INEQUALITIES FOR STOCHASTIC MODELS VIA SUPERMODULAR ORDERINGS

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ABSTRACT

The aim of this paper is to derive inequalities for random vectors by using the supermodular ordering. The properties of this ordering suggest to use it as a comparison for the "strength of dependence" in random vectors. In contrast to already established orderings of this type, the supermodular ordering has the advantage that it is not necessary to assume a common marginal distribution for the random vectors under comparison. As a consequence we obtain new inequalities by applying it to multivariate normal distributions, Markov chains and some stochastic models.

Key words: SUPERMODULAR ORDERING, POSITIVELY DEPENDENT RANDOM VECTORS, MAJORIZATION, MARKOV CHAINS, MULTIVARIATE NORMAL DISTRIBUTION, STOCHASTIC MODELS

1. INTRODUCTION

A useful method to derive inequalities for positively dependent random variables is to "measure" the strength of dependence. In their paper of 1989, Shaked and Tong [11] started to investigate some partial orderings of random vectors which yield a comparison of dependence. Most of the existing literature deals with the comparison of positive dependence of exchangeable random variables (for references see Shaked and Tong [11]) or random variables with a common marginal distribution (cf. Tong [17]). In this paper we consider the new supermodular ordering defined by Szekli et al. [13] (cf. also Chang [5]) which is stronger than the relations $\leq_K$ and $\leq_D$ defined in Bergmann [3]. With this ordering it is possible to compare random vectors with the same marginal distributions which are not necessarily equal. By doing so, we derive new inequalities for multivariate normal distributions, Markov chains and for system characteristics in stochastic models.

The paper is organized as follows: In section 2 we give some definitions of well-known orderings as well as the supermodular ordering. We discuss properties of these partial orderings and the relations between them. In section 3 we present three models of general type which are frequently encountered in applications and which yield the supermodular ordering. The first one is a modification of a model from Shaked and Tong [11], the second is taken from Tong [17]. In the third model the random variables X and Y evolve from a recursive equation. It appears to be useful in the comparison of Markov chains and for the comparison of system characteristics in stochastic models (cf. also Bäuerle and Rieder [2]). In section 4 we apply our results to normal distributions, Markov chains and stochastic models from biology, telecommunication and risk theory.

2. ORDERINGS

Throughout the paper we assume that two random vectors $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ are given with values in $I\!R^n$. By P^X and P^Y we denote the probability measures on the Borel-σ-algebra $\mathfrak{B}(I\!R^n)$ of $I\!R^n$ induced by X and Y respectively. Furtheron, we suppose that the marginal distributions of X and Y are the same i.e. $X_i \stackrel{d}{=} Y_i,\ i = 1, \dots, n$. Occasionally we even assume that X and Y have a common marginal distribution i.e. $X_1 \stackrel{d}{=} \dots \stackrel{d}{=} X_n \stackrel{d}{=} Y_1 \stackrel{d}{=} \dots \stackrel{d}{=} Y_n$. There are a lot of possibilities to "measure" the strength of dependence in random vectors. We will first recall two of the well-known (see e.g. Tong [17], Shaked and Tong [11], Bergmann [3]), noting that "increasing" and "decreasing" are used in a non-strict sense throughout the paper. A function which is either increasing or decreasing will be called monotonic.

Definition 2.1

a) For two random vectors X and Y with a common marginal distribution denote $X \leq_{pd} Y$ (the components of Y are more positively dependent than the components of X) if

$$E\Pi_{i=1}^{n} f(X_i) \leq E\Pi_{i=1}^{n} f(Y_i) \tag{1}$$

for all measurable functions $f : I\!R \to I\!R$ for which the expectations exist.

b) For two random vectors X and Y we denote $X \leq_{K(D)} Y$ if

$$E\Pi_{i=1}^{n} f_i(X_i) \leq E\Pi_{i=1}^{n} f_i(Y_i)$$

for all increasing (decreasing) functions $f_1, \ldots, f_n : I\!R \to I\!R_+$ for which the expectations exist.

Remarks:

a) If (1) holds for all measurable functions which are positive we write $\leq_{pd+}$.

b) $X \leq_{pd+} Y$ implies $P(X_1 \in B, \ldots, X_n \in B) \leq P(Y_1 \in B, \ldots, Y_n \in B)$ for all $B \in \mathfrak{B}(I\!R)$ since for indicator functions $I_{B \times \ldots \times B}(x) = \Pi_{i=1}^{n} I_B(x_i)$.

c) $X \leq_{pd} Y$ implies $cov(f(X_i), f(X_j)) \leq cov(f(Y_i), f(Y_j))$ for all i, j and measurable f.

d) An equivalent definition of $\le_K$ and $\le_D$ is

$$X \le_K Y \quad \text{if and only if} \quad P(X \ge t) \le P(Y \ge t) \ \text{ for all } \ t \in \mathbb{R}^n,$$

$$X \le_D Y \quad \text{if and only if} \quad P(X < t) \le P(Y < t) \ \text{ for all } \ t \in \mathbb{R}^n,$$

see Bergmann [3]. For another equivalent definition see Rüschendorf [9].

e) It is easily seen that $X \le_K Y$ and $X \le_D Y$ together imply $X_i \stackrel{d}{=} Y_i$.

f) $X \le_K Y$, $X \le_D Y$ together imply $cov(f(X_i), g(X_j)) \le cov(f(Y_i), g(Y_j))$ for all i, j and $f, g : \mathbb{R} \to \mathbb{R}_+$ which are both increasing or decreasing.

The properties (b) and (c) of $\le_{pd}$ and the properties (d) and (f) of $\le_K$, $\le_D$ are a justification for the use of these orderings to measure the strength of dependences. We will now introduce an ordering which was first defined in Szekli et al. [13] (cf. also Chang [5]) and which is based on the following definition for supermodular functions.

