Well-posedness for a General Class of Quasilinear Evolution Equations - With Applications to Maxwell's Equations

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“When I see a bird that walks like a duck and swims like a duck and quacks like a duck, I call that bird a duck.”

— James Whitcomb Riley (1849–1916)

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Chapter 1

Introduction

Macroscopic Maxwell's equations form the axiomatic foundation for all models describing electromagnetic phenomena. We specifically focus on applications in nonlinear optics, such as the propagation of light in fiber optic cables or photonic crystals. The corresponding nonlinear Maxwell equations still involve major mathematical challenges. These challenges start with the question of well-posedness, that is the existence and uniqueness of solutions, as well as their dependence on the input parameters, as they arise from experimental setups. Currently, these problems are addressed by a limited number of partial results from different subdisciplines of mathematics. The typical course of action therefore is to choose a specialized approach transforming these equations into quasilinear wave equations, which are then further simplified using physically motivated approximations. Applying this technique to the experiments mentioned above, yields problems of the Helmholz or nonlinear Schrödinger type.

Our objective is to undertake a first step towards a systematic investigation of the well-posedness for a wide class of nonlinear Maxwell equations. Our tool of choice is an approach outlined by Tosio Kato (1975) for the analysis of quasilinear evolution equations using operator semigroups. Inspired by this idea we will prove an abstract result, that can be applied to both, a wide class of nonlinear Maxwell equations involving problems from nonlinear optics, as well as to their corresponding quasilinear wave equations. By choosing this course of action, we succeed in presenting a unified theory uniting previously known results from different fields and improve upon them in some aspects, such as the required regularity assumptions for the input. Additionally, our proposed framework offers the possibility to further extend the class of systems under consideration. In the following, we discuss the fundamentals of electromagnetics in order to motivate the consideration of problems as mentioned above. Subsequently, we present our approach in detail.

1.1 The Fundamental Equations of Electromagnetism

The purpose of this section is to articulate the notion of the fundamental laws of electromagnetism, and to introduce the basic tools as well as the questions arising from this field. Our main references here are the classical treatises [18] and [16], where we mostly follow the presentation of the latter.

■ Macroscopic Maxwell's Equations

The differential version of Maxwell's equations in SI units are as follows:

$$\begin{aligned} \operatorname{rot} H &= \partial_t D + J && \text{(Ampère's Law)} \\ \operatorname{rot} E &= -\partial_t B && \text{(Faraday's Law)} \\ \operatorname{div} D &= \rho && \text{(Gauß' Law)} \\ \operatorname{div} B &= 0 && \text{(Gauß' Law of magnetic charge)} \end{aligned} \tag{Mw}$$

where

E = Electric field intensity vector,

B = Magnetic field flux density vector,

H = Magnetic field intensity vector,

D = Electric field flux density vector,

and the current and charge sources are described by

J = Electric current flux density vector,

ρ = Electric charge density.

By taking the divergence of the equation describing Ampère's Law and substituting in Gauß' Law, we obtain the conservation of charge, i.e.,

$$\operatorname{div} J + \partial_t \rho = 0.$$

Let $J = 0$ and $\rho = 0$. If D and B are solutions of (Mw), then we derive

$$\begin{aligned} \partial_t \operatorname{div} D(t) &= \operatorname{div} \operatorname{rot} H(t) = 0, \\ \partial_t \operatorname{div} B(t) &= -\operatorname{div} \operatorname{rot} E(t) = 0. \end{aligned} \tag{1.1}$$

Thus the Gaußian Laws are conserved quantities.

■ Constitutive Laws

In general, Maxwell's equations are insufficient for determining the electromagnetic field since there are six independent equations in twelve unknowns, namely the components of E, B, D and H. The first step to closing this gap is to introduce constitutive laws. For stationary media, these typically take the form

$$\begin{aligned} D &= \varepsilon_0 E + P, \\ H &= \mu_0^{-1} B - M \end{aligned} \tag{Cr}$$

and, if J is not fixed, but related to E by Ohm's law,

$$J = \sigma(E)E.$$

The new variables are

$$\begin{aligned}
\varepsilon_0 &= \text{permittivity of free space } (8.854 \times 10^{-12} \text{ farad/meter}),\\
\mu_0 &= \text{permeability of free space } (4\pi \times 10^{-7} \text{ henry/meter}),\\
\sigma &= \text{conductivity } (\sigma : \mathbb{R}^3 \to \mathbb{R}),\\
P &= \text{Polarisation},\\
M &= \text{Magnetisation}.
\end{aligned}$$

There are now several ways to characterize media by means of the polarisation and magnetisation vectors. Each physical experiment requires its own modelling which often is of interest in itself. The detailed study of such methods which are often founded on physical heuristics is beyond the scope of our interests. Instead we take a look at the most important models during the last decades which have influenced a broad range of mathematical fields, both analysis and numerics.

For example if one of the material laws (CR) is linear, e.g., $M = 0$, then one is often interested in solutions of the wave equation which arises by differentiating Ampère's law with respect to time and then substituting $\partial_t H = -\mu_0^{-1} \operatorname{rot} E$, i.e.,

$$\partial_{tt}(\varepsilon_0 E + P) = -\mu_0^{-1} \operatorname{rot}^2 E - \partial_t J = \mu_0^{-1} \Delta E - \partial_t J - \mu_0^{-1} \operatorname{grad} \operatorname{div} E. \tag{WE}$$

Now, we focus on the propagation of light through optic materials as for example fibres or photonic crystals. The core of such materials are insulators so that there are no free charges or currents, i.e., $\rho = 0$ and $J = 0$. These materials are further nonmagnetic which gives $M = 0$. A standard model used in the framework of nonlinear optics to describe the constitutive relation of E and P then is (cf. [5], Section 2.1)

$$\begin{aligned}
P =& \varepsilon_0 \int_{\mathbb{R}} \chi_1(t-s) E(x,s)\, \mathrm{d}s\\
&+ \varepsilon_0 \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_2(t-s_1, t-s_2) \cdot E(x,s_1) \otimes E(x,s_2)\, \mathrm{d}s_1 \mathrm{d}s_2\\
&+ \varepsilon_0 \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_3(t-s_1, t-s_2, t-s_3) \cdot\\
&\qquad\qquad E(x,s_1) \otimes E(x,s_2) \otimes E(x,s_3)\, \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3\\
&+ \varepsilon_0 \ldots,
\end{aligned} \tag{P}$$

where χ_j ($j \in \mathbb{N}$) is a tensor of order $j+1$, $\otimes$ is the usual tensor (or Kronecker) product and $\cdot$ denotes the contracted tensorial product. By further physically reasonable simplifications (cf. [6], Appendix A) one approximates (P) by the so called *Kerr nonlinearity*

$$P = \varepsilon_0 E(x,t) + \chi\, |E(x,t)|^2 E(x,t) \quad (\chi \in \mathbb{R}). \tag{KERR}$$

Different ansatzes for the electrical field E in the wave equation (WE) together with this nonlinear relation (KERR) usually lead either to a *Helmholtz equation* (cf. [35], Section 3.3) or a *nonlinear Schrödinger equation* (cf. [14], Section 1.4).

■ Boundary Conditions

In view of the propagation of light in an optical fibre (and of course in many other experimental situations), we surely have to deal with more than one material.

Therefore let Σ be a surface that separates two materials Ω_1 and Ω_2. Denote the unit normal vector on Σ which points from Ω_1 to Ω_2 by $\mathfrak{n}$. Let D_k, H_k, B_k and E_k be the fields considered in material $\Omega_k,\ k \in \{1,2\}$. With the help of Stokes's and Gauß' Theorem one can then deduce (cf. [8], Chapter I.4) the following *transmission conditions* on Σ:

$$\begin{aligned} (D_2 - D_1)\cdot \mathfrak{n} &= \rho_\Sigma,\\ (H_2 - H_1)\wedge \mathfrak{n} &= -J_\Sigma,\\ (B_2 - B_1)\cdot \mathfrak{n} &= 0,\\ (E_2 - E_1)\wedge \mathfrak{n} &= 0, \end{aligned} \tag{1.2}$$

where J_Σ and ρ_Σ denote the surface current density or the surface charge density respectively. We focus on the case in which one of the media (say Ω_2) is a *perfect conductor*. In such a medium the fields D_2, H_2, B_2 and E_2 vanish in Ω_2. Since the quantities J_Σ and ρ_Σ are in general unknown it is convenient to impose the reduced transmission conditions (cf. [8], Example I.2.4.3)

$$\begin{aligned} B\cdot \mathfrak{n} &= 0 \quad \text{on } \Sigma,\\ E\wedge \mathfrak{n} &= 0 \quad \text{on } \Sigma \end{aligned} \tag{Pc}$$

if we denote E_1 and B_1 simply by E and B. Suppose now that $J = 0$ and $\rho = 0$. If D and B are solutions of (Mw), then we derive

$$\partial_t(B(t)\cdot \mathfrak{n}) = -\operatorname{rot} E(t)\cdot \mathfrak{n} = \operatorname{div}(E(t)\wedge \mathfrak{n}) = 0. \tag{1.3}$$

Thus the boundary condition for B is a conserved quantity.

1.2 Strategy

In the following we are aiming for a unified theory of well-posedness, which covers a broad range of Maxwell's equations (in particular containing the Kerr nonlinearity) and the resulting wave equations (We). More precisely, we assume that there are no currents or charges, and that the material laws (Cr) are of the form

$$P = P(E), \quad M = M(H), \quad P,\ M \in C^{s+1}(\mathbb{R}^3, \mathbb{R}^3), \tag{Cr-L}$$

where $s \in \mathbb{N}_0$ will be specified later on. In the case of more than one material, we will impose either the perfect conduction boundary conditions (Pc) or, even more restrictive, full Dirichtlet boundary conditions on E, i.e.,

$$\begin{aligned} B\cdot \mathfrak{n} &= 0, \quad \text{on } \Sigma,\\ E &= 0, \quad \text{on } \Sigma. \end{aligned} \tag{Dr}$$

Addressing the second order problem (We), we further restrict our attention to the study of special divergence-free solutions, which arise by polarisation. We will work in mathematical units such that $\varepsilon_0 = \mu_0 = 1$. We therefore consider on the one hand the system

$$\begin{aligned}
\partial_t(E + P\circ E)(t,x) &= \operatorname{rot} H(t,x) && (t\in[0,T],\ x\in\Omega),\\
\partial_t(H + M\circ H)(t,x) &= -\operatorname{rot} E(t,x) && (t\in[0,T],\ x\in\Omega),\\
\operatorname{div}(P\circ E)(t,x) &= 0 && (t\in[0,T],\ x\in\Omega),\\
\operatorname{div}(H + H\circ M)(t,x) &= 0 && (t\in[0,T],\ x\in\Omega),\\
E(t,x)\wedge \mathfrak{n}(x) &= 0 && (t\in[0,T],\ x\in\partial\Omega),\\
M(H(t,x))\cdot \mathfrak{n}(x) &= 0 && (t\in[0,T],\ x\in\partial\Omega),\\
E(0,x) &= E_0(x) && (x\in\Omega),\\
H(0,x) &= H_0(x) && (x\in\Omega),
\end{aligned} \tag{1.4}$$

for some initial functions E_0, H_0 on $\Omega \subseteq \mathbb{R}^3$, and on the other hand the scalar problem

$$\begin{aligned}
\partial_{tt}u(t,x) + \partial_{tt}(K\circ u)(t,x) &= \Delta u(t,x) && (t\in[0,T],\ x\in\Omega),\\
u(t,x) &= 0 && (t\in[0,T],\ x\in\partial\Omega),\\
u(0,x) &= u_0(x) && (x\in\Omega),\\
\partial_t u(0,x) &= v_0(x) && (x\in\Omega),
\end{aligned} \tag{Cp-W}$$

where $K(u) = P((0,0,u)^\top\cdot(0,0,1)^\top)$ and a priori $\Omega\subseteq\mathbb{R}^2$. As we have seen earlier, by means of (1.1) and (1.3), the Gaußian laws as well as the nonlinear boundary condition in (1.4) are conserved quantities. Hence it is sufficient to impose these conditions for the initial values. Thus (1.4) reduces to

$$\begin{aligned}
\partial_t(E + P\circ E)(t,x) &= \operatorname{rot} H(t,x) && (t\in[0,T],\ x\in\Omega),\\
\partial_t(H + M\circ H)(t,x) &= -\operatorname{rot} E(t,x) && (t\in[0,T],\ x\in\Omega),\\
E(t,x)\wedge \mathfrak{n}(x) &= 0 && (t\in[0,T],\ x\in\partial\Omega),\\
E(0,x) &= E_0(x) && (x\in\Omega),\\
H(0,x) &= H_0(x) && (x\in\Omega),
\end{aligned} \tag{M-Pc}$$

in the case of a perfect conductor, or to

$$\begin{aligned}
\partial_t(E + P\circ E)(t,x) &= \operatorname{rot} H(t,x) && (t\in[0,T],\ x\in\Omega),\\
\partial_t(H + M\circ H)(t,x) &= -\operatorname{rot} E(t,x) && (t\in[0,T],\ x\in\Omega),\\
E(t,x) &= 0 && (t\in[0,T],\ x\in\partial\Omega),\\
E(0,x) &= E_0(x) && (x\in\Omega),\\
H(0,x) &= H_0(x) && (x\in\Omega),
\end{aligned} \tag{M-Dr}$$

for full Dirichtlet boundary conditions. If $\Omega = \mathbb{R}^3$, this reduces further to

$$\begin{aligned}
\partial_t(E + P\circ E)(t,x) &= \operatorname{rot} H(t,x) && (t\in[0,T],\ x\in\mathbb{R}^3),\\
\partial_t(H + M\circ H)(t,x) &= -\operatorname{rot} E(t,x) && (t\in[0,T],\ x\in\mathbb{R}^3),\\
E(0,x) &= E_0(x) && (x\in\mathbb{R}^3),\\
H(0,x) &= H_0(x) && (x\in\mathbb{R}^3).
\end{aligned} \tag{M-$\mathbb{R}^3$}$$

Thinking of the appearing nonlinearities as substitution operators in $L^2(\Omega)^3$, it is a priori not clear whether they will be Fréchet differentiable or not. Consequently we are not sure if we may apply the chain rule to expressions like

$$\partial_t(P \circ E)(t) = P'(E(t))\partial_t E(t).$$

Therefore we rewrite the above equations by differentiating their left hand sides. We thus consider the following Cauchy-problems

$$\begin{aligned} (I + P'(E(t,x)))\partial_t E(t,x) &= \operatorname{rot} H(t,x) && (t \in [0,T],\ x \in \Omega), \\ (I + M'(H(t,x)))\partial_t H(t,x) &= -\operatorname{rot} E(t,x) && (t \in [0,T],\ x \in \Omega), \\ E(t,x) \wedge \mathfrak{n}(x) &= 0 && (t \in [0,T],\ x \in \partial\Omega), \\ E(0,x) &= E_0(x) && (x \in \Omega), \\ H(0,x) &= H_0(x) && (x \in \Omega), \end{aligned} \tag{1.5}$$

in the case of a perfect conductor, or

$$\begin{aligned} (I + P'(E(t,x)))\partial_t E(t,x) &= \operatorname{rot} H(t,x) && (t \in [0,T],\ x \in \Omega), \\ (I + M'(H(t,x)))\partial_t H(t,x) &= -\operatorname{rot} E(t,x) && (t \in [0,T],\ x \in \Omega), \\ E(t,x) &= 0 && (t \in [0,T],\ x \in \partial\Omega), \\ E(0,x) &= E_0(x) && (x \in \Omega), \\ H(0,x) &= H_0(x) && (x \in \Omega), \end{aligned} \tag{1.6}$$

for full Dirichtlet boundary conditions. Finally, in the full space situation

$$\begin{aligned} (I + P'(E(t,x)))\partial_t E(t,x) &= \operatorname{rot} H(t,x) && (t \in [0,T],\ x \in \mathbb{R}^3), \\ (I + M'(H(t,x)))\partial_t H(t,x) &= -\operatorname{rot} E(t,x) && (t \in [0,T],\ x \in \mathbb{R}^3), \\ E(0,x) &= E_0(x) && (x \in \mathbb{R}^3), \\ H(0,x) &= H_0(x) && (x \in \mathbb{R}^3). \end{aligned} \tag{1.7}$$

The second order problem now reads

$$\begin{aligned} (1 + K'(u(t,x)))\partial_{tt} u(t,x) &+ [K''(u(t,x))\partial_t u(t,x)]\partial_t u(t,x) \\ &= \Delta u(t,x) && (t \in [0,T],\ x \in \Omega), \\ u(t,x) &= 0 && (t \in [0,T],\ x \in \partial\Omega), \\ u(0,x) &= u_0(x) && (x \in \Omega), \\ \partial_t u(0,x) &= v_0(x) && (x \in \Omega). \end{aligned} \tag{1.8}$$

■ State of the Art

By means of results for general symmetric hyperbolic systems, there are already several answers to the above posed Maxwell-type problems.

First, let $\Omega = \mathbb{R}^3$. If we consider nonlinearities P and M of the form (Cr-L), whose derivatives are not to negative (precisely if $P'(y) > -I$, $M'(y) > -I$ for all $y \in \mathbb{R}^3$), then we can apply Theorem II in [23] for initial values

$$u_0 := (E_0, H_0) \in H^{s+1}(\mathbb{R}^3)^6, \quad s > 3/2. \tag{1.9}$$

We then obtain unique local solutions of (1.7) belonging to $C([0,T],H^{s+1}(\mathbb{R}^3)^6)\cap C^1([0,T],H^s(\mathbb{R}^3)^6)$.

Now, let $\Omega\subseteq\mathbb{R}^3$ be a bounded domain whose boundary is of class C^∞. Suppose that P and M are of the form (CR-L) and satisfy $P'(0)>-I,\ M'(0)>-I$. Then we can apply Theorem 2.1 in [37] for initial values

$$u_0 := (E_0,H_0)\in H^{s+1}(\Omega)^6\cap H_0^1(\Omega)^6,\quad \mathbb{N}\ni s>19 \tag{1.10}$$

and obtain unique solutions $u\in C([0,T],H^{s+1}(\Omega)^6)$ of the Dirichlet Maxwell problem (1.6), with $\partial_t u(t)\in H^s(\Omega)^6$ $(0\leqslant t\leqslant T)$. But we also want to mention that for so called impedance boundary conditions (which do not fit into the framework 1.2), and the same class of nonlinearities, one gains solutions for nonlinear Maxwell's equations again for initial values of class H^3, see [32]. Somehow surprisingly, there do not exist positive answers for the problem (1.5) so far.

By means of Kato's approach to quasilinear hyperbolic evolution equations it was shown in [12] that if $K\in C^4(\mathbb{R})$ with $K'(0)>-1$, then for initial values

$$(u_0,v_0)\in\{u\in H^3(\Omega)\cap H_0^1(\Omega)\,:\,\Delta u\in H_0^1(\Omega)\}\times\big(H^2(\Omega)\cap H_0^1(\Omega)\big) \tag{1.11}$$

one obtains unique solutions

$$u\in C([0,T],H^3(\Omega))\cap C^1([0,T],H^2(\Omega))\cap C^2([0,T],H^1(\Omega))$$

for the wave-type Cauchy problem (CP-W).

In the following we will provide an abstract theorem (cf. Theorem 3.43) treating a general class of nonlinear evolution equations in a Hilbert space, which offers a unified approach for the above results. With the aid of this theorem we will reproduce the results (1.9) and (1.11). On the other hand we can significantly improve the regularity assumptions of (1.10) and further give examples of nonlinearities that leads to well-posedness also for the Maxwell problem with perfect conducting boundary conditions (1.5).

■ Our Approach

Rewriting the dynamical part of the Maxwell-type problems (1.5)-(1.7) as

$$\begin{pmatrix} I+P'(E) & 0\\ 0 & I+M'(H)\end{pmatrix}\partial_t\begin{pmatrix}E\\H\end{pmatrix}=\begin{pmatrix}0 & \operatorname{rot}\\ -\operatorname{rot} & 0\end{pmatrix}\begin{pmatrix}E\\H\end{pmatrix}$$

and introducing the coordinates $u:=(E,H)^\top$ as well as the quantities

$$\Lambda(u):=\begin{pmatrix} I+P'(E) & 0\\ 0 & I+M'(H)\end{pmatrix},\quad A:=\begin{pmatrix}0 & \operatorname{rot}\\ -\operatorname{rot} & 0\end{pmatrix},$$

we will interpret these equations as a nonlinear evolution equation taking the form

$$\Lambda(u(t))u'(t)=Au(t)\quad (t\in[0,T]),$$

where the unknown $u(t)$ is supposed to belong to some Hilbert space. Similarly, introducing the new variables

$$v := \partial_t u, \quad w := (u, v)^\top$$

and putting

$$\Lambda(w) := \begin{pmatrix} I & 0 \\ 0 & 1 + K'(u) \end{pmatrix}, \quad A := \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix},$$
$$Q(w) := \begin{pmatrix} 0 & 0 \\ 0 & -K''(u)v \end{pmatrix},$$

we will consider (Cp-W) as an evolution equation of the form

$$\Lambda(w(t))w'(t) = Aw(t) + Q(w(t)) \quad (t \in [0, T]),$$

where again the unknown $w(t)$ takes values in some Hilbert space. We want to approach these types of abstract nonlinear evolution equations by means of Banach's fixed point theorem, invoking the analysis of nonautonomous linear evolution evolution equations. The main task therefore will be to find adequate assumptions on the parameters for the abstract equations (such as the class of spaces under consideration, estimates on the nonlinearities, requirements on the linear part), which on the one hand lead to positive well-posedness theorems and on the other hand allow us to apply these results to our concrete problems in an appropriate setting.

For tackling this handy interplay we thus start in Chapter 2 and 3 to introduce the basic results on the required L^2-type function spaces on the one hand, and the theory for the study of nonautonomous Cauchy problems on the other hand. In Chapter 4, the resulting Theorems 3.43 and 3.45 (which are of interested to their own) will then be applied to our desired Cauchy-Problems from electromagnetics.

Each of the following chapters starts with a brief overview of what material is covered therein, as well as of an allocation of the most important notation.

Chapter **2**

Function Spaces and Differential Operators Related to Maxwell's Equations

We begin with the basic objects in our analysis, namely hyperbolic differential operators in L^2 and the associated classes of functions and vector fields.

We assume that the reader is familiar with the basic concepts and most important results concerning weakly differentiable functions in L^p. If needed, however, we refer to appendix A as well as to the references therein.

In Section 2.1, we introduce the spaces $H(\operatorname{div}, \Omega)$ and $H(\operatorname{rot}, \Omega)$ for a domain Ω in $\mathbb{R}^d$ and state the main results concerning, on the one hand, approximation by smooth functions, and on the other hand, trace theorems for tangential and normal traces. We provide these results in the L^p-framework, since there is no additional effort necessary. Thereafter, in Section 2.2, we study the regularity properties of vector fields contained in the intersection of the above spaces, whose tangential or normal component either vanish on the boundary, or satisfy additional smoothness assumptions. Finally, in the last section we put all these insights together and introduce Theorem 2.43, which supplies the main properties for our well-posedness results for Maxwell's equations in Chapter 4.

Notation. For two normed spaces X, Y we denote the space of bounded linear operators from X to Y by $\mathcal{B}(X, Y)$.

We use $a \lesssim b$ and $a \gtrsim b$ to denote the estimate $a \leqslant c\, b$ or $a \geqslant c\, b$ for some quantity c, which we call the implied constant. We will further write $a \sim b$ if both $a \lesssim b$ and $a \gtrsim b$ hold. If we need the implied constant to depend on parameters (e.g. p, d) we will indicate this by using subscripts, i.e., $a \lesssim_{p,d} b$ and so on. Sometimes we will also write $a \leqslant c_{p,d} b$ in such situations.

Given an exponent $1 \leqslant p \leqslant \infty$ we will denote its dual by p', i.e., $p' = p/(p-1)$ if $p < \infty$ and $p' = 1$ else.

Given a bounded subset Ω' of $\Omega \subseteq \mathbb{R}^d$ with $\overline{\Omega'} \subseteq \Omega$, we will write $\Omega' \subseteq\subseteq \Omega$. ⋊

2.1 Definition and Elementary Properties

For an open and non-empty set $\Omega \subseteq \mathbb{R}^d$, let $\mathcal{D}(\Omega)$ denote the locally convex space of test functions. If nothing else is said, then Ω shall always be a non-empty domain (open set) in some $\mathbb{R}^d$.

The Classical Operators 2.1 The operators **grad**, **div** and **rot** defined by

$$\begin{aligned}
\mathbf{grad}: \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)^d, &\quad \mathbf{grad}\,\varphi = (\partial_1\varphi, \dots, \partial_d\varphi)^\top, \\
\mathbf{div}: \mathcal{D}(\Omega)^d \to \mathcal{D}(\Omega), &\quad \mathbf{div}\,(\varphi_1, \dots, \varphi_d) = \sum_{k=1}^{d} \partial_k\varphi_k, \\
\mathbf{rot}: \mathcal{D}(\Omega)^3 \to \mathcal{D}(\Omega)^3, &\quad \mathbf{rot}\,(\varphi_1, \varphi_2, \varphi_3) = \begin{pmatrix} \partial_2\varphi_3 - \partial_3\varphi_2 \\ \partial_3\varphi_1 - \partial_1\varphi_3 \\ \partial_1\varphi_2 - \partial_2\varphi_1 \end{pmatrix},
\end{aligned}$$

are continuous with respect to the corresponding locally convex topologies on $\mathcal{D}(\Omega)^k$, where $k \in \{1, 3, d\}$. ⋊

Therefore the related adjoint operators are well defined and weakly continuous on the corresponding spaces of distributions $\mathcal{D}'(\Omega)^k$ $(k \in \{1, 3, d\})$.

Definition 2.2 We set

$$\begin{aligned}
\operatorname{grad}: \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)^d, &\quad \operatorname{grad} := -\mathbf{div}', \\
\operatorname{div}: \mathcal{D}'(\Omega)^d \to \mathcal{D}'(\Omega), &\quad \operatorname{div} := -\mathbf{grad}', \\
\operatorname{rot}: \mathcal{D}'(\Omega)^3 \to \mathcal{D}'(\Omega)^3, &\quad \operatorname{rot} := \mathbf{rot}',
\end{aligned}$$

which yields

$$\begin{aligned}
&\operatorname{grad} T = (\partial_1 T, \dots, \partial_d T), \quad T \in \mathcal{D}'(\Omega), \\
&\operatorname{div}(T_1, \dots, T_d) = \partial_1 T_1 + \dots + \partial_d T_d, \\
&\operatorname{rot}(T_1, T_2, T_3) = (\partial_2 T_3 - \partial_3 T_2,\ \partial_3 T_1 - \partial_1 T_3,\ \partial_1 T_2 - \partial_2 T_1), \quad T_i \in \mathcal{D}'(\Omega).
\end{aligned}$$ ⋊

Remark 2.3 We denote the canonical basis of $\mathbb{R}^d$ by $\{e_k : 1 \leqslant k \leqslant d\}$ and introduce the three dimensional skew symmetric matrices

$$J_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note, that

$$x \wedge y = \sum_{k=1}^{3} x_k J_k\, y \quad \text{for all } x, y \in \mathbb{R}^3.$$

Then we can rewrite the operators from the above definition as

$$\begin{aligned}
\operatorname{grad} T &= \sum_{k=1}^{d} e_k \, \partial_k T = \sum_{k=1}^{d} \partial_k (e_k T)\,, \\
\operatorname{div} T &= \sum_{k=1}^{d} e_k^\top \, \partial_k T = \sum_{k=1}^{d} \partial_k (e_k^\top T)\,, \quad T = (T_1, \dots, T_d)\,, \\
\operatorname{rot} T &= \sum_{k=1}^{3} J_k \, \partial_k T = \sum_{k=1}^{3} \partial_k (J_k T)\,, \quad T = (T_1, T_2, T_3)\,.
\end{aligned}$$

Therefore these operators are all first order partial differential operators in so called divergence form

$$\mathcal{L} : \mathcal{D}'(\Omega)^n \to \mathcal{D}'(\Omega)^m\,,\ T \mapsto \sum_{k=1}^{d} \partial_k (A_k T)\,,$$

where $m, n \in \mathbb{N}$ and $A_k \in \mathbb{R}^{m \times n}$. ⋊

In the following, we will not use the bold letters for the classical operators, since it is obvious from context which operator is supposed to be utilized.

For $d \in \mathbb{N}$ and $v = (v_1, \dots, v_d) \in L^1_{\text{loc}}(\Omega)^d$ we put

$$T_v \coloneqq (T_{v_1}, \dots, T_{v_d})$$

and define $T_{v_k}(\varphi) = \int_\Omega v_k \, \varphi \, \mathrm{d}x$, where $\varphi \in \mathcal{D}(\Omega)$. Hence T_v belongs to $\mathcal{D}'(\Omega)^d$. Distributions of this type are called *regular*. Given any distribution T in $\mathcal{D}'(\Omega)^d$, we say that T belongs to $L^p(\Omega)^d$ if there is a vector field $v \in L^p(\Omega)^d$ such that $T = T_v$. From the fundamental lemma of the calculus of variations it follows that such a function v is uniquely determined. Usually we will abbreviate T_v by v.

Definition 2.4 Let $1 \leqslant p < \infty$. We then set

(a) $W^p(\operatorname{grad}, \Omega) \coloneqq \left\{ v \in L^p(\Omega) \,:\, \operatorname{grad} T_v \in L^p(\Omega)^d \right\}$, and

$$\|v\|_{\text{grad}} \coloneqq \left(\|v\|_p^p + \|\operatorname{grad} v\|_p^p \right)^{1/p}\,, \quad T_{\operatorname{grad} v} = \operatorname{grad} T_v\,.$$

(b) $W^p(\operatorname{div}, \Omega) \coloneqq \left\{ v \in L^p(\Omega)^d \,:\, \operatorname{div} T_v \in L^p(\Omega) \right\}$, and

$$\|v\|_{\text{div}} \coloneqq \left(\|v\|_p^p + \|\operatorname{div} v\|_p^p \right)^{1/p}\,, \quad T_{\operatorname{div} v} = \operatorname{div} T_v\,.$$

(c) $W^p(\operatorname{rot}, \Omega) \coloneqq \{ v \in L^p(\Omega)^3 \,:\, \operatorname{rot} T_v \in L^p(\Omega)^3 \}$, and

$$\|v\|_{\text{rot}} \coloneqq \left(\|v\|_p^p + \|\operatorname{rot} v\|_p^p \right)^{1/p}\,, \quad T_{\operatorname{rot} v} = \operatorname{rot} T_v\,.$$

If $p = 2$, then we write $H(\operatorname{div}, \Omega) \coloneqq W^2(\operatorname{div}, \Omega)$ and $H(\operatorname{rot}, \Omega) \coloneqq W^2(\operatorname{div}, \Omega)$. ⋊

Remark 2.5 Unless something else is said, we equip $L^p(\Omega)^n$ $(n \in \mathbb{N})$ with the norm

$$\|v\|_p := \left(\int_\Omega |v(x)|_p^p \, dx\right)^{1/p} \qquad (v \in L^p(\Omega)^n),$$

where $|a|_p^p = \sum_{i=1}^n |a_i|^p$ for $a = (a_1, \dots, a_n)^\top \in \mathbb{C}^n$. ⋊

The above classes of spaces are very special types of so called *graph spaces of first order partial differential operators in divergence form*. These are the maximal domains of operators of the form

$$\mathcal{L} : L^p(\Omega)^m \subseteq \mathcal{D}'(\Omega)^n \to \mathcal{D}'(\Omega)^m \,,\; T \mapsto \sum_{k=1}^d \partial_k(A_k T) + BT \,,$$

where $m, n \in \mathbb{N}$, $A_k \in W^{1,\infty}(\Omega)^{m\times n}$ and $B \in L^\infty(\Omega)^{m\times n}$. We put

$$W^p(\mathcal{L}, \Omega) := \{v \in L^p(\Omega)^m \,:\, \mathcal{L}T_v \in L^p(\Omega)^n\} \,,$$

and endow this space with the norm

$$\|v\|_{\mathcal{L}} := \left(\|v\|_p^p + \|\mathcal{L}v\|_p^p\right)^{1/p}, \quad T_{\mathcal{L}v} = \mathcal{L}T_v \,.$$

There already is vast and continuously growing literature[1] concerning these spaces.

Remark 2.6 As an immediate consequence we obtain the following characterization of the spaces $W^p(\mathrm{grad}, \Omega)$, $W^p(\mathrm{div}, \Omega)$ and $W^p(\mathrm{rot}, \Omega)$, which is frequently used in the former literature as a definition.

(a) A function $u \in L^p(\Omega)$ belongs to $W^p(\mathrm{grad}, \Omega)$ if and only if there is some $w \in L^p(\Omega)^d$ such that

$$\int_\Omega u \,\mathrm{div}\, \varphi \, dx = -\int_\Omega w \cdot \varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega)^d. \tag{2.1}$$

In this case we have $w = \mathrm{grad}\, u$.

(b) A vector field $v \in L^p(\Omega)^d$ belongs to $W^p(\mathrm{div}, \Omega)$ if and only if there is some $f \in L^p(\Omega)$ such that

$$\int_\Omega v \cdot \mathrm{grad}\, \varphi \, dx = -\int_\Omega f \varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{2.2}$$

In this case we have $f = \mathrm{div}\, v$.

(c) A vector field $v \in L^p(\Omega)^3$ belongs to $W^p(\mathrm{rot}, \Omega)$ if and only if there is some $w \in L^p(\Omega)^3$ such that

$$\int_\Omega v \cdot \mathrm{rot}\, \varphi \, dx = \int_\Omega w \cdot \varphi \, dx \quad \text{for all } \varphi \in \mathcal{D}(\Omega)^3. \tag{2.3}$$

In this case we have $w = \mathrm{rot}\, v$. ⋊

[1] A detailed introduction is given in [19].

The following lemmas are direct consequences of the respective definitions.

LEMMA 2.7 *We have $W^p(\mathrm{grad},\Omega) = W^{1,p}(\Omega)$ with identical norms.* $\rtimes$

LEMMA 2.8 *The operators* $\mathrm{div} : W^p(\mathrm{div},\Omega) \subseteq L^p(\Omega)^d \to L^p(\Omega)$, $v \mapsto \mathrm{div}\, v$ *and* $\mathrm{rot} : W^p(\mathrm{rot},\Omega) \subseteq L^p(\Omega)^3 \to L^p(\Omega)^3$, $v \mapsto \mathrm{rot}\, v$ *are closed.* $\rtimes$

By definition, the spaces $W^p(\mathrm{div},\Omega)$ and $W^p(\mathrm{rot},\Omega)$ are nothing more than the domains of the closed operators div and rot endowed with the respective graph norms. Through this we obtain the following lemma.

LEMMA 2.9 *The sets $W^p(\mathrm{div},\Omega)$ and $W^p(\mathrm{rot},\Omega)$ are Banach spaces. In particular $H(\mathrm{div},\Omega)$ and $H(\mathrm{rot},\Omega)$ are Hilbert spaces when endowed with the inner products*

$$(u \mid v)_{\mathrm{div}} := (u \mid v)_2 + (\mathrm{div}\, u \mid \mathrm{div}\, v)_2 \qquad (u,\, v \in H(\mathrm{div},\Omega)),$$
$$(u \mid v)_{\mathrm{rot}} := (u \mid v)_2 + (\mathrm{rot}\, u \mid \mathrm{rot}\, v)_2 \qquad (u,\, v \in H(\mathrm{rot},\Omega)). \quad \rtimes$$

LEMMA 2.10 *If $u \in W^{1,p}(\Omega)^d$, then $u \in W^p(\mathrm{div},\Omega)$ and $u \in W^p(\mathrm{rot},\Omega)$, with*

$$\mathrm{div}\, u = \sum_{k=1}^{d} \partial_k u_k\,, \quad \mathrm{rot}\, u = \sum_{k=1}^{3} J_k \partial_k u\,,$$

where ∂_k denotes the respective weak partial derivative. Moreover, there are constants such that

$$\|\mathrm{div}\, u\|_p \leqslant c_{d,p}\, \|\mathrm{grad}\, u\|_p, \quad \|\mathrm{rot}\, u\|_p \leqslant c_p\, \|\mathrm{grad}\, u\|_p,$$

and therefore $W^{1,p}(\Omega)^d \hookrightarrow W^p(\mathrm{div},\Omega)$, as well as $W^{1,p}(\Omega)^3 \hookrightarrow W^p(\mathrm{rot},\Omega)$. $\rtimes$

■ Mollification and Approximation

Using the techniques introduced by Meyers and Serrin in 1964 (see [1], page 67) we derive a basic density result.

THEOREM 2.11 *Let $\Omega \subseteq \mathbb{R}^d$ be open and non-empty and let $1 \leqslant p < \infty$. Then $C^\infty(\Omega)^d \cap W^p(\mathrm{div},\Omega)$ is dense in $W^p(\mathrm{div},\Omega)$ and $C^\infty(\Omega)^3 \cap W^p(\mathrm{rot},\Omega)$ is dense in $W^p(\mathrm{rot},\Omega)$.* $\rtimes$

Proof. A detailed proof, even concerning the more general spaces $W^p(\mathcal{L},\Omega)$, can be found in Theorem 1.2 of [19]. ∎

DEFINITION 2.12 For $\emptyset \neq \Omega \subseteq \mathbb{R}^d$ we put

$$\mathcal{D}(\overline{\Omega})^d := \left\{ \varphi|_\Omega \, : \, \varphi \in \mathcal{D}(\mathbb{R}^d)^d \right\}. \quad \rtimes$$

Obviously $\mathcal{D}(\Omega)^d \subseteq \mathcal{D}(\overline{\Omega})^d \subseteq W^{1,p}(\Omega)^d$ for all $1 \leqslant p < \infty$. In particular $\mathcal{D}(\overline{\Omega})^d$ is contained in $W^p(\mathrm{div},\Omega)$, and $\mathcal{D}(\overline{\Omega})^3$ belongs to $W^p(\mathrm{rot},\Omega)$.

Definition 2.13 For $1 \leqslant p < \infty$ we define

$$\begin{aligned} W_0^p(\operatorname{div}, \Omega) &\coloneqq \text{closure of } \mathcal{D}(\Omega)^d \text{ in } W^p(\operatorname{div}, \Omega)\,, \\ W_0^p(\operatorname{rot}, \Omega) &\coloneqq \text{closure of } \mathcal{D}(\Omega)^3 \text{ in } W^p(\operatorname{rot}, \Omega)\,. \end{aligned}$$

Equipped with the norms $\|\cdot\|_{\operatorname{div}}, \|\cdot\|_{\operatorname{rot}}$ these are also Banach spaces. If p equals two, we denote the resulting Hilbert spaces by $H_0(\operatorname{div}, \Omega)$ and $H_0(\operatorname{rot}, \Omega)$. ⋊

The approximating functions in Theorem 2.11 do not need to be regular up to the boundary of the domain. To obtain stronger density results, one needs a regularity property of the boundary.

Definition 2.14 We say that a non-empty domain $\Omega \subseteq \mathbb{R}^d$ satisfies the *segment condition* if every $y \in \partial\Omega$ has a neighbourhood U_y and a non-zero vector v_y such that if $x \in \overline{\Omega} \cap U_y$, then $x + t v_y \in \Omega$ for all $0 < t < 1$. ⋊

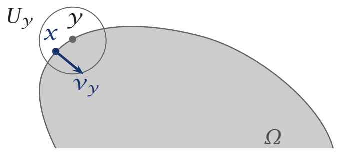

Figure 1: The segment property.

The boundary of a domain satisfying the segment condition must be $d-1$-dimensional, and the domain cannot lie on both sides of any part of its boundary. We can actually characterize such sets as follows (see Theorem 10.24 in [25]).

Lemma 2.15 *A domain $\Omega \subseteq \mathbb{R}^d$ satisfies the segment property if and only if $\partial\Omega$ is continuous, cf. Remark A.10.* ⋊

Before we state stronger approximation results for $W^p(\operatorname{div}, \Omega)$ and $W^p(\operatorname{rot}, \Omega)$, we want to put a few considerations first to get a better understanding of these insights. We thus recall the following approximation results for Sobolev spaces. For u belonging to $L^p(\Omega)$, we denote by $\tilde{u}$ the zero extension in $L^p(\mathbb{R}^d)$.

Proposition 2.16 *Let the domain $\Omega \subseteq \mathbb{R}^d$ satisfy the segment property and let $m \in \mathbb{N}$ and $1 \leqslant p < \infty$. Then the following assertions hold.*

(a) *If $\tilde{u} \in W^{m,p}(\mathbb{R}^d)$, then $u \in W_0^{m,p}(\Omega)$.*

(b) *$\mathcal{D}(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$.* ⋊

Proof. An elementary proof of (a) can be found in [1] Theorem 5.29, and for (b) we refer to [1] Theorem 3.22. ∎

We will see soon that the same approximation assertions will also hold for the spaces $W^p(\mathrm{div}, \Omega)$ and $W^p(\mathrm{rot}, \Omega)$, or even more generally for the spaces $W^p(\mathcal{L}, \Omega)$. But first we want to point out why the segment property is a kind of canonical assumption on the geometry of the boundary for such approximation results.

The crucial point is that mollification by convolution with a smooth function requires an epsilon of space (see the picture below). More precisely: Let J_ρ denote a mollifier for arbitrary $\rho > 0$ and choose for example $u \in W^{m,p}(\Omega)$ for some $m \in \mathbb{N}$ and $1 \leqslant p < \infty$. If $\Omega' \Subset \Omega$, then

$$(J_\rho \star \partial^\alpha v)(x) = \partial^\alpha (J_\rho \star v)(x) \quad (x \in \Omega',\ |\alpha| \leqslant m)$$

for all $\rho < \mathrm{dist}(\Omega', \partial\Omega)$.

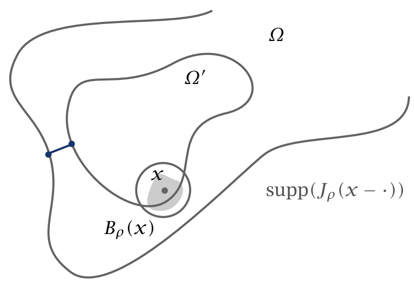

Figure 2: Mollification requires space.

As we may expect by Figure 1, the definition of the segment condition allows us to gain such a desired epsilon of space to the boundary.

Demanding the segment condition, one can prove an analog result of Proposition 2.16 for the spaces $W^p(\mathcal{L}, \Omega)$, thus particularly for the spaces $W^p(\mathrm{div}, \Omega)$ and $W^p(\mathrm{rot}, \Omega)$. In contrast to the proof for the classical Sobolev spaces, there arise further technical problems from the loss of strong convergence in L^p.

Proposition 2.17 *Let the domain $\Omega \subseteq \mathbb{R}^d$ satisfy the segment property and let $1 \leqslant p < \infty$. Then the following assertions hold.*

(a) *If $\tilde{u} \in W^p(\mathrm{div}, \mathbb{R}^d)$ and $\tilde{v} \in W^p(\mathrm{rot}, \mathbb{R}^3)$, then $u \in W_0^p(\mathrm{div}, \Omega)$ and $v \in W_0^p(\mathrm{rot}, \Omega)$.*

(b) *$\mathcal{D}(\overline{\Omega})^d$ is dense in $W^p(\mathrm{div}, \Omega)$ and $\mathcal{D}(\overline{\Omega})^3$ is dense in $W^p(\mathrm{rot}, \Omega)$.* ⋊

Proof. For a proof we refer to Theorem 4 and 5 in [4]. ∎

Remark 2.18 As an immediate consequence we have $W^p(\mathrm{div}, \mathbb{R}^d) = W_0^p(\mathrm{div}, \mathbb{R}^d)$ as well as $W^p(\mathrm{rot}, \mathbb{R}^3) = W_0^p(\mathrm{rot}, \mathbb{R}^3)$, i.e., the test functions are dense in $W^p(\mathrm{div}, \mathbb{R}^d)$ and $W^p(\mathrm{rot}, \mathbb{R}^3)$. ⋊

■ Trace Theorems

First, we recall the trace theorems concerning the Sobolev spaces $W^{1,p}(\Omega)$ on the reflexive scale $p > 1$. Note, that $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p}(\Omega)$ if Ω satisfies the segment property.

Theorem 2.19 *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain with compact boundary and let $1 < p < \infty$. Then the following assertions hold.*

(a) *Setting $\operatorname{tr}_p v \coloneqq v|_{\partial\Omega}$ for $v \in \mathcal{D}(\overline{\Omega})$, we obtain a mapping*

$$\operatorname{tr}_p : \mathcal{D}(\overline{\Omega}) \subseteq W^{1,p}(\Omega) \to W^{1-1/p,p}(\partial\Omega) = W^{1/p',p}(\partial\Omega)$$

which satisfies

$$\|\operatorname{tr}_p v\|_{W^{1/p',p}(\partial\Omega)} \leqslant c_{d,p} \|v\|_{W^{1,p}(\Omega)} \,.$$

We will denote the unique continuous extension to $W^{1,p}(\Omega)$ by tr_p, too. We have

$$\ker(\operatorname{tr}_p) = W_0^{1,p}(\Omega) \,.$$

(b) *There is a linear and continuous operator*

$$\operatorname{ex}_p : W^{1-1/p,p}(\partial\Omega) \to W^{1,p}(\Omega)$$

which satisfies $\operatorname{tr}_p \circ \operatorname{ex}_p = \operatorname{Id}_{W^{1-1/p,p}(\partial\Omega)}$. ⋊

Proof. For an elementary approach to these results we refer to Section 16 and 17 of Chapter IX in [11], or alternatively to Chapter 3 in [10]. ∎

Remark 2.20 (a) Given $v \in W^{1,p}(\Omega)$ we will usually write $\operatorname{tr}_p v \eqqcolon v|_{\partial\Omega}$. In the case of $p = 2$, we will abbreviate tr_2 by tr as well as ex_2 by ex.

(b) In particular $\operatorname{tr}_p(W^{1,p}(\Omega)) = W^{1-1/p,p}(\partial\Omega)$, and therefore the quotient norm

$$\|\varphi\|_{\operatorname{tr}_p} \coloneqq \inf\left\{\|v\|_{W^{1,p}(\Omega)} : \varphi = v|_{\partial\Omega},\, v \in W^{1,p}(\Omega)\right\} = \|\widehat{\operatorname{tr}}_p^{-1}\varphi\|_{W^{1,p}(\Omega)/\ker(\operatorname{tr}_p)}$$

on $W^{1-1/p,p}(\partial\Omega)$ is equivalent to its intrinsic[2] norm. Here $\widehat{\operatorname{tr}}_p$ denotes the canonical algebraic isomorphism between $W^{1,p}(\Omega)/\ker(\operatorname{tr}_p)$ and $\operatorname{tr}_p(W^{1,p}(\Omega))$.

(c) Sometimes it is convenient to denote the duality brackets $\langle\cdot,\cdot\rangle_{\partial\Omega}$ of $W^{1/p,p'}(\partial\Omega)$ and $W^{1/p,p'}(\partial\Omega)'$ by $\int_{\partial\Omega} \mathrm{d}\sigma$, i.e.,

$$\int_{\partial\Omega} v\, T \,\mathrm{d}\sigma \coloneqq \langle v, T\rangle_{\partial\Omega} = T(v) \quad (v \in W^{1/p,p'}(\partial\Omega),\ T \in W^{1/p,p'}(\partial\Omega)').$$

We further put $W^{-1/p,p}(\partial\Omega) \coloneqq W^{1/p,p'}(\partial\Omega)'$. ⋊

[2] Cf. Definition A.19

We point out that in the Hilbert space situation Theorem 2.19 states

$$\operatorname{tr} : H^1(\Omega) \to H^{1/2}(\partial\Omega)\,, \quad \operatorname{ex} : H^{1/2}(\partial\Omega) \to H^1(\Omega)\,.$$

In particular $H^{1/2}(\partial\Omega) = \operatorname{tr}\left(H^1(\Omega)\right)$, so that for $\varphi \in H^{1/2}(\partial\Omega)$ we have

$$\|\varphi\|_{H^{1/2}(\partial\Omega)} \sim \inf\left\{\|v\|_{H^1(\Omega)} \,:\, \varphi = v\,|_{\partial\Omega},\, v \in H^1(\Omega)\right\}\,. \tag{2.4}$$

By means of these classical trace theorems we are able to derive the important trace theorems for $W^p(\operatorname{div},\Omega)$ and $W^p(\operatorname{rot},\Omega)$. Since it is not easy to find (detailed) proofs of these results in the literature, we decided to provide them here.

Trace Theorem for $W^p(\operatorname{div},\Omega)$ 2.21 *Let $1 < p < \infty$. Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain with compact boundary and denote the unit outward normal by $\mathfrak{n} \in L^\infty(\partial\Omega)^d$. Then the following assertions hold.*

(a) *Setting $\gamma_{\mathfrak{n}}(v) \coloneqq T_{v|_{\partial\Omega}\cdot\mathfrak{n}}$ for $v \in \mathcal{D}(\overline{\Omega})^d$, we obtain a mapping*

$$\gamma_{\mathfrak{n}} : \mathcal{D}(\overline{\Omega})^d \subseteq W^p(\operatorname{div},\Omega) \to W^{1/p,p'}(\partial\Omega)'$$

which satisfies

$$|\gamma_{\mathfrak{n}}(v)(\mu)| \leqslant c_{d,p}\,\|v\|_{\operatorname{div}}\,\|\mu\|_{W^{1/p,p'}(\partial\Omega)} \quad (\mu \in W^{1/p,p'}(\partial\Omega)).$$

We will denote the unique continuous extension to $W^p(\operatorname{div},\Omega)$ by $\gamma_{\mathfrak{n}}$, too.

(b) *If $v \in W^p(\operatorname{div},\Omega)$ and $\varphi \in W^{1,p'}(\Omega)$, then the following Green's formula holds*

$$\int_\Omega v\cdot\operatorname{grad}\varphi\,\mathrm{d}x = -\int_\Omega(\operatorname{div}v)\varphi\,\mathrm{d}x + \left\langle \operatorname{tr}_{p'}\varphi, \gamma_{\mathfrak{n}}(v)\right\rangle_{\partial\Omega}. \tag{2.5}$$

(c) *The operator $\gamma_{\mathfrak{n}}$ in $\mathcal{B}(W^p(\operatorname{div},\Omega), W^{-1/p,p'}(\partial\Omega))$ satisfies*

$$\ker(\gamma_{\mathfrak{n}}) = W^p_0(\operatorname{div},\Omega).$$

(d) *The mapping $\gamma_{\mathfrak{n}} : H(\operatorname{div},\Omega) \to H^{-1/2}(\partial\Omega)$ is onto.* ⋊

Proof. (a) : Choose $v \in \mathcal{D}(\overline{\Omega})^d$ and let $\varphi \in \mathcal{D}(\overline{\Omega})$. Applying Gauß' Theorem A.18, we obtain

$$\int_\Omega v\cdot\operatorname{grad}\varphi\,\mathrm{d}x + \int_\Omega(\operatorname{div}v)\varphi\,\mathrm{d}x = \int_\Omega \operatorname{div}(\varphi\, v)\,\mathrm{d}x = \int_{\partial\Omega}\varphi\, v\,|_{\partial\Omega}\cdot\mathfrak{n}\,\mathrm{d}\sigma\,.$$

Now, let $\varphi \in W^{1,p'}(\Omega)$. Since $\mathcal{D}(\overline{\Omega})$ is dense in $W^{1,p'}(\Omega)$ there is a sequence $(\varphi_n)_n$ in $\mathcal{D}(\overline{\Omega})$ which converges to φ in $W^{1,p'}(\Omega)$ so that $\varphi_n \to \varphi$ and $\operatorname{grad}\varphi_n \to \operatorname{grad}\varphi$ in $L^{p'}$ as $n \to \infty$. The continuity of the L^p-$L^{p'}$ duality yields

$$\begin{aligned}\int_\Omega v\cdot\operatorname{grad}\varphi\,\mathrm{d}x + \int_\Omega(\operatorname{div}v)\varphi\,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega v\cdot\operatorname{grad}\varphi_n\,\mathrm{d}x + \int_\Omega(\operatorname{div}v)\varphi_n\,\mathrm{d}x\\ &= \lim_{n\to\infty}\int_{\partial\Omega}\varphi_n\, v\,|_{\partial\Omega}\cdot\mathfrak{n}\,\mathrm{d}\sigma\,.\end{aligned}$$

Since $\varphi_n|_{\partial\Omega} = \operatorname{tr}_{p'}\varphi_n$ and $\|\operatorname{tr}_{p'}\varphi_n - \operatorname{tr}_{p'}\varphi\|_{W^{1/p,p'}(\partial\Omega)} \leqslant c\,\|\varphi_n - \varphi\|_{W^{1,p'}(\Omega)}$, we further deduce from Hölders' inequality that

$$\left|\int_{\partial\Omega}(\operatorname{tr}_{p'}\varphi)\,\nu\cdot\mathfrak{n}\,\mathrm{d}\sigma - \int_{\partial\Omega}\varphi_n\,\nu\cdot\mathfrak{n}\,\mathrm{d}\sigma\right| \leqslant c\,\|\varphi_n-\varphi\|_{W^{1,p'}(\Omega)}\,\|\nu|_{\partial\Omega}\cdot\mathfrak{n}\|_{L^p(\partial\Omega)},$$

and hence

$$\int_\Omega \nu\cdot\operatorname{grad}\varphi\,\mathrm{d}x + \int_\Omega(\operatorname{div}\nu)\varphi\,\mathrm{d}x = \int_{\partial\Omega}(\operatorname{tr}_{p'}\varphi)\,\nu\cdot\mathfrak{n}\,\mathrm{d}\sigma = \gamma_{\mathfrak{n}}(\nu)(\operatorname{tr}_{p'}\varphi).$$

It follows

$$|\gamma_{\mathfrak{n}}(\nu)(\operatorname{tr}_{p'}\varphi)| \leqslant \|\nu\|_p\,\||\operatorname{grad}\varphi|_{p'}\|_{p'} + \|\operatorname{div}\nu\|_p\,\|\varphi\|_{p'} \leqslant \|\nu\|_{\operatorname{div}}\,\|\varphi\|_{1,p'}.$$

Finally, let $\mu \in W^{-1/p,p'}(\partial\Omega)$. For all $\varphi \in W^{1,p'}(\Omega)$ with $\mu = \operatorname{tr}_{p'}\varphi$ we then know $|\gamma_{\mathfrak{n}}(\nu)(\mu)| \leqslant \|\nu\|_{\operatorname{div}}\,\|\varphi\|_{1,p'}$ and therefore

$$|\gamma_{\mathfrak{n}}(\nu)(\mu)| \leqslant \|\nu\|_{\operatorname{div}} \inf\left\{\|\varphi\|_{1,p'} : \mu = \operatorname{tr}_{p'}\varphi\right\} \leqslant c\,\|\nu\|_{\operatorname{div}}\,\|\mu\|_{W^{1/p,p'}(\partial\Omega)}.$$

(b) : We have seen in (a) that formula (2.5) holds for all $\nu \in \mathcal{D}(\overline{\Omega})^d$ and $\varphi \in W^{1,p'}(\Omega)$. Now, let $\nu \in W^p(\operatorname{div},\Omega)$. Since $\mathcal{D}(\overline{\Omega})^d$ is dense in $W^p(\operatorname{div},\Omega)$ by Proposition 2.17, there is a sequence $(\nu_n)_n$ in $\mathcal{D}(\overline{\Omega})^d$ which converges to ν in $W^p(\operatorname{div},\Omega)$, i.e., $\nu_n \to \nu$ and $\operatorname{div}\nu_n \to \operatorname{div}\nu$ in L^p as $n\to\infty$. The continuity of the L^p-$L^{p'}$-duality thus yields

$$\begin{aligned}\int_\Omega \nu\cdot\operatorname{grad}\varphi\,\mathrm{d}x + \int_\Omega(\operatorname{div}\nu)\varphi\,\mathrm{d}x &= \lim_{n\to\infty}\int_\Omega \nu_n\cdot\operatorname{grad}\varphi\,\mathrm{d}x + \int_\Omega(\operatorname{div}\nu_n)\varphi\,\mathrm{d}x\\ &= \lim_{n\to\infty}\gamma_{\mathfrak{n}}(\nu_n)(\operatorname{tr}_{p'}\varphi).\end{aligned}$$

Further, from (a) we know that $\nu_n \to \nu$ in $W^p(\operatorname{div},\Omega)$ implies $\gamma_{\mathfrak{n}}(\nu_n) \to \gamma_{\mathfrak{n}}(\nu)$ in $W^{-1/p,p}(\partial\Omega)$ and hence $\gamma_{\mathfrak{n}}(\nu_n)(\mu) \to \gamma_{\mathfrak{n}}(\nu)(\mu)$ for all $\mu \in W^{1/p,p'}(\partial\Omega)$, which gives the claim.

