormalization constant which is independent of all occurring masses. Thus they can be set to zero and one obtains

$$\delta\widetilde{Z}_2^{(1)} = -\frac{1}{p^2}\Sigma_{11}^{(1)}\left(p^2\right)\Big|_{\text{pp}} = -\frac{1}{p^2}\Sigma_{22}^{(1)}\left(p^2\right)\Big|_{\text{pp}}. \tag{92}$$

Once $\delta\widetilde{Z}_2^{(1)}$ is known (91) is used to obtain $\delta Z_{m_{\tilde{t}_i}}^{(1)}$ keeping the mass dependence in $\Sigma_{ii}^{(1)}$ (see below for more details). The second equation of (91) is used to obtain the renormalization constant of the mixing angle via

$$\delta\theta_t^{(1)} = \left.\frac{\Sigma_{12}^{(1)}}{m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2}\right|_{\text{pp}}. \tag{93}$$

Proceeding to two loops we obtain the equations

$$\Big[\Sigma_{ii}^{(2)} + \delta\widetilde{Z}_2^{(1)}\Sigma_{ii}^{(1)} - m_{\tilde{t}_i}^2\left(\delta\widetilde{Z}_2^{(2)} + \delta\widetilde{Z}_2^{(1)}\delta Z_{m_{\tilde{t}_i}}^{(1)} + \delta Z_{m_{\tilde{t}_i}}^{(2)}\right) + \delta\widetilde{Z}_2^{(2)}p^2 + (-1)^{(i+1)}\delta\theta_t^{(1)} \times\left(-2\Sigma_{12}^{(1)} + \delta\theta_t^{(1)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)\right)\Big]\Big|_{\mathrm{pp}} = 0, \quad i = 1, 2,$$

$$\Big[-\delta\theta_t^{(2)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) - \delta\theta_t^{(1)}\delta\widetilde{Z}_2^{(1)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) - \delta\theta_t^{(1)}\delta Z_{m_{\tilde{t}_1}}^{(1)}m_{\tilde{t}_1}^2 + \delta\theta_t^{(1)}\delta Z_{m_{\tilde{t}_2}}^{(1)}m_{\tilde{t}_2}^2 + \delta\theta_t^{(1)}\Sigma_{11}^{(1)} - \delta\theta_t^{(1)}\Sigma_{22}^{(1)} + \delta\widetilde{Z}_2^{(1)}\Sigma_{12}^{(1)} + \Sigma_{12}^{(2)}\Big]\Big|_{\mathrm{pp}} = 0, \tag{94}$$

which are solved for $\widetilde{Z}_2^{(2)}$, $\delta Z_{m_{\tilde{t}_i}}^{(2)}$, and $\delta\theta_t^{(2)}$ using the same strategy as at one-loop level.

Similarly, at three-loop order we have

$$\begin{aligned}
&\Big[(-1)^{i+1}\Big\{\left(\delta\theta_t^{(1)}\right)^2 \times\left(\delta\widetilde{Z}_2^{(1)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + \delta Z_{m_{\tilde{t}_1}}^{(1)}m_{\tilde{t}_1}^2 - \delta Z_{m_{\tilde{t}_2}}^{(1)}m_{\tilde{t}_2}^2 - \Sigma_{11}^{(1)} + \Sigma_{22}^{(1)}\right) \\
&\quad + \delta\theta_t^{(1)}\left(2\delta\theta_t^{(2)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) - 2\delta\widetilde{Z}_2^{(1)}\Sigma_{12}^{(1)} - 2\Sigma_{12}^{(2)}\right) - 2\delta\theta_t^{(2)}\Sigma_{12}^{(1)}\Big\} \\
&\quad + \delta\widetilde{Z}_2^{(1)}\left(\Sigma_{ii}^{(2)} - \delta Z_{m_{\tilde{t}_i}}^{(2)}m_{\tilde{t}_i}^2\right) - \delta\widetilde{Z}_2^{(2)}\delta Z_{m_{\tilde{t}_i}}^{(1)}m_{\tilde{t}_i}^2 + \delta\widetilde{Z}_2^{(2)}\Sigma_{ii}^{(1)} - \delta\widetilde{Z}_2^{(3)}m_{\tilde{t}_i}^2 + \delta\widetilde{Z}_2^{(3)}p^2 - \delta Z_{m_{\tilde{t}_i}}^{(3)}m_{\tilde{t}_i}^2 + \Sigma_{ii}^{(3)}\Big]\Big|_{\mathrm{pp}} = 0, \quad i = 1, 2, \\
&\Big[\delta\theta_t^{(1)}\left(-\delta\widetilde{Z}_2^{(1)}\delta Z_{m_{\tilde{t}_1}}^{(1)}m_{\tilde{t}_1}^2 + \delta\widetilde{Z}_2^{(1)}\delta Z_{m_{\tilde{t}_2}}^{(1)}m_{\tilde{t}_2}^2 + \delta\widetilde{Z}_2^{(1)}\Sigma_{11}^{(1)} - \delta\widetilde{Z}_2^{(1)}\Sigma_{22}^{(1)} - \delta\widetilde{Z}_2^{(2)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) - \delta Z_{m_{\tilde{t}_1}}^{(2)}m_{\tilde{t}_1}^2 + \delta Z_{m_{\tilde{t}_2}}^{(2)}m_{\tilde{t}_2}^2 + \Sigma_{11}^{(2)} - \Sigma_{22}^{(2)}\right) \\
&\quad + \delta\theta_t^{(2)}\left(-\delta\widetilde{Z}_2^{(1)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) - \delta Z_{m_{\tilde{t}_1}}^{(1)}m_{\tilde{t}_1}^2 + \delta Z_{m_{\tilde{t}_2}}^{(1)}m_{\tilde{t}_2}^2 + \Sigma_{11}^{(1)} - \Sigma_{22}^{(1)}\right) \\
&\quad - \delta\theta_t^{(3)}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + \delta\widetilde{Z}_2^{(1)}\Sigma_{12}^{(2)} + \delta\widetilde{Z}_2^{(2)}\Sigma_{12}^{(1)} + \Sigma_{12}^{(3)} + \frac{2}{3}\left(\delta\theta_t^{(1)}\right)^3\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) - 2\left(\delta\theta_t^{(1)}\right)^2\Sigma_{12}^{(1)}\Big]\Big|_{\mathrm{pp}} = 0.
\end{aligned} \tag{95}$$

Sample diagrams contributing to Σ_{11} up to three loops can be found in Figure 3; the contributions to Σ_{12}, and Σ_{22} look very similar. Once the quantities Σ_{11}, Σ_{12} and Σ_{22} are known to three-loop order it is possible to extract the renormalization constants for the squark wave function and mass and the mixing angle from (95).

As compared to the corresponding self-energy contributions for fermions or gauge bosons, which after proper projection only lead to logarithmically divergent integrals, the quantities in the above equations have mass dimension two. As a consequence the renormalization constants of the squark masses and the mixing angles depend on the occurring masses, even in a minimal subtraction scheme like $\overline{\mathrm{DR}}$. At three-loop order an exact evaluation of the corresponding integrals is not possible. It is nevertheless possible to reconstruct the complete dependence on the occurring masses using repeated asymptotic expansions and in addition some knowledge about the structure of the final result. Thus, one has to keep during the calculation nonvanishing squark, gluino, and the top quark masses and chose convenient hierarchies between them. For the asymptotic expansion (see, e.g., [150]) one can use `exp` [142, 143]. As a result only one-scale integrals up to three loops appear which can be evaluated with the packages `MINCER` [115] and `MATAD` [124].

After the calculation of the bare self-energies one has to renormalize all occurring parameters in the $\overline{\mathrm{DR}}$ scheme. For the three-loop calculation one needs the counterterms for α_s, m_t, $m_{\tilde{g}}$, $m_{\tilde{t}_i}$, θ_t, and m_ϵ to two-loop order and the one for $m_{\tilde{q}}$ to one-loop approximation. Furthermore, also the QCD gauge parameter has to be renormalized to two loops since it appears in the results for the wave function anomalous dimensions.

At this point some comments on the treatment of the ϵ-scalar mass, m_ϵ, are in order. In practice there are two renormalization schemes for m_ϵ which are frequently used, the $\overline{\mathrm{DR}}$ and on-shell scheme. In the latter one requires that the renormalized mass vanishes in each order in perturbation theory whereas in the $\overline{\mathrm{DR}}$ prescription only the pole parts are subtracted by the renormalization constant. In the $\overline{\mathrm{DR}}$ scheme it is important to keep m_ϵ different from zero since the renormalization group equations for the squark masses and m_ϵ are coupled. A nonvanishing ϵ-scalar mass in intermediate steps is also required for the computation of the anomalous dimensions in the $\overline{\mathrm{DR}}'$ scheme [164] (see below) which was constructed in order to disentangle the running of m_ϵ from the one of the squark parameters.

In the following, we present only the results derived in the scheme where the ϵ-scalar mass is renormalized in

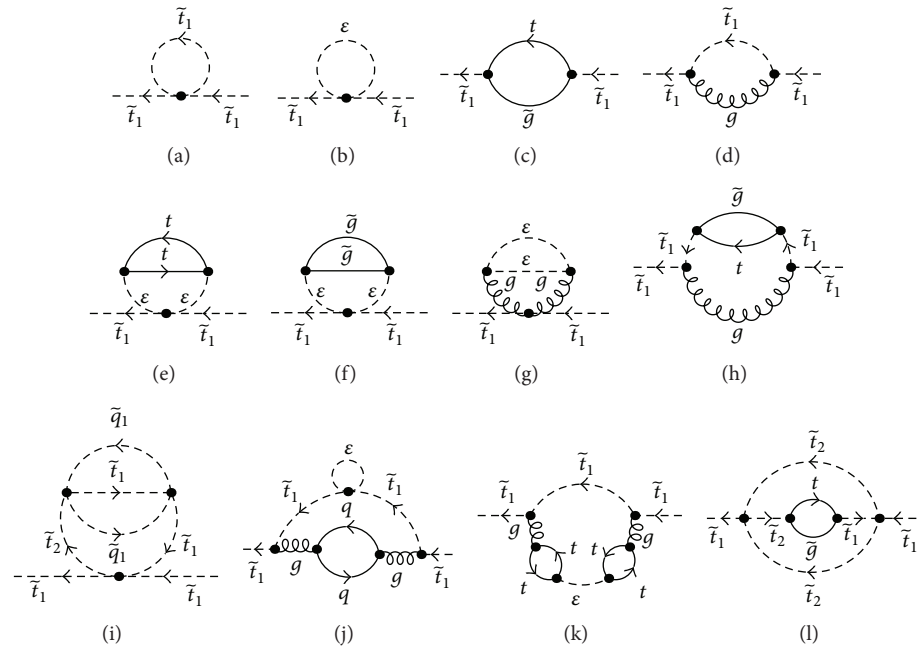

FIGURE 3: Sample diagrams contributing to Σ_{11} at one, two, and three loops. The symbols t, $\tilde{t}_i$, g, $\tilde{g}$, and ϵ denote top quarks, top squarks, gluons, gluinos, and ϵ-scalars, respectively.

$\overline{\text{DR}}$ scheme. The two-loop results for the renormalization constants of the top squark mass $m_{\tilde{t}_1}$ read

$$m_{\tilde{t}_1}^2 \delta Z_{m_{\tilde{t}_1}}^{(1)} = C_R \left(-m_{\tilde{g}}^2 - m_t^2 + m_{\tilde{g}} m_t s_{2t} + \frac{m_{\tilde{t}_2}^2 - m_{\tilde{t}_1}^2}{4} s_{2t}^2 \right) \frac{1}{\epsilon},$$

$$\begin{aligned} m_{\tilde{t}_1}^2 \delta Z_{m_{\tilde{t}_1}}^{(2)} &= \left\{ C_R^2 \left[\frac{c_{2t}^2 m_{\tilde{g}}^2 m_t^2}{m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2} + \frac{\left(1 + c_{2t}^2\right) s_{2t}^2 \left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + 8 m_t^2}{16} \right.\right. \\ &\quad \left.\left. - \frac{\left(1 + c_{2t}^2\right) m_{\tilde{g}} m_t s_{2t}}{2} \right] \right. \\ &\quad + C_A C_R \\ &\quad \times \left[\frac{9 m_{\tilde{g}}^2}{8} + 3 \frac{s_{2t}^2 \left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + 4 m_t^2}{32} - \frac{3 m_{\tilde{g}} m_t s_{2t}}{4} \right] \\ &\quad + C_R T_f \left[\frac{-3 m_{\tilde{g}}^2}{4} - \frac{s_{2t}^2 \left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + 4 m_t^2}{16} \right. \\ &\quad \left.\left. + \frac{m_{\tilde{g}} m_t s_{2t}}{2} \right] \right\} \frac{1}{\epsilon^2} \\ &\quad + \left\{ C_R^2 \left[\frac{3 m_{\tilde{g}}^2}{4} + \frac{s_{2t}^2 \left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + 4 m_t^2}{16} - \frac{m_{\tilde{g}} m_t s_{2t}}{2} \right] \right. \\ &\quad + C_A C_R \left[\frac{-11 m_{\tilde{g}}^2}{8} - 3 \frac{s_{2t}^2 \left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + 4 m_t^2}{32} \right. \\ &\quad \left. + \frac{3 m_{\tilde{g}} m_t s_{2t}}{4} \right] \\ &\quad + C_R T_q \left[\frac{3 m_{\tilde{g}}^2}{4} + \frac{s_{2t}^2 \left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + 8 m_{\tilde{q}}^2 + 4 m_t^2}{16} \right. \\ &\quad \left. - \frac{m_{\tilde{g}} m_t s_{2t}}{2} \right] \\ &\quad + C_R T_t \\ &\quad \times \left[\frac{3 m_{\tilde{g}}^2}{4} + \frac{s_{2t}^2 \left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right) + 4 m_{\tilde{t}_1}^2 + 4 m_{\tilde{t}_2}^2 - 4 m_t^2}{16} \right. \\ &\quad \left.\left. - \frac{m_{\tilde{g}} m_t s_{2t}}{2} \right] \right\} \frac{1}{\epsilon} \\ &\quad + m_\varepsilon^2 \left(-C_A C_R \frac{3}{8} + C_R T_f \frac{1}{4} \right) \frac{1}{\epsilon}, \end{aligned} \tag{96}$$

where we have introduced the abbreviations $T_l = n_l I_2(R)$, with $l = f, q, t$ and $c_{nt} = \cos(n\theta_t)$ and $s_{nt} = \sin(n\theta_t)$. n_q denotes the number of light quark flavors and takes in this

case the value $n_q = 5$. $n_t = 1$ has been introduced for convenience and it holds $n_f = n_q + n_t$. Furthermore m_ε denotes the $\overline{\text{DR}}$ renormalized ε-scalar mass. The corresponding results for $m_{\tilde{t}_2}$ can be derived from (96) by interchanging $m_{\tilde{t}_1}$ and $m_{\tilde{t}_2}$ and changing the sign of θ_t.

Finally, for the mixing angle we have

$$\begin{aligned}
&\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)\delta\theta_t^{(1)} = C_R c_{2t}\left(m_{\tilde{g}} m_t - \frac{s_{2t}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)}{4}\right)\frac{1}{\epsilon},\\
&\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)\delta\theta_t^{(2)}\\
&= \left\{C_R^2 c_{2t}\left[\left(s_{2t}^2 - c_{2t}^2\right)\left(\frac{m_{\tilde{g}} m_t}{2} - \frac{s_{2t}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)}{16}\right) - \frac{2 s_{2t} m_{\tilde{g}}^2 m_t^2}{m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2}\right]\right.\\
&\quad + C_R C_A c_{2t}\left[\frac{-3 m_{\tilde{g}} m_t}{4} + \frac{3 s_{2t}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)}{32}\right]\\
&\quad \left. + C_R T_f c_{2t}\left[\frac{m_{\tilde{g}} m_t}{2} - \frac{s_{2t}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)}{16}\right]\right\}\frac{1}{\epsilon^2}\\
&\quad + \left\{C_R^2 c_{2t}\left[-\frac{m_{\tilde{g}} m_t}{2} + \frac{s_{2t}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)}{16}\right]\right.\\
&\quad + C_R C_A c_{2t}\left[\frac{3 m_{\tilde{g}} m_t}{4} - \frac{3 s_{2t}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)}{32}\right]\\
&\quad \left. + C_R T_f c_{2t}\left[-\frac{m_{\tilde{g}} m_t}{2} + \frac{s_{2t}\left(m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2\right)}{16}\right]\right\}\frac{1}{\epsilon}.
\end{aligned} \tag{97}$$

The three-loop results are also available in electronic form [161], but they are too lengthy to be explicitly given in this review.

In the case of degenerate squark masses, one can take naively the limit $m_{\tilde{t}_2} \to m_{\tilde{t}_1}$ in (96). Furthermore one has to nullify the mixing angle. The quantities $\delta\theta_t^{(1,2)}$ are not defined in the mass-degenerate case which is reflected by the fact that the limit $m_{\tilde{t}_2} \to m_{\tilde{t}_1}$ does not exist in (97).

For completeness let us also provide the three-loop result for mass-degenerate squarks which is given by

$$\begin{aligned}
m_{\tilde{q}}^2 Z_{m_{\tilde{q}}} &= m_{\tilde{q}}^2 - \frac{\alpha_s}{\pi}\frac{1}{\epsilon} C_R m_{\tilde{g}}^2 + \left(\frac{\alpha_s}{\pi}\right)^2 \frac{1}{16}\\
&\quad \times \left\{\frac{2}{\epsilon^2} C_R\left(9C_A - 6T_f\right) m_{\tilde{g}}^2\right.\\
&\quad + \frac{1}{\epsilon}\left[4C_R\left[2T_q m_{\tilde{q}}^2 + T_t\left(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2 - 2M_t^2\right)\right]\right.\\
&\quad \left. + \left[2C_R\left(-11C_A + 6C_R\right) + 12 C_R T_f\right] m_{\tilde{g}}^2\right.\\
&\quad \left.\left. + \left(-6C_A C_R + 4C_R T_f\right) m_\varepsilon^2\right]\right\}\\
&\quad \times \left(\frac{\alpha_s}{\pi}\right)^3 \frac{1}{64}\\
&\quad \times \left\{\frac{8}{\epsilon^3} C_R\left[-9C_A^2 + 12 C_A T_f - 4T_f^2\right] m_{\tilde{g}}^2\right.\\
&\quad + \frac{1}{\epsilon^2}\left[8C_R T_q\left(-3C_A + 2T_f\right) M_{\tilde{q}}^2 + 4C_R T_f\right.\\
&\quad \times \left(-3C_A + 2T_f\right)\left(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2 - 2M_t^2\right)\\
&\quad \left. + 2C_R\left(3C_A - 2T_f\right)^2 m_\varepsilon^2\right]\\
&\quad + \frac{1}{\epsilon}\left[C_R\left(5C_A - 2C_R + 2T_f\right)\right.\\
&\quad \times \left[T_q M_{\tilde{q}}^2 + 4T_t\left(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2 - 2M_t^2\right)\right]\\
&\quad + 8C_R m_{\tilde{g}}^2\left[-10C_A^2 + 7C_A C_R - 8C_R^2\right.\\
&\quad \left. + 4T_f^2 + 6T_f C_R\left(3 - 4\zeta(3)\right)\right.\\
&\quad \left. + 24 T_f C_A \zeta(3)\right]\\
&\quad + 2C_R\left(3C_A - 2T_f\right)\\
&\quad \left.\left. \times \left(-5C_A + 2C_R + 2T_f\right) m_\varepsilon^2\right]\right\},
\end{aligned} \tag{98}$$

where we have used the above mentioned abbreviations $T_l = n_l I_2(R)$, with $l = f, q, t$. The terms that do not involve T_t can be obtained from $Z_{m_{\tilde{t}_1}}$ by setting $m_{\tilde{t}_2} = m_{\tilde{t}_1}$, $m_t = 0$, and $\theta_t = 0$.

As mentioned before, the ϵ-scalar mass needs to be renormalized at two loops within the $\overline{\text{DR}}$ scheme, in order to obtain the three-loop renormalization constants for squark masses and mixing angles. The corresponding renormalization constant is given by

$$\begin{aligned}
Z_{m_\varepsilon} &= 1 + \frac{\alpha_s}{\pi}\frac{1}{\epsilon}\\
&\quad \times \left\{\frac{-3}{4} C_A + \frac{1}{2} T_f\right.\\
&\quad + \left[-\frac{C_A}{2} m_{\tilde{g}}^2\right.\\
&\quad \left.\left. + 2T_q m_{\tilde{q}}^2 + T_t\left(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2 - 2m_t^2\right)\right]\frac{1}{2m_\varepsilon^2}\right\}\\
&\quad + \left(\frac{\alpha_s}{\pi}\right)^2
\end{aligned}$$

$$\times \left\{ \frac{1}{\epsilon^2} \left[\frac{9}{16} C_A^2 - \frac{3}{4} C_A T_f + \frac{1}{4} T_f^2 + \left(\frac{3C_A^2 - 2C_A T_f - 2C_R T_f}{4} m_{\tilde{g}}^2 - \frac{3C_A T_q - 2T_f T_q}{4} m_{\tilde{q}}^2 - \frac{3C_A T_t - 2T_f T_t}{8} \times \left(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2 - 2m_t^2 \right) \right) \frac{1}{m_\epsilon^2} \right] \times \frac{1}{\epsilon} \left[-\frac{3}{8} C_A^2 + \frac{1}{4} C_A T_f + \left(-\frac{5C_A^2 - 2C_A T_f - 4C_R T_f}{8} m_{\tilde{g}}^2 + \frac{C_A T_q}{2} m_{\tilde{q}}^2 + \frac{C_A T_t}{4} \left(m_{\tilde{t}_1}^2 + m_{\tilde{t}_2}^2 - 2m_t^2 \right) \right) \frac{1}{m_\epsilon^2} \right] \right\}. \tag{99}$$

Let us detail at this point on the choice of scheme. When computing the anomalous dimensions for the physical parameters, one has to consider the combined set of differential equations of all $\overline{\text{DR}}$ parameters appearing in the corresponding renormalization constants. This concerns in particular the unphysical ϵ-scalar mass which means that although m_ϵ is set to zero at one scale it is different from zero once this scale is changed. A way out from this situation is to renormalize the ϵ-scalar mass on shell and set the renormalized mass M_ϵ to zero. However, this scheme might become quite involved in practice, because of the on-shell two-loop diagrams that have to be computed. Alternatively one could shift the squark masses by a finite term which is chosen such that the ϵ-scalar decouples from the system of differential equations. The resulting renormalization scheme is called $\overline{\text{DR}}'$ scheme and has been suggested in [164]. For this calculation the finite shift is needed up to two loops and is given by [152, 164]

$$m_{\tilde{f}}^2 \longrightarrow m_{\tilde{f}}^2 - \frac{\alpha_s}{\pi} \frac{1}{2} C_F m_\epsilon^2 + \left(\frac{\alpha_s}{\pi} \right)^2 \times C_F m_\epsilon^2 \left(\frac{1}{4} T_f \left(n_q + n_t \right) + \frac{1}{4} C_F - \frac{3}{8} C_A \right), \tag{100}$$

where $f = t$ or $f = q$.

At the end of this section we want to discuss briefly the numerical impact of the higher order corrections on the squark masses. If one chooses the SUSY mass parameters of the order of $\mathcal{O}(1\,\text{TeV})$, one observes a moderate shift of a few GeV when going from one to two loops. After switching on the three-loop terms, however, the squark masses are decreased by about 40 GeV which is approximately an order of magnitude larger than the two-loop corrections. Nevertheless it corresponds to a shift in the masses of about 3% which is a reasonable amount for a three-loop SUSY-QCD term. Our observation coincides with the findings of [158, 159] where also relatively large three-loop corrections for the squarks have been identified.

5. The SM Gauge Beta Functions to Three Loops

In this section we report on the recent calculation of the three-loop gauge beta functions of the SM. In contrast to the supersymmetric theories, the SM beta functions to three loops have been computed only last year. At this point, it becomes probably clear the importance of all-order relations for the anomalous dimensions of supersymmetric theories valid in special regularization schemes. In the absence of SUSY and its holomorphic properties, one has to derive the anomalous dimension from a pure diagrammatic computation, which at the three-loop level becomes computationally quite involved.

The SM beta functions are important tools that allow us to relate theory predictions for various parameters at different energy scales. An important example in this respect is the inspection of the gauge coupling unification at high energies, for which precise experimental data of the couplings at the electroweak scale combined with accurate calculations of the RGEs yields precise predictions.

The computation of the beta functions of gauge theories has a long history. The one-loop beta functions in gauge theories along with the discovery of asymptotic freedom have been presented in [6, 7, 165]. The computation of the corresponding two-loop corrections followed a few years later in a series of papers. Namely, for gauge theories without fermions the results were computed in [166, 167], with those for gauge theories with fermions neglecting Yukawa couplings in [168–170] and considering also Yukawa couplings in [171]. The two-loop gauge coupling beta functions in an arbitrary quantum field theory have been considered in [172–175]. At the three-loop order, the first computed contributions to the gauge beta functions were those induced through the scalar self-interactions in [122, 123]. An important contribution to the field was the computation of the three-loop beta function in QCD [111, 112]. Yukawa contributions to it have been obtained in [176]. The generalization of these results to a general quantum field theory based on a single gauge group has been achieved in [101, 102]. For QCD, even the four-loop corrections are known from [119, 121]. In the following we concentrate on the calculation of the beta functions for the three gauge couplings of the SM up to three loops in the $\overline{\text{MS}}$ scheme. They have been computed for the first time in [49] and confirmed by an independent calculation in [177].

Let us in a first step fix the notation. We denote the three gauge couplings by α_1, α_2, and α_3 and adopt a SU(5)-like normalization. They are related to the quantities usually used in the SM by the all-order relations

$$\begin{aligned} \alpha_1 &= \frac{5}{3} \frac{\alpha_{\text{QED}}}{\cos^2\theta_W}, \\ \alpha_2 &= \frac{\alpha_{\text{QED}}}{\sin^2\theta_W}, \\ \alpha_3 &= \alpha_s, \end{aligned} \tag{101}$$

where α_{QED} is the fine structure constant, θ_W the weak mixing angle, and α_s the strong coupling.

The SM Yukawa interactions are described by (see, e.g., Chapter 11 of [178])

$$\begin{aligned}&\mathscr{L}_{\mathrm{Yukawa}}\\&\quad=-\overline{Q}_i^L Y_{ij}^U \epsilon H^\star u_j^R-\overline{Q}_i^L Y_{ij}^D H d_j^R-\overline{L}_i^L Y_{ij}^L H l_j^R+\mathrm{h.c.},\end{aligned}\tag{102}$$

where $Y^{U,D,L}$ are complex 3×3 matrices, i, j are generation labels, H denotes the Higgs field, and ϵ is the 2×2 antisymmetric tensor. Q^L, L^L are the left-handed quark and lepton doublets; and u^R, d^R, l^R are the right-handed up- and down-type quark and lepton singlets, respectively. The physical mass eigenstates are obtained by diagonalizing $Y^{U,D,L}$ by six unitary matrices $V_{L,R}^{U,D,L}$ as follows:

$$\widetilde{Y}_{\mathrm{diag}}^f=V_L^f Y^f V_R^{f\dagger},\quad f=U,D,L.\tag{103}$$

As a result the charged-current $W^\pm$ couples to the physical quark states with couplings parametrized by the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{\mathrm{CKM}}\equiv V_L^U V_L^{D\dagger}$. We furthermore introduce the notation

$$\widehat{T}=\frac{1}{4\pi}Y^U Y^{U\dagger},\qquad \widehat{B}=\frac{1}{4\pi}Y^D Y^{D\dagger},\qquad \widehat{L}=\frac{1}{4\pi}Y^L Y^{L\dagger}.\tag{104}$$

Of course, only traces over products of Yukawa matrices can occur because they arise from closed fermion loops. Using (103) and (104) it is straightforward to see that only traces of diagonal matrices have to be taken except for $\mathrm{tr}\widehat{T}\widehat{B}$ which is given by

$$\mathrm{tr}\widehat{T}\widehat{B}=\mathrm{tr}\left[\begin{pmatrix}\alpha_u&0&0\\0&\alpha_c&0\\0&0&\alpha_t\end{pmatrix}V_{\mathrm{CKM}}\begin{pmatrix}\alpha_d&0&0\\0&\alpha_s&0\\0&0&\alpha_b\end{pmatrix}V_{\mathrm{CKM}}^\dagger\right].\tag{105}$$

The Yukawa couplings are related to the SM parameters via the tree-level relations

$$\alpha_x=\frac{\alpha_{\mathrm{QED}}m_x^2}{2\sin^2\theta_W M_W^2},\quad \text{with } x=t,b,\tau,c,s,\ldots,\tag{106}$$

where m_x and M_W are the fermion and W boson mass, respectively.

We denote the Higgs boson self-coupling by $\widehat{\lambda}$, where the Lagrange density contains the following term:

$$\mathscr{L}=\cdots-\left(4\pi\widehat{\lambda}\right)\left(H^\dagger H\right)^2+\cdots\tag{107}$$

describing the quartic Higgs boson self-interaction.

The beta functions are obtained by calculating the renormalization constants relating bare and renormalized couplings via the relation

$$\alpha_i^{\mathrm{bare}}=\mu^{2\epsilon}Z_{\alpha_i}\left(\left\{\alpha_j\right\},\epsilon\right)\alpha_i.\tag{108}$$

Taking into account that α_i^{bare} does not depend on μ and taking into account that Z_{α_i} may depend on all couplings lead to the following formula:

$$\beta_i=-\left[\epsilon\frac{\alpha_i}{\pi}+\frac{\alpha_i}{Z_{\alpha_i}}\sum_{j=1,j\neq i}^{7}\frac{\partial Z_{\alpha_i}}{\partial\alpha_j}\beta_j\right]\left(1+\frac{\alpha_i}{Z_{\alpha_i}}\frac{\partial Z_{\alpha_i}}{\partial\alpha_i}\right)^{-1},\tag{109}$$

where $i=1$, 2 or 3. We furthermore set $\alpha_4=\alpha_t$, $\alpha_5=\alpha_b$, $\alpha_6=\alpha_\tau$, and $\alpha_7=\widehat{\lambda}$ and neglect the rest of Yukawa couplings.

The first term in the first factor of (109) originates from the term $\mu^{2\epsilon}$ in (108) and vanishes in four space time dimensions. The second term in the first factor contains the beta functions of the remaining six couplings of the SM. Note that (for the gauge couplings) the one-loop term of Z_{α_i} only contains α_i, whereas at two loops all couplings are present except $\widehat{\lambda}$. The latter appears for the first time at three-loop level. As a consequence, it is necessary to know β_j for $j=4,5,6$ to one-loop order and only the ϵ-dependent term for β_7, namely, $\beta_7=-\epsilon\alpha_7/\pi$. From the second term in the first factor and the second factor of (109) one can read off that three-loop corrections to Z_{α_i} are required for the computation of β_i to the same loop order.

In the $\overline{\mathrm{MS}}$ scheme the beta functions are mass independent. This allows us to use the SM in the unbroken phase as a framework for our calculation. In principle each vertex containing the coupling $g_i=\sqrt{4\pi\alpha_i}$ can be used in order to determine the corresponding renormalization constant via the relation (42). In order to compute the individual renormalization constants entering equation (42) one can proceed as outlined in the previous sections.

A second method that can be used to get an independent result for the renormalization constants of the gauge couplings is a calculation in the background field gauge (BFG) [179, 180]. The basic idea of the BFG is the splitting of all gauge fields in a "quantum" and a "classical" part where in practical calculations the latter only occurs as external particle.

The BFG has the advantage that Ward identities guarantee that renormalization constants for gauge couplings can be obtained from the exclusive knowledge of the corresponding wave function renormalization constant. Thus we have the following formula:

$$Z_{\alpha_i}=\frac{1}{Z_{A_i,\mathrm{wf}}},\tag{110}$$

where A denotes the gauge boson corresponding to the gauge coupling α_i.

In the BFG calculation it is advisable to adopt Landau gauge in order to avoid the renormalization of the gauge parameters ξ_i. However, it is not possible to choose Landau gauge from the very beginning since some Feynman rules for vertices involving a background gauge boson contain terms proportional to $1/\xi_i$ where $\xi_i=0$ corresponds to Landau gauge. To circumvent this problem one has to evaluate the bare integrals for arbitrary gauge parameters. In the final result all inverse powers of ξ_i cancel and thus the limit $\xi_i=0$ can be taken at the bare level.

An important issue in the present calculation is the treatment of γ_5 within dimensional regularization. Nontrivial contributions may arise if in the course of the calculation two fermion traces occur where both of them contain an odd number of γ_5-matrices and four or more γ-matrices. It is straightforward to see that the three-point Green functions that are required for this computation contain at most one-loop triangle subdiagrams (three-point Green functions involving external fermion lines are not considered here). This could potentially lead to contributions where a careful treatment of γ_5 is required. However, all these contributions vanish identically due to anomaly cancellations in the SM (see, e.g., [181]). This can also be checked by an explicit calculation using the seminaive regularization prescription for γ_5 as discussed in Section 3.1. Due to the $\mathcal{O}(\epsilon)$ ambiguity of (27), this approach can be directly applied only to diagrams that contain at most simple poles in ϵ. Otherwise, finite counterterms have to be introduced in order to restore Ward identities [86]. However, the diagrams contributing to this calculation that contain one-loop triangle subdiagrams have at most simple poles in ϵ. This explains why one obtains correct results for the three-loop beta functions even without implementing the 't Hooft-Veltman scheme for the regularization of γ_5.

From the technical point of view, all the methods and programs discussed in the previous section can also be applied in this computation. The main difficulty of this calculation is the enormous number of diagrams (of about a million diagrams) that contribute to the individual renormalization factors. In order to handle such an enormous amount of diagrams in a reasonable wall-clock time, one needs to parallelize the calculation.