Definition 2.2

a) A function $f : \mathbb{R}^n \to \mathbb{R}$ is called supermodular, if for all $x, y \in \mathbb{R}^n$:

$$f(x \vee y) + f(x \wedge y) \ \ge \ f(x) + f(y)$$

where $x \vee (\wedge) y$ denotes the componentwise maximum (minimum) of x, y.

b) For two random vectors X and Y we denote $X \le_{sm} Y$ if

$$Ef(X) \le Ef(Y) \tag{2}$$

for all measurable, supermodular functions $f : \mathbb{R}^n \to \mathbb{R}$ for which the expectations exist.

Shaked and Tong [11] mentioned that especially employing symmetric functions in (2) leads to useful orderings (a function $f : \mathbb{R}^n \to \mathbb{R}$ which satisfies $f(x) = f(\Pi x)$ for all permutations Πx of x is called symmetric). This is mainly due to the fact that symmetric functions imply an analogous relation for the order statistics. Therefore, we will also concern the relation which is obtained by requiring (2) for all symmetric and supermodular functions, denoted by $\leq_{symsm}$ (notice that supermodular functions are not necessarily symmetric in general). Nevertheless it will turn out that the supermodular ordering is very useful to compare the strength of dependence in sequences of random variables which bear a certain time structure (e.g. Markov chains) and are not necessarily exchangeable (a random vector is called exchangeable, if the joint distribution is invariant under permutations). Important properties of supermodular functions are summarized in the following lemma. Proofs, if not given here, can be found in Marshall and Olkin [7] or Heyman and Sobel [6].

Lemma 2.1

a) *Let $f : I\!R^n \to I\!R$ be supermodular and $\phi, \phi_1, \ldots, \phi_n : I\!R \to I\!R$.*

   (i) If f is increasing then $\max\{f, c\}$ is supermodular for all $c \in I\!R$.

   (ii) If f is monotonic and ϕ increasing and convex then $\phi \circ f$ is monotonic and supermodular.

   (iii) If $\phi_1, \ldots, \phi_n$ are monotonic in the same direction then the composition $f(\phi_1(.), \ldots, \phi_n(.))$ is supermodular.

d) *Let $\phi_i : I\!R \to I\!R_+$, $i = 1, \ldots, n$ and define $f(x_1, \ldots, x_n) := \Pi_{i=1}^n \phi_i(x_i)$. f is supermodular if and only if $\phi_1, \ldots, \phi_n$ are monotonic in the same direction.*

Proof of part b): Let $n = 2$, $f(x_1, x_2) := \phi_1(x_1)\phi_2(x_2)$ and assume w.l.o.g. $x_1 \leq y_1$, $x_2 \geq y_2$. f is supermodular if and only if

$$\phi_1(x_1)\phi_2(x_2) + \phi_1(y_1)\phi_2(y_2) \leq \phi_1(x_1)\phi_2(y_2) + \phi_1(y_1)\phi_2(x_2) \Leftrightarrow$$
$$\Leftrightarrow (\phi_2(x_2) - \phi_2(y_2))(\phi_1(y_1) - \phi_1(x_1)) \geq 0$$

which is true if and only if ϕ_1 and ϕ_2 are both increasing or decreasing. The statement for arbitrary n follows easily by induction. □

We note in passing that if $f \in C^2(I\!R^n)$ (i.e. f is twice continuous differentiable) then f is supermodular if and only if $\frac{\partial^2}{\partial x_i \partial x_j} f(x) \geq 0, \quad \forall i \neq j, \ i,j = 1,\ldots,n$. This is sometimes a convenient characterization of supermodular functions. These properties imply of course a lot of interesting properties for the $\leq_{sm}$ and $\leq_{symsm}$ ordering. Beforehand, we note that $\leq_{st}$ ($\leq_{ic}$) is used for the stochastic (increasing convex) ordering i.e. $X \leq_{st} Y$ ($X \leq_{ic} Y$) if and only if $Ef(X) \leq Ef(Y)$ for all $f : I\!R^n \to I\!R$ increasing (increasing and convex) for which the expectations exist. (See e.g. Shaked and Shanthikumar [10] or Szekli [14]).

Lemma 2.2 *If two random vectors satisfy $X \leq_{sm} Y$ then*

a) $X_i \stackrel{d}{=} Y_i, \ i = 1,\ldots,n.$

b) $cov(f(X_i), g(X_j)) \leq cov(f(Y_i), g(Y_j))$ *for all* $f, g : I\!R \to I\!R_+$ *both increasing or decreasing.*

c) $\max(X_1,\ldots,X_n) \geq_{st} \max(Y_1,\ldots,Y_n)$, $\min(X_1,\ldots,X_n) \leq_{st} \min(Y_1,\ldots,Y_n)$.

d) $(X_{i_1},\ldots,X_{i_k}) \leq_{sm} (Y_{i_1},\ldots,Y_{i_k})$ *for all* $1 \leq i_1 < \ldots < i_k \leq n$.

Proof: a), d) are direct consequences of the definition.
b) Follows from part a) and Lemma 2.1 b).
c) Let $f : I\!R \to I\!R$ be decreasing, $x, y \in I\!R^n$ and assume w.l.o.g. that $\max(x_1,\ldots,x_n,y_1,\ldots,y_n) = x_i$ for $i \in \{1,\ldots,n\}$. Then

$$f(\max(x_1,\ldots,x_n)) + f(\max(y_1,\ldots,y_n)) = f(\max(x_1 \vee y_1,\ldots,x_n \vee y_n)) +$$

$$+ f(\max(y_1,\ldots,y_n)) \leq f(\max(x_1 \vee y_1,\ldots,x_n \vee y_n)) + f(\max(x_1 \wedge y_1,\ldots,x_n \wedge y_n)).$$

Hence $f \circ \max(.,\ldots,.)$ is supermodular for all f decreasing which yields $\max(X_1,\ldots,X_n) \geq_{st} \max(Y_1,\ldots,Y_n)$. The second statement follows analogously. □

For the next lemma we need the notion of majorization (see e.g. Marshall and Olkin [7]): let $x, y \in I\!R^n$. Denote $x \prec y$ (y majorizes x) if and only if $\sum_{i=1}^r x_{[i]} \leq \sum_{i=1}^r y_{[i]}$, $r = 1, \ldots, n-1$, $\sum_{i=1}^n x_{[i]} = \sum_{i=1}^n y_{[i]}$, where $x_{[1]} \geq \ldots \geq x_{[n]}$ denotes the increasing rearrangement of x. For random variables, $(X_{[1]}, \ldots, X_{[n]})$ denotes the order statistic.