(c) : We show both inclusions separately. For $\nu \in \mathcal{D}(\Omega)^d$ and $\varphi \in W^{1,p'}(\Omega)$ we infer from Green's formula (2.5) that

$$\gamma_{\mathfrak{n}}(\nu)(\operatorname{tr}_{p'}\varphi) = \int_\Omega \nu\cdot\operatorname{grad}\varphi\,\mathrm{d}x + \int_\Omega(\operatorname{div}\nu)\varphi\,\mathrm{d}x = 0,$$

so that ν is contained in $\ker(\gamma_{\mathfrak{n}})$ in this case. For $\nu \in W_0^p(\operatorname{div},\Omega)$, choose a sequence $(\nu_n)_n$ in $\mathcal{D}(\Omega)^d$ with $\nu_n \to \nu$ in $W^p(\operatorname{div},\Omega)$. Then the continuity of $\gamma_{\mathfrak{n}}$ in $W^p(\operatorname{div},\Omega)$ yields $\gamma_{\mathfrak{n}}(\nu_n) \to \gamma_{\mathfrak{n}}(\nu)$ in $W^{-1/p,p}(\partial\Omega)$ and therefore $\gamma_{\mathfrak{n}}(\nu)(\mu) = 0$ for all $\mu \in W^{1/p,p'}(\partial\Omega)$ by approximation, which proves the first inclusion "$\supset$".

Conversely, let $\nu \in \ker(\gamma_{\mathfrak{n}}) \subseteq W^p(\operatorname{div},\Omega)$. We will show $\tilde{\nu} \in W^p(\operatorname{div},\mathbb{R}^d)$ so that the assertion follows from Proposition 2.17. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)^d$. Because of

$\gamma_{\mathfrak{n}}(v) = 0$ and $\operatorname{grad} \varphi\,|_{\Omega} \in \mathcal{D}(\overline{\Omega})^d \subseteq W^{1,p'}(\Omega)^d$, we compute

$$\begin{aligned}\int_{\mathbb{R}^d} \tilde{v} \cdot \nabla\varphi \,\mathrm{d}x &= \int_{\Omega} v \cdot \nabla\varphi \,\mathrm{d}x = -\int_{\Omega} (\operatorname{div} v)\varphi \,\mathrm{d}x + \left\langle \operatorname{tr}_{p'}\varphi, \gamma_{\mathfrak{n}}(v)\right\rangle_{\partial\Omega} \\ &= -\int_{\Omega} (\operatorname{div} v)\varphi \,\mathrm{d}x = -\int_{\mathbb{R}^d} (\widetilde{\operatorname{div} v})\varphi \,\mathrm{d}x.\end{aligned}$$

In view of (2.2) the function $\tilde{v}$ belongs to $W^p(\operatorname{div}, \mathbb{R}^d)$ with $\operatorname{div}\tilde{v} = \widetilde{\operatorname{div} v}$.

(d) : Follows from a result from elliptic partial differential equations. Since we will not make use of this statement, we refer to Theorem 1 of Chapter 9 in [9] for a proof. ∎

Corollary 2.22 *Let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain with compact boundary.*

(a) *If $v \in W^p_0(\operatorname{div}, \Omega)$ and $\varphi \in W^{1,p'}(\Omega)$, or $v \in W^p(\operatorname{div}, \Omega)$ and $\varphi \in W^{1,p'}_0(\Omega)$, then*

$$\int_{\Omega} v \cdot \operatorname{grad}\varphi \,\mathrm{d}x = -\int_{\Omega} (\operatorname{div} v)\varphi \,\mathrm{d}x\,. \quad \rtimes$$

(b) *If $v \in L^p(\Omega)^d$, then $v \in W^p_0(\operatorname{div}, \Omega)$ if and only if there exists $w \in L^p(\Omega)$ such that*

$$\int_{\Omega} v \cdot \operatorname{grad}\varphi \,\mathrm{d}x = -\int_{\Omega} w\,\varphi \,\mathrm{d}x \quad \textit{for all } \varphi \in \mathcal{D}(\overline{\Omega}).$$

In this case we have $w = \operatorname{div} v$. $\rtimes$

Proof. (a) : This is an immediate consequence of Green's formula (2.5) and the fact that $\ker(\operatorname{tr}_{p'}) = W^{1,p'}_0(\Omega)$ as well as $\ker(\gamma_{\mathfrak{n}}) = W^p_0(\operatorname{div}, \Omega)$.

(b) : First, let $v \in W^p_0(\operatorname{div}, \Omega)$. Green's formula (2.5) and $\gamma_{\mathfrak{n}}(v) = 0$ imply

$$\int_{\Omega} v \cdot \operatorname{grad}\varphi \,\mathrm{d}x = -\int_{\Omega} (\operatorname{div} v)\varphi \,\mathrm{d}x + \left\langle \operatorname{tr}_{p'}\varphi, \gamma_{\mathfrak{n}}(v)\right\rangle_{\partial\Omega} = -\int_{\Omega} (\operatorname{div} v)\varphi \,\mathrm{d}x$$

for every $\varphi \in \mathcal{D}(\overline{\Omega})$. Conversely, choose $v \in L^p(\Omega)^d$ as in the statement of the corollary. Due to (2.2), the function v is contained in $W^p(\operatorname{div}, \Omega)$ with $\operatorname{div} v = w$. Further, for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ we obtain

$$\int_{\mathbb{R}^d} \tilde{v} \cdot \operatorname{grad}\varphi \,\mathrm{d}x = -\int_{\mathbb{R}^d} \tilde{w} \cdot \varphi \,\mathrm{d}x\,,$$

which yields $\tilde{v} \in W^p(\operatorname{div}, \mathbb{R}^d)$. Hence v belongs to $W^p_0(\operatorname{div}, \Omega)$, as claimed. ∎

Trace Theorem for $W^p(\operatorname{rot}, \Omega)$ 2.23 *Let $1 < p < \infty$. Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain with compact boundary and denote the unit outward normal by $\mathfrak{n} \in L^\infty(\partial\Omega)^3$. Then the following assertions hold.*

(a) *Setting* $\gamma_{\mathrm{t}}(v) \coloneqq T_{v|_{\partial\Omega}\wedge\mathfrak{n}}$ *for* $v \in \mathcal{D}(\overline{\Omega})^3$*, we obtain a mapping*

$$\gamma_{\mathrm{t}} : \mathcal{D}(\overline{\Omega})^d \subseteq W^p(\mathrm{rot}, \Omega) \to W^{-1/p,p}(\partial\Omega)^3$$

which satisfies

$$|\gamma_{\mathrm{t}}(v)(\mu)| \leqslant c_{d,p} \, \|v\|_{\mathrm{rot}} \, \|\mu\|_{W^{1/p,p'}(\partial\Omega)^3} \quad (\mu \in W^{1/p,p'}(\partial\Omega)^3).$$

We will denote the unique continuous extension to $W^p(\mathrm{rot}, \Omega)$ *by* γ_{t}*, too.*

(b) *If* $v \in W^p(\mathrm{rot}, \Omega)$ *and* $\varphi \in W^{1,p'}(\Omega)^3$*, then the following Green's formula*

$$\int_\Omega v \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x = \int_\Omega \mathrm{rot}\, v \cdot \varphi \, \mathrm{d}x + \left\langle \mathrm{tr}_{p'} \varphi, \gamma_{\mathrm{t}}(v) \right\rangle_{\partial\Omega} \tag{2.6}$$

holds, where $\mathrm{tr}_{p'}$ *is understood component wise.*

(c) *The operator* $\gamma_{\mathrm{t}} \in \mathcal{B}(W^p(\mathrm{rot}, \Omega), W^{-1/p,p}(\partial\Omega)^3)$ *satisfies*

$$\ker(\gamma_{\mathrm{t}}) = W^p_0(\mathrm{rot}, \Omega). \quad \rtimes$$

Proof. (a) : Choose $v \in \mathcal{D}(\overline{\Omega})^3$ and let $\varphi \in \mathcal{D}(\overline{\Omega})^3$. Applying Gauß' Theorem A.18 we obtain

$$\begin{aligned} &\int_\Omega v \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x - \int_\Omega \mathrm{rot}\, v \cdot \varphi \, \mathrm{d}x \\ &\quad = \int_\Omega \mathrm{div}(\varphi \wedge v) \, \mathrm{d}x = \int_{\partial\Omega} (\varphi \wedge v) \cdot \mathfrak{n} \, \mathrm{d}\sigma = \int_{\partial\Omega} (v \wedge \mathfrak{n}) \cdot \varphi \, \mathrm{d}\sigma \,. \end{aligned}$$

Now, let $\varphi \in W^{1,p'}(\Omega)^3$. Since $\mathcal{D}(\overline{\Omega})^3$ is dense in $W^{1,p'}(\Omega)^3$, there is a sequence $(\varphi_n)_n$ in $\mathcal{D}(\overline{\Omega})^3$ which converges to φ in $W^{1,p'}(\Omega)^3$, i.e., $\varphi_n \to \varphi$ and $\mathrm{rot}\, \varphi_n \to \mathrm{rot}\, \varphi$ in $L^{p'}$ as $n \to \infty$. The continuity of the L^p-$L^{p'}$ duality thus yields

$$\begin{aligned} \int_\Omega v \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x - \int_\Omega \mathrm{rot}\, v \cdot \varphi \, \mathrm{d}x &= \lim_{n\to\infty} \int_\Omega v \cdot \mathrm{rot}\, \varphi_n \, \mathrm{d}x - \int_\Omega \mathrm{rot}\, v \cdot \varphi_n \, \mathrm{d}x \\ &= \lim_{n\to\infty} \int_{\partial\Omega} (v \wedge \mathfrak{n}) \cdot \varphi_n \, \mathrm{d}\sigma \,. \end{aligned}$$

Since $\varphi_n\big|_{\partial\Omega} = \mathrm{tr}_{p'} \varphi_n$ and $\|\mathrm{tr}_{p'} \varphi_n - \mathrm{tr}_{p'} \varphi\|_{W^{1/p,p'}(\partial\Omega)^3} \leqslant c \, \|\varphi_n - \varphi\|_{W^{1,p'}(\Omega)^3}$, we further derive from Hölder's inequality that

$$\begin{aligned} &\left| \int_{\partial\Omega} \mathrm{tr}_{p'} \varphi \cdot (\varphi \wedge \mathfrak{n}) \, \mathrm{d}\sigma - \int_{\partial\Omega} \varphi_n \cdot (\varphi \wedge \mathfrak{n}) \, \mathrm{d}\sigma \right| \\ &\qquad \leqslant c \, \|\varphi_n - \varphi\|_{W^{1,p'}(\Omega)^3} \, \|v \wedge \mathfrak{n}\|_{L^p(\partial\Omega)^3}, \end{aligned}$$

and hence

$$\int_\Omega v \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x - \int_\Omega \mathrm{rot}\, v \cdot \varphi \, \mathrm{d}x = \int_{\partial\Omega} (v \wedge \mathfrak{n}) \cdot \varphi \, \mathrm{d}\sigma = \gamma_{\mathrm{t}}(v)(\mathrm{tr}_{p'} \varphi).$$

Using also Lemma 2.10 we derive

$$|\gamma_t(v)(\mathrm{tr}_{p'}\varphi)| \leqslant \|v\|_p\,\|\mathrm{rot}\,\varphi\|_{p'} + \|\mathrm{rot}\,v\|_p\,\|\varphi\|_{p'} \leqslant \|v\|_{\mathrm{rot}}\,\|\varphi\|_{\mathrm{rot}} \leqslant c\,\|v\|_{\mathrm{rot}}\,\|\varphi\|_{W^{1,p'}(\Omega)^3}.$$

Finally, let $\mu \in W^{1/p,p'}(\partial\Omega)^3$. Take $\varphi \in W^{1,p'}(\Omega)^3$ with $\mu = \mathrm{tr}_{p'}\varphi$. It follows $|\gamma_t(v)(\mu)| \leqslant c\,\|v\|_{\mathrm{rot}}\,\|\varphi\|_{W^{1,p'}(\Omega)^3}$ and therefore

$$|\gamma_t(v)(\mu)| \leqslant c\,\|v\|_{\mathrm{rot}}\,\inf\Big\{\|\varphi\|_{W^{1,p'}(\Omega)^3} : \mu = \mathrm{tr}_{p'}\varphi\Big\} \leqslant c\,\|v\|_{\mathrm{rot}}\,\|\mu\|_{W^{1/p,p'}(\partial\Omega)^3}.$$

(b) : We have seen in (a) that formula (2.6) holds for all $v \in \mathcal{D}(\overline{\Omega})^3$ and $\varphi \in W^{1,p'}(\Omega)^3$. Now, let $v \in W^p(\mathrm{rot},\Omega)$. Since $\mathcal{D}(\overline{\Omega})^3$ is dense in $W^p(\mathrm{rot},\Omega)$ there is a sequence $(v_n)_n$ in $v \in \mathcal{D}(\overline{\Omega})^3$ which converges to v in $W^p(\mathrm{rot},\Omega)$, i.e., $v_n \to v$ and $\mathrm{rot}\,v_n \to \mathrm{rot}\,v$ in L^p as $n \to \infty$. By means of the continuity of the L^p-$L^{p'}$-duality we arrive at

$$\int_\Omega v \cdot \mathrm{rot}\,\varphi\,\mathrm{d}x - \int_\Omega \mathrm{rot}\,v \cdot \varphi\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega v_n \cdot \mathrm{rot}\,\varphi\,\mathrm{d}x - \int_\Omega \mathrm{rot}\,v_n \cdot \varphi\,\mathrm{d}x = \lim_{n\to\infty} \gamma_t(v_n)(\mathrm{tr}_{p'}\varphi).$$

Further, we know from (a) that $v_n \to v$ in $W^p(\mathrm{rot},\Omega)$ implies $\gamma_t(v_n) \to \gamma_t(v)$ in $W^{-1/p,p}(\partial\Omega)^3$ and therefore $\gamma_t(v_n)(\mu) \to \gamma_t(v)(\mu)$ for all $\mu \in W^{1/p,p'}(\partial\Omega)^3$, which gives the claim.

(c) : We show both inclusions separately. For $v \in \mathcal{D}(\Omega)^3$ and $\varphi \in W^{1,p'}(\Omega)^3$ we infer from Green's formula in (b)

$$\gamma_t(v)(\mathrm{tr}_{p'}\varphi) = \int_\Omega v \cdot \mathrm{rot}\,\varphi\,\mathrm{d}x - \int_\Omega \mathrm{rot}\,v \cdot \varphi\,\mathrm{d}x = 0,$$

so that v belongs to $\ker(\gamma_t)$ in this case. For $v \in W^p_0(\mathrm{rot},\Omega)$, choose a sequence $(v_n)_n$ in $\mathcal{D}(\Omega)^3$ with $v_n \to v$ in $W^p(\mathrm{rot},\Omega)$. Then the continuity of γ_t in $W^p(\mathrm{rot},\Omega)$ yields $\gamma_t(v_n) \to \gamma_t(v)$ in $W^{-1/p,p}(\partial\Omega)^3$ and therefore $\gamma_t(v)(\mu) = 0$ for all $\mu \in W^{1/p,p'}(\partial\Omega)^3$ by approximation, which proves the first inclusion "$\supset$".

Conversely, let $v \in \ker(\gamma_t) \subseteq W^p(\mathrm{rot},\Omega)$. We will show $\tilde{v} \in W^p(\mathrm{rot},\mathbb{R}^3)$ so that the assertions follows from Proposition 2.17. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)^3$. Because of $\gamma_t(v) = 0$ and $\mathrm{rot}\,\varphi|_\Omega \in \mathcal{D}(\overline{\Omega})^3 \subseteq W^{1,p'}(\Omega)^3$, we calculate

$$\int_{\mathbb{R}^d} \tilde{v} \cdot \mathrm{rot}\,\varphi\,\mathrm{d}x = \int_\Omega v \cdot \mathrm{rot}\,\varphi\,\mathrm{d}x = \int_\Omega \mathrm{rot}\,v \cdot \varphi\,\mathrm{d}x + \Big\langle \mathrm{tr}_{p'}\varphi, \gamma_t(v)\Big\rangle_{\partial\Omega} = \int_\Omega \mathrm{rot}\,v \cdot \varphi\,\mathrm{d}x = \int_{\mathbb{R}^3} \widetilde{\mathrm{rot}\,v} \cdot \varphi\,\mathrm{d}x.$$

Due to (2.3), the vector field $\tilde{v}$ is contained in $W^p(\mathrm{rot},\mathbb{R}^3)$ with $\mathrm{rot}\,\tilde{v} = \widetilde{\mathrm{rot}\,v}$. ∎

Corollary 2.24 (a) *If $v \in W^p(\mathrm{rot},\Omega)$ and $u \in W^{p'}_0(\mathrm{rot},\Omega)$, then*

$$\int_\Omega v \cdot \mathrm{rot}\,u\,\mathrm{d}x = \int_\Omega \mathrm{rot}\,v \cdot u\,\mathrm{d}x.$$

(b) *If* $v \in L^p(\Omega)^3$*, then* $v \in W_0^p(\mathrm{rot}, \Omega)$ *if and only if there exists* $w \in L^p(\Omega)^3$ *such that*

$$\int_\Omega v \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x = \int_\Omega w \cdot \varphi \, \mathrm{d}x \quad \textit{for all } \varphi \in \mathcal{D}(\overline{\Omega})^3.$$

In this case we have $w = \mathrm{rot}\, v$*.* ⋊

Proof. (a) : For $v \in W^p(\mathrm{rot}, \Omega)$ and $u \in \mathcal{D}(\Omega)^3 \subseteq W_0^{p'}(\mathrm{rot}, \Omega)$ we obtain from Green's formula (2.6) that

$$\int_\Omega v \cdot \mathrm{rot}\, u \, \mathrm{d}x = \int_\Omega \mathrm{rot}\, v \cdot u \, \mathrm{d}x + \big\langle \mathrm{tr}_{p'} u, \gamma_t(v) \big\rangle_{\partial\Omega} = \int_\Omega \mathrm{rot}\, v \cdot u \, \mathrm{d}x \,.$$

Now, let $u \in W_0^{p'}(\mathrm{rot}, \Omega)$. Choose $(u_n)_n$ in $\mathcal{D}(\Omega)^3$ such that $u_n \to u$ in $W^{p'}(\mathrm{rot}, \Omega)$. Then the claim follows by approximation.

(b) : First, let $v \in W_0^p(\mathrm{rot}, \Omega)$. For all $\varphi \in \mathcal{D}(\Omega)^3$, Green's formula (2.6) and $\ker \gamma_t = W_0^p(\mathrm{rot}, \Omega)$ then yield

$$\int_\Omega v \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x = \int_\Omega \mathrm{rot}\, v \cdot \varphi \, \mathrm{d}x + \gamma_t(v)(\mathrm{tr}_{p'} \varphi) = \int_\Omega \mathrm{rot}\, v \cdot \varphi \, \mathrm{d}x \,.$$

Conversely, let $v \in L^p(\Omega)^3$ be as in the assertion. Due to (2.3), the vector field v belongs to $W^p(\mathrm{rot}, \Omega)$ with $\mathrm{rot}\, v = w$. For every $\varphi \in \mathcal{D}(\mathbb{R}^3)^3$ we further calculate

$$\int_{\mathbb{R}^3} \tilde{v} \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x = \int_\Omega v \cdot \mathrm{rot}\, \varphi \, \mathrm{d}x = \int_\Omega w \cdot \varphi \, \mathrm{d}x = \int_{\mathbb{R}^3} \tilde{w} \cdot \varphi \, \mathrm{d}x \,.$$

By means of (2.3), we conclude $\tilde{v} \in W^p(\mathrm{rot}, \mathbb{R}^3)$ and $\mathrm{rot}\, \tilde{v} = \widetilde{\mathrm{rot}\, v}$. Hence the vector field v is contained in $W_0^p(\mathrm{rot}, \Omega)$. ∎

■ Connection to the Fourier Transformation

Similar to the characterization of the Sobolev spaces $H^m(\mathbb{R}^d)$ as Bessel-Potential spaces, we will describe the spaces $H(\mathrm{div}, \mathbb{R}^d)$ and $H(\mathrm{rot}, \mathbb{R}^3)$ using the Fourier transformation. We start with the Fourier transform of our main operators.

Lemma 2.25 (a) *If* $v \in H(\mathrm{div}, \mathbb{R}^d)$*, then*

$$\mathcal{F}(\mathrm{div}\, v)(\xi) = i\, \xi \cdot (\mathcal{F} v)(\xi) \quad \textit{for almost every } \xi \in \mathbb{R}^d \,.$$

(b) *If* $v \in H(\mathrm{rot}, \mathbb{R}^3)$*, then*

$$\mathcal{F}(\mathrm{rot}\, v)(\xi) = i\, \xi \wedge (\mathcal{F} v)(\xi) \quad \textit{for almost every } \xi \in \mathbb{R}^3 \,. \quad ⋊$$

Proof. (a) : Let φ belong to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. We first show that

$$(\mathrm{div}\, v \mid \varphi)_2 = -(v \mid \mathrm{grad}\, \varphi)_2 \,.$$

Since v belongs to $H(\operatorname{div}, \mathbb{R}^d)$, we know from (2.2) that

$$(\operatorname{div} v \mid \phi)_2 = -(v \mid \operatorname{grad} \phi)_2 \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}^d)\,.$$

Choose $\phi \in \mathcal{D}(\mathbb{R}^d)$ with

$$\phi(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2, \end{cases}$$

and $\|\partial_k \phi\|_\infty \leqslant c$ for all $1 \leqslant k \leqslant d$, and define $\varphi_n(x) \coloneqq \phi(x/n)\,\varphi(x)$ for $x \in \mathbb{R}^d$. Obviously $\varphi_n \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi_n(x) \to \varphi(x)$ for every $x \in \mathbb{R}^d$. We further have

$$\partial_k \varphi_n(x) = \frac{1}{n}(\partial_k \phi)(x/n)\,\varphi(x) + \phi(x/n)\,\partial_k \varphi(x) \to \partial_k \varphi(x) \quad (n \to \infty)$$

for every $x \in \mathbb{R}^d$ and all $1 \leqslant k \leqslant d$. Since φ, $\partial_k \varphi \in L^r(\mathbb{R}^d)$ for every $r \geqslant 1$, Lebesque's theorem yields $\varphi_n \to \varphi$ and $\operatorname{grad} \varphi_n \to \operatorname{grad} \varphi$ in $L^2(\mathbb{R}^d)$. We thus obtain

$$(v \mid \operatorname{grad} \varphi)_2 = \lim_{n\to\infty} (v \mid \operatorname{grad} \varphi_n)_2 = -\lim_{n\to\infty} (\operatorname{div} v \mid \varphi_n)_2 = -(\operatorname{div} v \mid \varphi)_2,$$

as claimed. Taking into account $v \in H(\operatorname{div}, \mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, we conclude

$$\begin{aligned}(\mathcal{F}(\operatorname{div} v) \mid \mathcal{F}\varphi)_2 &= (\operatorname{div} v \mid \varphi)_2 = -(v \mid \operatorname{grad} \varphi)_{L^2(\mathbb{R}^d)^d} \\ &= -(\mathcal{F}v \mid \mathcal{F}(\operatorname{grad} \varphi))_{L^2(\mathbb{R}^d)^d} = (\mathcal{F}v \mid i\,\xi\,\mathcal{F}\varphi)_{L^2(\mathbb{R}^d)^d} \\ &= (i\,\xi \cdot \mathcal{F}v \mid \mathcal{F}\varphi)_2\,.\end{aligned}$$

Because $\mathcal{F}$ is bijective on $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d) \subseteq \mathcal{S}(\mathbb{R}^d)$, the claim follows by the fundamental lemma of calculus of variations.

(b) : Let $\varphi \in \mathcal{S}(\mathbb{R}^3)^3$. We start with the observation that

$$\begin{aligned}\mathcal{F}(\operatorname{rot} \varphi)(\xi) &= \mathcal{F}\left(\sum_{k=1}^3 J_k\,\partial_k \varphi\right)(\xi) = \sum_{k=1}^3 J_k \mathcal{F}\,(\,\partial_k \varphi)\,(\xi) \\ &= \sum_{k=1}^3 i\xi_k\,J_k\,(\mathcal{F}\varphi)(\xi) = i\xi \wedge (\mathcal{F}\varphi)(\xi)\,.\end{aligned}$$

Let $v \in H(\operatorname{rot}, \mathbb{R}^3)$. Similar to (a) we prove that

$$(\operatorname{rot} v \mid \varphi)_2 = (v \mid \operatorname{rot} \varphi)_2 \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^3)^3\,.$$

We thus compute

$$\begin{aligned}(\mathcal{F}(\operatorname{rot} v) \mid \mathcal{F}\varphi)_2 &= (\operatorname{rot} v \mid \varphi)_2 = (v \mid \operatorname{rot} \varphi)_2 = (\mathcal{F}v \mid \mathcal{F}(\operatorname{rot} \varphi))_2 \\ &= \Big(\mathcal{F}v \mid i \textstyle\sum_{k=1}^3 \xi_k\,J_k\,(\mathcal{F}\varphi)(\xi)\Big)_2 \\ &= \Big(-i \textstyle\sum_{k=1}^3 \xi_k\,J_k^\top(\mathcal{F}v) \mid \mathcal{F}\varphi\Big)_2 \\ &= (i\xi \wedge (\mathcal{F}v)(\xi) \mid \mathcal{F}\varphi)_2\,.\end{aligned}$$

Due to $\mathcal{F}$ being bijective on $\mathcal{S}(\mathbb{R}^3)^3$ and $\mathcal{D}(\mathbb{R}^3)^3 \subseteq \mathcal{S}(\mathbb{R}^3)^3$, the assertion is a consequence of the fundamental lemma of calculus of variations. ∎

The next result follows from the unitarity of the Fourier transformation on $L^2(\mathbb{R}^d)$.

COROLLARY 2.26 *For $x \in \mathbb{R}^d$ we define $\ell(x) = x$. Then*

(a) $H(\mathrm{div}, \mathbb{R}^d) = \left\{ v \in L^2(\mathbb{R}^d)^d : \ell \cdot \mathcal{F}v \in L^2(\mathbb{R}^d) \right\}$ *and*

$$\mathrm{div}\, v = \mathcal{F}^{-1}\left(i\ell \cdot \mathcal{F}v\right).$$

(b) $H(\mathrm{rot}, \mathbb{R}^3) = \{ v \in L^2(\mathbb{R}^3)^3 : \ell \wedge \mathcal{F}v \in L^2(\mathbb{R}^3)^3 \}$ *and*

$$\mathrm{rot}\, v = \mathcal{F}^{-1}\left(i\ell \wedge \mathcal{F}v\right).$$

(c) $H(\mathrm{div}, \mathbb{R}^3) \cap H(\mathrm{rot}, \mathbb{R}^3) = H^1(\mathbb{R}^3)^3$ *and*

$$\|v\|_{H^1(\mathbb{R}^3)^3} \sim \left(\|v\|_2^2 + \|\,\mathrm{div}\, v\|_2^2 + \|\,\mathrm{rot}\, v\|_2^2\right)^{1/2} \quad (v \in H^1(\mathbb{R}^3)^3). \quad ⋊$$

Proof. (c) : From Lemma 2.10 we already know $H^1(\mathbb{R}^3)^3 \subseteq H(\mathrm{div}, \mathbb{R}^3) \cap H(\mathrm{rot}, \mathbb{R}^3)$ and the corresponding estimate. The other direction is now a direct consequence of $(a), (b)$ and the Lagrange identity

$$|a \cdot b|^2 + |a \wedge b|^2 = |a|^2\, |b|^2 \quad (a, b \in \mathbb{R}^3),$$

which yields

$$\begin{aligned}\|\mathrm{div}\, v\|_2^2 + \|\mathrm{rot}\, v\|_2^2 &= \|\ell \cdot \mathcal{F}v\|_2^2 + \|\ell \wedge \mathcal{F}v\|_2^2 \\ &= \|\,|\ell|\,|\mathcal{F}v|\,\|_2^2 \geqslant \|\partial_j v_k\|_2^2 \qquad \text{for all } j, k \in \{1,2,3\}. \quad ∎\end{aligned}$$

2.2 Relations between $H(\mathrm{div}), H(\mathrm{rot})$ and H^1

In the following we will restrict our investigations to the Hilbert space situation, which builds the major framework for our upcoming investigations. The stated results in this section will mostly remain valid also in the L^p-setting, but in contrast to the previous section, the proofs would require real additional effort and also the usage of alternative methods.

DEFINITION 2.27 We put

$$H(\mathrm{div}, \mathrm{rot}, \Omega) \coloneqq H(\mathrm{div}, \Omega) \cap H(\mathrm{rot}, \Omega),$$

and equip this space with the inner product

$$(u \mid v)_{\mathrm{div,rot}} \coloneqq (u \mid v)_2 + (\mathrm{div}\, u \mid \mathrm{div}\, v)_2 + (\mathrm{rot}\, u \mid \mathrm{rot}\, v)_2,$$

where $u, v \in H(\mathrm{div}, \mathrm{rot}, \Omega)$. We will further consider the subspaces

$$\begin{aligned} H_{\mathfrak{n}0}(\mathrm{div}, \mathrm{rot}, \Omega) &\coloneqq H_0(\mathrm{div}, \Omega) \cap H(\mathrm{rot}, \Omega), \\ H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega) &\coloneqq H(\mathrm{div}, \Omega) \cap H_0(\mathrm{rot}, \Omega), \\ H_0(\mathrm{div}, \mathrm{rot}, \Omega) &\coloneqq H_0(\mathrm{div}, \Omega) \cap H_0(\mathrm{rot}, \Omega). \quad ⋊ \end{aligned}$$

The following remarks are a direct consequence of the considerations from the previous section.

Remark 2.28 If Ω satisfies the segment property, then $\mathcal{D}(\overline{\Omega})^3$ is a dense subset of $H(\mathrm{div}, \mathrm{rot}, \Omega)$, and if Ω is even a Lipschitz domain with compact boundary, then

$$H_{\mathfrak{n}0}(\mathrm{div}, \mathrm{rot}, \Omega) = \{v \in H(\mathrm{div}, \mathrm{rot}, \Omega) : \gamma_{\mathfrak{n}}(v) = 0\}$$

as well as

$$H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega) = \{v \in H(\mathrm{div}, \mathrm{rot}, \Omega) : \gamma_{\mathfrak{t}}(v) = 0\} . \quad \rtimes$$

The first aim is to understand in which situations we have additional regularity for the above spaces. More precisely, we ask if there are certain regularity assumptions on the boundary such that both spaces $H_{\mathfrak{n}0}(\mathrm{div}, \mathrm{rot}, \Omega)$ and $H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega)$ can be embedded into $H^1(\Omega)^3$. Therefore we state the following density result.

Lemma 2.29 *Let Ω be a bounded $C^{1,1}$-domain. Then $H_{\mathfrak{n}0}(\mathrm{div}, \mathrm{rot}, \Omega) \cap H^1(\Omega)^3$ is dense in $H_{\mathfrak{n}0}(\mathrm{div}, \mathrm{rot}, \Omega)$ and $H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega) \cap H^1(\Omega)^3$ is dense in $H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega)$.* $\rtimes$

Proof. For a proof which does not depend on the existence of vector potentials and corresponding orthogonal decompositions of $L^2(\Omega)^3$, we refer to Lemma 2.10 and Lemma 2.13 in [2]. There, a regularity result for certain associated Neumann problems - this is where the regularity assumptions on the boundary enters - is used to achieve the desired H^1 approximation. To give a deeper impression of this approach, we sketch it for the space $H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega)$, adopting the notation of [2].

Given $v \in H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega)$ there is a sequence $(v_k)_k$ in $\mathcal{D}(\overline{\Omega})^3$ converging to v in $H^1_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega)$. For each k we consider the resulting Neumann problem

$$\begin{aligned} \Delta\chi_k &= \mathrm{div}\, v_k && \text{in } \Omega, \\ \partial_{\mathfrak{n}}\chi_k &= v_k \cdot \mathfrak{n} && \text{on } \partial\Omega \end{aligned}$$

in H^1. Due to the regularity assumption on the boundary we infer that the solution χ_k already belongs to $H^2(\Omega)$ so that $v_k - \mathrm{grad}\,\chi_k$ is contained in $H^1(\Omega)^3$. Further, by a straight forward calculation one finds that $(v_k)_k$ converges in H^1 to the solution χ of the Neumann problem associated to v, i.e.,

$$\begin{aligned} \Delta\chi &= \mathrm{div}\, v && \text{in } \Omega, \\ \partial_{\mathfrak{n}}\chi &= v \cdot \mathfrak{n} && \text{on } \partial\Omega. \end{aligned}$$

Thus χ also belongs to $H^2(\Omega)$. Finally, one can show that the sequence $(v_k - \mathrm{grad}\,\chi_k + \mathrm{grad}\,\chi)_k$ belongs to $H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega) \cap H^1(\Omega)^3$ and does the job. ∎

The following technical lemma is the key to the desired embedding results of $H_{\mathfrak{n}0}(\mathrm{div}, \mathrm{rot}, \Omega)$ and $H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega)$ into $H^1(\Omega)^3$.

LEMMA 2.30 (a) *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain with compact boundary. For all $v \in \mathcal{D}(\overline{\Omega})^3$ we have*

$$\int_\Omega |\mathrm{div}\, v|^2 + |\mathrm{rot}\, v|^2 \,\mathrm{d}x = \int_\Omega |\mathrm{grad}\, v|^2 \,\mathrm{d}x + I_{\partial\Omega}\,,$$

where the remaining term is given by

$$I_{\partial\Omega} = \int_{\partial\Omega} \left\{ \mathrm{div}\,\overline{v}\,(v \cdot \mathfrak{n}) - \mathrm{rot}\,\overline{v} \cdot (v \wedge \mathfrak{n}) - \sum_{i=1}^{3} v_i\,(\nabla \overline{v_i} \cdot \mathfrak{n}) \right\} \mathrm{d}\sigma\,.$$

(b) *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded $C^{1,1}$-domain and $v \in \mathcal{D}(\overline{\Omega})^3$. If $v \wedge \mathfrak{n} = 0$ on $\partial\Omega$, then*

$$I_{\partial\Omega} = \int_{\partial\Omega} 2\,H\,|v \cdot \mathfrak{n}|^2 \,\mathrm{d}\sigma,$$

with the mean curvature H of $\partial\Omega$. If $v \cdot \mathfrak{n} = 0$ on $\partial\Omega$, then

$$I_{\partial\Omega} = \int_{\partial\Omega} \mathrm{II}(v \wedge \mathfrak{n}, v \wedge \mathfrak{n}) \,\mathrm{d}\sigma,$$

with the second fundamental form II *of $\partial\Omega$.* ⋊

Proof. (a) : We recall from (2.5) and (2.6) the Green's formulas

$$\int_\Omega (\mathrm{div}\, v)\varphi \,\mathrm{d}x = -\int_\Omega v \cdot \mathrm{grad}\,\varphi \,\mathrm{d}x + \int_{\partial\Omega} \varphi(v \cdot \mathfrak{n}) \,\mathrm{d}\sigma,$$

for v, $\varphi \in \mathcal{D}(\overline{\Omega})^d$, and

$$\int_\Omega \mathrm{rot}\, v \cdot \varphi \,\mathrm{d}x = \int_\Omega v \cdot \mathrm{rot}\,\varphi \,\mathrm{d}x - \int_{\partial\Omega} \varphi \cdot (v \wedge \mathfrak{n}) \,\mathrm{d}\sigma,$$

for v, $\varphi \in \mathcal{D}(\overline{\Omega})^3$. We deduce

$$\begin{aligned}
&\int_\Omega \mathrm{rot}\, v \cdot \mathrm{rot}\,\overline{v} \,\mathrm{d}x + \int_\Omega \mathrm{div}\, v\,(\mathrm{div}\,\overline{v}) \,\mathrm{d}x \\
&= \int_\Omega (v \cdot \mathrm{rot}\,\mathrm{rot}\,\overline{v} - v \cdot \mathrm{grad}\,\mathrm{div}\,\overline{v}) \,\mathrm{d}x + \int_{\partial\Omega} (\mathrm{div}\,\overline{v}\,(v \cdot \mathfrak{n}) - \mathrm{rot}\,\overline{v} \cdot (v \wedge \mathfrak{n}))\mathrm{d}\sigma \\
&= -\int_\Omega v \cdot \Delta\overline{v} \,\mathrm{d}x + \int_{\partial\Omega} (\mathrm{div}\,\overline{v}\,(v \cdot \mathfrak{n}) - \mathrm{rot}\,\overline{v} \cdot (v \wedge \mathfrak{n}))\mathrm{d}\sigma.
\end{aligned}$$

Applying again the first mentioned Green's formula, we further obtain

$$-\int_\Omega v_i\,\Delta\overline{v_i} \,\mathrm{d}x = \int_\Omega \nabla v_i \cdot \nabla\overline{v_i} \,\mathrm{d}x - \int_{\partial\Omega} v_i(\nabla\overline{v_i} \cdot \mathfrak{n}) \,\mathrm{d}\sigma\,,$$

which yields the desired expression.

(b) : The proof is beyond the scope of this thesis, since it is based on advanced methods in differential geometry. We refer the interested reader to Lemma 2.13 in [2], as well as the references therein. ∎

COROLLARY 2.31 *We have* $H_0(\mathrm{div},\mathrm{rot},\Omega) = H_0^1(\Omega)^3$ *with equivalent norms.* ⋊

Proof. Let $v \in \mathcal{D}(\Omega)^3$. In view of the formula in Lemma 2.30 (a), we deduce

$$\|v\|_2 + \|\mathrm{div}\, v\|_2 + \|\mathrm{rot}\, v\|_2 = \|v\|_{H^1(\Omega)^3}.$$

The claim then follows by approximation, as executed in the proof of Theorem 2.33. ∎

Remark 2.32 The above corollary is also a simple consequence of Corollary 2.26 (c). But now, with the above lemma, we have gained an alternative proof, which does not make use of the Fourier transform. ⋊

THEOREM 2.33 *Let* Ω *be a bounded* $C^{1,1}$*-domain. Then the spaces* $H_{\mathfrak{n}0}(\mathrm{div},\mathrm{rot},\Omega)$ *and* $H_{\mathfrak{t}0}(\mathrm{div},\mathrm{rot},\Omega)$ *are continuously embedded in* $H^1(\Omega)^3$. ⋊

Proof. We prove this theorem by deducing it from Lemma 2.29 and Lemma 2.30. Let $v \in \mathcal{D}(\overline{\Omega})^3$. By Theorem 2.19 we can estimate

$$\left| \int_{\partial\Omega} H\,|v \cdot \mathfrak{n}|^2 \,\mathrm{d}\sigma \right| \leqslant \|H\|_{L^\infty(\partial\Omega)}\, \||v|\|_{L^2(\partial\Omega)} \leqslant c\, \|v\|_{H^1(\Omega)^3},$$

and similarly for $\int_{\partial\Omega} \mathrm{II}(v \wedge \mathfrak{n}, v \wedge \mathfrak{n})\,\mathrm{d}\sigma$. Applying Lemma 2.30 (b) thus yields

$$\|v\|_{\mathrm{div},\mathrm{rot}} \leqslant c\, \|v\|_{H^1(\Omega)^3} \tag{$\star$}$$

in both cases. Recall that $\mathcal{D}(\overline{\Omega})^3$ is dense in $H_{\mathfrak{n}0}(\mathrm{div},\mathrm{rot},\Omega)$ and $H_{\mathfrak{t}0}(\mathrm{div},\mathrm{rot},\Omega)$ as well as in $H^1(\Omega)^3$. By approximation we then find that ($\star$) is valid for all vector fields belonging to $H_{\mathfrak{n}0}(\mathrm{div},\mathrm{rot},\Omega) \cap H^1(\Omega)^3$ or $H_{\mathfrak{t}0}(\mathrm{div},\mathrm{rot},\Omega) \cap H^1(\Omega)^3$. Now, let v be an arbitrary vector field in $H_{\mathfrak{n}0}(\mathrm{div},\mathrm{rot},\Omega) \cup H_{\mathfrak{t}0}(\mathrm{div},\mathrm{rot},\Omega)$. By means of Lemma 2.29, there is a sequence $(v_k)_k$ belonging to $H_{\mathfrak{n}0}(\mathrm{div},\mathrm{rot},\Omega) \cap H^1(\Omega)^3$ or $H_{\mathfrak{t}0}(\mathrm{div},\mathrm{rot},\Omega) \cap H^1(\Omega)^3$ which converges to v in $H_{\mathfrak{n}0}(\mathrm{div},\mathrm{rot},\Omega)$ or $H_{\mathfrak{t}0}(\mathrm{div},\mathrm{rot},\Omega)$ respectively. Applying the estimate ($\star$) to each v_k, we see that the sequence $(v_k)_k$ is bounded in $H^1(\Omega)^3$. Hence, it admits a subsequence which converges weakly in $H^1(\Omega)^3$. Of course, this limit is nothing else but v, which proofs the claim. ∎

Remark 2.34 The embeddings of $H_{\mathfrak{n}0}(\mathrm{div},\mathrm{rot},\Omega)$ and $H_{\mathfrak{t}0}(\mathrm{div},\mathrm{rot},\Omega)$ in $H^1(\Omega)^3$ are no longer valid in general for Lipschitz domains. A counterexample can be found in [2] on page 832. However, a comparable regularity result, first shown in [7], states that for a bounded Lipschitz domain these spaces are continuously embedded in $H^{1/2}(\Omega)^3$. ⋊

Another interesting relation is the following

LEMMA 2.35 *Let Ω be a non-empty domain in $\mathbb{R}^3$. If $v \in C^1(\Omega, \mathbb{R}^3)$, then for any $r > 0$ and $x \in \Omega$ with $B(x,r) \subseteq \Omega$ we have*

$$\fint_{\partial B(x,r)} v(y) \mathrm{d}\sigma(y) = v(x) + \int_0^r \left(\fint_{\partial B(x,s)} \mathrm{rot}\, v(y) \wedge \mathfrak{n}(y)\, \mathrm{d}\sigma(y) \right) \mathrm{d}s + \int_0^r \left(\fint_{\partial B(x,s)} \mathrm{div}\, v(y)\, \mathfrak{n}(y)\, \mathrm{d}\sigma(y) \right) \mathrm{d}s\,,$$

or equivalently

$$\int_0^r \left(\fint_{\partial B(x,s)} \mathrm{grad}\, v(y) \cdot \mathfrak{n}(y)\, \mathrm{d}\sigma(y) \right) \mathrm{d}s = \int_0^r \left(\fint_{\partial B(x,s)} \mathrm{rot}\, v(y) \wedge \mathfrak{n}(y)\, \mathrm{d}\sigma(y) \right) \mathrm{d}s + \int_0^r \left(\fint_{\partial B(x,s)} \mathrm{div}\, v(y)\, \mathfrak{n}(y)\, \mathrm{d}\sigma(y) \right) \mathrm{d}s\,,$$

where as usual

$$\fint_\Sigma f(y)\, \mathrm{d}\sigma(y) \coloneqq \frac{1}{|\Sigma|} \int_\Sigma f(y)\, \mathrm{d}\sigma(y)\,. \quad \rtimes$$

Proof. To give an impression of how to get to this expressions, assume that v belongs to $C^2(\Omega, \mathbb{R}^3)$. Differentiating

$$\psi(R) \coloneqq \frac{1}{|\partial B(x,R)|} \int_{\partial B(x,R)} v(y)\, \mathrm{d}\sigma(y) = \frac{1}{|\partial B(0,1)|} \int_{\partial B(0,1)} v(x + Ry)\, \mathrm{d}\sigma(y) \quad (0 < R \leqslant r)$$

and using Gauß' theorem implies

$$\fint_{\partial B(x,r)} v(y)\, \mathrm{d}\sigma = v(x) + \frac{1}{d} \int_0^r s \left(\fint_{B(x,s)} \Delta v(y)\, \mathrm{d}y \right) \mathrm{d}s.$$

The conclusion thus follows from

$$\Delta v = \mathrm{grad}\,\mathrm{div}\, v - \mathrm{rot}\,\mathrm{rot}\, v. \quad \blacksquare$$

Remark 2.36 As we have already mentioned, one needs a different approach in the case $p \neq 2$ to achieve analogous results. This is because there is no such relation involving $\|\mathrm{div}\, v\|_p, \|\mathrm{rot}\, v\|_p$, and $\|\mathrm{grad}\, v\|_p$ as in Lemma 2.30. Invoking the fundamental solution of the Laplace operator in $\mathbb{R}^3$, which is given by

$$\frac{-1}{4\pi\, |x|} \quad (x \in \mathbb{R}^3 \setminus \{0\}),$$

one can represent any vector field in $\mathcal{D}(\overline{\Omega})^3$ through

$$v(x) = \frac{-1}{4\pi}\, \mathrm{grad}_x \int_\Omega \frac{\mathrm{div}\, v(y)}{|x-y|}\, \mathrm{d}y + \frac{1}{4\pi}\, \mathrm{rot}_x \int_\Omega \frac{\mathrm{rot}\, v(y)}{|x-y|}\, \mathrm{d}y.$$

This representation is (primal in physics) referred to as the *Helmholtz decomposition*. Employing the Calderón-Zygmund inequality (cf. [42]) one can now prove analogous results of Theorem 2.33 in the L^p-framework. For the corresponding statements we refer the reader to Section 1 from [3]. ⋊

■ Inhomogeneous Boundary Conditions

In the following we show that the results of Theorem 2.33 can be extended to the case in which the boundary conditions $v \cdot \mathfrak{n} = 0$ or $v \wedge \mathfrak{n} = 0$ on $\partial\Omega$ are replaced by inhomogeneous ones. Therefore we introduce the following spaces.

DEFINITION 2.37 Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain with compact boundary. For $s \in \mathbb{N}$ we define

$$\begin{aligned} H^s_{\mathfrak{n}}(\operatorname{div}, \operatorname{rot}, \Omega) := \Big\{ & v \in L^2(\Omega)^3 : \\ & \operatorname{div} v \in H^{s-1}(\Omega), \operatorname{rot} v \in H^{s-1}(\Omega)^3, \gamma_{\mathfrak{n}}(v) \in H^{s-1/2,}(\partial\Omega) \Big\} \end{aligned}$$

and

$$\begin{aligned} H^s_{\mathfrak{t}}(\operatorname{div}, \operatorname{rot}, \Omega) := \Big\{ & v \in L^2(\Omega)^3 : \\ & \operatorname{div} v \in H^{s-1}(\Omega), \operatorname{rot} v \in H^{s-1}(\Omega)^3, \gamma_{\mathfrak{t}}(v) \in H^{s-1/2}(\partial\Omega)^3 \Big\}. \end{aligned}$$

We further endow this spaces with the norms

$$\begin{aligned} \|v\|_{H^s_{\mathfrak{n}}} &:= \|v\|_2 + \|\operatorname{div} v\|_{H^{s-1}(\Omega)} + \|\operatorname{rot} v\|_{H^{s-1}(\Omega)^3} + \|\gamma_{\mathfrak{n}}(v)\|_{L^2(\partial\Omega)}, \\ \|v\|_{H^s_{\mathfrak{t}}} &:= \|v\|_2 + \|\operatorname{div} v\|_{H^{s-1}(\Omega)} + \|\operatorname{rot} v\|_{H^{s-1}(\Omega)^3} + \|\gamma_{\mathfrak{t}}(v)\|_{L^2(\partial\Omega)^3}, \end{aligned}$$

for $v \in H^s_{\mathfrak{n}}(\operatorname{div}, \operatorname{rot}, \Omega)$ or $v \in H^s_{\mathfrak{t}}(\operatorname{div}, \operatorname{rot}, \Omega)$ respectively. ⋊

THEOREM 2.38 *Let $s \in \mathbb{N}$, and assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded $C^{s,1}$-domain. Then the spaces $H^s_{\mathfrak{n}}(\operatorname{div}, \operatorname{rot}, \Omega)$ and $H^s_{\mathfrak{t}}(\operatorname{div}, \operatorname{rot}, \Omega)$ are both continuously embedded in $H^s(\Omega)^3$.* ⋊

Proof. The proof heavily depends on the following Helmholtz decomposition of weakly differentiable functions in L^2. For any $u \in H^s(\Omega)^3$, provided $\partial\Omega$ is of class $C^{s,1}$, there is a vector potential $p \in H^{s+1}(\Omega)^3$ and a function $v \in H^s(\Omega)^3$ satisfying $\operatorname{div} v = 0$ and $\gamma_{\mathfrak{n}}(v) = 0$ such that

$$u = \operatorname{grad} p + v.$$

A further inspection of these functions then leads to the desired claim. We refer to Propositions 6 and 6' in [9]. ∎

COROLLARY 2.39 *Let $s \in \mathbb{N}$, and assume that $\Omega \subseteq \mathbb{R}^3$ is a bounded $C^{s,1}$-domain. Then the spaces*

$$\begin{aligned}\mathcal{T}_s &:= \Big\{ v \in L^2(\Omega)^3 : \operatorname{div} v = 0,\ \operatorname{rot}^k v \in L^2(\Omega)^3\ (1 \leqslant k \leqslant s), \\ &\qquad \gamma_{\mathfrak{t}}(v) = 0,\ \gamma_{\mathfrak{t}}(\operatorname{rot}^{2k} v) = 0\ (1 \leqslant k \leqslant [(s-1)/2]) \Big\}, \\ \mathcal{N}_s &:= \Big\{ v \in L^2(\Omega)^3 : \operatorname{div} v = 0,\ \operatorname{rot}^k v \in L^2(\Omega)^3\ (1 \leqslant k \leqslant s), \\ &\qquad \gamma_{\mathfrak{n}}(v) = 0,\ \gamma_{\mathfrak{t}}(\operatorname{rot}^{2k-1} v) = 0\ (1 \leqslant k \leqslant [s/2]) \Big\},\end{aligned}$$

are closed subspaces of $H^s(\Omega)^3$. ⋊

Proof. We prove the assertion by induction. For $s = 1$, we have

$$\begin{aligned}\mathcal{T}_1 &= \{ v \in H_{\mathfrak{t}0}(\operatorname{div}, \operatorname{rot}, \Omega) : \operatorname{div} v = 0 \}, \\ \mathcal{N}_1 &= \{ v \in H_{\mathfrak{n}0}(\operatorname{div}, \operatorname{rot}, \Omega) : \operatorname{div} v = 0 \}.\end{aligned}$$

Hence $\mathcal{T}_1,\ \mathcal{N}_1 \subseteq H^1(\Omega)^3$ due to Theorem 2.33 and the closedness of the divergence operator. If $s > 1$, then

$$\begin{aligned}\mathcal{T}_s &= \{ v \in \mathcal{T}_{s-1} : \operatorname{rot} v \in \mathcal{N}_{s-1} \}, \\ \mathcal{N}_s &= \{ v \in \mathcal{N}_{s-1} : \operatorname{rot} v \in \mathcal{T}_{s-1} \}.\end{aligned}$$

First, let $v \in \mathcal{T}_s$. Owing to the recurrence hypotheses, we know that v and $\operatorname{rot} v$ belong to $H^{s-1}(\Omega)^3$ and satisfy $\operatorname{div} v = 0$ as well as $\gamma_{\mathfrak{t}}(v) = 0$. Thus we infer from Theorem 2.38 that $v \in H^s(\Omega)^3$. Moreover, $\mathcal{T}_s$ is a closed subspace of $H^1(\Omega)^3$ because of the closedness of the divergence- and the tangential trace operator $\gamma_{\mathfrak{t}}$. Now, if $v \in \mathcal{N}_s$, we argue exactly in the same way, only replacing γ_t by $\gamma_{\mathfrak{n}}$. ∎

2.3 Properties of the Maxwell Operator

We are now able to state the main result of this chapter, concerning the differential operators which naturally appear in the analysis of Maxwell's equations. First, we recall some elementary facts about matrix valued functions.

