OPTIMAL CONTROL OF STOCHASTIC NETWORKS - AN APPROACH VIA FLUID MODELS

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Abstract

We consider a general control problem for networks which includes the special cases of scheduling in multiclass queueing networks and routing problems. The fluid approximation of the network is used to derive new results about the optimal control for the stochastic network. The main emphasis lies on the average cost criterion, however the β-discounted as well as the finite cost problem are also investigated. One of our main results states that the fluid problem provides a lower bound to the stochastic network problem. For scheduling problems in multiclass queueing networks we show the existence of an average cost optimal decision rule, if the usual traffic conditions are satisfied. Moreover, we give under the same condition a simple stabilizing scheduling policy. Another important issue that we address is the construction of simple asymptotically optimal decision rules. Asymptotic optimality is here seen w.r.t. fluid scaling. We show that every minimizer of the optimality equation is asymptotically optimal. And what is more important for practical purposes, we outline a general way to identify fluid optimal feedback rules as asymptotically optimal ones. Last but not least for routing problems an asymptotically optimal decision rule is given explicitly, namely a so-called least-loaded-routing rule.

Keywords

Stochastic network, average cost optimality equation, asymptotic optimality, deterministic control problem, stability

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1 Introduction

Optimal control of stochastic networks has been studied extensively over the last decade. The usual approach to solve these problems is to use the theory of *Markov decision processes.* The procedure is here as follows: we first have to check the validity of the optimality equation which reads in the average cost case

$$G = \min_{u \in D(x)} \left[c(x) + \sum_{x'} q(x,u,x')h(x') \right],$$

where $c(x)$ are the holding cost and $q(x,u,x')$ are the controlled intensities of the network process. Next we can try to find a solution (G,h) of the optimality equation. The minimizer of the right-hand side then gives an average cost optimal decision rule. However, there are only a few problems which can be solved explicitly (cf. Sennott (1998), Kitaev/Rykov (1995), Stidham/Weber (1993)). Another possibility is to solve the optimality equation numerically by policy iteration. The problem here is the large - often unbounded state space. This has led in recent years to study approximations of these control problems. One possibility is to look at the *Brownian approximation* of the network. See e.g. the survey papers by Harrison (1996) and Williams (1998). The control problem of the approximation is sometimes easier to solve and gives a policy which is asymptotically optimal when the work load of the system reaches its capacity limit. Another approximation which has been studied more intensively in recent years and will be used in this paper is the *fluid approximation.* The fluid approximation is a very simple deterministic first-order approximation of the stochastic network. It has initially been used to investigate questions of stability of networks. However, it turned out that the optimal control of the fluid approximation and the optimal policy of the stochastic network are very similar (cf. Bäuerle/Rieder (2000), Atkins/Chen (1995)). Therefore, recent investigations focused on finding connections between them. This is an important question because the optimal control in the fluid problem can be obtained rather easily (cf. Weiss (1996), Avram et al. (1995)). At least there exist efficient algorithms for the solution of the fluid problem, since it is of a special type, namely a so-called *separated continous linear program* (cf. Luo/Bertsimas (1998), Pullan (1995)). There are already a number of interesting results in the literature dealing with the connection of the fluid problem and the stochastic network problem. In particular it has been shown by Meyn (1997a) that a sequence of decision rules for the network problem which is generated by policy iteration converges against an asymptotically optimal decision rule, provided the initial decision rule is stable. Asymptotic optimality is here seen w.r.t. fluid scaling. Unfortunately a naive one-to-one translation of the fluid optimal control is in general not asymptotically optimal. However, it has been shown by Maglaras (1998) and Bäuerle (2000a) that asymptotically optimal policies can always be constructed. The construction in Maglaras (1998) is such that the state of the network is reviewed at discrete time points and the actions which have to be carried out over the next planning period are computed from a linear program. The procedure needs safty stock requirements to ensure that the plans can be processed properly. The proposal of Bäuerle (2000a) relies on the fact that the optimal control of the fluid problem is piecewise constant. A simple modification of the fluid optimal control on these pieces gives then an asymptotically optimal control. The disadvantage of both approaches is that the proposed policies are instationary.

In the present paper we investigate a rather general control problem which contains scheduling problems in multiclass queueing networks and routing problems as special cases. We are mainly interested in the average cost criterion, however we will also deal with the β-discounted cost criterion and with finite cost. The main results of this paper are the following.

(i) We show that the value function of the fluid problem always provides a lower bound to the value function of the stochastic network problem in the case of β-discounted and finite cost.

(ii) For scheduling problems in multiclass queueing networks we show the existence of an average cost optimal decision rule if the usual traffic conditions are satisfied.

(iii) For scheduling problems in multiclass queueing networks we give a simple stabilizing scheduling rule under the assumption that the usual traffic conditions are satisfied.

(iv) We outline a general way to identify fluid optimal feedback rules as asymptotically optimal ones and we show that every minimizer of the average cost optimality equation is asymptotically optimal.

(v) For routing problems an asymptotically optimal decision rule is given explicitly, namely a so-called least-loaded-routing rule.

The fluid problem as a lower bound is an interesting result, since such a behavior has been conjectured due to the results in various special cases (see e.g. Ott/Shanthikumar (1996), Bäuerle (1999), Altman et al. (1999)). Also the stabilizing scheduling rule for multiclass queueing networks is of importance since it is much simpler than other policies which have been suggested so far (e.g. Maglaras (1999), Dai (1998) section 2.9 gives a survey).