Lemma 2.3 *If X, Y are exchangeable with same marginal distribution and $X \leq_{symsm} Y$ then*

$$(Ef(X_{[1]}), \ldots, Ef(X_{[n]})) \succ (Ef(Y_{[1]}), \ldots, Ef(Y_{[n]}))$$

for all monotonic functions $f : I\!R \to I\!R$ for which the expectations exist.

Proof: We have to show that $E(-\sum_{i=1}^k f(X_{[i]})) \leq E(-\sum_{i=1}^k f(Y_{[i]}))$ for all monotonic functions $f : I\!R \to I\!R$ and $k = 1, \ldots, n-1$ (equality for $k = n$ holds since X and Y have the same marginal distribution).
Define $h(x) := -\max_{1 \leq i_1 < \ldots < i_k \leq n}(\sum_{j=1}^k x_{i_j})$ and $g(x) = (f(x_1), \ldots, f(x_n))$. Chang [5] showed that $h(x)$ is supermodular. From Lemma 2.1 a) we know that $h(g(x))$ is supermodular, too which yields the result. □

Important applications of the supermodular ordering are the following (see e.g. Tchen [15] and Meester and Shanthikumar [8]).

Lemma 2.4

a) **(Lorentz-inequality)** *Let $X_1, \ldots, X_n$ be identically distributed random variables. Then $(X_1, \ldots, X_n) \leq_{sm} (X_1, \ldots, X_1)$.*

b) *Let $\{X_n\}$ be a sequence of independent random variables. If $\{Y_n\}$ is*

sequentially stochastic increasing *(i.e.* $P(Y_n \in . \mid Y_1 = t_1, \ldots, Y_{n-1} = t_{n-1})$ *is stochastically increasing in* $(t_1, \ldots, t_{n-1})$ *for all* n*) and* $X_n \stackrel{d}{=} Y_n$ *then*

$$(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n).$$

At the end of this section we examine briefly the relations between the introduced orderings. From Lemma 2.1 b) it follows directly that $X \leq_{sm} Y$ implies $X \leq_K Y$ and $X \leq_D Y$. In the case of $n = 2$ it even holds that $(X_1, X_2) \leq_{sm} (Y_1, Y_2)$ if and only if $(X_1, X_2) \leq_K (Y_1, Y_2)$ and $(X_1, X_2) \leq_D (Y_1, Y_2)$. This can be derived from Theorem 3 in Rüschendorf [9] or Theorem 1 in Tchen [15].

As far as the orderings $\leq_{sm}$ and $\leq_{pd}$ (resp. $\leq_{pd+}$) are concerned, there are random vectors which satisfy $X \leq_{sm} Y$ but not $X \leq_{pd} Y$ ($X \leq_{pd+} Y$) and vice versa. This shows the following examples:

Example 2.1 Let $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, with $P(X = (i,j)) = \frac{1}{16}$, $i, j \in \{1,2,3,4\}$ and $P(Y = (1,4)) = P(Y = (2,3)) = 0$, $P(Y = (2,4)) = P(Y = (1,3)) = \frac{1}{8}$ and $P(Y = (i,j)) = \frac{1}{16}$ otherwise $(i, j \in \{1,2,3,4\})$. Then it is easily seen that $X \leq_{sm} Y$. But choosing $f : I\!R \to I\!R_+$ as $f(1) = f(4) = 1$, $f(2) = f(3) = 0$ we obtain $Ef(X_1)f(X_2) \geq Ef(Y_1)f(Y_2)$ which implies that $X \not\leq_{pd+} Y$, hence $X \not\leq_{pd} Y$.

Example 2.2 Let X, Y be independent and identically distributed (iid) random variables. We will see in section 3 that $(X,X,X,X,Y,Y) \leq_{pd} (X,X,Y,Y,Y,Y)$. But they are surely not $\leq_{sm}$ ordered, since then by Lemma 2.2 d) $(X,X) \leq_{sm} (X,Y)$ which contradicts the Lorentz-inequality. Hence $X \not\leq_{sm} Y$.

3. THE MODELS

In this section we investigate three different constructions of random vectors X and Y which can be compared with respect to their strength of dependence.

The first model is a modification of a model by Shaked and Tong [11]. Model 3.2 was proposed by Tong [17] where he showed that the random vectors X and Y can be compared with respect to $\leq_{pd+}$ ($\leq_{pd}$). We will now show that they are supermodular ordered, too. The third model is of recursive nature and leads to the comparison of Markov chains and system characteristics in stochastic models.

Model 3.1 (cf. Shaked and Tong [11])
In this model we suppose that the n-dimensional random vectors X and Y are of the following specific structure

$$\begin{aligned}(X_1,\ldots,X_n) &= (g_1(Z_1,W),\ldots,g_n(Z_n,W))\\ (Y_1,\ldots,Y_n) &= (\tilde{g}_1(U_1,V,W),\ldots,\tilde{g}_n(U_n,V,W))\end{aligned}$$

where $Z_1,\ldots,Z_n$ are iid random variables, $U_1,\ldots,U_n$ are iid random variables and (V,W) is a random vector independent of $\{Z_i\}$ and $\{U_i\}$. Moreover, we assume for the measurable functions $g_i : I\!R^2 \to I\!R$ and $\tilde{g}_i : I\!R^3 \to I\!R$ that for every fixed w in the support of W

$$g_i(Z_i,w) \stackrel{d}{=} \tilde{g}_i(U_i,V,w) \quad i=1,\ldots,n.$$

Shaked and Tong [11] showed that we obtain $X \leq_{pd} Y$ if g_i and $\tilde{g}_i$ are independent of i. Under a different assumption it is possible to prove