DEFINITION 2.40 We endow $L^\infty(\Omega)^{3\times3}$ with the norm

$$\|\varepsilon\|_\infty := \operatorname*{ess-sup}_{x\in\Omega} |\varepsilon(x)| \quad (\varepsilon \in L^\infty(\Omega)^{3\times3}),$$

where $|\cdot|$ denotes the matrix norm induced by the euclidean norm on $\mathbb{R}^3$. We further say that $\varepsilon \in L^\infty(\Omega)^{3\times3}$ is bounded from below by $\delta \geqslant 0$, and write $\varepsilon \geqslant \delta$ if

$$\varepsilon(x)u \cdot u \geqslant \delta\,|u|^2 \quad \text{for every } u \in \mathbb{R}^3 \text{ and for almost every } x \in \Omega.$$

If we even have a strict inequality, then we say that ε is strictly bounded from below by δ and write $\varepsilon > \delta$. ⋊

Remark 2.41 If $\varepsilon \in L^\infty(\Omega)^{3\times 3}$ is invertible and bounded from below by $\delta > 0$, then ε^{-1} is bounded from below by $\delta \,\|\varepsilon\|_\infty^{-1}$ and we have $\|\varepsilon^{-1}\|_\infty \leqslant \delta^{-1}$. ⋊

Definition 2.42 The *Maxwell operator* in the phase space $L^2(\Omega)^3 \times L^2(\Omega)^3$ is defined by

$$\mathbf{A} \coloneqq \begin{pmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{pmatrix}, \quad D(\mathbf{A}) \coloneqq H_0(\operatorname{rot}, \Omega) \times H(\operatorname{rot}, \Omega). \quad ⋊$$

Theorem 2.43 *Let $\Omega \subseteq \mathbb{R}^3$ be either the full space or a Lipschitz domain with compact boundary and put $X \coloneqq L^2(\Omega)^3 \times L^2(\Omega)^3$. Then we have:*

(a) *If $\varepsilon,\ \mu \in L^\infty(\Omega)^{3\times 3}$ are symmetric and bounded from below by some $\delta > 0$, then the operator*

$$\begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \mathbf{A} = \begin{pmatrix} 0 & \varepsilon^{-1} \operatorname{rot} \\ -\mu^{-1} \operatorname{rot} & 0 \end{pmatrix}$$

endowed with its maximal domain $D(\mathbf{A})$ is skew-adjoint in X with respect to the weighted inner product

$$\Big((E,H) \mid (\tilde{E},\tilde{H})\Big)_{\varepsilon,\mu} \coloneqq \Big(\varepsilon E \mid \tilde{E}\Big)_2 + \Big(\mu H \mid \tilde{H}\Big)_2,$$

which itself is equivalent to the canonical L^2-inner product.

(b) *The space*

$$X_0 \coloneqq \{(E,H) \in X \,:\, \operatorname{div}(E) = \operatorname{div}(H) = 0,\ \gamma_{\mathfrak{n}}(H) = 0\}$$

is a closed subspace of X and $\mathbf{A}$ maps $D(\mathbf{A})$ into X_0, i.e., $\mathbf{A}D(\mathbf{A}) \subseteq X_0$. The resulting restriction $\mathbf{A}_0 \coloneqq \mathbf{A}|_{X_0}$, $D(\mathbf{A}_0) \coloneqq D(\mathbf{A}) \cap X_0$ of $\mathbf{A}$ to X_0 is skew-adjoint in X_0. If in addition Ω is a bounded $C^{1,1}$-domain, then the domain of $\mathbf{A}_0$ endowed with the graph norm of $\mathbf{A}_0$ is a closed subspace of $H^1(\Omega)^3 \times H^1(\Omega)^3$.

(c) *If Ω is a bounded $C^{1,1}$-domain, then the spectrum of $\mathbf{A}_0$ is an imaginary point spectrum, with no finite accumulation point.*

(d) *Let $s \in \mathbb{N}$. If Ω is a bounded $C^{s,1}$-domain, then*

$$D(\mathbf{A}_0^s) = \Big\{(E,H) \in D(\mathbf{A}_0) \,:\, \mathbf{A}x \in D(\mathbf{A}_0^{s-1})\Big\} = D(\mathbf{A}^s) \cap X_0$$

endowed with the graph norm is a closed subspace of $H^s(\Omega)^3 \times H^s(\Omega)^3$. ⋊

Proof. (a) : We prove the assertion by showing that this operator is closed and skew-symmetric in X and that the sum of this operator with $\pm I$ has dense range in X. The closedness is due to the closedness of rot in $L^2(\Omega)^3$ and the continuity of the tangential trace operator γ_t. In the following we denote the considered operator by $\mathbf{A}_{\varepsilon,\mu}$. Aiming for the skew-symmetry, we choose $(E,H),\ (\tilde{E},\tilde{H}) \in D(\mathbf{A})$ and compute, invoking Corollary 2.24 (a),

$$\begin{aligned}\Big(\mathbf{A}_{\varepsilon,\mu}(E,H) \mid (\tilde{E},\tilde{H})\Big)_{\varepsilon,\mu} &= \Big(\varepsilon^{-1} \operatorname{rot} H \mid \tilde{E}\Big)_{\varepsilon} - \Big(\mu^{-1} \operatorname{rot} E \mid \tilde{H}\Big)_{\mu} \\ &= \Big(H \mid \operatorname{rot} \tilde{E}\Big)_2 - \Big(E \mid \operatorname{rot} \tilde{H}\Big)_2 \\ &= -\Big(H \mid -\mu^{-1} \operatorname{rot} \tilde{E}\Big)_{\mu} - \Big(E \mid \varepsilon^{-1} \operatorname{rot} \tilde{H}\Big)_{\varepsilon} \\ &= -\Big((E,H) \mid \mathbf{A}_{\varepsilon,\mu}(\tilde{E},\tilde{H})\Big)_{\varepsilon,\mu}.\end{aligned}$$

Finally, we will show that the dense subset $L^2(\Omega)^3 \times H(\operatorname{rot},\Omega)$ of X is contained in $\operatorname{ran}(I \pm A_{\varepsilon,\mu})$. Thus given $f \in L^2(\Omega)^3$ and $g \in H(\operatorname{rot},\Omega)$, we have to solve the equations

$$E \pm \varepsilon^{-1} \operatorname{rot} H = f, \quad H \mp \mu^{-1} \operatorname{rot} E = g, \tag{$\star$}$$

with unknowns $E \in H_0(\operatorname{rot},\Omega)$ and $H \in H(\operatorname{rot},\Omega)$. Inserting, a priori formally, the second equation of $(\star)$ in the first one, we are now interested in solving

$$\varepsilon E + \operatorname{rot}\Big(\mu^{-1} \operatorname{rot} E\Big) = \varepsilon f \mp \operatorname{rot} g. \tag{$\star\star$}$$

Note, that $h := \varepsilon f \mp \operatorname{rot} g \in L^2(\Omega)^3$ by assumption. Applying test functions φ on $(\star\star)$ and integrating by parts leads to

$$(\varepsilon E \mid \varphi)_2 + (\mu^{-1} \operatorname{rot} E \mid \operatorname{rot} \varphi)_2 = (h \mid \varphi)_2.$$

This motivates us to consider the symmetric bilinear form

$$a(E,u) := (\varepsilon E \mid u)_2 + (\mu^{-1} \operatorname{rot} E \mid \operatorname{rot} u)_2 \quad (E,\ u \in H_0(\operatorname{rot},\Omega)).$$

It is readily seen that a is continuous. For every $E \in H_0(\operatorname{rot},\Omega)$, we further calculate

$$a(E,E) \geqslant \delta \|E\|_2^2 + \frac{\delta}{\|\mu\|_{L^\infty(\Omega)^{3\times 3}}} \|\operatorname{rot} E\|_2^2 \geqslant \delta \min\{1, \|\mu\|^{-1}_{L^\infty(\Omega)^{3\times 3}}\} \|E\|^2_{\operatorname{rot}},$$

which means that a is also coercive. The Lax-Milgram lemma thus provides a vector field $E \in H_0(\operatorname{rot},\Omega)$ such that $a(E,u) = (h \mid u)_2$ for all $u \in H_0(\operatorname{rot},\Omega)$. In particular $(\mu^{-1} \operatorname{rot} E \mid \operatorname{rot} \varphi)_2 = (h - \varepsilon E \mid \varphi)_2$ for all $\varphi \in \mathcal{D}(\Omega)^3$. Since the vector field $h - \varepsilon E$ is square integrable this just means that $\mu^{-1} \operatorname{rot} E$ belongs to $H(\operatorname{rot},\Omega)$ with $\operatorname{rot}(\mu^{-1} \operatorname{rot} E) = h - \varepsilon E$, cf. (2.3). Hence E satisfies equation $(\star\star)$. Putting

$$H := g \pm \mu^{-1} \operatorname{rot} E \in H(\operatorname{rot},\Omega)$$

we have constructed $(E,H) \in D(\mathbf{A})$ such that $(I \pm \mathbf{A}_{\varepsilon,\mu})(E,H) = (f,g)$ and we are done.

(b) : The subspace X_0 is closed in X due to the closedness of div in L^2 and the continuity of the normal trace operator $\gamma_{\mathfrak{n}}$. Now, we will prove that if a vector field E belongs to $H(\mathrm{rot}, \Omega)$, then $\mathrm{rot}\, E$ is already contained in $H(\mathrm{div}, \Omega)$ and satisfies

$$\mathrm{div}(\mathrm{rot}\, E) = 0, \quad \gamma_{\mathfrak{n}}(\mathrm{rot}\, E) = 0.$$

In this sense $\mathrm{div}\,\mathrm{rot} = 0$ is also valid for vector fields whose distributional rotation is square integrable. In particular this proves that $\mathbf{A}\, D(\mathbf{A}) \subseteq X_0$. Let $E \in H(\mathrm{rot}, \Omega)$ and $\varphi \in \mathcal{D}(\Omega)$. Since $\mathrm{grad}\, \varphi \in \mathcal{D}(\Omega)^3$, we calculate

$$(\mathrm{rot}\, E \mid \mathrm{grad}\, \varphi)_2 = (E \mid \mathrm{rot}\,\mathrm{grad}\, \varphi)_2 = 0$$

so that, by means of (2.2), we deduce $\mathrm{rot}\, E \in H(\mathrm{div}, \Omega)$ with $\mathrm{div}(\mathrm{rot}\, E) = 0$. Now, let $\varphi \in H^1(\Omega)$ and choose a sequence $(\varphi_k)_k$ in $\mathcal{D}(\overline{\Omega})$ which converges to φ in $H^1(\Omega)$. Due to the continuity of the trace operator tr, cf. Theorem 2.19, the sequence $(\mathrm{tr}\, \varphi_k)_k$ converges to $\mathrm{tr}\, \varphi$ in $H^{1/2}(\partial\Omega)$. As $\gamma_{\mathfrak{n}}(\mathrm{tr}\, \varphi)$ is contained in $H^{-1/2}(\partial\Omega)$, we conclude

$$\gamma_{\mathfrak{n}}(\mathrm{rot}\, E)(\mathrm{tr}\, \varphi) = \lim_{k\to\infty} \gamma_{\mathfrak{n}}(\mathrm{rot}\, E)(\mathrm{tr}\, \varphi_k).$$

Applying Green's formula 2.5 on each $\gamma_{\mathfrak{n}}(\mathrm{rot}\, E)(\mathrm{tr}\, \varphi_k)$, we obtain

$$\gamma_{\mathfrak{n}}(\mathrm{rot}\, E)(\mathrm{tr}\, \varphi_k) = (\mathrm{rot}\, E \mid \mathrm{grad}\, \varphi_k)_2 + (\mathrm{div}\,\mathrm{rot}\, E \mid \varphi_k)_2 = 0,$$

and therefore $\gamma_{\mathfrak{n}}(\mathrm{rot}\, E)(\mathrm{tr}\, \varphi) = 0$, as claimed. The skew-adjointness of $\mathbf{A}_0$ can be deduced analogously to the procedure in (a). Further, by definition, we can write

$$D(\mathbf{A}_0) = \{E \in H_{\mathfrak{t}0}(\mathrm{div}, \mathrm{rot}, \Omega) \,:\, \mathrm{div}\, E = 0\} \times \{H \in H_{\mathfrak{n}0}(\mathrm{div}, \mathrm{rot}, \Omega) \,:\, \mathrm{div}\, H = 0\}.$$

Thus it follows from Theorem 2.33 and the closedness of the divergence operator that $D(\mathbf{A}_0)$ is a closed subspace of $H^1(\Omega)^3 \times H^1(\Omega)^3$.

(c) : Since Ω is a bounded $C^{1,1}$-domain it follows from Rellich's theorem that the embedding $H^1(\Omega) \to L^2(\Omega)$ is compact. Due to the last assertion in (b), we infer that the embedding $D(\mathbf{A}_0) \to X_0$ is also compact and hence the operator $\mathbf{A}_0$ has a compact resolvent.

(d) : By definition, we have for example

$$\begin{aligned} D(\mathbf{A}_0^2) =& \Big\{E \in L^2(\Omega)^3 \,:\, \mathrm{div}\, E = 0,\ \mathrm{rot}^k E \in L^2(\Omega)^3 \ (1 \leqslant k \leqslant 2), \\ &\qquad \gamma_{\mathfrak{n}}(\mathrm{rot}\, E) = 0,\ \gamma_{\mathfrak{t}}(E) = 0\Big\} \times \\ &\Big\{H \in L^2(\Omega)^3 \,:\, \mathrm{div}\, H = 0,\ \mathrm{rot}^k H \in L^2(\Omega)^3 \ (1 \leqslant k \leqslant 2), \\ &\qquad \gamma_{\mathfrak{t}}(\mathrm{rot}\, H) = 0,\ \gamma_{\mathfrak{n}}(H) = 0\Big\} \\ =& \mathcal{T}_2 \times \mathcal{N}_2, \end{aligned}$$

with $\mathcal{T}_s$, $\mathcal{N}_s$ $(s \in \mathbb{N})$ as in Corollary 2.39. Continuing inductively one sees that actually $D(\mathbf{A}_0^s) = \mathcal{T}_s \times \mathcal{N}_s$. Thus the claim is a direct consequence of Corollary 2.39. ∎

Chapter 3

Evolution Equations

In this chapter we shall investigate the quasilinear evolution equations which arise in the study of nonlinear Maxwell's equations as introduced in Chapter 1. We are thus interested in well-posedness results for Cauchy problems of the form

$$\begin{aligned} \Lambda(u(t))u'(t) &= Au(t) + Q(u(t))u(t) \quad (t \in [0,T]), \\ u(0) &= u_0. \end{aligned} \tag{Q}$$

Here the unknown u takes values $u(t)$ in a Hilbert space. To construct solutions of this nonlinear Cauchy problem we use an approach introduced by T. Kato in [24]. Roughly speaking this means that we want to approach this problem in the following way. For linearization, we fix (certain) functions $t \mapsto \varphi(t)$ such that $\Lambda(\varphi(t))$ is invertible, and consider the resulting linear, but nonautonomous evolution equation

$$\begin{aligned} u'(t) &= \Lambda(\varphi(t))^{-1}\{A + Q(\varphi(t))\}u(t) =: A_\varphi(t)\, u(t) \quad (t \in [0,T]), \\ u(0) &= u_0\,. \end{aligned}$$

If this admits a unique solution u_φ, then we may consider the solution operator $\Phi : \varphi \mapsto u_\varphi$. Now, every φ with $\Phi(\varphi) = \varphi$ is a solution of (Q). We thus have transformed the problem of finding a solution of the Cauchy problem (Q) into searching for fixed points of Φ.

So, the strategy of the upcoming chapter is as follows. First, we will introduce from scratch the basic concepts and ideas for tackling nonautonomous evolution equations in Section 1 and Section 2. These two sections mainly provide a summary of the existing literature working in this field, in principal we have used the early pioneering works of Kato [20–22] and some more later works like [28–31, 36, 39]. The main result is Theorem 3.35, but we also want to mention Lemma 3.30, which is non-standard and facilitates the control of several constants during some exhausting calculations in Section 3. Using these insights, we will solve the above mentioned fixed point problem by invoking the contraction principle on adequate complete metric spaces and gain the existence of solutions for the Cauchy problem (Q). Further, we will extend these basic existence results to the local well-posedness Theorems 3.43 and 3.43, which are in the center of Section 3.

Finally, in Section 4, we address second order evolution equations. By the usual order reduction procedure, we will upgrade the theorems from Section 3 to Theorem 3.45.

Notation. For two normed spaces X, Y we denote the space of bounded linear operators from X to Y by $\mathcal{B}(X, Y)$. Given $R > 0$, we denote by $\overline{B}_X(0, R)$, $\overline{B}_Y(0, R)$ the closed balls of radius R in X or Y respectively.

We assume that the reader is familiar with the basic concepts and most important results concerning autonomous evolution equations and operator semigroups. For an introduction we refer to the monographs [13, 31, 39].

3.1 Linear Nonautonomous Equations

Though we will only consider Hilbert spaces in the later applications, we will work out the following theory on Banach spaces, since it does not make additional work. We first introduce the basic concepts for well-posedness of nonautonomous Cauchy problems

$$\begin{aligned} u'(t) &= A(t)u(t) \quad (t,s \in J,\ t \geqslant s), \\ u(s) &= u_s, \end{aligned} \tag{CP}$$

where $A(\cdot) \coloneqq \{A(t) : D(A(t)) \subseteq X \to X : t \in J\}$ is a family of linear operators on some Banach space X, and J is a non-trivial interval.

Notation 3.1 Let $J \subseteq \mathbb{R}$ be an interval. We then put

$$\Delta_J \coloneqq \{(t,s) \in J \times J : t \geqslant s\}.$$

For $t \in \mathbb{R}$ we define

$$J_{\geqslant t} \coloneqq \{s \in J : s \geqslant t\} = J \cap [t,\infty),$$

and analogously $J_{>t},\ J_{\leqslant t},\ J_{<t}$. ⋊

Definition 3.2 Given $s \in J$ and $u_s \in D(A(s))$, we say that a continuous function $u : J_{\geqslant s} \to X$ is a *naive solution* of the associated Cauchy problem if $u \in C^1(J_{\geqslant s}, X)$, $u(t) \in D(A(t))$ for all $t \in J_{\geqslant s}$, and u solves (CP). ⋊

The definition of well-posedness is not so straightforward as in the autonomous case. However, the following definition, compare [13, 28, 36], seems to be appropriate.

Well-posedness 3.3 The nonautonomous Cauchy Problem (CP) for a family of linear operators $\{A(t) : t \in J\}$ on the Banach space X, is called *well-posed on spaces Y_s* if the following holds:

(a) There are dense subspaces Y_s $(s \in J)$ of X with $Y_s \subseteq D(A(s))$ such that for each $y \in Y_s$ there is a unique naive solution $u(\cdot\,; s, y)$ of (CP) with $u(t\,; s,y) \in Y_t$ for $t \in J_{\geqslant s}$.

(b) If $s_n \to s$ and $y_n \to y$ for $s_n,\ s \in J$ and $y_n \in Y_{s_n},\ y \in Y_s$, then we have

$$\tilde{u}(t\,; s_n, y_n) \to \tilde{u}(t\,; s, y) \quad \text{in } X$$

uniformly for t in compact subsets of J, where $\tilde{u}(t\,; r,y) = u(t\,;r,y)$ if $t \geqslant r$ and $\tilde{u}(t\,; r,y) = y$ if $t \leqslant r$.

In other words, we require that there exists a unique solution for sufficiently many initial values, and that the solutions depend continuously on the initial data.

(c) If in addition there are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|u(t;s,y)\| \leqslant Me^{\omega(t-s)}\|y\|$$

for all $y \in Y_s$ and $t \in J_{\geqslant s}$, then the Cauchy problem is called *well-posed with exponentially bounded solutions.* ⋊

Remark 3.4 Concerning part (a) and (b) of Definition 3.3, we want to emphasise that there are examples (cf. Example 3 in [30]) in which it is not possible to choose $Y_s = D(A(s))$, even if each operator $A(t)$ $(t \in J)$ is the generator of a strongly continuous semigroup on X. ⋊

Now, suppose that (CP) is well-posed. Then we may define

$$U(t,s)y \coloneqq u(t;s,y) \quad \big((t,s) \in \Delta_J,\ y \in Y_s\big).$$

By using the continuous dependence on the data, similar as it is done in the autonomous case, we can extend each $U(t,s)$ to a bounded operator $U(t,s) \in \mathcal{B}(X)$. The resulting family satisfies the following properties (cf. [30], Proposition 3.10.)

$U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = I$ for all $s,r,t \in J$ with $s \leqslant r \leqslant t$,

$\Delta_J \to \mathcal{B}(X),\ (t,s) \mapsto U(t,s)$ is strongly continuous.

This motivates the following definitions.

Definition 3.5 (a) An *evolution family*, or *propagator,* on a Banach space X (with parameter interval J) is a strongly continuous mapping $U : \Delta_J \to \mathcal{B}(X)$ which satisfies the *chain condition*

$$U(t,s) = U(t,r)U(r,s), \quad U(s,s) = I \quad (t \geqslant r \geqslant s \text{ in } J).$$

Further, the evolution family U is called *exponentially bounded* if there are $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|U(t,s)\|_{\mathcal{B}(X)} \leqslant Me^{\omega(t-s)} \quad \big((t,s) \in \Delta_J\big).$$

(b) We say that an evolution family $U(\cdot,\cdot) \coloneqq \{U(t,s) : (t,s) \in \Delta_J\}$ *solves the nonautonomous Cauchy problem on spaces* Y_s, or that $\{A(t) : t \in J\}$ *generates* the evolution family $U(\cdot,\cdot)$ *on spaces* Y_s if there are dense subspaces Y_s $(s \in J)$ of X with $Y_s \subseteq D(A(s))$ such that

$$U(t,s)Y_s \subseteq Y_t \quad (t \in J_{\geqslant s}),$$

and the map $t \mapsto U(t,s)y$ is a naive solution for each $s \in J$ and $y \in Y_s$. ⋊

Remark 3.6 If $\{A(t) : t \in J\}$ generates the evolution family $\{U(t,s) : (t,s) \in \Delta_J\}$ on spaces Y_s, then the map $s \mapsto U(t,s)y$ $(y \in Y_s)$ will already be differentiable from the right for $s \in J_{<t}$ with

$$\frac{\mathrm{d}^+}{\mathrm{d}s^+} U(t,s)y = -U(t,s)A(s)y.$$

For a proof we refer to [30], Lemma 3.9. ⋊

The connection of these two fundamental concepts, i.e., the language of well-posedness as introduced in Definition 3.3, and the theory of evolution families as indicated above, is given through the following proposition, which states that these two things are merely two sides of the same coin.

Proposition 3.7 *The nonautonomous Cauchy problem* (CP) *is well-posed on spaces Y_s if and only if there is an evolution family which solves* (CP) *on the spaces Y_s.* ⋊

Proof. A detailed proof is given in [30], Proposition 3.10. ∎

If $U : \Delta_J \to \mathcal{B}(X)$ is strongly continuous, then it is obviously separately strongly continuous and the uniform boundedness principle yields $\sup_{(t,s)\in C} \|U(t,s)\| < \infty$, for each compact set $C \subseteq \Delta_J$. The following lemma states that the converse is also true, provided U satisfies the chain property.

Lemma 3.8 *Let $U : \Delta_J \to \mathcal{B}(X)$ satisfy the chain property and suppose that:*

(a) *For any $s \in J$ the map $J_{\geqslant s} \to \mathcal{B}(X)$, $t \mapsto U(t,s)$, is strongly continuous.*

(b) *For any $t \in J$ the map $J_{\leqslant t} \to \mathcal{B}(X)$, $s \mapsto U(t,s)$, is strongly continuous at $s = t$.*

(c) *U is locally bounded.*

Then U is strongly continuous and hence an evolution family. ⋊

Proof. Given an arbitrary $(t_0, s_0) \in \Delta_J$, we want to show that for every $x \in X$, $U(t,s)x \to U(t_0,s_0)x$ as $(t,s) \to (t_0,s_0)$. Thus fix $(t_0,s_0) \in \Delta_J$ and put $J_0 := J_{<s_0}$ if $s_0 \neq \inf J$ and $J_0 = \{s_0\}$ else. Condition (b) implies that

$$D := \bigcup_{r \in J_0} U(s_0, r)X$$

is dense in X. Since $U(\cdot,\cdot)$ is locally bounded, it thus suffices to show that $U(t,s)x \to U(t_0,s_0)x$ for all $x \in D$, i.e., for all x of the form $x = U(s_0,r)\tilde{x}$ $(r \in J_0,\ \tilde{x} \in X)$. Now, let $(t_n,s_n) \in \Delta_J$ with $t_n \to t_0$, $s_n \to s_0$, and without loss of generality $s_n \geqslant r$ for all $n \in \mathbb{N}$. Using condition (a) and (c), we conclude

$$\begin{aligned} U(t_n,s_n)x &= U(t_n,s_n)\big(x - U(s_n,r)\tilde{x}\big) + U(t_n,s_n)U(s_n,r)\tilde{x} \\ &= U(t_n,s_n)\big(U(s_0,r)\tilde{x} - U(s_n,r)\tilde{x}\big) + U(t_n,r)\tilde{x} \\ &\longrightarrow 0 + U(t_0,r)\tilde{x} = U(t_0,s_0)U(s_0,r)\tilde{x} = U(t_0,s_0)x\,. \end{aligned}$$ ∎

Remark 3.9 Condition (c) of the above lemma is really needed. Separate strong continuity of a family satisfying the chain property is not sufficient to obtain local boundedness. A counterexample can be found in [41] on page 11. Note the difference to the autonomous case. There the local boundedness of the family $T(\cdot)$ was indeed a consequence of the strong continuity of $t \mapsto T(t)$ at $t = 0$. ⋊

■ A Stronger Solution Concept

Of special interest for us will be the case where there is a common dense (in X) subspace of the domains $D(A(t))$, $t \in J$. More precisely:

Assumption 3.10 Suppose that there is a Banach space $(Y, \|\cdot\|_Y)$ such that $Y \subseteq X$ and the corresponding embedding is continuous and dense. In this situation we say that (CP) is *well-posed on* Y if (CP) is well-posed on the spaces $\{Y_s : s \in J\}$, where $Y_s = Y$. ⋊

Unfortunately, even in this simple situation we still do not know any simple conditions that guarantee the existence of naive solutions. In order to obtain such solutions under reasonable conditions we introduce a stronger concept of solutions, which for example could be motivated by Example 3.12 below and the related constructing concept in the following Section 3.2.

Definition 3.11 Let Y be as in Assumption 3.10. For $u_s \in Y$, a function $u \in C(J_{\geqslant s}, Y) \cap C^1(J_{\geqslant s}, X)$ which satisfies (CP) is called a Y*-valued solution* of (CP). Further, we say that an evolution family $\{U(t,s) : (t,s) \in \Delta_J\}$ *solves the nonautonomous Cauchy problem on* Y, or that $\{A(t) : t \in J\}$ *generates* the evolution family $\{U(t,s) : (t,s) \in \Delta_J\}$ *on* Y if

$$U(t,s)Y \subseteq Y \quad (t \in J_{\geqslant s}),$$

and the map $t \mapsto U(t,s)y$ is a Y-valued solution for each $s \in J$ and $y \in Y_s$. ⋊

Example 3.12 Assume $J = [0, \infty)$ and $A(t) = A$ for all $t \in J$, where A is the generator of a C_0-semigroup. Then $\{A(t) : t \in J\}$ generates the evolution family

$$U(t,s) = e^{(t-s)A} \quad ((t,s) \in \Delta_J)$$

on the space $(Y, \|\cdot\|_Y) = (D(A), \|\cdot\|_A)$. In particular, each solution u of the Cauchy problem

$$u'(t) = Au(t) \quad (t \geqslant s), \quad u(s) = y \in D(A),$$

is a Y-valued solution and it is given by $u(t) = e^{(t-s)A}y$. ⋊

For an inhomogeneity $f \in C(J, X)$, we also study the inhomogeneous Cauchy problem

$$\begin{aligned} u'(t) &= A(t)u(t) + f(t) \quad (t \geqslant s \text{ in } J), \\ u(s) &= y \in Y. \end{aligned} \tag{iCP}$$

Again a function $u \in C(J_{\geqslant s}, Y) \cap C^1(J_{\geqslant s}, X)$ which satisfies (iCP) is called a *Y-valued solution of* (iCP). Similar to the autonomous case if $\{A(t) : t \in J\}$ generates an evolution family, then any Y-valued solution is given by the variation of constants formula.

Theorem 3.13 *Let $f \in C(J, X)$ and assume that $\{A(t) : t \in J\}$ generates an evolution family $\{U(t,s) : (t, s \in \Delta_J)\}$ on Y. Then each Y-valued solution u of* (iCP) *is given through*

$$u(t) = U(t,s)y + \int_s^t U(t,r)f(r)\,\mathrm{d}r.$$

In particular, each Y-valued solution of (iCP) *is unique in this case.* ⋊

Proof. Let $u \in C^1(J_{\geqslant s}, X)$ solve (iCP). From Remark 3.6 we infer that the function $v(t,r) := U(t,r)u(r)$ is differentiable from the right for every r in $J_{<t}$ with

$$\begin{aligned} \frac{\mathrm{d}^+}{\mathrm{d}r^+} v(t,r) &= -U(t,r)A(r)u(r) + U(t,r)A(r)u(r) + U(t,r)f(r) \\ &= U(t,r)f(r). \end{aligned}$$

Thus integrating over $[s,t]$ yields

$$u(t) - U(t,s)y = \int_s^t U(t,s)f(r)\,\mathrm{d}r. \quad \blacksquare$$

3.2 General Construction of an Evolution Family

In the following let $\{A(t) : t \in J = [a,b]\}$ be a family of generators in a Banach space X (i.e., each $A(t)$ generates a strongly continuous semigroup in X) such that there is a Banach space Y which is densely and continuously embedded in X and which satisfies $Y \subseteq D(A(t))$ for all $t \in J$, cf. Assumption 3.10. We want to construct an evolution family $U : \Delta_J \to \mathcal{B}(X)$ that solves

$$\begin{aligned} u'(t) &= A(t)u(t) \quad (t \geqslant s \text{ in } J), \\ u(s) &= y, \end{aligned}$$

on the space Y. To this aim, we pursue an idea going back to Tosio Kato [21]:

Approximate $A(\cdot)$ by step functions $A_{\mathcal{P}}(\cdot)$ for a partition $\mathcal{P} := \{t_0, t_1, \ldots, t_n\}$ of the interval $J = [a,b]$, i.e.,

$$A_{\mathcal{P}}(t) := \sum_{k=1}^{n} \mathbf{I}_{[t_{k-1},t_k)}(t)\, A(t_{k-1}) + \mathbf{I}_{\{b\}}\, A(b),$$

where $\mathbf{I}_M$ denotes the characteristic function of the measurable set $M \subseteq \mathbb{R}$.

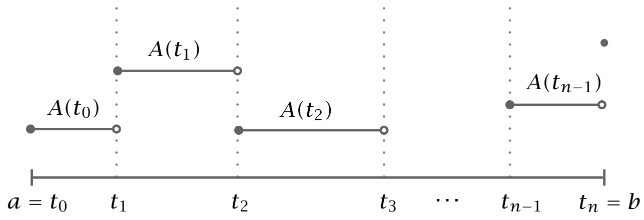

Figure 3: Approximation scheme for a family of operators $A(\cdot)$

Then it follows (under mild additional assumptions) that $A_{\mathcal{P}}(\cdot)$ generates (except of a finite number of values) a unique evolution family $U_{\mathcal{P}} : \Delta_J \to \mathcal{B}(X)$ which is given by

$$U_{\mathcal{P}}(t,s) := e^{(t-s)A(t_{k-1})}, \quad t_{k-1} \leqslant s \leqslant t \leqslant t_k\,,$$
$$U_{\mathcal{P}}(t,s) := e^{(t-t_l)A(t_l)} \prod_{j=k+1}^{l} e^{(t_j-t_{j-1})A(t_{j-1})}\, e^{(t_k-s)A(t_{k-1})}, \quad t_{k-1} \leqslant s < t_k \leqslant t_l < t \leqslant t_{l+1}\,.$$

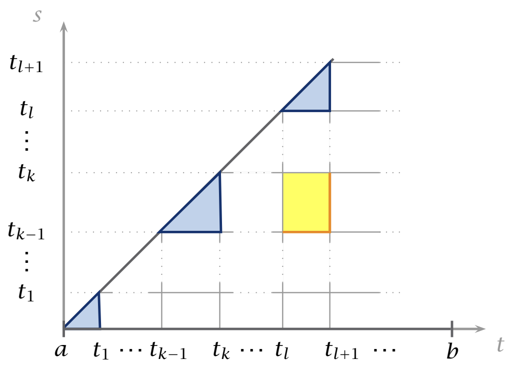

Figure 4: Definition of $U_{\mathcal{P}}(\cdot,\cdot)$ on the triangle Δ_J.

Remark 3.14 The following heuristic might serve as a motivation for the expression of $U_{\mathcal{P}}$: Take a partition $\mathcal{P} = \{a, t_1, b\}$ and assume first that $s \in [a, t_1)$. Then we want to solve the two systems

$$u_0'(t) = A(a)u_0(t) \quad (t \geqslant s,\ t \in [a, t_1]),$$
$$u_0(s) = y\,,$$

and

$$u_1'(t) = A(t_1)u_1(t) \quad (t \geqslant s,\ t \in [t_1, b]),$$
$$u_1(t_1) = u_0(t_1)\,.$$

The candidate for the desired solution according to $A_{\mathcal{P}}(\cdot)$ is then $u = \mathbf{I}_{[a,t_1)}u_0 + \mathbf{I}_{[t_1,b]}u_1$. Using Example 3.12 we obtain

$$u_0(t) = e^{(t-s)A(a)}y \quad (t \geqslant s,\ t \in [a,t_1)),$$
$$u_1(t) = e^{(t-t_1)A(t_1)}u_0(t_1) = e^{(t-t_1)A(t_1)}e^{(t_1-s)A(a)}y \quad (t \geqslant s,\ t \in [t_1,b]).$$

Continuing inductively we shall find the above expressions for $U_{\mathcal{P}}$. ⋊

Now, we start to examine the properties of $U_{\mathcal{P}}$. The mapping $U_{\mathcal{P}} : \Delta_J \to \mathcal{B}(X)$ immediately satisfies

- $U_{\mathcal{P}}(t,r)U_{\mathcal{P}}(r,s) = U_{\mathcal{P}}(t,s), \quad U_{\mathcal{P}}(s,s) = I \quad (t \geqslant r \geqslant s)$ in J.
- $U_{\mathcal{P}}$ is strongly continuous.

Where the latter can easily shown with Lemma 3.8. Hence each $U_{\mathcal{P}}(\cdot,\cdot)$ is an evolution family. In the following we want to find out under which circumstances this evolution family (at least almost everywhere) is generated by $A_{\mathcal{P}}(\cdot)$, and when does the strong limit

$$U(t,s) \coloneqq \operatorname*{s-lim}_{\|\mathcal{P}\|\to 0} U_{\mathcal{P}}(t,s)$$

exists locally uniformly in (t,s). This limit will then serve as the candidate for our desired evolution family generated by $A(\cdot)$. Abbreviate $U_n \coloneqq U_{\mathcal{P}_n}$ for a partition $\mathcal{P}_n \coloneqq \{a, t_1^n, \dots, t_{n-1}^n, b\}$ of $J = [a,b]$.

We expect that the derivatives of $U_n(t,s)$ are given by $A_n(t)$, respective $A_n(s)$, at least for almost every t or s. If this holds, we may calculate

$$\begin{aligned} U_n(t,s)y - U_m(t,s)y &= \int_s^t \frac{\mathrm{d}}{\mathrm{d}r} U_m(t,r)U_n(r,s)y \,\mathrm{d}r \\ &= \int_s^t U_m(t,r)(A_m(r) - A_n(r))U_n(r,s)y \,\mathrm{d}r. \end{aligned}$$

In order to estimate the integral, we need to control arbitrary products of semigroups $e^{s_j A(t_j)}$ relative to a given partition of the interval J. The next definition, again going back to Kato, cf. [21], seems to be adequate.

Definition 3.15 A family of generators $\{A(t) : t \in [a,b]\}$ is called *stable (or Kato-stable)* if there are numbers $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $(\omega,\infty) \subseteq \rho(A(t))$ for all $t \in [a,b]$, and

$$\left\| e^{s_k A(t_k)} e^{s_{k-1}A(t_{k-1})} \dots e^{s_1 A(t_1)} \right\|_{\mathcal{B}(X)} \leqslant M e^{\omega(s_k + s_{k-1} + \dots + s_1)}$$

for all $s_j \geqslant 0$ and all $a \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_k \leqslant b$. In this case we will write

$$A(\cdot) \in \operatorname{stab}(X, M, \omega). \quad ⋊$$

Before continuing we also want to recall the concept of A-admissible subspaces of a Banach space X.

Definition 3.16 Let $A : D(A) \subseteq X \to X$ be a linear operator in a Banach space X. If Y is a subspace of X, then the *part of A in Y* is the linear operator $A|_Y$ given by

$$D(A|_Y) \coloneqq \{x \in D(A) \cap Y : Ax \in Y\}, \quad A|_Y x \coloneqq Ax \quad (x \in D(A|_Y)).$$

If Y is an invariant subspace of A, i.e., $A(D(A) \cap Y) \subseteq Y$, then the part of A coincides with the restriction of A to Y. Now, let A be the generator of a C_0-semigroup $T(\cdot)$. Then, a subspace Y of X is called *A-admissible* if it is an invariant subspace of $T(t)$ for all $t \geqslant 0$, and the restrictions of $T(t)$ to Y again form a C_0-semigroup on Y. ⋊

It is well known, cf. Proposition 2.3 in [21], or Theorem 5.5 in [31] that

Lemma 3.17 *Let A be the generator of a C_0-semigroup $T(\cdot)$ and Y be a subspace of X. Then Y is A-admissible if and only if the following assertions hold.*

(a) There is some $\omega \in \mathbb{R}$ such that Y is an invariant subspace of $R(\lambda, \omega)$ for all $\lambda > \omega$, and

(b) $A|_Y$, the part of A in Y, is the generator of a C_0-semigroup on Y.

In this case $T(\cdot)|_Y$ is generated by $A|_Y$, briefly

$$e^{tA}|_Y = e^{tA|_Y}. \quad ⋊$$

Given a partition $\mathcal{P}_n = \{0, t_1^n, \ldots, t_{n-1}^n, T\}$ of $[0,T]$ let $U_n(t,s) \coloneqq U_{\mathcal{P}_n}(t,s)$ denote the family of operators defined at the beginning of Section 2.2. One can now show, under the conditions $(H1)$-$(H3)$ stated below, that for every $x \in X$ the limit

$$U(t,s)x \coloneqq \lim_{n\to\infty} U_n(t,s)x \tag{3.1}$$

exists uniformly on $0 \leqslant s \leqslant t \leqslant T$, as $n \to \infty$. Theorem 3.18 indicates that the resulting evolution family $U(\cdot,\cdot)$ indeed is a promising candidate for solving the nonautonomous Cauchy problem on Y. In view of their applications of abstract results to partial differential equations, the conditions $(H1)$-$(H3)$ are usually referred to as the *hyperbolic* case. The following theorem is adopted from [31], Theorem 5.3.1.

Theorem 3.18 *Suppose that there is some Banach space $Y \subseteq X$ which is densely and continuously embedded in X. Further assume that the family of generators $\{A(t) : t \in J = [a,b]\}$ satisfies the following conditions:*

(H1) $A(\cdot) \in \mathrm{stab}(X, M, \omega)$.

(H2) *Y is $A(t)$-admissible for any $t \in J$, and the family of parts $A(t)\big|_Y$ is stable in Y, say $A(\cdot)\big|_Y \in \operatorname{stab}(Y, \tilde{M}, \tilde{\omega})$.*

(H3) *$Y \subseteq D(A(t))$ and $A(t) \in \mathcal{B}(Y, X)$ for all $t \in J$, and the map $t \mapsto A(t)$, $J \to \mathcal{B}(Y, X)$, is continuous.*

Then there exists a unique evolution family $U : \Delta_J \to \mathcal{B}(X)$ satisfying

(E1) $\|U(t,s)\|_{\mathcal{B}(X)} \leqslant Me^{\omega(t-s)} \quad ((t,s) \in \Delta_J)$.

(E2) *For every $y \in Y$ and $s \in J$ the map $U(\cdot, s)y : J_{\geqslant s} \to X$, is differentiable from the right at s with*

$$\frac{\mathrm{d}^+}{\mathrm{d}t^+} U(t,s)y\Big|_{t=s} = A(s)y.$$

(E3) *For every $y \in Y$ and $t \in J$ the map $U(t, \cdot)y : J_{\leqslant t} \to X$, is differentiable with*

$$\frac{\mathrm{d}}{\mathrm{d}s} U(t,s)y = -U(t,s)A(s)y \quad (s \in J_{\leqslant t}). \quad \rtimes$$

As a direct consequence we can rephrase Theorem 3.13 for Y-valued solutions of the inhomogeneous Cauchy problem.

THEOREM 3.19 *Let $\{A(t) : t \in J\}$ be a family of generators in X which satisfies the assumptions $(H1)$-$(H3)$ of Theorem 3.18 and let $f \in C(J, X)$. Then any Y-valued solution u of (iCP) is given through*

$$u(t) = U(t,s)y + \int_s^t U(t,r)f(r)\,\mathrm{d}r\,.$$

In particular, each Y-valued solution of (iCP) is unique in this case. $\rtimes$

Proof. See Theorem 4.2 in [31]. ∎

Before we continue studying generation theorems concerning evolution equations, we try to find suitable conditions implying assumptions $(H1)$-$(H3)$ for a broad range of applications. First, we give a characterization of the Kato stability in terms of resolvents.

LEMMA 3.20 *A family $\{A(t) : t \in [a,b]\}$ is stable if and only if one of the following assertions hold.*

(a) *There are constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A(t))$ for all $t \in J$ and the estimate*

$$\|R(\lambda, A(t_k))R(\lambda, A(t_{k-1}))\dots R(\lambda, A(t_1))\|_{\mathcal{B}(X)} \leqslant \frac{M}{(\lambda-\omega)^k}$$

holds for all $\lambda > \omega$ and all $a \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_k \leqslant b$.

(b) *There are constants $M \geqslant 1$, $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A(t))$ for all $t \in J$ and the estimate*

$$||R(\lambda_k, A(t_k))R(\lambda_{k-1}, A(t_{k-1})) \dots R(\lambda_1, A(t_1))||_{\mathcal{B}(X)} \leqslant \frac{M}{\prod_{j=1}^{k}(\lambda_j - \omega)}$$

holds for all $\lambda_j > \omega$ and all $a \leqslant t_1 \leqslant t_2 \leqslant \dots \leqslant t_k \leqslant b$. ⋊

Proof. Proofs can be found in [21], Proposition 3.3, or [31], Theorem 5.2.2. ∎

If for some $\omega \in \mathbb{R}$ we know that $\omega I - A(t)$ generates a C_0-semigroup of contractions for all $t \in J$ (which is usually denoted by $A(t) \in G(X, 1, \omega)$), then $A(\cdot)$ clearly belongs to $\operatorname{stab}(X, 1, \omega)$. Unfortunately, this condition is in general too restrictive for the applications we are interested in. But it can be relaxed by the following lemma, which indeed seems to be the only practical way to show that a family $\{A(t) : t \in J\}$ is stable. Because of its importance for our main results, and since the proof of this lemma is merely sketched in the literature we provide a detailed proof.

Lemma 3.21 *Suppose that for each $t \in J := [a, b]$ there is a norm $\|\cdot\|_t$ on X, equivalent to $\|\cdot\|$ with constants $k_t,\ K_t > 0$, i.e.,*

$$k_t \, \|x\|_t \leqslant \|x\| \leqslant K_t \, \|x\|_t \quad (x \in X),$$

such that the family $\{\|\cdot\|_t : t \in J\}$ depends smoothly on t in the sense that

$$\|x\|_t \leqslant e^{c|t-s|} \, \|x\|_s \quad (x \in X,\ t, s \in J)$$

for some constant c. Denote by X_t the space X endowed with $\|\cdot\|_t$. Now, if for some $\omega \in \mathbb{R}$ we can show $A(t) \in G(X_t, 1, \omega)$ $(t \in J)$, then

$$A(\cdot) \in \operatorname{stab}\Big(X, \frac{K_t}{k_t} e^{2c \max J}, \omega\Big) \quad \text{for any } t \in J. \quad ⋊$$

Proof. First, we will show that $A(\cdot) \in \operatorname{stab}(X_t, e^{2c \max J}, \omega)$ for any $t \in J$. Since $A(t) \in G(X_t, 1, \omega)$, and the norm $\|\cdot\|_t$ is equivalent to the one on X, we know that each $A(t)$ is a generator on X and that $(\omega, \infty) \subseteq \rho(A(t))$ for all $t \in J$. Let $b \geqslant t_k \geqslant \dots \geqslant t_1 \geqslant a$ and $t \in J$ be arbitrary. For every $x \in X$ and $\lambda > \omega$ we obtain the estimate

$$\begin{aligned}
&\|R(\lambda, A(t_k)) \dots R(\lambda, A(t_1))x\|_{t_k} \\
&\qquad \leqslant (\lambda - \omega)^{-1} ||R(\lambda, A(t_{k-1})) \dots R(\lambda, A(t_1))x||_{t_k} \\
&\qquad \leqslant (\lambda - \omega)^{-1} \, e^{c(t_k - t_{k-1})} \, ||R(\lambda, A(t_{k-1})) \dots R(\lambda, A(t_1))x||_{t_{k-1}} \\
&\qquad \vdots \\
&\qquad \leqslant (\lambda - \omega)^{-k} \, e^{c(t_k - t_1)} \, \|x\|_{t_1}.
\end{aligned}$$

Thus for $t \in J$ from above we get

$$\begin{aligned}
&\|R(\lambda, A(t_k)) \dots R(\lambda, A(t_1))x\|_t \\
&\qquad \leqslant e^{c|t-t_k|} \, (\lambda - \omega)^{-k} \, e^{c(t_k - t_1)} \, \|x\|_{t_1} \\
&\qquad \leqslant e^{c(|t-t_k| + t_k - t_1 + |t_1 - t|)} \, (\lambda - \omega)^{-k} \, \|x\|_t.
\end{aligned}$$

We consider the three cases $t \geqslant t_k$, $t_k \geqslant t \geqslant t_1$ and $t_1 \geqslant t$, where

$$\begin{aligned}
|t - t_k| + t_k - t_1 + |t_1 - t| &= 2(t - t_1) \;\; \leqslant 2 \max J, \\
|t - t_k| + t_k - t_1 + |t_1 - t| &= 2(t_k - t_1) \leqslant 2 \max J, \\
|t - t_k| + t_k - t_1 + |t_1 - t| &= 2(t_k - t) \;\; \leqslant 2 \max J,
\end{aligned}$$

respectively, so that

$$\|R(\lambda, A(t_k)) \dots R(\lambda, A(t_1))x\|_t \leqslant e^{2c \max J} \, (\lambda - \omega)^{-k} \, \|x\|_t.$$

The above claim now follows from Lemma 3.20 (b). Finally, calculating

$$\begin{aligned}
&\|R(\lambda, A(t_k)) \dots R(\lambda, A(t_1))x\| \\
&\qquad \leqslant K_t \, \|R(\lambda, A(t_k)) \dots R(\lambda, A(t_1))x\|_t \\
&\qquad \leqslant K_t \, e^{2c \max J} \, (\lambda - \omega)^{-k} \, \|x\|_t \\
&\qquad \leqslant \frac{K_t}{k_t} \, e^{2c \max J} \, (\lambda - \omega)^{-k} \, \|x\| \, ,
\end{aligned}$$

we achieve the desired statement of the lemma. ∎

The following perturbation result for stable families is also an encouraging criterion for verifying stability. It is an extension of the well known bounded perturbation theorem for C_0-semigroups (see Theorem III.1.3 in [13]).

Theorem 3.22 *Suppose* $A(\cdot) \in \operatorname{stab}(X, M, \omega)$ *and let* $B(t) \in \mathcal{B}(X)$ $(t \in J)$ *with* $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$ *for some* $b > 0$*. Then*

$$A + B(\cdot) \in \operatorname{stab}(X, M, \omega + bM) \, . \quad \rtimes$$

Proof. See for example Theorem 5.2.3 in [31]. ∎

We sketch a typical situation, in which we want to apply these results. Suppose that we want to verify the stability of a family of operators taking the form $A(t) + B(t)$ $(t \in J)$. Assume further that $A(\cdot) \in G(X_t, 1, \omega)$ for a family of equivalent norms $\|\cdot\|_t$ as in Lemma 3.21, and that $B(t) \in \mathcal{B}(X)$ with $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$ for some positive constant b. Applying first Lemma 3.21, and then Theorem 3.22, we derive

$$A + B(\cdot) \in \operatorname{stab}(X, \frac{K_t}{k_t} e^{2c \max J}, \omega + b \, \frac{K_t}{k_t} e^{2c \max J}).$$

As stated in the introduction, a precise knowledge of such stability results would be desirable for sharp estimates in the later quasilinear problems. To this aim, we think that it is worth mentioning that the annoying exponent $\omega + b\,\frac{K_t}{k_t}e^{2c\max J}$ is only a consequence of the manner of applying the above results. Exactly as in Lemma 3.21, we obtain a much smaller exponent.

Lemma 3.23 *As in Lemma 3.21 suppose that for each $t \in J \coloneqq [a,b]$ there is a norm $\|\cdot\|_t$ on X, which is equivalent to $\|\cdot\|$ with constants $k_t,\ K_t > 0$ such that the family $\{\|\cdot\|_t : t \in J\}$ depends smoothly on t, i.e.,*

$$\|x\|_t \leqslant e^{c|t-s|}\,\|x\|_s \quad (x \in X,\ t,s \in J)$$

for some constant c. Denote by X_t the space X endowed with $\|\cdot\|_t$. Assume further that $A(t) \in G(X_t, 1, \omega)$ $(t \in J)$ for some $\omega \in \mathbb{R}$, and suppose that there are bounded, linear operators $B(t)$ $(t \in J)$ and some constant b such that $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$. Then

$$A + B(\cdot) \in \operatorname{stab}(X, \frac{K_t}{k_t}e^{2c\max J}, \omega + b). \quad \rtimes$$

We now want to analyse the condition $(H2)$ of Theorem 3.18. The crucial part here is already to decide whether a given Banach space $Y \subseteq X$ is A-admissible for some generator A, or not. So, assume that $(Y, \|\cdot\|_Y)$ is a Banach space which is continuously and densely embedded in X. From Lemma 3.17 we already know that Y is A-admissible if and only if the Cauchy problem

$$v'(t) = Av(t) \quad (t \geqslant 0),$$
$$v(0) = y,$$

is well-posed in Y. Suppose that there is an isomorphism S of the spaces Y and X, i.e., a continuous mapping $S : Y \to X$ which is onto and one-to-one. Then we can consider the new coordinates

$$u(t) \coloneqq Sv(t) \in X.$$

Differentiation yields

$$u'(t) = Sv'(t) = SAv(t) = SAS^{-1}\,u(t) \quad (t \geqslant 0)$$

and of course

$$u(0) = Sy,$$

which is now an evolution equation in X. These considerations motivate the following result.

Proposition 3.24 *Let A be the generator of a C_0-semigroup on X, and suppose that the Banach space $Y \subseteq X$ is continuously and densely embedded in X. Assume* *further that there is an isomorphism S from Y onto X. Then Y is A-admissible if and only if SAS^{-1} on its maximal domain*

$$D(SAS^{-1}) = \left\{x \in X : S^{-1}x \in D(A),\ AS^{-1}x \in Y\right\},$$

is the generator of a C_0-semigroup on X. In this case

$$e^{tSAS^{-1}} = S\, e^{tA}|_Y\, S^{-1}. \quad \rtimes$$

Proof. See Theorem 4.5.8 in [31]. ∎

Corollary 3.25 *Let A be the generator of a C_0-semigroup on X and suppose that the Banach space $Y \subseteq X$ is continuously and densely embedded in X. Assume further that there is an isomorphism S from Y onto X. If there is an operator $B \in \mathcal{B}(X)$ such that*

$$D(SAS^{-1}) = D(A), \quad SAS^{-1}x = Ax + Bx \quad (x \in D(A)),$$

then Y is A-admissible. $\rtimes$

Finally, the following lemma allows us to verify whether a given subspace Y of X is A-admissible or not, by estimating the commutator of A and S on a "nice" subset of $D(A)$. Again, since there is actually no proof in the cited literature we will provide one here.