We are now in the position to present the results for the beta functions of the gauge couplings which are given by

$$\begin{aligned}
\beta_1 = {} & \frac{\alpha_1^2}{(4\pi)^2}\left\{\frac{2}{5}+\frac{16n_G}{3}\right\}+\frac{\alpha_1^2}{(4\pi)^3}\\
&\times\left\{\frac{18\alpha_1}{25}+\frac{18\alpha_2}{5}-\frac{34\mathrm{tr}\widehat{T}}{5}-2\mathrm{tr}\widehat{B}\right.\\
&\qquad\left.-6tr\widehat{L}+n_G\left[\frac{76\alpha_1}{15}+\frac{12\alpha_2}{5}+\frac{176\alpha_3}{15}\right]\right\}\\
&+\frac{\alpha_1^2}{(4\pi)^4}\\
&\times\left\{\frac{489\alpha_1^2}{2000}+\frac{783\alpha_1\alpha_2}{200}+\frac{3401\alpha_2^2}{80}\right.\\
&\quad+\frac{54\alpha_1\widehat{\lambda}}{25}+\frac{18\alpha_2\widehat{\lambda}}{5}-\frac{36\widehat{\lambda}^2}{5}-\frac{2827\alpha_1\mathrm{tr}\widehat{T}}{200}\\
&\quad-\frac{471\alpha_2\mathrm{tr}\widehat{T}}{8}-\frac{116\alpha_3\mathrm{tr}\widehat{T}}{5}-\frac{1267\alpha_1\mathrm{tr}\widehat{B}}{200}\\
&\quad-\frac{1311\alpha_2\mathrm{tr}\widehat{B}}{40}-\frac{68\alpha_3\mathrm{tr}\widehat{B}}{5}-\frac{2529\alpha_1\mathrm{tr}\widehat{L}}{200}\\
&\quad-\frac{1629\alpha_2\mathrm{tr}\widehat{L}}{40}+\frac{183\mathrm{tr}\widehat{B}^2}{20}+\frac{51\left(\mathrm{tr}\widehat{B}\right)^2}{10}\\
&\quad+\frac{157\mathrm{tr}\widehat{B}\mathrm{tr}\widehat{L}}{5}+\frac{261\mathrm{tr}\widehat{L}^2}{20}+\frac{99\left(\mathrm{tr}\widehat{L}\right)^2}{10}\\
&\quad+\frac{3\mathrm{tr}\widehat{T}\widehat{B}}{2}+\frac{339\mathrm{tr}\widehat{T}^2}{20}+\frac{177\mathrm{tr}\widehat{T}\mathrm{tr}\widehat{B}}{5}\\
&\quad+\frac{199\mathrm{tr}\widehat{T}\mathrm{tr}\widehat{L}}{5}+\frac{303\left(\mathrm{tr}\widehat{T}\right)^2}{10}\\
&\quad+n_G\left[-\frac{232\alpha_1^2}{75}-\frac{7\alpha_1\alpha_2}{25}+\frac{166\alpha_2^2}{15}\right.\\
&\qquad\left.-\frac{548\alpha_1\alpha_3}{225}-\frac{4\alpha_2\alpha_3}{5}+\frac{1100\alpha_3^2}{9}\right]\\
&\quad+n_G^2\left[-\frac{836\alpha_1^2}{135}-\frac{44\alpha_2^2}{15}-\frac{1936\alpha_3^2}{135}\right]\Bigg\},\\
\beta_2 = {} & \frac{\alpha_2^2}{(4\pi)^2}\left\{-\frac{86}{3}+\frac{16n_G}{3}\right\}+\frac{\alpha_2^2}{(4\pi)^3}\\
&\times\left\{\frac{6\alpha_1}{5}-\frac{518\alpha_2}{3}-6\mathrm{tr}\widehat{T}-6\mathrm{tr}\widehat{B}-2\mathrm{tr}\widehat{L}\right.\\
&\qquad\left.+n_G\left[\frac{4\alpha_1}{5}+\frac{196\alpha_2}{3}+16\alpha_3\right]\right\}+\frac{\alpha_2^2}{(4\pi)^4}\\
&\times\left\{\frac{163\alpha_1^2}{400}+\frac{561\alpha_1\alpha_2}{40}-\frac{667111\alpha_2^2}{432}\right.\\
&\quad+\frac{6\alpha_1\widehat{\lambda}}{5}+6\alpha_2\widehat{\lambda}-12\widehat{\lambda}^2-\frac{593\alpha_1\mathrm{tr}\widehat{T}}{40}\\
&\quad-\frac{729\alpha_2\mathrm{tr}\widehat{T}}{8}-28\alpha_3\mathrm{tr}\widehat{T}-\frac{533\alpha_1\mathrm{tr}\widehat{B}}{40}\\
&\quad-\frac{729\alpha_2\mathrm{tr}\widehat{B}}{8}-28\alpha_3\mathrm{tr}\widehat{B}-\frac{51\alpha_1\mathrm{tr}\widehat{L}}{8}\\
&\quad-\frac{243\alpha_2\mathrm{tr}\widehat{L}}{8}+\frac{57\mathrm{tr}\widehat{B}^2}{4}+\frac{45\left(\mathrm{tr}\widehat{B}\right)^2}{2}\\
&\quad+15\mathrm{tr}\widehat{B}\mathrm{tr}\widehat{L}+\frac{19\mathrm{tr}\widehat{L}^2}{4}+\frac{5\left(\mathrm{tr}\widehat{L}\right)^2}{2}+\frac{27\mathrm{tr}\widehat{T}\widehat{B}}{2}\\
&\quad+\frac{57\mathrm{tr}\widehat{T}^2}{4}+45\mathrm{tr}\widehat{T}\,\mathrm{tr}\,\widehat{B}+15\,\mathrm{tr}\,\widehat{T}\,\mathrm{tr}\,\widehat{L}+\frac{45\left(\mathrm{tr}\,\widehat{L}\right)^2}{2}\\
&\quad+n_G\left[-\frac{28\alpha_1^2}{15}+\frac{13\alpha_1\alpha_2}{5}+\frac{25648\alpha_2^2}{27}\right.\\
&\qquad\left.-\frac{4\alpha_1\alpha_3}{15}+52\alpha_2\alpha_3+\frac{500\alpha_3^2}{3}\right]
\end{aligned}$$

$$+n_G^2\left[-\frac{44\alpha_1^2}{45}-\frac{1660\alpha_2^2}{27}-\frac{176\alpha_3^2}{9}\right]\Bigg\},$$

$$\beta_3=\frac{\alpha_3^2}{(4\pi)^2}\left\{-44+\frac{16n_G}{3}\right\}+\frac{\alpha_3^2}{(4\pi)^3}$$
$$\times\Bigg\{-408\alpha_3-8\mathrm{tr}\widehat{T}-8\mathrm{tr}\widehat{B}$$
$$+n_G\left[\frac{22\alpha_1}{15}+6\alpha_2+\frac{304\alpha_3}{3}\right]\Bigg\}$$
$$+\frac{\alpha_3^2}{(4\pi)^4}$$
$$\times\Bigg\{-5714\alpha_3^2-\frac{101\alpha_1\mathrm{tr}\widehat{T}}{10}-\frac{93\alpha_2\mathrm{tr}\widehat{T}}{2}$$
$$-160\alpha_3\mathrm{tr}\widehat{T}-\frac{89\alpha_1\mathrm{tr}\widehat{B}}{10}-\frac{93\alpha_2\mathrm{tr}\widehat{B}}{2}$$
$$-160\alpha_3\mathrm{tr}\widehat{B}+18\mathrm{tr}\widehat{B}^2+42\left(\mathrm{tr}\widehat{B}\right)^2$$
$$+14\mathrm{tr}\widehat{B}\mathrm{tr}\widehat{L}-12\mathrm{tr}\widehat{T}\widehat{B}+18\mathrm{tr}\widehat{T}^2$$
$$+84\mathrm{tr}\widehat{T}\mathrm{tr}\widehat{B}+14\mathrm{tr}\widehat{T}\mathrm{tr}\widehat{L}+42\left(\mathrm{tr}\widehat{T}\right)^2$$
$$+n_G\left[-\frac{13\alpha_1^2}{30}-\frac{\alpha_1\alpha_2}{10}+\frac{241\alpha_2^2}{6}\right.$$
$$\left.+\frac{308\alpha_1\alpha_3}{45}+28\alpha_2\alpha_3+\frac{20132\alpha_3^2}{9}\right]$$
$$+n_G^2\left[-\frac{242\alpha_1^2}{135}-\frac{22\alpha_2^2}{3}-\frac{2600\alpha_3^2}{27}\right]\Bigg\}. \tag{111}$$

In the above formulas n_G denotes the number of fermion generations. It is obtained by labeling the closed quark and lepton loops present in the diagrams.

Let us finally briefly discuss the numerical impact of the new three-loop corrections. In Figure 4 from [49] we reproduce the running of α_1 and α_2 from $\mu = M_Z$ to the energy scales where these two couplings become equal. The dotted and dashed lines correspond to one- and two-loop running, respectively. One observes a significant change of the curves, which is in particular much bigger than the experimental uncertainty indicated by the dashed band. Thus in case only one- and two-loop perturbative corrections are included the theory uncertainty is much bigger than the experimental one. This changes with the inclusion of the three-loop terms. The results are shown as solid lines which are closed to the corresponding dashed curves. The effect is small, however, still of the order of the experimental uncertainty, in particular for α_2. The three-loop effects on α_3 predictions are, as expected, much smaller than the experimental uncertainty. For this reason, the strong coupling was not displayed in Figure 4. Let us briefly point out that the energy scale at which the electroweak couplings meet each other is of about 10^{13} GeV. Coupling unification at such a low-energy scale would imply a too rapid proton decay, in contrast to the experimental results. Thus, even from this partial analysis, we can conclude that the statement that gauge coupling unification cannot be achieved within the SM remains valid even after the inclusion of the three-loop radiative corrections. More details about this topic can be found in the next section.

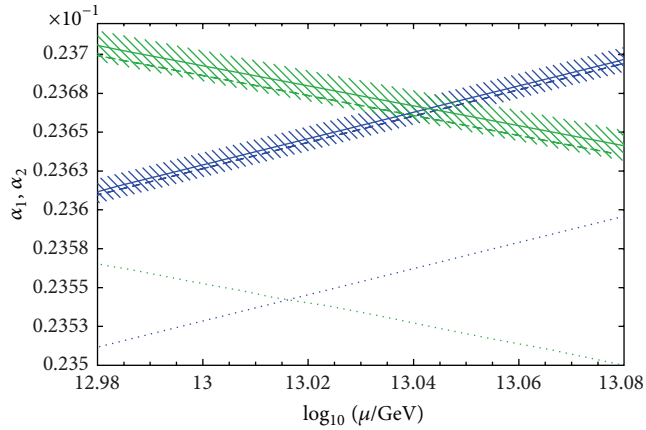

Figure 4: The running of the electroweak gauge couplings in the SM from [49]. The lines with positive slope correspond to α_1, the lines with negative slope to α_2. The dotted, dashed, and solid lines correspond to one-, two-, and three-loop precision, respectively. The bands around the three-loop curves visualize the experimental uncertainty.

6. Gauge Coupling Unification in Supersymmetric Models

An appealing hint in favor of supersymmetry is the apparent unification of gauge couplings at a scale of about 10^{16} GeV [20–22]. Gauge coupling unification is highly sensitive to the heavy particle mass spectrum. This property allows us to probe unification through precision measurements of low-energy parameters like the gauge couplings at the electroweak scale and the supersymmetric mass spectrum. The current precision of the experimental data for the relevant input parameters [178, 182] and the substantial progress on the theory side [50, 103, 127, 158, 159, 183] require renormalization group analyses even at three-loop accuracy. Within this method, one needs n-loop RGEs and $(n-1)$-loop threshold corrections to achieve n-loop precision. We have discussed in detail the derivation of RGEs in the previous sections. The first part of this section is devoted to the calculation of threshold corrections. As an example, the determination of the two-loop SUSY-QCD threshold corrections for the strong coupling α_s and the bottom-quark mass m_b will be presented. In the second part of this section, we outline the phenomenological analysis of gauge coupling unification within the minimal SUSY SU(5) model.

6.1. Effective Field Theory Approach: Decoupling Coefficients. As already stated above, the underlying motivation for the running analysis is to relate physical parameters measured at the electroweak scale with the Lagrange parameters at the GUT scale. The running parameters are most conveniently defined in mass-independent renormalization schemes such as $\overline{\text{MS}}$ for the SM parameters and $\overline{\text{DR}}$ for the MSSM parameters. These schemes have the advantage that the gauge beta functions are mass independent and their computation is much easier than in physical mass dependent schemes. However, quantum corrections to low-energy processes contain logarithmically enhanced contributions from heavy particles with masses much greater than the energy-scale of the process under consideration. In other words in such "unphysical" renormalization schemes the Appelquist-Carazzone decoupling theorem [184] does not hold in its naive form. An elegant approach to get rid of this unwanted behavior in the $\overline{\text{MS}}$ or $\overline{\text{DR}}$ scheme is to formulate an effective theory (ET) (for more details see [134, 185]) integrating out all heavy particles. The parameters of the ET must be modified ("rescaled") in order to take into account the effects of the heavy fields. The ET parameters are related to the parameters of the full theory by the so-called matching or decoupling relations.

They have been computed in QCD including corrections up to the four-loop order for the strong coupling [186, 187] and three-loop order for quark masses [185]. In the MSSM the two-loop SUSY-QCD [50, 183, 188] and SUSY-EW [189–191] expressions are known. Very recently, even the three-loop SUSY-QCD corrections to decoupling coefficient of the strong coupling were computed [192].

In the following, we concentrate on the calculation of the decoupling coefficients for the strong coupling and the bottom-quark mass within SUSY-QCD. They are the most interesting quantities from the phenomenological point of view because they are on the one hand the main ingredients for the study of the gauge and Yukawa coupling unification. On the other hand they are the quantities that receive the largest radiative corrections, for which next-to-next-to-leading-order corrections are essential for high precision predictions.

6.1.1. Framework. We consider SUSY-QCD with n_f active quark and $n_s = n_f$ active squark flavors and $n_{\tilde{g}} = 1$ gluinos. Furthermore, we assume that $n_l = 5$ quarks are light (among which the bottom quark) and that the top quark and all squarks and the gluino are heavy. Integrating out the heavy fields from the full SUSY-QCD Lagrangian, we obtain the Lagrange density corresponding to the effective QCD with n_l light quarks plus nonrenormalizable interactions. The latter are suppressed by negative powers of the heavy masses and will be neglected here. The effective Lagrangian can be written as follows:

$$\begin{aligned}&\mathscr{L}_{\text{eff}}\left(g_s^0, m_q^0, \xi^0; q^0, G_\mu^{0,a}, c^{0,a}; \zeta_i^0\right)\\&\quad= \mathscr{L}^{\text{QCD}}\left(g_s^{0\prime}, m_q^{0\prime}, \xi^{0\prime}; q^{0\prime}, G_\mu^{0\prime,a}, c^{0\prime,a}\right),\end{aligned} \tag{112}$$

where q, G_μ^a, c^a denote the light-quark, the gluon, and the ghost fields, respectively, m_q stands for the light quark masses, ξ is the gauge parameter, and $g_s = \sqrt{4\pi\alpha_s}$ is the strong coupling. The index 0 marks bare quantities and $\mathscr{L}^{\text{QCD}}$ is the usual QCD Lagrangian from which all heavy fields have been discarded. As a result the fields, masses, and couplings associated with light particles have to be rescaled. They are labeled by a prime in (112) and are related to the original parameters through decoupling relations:

$$\begin{aligned}&g_s^{0\prime} = \zeta_g^0 g_s^0, \qquad m_q^{0\prime} = \zeta_m^0 m_q^0, \qquad \xi^{0\prime} - 1 = \zeta_3^0\left(\xi^0 - 1\right),\\&q^{0\prime} = \sqrt{\zeta_2^0}\, q^0, \qquad G_\mu^{0\prime,a} = \sqrt{\zeta_3^0}\, G_\mu^{0,a}, \qquad c^{0\prime,a} = \sqrt{\tilde{\zeta}_3^0}\, c^{0,a}.\end{aligned} \tag{113}$$

Since the decoupling coefficients are universal quantities, they are independent of the momenta carried by the incoming and outgoing particles. The authors of [185] showed that the bare decoupling coefficients ζ_m^0, ζ_2^0, ζ_3^0, $\tilde{\zeta}_3^0$ can be derived from the quark, the gluon, and the ghost propagators, all evaluated at vanishing external momenta, via the relations

$$\begin{aligned}\zeta_3^{(0)} &= 1 + \Pi^{0,h}(0),\\ \zeta_2^{(0)} &= 1 + \Sigma_v^{0,h}(0),\\ \zeta_m^{(0)} &= \frac{1 - \Sigma_s^{0,h}(0)}{1 + \Sigma_v^{0,h}(0)}.\end{aligned} \tag{114}$$

The superscript h indicates that in the framework of DREG or DRED only diagrams containing at least one heavy particle inside the loops contribute and that only the hard regions in the asymptotic expansion of the diagrams are taken into account.

In Figure 5 are shown sample Feynman diagrams contributing to the decoupling coefficients for the strong coupling (a) and the bottom-quark mass (b).

For the computation of ζ_g one has to consider in addition one vertex involving the strong coupling. A convenient choice is the relation

$$\zeta_g^0 = \frac{\tilde{\zeta}_1^0}{\tilde{\zeta}_3^0\sqrt{\zeta_3^0}}, \tag{115}$$

where $\tilde{\zeta}_1^0$ denotes the decoupling constant for the ghost-gluon vertex.

The finite decoupling coefficients are obtained upon the renormalization of the bare parameters. They are given by

$$\zeta_g = \frac{Z_g}{Z_g'}\zeta_g^0, \qquad \zeta_m = \frac{Z_m}{Z_m'}\zeta_m^0, \tag{116}$$

where Z_g' and Z_m' correspond to the renormalization constants in the effective theory, and Z_g and Z_m denote the same quantities in the full theory. Since we are interested in the two-loop results for ζ_i, $i = g, m$, the corresponding renormalization constants for SUSY-QCD and QCD have to be implemented with the same accuracy. Analytical results for the latter up to the three-loop order can be found in the previous sections and the references cited therein, for example, [134, 193, 194].

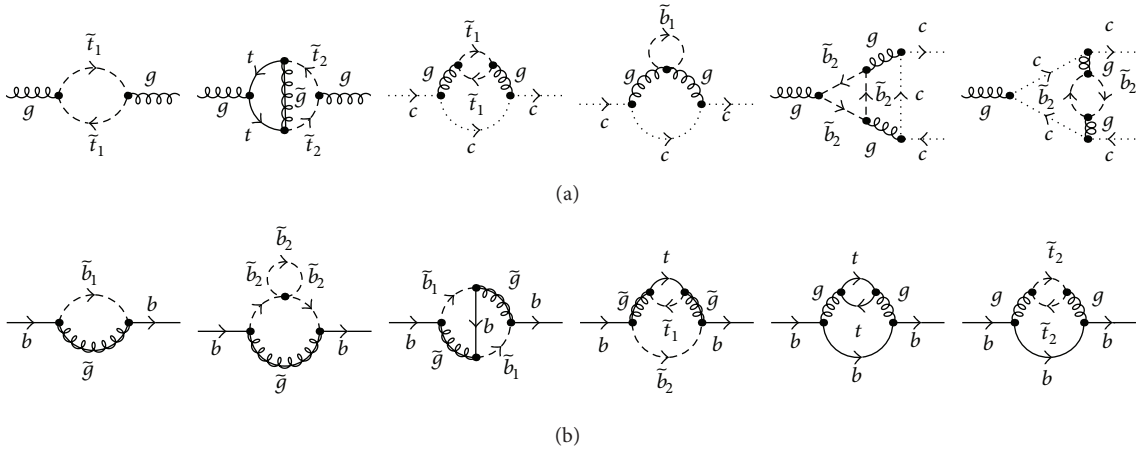

FIGURE 5: Sample diagrams contributing to ζ_3, $\widetilde{\zeta}_3$, $\widetilde{\zeta}_1$, and ζ_m with gluons (g), ghosts (c), bottom/top quarks (b/t), bottom/top squarks ($\widetilde{b}/\widetilde{t}$), and gluinos ($\widetilde{g}$).

6.1.2. Renormalization Scheme. Apart from the renormalization constants of the external fields, also the renormalization of the input parameters is required. However, for the renormalization of the gluino and squark masses and the squark mixing angle we choose the on-shell scheme. This scheme allows us to use directly the physical parameters in the running analyses making the implementation very simple. The explicit formulae at the one-loop order can be found in [195, 196]. The two-loop counterterms are known analytically only for specific mass hierarchies [156] and numerically for arbitrary masses [197].

For the computation of the decoupling coefficient for the bottom-quark mass at order $\mathcal{O}(\alpha_s^2)$ one needs to renormalize in addition the bottom-quark mass and the trilinear coupling A_b as well as the ϵ-scalar mass. As the bottom-quark mass is neglected with respect to heavy particle masses, an explicit dependence of the radiative corrections on m_b can occur only through bottom Yukawa coupling. In order to avoid the occurrence of large logarithms of the form $\alpha_s^2 \log(\mu^2/m_b^2)$ with $\mu \simeq \widetilde{M}$, one has to renormalize the bottom Yukawa coupling in the $\overline{\text{DR}}$ scheme. In this way, the large logarithms are absorbed into the running mass and the higher order corrections are maintained small.

The renormalization prescription for the trilinear coupling A_b is fixed by the tree-level relation

$$\sin 2\theta_b = \frac{2m_b\left(A_b - \mu \tan\beta\right)}{m_{\widetilde{b}_1}^2 - m_{\widetilde{b}_2}^2}. \tag{117}$$

The parameters μ and $\tan\beta$ do not acquire $\mathcal{O}(\alpha_s)$ corrections to the one-loop level. Generically, the counterterm for A_b can be expressed as

$$\delta A_b = \left(2\cos 2\theta_b \delta\theta_b + \sin 2\theta_b \frac{\delta m_{\widetilde{b}_1}^2 - \delta m_{\widetilde{b}_2}^2}{m_{\widetilde{b}_1}^2 - m_{\widetilde{b}_2}^2} - \sin 2\theta_b \frac{\delta m_b}{m_b}\right) \times \frac{m_{\widetilde{b}_1}^2 - m_{\widetilde{b}_2}^2}{2m_b}, \tag{118}$$

where δm_b and $\delta m_{\widetilde{b}_{1,2}}^2$ are the counterterms corresponding to bottom-quark and squark masses, respectively. Due to the use of different renormalization prescriptions for the bottom-quark and squark masses and mixing angle, the parameter A_b is renormalized in a *mixed* scheme.

Finally, the last parameter to be renormalized is the ϵ-scalar mass. In softly broken SUSY theories, as it is the case of MSSM or SUSY-QCD, they get a radiatively induced mass. As already discussed in the previous sections, there are different approaches in the literature to perform the renormalization in such a case. To obtain decoupling coefficients independent of the unphysical parameter m_ϵ, one has to modify the bottom squark masses by finite quantities [152, 164] according to the relation (100). Such finite shifts have to be performed for both renormalization schemes for squark masses $\overline{\text{DR}}$ and on shell.

6.1.3. Analytical Results. The exact one-loop results for the decoupling coefficients of the strong coupling constant ζ_s and bottom-quark mass ζ_m can be found in [50, 188, 195]. We list them below up to order $\mathcal{O}(\epsilon)$:

$$\begin{aligned}\zeta_s &= 1 + \frac{\alpha_s^{(\text{SQCD})}}{\pi} \\ &\quad\times\Bigg[-\frac{1}{6}C_A L_{\widetilde{g}} - \frac{1}{6}L_t - \sum_q \sum_{i=1,2} \frac{1}{24} L_{\widetilde{q}_i} \\ &\qquad - \epsilon\left(\frac{C_A}{12}\left(L_{\widetilde{g}}^2 + \zeta(2)\right) + \frac{1}{12}\left(L_t^2 + \zeta(2)\right)\right. \\ &\qquad\left. - \frac{1}{48}\sum_q \sum_{i=1,2}\left(L_{\widetilde{q}_i}^2 + \zeta(2)\right)\right)\Bigg],\end{aligned} \tag{119}$$

$$\zeta_{m_b} = 1 + \frac{\alpha_s^{(\mathrm{SQCD})}}{\pi} C_F \times \sum_{i=1,2} \left\{ -\frac{\left(1 + L_{\tilde{b}_i}\right)}{4} \frac{m_{\tilde{b}_i}^2}{\left(m_{\tilde{b}_i}^2 - m_{\tilde{g}}^2\right)} + \frac{\left(3 + 2L_{\tilde{b}_i}\right) m_{\tilde{b}_i}^4 - \left(3 + 2L_{\tilde{g}}\right) m_{\tilde{g}}^4}{16\left(m_{\tilde{b}_i}^2 - m_{\tilde{g}}^2\right)^2} - \frac{(-1)^i \, X_b m_{\tilde{g}}}{m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2} \frac{m_{\tilde{b}_i}^2 L_{\tilde{b}_i} - m_{\tilde{g}}^2 L_{\tilde{g}}}{2\left(m_{\tilde{b}_i}^2 - m_{\tilde{g}}^2\right)} + \epsilon \left[-\frac{m_{\tilde{b}_i}^2 \left(2 + L_{\tilde{b}_i}\left(2 + L_{\tilde{b}_i}\right) + \zeta(2)\right)}{8\left(m_{\tilde{b}_i}^2 - m_{\tilde{g}}^2\right)} + \left(\left(m_{\tilde{b}_i}^4 \left(7 + 2L_{\tilde{b}_i}\left(3 + L_{\tilde{b}_i}\right) + 2\zeta(2)\right) - m_{\tilde{g}}^4 \left(7 + 2L_{\tilde{g}}\left(3 + L_{\tilde{g}}\right) + 2\zeta(2)\right)\right) \times \left(32\left(m_{\tilde{b}_i}^2 - m_{\tilde{g}}^2\right)^2\right)^{-1} \right) + \left(\frac{(-1)^i \, X_b m_{\tilde{g}}}{m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2} \times \left(m_{\tilde{g}}^2 L_{\tilde{g}}\left(2 + L_{\tilde{g}}\right) - m_{\tilde{b}_i}^2 L_{\tilde{b}_i}\left(2 + L_{\tilde{b}_i}\right)\right) \times \left(4\left(m_{\tilde{b}_i}^2 - m_{\tilde{g}}^2\right)\right)^{-1} \right) \right] \right\}, \tag{120}$$

where $\zeta(2)$ is Riemann's zeta function with $\zeta(2) = \pi^2/6$. In the above equations we have adopted the abbreviations

$$L_i = \ln \frac{\mu^2}{m_i^2}, \quad i \in \left\{t, \tilde{g}, \tilde{q}_{1,2}, \tilde{b}_{1,2}\right\}, \qquad X_b = A_b - \mu_{\mathrm{SUSY}} \tan\beta. \tag{121}$$

$\alpha_s^{(\mathrm{SQCD})}$ denotes the strong coupling constant in SUSY-QCD.

The presence of the terms proportional to the parameter X_b is a manifestation of the supersymmetry breaking. They are generated by the Yukawa interaction between left- and right-handed bottom squarks and the CP-neutral Higgs fields. Their contribution to the decoupling coefficient of the bottom-quark mass can be related through the low-energy theorem [198] to the decay rate of the Higgs boson to $b\bar{b}$ pairs. To one-loop order, the X_b term of (120) coincides with the SUSY-QCD corrections to the decay rate $\phi \to b\bar{b}$ [199]. To higher orders, the relation between the two parameters becomes more involved (for details see Section 8). These Yukawa-coupling-induced contributions attracted a lot of attention due to the fact that they are the dominant corrections for large values of $\tan\beta$. They may in general become comparable with the tree-level bottom-quark mass. Thus, they need to be resumed even at the two-loop level.

The analytical two-loop results for the decoupling coefficients are too lengthy to be displayed here. They are available in [50] together with their expressions for some phenomenologically motivated mass hierarchies. The dominant two-loop contributions to ζ_{m_b}, that is, the terms enhanced by $\tan\beta$, have been confirmed by the independent computation of [190, 191]. Also the dominant SUSY-QCD-EW corrections to ζ_{m_b} at the two-loop order have been computed in [190, 191].

6.1.4. Numerical Analysis. In this section we discuss briefly the numerical impact of the two-loop decoupling coefficients derived above on the prediction of the strong coupling constant at the GUT scale. As already pointed out, the scale μ_{dec} at which the decoupling of the heavy particles is performed is not fixed by the theory. The dependence of physical observables on this unphysical parameter is a measure of the theoretical uncertainty left over. At fixed order perturbation theory, it is expected that the relations between the running parameters evaluated at high-energy scale and their low-energy values become less sensitive to the choice of μ_{dec}, once higher order radiative corrections are considered.

In [200] a consistent method for the calculation of the energy evolution of physical parameters was proposed. For example, one derives the SM values $\alpha_s^{(5)}(\mu_{\mathrm{dec}})$ and at the heavy scale μ_{dec} from the n-loop SM RGEs. Here μ_{dec} denotes the energy scale at which the heavy particles are supposed to become active, that is, the scale where the matching between the SM and the MSSM is performed. Before the matching procedure, one has to perform also the change of regularization scheme from $\overline{\mathrm{MS}}$ to $\overline{\mathrm{DR}}$. For consistency, the n-loop running parameters have to be folded with $(n-1)$-loop conversion and decoupling relations. The latter are known in SUSY-QCD up to two-loop order [201] and within MSSM to one-loop order [202]. Above the decoupling scale, the energy dependence of the running parameters is governed by the n-loop MSSM RGEs.

The dependence of $\alpha_s(\mu_{\mathrm{GUT}})$ on the decoupling scale is displayed in Figure 6 from [50]. The dotted, dashed, and solid lines denote the one-, two-, and three-loop running, where the corresponding exact results for the decoupling coefficients have been implemented. One can see the improved stability of the three-loop results with respect to the decoupling-scale variation. The uncertainty induced by the current experimental accuracy on $\alpha_s(M_Z)$, $\delta\alpha_s = 0.001$ [203], is indicated by the hatched band.

In order to get an idea of the effects induced by the SUSY mass parameters on $\alpha_s(\mu_{\mathrm{GUT}})$, two different mass spectra are shown. As reference was chosen the so-called Snowmass Point SPS1a′ scenario [204] for which rather low SUSY mass

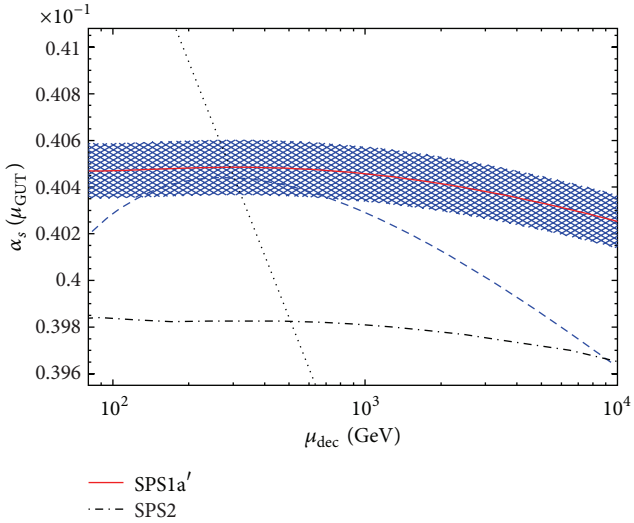

FIGURE 6: $\alpha_s(\mu_{\rm GUT})$ as a function of $\mu_{\rm dec}$ from [50]. Dotted, dashed, and solid lines denote the one-, two-, and three-loop contributions, respectively, obtained using for the input parameters their values for the SPS1a' benchmark point. The dash-dotted line shows the three-loop running corresponding to the SPS2 point.

parameters are required: $m_{\tilde{g}} = 607.1$ GeV, $m_{\tilde{t}_1} = 366.5$ GeV, $m_{\tilde{t}_2} = 585.5$ GeV, $m_{\tilde{b}_1} = 506.3$ GeV, $m_{\tilde{b}_2} = 545.7$ GeV, $A_t^{\overline{\rm DR}}(1\,{\rm TeV}) = -565.1$ GeV, $A_b^{\overline{\rm DR}}(1\,{\rm TeV}) = -943.4$ GeV, $\mu = 396.0$ GeV, and $\tan\beta = 10.0$. In addition the dash-dotted line shows the three-loop results when the SUSY parameters corresponding to the SPS2 [205] scenario are adopted. Their explicit values are $m_{\tilde{g}} = 784.4$ GeV, $m_{\tilde{t}_1} = 1003.9$ GeV, $m_{\tilde{t}_2} = 1307.4$ GeV, $m_{\tilde{b}_1} = 1296.6$ GeV, $m_{\tilde{b}_2} = 1520.1$ GeV, and $\tan\beta = 10.0$. One clearly notices the great impact of the SUSY mass pattern on the predicted value of the strong coupling at high energies. Accordingly, for precision studies concerning gauge coupling unification the explicit mass pattern of heavy particles must be taken into account.

At this point, a comment on the chosen mass spectrum is in order. The nature of this plot is rather academic and aims to quantify the mass dependence of the strong coupling constant at high energies. The two mass spectra are already excluded by the direct searches at the LHC. Nevertheless, one can easily estimate that for heavier SUSY particles $M_{\rm SUSY} > 1.5$ TeV the value of $\alpha_s(\mu_{\rm GUT})$ decreases below the value 0.398. Its implication on the quality of the unification will be discussed in the next section.

6.2. Gauge Coupling Unification in the Minimal SUSY SU(5) Model. The gauge coupling unification might be predicted, even under the assumption of a minimal particle content of the underlying GUT like in the so-called minimal SUSY SU(5) model [206, 207]. This is the most predictive model among the currently known candidates for SUSY GUTs. However, immediately after its formulation it has been noticed that new dimension-five operators may cause rapid proton decay. Together with the requirement of gauge coupling unification this aspect was used to even rule out the SUSY SU(5) model [208, 209]. However, subsequent careful analyses have shown that the proton decay rate for the dominant channel $p \to K^+\overline{\nu}$ can be suppressed either by sfermion mixing [210] or by taking into account higher dimensional operators induced at the Planck scale [211–213].

In the following, we review the latest analysis on the gauge coupling unification in the renormalizable version of minimal SUSY SU(5). This model is not the best motivated phenomenologically, but it requires the most severe constraints on the GUT parameters. More precisely, we outline the constraints on the mass of the color triplet Higgs M_{H_c} and the grand unification scale (see below the exact definition of M_{H_c} and M_G) M_G taking into account the latest experimental data for the weak scale parameters and the most precise theoretical predictions currently available. The two parameters are predicted in the "bottom-up" approach, taking into account threshold corrections generated by the superpartners of the SM particles as well as those due to the superheavy SUSY-GUT particles. In addition, the gauge coupling constants of the SM at the electroweak scale and the MSSM mass spectrum are required as input parameters. The predicted values for the two parameters have to be compared with the constraints derived from the nonobservation of proton decay.