Theorem 3.1 *If the functions $\tilde{g}_i$ are increasing in the second component, we obtain under the assumptions of model 3.1*

$$X \leq_{sm} Y.$$

Proof: Let $f : I\!R^n \to I\!R$ be supermodular and let w be a realization of W and $u_1,\ldots,u_n$ a realization of $(U_1,\ldots,U_n)$. Denote by $V_1,\ldots,V_n$ a sequence of iid random variables with the conditional distribution of V given $W = w$. From the assumptions we know that $\tilde{g}_1(u_1,V_1,w),\ldots,\tilde{g}_n(u_n,V_n,w)$ are independent and $\tilde{g}_i(u_i,V_i,w) \stackrel{d}{=} \tilde{g}_i(u_i,V,w)$. Denote by $G_i = (\tilde{g}_1(u_1,V,w),\ldots,\tilde{g}_i(u_i,V,w))$. Take x

and y from the support of G_{i-1} such that $x \leq y$. Since $\tilde{g}$ is increasing in the second component we obtain for all $\omega \in \{\omega \mid G_{i-1}(\omega) = x\}$ and $\omega' \in \{\omega \mid G_{i-1}(\omega) = y\}$ that $V(\omega) \leq V(\omega')$. Therefore, the conditional distribution $P(\tilde{g}_i(u_i, V, w) \in . \mid G_{i-1} = x)$ is increasing in x. Hence we get with Lemma 2.4

$$\begin{aligned}
&Ef(g_1(Z_1, W), \ldots, g_n(Z_n, W)) = EE[f(g_1(Z_1, W), \ldots, g_n(Z_n, W)) \mid W] = \\
&= EE[f(\tilde{g}_1(U_1, V_1, W), \ldots, \tilde{g}_n(U_n, V_n, W)) \mid W, U_1, \ldots, U_n] \leq \\
&\leq EE[f(\tilde{g}_1(U_1, V, W), \ldots, \tilde{g}_n(U_n, V, W)) \mid W, U_1, \ldots U_n] = \\
&= Ef(\tilde{g}_1(U_1, V, W), \ldots, \tilde{g}_n(U_n, V, W))
\end{aligned}$$

which completes the proof. □

Model 3.2 (cf. Tong [17])

Let k and k' be two n-dimensional vectors with

$$k = (k_1, \ldots, k_r, 0, \ldots, 0), \quad k' = (k'_1, \ldots, k'_l, 0, \ldots, 0)$$

$1 \leq r, l \leq n, \quad k_i, k'_i \in I\!N$ for all i and $\sum_{i=1}^n k_i = \sum_{i=1}^n k'_i = n$.
Suppose that k' majorizes k. Notice that $k \prec k'$ implies $l \leq r$. The n-dimensional random vectors X and Y satisfy

$$\begin{array}{rclrcl}
X_1 & = & g(U_1, V_1, W) & Y_1 & = & g(U_1, V_1, W) \\
& \vdots & & & \vdots & \\
X_{k_1} & = & g(U_{k_1}, V_1, W) & Y_{k'_1} & = & g(U_{k'_1}, V_1, W) \\
X_{k_1+1} & = & g(U_{k_1+1}, V_2, W) & Y_{k'_1+1} & = & g(U_{k'_1+1}, V_2, W) \\
& \vdots & & & \vdots & \\
X_{k_1+k_2} & = & g(U_{k_1+k_2}, V_2, W) & Y_{k'_1+k'_2} & = & g(U_{k'_1+k'_2}, V_2, W) \\
& \vdots & & & \vdots & \\
X_n & = & g(U_n, V_r, W) & Y_n & = & g(U_n, V_l, W)
\end{array}$$

where $U_1, \ldots, U_n$ are iid random variables, $V_1, \ldots, V_r$ are iid random variables and W is a random variable independent of $\{U_i\}$ and $\{V_i\}$. $g : I\!R^3 \to I\!R$ is an arbitrary, measurable function. I.e. the random variables $X_1, \ldots, X_n$ $(Y_1, \ldots, Y_n)$ are partitioned into r (l) groups. Each random variable depends on a common random variable W, a group specific random variable V_i and an individual random variable U_i.

Tong [17] proved under these assumptions that $X \leq_{pd+} Y$ and $X \leq_{pd} Y$ if in addition k and k' contain only even natural numbers. We will now show

Theorem 3.2 *If the function g is monotonic in the second component, we obtain under the assumptions of model 3.2*

$$X \leq_{symsm} Y.$$

In order to prove Theorem 3.2 we need the following lemma:

Lemma 3.3 *Let $\{X_i\}$ be a sequence of iid random variables and define the n-dimensional vectors k, k' as above. Denote*

$$X = (X_1, \ldots, X_1, X_2, \ldots, X_2, \ldots, X_r, \ldots, X_r)$$
$$Y = (X_1, \ldots, X_1, X_2, \ldots, X_2, \ldots, X_l, \ldots, X_l)$$

where the block of X_i's in X (Y) has length k_i (k_i'). If $k \prec k'$ then

$$X \leq_{symsm} Y.$$

Proof: It is known that if $k \prec k'$ then there exists a sequence $z^1, \ldots, z^n \in I\!N_0^n$ such that $k \prec z^1 \prec \ldots \prec z^n \prec k'$ and each two vectors $z^i,\ z^{i+1},\ i = 1, \ldots, n-1$ differ only in two components (see e.g. Marshall and Olkin [7]). Therefore it suffices to prove the statement for $n = 2$. Let $(k_1, k_2) \prec (k_1', k_2')$, (assume w.l.o.g. $k_1 \geq k_2, k_1' \geq k_2'$) i.e. $k_1 \leq k_1'$ and $k_1 + k_2 = k_1' + k_2' = m$.