Lemma 3.26 *Let A be the generator of a C_0-semigroup on X, and suppose that the Banach space $Y \subseteq X$ is continuously and densely embedded in X. Further assume that there is an isomorphism S from Y onto X. If there is a core $D \subseteq D(A)$ for A such that*

(a) $D \subseteq D(SAS^{-1})$,

(b) $\|SAS^{-1}x - Ax\|_X \leqslant c\,\|x\|_X \quad (x \in D,$ *for some positive constant* $c)$,

then Y is A-admissible. $\rtimes$

Proof. The assertions allow us to extend $SAS^{-1} - A$ (defined on D) to a bounded operator B on X. Of course, $SAS^{-1}x = Ax + Bx$ for all $x \in D$. We will show that already $D(SAS^{-1}) = D(A)$ and $SAS^{-1}x = Ax + Bx$ for all $x \in D(A)$. First, let $x \in D(A)$. Since D is a core for A there is a sequence $(x_n)_n$ in D such that $x_n \to x$ and $Ax_n \to Ax$ in X. The continuity of B and S^{-1} yields $Bx_n \to Bx$ in X and $S^{-1}x_n \to S^{-1}x$ in Y. Because Y is continuously embedded in X, we also obtain $S^{-1}x_n \to S^{-1}x$ in X. Further, for the core elements x_n we see

$$AS^{-1}x_n = S^{-1}(Ax_n + Bx_n) \to S^{-1}(Ax + Bx) \quad \text{in } Y,$$

and again, since the norm in Y is stronger then the norm X, this limit also exists in X. Since A is closed, it thus follows that $S^{-1}x \in D(A)$ and $AS^{-1}x = S^{-1}(Ax + Bx) \in Y$ which means $x \in D(SAS^{-1})$ and $SAS^{-1}x = Ax + Bx$. For the remaining inclusion, we take $\lambda \in \rho(A)$. For every $x \in D(SAS^{-1})$ we have

$$S(\lambda I_X - A)S^{-1}x = (\lambda I_X - SAS^{-1})x,$$

and hence

$$[SR(\lambda, A)S^{-1}](\lambda I_X - SAS^{-1})x = x.$$

Conversely, let $x \in X$. Clearly,

$$S^{-1}[SR(\lambda, A)S^{-1}]x = R(\lambda, A)S^{-1}x \in D(A)$$

and $(\lambda - A)S^{-1}[SR(\lambda, A)S^{-1}] = S^{-1}x$. We then obtain

$$AS^{-1}[SR(\lambda, A)S^{-1}]x = -S^{-1}x + \lambda S^{-1}[SR(\lambda, A)S^{-1}]x \in Y\,.$$

Hence we have shown that $SR(\lambda, A)S^{-1}x \in D(SAS^{-1})$ so that we may calculate

$$(\lambda I_X - SAS^{-1})[SR(\lambda, A)S^{-1}]x = S(\lambda I_X - A)S^{-1}[SR(\lambda, A)S^{-1}]x = x.$$

Therefore $\lambda \in \rho(SAS^{-1})$ and $R(\lambda, SAS^{-1}) = SR(\lambda, A + B)S^{-1}$. Since B is a bounded perturbation of A, also $A + B$ generates a C_0-semigroup and therefore $\rho(A + B) \cap \rho(A) \neq \emptyset$. As a result, the larger set $\rho(A + B) \cap \rho(SAS^{-1})$ is not empty, which finally implies that $SAS^{-1} = A + B$. Thus the lemma is a consequence of Corollary 3.25. ∎

Corollary 3.27 *Let the assumptions of Lemma 3.26 hold, and let $D \subseteq D(A)$ be a core for A such that*

(a) $S^{-1}D \subseteq D(SA) \cap D(AS)$,

(b) $\|(SA - AS)S^{-1}x\|_X \leqslant c\,\|x\|_X \quad (x \in D,$ *for some positive constant* $c)$.

Then Y is A-admissible. ⋊

Proof. Assertion (a) implies $D \subseteq D(SAS^{-1})$ and then (b) also yields $\|SAS^{-1}x - Ax\|_X \leqslant c\,\|x\|_X$ so that we can apply Lemma 3.26. ∎

Remark 3.28 Instead of verifying (b) in Corollary 3.27, it is already sufficient to show that

$$\left\|(AS - SA)S^{-1}x\right\|_X \leqslant c\,\|S^{-1}x\|_X \quad (x \in D, \text{ for some positive constant } c),$$

since automatically $\|S^{-1}x\|_X \leqslant c_{\mathrm{em}}\|S^{-1}x\|_Y \leqslant c_{\mathrm{em}}\|S^{-1}\|_{Y,X}\|x\|_X$. Denoting by

$$[S, A] \coloneqq SA - AS, \quad D([S, A]) = D(SA) \cap D(AS)$$

the *commutator* of S and A, we have thus shown that the commutator estimate

(a) $S^{-1}D \subseteq D([S,A])$,

(b) $\|[S,A]u\|_X \leqslant c\,\|u\|_X \quad (u \in S^{-1}D,$ for some positive constant $c)$,

for some core D of A implies that Y is A-admissible. ⋊

We will use these facts to analyse the condition $(H2)$ for a given operator family $\{A(t) : t \in J\}$. There are basically two possibilities to generalize the above results to this nonautonomous situation. We may assume that for each $A(t)$ $(t \in J)$ there is some isomorphism $S(t)$ of Y onto X, or that there is some uniform isomorphism S of Y onto X. We will start with the latter possibility.

Lemma 3.29 *Suppose $A(\cdot) \in \operatorname{stab}(X,M,\omega)$. If there is an isomorphism S of Y onto X and if there are $B(t) \in \mathcal{B}(X)$ with $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$ for some positive b such that*

$$D(SA(t)S^{-1}) = D(A(t)),$$
$$SA(t)S^{-1}x = A(t)x + B(t)x \quad (t \in J,\ x \in D(A(t))),$$

then Y is $A(t)$-admissible for every $t \in J$ and

$$A(\cdot)\big|_Y \in \operatorname{stab}(Y,\ \|S\|\,\|S^{-1}\|\,M,\ \omega + bM)\,.$$

In particular, $(H2)$ holds. ⋊

Proof. It is an immediate consequence of Corollary 3.25 that Y is $A(t)$-admissible for each $t \in J$. From Theorem 3.24 we also know that

$$e^{\tau A(t)}\big|_Y = S^{-1}e^{\tau SA(t)S^{-1}}S \quad (t \in J,\ \tau \geqslant 0).$$

Further, the family $SA(\cdot)S^{-1} = A(\cdot) + B(\cdot)$ is stable in X with stability constants M and $\omega + bM$, by Theorem 3.22. Let $t_k \geqslant \ldots \geqslant t_1$ and $s_j \geqslant 0$. We then conclude

$$\begin{aligned}\left\|e^{s_k A(t_k)}\big|_Y \ldots e^{s_1 A(t_1)}\big|_Y\right\|_{\mathcal{B}(Y)} &= \left\|S^{-1}e^{s_k SA(t_k)S^{-1}}S \ldots S^{-1}e^{s_1 SA(t_1)S^{-1}}S\right\|_{\mathcal{B}(Y)} \\ &= \left\|S^{-1}e^{s_k SA(t_k)S^{-1}} \ldots e^{s_1 SA(t_1)S^{-1}}S\right\|_{\mathcal{B}(Y)} \\ &\leqslant \|S^{-1}\|\,\|e^{s_k(A(t_k)+B(t_k))} \ldots e^{s_1(A(t_1)+B(t_1))}\|_{\mathcal{B}(X)}\|S\|\,,\end{aligned}$$

which closes the proof. ∎

Again, in the situation of Lemma 3.23 we can obtain a sharper constant than $\omega + bM$.

Lemma 3.30 *As in Lemma 3.21 suppose that for each $t \in J := [a,b]$ there is a norm $\|\cdot\|_t$ on X, equivalent to $\|\cdot\|$ with constants $k_t,\ K_t > 0$ such that the family $\{\|\cdot\|_t : t \in J\}$ depends smoothly on t, i.e.,*

$$\|x\|_t \leqslant e^{c|t-s|}\,\|x\|_s \quad (x \in X,\ t,s \in J),$$

for some constant c. Denote by X_t the space X endowed with $\|\cdot\|_t$. Assume futher that $A(t) \in G(X_t, 1, \omega)$ $(t \in J)$ for some $\omega \in \mathbb{R}$. If there is an isomorphism S of Y onto X, and if there are $B(t) \in \mathcal{B}(X)$ with $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$ for some positive b such that

$$D(SA(t)S^{-1}) = D(A(t)),$$
$$SA(t)S^{-1}x = A(t)x + B(t)x \quad (t \in J,\ x \in D(A(t))),$$

then Y is $A(t)$-admissible for every $t \in J$ and

$$A(\cdot)\big|_Y \in \mathrm{stab}(Y,\ \|S\|\,\|S^{-1}\|\,\frac{K_t}{k_t} e^{2c\max J},\ \omega + b)\,.$$

In particular, $(H2)$ holds.

Of course all the corollaries from the autonomous case carry over to the nonautonomous case, we will only formulate one of these as an example.

Corollary 3.31 *Suppose $A(\cdot) \in \mathrm{stab}(X, M, \omega)$ and that there is an isomorphism S of Y onto X. If for each $t \in J$ there is core $D(t)$ for $A(t)$ such that*

(a) $S^{-1}D(t) \subseteq D([S, A(t)])$,

(b) $\|[S, A(t)]u\|_X \leqslant c\,\|u\|_X \quad (u \in S^{-1}D(t),$ *for some uniform constant* $c)$,

then $(H2)$ is satisfied and $A(\cdot)\big|_Y \in \mathrm{stab}(Y,\ \|S\|\,\|S^{-1}\|\,M,\ \omega + c_{\mathrm{em}}\|S^{-1}\|M)$. ⋊

We consider the case in which there is a family of isomorphisms $S(t)$ $(t \in J)$ from Y to X. To obtain $A(t)$-admissibility for every $t \in J$, one needs additional assumptions concerning the map $t \mapsto S(t)$.

Lemma 3.32 *Suppose $A(\cdot) \in \mathrm{stab}(X, M, \omega)$. If for each $t \in J$ there is an isomorphism $S(t)$ of Y onto X such that*

$$\|S(t)\|_{\mathcal{B}(Y,X)},\ \|S(t)^{-1}\|_{\mathcal{B}(X,Y)} \leqslant c \quad (t \in J,\ \text{for some constant } c),$$
$$J \to \mathcal{B}(Y, X),\ t \mapsto S(t) \text{ is of bounded variation,}$$

and if there are $B(t) \in \mathcal{B}(X)$ with $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$ for some positive b such that

$$D(S(t)A(t)S(t)^{-1}) = D(A(t)),$$
$$S(t)A(t)S(t)^{-1}x = A(t)x + B(t)x \quad (t \in J,\ x \in D(A(t))),$$

then Y is $A(t)$-admissible for every $t \in J$ and

$$A(\cdot)\big|_Y \in \mathrm{stab}(Y,\ c^2\,M\,e^{cM\mathrm{var}(S)},\ \omega + bM)\,.$$

In particular, $(H2)$ holds. ⋊

Proof. We can copy the proof of Lemma 3.29 by replacing S with $S(t)$ until the estimation of the products $\prod_j S(t_j)^{-1} e^{s_j(A(t_j)+B(t_j))} S(t_j)$, since now, because of the time dependence, we have no cancellation of the isomorphism. The remaining part of the proof can for example be found in [21], Proposition 4.4. ∎

We now return to the question under which additional assumptions to $(H1)$-$(H3)$ we can achieve that the evolution family $U(\cdot,\cdot)$ in Theorem 3.18 is actually generated by $A(\cdot)$ on Y, where we will invoke the above conditions for $(H2)$. We first observe that it remains to establish the invariance of Y under the family $U(t,s)$ $((t,s) \in \Delta_J)$.

Lemma 3.33 *Let $\{A(t) : t \in J = [a,b]\}$ satisfy the conditions $(H1)$-$(H3)$ of Theorem 3.18, and let $\{U(t,s) : (t,s) \in \Delta_J\}$ be the corresponding evolution family given in Theorem 3.18. If in addition*

(E4) $U(t,s)Y \subseteq Y \quad ((s,t) \in \Delta_J)$,

(E5) for any $y \in Y$ the map $\Delta_J \to Y,\ (t,s) \mapsto U(t,s)y$ is continuous,

then $\{U(t,s) : (t,s) \in \Delta_J\}$ is generated by $\{A(t) : t \in J\}$ on Y. ⋊

Proof. We refer to [31], Theorem 5.4.3. ∎

Thus we only need to look for further assumptions on $A(\cdot)$ that guarantees that $(E4)$ and $(E5)$ are satisfied.

■ If Y is a Hilbert Space

By means of the Banach-Alaoglu theorem one can show (cf. [21] Theorem 5.1) that for reflexive Y, the evolution family from Theorem 3.18 already leaves Y invariant and is weakly continuous in Y. Now, let Y be a Hilbert space. If we further assume that the stability condition $(H2)$ originates by a family of equivalent norms on Y, as stated in Lemma 3.21, then we obtain the following theorem, which states that (up to a countable number of exceptions) the evolution family constructed in Theorem 3.18 solves the nonautonomous Cauchy problem (CP) on Y.

Theorem 3.34 *Let X be a Hilbert space and suppose that there is another Hilbert space $Y \subseteq X$ which is densely and continuously embedded in X. Assume further that the family of generators $\{A(t) : t \in J = [0,T]\}$ satisfies $(H1), (H2), (H3)$ and*

(H4) For each $t \in J$ there is an inner product $(\cdot \mid \cdot)_{Y,t}$ on Y such that $\|\cdot\|_{Y,t}$ is equivalent to $\|\cdot\|_Y = \sqrt{(\cdot \mid \cdot)_Y}$ and the resulting family $\{\|\cdot\|_{Y,t} : t \in J\}$ depends smoothly on t, i.e., there is some constant c such that

$$\|u\|_{Y,t} \leqslant e^{c|t-s|} \|u\|_{Y,s} \quad (u \in Y,\ t,s \in J).$$

The semigroup $(e^{\tau A(t)})_{\tau \geqslant 0}$ is quasi contractive on $(Y, \|\cdot\|_{Y,t})$ for every $t \in J$.

Then the evolution family $\{U(t,s) : (t,s) \in \Delta_J\}$ *from Theorem 3.18 satisfies, in addition to* $(E1)$-$(E3)$*:*

- $U(t,s)Y \subseteq Y$, $\|U(t,s)\|_{\mathcal{B}(Y)} \leqslant \tilde{M}e^{\tilde{\omega}(t-s)}$, *and* $(t,s) \mapsto U(t,s)$ *is weakly continuous in* Y.
- *For each* $t \in J$ *and* $y \in Y$ *the mapping* $J_{\leqslant t} \to Y$, $s \mapsto U(t,s)y$, *is continuous.*
- *For each* $s \in J$ *and* $y \in Y$ *the mapping* $J_{\geqslant s} \to Y$, $s \mapsto U(t,s)y$, *is right-continuous, and continuous except possibly for countable values of* t.
- *For each* $s \in J$ *and* $y \in Y$ *the derivative of the mapping* $J_{\geqslant s} \to Y$, $s \mapsto U(t,s)y$, *exists except possibly for countable values of* t, *equals* $-A(t)U(t,s)y$, *and is continuous in* X *with similar exceptions.* ⋊

Proof. See [21], Theorem 5.2. ∎

In Remark 5.3 of [21], Kato describes one way to overcome these annoying (possible) exceptions in the above theorem, which will indeed fit into the setting of our desired applications. He assumed that the *time reversed family* $A_\sigma(\cdot)$ (where $A_\sigma(t) \coloneqq -A(T-t)$ $(t \in [0,T])$) also satisfies the conditions of the above theorem. The following result will be crucial for our investigations of quasilinear problems in the next section.

Theorem 3.35 *Let* X *be a Hilbert space and suppose that there is another Hilbert space* $Y \subseteq X$ *which is densely and continuously embedded in* X. *Assume further that the family of generators* $\{A(t) : t \in J = [0,T]\}$ *as well as* $\{-A(T-t) : t \in J\}$ *satisfy* $(H1), (H2), (H3)$ *and* $(H4)$. *Then there exists a unique evolution family* $U : \Delta_J \to \mathcal{B}(X)$ *satisfying the following assertions:*

(a) $U(t,s)Y \subseteq Y$ *for each* $(t,s) \in \Delta_J$.

(b) *For each* $s \in [0,T]$ *and each* $y \in Y$, *the map* $t \mapsto U(t,s)y$ *belongs to the space* $C([s,T],Y) \cap C^1([s,T],X)$ *and solves the nonautonomous Cauchy problem* (CP) *with initial value* y. *In particular, the derivative*

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t,s)y = A(t)U(t,s)y$$

exists in X.

(c) *For each* $s \in [0,T]$ *and each* $y \in Y$ *the derivative*

$$\frac{\mathrm{d}}{\mathrm{d}s}U(t,s)y = -U(t,s)A(s)y$$

exists in X *and is continuous in* (t,s).

(d) *The evolution family satisfies the following estimates*

$$\|U(t,s)\|_{\mathcal{B}(X)} \leqslant M\, e^{\omega(t-s)}, \quad \|U(t,s)\|_{\mathcal{B}(Y)} \leqslant \tilde{M}\, e^{\tilde{\omega}(t-s)} \quad ((t,s) \in \Delta_J),$$

where M, ω *and* $\tilde{M}, \tilde{\omega}$ *are the stability constants of assumption* $(H1)$ *and* $(H2)$ *respectively.*

In particular, $U(\cdot,\cdot)$ *is generated by* $A(\cdot)$ *in* Y. ⋊

Proof. Combining Lemma 3.33 and Theorem 3.34, it only remains to show that $\Delta_J \to Y$, $(t,s) \mapsto U(t,s)y$ is continous for every initial value $y \in Y$. Let $y \in Y$. From Theorem 3.34 we further know that $s \mapsto U(t,s)y$ and $t \mapsto U(t,s)y$ are continuous from the right and that $U(\cdot,\cdot)$ is locally bounded in $\mathcal{B}(Y)$. Thus $(t,s) \mapsto U(t,s)y$ is right continuous due to Lemma 3.8. So, we will prove Theorem 3.35 by showing that $(t,s) \mapsto U(t,s)y$ is also left continuous. Applying Theorem 3.35 to the time reversed family $A_\sigma(\cdot)$ we obtain an evolution family $\Delta_J \to Y$, $(t,s) \mapsto V(t,s)y$ which is also continuous from the right. We will show that

$$V(t,s)y = U(T-t, T-s)y \quad ((t,s) \in \Delta_J), \tag{$\star$}$$

which immediately gives the claim. Let $\mathcal{P}_n$ be a partition of the interval $[a,b]$ and let U_n and V_n denote the evolution families approximating U and V from (3.1). It is readily seen by rescaling that

$$V_n(t,s)y = U_n(T-t, T-s)y \quad ((t,s) \in \Delta_J),$$

and therefore $(\star)$ follows by approximation. ∎

■ If Y is a General Banach Space

In the Hilbert space scenario the application of Theorem 3.35 to the linearizations of Maxwell's equations and the quasilinear wave equation will depend, besides the existence of a smoothly depending family of equivalent norms, basically on the fact that the operator A even generates a C_0-group and that we are working on interpolation spaces. Since we will work out theorems which cover somewhat more general situations, at least in this abstract framework, we will also state the corresponding generation theorems taken from [22].

To give an idea how to proceed here, we suppose that there is an isomorphism S of Y onto X such that

$$SA(t)S^{-1} = A(t) + B(t)$$

for some $B(t) \in \mathcal{B}(X)$. The idea is the following:

If we can find a strongly continuous mapping $V : \Delta_J \to \mathcal{B}(X)$ such that

$$U(t,s) = S^{-1}V(t,s)S \quad ((t,s) \in \Delta_J),$$

then $U(\cdot,\cdot)$ automatically satisfies $(E4)$ and $(E5)$, i.e., we are done.

So, how to find such a family of bounded operators? Formally we want to use $V(t,s) = SU(t,s)S^{-1}$ which does not make sense, since we do not know whether $U(t,s)$ maps Y into itself. But the expressions $U(t,s)S^{-1}$ and $S^{-1}U(t,s)$ are well defined so that we may consider their difference

$$\begin{aligned} U(t,s)S^{-1} - S^{-1}U(t,s) &= -U(t,r)S^{-1}U(r,s)\Big|_{r=s}^{t} \\ &= -\int_s^t \frac{\mathrm{d}}{\mathrm{d}r} U(t,r)S^{-1}U(r,s)\,\mathrm{d}r. \end{aligned}$$

We calculate

$$\frac{\mathrm{d}}{\mathrm{d}r} U(t,r)S^{-1}U(r,s) = -U(t,r)A(r)S^{-1}U(r,s) + U(t,r)S^{-1}A(r)U(r,s).$$

By inserting $A(t)S^{-1} = S^{-1}(A(t) + B(t))$, we derive

$$\frac{\mathrm{d}}{\mathrm{d}r} U(t,r)S^{-1}U(r,s) = -U(t,r)S^{-1}B(r)U(r,s).$$

We thus have shown

$$U(t,s)S^{-1} - S^{-1}U(t,s) = \int_s^t U(t,r)S^{-1}B(r)U(r,s)\,\mathrm{d}r.$$

Hence by formally multiplying this equation with S from the left and thinking of $V(t,s) = SU(t,s)S^{-1}$, we see that the desired operators $V(t,s)$ should satisfy the integral equation

$$V(t,s) = U(t,s) + \int_s^t V(t,r)B(r)U(r,s)\,\mathrm{d}r \tag{3.2}$$

in $\mathcal{B}(X)$. Now, the strategy is clear. Try to find suitable conditions on $B(\cdot)$ so that (3.2) admits a unique solution $V(t,s) \in \mathcal{B}(X)$, and try to show that the resulting family $\{V(t,s) : (t,s) \in \Delta_J\}$ actually satisfies

$$U(t,s) = S^{-1}V(t,s)S\,.$$

In [21], Theorem 6.1 it was shown that if $B : J \to \mathcal{B}(X)$ is strongly continuous, then this can be done more or less easily. But unfortunately, this condition on $B(\cdot)$ is to restrictive for the desired applications. Therefore we state the generalization from Theorem I in [22], where it was shown that it is enough to require that $B : J \to \mathcal{B}(X)$ is strongly measurable and that for example $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$.

Theorem 3.36 *Suppose that there is some Banach space $Y \subseteq X$ which is densely and continuously embedded in X. Assume further that the family of generators $\{A(t) : t \in J = [a,b]\}$ satisfies the following conditions:*

(H1) $A(\cdot) \in \mathrm{stab}(X, M, \omega)$.

(H2)' *There is an isomorphism S of Y onto X and there are $B(t) \in \mathcal{B}(X)$ with $\|B(t)\|_{\mathcal{B}(x)} \leqslant b$ for some positive b such that*

$$D(SA(t)S^{-1}) = D(A(t)),$$
$$SA(t)S^{-1}x = A(t)x + B(t)x \quad (t \in J,\ x \in D(A(t))),$$

and such that $t \to B(t)$ is strongly measurable.

(H3) *$Y \subseteq D(A(t))$ and $A(t) \in \mathcal{B}(Y, X)$ for all $t \in J$, and the map $t \mapsto A(t)$, $J \to \mathcal{B}(Y, X)$, is continuous.*

Then there exists a unique evolution family $U : \Delta_J \to \mathcal{B}(X)$ which is generated by $\{A(t) : t \in J\}$ in Y. The evolution family can further be estimated by

$$\|U(t,s)\|_{\mathcal{B}(X)} \leqslant M\, e^{\omega(t-s)} \qquad ((t,s) \in \Delta_J),$$
$$\|U(t,s)\|_{\mathcal{B}(Y)} \leqslant M\, \|S\|\, \|S^{-1}\|\, e^{(\omega+Mb)(t-s)} \qquad ((t,s) \in \Delta_J). \quad \rtimes$$

Instead of requiring the rather restrictive assumption that there is a uniform isomorphism S and a family of operators $V(\cdot,\cdot)$ such that $U(t,s) = S^{-1}V(t,s)S$, one can also use the ansatz

$$U(t,s) = S(t)^{-1}V(t,s)S(s) \quad ((t,s) \in \Delta_J),$$

for a family of isomorphism $S(t)$ $(t \in J)$. This leads, in the same way as done above, to the integral equation

$$V(t,s) = U(t,s) + \int_s^t V(t,r)\left(B(r) + S'(r)S(r)^{-1}\right)U(r,s)\,\mathrm{d}r\,,$$

provided the existence of $S'(r)S(r)^{-1} \in \mathcal{B}(X)$. As a result we obtain the next result (cf. [31], Theorem 5.4.6).

THEOREM 3.37 *Suppose that there is some Banach space $Y \subseteq X$ which is densely and continuously embedded in X. Assume further that the family of generators $\{A(t) : t \in J = [a,b]\}$ satisfies the following conditions:*

(H1) $A(\cdot) \in \operatorname{stab}(X, M, \omega)$.

(H2)" *There is a family of isomorphism $S(t)$ $(t \in J)$ of Y onto X such that*

$$\|S(t)\|_{\mathcal{B}(Y,X)}, \|S(t)^{-1}\|_{\mathcal{B}(X,Y)} \leqslant c \quad (t \in J,\ \textit{for some constant } c),$$
$$J \to X,\ t \mapsto S(t)x \quad \textit{is differentiable for any } x \in X.$$

Further there is a strongly measurable map $B : J \to \mathcal{B}(X)$ with $\|B(t)\|_{\mathcal{B}(X)} \leqslant b$ for some positive b such that

$$D(S(t)A(t)S(t)^{-1}) = D(A(t)),$$
$$S(t)A(t)S(t)^{-1}x = A(t)x + B(t)x \quad (t \in J,\ x \in D(A(t))).$$

(H3) $Y \subseteq D(A(t))$ *and* $A(t) \in \mathcal{B}(Y,X)$ *for all* $t \in J$*, and the map* $t \mapsto A(t)$*,* $J \to \mathcal{B}(Y,X)$*, is continuous.*

Then there exists a unique evolution family $U : \Delta_J \to \mathcal{B}(X)$ *which is generated by* $\{A(t) : t \in J\}$ *in Y. The evolution family can further be estimated by*

$$\|U(t,s)\|_{\mathcal{B}(X)} \leqslant M\, e^{\omega(t-s)} \qquad ((t,s) \in \Delta_J),$$
$$\|U(t,s)\|_{\mathcal{B}(Y)} \leqslant M\, c^2\, e^{cM\operatorname{var}(S)}\, e^{(\omega+Mb)(t-s)} \quad ((t,s) \in \Delta_J). \quad \rtimes$$

Remark 3.38 Obviously Theorem 3.37 is a generalization of Theorem 3.36. We just have to choose $S(t) = S\ (t \in J)$. $\rtimes$

One special case in which the conditions of Theorem 3.36 and 3.37 can be easily verified is the case where $D(A(t)) = D$ is independent of t. In this case we equip D with the graph norm of $A(t)$ for some $t \in J$, e.g. $t = 0$. Then the space $(Y, \|\cdot\|_Y) := (D, \|\cdot\|_{A(0)})$ is a Banach space which is continuously and densely embedded in X.

Theorem 3.39 *Let* $\{A(t) : t \in J = [a,b]\}$ *be a stable family of generators in X, and let* $D(A(t)) = D,\ t \in J$*, be independent of* t*. Assume further that the map*

$$J \to X,\ t \mapsto A(t)x, \quad \text{is differentiable for every } x \in D\,.$$

Then there exists a unique evolution family $U : \Delta_J \to \mathcal{B}(X)$ *which is generated by* $\{A(t) : t \in J\}$ *in* Y. $\rtimes$

Proof. Let $A(\cdot) \in \operatorname{stab}(X,M,\omega)$. By the assumptions we can choose $S(t) := \lambda I - A(t)\ (t \in J)$, for some $\lambda > \omega$ and apply Theorem 3.37. ∎

3.3 Quasilinear Equations

In this section the linear theory developed so far will be used for the investigation of the quasilinear system

$$\begin{aligned} \Lambda(u(t))u'(t) &= Au(t) + Q(u(t))u(t) \quad (t \in [0,T]), \\ u(0) &= u_0 \end{aligned} \tag{Q}$$

that we have brought up in the introduction. As mentioned there, we first try to find solutions of this initial value problem by substituting certain functions φ into the quasilinear parts $\Lambda(\cdot)$ and $Q(\cdot)$ such that $\Lambda(\varphi(t))\ (t \in J)$ is invertible. We then consider the resulting nonautonomous Cauchy problem

$$\begin{aligned} &u'(t) = \Lambda(\varphi(t))^{-1}(A + Q(\varphi(t)))u(t) =: A_\varphi(t)u(t) \quad (t \in [0,T]), \\ &u(0) = u_0. \end{aligned} \tag{Q$_\mathrm{L}$}$$

Hence, we have to find "amenable" (and in view of our desired applications verifiable) conditions on the parameters of this equation which allows us to apply the theorems from the previous sections and obtain a unique solution u_φ of ($\mathrm{Q_L}$). This leads to a solution operator $\Phi : u \mapsto u_\varphi$ for a given initial value u_0, i.e.,

$$\Phi(\varphi)(t) = U_\varphi(t,0)u_0 = u_\varphi(t).$$

Fixed points of Φ will be solutions of (Q). Recall that by definition

$$\Phi(\varphi)'(t) = A_\varphi(t)\Phi(\varphi)(t), \quad \Phi(\varphi)(0) = u_0.$$

We thus want to find suitable spaces and estimates on $U_\varphi(\cdot,\cdot)$ to apply the contraction mapping principle on Φ. For stating the final theorem we recall the concept of interpolation spaces.

Definition 3.40 Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces with $Z \subseteq Y \subseteq X$ such that the corresponding canonical embeddings are continuous. Then Y is called *interpolation space between* Z and X if the following holds:

If $T \in \mathcal{B}(X)$ leaves Z invariant and $T\big|_Z \in \mathcal{B}(Z)$, then T leaves Y invariant and $T\big|_Y \in \mathcal{B}(Y)$.

Consequently (cf. [26], Lemma 0.1) there is a constant c such that

$$\|T\|_{\mathcal{B}(Y)} \leqslant c \max\{\|T\|_{\mathcal{B}(Z)}, \|T\|_{\mathcal{B}(X)}\}.$$

In most situations this constant equals one, particularly in our later applications. Thus we will assume that $c = 1$ throughout the remaining chapter. ⋊

Theorem 3.41 *Let $(X, (\cdot\,|\,\cdot)_X)$ be a Hilbert space and $A : D(A) \subseteq X \to X$ be a skew-adjoint operator in X. Assume that there are Hilbert spaces $(Y, (\cdot\,|\,\cdot)_Y)$ and $(Z, (\cdot\,|\,\cdot)_Z)$ with $Z \subseteq Y \subseteq D(A) \subseteq X$, such that Z is densely and continuously embedded in Y, and Y is densely and continuously embedded in X. Further, let Y be an interpolation space between Z and X and let $A \in \mathcal{B}(Z, Y)$.*

Let $\{Q(x) : x \in X\}$ be a family of bounded linear operators on X and suppose that there is a ball $W \coloneqq \overline{B}_Y(0, R)$ in Y and a family of linear operators $\{\Lambda(y) : y \in W\}$ in X such that the following assumptions are satisfied.

(PD) $\Lambda(y) \in \mathcal{B}(X)$ *for each $y \in W$ and there is a constant $\delta > 0$ such that*

$$\Lambda(y) \geqslant \delta I \quad (y \in W).$$

(G) $\operatorname{Ran}(I \mp \Lambda(y)^{-1}A)$ *is dense in X for all $y \in W$.*

(LC) There is a positive constant L such that

$$\|\Lambda(y) - \Lambda(\tilde{y})\|_{\mathcal{B}(X)} \leqslant L\,\|y - \tilde{y}\|_Y \quad (y, \tilde{y} \in W).$$

(LC-i) $\Lambda(y)^{-1} \in \mathcal{B}(Y)$ *for each $y \in W$, and there is a constant l_0 such that*

$$\left\|\Lambda(y)^{-1} - \Lambda(\tilde{y})^{-1}\right\|_Y \leqslant l_0\,\|y - \tilde{y}\|_Y \quad (y, \tilde{y} \in W).$$

(LC-q) *Let $r > 0$ be arbitrary. There is a constant $q = q(r)$ such that $\|Q(y)\|_{\mathcal{B}(X)} \leqslant q$ for all $y \in W \cap \overline{B}_Z(0,r)$. We further suppose that $Q(y) \in \mathcal{B}(Z,Y)$ for all $y \in W$, and that there is a constant l_1 such that*

$$\|Q(y) - Q(\tilde{y})\|_{\mathcal{B}(Z,Y)} \leqslant l_1 \|y - \tilde{y}\|_Y \quad (y, \tilde{y} \in W).$$

(CE) *There is a continuous isomorphism $S : Z \to X$ and for each $r > 0$ there are linear operators $B(z) \in \mathcal{B}(X) (z \in W \cap \overline{B}_Z(0,r))$, with $\|B(z)\|_{\mathcal{B}(X)} \leqslant b$ for some positive b such that*

$$S\Lambda(z)^{-1}(A + Q(z))S^{-1} = \Lambda(z)^{-1}(A + Q(z)) + B(z) \quad (z \in W \cap \overline{B}_Z(0,r)).$$

Note that this includes the domain relation $D(S\Lambda(z)^{-1}(A+Q(z))S^{-1}) = D(A)$.

Let $\kappa \in (0,1)$ and $r_0 > 0$ be arbitrary and define $c_0 = c_0(R)$ and $c_1 = c_1(R)$ by

$$c_0 \coloneqq \|S\|_{\mathcal{B}(Z,X)} \|S^{-1}\|_{\mathcal{B}(X,Z)} (\delta^{-1} \|\Lambda(0)\|_{\mathcal{B}(X)})^{1/2} e^{-(1/2)\|\Lambda(0)\|^{-1}_{\mathcal{B}(X)} LR},$$
$$c_1 \coloneqq l_0(1 + l_0 R + \|P(0)\|_{\mathcal{B}(Z,Y)}) \|A\|_{\mathcal{B}(Z,Y)} + l_1(l_0 R + \|\Lambda(0)^{-1}\|_{\mathcal{B}(Y)}).$$

Then the following assertions hold.

(a) *For each $u_0 \in \overline{B}_Y(0, \kappa \frac{R}{c_0}) \cap \overline{B}_Z(0, r_0)$ there exists a time $T = T(\kappa, r_0, R) > 0$ and a solution*

$$u(\cdot, u_0) = u \in C([0,T], Z) \cap C^1([0,T], Y)$$

of (Q) *with $u(t) \in W$ for all $t \in [0,T]$. Moreover, we know that*

$$\|u(t)\|_Z \leqslant \frac{c_0 \, r_0}{\kappa} \quad (0 \leqslant t \leqslant T)$$

and further that

$$T \geqslant \min\left\{ \frac{-\log(\kappa)}{\delta^{-1/2} L r + \delta^{-1} q + b}, \frac{\|S\| \|S^{-1}\| \, \kappa^2}{r_0 \, c_0^2 \, c_1} \right\} \eqqcolon T_0(\kappa, r_0, R).$$

(b) *If $v \in C([0,T'], Z) \cap C^1([0,T'], Y)$ is another solution of* (Q) *which also satisfies $v(t) \in W$ for each $t \in [0,T']$, then $v(t) = u(t)$ for all times t between 0 and $\min\{T, T'\}$.*

(c) *The mapping*

$$\overline{B}_Y\left(0, \kappa \frac{R}{c_0}\right) \cap \overline{B}_Z(0, r_0) \subseteq Y \to C([0, T_0], Y), \quad u_0 \mapsto u(\cdot, u_0),$$

is Lipschitz continuous.

(d) Suppose $Q = 0$. *If there is a constant* $\varepsilon > 0$ *and a Fréchet differentiable operator* $\lambda : B_Y(0, R+\varepsilon) \subseteq Y \to Y$ *such that* $\Lambda(y) = D\lambda(y)$ $(y \in W)$*, then the assertions from* (a)*-*(c) *remain true if we replace* (Q) *by the evolution equation*

$$(\lambda \circ u)'(t) = Au(t) \quad (t \in [0,T]),$$
$$u(0) = u_0. \quad \rtimes$$

Proof. (a) : We divide the proof into five parts.

Step 1. Preliminaries.

Let $y \in W$. It is an immediate consequence of (PD) that each $\Lambda(y)$ is invertible in X with uniform bounded inverse $\|\Lambda(y)^{-1}\|_{\mathcal{B}(X)} \leqslant \delta^{-1}$. Condition $(LC\text{-}i)$ further yields

$$\begin{aligned}\|\Lambda(y)^{-1}v\|_Y &\leqslant \|\Lambda(y)^{-1}v - \Lambda(0)^{-1}v\|_Y + \|\Lambda(0)^{-1}v\|_Y \\ &\leqslant \left\{l_0\|y\|_Y + \|\Lambda(0)^{-1}\|_{\mathcal{B}(Y)}\right\}\|v\|_Y \qquad (v \in Y)\end{aligned}$$

so that

$$\|\Lambda(y)^{-1}\|_{\mathcal{B}(Y)} \leqslant l_0 R + \|\Lambda(0)^{-1}\|_{\mathcal{B}(Y)} =: \lambda_1.$$

For $T > 0$ and $r, \gamma > 0$ we define

$$\begin{aligned}&E(T,r,\gamma)\\ &\quad := \left\{\varphi \in C([0,T],Z) : \|\varphi(t)\|_Y \leqslant R,\ \|\varphi(t)\|_Z \leqslant r,\ [\varphi]_{\mathrm{Lip}([0,T],Y)} \leqslant \gamma\right\}.\end{aligned}$$

A function $\varphi \in E(T,r,\gamma)$ thus belongs to $C([0,T],Y)$ and $\varphi(t) \in W$ $(t \in [0,T])$ so that $\Lambda(\varphi(t))$ is invertible for all $t \in [0,T]$. Let $\varphi \in E(T,r,\gamma)$. First we will show that the family of linear operators $\{\mathcal{A}_\varphi(t) : t \in [0,T]\}$ given by

$$\begin{aligned}\mathcal{A}_\varphi(t) &:= A_\varphi(t) + Q_\varphi(t)\\ &:= \Lambda(\varphi(t))^{-1}A + \Lambda(\varphi(t))^{-1}Q(\varphi(t)) \quad (0 \leqslant t \leqslant T),\end{aligned}$$

generates an evolution family in Z. Recall that $A \in \mathcal{B}(Z,Y)$ by assumption. To apply Theorem 3.35 we introduce new inner products on X by setting

$$(u \mid v)_{X,t} := (\Lambda(\varphi(t))u \mid v)_X \quad (u, v \in X),$$

for $t \in [0,T]$. We denote by X_t the space X endowed with this inner product. Further, we write $\|\cdot\|_{X,t}$ for the norm associated to $(\cdot \mid \cdot)_{X,t}$. We next show that each of this norms is equivalent to the norm $\|\cdot\|_X$ on X. Let $u \in X$ and $t \in [0,T]$. Condition (PD) directly gives the lower bound

$$\|u\|_{X,t}^2 = (\Lambda(\varphi(t))u \mid u)_X \geqslant \delta^2\,\|u\|_X^2.$$

For the upper estimate we first note that

$$\begin{aligned}\|\Lambda(y)\|_{\mathcal{B}(X)} &\leqslant \|\Lambda(y)-\Lambda(0)\|_{\mathcal{B}(X)} + \|\Lambda(0)\|_{\mathcal{B}(X)}\\ &\leqslant LR + \|\Lambda(0)\|_{\mathcal{B}(X)}.\end{aligned}$$

Using this inequality, $\varphi \in E(T,r,y)$, and Cauchy-Schwarz we derive

$$\begin{aligned}\|u\|_{X,t}^2 &= (\Lambda(\varphi(t))u \mid u)_X \leqslant \|\Lambda(\varphi(t))\|_{\mathcal{B}(X)}\,\|u\|_X^2\\ &\leqslant \{\|\Lambda(0)\|_{\mathcal{B}(X)} + LR\}\,\|u\|_X^2 = \|\Lambda(0)\|_{\mathcal{B}(X)}\left\{1+\|\Lambda(0)\|_{\mathcal{B}(X)}^{-1}LR\right\}\,\|u\|_X^2\\ &\leqslant \|\Lambda(0)\|_{\mathcal{B}(X)}\,e^{\|\Lambda(0)\|_{\mathcal{B}(X)}^{-1}LR}\,\|u\|_X^2.\end{aligned}$$

Putting $\lambda_0 \coloneqq \|\Lambda(0)\|_{\mathcal{B}(X)}$, we have thus found

$$\lambda_0^{-1/2}\,e^{-(1/2)\,\lambda_0^{-1}LR}\,\|u\|_{X,t} \leqslant \|u\|_X \leqslant \delta^{-1/2}\,\|u\|_{X,t} \quad (t\in[0,T],\ u\in X). \tag{3.3}$$

Step 2. Stability of the unperturbed linear problem in X.

To obtain well-posedness for the linear system with respect to the family $\mathcal{A}_\varphi(\cdot)$, we first study the family $A_\varphi(\cdot)$ and then continue by switching on the perturbation $Q_\varphi(\cdot)$. We now want to establish that $\pm A_\varphi(t)$ generate contraction semigroups on X for every $t\in[0,T]$. Because of (3.3) each $\pm A_\varphi(t)$ then generates a C_0-semigroup on X. Note that

$$\mathrm{Re}\big(\pm A_\varphi(t)u \mid u\big)_{X,t} = \pm\mathrm{Re}\big(\Lambda(\varphi(t))A_\varphi(t)u \mid u\big)_X = \pm\mathrm{Re}(Au \mid u)_X = 0$$

for $u\in X$, which means that each $\pm A_\varphi(t)$ is dissipative in X_t. Since A is closed in X and $\Lambda(\varphi(t))^{-1}$ is bounded on X, we obtain that each $\pm A_\varphi(t)$ is closed in X and therefore also in X_t.

Properties (G) and (3.3) yield the density of $\mathrm{Ran}(I\pm A_\varphi(t))$ in X_t. As a result each $\pm A_\varphi(t)$ is maximal dissipative in X_t, i.e., it defines a contraction semigroup on X_t for every $t\in[0,T]$. Our next aim is the stability of both families $\pm A_\varphi(\cdot)$ in X. Let $u\in X$, t, $s\in[0,T]$ and $\varphi\in E(T,r,y)$. By means of (LC) and (3.3) we estimate

$$\begin{aligned}\|u\|_{X,t}^2 &= (\{\Lambda(\varphi(t))-\Lambda(\varphi(s))\}u \mid u)_X + (\Lambda(\varphi(s)u \mid u)_X\\ &\leqslant \|\Lambda(\varphi(t))-\Lambda(\varphi(s))\|_{\mathcal{B}(X)}\|u\|_X^2 + \|u\|_{X,s}^2\\ &\leqslant \{L\|\varphi(t)-\varphi(s)\|_Y\delta^{-1/2}+1\}\,\|u\|_{X,s}^2\\ &\leqslant \{Ly|t-s|\delta^{-1/2}+1\}\,\|u\|_{X,s}^2\\ &\leqslant e^{\delta^{-1/2}Ly\,|t-s|}\,\|u\|_{X,s}^2,\end{aligned}$$

so that

$$\|u\|_{X,t} \leqslant e^{1/2\,\delta^{-1/2}Ly\,|t-s|}\,\|u\|_{X,s}.$$

In view of this relation and (3.3), Lemma 3.21 implies

$$\pm A_\varphi(\cdot) \in \operatorname{stab}\left(X,\, \delta^{-1/2}\lambda_0^{1/2} e^{(1/2)\lambda_0^{-1}LR}\, e^{\delta^{-1/2}L\gamma T},\, 0\right).$$

Writing

$$k_0 \coloneqq \delta^{-1/2}\lambda_0^{1/2} e^{1/2\,\lambda_0^{-1}LR}, \quad k_1 \coloneqq \delta^{-1/2}L\gamma, \quad k_T \coloneqq c_0\, e^{c_1 T}, \tag{3.4}$$

this reads as

$$\pm A_\varphi(\cdot) \in \operatorname{stab}(X,\, k_T,\, 0),$$

i.e.,

$$\left\| e^{\pm s_k A_\varphi(t_k)} \dots e^{\pm s_1 A_\varphi(t_1)} \right\|_{\mathcal{B}(X)} \leqslant k_T,$$

for every $s_j \geqslant 0$ and $T \geqslant t_k \geqslant \dots \geqslant t_1 \geqslant 0$.

Step 3. Well-posedness of the linear problem in Y.

In the following we will concentrate only on the operators $\mathcal{A}_\varphi(t)$, since all the upcoming calculations can easily be transformed to the operators $-\mathcal{A}_\varphi(t)$.

We next add the operators $Q_\varphi(t)$. Since $\varphi(t)$ is contained in $W \cap \overline{B}_Z(0,r)$, assumptions (PD) and $(LC\text{-}q)$ yield

$$\|Q_\varphi(t)\|_{\mathcal{B}(X)} \leqslant \|\Lambda(\varphi(t))^{-1}\|_{\mathcal{B}(X)} \|Q(\varphi(t))\|_{\mathcal{B}(X)} \leqslant \frac{q}{\delta}.$$

Using Lemma 3.23 we thus obtain

$$\pm\mathcal{A}_\varphi(\cdot) = \pm(A_\varphi(\cdot) + Q_\varphi(\cdot)) \in \operatorname{stab}\left(X,\, k_T,\, \frac{q}{\delta}\right).$$

From (CE) we further know that

$$S\mathcal{A}_\varphi(t)S^{-1} = \mathcal{A}_\varphi(t) + B_\varphi(t),$$

where $B_\varphi(t) \coloneqq B(\varphi(t)) \in \mathcal{B}(X)$, and $\|B_\varphi(t)\|_{\mathcal{B}(X)} \leqslant b$ for all $t \in [0,T]$. Due to Lemma 3.30 it follows that Z is $\mathcal{A}_\varphi(t)$-admissible for each $0 \leqslant t \leqslant T$ and

$$\mathcal{A}_\varphi(\cdot)\big|_Z \in \operatorname{stab}\left(Z,\, \|S\|_{\mathcal{B}(Z,X)}\|S^{-1}\|_{\mathcal{B}(X,Z)}k_T,\, \frac{p}{\delta} + b\right). \tag{$\star$}$$

Putting $\omega \coloneqq \frac{q}{\delta} + b$, we have $e^{s\mathcal{A}_\varphi(t)}\big|_Z = e^{s\mathcal{A}_\varphi(t)|_Z} = S^{-1}e^{s\mathcal{A}_\varphi(t)}S \in \mathcal{B}(Z)$ and

$$\left\| e^{s_k\mathcal{A}_\varphi(t_k)}\big|_Z \dots e^{s_1\mathcal{A}_\varphi(t_1)}\big|_Z \right\|_{\mathcal{B}(Z)} \leqslant \|S\|_{\mathcal{B}(Z,X)}\|S^{-1}\|_{\mathcal{B}(X,Z)}\, k_T\, e^{\omega(s_k+\dots+s_1)}$$

for each $s_j \geqslant 0$ and $T \geqslant t_k \geqslant \ldots \geqslant t_1 \geqslant 0$. Because Y is an interpolation space between Z and X, cf. Definition 3.40, it follows that also Y is $\mathcal{A}_\varphi(t)$-admissible for any $0 \leqslant t \leqslant T$ and that

$$\left\| e^{s\mathcal{A}_\varphi(t)}\,|_Y \right\|_{\mathcal{B}(Y)} \leqslant \max\left\{ \left\| e^{s\mathcal{A}_\varphi(t)} \right\|_{\mathcal{B}(X)}, \left\| e^{s\mathcal{A}_\varphi(t)}\,|_Z \right\|_{\mathcal{B}(Z)} \right\}$$
$$\leqslant \|S\|_{\mathcal{B}(Z,X)} \|S^{-1}\|_{\mathcal{B}(X,Z)}\, k_T\, e^{\omega s}\,.$$

Moreover,

$$\left\| e^{s_k \mathcal{A}_\varphi(t_k)}\,|_Y \ldots e^{s_1 \mathcal{A}_\varphi(t_1)}\,|_Y \right\|_{\mathcal{B}(Y)} \leqslant \|S\|_{\mathcal{B}(Z,X)} \|S^{-1}\|_{\mathcal{B}(X,Z)}\, k_T\, e^{\omega(s_k + \ldots + s_1)}$$

for each $s_j \geqslant 0$ and $T \geqslant t_k \geqslant \ldots \geqslant t_1 \geqslant 0$, which means

$$\mathcal{A}_\varphi(\cdot)\,|_Y \in \operatorname{stab}\left(Y, \|S\|_{\mathcal{B}(Z,X)} \|S^{-1}\|_{\mathcal{B}(X,Z)}\, k_T, \omega\right). \tag{3.5}$$

We put

$$M_T \coloneqq \|S\|_{\mathcal{B}(Z,X)} \|S^{-1}\|_{\mathcal{B}(X,Z)}\, k_T.$$

From now on, we will consider Y (instead of X) as the new phase space on which the operators

$$\mathbf{A}_\varphi(t) \coloneqq \mathcal{A}_\varphi(t)\,|_Y, \quad D(\mathbf{A}_\varphi(t)) = \{y \in D(A) \cap Y : \mathcal{A}_\varphi(t)y \in Y\}$$

act. We have already shown that each $\mathbf{A}_\varphi(t)$ generates a strongly continuous semigroup on Y and that

$$\mathbf{A}_\varphi(\cdot) \in \operatorname{stab}(Y, M_T, \omega).$$

Reinterpreting the previous results in this way, we obtain that Z is $\mathbf{A}_\varphi(t)$-admissible for all $0 \leqslant t \leqslant T$ and that $\mathbf{A}_\varphi(t)\,|_Z = \mathcal{A}_\varphi(t)\,|_Z$, which yields

$$\mathbf{A}_\varphi(\cdot)\,|_Z \in \operatorname{stab}(Z, M_T, \omega)\,. \tag{3.6}$$

In other words $\{\mathbf{A}_\varphi(t) : t \in [0,T]\}$ satisfies $(H1)$ and $(H2)$ of Theorem 3.35. We now use the isomorphism $S : Z \to X$ to introduce the inner products

$$(v \mid w)_{Z,t} \coloneqq (Sv \mid Sw)_{X,t} \quad (v, w \in Z)$$

on Z. Thus $\|v\|_{Z,t} = \|Sv\|_{X,t}$ and we will write Z_t for Z endowed with this norm. Since S and S^{-1} are continuous it follows that each norm $\|\cdot\|_{Z,t}$ is equivalent to $\|\cdot\|_Z$, and we also have

$$\|v\|_{Z,t} \leqslant e^{1/2\,\delta^{-1/2} L_Y |t-s|}\, \|v\|_{Z,s} \quad (t, s \in [0,T],\ v \in Z).$$

Further, a second inspection of the previous calculations in X, concerning the norms $\|\cdot\|_{X,t}$ and the operators $\mathcal{A}_\varphi(t)$, shows that each $\mathbf{A}_\varphi(t)\big|_Z$ is quasi contractive in Z_t, which proves $(H4)$.

It only remains to verify condition $(H3)$ for $\mathbf{A}_\varphi(\cdot)$. Therefore we invoke assumption $(LC\text{-}q)$ to bound the operators $P(y) \in \mathcal{B}(Z,Y)$ on W. Given $y \in W$ we estimate

$$\begin{aligned}\|P(y)\|_{\mathcal{B}(Z,Y)} &\leqslant \|P(y) - P(0)\|_{\mathcal{B}(Z,Y)} + \|P(0)\|_{\mathcal{B}(Z,Y)}\\ &\leqslant l_1\|y\|_Y + \|P(0)\|_{\mathcal{B}(Z,Y)}\end{aligned}$$

so that

$$\|P(y)\|_{\mathcal{B}(Z,Y)} \leqslant l_1 R + \|P(0)\|_{\mathcal{B}(Z,Y)} =: \lambda_2. \tag{3.7}$$

These observations, $\Lambda(y)^{-1} \in \mathcal{B}(Y)$, and $A \in \mathcal{B}(Z,Y)$ imply that $Z \subseteq D(\mathbf{A}_\varphi(t))$ as well as $\mathbf{A}_\varphi(t) \in \mathcal{B}(Z,Y)$. More precisely, for every $v \in Z$ we obtain

$$\begin{aligned}\|\mathbf{A}_\varphi(t)v\|_Y \leqslant{}& \|\Lambda(\varphi(t))^{-1}\|_{\mathcal{B}(Y)}\|A\|_{\mathcal{B}(Z,Y)}\,\|v\|_Z\\ &+ \|\Lambda(\varphi(t))^{-1}\|_{\mathcal{B}(Y)}\|P(\varphi(t))\|_{\mathcal{B}(Z,Y)}\|v\|_Z\end{aligned}$$

so that

$$\|\mathbf{A}_\varphi(t)\|_{\mathcal{B}(Z,Y)} \leqslant \lambda_1(\|A\|_{\mathcal{B}(Z,Y)} + \lambda_2). \tag{3.8}$$

Concerning the map $\mathbf{A}_\varphi : [0,T] \to \mathcal{B}(Z,Y)$ we do the following calculation. For $t,s \in [0,T]$ and $v \in Z$ we have

$$\|\mathbf{A}_\varphi(t)v - \mathbf{A}_\varphi(s)v\|_Y \leqslant \|A_\varphi(t)v - A_\varphi(s)v\|_Y + \|P_\varphi(t)v - P_\varphi(s)v\|_Y, \tag{3.9}$$

where we can further estimate

$$\begin{aligned}\|A_\varphi(t)v - A_\varphi(s)v\|_Y &\leqslant \|\Lambda(\varphi(t))^{-1} - \Lambda(\varphi(t))^{-1}\|_{\mathcal{B}(Y)}\,\|Av\|_Y\\ &\leqslant l_0\,\|A\|_{\mathcal{B}(Z,Y)}\,\|\varphi(t)-\varphi(s)\|\,\|v\|_Z\\ &\leqslant l_0\gamma\,\|A\|_{\mathcal{B}(Z,Y)}\,|t-s|\,\|v\|_Z.\end{aligned}$$

The second summand in (3.9) we estimate by

$$\begin{aligned}\|P_\varphi(t)v - P_\varphi(s)v\|_Y \leqslant{}& \left\|\Lambda(\varphi(t))^{-1}P(\varphi(t))v - \Lambda(\varphi(t))^{-1}P(\varphi(s))v\right\|_Y\\ &+ \left\|\Lambda(\varphi(t))^{-1}P(\varphi(s))v - \Lambda(\varphi(s))^{-1}P(\varphi(s))v\right\|_Y\\ \leqslant{}& \left\|\Lambda(\varphi(t))^{-1}\right\|_{\mathcal{B}(Y)}\|P(\varphi(t))v - P(\varphi(s))v\|_Y\\ &+ \left\|\Lambda(\varphi(t))^{-1} - \Lambda(\varphi(s))^{-1}\right\|_{\mathcal{B}(Y)}\|P_\varphi(s)v\|_Y\\ \leqslant{}& l_1\lambda_1\|\varphi(t)-\varphi(s)\|_Y\|v\|_Z + l_0\lambda_2\|\varphi(t)-\varphi(s)\|_Y\|v\|_Z.\end{aligned}$$

Since $\varphi \in E(T, r, \gamma)$, we have shown that

$$\|P_\varphi(t) - P_\varphi(s)\|_{\mathcal{B}(Z,Y)} \leqslant \gamma(l_0\lambda_2 + l_1\lambda_1)|t-s|$$

and therefore

$$\|\mathbf{A}_\varphi(t) - \mathbf{A}_\varphi(s)\|_{\mathcal{B}(Z,Y)} \leqslant \gamma(l_0\|A\|_{\mathcal{B}(Z,Y)} + l_0\lambda_2 + l_1\lambda_1)|t-s|,$$

which means that $\mathbf{A}_\varphi$ is even Lipschitz continuous, and $(H3)$ holds.