The paper is organized as follows. In section 2 we present the general mathematical control problem and define the special cases of scheduling problems in multiclass queueing networks and routing problems. The next section contains some results about the deterministic fluid problem itself and in section 4 we prove the statement about the fluid problem as a lower bound for the network problem. The scheduling problem in multiclass queueing networks is investigated in section 5. Under the usual traffic conditions it will be shown that an average cost optimal decision rule exists and a simple stabilizing decision rule is given. The remaining two sections deal with asymptotic optimality. In particular an asymptotically optimal decision rule for the routing problem will be given.

2 The stochastic network problem

Typical control problems which appear for stochastic networks are scheduling and routing problems. In this section we first present a rather general control problem. At the end of this section we show that scheduling problems in open multiclass queueing networks and routing problems are covered by this formulation.

The state process of the network is supposed to be a continuous-time Markov chain $X(t) = (X_1(t), \dots, X_K(t))$ in N_0^K, where the k-th component of $X(t)$ gives the number of jobs of class k at time t. We assume that interarrival and service times are independent and exponentially distributed. The model allows to control the transition rates of the process continuously over time. From the theory of Markov decision processes we know that we can restrict to controls which change at jump time points of the state process only. This leads to the following Markov decision process: there are K queues, hence the *state space* is $S = \mathbb{N}_0^K$. The *action space* $U \subset \mathbb{R}^L$ has to be compact and convex. The generator $Q = (q(x,u,x'))$ of $\{X(t),\ t \geq 0\}$ should satisfy the following conditions for all $x, x' \in S$:

(i) $D(x) := \{u \in U \mid q(x,u,x') = 0, \text{ if } x' \notin S\} \neq \emptyset$.

(ii) there exists a linear function $b : U \to \mathbb{R}^K$ such that for all $u \in D(x)$

$$\sum_{x' \in S} (x' - x) q(x,u,x') = b(u).$$

(iii) there exists a $q \in I\!R_+$ with $\sup_{u \in U} \sup_{x,x' \in S} |q(x,u,x')| < q$.

The set $D(x)$ is the *set of admissible actions* in state x. The function $b(\cdot)$ gives the expected drift of the network. $b(\cdot)$ is typically linear for these network problems. The *cost rate function* $c : S \to I\!R_+$ is assumed to be linear, i.e. $c(x) = c^T x$ with $c \in I\!R_+^K$. Denote by (T_n), $T_0 := 0$ the sequence of jump times of the Markov chain $\{X(t),\ t \geq 0\}$. A (stationary) policy for the Markov decision process is given by a decision rule $f : S \to U$ with $f(x) \in D(x)$, where f is applied at time T_n. For a fixed decision rule f and initial state $x \in S$, there exists a family of probability measures P_x^f on a measurable space $(\Omega, \mathcal{F})$ and stochastic processes $\{X(t),\ t \geq 0\}$ and $\{\pi(t),\ t \geq 0\}$ such that for $0 =: T_0 < T_1 < T_2 < \dots$

$$X(t) = X(T_n),\ T_n \leq t < T_{n+1}$$

$$\pi(t) = f(X(T_n)),\ T_n \leq t < T_{n+1}$$

and with $q(x,u) := \sum_{x' \neq x} q(x,u,x')$

(i) $P_x^f(X(0) = x) = P_x^f(T_0 = 0) = 1$ for all $x \in S$.

(ii) $P_{x_0}^f(T_{n+1} - T_n > t \mid T_0, X(T_0), \dots, T_n, X(T_n) = x) = e^{-q(x,f(x))t}$ for all $x \in S, t \geq 0$.

(iii) $P_{x_0}^f(X(T_{n+1}) = x' \mid T_0, X(T_0), \dots, T_n, X(T_n) = x, T_{n+1}) = \frac{q(x,f(x),x')}{q(x,f(x))}$ for $x, x' \in S, x \neq x'$ and zero, if $x = x'$.

As far as the optimization criterion is concerned, we are mainly interested in the average cost criterion, however we will also deal with the β-discounted cost criterion and with finite cost. For a fixed initial state $x \in S$ and a given decision rule f we define for an interest rate $\beta > 0$ and time horizon $T > 0$

$$G_f(x) = \limsup_{T \to \infty} \frac{1}{T} E_x^f \left[\int_0^T c(X(t))\, dt \right]$$

$$V_{\beta,f}(x) = E_x^f \left[\int_0^\infty e^{-\beta t} c(X(t))\, dt \right]$$

$$V_{T,f}(x) = E_x^f \left[\int_0^T c(X(t))\, dt \right]$$

The *optimization problems* are then given by

$$G(x) = \inf_f G_f(x)$$

$$V_\beta(x) = \inf_f V_{\beta,f}(x)$$

$$V_T(x) = \inf_f V_{T,f}(x)$$

respectively. Throughout we assume that $D(x)$ is compact for all $x \in S$ and the mapping $u \to q(x,u,x')$ is continuous for all $x, x' \in S$. Under these assumptions, there exists an optimal decision rule for the β-discounted problem. Moreover, this decision rule is optimal among all non-anticipating policies.

The average cost optimality equation for the network problem is of the form

$$G = \min_{u \in D(x)} \left[c(x) + \sum_{x'} q(x,u,x')h(x') \right].$$

With its help it is possible to solve the average cost problem. Let us define for $x \in S$, $\beta > 0$ and a fixed state $\xi \in S$

$$h_\beta(x) = V_\beta(x) - V_\beta(\xi).$$

h_β is called *relative value function.* Under the following conditions which are due to Sennott (1989), there exists a non-negative solution (G, h) of the average cost optimality equation and every minimizer of the right-hand-side gives an average optimal decision rule:

(i) There exists a decision rule f such that $G_f(x) < \infty$ for all $x \in S$.