For completeness, we present below our notation in the framework of minimal SUSY SU(5). The superpotential of the model [206] is given by

$$\begin{aligned}\mathscr{W} &= M_1\,{\rm Tr}\left(\Sigma^2\right) + \lambda_1\,{\rm Tr}\left(\Sigma^3\right) + \lambda_2\overline{\rm H}\Sigma{\rm H} + M_2\overline{\rm H}{\rm H}\\ &\quad + \sqrt{2}Y_d^{ij}\Psi_i\phi_j\overline{H} + \frac{1}{4}Y_u^{ij}\Psi_i\Psi_j H,\end{aligned} \tag{122}$$

where Ψ_i and ϕ_i ($i = 1, 2, 3$ is a generation index) are matter multiplets in the **10**- and $\overline{\mathbf{5}}$-dimensional representation of SU(5). Their decomposition with respect to the SM gauge group SU(3) × SU(2) × U(1) reads

$$\begin{aligned}\mathbf{5} &= \left(3, 1, -\frac{1}{3}\right) \oplus \left(1, 2, +\frac{1}{2}\right),\\ \mathbf{10} &= \left(\overline{3}, 1, -\frac{2}{3}\right) \oplus \left(3, 2, \frac{1}{6}\right) \oplus (1, 1, 1).\end{aligned} \tag{123}$$

The field H ($\overline{H}$) is realized in the **5** ($\overline{\mathbf{5}}$) representation. The gauge group SU(5) is broken to the SM gauge group if the adjoint Higgs boson $\Sigma \equiv \Phi^a T^a$ ($a = 1, \ldots, 24$) living in the **24** representation gets the vacuum expectation value $\langle\Sigma\rangle = V/(2\sqrt{30}) \times {\rm diag}(-2, -2, -2, 3, 3)$, with $V = -4\sqrt{30}M_1/(3\lambda_1)$ (here, we parametrize as usual the **24** representation like a 5×5 matrix). Its decomposition with respect to the SM gauge group reads

$$\mathbf{24} = (1, 1, 0) \oplus (1, 3, 0) \oplus (8, 1, 0) \oplus \left(3, 2, -\frac{5}{6}\right) \oplus \left(\overline{3}, 2, \frac{5}{6}\right). \tag{124}$$

Choosing $\langle\overline{H}\rangle = \langle H\rangle \ll V$ and in addition imposing the (tree-level-) fine-tuning condition $M_2 = -\sqrt{3}\lambda_2 V/\sqrt{40}$, the

isodoublets in H and $\overline{H}$ remain massless. Furthermore, one gets the following superheavy mass spectrum:

$$M_X^2 = \frac{5}{12}g^2V^2, \qquad M_{H_c}^2 = \frac{5}{24}\lambda_2^2V^2,$$
$$M_\Sigma^2 \equiv M_{(8,1)}^2 = M_{(1,3)}^2 = 25M_{(1,1)}^2 = \frac{15}{32}\lambda_1^2V^2, \tag{125}$$

where the indices in round brackets refer to the SU(3) and SU(2) quantum numbers. Here M_Σ denotes the mass of the color octet part of the adjoint Higgs boson Σ and M_{H_c} stands for the mass of the color triplets of H and $\overline{H}$. M_X is the mass of the gauge bosons and g is the gauge coupling. The equality $M_{(8,1)}^2 = M_{(1,3)}^2$ holds only if one neglects operators that are suppressed by $1/M_{\mathrm{Pl}}$ as we do here.

In the study of the energy evolution of the gauge couplings up to scales of the order $\mathcal{O}(10^{16})$ GeV, one has to apply the effective field theory (EFT) approach twice: once at an energy scale comparable with the SUSY particle masses and once at the GUT scale. In practice, this translates into the following steps.

(1) Running within the SM from $\mu = M_Z$ to the SUSY scale μ_{SUSY}.

In this step, the three-loop beta function of QCD [111, 112] and up to three-loop RGEs in the electroweak sector [49, 114] are necessary in order to obtain the values of the gauge couplings at $\mu_{\mathrm{SUSY}} \approx 1$ TeV. At this point we want to stress once again that the value of μ_{SUSY} is a free parameter. Let us mention that the top quark threshold effects are taken into account in the determination of the input parameters (for details see next section) and the running analysis is performed in SM with six active quark flavors.

(2) SUSY threshold corrections.

In order to still cure the naturalness problem of the SM, the SUSY mass spectrum has to be at most in the TeV range. Thus, for energies of this order of magnitude, it is expected that the SUSY particles become active and the proper matching between the SM and the MSSM has to be performed. The one-loop decoupling relations for α_1 and α_2 [214] and the Yukawa couplings [195] are known since long time. The SUSY-QCD decoupling effects for α_3 and m_b are known to three- and two-loop order, respectively, as discussed in the previous section. A fully consistent approach would require two-loop threshold corrections not only in the strong but also in the electroweak sector. They are not yet available, nevertheless it is expected that their numerical impact is relatively small.

At this stage also the change of renormalization scheme from $\overline{\mathrm{MS}}$ to $\overline{\mathrm{DR}}$ has to be taken into account. To establish the conversion relations between the running parameters in the two schemes, one can use the method discussed in Section 3.3.3, where such relations have been derived in the context of nonsupersymmetric theories (for more details see [201]). The conversion relations that are of interest for the numerical analysis discussed in this section are those involving the gauge couplings and the quark masses of the third generation, as only their Yukawa couplings give sizable effects. They are known up to the two-loop order in SUSY-QCD [201]. For the convenience of the reader we cite them below

$$\alpha_s^{\overline{\mathrm{MS}}} = \alpha_s^{\overline{\mathrm{DR}}}\left[1 - \frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\frac{C_A}{3} + \left(\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right)^2\left(-\frac{11}{9}C_A^2 + 2T_fC_R\right)\right], \tag{126}$$

$$m_q^{\overline{\mathrm{MS}}} = m_q^{\overline{\mathrm{DR}}}\left[1 + \frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}C_R + \left(\frac{\alpha_s^{\overline{\mathrm{DR}}}}{\pi}\right)^2\left(\frac{7}{12}C_AC_R + \frac{7}{4}C_R^2 - \frac{1}{2}T_fC_R\right)\right], \tag{127}$$

where the group invariants are defined as in Appendix A and $T_f = I_2(R)n_f$, with n_f being the number of active fermions.

(3) Running within the MSSM from μ_{SUSY} to the high-energy scale μ_{GUT}.

In this step the three-loop RGEs of the MSSM [103, 127] are required to evolve the gauge and Yukawa couplings from μ_{SUSY} to some very high scale of the order of 10^{16} GeV, that we denote μ_{GUT}. At this energy scale it is expected that SUSY-GUT particles become active.

(4) SUSY-GUT threshold effects.

At the energy scale μ_{GUT}, threshold corrections induced by the nondegenerate SUSY-GUT spectrum have to be taken into account. The one-loop formulas of the decoupling coefficients for a general gauge group have been known for a long time [215–217]. The specification to the minimal SUSY SU(5) reads [218, 219]

$$\zeta_{\alpha_1}(\mu) = 1 + \frac{\alpha^{\mathrm{SU}(5)}(\mu)}{4\pi}\left(-\frac{2}{5}L_{\mu H_c} + 10L_{\mu X}\right),$$
$$\zeta_{\alpha_2}(\mu) = 1 + \frac{\alpha^{\mathrm{SU}(5)}(\mu)}{4\pi}\left(-2L_{\mu\Sigma} + 6L_{\mu X}\right), \tag{128}$$
$$\zeta_{\alpha_3}(\mu) = 1 + \frac{\alpha^{\mathrm{SU}(5)}(\mu)}{4\pi}\left(-L_{\mu H_c} - 3L_{\mu\Sigma} + 4L_{\mu X}\right),$$

where $L_{\mu x} = \ln(\mu^2/M_x^2)$ and for simplicity we keep from the list of arguments of the coefficients ζ_{α_i} only the decoupling scale. $\alpha^{\mathrm{SU}(5)}(\mu)/4\pi$ is the gauge coupling constant of the unified theory, that is, of the SUSY SU(5) model.

A suitable linear combination of the three equations above leads to the following two relations:

$$4\pi\left(-\frac{1}{\alpha_1(\mu)} + 3\frac{1}{\alpha_2(\mu)} - 2\frac{1}{\alpha_3(\mu)}\right) = -\frac{12}{5}L_{\mu H_c},$$
$$4\pi\left(5\frac{1}{\alpha_1(\mu)} - 3\frac{1}{\alpha_2(\mu)} - 2\frac{1}{\alpha_3(\mu)}\right) = -24\left(L_{\mu X} + \frac{1}{2}L_{\mu\Sigma}\right), \tag{129}$$

where $\alpha^{SU(5)}$ has been eliminated. These equations allow the prediction of the colored triplet Higgs boson mass M_{H_c} from the knowledge of the MSSM gauge couplings at the energy scale $\mu = \mu_{GUT}$. It is furthermore common to define a new mass parameter $M_G = \sqrt[3]{M_X^2 M_\Sigma}$, the so-called grand unified mass scale. It can also be determined from the knowledge of the MSSM gauge couplings at μ_{GUT} through (129). These observations makes it quite easy to test the minimal SUSY SU(5) model once the required experimental data are available in combination with a high-order analysis.

(5) Running from μ_{GUT} to the Planck scale M_{Pl}.

The last sequence of this approach consists in the running within the SUSY-SU(5) model. The three-loop RGEs for the gauge [64], and the one-loop formulas for the Yukawa, and Higgs self-couplings [220] are available in the literature. In addition, the perturbativity constraints (i.e., all couplings of the theory are smaller than unity) have to be imposed.

6.2.1. Input Parameters. As can be inferred from the discussion above, to constrain the GUT parameters one needs in addition to the precise running analysis also precise input parameters. Explicitly, one needs the values of weak mixing angle in the $\overline{MS}$ scheme [221], the QED coupling constant at zero-momentum transfer, and its hadronic contribution [222] in order to obtain its counterpart at the Z-boson scale, and the strong coupling constant [182]. (We adopt the central value from [182]; however, we use it as our default choice for the uncertainty 0.0020 instead of 0.0007.) Their numerical values and uncertainties are

$$\begin{aligned} \sin^2\Theta^{\overline{MS}} &= 0.23119 \pm 0.00014, \\ \alpha &= \frac{1}{137.036}, \\ \Delta\alpha^{(5)}_{had} &= 0.02761 \pm 0.00015, \\ \alpha_s\left(M_Z\right) &= 0.1184 \pm 0.0020. \end{aligned} \tag{130}$$

To determine the value of α in the $\overline{MS}$ scheme, it is necessary to take into account the hadronic $\Delta\alpha^{(5)}_{had}$, leptonic $\Delta\alpha^{(5)}_{lep}$ [223], and top quark $\Delta\alpha^{(5)}_{top}$ [224] contributions to the on-shell value. In addition, the conversion formula to the $\overline{MS}$ scheme has to be taken into account. Thus, one obtains

$$\begin{aligned} &\alpha^{\overline{MS}}\left(M_Z\right) \\ &= \frac{\alpha}{1 - \Delta\alpha^{(5)}_{lep} - \Delta\alpha^{(5)}_{had} - \Delta\alpha^{(5)}_{top} - \left(\Delta\alpha^{(5),\overline{MS}} - \Delta\alpha^{(5),OS}\right)} \\ &= \frac{1}{127.960 \pm 0.021}. \end{aligned} \tag{131}$$

For supersymmetric particle masses of order $\mathcal{O}(1\,\text{TeV})$ it is appropriate to take into account top quark threshold effects in a separate step. For convenience, we choose the scale at which we decouple the top quark to be $\mu_{dec} = M_Z$. The corresponding threshold corrections are available from [221, 225, 226] and give the following contributions:

$$\begin{aligned} \alpha^{(6),\overline{MS}}\left(M_Z\right) &= \frac{1}{(128.129 \pm 0.021)}, \\ \sin^2\Theta^{(6),\overline{MS}}\left(M_Z\right) &= 0.23138 \pm 0.00014, \\ \alpha_s^{(6)}\left(M_Z\right) &= 0.1173 \pm 0.0020. \end{aligned} \tag{132}$$

Even more, the supersymmetric particles can induce sizeable effects in the extraction of the weak mixing angle from experimental data. Such effects are by construction suppressed by the square of the supersymmetric mass scale [214, 227]. For a typical supersymmetric mass scale ≥ 1 TeV such corrections can lead to shifts in M_{H_c} of the order of $\leq 10\%$.

6.2.2. Numerical Analysis. For illustration of the numerical effects we adopt the `mSUGRA` [228] scenario for the SUSY breaking mechanism with the following initial parameters:

$$\begin{aligned} m_0 = m_{1/2} = -A_0 &= 1000\,\text{GeV}, \\ \tan\beta = 3, \qquad \mu &> 0 \end{aligned} \tag{133}$$

and generate with the help of the code `SOFTSUSY` [229] the supersymmetric mass spectrum. This results in squark masses of the order of 2 TeV, thus beyond the above exclusion bounds currently established by direct searches at the LHC.

In Figure 7 from we visualize the running (and decoupling) of the gauge couplings where the parameters of (133) together with $\mu_{SUSY} = 1000\,\text{GeV}$ and $\mu_{GUT} = 10^{16}\,\text{GeV}$ have been adopted. In addition we have chosen $M_\Sigma = 1 \cdot 10^{15}$ GeV which leads via (129) to $M_{H_c} = 1.7 \cdot 10^{15}$ GeV and $M_X = 4.6 \cdot 10^{16}$ GeV. One can clearly see the discontinuities at the matching scales and the change of the slopes when passing them. In Figure 7(b) the region around $\mu = 10^{16}$ GeV is enlarged which allows for a closer look at the unification region. The bands indicate 1σ uncertainties of α_i at the electroweak scale (cf. (132)). In Figure 7(b) we furthermore perform the decoupling of the superheavy masses for two different values of μ_{GUT}. One observes quite different threshold corrections leading to a nice agreement of $\alpha^{SU(5)}$ above 10^{16} GeV. Figure 7 stresses again that the uncertainty of α_s is the most important one for the constraints that one can set on GUT models from low-energy data. Furthermore, it illustrates the size of the GUT threshold corrections and emphasizes the importance of precision calculations.

In the following, we discuss the dependence of M_{H_c} and M_G on various parameters entering our analysis. We start with varying the supersymmetric mass spectrum and use (129) in order to extract both M_{H_c} and M_G. The decoupling scales are fixed to $\mu_{SUSY} = 1000\,\text{GeV}$ and $\mu_{GUT} = 10^{16}\,\text{GeV}$, respectively, which ensures that the three-loop effect is rather small. In Figure 8 the parameter $m_{1/2}$ is varied up to 4 TeV. The solid and dashed lines correspond to M_{H_c} and M_G, respectively, which show a substantial variation. On the other hand, m_0, $\tan\beta$, and A_0 have only a minor influence on the GUT masses and thus we refrain from explicitly showing the dependence.

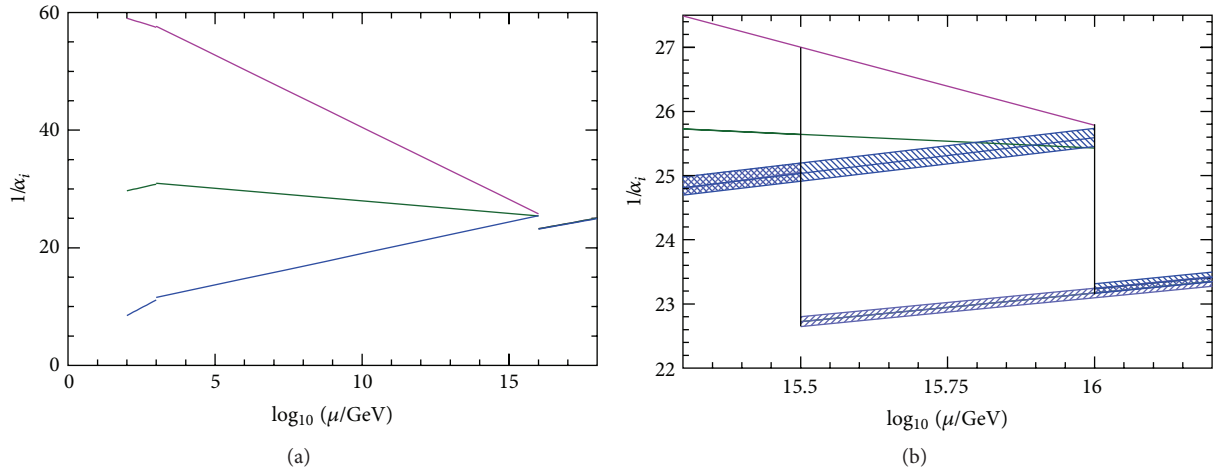

FIGURE 7: Running of the gauge couplings from the electroweak to the Planck scale from [51]. The discontinuity for $\mu = \mu_{\text{SUSY}}$ and $\mu = \mu_{\text{GUT}}$ is clearly visible. In panel (b) an enlargement of (a) for the region around $\mu = \mu_{\text{GUT}}$ is shown where for the decoupling the two values $\mu_{\text{GUT}} = 10^{15.5}$ GeV $\approx 3.2 \cdot 10^{15}$ GeV and $\mu_{\text{GUT}} = 10^{16}$ GeV have been chosen.

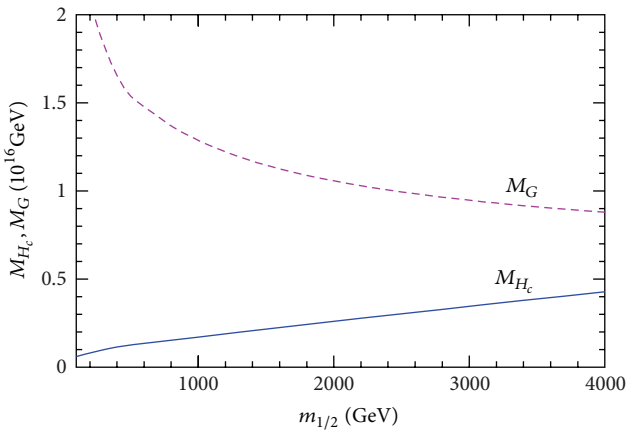

FIGURE 8: Dependence of M_{H_c} (solid) and M_G (dashed) on $m_{1/2}$ from [51].

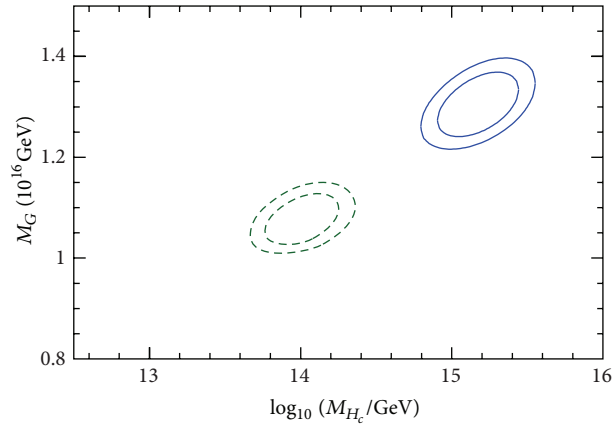

FIGURE 9: Ellipses in the M_{H_c}-M_G plane obtained from the uncertainties of the gauge coupling sat the electroweak scale. The input parameters of (130) have been used whereas $\delta\alpha_s = 0.0010$ has been chosen. Dashed and solid lines correspond to the two- and three-loop analysis, respectively.

An interesting aspect from the phenomenological point of view is the study of the effects of the experimental uncertainties of α_i (c.f. (130)) on the prediction of M_{H_c} and M_G. For this, we fix the SUSY spectrum as before (see (133)) and set $\mu_{\text{SUSY}} = M_Z$ which has often been common practice in similar analyses (see, e.g., [209]). Taking into account correlated errors and performing a χ^2 analysis lead to ellipses in the M_{H_c}-M_G plane. In Figure 9 from the results for the two- (dashed lines) and three-loop (continuous lines) analyses are shown. The two concentric ellipses correspond to 68% and 90% confidence level, respectively, where only parametric uncertainties from (132) have been taken into account. Let us, however, stress that an optimistic uncertainty of $\delta\alpha_s = 0.0010$ has been adopted for this plot. As expected, the uncertainty of α_s induces the largest contributions to the uncertainties on M_{H_c} and M_G. In particular, it essentially determines the semimajor axis of the ellipses. The three-loop corrections induce a significant shift to higher masses of about an order of magnitude for M_{H_c}. In the same time M_G increases by about $2 \cdot 10^{15}$ GeV. This demonstrates the importance of the precision calculations in such type of analyses. As has been discussed in the original paper [51] they are also essential in order to remove the dependence on μ_{SUSY}. In fact, choosing μ_{SUSY} close to the supersymmetric mass scale leads to small three-loop effects, since the two-loop ellipses are essentially shifted on top of the three-loop ones.

At this point a discussion about the additional constraint on the Higgs triplet mass M_{H_c} that can be derived from the nonobservation of the proton decay is in order. The latest upper bound on the proton decay rate for the channel $p \to K^+\overline{\nu}$ [230] is $\Gamma_{\text{exp}} = 4.35 \times 10^{-34}/y$. In order to translate it into a lower bound for the Higgs triplet mass, one needs an additional assumption about the Yukawa couplings that enter

the expression of the decay rate $\Gamma(p \to K^+\overline{\nu})$. As pointed out in [211] this is because down quark and lepton Yukawa couplings fail to unify within the minimal renormalizable SUSY SU(5) model and so a completely consistent treatment is not possible. Therefore one could either choose (Y_{ql} is the Yukawa coupling of the quark and lepton doublets to the Higgs color triplet; Y_{ud} is the corresponding coupling for the up and down quark singlets) (i) $Y_{ql} = Y_{ud} = Y_d$ or (ii) $Y_{ql} = Y_{ud} = Y_e$, which leads to completely different phenomenological consequences. Both choices are equally justified once higher dimensional operators are included. Since these operators further weaken the bounds presented below, we refrain from including these bounds into the analysis presented here. For the case (i) and supersymmetric particle masses around 1 TeV the lower bound for the Higgs triplet mass can be read off from Figure 2 of [211] and amounts to $M_{H_c} \geq 1.05 \times 10^{17}$ GeV whereas for the second choice it becomes $M_{H_c} \geq 5.25 \times 10^{15}$ GeV. From our phenomenological analysis presented above it turns out that within the minimal SUSY SU(5) model the upper bound for M_{H_c} is of about 10^{16} GeV. Thus, the substantial increase of about one order of magnitude for the upper bound on M_{H_c} induced by the three-loop order running analysis attenuates the tension between the theoretical predictions made under the assumption (i) and the experimental data. The choice (ii) for the Yukawa couplings clearly shows that the minimal SUSY SU(5) model cannot be ruled out by the current experimental data on proton decay rates. More experimental information about the SUSY mass spectrum and proton decay rates is required in order to be able to draw a firm conclusion.

7. The Mass of the Lightest Higgs Boson in the MSSM

7.1. Higgs Boson Mass in the SM. Spontaneous symmetry breaking was introduced into the particle physics in the seminal papers [231–233] and the existence of the Higgs boson was postulated by Higgs in 1964 in [234]. The next important step was the incorporation of the spontaneous symmetry breaking into the unified model of the weak and electromagnetic interactions [2, 3]. The breakthrough of these ideas came with the proof of the renormalizability of spontaneously broken gauge theories by' t Hooft and Veltman [69, 235].

The direct Higgs boson searches performed at LEP 1 in $Z^0 \to H + \overline{f}f$ and at LEP 2 in $e^+e^- \to Z^0 + H$ channels provided us with a lower bound on its mass of $M_h > 114.4$ GeV at the 95% CL [236]. In parallel to the direct searches, the high precision electroweak data obtained at LEP allowed us to estimate the possible mass range of the Higgs boson within the SM, namely, $M_h = 96^{+31}_{-21}$ GeV [237]. Moreover, the CDF and D0 experiments at the Tevatron [238] excluded the range of Higgs masses $156 < M_h < 177$ GeV as well as lower masses in the range already excluded by LEP. Only very recently, the existence of the Higgs boson could have been experimentally confirmed by the ATLAS and CMS collaborations at the LHC [31, 32]. Its mass is around 125-126 GeV. Currently, dedicated analysis is performed in order to establish if the observed boson is just the one predicted by the SM or hints towards new physics.

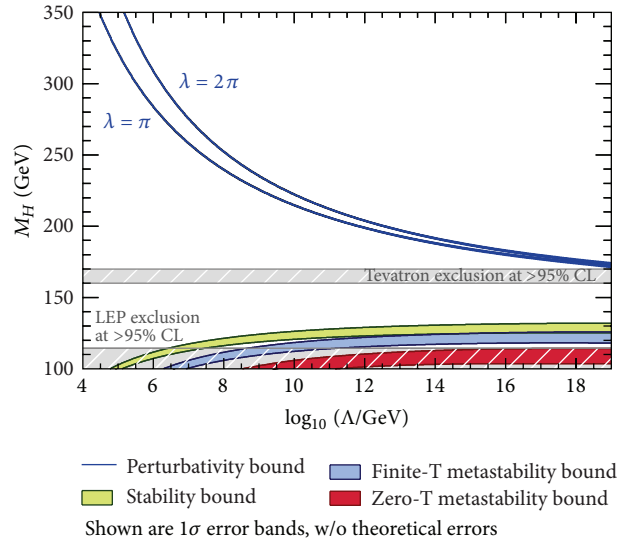

FIGURE 10: The Higgs boson mass as a function of the scale Λ up to which the SM may remain valid was obtained from perturbativity (solid dark blue line) and the stability of the electroweak vacuum (shaded regions). The figure is taken from [52].

The Higgs boson mass itself is a fundamental parameter of the SM. Together with the top quark mass and the strong coupling constant, it plays a crucial role in determining the stability bounds for the SM electroweak vacuum. The usual way to present this interplay is to display the allowed domains for M_h as a function of Λ, the scale up to which the SM may remain valid. If M_h is too large, the RGEs of the SM drive the Higgs self-coupling into the nonperturbative regime at some scale $\Lambda < M_{\text{Planck}}$. This is shown as the upper pair bold lines in Figure 10 that is taken from [52]. In this case new physics at a scale Λ will be required in order to prevent the Higgs self-coupling to blow up. On the other hand if M_h is too small, the RGEs drive the Higgs self-coupling to a negative value. In this case the Higgs potential can develop an instability at high field values $> \Lambda$, unless there is new physics at some scale $< \Lambda$ that prevents the occurrence of an additional minimum in the potential. This is shown as light shaded bands in Figure 10. Between the blow-up and the stability regions, there is a range of intermediate values of M_h for which the SM can survive up to the Planck scale. Taking into account the current theoretical and experimental errors on M_h, M_t, and α_s, stability up to the Planck scale cannot be yet excluded [53].

Nevertheless, as shown in Figure 11 from [53] and confirmed by [239], the range of M_h as revealed by the present searches at the LHC lies right at the edge between electroweak stability and instability regions. The possibility that the SM potential becomes unstable at large field values, below the Planck scale, does not contradict any experimental observation, provided its lifetime is longer than the age of the Universe. Indeed, the authors of [240] found that for $M_h =$

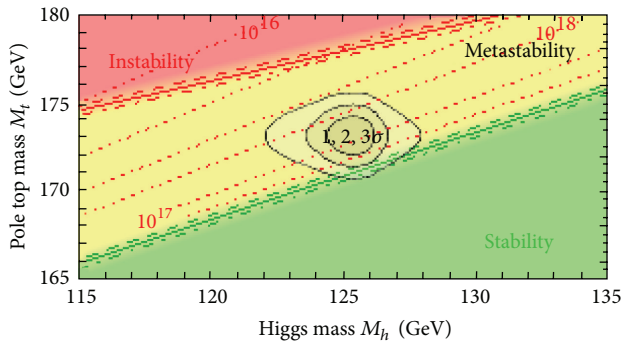

FIGURE 11: Measured value of the top mass and preferred range of m_h as revealed by the present searches at the LHC, compared to the regions corresponding to absolute stability, metastability, and instability of the SM vacuum [53]. The three boundaries lines correspond to $\alpha_s(M_Z) = 0.1184 \pm 0.0007$, and the grading of the colors indicates the size of the theoretical errors. The dotted contour lines show the instability scale Λ in GeV assuming $\alpha_s(M_Z) = 0.1184$.

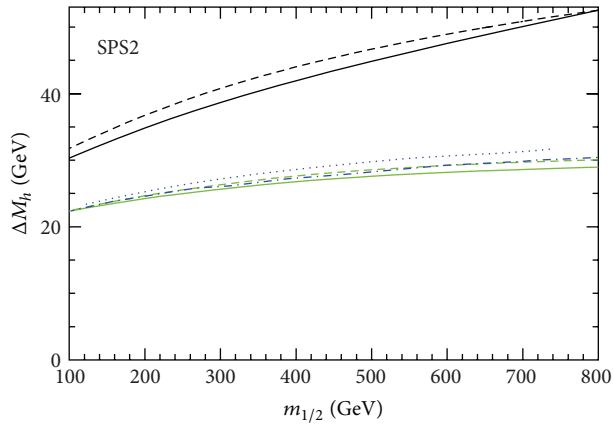

FIGURE 12: Comparison of complete and approximate one- and two-loop corrections to the Higgs boson mass for SPS2 scenario. The solid (full result) and dashed lines (m_t^4 approximation) represent the results in the on-shell scheme where the black and gray curves correspond to the one- and two-loop results, respectively. For comparison, the two-loop $\overline{\text{DR}}$ results are shown as dash-dotted (full result) and dotted (m_t^4 approximation) curves.

125 GeV, the instability scale lies around $10^{11\pm1}$ GeV. In this case, tunneling through quantum fluctuations is slow enough to ensure at least metastability of the electroweak vacuum.

It is also interesting to note that the SM extrapolation of the Higgs parameters (the mass parameter m^2 and quartic coupling λ) corresponds to near vanishing λ and its beta function at the Planck scale. The coupling $\lambda = 0$ is the critical value for the electroweak stability. Moreover, the coefficient m^2 of the Higgs bilinear in the scalar potential is also approximately zero (at the Planck scale). This is again a critical value that separates the symmetric phase ($m^2 > 0$) from the broken phase ($m^2 < 0$). At present, we do not know if this is just a numerical coincidence or the consequence of an underlying symmetry.

There are different interpretations in the literature for the near criticality of the SM parameters. For instance, SUSY implies that $m^2 = 0$. If SUSY is softly broken, m^2 would remain near zero, solving the hierarchy problem. Nevertheless, the analysis performed in [240] shows that the usual low-scale SUSY scenario can accommodate a Higgs mass around 125 GeV only for extreme values of the parameters, for example, large $\tan\beta$, heavy stops, or maximal stop mixing. Other explanations of the near criticality can be given via interpreting the Higgs as a Goldstone boson (composite Higgs modes) or as a consequence of transplanckian dynamics (like in multiverse models). In the following we concentrate on the SUSY explanation.

7.2. Higgs Boson Mass in the MSSM. A natural possibility to counterbalance the effects of the top quark on the evolution of the Higgs self-coupling was found within SUSY models, via the opposite effects induced by the top quark superpartners. The mass of the Higgs boson within SUSY models is linked to the magnitude of its self-coupling, which in turn is fixed by SUSY in terms of the electroweak gauge couplings. Compared to the SM, the MSSM Higgs sector is described by two additional parameters, usually chosen to be the pseudoscalar mass M_A and the ratio of the vacuum expectation values of the two Higgs doublets, $\tan\beta = v_2/v_1$. The masses of the other Higgs bosons are then fixed by SUSY constraints. In particular, the mass of the light CP-even Higgs boson, M_h, is bounded from above. At tree level, this constraint reads $M_h < M_Z$. Radiative corrections to the Higgs pole mass raise this bound substantially to values that were inaccessible at LEP [241–243]. The dominant radiative corrections are given by the contribution $\sim \alpha_t m_t^2 \sim m_t^4$ coming from top- and top squark loops (m_t is the top quark mass and $\sqrt{\alpha_t}$ is proportional to the top Yukawa coupling). For illustration, complete and approximate (i.e., only contributions $\sim m_t^4$) one- and two-loop corrections to the lightest Higgs boson mass are shown in Figure 12 from [156]. In this figure, the mass difference $\Delta M_h = M_h^{i-\text{loop}} - M_h^{\text{tree}}$ is shown as a function of the parameter $m_{1/2}$ in the scenario SPS2 [244, 245]. The small differences between the solid (full result) and dashed (m_t^4 approximation) lines demonstrate that the leading term $\sim m_t^4$ approximates the full result to a high accuracy. This motivates the computation of higher order corrections taking into account only the contributions that scale like $\sim m_t^4$.

From the one-loop corrections to the Higgs pole masses, that are known without any approximations [195, 246–248], one can show that a second approximation is appropriate. The bulk of the numerical effects can be obtained in the so-called effective-potential approach, for which the external momentum of the Higgs propagator is set to zero. Most of the relevant two-loop corrections have been evaluated in this approach (for reviews, see, e.g., [154, 249]). In addition, two-loop corrections including even CP-violating couplings and improvements from renormalization group considerations have been computed in [154, 249, 250]. In particular CP violating phases can lead to a shift of a few GeV in M_h; see, for example, [251, 252]. In [253] a large class of subdominant two-loop corrections to the lightest Higgs boson mass have been

considered. Furthermore, leading logarithmic corrections at three-loop order have been computed in [254]. The first complete three-loop calculation of the leading quartic top quark mass terms within supersymmetric QCD has been performed in [156, 255].

There are by now three computer programs publicly available which include most of the higher order corrections. `FeynHiggs` has been available already since 1998 [250, 256, 257] and has been continuously improved since then [258]. In particular, it contains all numerically important two-loop corrections and accepts both real and complex MSSM input parameters. The second program, `CPSuperH` [259, 260], is based on a renormalization group improved calculation and allows for explicit CP violation. Both programs compute the masses as well as the decay widths of the neutral and charged Higgs bosons. The third program, `H3m` [261], contains all currently available three-loop results. Furthermore, `H3m` constitutes an interface to `FeynHiggs` [262] and various SUSY spectrum generators, which allows for precise predictions of M_h on the basis of realistic SUSY scenarios.