With obvious modifications, one also shows that $-\mathbf{A}_\varphi(T-t)$ $(t \in [0,T])$, satisfies $(H1)$-$(H4)$.

Theorem 3.36 thus yields for any $\varphi \in E(T, r, \gamma)$ that the family of operators $\mathbf{A}_\varphi(\cdot)$ generates an evolution family $\{U_\varphi(t,s) : (t,s) \in \Delta_{[0,T]}\}$ in Z satisfying

$$\begin{aligned} &\|U_\varphi(t,s)\|_{\mathcal{B}(Y)} \leqslant M_T\, e^{\omega(t-s)} \\ &\|U_\varphi(t,s)\|_{\mathcal{B}(Z)} \leqslant M_T\, e^{\omega(t-s)} \quad \big((t,s) \in \Delta_{[0,T]}\big), \end{aligned} \tag{3.10}$$

where $M_T = \|S\|\|S^{-1}\|k_0\, e^{k_1 T}$ and $\omega = \frac{q}{\delta} + b$. In particular for each $z \in Z$ and $s \in [0,T]$ the function

$$U_\varphi(\cdot,s)z \in C([s,T],Z) \cap C^1([s,T],Y)$$

is the unique Z-valued solution of the Cauchy problem

$$u'(t) = \mathbf{A}_\varphi(t)u(t) = \mathcal{A}_\varphi(t)u(t) \quad (t \in [0,T]), \quad u(s) = z\,,$$

cf. Definition 3.11. For $\varphi \in E(T, r, \gamma)$ and $u_0 \in Z$, we define the operator Φ_{u_0} by

$$\Phi_{u_0}(\varphi)(t) \coloneqq U_\varphi(t,0)u_0\,.$$

In the next step we show that Φ_{u_0} defines a contraction on $E(T, r, \gamma)$ for suitable $u_0 \in Z$ and r, γ, $T > 0$ when it is endowed with a suitable metric.

Step 4. Solving the fixed point problem.

For $\varphi, \psi \in E(T, r, \gamma)$ we define

$$d(\varphi,\psi) \coloneqq \|\varphi(t) - \psi(t)\|_{C([0,T],Y)} = \sup_{0 \leqslant t \leqslant T} \|\varphi(t) - \psi(t)\|_Y.$$

Since Z is reflexive, $\big(E(T, r, \gamma), d\big)$ is a complete metric space. To see this, we first observe that $(C([0,T],Y), d)$ is a complete normed space, and the subset of $\varphi \in C([0,T],Y)$ satisfying $\|\varphi(t)\|_Y \leqslant R$ for all $0 \leqslant t \leqslant T$ and $[\varphi]_{\mathrm{Lip}([0,T],Y)} \leqslant \gamma$ is closed in $(C([0,T],Y), d)$. The reflexivity of Z implies that every ball in Z is weakly closed. Thus, if a sequence $(\varphi_k)_k$ in Z with $\|\varphi_k\|_Y \leqslant R$ and $\|\varphi_k\|_Z \leqslant r$ for every k, converges in Y to some φ, then $\|\varphi\|_Y \leqslant R$ and $\|\varphi\|_Y \leqslant r$. As a result, $(E(T,r,\gamma),d)$ is complete.

We next establish the estimates which imply that Φ_{u_0} leaves $E(T,r,\gamma)$ invariant and that Φ_{u_0} is a strict contraction. Throughout, let $\varphi,\psi \in E(T,r,\gamma)$ and $u_0 \in Z$. We choose r,γ,T below.

Since $U_\varphi(\cdot,0)u_0$ is a Z-valued solution the function $\Phi_{u_0}(\varphi)$ is contained in $C([0,T],Z)$. Inequality 3.10 further yields

$$\begin{aligned}\left\|\Phi_{u_0}(\varphi)(t)\right\|_Y &= \|U_\varphi(t,0)u_0\|_Y \leqslant \|U_\varphi(t,0)\|_{\mathcal{B}(Y)}\|u_0\|_Y \\ &\leqslant \|S\|\|S^{-1}\|\, k_0\, e^{(k_1+\omega)\,T}\,\|u_0\|_Y,\end{aligned}$$

$$\begin{aligned}\left\|\Phi_{u_0}(\varphi)(t)\right\|_Z &= \|U_\varphi(t,0)u_0\|_Z \leqslant \|U_\varphi(t,0)\|_{\mathcal{B}(Z)}\|u_0\|_Z \\ &\leqslant \|S\|\|S^{-1}\|\, k_0\, e^{(k_1+\omega)\,T}\,\|u_0\|_Z.\end{aligned}$$

Let $t,s \in [0,T]$. We write

$$\begin{aligned}U_\varphi(t,0)u_0 - U_\varphi(s,0)u_0 = U_\varphi(r,0)u_0\Big|_s^t &= \int_s^t \frac{\mathrm{d}}{\mathrm{d}r}U_\varphi(r,0)u_0\,\mathrm{d}r \\ &= \int_s^t \mathbf{A}_\varphi(r)U_\varphi(r,0)u_0\,\mathrm{d}r.\end{aligned}$$

Thus, using (3.8), we can estimate

$$\begin{aligned}\|\Phi_{u_0}(\varphi)(t) - \Phi_{u_0}(\varphi)(s)\|_Y &\leqslant \int_s^t \|\mathbf{A}_\varphi(r)\|_{\mathcal{B}(Z,Y)}\,\|U_\varphi(t,s)\|_{\mathcal{B}(Z)}\,\|u_0\|_Z\,\mathrm{d}r \\ &\leqslant \|S\|\|S^{-1}\|\lambda_1(\|A\|_{\mathcal{B}(Z,Y)} + \lambda_2)k_0e^{(k_1+\omega)T}\|u_0\|_Z\,|t-s|,\end{aligned}$$

so that

$$[\Phi_{u_0}(\varphi)]_{\mathrm{Lip}([0,T],Y)} \leqslant \|S\|\|S^{-1}\|\lambda_1(\|A\|_{\mathcal{B}(Z,Y)} + \lambda_2)k_0e^{(k_1+\omega)T}\|u_0\|_Z.$$

To show contractivity of $\varphi \mapsto \Phi_{u_0}(\varphi)$, we calculate

$$\begin{aligned}U_\varphi(t,0)u_0 - U_\psi(t,0)u_0 &= U_\varphi(t,0)U_\psi(0,0)u_0 - U_\varphi(t,t)U_\psi(t,0)u_0 \\ &= -\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}U_\varphi(t,s)U_\psi(s,0)u_0\,\mathrm{d}s \\ &= \int_0^t U_\varphi(t,s)\{\mathbf{A}_\varphi(s) - \mathbf{A}_\psi(s)\}U_\psi(s,0)u_0\,\mathrm{d}s,\end{aligned}$$

where we use that $U_\psi(s,0)Z \subseteq Z$ and Theorem 3.35. We can further estimate

$$\begin{aligned}&\|\Phi_{u_0}(\varphi)(t) - \Phi_{u_0}(\psi)(t)\|_Y \\ &\quad\leqslant \int_0^t \|U_\varphi(t,s)\|_{\mathcal{B}(Y)}\,\|\mathbf{A}_\varphi(s) - \mathbf{A}_\psi(s)\|_{\mathcal{B}(Z,Y)}\,\|U_\psi(s,0)\|_{\mathcal{B}(Z)}\,\mathrm{d}s\,\|u_0\|_Z\,.\end{aligned}$$

By definition

$$\|\mathbf{A}_\varphi(s) - \mathbf{A}_\psi(s)\|_{\mathcal{B}(Z,Y)} \leqslant \|A_\varphi(s) - A_\psi(s)\|_{\mathcal{B}(Z,Y)} + \|P_\varphi(s) - P_\psi(s)\|_{\mathcal{B}(Z,Y)}.$$

Assumption (LC-i) leads to

$$\begin{aligned}\|A_\varphi(s) - A_\psi(s)\|_{\mathcal{B}(Z,Y)} &\leqslant \|\Lambda(\varphi(s))^{-1} - \Lambda(\psi(s))^{-1}\|_{\mathcal{B}(Y)}\|A\|_{\mathcal{B}(Z,Y)} \\ &\leqslant l_0\,\|A\|_{\mathcal{B}(Z,Y)}\,\|\varphi(s) - \psi(s)\|_Y.\end{aligned}$$

For the remaining piece we use the assumptions (LC-i), (LC-q) as well as (3.7) and estimate

$$\begin{aligned}\|P_\varphi(t) - P_\psi(s)\|_{\mathcal{B}(Y,Z)} &\leqslant \left\|\Lambda(\varphi(s))^{-1}P(\varphi(s)) - \Lambda(\varphi(s))^{-1}P(\psi(s))\right\|_{\mathcal{B}(Y,Z)} \\ &\quad + \left\|\Lambda(\varphi(s))^{-1}P(\psi(s)) - \Lambda(\psi(s))^{-1}P(\psi(s))\right\|_{\mathcal{B}(Y,Z)} \\ &\leqslant \left\|\Lambda(\varphi(s))^{-1}\right\|_{\mathcal{B}(Y)}\|P(\varphi(s)) - P(\psi(s))\|_{\mathcal{B}(Y,Z)} \\ &\quad + \left\|\Lambda(\varphi(s))^{-1} - \Lambda(\psi(s))^{-1}\right\|_{\mathcal{B}(Y)}\|P_\psi(s)\|_{\mathcal{B}(Y,Z)} \\ &\leqslant (l_1\lambda_1 + l_0\lambda_2)\|\varphi(s) - \psi(s)\|_Y.\end{aligned}$$

Combining these estimates we obtain

$$\|\mathbf{A}_\varphi(s) - \mathbf{A}_\psi(s)\|_{\mathcal{B}(Z,Y)} \leqslant \big(l_0(\|A\|_{\mathcal{B}(Y,Z)} + \lambda_2) + l_1\lambda_1\big)\|\varphi(s) - \psi(s)\|_Y \tag{3.11}$$

and therefore

$$\begin{aligned}&\|\Phi_{u_0}(\varphi)(t) - \Phi_{u_0}(\psi)(t)\|_Y \\ &\qquad \leqslant T\,\|S\|\|S^{-1}\|k_0(l_0\|A\|_{\mathcal{B}(Y,Z)}(1 + \lambda_2) + l_1\lambda_1)k_0e^{(k_1+\omega)T}\|u_0\|_Z\,d(\varphi,\psi),\end{aligned}$$

so that

$$\begin{aligned}&d(\Phi_{u_0}(\varphi), \Phi_{u_0}(\psi)) \\ &\quad \leqslant T\,\|S\|\|S^{-1}\|k_0(l_0\|A\|_{\mathcal{B}(Y,Z)}(1 + \lambda_2) + l_1\lambda_1)k_0e^{(k_1+\omega)T}\|u_0\|_Z\,d(\varphi,\psi).\end{aligned}$$

Recalling 3.4, we define

$$\begin{aligned}c_0 &\coloneqq \|S\|\|S^{-1}\|k_0, \\ c_1 &\coloneqq c_0\,\lambda_1(\|A\|_{\mathcal{B}(Z,Y)} + \lambda_2), \\ c_2 &\coloneqq c_0\,k_0(l_0\|A\|_{\mathcal{B}(Z,Y)}(1 + \lambda_2) + l_1\lambda_1)\end{aligned}$$

Then we have shown that we can control Φ_{u_0} through

$$\|\Phi_{u_0}(\varphi)(t)\|_Y \leqslant c_0e^{(k_1+\omega)T}\,\|u_0\|_Y, \tag{3.12}$$

$$\|\Phi_{u_0}(\varphi)(t)\|_Z \leqslant c_0e^{(k_1+\omega)T}\,\|u_0\|_Z, \tag{3.13}$$

$$[\Phi_{u_0}(\varphi)]_{\mathrm{Lip}([0,T],Y)} \leqslant c_1e^{(k_1+\omega)T}\,\|u_0\|_Z, \tag{3.14}$$

$$d(\Phi_{u_0}(\varphi), \Phi_{u_0}(\psi)) \leqslant T\,c_2\,e^{(k_1+\omega)T}\,\|u_0\|_Z. \tag{3.15}$$

Fix $\kappa \in (0,1)$. We take $u_0 \in \overline{B}_Y(0, \kappa \frac{R}{c_0}) \cap \overline{B}_Z(0, r_0)$, where $r_0 > 0$ is arbitrary. We then fix

$$r \coloneqq \frac{c_0 r_0}{\kappa}, \quad \gamma \coloneqq \frac{c_1 r_0}{\kappa}, \quad T \coloneqq \min\left\{\frac{-\log \kappa}{k_1 + \omega}, \frac{\kappa^2}{c_2 r_0}\right\} > 0.$$

Observe that then $\kappa e^{(k_1+\omega)T} \leqslant 1$. With these numbers we define our metrical function space $E(T, r, \gamma)$.

The inequalities (3.12)-(3.14) imply that

$$\|\Phi_{u_0}(\varphi)(t)\|_Y \leqslant R, \quad \|\Phi_{u_0}(\varphi)(t)\|_Z \leqslant r, \quad [\Phi_{u_0}(\varphi)]_{\mathrm{Lip}([0,T],X)} \leqslant \gamma.$$

Thus Φ_{u_0} maps $E(T, r, \gamma)$ into itself for these choices. Further, from (3.15) we obtain

$$d(\Phi_{u_0}(\varphi), \Phi_{u_0}(\psi)) \leqslant T \frac{c_2 r_0}{\kappa} d(\varphi, \psi) \leqslant \kappa\, d(\varphi, \psi),$$

so that for our choices of u_0 and T, the mapping Φ_{u_0} actually defines a strict contraction on the complete metric space $(E(T, r, \gamma), d)$.

Step 5. Closing the proof.

By Banach's fixed point theorem there is exactly one fixed point $\varphi \in E(T, r, \gamma)$ of Φ_{u_0}, this means

$$\varphi(t) = \Phi_{u_0}(\varphi)(t) = U_\varphi(t,0)u_0 \quad (t \in [0,T]).$$

In particular, $\varphi \in C([0,T], Z) \cap C^1([0,T], Y)$ and

$$\varphi'(t) = \mathbf{A}_\varphi(t)\varphi(t) = \Lambda(\varphi(t))^{-1}A\varphi(t) + \Lambda(\varphi(t))^{-1}P(\varphi(t))\varphi(t) \quad (t \in [0,T]),$$
$$\varphi(0) = u_0,$$

so that indeed

$$\Lambda(\varphi(t))\varphi'(t) = A\varphi(t) + P(\varphi(t))\varphi(t) \quad (t \in [0,T]), \quad \varphi(0) = u_0.$$

We have thus shown (a).

(b) : Take another solution $v \in C([0,T'], Z) \cap C^1([0,T'], Y)$ of (Q) which satisfies $v(t) \in W$ $(t \in [0,T'])$, and hence

$$v'(t) = \Lambda(v(t))^{-1}Av(t) + \Lambda(v(t))^{-1}P(v(t))v(t) = \mathcal{A}_v(t), \quad v(0) = u_0.$$

For $0 \leqslant t \leqslant \min\{T, T'\} \eqqcolon \tau$ we then compute

$$\begin{aligned} v(t) - \varphi(t) &= U_\varphi(t,t)v(t) - U_\varphi(t,0)v(0) = \int_0^t \frac{\mathrm{d}}{\mathrm{d}s} U_\varphi(t,s)v(s)\,\mathrm{d}s \\ &= \int_0^t U_\varphi(t,s)\{\mathcal{A}_v(s) - \mathcal{A}_\varphi(s)\}v(s)\,\mathrm{d}s. \end{aligned}$$

Estimates 3.10 and 3.11 imply

$$\begin{aligned}\|v(t)-\varphi(t)\|_Y &\leqslant \int_0^t \|U_\varphi(t,s)\|_{\mathcal{B}(Y)}\|\mathcal{A}_v(s)-\mathcal{A}_\varphi(s)\|_{\mathcal{B}(Z,Y)}\|v(s)\|_Z\,\mathrm{d}s\\ &\leqslant M_T\, e^{\omega\tau}c_3 \sup_{0\leqslant s\leqslant\tau}\|v(s)\|_Z \int_0^t \|v(s)-\varphi(s)\|_Y\,\mathrm{d}s,\end{aligned}$$

where $c_3 = l_0(\|A\|_{\mathcal{B}(Z,Y)}+\lambda_2)+l_1\lambda_1$. From Gronwall's estimate we conclude

$$\|v(t)-\varphi(t)\|_Y = 0 \quad (0\leqslant t\leqslant \min\{T,T'\}).$$

(c) : Take $u_0,\tilde{u}_0 \in \overline{B}_Y(0,\kappa\frac{R}{c_0})\cap\overline{B}_Z(0,r_0)$. By the proof of part (a) there are fixed points $u,\tilde{u}\in E(T_0,r,\gamma)$ of Φ_{u_0} and $\Phi_{\tilde{u}_0}$ respective such that

$$\begin{aligned}u(t,u_0) &= u(t) = \Phi_{u_0}(u)(t),\\ u(t,\tilde{u}_0) &= \tilde{u}(t) = \Phi_{\tilde{u}_0}(\tilde{u})(t) \quad \text{for all } 0\leqslant t\leqslant T_0.\end{aligned}$$

The strict contractivity of Φ and estimate 3.10 then yield

$$\begin{aligned}\|u(t,u_0)-u(t,\tilde{u}_0)\|_Y &= \|\Phi_{u_0}(u)(t)-\Phi_{\tilde{u}_0}(\tilde{u})(t)\|_Y\\ &\leqslant \|\Phi_{u_0}(u)(t)-\Phi_{u_0}(\tilde{u})(t)\|_Y + \|\Phi_{u_0}(\tilde{u})(t)-\Phi_{\tilde{u}_0}(\tilde{u})(t)\|_Y\\ &= \|\Phi_{u_0}(u)(t)-\Phi_{u_0}(\tilde{u})(t)\|_Y + \|U_{\tilde{u}}(t,0)u_0-U_{\tilde{u}}(t,0)\tilde{u}_0\|_Y\\ &\leqslant \frac{1}{2}\|u(t,u_0)-u(t,\tilde{u}_0)\|_Y + M_Te^{\omega T}\|u_0-\tilde{u}_0\|_Y,\end{aligned}$$

so that $\|u(t,u_0)-u(t,\tilde{u}_0)\|_Y\leqslant 2M_Te^{\omega T}\|u_0-\tilde{u}_0\|_Y$.

(d) : Assertions (a)-(c) imply that there is a solution

$$u = u(\cdot,u_0)\in C([0,T],Z)\cap C^1([0,T],Y)$$

of the system

$$\begin{aligned}D\lambda(u(t))u'(t) &= Au(t) \quad (t\in[0,T]),\\ u(0) &= u_0,\end{aligned}$$

satisfying $u(t)\in\overline{B}_Y(0,R)$. Since $\lambda : B_Y(0,R+\varepsilon)\subseteq Y\to Y$ is differentiable, the chain rule yields that

$$D\lambda(u(t))u'(t) = (\lambda\circ u)'(t) \quad (t\in[0,T]).$$

Thus u actually is the desired solution. ∎

The subtle part in the verification of the assumptions of Theorem 3.43 is to find an isomorphism S which satisfies the commutator estimate (CE). Therefore we recall the considerations from Section 3.2 concerning admissible subspaces.

Remark 3.42 Let $r > 0$. Using Remark 3.28 we see that condition (CE) is fulfilled if there is a core D for A in X such that

(i) $S^{-1}D \subseteq D(A)$,

(ii) for all $u \in D$ and all $z \in \overline{B}_Z(0,r) \cap W$ we can estimate

$$\left\|S\Lambda(z)^{-1}AS^{-1}u - \Lambda(z)^{-1}Au\right\|_X \lesssim_{r,R} \|u\|_X. \quad ⋊$$

▸ **Nonlinearities of Defocusing Type**

If we can replace $W = \overline{B}_Y(0,R)$ by Y in Theorem 3.43, which corresponds to the limit $R \to \infty$, we obtain for any initial value $u_0 \in Z$ the existence of a time

$$T = T(\|u_0\|_Z) \geqslant T_0(\|u_0\|_Z) \coloneqq \frac{1}{\delta^{-1}q + b + \delta^{-1/2}L\,\|u_0\|_Z} > 0 \tag{3.16}$$

and a solution $u \in C([0,T],Y) \cap C^1([0,T],Z)$ of (Q). So, we define for an arbitrary $u_0 \in Z$ the *maximal existence time*

$$T^+(u_0) \coloneqq \left\{T > 0 : \ \exists \ \text{a solution } u \in C([0,T],Y) \cap C^1([0,T],Z) \text{ of (Q)}\right\}.$$

We have already shown that $T^+(u_0) \in (0,\infty]$ and further we want to prove the following well-posedness result for the evolution equation (Q) in this special case.

Theorem 3.43 *Assume that all the conditions of Theorem 3.43 are satisfied with $W = Y$. Let $u_0 \in Z$ and let $T_0(\|u_0\|_Z) > 0$ be given as in* (3.16)*. Then the following assertions hold:*

(a) *There is a unique maximal solution $u = u(\cdot,u_0) \in C([0,T^+(u_0)),Z) \cap C^1([0,T^+(u_0)),Y)$ of* (Q)*, where $T^+(u_0) \in (T_0(\|u_0\|_Z),\infty]$.*

(b) *If $T^+(u_0) < \infty$, then $\lim_{t\uparrow T^+(u_0)} \|u(t)\|_Z = \infty$.*

(c) *Choose any $T \in (0,T^+(u_0))$. Then there exists a radius $\eta > 0$ such that $T^+(v_0) > T$ for all $v_0 \in \overline{B}_Z(u_0,\eta)$. Further, the map*

$$\overline{B}_Z(u_0,\eta) \subseteq Y \to C([0,T],Y), \quad v_0 \mapsto u(\cdot,v_0)$$

is Lipschitz continuous. ⋊

Proof. We start with the following observations. Assume that $u \in C([0,T_1],Z) \cap C^1([0,T_1],Y)$ is a solution of (Q) on $[0,T_1]$ with initial value u_0. Then the following assertions hold.

i) If $v \in C([0,T_2],Z) \cap C^1([0,T_2],Y)$ is a solution of (Q) on $[0,T_2]$ with initial value $u(T_1)$, then the function w given by

$$w(t) := \begin{cases} u(t), & 0 \leqslant t \leqslant T_1, \\ v(t-T_1), & 0 \leqslant t \leqslant T_1 + T_2, \end{cases}$$

belongs to $C([0,T_1+T_2],Z) \cap C^1([0,T_1+T_2],Y)$ and solves (Q) with initial value u_0.

ii) Let $\tau \in (0,T_1)$. Then the function $u(\cdot+\tau)$ belongs to $C([0,T_1-\tau],Z) \cap C^1([0,T_1-\tau],Y)$ and solves (Q) with initial value $u(\tau)$.

In other words we may shift and "glue together" solutions of (Q). Using these insights, we now can copy the proof Theorem 8.6 in [17] with obvious modifications arising from Theorem 3.43. ∎

3.4 Second Order Equations

Motivated by (1.8) from Chapter 1, this section is addressed to abstract second order Cauchy-problems of the form

$$\begin{aligned} \gamma(u(t))u''(t) + [\Gamma(u(t))u'(t)]u'(t) &= -C^*Cu(t) \quad (t \in [0,T]), \\ u(0) &= u_0, \\ u'(0) &= v_0. \end{aligned} \tag{3.17}$$

Here the unknown u takes values $u(t)$ in a Hilbert space $(X,(\cdot\,|\,\cdot))$ and C is a densely defined, closed and invertible linear operator in X. Recall that consequently $C^{-1} \in \mathcal{B}(X)$ and

$$L := -C^*C$$

is self-adjoint with $0 \notin \sigma(L)$. Introducing the new variable

$$v := u',$$

we obtain the reduced first order system

$$\begin{pmatrix} I & 0 \\ 0 & \gamma(u(t)) \end{pmatrix} \begin{pmatrix} u'(t) \\ v'(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ L & -\Gamma(u(t))v(t) \end{pmatrix} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$$

for $t \in [0,T]$, with initial values $u(0) = u_0$ and $v(0) = v_0$. We thus set $w := (u,v)^\top$ and define

$$\Lambda(w) := \begin{pmatrix} I & 0 \\ 0 & \gamma(u) \end{pmatrix}, \quad Q(w) := \begin{pmatrix} 0 & 0 \\ 0 & -\Gamma(u)v \end{pmatrix}, \quad A := \begin{pmatrix} 0 & I \\ L & 0 \end{pmatrix}.$$

The Cauchy problem (3.17) then reads

$$\begin{aligned} \Lambda(w(t))w'(t) &= Aw(t) + P(w(t))w(t) \quad (t \in [0,T]), \\ w(0) &= w_0, \end{aligned} \tag{3.18}$$

where of course $w_0 = (u_0, v_0)^\top$. For stating the final result we recall the following elementary definition.

Definition 3.44 Let C be an densely defined, closed linear Operator and let X be a Hilbert space. We will write X_C^n $(n \in \mathbb{N})$ for the domain of the nth power of the operator C endowed with the inner product

$$(u \mid v)_{X_C^n} \coloneqq (C^n u \mid C^n v)_X.$$

In particular $X_C^n = D(C^n)$ and $(X_C^n, (\cdot \mid \cdot)_{X_C^n})$ is a Hilbert space. ⋊

Theorem 3.45 *Let $(H, (\cdot \mid \cdot)_H)$ be a Hilbert space and $C : D(C) \subseteq H \to H$ be a self-adjoint and invertible operator in H. Suppose that there is a ball $W \coloneqq \overline{B}_{H_C^2}(0,R)$ and families of linear operators $\{\gamma(y) : y \in W\}$ and $\{\Gamma(y) : y \in W\}$ satisfying the following assumptions.*

(PD) $\gamma(y) = \gamma(y)^* \in \mathcal{B}(H)$ *for every* $y \in W$ *and there is a constant* $\delta > 0$ *such that*

$$\gamma(y) \geqslant \delta I_H \quad (y \in W).$$

(G) $\operatorname{Ran}(I \pm \gamma(y)C^2) = H$ *for each* $y \in W$.

(LC-f) *There is a positive constant L such that*

$$\|\gamma(y) - \gamma(\tilde{y})\|_{\mathcal{B}(H)} \leqslant L \,\|y - \tilde{y}\|_{H_C^2} \quad (y, \tilde{y} \in W).$$

(LC-fi) $\gamma(y)^{-1} \in \mathcal{B}(H_C^1)$ *for any* $y \in W$ *and there is a constant* l_0 *such that*

$$\left\|\gamma(y)^{-1} - \gamma(\tilde{y})^{-1}\right\|_{\mathcal{B}(H_C^1)} \leqslant l_0 \,\|y - \tilde{y}\|_{H_C^2} \quad (y, \tilde{y} \in W).$$

(LC-s) *Let $r > 0$ be arbitrary. Then $\Gamma(y_1)y_2 \in \mathcal{B}(H)$ for each $(y_1, y_2) \in \overline{B}_{H_C^3 \times H_C^2}(0,r)$ and there is a constant $b_0 = b_0(r)$ such that*

$$\|\Gamma(y_1)y_2\|_{\mathcal{B}(H)} \leqslant b_0 \quad ((y_1, y_2) \in \overline{B}_{H_C^3 \times H_C^2}(0,r)).$$

Further, $\Gamma(y_1)y_2 \in \mathcal{B}(H_C^2, H_C^1) \eqqcolon \mathcal{B}$ for each $y = (y_1, y_2) \in \overline{B}_{H_C^2 \times H_C^1}(0,R)$ and there is a constant l_1 such that

$$\|\Gamma(y_1)y_1 - \Gamma(\tilde{y}_1)\tilde{y}_2\|_{\mathcal{B}} \leqslant l_1 \,\|y - \tilde{y}\|_{H_C^2 \times H_C^1} \quad (y,\ \tilde{y} \in \overline{B}_{H_C^2 \times H_C^1}(0,R)).$$

(CE) *Let* $r > 0$. *For all* $z = (z_1, z_2) \in \overline{B}_{H_C^2 \times H_C^1}(0, R) \cap \overline{B}_{H_C^3 \times H_C^2}(0, r)$ *there are linear operators* $B_1(z_1) \in \mathcal{B}(H_C^1, H)$ *and* $B_2(z) \in \mathcal{B}(H)$, *and constants* b_1, b_2 *with*

$$\|B_1(z_1)\|_{\mathcal{B}(H_C^1, H)} \leqslant b_1, \quad \|B_2(z)\|_{\mathcal{B}(H)} \leqslant b_2$$

such that

$$C^2 \gamma(z_1)^{-1} = \gamma(z_1)^{-1} C^2 + B_1(z_1),$$
$$C^2 \{\gamma(z_1)^{-1}[\Gamma(z_1) z_2]\} C^{-2} = \gamma(z_1)^{-1}[\Gamma(z_1) z_2] + B_2(z).$$

Note, that this includes the corresponding domain relations.

Let $\kappa \in (0,1)$ *and* $r_0 > 0$ *be arbitrary. Then the following assertions hold.*

(a) *There is a constant* $c_0 = c_0(R) > 0$ *such that for each*

$$(u_0, v_0) \in \overline{B}_{H_C^2 \times H_C^1}\left(0, \kappa \frac{R}{c_0}\right) \cap \overline{B}_{H_C^3 \times H_C^2}(0, r_0)$$

there exists a time $T = T(\kappa, r_0, R) > 0$ *and a solution*

$$u(\cdot, u_0, v_0) = u \in C([0,T], H_C^3) \cap C^1([0,T], H_C^2) \cap C^2([0,T], H_C^1)$$

of (3.18) *with* $\|(u(t), u'(t))\|_{H_C^2 \times H_C^1} \leqslant R$ *for all* $0 \leqslant t \leqslant T$.

(b) *If* $v \in C([0,T'], H_C^3) \cap C^1([0,T'], H_C^2) \cap C^2([0,T'], H_C^1)$ *is another solution of* (3.18) *which also satisfies* $\|(v(t), v'(t))\|_{H_C^2 \times H_C^1} \leqslant R$ *for each* $t \in [0, T']$, *then* $v(t) = u(t)$ *for all* $0 \leqslant t \leqslant \min\{T, T'\}$.

(c) *The mapping*

$$\overline{B}_{H_C^2 \times H_C^1}\left(0, \kappa \frac{R}{c_0}\right) \cap \overline{B}_{H_C^3 \times H_C^2}(0, r_0) \subseteq H_C^2 \times H_C^1 \to C([0, T_0], H_C^2 \times H_C^1),$$
$$(u_0, v_0) \mapsto u(\cdot, u_0, v_0),$$

is Lipschitz continuous.

(d) *If there is a constant* $\varepsilon > 0$ *and a twice Fréchet differentiable operator* $\lambda : B_{H_C^2}(0, R + \varepsilon) \subseteq H_C^2 \to H_C^2$ *such that* $\gamma(y) = D\lambda(y)$ $(y \in W)$ *and* $\Gamma(y) = D^2\lambda(y)$ $(y \in W)$, *then the assertions from* (a)-(c) *remain true if we replace* (3.18) *by the evolution equation*

$$(\lambda \circ u)''(t) = -C^2 u(t) \quad (t \in [0,T]),$$
$$u(0) = u_0,$$
$$u'(0) = v_0. \quad \rtimes$$

Proof. First, we apply Theorem 3.43 to the first order problem (3.18), by the following choice of Hilbert spaces

$$\begin{aligned} X &\coloneqq H^1_C \times H, && ((u,v) \mid (\tilde u,\tilde v))_X \coloneqq (u \mid \tilde u)_{H^1_C} + (v \mid \tilde v)_H, \\ Y &\coloneqq H^2_C \times H^1_C, && ((u,v) \mid (\tilde u,\tilde v))_Y \coloneqq (u \mid \tilde u)_{H^2_C} + (v \mid \tilde v)_{H^1_C}, \\ Z &\coloneqq H^3_C \times H^2_C, && ((u,v) \mid (\tilde u,\tilde v))_Z \coloneqq (u \mid \tilde u)_{H^3_C} + (v \mid \tilde v)_{H^2_C}. \end{aligned}$$

It is well known that the embeddings $Z \hookrightarrow Y \hookrightarrow X$ are continuous and dense, and that Y is an interpolation space between Z and X (cf. [26], Theorem 4.36). Recall

$$\Lambda(w) \coloneqq \begin{pmatrix} I & 0 \\ 0 & \gamma(u) \end{pmatrix}, \quad Q(w) \coloneqq \begin{pmatrix} 0 & 0 \\ 0 & -\Gamma(u)v \end{pmatrix}, \quad A \coloneqq \begin{pmatrix} 0 & I \\ -C^2 & 0 \end{pmatrix}$$

for $w = (u,v)$. Endowing A with the domain $D(A) \coloneqq H^1_C \times H$ it becomes skew-adjoint in X. To see this, we first calculate

$$\begin{aligned} (A(u,v) \mid (\tilde u,\tilde v))_X &= (v \mid \tilde u)_{H^1_C} - (Cu \mid C\tilde v)_H \\ &= (v \mid \tilde u)_{H^1_C} - (u \mid \tilde v)_{H^1_C} \\ &= -\{(u \mid \tilde v)_{H^1_C} - (v \mid \tilde u)_{H^1_C}\} \\ &= -\{(u \mid \tilde v)_{H^1_C} - (Cv \mid C\tilde u)_H\} \\ &= -((u,v) \mid A(\tilde u,\tilde v))_X, \end{aligned}$$

i.e., A is skew-symmetric in X. It is readily seen that A is invertible and the inverse is given by

$$A^{-1} = \begin{pmatrix} 0 & -C^{-2} \\ I & 0 \end{pmatrix},$$

which is therefore bounded. In particular $0 \in \rho(A)$, so that $\pm\mu \in \rho(A)$ for sufficiently small $\mu > 0$. Thus A is skew-adjoint. As a direct consequence of the definition of H^k_C we have $A \in \mathcal{B}(Z,Y)$ and further $Y \subseteq D(A)$. We now start to verify the conditions of Theorem 3.43 for the above choices of spaces and operators.

/// We put $\mathcal{W} \coloneqq \overline{B}_Y(0,R)$, with $R > 0$ from the assumption.

(PD) : Let $y = (y_1,y_2) \in \mathcal{W}$ and $w = (u,v), \tilde w = (\tilde u,\tilde v) \in X$. Then we estimate

$$\|\Lambda(y)w\|^2_X = \|u\|^2_{H^1_C} + \|\gamma(y_1)v\|^2_H \leqslant \max\{1, \|\gamma(y_1)\|_{\mathcal{B}(H)}\}\|w\|^2_X,$$

and derive

$$\begin{aligned} (\Lambda(y)w \mid \tilde w)_X &= (u \mid \tilde u)_{H^1_C} + (\gamma(y_1)v \mid \tilde v)_H \\ &= (u \mid \tilde u)_{H^1_C} + (v \mid \gamma(y_1)\tilde v)_H \\ &= (w \mid \Lambda(y)\tilde w)_X. \end{aligned}$$

Hence $\Lambda(y)$ is bounded and self-adjoint in X. We further estimate

$$(\Lambda(y)w \mid w)_X = (u \mid u)_{H_C^1} + \big(\gamma(y_1)v \mid v\big)_H \geqslant \min\{1,\delta\}\|w\|_X^2,$$

so that $\Lambda(y) \geqslant \min\{1,\delta\}\, I_H$.

(G) : For $w = (u,v) \in X$ and $y = (y_1, y_2) \in \mathcal{W}$ we have

$$(I \pm \Lambda(y)^{-1}A)w = \begin{pmatrix} u + v \\ \pm\gamma(y_1)C^2\, u + v \end{pmatrix}.$$

Given $\tilde{w} = (\tilde{u}, \tilde{v}) \in X$ we thus want solve the equations

$$\begin{aligned} u + v &= \tilde{u}, \\ \pm\gamma(y_1)^{-1}C^2\, u + v &= \tilde{v}. \end{aligned} \tag{3.19}$$

Because of assumption (G) of this theorem there is an element $u \in D(C^2)$ such that

$$(I \pm \gamma(y_1)^{-1}C^2)u = \tilde{u} - \tilde{v}.$$

Putting $v := \tilde{u} - w$ thus yields a solution (u,v) of system (3.19). Therefore the operator $I \pm \Lambda(y)^{-1}A$ is even onto.

(LC) : Let $w = (u,v)$ belong to X and let $y = (y_1,y_2)$, $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)$ be contained in $\mathcal{W}$. Then we derive

$$\begin{aligned} \|\Lambda(y)w - \Lambda(\tilde{y})w\|_X^2 &= \|\gamma(y_1)v - \gamma(\tilde{y}_1)v\|_H^2 \\ &\leqslant \|\gamma(y_1) - \gamma(\tilde{y}_1)\|_{\mathcal{B}(H)}^2\, \|v\|_H^2 \\ &\leqslant L^2\, \|y_1 - \tilde{y}_1\|_{H_C^2}^2\, \|v\|_H^2 \\ &\leqslant L^2\, \|y - \tilde{y}\|_Y^2\, \|w\|_X^2. \end{aligned}$$

Thus (LC) holds.

$(LC\text{-}i)$: Let $y = (y_1, y_2)$ belong to $\mathcal{W}$. The boundedness of $\gamma(y_1)^{-1}$ in H_C^1 implies

$$\Lambda(y)^{-1} = \begin{pmatrix} I & 0 \\ 0 & \gamma(y_1)^{-1} \end{pmatrix} \in \mathcal{B}(Y).$$

For $w = (u,v) \in Y$ we further calculate

$$\begin{aligned} \|\Lambda(y)^{-1}w - \Lambda(\tilde{y})^{-1}w\|_Y^2 &= \|\gamma(y_1)^{-1}v - \gamma(\tilde{y}_1)^{-1}v\|_{H_C^1}^2 \\ &\leqslant \|\gamma(y_1)^{-1} - \gamma(\tilde{y}_1)^{-1}\|_{\mathcal{B}(H_C^1)}^2\, \|v\|_{H_C^1}^2 \\ &\leqslant l_0^2\, \|y_1 - \tilde{y}_1\|_{H_C^2}^2\, \|v\|_{H_C^1}^2 \\ &\leqslant l_0^2\, \|y - \tilde{y}\|_Y^2\, \|w\|_Y^2, \end{aligned}$$

which gives $(LC\text{-}i)$.

$(LC\text{-}q)$: For $y = (y_1, y_2) \in \mathcal{W} \cap \overline{B}_Z(0, r)$ we derive

$$\begin{aligned}\|Q(y)w\|_X^2 &= \|\{\Gamma(y_1)y_2\}v\|_H^2 \leqslant \|\Gamma(y_1)y_2\|_{\mathcal{B}(H)}^2 \, \|v\|_H^2 \\ &\leqslant b_0^2 \, \|w\|_X^2,\end{aligned}$$

and therefore $\|Q(y)\|_{\mathcal{B}(X)} \leqslant b_0$. Now, let $z = (z_1, z_2)$ belong to Z. Then

$$\begin{aligned}\|Q(y)z\|_X^2 &= \|[\Gamma(y_1)y_2]z_2\|_{H_C^1}^2 \leqslant \|\Gamma(y_1)y_2\|_{\mathcal{B}}^2 \, \|z_2\|_{H_C^2}^2 \\ &\leqslant \|\Gamma(y_1)y_2\|_{\mathcal{B}}^2 \, \|z\|_Z^2\end{aligned}$$

so that $Q(y) \in \mathcal{B}(Z, Y)$ for each $y \in \mathcal{W}$. Given $y = (y_1, y_2),\ \tilde{y} = (\tilde{y}_1, \tilde{y}_2) \in \mathcal{W}$ we further estimate

$$\begin{aligned}\|Q(y)z - Q(\tilde{y})z\|_Y &\leqslant \|[\Gamma(\tilde{y}_1)\tilde{y}_2]\tilde{y}_2 - [\Gamma(y_1)y_2]y_2\|_{\mathcal{B}(H_C^2, H_C^1)} \, \|z\|_Z \\ &\leqslant l_1 \, \|y - \tilde{y}\|_Y \, \|z\|_Z,\end{aligned}$$

i.e., $(LC\text{-}q)$ holds.

(CE) : Given $r > 0$, let $z = (z_1, z_2)$ belong to $\mathcal{W} \cap \overline{B}_Z(0, r)$. As the continuous isomorphism we choose

$$S := \begin{pmatrix} -C^2 & 0 \\ 0 & -C^2 \end{pmatrix} : Z \to X.$$

Because of

$$\begin{aligned}&S\Lambda(z)^{-1}(A + Q(z))S^{-1} \\ &\quad = \begin{pmatrix} 0 & I \\ C^2\gamma(z_1)^{-1} & -C^2[\gamma(z_1)^{-1}\{\Gamma(z_1)z_2\}z_2]C^{-2} \end{pmatrix}\end{aligned}$$

it follows from the assumptions that

$$S\Lambda(z)^{-1}(A + Q(z))S^{-1} = \Lambda(z)^{-1}(A + Q(z)) + B(z)$$

if we put

$$B(z) := \begin{pmatrix} 0 & I \\ B_1(z) & B_2(z) \end{pmatrix} \in \mathcal{B}(X).$$

A straight forward calculation further yields $\|B(z)\|_{\mathcal{B}(X)} \leqslant (1 + \max\{b_1^2, b_2^2\})^{1/2}$, which proves (CE). ///

Let $\kappa \in (0, 1)$ and $r_0 > 0$. By means of Theorem 3.43 there is a constant $c_0 = c_0(R) > 0$ such that for each initial value

$$w_0 = (u_0, v_0) \in \overline{B}_Y\left(0, \kappa \frac{R}{c_0}\right) \cap \overline{B}_Z(0, r_0)$$

there is a time $T = T(\kappa, r_0, R) > 0$ and a function

$$w = (u, v) \in C([0,T], H^3_C \times H^2_C) \cap C^1([0,T], H^2_C \times H^1_C)$$

satisfying

$$\begin{aligned} u'(t) &= v(t), \\ \gamma(u(t))u''(t) &= -C^2 u(t) - \{[\Gamma(u(t))u'(t)]u'(t)\}u(t) \end{aligned}$$

and $\|u(t), u'(t)\|_Y \leqslant R$. Consequently

$$u = u(\cdot, u_0, v_0) \in C([0,T], H^3_C) \cap C^1([0,T], H^2_C) \cap C^2([0,T], H^1_C).$$

The remaining assertions are now direct consequences of the corresponding results of Theorem 3.43, (b), (c) and the chain rule. ∎

Chapter **4**

Analysis of Quasilinear Maxwell's and Wave Equations

In this final chapter, we apply the theory developed in the previous chapter to the problems (M-PC)-(M-$\mathbb{R}^3$) and (CP-W), which we will interpret as Cauchy problems in the Hilbert space of square integrable functions.

Notation. For two normed spaces X, Y we denote the space of bounded linear operators from X to Y by $\mathcal{B}(X,Y)$. Given $R > 0$, we denote by $\overline{B}_X(0,R)$, $\overline{B}_Y(0,R)$ the closed balls of radius R in X or Y respectively.

We use $a \lesssim b$ and $a \gtrsim b$ to denote the estimate $a \leqslant c\,b$ or $a \geqslant c\,b$ for some quantity c, which we call the implied constant. We will further write $a \sim b$ if both $a \lesssim b$ and $a \gtrsim b$ hold. If we need the implied constant to depend on parameters (e.g. p, d) we will indicate this by using subscripts, i.e., $a \lesssim_{p,d} b$ and so on. All functions we will consider in this chapter are taking values in real vector spaces. In particular we will write $L^p(\Omega)^m$, $H^s(\Omega)^n$ $(m, n \in \mathbb{N})$ etc. for the corresponding spaces of real valued functions or vector fields.

4.1 Maxwell's Equations

Let $\Omega \subseteq \mathbb{R}^3$ be either a bounded domain with boundary $\partial\Omega$ or the full space $\mathbb{R}^3$. We consider the quasilinear Maxwell's equations without external sources or charges given by

$$\begin{aligned} \partial_t D(t,x) &= \operatorname{rot} H(t,x) && (t \in [0,T],\ x \in \Omega), \\ \partial_t B(t,x) &= -\operatorname{rot} E(t,x) && (t \in [0,T],\ x \in \Omega), \\ \operatorname{div} D(t,x) &= 0 && (t \in [0,T],\ x \in \Omega), \\ \operatorname{div} B(t,x) &= 0 && (t \in [0,T],\ x \in \Omega). \end{aligned}$$

If $\Omega \neq \mathbb{R}^3$, we impose either perfect conducting boundary conditions

$$\begin{aligned} E(t,x) \wedge \mathfrak{n}(x) &= 0 \quad (t \in [0,T],\ x \in \partial\Omega), \\ B(t,x) \bullet \mathfrak{n}(x) &= 0 \quad (t \in [0,T],\ x \in \partial\Omega), \end{aligned} \tag{Pc}$$

or Dirichlet boundary conditions

$$\begin{aligned} E(t,x) &= 0 \quad (t \in [0,T],\ x \in \partial\Omega), \\ B(t,x) \bullet \mathfrak{n}(x) &= 0 \quad (t \in [0,T],\ x \in \partial\Omega). \end{aligned} \tag{Dr}$$

We restrict our studies to the local constitutive relations (Cr-L).

Material Law 4.1 We consider nonlinearities taking the form

$$D(t,x) = E(t,x) + P\big(E(t,x)\big), \quad B(t,x) = H(t,x) + M\big(H(t,x)\big),$$

for some vector fields $P, M : \mathbb{R}^3 \to \mathbb{R}^3$. We further assume that they are sufficiently smooth and not to negative, more precisely

(N1) $P, M \in C^{s+1}(\mathbb{R}^3, \mathbb{R}^3)$, where $s \in \mathbb{N}_0$ is specified later on.

(N2) $P'(0) > -I, \quad M'(0) > -I.$ ⋊

Consequences 4.2 *Since P' and M' are continuous it follows from (N2), that there are $\rho > 0$ and $\eta > -1$ such that*

$$P'(y_1) \geqslant \eta I, \quad M'(y_2) \geqslant \eta I \quad \big(y_1,\ y_2 \in \overline{B}_{\mathbb{R}^3}(0,\rho) \subseteq \mathbb{R}^3\big).$$

Putting $\delta := \eta + 1 > 0$ we thus obtain

$$I + P'(y_1) \geqslant \delta I, \quad I + M'(y_2) \geqslant \delta I \quad \big(y_1,\ y_2 \in \overline{B}_{\mathbb{R}^3}(0,\rho)\big), \tag{4.1}$$

and hence $I + P'(y_1)$ and $I + M'(y_2)$ are invertible with uniformly bounded inverses

$$\begin{aligned} &\left| (I + P'(y_1))^{-1} \right|_{\mathbb{R}^{3\times 3}} \leqslant \frac{1}{\delta}, \\ &\left| (I + M'(y_2))^{-1} \right|_{\mathbb{R}^{3\times 3}} \leqslant \frac{1}{\delta} \quad \big(y_1,\ y_2 \in \overline{B}_{\mathbb{R}^3}(0,\rho)\big). \end{aligned}$$

By means of the mean value theorem we further get constants $L_k,\ k \in \{0,\dots,s\}$ *such that*

$$\begin{aligned} \left| D^k P(y_1) - D^k P(\tilde{y}_1) \right|_{\mathcal{B}_k} &\leqslant L_k\,|y_1 - \tilde{y}_1|, \\ \left| D^k M(y_2) - D^k M(\tilde{y}_2) \right|_{\mathcal{B}_k} &\leqslant L_k\,|y_2 - \tilde{y}_2| \qquad \left(y_j,\ \tilde{y}_j \in \overline{B}_{\mathbb{R}^3}(0,\rho^*)\right), \end{aligned} \tag{4.2}$$

where $\rho^* > 0$ *is arbitrary and* $\mathcal{B}_k$ *is recursively defined via*

$$\mathcal{B}_0 \coloneqq \mathbb{R}^3, \quad \mathcal{B}_{k+1} \coloneqq \mathcal{B}(\mathbb{R}^3, \mathcal{B}_k). \quad \rtimes$$

If we differentiate D and B with respect to t and the material laws 3.1, we obtain

$$\begin{pmatrix} I + P'(E(t,x)) & 0 \\ 0 & I + M'(H(t,x)) \end{pmatrix} \partial_t \begin{pmatrix} E(t,x) \\ H(t,x) \end{pmatrix} = \begin{pmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{pmatrix} \begin{pmatrix} E(t,x) \\ H(t,x) \end{pmatrix}.$$

We introduce

$$u : J \times \Omega \subseteq \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^6, \quad u(t,x) = (E(t,x), H(t,x))^\top,$$

and define

$$\begin{aligned} &\lambda_1(y_1) \coloneqq I + P'(y_1), \quad \lambda_2(y_2) \coloneqq I + M'(y_2) \qquad (y_1,\ y_2 \in \mathbb{R}^3), \\ &\lambda(y_1, y_2) \coloneqq \begin{pmatrix} \lambda_1(y_1) & \\ & \lambda_2(y_2) \end{pmatrix}, \quad \mathbf{A} \coloneqq \begin{pmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{pmatrix}. \end{aligned}$$

The equations now become

$$\lambda(u(t,x))\,\partial_t u(t,x) = \mathbf{A}u(t,x) \quad (t \in [0,T],\ x \in \Omega). \tag{4.3}$$

Phase space 4.3 In the following we will consider (4.3) as an evolution equation in the Hilbert space

$$X \coloneqq X_0 \times X_0 \coloneqq L^2(\Omega)^3 \times L^2(\Omega)^3,$$

endowed with the scalar product

$$(u \mid v)_X \coloneqq (u_1 \mid v_1)_{X_0} + (u_2 \mid v_2)_{X_0} \quad (u = (u_1, u_2),\ v = (v_1, v_2) \in X).$$

We frequently use the identification of $L^2(\Omega, \mathbb{R}^3)$ with $L^2(\Omega)^3$, which particularly means

$$(u \mid v)_{L^2(\Omega,\mathbb{R}^3)} = \int_\Omega u(x) \cdot v(x)\,\mathrm{d}x = \sum_{k=1}^{3} \int_\Omega u_k(x) v_k(x)\,\mathrm{d}x = (u \mid v)_{L^2(\Omega)^3}.$$

We will denote both of these inner products and the associated norms simply by $(\cdot \mid \cdot)_{X_0}$ and $\|\cdot\|_{X_0}$ respectively.