(ii) There exist constants $L \in I\!R$, $\bar{\beta} > 0$ and a function $M : S \to I\!R_+$ with

$$L \le h_\beta(x) \le M(x)$$

for all $x \in S$ and $0 < \beta \le \bar{\beta}$.

However, there are only very few examples, where the average cost optimality equation can be solved or the structure of the minimizer can be determined. Also, a numerical iteration is often intractable. In section 7 we will show how to construct good decision rules in these cases which are asymptotically optimal with respect to fluid scaling.

Scheduling problems in multiclass queueing networks and routing problems are special cases of our general formulation which is outlined below.

A. Scheduling problems

A description of the model is as follows: there are J service stations in the network. Each station has one server. There are $K \ge J$ job classes - at least one at each station - with infinite waiting room. We denote by $C(j)$ the set of job classes which are processed at station j. The matrix $C = (c_{kj})$, with

$$c_{kj} = \begin{cases} 1, & \text{if } k \in C(j) \\ 0, & \text{else} \end{cases}$$

is the so-called *constituency matrix.* The external arrival processes are independent Poisson processes with intensities $\alpha_1, \ldots, \alpha_K$ respectively. We denote $\alpha = (\alpha_1, \ldots, \alpha_K)$. The service times of class k jobs are independent and identically distributed according to an exponential distribution with parameter $\mu_k > 0$. The decision is now how to assign the servers to the job classes in order to minimize the cost. We allow that the service capacity can be splitted. Hence the action space is given by $U = \{u \in [0,1]^K \mid \sum_{k \in C(j)} u_k \leq 1,\ j = 1, \ldots, J\}$, where for $u = (u_1, \ldots, u_K) \in U$, u_k gives the fraction of the responsible server which is assigned to class k. The set of admissible actions in state x is given by $D(x) := \{u \in U \mid x_k = 0 \Rightarrow u_k = 0\}$. Once a job of class k has been processed, it does not necessarily leave the system. With probability p_{ki} the job is routed to class i and has to be processed further. Throughout this paper we suppose that the routing matrix $P = (p_{ki})$ is transient, i.e. $P^n \to 0$ for $n \to \infty$. This implies in particular that $(I - P)^{-1} = \sum_{n=0}^{\infty} P^n \geq 0$, where I is the identity matrix. Let us denote by $D := diag(\mu_k)$ the diagonal matrix with the potential service rates μ_k on the diagonal and define $A := D(I - P)$. The state process is obviously a continuous-time Markov chain and the intensities are given for $x' \neq x$ by

$$q(x, u, x') = \begin{cases} \alpha_k, & x' = x + e_k \\ \mu_k u_k p_{ki}, & x' = x - e_k + e_i \\ \mu_k u_k (1 - \sum_{i=1}^{K} p_{ki}), & x' = x - e_k \end{cases}$$

where $e_1, \ldots, e_K$ are the unit vectors in $\mathbb{R}^K$. Hence the expected drift of the network for action $u \in U$ is $b(u) = \alpha - uA$.

B. Routing problems

The classical routing problem is as follows: there is an external stream of jobs arriving with intensity α at a controller who has to decide to which of K queues he/she routes the next job. Each queue has a server and the service times of queue k are independent and identically distributed according to an exponential distribution with parameter $\mu_k > 0$. The action space is given by $U = \{(u, v) \in [0,1]^{2K} \mid \sum_{k=1}^{K} u_k = 1\}$ where u_k gives the fraction of jobs routed to queue k and v_k gives the activation of the server at queue k. The set of admissible actions in state x is given by $D(x) = \{(u,v) \in U \mid x_k = 0 \ \Rightarrow v_k = 0\}$. The state process is a continuous-time Markov chain and the intensities are given for $x' \neq x$ by

$$q(x,u,x') = \begin{cases} \alpha u_k, & x' = x + e_k \\ \mu_k v_k, & x' = x - e_k \end{cases}$$

The expected drift, given action $(u,v) \in U$ is of the form $b(u) = \alpha u - Dv$.

3 The fluid problem

In this section we introduce the associated fluid problem to the stochastic network problem of the previous section and investigate its properties. Recall that $b(\cdot)$ is the expected drift of the stochastic network. A solution $\{(x(t), u(t)),\ t \geq 0\}$ of the conditions

$$\begin{aligned} &\text{(i)}\ \ x(t) = x_0 + \textstyle\int_0^t b(u(s))\, ds \\ &\text{(ii)}\ \ x(t) \geq 0 \\ &\text{(iii)}\ \ u(t) \in U \end{aligned} \tag{1}$$

is called a *fluid model solution*. The fluid problem itself is defined as

$$(F) \begin{cases} \int_0^\infty c(x(t))\, dt \to \min \\ x(t) = x_0 + \int_0^t b(u(s))\, ds \\ x(t) \geq 0 \\ u(t) \in U \end{cases}$$

We denote by $V^F(x_0)$ the value function of (F). Analogously we write $V_\beta^F(x_0)$ and $V_T^F(x_0)$ if the objectives are $\int_0^\infty e^{-\beta t} c(x(t))\, dt \to \min$ and $\int_0^T c(x(t))\, dt \to \min$ respectively. First it is important to note that a non-randomized optimal control $u^*(t)$ with associated optimal trajectory $x^*(t)$ always exists. This is due to the linearity of the control problem and follows from well-known existence theorems like e.g. given in section 2.8 of Seierstad/Sydsæter (1987). From Lemma 5 in Bäuerle (2000b)we obtain

Lemma 1:

The value functions $V^F(x), V^F_\beta(x)$ and $V^F_T(x)$ are convex in x.