Let X and Y be independent and identically distributed. Denote by $X^{(i)}$ ($Y^{(i)}$) the i-th component of a vector with $X^{(i)} = X$ ($Y^{(i)} = Y$). We have to show

$$(X^{(1)}, \ldots, X^{(k_1)}, Y^{(k_1+1)}, \ldots, Y^{(m)}) \leq_{symsm} (X^{(1)}, \ldots, X^{(k_1')}, Y^{(k_1'+1)}, \ldots, Y^{(m)})$$

It can be easily verified by induction that it suffices to prove the above inequality for $k_1' = k_1 + 1$, $k_2' = k_2 - 1$. Let $f : I\!R^n \to I\!R$ be symmetric and supermodular. Then

$$\begin{aligned}
& Ef(X^{(1)}, \ldots, X^{(k_1)}, Y^{(k_1+1)}, \ldots, Y^{(m)}) = \\
= \; & \frac{1}{2} \int_{I\!R} \int_{I\!R} [f(x^{(1)}, \ldots, x^{(k_1)}, y^{(k_1+1)}, \ldots, y^{(m)}) + \\
& \qquad f(y^{(1)}, \ldots, y^{(k_1)}, x^{(k_1+1)}, \ldots, x^{(m)})] \, P^X(dx) P^Y(dy) \leq \\
\leq \; & \frac{1}{2} \int_{I\!R} \int_{I\!R} [f(x^{(1)}, \ldots, x^{(k_1+1)}, y^{(k_1+2)}, \ldots, y^{(m)}) + \\
& \qquad f(y^{(1)}, \ldots, y^{(k_1+1)}, x^{(k_1+2)}, \ldots, x^{(m)})] \, P^X(dx) P^Y(dy) = \\
= \; & Ef(X^{(1)}, \ldots, X^{(k_1+1)}, Y^{(k_1+2)}, \ldots, Y^{(m)})
\end{aligned}$$

where the inequality holds due to the following observation (using the assumption that f is symmetric and supermodular): let $x, y \in I\!R$. Then

$$\begin{aligned}
& f(x^{(1)}, \ldots, x^{(k_1)}, y^{(k_1+1)}, \ldots, y^{(m)}) + f(y^{(1)}, \ldots, y^{(k_1)}, x^{(k_1+1)}, \ldots, x^{(m)}) = \\
= \; & f(x^{(1)}, \ldots, x^{(k_1)}, y^{(k_1+1)}, \ldots, y^{(m)}) + \\
& f(y^{(1)}, \ldots, y^{(k_1-k_2+1)}, x^{(k_1-k_2+2)}, \ldots, x^{(k_1+1)}, y^{(k_1+2)}, \ldots, y^{(m)}) \leq \\
\leq \; & f(x^{(1)}, \ldots, x^{(k_1+1)}, y^{(k_1+2)}, \ldots, y^{(m)}) + \\
& f(y^{(1)}, \ldots, y^{(k_1-k_2+1)}, x^{(k_1-k_2+2)}, \ldots, x^{(k_1)}, y^{(k_1+1)}, \ldots, y^{(m)}) = \\
= \; & f(x^{(1)}, \ldots, x^{(k_1+1)}, y^{(k_1+2)}, \ldots, y^{(m)}) + f(y^{(1)}, \ldots, y^{(k_1+1)}, x^{(k_1+2)}, \ldots, x^{(m)})
\end{aligned}$$

which completes the proof. □

Proof of Theorem 3.2: Define the n-dimensional random vectors V and V' as the sequence of V_i's respectively which appear in the construction of X and Y in model 3.2 i.e. V and V' are defined in the same way as X and Y in Lemma 3.3, where

$\{X_i\}$ is replaced by $\{V_i\}$. Let $f : I\!R^n \to I\!R$ be symmetric and supermodular. Then

$$\begin{aligned} Ef(X) &= \int_{I\!R}\int_{I\!R^n}\left[\int_{I\!R}\cdots\int_{I\!R} f(g(u_1,v_1,w),\ldots,g(u_n,v_n,w))\, P^U(du_1)\ldots\right. \\ &\qquad \left.\ldots P^U(du_n)\right] P^V(d(v_1,\ldots,v_n))P^W(dw) \end{aligned}$$

For an arbitrary fixed w from the support of W we define the function $h : I\!R^n \to I\!R$ as the inner integral

$$h(v_1,\ldots,v_n) = \int_{I\!R}\cdots\int_{I\!R} f(g(u_1,v_1,w),\ldots,g(u_n,v_n,w))\, P^U(du_1)\ldots P^U(du_n).$$

From Lemma 3.3 we know that $V \leq_{symsm} V'$. Therefore, it remains to show that h is symmetric and supermodular. Symmetry follows directly from the assumption that f is symmetric and supermodularity can be shown with Lemma 2.1 a). □

Model 3.3

Let $V_1,\ldots,V_n$ be a sequence of iid random variables and $\tilde V_1,\ldots,\tilde V_n$ a sequence of random variables such that $V_i \stackrel{d}{=} \tilde V_i$ for all i and $\{\tilde V_i\}$ is sequentially stochastic increasing. Let $U_1,\ldots,U_n$ be a sequence of iid random variables which are uniformly distributed over the interval $[0,1]$. For $p \in (0,1)$ let $X_1 = V_1$, $Y_1 = \tilde V_1$ and

$$\begin{aligned} X_{i+1} &= X_i + I_{[0,p]}(U_i)(V_i - X_i) \\ Y_{i+1} &= Y_i + I_{[0,p]}(U_i)(\tilde V_j - Y_i), \quad \text{if } Y_i = \tilde V_{j-1}, \quad i = 1,\ldots,n-1 \end{aligned}$$

and denote $X = (X_1,\ldots,X_n),\ Y = (Y_1,\ldots,Y_n)$.