The substitution operators corresponding to λ_j and λ are denoted by Λ_j and Λ, respectively. So we have

$$[\Lambda_j(y_j)u_j](x) = \lambda_j(y_j(x))u_j(x) \quad (x \in \Omega,\ j = 1,2)$$

for $y_j,\ u_j \in X_0,\ j = 1,2$. Finally, we consider $\mathbf{A}$ as a the Maxwell operator from Definition 2.42, i.e., $D(\mathbf{A}) = H_0(\mathrm{rot}, \Omega) \times H(\mathrm{rot}, \Omega)$. ⋊

A further important ingredient for our upcoming analysis is the connection between the substitution operators Λ_j from above and the substitution operators indicated by the nonlinearties P and M, i.e.,

$$\mathbf{P}(E) \coloneqq P \circ E, \quad \mathbf{M}(H) \coloneqq M \circ H.$$

More precisely, we want to know in which situations $\mathbf{P}$ and $\mathbf{M}$ are differentiable and their derivatives satisfy

$$I + \mathbf{P}'(E) = \Lambda_1(E), \quad I + \mathbf{M}'(H) = \Lambda_2(H).$$

Proposition 4.4 *Let Ω be a Lipschtiz domain and suppose that the vector fields P and M belong to $C^3(\mathbb{R}^3, \mathbb{R}^3)$. Then for every $r > 0$ the operators $\mathbf{P}$, $\mathbf{M}$ given by*

$$\begin{aligned} \mathbf{P} &: B_{H^2(\Omega)^3}(0,r) \subseteq H^2(\Omega)^3 \to H^2(\Omega)^3, \quad \mathbf{P}(y) \coloneqq P \circ y, \\ \mathbf{M} &: B_{H^2(\Omega)^3}(0,r) \subseteq H^2(\Omega)^3 \to H^2(\Omega)^3, \quad \mathbf{M}(y) \coloneqq M \circ y, \end{aligned}$$

are Fréchet differentiable and their derivatives are given by

$$\mathbf{P}'(y)h = (P' \circ y)h, \quad \mathbf{M}'(y)h = (M' \circ y)h. \quad ⋊$$

Proof. It surely suffices to consider the operator $\mathbf{P}$. Let y, h belong to $B_{H^2(\Omega)^3}(0,r)$ such that also $y + h \in B_{H^2(\Omega)^3}(0,r)$. By means of Taylor's formula we obtain for almost every $x \in \Omega$ the equality

$$P(y(x) + h(y)) - P(y(x)) - P'(y(x))h(x) = \int_0^1 [P'(y(x) + th(x)) - P'(y(x))]h(x)\,\mathrm{d}t \eqqcolon R_h(x).$$

We will now show that

$$\|R_h\|_{H^2(\Omega)^3} \lesssim_r \|h\|^2_{H^2(\Omega)^3},$$

which proves the claim. In the following we suppress the variable x for simplicity. Because of (4.2) there are constants $L_k,\ k \in \{1,2,3\}$ such that

$$\left| D^k P(y + th) - D^k P(y) \right|_{\mathcal{B}_k} \leqslant t\, L_k\, |h|\,. \tag{$\star$}$$

The Sobolev embedding $H^2(\Omega)^3 \hookrightarrow L^\infty(\Omega)^3$ further implies that there is some $\rho > 0$ so that

$$\|y\|_{H^2(\Omega)^3} \leqslant r \quad \Rightarrow \quad \|y\|_{L^\infty(\Omega)^3} \leqslant \rho.$$

Consequently if $y, h \in B_{H^2(\Omega)^3}(0, r)$, then $\|y + th\|_{L^\infty(\Omega)^3} \leqslant 2\rho$ for each $0 \leqslant t \leqslant 1$. The continuity assumption on the derivatives of P then implies that there are constants $c_k,\ k \in \{1, 2, 3\}$ such that

$$\left| D^k P(y + th) \right|_{\mathcal{B}_k} \leqslant c_k. \tag{$\star\star$}$$

First, we start estimating $\|R_h\|_{L^2(\Omega)^3}$. By means of $(\star)$ we estimate pointwise

$$|R_h| \leqslant L_1 \, |h|^2.$$

Invoking the Sobolev embedding $H^2(\Omega)^3 \hookrightarrow L^\infty(\Omega)^3$, we derive

$$\begin{aligned} \|R_h\|_{L^2(\Omega)^3} &\leqslant L_1 \, \| \, |h| \, |h| \, \|_{L^2(\Omega)} \leqslant L_1 \, \|h\|_{L^\infty(\Omega)^3} \, \|h\|_{L^2(\Omega)} \\ &\lesssim \|h\|^2_{H^2(\Omega)^3}. \end{aligned}$$

By means of the product and chain rule for weak derivatives we (a priori formally) calculate

$$\begin{aligned} \partial_k R_h = &\int_0^1 [P''(y + th)\{\partial_k y + t\partial_k h\} - P''(y)\partial_k y] h \, \mathrm{d}t \\ &+ \int_0^1 [P'(y + th) - P'(y)]\partial_k h \, dt. \end{aligned} \tag{4.4}$$

We see that

$$\partial_k R_h(x) = \int_0^1 f(t, x) \, \mathrm{d}t,$$

where, due to $(\star)$ and $(\star\star)$, the integrand is dominated by a constant, which is integrable with respect to the interval $[0, 1]$. This justifies the prior interchanging of the integral and the partial derivative. The second summand in (4.4) equals $R_{\partial_k h}$. Using again the Sobolev embedding $H^2 \hookrightarrow L^\infty$, we get

$$\begin{aligned} \|R_{\partial_k h}\|_{L^2(\Omega)^3} &\leqslant L_1 \, \| \, |h| \, |\partial_k h| \, \|_{L^2(\Omega)} \leqslant L_1 \, \|h\|_{L^\infty(\Omega)^3} \, \|\partial_k h\|_{L^2(\Omega)^3} \\ &\lesssim \|h\|^2_{H^2(\Omega)^3}. \end{aligned}$$

Rewriting the first integral as

$$\int_0^1 [\{P''(y + th) - P''(y)\}\partial_l y] h \, \mathrm{d}t + \int_0^1 t[P''(y + th)\partial_k h] h \, \mathrm{d}t =: R_1$$

and invoking the estimates $(\star)$ and $(\star\star)$, we deduce

$$|R_1| \leqslant L_2\,|h|\,|\partial_k y|\,|h| + c_2\,|\partial_k h|\,|h|.$$

The Sobolev embedding $L^\infty \hookrightarrow H^2$ thus yields

$$\begin{aligned}\|R_1\|_{L^2(\Omega)^3} &\lesssim \|\,|h|\,|\partial_k y|\,|h|\,\|_{L^2(\Omega)} + \|\,|\partial_k h|\,|h|\,\|_{L^2(\Omega)}\\ &\lesssim \|h\|^2_{H^2(\Omega)^3}\,\|y\|_{H^2(\Omega)^3} + \|h\|^2_{H^2(\Omega)^3}\\ &\lesssim_r \|h\|^2_{H^2(\Omega)^3},\end{aligned}$$

so that in the end

$$\|\partial_k R_h\|_{L^2(\Omega)^3} \lesssim_r \|h\|^2_{H^2(\Omega)^3} \quad (1 \leqslant k \leqslant 3).$$

For the second order derivatives we obtain

$$\begin{aligned}
&\partial_l\partial_k R_h = \\
&\quad \int_0^1 [\{P'''(y+th)\partial_l y\}\partial_k y - \{P'''(y)\partial_l y\}\partial_k y]h\,dt && =: T_1\\
&\quad \int_0^1 [P''(y+th)\partial_l\partial_k y - P''(y)\partial_l\partial_k y]h\,dt && =: T_2\\
&\quad \int_0^1 t[\{P'''(y+th)\partial_l y\}\partial_k h + \{P'''(y+th)\partial_l h\}\partial_k y]h\,dt && =: T_3\\
&\quad \int_0^1 [t^2\{P'''(y+th)\partial_l h\}\partial_k h + t\,P''(y+th)\partial_l\partial_k h]h\,dt && =: T_4\\
&\quad \int_0^1 [P''(y+th)\{\partial_k y + t\partial_k h\} - P''(y)\partial_k y]\partial_l h\,dt && =: T_5\\
&\quad \int_0^1 [P''(y+th)\{\partial_l y + t\partial_l h\} - P''(y)\partial_l y]\partial_k h\,dt && =: T_6\\
&\quad \int_0^1 [P'(y+th) - P'(y)]\partial_l\partial_k h\,dt && =: T_7.
\end{aligned}$$

By means of $(\star)$ and $(\star\star)$, we collect the following pointwise estimates

$$\begin{aligned}
|T_1| &\lesssim |h|\,|\partial_l y|\,|\partial_k y|\,|h|,\\
|T_2| &\lesssim |h|\,|\partial_l\partial_k y|\,|h|,\\
|T_3| &\lesssim |\partial_l y|\,|\partial_k h|\,|h| + |\partial_l h|\,|\partial_k y|\,|h|,\\
|T_4| &\lesssim |\partial_l h|\,|\partial_k h|\,|h| + |\partial_l\partial_k h|\,|h|,\\
|T_5| &\lesssim |h|\,|\partial_k y|\,|\partial_l h| + |\partial_l h|\,|\partial_k h|,\\
|T_6| &\lesssim |h|\,|\partial_l y|\,|\partial_k h| + |\partial_k h|\,|\partial_l h|,\\
|T_7| &\lesssim |h|\,|\partial_l\partial_k h|.
\end{aligned}$$

Invoking the Sobolev embeddings $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^p$ $(3 \leqslant p \leqslant 6)$ and also Hölder's inequality with $1/6 + 1/3 = 1/2$, we deduce

$$\begin{aligned}\|T_1\|_{L^2(\Omega)^3} &\lesssim \| |h|\,|\partial_l y|\,|\partial_k y|\,|h| \|_{L^2(\Omega)} \lesssim \|h\|^2_{L^\infty(\Omega)^3} \|\partial_l y\|_{L^6(\Omega)^3} \|\partial_k y\|_{L^3(\Omega)^3} \\ &\lesssim \|h\|^2_{H^2(\Omega)^3} \|\partial_l y\|_{H^1(\Omega)^3} \|\partial_k y\|_{H^1(\Omega)^3} \\ &\lesssim_{r^2} \|h\|^2_{H^2(\Omega)^3},\end{aligned}$$

$$\begin{aligned}\|T_2\|_{L^2(\Omega)^3} &\lesssim \| |h|\,|\partial_l\partial_k y|\,|h| \|_{L^2(\Omega)} \lesssim \|h\|^2_{H^2(\Omega)^3} \|\partial_l\partial_k y\|_{L^2(\Omega)^3} \\ &\lesssim_r \|h\|^2_{H^2(\Omega)^3}.\end{aligned}$$

Concerning the remaining terms T_3-T_7 the only parts we have not controlled so far are

$$\| |\partial_l y|\,|\partial_k h|\,|h| \|_{L^2(\Omega)}, \quad \| |\partial_l h|\,|\partial_k h|\,|h| \|_{L^2(\Omega)}, \quad \| |\partial_l\partial_k h|\,|h| \|_{L^2(\Omega)}.$$

Using exactly the same procedure from above, we obtain

$$\begin{aligned}\| |\partial_l y|\,|\partial_k h|\,|h| \|_{L^2(\Omega)} &\lesssim \|h\|_{L^\infty(\Omega)^3} \|\partial_l y\|_{L^6(\Omega)^3} \|h\|_{L^3(\Omega)^3} \\ &\lesssim \|h\|^2_{H^2(\Omega)^3} \|y\|_{H^2(\Omega)^3} \lesssim_r \|h\|^2_{H^2(\Omega)^3},\end{aligned}$$

$$\begin{aligned}\| |\partial_l h|\,|\partial_k h|\,|h| \|_{L^2(\Omega)} &\lesssim \|h\|_{L^\infty(\Omega)^3} \|\partial_l h\|_{L^6(\Omega)^3} \|h\|_{L^3(\Omega)^3} \\ &\lesssim \|h\|^2_{H^2(\Omega)^3} \|h\|_{H^2(\Omega)^3} \lesssim_r \|h\|^2_{H^2(\Omega)^3},\end{aligned}$$

$$\begin{aligned}\| |\partial_l\partial_k h|\,|h| \|_{L^2(\Omega)} &\lesssim \|h\|_{L^\infty(\Omega)^3} \|\partial_l\partial_k h\|_{L^2(\Omega)^3} \\ &\lesssim \|h\|^2_{H^2(\Omega)^3}.\end{aligned}$$

Putting all these estimates together we have shown that

$$\|\partial_l\partial_k R_h\|_{L^2(\Omega)^3} \lesssim_{r^2} \|h\|^2_{H^2(\Omega)^3} \quad (1 \leqslant l, k \leqslant 3),$$

which finally implies $\|R_h\|_{H^2(\Omega)^3} \lesssim_{r^2} \|h\|^2_{H^2(\Omega)^3}$. ∎

To include the constraints given by the Gaussian laws and the imposed boundary conditions, we use the following observations.

Lemma 4.5 *Let $k \in \mathbb{N}$ with $k \geqslant 3$, and $r > 0$.*

(a) *Let $u_0 = (E_0, H_0) \in X$ satisfy $D_0 \coloneqq E_0 + P(E_0) \in H(\mathrm{div}, \Omega)$ and $B_0 \coloneqq H_0 + M(H_0) \in H(\mathrm{div}, \Omega)$ with*

$$\mathrm{div}\, D_0 = \mathrm{div}\, B_0 = 0\,.$$

If $u = (E, H) \in C([0,T], H^k(\Omega)^6) \cap C^1([0,T], H^2(\Omega)^6)$ *is a solution of the initial value problem*

$$\Lambda(u(t))u'(t) = \mathbf{A}u(t) \quad (t \in [0,T]), \quad u(0) = u_0,$$

which also satisfies $\|u(t)\|_{H^k(\Omega)^6} \leqslant r$*, then* $D(t) \coloneqq E(t) + P(E(t))$ *and* $B(t) \coloneqq H(t) + M(H(t))$ *are contained in* $H(\mathrm{div}, \Omega)$ *for all* $t \in [0,T]$*, and*

$$\mathrm{div}\, D(t) = \mathrm{div}\, B(t) = 0 \quad (t \in [0,T]).$$

(b) *Let* $u_0 = (E_0, H_0) \in X$ *satisfy* $B_0 \coloneqq H_0 + M(H_0) \in H_0(\mathrm{div}, \Omega)$ *and let* Ω *be a compact Lipschitz domain. If* $u = (E, H) \in C([0,T], H^k(\Omega)^6) \cap C^1([0,T], H^2(\Omega)^6)$ *with* $u(t) \in H(\mathrm{rot}, \Omega)^2$ *is a solution of the initial value problem*

$$\Lambda(u(t))u'(t) = \mathbf{A}u(t) \quad (t \in [0,T]), \quad u(0) = u_0,$$

which also satisfies $\|u(t)\|_{H^k(\Omega)^6} \leqslant r$*, then* $B(t) = H(t) + M(H(t)) \in H_0(\mathrm{div}, \Omega)$ *for all* $t \in [0,T]$. ⋊

Proof. (a) : By means of Proposition 4.4, we may apply the chain rule to the mapping $t \mapsto D(t)$ and get

$$\frac{\mathrm{d}}{\mathrm{d}t} D(t) = (I + DP(E(t)))E'(t) = \mathrm{rot}\, H(t).$$

Let $\varphi \in \mathcal{D}(\Omega)$. Integration by parts (a priori formally) implies

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega D(t) \cdot \mathrm{grad}\, \varphi \, \mathrm{d}x &= \int_\Omega \mathrm{rot}\, H(t) \cdot \mathrm{grad}\, \varphi \, \mathrm{d}x \\ &= \int_\Omega H(t) \cdot \mathrm{rot}\, \mathrm{grad}\, \varphi \, \mathrm{d}x = 0. \end{aligned}$$

Hence

$$\int_\Omega D(t) \cdot \mathrm{grad}\, \varphi \, \mathrm{d}x = \int_\Omega D_0 \cdot \mathrm{grad}\, \varphi \, \mathrm{d}x = 0 \quad (t \in [0,T])$$

and the first assertion follows from (2.2). We can do exactly the same calculations for B. So, it only remains to justify the interchanging of the differential $\mathrm{d}/\mathrm{d}t$ and the integral. Invoking the Sobolev embedding $H^2(\Omega)^3 \hookrightarrow L^\infty(\Omega)^3$, we estimate

$$\begin{aligned} |\mathrm{rot}\, H(t) \cdot \mathrm{grad}\, \varphi| &\leqslant \|\mathrm{rot}\, H(t)\|_{L^\infty(\Omega)^3} |\mathrm{grad}\, \varphi| \\ &\lesssim \|H(t)\|_{H^k(\Omega)^3} |\mathrm{grad}\, \varphi| \lesssim r\, |\mathrm{grad}\, \varphi|. \end{aligned}$$

The integrand is thus dominated by a integrable function with respect to Ω.

(b) : For $\varphi \in \mathcal{D}(\overline{\Omega})$, we calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega B(t) \cdot \mathrm{grad}\, \varphi \, \mathrm{d}x &= -\int_\Omega \mathrm{rot}\, E(t) \cdot \mathrm{grad}\, \varphi \, \mathrm{d}x \\ &= -\int_\Omega E(t) \cdot \mathrm{rot}\, \mathrm{grad}\, \varphi \, \mathrm{d}x = 0, \end{aligned}$$

so that the assertion follows from Corollary 2.22 (b). Note, that we have skipped the technical details, since they are exactly the same as in (a). ∎

■ Dirichlet Boundary Problems

Now, we are ready to tackle the Maxwell-type Cauchy problems (M-Pc)-(M-$\mathbb{R}^3$) in the promised evolution framework. For treating the initial values in the way indicated above, we define the set

$$I_C := \Big\{(E,H) \in L^2(\Omega)^6 \,:\, E_0 + P(E_0) \in H(\operatorname{div},\Omega),\ \operatorname{div}(E_0 + P(E_0)) = 0,$$
$$H_0 + M(H_0) \in H_0(\operatorname{div},\Omega),\ \operatorname{div}(H_0 + M(H_0)) = 0\Big\}.$$

Given real parameters r and α, we further put

$$W(\alpha, r) := \overline{B}_{H^2(\Omega)^6 \cap H_0^1(\Omega)^6}(0,\alpha) \cap \overline{B}_{\left\{u \in H^4(\Omega)^6 \cap H_0^1(\Omega)^6 \,:\, \Delta u \in H_0^1(\Omega)^6\right\}}(0,r) \cap I_C.$$

We start with the Dirichlet problem (M-Dr).

Theorem 4.6 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded C^4-domain, and let the vector fields $P, M : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy the assumptions (N1) and (N2) for $s = 4$, i.e.,*

$$P,\ M \in C^5(\mathbb{R}^3, \mathbb{R}^3), \quad P'(0) > -I, \quad M'(0) > -I.$$

Further, let $\kappa \in (0,1)$ and $r_0 > 0$ be arbitrary. Then the following assertions hold.

(a) *There is a radius $R > 0$ and an associated constant $c_0 = c_0(R) > 0$, satisfying $c_0(R) \to \infty$ $(R \to \infty)$, such that for each*

$$u_0 = (E_0, H_0) \in W(\kappa\, c_0, r_0)$$

there exists a time $T = T(R, r_0, \kappa) > 0$ and a function

$$u(\cdot, u_0) = (E,H) \in C([0,T], H^4(\Omega)^6) \cap C^1([0,T], H^2(\Omega)^6),$$

with $\|u(t)\|_{H^2} \leqslant R$ $(t \in [0,T])$ which solves the Maxwell-type Cauchy problem (M-Dr) *by means of*

$$\begin{aligned}
(D \circ E)'(t) &= \operatorname{rot} H(t) && (t \in [0,T]),\\
(B \circ H)'(t) &= -\operatorname{rot} E(t) && (t \in [0,T]),\\
\operatorname{div} D(E(t)) &= 0 && (t \in [0,T]),\\
\operatorname{div} B(H(t)) &= 0 && (t \in [0,T]),\\
E(t) &\in H_0^1(\Omega) && (t \in [0,T]),\\
B(H(t)) &\in H_0(\operatorname{div},\Omega) && (t \in [0,T]),\\
u(0) &= u_0\,,
\end{aligned}$$

where $B(H) = H + M(H)$ and $D(E) = E + P(E)$. Moreover, we know that $u(t) \in H^2(\Omega)^6 \cap H_0^1(\Omega)^6$ for any $t \in [0,T]$.

(b) *If* $v \in C([0,T'],H^4(\Omega)^6) \cap C^1([0,T'],H^2(\Omega)^6)$ *is another solution of this system with* $\|v(t)\|_{H^2} \leqslant R$ *for all* $0 \leqslant t \leqslant T'$*, then* v *coincides with* u *on the interval* $[0,\min\{T,T'\}]$*. Further, the map*

$$W(\kappa c_0, r_0) \subseteq H^2(\Omega)^6 \to C([0,T],H^2(\Omega)^6), \quad u_0 \mapsto u(\cdot,u_0)$$

is Lipschitz continuous. $\rtimes$

Proof. The results will follow from Theorem 3.43. We start with some preparations. We denote by Δ_D the Dirichlet Laplace operator in $X_0 = L^2(\Omega)^3$ with domain $D(\Delta_D) = H^2(\Omega)^3 \cap H_0^1(\Omega)^3$. The resulting positive operator $-\Delta_D$ gives rise to the scale of spaces

$$X_0 = L^2(\Omega)^3, \quad X_n := D((-\Delta_D)^{n/2}) \quad (n \in \mathbb{N}),$$

with inner products given by

$$(u \mid v)_{X_n} := ((-\Delta_D)^{n/2}u \mid (-\Delta_D)^{n/2}v)_{X_0}.$$

The maps $\Delta_D : X_{n+2} \to X_n$ and $(-\Delta_D)^{n/2} : X_n \to X_0$ are isometric isomorphisms for every $n \in \mathbb{N}_0$. We further introduce the Hilbert spaces

$$Y := Y_0 \times Y_0 := D(-\Delta_D) \times D(-\Delta_D),$$
$$Z := Z_0 \times Z_0 := D((-\Delta_D)^2) \times D((-\Delta_D)^2),$$

and recall the isomorphisms

$$Y_0 = H^2(\Omega)^3 \cap H_0^1(\Omega)^3,$$
$$Z_0 = \left\{u \in H^4(\Omega)^3 \cap H_0^1(\Omega)^3 \,:\, \Delta u \in H_0^1(\Omega)^3\right\},$$

cf. [15], Theorem 8.13, or [12]. It is well known that the embeddings $Z \hookrightarrow Y \hookrightarrow X$ are continuous and dense, and that Y is an interpolation space between Z and X (cf. [26], Theorem 4.36). For the Maxwell operator $\mathbf{A}$ we have

$$D(\mathbf{A}) = H_0(\mathrm{rot},\Omega) \times H(\mathrm{rot},\Omega)$$

so that it becomes skew adjoint in X by Theorem 2.43 (a), and obviously $Y \subseteq D(\mathbf{A})$. For $1 \leqslant k \leqslant 3$ let J_k be the 3×3 matrices from Remark 2.3. Then we put

$$A_k := \begin{pmatrix} 0 & J_k \\ -J_k & 0 \end{pmatrix} \in \mathbb{R}^{6\times 6}$$

so that $\mathbf{A}$ can be written as

$$\mathbf{A} = \sum_{k=1}^{3} A_k \, \partial_{x_k},$$

and therefore obviously $\mathbf{A} \in \mathcal{B}(Z, Y)$. We also recall the Sobolev embeddings

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega), \quad H^1(\Omega) \hookrightarrow L^p(\Omega) \quad (p \in [3,6]).$$

Now, let $\rho > 0$ be the radius from the consequences 4.2 of assumption $(N2)$. Because of the first Sobolev embedding we can fix some Radius $R = R(\rho) > 0$ such that

$$\|u\|_{Y_0} \leqslant R \quad \Rightarrow \quad \|u\|_{L^\infty(\Omega)^3} \leqslant \rho\,. \tag{4.5}$$

We define

$$W := \overline{B}_Y(0,R) = \left\{(y_1, y_2) \in Y \,:\, \|y_1\|_{Y_0}^2 + \|y_2\|_{Y_0}^2 \leqslant R^2\right\}.$$

In particular $\|y_j\|_{L^\infty(\Omega)^3} \leqslant \rho$ for $j = 1, 2$ and $y = (y_1, y_2) \in W$. In view of Proposition 4.4, we already know that $\operatorname{diag}(\mathbf{P}, \mathbf{M}) : B_Y(0, R + \varepsilon) \subseteq Y \to Y$ is Fréchet differentiable for each $\varepsilon > 0$ and that the derivative plus the identity equals Λ.

We will now start to verify the assumptions of the mentioned theorem. We recall that $a \lesssim_x b$, for $a, b \in \mathbb{R}$, $x \in \mathbb{R}^d$, means that there is some constant $c = c(x)$ depending only on $x \in \mathbb{R}^d$ such that $a \leqslant c(x)\, b$.

/// (PD) : For $y = (y_1, y_2) \in W$ and $u = (u_1, u_2) \in X$ we obtain

$$\begin{aligned}\|\Lambda(y)u\|_X^2 &= \int_\Omega |\lambda_1(y_1(x))u_1(x)|^2\,\mathrm{d}x + \int_\Omega |\lambda_2(y_2(x))u_2(x)|^2\,\mathrm{d}x \\ &\leqslant \int_\Omega |I + P'(y_1(x))|_{\mathbb{R}^{3\times3}}^2 |u_1(x)|^2\,\mathrm{d}x \\ &\qquad + \int_\Omega |I + M'(y_2(x))|_{\mathbb{R}^{3\times3}}^2 |u_2(x)|^2\,\mathrm{d}x.\end{aligned}$$

Estimates (4.2) and (4.5) then imply

$$|I + P'(y_1(x))|_{\mathbb{R}^{3\times3}} \leqslant 1 + L_1, \quad |I + M'(y_2(x))|_{\mathbb{R}^{3\times3}}^2 \leqslant 1 + L_1,$$

so that $\Lambda(y) \in \mathcal{B}(X)$ with $\|\Lambda(y)\|_{\mathcal{B}(X)} \leqslant 1 + L_1$. By means of (4.1), we also estimate

$$\begin{aligned}(\Lambda(y)u \,|\, u)_X &= \int_\Omega \lambda_1(y_1(x))u_1(x) \cdot u_1(x)\,\mathrm{d}x + \int_\Omega \lambda_2(y_2(x)) \cdot u_2(x)\,\mathrm{d}x \\ &\geqslant \delta\,\|u_1\|_{X_0}^2 + \delta\,\|u_2\|_{X_0}^2 = \delta\,\|u\|_X^2,\end{aligned}$$

which means $\Lambda(y) \geqslant \delta I$.

(G) : Follows from Theorem 2.43, (a).

(LC) : For $y, \tilde{y} \in W$ and $u \in X$ we get

$$\begin{aligned}\|\Lambda(y)u - \Lambda(y)u\|_X \leqslant \sqrt{2}\big(&\|\Lambda_1(y_1)u_1 - \Lambda_1(\tilde{y}_1)u_1\|_{X_0} \\ &+ \|\Lambda_2(y_2)u_2 - \Lambda_2(\tilde{y}_2)u_2\|_{X_0}\big).\end{aligned}$$

Since each argument for P is exactly the same for M, it is enough to consider Λ_1. Using (4.2) and the Sobolev embedding, we estimate

$$\begin{aligned}
&\|\Lambda_1(y_1)u_1 - \Lambda_1(\tilde{y}_1)u_1\|_{X_0}^2 \\
&\qquad \leqslant \int_\Omega |P'(y_1(x)) - P'(\tilde{y}_1(x))|_{\mathbb{R}^{3\times 3}}^2 \, |u_1(x)|^2 \, \mathrm{d}x \\
&\qquad \leqslant L_2^2 \int_\Omega |y_1(x) - \tilde{y}_1(x)|^2 \, |u_1(x)|^2 \, \mathrm{d}x \\
&\qquad \lesssim \|y_1 - \tilde{y}_1\|_{Y_0}^2 \, \|u_1\|_{X_0}^2
\end{aligned}$$

so that

$$\|\Lambda(y)u - \Lambda(y)u\|_X \lesssim \|y - \tilde{y}\|_Y \, \|u\|_X .$$

$(LC\text{-}i)$: We make some observations first. Let $u \in Y$. For $y, \tilde{y} \in \overline{B}_{\mathbb{R}^3}(0,\rho)$ we have

$$(I + P'(y))^{-1} - (I + P'(\tilde{y}))^{-1} = (I + P'(y))^{-1}\Big\{P'(\tilde{y}) - P'(y)\Big\}(I + P'(\tilde{y}))^{-1}.$$

Note, that we will use $P', P'', \ldots, P^{(k)}$, for the kth derivative of the vector field P. Therefore (4.1) and (4.2) yield

$$\Big|(I + P'(y))^{-1} - (I + P'(\tilde{y}))^{-1}\Big|_{\mathbb{R}^{3\times 3}} \leqslant \Big\{\delta^{-2}L_2\Big\} \, |y - \tilde{y}|. \tag{4.6}$$

The same estimate is true if we replace P by M. We further use the derivative of the matrix valued functions $T_P, T_M : B_{\mathbb{R}^3}(0,\rho) \to \mathbb{R}^3$ given by

$$T_P(y) := (I + P'(y))^{-1}, \quad T_M(y) := (I + M'(y))^{-1}.$$

Since e.g., $T_P = \mathrm{inv} \circ (I + P')$ where

$$\begin{aligned}
I + P' : B_{\mathbb{R}^3}(0,\rho) \subseteq \mathbb{R}^3 \to \mathbb{R}^{3\times 3}_{\mathrm{inv}}, &\quad y \mapsto I + P'(y) \\
\mathrm{inv} : \mathbb{R}^{3\times 3}_{\mathrm{inv}} \to \mathbb{R}^{3\times 3}, &\quad A \mapsto A^{-1},
\end{aligned}$$

the chain rule implies $DT_P(y) = [D\mathrm{inv}(I{+}P'(y))]D(I{+}P')(y)$. Recalling $D\mathrm{inv}(A)H = -A^{-1}HA^{-1}$ for all $H \in \mathbb{R}^{3\times 3}$ and using $D(I + P')(y) = P''(y) \in \mathcal{B}(\mathbb{R}^3, \mathbb{R}^{3\times 3})$, we derive

$$DT_P(y)v = -(I + P'(y))^{-1} \, [P''(y)v] \, (I + P'(y))^{-1} \quad (v \in \mathbb{R}^3). \tag{4.7}$$

An analogous formula holds for T_M.

For $y = (y_1, y_2) \in W$ we already know that $\Lambda(y) \in \mathcal{B}(X)$ is invertible with $\|\Lambda(y)^{-1}\|_{\mathcal{B}(X)} \leqslant \delta^{-1}$. We actually have

$$[\Lambda(y)^{-1}u](x) = \begin{pmatrix} (I + P'(y_1(x)))^{-1}u_1(x) \\ (I + M'(y_2(x)))^{-1}u_2(x) \end{pmatrix}.$$

In particular,

$$[\Lambda(0)^{-1}u](x) = \begin{pmatrix} (I + P'(0))^{-1}u_1(x) \\ (I + M'(0)))^{-1}u_2(x) \end{pmatrix},$$

which means $\Lambda(0)^{-1}$ just multiplies u with a constant matrix. Therefore it follows immediately that $\Lambda(0)^{-1} \in \mathcal{B}(Y)$ with $\|\Lambda(0)^{-1}\|_{\mathcal{B}(Y)} \leqslant \delta^{-1}$. Because of (a priori formally)

$$\|\Lambda(y)^{-1}u - \Lambda(\tilde{y})^{-1}u\|_Y^2 = \sum_{j=1}^{2} \|\Lambda_j(y_j)^{-1}u_j - \Lambda_j(\tilde{y}_j)^{-1}u_j\|_{Y_0}^2,$$

we again focus on Λ_1 and drop the indicees for simplicity. We have

$$\begin{aligned}
\|\Lambda(y)^{-1}u - \Lambda(\tilde{y})^{-1}u\|_{Y_0}^2 = & \\
& \|T_P(y)u - T_P(\tilde{y})u\|_{X_0}^2 && =: I_1 \\
& + \sum_{k=1}^{3} \|\partial_k T_P(y)u - \partial_k T_P(\tilde{y})u\|_{X_0}^2 && =: I_2 \\
& + \sum_{k,l=1}^{3} \|\partial_l\partial_k T_P(y)u - \partial_l\partial_k T_P(\tilde{y})u\|_{X_0}^2 && =: I_3\,.
\end{aligned}$$

Applying estimate (4.6) and the Sobolev embedding, we obtain

$$I_1 \lesssim \|y - \tilde{y}\|_{Y_0}^2 \, \|u\|_{Y_0}^2.$$

The product and chain rule and formula (4.7) for DT_P imply that the weak partial derivatives in I_2 are given by

$$\begin{aligned}
&\partial_k\Big(T_P(y(x))u(x)\Big) = \\
&\quad T_P(y(x))\partial_k u(x) - T_P(y(x))[P''(y)\partial_k y(x)]T_P(y(x))u(x),
\end{aligned} \tag{4.8}$$

where $\{e_k : 1 \leqslant k \leqslant 3\}$ denotes the standard basis in $\mathbb{R}^3$. Writing $I_2 = \sum_k I_{2,k}$ and expanding

$$\begin{aligned}
I_{2,k}^{1/2} \leqslant\ & ||T_P(y)\partial_k u - T_P(\tilde{y})\partial_k u||_{X_0} && =: J_1 \\
& + ||T_P(\tilde{y})[\{P''(\tilde{y}) - P''(y)\}\partial_k\tilde{y}]T_P(\tilde{y})u||_{X_0} && =: J_2 \\
& + ||T_P(\tilde{y})[P''(y)\partial_k\tilde{y}]\{T_P(\tilde{y}) - T_P(y)\}u||_{X_0} && =: J_3 \\
& + ||\{T_P(\tilde{y}) - T_P(y)\}[P''(y)\partial_k\tilde{y}]T_P(y)u||_{X_0} && =: J_4 \\
& + ||T_P(y)[P''(y)\{\partial_k\tilde{y} - \partial_k y\}]T_P(y)u||_{X_0} && =: J_5
\end{aligned}$$

we continue estimating J_1-J_5. First, J_1 can be controlled in the same way as I_1, i.e.,

$$J_1 \lesssim \|y - \tilde{y}\|_{Y_0} \, \|\partial_k u\|_{X_0} \lesssim \|y - \tilde{y}\|_{Y_0} \, \|u\|_{Y_0}.$$

By means of (4.2) we estimate

$$
\begin{aligned}
&\Big| T_P(\tilde{y})\Big\{P''(\tilde{y})\partial_k\tilde{y} - P''(y)\partial_k\tilde{y}\Big\}T_P(y)u\Big| \\
&\qquad \leqslant \delta^{-2}\,|P''(\tilde{y}) - P''(y)|_{\mathcal{B}(\mathbb{R}^3,\mathbb{R}^{3\times 3})}\,|\partial_k\tilde{y}|\,|u| \\
&\qquad \leqslant \delta^{-2}L_3\,|y-\tilde{y}|\,|\partial_k\tilde{y}|\,|u|.
\end{aligned}
$$

Using Hölder's inequality with $1/3 + 1/6 = 1/2$ together with the Sobolev embeddings $H^1 \hookrightarrow L^p$ $(p \in [3,6])$ and $H^2 \hookrightarrow L^\infty$ we derive

$$
\begin{aligned}
J_2 &\lesssim \|y-\tilde{y}\|_{Y_0}\,\big\|\,|\partial_k\tilde{y}|\,|u|\,\big\|_{X_0} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\,\big\|\partial_k\tilde{y}\big\|_{L^3}\,\|u\|_{L^6} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\,\big\|\partial_k\tilde{y}\big\|_{H^1}\,\|u\|_{H^1} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\,\big\|\tilde{y}\big\|_{H^2}\,\|u\|_{H^2} \\
&\lesssim_R \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.
\end{aligned}
$$

In a similar way we estimate

$$
\Big|\{T_P(y) - T_P(\tilde{y})\}[P''(y)\partial_k\tilde{y}]T_P(y)u\Big| \lesssim |y-\tilde{y}|\,|\partial_l\tilde{y}|\,|u|
$$

and therefore again

$$
J_3, J_4 \lesssim_R \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.
$$

Finally, for J_5 we conclude, again using Hölder and Sobolev as done before,

$$
J_5 \lesssim \big\|\,|\partial_k y - \partial_k\tilde{y}|\,|u|\,\big\|_{X_0} \lesssim \big\|\partial_k y - \partial_k\tilde{y}\big\|_{H^1}\,\|u\|_{H^1} \lesssim \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.
$$

So, we have shown

$$
I_2 \lesssim_R \|y-\tilde{y}\|_{Y_0}^2\,\|u\|_{Y_0}^2,
$$

and it only remains to control I_3. Thus we need the second derivatives

$$
\partial_l\partial_k(T_P(y(x))u(x)) = \partial_l\Big(\partial_k(T_P(y(x))u(x))\Big).
$$

Therefore we have to differentiate expressions of the form $T_P(y)AT_P(y)u$ for a sufficiently regular matrix-valued function A. We have

$$
\begin{aligned}
\partial_l\,(T_P(y)AT_P(y)u) = &- T_P(y)[P''(y)\partial_l y]T_P(y)AT_P(y)u \\
&+ T_P(y)[\partial_l A]T_P(y)u \\
&- T_P(y)AT_P(y)[P''(y)\partial_l y]T_P(y)u \\
&+ T_P(y)AT_P(y)\partial_l u.
\end{aligned}
\tag{4.9}
$$

Introducing the matrix

$$\begin{aligned}\mathbf{d}_l(y,A) &\coloneqq -[P''(y)\partial_l y]T_P(y)A + \partial_l A - AT_P(y)[P''(y)\partial_l y]\\ &= \partial_l A - \Big\{[P''(y)\partial_l y]T_P(y)A + AT_P(y)[P''(y)\partial_l y]\Big\}\\ &\eqqcolon \partial_l A - [\![A, P_y]\!]\partial_l y,\end{aligned}$$

we can rewrite (4.9) as

$$\partial_l\left(T_P(y)AT_P(y)u\right) = T_P(y)\mathbf{d}_l(y,A)T_P(y)u + T_P(y)AT_P(y)\partial_l u.$$

Applying this together with (4.8) we arrive at

$$\begin{aligned}\partial_l\partial_k(T_P(y)u) = \;& T_P(y)\partial_l\partial_k u\\ &- T_P(y)[P''(y)\partial_l y]T_P(y)\partial_k u\\ &- T_P(y)\mathbf{d}_l(y, P''(y)\partial_k y)T_P(y)u\\ &- T_P(y)[P''(y)\partial_k y]T_P(y)\partial_l u,\end{aligned}$$

where

$$\begin{aligned}\mathbf{d}_l(y, P''(y)\partial_k y) &= \partial_l[P''(y)\partial_k y] - [\![[P''(y)\partial_k y], P_y]\!]\partial_l y\\ &= [P'''(y)\partial_l y]\partial_k y + P''(y)\partial_l\partial_k y\\ &\quad - [P''(y)\partial_l y]T_P(y)[P''(y)\partial_k y]\\ &\quad - [P''(y)\partial_k y]T_P(y)[P''(y)\partial_l y].\end{aligned}$$

Regrouping the resulting terms we thus obtain

$$\begin{aligned}\partial_l\partial_k(T_P(y)u) =& T_P(y)\partial_l\partial_k u\\ &- T_P(y)\Big[P''(y)\partial_l y\Big]T_P(y)\partial_k u\\ &- T_P(y)\Big[P''(y)\partial_k y\Big]T_P(y)\partial_l u\\ &- T_P(y)\Big[P''(y)\partial_l\partial_k y\Big]T_P(y)u\\ &- T_P(y)\Big[\{P'''(y)\partial_l y\}\partial_k y\Big]T_P(y)u\\ &+ T_P(y)\Big\{[P''(y)\partial_l y]T_P(y)[P''(y)\partial_k y]\Big\}T_P(y)u\\ &+ T_P(y)\Big\{[P''(y)\partial_k y]T_P(y)[P''(y)\partial_l y]\Big\}T_P(y)u.\end{aligned}\tag{4.10}$$

We can repeat the procedure as done for I_2. The first part $T_P(y)\partial_l\partial_k u - T_P(\tilde{y})\partial_l\partial_k u$ can be treated exactly in the same way as in the corresponding estimate concerning I_2. We consider the difference

$$\tilde{I}_1 \coloneqq T_P(\tilde{y})\Big[P''(\tilde{y})\partial_l\tilde{y}\Big]T_P(\tilde{y})\partial_k u - T_P(y)\Big[P''(y)\partial_l y\Big]T_P(y)\partial_k u.$$

We expand $\tilde{I}_1$ to

$$\begin{aligned}\tilde{I}_1 =& T_P(\tilde{y})[P''(\tilde{y})\partial_l\tilde{y}]\{T_P(\tilde{y}) - T_P(y)\}\partial_k u\\ &+ T_P(\tilde{y})\{P''(\tilde{y})[\partial_l\tilde{y} - \partial_l y]\}T_P(y)\partial_k u\\ &+ T_P(\tilde{y})\{[P''(\tilde{y}) - P''(y)]\partial_l y\}T_P(y)\partial_k u\\ &+ \{T_P(\tilde{y}) - T_P(y)\}[P''(y)\partial_l y]T_P(y)\partial_k u\end{aligned}$$

and enumerate the four resulting terms by $\tilde{J}_1$-$\tilde{J}_4$. Using (4.6), Hölder's inequality with $1/3 + 1/6 = 1/2$ and the Sobolev embeddings, we estimate

$$\begin{aligned}\|\tilde{J}_1\|_{X_0} &\lesssim \|\tilde{y} - y\|_{L^\infty} \,\|\, |\partial_l\tilde{y}|\,|\partial_k u|\,\|_{X_0}\\ &\lesssim \|y - \tilde{y}\|_{Y_0}\,\|\partial_l\tilde{y}\|_{L^3}\,\|\partial_k u\|_{L^6}\\ &\lesssim \|y - \tilde{y}\|_{Y_0}\,\|\partial_l\tilde{y}\|_{H^1}\,\|\partial_k u\|_{H^1}\\ &\lesssim \|y - \tilde{y}\|_{Y_0}\,\|\tilde{y}\|_{Y_0}\,\|u\|_{Y_0}\\ &\lesssim_R \|y - \tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.\end{aligned}$$

Similarly, (also invoking (4.2)) we obtain

$$\begin{aligned}\|\tilde{J}_2\|_{X_0} &\lesssim \|\, |\partial_l\tilde{y} - \partial_l y|\,|\partial_k u|\,\|_{X_0} \lesssim \|\partial_l y - \partial_l\tilde{y}\|_{H^1}\,\|\partial_k u\|_{H^1}\\ &\lesssim \|y - \tilde{y}\|_{Y_0}\,\|u\|_{Y_0},\\ \|\tilde{J}_3\|_{X_0} &\lesssim \|\tilde{y} - y\|_{L^\infty}\,\|\, |\partial_l y|\,|\partial_k u|\,\|_{X_0} \lesssim_R \|y - \tilde{y}\|_{Y_0}\,\|u\|_{Y_0},\\ \|\tilde{J}_4\|_{X_0} &\lesssim \|\tilde{y} - y\|_{L^\infty}\,\|\, |\partial_l y|\,|\partial_k u|\,\|_{X_0} \lesssim_R \|y - \tilde{y}\|_{Y_0}\,\|u\|_{Y_0},\end{aligned}$$

so that in the end

$$\|\tilde{I}_1\|_{X_0} \lesssim_R \|y - \tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.$$

Copying the above estimates and interchanging k with l, we see that

$$\tilde{I}_2 := T_P(\tilde{y})\Big[P''(\tilde{y})\partial_k\tilde{y}\Big]T_P(\tilde{y})\partial_l u - T_P(y)\Big[P''(y)\partial_k y\Big]T_P(y)\partial_l u$$

also fulfills

$$\|\tilde{I}_2\|_{X_0} \lesssim_R \|y - \tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.$$

Next, we investigate

$$\tilde{I}_3 := T_P(\tilde{y})\Big[P''(\tilde{y})\partial_l\partial_k\tilde{y}\Big]T_P(\tilde{y})u - T_P(y)\Big[P''(y)\partial_l\partial_k y\Big]T_P(y)u.$$

Expanding (as usual)

$$\begin{aligned}\tilde{I}_3 =& \{T_P(\tilde{y}) - T_P(y)\}[P''(\tilde{y})\partial_l\partial_k\tilde{y}]T_P(\tilde{y})u\\ &+ T_P(y)[\{P''(\tilde{y}) - P''(y)\}\partial_l\partial_k\tilde{y}]T_P(\tilde{y})u\\ &+ T_P(y)[P''(y)\{\partial_l\partial_k\tilde{y} - \partial_l\partial_k y\}]T_P(\tilde{y})u\\ &+ T_P(y)[P''(y)\partial_l\partial_k y]\{T_P(\tilde{y}) - T_P(y)\}u\end{aligned}$$

and enumerating the resulting terms by $\overline{J}_1$ to $\overline{J}_4$, we can estimate as follows. Using (4.6), the Sobolev embeddings and (4.2), we estimate

$$\begin{aligned}
\|\overline{J}_1\|_{X_0} &\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\, |\partial_l\partial_k\tilde{y}|\,|u|\,\|_{X_0} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\partial_l\partial_k\tilde{y}\|_{X_0}\, \|u\|_{L^\infty} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\tilde{y}\|_{Y_0}\, \|u\|_{Y_0} \\
&\lesssim_R \|y-\tilde{y}\|_{Y_0}\, \|u\|_{Y_0}
\end{aligned}$$

and

$$\begin{aligned}
\|\overline{J}_2\|_{X_0} &\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\, |\partial_l\partial_k\tilde{y}|\,|u|\,\|_{X_0} \lesssim_R \|y-\tilde{y}\|_{Y_0}\, \|u\|_{Y_0}, \\
\|\overline{J}_3\|_{X_0} &\lesssim \|\, |\partial_l\partial_k\tilde{y}-\partial_l\partial_k y|\,|u|\,\|_{X_0} \lesssim \|y-\tilde{y}\|_{Y_0}\, \|u\|_{Y_0}, \\
\|\overline{J}_4\|_{X_0} &\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\, |\partial_l\partial_k\tilde{y}|\,|u|\,\|_{X_0} \lesssim_R \|y-\tilde{y}\|_{Y_0}\, \|u\|_{Y_0}.
\end{aligned}$$

Summing up, we obtain

$$\|\tilde{I}_3\|_{X_0} \lesssim_R \|y-\tilde{y}\|_{Y_0}\, \|u\|_{Y_0}.$$

Now, we consider

$$\tilde{I}_4 \coloneqq T_P(\tilde{y})\Big[\{P'''(\tilde{y})\partial_l\tilde{y}\}\partial_k\tilde{y}\Big]T_P(\tilde{y})u - T_P(y)\Big[\{P'''(y)\partial_l y\}\partial_k y\Big]T_P(y)u.$$

We use our canonical expanding procedure

$$\begin{aligned}
\tilde{I}_4 =& \{T_P(\tilde{y})-T_P(y)\}[\{P'''(\tilde{y})\partial_l\tilde{y}\}\partial_k\tilde{y}]T_P(\tilde{y})u \\
&+ T_P(y)[\{[P'''(\tilde{y})-P'''(y)]\partial_l\tilde{y}\}\partial_k\tilde{y}]T_P(\tilde{y})u \\
&+ T_P(y)[\{P'''(y)[\partial_l\tilde{y}-\partial_l y]\}\partial_k\tilde{y}]T_P(\tilde{y})u \\
&+ T_P(y)[\{P'''(y)\partial_l y\}\{\partial_k\tilde{y}-\partial_k y\}]T_P(\tilde{y})u \\
&+ T_P(y)[\{P'''(y)\partial_l y\}\partial_k y]\{T_P(\tilde{y})-T_P(y)\}u
\end{aligned}$$

and enumerate the resulting pieces from $\hat{J}_1$ to $\hat{J}_5$. By means of (4.6), Hölder's inequality with $1/6+1/6+1/6=1/2$ and the Sobolev embeddings, we estimate

$$\begin{aligned}
\|\hat{J}_1\|_{X_0} &\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\, |\partial_l\tilde{y}|\,|\partial_k\tilde{y}|\,|u|\,\|_{X_0} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\partial_l\tilde{y}\|_{L^6}\, \|\partial_k\tilde{y}\|_{L^6}\, \|u\|_{L^6} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\partial_l\tilde{y}\|_{H^1}\, \|\partial_k\tilde{y}\|_{H^1}\, \|u\|_{H^1} \\
&\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\tilde{y}\|_{Y_0}^2\, \|u\|_{Y_0} \\
&\lesssim_{R^2} \|y-\tilde{y}\|_{Y_0}\, \|u\|_{Y_0}.
\end{aligned}$$

In a similar way (also using (4.2)), we further derive

$$\begin{aligned}
\|\hat{J}_2\|_{X_0} &\lesssim \|y-\tilde{y}\|_{Y_0}\, \|\,|\partial_l\tilde{y}|\,|\partial_k\tilde{y}|\,|u|\,\|_{X_0} \lesssim_{R^2} \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0},\\
\|\hat{J}_3\|_{X_0} &\lesssim \|\,|\partial_l\tilde{y}-\partial_l y|\,|\partial_k y|\,|u|\,\|_{X_0} \lesssim \|\partial_l\tilde{y}-\partial_l y\|_{L^6}\,\|\partial_k y\|_{L^6}\,\|u\|_{L^6}\\
&\lesssim_R \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0},\\
\|\hat{J}_4\|_{X_0} &\lesssim \|\,|\partial_l y|\,|\partial_k\tilde{y}-\partial_k y|\,|u|\,\|_{X_0} \lesssim \|\partial_l y\|_{L^6}\,\|\partial_k\tilde{y}-\partial_k y\|_{L^6}\,\|u\|_{L^6}\\
&\lesssim_R \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0},\\
\|\hat{J}_5\|_{X_0} &\lesssim \|y-\tilde{y}\|_{Y_0} \lesssim_{R^2} \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.
\end{aligned}$$

We thus have shown that

$$\|\tilde{I}_5\|_{X_0} \lesssim_R \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0}.$$

Finally, it only remains to analyse

$$\begin{aligned}
\tilde{I}_6 \coloneqq T_P(y)\Big\{[P''(y)\partial_l y]T_P(y)[P''(y)\partial_k y]\Big\}T_P(y)u\\
- T_P(\tilde{y})\Big\{[P''(\tilde{y})\partial_l\tilde{y}]T_P(\tilde{y})[P''(\tilde{y})\partial_k\tilde{y}]\Big\}T_P(\tilde{y})u,
\end{aligned}$$

since the remaining part only differs from this one in k and l. For the sake of clarity we put

$$A^{\#}(y) \coloneqq P''(y)\partial_l y, \quad \# \in \{k,l\}$$

and expand (one last time)

$$\begin{aligned}
\tilde{I}_6 =& \{T_P(y)-T_P(\tilde{y})\}[A^l(y)T_P(y)A^k(y)]T_P(y)u\\
&+ T_P(\tilde{y})[\{A^l(y)-A^l(\tilde{y})\}T_P(y)A^k(y)]T_P(y)u\\
&+ T_P(\tilde{y})[A^l(\tilde{y})\{T_P(y)-T_P(\tilde{y})\}A^k(y)]T_P(y)u\\
&+ T_P(\tilde{y})[A^l(\tilde{y})T_P(\tilde{y})\{A^k(y)-A^k(\tilde{y})\}]T_P(y)u\\
&+ T_P(\tilde{y})[A^l(y)T_P(y)A^k(y)]\{T_P(y)-T_P(\tilde{y})\}u,
\end{aligned}$$

where further

$$A^{\#}(y)-A^{\#}(\tilde{y}) = \{P''(y)-P''(\tilde{y})\}\partial_{\#}y + P''(\tilde{y})\{\partial_{\#}y-\partial_{\#}\tilde{y}\}.$$

We enumerate the above summands from $\mathcal{J}_1$ to $\mathcal{J}_5$. By means of (4.6), Hölder's inequality with $1/6+1/6+1/6=1/2$ and the Sobolev embeddings, we estimate $\mathcal{J}_1$, $\mathcal{J}_3$ and $\mathcal{J}_5$ (cf. the corresponding estimates of $\hat{J}_2$) by

$$\begin{aligned}
\|\mathcal{J}_n\|_{X_0} &\lesssim \|y-\tilde{y}\|_{Y_0}\,\|\,|\partial_l\tilde{y}|\,|\partial_k\tilde{y}|\,|u|\,\|_{X_0}\\
&\lesssim_{R^2} \|y-\tilde{y}\|_{Y_0}\,\|u\|_{Y_0} \quad (n\in\{1,3,5\}).
\end{aligned}$$

The estimates for $\mathcal{J}_2$ and $\mathcal{J}_4$ are identical, we only have to switch the roles of k and l. Therefore we only consider $\mathcal{J}_2$ in greater detail. Using (4.2), Hölder's inequality with $1/6 + 1/6 + 1/6 = 1/2$ and the Sobolev embeddings, we estimate

$$\begin{aligned}\|\mathcal{J}_2\|_{X_0} &\lesssim \|y - \tilde{y}\|_{Y_0} \| \, |\partial_l y| \, |\partial_k y| \, |u| \, \|_{X_0} + \| \, |\partial_l y - \partial_l \tilde{y}| \, |\partial_k y| \, |u| \, \|_{X_0} \\ &\lesssim_{R^2} \|y - \tilde{y}\|_{Y_0} \|u\|_{Y_0} + \|\partial_l \tilde{y} - \partial_l y\|_{L^6} \|\partial_k y\|_{L^6} \|u\|_{L^6} \\ &\lesssim_{R^2} \|y - \tilde{y}\|_{Y_0} \|u\|_{Y_0}.\end{aligned}$$

Consequently

$$\|\tilde{I}_6\|_{X_0} \lesssim_{R^2} \|y - \tilde{y}\|_{Y_0} \|u\|_{Y_0}$$

so that finally

$$I_3 \lesssim_{R^2} \|y - \tilde{y}\|_{Y_0}^2 \|u\|_{Y_0}^2.$$

Putting all these these things together we have shown that

$$\left\|\Lambda(y)^{-1}u - \Lambda(\tilde{y})^{-1}u\right\|_Y \leqslant l_0 \left\|y - \tilde{y}\right\|_Y \|u\|_Y \quad (y, \tilde{y} \in W, \; u \in Y), \tag{4.11}$$

for some $l_0 = \mathcal{O}(R^2)$ and further $\Lambda(0)^{-1} \in \mathcal{B}(Y)$. In particular

$$\Lambda(y)^{-1}u - \Lambda(\tilde{y})^{-1}u \in Y \quad (y, \tilde{y} \in W, \; u \in Y).$$

Hence for each $y \in W$ and $u \in Y$, we derive

$$\begin{aligned}\|\Lambda(y)^{-1}u\|_Y &\leqslant \|\Lambda(y)^{-1}u - \Lambda(0)^{-1}\|_Y + \|\Lambda(0)^{-1}u\|_Y \\ &\leqslant (\|y\|_Y + \|\Lambda(0)^{-1}\|_{\mathcal{B}(Y)})\|u\|_Y \\ &\leqslant (R + \|\Lambda(0)^{-1}\|_{\mathcal{B}(Y)})\|u\|_Y.\end{aligned}$$

Consequently $\Lambda(y)^{-1} \in \mathcal{B}(Y)$ for each $y \in W$ and by means of (4.11) it follows

$$\left\|\Lambda(y)^{-1} - \Lambda(\tilde{y})^{-1}\right\|_{\mathcal{B}(Y)} \leqslant l_0 \left\|y - \tilde{y}\right\|_Y \quad (y, \tilde{y} \in W),$$

i.e., $(LC\text{-}i)$ holds.

(CE) : Recall that Ω is a bounded C^4-domain. We choose the isometric isomorphism

$$S := \begin{pmatrix} \Delta_D^2 & 0 \\ 0 & \Delta_D^2 \end{pmatrix} : Z \to X$$

and will show that for each $z \in W \cap \overline{B}_Z(0, r)$, where $r > 0$ is arbitrary, we can estimate

$$\left\|S\Lambda(z)^{-1}\mathbf{A}S^{-1}u - \Lambda(z)^{-1}\mathbf{A}u\right\|_X \lesssim_{r,R} \|u\|_X$$

for all $u = (E, H)$ in a core of $\mathbf{A}$ which contains C^∞-functions only, i.e., for $\mathcal{D}(\overline{\Omega})^3 \times \mathcal{D}(\Omega)^3$. Remark 3.42 then implies (CE).

