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Remark on linear spaces

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Let Λ be a field. A linear space over Λ is a commutative group V together with a multiplication $\cdot : \Lambda \times V \to V$ satisfying the following:

1) $\lambda \cdot (x+y) = \lambda \cdot x + \lambda \cdot y \quad (\lambda \in \Lambda;\ x, y \in V)$,

2) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x \quad (\lambda, \mu \in \Lambda;\ x \in V)$,

3) $(\lambda\mu) \cdot x = \lambda \cdot (\mu \cdot x) \quad (\lambda, \mu \in \Lambda;\ x \in V)$,

4) $1 \cdot x = x \quad (x \in V)$.

For proving that a subset W of a linear space V is a linear subspace, it is sufficient to show

$$\begin{aligned} &W \neq \emptyset, \\ &x, y \in W \Rightarrow x + y \in W, \\ &\lambda \in \Lambda, x \in W \Rightarrow \lambda \cdot x \in W. \end{aligned}$$

In particular, it is not necessary to check explicitely the group-structure of W. Now we shall see that, analogously, it is not necessary to use the group-notion explicitely in the definition of a linear space.

Theorem. *Let V be a set, Λ a field, and let $+ : V \times V \to V$, $\cdot : \Lambda \times V \to V$ be given. Then V is a linear space over Λ if and only if the following conditions hold:*

I) $V \neq \emptyset$,

II) $(x+y)+z = x+(y+z) \quad (x, y, z \in V)$,

III) $x + y = y + x \quad (x, y \in V)$,

1), 2), 3), 4) *from above, and*

5) $0 \cdot x = 0 \cdot y \quad (x, y \in V)$,

where 0 denotes the zero of Λ.

Proof. It is sufficient to show that I) - III) and 1) - 5) imply V to be a group: Because of I) and 5),

$$\theta = 0 \cdot x \quad (x \in V)$$

is a well defined element of V, and we only need to verify

$$x + \theta = x \quad (x \in V), \qquad x + (-1) \cdot x = \theta \qquad (x \in V).$$

Both formulas easily follow from 2), 4), and the definition of θ:

$$\begin{aligned} x + \theta &= 1 \cdot x + 0 \cdot x = (1+0) \cdot x = 1 \cdot x = x, \\ x + (-1) \cdot x &= 1 \cdot x + (-1) \cdot x = (1 + (-1)) \cdot x = 0 \cdot x = \theta. \end{aligned}$$

Example showing the above theorem to be false, when 5) is not required: Let $+ : V \times V \to V$ satisfy I), II), III), and

$$\text{IV)} \quad x + x = x \quad (x \in V)$$

(e.g., $V = 2^E = \{x \mid x \subseteq E\}$ being the power-set of a set E, where $+$ means the union of subsets of E, i.e., $x + y = x \cup y$ for $x, y \in V$). When defining $\cdot : \Lambda \times V \to V$ by

$$\lambda \cdot x = x \quad (\lambda \in \Lambda,\ x \in V),$$

then 1), 2), 3), 4) are fulfilled, but V is not a linear space, unless V is a singleton. (In a linear space V, condition IV) forces every element to be zero.)

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