Theorem 3.4: *For model 3.3 we obtain*

$$X \leq_{sm} Y.$$

Proof: The definition of the random variables $\{V_i\}$ and $\{\tilde V_i\}$ implies that $(V_1,\ldots,V_n) \leq_{sm} (\tilde V_1,\ldots,\tilde V_n)$ (see Lemma 2.4). Therefore, the proof follows directly from the following observation: let $k_1,\ldots,k_r \in I\!N$ be fixed. Then it is easily seen that $(V_1,\ldots,V_n) \le_{sm} (\tilde V_1,\ldots,\tilde V_n)$ implies $(V_1^{(1)},\ldots,V_1^{(k_1)},V_2^{(1)},\ldots,V_2^{(k_2)},\ldots,V_r^{(k_r)}) \le_{sm} (\tilde V_1^{(1)},\ldots,\tilde V_1^{(k_1)},\tilde V_2^{(1)},\ldots,\tilde V_2^{(k_2)},\ldots,\tilde V_r^{(k_r)})$, where $k_i \in I\!N_0$. Since the random variables X and Y are of this structure with random k_i (independent of $\{X_i\}$ and $\{Y_i\}$), the proof is complete. □

4. EXAMPLES AND APPLICATIONS

The theorems of section 3 for the general models 3.1-3.3 can be used to derive inequalities for certain distributions and system characteristics of stochastic models. This is particularly of interest in applied problems, where we may be able to control the amount of dependence among some random variables. We will restrict ourselves to applications concerning the supermodular ordering. Applications for the $\le_{pd}$ ordering can be found in Shaked and Tong [11] and Tong [17].

In a first paragraph we deal with inequalities for normal distributions due to dependence. A second one is dedicated to the comparison of Markov chains and the last paragraph contains applications for stochastic models.

4.1. Inequalities for normal distributions

Example 4.1 (Equicorrelated normal variables)
Let X be $\mathcal{N}(\boldsymbol{\mu},\Sigma(\rho))$-distributed with $\boldsymbol{\mu}= (\mu_1,\ldots,\mu_n)'$ $\mu_i \in I\!R$ and $\Sigma(\rho) = (\sigma_{ij})$ such that $\sigma_{ii} = \sigma^2,\ \sigma_{ij} = \rho\sigma^2$ for $1 \le i < j \le n,\ \sigma^2 \in I\!R_+,\ \rho \in [0,1]$.

Theorem 4.1 *If X is $\mathcal{N}(\boldsymbol{\mu},\Sigma(\rho))$-distributed and Y is $\mathcal{N}(\boldsymbol{\mu},\Sigma(\rho'))$-distributed as above with $\rho,\rho' \in [0,1]$ then*

$$X \le_{sm} Y \quad \textit{if and only if} \quad \rho \le \rho'.$$

Proof: The "only if"-part follows from Lemma 2.2 b). For the "if"-part let $U_1,\ldots,U_n,V_1,\ldots,V_n,V,W$ be iid $\mathcal{N}(0,1)$ random variables and define $Z_i = \sqrt{1-\rho'}\ U_i + \sqrt{\rho'-\rho}\ V_i,\ i = 1,\ldots,n$. Then $Z_1,\ldots,Z_n$ are iid. With $\tilde g_i(u,v,w) = \sigma(\sqrt{1-\rho'}\ u + \sqrt{\rho'-\rho}\, v + \sqrt{\rho}\, w) + \mu_i$ and $g_i(z,w) = \sigma(z + \sqrt{\rho}\, w) + \mu_i$

we obtain $g_i(Z_i, w) \stackrel{d}{=} \tilde{g}_i(U_i, V, w),\ i = 1, \ldots, n$ for all $w \in I\!R$.

From Tong [18] p.120 we know that $(X_1, \ldots, X_n) \stackrel{d}{=} (g_1(Z_1, W), \ldots, g_n(Z_n, W))$ and $(Y_1, \ldots, Y_n) \stackrel{d}{=} (\tilde{g}_1(U_1, V, W), \ldots, \tilde{g}_n(U_n, V, W))$. Since $\tilde{g}$ is increasing in the second component we can apply Theorem 3.1 which yields the result. □

Example 4.2 (Multivariate normal distribution)

Let k be an n-dimensional integer vector as given in model 3.2 and let $X(k)$ be $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$-distributed with $\boldsymbol{\mu} = (\mu, \ldots, \mu)'$ and $\Sigma(k) = (\sigma_{ij})$ where $\sigma_{ii} = \sigma^2$ and for $i \neq j$

$$\sigma_{ij} = \begin{cases} \sigma^2 \rho_2, & \text{if } 1 \leq i, j \leq k_1,\ k_1 + 1 \leq i, j \leq k_1 + k_2, \ldots \\ & \ldots, \sum_{m=1}^{r-1} k_m < i,\ j \leq n \\ \sigma^2 \rho_1, & \text{else} \end{cases}$$

with $0 \leq \rho_1 \leq \rho_2 \leq 1,\ \mu \in I\!R,\ \sigma^2 \in I\!R_+$.

Theorem 4.2 *If X is $\mathcal{N}(\boldsymbol{\mu}, \Sigma(k))$-distributed and Y is $\mathcal{N}(\boldsymbol{\mu}, \Sigma(k'))$-distributed with two n-dimensional integer vectors k and k' as given in model 3.2 and $\Sigma(k)$ is defined as above, then*

$$k \prec k' \quad \textit{implies} \quad X(k) \leq_{symsm} X(k').$$

The proof follows directly from Theorem 3.2 and a similar construction of $X(k)$ and $X(k')$ as in the proof of Theorem 4.1.

Remark: In Theorem 4.2 the random variables $(X_1, \ldots, X_n) = X(k)$ are partitioned into r groups of sizes $k_1, \ldots, k_r$. The correlations of the variables within the same group are ρ_2 and the correlations of variables from different groups are ρ_1.

4.2. Comparison of Markov chains

With the help of model 3.3 it is possible to compare the strength of dependence in certain Markov chains (cf. Bäuerle [1]). We will first consider discrete-time Markov chains.