Because of

$$S\Lambda(z)^{-1}\mathbf{A}S^{-1} = \begin{pmatrix} 0 & \Delta_D^2 T_P(z) \operatorname{rot} \Delta_D^{-2} \\ -\Delta_D^2 T_M(z) \operatorname{rot} \Delta_D^{-2} & 0 \end{pmatrix}$$

it is enough to show that

$$\left\| \Delta_D^2 T_P(z) \operatorname{rot} \Delta_D^{-2} E - T_P(z) \operatorname{rot} E \right\|_{X_0} \lesssim_{r,R} \|E\|_{X_0}.$$

Since we are working on smooth functions, we have $\operatorname{rot} = \sum_{1 \leqslant k \leqslant 3} J_k \partial_k$ and therefore $\operatorname{rot} \Delta_D^{-2} E = \Delta_D^{-2} \operatorname{rot} E$, for these smooth functions E. Further, we can calculate Δ_D^2 through $\sum_{k,j} \partial_k \partial_k \partial_j \partial_j$. We write $v \coloneqq \operatorname{rot} \Delta_D^{-2} E$ and first consider the derivatives $\partial_m \partial_l \partial_k T_P(z) v$. We recall (4.10) and introduce the following notation. Given functions $z_1, \dots, z_n \in X_0$, we write

$$\Pi(z_1, \dots, z_n)$$

if there are operators $B_0^j, \dots, B_n^j \in \mathcal{B}(X_0)$, $j \in \{1, \dots, N\}$ such that

(i) $\Pi(z_1, \dots, z_n) = \sum_{j=1}^N B_0^j z_{\pi_j(1)} B_1^j z_{\pi_j(2)} B_2^j \dots B_{n-1}^j z_{\pi_j(n)} B_n^j$ for permutations π_j of $\{1, \dots, n\}$,

(ii) $\|\Pi(z_1, \dots, z_n) v\|_{X_0} \lesssim \| \, |z_1| \, \dots \, |z_n| \, |v| \, \|_{X_0} \quad (v \in X_0)$.

Note, that in particular

$$\Pi(z_{\pi(1)}, \dots, z_{\pi(n)}) = \Pi(z_1, \dots, z_n) \tag{4.12}$$

for each permutation π of $\{1, \dots, n\}$, and

$$\Pi(z_1, \dots, z_n) + \Pi(z_1, \dots, z_n) = \Pi(z_1, \dots, z_n). \tag{4.13}$$

Using this notation we can for example rewrite (4.10) as

$$\begin{aligned} \partial_l \partial_k T_P(y) u = & T_P(y) \partial_l \partial_k u \\ & + \Pi(\partial_l y) \, \partial_k u + \Pi(\partial_k y) \, \partial_l u \\ & + \{\Pi(\partial_k y, \partial_l y) + \Pi(\partial_l \partial_k y)\} u. \end{aligned}$$

Now, we start to derive the derivatives of order three. The starting point (as already mentioned) is the formula for the second order derivatives (4.10), which we will denote in the following way

$$
\begin{aligned}
\partial_l\partial_k(T_P(z)v) =& T_P(z)\partial_l\partial_k v && =: T_1\\
&- T_P(z)\Big[P''(z)\partial_l z\Big]T_P(z)\partial_k v && =: T_2\\
&- T_P(z)\Big[P''(z)\partial_k z\Big]T_P(z)\partial_l v && =: T_3\\
&- T_P(z)\Big[P''(z)\partial_l\partial_k z\Big]T_P(z)v && =: T_4\\
&- T_P(z)\Big[\{P'''(z)\partial_l z\}\partial_k z\Big]T_P(z)v && =: T_5\\
&+ T_P(z)\Big\{[P''(z)\partial_l z]T_P(z)[P''(z)\partial_k z]\Big\}T_P(z)v && =: T_6\\
&+ T_P(z)\Big\{[P''(z)\partial_k z]T_P(z)[P''(z)\partial_l z]\Big\}T_P(z)v && =: T_7.
\end{aligned}
$$

We will apply the formulas (4.8) and (4.9) to each term T_j $(1 \leqslant j \leqslant 7)$. First,

$$
\begin{aligned}
\partial_m T_1 &= T_P(z)\partial_m\partial_l\partial_k v - T_P(z)[P''(z)\partial_m z]T_P(z)\partial_l\partial_k v\\
&= T_P(z)\partial_m\partial_l\partial_k v + \Pi(\partial_m z)\,\partial_l\partial_k v.
\end{aligned}
$$

For T_2 we find

$$
\begin{aligned}
-\partial_m T_2 =& T_P(z)[\{P'''(z)\partial_m z\}\partial_l z]T_P(z)\partial_k v\\
&+ T_P(z)[P''(z)\partial_m\partial_l z]T_P(z)\partial_k v\\
&+ T_P(z)[P''(z)\partial_l z]T_P(z)\partial_m\partial_k v\\
&+ T_P(z)[[\![P''(z)\partial_l z, P_z]\!]\partial_m z]T_P(z)\partial_k v.
\end{aligned}
$$

Because of

$$
\| \, [[\![P''(z)\partial_l z, P_z]\!]\partial_m z]u \, \|_{X_0} \lesssim \| \, |\partial_l z| \, |\partial_m z| \, |u| \, \|_{X_0} \quad (u \in X_0),
$$

we can use (4.12) and (4.13) to rewrite $\partial_m T_2$ as

$$
\partial_m T_2 = \Pi(\partial_l z)\,\partial_m\partial_k v + \{\Pi(\partial_l z, \partial_m z) + \Pi(\partial_m\partial_l z)\}\partial_k v.
$$

Switching the roles of k and l transforms T_3 into T_2. Thus we obtain

$$
\partial_m T_3 = \Pi(\partial_k z)\,\partial_m\partial_l v + \{\Pi(\partial_k z, \partial_m z) + \Pi(\partial_m\partial_k z)\}\partial_l v.
$$

Now, for T_4 we derive

$$
\begin{aligned}
-\partial_m T_4 =& T_P(z)[\{P'''(z)\partial_m z\}\partial_l\partial_k z]T_P(z)v\\
&+ T_P(z)[P''(z)\partial_m\partial_l\partial_k z]T_P(z)v\\
&+ T_P(z)[P''(z)\partial_l\partial_k z]T_P(z)\partial_m v\\
&+ T_P(z)[[\![P''(z)\partial_l\partial_k z, P_z]\!]\partial_m z]T_P(z)v.
\end{aligned}
$$

Applying the Π-notation and regrouping the resulting terms yield

$$\partial_m T_4 = \Pi(\partial_l \partial_k z)\,\partial_m \nu + \{\Pi(\partial_m z, \partial_l \partial_k z) + \Pi(\partial_m \partial_l \partial_k z)\}\nu.$$

For T_5 we further find

$$\begin{aligned}
-\partial_m T_5 =& T_P(z)[\{[P^{(4)}(z)\partial_m z]\partial_l z\}\partial_k z]T_P(z)\nu \\
&+ T_P(z)[\{P'''(z)\partial_m \partial_l z\}\partial_k z]T_P(z)\nu \\
&+ T_P(z)[\{P'''(z)\partial_l z\}\partial_m \partial_k z]T_P(z)\nu \\
&+ T_P(z)[\{P'''(z)\partial_l z\}\partial_k z]T_P(z)\partial_m \nu \\
&+ T_P(z)[[\![\{P''(z)\partial_l z\}\partial_k z, P_z]\!]\partial_m z]T_P(z)\nu.
\end{aligned}$$

Again Π-notation with its rules (4.12) and (4.13) and regrouping lead to

$$\begin{aligned}
\partial_m T_5 =& \Pi(\partial_k z, \partial_l z)\,\partial_m \nu \\
&+ \{\Pi(\partial_k z, \partial_m \partial_l z) + \Pi(\partial_l z, \partial_m \partial_k z) + \Pi(\partial_k z, \partial_l z, \partial_m z)\}\nu.
\end{aligned}$$

Deriving the desired expressions for T_6 is somehow more exhausting, but the procedure is still the same. We get (the terms are already ordered with respect to the derivatives concerning ν)

$$\begin{aligned}
\partial_m T_6 =& T_P(z)[P''(z)\partial_k z]T_P(z)[P''(z)\partial_l z]T_P(z)\partial_m \nu \\
&+ T_P(z)[\{P'''(z)\partial_m z\}\partial_k z]T_P(z)[P''(z)\partial_l z]T_P(z)\nu \\
&+ T_P(z)[P''(z)\partial_k z]T_P(z)[\{P'''(z)\partial_m z\}\partial_l z]T_P(z)\nu \\
&+ T_P(z)[P''(z)\partial_m \partial_k z]T_P(z)[P''(z)\partial_l z]T_P(z)\nu \\
&+ T_P(z)[P''(z)\partial_k z]T_P(z)[P''(z)\partial_m \partial_l z]T_P(z)\nu \\
&- T_P(z)[P''(z)\partial_k z]T_P(z)[P''(z)\partial_m z]T_P(z)[P''(z)\partial_l z]T_P(z)\nu \\
&+ T_P(z)[[\![\{P''(z)\partial_k z\}T_P(z)\{P''(z)\partial_l z\}, P_z]\!]\partial_m z]T_P(z)\nu.
\end{aligned}$$

Since interchanging k with l transforms T_6 into T_7, applying the Π-procedure from above yields

$$\begin{aligned}
\partial_m T_6 =& \Pi(\partial_k z, \partial_l z)\,\partial_m \nu \\
&+ \{\Pi(\partial_k z, \partial_m \partial_l z) + \Pi(\partial_l z, \partial_m \partial_k z) + \Pi(\partial_k z, \partial_l z, \partial_m z)\}\nu \\
=& \partial_m T_7.
\end{aligned}$$

Combining these results we obtain

$$
\begin{aligned}
\partial_m\partial_l\partial_k\, T_P(z)v =& T_P(z)\partial_m\partial_l\partial_k v \\
&+ \Pi(\partial_m z)\,\partial_l\partial_k v + \Pi(\partial_l z)\,\partial_m\partial_k v + \Pi(\partial_k z)\,\partial_m\partial_l v \\
&+ \{\Pi(\partial_l z, \partial_m z) + \Pi(\partial_m\partial_l z)\}\partial_k v \\
&+ \{\Pi(\partial_k z, \partial_m z) + \Pi(\partial_m\partial_k z)\}\partial_l v \\
&+ \{\Pi(\partial_k z, \partial_l z) + \Pi(\partial_l\partial_k z)\}\partial_m v \\
&+ \{\Pi(\partial_k z, \partial_l z, \partial_m z) + \Pi(\partial_k z, \partial_m\partial_l z) \\
&\quad + \Pi(\partial_l z, \partial_m\partial_k z) + \Pi(\partial_m z, \partial_l\partial_k z) + \Pi(\partial_m\partial_l\partial_k z)\}v.
\end{aligned}
$$

Repeating this procedure with this new starting point, we obtain a corresponding formula for the desired derivatives of order four. Since our presentation of this routine has been very detailed, we think that it is convenient to directly state the result. We have

$$
\begin{aligned}
\partial_n\partial_m\partial_l\partial_k\, T_P(z)v =\;& T_P(z)\partial_n\partial_m\partial_l\partial_k v \\
&+ \Pi(\partial_n z)\,\partial_m\partial_l\partial_k v \\
&+ \Pi(\partial_m z)\,\partial_n\partial_m\partial_k v \\
&+ \Pi(\partial_l z)\,\partial_n\partial_m\partial_k v \\
&+ \Pi(\partial_k z)\,\partial_n\partial_m\partial_l v \\
&+ \{\Pi(\partial_m z, \partial_n z) + \Pi(\partial_n\partial_m z)\}\partial_l\partial_k v \\
&+ \{\Pi(\partial_l z, \partial_n z) + \Pi(\partial_n\partial_l z)\}\partial_m\partial_k v \\
&+ \{\Pi(\partial_l z, \partial_m z) + \Pi(\partial_m\partial_l z)\}\partial_n\partial_k v \\
&+ \{\Pi(\partial_l z, \partial_n z) + \Pi(\partial_n\partial_k z)\}\partial_m\partial_l v \\
&+ \{\Pi(\partial_k z, \partial_m z) + \Pi(\partial_m\partial_k z)\}\partial_n\partial_l v \\
&+ \{\Pi(\partial_k z, \partial_l z) + \Pi(\partial_l\partial_k m)\}\partial_n\partial_m v \\
&+ \{\Pi(\partial_l z, \partial_m z, \partial_n z) + \Pi(\partial_l z, \partial_n\partial_m z) \\
&\quad + \Pi(\partial_m z, \partial_n\partial_l z) + \Pi(\partial_n z, \partial_m\partial_l z) + \Pi(\partial_n\partial_m\partial_l z)\}\partial_k v \\
&+ \{\Pi(\partial_k z, \partial_m z, \partial_n z) + \Pi(\partial_k z, \partial_n\partial_m z) \\
&\quad + \Pi(\partial_m z, \partial_n\partial_k z) + \Pi(\partial_n z, \partial_m\partial_k z) + \Pi(\partial_n\partial_m\partial_k z)\}\partial_l v \\
&+ \{\Pi(\partial_k z, \partial_l z, \partial_n z) + \Pi(\partial_l z, \partial_n\partial_k z) \\
&\quad + \Pi(\partial_k z, \partial_n\partial_l z) + \Pi(\partial_n z, \partial_k\partial_l z) + \Pi(\partial_n\partial_l\partial_k z)\}\partial_m v \\
&+ \{\Pi(\partial_k z, \partial_l z, \partial_m z) + \Pi(\partial_k z, \partial_m\partial_l z) \\
&\quad + \Pi(\partial_l z, \partial_m\partial_k z) + \Pi(\partial_m z, \partial_l\partial_k z) + \Pi(\partial_m\partial_l\partial_k z)\}\partial_n v \\
&+ \Pi(\star)\,v,
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
\Pi(\star) = &\Pi(\partial_k z, \partial_l z, \partial_m z, \partial_n z)\\
&+ \Pi(\partial_k z, \partial_l z, \partial_n\partial_m z) + \Pi(\partial_k z, \partial_m z, \partial_n\partial_l z) + \Pi(\partial_k z, \partial_n z, \partial_m\partial_l z)\\
&+ \Pi(\partial_l z, \partial_m z, \partial_n\partial_k z) + \Pi(\partial_l z, \partial_n z, \partial_m\partial_k z) + \Pi(\partial_m z, \partial_n z, \partial_l\partial_k z)\\
&+ \Pi(\partial_l\partial_k z, \partial_n\partial_m z) + \Pi(\partial_m\partial_k z, \partial_n\partial_l z) + \Pi(\partial_n\partial_k z, \partial_m\partial_l z)\\
&+ \Pi(\partial_k z, \partial_n\partial_m\partial_l z) + \Pi(\partial_l z, \partial_n\partial_m\partial_k z)\\
&+ \Pi(\partial_m z, \partial_n\partial_l\partial_k z) + \Pi(\partial_n z, \partial_m\partial_l\partial_k z)\\
&+ \Pi(\partial_n\partial_m\partial_l\partial_k z).
\end{aligned}
$$

We recall that $v = \operatorname{rot}\Delta_D^{-1}E$, where E is contained in the above mentioned core of $\mathbf{A}$ and $\operatorname{rot} = \sum_{1\leqslant k\leqslant 3} J_k\partial_k$. Further, $z \in W \cap \overline{B}_Z(0,r)$ so that $\|z\|_{H^2} \lesssim R$ and $\|z\|_{H^4} \lesssim r$. By means of (4.14), we see that in order to estimate the remainder term $\Pi_{\mathcal{R}}$ of the difference

$$
\Delta_D^2 T_P(z)\operatorname{rot}\Delta_D^{-2}E - T_P(z)\operatorname{rot}E = \Pi_{\mathcal{R}},
$$

it suffices to control the following quantities

$$
\begin{aligned}
&\|\,|\partial_{j_1}z|\,|\partial_{j_2}\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0}, && \|\,|\partial_{j_1}z|\,|\partial_{j_2}z|\,|\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0},\\
&\|\,|\partial_{j_1}\partial_{j_2}z|\,|\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0}, && \|\,|\partial_{j_1}z|\,|\partial_{j_2}z|\,|\partial_{j_3}z|\,|\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0},\\
&\|\,|\partial_{j_1}z|\,|\partial_{j_2}\partial_{j_3}z|\,|\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0}, && \|\,|\partial_{j_1}\partial_{j_2}\partial_{j_3}z|\,|\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0},\\
&\|\,|\partial_{j_1}z|\,|\partial_{j_2}z|\,|\partial_{j_3}z|\,|\partial_{j_4}z|\,|\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0}, && \|\,|\partial_{j_1}z|\,|\partial_{j_2}z|\,|\partial_{j_3}\partial_{j_4}z|\,|\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0},\\
&\|\,|\partial_{j_1}\partial_{j_2}z|\,|\partial_{j_3}\partial_{j_4}z|\,|\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0}, && \|\,|\partial_{j_1}z|\,|\partial_{j_2}\partial_{j_3}\partial_{j_4}z|\,|\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0},\\
&\|\,|\partial_{j_1}\partial_{j_2}\partial_{j_3}\partial_{j_4}z|\,|\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0}
\end{aligned}
$$

for arbitrary $j_k \in \{1,2,3,4\}$, $1 \leqslant k \leqslant 5$. Now, using the Sobolev embedding $H^2 \hookrightarrow L^\infty$ we estimate

$$
\begin{aligned}
\|\,|\partial_{j_1}z|\,|\partial_{j_2}\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0} &\lesssim \|\partial_{j_1}z\|_{H^2}\,\|\partial_{j_2}\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E\|_{X_0}\\
&\lesssim \|z\|_{H^4}\,\|\Delta_D^{-2}E\|_{Z_0}\\
&\lesssim_r \|E\|_{X_0},
\end{aligned}
$$

$$
\begin{aligned}
\|\,|\partial_{j_1}z|\,|\partial_{j_2}z|\,|\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0} &\lesssim \|\partial_{j_1}z\|_{H^2}\,\|\partial_{j_2}z\|_{H^2}\,\|\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E\|_{X_0}\\
&\lesssim \|z\|_{H^4}^2\,\|\Delta_D^{-2}E\|_{Z_0}\\
&\lesssim_{r^2} \|E\|_{X_0},
\end{aligned}
$$

$$
\begin{aligned}
\|\,|\partial_{j_1}\partial_{j_2}z|\,|\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E|\,\|_{X_0} &\lesssim \|\partial_{j_1}\partial_{j_2}z\|_{H^2}\,\|\partial_{j_3}\partial_{j_4}\partial_{j_5}\Delta_D^{-2}E\|_{X_0}\\
&\lesssim \|z\|_{H^4}\,\|\Delta_D^{-2}E\|_{Z_0}\\
&\lesssim_r \|E\|_{X_0},
\end{aligned}
$$

$$\begin{aligned}
\| \, |\partial_{j_1} z| \, |\partial_{j_2} z| \, |\partial_{j_3} z| \, |\partial_{j_4} \partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|\partial_{j_1} z\|_{H^2} \|\partial_{j_2} z\|_{H^2} \|\partial_{j_3} z\|_{H^2} \|\partial_{j_4} \partial_{j_5} \Delta_D^{-2} E\|_{X_0} \\
&\lesssim \|z\|_{H^4}^3 \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r^3} \|E\|_{X_0},
\end{aligned}$$

$$\begin{aligned}
\| \, |\partial_{j_1} z| \, |\partial_{j_2} \partial_{j_3} z| \, |\partial_{j_4} \partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|\partial_{j_1} z\|_{H^2} \|\partial_{j_2} \partial_{j_3} z\|_{H^2} \|\partial_{j_4} \partial_{j_5} \Delta_D^{-2} E\|_{X_0} \\
&\lesssim \|z\|_{H^4}^2 \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r^2} \|E\|_{X_0},
\end{aligned}$$

$$\begin{aligned}
\| \, |\partial_{j_1} \partial_{j_2} \partial_{j_3} z| \, |\partial_{j_4} \partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|\partial_{j_1} \partial_{j_2} \partial_{j_3} z\|_{X_0} \, \|\partial_{j_4} \partial_{j_5} \Delta_D^{-2} E\|_{H^2} \\
&\lesssim \|z\|_{H^4} \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r} \|E\|_{X_0},
\end{aligned}$$

$$\begin{aligned}
\| \, |\partial_{j_1} z| \, |\partial_{j_2} z| \, |\partial_{j_3} z| \, |\partial_{j_4} z| \, |\partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|z\|_{H^4}^4 \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r^4} \|E\|_{X_0},
\end{aligned}$$

$$\begin{aligned}
\| \, |\partial_{j_1} z| \, |\partial_{j_2} z| \, |\partial_{j_3} \partial_{j_4} z| \, |\partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|\partial_{j_1} z\|_{H^2} \|\partial_{j_2} z\|_{H^2} \|\partial_{j_3} \partial_{j_4} z\|_{H^2} \|\partial_{j_5} \Delta_D^{-2} E\|_{X_0} \\
&\lesssim \|z\|_{H^4}^3 \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r^3} \|E\|_{X_0},
\end{aligned}$$

$$\begin{aligned}
\| \, |\partial_{j_1} \partial_{j_2} z| \, |\partial_{j_3} \partial_{j_4} z| \, |\partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|\partial_{j_1} \partial_{j_2} z\|_{H^2} \|\partial_{j_3} \partial_{j_4} z\|_{H^2} \|\partial_{j_5} \Delta_D^{-2} E\|_{X_0} \\
&\lesssim \|z\|_{H^4}^2 \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r^2} \|E\|_{X_0},
\end{aligned}$$

$$\begin{aligned}
\| \, |\partial_{j_1} z| \, |\partial_{j_2} \partial_{j_3} \partial_{j_4} z| \, |\partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|\partial_{j_1} z\|_{H^2} \|\partial_{j_2} \partial_{j_3} \partial_{j_4} z\|_{X_0} \|\partial_{j_5} \Delta_D^{-2} E\|_{H^2} \\
&\lesssim \|z\|_{H^4}^2 \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r^2} \|E\|_{X_0},
\end{aligned}$$

$$\begin{aligned}
\| \, |\partial_{j_1} \partial_{j_2} \partial_{j_3} \partial_{j_4} z| \, |\partial_{j_5} \Delta_D^{-2} E| \, \|_{X_0} &\lesssim \|\partial_{j_1} \partial_{j_2} \partial_{j_3} \partial_{j_4} z\|_{X_0} \|\partial_{j_5} \Delta_D^{-2} E\|_{H^2} \\
&\lesssim \|z\|_{H^4} \, \|\Delta_D^{-2} E\|_{Z_0} \\
&\lesssim_{r} \|E\|_{X_0}.
\end{aligned}$$

Putting all these estimates together we gain

$$\|\Delta_D^2 T_P(z) \operatorname{rot} \Delta_D^{-2} E - T_P(z) \operatorname{rot} E\|_{X_0} \lesssim_{r^4} \|E\|_{X_0},$$

and we are done. ///

Now, the claim follows by combining Theorem 3.45 and Lemma 4.5. ∎

Remark 4.7 (a) First, we want to give interesting examples of nonlinearities P, M which fulfill the assumptions of Theorem 4.9. Let $p, m \in C^4(\mathbb{R}, \mathbb{R})$ satisfy $p(0) > -1$ and $m(0) > -1$. Then one can choose P, M to be

$$P(y) \coloneqq p(|y|^2)y \quad (y \in \mathbb{R}^3),$$
$$M(z) \coloneqq m(|z|^2)z \quad (z \in \mathbb{R}^3).$$

To demonstrate the conformity of this ansatz, we just have to differentiate, which yields

$$P'(y) = p(|y|^2)I + 2p'(|y|^2)\, yy^\top,$$
$$M'(z) = m(|z|^2)I + 2m'(|z|^2)\, zz^\top.$$

Thus $P'(0) = p(0)I > -I$ and $M'(0) = m(0)I > -I$. The regularity condition is an immediate consequence of the differentiability assumption for p and m. Choosing $p(s) = \varepsilon_0 + \alpha\, s$ ($\varepsilon_0 > 0$, α, $s \in \mathbb{R}$) and $m \equiv 0$, we see that these types of nonlinearities particularly cover the Kerr-Nonlinearity

$$P(y) = \varepsilon_0 y + \alpha\, |y|^2 y.$$

(b) Besides $S = \operatorname{diag}(\Delta_D^2, \Delta_D^2)$ as chosen in the above proof, we could have also used $S = \operatorname{diag}(\Delta_D^{2n}, \Delta_D^{2n})$ for each $n \in \mathbb{N}$. This yields solutions

$$u(\cdot, u_0) \in C([0,T], H^{2n+2}(\Omega)^6) \cap C^1([0,T], H^2(\Omega)^6),$$

for adequate initial values in $H^{2n+2}(\Omega)^6$, cf. Theorem 4.9. As for the proof, we simply have to use generalisations of (4.14) for derivatives of higher orders. If we also adopted the remaining parts of the proof, we would further obtain solutions

$$u(\cdot, u_0) \in C([0,T], H^{2n+2}(\Omega)^6) \cap C^1([0,T], H^{2n}(\Omega)^6),$$

for corresponding initial values. If we want to additionally close the gap concerning the regularity in space from $2n+2$ to $2n+1$, we propose to choose the fractional operators

$$S = \operatorname{diag}(\Delta_D^{n/2}, \Delta_D^{n/2}) \quad (n \in \mathbb{N},\ n \geqslant 3).$$

Due to the loss of formulas of type (4.14), the problem then is to find the right commutator estimates for this operator. More precisely, choose $n = 3$, we then would need estimates

$$\left\| \Delta_D^{3/2}(I + P'(z))^{-1} \operatorname{rot} \Delta_D^{-3/2} u - (I + P'(z))^{-1} \operatorname{rot} u \right\|_{L^2(\Omega)^3} \lesssim_r \|u\|_{L^2(\Omega)^3}$$

for certain z and u. We believe this to be true but were unable to find quotes in the literature. We further think that the gained statement does not justify the effort of the proof, thus it remains a claim. ⋊

■ The Perfect Conductor

Next, we want to approach the perfect conducting boundary conditions (Pc). Therefore we have to choose a different isomorphism S and also new spaces Z and Y. In view of Theorem 2.43, we choose

$$S \coloneqq \mathbf{A}_0, \quad D(S) = D(\mathbf{A}) \cap X_0.$$

Recall that

$$\mathbf{A} = \begin{pmatrix} 0 & \operatorname{rot} \\ -\operatorname{rot} & 0 \end{pmatrix}, \quad D(\mathbf{A}) = H_0(\operatorname{rot}, \Omega) \times H(\operatorname{rot}, \Omega),$$
$$X_0 = \Big\{(E,H) \in X \,:\, \operatorname{div}(E) = \operatorname{div}(H) = 0,\ \gamma_{\mathfrak{n}}(H) = 0\Big\}.$$

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded $C^{3,1}$-domain and endow $D(S^k),\ k \in \{1,2,3\}$ with the corresponding graph norm. Then Theorem 2.43 (c) yields the following isomorphism

$$\begin{aligned} D(S^2) &= \Big\{(E,H) \in H^2(\Omega)^3 \times H^2(\Omega)^3 \,:\, \gamma_{\mathfrak{t}}(E) = 0\Big\} \cap X_0, \\ D(S^3) &= \Big\{(E,H) \in H^2(\Omega)^3 \times H^2(\Omega)^3 \,:\, \gamma_{\mathfrak{t}}(\operatorname{rot}^2 E) = 0, \\ &\qquad \gamma_{\mathfrak{n}}(H) = 0,\ \gamma_{\mathfrak{n}}(\operatorname{rot} H) = 0\Big\} \cap X_0, \end{aligned} \tag{4.15}$$

where the right hand sides are equipped with the usual Sobolev norms respectively. We thus put

$$Z \coloneqq Z_0 \times Z_0 \coloneqq D(S^3), \quad Y \coloneqq Y_0 \times Y_0 \coloneqq D(S^2).$$

Consequently, we will only look for solutions which in addition to the desired boundary conditions satisfy

$$\begin{aligned} &\operatorname{div} E = \operatorname{div} H = 0 && \text{in } \Omega, \\ &H \cdot \mathfrak{n} = \operatorname{rot} H \cdot \mathfrak{n} = 0 && \text{on } \partial\Omega, \\ &\operatorname{rot}^2 E \wedge \mathfrak{n} = 0 && \text{on } \partial\Omega. \end{aligned}$$

Due to (4.15) the verification of all assumptions in Theorem 3.43, besides the commutator estimate (CE), can be done in exactly the same way as in the proof of Theorem 4.9. Adopting the notation of the latter one, we now take a look at the remaining condition (CE). Let u be a smooth function. Denoting

$$v \coloneqq (v_1, v_2) \coloneqq \mathbf{A}S^{-1}u$$

we can rewrite

$$S\Lambda(z)^{-1}\mathbf{A}S^{-1}u = \begin{pmatrix} \operatorname{rot}^3 T_P(z)v_1 \\ -\operatorname{rot}^3 T_M(z)v_2 \end{pmatrix}.$$

Because of

$$\operatorname{rot}^3 = \sum_{1 \leqslant m,l,k \leqslant 3} J_m J_l J_k\, \partial_m \partial_l \partial_k,$$

we can use the same calculations as in the proof of part (CE) in Theorem 4.9 to obtain

$$\begin{aligned} \operatorname{rot}^3 T_P(z)v_1 &= T_P(z)\operatorname{rot}^3 v_1 + \sum_{1\leqslant m,l,k\leqslant 3} [J_mJ_lJ_k, T_P(z)]\,\partial_m\partial_l\partial_k v_1 + R_1 \\ -\operatorname{rot}^3 T_M(z)v_2 &= -T_M(z)\operatorname{rot}^3 v_2 - \sum_{1\leqslant m,l,k\leqslant 3} [J_mJ_lJ_k, T_P(z)]\,\partial_m\partial_l\partial_k v_2 + R_2, \end{aligned}$$

where (in view of the Π-notation) both R_1 and R_2 are of the same Π-class, namely

$$\begin{aligned} &\Pi(\partial_m z)\,\partial_l\partial_k v_\alpha + \Pi(\partial_l z)\,\partial_m\partial_k v_\alpha + \Pi(\partial_k z)\,\partial_m\partial_l v_\alpha \\ &+ \{\Pi(\partial_l z, \partial_m z) + \Pi(\partial_m\partial_l z)\}\partial_k v_\alpha \\ &+ \{\Pi(\partial_k z, \partial_m z) + \Pi(\partial_m\partial_k z)\}\partial_l v_\alpha \\ &+ \{\Pi(\partial_k z, \partial_l z) + \Pi(\partial_l\partial_k z)\}\partial_m v_\alpha \\ &+ \{\Pi(\partial_k z, \partial_l z, \partial_m z) + \Pi(\partial_k z, \partial_m\partial_l z) \\ &\quad + \Pi(\partial_l z, \partial_m\partial_k z) + \Pi(\partial_m z, \partial_l\partial_k z) + \Pi(\partial_m\partial_l\partial_k z)\}v_\alpha \quad (\alpha \in \{1,2\}). \end{aligned}$$

Hence we see that

$$S\Lambda(z)^{-1}\mathbf{A}S^{-1} = \Lambda(z)^{-1}\mathbf{A}u + C(z)u + B(z)u,$$

where $\|B(z)u\|_X \lesssim_r \|u\|_X$ for $z \in \overline{B}_Z(0,r)$ and

$$C(z)u = \sum_{m,l,k,j} \begin{pmatrix} 0 & [J_mJ_lJ_k, T_P(z)]J_j \\ -[J_mJ_lJ_k, T_M(z)]J_j & 0 \end{pmatrix} \partial_m\partial_l\partial_k\partial_j\, S^{-1}u.$$

Since $S^{-1} \in \mathcal{B}(Z,X)$, the estimate

$$\|C(z)u\|_{L^2} \lesssim_r \|u\|_{H^1},$$

is sharp, unless the matrix coefficients will vanish. Computing the matrix products $J_mJ_lJ_k$ one checks that if $m \neq l \neq k$, then

$$J_mJ_lJ_k = e_i e_{i+1}^\top,$$

where $\{e_1, e_2, e_3\}$ denotes the standard basis in $\mathbb{R}^3$. Therefore

$$[J_mJ_lJ_k, T_P(z)] = 0 \ \text{ for all } m,l,k \quad \Leftrightarrow \quad T_P(z) = f(z)\,I$$

for some scalar function f. Inserting $T_P(z) = (I + P'(z))^{-1}$ this yields

$$I = f(z)I(I + P'(z)).$$

Thus the choice of $S = \mathbf{A}_0^3$ only applies to nonlinearities $N \in \{P, M\}$ satisfying

$$\begin{aligned} &\partial_1 N_1 = \partial_2 N_2 = \partial_3 N_3, \\ &\partial_i N_j = 0 \quad \text{if } i \neq j. \end{aligned}$$

Taking also the assumptions (N1) and (N2) into account this means that we can treat polarisations and magnetisations of the form

$$\begin{aligned} P(y) &= p(y_k)(1,1,1)^\top + P_0, \\ M(y) &= m(y_l)(1,1,1)^\top + M_0 \quad (y \in \mathbb{R}^3), \end{aligned} \tag{4.16}$$

where $k, l \in \{1,2,3\}$ and $P_0, M_0 \in \mathbb{R}^3$ are arbitrary and

$$p, m \in C^3(\mathbb{R}, \mathbb{R}), \quad p'(0) > -1, \quad m'(0) > -1.$$

We summarize these results in the following proposition.

Given real parameters r and α, we put

$$W(\alpha, r) := \overline{B}_Y(0,\alpha) \cap \overline{B}_Z(0,r) \cap I_C,$$

with I_C from the definition prior to Theorem 4.9.

Proposition 4.8 *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded $C^{3,1}$-domain, and let the vector fields $P, M : \mathbb{R}^3 \to \mathbb{R}^3$ be of the form* (4.16)*. Further, let $\kappa \in (0,1)$ and $r_0 > 0$ be arbitrary. Then the following assertions hold.*

(a) *There is a radius $R > 0$ and an associated constant $c_0 = c_0(R) > 0$, satisfying $c_0(R) \to \infty$ $(R \to \infty)$, such that for each*

$$u_0 = (E_0, H_0) \in W(\kappa\, c_0, r_0)$$

there exists a time $T = T(R, r_0, \kappa) > 0$ and a function

$$u(\cdot, u_0) = (E, H) \in C([0,T], H^3(\Omega)^6) \cap C^1([0,T], H^2(\Omega)^6),$$

with $\|u(t)\|_{H^2} \leqslant R$ $(t \in [0,T])$ which solves the Maxwell-type Cauchy problem (M-Pc) *by means of*

$$\begin{aligned} (D \circ E)'(t) &= \operatorname{rot} H(t) && (t \in [0,T]), \\ (B \circ H)'(t) &= -\operatorname{rot} E(t) && (t \in [0,T]), \\ \operatorname{div} D(E(t)) &= 0 && (t \in [0,T]), \\ \operatorname{div} B(H(t)) &= 0 && (t \in [0,T]), \\ \operatorname{div} E(t) &= 0 && (t \in [0,T]), \\ \operatorname{div} H(t) &= 0 && (t \in [0,T]), \\ E(t), \operatorname{rot}^2 E(t) &\in H_0(\operatorname{rot}, \Omega) && (t \in [0,T]), \\ B(H(t)) &\in H_0(\operatorname{div}, \Omega) && (t \in [0,T]), \\ H(t), \operatorname{rot} H(t) &\in H_0(\operatorname{div}, \Omega) && (t \in [0,T]), \\ u(0) &= u_0\,, \end{aligned}$$

where $B(H) = H + M(H)$ and $D(E) = E + P(E)$.

(b) *If* $v \in C([0,T'],H^3(\Omega)^6) \cap C^1([0,T'],H^2(\Omega)^6)$ *is another solution of this system with* $\|v(t)\|_{H^2} \leqslant R$ *for all* $0 \leqslant t \leqslant T'$*, then* v *coincides with* u *on the interval* $[0,\min\{T,T'\}]$*. Further, the map*

$$W(\kappa c_0, r_0) \subseteq H^2(\Omega)^6 \to C([0,T],H^2(\Omega)^6), \quad u_0 \mapsto u(\cdot,u_0)$$

is Lipschitz continuous. ⋊

Obviously the nonlinearities of type (4.16) do not cover the Kerr nonlinearity

$$P(y) = \varepsilon_0\, y + \alpha\, |y|^2\, y, \quad M \equiv 0 \quad (\varepsilon_0 > 0,\ \alpha \in \mathbb{R},\ y \in \mathbb{R}^3).$$

Therefore, as a first step, we try to specify coefficients $A_{mlk} \in \mathbb{R}^{6\times 6}$ such that

$$S = \sum_{1\leqslant m,l,k\leqslant 3} A_{mlk}\partial_m\partial_l\partial_k$$

leads to the identity (a priori formally)

$$S\Lambda(z)^{-1}\mathbf{A}S^{-1}u = \Lambda(z)^{-1}\mathbf{A}u + B(z)S^{-1}u,$$

where u is sufficiently smooth and $B(z)$ consists of expressions having derivatives of order less or equal than two. Denoting

$$v \coloneqq (v_1, v_2) \coloneqq \mathbf{A}S^{-1}u$$

we compute (using again the same methods as in the case $S = \mathbf{A}_0^3$)

$$\begin{aligned} S\begin{pmatrix} T_P(z)v_1 \\ v_2 \end{pmatrix} &= \sum_{m,l,k} A_{mlk}\begin{pmatrix} T_P(z)\partial_m\partial_l\partial_k v_1 + R_1 \\ \partial_m\partial_l\partial_k v_2 + R_2 \end{pmatrix} \\ &= \sum_{m,l,k} A_{mlk}\begin{pmatrix} T_P(z)\partial_m\partial_l\partial_k v_1 \\ \partial_m\partial_l\partial_k v_2 \end{pmatrix} + B(z)S^{-1} \\ &= \Lambda(z)^{-1}\mathbf{A}u + C(z)u + B(z)S^{-1}u, \end{aligned}$$

where

$$C(z)u = \sum_{m,l,k}[A_{mlk}, \operatorname{diag}(T_P(z), I)]\begin{pmatrix} 0 & J_j \\ -J_j & 0 \end{pmatrix}\partial_m\partial_k\partial_l\partial_j S^{-1}u,$$

and $B(z)$ only consists of expressions having the desired shape. With the same arguments from the previous observations concerning the operator $S = \mathbf{A}_0^3$, we need the coefficients A_{mlk} to satisfy

$$[A_{mlk}, \operatorname{diag}(T_P(z), I)] = 0 \quad \text{for all } 1 \leqslant m,l,k \leqslant 3.$$

Writing

$$A_{mlk} = \begin{pmatrix} A_{mlk}^{11} & A_{mlk}^{12} \\ A_{mlk}^{21} & A_{mlk}^{22} \end{pmatrix}$$

this implies

$$\begin{aligned} A^{11}_{mlk} T_P(z) &= T_P(z) A^{11}_{mlk}, \\ T_P(z) A^{12}_{mlk} &= A^{12}_{mlk}, \\ A^{21}_{mlk} T_P(z) &= A^{21}_{mlk} \qquad \text{for all } 1 \leqslant m, l, k \leqslant 3. \end{aligned}$$

In particular $A^{12}_{mlk} = A^{21}_{mlk} = 0$. By means of the Sherman-Morisson-Formula (cf. [33], Problem 4.20) we then compute

$$\begin{aligned} (I + P'(z))^{-1} &= \big((1+\varepsilon_0)I + 2\alpha z z^\top\big)^{-1} \\ &= \frac{1}{1+\varepsilon_0} I - \frac{2\alpha z z^\top}{1+\varepsilon_0 + 2\alpha |z|^2}. \end{aligned}$$

Putting

$$f(s) \coloneqq \frac{-2\alpha}{1+\varepsilon_0+2\alpha s} \quad (s > 0 \text{ small}),$$

this reads $T_P(z) = (1+\varepsilon_0)^{-1} I + f(|z|^2) z z^\top$. Therefore we need A^{11}_{mlk} to satisfy

$$A^{11}_{mlk} x x^\top = x x^\top A^{11}_{mlk} \quad \text{for all } x \in \mathbb{R}^3,\ 1 \leqslant m, l, k \leqslant 3,$$

i.e., all the matrices A^{11}_{mlk} have to be diagonal. Thus there is unfortunately no hope to include the vector valued mixing boundary conditions for a perfect conductor using this approach.

■ Full Space Framework

Concluding we consider the full space situation, where besides the loss of the boundary conditions, also the characterisation of Sobolev spaces in terms of the Fourier Transformation makes the analysis more comfortable.

Given real parameters r and α, we put

$$W(\alpha, r) \coloneqq \overline{B}_{H^s(\mathbb{R}^3)^6}(0,\alpha) \cap \overline{B}_{H^{s+1}(\mathbb{R}^3)^6}(0,r) \cap I_C,$$

where I_C is defined as in the Dirichlet case.

Theorem 4.9 *Let $s > 3/2$ and assume that the vector fields $P, M : \mathbb{R}^3 \to \mathbb{R}^3$ satisfy*

$$P,\ M \in C^{[s]+1}(\mathbb{R}^3, \mathbb{R}^3), \quad P'(0) > -I, \quad M'(0) > -I.$$

Further, let $\kappa \in (0,1)$ and $r_0 > 0$ be arbitrary. Then the following assertions hold.

(a) *There is a radius $R > 0$ and an associated constant $c_0 = c_0(R) > 0$, satisfying $c_0(R) \to \infty$ $(R \to \infty)$, such that for each*

$$u_0 = (E_0, H_0) \in W(\kappa\, c_0, r_0)$$

there exists a time $T = T(R, r_0, \kappa) > 0$ *and a function*

$$u(\cdot, u_0) = (E, H) \in C\big([0,T], H^{s+1}(\mathbb{R}^3)^6\big) \cap C^1\big([0,T], H^s(\mathbb{R}^3)^6\big),$$

with $\|u(t)\|_{H^s} \leqslant R$ $(t \in [0,T])$ *which solves the Maxwell-type Cauchy problem* (M-$\mathbb{R}^3$) *by means of*

$$\begin{aligned} (D\circ E)'(t) &= \operatorname{rot} H(t) && \big(t\in[0,T]\big),\\ (B\circ H)'(t) &= -\operatorname{rot} E(t) && \big(t\in[0,T]\big),\\ \operatorname{div} D(E(t)) &= 0 && \big(t\in[0,T]\big),\\ \operatorname{div} B(H(t)) &= 0 && \big(t\in[0,T]\big),\\ u(0) &= u_0\,, \end{aligned}$$

where $B(H) = H + M(H)$ *and* $D(E) = E + P(E)$.

(b) *If* $v \in C\big([0,T'], H^{s+1}(\mathbb{R}^3)^6\big) \cap C^1\big([0,T'], H^s(\mathbb{R}^3)^6\big)$ *is another solution of this system with* $\|v(t)\|_{H^s} \leqslant R$ *for all* $0 \leqslant t \leqslant T'$, *then* v *coincides with* u *on the interval* $[0, \min\{T, T'\}]$. *Further, the map*

$$W(\kappa\, c_0, r_0) \subseteq H^s(\mathbb{R}^3)^6 \to C\big([0,T], H^s(\mathbb{R}^3)^6\big), \quad u_0 \mapsto u(\cdot, u_0)$$

is Lipschitz continuous. ⋊

Proof. The proof follows the same pattern as the proof of Theorem 4.9. In particular we adopt the notation concerning the substitution operators. As the phase space triple we choose

$$\begin{aligned} X &:= X_0 \times X_0 := L^2(\mathbb{R}^3)^3 \times L^2(\mathbb{R}^3)^3,\\ Y &:= Y_0 \times Y_0 := H^s(\mathbb{R}^3)^3 \times H^s(\mathbb{R}^3)^3,\\ Z &:= Z_0 \times Z_0 := H^{s+1}(\mathbb{R}^3)^3 \times H^{s+1}(\mathbb{R}^3)^3. \end{aligned}$$

It is thus well known that the embeddings $Z \hookrightarrow Y \hookrightarrow X$ are continuous and dense, and that Y is an interpolation space between Z and X. In the following we will write H^s, L^p and so on, instead of $H^s(\mathbb{R}^3)^m$, $L^p(\mathbb{R}^3)^n$ $(m, n \in \mathbb{N})$, since it is clear from context which dimension is needed. Again, as already done in the Dirichlet case, by rewriting $\mathbf{A}$ as a first order differential operator, we see that $\mathbf{A} \in \mathcal{B}(Z, Y)$. Recalling the Sobolev embedding

$$H^s \hookrightarrow L^\infty,$$

we can fix some radius $R > 0$ such that

$$\|u\|_{Y_0} \leqslant R \quad \Rightarrow \quad \|u\|_{L^\infty} \leqslant \rho.$$

We put $W \coloneqq \overline{B}_Y(0,R)$ and start to verify the assumptions from Theorem 3.43.

First, the verification of (PD) and (G) can be done exactly in the same way as in the Dirichlet case. We thus start with (LC). Let $y = (y_1, y_2), \tilde{y} = (\tilde{y}_1, \tilde{y}_2) \in W$ and let u belong to X.

$$\|\Lambda(y)u - \Lambda(y)u\|_X \leqslant \sqrt{2}\big(\|\Lambda_1(y_1)u_1 - \Lambda_1(\tilde{y}_1)u_1\|_{X_0} \\ + \|\Lambda_2(y_2)u_2 - \Lambda_2(\tilde{y}_2)u_2\|_{X_0}\big).$$

Since each argument for P is exactly the same for M, it is enough to consider Λ_1. Using (4.2) and the Sobolev embedding, we estimate

$$\begin{aligned} &\|\Lambda_1(y_1)u_1 - \Lambda_1(\tilde{y}_1)u_1\|_{X_0}^2 \\ &\qquad \leqslant \int_\Omega \big|P'(y_1(x)) - P'(\tilde{y}_1(x))\big|^2 |u_1(x)|^2 \,\mathrm{d}x \\ &\qquad \lesssim \|y_1 - \tilde{y}_1\|_{L^\infty}^2 \|u_1\|_{L^2}^2 \\ &\qquad \lesssim \|y_1 - \tilde{y}_1\|_{Y_0}^2 \|u_1\|_{X_0}^2 \end{aligned}$$

so that

$$\|\Lambda(y)u - \Lambda(y)u\|_X \lesssim \|y - \tilde{y}\|_Y \|u\|_X .$$

$(LC\text{-}i)$: Let $y = (y_1, y_2), \tilde{y} = (\tilde{y}_1, \tilde{y}_2) \in W$ and $u \in Y$. As in the Dirichlet case it is readily seen that

$$\Lambda(y)^{-1}u = \begin{pmatrix} T_P(y_1)u_1 \\ T_M(y_2)u_2 \end{pmatrix},$$

where $T_P = (I + P'(\cdot))^{-1}$ and $T_M = (I + M'(\cdot))^{-1}$. In particular $\Lambda(0)^{-1} \in \mathcal{B}(Y)$. By means of Proposition A.5, we start estimating

$$\begin{aligned} &\|\Lambda_1(y_1)^{-1}u_1 - \Lambda_2(\tilde{y}_1)^{-1}u_1\|_{Y_0} \\ &\qquad = \|[T_P(y_1) - T_P(\tilde{y}_1)]u_1\|_{Y_0} \\ &\qquad \lesssim \|T_P(y_1) - T_P(\tilde{y}_1)\|_{L^\infty} \|u_1\|_{H^s} + \|T_P(y_1) - T_P(\tilde{y}_1)\|_{H^s} \|u_1\|_{L^\infty}. \end{aligned}$$

Due to the local Lipschitz continuity of T_P and the Sobolev embedding $H^s \hookrightarrow L^\infty$, we further estimate

$$\begin{aligned} \|T_P(y_1) - T_P(\tilde{y}_1)\|_{L^\infty} &\lesssim_R \|y_1 - \tilde{y}_1\|_{H^s}, \\ \|u_1\|_{L^\infty} &\lesssim \|u\|_{H^s}, \end{aligned}$$

so that it only remains to show that

$$\|T_P(y_1) - T_P(\tilde{y}_1)\|_{H^s} \lesssim_R \|y_1 - \tilde{y}_1\|_{H^s}.$$

We write $s = [s] + \{s\}$ with $[s] \in \mathbb{N}_0$ and $\{s\} \in (0,1)$. Putting

$$F \coloneqq T_P(y_1) - T_P(\tilde{y}_1),$$

we use the following representation of the H^s-norm (cf. Corollary A.4)

$$\begin{aligned}\|F\|^2_{H^s} &\sim \|F\|^2_{H^{[s]}} + \sum_{|\alpha|=[s]} |\partial^\alpha F|^2_{s,2}\\ &= \|F\|^2_{H^{[s]}} + \sum_{|\alpha|=[s]} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\partial^\alpha F(x_1) - \partial^\alpha F(x_2)|^2}{|x_1 - x_2|^{3+2\{s\}}}\,\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

Now, the estimate of the first summand for $[s] \leqslant 2$ was shown at the corresponding part in the Dirichlet case. For integers $[s] \geqslant 3$ this method can be extended inductively, though the calculation of the higher order partial derivatives of $T_P \circ y$ are exhausting. Nevertheless this proof is for example executed on page 202 in [23]. So, we turn to the second summand, where we again restrict our calculations to the case $[s] = 1$, since the cases $[s] \geqslant 2$ then follow inductively. Using

$$\partial_k T_P(y) = -T_P(y)[P''(y)\partial_k y]T_P(y),$$

the L^∞-boundedness as well as the local Lipschitz continuity of the quantities $T_P(y)$ and $P''(y)$, and the usual quadratic expansion of the involved terms it follows that

$$\begin{aligned}|\partial_k F(x_1) - \partial_k F(x_2)|^2 \lesssim_R &|\partial_k(y_1 - \tilde{y}_1)(x_1) - \partial_k(y_1 - \tilde{y}_1)(x_2)|^2\\ &+ |(y_1 - \tilde{y}_1)(x_1) - (y_1 - \tilde{y}_1)(x_2)|^2,\end{aligned}$$

for almost every x_1, x_2. Hence we estimate

$$\sum_{|\alpha|=[s]} |\partial^\alpha F|^2_{\{s\},2} \lesssim_R \sum_{|\alpha|=[s]} |\partial^\alpha(y_1 - \tilde{y}_1)|^2_{\{s\},2} + \|y_1 - \tilde{y}_1\|^2_{H^{[s]}},$$

due to Lemma A.3. Putting all these things together, we have thus shown that

$$\begin{aligned}\|F\|^2_{H^s} &\lesssim_R \|y_1 - \tilde{y}_1\|^2_{H^{[s]}} + \sum_{|\alpha|=[s]} |\partial^\alpha(y_1 - \tilde{y}_1)|^2_{\{s\},2}\\ &\sim \|y_1 - \tilde{y}_1\|^2_{H^s},\end{aligned}$$

and we are done.

(CE) : Let $\Lambda^s = (I - \Delta)^{s/2}$ denote the Bessel-Potential $\mathcal{F}^{-1}\langle\cdot\rangle^s\mathcal{F}$ and put

$$S \coloneqq \operatorname{diag}(S_0, S_0) \coloneqq \operatorname{diag}(\Lambda^s, \Lambda^s, \Lambda^s, \Lambda^s, \Lambda^s, \Lambda^s).$$

Then (cf. Section 1 in Appendix A) S is an isomorphic isomorphism from Z to X. For z belonging to $\overline{B}_Z(0,r)$, we can write

$$\Lambda(z)^{-1}\mathbf{A} = \sum_{k=1}^{3} A_k(z)\,\partial_k, \quad A_k(z) = \begin{pmatrix} 0 & T_P(z)J_k \\ -T_M(z)J_k & 0 \end{pmatrix}.$$

Using this representation it was now shown in [24] (starting at page 51) that

$$S\Lambda(z)^{-1}\mathbf{A}S - \Lambda(z)^{-1}\mathbf{A}$$

can be extended to a bounded operator $B(z) \in \mathcal{B}(X)$ for each $z \in \overline{B}_Z(0, r)$, and that there is a constant $b = b(r)$ such that $\|B(z)\|_{\mathcal{B}(z)} \leqslant b$, i.e., (CE) holds. ∎

Finally, at the end of this chapter we want to give a conclusion on how to possibly weaken the regularity assumption on the initial data u_0. So far, we have shown that we get solutions for adequate initial values

$$u_0 \in H^{s+1}, \quad s > 3/2.$$

Now, we suggest to attempt a linearization different from the one described in Section 3.3. For this purpose let u_0 be an arbitrary function in $X = L^2$ such that $Q(u_0)^{-1}$ exists. Inspired by an approach of C. Sogge to a class of quasilinear wave equations (cf. [38]), we search for functions u close to u_0 (in way that has to be specified later on, since it depends on later choices of function spaces) solving the system

$$\begin{aligned} (\partial_t - Q(u_0)^{-1}\mathbf{A})u &= (Q(u)^{-1} - Q(u_0)^{-1})\mathbf{A}u \quad (t \in [0, T]), \\ u(0) &= u_0. \end{aligned} \tag{4.17}$$

We then linearize (4.17) by freezing u on the right hand side of this equation, i.e., we substitute u by a function v (of a possibly large class) and consider the resulting system

$$\begin{aligned} (\partial_t - Q(u_0)^{-1}\mathbf{A})u &= (Q(v)^{-1} - Q(u_0)^{-1})\mathbf{A}v =: F_v(t) \quad (t \in [0, T]), \\ u(0) &= u_0. \end{aligned}$$

The (a priori formal) solution u_v is then given by the variation of constants formula

$$u_v(t) = e^{tQ(u_0)^{-1}\mathbf{A}}u_0 + \int_0^t e^{(t-s)Q(u_0)^{-1}\mathbf{A}}F_v(s)\,\mathrm{d}s =: \Phi_{u_0}(v)(t).$$

A starting point for a promising analysis of the so defined solution operator Φ_{u_0} would thus be the establishment of Strichartz estimates for the operator

$$Q(u_0)^{-1}\mathbf{A} = \begin{pmatrix} 0 & \varepsilon(u_0)^{-1}\operatorname{rot} \\ -\mu(u_0)^{-1}\operatorname{rot} & \end{pmatrix},$$

i.e., for the Maxwell operator with bounded and symmetric coefficients.