Classical control theory provides sufficient conditions for the optimal solution of these kind of problems. With the Hamiltonian $H: I\!R^K \times U \times I\!R^K \to I\!R$ defined as $H(x,u,p) = c(x) + p \cdot b(u)$ we obtain (cf. Seierstad/Sydsæter (1987) Theorem 1, Chapter 5)

Lemma 2:

The admissible control $u^*(t)$ with the associated trajectory $x^*(t)$ is optimal for the finite horizon problem if there exists a continuous and piecewise continuously differentiable vector function $p(t) = (p_1(t), \ldots, p_K(t))$ as well as a piecewise continuous vector function $\eta(t) = (\eta_1(t), \ldots, \eta_K(t))$ such that for all $t \in [0,T]$

(i) $u^*(t)$ minimizes $u(t) \mapsto H(x^*(t), u(t), p(t)),\ u(t) \in U$.

(ii) $\dot{p}(t) = -c + \eta(t)$ except at points of discontinuity of $u^*(t)$.

(iii) $\eta(t) \geq 0,\ \eta(t)x^*(t) = 0$.

(iv) $p(T) = 0$.

The routing problem will be solved explicitly in section 7. Therefore, the remainder of this section is restricted to scheduling problems, i.e. we have $b(u) = \lambda - uA^T$. A non-trivial result is the existence of an optimal feedback control. A feedback control is a function $\varphi: I\!R^K_+ \to U$ such that the equation

$$\dot{x}(t) = b(\varphi(x(t))),\ t \geq 0 \quad x(0) = x_0$$

has a unique solution $\{x(t),\ t \geq 0\}$ which is admissible, i.e. $x(t) \geq 0$.

Theorem 3:

Suppose the scheduling problem of section 2 is given and the solution of (F) is unique. Then there exists an optimal feedback control φ^*, which satisfies $\varphi^*(\gamma x) = \varphi^*(x)$ for all $x \in I\!R^K_+$ and $\gamma > 0$.

Proof: The existence of an optimal feedback control follows as in Sethi/Zhang (1994) Lemma 4.1.(ii) since due to our assumptions A^{-1} exists. Now let $x \in I\!R^K_+, x \neq 0$. The assertion $f^*(\gamma x) = f^*(x)$ follows from the scaling property of the fluid model (cf. Chen (1995) p.642).

Suppose that $\{(u(t), x(t)),\ t \geq 0\}$ is an arbitrary pair of admissible control and associated trajectory. Define $\hat{u}(t) = u(\frac{t}{\gamma})$ as control for start in $\gamma x,\ \gamma > 0$. Then

$$\hat{x}(t) = \gamma x + \int_0^t b(\hat{u}(s))\, ds = \gamma x + \gamma \int_0^{\frac{t}{\gamma}} b(u(s))\, ds = \gamma x(\frac{t}{\gamma})$$

is the associated trajectory. Hence $\{(\hat{u}(t), \hat{x}(t)),\ t \geq 0\}$ is an admissible pair for start in γx. Vice versa if $\{(\hat{u}(t), \hat{x}(t)),\ t \geq 0\}$ is an admissible pair for start in γx, $\{(u(t) = \hat{u}(\gamma t), x(t) = \frac{1}{\gamma}\hat{x}(\gamma t)),\ t \geq 0\}$ is an admissible pair for start in x and we obtain

$$\int_0^\infty c(\hat{x}(t))\, dt = \gamma \int_0^\infty c(x(\frac{t}{\gamma}))\, dt = \gamma^2 \int_0^\infty c(x(t))\, dt.$$

Thus in particular $V^F(x) = \frac{1}{\gamma^2} V^F(\gamma x)$ and $\varphi^*(\gamma x) = \varphi^*(x)$. □

4 A lower bound

In this section we will establish a first relation between the fluid problem and the stochastic network problem. Namely, the value function of the fluid problem provides a lower bound for the value function of the stochastic network. This phenomenon has already been encountered in various special cases (see e.g. Ott/Shanthikumar (1996), Bäuerle (1999), Altman et al. (1999)).

Theorem 4:

For all initial states $x \in S$ and $T \in I\!R_+ + \{\infty\}$ it holds that

$$V_T^F(x) \leq V_T(x), \quad V_\beta^F(x) \leq V_\beta(x)$$

Proof: Suppose that $x \in S$ is the initial state. Let f be an arbitrary decision rule for the stochastic network. The induced state process $\{X(t),\ t \geq 0\}$ is a Markov process with generator $Q = (q(x, f(x), x'))$. Hence it holds that

$$X(t) - x - \int_0^t b(f(X(s)))\, ds = M(t)$$

where $\{M(t),\ t \geq 0\}$ is a Martingale and $M(0) = 0$. Taking expectation on both sides and exploiting the fact that $b(\cdot)$ is linear we obtain for $t \geq 0$

$$E[X(t)] = x + \int_0^t b(E[f(X(s))])\, ds \tag{2}$$

which is a variation of the so-called Dynkin formula. Moreover, since f is a decision rule we get for all $t \geq 0$ a.s.

$$X(t) \geq 0 \quad \text{and} \quad f(X(t)) \in U.$$

When we define $x(t) := E[X(t)]$ and $u(t) := E[f(X(t))]$ we get $x(t) \geq 0$ and $u(t) \in U$ for all $t \geq 0$ since U is convex. And from (2) we have

$$x(t) = x + \int_0^t b(u(s))\, ds.$$

Thus the pair $\{(x(t), u(t)),\ t \geq 0\}$ is admissible for the fluid problem for every decision rule f. Moreover,

$$V_T(x) = E_x^{f^*}[\int_0^T cX(t)\, dt] = \int_0^T cx(t)\, dt$$
$$V_\beta(x) = E_x^{f^\beta}[\int_0^\infty e^{-\beta t} cX(t)\, dt] = \int_0^\infty e^{-\beta t} cx(t)\, dt$$

where f^*, f^β are the corresponding optimal decision rules. Thus, the statement follows. $\square$

Theorem 4 strengthens results in Bäuerle (2000a) where the lower bound was only stated in an asymptotic way. Since the value function of the fluid problem can be computed rather easily we can evaluate the performance of benchmark policies in the stochastic network by comparison with the lower bound.