Application 4.3 (Discrete-time Markov chains)
Let $0 \le p_1, \ldots, p_m \le 1$ and define $p = \sum_{j=1}^m p_j$. Suppose $p \le 1$. Let $\{V_i\}$ be iid random variables with $P(V_i = j) = \frac{p_j}{p}$, $j = 1, \ldots, m$ and define $\{X_n\}$ as in model 3.3. Then

$$P(X_{n+1} = j \mid X_n = i) = P(i + I_{[0,p]}(U_n)(V_n - i) = j) = pP(V_n = j) = p_j$$

if $i \neq j$ and if $i = j$

$$P(X_{n+1} = i \mid X_n = i) = pP(V_n = i) + 1 - p = 1 - \sum_{\nu \neq i} p_\nu$$

i.e. $\{X_n\}$ forms a Markov chain with state space $\{1, \ldots, m\}$ and transition matrix $P = (p_{ij})$, where $p_{ij} = p_j$ if $i \neq j$ and $p_{ii} = 1 - \sum_{\nu \neq i} p_\nu$.
Let $\{\tilde{V}_n\}$ be a Markov chain with $\tilde{V}_1 \stackrel{d}{=} V_1$ and transition matrix $Q = (q_{ij})$, where $q_{ij} = \frac{cp_j}{p}$ if $i \neq j$ and $q_{ii} = 1 - \frac{c}{p} \sum_{\nu \neq i} p_\nu$, with $c \in [0, 1]$. It is not difficult to show that $\{\tilde{V}_n\}$ is sequentially stochastic increasing and $\tilde{V}_n \stackrel{d}{=} V_n$ for all $n \in I\!N$. If we define $\{Y_n\}$ as in model 3.3 we obtain (suppose $Y_n = \tilde{V}_{k-1}$)

$$P(Y_{n+1} = j \mid Y_n = i) = P(i + I_{[0,p]}(U_n)(\tilde{V}_k - i) = j) = pP(\tilde{V}_k = j) = cp_j$$

if $i \neq j$ and if $i = j$

$$P(Y_{n+1} = i \mid Y_n = i) = pP(\tilde{V}_k = i) + 1 - p = 1 - c \sum_{\nu \neq i} p_\nu$$

i.e. $\{Y_n\}$ forms a Markov chain with transition matrix $P' = (p'_{ij})$, where $p'_{ij} = cp_j$ if $i \neq j$ and $p_{ii} = 1 - c \sum_{\nu \neq i} p_\nu$. By applying Theorem 3.4 we obtain

Theorem 4.3 *For the above defined Markov chains and $n \in I\!N$ it holds that*

$$(X_1, \dots, X_n) \leq_{sm} (Y_1, \dots, Y_n)$$

Application 4.4 (Continuous-time Markov chains)

Let $\{X(t), t \geq 0\}$ be a continuous-time Markov chain with state space $\{1, \dots, m\}$ and intensity matrix $Q = (q_{ij})$, with $q_{ij} = \alpha_j \in I\!R_+$ if $i \neq j$ and $q_i = -q_{ii} = \sum_{\nu \neq i} \alpha_\nu$. Denote $\alpha = \sum_j \alpha_j$. It is well known that $\{X(t), t \geq 0\}$ can be constructed in the following way (cf. Stoyan [12]):

Define $\{\Lambda_n\}$ as a Markov chain with initial distribution $(\frac{\alpha_1}{\alpha}, \dots, \frac{\alpha_m}{\alpha})$ (i.e. $P(\Lambda_0 = j) = \frac{\alpha_j}{\alpha}$) and transition matrix $P = (p_{ij})$, with $p_{ij} = \frac{\alpha_j}{\alpha}$ if $i \neq j$ and $p_{ii} = 1 - \sum_{\nu \neq i} \frac{\alpha_\nu}{\alpha}$. Denote by $\{\sigma_n\}$ a sequence of iid random variables which are exponentially distributed with parameter α. If we define $Z(t) = \Lambda_n$ if $\sigma_0 + \ldots + \sigma_{n-1} < t \leq \sigma_0 + \ldots + \sigma_n$ where $\sigma_0 = 0$ then $\{Z(t),\ t \geq 0\} \stackrel{d}{=} \{X(t),\ t \geq 0\}$. Furtheron, let $\{Y(t), t \geq 0\}$ be a continuous-time Markov chain with intensity matrix $cQ,\ c \in (0, 1]$. The corresponding discrete-time Markov chain $\{\Lambda'_n\}$ in the construction above has transition matrix $P' = (p'_{ij})$, where $p'_{ij} = c\frac{\alpha_j}{\alpha}$ if $i \neq j$ and $p'_{ii} = 1 - c\sum_{\nu \neq i} \frac{\alpha_\nu}{\alpha}$, whereas $\{\sigma_n\}$ stays the same. I.e. the Markov chains $\{\Lambda_n\}$ and $\{\Lambda'_n\}$ are of the same form as in application 4.3. Therefore Theorem 4.3 together with Lemma 2.2 d) implies

Theorem 4.4 *For the above defined Markov chains and $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n < \infty$, $n \in I\!N$ we obtain*

$$(X(t_1), \dots, X(t_n)) \leq_{sm} (Y(t_1), \dots, Y(t_n))$$

4.3 Applications to stochastic models

Application 4.5 (Genetic selection) cf. Tong [16]

Suppose that there are r families of animals of sizes $k_1, \dots, k_r$. The phenotype X_i of the i-th animal (belonging to family j) is effected by a common random variable

W, a family-specific random variable V_j and an individual random variable U_i as in model 3.2 (It is often assumed that $X_i = \mu + V_j + U_i$, where $\mu \in I\!R$ is a constant and V_j and U_i are $\mathcal{N}(0, \sigma_1^2)$ and $\mathcal{N}(0, \sigma_2^2)$-distributed). We want to keep the animal with maximal X_i for breeding. From Theorem 3.2 we know that if $(k_1, \ldots, k_r, 0, \ldots, 0) \prec (k'_1, \ldots, k'_l, 0, \ldots, 0)$ then $X \leq_{symsm} X'$. In particular $\max_i X_i \geq_{st} \max_i X'_i$. I.e. if we are only able to observe the phenotypes of n animals, we should choose these animals from different families, since by then the expected maximal phenotype will be the greatest. (Each other choice will take at most two animals from the same family which leads to smaller $E \max X_i$.)