7.2.1. Calculation of $\mathcal{O}(\alpha_t\alpha_s^2)$ Corrections in the MSSM. In this section we focus on details of the calculation of the lightest Higgs boson mass to three-loop accuracy in SUSY-QCD. It was the first calculation of an observable at this order of accuracy in the framework of SUSY-QCD and it raised technical difficulties specific to higher order calculations.

At tree level, the mass matrix of the neutral, CP-even Higgs bosons h, H has the following form:

$$\mathscr{M}^2_{H,\text{tree}} = \frac{\sin 2\beta}{2} \times \begin{pmatrix} M_Z^2\cot\beta + M_A^2\tan\beta & -M_Z^2 - M_A^2 \\ -M_Z^2 - M_A^2 & M_Z^2\tan\beta + M_A^2\cot\beta \end{pmatrix}. \tag{134}$$

The diagonalization of $\mathscr{M}^2_{H,\text{tree}}$ gives the tree-level result for M_h and M_H and leads to the well-known bound $M_h < M_Z$ which is approached in the limit $\tan\beta \to \infty$.

The mass matrix $\mathscr{M}^2_H$ is obtained from the quadratic terms in the Higgs boson potential constructed from the fields ϕ_1 and ϕ_2. They are related to the physical Higgs mass eigenstates via the mixing angle α

$$\begin{pmatrix} H \\ h \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{135}$$

As usual, h stands for the lightest Higgs boson. The mixing angle α is determined at the leading order through

$$\tan 2\alpha = \tan 2\beta \frac{M_A^2 + M_Z^2}{M_A^2 - M_Z^2}, \quad -\frac{\pi}{2} < \alpha < 0, \tag{136}$$

where M_Z is the mass of the Z boson and $\tan\beta = v_2/v_1$. Since ϕ_1 does not couple directly to top quarks, it is convenient to perform the calculations of the Feynman diagrams in the (ϕ_1, ϕ_2) basis.

Including higher order corrections, one obtains the Higgs boson mass matrix

$$\mathscr{M}^2_H = \mathscr{M}^2_{H,\text{tree}} - \begin{pmatrix} \widehat{\Sigma}_{\phi_1} & \widehat{\Sigma}_{\phi_1\phi_2} \\ \widehat{\Sigma}_{\phi_1\phi_2} & \widehat{\Sigma}_{\phi_2} \end{pmatrix}, \tag{137}$$

which again gives the physical Higgs boson masses upon diagonalization. The renormalized quantities $\widehat{\Sigma}_{\phi_1}$, $\widehat{\Sigma}_{\phi_2}$, and $\widehat{\Sigma}_{\phi_1\phi_2}$ are obtained from the self-energies of the fields ϕ_1, ϕ_2, and A, evaluated at zero external momentum, as well as from tadpole contributions of ϕ_1 and ϕ_2 (see, e.g., [154]). One has

$$\begin{aligned} \widehat{\Sigma}_{\phi_1} &= \Sigma_{\phi_1} - \Sigma_A \sin^2\beta \\ &\quad + \frac{e}{2M_W \sin\vartheta_W} t_{\phi_1} \cos\beta \left(1 + \sin^2\beta\right) \\ &\quad - \frac{e}{2M_W \sin\vartheta_W} t_{\phi_2} \cos^2\beta \sin\beta, \\ \widehat{\Sigma}_{\phi_2} &= \Sigma_{\phi_2} - \Sigma_A \cos^2\beta \\ &\quad - \frac{e}{2M_W \sin\vartheta_W} t_{\phi_1} \sin^2\beta \cos\beta \\ &\quad + \frac{e}{2M_W \sin\vartheta_W} t_{\phi_2} \sin\beta \left(1 + \cos^2\beta\right), \\ \widehat{\Sigma}_{\phi_1\phi_2} &= \Sigma_{\phi_1\phi_2} + \Sigma_A \sin\beta\cos\beta \\ &\quad + \frac{e}{2M_W \sin\vartheta_W} t_{\phi_1} \sin^3\beta \\ &\quad + \frac{e}{2M_W \sin\vartheta_W} t_{\phi_2} \cos^3\beta. \end{aligned} \tag{138}$$

In this equation, ϑ_W is the weak mixing angle, Σ_A denotes the self-energy of the pseudoscalar Higgs boson, and t_{ϕ_i} denotes the tadpole contributions of the field ϕ_i. Typical diagrams to the individual contributions can be found in Figure 13.

Considering the many different mass parameters entering the formula for the Higgs boson mass an exact calculation of the three-loop corrections is currently not feasible. However, it is possible to apply expansion techniques [150] for various limits which allow to cover a large part of the supersymmetric parameter space. After the application of the asymptotic expansion the resulting integrals have to be reduced to an independent set of master integrals. For the case of the Higgs mass corrections there will be only three-loop tadpole integrals that can be handled with the program MATAD [124].

A technical subtlety arises when calculating diagrams like those shown in Figure 14. If both the external momentum and the ε-scalar mass are set to zero from the beginning, an infrared divergence occurs and cancels the ultraviolet divergence of the integral. In effect, the diagram will be of order $(d-4)$ due to the ε-scalar algebra. In order to avoid this, one can keep the external momentum q nonzero, though much smaller than all other scales. The ultraviolet pole multiplied by the algebraic factor of $(d-4)$ then produces a finite contribution, while the infrared divergence leads to a

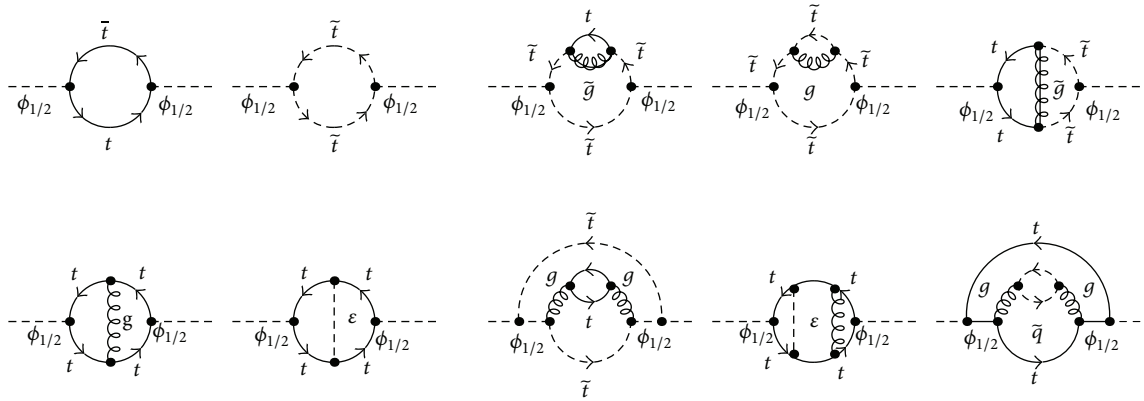

FIGURE 13: Sample diagrams contributing to Σ_{ϕ_1}, Σ_{ϕ_2}, $\Sigma_{\phi_1\phi_2}$, and so forth to one, two, and three loops. Internal solid, dashed, dotted, and curly lines correspond to top quarks, top squarks, ε-scalar, and gluons, respectively. Gluinos are depicted with curly lines with an additional solid line in the middle. The external dashed line corresponds to the Higgs bosons.

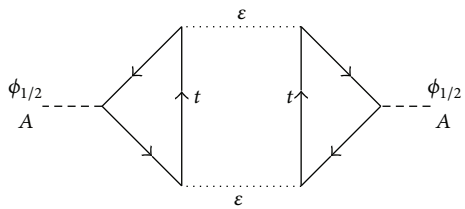

FIGURE 14: Sample diagram contributing a finite term to Σ_{ϕ_1}, Σ_{ϕ_2}, and so forth when the infrared divergence are regulated through a small external momentum or a finite ε-scalar mass.

contribution $(d-4)\ln(q^2)$ that vanishes as $d \to 4$. Instead of the requirement $q \neq 0$ one could also introduce a nonzero mass for the ε-scalars in order to regulate the infrared divergences. In the final result one observes that the regulator is multiplied by an additional factor $(d-4)$ leading to a finite result for $M_\varepsilon \to 0$. Alternatively, one can allow a nonzero ε-scalar mass and shift the squark mass counterterms so that all M_ε dependent contributions in the final result cancel out (this renormalization scheme is equivalent to the $\overline{\text{DR}}'$ scheme discussed in Section 4.2; however it is not identical). All these renormalization prescriptions lead to identical results for the corrections to the Higgs boson mass M_h, that is a nontrivial check of the calculation.

Concerning the renormalization, it is well known that the perturbative series can exhibit a bad convergence behavior in case it is parametrized in terms of the on-shell quark masses (for a typical example we refer to the electroweak ρ parameter; using the on-shell top quark mass the four-loop corrections [263–265] are larger by a factor 50 as compared to the $\overline{\text{MS}}$ scheme) which is due to intrinsically large contributions related to the infrared behavior of the theory. Thus, it is tempting to reparametrize the results for the Higgs boson mass in terms of the top quark mass renormalized in the $\overline{\text{DR}}$ scheme. Moreover, the two-loop renormalization constants for the masses of the SUSY particles and the top squark mixing angle, that are required for this calculation, are much more complicated in the on-shell scheme as compared to the $\overline{\text{DR}}$ ones. Thus, it is preferable to adopt the $\overline{\text{DR}}$ scheme also for these parameters. The renormalization constants for the gluino and ε-scalar masses are needed only at the one-loop order. For them, both schemes are accessible. Nevertheless, the ε-scalar mass renormalized in the on-shell scheme might be better suited for this type of calculations. In this case, it can be set equal to zero in the three-loop diagrams, which makes the calculation less involved. An extensive discussion about the calculation of the two-loop renormalization constants required in this computation as well as explicit formulae can be found in Section 4. In the remainder of this section we will refer to this renormalization scheme as $\overline{\text{DR}}$ scheme although it contains a mixture of on-shell and $\overline{\text{DR}}$ parameters in order to distinguish between it and the genuine on-shell scheme.

At this point a comment concerning the minimal $\overline{\text{DR}}$ renormalization constants for the masses of the top squarks is in order. Due to diagrams involving heavy squarks $\tilde{q}$, for example, Figure 15(a), the squared Higgs boson mass receives contributions which are proportional to $m_{\tilde{q}}^2$ and thus can lead to unnatural large corrections. For this reason the on-shell scheme for these contributions is better suited, because it avoids the potentially large terms $\sim m_{\tilde{q}}^2$ from the three-loop diagrams. The renormalization of the mixing angle is free of such enhanced contributions and can be done in the pure $\overline{\text{DR}}$ scheme. A similar behavior is observed when the gluino is much heavier than the top squarks [130, 266]. In this case, the two- and three-loop corrections to the Higgs masses contain terms proportional to $m_{\tilde{g}}$ and $m_{\tilde{g}}^2$. These contributions are canceled when the masses are renormalized in the on-shell scheme by the finite parts of the relevant counterterms. Thus, in order to avoid unnatural large radiative corrections to the Higgs mass for scenarios with heavy gluinos a modified nonminimal renormalization scheme for the top squark masses is required. The additional finite shifts of top squark masses are chosen such that they cancel the power-like behavior of the gluino contributions. Again, the renormalization of the mixing angle will

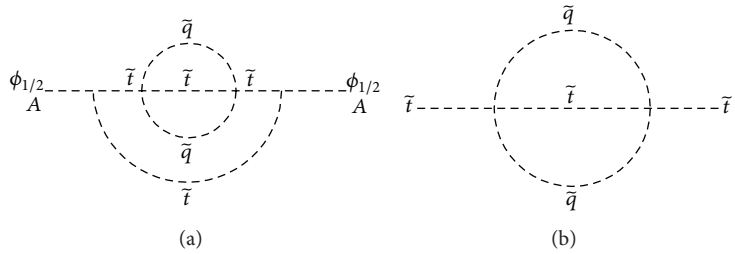

FIGURE 15: (a) Feynman diagram involving a heavy virtual squark contributing to the Higgs boson self-energy. (b) Counterterm diagram related to the diagram in (a). The same notation as in Figure 13 has been adopted.

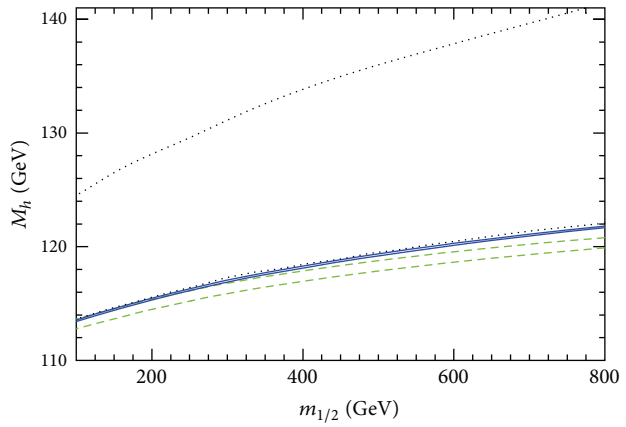

FIGURE 16: Renormalization scheme dependence of M_h as a function of $m_{1/2}$ adopting SPS2. Dotted, dashed, and solid curves correspond to one-, two-, and three-loop results. The $\overline{\text{DR}}$ (on-shell) results correspond to the lower (upper) curves. The three-loop curves obtained in the two renormalization schemes lay on top of each other.

not be modified as compared to the genuine $\overline{\text{DR}}$ scheme. The relevant finite shifts for commonly adopted scenarios are explicitly given in Appendix B.

As an illustration of the renormalization scheme issue, we show in Figure 16 from [156] the renormalization scheme dependence of M_h as a function of the parameter $m_{1/2}$ for the SPS2 scenario. In the left panel of Figure 16 the upper dotted, dashed, and solid curves correspond to the one-, two-, and three-loop prediction of M_h in the on-shell scheme whereas the corresponding lower three curves are obtained in the $\overline{\text{DR}}$ scheme. In the on-shell scheme one observes large positive one-loop corrections which get reduced by 10 to 20 GeV after including the two-loop terms. The three-loop corrections amount to several hundred MeV. They are positive or negative depending on the value of $m_{1/2}$. The situation is completely different for $\overline{\text{DR}}$ mass parameters: the one-loop corrections are significantly smaller and lead to values of M_h which are already of the order of the two- and three-loop on-shell prediction. The two-loop term leads to a small shift of the order of -1 GeV and the three-loop term to a positive shift of about the same order of magnitude. The final prediction for M_h is very close to the one obtained after incorporating three-loop on-shell results. (There are regions in the parameter space where the two-loop corrections are accidentally small in the $\overline{\text{DR}}$ scheme leading to relatively large three-loop terms. Nevertheless the overall size of the two- and three-loop corrections is small).

The three-loop results have in general very long expressions. However, for simplifying assumptions about the supersymmetric mass spectrum, like for example the *natural* SUSY, for which the superpartners of the first and second generations of quarks are much heavier than the gluino and third generations of squarks, that is, $m_{\tilde{q}} \gg m_{\tilde{t}_1} \approx m_{\tilde{t}_2} \approx m_{\tilde{g}}$, the analytical expressions for the dominant contributions have a quite compact form. Let us mention that in general, for the case of quasi-degenerate masses a naive Taylor expansion in the mass differences is sufficient, while for large mass ratios an asymptotic expansion is necessary. For illustration, we give below the three-loop results for the two-point functions contributing to the Higgs boson mass, where for the renormalization of the stop quark masses the modified $\overline{\text{DR}}$ scheme as given in (B.1) was adopted. One has

$$\begin{aligned}\widehat{\Sigma}_{\phi_1} = {} & \frac{G_F m_t^4 \sqrt{2}}{\pi^2 \cos^2\beta}\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{A_t^2}{m_{\text{SUSY}}^2} \\ & \times \left[-\frac{349}{9} + \frac{32}{9}L_{\mu t} + \frac{32}{9}L_{\mu t}^2 + \left(\frac{56}{9} + \frac{64}{9}L_{\mu t}\right)L_{tS} \right. \\ & \left. + \frac{32}{9}L_{tS}^2 + \frac{94}{3}\zeta(3) + \mathcal{O}\left(\frac{m_{\text{SUSY}}^4}{m_{\tilde{q}}^4}\right)\right],\end{aligned}$$

$$\begin{aligned}\widehat{\Sigma}_{\phi_{12}} = {} & \frac{G_F m_t^4 \sqrt{2}}{\pi^2 \cos\beta \sin\beta} \\ & \times \left[\frac{\alpha_s}{4\pi}\frac{A_t}{m_{\text{SUSY}}}\left(-2 - 4L_{\mu t} - 2L_{tS}\right) + \left(\frac{\alpha_s}{4\pi}\right)^2 \right. \\ & \times \left\{ \frac{A_t^2}{m_{\text{SUSY}}^2} \right. \\ & \times \left(\frac{349}{9} - \frac{32}{9}L_{\mu t} - \frac{32}{9}L_{\mu t}^2 \right. \\ & + \left(-\frac{56}{9} - \frac{64}{9}L_{\mu t}\right)L_{tS} \\ & \left. \left. \left. - \frac{32}{9}L_{tS}^2 - \frac{94}{3}\zeta(3)\right)\right.\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{A_t m_{\mathrm{SUSY}}}{m_{\tilde{q}}^2}\\
&\times\left(40-\frac{160}{9}L_{t\tilde{q}}-\frac{40}{3}L_{t\tilde{q}}^2\right.\\
&\quad\left.+L_{tS}\left(\frac{160}{9}+\frac{40}{3}L_{t\tilde{q}}\right)-\frac{80}{3}\zeta(2)\right)\\
&+\frac{A_t}{m_{\mathrm{SUSY}}}\\
&\times\left(-\frac{416}{27}-\frac{364}{27}L_{\mu t}\right.\\
&\quad-\frac{100}{3}L_{\mu t}^2-\frac{304}{9}L_{tS}^2+\frac{200}{9}L_{t\tilde{q}}-20L_{t\tilde{q}}^2\\
&\quad+L_{tS}\left(-\frac{628}{27}-\frac{400}{9}L_{\mu t}+\frac{80}{3}L_{t\tilde{q}}\right)\\
&\quad\left.-40\zeta(2)+\frac{106}{3}\zeta(3)\right)\Bigg\}\\
&+\mathcal{O}\left(\frac{m_{\mathrm{SUSY}}^4}{m_{\tilde{q}}^4}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Sigma}_{\phi_2}=&\frac{G_F m_t^4\sqrt{2}}{\pi^2\sin^2\beta}\\
&\times\left[\frac{3}{2}L_{tS}+\frac{\alpha_s}{4\pi}\right.\\
&\quad\times\left(4+\left(4+16L_{\mu t}\right)L_{tS}+4L_{tS}^2\right.\\
&\qquad\left.+\frac{A_t}{m_{\mathrm{SUSY}}}\left(4+8L_{\mu t}+4L_{tS}\right)\right)\\
&\quad+\left(\frac{\alpha_s}{4\pi}\right)^2\\
&\quad\times\left\{\frac{2764}{9}-\frac{116}{27}L_{\mu t}-\frac{136}{3}L_{\mu t}^2\right.\\
&\qquad+\left(-\frac{644}{9}+\frac{164}{3}L_{\mu t}\right)L_{tS}^2\\
&\qquad+24L_{tS}^3+\frac{400}{3}L_{t\tilde{q}}-\frac{200}{3}L_{t\tilde{q}}^2-\frac{20}{3}L_{t\tilde{q}}^3\\
&\qquad-120\zeta(2)-80L_{t\tilde{q}}\zeta(2)+\frac{8}{3}\zeta(3)\\
&\qquad-\left(\frac{2216}{27}+\frac{644}{9}L_{\mu t}-\frac{328}{3}L_{\mu t}^2-40L_{t\tilde{q}}\right.\\
&\qquad\left.-20L_{t\tilde{q}}^2-40\zeta(2)+16\zeta(3)\right)L_{tS}\\
&\qquad+\frac{m_{\mathrm{SUSY}}^2}{m_{\tilde{q}}^2}\\
&\qquad\times\left(\frac{42356}{225}+8L_{tS}^2-\frac{2128}{45}L_{t\tilde{q}}-\frac{176}{3}L_{t\tilde{q}}^2\right.\\
&\qquad\quad\left.+\left(\frac{3928}{45}+\frac{152}{3}L_{t\tilde{q}}\right)L_{tS}-\frac{400}{3}\zeta(2)\right)\\
&\qquad+\frac{A_t m_{\mathrm{SUSY}}}{m_{\tilde{q}}^2}\\
&\qquad\times\left(-80+L_{tS}\left(-\frac{320}{9}-\frac{80}{3}L_{t\tilde{q}}\right)\right.\\
&\qquad\quad\left.+\frac{320}{9}L_{t\tilde{q}}+\frac{80}{3}L_{t\tilde{q}}^2+\frac{160}{3}\zeta(2)\right)\\
&\qquad+\frac{A_t}{m_{\mathrm{SUSY}}}\\
&\qquad\times\left(\frac{832}{27}+\frac{728}{27}L_{\mu t}+\frac{200}{3}L_{\mu t}^2+\frac{608}{9}L_{tS}^2\right.\\
&\qquad\quad+L_{tS}\left(\frac{1256}{27}+\frac{800}{9}L_{\mu t}-\frac{160}{3}L_{t\tilde{q}}\right)\\
&\qquad\quad-\frac{400}{9}L_{t\tilde{q}}+40L_{t\tilde{q}}^2\\
&\qquad\quad\left.+80\zeta(2)-\frac{212}{3}\zeta(3)\right)\\
&\qquad+\frac{A_t^2}{m_{\mathrm{SUSY}}^2}\\
&\qquad\times\left(-\frac{349}{9}+\frac{32}{9}L_{\mu t}+\frac{32}{9}L_{\mu t}^2\right.\\
&\qquad\quad+\left(\frac{56}{9}+\frac{64}{9}L_{\mu t}\right)L_{tS}+\frac{32}{9}L_{tS}^2\\
&\qquad\quad\left.\left.+\frac{94}{3}\zeta(3)\right)\right\}+\mathcal{O}\left(\frac{m_{\mathrm{SUSY}}^4}{m_{\tilde{q}}^4}\right)\Bigg],
\end{aligned}
\tag{139}
$$

with $m_t = m_t(\mu_r)$, $m_{\mathrm{SUSY}} = m_{\mathrm{SUSY}}(\mu_r) = m_{\tilde{t}_1}(\mu_r) = m_{\tilde{t}_2}(\mu_r) = m_{\tilde{g}}(\mu_r)$, $L_{\mu t} = \ln(\mu_r^2/m_t^2)$, $L_{tS} = \ln(m_t^2/m_{\mathrm{SUSY}}^2)$, and $L_{t\tilde{q}} = \ln(m_t^2/m_{\tilde{q}}^2)$, where μ_r is the renormalization scale.

7.2.2. Phenomenological Analysis. In order to quantify the phenomenological significance of the three-loop contributions, it is interesting to investigate the dependence of M_h on SUSY parameters. In the following, we adopt the "modified $m_h^{\max}$" scenario as defined in [54]. The relevant MSSM parameters for our analysis are the top squark masses $m_{\tilde{t}_1}$ = 370 GeV and $m_{\tilde{t}_2}$ = 1045 GeV, the gluino mass $m_{\tilde{g}}$ = 860 GeV, the squark mass scale $m_{\tilde{q}}$ = 1042 GeV, the top trilinear coupling A_t = 1500 GeV, and the mass of the pseudoscalar Higgs M_A = 1000 GeV.

In Figure 17 from [54] is shown the comparison between the two- (dashed line) and three-loop (full line) predictions for the Higgs boson mass as a function of tan β parameter. As can be read from the plot, the genuine three-loop corrections amount to around 2 GeV for the given mass spectrum, independently of the value of tan β. Let us remind the reader

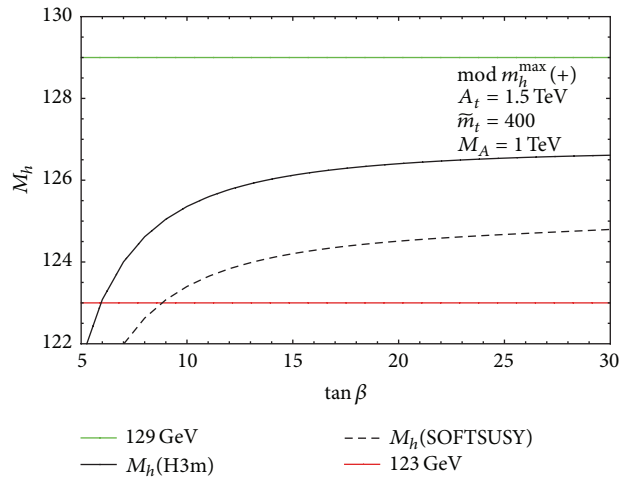

FIGURE 17: The light Higgs boson mass as a function of tan β to three-loop accuracy from [54].

that the experimental accuracy on M_h expected at the LHC is almost an order of magnitude smaller. It is also worth mentioning that the three-loop corrections are positive and increase the predicted value for M_h beyond 125 GeV. For increasing gluino and third generation squark masses, the light Higgs boson mass becomes larger and values well above 120 GeV can be reached.

We can infer from the above analysis that for a precise comparison with the experimental data expected from the LHC experiments, the three-loop corrections are indispensable. Moreover, the MSSM predictions can easily accommodate a light Higgs boson mass in the range $125 < M_h < 127$ GeV as observed in the current experiments at the LHC.

8. Hadronic Higgs Production and Decay in Susy Models

After the discovery of the new scalar particle with the mass around 125 GeV the most important question to be answered is whether it is indeed the Higgs boson predicted by the SM or it has another origin. To obtain the answer one has to study in detail the interaction properties of this new scalar with the SM particles. This task requires the comparison of the theory predictions for the production cross sections and the decay rates of the newly discovered scalar particle with the experimental data. In most of the cases, a precision of the theory predictions at the percent level is required in order to cope with the experimental accuracy. This implies that radiative corrections even at next-to-next to leading order (NNLO) have to be taken into account.

In the current section we concentrate on the radiative corrections up to NNLO to the hadronic Higgs production and decays within the MSSM. It turns out that in most of the cases only the NNLO SUSY-QCD corrections have to be taken into account. If available in the literature, also the NNLO top-Yukawa corrections, expected to be the next dominant contributions, will be discussed. As the exact analytic calculations are not always feasible, several theoretical methods employing phenomenologically well-motivated simplifying assumptions will be presented.

8.1. Effective Field Theory Formalism. In this section we want to derive the effective field theory formalism (EFT) following the method of operator product expansion (OPE) introduced by Wilson [267] (for a pedagogical overview of the method see also [75]). The main idea is again to disentangle the long and short distance physics from each other. Precisely, the long distance physics is described by local operators constructed from light degrees of freedom $\mathcal{O}_i$, whereas the effects of the heavy degrees of freedom are absorbed into coefficient functions of the operators. For QCD the relevant local operators have dimension four. Their renormalization and the issue of operator mixing under renormalization have been studied in detail in the literature [268–270]. For all processes studied in this section, the low-energy effective theory is QCD with five active flavors supplemented with a light Higgs boson. For completeness, we briefly review the main results concerning the renormalization of the local dimension four operators below.

In the following we assume for simplicity that the fundamental theory is described by the SUSY-QCD particle content together with the two Higgs doublets of the MSSM, ϕ_i, with $i = 1, 2$. The corresponding interactions are described by the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{SQCD}} + \sum_{i=1,2} \mathcal{L}_{q\phi_i} + \sum_{i=1,2} \mathcal{L}_{\tilde{q}\phi_i}, \tag{140}$$

where

$$\begin{aligned} \mathcal{L}_{q\phi_i} &= -\sum_{q=1}^{6} \frac{m_q}{v} g_q^{\phi_i} \bar{q} q \phi_i, \\ \mathcal{L}_{\tilde{q}\phi_i} &= -\sum_{q=1}^{6} \sum_{r,k=1,2} \frac{m_q}{v} g_{\tilde{q};kr}^{\phi_i} \tilde{q}_k^{\star} \tilde{q}_r \phi_i. \end{aligned} \tag{141}$$

$\mathcal{L}_{\text{QCD}} + \mathcal{L}_{\text{SQCD}}$ denotes the supersymmetric extension of the full QCD Lagrangian with six quark flavors. The couplings $g_q^{\phi_i}$ and $g_{\tilde{q};kr}^{\phi_i}$ are defined in Table 1, where $v = \sqrt{v_1^2 + v_2^2}$, with v_i, $i = 1, 2$, is obtained from the vacuum expectation values of the two Higgs doublets of the MSSM. The fields $\tilde{q}_i$, with $i = 1, 2$, denote as before the squark mass eigenstates, while θ_q stands for the mixing angle defined through

$$\begin{aligned} \sin 2\theta_q &= \frac{2 m_q X_q}{m_{\tilde{q}_1}^2 - m_{\tilde{q}_2}^2}, \\ X_q &= A_q - \mu_{\text{SUSY}} \begin{cases} \tan\beta, & \text{for down-type quarks}, \\ \cot\beta, & \text{for up-type quarks}, \end{cases} \end{aligned} \tag{142}$$

where A_q is the trilinear coupling and μ_{SUSY} the Higgs-Higgsino bilinear coupling.

We assume further the mass of the lightest Higgs boson h to be much smaller than the mass of the top quark and of the SUSY particles, as well as all the other Higgs bosons.

Table 1: Yukawa coupling coefficients for up and down type quark and squark, where $S_q = \sin 2\theta_q$ and $C_q = \cos 2\theta_q$, and $S_\beta = \sin\beta$ and $C_\beta = \cos\beta$.

f	$g_q^{\phi_1}$	$g_{\tilde{q}_1 11}^{\phi_1}$	$g_{\tilde{q}_1 12}^{\phi_1} = g_{\tilde{q}_1 21}^{\phi_1}$	$g_{\tilde{q}_1 22}^{\phi_1}$
up	0	$-\mu S_q/S_\beta$	$-\mu C_q/S_\beta$	$\mu S_q/S_\beta$
down	$1/C_\beta$	$(2m_q + A_q S_q)/C_\beta$	$A_q C_q/C_\beta$	$(2m_q - A_q S_q)/C_\beta$
f	$g_q^{\phi_2}$	$g_{\tilde{q}_2 11}^{\phi_2}$	$g_{\tilde{q}_2 12}^{\phi_2} = g_{\tilde{q}_2 21}^{\phi_2}$	$g_{\tilde{q}_2 22}^{\phi_2}$
up	$1/S_\beta$	$(2m_q + A_q S_q)/S_\beta$	$A_q C_q/S_\beta$	$(2m_q - A_q S_q)/S_\beta$
down	0	$-\mu S_q/C_\beta$	$-\mu C_q/C_\beta$	$\mu S_q/C_\beta$

In this case, the physical phenomena at low energies can be described by an effective theory containing five quark flavors and the light Higgs

$$\mathscr{L} \longrightarrow \mathscr{L}_Y^{\text{eff}} + \mathscr{L}_{\text{QCD}}^{(5)}, \tag{143}$$

where $\mathscr{L}_{\text{QCD}}^{(5)}$ denotes the Lagrangian of QCD with five active flavors.

At leading order in the heavy masses, the effective Lagrangian $\mathscr{L}_Y^{\text{eff}}$ can be written as a linear combination of three physical, gauge independent operators [185, 270] constructed from the light degrees of freedom

$$\mathscr{L}_Y^{\text{eff}} = -\frac{h^{(0)}}{v^{(0)}}\left[C_1^0 \mathcal{O}_1^0 + \sum_q \left(C_{2q}^0 \mathcal{O}_{2q}^0 + C_{3q}^0 \mathcal{O}_{3q}^0\right)\right], \tag{144}$$

where the coefficient functions C_i, $i = 1, 2q, 3q$, parametrize the effects of the heavy particles on the low-energy phenomena. The superscript 0 labels bare quantities. The three operators are defined as

$$\begin{aligned} \mathcal{O}_1^0 &= \left(G_{\mu,\nu}^{0,\prime,a}\right)^2, \\ \mathcal{O}_{2q}^0 &= m_q^{0,\prime}\bar{q}^{0,\prime} q^{0,\prime}, \\ \mathcal{O}_{3q}^0 &= \bar{q}^{0,\prime}\left(i\not{D}^{0,\prime} - m_q^{0,\prime}\right) q^{0,\prime}, \end{aligned} \tag{145}$$

where $G_{\mu,\nu}^{0,\prime,a}$ and $D_\mu^{0,\prime}$ are the gluon field strength tensor and the covariant derivative, respectively. The primes label the quantities in the effective theory. The relations between the parameters and fields in the full and effective theories have been derived in Section 6.1. The explicit formulae can be read off from (113). The operator $\mathcal{O}_{3q}$ vanishes by the fermionic equation of motion and it will not contribute to physical observables. Thus, the last term in (144) might be omitted, once the coefficients C_1^0, C_{2q}^0 are determined.

For convenience of the reader we reproduce the results for the renormalization constants of the operators $\mathcal{O}_1^0$ and $\mathcal{O}_{2q}^0$ that are of interest

$$\mathcal{O}_1 = Z_{11}\mathcal{O}_1^0 + Z_{12}\mathcal{O}_{2q}^0, \qquad \mathcal{O}_{2q} = Z_{22}\mathcal{O}_{2q}^0, \tag{146}$$

where

$$\begin{aligned} Z_{11} &= \left(1 - \frac{\pi}{\alpha_s'}\frac{\beta\left(\alpha_s'\right)}{\epsilon}\right)^{-1}, \\ Z_{12} &= -\frac{4\gamma_m\left(\alpha_s'\right)}{\epsilon}\left(1 - \frac{\pi}{\alpha_s'}\frac{\beta\left(\alpha_s'\right)}{\epsilon}\right)^{-1}, \\ Z_{22} &= 1, \\ C_1 &= Z_{11}^{-1} C_1^0, \qquad C_{2q} = C_{2q}^0 - \frac{Z_{12}}{Z_{11}} C_1^0. \end{aligned} \tag{147}$$

In the above equations the beta function and quark mass anomalous dimension γ_m refer to QCD with $n_l = 5$ active flavors evaluated in the $\overline{\text{MS}}$ scheme. They are needed up to three-loop order and have been given explicitly in Section 3.