5 Existence of average cost optimal policies and simple stabilizing policies

In this section we will show that under the usual traffic conditions an average cost optimal decision rule (with finite cost) exists in the scheduling problem for multiclass queueing networks. Moreover, we give a simple decision rule under which the network process is then positive recurrent. The question of constructing so-called stabilizing policies has been treated in several papers (cf. Dai (1998) for a survey). However the decision rule we suggest is much simpler.

First it is convenient to denote by $\lambda = (\lambda_1, \ldots, \lambda_K)$ the nominal total arrival rate to the different classes. λ is the solution of the traffic equation $\lambda_i = \alpha_i + \sum_{k=1}^K \lambda_k p_{ki},\ i = 1, \ldots, K$ or in matrix notation $\lambda = \alpha + \lambda P$. Since P is transient we obtain

$$\lambda = \alpha(I - P)^{-1}.$$

The traffic intensity ρ_j at station j is then given by

$$\rho_j = \sum_{k \in C(j)} \frac{\lambda_k}{\mu_k}$$

and the usual traffic condition is $\rho_j < 1,\ j = 1, \dots, J$. When we denote $\rho = (\rho_1, \dots, \rho_J)$ we obtain $\rho = \lambda D^{-1} C = \alpha (I - P)^{-1} D^{-1} C = \alpha A^{-1} C$. In particular, if $\rho_j < 1,\ j = 1, \dots, J$, there exists an $\varepsilon > 0$ such that $(\varepsilon 1\!\!1 + \alpha A^{-1}) C \le 1\!\!1$ still holds. $1\!\!1$ is a vector containing 1's only, the dimension should be clear from the context. For $x, y \in I\!R^K$ we denote by $x \wedge y := (\min\{x_1, y_1\}, \dots, \min\{x_K, y_K\})$ the componentwise minimum of x and y.

Theorem 5:

Suppose that the multiclass queueing network of section 2 is given and the usual traffic conditions are satisfied, i.e. $\rho_j < 1,\ j = 1, \dots, J$. The fixed server assignment

$$f^s(x) = (\varepsilon 1\!\!1 + \alpha A^{-1}) \wedge x, \quad x \in S$$

with $\varepsilon > 0$ such that $(\varepsilon 1\!\!1 + \alpha A^{-1}) C \le 1\!\!1$ stabilizes the network. I.e. the network process which is induced by f^s is positive recurrent.

Proof: Note that $f^s(x) \in D(x)$ for all $x \in S$, due to our assumption on ρ_j and ε. If we use the fixed server assignment f^s, our network behaves like a Jackson network with service rates $\hat{\mu}_k := \mu_k (\varepsilon 1\!\!1 + \alpha A^{-1})_k,\ k = 1, \dots, K$. It is well-known (cf. Asmussen (1987) Theorem 5.2) that the Jackson network is stable, if $\lambda_k < \hat{\mu}_k,\ k = 1, \dots, K$, where λ_k is as before. Since we have $\hat{\mu} = (\varepsilon 1\!\!1 + \alpha A^{-1}) D = \varepsilon \mu + \alpha (I - P)^{-1} = \varepsilon \mu + \lambda$ we obtain stability. $\square$

Corollary 6:

Suppose that the multiclass queueing network of section 2 is given and the usual traffic conditions are satisfied, i.e. $\rho_j < 1,\ j = 1, \dots, J$.

a) Then there exists a non-negative solution (G, h) of the average cost optimality equation

$$G = \min_{u \in D(x)} \{ c(x) + \sum_{x'} q(x, u, x') h(x') \}. \tag{3}$$

b) Every minimizer of (3) is an average cost optimal decision rule with finite average cost G.

Proof: From Theorem 5 and Theorem 3.2. in Meyn (1997b) it follows that under decision rule f^s we have $E_x^{f^s}[\int_0^\tau c(X(t))\,dt] < \infty$, where $\tau = \inf\{t \geq 0 \mid X(t) = 0\}$. If we define the policy

$$\hat{\pi}_\beta(t) = \begin{cases} f^s(X(t)), & \text{if } t < \tau \\ \pi_\beta(t-\tau), & \text{if } t \geq \tau \end{cases}$$

where π_β is the optimal control in the β-discounted case, then

$$V_\beta(x) \leq V_{\beta,\hat{\pi}_\beta}(x) \leq E_x^{f^s}[\int_0^\tau c(X(t))\,dt] + V_\beta(0).$$

This implies that the relative value function with $\xi = 0$ satisfies $h_\beta(x) = V_\beta(x) - V_\beta(0) \leq M(x) := E_x^{f^s}[\int_0^\tau c(X(t))dt]$ for all $\beta > 0$. Moreover, it can be shown by induction that $V_\beta(x)$ is increasing, i.e. if $x' \geq x$ then $V_\beta(x') \geq V_\beta(x)$ which implies in particular that $h_\beta(x) \geq 0$ for all $\beta > 0$. Thus, the conditions of Sennott (1989) are fulfilled which imply the statements (cf. section 2). $\square$