4.2 Wave Equations

Let $\Omega \subseteq \mathbb{R}^d$ $(d \leqslant 3)$, be a bounded domain with boundary $\partial\Omega$. We consider the quasilinear Wave-type Cauchy problem from the introduction, i.e.,

$$\begin{aligned} \partial_{tt}u(t,x) + \partial_{tt}(K \circ u)(t,x) &= \Delta u(t,x) && (t \in [0,T],\ x \in \Omega), \\ u(t,x) &= 0 && (t \in [0,T],\ x \in \partial\Omega), \\ u(0,x) &= u_0(x) && (x \in \Omega), \\ \partial_t u(0,x) &= v_0(x) && (x \in \Omega). \end{aligned} \tag{CP-W}$$

Assumption 4.10 In the following we suppose that

$$K \in C^4(\mathbb{R},\mathbb{R}), \quad K'(0) > -1.$$

By continuity there are numbers $\rho > 0$ and $\delta > 0$ such that

$$1 + K'(y) \geqslant \delta \quad (|y| \leqslant \rho). \tag{4.18}$$

By means of the mean value theorem we further obtain constants $L_k,\ k \in \{0,\dots,4\}$ such that

$$|D^k K(y) - D^k K(\tilde{y})| \leqslant L_k\, |y - \tilde{y}| \quad (y,\tilde{y} \in \overline{B}_{\mathbb{R}}(0,\rho^*)), \tag{4.19}$$

where $\rho^* > 0$ is arbitrary. In particular

$$\sup_{|y|\leqslant r} |D^k K(y)| =: c_k < \infty \quad \text{for all } r > 0,\ k \in \{0,\dots,4\}. \tag{4.20}$$

Finally, we assume that $\Omega \subseteq \mathbb{R}^d$ $(d \leqslant 3)$ is a bounded C^3-domain. ⋊

If we differentiate $K \circ u$ twice with respect to t, we obtain

$$(1 + K'(u(t,x)))\partial_{tt}u(t,x) + [K''(u(t,x))\partial_t u(t,x)]\partial_t u(t,x) = \Delta u(t,x).$$

Introducing the operators

$$\gamma(u) := 1 + K'(u), \quad \Gamma(u) := K''(u),$$

this equation becomes

$$\gamma(u(t,x))\partial_{tt}u(t,x) + [\Gamma(u(t,x))\partial_t u(t,x)]\partial_t u(t,x) = \Delta u(t,x). \tag{4.21}$$

Phase Space 4.11 In the following we will consider (4.21) as an evolution equation in the Hilbert space $H := L^2(\Omega)$, equipped with the canonical inner product. We further consider Δ as the Dirichlet Laplace operator in H, which we will denote by Δ_D. We have already seen in the previous section that $-\Delta_D$ endowed with the domain $H^2(\Omega) \cap H_0^1(\Omega)$ is positive and induces a scale of Hilbert spaces given by

$$H_n := D((-\Delta_D)^{n/2}), \quad (u \mid v)_{H_n} = \big((-\Delta_D)^{n/2}u \mid (-\Delta_D)^{n/2}v\big)_H.$$

Putting

$$C := (-\Delta_D)^{1/2}, \quad D(C) = H_0^1(\Omega) \subseteq H$$

yields a positive and invertible operator satisfying $\Delta_D = -C^2$. We further obtain

$$H_1 = H_C^1, \quad H_2 = H_C^2, \quad H_3 = H_C^3,$$

where we recall the isomorphism (cf. Section 4.1)

$$\begin{aligned} H_C^1 &= H_0^1(\Omega), \\ H_C^2 &= H^2(\Omega) \cap H_0^1(\Omega), \\ H_C^3 &= \left\{u \in H^3(\Omega) \cap H_0^1(\Omega) \,:\, \Delta u \in H_0^1(\Omega)\right\}. \end{aligned}$$

Note, that the boundedness of Ω implies that

$$\|u\|_{H_C^1} = \|\,|\nabla u|\,\|_{L^2(\Omega)},$$

due to Poincarè's inequality. ⋊

We are now in the position to state the well-posedness result for the second order Cauchy-problem (Cp-W).

Theorem 4.12 *Let $\Omega \subseteq \mathbb{R}^d$ $(d \leqslant 3)$ be a bounded C^3-domain, and let the function $K : \mathbb{R} \to \mathbb{R}$ satisfy Assumption 4.10, i.e.,*

$$K \in C^4(\mathbb{R}, \mathbb{R}), \quad K'(0) > -1.$$

Further, let $\kappa \in (0,1)$ and $r_0 > 0$ be arbitrary and let H_C^k, $k \in \{1,2,3\}$ denote the Hilbert spaces from Assumption 4.11. Then the following assertions hold.

(a) *There is a constant $c_0 = c_0(R) > 0$ such that for each*

$$(u_0, v_0) \in \overline{B}_{H_C^2 \times H_C^1}\left(0, \kappa \frac{R}{c_0}\right) \cap \overline{B}_{H_C^3 \times H_C^2}(0, r_0)$$

there exists a time $T = T(\kappa, r_0, R) > 0$ and a function

$$u(\cdot, u_0, v_0) = u \in C([0,T], H_C^3) \cap C^1([0,T], H_C^2) \cap C^2([0,T], H_C^1)$$

which solves (Cp-W) *by means of*

$$\begin{aligned} u''(t) + (K \circ u)''(t) &= -\Delta u(t) \quad (t \in [0,T]), \\ u(t) &\in H_0^1(\Omega) \quad (t \in [0,T]), \\ u(0) &= u_0, \\ u'(0) &= v_0. \end{aligned}$$

Moreover, we know that $\|(u(t), u'(t))\|_{H_C^2 \times H_C^1} \leqslant R$ for all $0 \leqslant t \leqslant T$.

(b) *If $v \in C([0,T'], H_C^3) \cap C^1([0,T'], H_C^2) \cap C^2([0,T'], H_C^1)$ is another solution of this system which also satisfies $\|(v(t), v'(t))\|_{H_C^2 \times H_C^1} \leqslant R$ for each $t \in [0,T']$, then $v(t) = u(t)$ for all $0 \leqslant t \leqslant \min\{T, T'\}$.*

(c) *The mapping*

$$\overline{B}_{H^2_C \times H^1_C}\left(0, \kappa \frac{R}{c_0}\right) \cap \overline{B}_{H^3_C \times H^2_C}(0, r_0) \subseteq H^2_C \times H^1_C \to C([0, T_0], H^2_C \times H^1_C),$$
$$(u_0, v_0) \mapsto u(\cdot, u_0, v_0),$$

is Lipschitz continuous. ⋊

Proof. We prove this theorem by deducing it from Theorem 3.45. Let $\rho > 0$ be the radius from Assumption 4.10. Recalling the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ we can fix some $R = R(\rho) > 0$ such that

$$\|u\|_{H^2_C} \leqslant R \quad \Rightarrow \quad \|u\|_{L^\infty(\Omega)} \leqslant \rho.$$

We thus put

$$W \coloneqq \overline{B}_{H^2_C}(0, R),$$

and start to verify the assumptions from Theorem 3.45 for the operator families $\{\gamma(y) : y \in W\}$ and $\{\Gamma(y) : y \in W\}$, where

$$\gamma(y) = 1 + K'(y), \quad \Gamma(y) = K''(y).$$

/// (PD) : Let $y \in W$ and $u, v \in H$. Then

$$\|\gamma(y)u\|_H = \|(1 + K'(y))u\|_{L^2} \leqslant (1 + c_1)\|u\|_H,$$

due to (4.20). Thus $\gamma(y) \in \mathcal{B}(H)$. We further derive

$$(\gamma(y)u \mid v)_H = \int_\Omega (1 + K'(y))uv \, \mathrm{d}x = \int_\Omega u(1 + K'(y))v \, \mathrm{d}x = (u \mid \gamma(y)v)_H$$

so that $\gamma(y) = \gamma(y)^*$. By means of (4.18) we estimate

$$(\gamma(y)u \mid u)_H \geqslant \delta\|u\|^2_H,$$

which finally implies (PD).

(G) : Given $y \in W$ we put

$$a \coloneqq 1 + K' \circ y \in L^\infty(\Omega).$$

Thus we have to show that for each $\varphi \in L^2(\Omega)$ there is a solution $u \in H^2(\Omega) \cap H^1_0(\Omega)$ of the Dirichlet problem

$$u \pm a\Delta u = \varphi \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

But this is a standard result in the theory of elliptic partial differential equations, cf. [15], Theorem 8.13.

$(LC\text{-}f)$: Let $y, \tilde{y} \in W$ and let u belong to H. Using (4.19) and the Sobolev embedding $H^2_C \hookrightarrow L^\infty$ we compute

$$\begin{aligned}\|\gamma(y)u - \gamma(\tilde{y})u\|_H^2 &= \int_\Omega |K'(y) - K'(\tilde{y})|^2 \, |u|^2 \,\mathrm{d}x \\ &\leqslant L_1^2 \, \|y - \tilde{y}\|_{L^\infty}^2 \, \|u\|_H^2 \\ &\lesssim \|y - \tilde{y}\|_{H^2_C}^2 \, \|u\|_H^2,\end{aligned}$$

which yields $(LC\text{-}f)$.

$(LC\text{-}fi)$: Given $y \in W$ it is readily seen that $\gamma(y)$ is invertible with

$$\gamma(y)^{-1} = \frac{1}{1 + K'(y)} \in \mathcal{B}(H),$$

so that $\|\gamma(y)^{-1}\|_{\mathcal{B}(H)} \leqslant \delta^{-1}$. In particular $\gamma(0)^{-1}$ corresponds to a multiplication with a constant, thus $\gamma(0)^{-1} \in \mathcal{B}(H^1_C)$. Now, let $y, \tilde{y} \in W$ and u belong to H^1_C. Then we want to control

$$\|\gamma(y)^{-1}u - \gamma(\tilde{y})^{-1}u\|_{H^1_C}^2 = \sum_{k=1}^{d} \|\partial_k\big(\gamma(y)^{-1}u - \gamma(\tilde{y})^{-1}u\big)\|_{L^2}^2.$$

By means of the product and chain rule for weak derivatives, we derive

$$\partial_k \, \gamma(y)^{-1}u = \frac{-K''(y)\partial_k y}{(1 + K'(y))^2} u + \frac{\partial_k u}{1 + K'(y)}.$$

Hence

$$\begin{aligned}\partial_k\big(\gamma(y)^{-1}u - \gamma(\tilde{y})^{-1}u\big) = &\left(\frac{1}{1 + K'(y)} - \frac{1}{1 + K'(y)}\right)\partial_k u \\ &+ \left(\frac{K''(\tilde{y})\partial_k \tilde{y}}{(1 + K'(\tilde{y}))^2} - \frac{K''(y)\partial_k y}{(1 + K'(y))^2}\right) u.\end{aligned}$$

We first use (4.18) and (4.19) to estimate

$$\begin{aligned}\left|\frac{1}{1 + K'(y)} - \frac{1}{1 + K'(\tilde{y})}\right| &= \frac{|K'(\tilde{y}) - K'(y)|}{|1 + K'(y)|\,|1 + K'(y)|} \\ &\leqslant \delta^{-2} L_1 \, |y - \tilde{y}|.\end{aligned}$$

Invoking the Sobolev embedding $H^2_C \hookrightarrow L^\infty$ then implies

$$\begin{aligned}\|\big((1 + K'(y))^{-1} - (1 + K'(\tilde{y}))^{-1}\big)\partial_k u\|_{L^2} &\lesssim \|y - \tilde{y}\|_{L^\infty} \, \|\partial_k u\|_{L^2} \\ &\lesssim \|y - \tilde{y}\|_{H^2_C} \, \|u\|_{H^1_C}.\end{aligned}$$

Because of

$$(1 + K'(y))^2 - (1 + K'(\tilde{y}))^2 = \{K'(y) - K'(\tilde{y})\}(2 + K'(y) + K'(\tilde{y}))$$

we can use (4.18)-(4.20) to estimate

$$\begin{aligned}\left|\frac{K''(\tilde{y})\partial_k\tilde{y}}{(1+K'(\tilde{y}))^2}-\frac{K''(y)\partial_k y}{(1+K'(y))^2}\right| &= \frac{1}{(1+K'(\tilde{y}))^2\,(1+K'(\tilde{y}))^2}\\ &\quad \Big|\{K''(\tilde{y})-K''(y)\}\partial_k y\,(1+K'(y))^2\\ &\quad + K''(y)\{\partial_k\tilde{y}-\partial_k y\}\,(1+K'(y))^2\\ &\quad + K''(y)\partial_k y\{(1+K'(y))^2-(1+K'(\tilde{y}))^2\}\Big|\\ &\leqslant \delta^{-2}L_2\,|y-\tilde{y}|\,|\partial_k\tilde{y}| + c_2\,|\partial_k y-\partial_k\tilde{y}|\\ &\quad + c_2(2+2c_1)L_0\,|\partial_k y|\,|y-\tilde{y}|.\end{aligned}$$

Using the Sobolev embedding $H^1 \hookrightarrow L^p$ $(3 \leqslant p \leqslant 6)$ and Hölder's inequality with $1/6+1/6+1/6=1/2$ and $1/3+1/6=1/2$, then yields

$$\begin{aligned}&\|(K''(\tilde{y})\partial_k\tilde{y}(1+K'(y))^{-2}-K''(y)\partial_k y(1+K'(\tilde{y}))^{-2})u\|_{L^2}\\ &\quad\lesssim \|\,|y-\tilde{y}|\,|\partial_k\tilde{y}|\,|u|\,\|_{L^2} + \|\,|\partial_k y-\partial_k\tilde{y}|\,|u|\,\|_{L^2}\\ &\qquad + \|\,|y-\tilde{y}|\,|\partial_k y|\,|u|\,\|_{L^2}\\ &\quad\lesssim \|y-\tilde{y}\|_{L^6}\,\|\partial_k\tilde{y}\|_{L^6}\,\|u\|_{L^6} + \|\partial_k y-\partial_k\tilde{y}\|_{L^3}\|u\|_{L^6}\\ &\qquad + \|y-\tilde{y}\|_{L^6}\,\|\partial_k y\|_{L^6}\,\|u\|_{L^6}\\ &\quad\lesssim \|y-\tilde{y}\|_{H^1}\,\|\partial_k\tilde{y}\|_{H^1}\,\|u\|_{H^1_C} + \|\partial_k y-\partial_k\tilde{y}\|_{H^1}\|u\|_{H^1_C}\\ &\qquad + \|y-\tilde{y}\|_{H^1}\,\|\partial_k y\|_{H^1}\,\|u\|_{H^1_C}\\ &\quad\lesssim \|y-\tilde{y}\|_{H^2_C}\,(\|\tilde{y}\|_{H^2_C}+\|y\|_{H^2_C}+1)\,\|u\|_{H^1_C}\\ &\quad\lesssim_R \|y-\tilde{y}\|_{H^2_C}\,\|u\|_{H^1_C}.\end{aligned}$$

Putting these two estimates together we have shown that

$$\|\gamma(y)^{-1}u-\gamma(\tilde{y})^{-1}u\|_{H^1_C} \lesssim_R \|y-\tilde{y}\|_{H^2_C}\,\|u\|_{H^1_C}. \tag{4.22}$$

In particular

$$\gamma(y)^{-1}u-\gamma(\tilde{y})^{-1}u \in H^1_C \quad (y,\tilde{y}\in W,\ u\in H^1_C).$$

Hence for each $y \in W$ and $u \in H^1_C$ we obtain

$$\begin{aligned}\|\gamma(y)^{-1}u\|_{H^1_C} &\leqslant \|\gamma(y)^{-1}u-\gamma(0)^{-1}u\|_{H^1C} + \|\gamma(0)^{-1}\|\,\|u\|_{H^1_C}\\ &\lesssim_R \{R+\|\gamma(0)^{-1}\|\}\,\|u\|_{H^1_C}.\end{aligned}$$

Consequently $\gamma(y)^{-1} \in \mathcal{B}(H^1_C)$ for each $y \in W$ and by means of (4.22) it follows

$$\|\gamma(y)^{-1}-\gamma(\tilde{y})^{-1}\|_{\mathcal{B}(H^1_C)} \lesssim_R \|y-\tilde{y}\|_{H^2_C} \quad (y,\tilde{y}\in W),$$

i.e., *(LC-fi)* holds.

$(LC\text{-}s)$: Let y belong to $\overline{B}_Z(0,r)$ and u be contained in H. By means of (4.20) and the Sobolev embedding $H^2 \hookrightarrow L^\infty$ we derive

$$\begin{aligned}\|[\Gamma(y_1)y_2]u\|_H^2 &= \int_\Omega |K''(y_1)y_2|^2\,|u|^2\,\mathrm{d}x \\ &\lesssim \|y_2\|_{H^2}^2\,\|u\|_{L^2}^2 \\ &\lesssim \|y\|_Z^2\,\|u\|_H^2.\end{aligned}$$

We thus obtain $\|\Gamma(y_1)y_2\|_{\mathcal{B}(H)} \lesssim r$. Now, let $y, \tilde{y} \in \mathcal{W}$ and $u \in H_C^2$. We then want to control

$$\begin{aligned}&\|[\Gamma(y_1)y_2]u - [\Gamma(y_1)y_2]u\|_{H_C^1}^2 \\ &\qquad = \sum_{k=1}^d \int_\Omega |\partial_k(K''(y_1)y_2 u - K''(\tilde{y}_1)\tilde{y}_2 u)|^2\,\mathrm{d}x.\end{aligned}$$

Applying the product yields

$$\partial_k(K''(y_1)y_2u) = [\partial_k(K''(y_1)y_2)]u + [K''(y_1)y_2]\partial_k u. \tag{4.23}$$

First, we expand

$$\begin{aligned}&[\partial_k(K''(y_1)y_2) - \partial_k K''(\tilde{y}_1)\tilde{y}_2]u \\ &\quad = \{K'''(y_1) - K'''(\tilde{y}_1)\}\partial_k y_1\, y_2\, u \\ &\qquad + K'''(\tilde{y}_1)\{\partial_k y_1 - \partial_k\tilde{y}_1\}\, y_2\, u \\ &\qquad + K'''(\tilde{y}_1)\partial_k\tilde{y}_1\,\{y_2 - \tilde{y}_2\}\, u \\ &\qquad + \{K''(y_1) - K''(\tilde{y}_1)\}\partial_k y_2\, u \\ &\qquad + K''(\tilde{y}_1)\,\{\partial_k y_2 - \partial_k\tilde{y}_2\}\, u.\end{aligned}$$

Invoking (4.19) and (4.20), the Sobolev embeddings $H^2 \hookrightarrow L^\infty$ and $H^1 \hookrightarrow L^p$ $(3 \leqslant p \leqslant 6)$, and Hölder's inequality with $1/6 + 1/6 + 1/6 = 1/2$ and $1/3 + 1/6 = 1/2$ then yields

$$\begin{aligned}&\|[\partial_k(K''(y_1)y_2) - \partial_k(K''(\tilde{y}_1)\tilde{y}_2)]u\|_{L^2} \\ &\quad\lesssim \|y_1 - \tilde{y}_1\|_{L^\infty}\,\|\,|\partial_k y_1|\,|y_2|\,|u|\|_{L^2} \\ &\qquad + \|\,|\partial_k y_1 - \partial_k\tilde{y}_1|\,|y_2|\,|u|\,\|_{L^2} + \|\,|\partial_k\tilde{y}_1|\,|y_2 - \tilde{y}_2|\,|u|\,\|_{L^2} \\ &\qquad + \|y_1 - \tilde{y}_1\|_{L^\infty}\,\|\,|\partial_k y_2|\,|u|\|_{L^2} + \|\,|\partial_k y_2 - \partial_k\tilde{y}_2|\,|u|\|_{L^2} \\ &\quad\lesssim \|y_1 - \tilde{y}_1\|_{H_C^2}\,\|\partial_k y_1\|_{H^1}\,\|y_2\|_{H^1}\,\|u\|_{H^1} \\ &\qquad + \|\partial_k y_1 - \partial_k\tilde{y}_1\|_{H^1}\,\|y_2\|_{H^1}\,\|u\|_{H^1} + \|\partial_k\tilde{y}_1\|_{H^1}\,\|y_2 - \tilde{y}_2\|_{H^1}\,\|u\|_{H^1} \\ &\qquad + \|y_1 - \tilde{y}_1\|_{H_C^2}\,\|\partial_k y_2\|_{H^1}\,\|u\|_{H^1} + \|\partial_k y_2 - \partial_k\tilde{y}_2\|_{L^2}\,\|u\|_{L^\infty} \\ &\quad\lesssim \|y - \tilde{y}\|_Y\,(\|y\|_Y^2 + 3\|y\|_Y + 1)\,\|u\|_{H_C^2} \\ &\quad\lesssim_{R^2} \|y - \tilde{y}\|_Y\,\|u\|_{H_C^2}.\end{aligned}$$

Rewriting the difference related to the second summand in (4.23) as

$$\begin{aligned}(K''(y_1)y_2 - K''(\tilde{y}_1)\tilde{y}_2)\partial_k u \\ &= K''(y_1)\{y_2 - \tilde{y}_2\}\partial_k u + \{K''(y_1) - K''(\tilde{y}_1)\}\tilde{y}_2\,\partial_k u,\end{aligned}$$

and concluding correspondingly, we obtain

$$\begin{aligned}\|(K''(y_1)y_2 - K''(\tilde{y}_1)\tilde{y}_2)\partial_k u\|_{L^2} \\ &\lesssim \|y_2 - \tilde{y}_2\|_{H^1_C}\,\|\partial_k u\|_{H^1} + \|y_1 - \tilde{y}_1\|_{H^2_C}\,\|\tilde{y}_2\|_{H^1_C}\,\|\partial_k u\|_{H^1} \\ &\lesssim \|y\tilde{y}\|_Y(1 + \|y\|_Y)\|u\|_{H^2_C} \\ &\lesssim_R \|y - \tilde{y}\|\,\|u\|_{H^2_C}.\end{aligned}$$

Putting these two estimates together and arguing as in the end of part $(LC\text{-}fi)$ then yields

$$\|\Gamma(y_1)y_2 - \Gamma(y_1)y_2\|_{\mathcal{B}(H^2_C,H^1_C)} \lesssim_{R^2} \|y - \tilde{y}\|_Y \quad (y,\tilde{y} \in \mathcal{W}),$$

i.e., $(LC\text{-}s)$ holds.

(CE) : Given $z = (z_1, z_2) \in \mathcal{W} \cap \overline{B}_Z(0,r)$ we define

$$\begin{aligned}\mathcal{B}_1(z_1) &\coloneqq \Delta_D\left(\frac{1}{1 + K'(z_1)}\cdot\right) - \frac{1}{1 + K'(z_1)}\Delta_D, \\ \mathcal{B}_2(z) &\coloneqq -\Delta_D\left(\frac{K''(z_1)z_2}{1 + K'(z_1)}\Delta_D^{-1}\cdot\right) + \frac{K''(z_1)z_2}{1 + K'(z_1)}.\end{aligned}$$

We will now show that $\mathcal{B}_1(z_1)$ extends to a bounded operator from H^1_C to H and $\mathcal{B}_2(z)$ extends to a bounded operator on H, both extensions having norms which are uniformly bounded on $\overline{B}_Z(0,r)$. Thus (CE) holds. In the following we will omit the subscript D. First, let $u \in H^2_C$. Putting

$$f(z_1) \coloneqq \frac{1}{1 + K'(z_1)},$$

we derive

$$\mathcal{B}(z_1)u = f''(z_1)\,|\nabla z_1|^2\,u + 2f'(z_1)\,\nabla z_1 \cdot \nabla u + f'(z_1)\,\Delta z_1\,u, \tag{4.24}$$

where

$$f'(z_1) = \frac{-K''(z_1)}{(1 + K'(z_1))^2}, \quad f''(z_1) = \frac{2K''(z_1)^2 - (1 + K'(z_1))K'''(z_1)^2}{(1 + K'(z_1))^3}.$$

Obviously (4.24) extends to H^1_C and by means of (4.18) and (4.20), we estimate

$$\|f'(z_1)\|_{L^\infty} \leqslant c_2\delta^{-2}, \quad \|f''(z_1)\|_{L^\infty} \leqslant (2c_2^2 + (1 + c_1)c_2^2)\delta^{-3}.$$

Using the Sobolev embeddings $H^1 \hookrightarrow L^p$ $(3 \leqslant p \leqslant 6)$ and $H^2 \hookrightarrow L^\infty$, and applying Hölder's inequality with $1/3 + 1/6 = 1/2$ then yields

$$\begin{aligned}\|\mathcal{B}_1(z_1)u\|_H &\lesssim_r \| \, |\nabla z_1|^2 \, |u| \, \|_{L^2} + \| \, |\nabla z_1| \, |u| \, \|_{L^2} + \| \, |\Delta z_1| \, |u| \, \|_{L^2} \\ &\lesssim_r (\| |\nabla z_1| \|_{H^2}^2 + \| |\nabla z_1| \|_{H^2} + \|\Delta z_1\|_{H^2}) \|u\|_{H^1_C} \\ &\lesssim_r (\|z_1\|_{H^3_C}^2 + 2\|z_1\|_{H^3_C}) \|u\|_{H^1_C}.\end{aligned}$$

Hence $\mathcal{B}_1(z_1) \in \mathcal{B}(H^1_C, H)$ and there is a constant $b_1 = b_1(r)$ such that

$$\|\mathcal{B}_1(z_1)\|_{\mathcal{B}(H^1_C, H)} \leqslant b_0.$$

Now, let $u \in H$. Denoting

$$g(z_1) := -K''(z_1) f(z_1),$$

we obtain

$$\begin{aligned}\mathcal{B}_2(z)u = &\Big(g'(z_1)\, \Delta z_1 \, z_2 + g''(z_1)\, |\nabla z_1|^2 \, z_2 + 2g'(z_1)\, \nabla z_1 \cdot \nabla z_2 \\ &+ g(z_1)\, \Delta z_1\Big) \Delta^{-1} u \\ &+ 2\big(g(z_1)\, \nabla z_2 + g'(z_1)\, z_2 \, \nabla z_1\big) \cdot \nabla \Delta^{-1} u,\end{aligned}$$

where

$$\begin{aligned}g'(z_1) &= \frac{-K'''(z_1)(1 + K'(z_1)) + K''(z_1)^2}{(1 + K'(z_1))^2}, \\ g''(z_1) &= \frac{-K^{(4)}(z_1)(1 + K'(z_1))^2 + K'''(z_1)K''(z_1)(1 + K'(z_1))}{(1 + K'(z_1))^3}.\end{aligned}$$

Thus

$$\begin{aligned}\|g'(z_1)\|_{L^\infty} &\leqslant \delta^{-2}(c_3(1 + c_1) + c_2^2), \\ \|g''(z_1)\|_{L^\infty} &\leqslant \delta^{-3}(c_4(1 + c_1)^2 + c_3 c_2 (1 + c_1)),\end{aligned}$$

due to (4.18) and (4.20). We thus can estimate

$$\begin{aligned}\|\mathcal{B}_2(z)u\|_H \lesssim_r &\| \, |\Delta z_1| \, |z_2| \, |\Delta^{-1} u| \|_{L^2} + \| \, |\nabla z_1|^2 \, |z_2| \, |\Delta^{-1} u| \|_{L^2} \\ &+ \| \, |\nabla z_1| \, |\nabla z_2| \, |\Delta^{-1} u| \|_{L^2} + \| \, |\Delta z_1| \, |\Delta^{-1} u| \|_{L^2} \\ &+ \| \, |\nabla z_1| \, |\nabla \Delta^{-1} u| \|_{L^2} + \| \, |\nabla z_1| \, |z_2| \, |\nabla \Delta^{-1} u| \|_{L^2}.\end{aligned}$$

So, once again Using the Sobolev embeddings $H^1 \hookrightarrow L^p$ $(3 \leqslant p \leqslant 6)$ and $H^2 \hookrightarrow L^\infty$, and applying Hölder's inequality with $1/6 + 1/6 + 1/6 = 1/2$ and $1/3 + 1/6 = 1/2$

implies

$$\begin{aligned}
\|\mathcal{B}_2(z)u\|_H \lesssim_r \Big(&\|\Delta z_1\|_{L^6} \|z_2\|_{L^6} + \||\nabla z_1|\|_{L^6} \|z_2\|_{L^\infty} \\
&+ \||\nabla z_1|\|_{L^6} \||\nabla z_2|\|_{L^6}\Big) \|\Delta^{-1}u\|_{L^6} \\
&+ \|\Delta z_1\|_{L^2} \|\Delta^{-1}u\|_{L^\infty} \\
&+ \Big(\||\nabla z_1|\|_{L^3} + \||\nabla z_1|\|_{L^6} \|z_2\|_{L^6}\Big) \||\nabla \Delta^{-1}u|\|_{L^6} \\
\lesssim_r \Big(&\|z_1\|_{H^3_C} \|z_2\|_{H^2_C} + \|z_1\|_{H^3_C} \|z_2\|_{H^2_C} \\
&+ \|z_1\|_{H^3_C} \|z_2\|_{H^2_C}\Big) \|\Delta^{-1}u\|_{H^2_C} \\
&+ \|z_1\|_{H^3_C} \|\Delta^{-1}u\|_{H^2_C} \\
&+ \Big(\|z_1\|_{H^3_C} + \|z_1\|_{H^3_C} \|z_2\|_{H^2_C}\Big) \|\Delta^{-1}u\|_{H^2_C}.
\end{aligned}$$

Recalling $\|\Delta^{-1}u\|_{H^2_C} \lesssim \|u\|_H$ we thus have shown that $\mathcal{B}_2(z) \in \mathcal{B}(H)$ and that there is a constant $b_2 = b_2(r^3)$ such that

$$\|\mathcal{B}_2(z)\|_{\mathcal{B}(H)} \leqslant b_2,$$

and we are done. ///

In view of Theorem 3.45 it only remains to find some twice differentiable operator λ such that $D\lambda(y) = \gamma(y)$ and $D^2\lambda(y) = \Gamma(y)$ for $y \in \overline{B}_{H^2_C}(0, R)$. Therefore let $r > R$ be arbitrary and define

$$\lambda : B_{H^2_C}(0, r) \subseteq H^2_C \to H^2_C, \quad \lambda(y) \coloneqq y + K(y).$$

Let y, h belong to $B_{H^2(\Omega)^3}(0, r)$ such that also $y + h \in B_{H^2(\Omega)^3}(0, r)$. By means of Taylor's formula we obtain for almost every $x \in \Omega$ the equalities

$$\begin{aligned}
K(y(x) + h(y)) - &K(y(x)) - K'(y(x))h(x) \\
&= \int_0^1 [K'(y(x) + th(x)) - K'(y(x))]h(x)\,\mathrm{d}t \eqqcolon R^1_h(x),
\end{aligned}$$

and

$$\begin{aligned}
K'(y(x) + h(y)) - &K'(y(x)) - K''(y(x))h(x) \\
&= \int_0^1 [K''(y(x) + th(x)) - K''(y(x))]h(x)\,\mathrm{d}t \eqqcolon R^2_h(x).
\end{aligned}$$

Now, we conclude exactly in the same way as we have done in the proof of Proposition 4.4 that

$$\|R^k_h\|_{H^2_C} \lesssim_r \|h\|^2_{H^2_C} \quad (k \in \{0, 1\}).$$

We have thus shown that λ is twice differentiable and that

$$\begin{aligned}
D\lambda(y) &= 1 + K'(y) = \gamma(y), \\
D^2\lambda(y) &= K''(y) = \Gamma(y) \quad (y \in B_{H^2_C}(0, r)). \quad \blacksquare
\end{aligned}$$

Appendix **A**

Function Spaces

For convenience of the reader we include the elementary definitions and basic properties of the Sobolev spaces relevant to this treatise. For more detailed information we refer to [1, 10, 11, 15, 25, 27, 40].

A.1 Sobolev Spaces and Fourier Transformation

Definition A.1 Let $\Omega \subseteq \mathbb{R}^d$ be open and assume that $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}_0$. The Sobolev space $W^{k,p}(\Omega)$ is then defined by

$$W^{k,p}(\Omega) \coloneqq \{v \in L^p(\Omega) : \partial^\alpha v \in L^p(\Omega) \text{ for all } |\alpha| \leqslant k\},$$

and endowed with the norm

$$\|u\|_{W^{k,p}(\Omega)} \coloneqq \left(\sum_{|\alpha|\leqslant k} \|\partial^\alpha v\|^p_{L^p(\Omega)}\right)^{1/p}, \quad p < \infty,$$

$$\|u\|_{W^{k,\infty}(\Omega)} \coloneqq \max_{|\alpha|\leqslant k} \|\partial^\alpha v\|_{L^\infty(\Omega)}.$$

We will usually write $H^k(\Omega)$ instead of $W^{k,2}(\Omega)$. ⋊

Now, we consider the case where $\Omega = \mathbb{R}^d$. Then by means of the Fourier Transformation we can characterize the spaces $H^k(\mathbb{R}^d)$ in the following way.

Proposition A.2 *If $k \in \mathbb{N}_0$, then*

$$H^k(\mathbb{R}^d) = \left\{v \in L^2(\mathbb{R}^d) : \xi \mapsto (1+\xi^2)^{k/2}\,\hat{v}(\xi) \in L^2(\mathbb{R}^d)\right\},$$

and the Sobolev-norm $\|\cdot\|_{W^k_2(\mathbb{R}^d)}$ is equivalent to the norm

$$\|v\|^2_{k,2} \coloneqq \int_{\mathbb{R}^d} (1+\xi^2)^k |\hat{v}(\xi)|^2\, \mathrm{d}\xi = \|\mathcal{F}^{-1}\langle\cdot\rangle^k \mathcal{F}v\|^2_{L^2(\mathbb{R}^d)}. \quad ⋊$$

Thus for arbitrary $s \geqslant 0$ we consider the Bessel potential

$$\Lambda^s \coloneqq (I - \Delta)^{s/2} \coloneqq \mathcal{F}^{-1} \langle \cdot \rangle^s \mathcal{F}$$

in $L^2(\mathbb{R}^d)$. We then define the Sobolev space $H^s(\mathbb{R}^d)$ by

$$H^s(\mathbb{R}^d) \coloneqq D(\Lambda^s) = \left\{ v \in L^2(\mathbb{R}^d) \,:\, \xi \mapsto (1 + \xi^2)^{s/2}\, \hat{v}(\xi) \in L^2(\mathbb{R}^d) \right\}$$

and equip this space with the Graph norm of Λ^s, i.e.,

$$\|u\|_{s,2}^2 \coloneqq \|\mathcal{F}^{-1} \langle \cdot \rangle^k \mathcal{F} v\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (1 + \xi^2)^s |\hat{v}(\xi)|^2 \,\mathrm{d}\xi.$$

In the following we will need an alternative characterisation of the Sobolev-norm $\|\cdot\|_{s,2}$ by means of so called Slobodetskii seminorms. These are given by

$$|v|_{s,2} \coloneqq \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|v(x) - v(y)|^2}{|x - y|^{d+2s}} \right)^{1/2}, \quad 0 < s < 1.$$

The key lemma to the desired description is the following, see [27], Lemma 3.15.

Lemma A.3 *If* $0 < s < 1$, *then there is a positive constant* c_s *such that*

$$|v|_{s,2}^2 = c_s \int_{\mathbb{R}^d} |\xi|^{2s}\, |\hat{v}(\xi)|^2 \,\mathrm{d}\xi,$$

for each $v \in H^s(\mathbb{R}^d)$. ⋊

Consequently we obtain the the following characterisation.

Corollary A.4 *Let* $s > 0$ *be of the form* $s = [s] + \{s\}$ *for some* $[s] \in \mathbb{N}_0$ *and* $0 < \{s\} < 1$. *Then the norm* $\|\cdot\|_s$ *on* $H^s(\mathbb{R}^d)$ *is equivalent to the norm*

$$\begin{aligned} \|v\|_{W_2^s(\mathbb{R}^d)}^2 &\coloneqq \sum_{|\alpha| \leqslant [s]} \int_{\mathbb{R}^d} |\partial^\alpha v(x)|^2 \,\mathrm{d}x + \sum_{|\alpha| = [s]} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^2}{|x - y|^{d+2\{s\}}} \,\mathrm{d}x\mathrm{d}y \\ &= \|v\|_{H^{[s]}(\mathbb{R}^d)}^2 + \sum_{|\alpha| = [s]} |\partial^\alpha v|_{\{s\},2}^2. \end{aligned}$$ ⋊

We will make use of the following estimate concerning products of functions, see [34], Section 4.6.4, Theorem 3.

Proposition A.5 *Let* $d \in \mathbb{N}$ *and* $s > d/2$. *Then* $H^s(\mathbb{R}^d)$ *forms an algebra and for all* $u, v \in H^s(\mathbb{R}^d)$ *there is some constant* $c = c_{s,d}$ *only depending on* s *and* d *such that*

$$\|uv\|_{H^s(\mathbb{R}^d)} \leqslant c \left(\|u\|_{H^s(\mathbb{R}^d)} \|v\|_{L^\infty(\mathbb{R}^d)} + \|u\|_{L^\infty(\mathbb{R}^d)} \|u\|_{H^s(\mathbb{R}^d)} \right).$$ ⋊

A.2 Sobolev Spaces on Domains

For $s > 0$ we write

$$s = [s] + \{s\} \quad \text{such that } [s] \in \mathbb{N}_0,\ \{s\} \in (0,1).$$

Given $s \geqslant 0$ and $1 \leqslant p < \infty$ we define

$$|v|_{\{s\},p} := \left(\int_\Omega \int_\Omega \frac{|v(x) - v(y)|^p}{|x-y|^{d+\{s\}p}} \mathrm{d}x\mathrm{d}y \right)^{1/p}$$

and

$$\|v\|_{W^{s,p}} := \left(\|v\|^p_{W_p^{[s]}} + \sum_{|\alpha|=[s]} |\partial^\alpha v|^p_{\{s\},p} \right)^{1/p}.$$

Definition A.6 Let $\Omega \subseteq \mathbb{R}^d$ be open and $s \geqslant 0$ and $1 \leqslant p < \infty$. Then we put

$$W^{s,p}(\Omega) := \left\{ v \in W_p^{[s]}(\Omega) \,:\, |\partial^\alpha v|_{s,p} < \infty \text{ for all } |\alpha| = [s] \right\},$$

and

$$\|v\|_{W_p^s(\Omega)} := \|v\|_{W^{s,p}}. \quad \rtimes$$

A.3 Sobolev Spaces on Manifolds

Definition A.7 Let $\mathbb{N} \ni d \geqslant 2$, $k \in \mathbb{N}_0$ and $0 \leqslant \alpha \leqslant 1$. A non empty domain $\Omega \subseteq \mathbb{R}^d$ is called *special $C^{k,\alpha}$-domain* if there exists $f \in C^{k,\alpha}(\mathbb{R}^{d-1})$ such that

$$\Omega = R\,\mathrm{Epi}(f),$$

for some rigid transformation R, i.e., $Rx = Ax + b$ $(x \in \mathbb{R}^{d-1})$ where A is some $d-1$-dimensional orthogonal matrix and b is a $d-1$-dimensional vector. $\rtimes$

Remark A.8 Let $k \in \mathbb{N}$ and $0 \leqslant \alpha \leqslant 1$. If Ω is a special $C^{k,\alpha}$-domain with $\Omega = R\,\mathrm{Epi}(f)$, then $\partial\Omega = R\mathrm{Graph}(f)$. Further, for every $x \in \mathbb{R}^{d-1}$ the unit outward normal vector $\mathfrak{n}$ at $R(x, f(x)) \in \partial\Omega$ is given by

$$\mathfrak{n}(x) = \frac{1}{\sqrt{1 + |\nabla f(x)|^2}} R \begin{pmatrix} -\nabla f(x) \\ 1 \end{pmatrix}.$$

If $k = 0$ and $\alpha = 1$, then by the Rademacher theorem this remains true at least for almost every $x \in \mathbb{R}^{d-1}$. $\rtimes$

Definition A.9 A domain $\Omega \subseteq \mathbb{R}^d$ is called *$C^{k,\alpha}$-domain*, or we say that $\partial\Omega$ is of class $C^{k,\alpha}$ if for every $x \in \partial\Omega$ there exist an $\varepsilon = \varepsilon_x > 0$ and a special $C^{k,\alpha}$-domain Ω_x such that

$$B_\varepsilon(x) \cap \Omega = B_\varepsilon(x) \cap \Omega_x. \quad \rtimes$$

Remark A.10 A $C^{0,1}$-domain is usually called Lipschitz domain. Further, if Ω is a $C^{0,0}$-domain we say that $\partial\Omega$ is continuous, or that Ω has a continuous boundary. ⋊

Proposition A.11 *A domain $\Omega \subseteq \mathbb{R}^d$, whose boundary is compact, defines a $C^{k,\alpha}$-domain if and only if there are finitely many open sets $U_j \subseteq \mathbb{R}^d, 1 \leqslant j \leqslant N$ with $\partial\Omega \subseteq \bigcup_j U_j$ and corresponding $C^{k,\alpha}$-diffeomorphism $g_j : U_j \to B_1(0) \subseteq \mathbb{R}^d$ such that*

$$
\begin{aligned}
g_j(U_j \cap \Omega) &= B_1^+(0) = \{(x', x_d) \in B_1(0) : x_d > 0\}, \\
g_j(U_j \cap \partial\Omega) &= B_1^0(0) = \{(x', x_d) \in B_1(0) : x_d = 0\}, \\
g_j(U_j \cap \Omega) &= B_1^-(0) = \{(x', x_d) \in B_1(0) : x_d < 0\}. \quad ⋊
\end{aligned}
$$

Proof. Since $\overline{\Omega}$ is compact there are finitely many $x_1, \dots, x_N \in \partial\Omega$ and $\varepsilon_1, \dots, \varepsilon_N > 0$ such that $\partial\Omega \subseteq \bigcup_j B_{\varepsilon_j}(x_j)$ and

$$B_{\varepsilon_j}(x_j) \cap \Omega = B_{\varepsilon_j}(x_j) \cap R_j \mathrm{Epi}(f_j),$$

with R_j and $f_j \in C^{k,\alpha}(\mathbb{R}^{d-1})$ from the definition of a bounded Lipschitz domain. We put $U_j \coloneqq B_{\varepsilon_j}(x_j)$ and construct the maps g_j as follows.

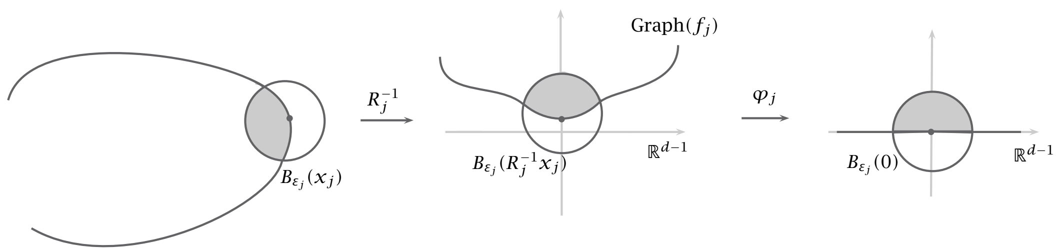

Figure 5: Constructing charts.

First, we rotate and translate back with R_j^{-1} where

$$R_j^{-1}(U_j \cap \Omega) = B_{\varepsilon_j}(R_j^{-1}x_j) \cap \mathrm{Epi}(f_j).$$

Then we continue with $\varphi_j : \mathbb{R}^d \to \mathbb{R}^d$, $\varphi_j(x', x_d) = (x', x_d - f_j(x'))$. Consequently

$$
\begin{aligned}
\varphi_j(B_{\varepsilon_j}(R_j^{-1}x_j) \cap \mathrm{Epi}(f_j)) &= B_{\varepsilon_j}^+(0), \\
\varphi_j(B_{\varepsilon_j}(R_j^{-1}x_j) \cap \mathrm{Graph}(f_j)) &= B_{\varepsilon_j}^0(0), \\
\varphi_j(B_{\varepsilon_j}(R_j^{-1}x_j) \cap \overline{\mathrm{Epi}}(f_j)^c) &= B_{\varepsilon_j}^-(0).
\end{aligned}
$$

Notice that each φ_j is a $C^{k,\alpha}$-diffeomorphism whose inverse is given by $\varphi_j^{-1}(y', y_d) = (y', y_d + f_j(y'))$. Finally, we rescale to $\mathbb{R}^d$ with ε_j^{-1}. ∎

Remark A.12 If we identify $B_1^0(0) \subseteq \mathbb{R}^d$ with $B_1(0) \subseteq \mathbb{R}^{d-1}$ we get a parametrisation of $\Gamma_j \coloneqq \partial\Omega \cap U_j$ by $\psi_j \coloneqq g_j^{-1} : B_1(0) \subseteq \mathbb{R}^{d-1} \to \partial\Omega$. Hence $\Gamma_j = \psi_j(B_1(0))$. ⋊

So, we first investigate how to integrate on surfaces of the type $\Gamma = \psi(U)$, $U \subseteq \mathbb{R}^{d-1}$ open, for some $C^{k,\alpha}$-diffeomorphism $\psi : U \subseteq \mathbb{R}^{d-1} \to \mathbb{R}^d$. Given such a diffeomorphism we define

$$g_\psi(x) \coloneqq \det D\psi(x)^\top D\psi(x) \geqslant 0,$$

for almost every $x \in \Omega$. Remember that $\operatorname{vol}(\Gamma) = \int_U \sqrt{g_\psi(x)}\,\mathrm{d}x$.

Definition A.13 Let $\Omega \subseteq \mathbb{R}^d$ be a bounded $C^{k,\alpha}$-domain, and $\Gamma \subseteq \partial\Omega$ with $\Gamma = \psi(U)$ for some $C^{k,\alpha}$-diffeomorphism $\psi : U \subseteq \mathbb{R}^{d-1} \to \mathbb{R}^d$. Let $\nu : \Gamma \subseteq \mathbb{R}^d \to \mathbb{C}$ be measurable. We say $\nu \in L^p(\Gamma)$ $(1 \leqslant p < \infty)$ if $\nu \circ \psi \in L^p(U, \sqrt{g_\psi}\,\mathrm{d}x)$ and put

$$\int_\Gamma |\nu|^p \,\mathrm{d}\sigma \coloneqq \int_U |\nu(\psi(x))|^p \sqrt{g_\psi(x)}\,\mathrm{d}x.$$

This definition is independent of the choice of ψ. ⋊

Lemma A.14 *Let $\Gamma = \psi(U)$ be part of the boundary of a bounded $C^{k,\alpha}$-domain. If $R : \mathbb{R}^d \to \mathbb{R}^d$ is a rigid transformation, then $g_{R\circ\psi} = g_\psi$ and $\nu \in L^1(\Gamma)$ if and only if $\nu \circ R^{-1} \in L^1(R\Gamma)$. In this case we have*

$$\int_{R\Gamma} (\nu \circ R^{-1})\,\mathrm{d}\sigma = \int_\Gamma \nu \,\mathrm{d}\sigma. \quad ⋊$$

Proof. Since $Rx = Ax + b$ with $A^\top A = I$, we get

$$g_{R\circ\psi} = \det D\psi^\top DT(\psi)^\top DT(\psi) D\psi = \det D\psi^\top D\psi = g_\psi.$$

As $R\Gamma = (R \circ \psi)(U)$, $R \circ \psi$ is a parametrisation of $R\Gamma$, hence we simply compute

$$\int_{R\Gamma} (\nu \circ R^{-1})\,\mathrm{d}\sigma = \int_U \nu(\psi(x))\sqrt{g_{R\circ\psi}(x)}\,\mathrm{d}x = \int_\Gamma \nu\,\mathrm{d}\sigma. \quad ∎$$

We focus again on a special $C^{k,\alpha}$-domain Ω. Then the canonical parametrisation ψ is of the form $\psi(x) = R(x, f(x))$ $(x \in B_\varepsilon(0))$, for some rigid transformation R. Further, from linear algebra we know that

$$\det(I + xy^\top) = 1 + y^\top x \quad (x, y \in \mathbb{R}^d).$$

Hence if $p(x) \coloneqq (x, f(x))$ $(x \in B_\varepsilon(0))$, then

$$g_p(x) = \det Dp(x)^\top Dp(x) = \det(I + \nabla f(x) \nabla f(x)^\top) = 1 + |\nabla f(x)|^2.$$

So, for $\Gamma = R\mathrm{Graph}(f)$, $f : B_\varepsilon(0) \subseteq \mathbb{R}^{d-1} \to \mathbb{R}$ we get, thanks to lemma A.14

$$\int_\Gamma \nu\,\mathrm{d}\sigma = \int_{R^{-1}\Gamma} (\nu \circ R)\,\mathrm{d}\sigma = \int_{B_\varepsilon(0)} \nu(R(x, f(x)))\sqrt{1 + |\nabla f(x)|^2}\,\mathrm{d}x.$$

Definition A.15 As we have already seen before, for a $C^{k,\alpha}$-domain $\Omega \subseteq \mathbb{R}^d$ with compact boundary, we can choose finitely many $\varepsilon_j > 0$, $x_j \in \partial\Omega$ $(1 \leqslant j \leqslant N)$ such that $\partial\Omega \subseteq \bigcup_j B_{\varepsilon_j}(x_j)$. We put $U_j \coloneqq B_{\varepsilon_j}(x_j)$ and chose an open $U_0 \subseteq\subseteq \Omega$ with $\overline{\Omega} \subseteq U_{j=0}^N$. Thus $\{U_j : 0 \leqslant j \leqslant N\}$ is an open cover of the compact set Ω and hence there is a partition of unity according to the U_j, i.e. there are test functions $\eta_j \in C_c^\infty(U_j)$, $0 \leqslant \eta_j \leqslant 1$, with

$$\sum_{j=0}^N \eta_j(x) = 1, \quad x \in \overline{\Omega}.$$

We call a family of pairs $(U_j, \eta_j)_{1\leqslant j \leqslant N}$ with the above properties a *localization of* Ω. ⋊

Definition A.16 Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain with localization $(U_j, \eta_j)_{1\leqslant j\leqslant N}$ and let $v : \partial\Omega \to \mathbb{C}$. For $1 \leqslant p < \infty$ we say that v belongs to $L^p(\partial\Omega)$ if and only if $v\eta_j \in L^p(\partial\Omega \cap U_j)$ for all $1 \leqslant j \leqslant N$. We define

$$\int_{\partial\Omega} v \,\mathrm{d}\sigma \coloneqq \sum_{j=1}^N \int_{\partial\Omega\cap U_j} v\eta_j \,\mathrm{d}\sigma,$$

where we know how to compute the terms on the right side, that is to say

$$\int_{\partial\Omega\cap U_j} v\eta_j \,\mathrm{d}\sigma = \int_{B_{\varepsilon_j}(0)} \eta_j(R_j(x, f_j(x)))v(R_j(x, f_j(x)))\sqrt{1 + |\nabla f_j(x)|^2}\,\mathrm{d}x,$$

for some rigid transformation R_j and $f_j \in C^{k,\alpha}(\mathbb{R}^{d-1})$. This Definition is independent of the choice of the localization. ⋊

Proposition A.17 *If we endow $L^p(\partial\Omega)$ with the norm*

$$\|v\|^p_{L^p(\partial\Omega)} \coloneqq \int_{\partial\Omega} |v|^p \,\mathrm{d}\sigma = \sum_{j=1}^N \int_{\partial\Omega\cap U_j} \eta_j|v|^p \,\mathrm{d}\sigma,$$

then $L^p(\partial\Omega)$ is a Banach space. In particular $L^2(\partial\Omega)$ is a Hilbert space with respect to the scalar product

$$(u \mid v)_{L^2(\partial\Omega)} \coloneqq \int_{\partial\Omega} u\overline{v}\,\mathrm{d}\sigma. \quad ⋊$$

Gauß Theorem A.18 *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain with unit outward normal vector $\mathfrak{n}$. If $v \in C(\overline{\Omega}, \mathbb{R}^d)$ with $v\big|_\Omega \in C_b^1(\Omega, \mathbb{R}^d)$, then*

$$\int_\Omega \operatorname{div} v \,\mathrm{d}x = \int_{\partial\Omega} v \cdot \mathfrak{n}\,\mathrm{d}\sigma. \quad ⋊$$

Definition A.19 Suppose $\partial\Omega$ is a $C^{k,\alpha}$-domain with compact boundary, and let $(U_j = B_{\varepsilon_j}(x_j), \eta_j)_{1\leqslant j\leqslant N}$ be the corresponding standard localization of Ω. For $s \geqslant 0$ and $1 \leqslant p < \infty$ we say

$$v \in W^{p,s}(\partial\Omega) \quad :\Leftrightarrow \quad \eta_j v \in W^{p,s}(B_{\varepsilon_j}(0)) \text{ for all } 1 \leqslant j \leqslant N,$$

and put

$$|v|_{\{s\},p,\partial\Omega} \coloneqq \left(\sum_{j=1}^{N} \int_{\partial\Omega\cap U_j} \int_{\partial\Omega\cap U_j} \frac{|(\eta_j v)(x) - (\eta_j v)(y)|^p}{|x-y|^{(d-1)+p\{s\}}} \, \mathrm{d}\sigma(x)\mathrm{d}\sigma(y) \right)^{1/p},$$

as well as

$$\|v\|_{W^{p,s}(\partial\Omega)} \coloneqq \left(\sum_{|\alpha|\leqslant[s]} \|\partial^\alpha v\|^p_{L^p(\partial\Omega)} + \sum_{|\alpha|=\{s\}} |\partial^\alpha v|^p_{\{s\},p,\partial\Omega} \right)^{1/p}. \quad \rtimes$$

Remark A.20 The values $|v|_{\{s\},p,\partial\Omega}$ in the definition above actually depend on the choice of the localization, but all the corresponding norms $\|v\|_{W^{p,s}(\partial\Omega)}$ are equivalent. $\rtimes$

We conclude with the Sobolev embedding theorems, see e.g. [10], Theorem 4.57.

Theorem A.21 *Let $\Omega \subseteq \mathbb{R}^d$ either be the full space or a bounded Lipschitz-domain. Then the following assertions hold.*

- *If $sp < d$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q \leqslant dp/(d-sp)$.*
- *If $sp = d$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $q < \infty$.*
- *If $sp > d$, then we have:*
  - *(i) If $s - d/p \notin \mathbb{N}$, then $W^{s,p}(\Omega) \hookrightarrow C_b^{[s-d/p],s-d/p-[s-d/p]}(\Omega)$.*
  - *(ii) If $s - d/p \in \mathbb{N}$, then $W^{s,p}(\Omega) \hookrightarrow C_b^{s-d/p-1,\lambda}(\Omega)$ for every $0 \leqslant \lambda < 1$.* $\rtimes$

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