The renormalized coefficient functions and operators are finite but not renormalization group (RG) invariant. In [271], a redefinition of the coefficient functions and operators was introduced so that they are separately renormalization group invariant. The RG invariant operators are defined as follows:

$$\begin{aligned} \mathcal{O}_g &= -\frac{2\pi}{\beta_0^{(5)}}\left(\frac{\pi\beta^{(5)}}{2\alpha_s^{(5)}}\mathcal{O}_1 - 2\gamma_m^{(5)}\sum_q \mathcal{O}_{2q}\right), \\ \mathcal{O}_q &= \mathcal{O}_{2q}, \end{aligned} \tag{148}$$

where the superscript (5) marks that there are five active quarks to be considered in the formulas for the beta function and the mass anomalous dimension γ_m. Accordingly, the associated coefficient functions are given by

$$\begin{aligned} C_g &= -\frac{\alpha_s^{(5)}\beta_0^{(5)}}{\pi^2\beta^{(5)}} C_1, \\ C_q &= \frac{4\alpha_s^{(5)}\gamma_m^{(5)}}{\pi\beta^{(5)}} C_1 + C_{2q}. \end{aligned} \tag{149}$$

This procedure allows us to choose independent renormalization scales for coefficient functions and operators. In practice, one chooses $\mu \approx M_h$ for the renormalization scale of the operators and $\mu \approx \widetilde{M}$ (where $\widetilde{M}$ denotes an averaged mass for the heavy supersymmetric particles) for the coefficient functions. Thus, (148) is to be evaluated at a low-scale $\mu \approx M_h$, whereas (149) is to be utilized at a high scale $\mu \approx \widetilde{M}$.

For the computation of the Higgs production and decay rates, it is however more convenient to reexpress the effective Lagrangian in terms of the operators $\mathcal{O}_1$ and $\mathcal{O}_{2q}$. However now, one keeps the separation of the scales for operators and

coefficient functions as given in (148) and (149). The new coefficient functions read [271]

$$C_1\left(\widetilde{M}, M_h\right) = \frac{\alpha_s'\left(\widetilde{M}\right)\beta^{(5)}\left(\alpha_s'\left(M_h\right)\right)}{\alpha_s'\left(M_h\right)\beta^{(5)}\left(\alpha_s'\left(\widetilde{M}\right)\right)} C_1\left(\widetilde{M}\right)$$
$$= -\frac{\pi^2\beta^{(5)}\left(\alpha_s'\left(M_h\right)\right)}{\left[\alpha_s'\left(M_h\right)\right]^2\beta_0^{(5)}} C_g\left(\widetilde{M}\right),$$
$$C_2\left(\widetilde{M}, M_h\right) = \frac{4\alpha_s'\left(\widetilde{M}\right)}{\pi\beta^{(5)}\left(\alpha_s'\left(\widetilde{M}\right)\right)} \times\left[\gamma_m^{(5)}\left(\alpha_s'\left(\widetilde{M}\right)\right) - \gamma_m^{(5)}\left(\alpha_s'\left(M_h\right)\right)\right] C_1\left(\widetilde{M}\right) + C_{2q}\left(\widetilde{M}\right). \tag{150}$$

The explicit computation of the coefficient functions will be discussed in detail in the next section.

8.2. Computation of the Coefficient Functions C_1 and C_{2q}. To calculate the coefficient functions one has to consider appropriate Green functions in the full and the effective theory and relate them via the decoupling relations. For example, the amputated Green function involving the $q\overline{q}$ pair and the zero-momentum insertion of the operator $\mathcal{O}_h$ which mediates the couplings to the light Higgs boson h contains both coefficient functions C_{2q} and C_{3q}

$$\Gamma^0_{\overline{q}q\mathcal{O}_h}(p,-p) = i^2\int \mathrm{d}x\mathrm{d}y e^{ip(x-y)}\left\langle Tq^0(x)\,\overline{q}^0(y)\,\mathcal{O}_h(0)\right\rangle^{1\mathrm{PI}} = -\zeta_2^{(0)}\int \mathrm{d}x\mathrm{d}y e^{ip(x-y)} \times\left\langle Tq'^{,0}(x)\,\overline{q}'^{,0}(y)\left(C_{2q}\mathcal{O}_{2q} + C_{3q}\mathcal{O}_{3q}\right)\right\rangle^{1\mathrm{PI}}, \tag{151}$$

where p is the outgoing momentum of the quark and we label the quantities in the effective theory with a prime.

Upon decomposition of the Green function $\Gamma^0_{\overline{q}q\mathcal{O}_h}$ into its scalar and vector components and taking the limit $p \to 0$, one obtains for the coefficient function C_{2q} the following expression:

$$C^0_{2q} = \frac{\Gamma^{0,h}_{\overline{q}q\mathcal{O}_h;s}(0,0)}{1-\Sigma_s^{0,h}(0)} + \frac{\Gamma^{0,h}_{\overline{q}q\mathcal{O}_h;v}(0,0)}{1+\Sigma_v^{0,h}(0)}. \tag{152}$$

The quantities $\Sigma_v^{0,h}(0)$ and $\Sigma_s^{0,h}(0)$ have been defined in (114). The superscript h in the above equation marks that only the hard parts of the Green functions survive when one sets the external momenta to zero $p^2 = p_h^2 = 0$.

From the technical point of view, to separate the vector and scalar contributions to the vertex Green function $\Gamma_{\overline{q}q\mathcal{O}_h}$ one has to perform a naive Taylor expansion up to linear order in the external momenta carried by quarks. After the projection on vector and scalar parts, the external momenta can be set to zero. Nevertheless, the light Higgs mass approximation $M_h^2 = p_h^2 \approx 0$ can be applied from the very beginning, which implies that the quark momenta can be chosen to be equal. As a consequence, vertex diagrams are reduced to two-point functions with vanishing external momenta, that can be further mapped to vacuum integrals.

Similarly, one can compute the coefficient function C_1 via the Green function formed by the coupling of the operators $\mathcal{O}_h$ to two gluons

$$\delta^{ab}\Gamma^{0,\mu\nu}_{GG\mathcal{O}_h}(p_1,p_2) = i^2\int \mathrm{d}x\mathrm{d}y e^{i(p_1\cdot x + p_2\cdot y)}\left\langle TG^{0,a,\mu}(x)\,G^{0,b,\nu}(y)\,\mathcal{O}_h(0)\right\rangle^{1\mathrm{PI}} = \delta^{ab}\left(-g^{\mu\nu}p_1\cdot p_2 + p_1^\nu p_2^\mu\right)\Gamma^0_{GG\mathcal{O}_h}(p_1,p_2), \tag{153}$$

where p_1 and p_2 denote the outgoing momenta of the gluons with the color indices a and b. As it was shown in [124, 271] the coefficient C_1 is given by the following relation:

$$C_1^0 = -\frac{1}{4}\frac{1}{\zeta_3^0}\Gamma^0_{GG\mathcal{O}_h}(0,0) = -\frac{1}{4}\frac{1}{\Pi^{0,h}(0)} \times\left(\frac{g_{\mu\nu}p_1\cdot p_2 - p_{1,\nu}p_{2,\mu} - p_{1,\mu}p_{2,\nu}}{(d-2)\left(p_1\cdot p_2\right)^2} \times\Gamma^{0,\mu\nu}_{GG\mathcal{O}_h}(p_1,p_2)\right)\Bigg|_{p_1^2=p_2^2=0}, \tag{154}$$

where d denotes as usual the number of space time dimensions in dimensional regularization scheme and $\Pi^{0,h}(0)$ has been defined in (114). Let us mention at this point that the projector given in (154) projects out the coefficient of the term proportional to $g^{\mu\nu}$ in (153). To explicitly verify the transversality of the Green function $\Gamma^{0,\mu\nu}_{GG\mathcal{O}_h}(p_1,p_2)$, one needs to compute also the coefficient of the Lorentz structure proportional to $p_1^\nu p_2^\mu$ using a second projector (for the explicit formula see, e.g., [129]).

In (154), one has to keep $p_1 \neq 0$ and $p_2 \neq 0$ until the projection is applied. When only heavy particles are running in the loops, a naive Taylor expansion to the linear order in the two external momenta is required. After the expansion, the factor $(p_1 \cdot p_2)^2$ in the denominator cancels and the two external momenta can be set to zero. In this way the vertex topologies implied in (154) are reduced to vacuum integrals. Nevertheless, when light particles are present in the loops, for example, bottom quarks, a naive Taylor expansion is not enough and one has to perform an asymptotic expansion. In this case the resulting Feynman integrals can be decomposed into massive vacuum integrals and vertex integrals with external momenta satisfying $p_1^2 = p_2^2 = 0$ and

$2p_1 \cdot p_2 = M_h^2$ and light quark masses present in the loops. Up to now, the light quark mass effects have been evaluated at NLO in [272, 273], which requires the computation of two-loop massive vacuum integrals and 1-loop vertex integrals.

As explained above the computation of the coefficient functions C_1 and C_{2q} involves vacuum integrals with several mass scales. Up to two-loop order such integrals are known exactly [274]. However, the three-loop multiscale integrals are not known and the computation of the coefficient C_1 at NNLO can be performed only for specific mass hierarchies between the SUSY particles, that requires application of the asymptotic expansion method (for details see [54, 156]).

In SM, the coefficient functions C_1 and C_{2q} are known up to the third order in perturbation theory. The first order QCD corrections to C_1 have been computed in [275–277], while the same order contribution to C_{2q} vanishes in the SM. The second order QCD corrections to the coefficients C_1 and C_2 can be found in [271]. The leading Yukawa corrections to the coefficient functions have been evaluated in [176]. For the coefficient function C_1 the fourth order QCD corrections have been computed recently [186, 187]. Using the low-energy theorem, the authors of [278] computed even the fifth order QCD corrections to the coefficient C_1 up to contributions originating in the n_l-dependent part of the five-loop QCD beta function, that are currently not known.

In the MSSM, the coefficient functions C_1 and C_{2q} are known at the NNLO. The NLO corrections to C_1 have been computed within SUSY-QCD for the first time in [196, 279] and confirmed analytically [280] and numerically [281] (see also [282]). In [283, 284] the squark loop contributions to Higgs boson production in the MSSM have been computed without assuming any mass hierarchy. In SUSY models with large values of $\tan\beta$, the radiative corrections due to the bottom sector can become large and they have been computed analytically at NLO in [272, 273] and confirmed numerically in [281]. For the coefficient function C_{2q} the NLO SUSY-QCD and top Yukawa corrections are known analytically since quite some time [199]. The dominant ($\tan\beta$ enhanced) NNLO SUSY-QCD and top Yukawa corrections to C_{2b} have been computed in [190, 191]. The SUSY-QCD contributions have been confirmed analytically in [285].

For completeness, we display here the one-loop order coefficients C_1 and C_{2b} providing also $\mathcal{O}(\epsilon)$ terms that are necessary for the higher order calculations:

$$
\begin{aligned}
C_1 = & -\frac{\alpha_s}{3\pi} \\
& \times \left\{ + \frac{\sin\alpha}{\cos\beta} \right. \\
& \quad \times \left[\frac{M_t^2 \mu_{\mathrm{SUSY}} X_t}{4 m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2 \tan\beta} \right. \\
& \qquad \left. - \epsilon \frac{M_t \mu_{\mathrm{SUSY}} \sin 2\theta_t}{8 \tan\beta} \left(\frac{L_{\mu\tilde{t}_1}}{m_{\tilde{t}_1}^2} - \frac{L_{\mu\tilde{t}_2}}{m_{\tilde{t}_2}^2} \right) \right] \\
& \quad - \frac{\cos\alpha}{\sin\beta} \\
& \quad \times \left[\frac{4 m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2 + m_{\tilde{t}_1}^2 M_t^2 + m_{\tilde{t}_2}^2 M_t^2 - A_t M_t^2 X_t}{4 m_{\tilde{t}_1}^2 m_{\tilde{t}_2}^2} \right. \\
& \qquad + \epsilon \frac{A_t M_t \sin 2\theta_t}{8} \left(\frac{L_{\mu\tilde{t}_1}}{m_{\tilde{t}_1}^2} - \frac{L_{\mu\tilde{t}_2}}{m_{\tilde{t}_2}^2} \right) \\
& \qquad \left. \left. + \epsilon \frac{M_t^2}{4} \left(\frac{4 L_{\mu t}}{M_t^2} + \frac{L_{\mu\tilde{t}_1}}{m_{\tilde{t}_1}^2} + \frac{L_{\mu\tilde{t}_2}}{m_{\tilde{t}_2}^2} \right) \right] \right\}, \qquad (155)
\end{aligned}
$$

$$
\begin{aligned}
C_{2b} = & -\frac{\sin\alpha}{\cos\beta} \\
& \times \left(\left(1 + \frac{\alpha_s}{2\pi} C_F A_b m_{\tilde{g}} \right. \right. \\
& \qquad \left. \times \left[F_1\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) + \epsilon F_2\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) \right] \right) \\
& \qquad \times \left(1 + \frac{\alpha_s}{2\pi} C_F X_b m_{\tilde{g}} \right. \\
& \qquad\quad \times \left[F_1\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) \right. \\
& \qquad\qquad \left. \left. \left. + \epsilon F_2\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) \right] \right)^{-1} \right) \\
& + \frac{\cos\alpha}{\sin\beta} \\
& \times \left(\frac{\alpha_s}{2\pi} C_F \left(-\mu_{\mathrm{SUSY}} \tan\beta \right) m_{\tilde{g}} \right. \\
& \qquad \left. \times \left[F_1\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) + \epsilon F_2\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) \right] \right) \\
& \times \left(1 + \frac{\alpha_s}{2\pi} C_F X_b m_{\tilde{g}} \right. \\
& \qquad \left. \times \left[F_1\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) + \epsilon F_2\left(m_{\tilde{b}_1}^2, m_{\tilde{b}_2}^2, m_{\tilde{g}}^2 \right) \right] \right)^{-1}, \qquad (156)
\end{aligned}
$$

where the functions F_1 and F_2 are defined through

$$
\begin{aligned}
F_1(x,y,z) &= -\frac{xy\ln(y/x) + yz\ln(z/y) + zx\ln xz}{(x-y)(y-z)(z-x)}, \\
F_2(x,y,z) &= \frac{1}{(x-y)(y-z)(z-x)} \\
&\quad \times \left[xy\ln\frac{y}{x}\left(1 + \frac{1}{2}\ln\frac{\mu^2}{\sqrt{xy}} \right) \right. \qquad (157) \\
&\quad + yz\ln\frac{z}{y}\left(1 + \frac{1}{2}\ln\frac{\mu^2}{\sqrt{yz}} \right) \\
&\quad \left. + zx\ln xz\left(1 + \frac{1}{2}\ln\frac{\mu^2}{\sqrt{xz}} \right) \right].
\end{aligned}
$$

The corresponding expression for up-type quarks can be easily obtained by replacing $\sin\alpha$ with $\cos\alpha$ and $\sin\beta$ with $\cos\beta$ and vice versa.

The approach outlined above has the advantage that it simplifies significantly the calculation, once the limit $M_h^2 = p_h^2 \approx 0$ is applied. The validity of this approximation has been proved within the SM at the NNLO [286, 287] (for the SM, it is known as the infinite top quark mass approximation). Since the SUSY particle masses are expected to be considerably heavier than the top quark mass, we expect that this approximation holds in the MSSM even with higher accuracy.

A second possibility to compute the coefficient functions is to relate them via the low-energy theorem (LET) to vacuum polarization and quark self-energy corrections. This approach resides heavily on the fact that the momenta carried by the Higgs boson can be set to zero. In this case, it was shown (within the SM) that the amplitude of a process containing $(N + 1)$ external particles, from which one is a Higgs boson with vanishing momenta, can be computed from the amplitude with N external particles, obtained in the absence of the Higgs external leg [288–293]:

$$\lim_{p_h \to 0} \Gamma^{h,A_1,A_2,\ldots,A_N}\left(p_h, p_{A_1}, p_{A_1}, \ldots, P_{A_N}\right) = \frac{\partial}{\partial v}\Gamma^{A_1,A_2,\ldots,A_N}\left(p_{A_1}, p_{A_1}, \ldots, P_{A_N}\right), \tag{158}$$

where v denotes the vacuum expectation value (VEV) of the theory. Beyond tree level, all kinematic parameters must be considered as bare quantities. For certain special theories and renormalization schemes the above equation holds even for renormalized parameters (for details see [294]). Within QCD all-order formulae relating the coefficient functions of dimension four operators with the decoupling coefficients for the strong coupling and the quark masses have been derived [185]. Within the MSSM, (158) has to be generalized to the case where two Higgs fields acquire VEVs. Nevertheless, it has been proved [192, 280, 285] that within SUSY-QCD the coefficient functions C_1 and C_{2q} can be derived up to NNLO from the decoupling coefficients ζ_s and ζ_{m_q} through the following relations:

$$C_1^0 = \left(-\sin\alpha \widehat{D}_{\phi_1}^0 + \cos\alpha \widehat{D}_{\phi_2}^0\right)\ln\zeta_s^0 \equiv \widehat{D}_h^0 \ln\zeta_s^0,$$
$$C_{2q}^0 = \left(-\sin\alpha \widehat{D}_{\phi_1}^0 + \cos\alpha \widehat{D}_{\phi_2}^0\right)\ln\zeta_{m_q}^0 \equiv \widehat{D}_h^0 \ln\zeta_{m_q}^0. \tag{159}$$

As usual, the superscript 0 labels bare quantities. The operators $\widehat{D}_{\phi_i}^0$, with $i = 1, 2$, contain the derivatives with respect to the two VEVs of the MSSM. They have been derived using the field dependent definitions of quark and squark masses and mixing angles in [280]. However, for the computation of the coefficient function C_1 at the NNLO, also the dependence of the ε-scalar mass on the VEVs through the loop induced Higgs-ε-scalar coupling has to be taken into account [192]. As can be understood from equations (21) and (22) in [280] the dominant contributions to the differential operators originate from the pure SUSY-QCD terms. For exemplification and to fix the normalization, we reproduce here the terms corresponding to the third generation quarks keeping only the linear terms in bottom quark masses (please note the sign difference in the definition of parameter μ_{SUSY} between [280] and [192, 285]):

$$\begin{aligned}
\widehat{D}_{\phi_1} &= \frac{1}{\cos\beta}\left(m_b A_b \mathscr{F}_b + m_b \mathscr{G}_b\right) \\
&\quad - \frac{1}{\sin\beta} m_t \mu_{\text{SUSY}} \sin 2\theta_t \mathscr{F}_t, \\
\widehat{D}_{\phi_2} &= \frac{1}{\cos\beta}\left(-m_b \mu_{\text{SUSY}} \mathscr{F}_b\right) \\
&\quad + \frac{1}{\sin\beta}\left(m_t A_t \sin 2\theta_t \mathscr{F}_t + 2m_t^2 \mathscr{G}_t\right), \\
\text{with } \mathscr{F}_b &= \frac{2}{m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2}\left(1 - \sin^2 2\theta_b\right)\frac{\partial}{\partial \sin 2\theta_b}, \\
\mathscr{G}_b &= \frac{\partial}{\partial m_b}, \\
\mathscr{F}_t &= \frac{\partial}{\partial m_{\tilde{t}_1}^2} - \frac{\partial}{\partial m_{\tilde{t}_2}^2} \\
&\quad + \frac{2}{m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2}\frac{\left(1 - \sin^2 2\theta_t\right)}{\sin 2\theta_t}\frac{\partial}{\partial \sin 2\theta_t}, \\
\mathscr{G}_t &= \frac{\partial}{\partial m_{\tilde{t}_1}^2} + \frac{\partial}{\partial m_{\tilde{t}_2}^2} + \frac{\partial}{\partial m_t^2}.
\end{aligned} \tag{160}$$

On the right hand side of the above equations, all parameters are the bare ones. We omitted the superscript "0" to avoid clumsy notation. For large values of $\tan\beta$ the dominant contributions to the coefficient functions, that is, the terms proportional to $\mu_{\text{SUSY}} \tan\beta$, are generated through the term containing the derivative $\mathscr{F}_b$ in $\widehat{D}_{\phi_2}$. Taking into account the parametric dependence of the quark self-energy $\Sigma^{0,h}$ on masses and mixing angles, one can easily derive these contributions from the terms proportional to $\sin 2\theta_b$ in $\Sigma_s^{0,h}$.

In this section we study the phenomenological applications of the computations discussed above. We concentrate on the calculation within the MSSM of the total decay rate into hadrons $\Gamma(h \to \text{hadrons})$, that is composed of the partial decay widths into quarks $\Gamma(h \to q\bar{q})$ and gluons $\Gamma(h \to gg)$. Although, the channel $\Gamma(h \to b\bar{b})$ gives the dominant contributions to the total Higgs decay rate, it was not used among the Higgs discovery channels at the LHC, due to its huge background. Nevertheless, it has a big impact on all branching ratios and is an important channel for the identification of the Higgs properties. Precisely, the uncertainties on the partial decay width $\Gamma(h \to b\bar{b})$ translate into significant systematic errors for all the other nonleading branching ratios. For illustration we show in Figure 18 from [55] the branching ratios of the Higgs boson in the SM at the LO. For precise analysis they have to be complemented by genuine SM radiative corrections together with corrections due to the supersymmetric particles, that can be embedded in the decoupling coefficients as discussed in the previous section.

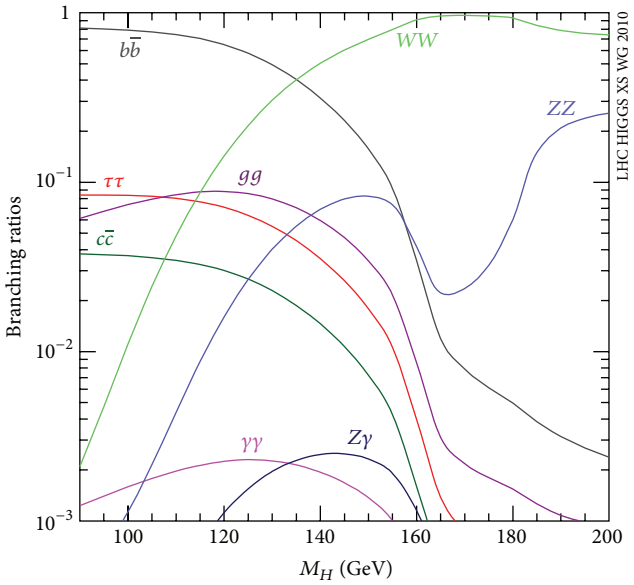

FIGURE 18: Higgs boson branching ratios in the SM at the LO from [55].

Starting from the effective Lagrangian (144) one can derive the following formula for the total decay width into hadrons:

$$\begin{aligned}\Gamma(h \longrightarrow \text{hadrons}) &= \left(1+\bar{\delta}_u\right)^2 \left\{\sum_q \Gamma^{(0)}_{q\bar{q}} \times \left[\left(1+\Delta_{22}\right)\mathscr{C}_2^2 + \Delta_{12}\mathscr{C}_1\mathscr{C}_2\right] + \Gamma^{(0)}_{gg}\left(1+\Delta_{11}\right)\mathscr{C}_1^2\right\},\end{aligned} \tag{161}$$

where the coefficient functions $\mathscr{C}_1$ and $\mathscr{C}_2$ have been defined in (150).

At the lowest order in perturbation theory, the first line corresponds to $\Gamma(h \to q\bar{q})$, whereas the second one stands for $\Gamma(h \to gg)$. At higher orders, however, the splitting of (161) into the decay widths to fermions and gluons is not straightforward anymore, due to the occurrence of diagrams contributing to both channels.

The LO expressions for the branching ratios are given by

$$\begin{aligned}\Gamma^{(0)}_{q\bar{q}} &= \frac{N_c G_F M_h m_q^2}{4\pi\sqrt{2}}\left(1-\frac{4m_q^2}{M_h^2}\right)^{3/2}, \\ \Gamma^{(0)}_{gg} &= \frac{N_c C_F G_F M_h^3}{\pi\sqrt{2}},\end{aligned} \tag{162}$$

where G_F denotes the Fermi constant. As is well known [207, 277, 295], the large logarithms of the type $\ln(M_h^2/m_q^2)$ can be resumed by taking m_q in (162) to be the $\overline{\text{MS}}$ mass $m_q^{\overline{\text{MS}}}(\mu)$ evaluated at the scale $\mu = M_h$.

The coefficients Δ_{11}, Δ_{12}, Δ_{22} describe the low-energy physics. Therefore, they have to be computed in the effective theory and are independent of the heavy masses. Using the method of operators described in the previous section, they can be related via the optical theorem to the absorptive parts of the scalar correlators Π_{jk}:

$$\begin{aligned}\Delta_{jk} &= \frac{1}{M_h}\,\text{Im}\left(\Pi_{jk}\right) \\ &= \frac{1}{M_h}\,\text{Im}\left(i\int \mathrm{d}x e^{ipx}\left\langle 0\left|T\left[\mathscr{O}_j(x)\,\mathscr{O}_k(0)\right]\right|0\right\rangle\Big|_{p^2=M_h^2}\right), \\ & \qquad j,k = 1,2,\end{aligned} \tag{163}$$

where p is the momentum of the external Higgs boson. They have been computed within SM up to three-loop order (see [134] for a comprehensive review on this topic). For the analysis discussed in this section, their one- and two-loop QCD corrections are required. The two-loop QCD contributions to the coefficients Δ_{22} and Δ_{11} are given by [296, 297]. One has

$$\begin{aligned}\Delta_{22} &= \frac{\alpha_s'(\mu)}{\pi}\left(\frac{17}{3}+2\ln\frac{\mu^2}{M_h^2}\right)+\left(\frac{\alpha_s'(\mu)}{\pi}\right)^2 \\ &\quad\times\left[\frac{10801}{144}-\frac{19}{2}\zeta(2)-\frac{39}{2}\zeta(3)\right. \\ &\quad+\frac{106}{3}\ln\frac{\mu^2}{M_h^2}+\frac{19}{4}\ln^2\frac{\mu^2}{M_h^2} \\ &\quad-n_l\left(\frac{65}{24}-\frac{1}{3}\zeta(2)-\frac{2}{3}\zeta(3)\right. \\ &\quad\left.\left.+\frac{11}{9}\ln\frac{\mu^2}{M_h^2}+\frac{1}{6}\ln^2\frac{\mu^2}{M_h^2}\right)\right],\end{aligned} \tag{164}$$

$$\begin{aligned}\Delta_{11} &= \frac{\alpha_s'(\mu)}{\pi}\left[\frac{73}{4}+\frac{11}{2}\ln\frac{\mu^2}{M_h^2}-n_l\left(\frac{7}{6}+\frac{1}{3}\ln\frac{\mu^2}{M_h^2}\right)\right] \\ &\quad+\left(\frac{\alpha_s'(\mu)}{\pi}\right)^2 \\ &\quad\times\left[\frac{37631}{96}-\frac{363}{8}\zeta(2)-\frac{495}{8}\zeta(3)\right. \\ &\quad+\frac{2817}{16}\ln\frac{\mu^2}{M_h^2}+\frac{363}{16}\ln^2\frac{\mu^2}{M_h^2} \\ &\quad-n_l\left(\frac{7189}{144}-\frac{11}{2}\zeta(2)-\frac{5}{4}\zeta(3)\right.\end{aligned}$$

$$+\frac{263}{12}\ln\frac{\mu^2}{M_h^2}+\frac{11}{4}\ln^2\frac{\mu^2}{M_h^2}\Bigg)+n_l^2$$
$$\times\left(\frac{127}{108}-\frac{1}{6}\zeta(2)+\frac{7}{12}\ln\frac{\mu^2}{M_h^2}+\frac{1}{12}\ln^2\frac{\mu^2}{M_h^2}\right)\Bigg], \tag{165}$$

with $\zeta(x)$ being the Riemann's zeta function.

The additional QCD correction Δ_{12} is generated through double-triangle topologies. It was first computed in [271] and it reads

$$\Delta_{12}=\frac{\alpha_s'(\mu)}{\pi}C_F\left(-19+6\zeta(2)-\ln^2\frac{m_q^2}{M_h^2}-6\ln\frac{\mu^2}{M_h^2}\right). \tag{166}$$

The universal corrections $\overline{\delta}_u$ of $\mathcal{O}(\alpha_s^n x_t)$, where $x_t = (\alpha_t/4\pi)^2 = G_F M_t^2/(8\pi^2\sqrt{2})$, with α_t being the top-Yukawa coupling, contain the contributions from the renormalization of the Higgs wave function and the vacuum expectation value [298]. It is given by

$$\overline{\delta}_u = x_t\left[\frac{7}{2}+\frac{\alpha_s'(\mu)}{\pi}\left(\frac{19}{3}-2\zeta(2)+7\ln\frac{\mu^2}{M_t^2}\right)+\mathcal{O}\left(\alpha_s^2\right)\right]. \tag{167}$$

Now, we are in a position to interpret the phenomenological significance of (161). In the following section we concentrate on the numerical effects of the radiative corrections to the hadronic Higgs decay.

The SM input parameters are the strong coupling constant at the Z-boson mass scale $\alpha_s(M_Z) = 0.1184$ [182], the top quark pole mass $M_t = 173.1\,\text{GeV}$ [178], and the running bottom-quark mass in the $\overline{\text{MS}}$ scheme $m_b(m_b) = 4.163\,\text{GeV}$ [299]. For the supersymmetric parameters we adopted the corresponding values of the "modified $m_h^{\max}$" scenario as described in Section 7.2.2 (for details see [54]).

In Figure 19 we focus on the decay channel $h \to b\bar{b}$ and display the decay width as a function of the Higgs boson mass M_h. We chose in this case $\tan\beta = 50$. The two-loop genuine QCD and electroweak corrections (i.e., computed in the effective theory) to the process $h \to b\bar{b}$, as well as the two-loop SUSY-QCD corrections to the Higgs boson mass, are depicted by the dotted line. More precisely, they are derived from (161), where the coefficient functions $\mathscr{C}_1$ and $\mathscr{C}_2$ are set to their tree-level values. The additional SUSY-QCD vertex corrections parametrized through the coefficient functions $\mathscr{C}_1$ and $\mathscr{C}_2$ are represented at the one- and two-loop order by the dashed and solid lines, respectively. We also take into account the one-loop SUSY-EW corrections to the coefficient function $\mathscr{C}_2$ and fix their renormalization scale at $\mu_{\text{SEW}} = (m_{\tilde{t}_1} + m_{\tilde{t}_2} + \mu_{\text{SUSY}})/15$, for which the two-loop SUSY-EW corrections become negligible [190, 191]. The genuine two-loop corrections are negligible. Nevertheless, they are essential tools for the proof of the convergence of the perturbative expansion.

The large one-loop SUSY-QCD radiative corrections to $\Gamma(h \to b\bar{b})$ have only a relatively small impact on the branching ratio $\text{BR}(h \to b\bar{b})$, but they can have a large impact on $\text{BR}(h \to \tau^+\tau^-)$. For sufficiently large $\tan\beta$ and μ_{SUSY}, the measurement of $\text{BR}(h \to \tau^+\tau^-)$ can provide information about the distinction between the SM and MSSM predictions.

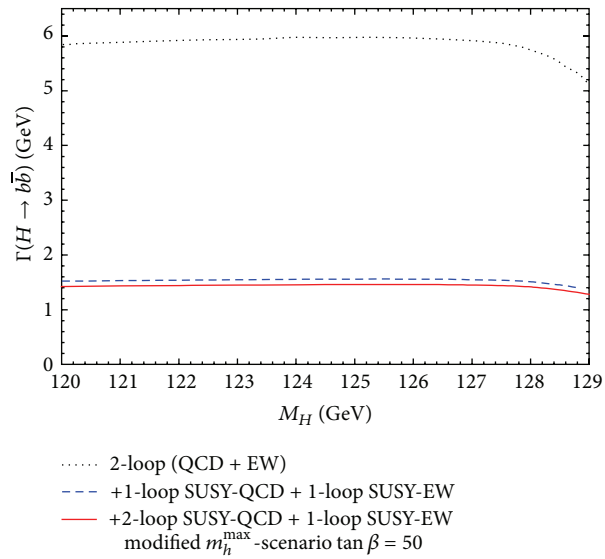

FIGURE 19: $\Gamma(h \to b\bar{b})$ for the "modified $m_h^{\max}$" scenario as a function of M_h. The dotted line displays the two-loop QCD and electroweak corrections together with two-loop corrections to the Higgs boson propagator. The dashed and solid lines depict in addition the one- and two-loop SUSY-QCD vertex corrections, respectively.

The gluonic Higgs decay rate can be directly measured only at e^+e^- colliders. At hadron colliders, they can be measured only indirectly with rather bad accuracy of the order of 20%. As it has been shown, the genuine SUSY-QCD corrections to the gluonic Higgs decay are rather small [300]. For the experimental analysis relevant at the LHC they can be neglected with respect to the standard quark contributions to the hadronic decay rate. The QCD corrections are known in the SM up to the NNNLO [186, 187, 278] in the heavy-top-mass limit (here, the mass of the Higgs boson is assumed to be much smaller than the mass of the top quark). Even the mixed QCD-electroweak corrections at the three-loop level are known [176] in the same approximations. The genuine NLO SUSY-QCD corrections have been evaluated in [196, 280] and amount to about −5% from the QCD corrections at NLO.

A much more interesting Higgs decay channel from the perspective of the ongoing experiments conducted at the LHC is the rare $h \to \gamma\gamma$ channel. In this case the coupling of the Higgs to photons is mediated by loops containing electrically charged particles. If the masses of the particles inside loops are generated through the Higgs mechanism, as in the case of the SM, the couplings to the Higgs boson grow with the masses, balancing the decrease due to rising loop masses. If the masses of the particles are generated by different mechanisms, as is the case in SUSY, the effect of the heavy particles on the $h\gamma\gamma$ coupling is in general small.