6 Asymptotic optimality

In this section we will introduce the concept of asymptotic optimality. To define this notion we have to reveal a second relation between the stochastic network and the fluid model. Namely, the stochastic network converges under fluid scaling to the fluid model. Let us first explain how fluid scaling works. By $\gamma > 0$ we denote the fluid scaling parameter which will always tend to infinity. Suppose that a fixed decision rule f and an initial state x is given. By $\{\hat{X}^\gamma(t),\ t \geq 0\}$ we denote the state process with initial state $\gamma \cdot x$ under decision rule f and $\{\hat{\pi}^\gamma(t),\ t \geq 0\}$ is the corresponding action process. The scaled processes $\{X^\gamma(t),\ t \geq 0\}$ and $\{\pi^\gamma(t),\ t \geq 0\}$ are then defined by

$$X^\gamma(t) := \frac{1}{\gamma}\hat{X}^\gamma(\gamma t)$$

$$\pi^\gamma(t) := \hat{\pi}^\gamma(\gamma t).$$

We understand the processes $\{X^\gamma(t),\ t \geq 0\}$ as random elements with values in $D^N[0,\infty)$, which is the space of $\mathbb{R}^K$-valued functions on $[0,\infty)$ that are right continuous and have left-hand limits and all endowed with the Skorokhod topology. By $\Rightarrow$ we denote the weak convergence of the processes as $\gamma \to \infty$. It can be shown (e.g. Dai (1995), Bäuerle (2000) Theorem 3) that every sequence $\{(X^\gamma(t), \pi^\gamma(t)),\ t \geq 0\}$ has a further subsequence $\{(X^{\gamma_n}(t), \pi^{\gamma_n}(t)),\ t \geq 0\}$ such that $\{(X^{\gamma_n}(t), \int_0^t \pi^{\gamma_n}(s)ds),\ t \geq 0\} \Rightarrow \{(X(t), \int_0^t \pi(s)ds),\ t \geq 0\}$ and the limit satisfies a.s.

(i) $X(t) = x + \int_0^t b(\pi(s))\, ds$.

(ii) $X(t) \in I\!R_+^K$.

(iii) $\pi(t) \in U$.

The corresponding value functions are given by

$$V_{T,f}^\gamma(x) := E_x^f \left[\int_0^T c(X^\gamma(t))\, dt \right] \quad V_{\beta,f}^\gamma(x) = E_x^f \left[\int_0^\infty e^{-\beta t} c(X^\gamma(t))\, dt \right].$$

Since the expected drift of the scaled process is again $b(\cdot)$ independent of γ, we obtain with Theorem 4 that $V_T^\gamma(x) \geq V_T^F(x)$ and $V_\beta^\gamma(x) \geq V_\beta^F(x)$ for all $\gamma \geq 0$, $x \in I\!R_+^K$. Adapting the definition in Meyn (1997a) (cf. also Meyn (2000)) we say

Definition:

A decision rule f^* is called *asymptotically optimal*, if

$$\liminf_{T\to\infty} \liminf_{\gamma\to\infty} E_x^{f^*} \left[\int_0^T c(X^\gamma(t))\, dt \right] = V^F(x) < \infty$$

In Bäuerle (2000) and Maglaras (1998) it has been shown that a (non-stationary) asymptotically optimal policy can always be constructed if $V^F(x) < \infty$. We will now show that the set of asymptotically optimal policies contain the average cost optimal policies which are solutions of the average cost optimality equation. Meyn (2000) has shown implicitly in Theorem 3.1 that every decision rule, obtained as limit point of the policy iteration is asymptotically optimal provided the initial decision rule is stable.

Theorem 7:

Suppose there exists a finite, non-negative solution (G, h) of the average cost optimality equation (3). Every average cost optimal policy which is obtained as a minimizer of (3) is asymptotically optimal.

Proof: Let f be a minimizer of (3), i.e. for all $x \in S$, it holds that $G = c(x)+\sum_{x'} q(x, f(x), x')h(x')$. Thus we obtain

$$\int_0^T E_x^f[c(X(s))]\, ds = GT - \int_0^T E_x^f\left[\sum_{x'} q(X(s), f(X(s)), x')h(x')\right]\, ds.$$

According to the Dynkin formula we have

$$\int_0^T E_x^f\left[\sum_{x'} q(X(s), f(X(s)), x')h(x')\right]\, ds = E_x^f[h(X(T))] - h(x).$$

Thus, since $h \geq 0$

$$\int_0^T E_x^f[c(X(s))]\, ds = GT - E_x^f[h(X(T))] + h(x) \leq GT + h(x).$$

Theorem 3.2 in Meyn (1997b) now implies that f is L_2-stable. In particular $V^F(x) < \infty$. Since f cannot be improved by policy iteration we obtain with Theorem 5.2.(ii) in Meyn (1997b) that f is asymptotically optimal. □

7 Asymptotic optimality of special decision rules

In this section we are interested in constructing asymptotically optimal decision rules. The main problem is that a naive one-to-one translation of the fluid optimal feedback control is in general not asymptotically optimal (cf. Meyn (1997b), Maglaras (1998)). Thus, it would be desirable to identify situations where a one-to-one translation is asymptotically optimal. The main idea of this approach is to find further conditions under which the fluid problem has only one feasible pair of state-action trajectory, namely the optimal one. Here the following Lemma 2.4. of Dai/Williams (1995) is helpful. Suppose that $m : \mathbb{R}_+^K \to \mathbb{R}$ is continuous and bounded, then it holds that

$$\int_0^t m(X^\gamma(s))d\left(\int_0^s \pi^\gamma(\tau)\, d\tau\right) \Rightarrow \int_0^t m(X(s))d\left(\int_0^s \pi(\tau)\, d\tau\right) \tag{4}$$

for $\gamma \to \infty$, where $\{(X(t), \pi(t)),\ t \geq 0\}$ is the fluid limit. In the sequel we will treat the scheduling and routing problems of section 2 separately. Other special cases where asymptotically optimal decision rules have been computed can be found e.g. in Gajrat/Hordijk

(1999,2000), Alanyali/Hajek (1998).

A. Scheduling Problems

In many fluid problems it turns out that so-called static buffer priority (SBP)-feedback rules are optimal for the scheduling problem (cf. Bäuerle/Rieder (2000), Gajrat/Hordijk (2000)). Under a SBP rule, classes are ordered in each station according to their priorities. Fluid of higher priority is then served first. In the stochastic network problem jobs with higher priority are served first. Suppose that classes with smaller number have higher priority within each station. When we assume that the numbers of classes in station j are $\{k_{j-1}+1,\ldots,k_j = k_{j-1}+|C(j)|\}$ with $k_0 = 0$, then the following equation is a.s. satisfied in the stochastic network for all $\gamma > 0$ under ths SBP decision rule

$$\int_0^\infty \left(X^\gamma_{k_{j-1}+1}(t)+\ldots+X^\gamma_{k_{j-1}+\nu}(t)\right)\wedge 1 d\left(t-\int_0^t \pi^\gamma_{k_{j-1}+1}(s)\,ds-\ldots-\int_0^t \pi^\gamma_{k_{j-1}+\nu}(s)\,ds\right)=0,$$

for $\nu = 1,\ldots,|C(j)|,\ j = 1,\ldots,J$. Since $x \wedge 1$ is bounded and continuous, this condition holds in the limit, i.e. we have that every fluid model solution satisfies

$$\int_0^\infty \left(X_{k_{j-1}+1}(t)+\ldots+X_{k_{j-1}+\nu}\right)\wedge 1\, d\left(t-\int_0^t \pi_{k_{j-1}+1}(s)\,ds-\ldots-\int_0^t \pi_{k_{j-1}+\nu}(s)\,ds\right)=0, \tag{5}$$

for $\nu = 1,\ldots,|C(j)|,\ j = 1,\ldots,J$ and we obtain

Theorem 8:

Suppose that a SBP decision rule is optimal for (F) and the solution of (1) together with (5) is unique. Then the SBP decision rule is asymptotically optimal.

Proof: The proof follows from the convergence results and the fact that the fluid limit is unique in this case.

Remark:

It is important to ensure the uniqueness of the solution of (1) and (5) since this is in general not the case. Choosing suitably defined cost rates, the example in section 2.7 of Dai (1998) provides a situation where the optimal feedback control of (F) is a SBP rule though it is not asymptotically optimal.

B. Routing Problems

Let us now consider the routing problem which has been introduced in section 2. We assume that we have equal holding costs $c_1 = \ldots = c_K = c$ for the jobs. It is possible to show that a so-called least-loaded routing rule is asymptotically optimal. Alanyali/Hajek (1998) have shown this statement for a different class of routing problems. Let us first look at the associated fluid problem

$$(F_T)\quad \begin{cases} \int\limits_0^T \sum_{k=1}^K c x_k(t)\, dt \to \min \\ x(t) = x(0) + \int\limits_0^t \alpha u(s) - Dv(s) ds \\ x(t) \geq 0 \\ (u(t), v(t)) \in U,\ t \geq 0 \end{cases}$$

where $U = \{(u,v) \in [0,1]^{2K} \mid \sum_{k=1}^K u_k = 1\}$. We suppose that $\alpha < \sum_{k=1}^K \mu_k$ to guarantee stability. T should be big enough such that the system can be drained within time T. We show that the least-loaded routing (LLR) feedback control is optimal for (F_T). The control is define in the following way. For $x \in I\!R_+^N$ denote by

$$I(x) = \{1 \leq k \leq K \mid \frac{x_k}{\mu_k} = \min_{1 \leq i \leq K} \frac{x_i}{\mu_i}\}$$

the buffer numbers with least load, if x gives the current buffer contents. The LLR-feedback controls $u^*(x)$ and $v^*(x)$ are then defined by

$$u_k^*(x) = \begin{cases} \mu_k (\sum_{i \in I(x)} \mu_i)^{-1} & , \text{ if } k \in I(x) \\ 0 & , \text{ if } k \notin I(x). \end{cases}$$

$$v_k^*(x) = \begin{cases} 1 & , \text{ if } x_k > 0 \\ \min\{1, \frac{\alpha u_k^*(x)}{\mu_k}\} & , \text{ if } x_k = 0 \end{cases}$$

for $k = 1, \ldots, K$.

Theorem 9:

The LLR feedback control is optimal for the fluid problem.

Proof: First, the differential equation

$$\dot{x}(t) = \alpha u^*(x(t)) - Dv^*(x(t))$$

has a unique solution $\{x^*(t),\ t \geq 0\}$ which we will show in the proof of Theorem 10. Thus, $\{x^*(t),\ t \geq 0\}$ is the trajectory under the LLR feedback control and $\{(u^*(t), v^*(t)),\ t \geq 0\}$ the corresponding optimal control. We use Lemma 2 to prove the statement, where (i) reads

$$(u^*(t), v^*(t)) \text{ minimizes } (u(t), v(t)) \mapsto p(t)u(t) - p(t)D\, v(t) \text{ for } (u(t), v(t)) \in U.$$

Let $x(0) = x \in I\!R_+^K$ be the initial state and denote by $T_k = \inf\{t > 0 \mid x_k^*(t) = 0\}$ the time it takes to empty buffer k, $k = 1, \ldots, K$ under the LLR control. Note that under our stability assumption $T_k < T$ for all k. As adjoint functions we take for $t \geq 0$ and $k = 1, \ldots, K$

$$p_k(t) := \begin{cases} c(T_k - t), & \text{if } T_k \geq t \\ 0, & \text{if } T_k < t \end{cases}$$

and define the Lagrange multipliers for $t \geq 0$ and $k = 1, \ldots, K$ as

$$\eta_k(t) := c1_{[t > T_k]}.$$

Obviously (ii)-(iv) of Lemma 2 are satisfied. For (i) note that $T_k \geq T_i$ implies $p_k(t) \geq p_i(t)$, for all $t \geq 0$ and $p_k(t) > 0$ if $x_k(t) > 0$. Hence $(u^*(t), v^*(t))$ solves the minimization problem in (i) and the assertion follows. □

Theorem 10:
The LLR feedback control is asymptotically optimal for the stochastic network.