In SM with the Higgs boson mass of about 125 GeV only the top quark and the W boson effectively contribute and

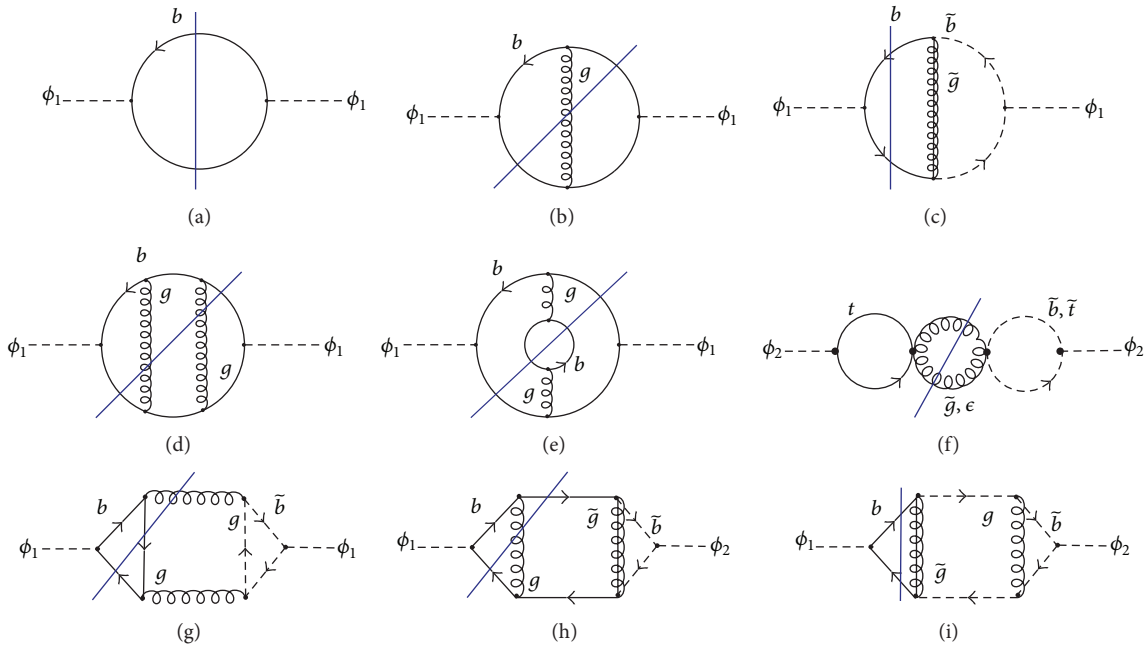

Figure 20: One-, two-, and three-loop Feynman diagrams contributing to the Higgs boson propagator in SUSY-QCD. Dashed lines denote Higgs bosons, whereas oriented dashed lines represent the squarks. For the other particles we use the same convention as before.

they interfere destructively. The radiative corrections are well under control. The QCD contributions are known up to NNLO [301] and the electroweak corrections to NLO [302–304]. The SUSY-QCD corrections to $\Gamma(h \to \gamma\gamma)$ are known with the same accuracy as in the case of $\Gamma(h \to gg)$. The NLO corrections have been computed in [300, 305] and the NNLO contributions can be found in [306]. Also for this channel, the SUSY corrections are small as compared to the SM ones.

For an intermediate Higgs mass of about 125 GeV it is legitimate to investigate the quality of the approximation discussed in the previous section. For accurate results, one has to take also into consideration in (144) operators of dimension six and higher, that are suppressed at least by a factor M_h^2/M_t^2. However, the application of higher dimensional operators in the context of SUSY is quite tedious. A more familiar method for this purpose is to use the optical theorem. Hereby, one has to consider corrections to the Higgs boson self-energy $\Pi_h(q^2)$. The imaginary part of this quantity provides us with the total decay rate of the Higgs boson

$$\Gamma_h = \frac{1}{M_h} \operatorname{Im} \Pi_h \left(M_h^2\right). \tag{168}$$

According to the Cutkosky cutting rules, nonvanishing contributions to the imaginary part of the Higgs boson self-energy will provide only those diagrams, that can be cut in such a way that all resulting final state particles can be set simultaneously on their mass shell. Sample diagrams contributing to the hadronic decay rate can be seen in Figure 20.

The imaginary parts originate from the $i\epsilon$-prescription for on-shell propagators. In the results obtained using DRED they are embedded in complex logarithms occurring in the ϵ-expansion of the expression

$$\left(\frac{-\mu^2}{q^2 + i\epsilon}\right)^{\epsilon} = 1 - \epsilon \log\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) + \frac{1}{2}\epsilon^2 \log^2\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) + \mathcal{O}\left(\epsilon^3\right). \tag{169}$$

After setting the external momenta on the Higgs mass shell $q^2 = M_h^2$, one obtains further

$$\log\left(\frac{-q^2 - i\epsilon}{\mu^2}\right) = \log\left(\frac{M_h^2}{\mu^2}\right) - i\pi. \tag{170}$$

The analytic calculation of the three-loop diagrams contributing to Γ_h in SUSY-QCD is not yet possible. Nevertheless, for fixed mass hierarchies between the occurring particles, the method of asymptotic expansion can be successfully applied. For illustration, we consider a degenerate SUSY mass spectrum satisfying the following inequality with respect to the SM particle masses:

$$m_q \ll M_h \ll m_t \ll M_S \equiv m_{\tilde{g}} = m_{\tilde{q}}. \tag{171}$$

Similar to the computation of three-loop SUSY-QCD corrections to the light Higgs bosons mass, also in this calculation one has to make an additional Taylor expansion of bottom squark propagator in bottom squark mass differences Δ_b defined like

$$\Delta_b = \frac{m_{\tilde{b}_1}^2 - m_{\tilde{b}_2}^2}{m_{\tilde{b}_1}^2}. \tag{172}$$

This procedure allows to correctly take into account the contributions generated by the bottom squark mixing angle renormalization.

In the following we consider the same renormalization scheme as in Section 7.2.1. The results for Γ_h including the dominant mass corrections at $O(\alpha_s^2)$ read [307]

$$\begin{aligned}
\Gamma_h &= \Gamma_{qq}^{(0)}\left(\frac{\sin\alpha}{\cos\beta}\right)^2 \\
&\quad\times\left\{1+\frac{4}{3}\frac{\alpha_s}{\pi}\right. \\
&\qquad\times\left[\frac{19}{4}+\frac{3}{2}L_{\mu h}-\frac{1}{2}L_{\mu S}\right. \\
&\qquad\quad+\left(-\frac{15}{2}-9L_{\mu h}+3L_{\mu S}\right)\frac{m_b^2}{M_h^2} \\
&\qquad\quad+\frac{5}{12}\frac{m_b^2}{M_S^2}+\frac{1}{15}\frac{m_b^2M_h^2}{M_S^4} \\
&\qquad\quad-\frac{A_b-\mu_{\mathrm{SUSY}}\cot\alpha}{M_S} \\
&\qquad\quad\left.\left.\times\left(\frac{1}{2}+\frac{1}{12}\frac{m_b^2}{M_S^2}+\frac{1}{24}\frac{M_h^2}{M_S^2}\right)\right] \\
&\qquad+\left(\frac{\alpha_s}{\pi}\right)^2 \\
&\qquad\times\left[\frac{14093}{216}+\frac{541}{18}L_{\mu h}+\frac{47}{12}L_{\mu h}^2-\frac{559}{36}L_{\mu S}\right. \\
&\qquad\quad-\frac{10}{3}L_{\mu h}L_{\mu S}+\frac{35}{36}L_{\mu S}^2-\frac{11}{9}L_{\mu t} \\
&\qquad\quad-\frac{1}{3}L_{\mu h}L_{\mu t}+\frac{1}{6}L_{\mu t}^2-\frac{97}{6}\zeta(3) \\
&\qquad\quad+\left(\frac{107}{675}+\frac{2}{45}L_{\mu h}-\frac{2}{45}L_{\mu t}\right)\frac{M_h^2}{m_t^2} \\
&\qquad\quad+\left(-\frac{529}{88200}-\frac{1}{420}L_{\mu h}+\frac{1}{420}L_{\mu t}\right)\frac{M_h^4}{m_t^4} \\
&\qquad\quad+\left(\frac{7}{108}+\frac{1}{9}L_{\mu S}-\frac{1}{9}L_{\mu t}\right)\frac{m_t^2}{M_S^2} \\
&\qquad\quad+\left(\frac{5821}{16200}+\frac{17}{135}L_{\mu h}-\frac{17}{135}L_{\mu S}\right)\frac{M_h^2}{M_S^2} \\
&\qquad\quad+\frac{\left(A_b-\mu_{\mathrm{SUSY}}\cot\alpha\right)^2}{M_S^2}\left(\frac{1}{9}+\frac{1}{54}\frac{M_h^2}{M_S^2}\right) \\
&\qquad\quad+\frac{A_b-\mu_{\mathrm{SUSY}}\cot\alpha}{M_S} \\
&\qquad\quad\times\left[-\frac{119}{18}-\frac{4}{3}L_{\mu h}-\frac{1}{18}L_{\mu S}\right. \\
&\qquad\qquad+\left(-\frac{7}{54}-\frac{1}{18}L_{\mu S}+\frac{1}{18}L_{\mu t}\right)\frac{m_t^2}{M_S^2} \\
&\qquad\qquad\left.+\left(-\frac{62}{81}-\frac{1}{9}L_{\mu h}-\frac{1}{216}L_{\mu S}\right)\frac{M_h^2}{M_S^2}\right] \\
&\qquad+\frac{\tan\beta}{\sin\alpha} \\
&\qquad\quad\times\left[-\frac{28}{9}-\frac{2}{3}L_{\mu h}+\frac{2}{3}L_{\mu t}+\frac{5}{54}\frac{X_t m_t^2}{M_S^2}\right. \\
&\qquad\qquad+\left(-\frac{2011}{24300}-\frac{41}{1620}L_{\mu h}+\frac{41}{1620}L_{\mu t}\right)\frac{M_h^2}{m_t^2} \\
&\qquad\qquad+\left(-\frac{28307}{4762800}-\frac{47}{22680}L_{\mu h}\right. \\
&\qquad\qquad\quad\left.+\frac{47}{22680}L_{\mu t}\right)\frac{M_h^4}{m_t^4} \\
&\qquad\qquad+\left(-\frac{85}{54}-\frac{1}{3}L_{\mu h}+\frac{1}{3}L_{\mu S}\right)\frac{m_t^2}{M_S^2} \\
&\qquad\qquad\left.\left.\left.-\frac{1}{27}\frac{M_h^2}{M_S^2}-\frac{7}{3240}\frac{M_h^4}{m_t^2M_S^2}\right]\right]\right\} \\
&\quad+\Gamma_{gg}^{(0)}\left(\frac{\cos\alpha}{\sin\beta}\right)^2\left(\frac{\alpha_s}{\pi}\right)^2 \\
&\quad\times\left[\frac{1}{144}+\frac{1}{144}\frac{m_t^2}{M_S^2}+\frac{7}{8640}\frac{M_h^2}{m_t^2}+\frac{7}{17280}\frac{M_h^2}{M_S^2}\right. \\
&\qquad\left.+\frac{169}{2073600}\frac{M_h^4}{m_t^4}+\frac{1}{24192}\frac{M_h^4}{m_t^2M_S^2}\right] \\
&\quad+\mathcal{O}\left(\frac{M_h^4}{M_S^4},\frac{M_t^4}{M_S^4},\frac{M_h^6}{m_t^6}\right). \qquad (173)
\end{aligned}$$

For a light Higgs mass $M_h = 125\,\mathrm{GeV}$ and SUSY masses of about 1 TeV, $\tan\beta = 40$, and SM parameters chosen as in the previous sections, the mass corrections at NLO and NNLO amount to below one percent from the dominant contribution (i.e., computed in the EFT) at the corresponding order in perturbation theory. They are beyond the reach of the LHC accuracy, but they might be of phenomenological interest at a future linear collider.

During the last years, a lot of effort has been devoted to precise predictions for Higgs production at hadron colliders (for reviews, see [36, 308–310]). They constituted basic ingredients for the discovery of the new scalar particle at the LHC. The main production channel for the SM Higgs boson at the LHC is the loop-induced gluon-fusion channel. For

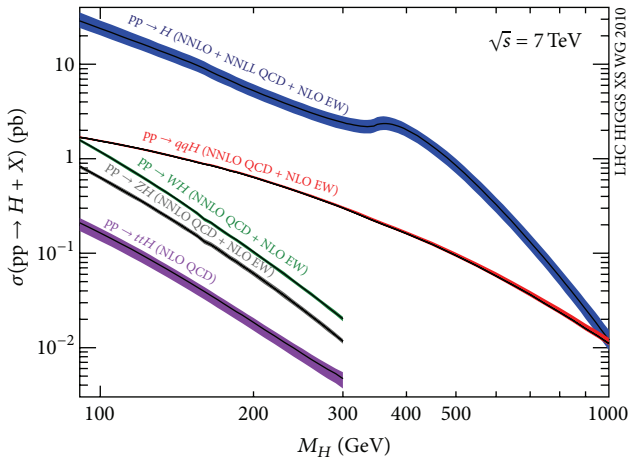

FIGURE 21: Higgs production cross sections at the LHC for $\sqrt{s}$ = 8 TeV together with the uncertainties from the missing higher order corrections and the parton density functions from [55].

illustration, we reproduce in Figure 21 from [55] the theoretical predictions for the main Higgs production channels together with the uncertainties due to missing higher order corrections and to the uncertainties on the parton density functions (PDFs).

An important application of the Higgs discovery is to constrain the parameters of theories predicting physics beyond the SM. This is also the case for SUSY theories. Given the high sensitivity of the Higgs observables (its mass, production cross sections, and decay rates) on the parameters of the top sector in the MSSM, one can derive lower bounds for top squark masses and set constraints for their mixing angle. For this task one needs, among other ingredients, precise predictions for the Higgs production cross section, including even NNLO SUSY-QCD corrections. As discussed in Section 8.2, exact analytic calculations at this order in perturbation theory are not yet feasible. Instead one has to use the EFT approach together with the method of asymptotic expansions. In the SM, it was shown [286, 287, 311–314] that the exact result for the hadronic cross section for intermediate Higgs masses ($M_h < 2m_t$) is approximated to better than 1% level by the result derived with EFT approach, if the full top mass dependence at LO is factored out.

Following the same reasoning for the case of the MSSM, one can write the hadronic cross section $\sigma \equiv \sigma(pp \to h + X)$ as as a function of the hadronic center-of-mass energy $\sqrt{s}$. It reads [196]

$$\sigma(z) = \rho_0 \sigma_0 \left(-3\pi \frac{C_g(\mu_h)}{c_1^{(0)}}\right)^2 \times \left[\Sigma^{(0)}(z) + \frac{\alpha_s(\mu_s)}{\pi}\Sigma^{(1)}(z) + \left(\frac{\alpha_s(\mu_s)}{\pi}\right)^2 \Sigma^{(2)}(z) + \cdots\right]. \quad (174)$$

The exact LO contribution, denoted here by σ_0, is factored out, as discussed above. The higher order corrections are computed within the EFT approach and the separation of short and long distance contributions is explicit in (174). For a better convergence of the perturbative expansion and to avoid the occurrence of large logarithms, one makes use of scale separation as discussed in Section 8.1. Thus, the coefficient functions C_g and $c_1^{(0)}$ that contain the radiative corrections due to heavy particles are evaluated at a heavy scale of the order of the SUSY particle masses $\mu_h = \mathcal{O}(\widetilde{M})$. The partonic cross sections $\Sigma^{(n)}(z)$ are computed at a low scale of the order of the Higgs mass $\mu_s = \mathcal{O}(M_h)$. The individual building blocks in (174) are discussed below.

The normalization coefficient ρ_0 is given by

$$\rho_0 = \frac{G_F[\alpha_s(\mu_S)]^2}{288\pi\sqrt{2}}, \quad (175)$$

where the presence of the strong coupling evaluated at the low-energy scale μ_s is due to the use of renormalization group invariant operators and coefficient functions as given in (148) and (150).

σ_0 contains the exact dependence on all masses and momenta at the LO. Its analytic expression is known for quite long time. For convenience of the reader, we reproduce it here, in the normalization of [54]:

$$\sigma_0 = \left|\frac{3}{2}\frac{\cos\alpha}{\sin\beta} \times \left\{A(\tau_t) + \sum_{i=1,2}(-1)^i \times \left[\frac{\sin(2\theta_t)}{2}\left(\tan\alpha + \frac{1}{\tan\beta}\right) \times \frac{m_t\mu_{\text{SUSY}}}{2m_{\tilde{t}_i}^2} + \frac{m_t^2}{8m_{\tilde{t}_i}^2} \times \left(\sin^2(2\theta_t)\frac{m_{\tilde{t}_1}^2 - m_{\tilde{t}_2}^2}{m_t^2} - 4(-1)^i\right)\right] \times \widetilde{A}(\tau_{\tilde{t}_i})\right\} + \mathcal{O}\left(\frac{M_Z^2}{m_{\tilde{t}_i}^2}\right)\right|^2, \quad (176)$$

with

$$\tau_i = \frac{4m_i^2}{M_h^2}, \qquad A(\tau) = \tau\left[1 + (1-\tau) f(\tau)\right],$$
$$\widetilde{A}(\tau) = \tau\left(1 - \tau f(\tau)\right), \tag{177}$$
$$f(\tau) = \begin{cases} \arcsin^2\left(\dfrac{1}{\sqrt{\tau}}\right), & \tau \geq 1, \\ -\dfrac{1}{4}\left(\ln\dfrac{1+\sqrt{1-\tau}}{1-\sqrt{1-\tau}} - i\pi\right)^2, & \tau < 1. \end{cases}$$

The coefficient c_1^0 is defined through the one-loop relation

$$c_1^0 = -\frac{3\pi}{\alpha_s} C_1^{(1-\text{loop})}. \tag{178}$$

Its SUSY-QCD part can be read off directly from (155). The coefficient c_1^0 is factored out because it is already contained in the LO contribution σ_0 as can be easily understood from (176). Indeed, in the limit of light Higgs masses $M_h \ll m_t$, $m_{\tilde{t}}$, $m_{\tilde{g}}$ and neglecting mass suppressed contributions of the order of $\mathcal{O}(M_h^2/m_t^2)$, $\mathcal{O}(M_h^2/m_{\tilde{t}}^2)$, and $\mathcal{O}(M_Z^2/m_{\tilde{t}}^2)$ the LO contribution σ_0 takes the form (we adopt here the normalization of [54])

$$\sigma_0 \longrightarrow \left|c_1^{(0)}\right|^2. \tag{179}$$

The coefficient C_g was defined in (149) and has to be evaluated at the heavy scale. Let us point out that the factor $-3\pi C_g(\mu_h)/c_1^0$ expanded in the strong coupling $\alpha_s(\mu_h)$ takes the form

$$-3\pi\frac{C_g(\mu_h)}{c_1^0} = 1 + \frac{\alpha_s(\mu_h)}{\pi} c_g^{(1)} + \left(\frac{\alpha_s(\mu_h)}{\pi}\right)^2 c_g^{(2)} + \cdots, \tag{180}$$

where the coefficients $c_g^{(i)}$, with $i = 1, 2$, are known, once the coefficient C_1 is computed up to the NNLO.

Finally, $\Sigma^{(n)}(z)$ is defined through the convolution

$$\Sigma^{(n)}(z) = \sum_{i,j\in\{q\bar{q}g\}} \int_z^1 dx_1 \times \int_{z/x_1}^1 dx_2 f_{i/p}(x_1) f_{j/p}(x_2) \widehat{\Sigma}_{ij}^{(n)}\left(\frac{z}{x_1 x_2}\right), \qquad z \equiv \frac{M_h^2}{s} \tag{181}$$

of $f_{j/p}(x)$, the density of parton i inside the proton, and $\widehat{\Sigma}_{ij}^{(n)}(x)$, the partonic cross section expanded up to the nth order in $\alpha_s(\mu_s)$, and computed in the effective-theory approach. At the LO, it reads

$$\widehat{\Sigma}_{ij}^{(0)}(x) = \delta_{ig}\delta_{jg}\delta(1-x). \tag{182}$$

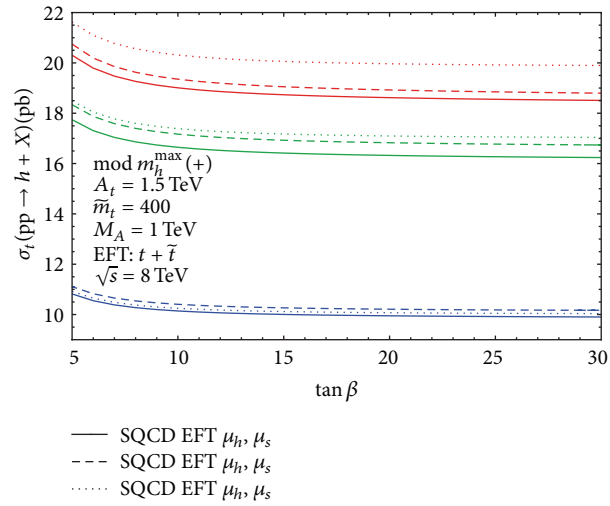

FIGURE 22: The cross section $\sigma_t^{\text{SUSY-QCD}}(\mu_s, \mu_h)$ as a function of tan β using the parameters in to LO (bottom), NLO (middle), and NNLO (top) from [54]. The dotted line corresponds to the SM and the dashed and solid lines to the MSSM.

The NLO and NNLO contributions contain the real and virtual corrections associated with the operator $\mathcal{O}_1$ and its mixture with the operator $\mathcal{O}_2$. Since they are computed within the effective theory, they can be taken over from the SM computations reported in [275, 276, 286, 314].

Let us mention that there is also a third scale present in (174), namely, the factorization scale μ_F embedded in the PDFs. Usually it is chosen to be equal to the low-scale μ_s, that is, $\mu_F = \mu_s$. The choice of scales plays an important role in precision calculations of the hadronic Higgs production cross section, especially when particles much heavier than the SM ones are present. We discuss in the next section the phenomenological impact of the NNLO corrections.

For the numerical analysis we choose a supersymmetric mass spectrum in the so-called "modified $m_h^{\max}$ scenario" as defined in [54]. It is a modification of the original "$m_h^{\max}$" scenario [315] such that one of the top squarks becomes light and the other one remains heavy, at the TeV scale. At the same time Higgs masses as large as 127 GeV can be achieved.

For illustration, we reproduce in Figure 22 the results of [54] that constitute the most precise prediction for the hadronic cross section in the gluon fusion channel in the framework of the MSSM. Here, $\sigma_t^{\text{SUSY-QCD}}(\mu_s, \mu_h)$ denotes the dominant contribution originating from the top sector. From bottom to top, the LO, NLO, and NNLO results are depicted for $5 \leq \tan\beta \leq 30$ and choosing $m_{\tilde{t}_1} = 400$ GeV. The dotted lines represent the SM results. The solid and the dashed lines show the MSSM predictions for two different scale choices: $\mu_h = M_t$ and $\mu_s = M_h/2$ and $\mu_h = \mu_s = M_h/2$, respectively. The MSSM results are reduced by a few percent as compared to the SM prediction. This effect increases when going from LO to NLO and finally to NNLO where a difference of about 5% is observed. This behavior is specific for supersymmetric mass spectra containing at least one light squark of the third generation. For the case

when all SUSY particles are heavy, at the TeV scale (the so-called decoupling limit) the genuine SUSY-QCD corrections to the production cross section become negligible. The fact that the difference between the SM and MSSM predictions increases when higher order radiative corrections are taken into account can be explained by the occurrence of many new SUSY contributions.

As can be seen from the figure, the effect of scale choice is not negligible: the results for $\mu_h = \mu_s$ are in general a few percent above the ones with $\mu_h \neq \mu_s$. At NLO, the scale dependence increases as compared to the LO case as a consequence of the special organisation of the perturbative series. Nevertheless, the scale dependence decreases when going from NLO to NNLO as expected.

9. Conclusions

In this review we report on precision calculations in supersymmetric theories. They are not only important ingredients for the development of quantum field theories in general, but they are also required by the current experimental analyses searching for indirect manifestations of SUSY in collider experiments at the TeV scale. The latter topic is of utmost importance for particle physics: the nonobservation of any supersymmetric particle at the TeV scale renders low-energy supersymmetric theories debatable. Obviously, to prove or disprove a theory for which enormous efforts both at theoretical and experimental level have been devoted over the last four decades is a very complex task. In this review, we concentrate on the indirect searches for SUSY that can be carried through precision tests of the gauge coupling unification hypothesis, the prediction of a light Higgs boson mass, and the interaction properties of the Higgs boson with the SM particles.

It turns out that the hypothesis of gauge coupling unification even in the framework of minimal SUSY SU(5) model cannot be falsified with the help of currently available experimental data. Let us mention that the contributions at the three-loop order in perturbation theory are essential in this analysis. The conclusion drawn from precision calculations reconfirm earlier results derived from model building arguments.

Furthermore, the theoretical prediction of the light Higgs boson mass within SUSY with an accuracy comparable with the one reached by the ongoing experimental analyses conducted at the LHC is an important tool for constraining the supersymmetric parameter space. For this purpose one needs to calculate even three-loop Feynman integrals involving many different mass scales. At present, an exact analytic computation is not feasible. Nevertheless, the method of asymptotic expansion can be applied successfully also in SUSY theories and provides us with precise results. Specifically, the lightest Higgs bosons mass within the MSSM can be predicted at present with an accuracy of about 1 GeV for the parameter space of phenomenological interest.

Moreover, after the recent discovery of the Higgs boson at the LHC, the natural question is whether it has the characteristics of the particle predicted by the SM or new theories are required to describe it. To answer this question from the perspective of supersymmetric theories, one needs predictions of the hadronic Higgs production cross section and its decay rates into SM particles with the same precision as in the SM. To achieve such an accuracy, again multiloop calculations up to the three-loop order are required.

Detailed analyses of the data taken or to be taken at the LHC running at energies up to 14 TeV are expected to provide us with new insights into the particle physics and hopefully with the answer to the question whether low-energy supersymmetry is the right theory to describe the phenomena at the TeV scale.

Appendices

A. Group Theory

We consider a gauge group $\mathcal{G}$ with generators R^a satisfying the Lie algebra (useful sources for some of the material in this section have included [119, 136, 139])

$$\left[R^a, R^b\right] = if^{abc}R^c. \tag{A.1}$$

We work throughout with a fermion representation consisting of N_f sets of Dirac fermions or $2N_f$ sets of two-component fermions, in irreducible representations with identical Casimir invariants, using R^a to denote the generators in one such representation. Thus R^aR^a is proportional to the unit matrix:

$$R^aR^a = C_R \cdot I. \tag{A.2}$$

For the adjoint representation we have

$$C_A\delta_{ab} = f_{acd}f_{bcd}. \tag{A.3}$$

$I_2(R)$ is defined by

$$\mathrm{Tr}\left[R^aR^b\right] = I_2(R)\,\delta^{ab}. \tag{A.4}$$

Thus we have

$$C_R d_R = I_2(R)\,N_A, \tag{A.5}$$

where N_A is the number of generators and d_R is the dimensionality of the representation R. Evidently $I_2(A) = C_A$. The fully symmetric tensors d_R^{abcd} and d_A^{abcd} are defined by

$$\begin{aligned} d_R^{abcd} &= \frac{1}{6}\,\mathrm{Tr}\left[R^{(a}R^bR^cR^{d)}\right], \\ d_A^{abcd} &= \frac{1}{6}\,\mathrm{Tr}\left[F^{(a}F^bF^cF^{d)}\right], \end{aligned} \tag{A.6}$$

where

$$\left(F^a\right)^{bc} = if^{bac}, \tag{A.7}$$

$$\begin{aligned} R^{(a}R^bR^cR^{d)} = {} & R^aR^bR^cR^d + R^aR^bR^dR^c + R^aR^cR^bR^d \\ & + R^aR^cR^dR^b + R^aR^dR^bR^c + R^aR^dR^cR^b, \end{aligned} \tag{A.8}$$

(similar to $F^{(a}F^bF^cF^{d)}$).

Table 2: SU(N) Group invariants (here R is the fundamental representation).

Group	SU(N)
C_A	bN
C_R	$b\left(\left(N^2-1\right)/2N\right)$
$I_2\left(A\right)$	bN
$I_2\left(R\right)$	$b/2$
N_A	N^2-1
$D_4\left(AA\right)$	$\left(b^4/24\right)\left(N^2+36\right)N^2$
$D_4\left(RA\right)$	$\left(b^4/48\right)N(N^2+6)$
$D_4\left(RR\right)$	$\left(b^4/96N^2\right)\left(18-6N^2+N^4\right)$
$D_4\left(AAA\right)$	$\left(b^6/216\right)N^2(324+135N^2+N^4)$
$D_4\left(RAA\right)$	$\left(b^6/432\right)N^3(51+N^2)$

Table 3: SO(N) Group invariants (here R is the fundamental representation).

Group	SO(N)
C_A	$b(N-2)$
C_R	$(b/2)\left(N-1\right)$
$I_2\left(A\right)$	$b(N-2)$
$I_2\left(R\right)$	b
N_A	$(1/2)\,N(N-1)$
$D_4\left(AA\right)$	$\left(b^4/24\right)(N-2)(-296+138N-15N^2+N^3)$
$D_4\left(RA\right)$	$\left(b^4/24\right)(N-2)(22-7N+N^2)$
$D_4\left(RR\right)$	$\left(b^4/24\right)\left(4-N+N^2\right)$
$D_4\left(AAA\right)$	$\left(b^6/432\right)(N-2)(-29440+23272N-7018N^2+971N^3-47N^4+2N^5)$
$D_4\left(RAA\right)$	$\left(b^6/432\right)(N-2)(2048-1582N+387N^2-31N^3+2N^4)$

Table 4: Sp(N) Group invariants (here R is the fundamental representation).

Group	Sp(N)
C_A	$b(N+2)$
C_R	$(b/4)\left(N+1\right)$
$I_2\left(A\right)$	$b(N+2)$
$I_2\left(R\right)$	$b/2$
N_A	$(1/2)\,N(N+1)$
$D_4\left(AA\right)$	$\left(b^4/384\right)(N+2)(296+138N+15N^2+N^3)$
$D_4\left(RA\right)$	$\left(b^4/384\right)(N+2)(22+7N+N^2)$
$D_4\left(RR\right)$	$\left(b^4/384\right)\left(4+N+N^2\right)$
$D_4\left(AAA\right)$	$\left(b^6/27648\right)(N+2)(29440+23272N+7018N^2+971N^3+47N^4+2N^5)$
$D_4\left(RAA\right)$	$\left(b^6/27648\right)(N+2)(2048+1582N+387N^2+31N^3+2N^4)$

The additional tensor invariants occurring in the results are defined as

$$
\begin{aligned}
D_3\left(RR\right) &= \frac{d_A^{abc}d_A^{abc}}{N_A},\\
D_4\left(AA\right) &= \frac{d_A^{abcd}d_A^{abcd}}{N_A},\\
D_4\left(RA\right) &= \frac{d_R^{abcd}d_A^{abcd}}{N_A},\\
D_4\left(AAA\right) &= \frac{d_A^{abcd}d_A^{cdef}d_A^{abef}}{N_A},\\
D_4\left(RAA\right) &= \frac{d_R^{abcd}d_A^{cdef}d_A^{abef}}{N_A}.
\end{aligned}
\tag{A.9}
$$

In Tables 2, 3, and 4 we present results for the various tensor invariants for the groups SU(N), SO(N), and Sp(N), when the fermion representation R is the fundamental representation.

The canonical choice of b is $b = 1$ for all groups, but sometimes different choices are more convenient [139].

B. Modification of the $\overline{\text{DR}}$ Scheme: $\overline{\text{MDR}}$

In the following we provide analytic expressions for the finite shifts introduced in the top squark mass counterterms as compared to the $\overline{\text{DR}}$ scheme. According to the discussion in Section 7, one can distinguish four cases for the mass hierarchies.

Case 1. $m_{\tilde{q}} \gg m_{\tilde{t}_i}\ (i=1,2)$

$$
\left(\frac{m_{\tilde{t}_i}^{\overline{\text{MDR}}}}{m_{\tilde{t}_i}}\right)^2 = 1-\left(\alpha_s\right)^2 C_R N_q I_2\left(R\right) \times\left(-\frac{1}{2}+L_{\mu\tilde{q}}+\zeta\left(2\right)\right)\frac{m_{\tilde{q}}^2}{m_{\tilde{t}_i}^2}.
\tag{B.1}
$$

The label $N_q = 5$ has been introduced for convenience and for the logarithms the abbreviation $L_{\mu\tilde{q}} = \ln(\mu^2/m_{\tilde{q}}^2)$ has been introduced.

Case 2. $m_{\tilde{t}_2} \gg m_{\tilde{t}_1}$

$$
\left(\frac{m_{\tilde{t}_1}^{\overline{\text{MDR}}}}{m_{\tilde{t}_1}}\right)^2 = 1-\left(\alpha_s\right)^2 C_R I_2\left(R\right) \times\left(-\frac{1}{4}+\frac{1}{2}L_{\mu\tilde{t}_2}+\frac{1}{2}\zeta\left(2\right)\right)\frac{m_{\tilde{t}_2}^2}{m_{\tilde{t}_1}^2}.
\tag{B.2}
$$

In this equation we have $L_{\mu\tilde{t}_2} = \ln(\mu^2/m_{\tilde{t}_2}^2)$.

Case 3. $m_{\tilde{g}} \gg m_{\tilde{t}_i}$, $(i = 1, 2)$ and $m_{\tilde{q}} \gg m_{\tilde{g}}$

$$\left(\frac{m_{\tilde{t}_i}^{\overline{\mathrm{MDR}}}}{m_{\tilde{t}_i}}\right)^2 = 1 + \alpha_s C_R\left[1 + L_{\mu\tilde{g}}\right]\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + (\alpha_s)^2 \times \left\{C_R^2\left[-\frac{11}{4} - \frac{3}{2}L_{\mu\tilde{g}} + \zeta(2)\right]\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + C_A C_R\left[\frac{21}{8} + \frac{7}{2}L_{\mu\tilde{g}} + \frac{9}{8}L_{\mu\tilde{g}}^2 - \frac{1}{4}\zeta(2)\right]\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + C_R N_t I_2(R) \times \left[-\left(2 + 2L_{\mu\tilde{g}} + \frac{3}{4}L_{\mu\tilde{g}}^2\right)\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + (1 - 2\zeta(2))\frac{m_{\tilde{g}}\left(m_{\tilde{g}} - m_{\tilde{t}_2}\right)}{m_{\tilde{t}_i}^2} + \left(\frac{1}{4} - \frac{1}{2}L_{\mu\tilde{t}_2} - \frac{1}{2}\zeta(2)\right)\frac{m_{\tilde{t}_2}^2}{m_{\tilde{t}_i}^2}\right] + C_R N_q I_2(R) \times \left[\left(-\frac{5}{8} - \frac{3}{4}L_{\mu\tilde{g}} - \frac{5}{4}L_{\mu\tilde{q}} - \frac{3}{2}L_{\mu\tilde{g}}L_{\mu\tilde{q}} + \frac{3}{4}L_{\mu\tilde{q}}^2 + \frac{3}{2}\zeta(2)\right)\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + \left(-\frac{43}{36} - \frac{5}{6}L_{\tilde{q}\tilde{g}}\right)\frac{m_{\tilde{g}}^4}{m_{\tilde{q}}^2 m_{\tilde{t}_i}^2} + \left(-\frac{67}{288} - \frac{7}{24}L_{\tilde{q}\tilde{g}}\right)\frac{m_{\tilde{g}}^6}{m_{\tilde{q}}^4 m_{\tilde{t}_i}^2} + \left(+\frac{1}{2} - L_{\mu\tilde{q}} - \zeta(2)\right)\frac{m_{\tilde{q}}^2}{m_{\tilde{t}_i}^2}\right]\right\}. \tag{B.3}$$

Here $N_t = 1$, $L_{\mu\tilde{g}} = \ln(\mu^2/m_{\tilde{g}}^2)$, and $L_{\tilde{q}\tilde{g}} = \ln(m_{\tilde{q}}^2/m_{\tilde{g}}^2)$.