Proof: Define $m(x) := \min\{\frac{x_1}{\mu_1}, \ldots \frac{x_K}{\mu_K}\}$.For the stochastic network process under the LLR control, we obtain a.s.

$$\int_0^\infty \left(\frac{X_k^\gamma(t)}{\mu_k} - m(X^\gamma(t)) \right) \wedge 1 \, d\left(\int_0^t \pi_k^\gamma(s)\, ds \right) = 0 \tag{6}$$

for $k = 1, \ldots, K$ and $\gamma > 0$ since at time t either k is a buffer with least load, in which case $\frac{X_k^\gamma(t)}{\mu_k} = m(X^\gamma(t))$ or $\pi_k^\gamma(t) = 0$. This condition is of the form (4). Hence under fluid scaling it is preserved in the limit. Thus, the limit $\{(X(t), u(t)),\ t \geq 0\}$ satisfies (1) together with (6) and it is easy to see that the pair of state-action trajectory under the LLR control in the fluid model also satisfies

$$\int_0^\infty \left(\frac{x_k(t)}{\mu_k} - m(x(t)) \right) \wedge 1 \, d\left(\int_0^t u_k(s)\, ds \right) = 0$$

for $k = 1, \ldots, K$. The only thing that is now left to obtain asymptotic optimality is to show that the solution of (1) together with (6) is unique. To do this we show that whenever

$x(0) \geq x'(0)$, then $x(t) \geq x'(t)$ for all $t \geq 0$ by contradiction. Suppose $t_0 := \sup\{t \geq 0 \mid x(s) \geq x'(s),\ 0 \leq s \leq t\} < \infty$. Due to the continuity there exist an $\varepsilon > 0$ and a set $M \subset \{1, \dots, K\}$ with $x'_k(t) > x_k(t),\ \dot{x}'(t) \geq \dot{x}_k(t)$ in $[t_0, t_0+\varepsilon)$ for all $k \in M$. This can only happen if $u_k(t) < u'_k(t)$ for $t \in [t_0, t_0+\varepsilon)$. Thus, $x'_k(t)$ has least load for $k \in M,\ t \in [t_0, t_0+\varepsilon)$. Since $x_j(t) \geq x'_j(t)$ for $j \notin M$, $x_k(t)$ has also least load for $t \in [t_0, t_0+\varepsilon)$ and

$$|\{j \mid x_j(t) = \mu_j m(x(t))\}| \leq |\{j \mid x'_j(t) = \mu_j m(x'(t))\}|.$$

Therefore, $u_k(t) \geq u'_k(t)$ for $t \in [t_0, t_0+\varepsilon)$ which is a contradiction. $\square$

This is an interesting result, since even in the case of two queues the optimal routing rule for the stochastic problem is not known in general. A lot of papers are dealing with this topic, among them e.g. Hajek (1984), Hordijk/Koole (1992), Chen/Xu (1993). At least the structure of the optimal routing rule f^* is known: there exists a switching curve $s : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(x_1, x_2) = \begin{cases} (1,0), & \text{if } x_2 > s(x_1) \\ (0,1), & \text{if } x_2 \leq s(x_1) \end{cases}$$

where s is increasing and unbounded (cf. Hajek (1984), Chen/Xu (1993)). The LLR feedback rule u^* in the two buffer case can also be described by a switching curve $s^F : \mathbb{R}_+ \to \mathbb{R}$ with $s^F(x_1) = \frac{\mu_2}{\mu_1} x_1$ as follows:

$$u^*(x_1, x_2) = \begin{cases} (1,0), & \text{if } x_2 > \frac{\mu_2}{\mu_1} x_1 \\ (0,1), & \text{if } x_2 < \frac{\mu_2}{\mu_1} x_1 \\ (\frac{\mu_1}{\mu_1+\mu_2}, \frac{\mu_2}{\mu_1+\mu_2}), & \text{if } x_2 = \frac{\mu_2}{\mu_1} x_1 \end{cases}$$

In the sequel we have computed the optimal switching curve of the average cost stochastic network problem numerically and compared it to the linear switching curve of the fluid problem. For different kind of parameter settings we obtained very good results. W.l.o.g. we always set $\alpha = 1$, since the optimal decision rule does not change when the intensities α, μ_1, μ_2 are multiplied by constants. The holding cost have been chosen $c_1 = c_2 = 1$. From Theorem 10 it follows that the switching curve $s(x_1)$ is asymptotically equal to $\frac{\mu_2}{\mu_1} x_1$, i.e. $\lim_{x_1 \to \infty} \frac{s(x_1)}{x_1} = \frac{\mu_1}{\mu_2}$. The gray region in the figures is the part of the state space where it is optimal to route to queue 2. The dotted line is the linear switching curve of the fluid model. Figure 1 shows the results for $\alpha = 1,\ \mu_1 = 2,\ \mu_2 = 1$ and figure 2 for $\alpha = 1,\ \mu_1 = 4,\ \mu_2 = 1$.

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