Case 4. $m_{\tilde{g}} \gg m_{\tilde{t}_1}$ and $m_{\tilde{q}} \approx m_{\tilde{g}}$

$$\left(\frac{m_{\tilde{t}_i}^{\overline{\mathrm{MDR}}}}{m_{\tilde{t}_i}}\right)^2 = 1 + \alpha_s C_R\left[1 + L_{\mu\tilde{g}}\right]\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + (\alpha_s)^2 \times \left\{C_R^2\left[-\frac{11}{4} - \frac{3}{2}L_{\mu\tilde{g}} + \zeta(2)\right]\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + C_A C_R\left[\frac{21}{8} + \frac{7}{2}L_{\mu\tilde{g}} + \frac{9}{8}L_{\mu\tilde{g}}^2 - \frac{1}{4}\zeta(2)\right]\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + C_R N_t I_2(R) \times \left[-\left(2 + 2L_{\mu\tilde{g}} + \frac{3}{4}L_{\mu\tilde{g}}^2\right)\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} + (1 - 2\zeta(2))\frac{m_{\tilde{g}}\left(m_{\tilde{g}} - m_{\tilde{t}_2}\right)}{m_{\tilde{t}_i}^2} + \left(\frac{1}{4} - \frac{1}{2}L_{\mu\tilde{t}_2} - \frac{1}{2}\zeta(2)\right)\frac{m_{\tilde{t}_2}^2}{m_{\tilde{t}_i}^2}\right] + C_R N_q I_2(R) \times \left[\left(-\frac{3}{4}L_{\mu\tilde{g}} - \frac{5}{4}L_{\mu\tilde{q}} - \frac{3}{2}L_{\mu\tilde{g}}L_{\mu\tilde{q}} + \frac{3}{4}L_{\mu\tilde{q}}^2 + \frac{3}{2}\zeta(2)\right)\frac{m_{\tilde{g}}^2}{m_{\tilde{t}_i}^2} - 4\zeta(2)\frac{m_{\tilde{g}}\left(m_{\tilde{g}} - m_{\tilde{t}_2}\right)}{m_{\tilde{t}_i}^2} - \left(\frac{7}{4} + L_{\mu\tilde{q}} + \zeta(2)\right)\frac{m_{\tilde{q}}^2}{m_{\tilde{t}_i}^2}\right]\right\}. \tag{B.4}$$

All the masses on the right hand side of (B.1), (B.2), (B.3), and (B.4) are $\overline{\mathrm{DR}}$ masses. Let us also mention that the above formulae are valid for the case $M_\varepsilon = 0$. The finite shifts given for the Cases 3 and 4 can also be used for other mass hierarchies like, for example, $m_{\tilde{q}} \gg m_{\tilde{t}_2} \approx m_{\tilde{g}} \gg m_{\tilde{t}_1}$ or $m_{\tilde{q}} \approx m_{\tilde{t}_2} \approx m_{\tilde{g}} \gg m_{\tilde{t}_1}$.

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References

[1] L. V. Avdeev and M. Y. Kalmykov, "Pole masses of quarks in dimensional reduction," *Nuclear Physics B*, vol. 502, no. 1-2, pp. 419–435, 1997.

[2] S. Weinberg, "A model of leptons," *Physical Review Letters*, vol. 19, no. 21, pp. 1264–1266, 1967.

[3] A. Salam, "Weak and electromagnetic interactions," in *Proceedings of the 8th Nobel Symposium*, pp. 367–377, Lerum, Sweden, May 1968.

[4] M. Gell-Mann, "Quarks," *Acta Physica Austriaca*, Supplemet 9, pp. 733–761, 1972.

[5] H. Fritzsch, M. Gell-Mann, and H. Leutwyler, "Advantages of the color octet gluon picture," *Physics Letters B*, vol. 47, no. 4, pp. 365–368, 1973.

[6] D. J. Gross and F. Wilczek, "Ultraviolet behavior of non-abelian gauge theories," *Physical Review Letters*, vol. 30, no. 26, pp. 1343–1346, 1973.

[7] D. J. Gross and F. Wilczek, "Asymptotically free gauge theories. I," *Physical Review D*, vol. 8, no. 10, pp. 3633–3652, 1973.

[8] S. Coleman and J. Mandula, "All possible symmetries of the S matrix," *Physical Review*, vol. 159, no. 5, pp. 1251–1256, 1967.

[9] R. Haag, J. T. Łopuszański, and M. Sohnius, "All possible generators of supersymmetries of the *S*-matrix," *Nuclear Physics B*, vol. 88, pp. 257–274, 1975.

[10] Y. A. Golfand and E. P. Likhtman, "Extension of the algebra of poincare group generators and violation of P invariance," *JETP Letters*, vol. 13, no. 8, p. 323, 1971, Translated from *Pis'ma Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, vol. 13 pp. 452, 1971.

[11] D. V. Volkov and V. P. Akulov, "Possible universal neutrino interaction," *JETP Letters*, vol. 16, p. 438, 1972, Translated from *Pis'ma Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, vol. 16 pp. 621, 1972.

[12] J. Wess and B. Zumino, "Supergauge transformations in four dimensions," *Nuclear Physics B*, vol. 70, pp. 39–50, 1974.

[13] A. Salam and J. Strathdee, "Super-gauge transformations," *Nuclear Physics B*, vol. 76, no. 3, pp. 477–482, 1974.

[14] B. Zumino, "Supersymmetry and the vacuum," *Nuclear Physics B*, vol. 89, no. 3, pp. 535–546, 1975.

[15] P. C. West, "The supersymmetric effective potential," *Nuclear Physics B*, vol. 106, pp. 219–227, 1976.

[16] M. T. Grisaru, W. Siegel, and M. Roček, "Improved methods for supergraphs," *Nuclear Physics B*, vol. 159, no. 3, pp. 429–450, 1979.

[17] L. Girardello and M. T. Grisaru, "Soft breaking of supersymmetry," *Nuclear Physics B*, vol. 194, no. 1, pp. 65–76, 1982.

[18] A. V. Gladyshev and D. I. Kazakov, "Is (low energy) dusy still alive?" *European School of High-Energy Physics*. In press http://arxiv.org/abs/1212.2548.

[19] http://lepewwg.web.cern.ch/LEPEWWG/.

[20] J. Ellis, S. Kelley, and D. V. Nanopoulos, "Probing the desert using gauge coupling unification," *Physics Letters B*, vol. 260, no. 1-2, pp. 131–137, 1991.

[21] U. Amaldi, W. de Boer, and H. Fürstenau, "Comparison of grand unified theories with electroweak and strong coupling constants measured at LEP," *Physics Letters B*, vol. 260, no. 3-4, pp. 447–455, 1991.

[22] P. Langacker and M. X. Luo, "Implications of precision electroweak experiments for m_t,ρ_0, $\sin^2\theta_W$, and grand unification," *Physical Review D*, vol. 44, no. 3, pp. 817–822, 1991.

[23] Y. Sofue and V. Rubin, "Rotation curves of spiral galaxies," *Annual Review of Astronomy and Astrophysics*, vol. 39, pp. 137–174, 2001.

[24] N. Kaiser and G. Squires, "Mapping the dark matter with weak gravitational lensing," *Astrophysics Journal*, vol. 404, no. 2, pp. 441–450, 1993.

[25] C. S. Kochanek, "Gravitational lensing limits on cold dark matter and its variants," *Astrophysics Journal*, vol. 453, p. 545, 1995.

[26] S. Eidelman, K. G. Hayes, K. A. Olive et al., "Review of particle physics," *Physics Letters B*, vol. 592, pp. 1–5, 2004.

[27] https://twiki.cern.ch/twiki/bin/view/AtlasPublic/SupersymmetryPublicResults.

[28] J. Angle, E. Aprile, F. Arneodo et al., "Search for light dark matter in XENON10 data," *Physical Review Letters*, vol. 107, no. 5, Article ID 051301, 5 pages, 2011.

[29] http://map.gsfc.nasa.gov/.

[30] G. W. Bennett, B. Bousquet, H. N. Brown et al., "Final report of the muon E821 anomalous magnetic moment measurement at BNL," *Physical Review D*, vol. 73, no. 7, Article ID 072003, 41 pages, 2006.

[31] https://twiki.cern.ch/twiki/bin/view/AtlasPublic/HiggsPublicResults.

[32] https://twiki.cern.ch/twiki/bin/view/CMSPublic/PhysicsResults.

[33] S. Heinemeyer, W. Hollik, and G. Weiglein, "Electroweak precision observables in the minimal supersymmetric standard model," *Physics Reports*, vol. 425, no. 5-6, pp. 265–368, 2006.

[34] D. Stockinger, "The muon magnetic moment and supersymmetry," *Journal of Physics G*, vol. 34, no. 2, article R45.

[35] F. del Aguila, J. A. Aguilar-Saavedra, B. C. Allanach et al., "Collider aspects of flavor physics at high Q," *European Physical Journal C*, vol. 57, no. 1-2, pp. 183–307, 2008.

[36] S. Dittmaier, C. Mariotti, G. Passarinos et al., "Handbook of LHC higgs cross sections: 1. Inclusive observables," http://arxiv.org/abs/1101.0593 .

[37] M. A. Shifman and A. I. Vainshtein, "Solution of the anomaly puzzle in SUSY gauge theories and the Wilson operator expansion," *Nuclear Physics B*, vol. 277, pp. 456–486, 1986, translated from *Soviet Physics—JETP*, vol. 64, p. 428, 1986, *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, vol. 91, pp. 723–744, 1986.

[38] N. Arkani-Hamed and H. Murayama, "Holomorphy, rescaling anomalies and exact functions in supersymmetric gauge theories," *Journal of High Energy Physics*, vol. 2000, no. 6, article 30, 2000.

[39] K. G. Wilson and J. Kogut, "The renormalization group and the ε expansion," *Physics Reports*, vol. 12, no. 2, pp. 75–199, 1974.

[40] M. Gell-Mann and F. E. Low, "Quantum electrodynamics at small distances," *Physical Review*, vol. 95, pp. 1300–1312, 1954.

[41] C. G. Callan, "Broken scale invariance in scalar field theory," *Physical Review D*, vol. 2, no. 8, pp. 1541–1547, 1970.

[42] K. Symanzik, "Small distance behaviour in field theory and power counting," *Communications in Mathematical Physics*, vol. 18, no. 3, pp. 227–246, 1970.

[43] M. T. Grisaru, B. Milewski, and D. Zanon, "The structure of UV divergences in SS YM theories," *Physics Letters B*, vol. 155, no. 5-6, pp. 357–363, 1985.

[44] J. Hisano and M. A. Shifman, "Exact results for soft supersymmetry-breaking parameters in supersymmetric gauge theories," *Physical Review D*, vol. 56, no. 9, pp. 5475–5482, 1997.

[45] N. Arkani-Hamed, G. F. Giudice, M. A. Luty, and R. Rattazzi, "Supersymmetry-breaking loops from analytic continuation into superspace," *Physical Review D*, vol. 58, no. 11, Article ID 115005, 1998.

[46] Y. Yamada, "Two-loop renormalization group equations for soft supersymmetry-breaking scalar interactions: supergraph method," *Physical Review D*, vol. 50, no. 5, pp. 3537–3545, 1994.

[47] L. V. Avdeev, D. I. Kazakov, and I. N. Kondrashuk, "Renormalizations in softly broken SUSY gauge theories," *Nuclear Physics B*, vol. 510, no. 1-2, pp. 289–312, 1998.

[48] D. I. Kazakov and V. N. Velizhanin, "Massive ghosts in softly broken SUSY gauge theories," *Physics Letters B*, vol. 485, no. 4, pp. 393–402, 2000.

[49] L. N. Mihaila, J. Salomon, and M. Steinhauser, "Gauge coupling β functions in the standard model to three loops," *Physical Review Letters*, vol. 108, no. 15, Article ID 151602, 4 pages, 2012.

[50] A. Bauer, L. Mihaila, and J. Salomon, "Matching coefficients for α_s and m_b to $O(\alpha_s^2)$ in the MSSM," *Journal of High Energy Physics*, vol. 2009, no. 2, article 37, 2009.

[51] W. Martens, L. Mihaila, J. Salomon, and M. Steinhauser, "Minimal supersymmetric SU(5) and gauge coupling unification at three loops," *Physical Review D*, vol. 82, no. 9, Article ID 095013, 11 pages, 2010.

[52] J. Ellis, J. R. Espinosa, G. F. Giudice, A. Hoecker, and A. Riotto, "The probable fate of the standard model," *Physics Letters B*, vol. 679, no. 4, pp. 369–375, 2009.

[53] J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori, A. Riotto, and A. Strumia, "Higgs mass implications on the stability of the electroweak vacuum," *Physics Letters B*, vol. 709, no. 3, pp. 222–228, 2012.

[54] A. Pak, M. Steinhauser, and N. Zerf, "Supersymmetric next-to-next-to-leading order corrections to Higgs boson production in gluon fusion," *Journal of High Energy Physics*, vol. 2012, no. 9, article 118, 2012.

[55] https://twiki.cern.ch/twiki/bin/view/LHCPhysics/CrossSections.

[56] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "The β function in supersymmetric gauge theories. Instantons versus traditional approach," *Physics Letters B*, vol. 166, no. 3, pp. 329–333, 1986, translated from *Soviet Journal of Nuclear Physics*, vol. 43, p. 294, 1986, *Yadernaya Fizika*, vol. 43, p. 459, 1986.

[57] W. Siegel, "Supersymmetric dimensional regularization via dimensional reduction," *Physics Letters B*, vol. 84, no. 2, pp. 193–196, 1979.

[58] W. Siegel, "Inconsistency of supersymmetric dimensional regularization," *Physics Letters B*, vol. 94, no. 1, pp. 37–40, 1980.

[59] L. V. Avdeev, G. A. Chochia, and A. A. Vladimirov, "On the scope of supersymmetric dimensional regularization," *Physics Letters B*, vol. 105, no. 4, pp. 272–274, 1981.

[60] L. V. Avdeev and A. A. Vladimirov, "Dimensional regularization and supersymmetry," *Nuclear Physics B*, vol. 219, no. 1, pp. 262–276, 1983.

[61] I. Jack and D. R. T. Jones, "The gaugino β-function," *Physics Letters B*, vol. 415, no. 4, pp. 383–389, 1997.

[62] I. Jack, D. R. T. Jones, and A. Pickering, "The soft scalar mass β-function," *Physics Letters B*, vol. 432, no. 1-2, pp. 114–119, 1998.

[63] I. Jack, D. R. T. Jones, and C. G. North, "$N = 1$ supersymmetry and the three-loop anomalous dimension for the chiral superfield," *Nuclear Physics B*, vol. 473, no. 1-2, pp. 308–322, 1996.

[64] I. Jack, D. R. T. Jones, and C. G. North, "$N = 1$ supersymmetry and the three-loop gauge β-function," *Physics Letters B*, vol. 386, no. 1–4, pp. 138–140, 1996.

[65] I. Jack, D. R. T. Jones, and C. G. North, "Scheme dependence and the NSVZ β-function," *Nuclear Physics B*, vol. 486, no. 1-2, pp. 479–499, 1997.

[66] I. Jack, D. R. T. Jones, and K. L. Roberts, "Equivalence of dimensional reduction and dimensional regularisation," *Zeitschrift für Physik C*, vol. 63, no. 1, pp. 151–159, 1994.

[67] L. V. Avdeev and O. V. Tarasov, "The three-loop β-function in the $N = 1$, 2, 4 supersymmetric Yang-Mills theories," *Physics Letters B*, vol. 112, no. 4-5, pp. 356–358, 1982.

[68] C. G. Bollini and J. J. Giambiagi, "Razmyernaya pyeryenormirovka: Chislo izmyeryeniicyrillic, short, kak ryegulyarizatsionnyicyrillic, short paramyetr," *Il Nuovo Cimento*, vol. 12, no. 1, pp. 20–26, 1972.

[69] G. 't Hooft and M. Veltman, "Regularization and renormalization of gauge fields," *Nuclear Physics B*, vol. 44, pp. 189–213, 1972.

[70] O. Piguet, K. Sibold, and M. Schweda, "General solution of the supersymmetry consistency conditions," *Nuclear Physics B*, vol. 174, no. 1, pp. 183–188, 1980.

[71] J. C. Ward, "An identity in quantum eletrodynamics," *Physical Review*, vol. 78, no. 2, p. 182, 1950.

[72] Y. Takahashi, "On the generalized ward identity," *Il Nuovo Cimento*, vol. 6, no. 2, pp. 371–375, 1957.

[73] A. A. Slavnov, "Ward identities in gauge theories," *Theoretical and Mathematical Physics*, vol. 10, no. 2, pp. 99–104, 1972, Translated from *Teoreticheskaya i Matematicheskaya Fizika*, vol. 10, no. 2, pp. 153–161, 1972.

[74] D. Stöckinger, "Regularization by dimensional reduction: consistency, quantum action principle, and supersymmetry," *Journal of High Energy Physics*, vol. 2005, no. 3, article 76, 2005.

[75] J. C. Collins, *Renormalization*, Cambridge University Press, Cambridge, UK, 1984.

[76] J. G. Korner and M. M. Tung, "Dimensional reduction methods in QCD," *Zeitschrift für Physik C*, vol. 64, no. 2, pp. 255–265, 1994.

[77] M. Misiak, "On dimensional methods in rare b decays," *Physics Letters B*, vol. 321, no. 1-2, pp. 113–120, 1994.

[78] Z. Kunszt, A. Signer, and Z. Trocsanyi, "One-loop helicity amplitudes for all $2 \to 2$ processes in QCD and $N = 1$ supersymmetric Yang-Mills theory," *Nuclear Physics B*, vol. 411, no. 2-3, pp. 397–442, 1994.

[79] J. Smith and W. L. van Neerven, "The difference between n-dimensional regularization and n-dimensional reduction in QCD," *European Physical Journal C*, vol. 40, no. 2, pp. 199–203, 2005.

[80] G. Altarelli, G. Curci, G. Martinelli, and S. Petrarca, "QCD non-leading corrections to weak decays as an application of regularization by dimensional reduction," *Nuclear Physics B*, vol. 187, no. 3, pp. 461–513, 1981.

[81] D. M. Capper, D. R. T. Jones, and P. van Nieuwenhuizen, "Regularization by dimensional reduction of supersymmetric and non-supersymmetric gauge theories," *Nuclear Physics B*, vol. 167, no. 3, pp. 479–499, 1980.

[82] H. Nicolai and P. K. Townsend, "Anomalies and supersymmetric regularization by dimensio nal reduction," *Physics Letters B*, vol. 93, no. 1-2, pp. 111–115, 1980.

[83] D. R. T. Jones and J. P. Leveille, "Dimensional regularization and the two-loop axial anomaly in abelian, non-abelian and

supersymmetric gauge theories," *Nuclear Physics B*, vol. 206, no. 3, pp. 473–495, 1982, Erratum in *Nuclear Physics B*, vol. 22, p. 517, 1983.

[84] S. L. Adler and W. A. Bardeen, "Absence of higher-order corrections in the anomalous axial-vector divergence equation," *Physical Review*, vol. 182, no. 5, pp. 1517–1536, 1969.

[85] T. L. Trueman, "Spurious anomalies in dimensional renormalization," *Zeitschrift für Physik C*, vol. 69, no. 1, pp. 525–536, 1996.

[86] S. A. Larin, "The renormalization of the axial anomaly in dimensional regularization," *Physics Letters B*, vol. 303, no. 1-2, pp. 113–118, 1993.

[87] K. G. Chetyrkin, B. A. Kniehl, M. Steinhauser, and W. A. Bardeen, "Effective QCD interactions of CP-odd higgs bosons at three loops," *Nuclear Physics B*, vol. 535, no. 1-2, pp. 3–18, 1998.

[88] S. L. Adler, "Axial-vector vertex in spinor electrodynamics," *Physical Review*, vol. 177, no. 5, pp. 2426–2438, 1969.

[89] J. S. Bell and R. Jackiw, "A PCAC puzzle: $\pi^\circ \to \gamma\gamma$ in the α-model," *Il Nuovo Cimento A*, vol. 60, no. 1, pp. 47–61, 1969.

[90] W. A. Bardeen, "Anomalous ward identities in spinor field theories," *Physical Review*, vol. 184, no. 5, pp. 1848–1859, 1969.

[91] W. A. Bardeen, R. Gastmans, and B. Lautrup, "Static quantities in Weinberg's model of weak and electromagnetic interactions," *Nuclear Physics B*, vol. 46, no. 1, pp. 319–331, 1972.

[92] M. S. Chanowitz, M. Furman, and I. Hinchliffe, "The axial current in dimensional regularization," *Nuclear Physics B*, vol. 159, no. 1-2, pp. 225–243, 1979.

[93] F. Jegerlehner, "Facts of life with γ_5," *European Physical Journal C*, vol. 18, no. 4, pp. 673–679, 2001.

[94] J. Fleischer, O. V. Tarasov, and F. Jegerlehner, "Two-loop heavy top corrections to the ρ parameter. A simple formula valid for arbitrary Higgs mass," *Physics Letters B*, vol. 319, no. 1–3, pp. 249–256, 1993.

[95] J. Fleischer, O. V. Tarasov, F. Jegerlehner, and P. Raczka, "Two-loop $O(\alpha_s G_\mu m_t^2)$ corrections to the partial decay width of the Z^0 into $b\bar{b}$ final states in the large top-mass limit," *Physics Letters B*, vol. 293, no. 3-4, pp. 437–444, 1992.

[96] A. Freitas, W. Hollik, W. Walter, and G. Weiglein, "Complete fermionic two-loop results for the $M_W - M_Z$ interdependence," *Physics Letters B*, vol. 495, no. 3-4, pp. 338–346, 2000, Erratum in *Physics Letters B*, vol. 570, no. 3-4, p. 265, 2003.

[97] A. Freitas, W. Hollik, W. Walter, and G. Weiglein, "Electroweak two-loop corrections to the $M_W - M_Z$ mass correlation in the standard model," *Nuclear Physics B*, vol. 632, no. 1–3, pp. 189–218, 2002, Erratum in *Nuclear Physics B*, vol. 666, no. 1-2, pp. 305–307, 2003.

[98] L. Avdeev, J. Fleischer, S. Mikhailov, and O. Tarasov, "$O(\alpha\alpha_s^2)$ correction to the electroweak ϱ parameter," *Physics Letters B*, vol. 336, no. 3-4, pp. 560–566, 1994, Erratum in *Physics Letters B*, vol. 349, pp. 597–589, 1995.

[99] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser, "Corrections of order $O(G_F M_t^2 \alpha_s^2)$ to the ϱ parameter," *Physics Letters B*, vol. 351, no. 1–3, pp. 331–338, 1995.

[100] S. Heinemeyer, D. Stockinger, and G. Weiglein, "Electroweak and supersymmetric two-loop corrections to $\mu(g-2)$," *Nuclear Physics B*, vol. 699, pp. 103–123, 2004.

[101] A. G. M. Pickering, J. A. Gracey, and D. R. T. Jones, "Three loop gauge β-function for the most general single gauge-coupling theory," *Physics Letters B*, vol. 510, pp. 347–354, 2001, Erratum in *Physics Letters B*, vol. 535, pp. 377, 2002.

[102] A. G. M. Pickering, J. A. Gracey, and D. R. T. Jones, "Three loop gauge β-function for the most general single gauge-coupling theory," *Physics Letters B*, vol. 512, pp. 230–238, 2002.

[103] R. V. Harlander, L. Mihaila, and M. Steinhauser, "The SUSY-QCD β function to three loops," *European Physical Journal C*, vol. 63, no. 3, pp. 383–390, 2009.

[104] G. 't Hooft, "Dimensional regularization and the renormalization group," *Nuclear Physics B*, vol. 61, pp. 455–468, 1973.

[105] W. A. Bardeen, A. J. Buras, D. W. Duke, and T. Muta, "Deep-inelastic scattering beyond the leading order in asymptotically free gauge theories," *Physical Review D*, vol. 18, no. 11, pp. 3998–4017, 1978.

[106] J. C. Collins, "Structure of counterterms in dimensional regularization," *Nuclear Physics B*, vol. 80, no. 2, pp. 341–348, 1974.

[107] J. C. Collins, "Normal products in dimensional regularization," *Nuclear Physics B*, vol. 92, no. 4, pp. 477–506, 1975.

[108] K. G. Chetyrkin and V. A. Smirnov, "R*-operation corrected," *Physics Letters B*, vol. 144, no. 5-6, pp. 419–424, 1984.

[109] A. A. Vladimirov, "Method of calculating renormalization-group functions in the scheme of dimensional regularization," *Theoretical and Mathematical Physics*, vol. 43, no. 2, pp. 417–422, 1980, Translated from *Teoreticheskaya i Matematicheskaya Fizika*, vol. 43, no. 2, pp. 210–217, 1980.

[110] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, "New approach to evaluation of multiloop Feynman integrals: the Gegenbauer polynomial x-space technique," *Nuclear Physics B*, vol. 174, no. 2-3, pp. 345–377, 1980.

[111] O. V. Tarasov, A. A. Vladimirov, and A. Y. Zharkov, "The gell-mann-low function of QCD in the three-loop approximation," *Physics Letters B*, vol. 93, no. 4, pp. 429–432, 1980.

[112] S. A. Larin and J. A. M. Vermaseren, "The three-loop QCD β-function and anomalous dimensions," *Physics Letters B*, vol. 303, no. 3-4, pp. 334–336, 1993.

[113] R. Harlander, P. Kant, L. Mihaila, and M. Steinhauser, "Dimensional reduction applied to QCD at three loops," *Journal of High Energy Physics*, vol. 2006, no. 9, article 53, 2006.

[114] K. G. Chetyrkin and M. F. Zoller, "Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the standard model," *Journal of High Energy Physics*, vol. 2012, no. 6, article 33, 2012.

[115] S. A. Larin, F. V. Tkachov, and J. A. M. Vermaseren, "The FORM version of MINCER," NIKHEF-H-91-18, 1991.

[116] J. A. M. Vermaseren, "New features of FORM," http://arxiv.org/abs/math-ph/0010025.

[117] K. G. Chetyrkin, M. Misiak, and M. Münz, "β functions and anomalous dimensions up to three loops," *Nuclear Physics B*, vol. 518, no. 1-2, pp. 473–494, 1998.

[118] K. G. Chetyrkin, "Quark mass anomalous dimension to $O(\alpha_s^4)$," *Physics Letters B*, vol. 404, no. 1-2, pp. 161–165, 1997.

[119] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, "The four-loop β-function in quantum chromodynamics," *Physics Letters B*, vol. 400, no. 3-4, pp. 379–384, 1997.

[120] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, "The 4-loop quark mass anomalous dimension and the invariant quark mass," *Physics Letters B*, vol. 405, no. 34, pp. 327–333, 1997.

[121] M. Czakon, "The four-loop QCD β-function and anomalous dimensions," *Nuclear Physics B*, vol. 710, no. 1-2, pp. 485–498, 2005.

[122] T. Curtright, "Three-loop charge renormalization effects due to quartic scalar self-interactions," *Physical Review D*, vol. 21, no. 6, pp. 1543–1549, 1980.

[123] D. R. T. Jones, "Comment on the charge-renormalization effects of quartic scalar self-interactions," *Physical Review D*, vol. 22, no. 12, pp. 3140–3141, 1980.

[124] M. Steinhauser, "MATAD: a program package for the computation of MAssive TADpoles," *Computer Physics Communications*, vol. 134, no. 3, pp. 335–364, 2001.

[125] K. G. Chetyrkin, "Corrections of order α_s^3 to R_{had} in pQCD with light gluinos," *Physics Letters B*, vol. 391, no. 3-4, pp. 402–412, 1997.

[126] K. G. Chetyrkin, "Correlator of the quark scalar currents and Γ_{tot} ($H \to$ hadrons) at $O(\alpha_s^3)$ in pQCD," *Physics Letters B*, vol. 390, no. 1–4, pp. 309–317, 1997.

[127] P. M. Ferreira, I. Jack, and D. R. T. Jones, "The three-loop SSM β-functions," *Physics Letters B*, vol. 387, no. 1, pp. 80–86, 1996.

[128] Z. Bern, A. de Freitas, L. Dixon, and H. L. Wong, "Supersymmetric regularization, two-loop QCD amplitudes, and coupling shifts," *Physical Review D*, vol. 66, no. 8, Article ID 085002, 25 pages, 2002.

[129] A. Pak, M. Steinhauser, and N. Zerf, "Towards Higgs boson production in gluon fusion to NNLO in the MSSM," *European Physical Journal C*, vol. 71, article 1602, 2011.

[130] P. Kant, *Higgs-Masse im MSSM und Dimensionale Reduktion in hohen Ordnungen der Störungstheorie [Ph.D. thesis]*, University of Karlsruhe, 2008.

[131] W. Siegel, P. K. Townsend, and P. van Nieuwenhuizen, "Supersymmetric dimensional regularization," in *Proceedings of the Cambridge Meeting on Supergravity*, 1980, ITP-SB-80-65.

[132] R. van Damme and G. 't Hooft, "Breakdown of unitarity in the dimensional reduction scheme," *Physics Letters B*, vol. 150, no. 1–3, pp. 133–138, 1985.

[133] I. Jack, D. R. T. Jones, and K. L. Roberts, "Dimensional reduction in non-supersymmetric theories," *Zeitschrift für Physik C*, vol. 62, no. 1, pp. 161–166, 1994.

[134] M. Steinhauser, "Results and techniques of multi-loop calculations," *Physics Reports*, vol. 364, no. 4, pp. 247–357, 2002.

[135] I. Jack, D. R. T. Jones, P. Kant, and L. Mihaila, "The four-loop DRED gauge β-function and fermion mass anomalous dimension for general gauge groups," *Journal of High Energy Physics*, vol. 2007, no. 9, article 58, 2007.

[136] P. Cvitanović, "Group theory for Feynman diagrams in non-Abelian gauge theories," *Physical Review D*, vol. 14, no. 6, pp. 1536–1553, 1976.

[137] P. Dittner, "Invariant tensors in SU(3). II," *Communications in Mathematical Physics*, vol. 27, pp. 44–52, 1972.

[138] A. J. Macfarlane, A. Sudbery, and P. H. Weisz, "On Gell-Mann's λ-matrices,d- and f-tensors, octets, and parametrizations of SU(3)," *Communications in Mathematical Physics*, vol. 11, no. 1, pp. 77–90, 1968.

[139] T. van Ritbergen, A. N. Schellekens, and J. A. M. Vermaseren, "Group theory factors for Feynman diagrams," *International Journal of Modern Physics A*, vol. 14, no. 1, pp. 41–96, 1999.

[140] T. P. Cheng, E. Eichten, and L. F. Li, "Higgs phenomena in asymptotically free gauge theories," *Physical Review D*, vol. 9, no. 8, pp. 2259–2273, 1974.

[141] P. Nogueira, "Automatic Feynman graph generation," *Journal of Computational Physics*, vol. 105, no. 2, pp. 279–289, 1993.

[142] T. Seidensticker, "Automatic application of successive asymptotic expansions of Feynman diagrams," in *Proceedings of the 6th International Workshop on New Computing Techniques in Physics Research (AIHENP '99)*, 1999, http://arxiv.org/abs/hep-ph/9905298.

[143] R. Harlander, T. Seidensticker, and M. Steinhauser, "Corrections of script $O(\alpha\alpha_s)$ to the decay of the Z boson into bottom quarks," *Physics Letters B*, vol. 426, no. 1-2, pp. 125–132, 1998.

[144] R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila, and M. Steinhauser, "Four-loop β function and mass anomalous dimension in dimensional reduction," *Journal of High Energy Physics*, vol. 2006, no. 12, article 24, 2006.

[145] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, "Order α_s^4 QCD corrections to Z and t decays," *Physical Review Letters*, vol. 101, no. 1, Article ID 012002, 4 pages, 2008.

[146] P. A. Baikov, K. G. Chetyrkin, and J. H. Kuhn, "Adler function, DIS sum rules and Crewther relations," *Nuclear Physics B*, vol. 205-206, pp. 237–241, 2010.

[147] I. Jack, D. R. T. Jones, and A. Pickering, "The connection between the DRED and NSVZ renormalisation schemes," *Physics Letters B*, vol. 435, no. 1-2, pp. 61–66, 1998.

[148] D. R. T. Jones, "More on the axial anomaly in supersymmetric Yang-Mills theory," *Physics Letters B*, vol. 123, no. 1-2, pp. 45–46, 1983.

[149] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Instanton effects in supersymmetric theories," *Nuclear Physics B*, vol. 229, no. 2, pp. 407–420, 1983.

[150] V. A. Smirnov, *Applied Asymptotic Expansions in Momenta and Masses*, vol. 177 of *Springer Tracts in Modern Physics*, Springer, New York, NY, USA, 2002.

[151] P. Marquard, L. Mihaila, J. H. Piclum, and M. Steinhauser, "Relation between the pole and the minimally subtracted mass in dimensional regularization and dimensional reduction to three-loop order," *Nuclear Physics B*, vol. 773, no. 1-2, pp. 1–18, 2007.

[152] S. P. Martin, "Two-loop effective potential for a general renormalizable theory and softly broken supersymmetry," *Physical Review D*, vol. 65, no. 11, Article ID 116003, 20 pages, 2002.

[153] http://www.fnal.gov/pub/science/experiments/energy/tevatron/.

[154] S. Heinemeyer, "MSSM Higgs physics at higher orders," *International Journal of Modern Physics A*, vol. 21, no. 13, pp. 2659–2772, 2006.

[155] G. Degrassi and P. Slavich, "On the radiative corrections to the neutral Higgs boson masses in the NMSSM," *Nuclear Physics B*, vol. 825, no. 1-2, pp. 119–150, 2010.

[156] P. Kant, R. V. Harlander, L. Mihaila, and M. Steinhauser, "Light MSSM Higgs boson mass to three-loop accuracy," *Journal of High Energy Physics*, vol. 2010, no. 8, article 104, 2010.

[157] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Exact Gell-Mann-low function of supersymmetric Yang-Mills theories from instanton calculus," *Nuclear Physics B*, vol. 229, no. 2, pp. 381–393, 1983.

[158] I. Jack, D. R. T. Jones, and A. F. Kord, "Three loop soft running, benchmark points and semi-perturbative unification," *Physics Letters B*, vol. 579, no. 1-2, pp. 180–188, 2004.

[159] I. Jack, D. R. T. Jones, and A. F. Kord, "Snowmass benchmark points and three-loop running," *Annals of Physics*, vol. 316, no. 1, pp. 213–233, 2005.

[160] L. N. Mihaila, J. Salomon, and M. Steinhauser, "Renormalization constants and β functions for the gauge couplings of the standard model to three-loop order," *Physical Review D*, vol. 86, no. 9, Article ID 096008, 27 pages, 2012.

[161] T. Hermann, L. Mihaila, and M. Steinhauser, "Three-loop anomalous dimensions for squarks in supersymmetric QCD," *Physics Letters B*, vol. 703, no. 1, pp. 51–59, 2011.

[162] W. Hollik, E. Kraus, M. Roth, C. Rupp, K. Sibold, and D. Stöckinger, "Renormalization of the minimal supersymmetric standard model," *Nuclear Physics B*, vol. 639, no. 1-2, pp. 3–65, 2002.

[163] W. Hollik, E. Kraus, and D. Stockinger, "Renormalization of supersymmetric Yang-Mills theories with soft supersymmetry breaking," *European Physical Journal C*, vol. 23, no. 4, pp. 735–747, 2002.

[164] I. Jack, D. R. T. Jones, S. P. Martin, M. T. Vaughn, and Y. Yamada, "Decoupling of the ε-scalar mass in softly broken supersymmetry," *Physical Review D*, vol. 50, no. 9, pp. R5481–R5483, 1994.

[165] H. D. Politzer, "Reliable perturbative results for strong interactions?" *Physical Review Letters*, vol. 30, no. 26, pp. 1346–1349, 1973.

[166] D. R. T. Jones, "Two-loop diagrams in Yang-Mills theory," *Nuclear Physics B*, vol. 75, no. 3, pp. 531–538, 1974.

[167] O. V. Tarasov and A. A. Vladimirov, "Two loop renormalization of the Yang-Mills theory in an arbitrary gauge," *Soviet Journal of Nuclear Physics*, vol. 25, p. 585, 1977, Translated from *Yadernaya Fizika*, vol. 25, pp. 1104–1106, 1977.

[168] W. E. Caswell, "Asymptotic behavior of non-abelian gauge theories to two-loop order," *Physical Review Letters*, vol. 33, no. 4, pp. 244–246, 1974.

[169] E. Egorian and O. V. Tarasov, "Two loop renormalization of the QCD in an arbitrary gaug," *Teoreticheskaya i Matematicheskaya Fizika*, vol. 41, pp. 26–32, 1979, Translated from *Theoretical and Mathematical Physics*, vol. 41, pp. 863–867, 1979.

[170] D. R. T. Jones, "Two-loop function for a $G_1 \times G_2$ gauge theory," *Physical Review D*, vol. 25, no. 2, pp. 581–582, 1982.

[171] M. S. Fischler and C. T. Hill, "Effects of large mass fermions on M_X and $\sin^2\theta_W$," *Nuclear Physics B*, vol. 193, no. 1, pp. 53–60, 1981.

[172] M. E. Machacek and M. T. Vaughn, "Two-loop renormalization group equations in a general quantum field theory: (I). Wave function renormalization," *Nuclear Physics B*, vol. 222, no. 1, pp. 83–103, 1983.

[173] M. E. Machacek and M. T. Vaughn, "Two-loop renormalization group equations in a general quantum field theory (II). Yukawa couplings," *Nuclear Physics B*, vol. 236, no. 1, pp. 221–232, 1984.

[174] M. E. Machacek and M. T. Vaughn, "Two-loop renormalization group equations in a general quantum field theory: (III). Scalar quartic couplings," *Nuclear Physics B*, vol. 249, no. 1, pp. 70–92, 1985.

[175] I. Jack and H. Osborn, "General background field calculations with fermion fields," *Nuclear Physics B*, vol. 249, no. 3, pp. 472–506, 1985.

[176] M. Steinhauser, "Higgs boson decay into gluons up to $o(\alpha_s^3 G_F m_t^2)$," *Physical Review D*, vol. 59, no. 5, Article ID 054005, 11 pages, 1999.

[177] A. V. Bednyakov, A. F. Pikelner, and V. N. Velizhanin, "Anomalous dimensions of gauge fields and gauge coupling β-functions in the standard model at three loops," *Journal of High Energy Physics*, vol. 2013, no. 1, article 17, 2013.

[178] K. Nakamura, K. Hagiwara, K. Hikasa et al., "Review of particle physics," *Journal of Physics G*, vol. 37, no. 7, Article ID 075021, 2010.

[179] L. F. Abbott, "The background field method beyond one loop," *Nuclear Physics B*, vol. 185, no. 1, pp. 189–203, 1981.

[180] A. Denner, G. Weiglein, and S. Dittmaier, "Application of the background-field method to the electroweak standard model," *Nuclear Physics B*, vol. 440, no. 1-2, pp. 95–128, 1995.

[181] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley, Reading, Mass, USA, 1995.

[182] S. Bethke, "The 2009 world average of α," *European Physical Journal C*, vol. 64, no. 4, pp. 689–703, 2009.

[183] R. Harlander, L. Mihaila, and M. Steinhauser, "Two-loop matching coefficients for the strong coupling in the minimal supersymmetric standard model," *Physical Review D*, vol. 72, no. 9, Article ID 095009, 11 pages, 2005.

[184] T. Appelquist and J. Carazzone, "Infrared singularities and massive fields," *Physical Review D*, vol. 11, no. 10, pp. 2856–2861, 1975.

[185] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, "Decoupling relations to $O(\alpha_s^3)$ and their connection to low-energy theorems," *Nuclear Physics B*, vol. 510, no. 1-2, pp. 61–87, 1998.

[186] Y. Schröder and M. Steinhauser, "Four-loop decoupling relations for the strong coupling," *Journal of High Energy Physics*, vol. 2006, no. 1, article 51, 2006.

[187] K. G. Chetyrkin, J. H. Kuhn, and C. Sturm, "QCD decoupling at four loops," *Nuclear Physics B*, vol. 744, no. 1-2, pp. 121–135, 2006.

[188] A. V. Bednyakov, "Running mass of the b-quark in QCD and SUSY QCD," *International Journal of Modern Physics A*, vol. 22, no. 29, p. 5245, 2007.

[189] A. V. Bednyakov, "On the two-loop decoupling corrections to τ-lepton and b-quark running masses in the MSSM," *International Journal of Modern Physics A*, vol. 25, no. 10, p. 2437, 2010.

[190] D. Noth and M. Spira, "Higgs boson couplings to bottom quarks: two-loop supersymmetry-QCD corrections," *Physical Review Letters*, vol. 101, no. 18, Article ID 181801, 4 pages, 2008.

[191] D. Noth and M. Spira, "Supersymmetric Higgs Yukawa couplings to bottom quarks at next-to-next-to-leading order," *Journal of High Energy Physics*, vol. 2011, no. 6, article 84, 2011.

[192] A. Kurz, M. Steinhauser, and N. Zerf, "Decoupling constant for α_s and the effective gluon-Higgs coupling to three loops in supersymmetric QCD," *Journal of High Energy Physics*, vol. 2012, no. 7, article 138, 2012.

[193] I. Jack and D. R. T. Jones, "Soft supersymmetry breaking and finiteness," *Physics Letters B*, vol. 333, no. 3-4, pp. 372–379, 1994.

[194] A. Bednyakov, A. Onishchenko, V. Velizhanin, and O. Veretin, "Two-loop $O(\alpha_s^2)$ MSSM corrections to the pole masses of heavy quarks," *European Physical Journal C*, vol. 29, no. 1, pp. 87–101, 2003.

[195] D. M. Pierce, J. A. Bagger, K. T. Matchev, and R. J. Zhang, "Precision corrections in the minimal supersymmetric standard model," *Nuclear Physics B*, vol. 491, no. 1-2, pp. 3–67, 1997.

[196] R. V. Harlander and M. Steinhauser, "Supersymmetric Higgs production in gluon fusion at next-to-leading order," *Journal of High Energy Physics*, vol. 2004, no. 9, article 66, 2004.

[197] S. P. Martin and D. G. Robertson, "TSIL: a program for the calculation of two-loop self-energy integrals," *Computer Physics Communications*, vol. 174, no. 2, pp. 133–151, 2006.

[198] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Remarks on Higgs-boson interactions with nucleons," *Physics Letters B*, vol. 78, no. 4, pp. 443–446, 1978.

[199] M. Carena, D. Garcia, U. Nierste, and C. E. M. Wagner, "Effective Lagrangian for the $\bar{t}bH^+$ interaction in the MSSM and charged Higgs phenomenology," *Nuclear Physics B*, vol. 577, no. 1-2, pp. 88–120, 2000.

[200] R. V. Harlander, L. Mihaila, and M. Steinhauser, "Running of α_s and m_b in the MSSM," *Physical Review D*, vol. 76, no. 5, Article ID 055002, 9 pages, 2007.

[201] L. Mihaila, "Two-loop parameter relations between dimensional regularization and dimensional reduction applied to SUSY-QCD," *Physics Letters B*, vol. 681, no. 1, pp. 52–59, 2009.

[202] D. Stockinger and P. Varso, "FeynArts model file for MSSM transition counterterms from DREG to DRED," *Computer Physics Communications*, vol. 183, no. 2, pp. 422–430, 2012.

[203] S. Bethke, "Experimental tests of asymptotic freedom," *Progress in Particle and Nuclear Physics*, vol. 58, no. 2, pp. 351–386, 2007.

[204] J. A. Aguilar-Saavedra, A. Ali, B. C. Allanach et al., "Supersymmetry parameter analysis: SPA convention and project," *European Physical Journal C*, vol. 46, no. 1, pp. 43–60, 2006.

[205] N. Ghodbane and H. U. Martyn, "Compilation of SUSY particle spectra from Snowmass 2001 benchmark models," in *Proceedings of the APS/DPF/DPB Summer Study on the Future of Particle Physics (Snowmass '01)*, N. Graf, Ed., http://arXiv:hep-ph/0201233.

[206] S. Dimopoulos and H. Georgi, "Softly broken supersymmetry and SU(5)," *Nuclear Physics B*, vol. 193, no. 1, pp. 150–162, 1981.

[207] N. Sakai, "Naturalnes in supersymmetric GUTS," *Zeitschrift für Physik C*, vol. 11, no. 2, pp. 153–157, 1981.

[208] T. Goto and T. Nihei, "Effect of an RRRR dimension 5 operator on the proton decay in the minimal SU(5) SUGRA GUT model," *Physical Review D*, vol. 59, no. 11, Article ID 115009, 14 pages, 1999.

[209] H. Murayama and A. Pierce, "Not even decoupling can save minimal supersymmetric SU(5)," *Physical Review D*, vol. 65, no. 5, Article ID 055009, 8 pages, 2002.

[210] B. Bajc, P. Fileviez Perez, and G. Senjanovic, "Proton decay in minimal supersymmetric SU(5)," *Physical Review D*, vol. 66, no. 7, Article ID 075005, 8 pages, 2002.

[211] D. Emmanuel-Costa and S. Wiesenfeldt, "Proton decay in a consistent supersymmetric SU(5) GUT model," *Nuclear Physics B*, vol. 661, pp. 62–82, 2003.

[212] S. Wiesenfeldt, *Proton decay in supersymmetric grand unified theories [Ph.D. thesis]*, University of Hamburg, 2004.

[213] B. Bajc, P. F. Perez, and G. Senjanovic, "Minimal supersymmetric SU(5) theory and proton decay: where do we stand?" in *Proceedings of the Physics Beyond the Standard Model: Beyond the Desert Conference*, p. 734, IOP, Bristol, UK, 2003, http://arxiv.org/abs/hep-ph/0210374.

[214] Y. Yamada, "SUSY and GUT threshold effects in SUSYSU(5) models," *Zeitschrift für Physik C*, vol. 60, pp. 83–93, 1993.

[215] L. Hall, "Grand unification of effective gauge theories," *Nuclear Physics B*, vol. 178, no. 1, pp. 75–124, 1981.

[216] S. Weinberg, "Effective gauge theories," *Physics Letters B*, vol. 91, no. 1, pp. 51–55, 1980.

[217] M. B. Einhorn and D. R. T. Jones, "The weak mixing angle and unification mass in supersymmetric SU(5)," *Nuclear Physics B*, vol. 196, no. 3, pp. 475–488, 1982.

[218] K. Hagiwara and Y. Yamada, "Grand unification threshold effects in supersymmetric SU(5) models," *Physical Review Letters*, vol. 70, no. 6, pp. 709–712, 1993.

[219] A. Dedes, A. B. Lahanas, J. Rizos, and K. Tamvakis, "Threshold effects and radiative electroweak symmetry breaking in SU(5) extensions of the MSSM," *Physical Review D*, vol. 55, no. 5, pp. 2955–2964, 1997.

[220] J. Hisano, H. Murayama, and T. Yanagida, "Nucleon decay in the minimal supersymmetric SU(5) grand unification," *Nuclear Physics B*, vol. 402, no. 1-2, pp. 46–84, 1993.

[221] C. Amsler, M. Doser, M. Antonelli et al., "Review of particle physics," *Physics Letters B*, vol. 667, no. 1–5, pp. 1–6, 2008.

[222] T. Teubner, K. Hagiwara, R. Liao, A. D. Martin, and D. Nomura, "Update of g—2 of the muon and $\Delta\alpha$," *Chinese Physics C*, vol. 34, article 728, 2010.

[223] M. Steinhauser, "Leptonic contribution to the effective electromagnetic coupling constant up to three loops," *Physics Letters B*, vol. 429, no. 1-2, pp. 158–161, 1998.

[224] J. H. Kuhn and M. Steinhauser, "A perturbation theory driven analysis of the effective QED coupling at M_Z," *Physics Letters B*, vol. 437, no. 3-4, pp. 425–431, 1998.

[225] S. Fanchiotti, B. Kniehl, and A. Sirlin, "Incorporation of QCD effects in basic corrections of the electroweak theory," *Physical Review D*, vol. 48, no. 1, pp. 307–331, 1993.

[226] K. G. Chetyrkin, J. H. Kuhn, and M. Steinhauser, "RunDec: a mathematica package for running and decoupling of the strong coupling and quark masses," *Computer Physics Communications*, vol. 133, no. 1, pp. 43–65, 2000.

[227] A. Dedes, A. B. Lahanas, and K. Tamvakis, "The effective weak mixing angle in the MSSM," *Physical Review D*, vol. 59, no. 1, Article ID 015019, 17 pages, 1999.

[228] A. H. Chamseddine, R. Arnowitt, and P. Nath, "Locally supersymmetric grand unification," *Physical Review Letters*, vol. 49, no. 14, pp. 970–974, 1982.

[229] B. C. Allanach, "SOFTSUSY: a program for calculating supersymmetric spectra," *Computer Physics Communications*, vol. 143, no. 3, pp. 305–331, 2002.

[230] K. Kobayashi, M. Earl, Y. Ashie et al., "Search for nucleon decay via modes favored by supersymmetric grand unification models in Super-Kamiokande-I," *Physical Review D*, vol. 72, no. 5, Article ID 052007, 15 pages, 2005.

[231] Y. Nambu, "Axial vector current conservation in weak interactions," *Physical Review Letters*, vol. 4, no. 7, pp. 380–382, 1960.

[232] F. Englert and R. Brout, "Broken symmetry and the mass of gauge vector mesons," *Physical Review Letters*, vol. 13, pp. 321–323, 1964.

[233] P. W. Higgs, "Broken symmetries, massless particles and gauge fields," *Physics Letters*, vol. 12, pp. 132–133, 1964.

[234] P. W. Higgs, "Broken symmetries and the masses of gauge bosons," *Physical Review Letters*, vol. 13, pp. 508–509, 1964.

[235] G. 't Hooft, "Renormalizable Lagrangians for massive Yang-Mills fields," *Nuclear Physics B*, vol. 35, no. 1, pp. 167–188, 1971.

[236] ALEPH Collaboration, DELPHI Collaboration, L3 Collaboration, OPAL Collaboration, and The LEP Working Group for Higgs Boson Searches, "Search for the standard model higgs boson at LEP," *Physics Letters B*, vol. 565, pp. 61–75, 2003.

[237] M. Baak, M. Goebel, J. Haller et al., "The electroweak fit of the standard model after the discovery of a new boson at the LHC," *European Physical Journal C*, vol. 72, article 2205, 2012.

[238] TEVNPH Working Group, CDF, and D0 Collaborations, "Combined CDF and D0 search for standard model higgs boson production with up to 10.0 fb-1 of data," http://arxiv.org/abs/1203.3774.

[239] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl, and M. Shaposhnikov, "Higgs boson mass and new physics," *Journal of High Energy Physics*, vol. 2012, no. 10, article 140, 2012.

[240] G. Degrassi, S. di Vita, J. Elias-Miro et al., "Higgs mass and vacuum stability in the standard model at NNLO," *Journal of High Energy Physics*, vol. 2012, no. 8, article 98, 2012.

[241] J. R. Ellis, G. Ridolfi, and F. Z. Zwirner, "On radiative corrections to supersymmetric Higgs boson masses and their implications for LEP searches," *Physics Letters B*, vol. 262, no. 4, pp. 477–484, 1991.

[242] Y. Okada, M. Yamaguchi, and T. Yanagida, "Welcome," *Progress of Theoretical Physics*, vol. 85, p. 1, 1991.

[243] H. E. Haber and R. Hempfling, "Can the mass of the lightest Higgs boson of the minimal supersymmetric model be larger than m_Z?" *Physical Review Letters*, vol. 66, no. 14, pp. 1815–1818, 1991.

[244] B. C. Allanach, M. Battaglia, G. A. Blair et al., "The snowmass points and slopes: benchmarks for SUSY searches," in *Proceedings of APS/DPF/DPB Summer Study on the Future of Particle Physics*, N. Graf, Ed., Snowmass, Colorado, June 2001.

[245] B. C. Allanach, M. Battaglia, G. A. Blair et al., "The snowmass points and slopes: benchmarks for SUSY searches," *European Physical Journal C*, vol. 25, no. 1, pp. 113–123, 2002.

[246] P. H. Chankowski, S. Pokorski, and J. Rosiek, "Charged and neutral supersymmetric Higgs boson masses. Complete one-loop analysis," *Physics Letters B*, vol. 274, no. 2, pp. 191–198, 1992.

[247] A. Brignole, "Radiative corrections to the supersymmetric neutral Higgs boson masses," *Physics Letters B*, vol. 281, no. 3-4, pp. 284–294, 1992.

[248] A. Dabelstein, "The one-loop renormalization of the MSSM Higgs sector and its application to the neutral scalar Higgs masses," *Zeitschrift für Physik C*, vol. 67, no. 3, pp. 495–511, 1995.

[249] B. C. Allanach, A. Djouadi, J. L. Kneur, W. Porod, and P. Slavich, "Precise determination of the neutral Higgs boson masses in the MSSM," *Journal of High Energy Physics*, vol. 2004, no. 9, article 44, 2004.

[250] M. Frank, T. Hahn, S. Heinemeyer, W. Hollik, H. Rzehak, and G. Weiglein, "The Higgs boson masses and mixings of the complex MSSM in the Feynman-diagrammatic approach," *Journal of High Energy Physics*, vol. 2007, no. 2, article 47, 2007.

[251] S. Heinemeyer, W. Hollik, H. Rzehak, and G. Weiglein, "The Higgs sector of the complex MSSM at two-loop order: QCD contributions," *Physics Letters B*, vol. 652, no. 5-6, pp. 300–309, 2007.

[252] M. S. Carena, J. R. Ellis, A. Pilaftsis, and C. E. M. Wagner, "Renormalization-group-improved effective potential for the MSSM Higgs sector with explicit CP violation," *Nuclear Physics B*, vol. 586, no. 1-2, pp. 92–140, 2000.

[253] S. P. Martin, "Complete two-loop effective potential approximation to the lightest Higgs scalar boson mass in supersymmetry," *Physical Review D*, vol. 67, no. 9, Article ID 095012, 5 pages, 2003.

[254] S. P. Martin, "Three-loop corrections to the lightest Higgs scalar boson mass in supersymmetry," *Physical Review D*, vol. 75, no. 5, Article ID 055005, 9 pages, 2007.

[255] R. V. Harlander, P. Kant, L. Mihaila, and M. Steinhauser, "Higgs boson mass in supersymmetry to three loops," *Physical Review Letters*, vol. 100, no. 19, Article ID 191602, 4 pages, 2008, Erratum in *Physical Review Letters*, vol. 101, no. 3. Article ID 039901, 2008.

[256] S. Heinemeyer, W. Hollik, and G. Weiglein, "FeynHiggs: a program for the calculation of the masses of the neutral C-even Higgs bosons in the MSSM," *Computer Physics Communications*, vol. 124, no. 1, pp. 76–89, 2000.

[257] G. Degrassi, S. Heinemeyer, W. Hollik, P. Slavich, and G. Weiglein, "Towards high-precision predictions for the MSSM Higgs sector," *European Physical Journal C*, vol. 28, no. 1, pp. 133–143, 2003.

[258] T. Hahn, S. Heinemeyer, W. Hollik, H. Rzehak, and G. Weiglein, "FeynHiggs: a program for the calculation of MSSM Higgs-boson observables—version 2.6.5," *Computer Physics Communications*, vol. 180, no. 8, pp. 1426–1427, 2009.

[259] J. S. Lee, A. Pilaftsis, M. S. Carena et al., "CPsuperH: a computational tool for Higgs phenomenology in the minimal supersymmetric standard model with explicit CP violation," *Computer Physics Communications*, vol. 156, pp. 283–317, 2004.

[260] J. S. Lee, M. Carena, J. Ellis, A. Pilaftsis, and C. E. M. Wagner, "CPsuperH2.0: an improved computational tool for Higgs phenomenology in the MSSM with explicit CP violation," *Computer Physics Communications*, vol. 180, no. 2, pp. 312–331, 2009.

[261] http://www-ttp.particle.uni-karlsruhe.de/Progdata/ttp10/ttp10-23/.

[262] http://www.feynhiggs.de/.

[263] Y. Schroder and M. Steinhauser, "Four-loop singlet contribution to the electroweak ρ parameter," *Physics Letters B*, vol. 622, no. 1-2124, p. 130, 2005.

[264] K. G. Chetyrkin, M. Faisst, J. H. Kuhn, P. Maierhofer, and C. Sturm, "Four-loop QCD corrections to the electroweak ρ parameter," *Physical Review Letters*, vol. 97, no. 10, Article ID 102003, 3 pages, 2006.

[265] R. Boughezal and M. Czakon, "Single scale tadpoles and $O(G_F m_t^2 \alpha_s^3)$ corrections to the ρ parameter," *Nuclear Physics B*, vol. 755, no. 1–3, pp. 221–238, 2006.

[266] G. Degrassi, P. Slavich, and F. Zwirner, "On the neutral Higgs boson masses in the MSSM for arbitrary stop mixing," *Nuclear Physics B*, vol. 611, no. 1–3, pp. 403–422, 2001.

[267] K. G. Wilson, "Non-Lagrangian models of current algebra," *Physical Review*, vol. 179, pp. 1499–1512, 1969.

[268] H. Kluberg-Stern and J. B. Zuber, "Renormalization of non-Abelian gauge theories in a background-field gauge. II. Gauge-invariant operators," *Physical Review D*, vol. 12, no. 10, pp. 3159–3180, 1975.

[269] N. K. Nielsen, "Gauge invariance and broken conformal symmetry," *Nuclear Physics B*, vol. 97, no. 3, pp. 527–540, 1975.

[270] V. P. Spiridonov, "Anomalous dimension of g**2 (Muon-neutrino) and beta function," Report No. INR P-0378, 1984.

[271] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, "Three-loop script capital $O(\alpha_s^2 G_F M_t^2)$ corrections to hadronic Higgs decays," *Nuclear Physics B*, vol. 490, no. 1-2, pp. 19–39, 1997.

[272] G. Degrassi and P. Slavich, "NLO QCD bottom corrections to Higgs boson production in the MSSM," *Journal of High Energy Physics*, vol. 2010, no. 11, article 44, 2010.

[273] R. V. Harlander, F. Hofmann, and H. Mantler, "Supersymmetric Higgs production in gluon fusion," *Journal of High Energy Physics*, vol. 2011, no. 2, article 55, 2011.

[274] A. I. Davydychev and J. B. Tausk, "Two-loop self-energy diagrams with different masses and the momentum expansion," *Nuclear Physics B*, vol. 397, no. 1-2, pp. 123–142, 1993.

[275] S. Dawson, "Radiative corrections to Higgs boson production," *Nuclear Physics B*, vol. 359, no. 2-3, pp. 283–300, 1991.

[276] A. Djouadi, M. Spira, and P. M. Zerwas, "Production of Higgs bosons in proton colliders. QCD corrections," *Physics Letters B*, vol. 264, no. 3-4, pp. 440–446, 1991.

[277] T. Inami, T. Kubota, and Y. Okada, "Effective gauge theory and the effect of heavy quarks in Higgs boson decays," *Zeitschrift für Physik C*, vol. 18, no. 1, pp. 69–80, 1983.

[278] P. A. Baikov and K. G. Chetyrkin, "Top quark mediated Higgs boson decay into hadrons to order α_s^5," *Physical Review Letters*, vol. 97, no. 6, Article ID 061803, 3 pages, 2006.

[279] R. V. Harlander and M. Steinhauser, "Hadronic Higgs production and decay in supersymmetry at next-to-leading order," *Physics Letters B*, vol. 574, no. 3-4, pp. 258–268, 2003.

[280] G. Degrassi and P. Slavich, "On the NLO QCD corrections to Higgs production and decay in the MSSM," *Nuclear Physics B*, vol. 805, no. 1-2, pp. 267–286, 2008.

[281] C. Anastasiou, S. Beerli, and A. Daleo, "Two-loop QCD Amplitude for $gg \to h$, H in the minimal supersymmetric standard model," *Physical Review Letters*, vol. 100, no. 24, Article ID 241806, 4 pages, 2008.

[282] M. Muhlleitner, H. Rzehak, and M. Spira, "MSSM Higgs boson production via gluon fusion: the large gluino mass limit," *Journal of High Energy Physics*, vol. 2009, no. 4, article 23, 2009.

[283] M. Muhlleitner and M. Spira, "Higgs boson production via gluon fusion: squark loops at NLO QCD," *Nuclear Physics B*, vol. 790, no. 1-2, pp. 1–27, 2008.

[284] R. Bonciani, G. Degrassi, and A. Vicini, "Scalar particle contribution to Higgs production via gluon fusion at NLO," *Journal of High Energy Physics*, vol. 2007, no. 11, article aritcle 95, 2007.

[285] L. Mihaila and C. Reisser, "$O(\alpha_s^2)$ corrections to fermionic Higgs decays in the MSSM," *Journal of High Energy Physics*, vol. 2010, no. 8, article 21, 2010.

[286] R. V. Harlander and K. J. Ozeren, "Finite top mass effects for hadronic Higgs production at next-to-next-to-leading order," *Journal of High Energy Physics*, vol. 2009, no. 11, article 88, 2009.

[287] A. Pak, M. Rogal, and M. Steinhauser, "Finite top quark mass effects in NNLO Higgs boson production at LHC," *Journal of High Energy Physics*, vol. 2010, no. 2, article 25, 2010.

[288] J. R. Ellis, M. K. Gaillard, and D. V. Nanopoulos, "A phenomenological profile of the Higgs boson," *Nuclear Physics B*, vol. 106, pp. 292–340, 1976.

[289] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, "Remarks on Higgs-boson interactions with nucleons," *Physics Letters B*, vol. 78, no. 4, pp. 443–446, 1978.

[290] M. A. Shifman, A. I. Vainshtein, M. B. Voloshin, and V. I. Zakharov, "Low-energy theorems for Higgs Boson couplings to photons," *Soviet Journal of Nuclear Physics*, vol. 30, p. 711, 1979, Translated from *Yadernaya Fizika*, vol. 30, p. 1368, 1979.

[291] A. I. Vainshtein, V. I. Zakharov, and M. A. Shifman, "Higgs particles," *Soviet Physics Uspekhi*, vol. 23, no. 8, article 429, 1980, Translated from *Uspekhi Fizicheskikh Nauk*, vol. 131, pp. 537–575, 1980.

[292] B. A. Kniehl and M. Spira, "Low-energy theorems in Higgs physics," *Zeitschrift für Physik C*, vol. 69, no. 1, pp. 77–87, 1995.

[293] M. Spira, A. Djouadi, D. Graudenz, and P. M. Zerwas, "Higgs boson production at the LHC," *Nuclear Physics B*, vol. 453, no. 1-2, pp. 17–82, 1995.

[294] W. Kilian, "Renormalized soft-higgs theorems," *Zeitschrift für Physik C*, vol. 69, no. 1, pp. 89–97, 1995.

[295] E. Braaten and J. P. Leveille, "Higgs-boson decay and the running mass," *Physical Review D*, vol. 22, no. 3, pp. 715–721, 1980.

[296] S. G. Gorishnii, A. L. Kataev, S. A. Larin, and L. R. Surguladze, "Three-loop QCD correction to the correlator of the Quark scalar currents and Γ_{tot} (H∘ → hadrons)," *Modern Physics Letters A*, vol. 5, no. 32, article 2703, 1990.

[297] K. G. Chetyrkin, B. A. Kniehl, and M. Steinhauser, "Strong coupling constant with flavor thresholds at four loops in the modified minimal-subtraction scheme," *Physical Review Letters*, vol. 79, no. 12, pp. 2184–2187, 1997.

[298] A. Kwiatkowski and M. Steinhauser, "Corrections of order $O(G_F\alpha_s m_t^2)$ to the Higgs decay rate $\Gamma(H \to b\bar{b})$," *Physics Letters B*, vol. 338, no. 1, pp. 66–70, 1994, Erratum in *Physics Letters B*, vol. 342, p. 455, 1995.

[299] K. G. Chetyrkin, J. H. Kuhn, A. Maier et al., "Charm and bottom quark masses: an update," *Physical Review D*, vol. 80, no. 7, Article ID 074010, 5 pages, 2009.

[300] M. Spira, A. Djouadi, D. Graudenz, and P. M. Zerwas, "Higgs boson production at the LHC," *Nuclear Physics B*, vol. 453, no. 1-2, pp. 17–82, 1995.

[301] M. Steinhauser, "Corrections of $O(\alpha_s^2)$ to the decay of an intermediate-mass Higgs boson into two photons," in *Proceedings of the Ringberg Workshop*, pp. 177–185, Tegernsee, Germany, 1996, http://arxiv.org/abs/hep-ph/9612395.

[302] U. Aglietti, R. Bonciani, G. Degrassi, and A. Vicini, "Two-loop light fermion contribution to Higgs production and decays," *Physics Letters B*, vol. 595, no. 1–4, pp. 432–441, 2004.

[303] F. Fugel, B. A. Kniehl, and M. Steinhauser, "Two-loop electroweak correction of $O(G_F M_t^2)$ to the Higgs-boson decay into photons," *Nuclear Physics B*, vol. 702, no. 1-2, pp. 333–345, 2004.

[304] G. Degrassi and F. Maltoni, "Two-loop electroweak corrections to the Higgs-boson decay $H \to \gamma\gamma$," *Nuclear Physics B*, vol. 724, no. 1-2, pp. 183–196, 2005.

[305] R. Harlander and P. Kant, "Higgs production and decay: analytic results at next-to-leading order QCD," *Journal of High Energy Physics*, vol. 2005, no. 12, article 15, 2005.

[306] A. Kurz, *Entkopplungseffekte zur Dreischleifenordnung in supersymmetrischer QCD [Diploma thesis]*, Karlsruhe Institute of Technology, 2010.

[307] J. Kleine, *Dimensionale Regularisierung und Dimensionale Reduktion in der supersymmetrischen QCD [Diploma thesis]*, Karlsruhe Institute of Technology, 2010.

[308] A. Djouadi, "The anatomy of electroweak symmetry breaking: tome I: the Higgs boson in the standard model," *Physics Reports*, vol. 457, no. 1–4, pp. 1–216, 2008.

[309] A. Djouadi, "The anatomy of electroweak symmetry breaking tome II: the Higgs bosons in the minimal supersymmetric model," *Physics Reports*, vol. 459, no. 1–6, pp. 1–241, 2008.

[310] R. Harlander, "Higgs production at the large Hadron collider: theoretical status," *Journal of Physics G*, vol. 35, no. 3, Article ID 033001, 2008.

[311] R. V. Harlander and K. J. Ozeren, "Top mass effects in Higgs production at next-to-next-to-leading order QCD: virtual corrections," *Physics Letters B*, vol. 679, no. 5, pp. 467–472, 2009.

[312] R. V. Harlander, H. Mantler, S. Marzani, and K. J. Ozeren, "Higgs production in gluon fusion at next-to-next-to-leading order QCD for finite top mass," *European Physical Journal C*, vol. 66, no. 3-4, pp. 359–372, 2010.

[313] A. Pak, M. Rogal, and M. Steinhauser, "Virtual three-loop corrections to Higgs boson production in gluon fusion for finite top quark mass," *Physics Letters B*, vol. 679, no. 5, pp. 473–477, 2009.

[314] A. Pak, M. Rogal, and M. Steinhauser, "Production of scalar and pseudo-scalar Higgs bosons to next-to-next-to-leading order at hadron colliders," *Journal of High Energy Physics*, vol. 2011, no. 9, article 88, 2011.

[315] M. Carena, S. Heinemeyer, C. E. M. Wagner, and G. Weiglein, "Suggestions for benchmark scenarios for MSSM Higgs boson searches at hadron colliders," *European Physical Journal C*, vol. 26, no. 4, pp. 601–607, 2003.