Application 4.6 (TDM-model) cf. Chang et al. [5]

Consider the following queueing model for a Time Division Multiplexing (TDM) system with voice/data integration: during the $(n+1)$-st time frame, D_n data packets and V_n voice packets are arriving. $\{D_n\}$ is supposed to be a sequence of iid random variables, independent of $\{V_n\}$. $\{V_n\}$ is supposed to be a superposition of N independent two-state Markov chains $\{J_n^{(i)}\}$, $i = 1, \ldots, N$ which alter between the silence state (state 0, i.e. no packets are transmitted by source i) and the talkspurt state (state 1, i.e. voice source i transmits one voice packet during time frame $n+1$). Hence $V_n = \sum_{i=1}^N J_n^{(i)}$. The transition matrix of the Markov chain $\{J_n^{(i)}\}$ is assumed to be

$$P = P(cp_0^{(i)}, cp_1^{(i)}) = \begin{pmatrix} 1 - cp_1^{(i)} & cp_1^{(i)} \\ cp_0^{(i)} & 1 - cp_0^{(i)} \end{pmatrix}$$

with $0 \le p_0^{(i)} + p_1^{(i)} \le 1$, $i = 1, \dots, N$ and $c \in [0,1]$.

The arriving data and voice packets can be processed by M channels or slots, each of which is able to handle one packet per time frame. The queue size S_n at the end of the n-th time frame is then given by the following recursive equation

$$S_{n+1} = \max\{0, S_n + V_n + D_n - M\}, \quad n \in I\!N_0.$$

The Markov chains $\{J_n^{(i)}\}$ are obviously of the same structure as in Application 4.3 (with m=2). Since they are independent, we obtain by inductive application of Theorem 4.3 $(V_0, \dots, V_n) \le_{sm} (V_0', \dots, V_n')$ if $0 \le c' \le c \le 1$ (where parameter c' belongs to $\{V_i'\}$). If we denote $s_1 = \Phi_1(v_0) = \max\{0, v_0 + d_0 - M\}$ and for $n \ge 1$, $s_{n+1} = \Phi_{n+1}(v_0, \dots, v_n) = \max\{0, \Phi_n(v_0, \dots, v_{n-1}) + v_n + d_n - M\}$ it is possible to show inductively with the help of Lemma 2.1 that Φ_{n+1} is supermodular (and increasing) in $(v_0, \dots, v_n)$. Therefore we obtain $E\Phi_n(V_0, \dots, V_n) \le E\Phi_n(V_0', \dots, V_{n-1}')$ if $c' \le c$ for fixed $(d_0, \dots, d_{n-1}) \in I\!R^n$. Since $\{D_n\}$ is independent of $\{V_n\}$ we get $ES_n \le ES_n'$ if $c' \le c$. I.e. the greater the dependence between on and off-periods, the greater is the expected number of waiting data packets at time n.

Application 4.7 (Ruin model)

Suppose that an insurance company is interested in the time Z of the next claim arrival. By $N(t)$ we denote the counting process of the claims (i.e. $N(t)$ denotes the number of claims which have arrived before time t). Assume that $N(t)$ is a Markov-modulated Poisson process with intensity $\lambda(t)$, i.e. $\{\lambda(t), t \ge 0\}$ is a Markov process with finite state space and given $\{\lambda(t), t \ge 0\}$, $N(t)$ is an inhomogeneous Poisson process with (deterministic) intensity $\{\lambda(t), t \ge 0\}$. Assume further that $\{\lambda(t), t \ge 0\}$ has an intensity matrix $Q = (cq_{ij})$, with $q_{ij} = \alpha_j \in I\!R_+$ if $i \ne j$ and $q_i = -q_{ii} = \sum_{\nu \ne i} \alpha_\nu$, $c \in (0,1]$ as in Application 4.4. Take $c, c' \in (0,1]$ such that $c \le c'$. Denote by $\{\Lambda_n\}$, $\{\Lambda_n'\}$ and $\{\sigma_n\}$ the constructive elements of the Markov chains as in Application 4.4 respectively. Hence we know that $(\Lambda_1, \dots, \Lambda_n) \ge_{sm} (\Lambda_1', \dots, \Lambda_n')$. Denote by $\{X_i\}$ and $\{X_i'\}$ independent random variables which are exponentially distributed with random parameter Λ_i. We obtain now $(X_1, \dots, X_n) \ge_{sm} (X_1', \dots, X_n')$ with a similar argument as in Theorem

3.3 of Bäuerle and Rieder [2]. Finally suppose that $\{s_i\}$ is a realization of $\{\sigma_i\}$ and $0 < s_1 < \ldots < s_1 + \ldots + s_n \leq t < s_1 + \ldots + s_{n+1}$ with $t \in I\!R_+$. Hence for the claim arrival times Z and Z' with parameter c and c'

$$\begin{aligned} &P(Z > t|\{\sigma_i\} = \{s_i\}) = P(X_1 > s_1, \ldots, X_n > s_n, X_{n+1} > t - s_1 - \ldots - s_n) = \\ &= EI_{[x>s_1]}(X_1) \ldots I_{[x>s_n]}(X_n) I_{[x>t-s_1-\ldots-s_n]}(X_{n+1}) \geq \\ &\geq EI_{[x>s_1]}(X_1') \ldots I_{[x>s_n]}(X_n') I_{[x>t-s_1-\ldots-s_n]}(X_{n+1}') = P(Z' > t \mid \{\sigma_i\} = \{s_i\}) \end{aligned}$$

since $I_{[x>s_1]}(.) \ldots I_{[x>s_n]}(.) I_{[x>t-s_1-\ldots-s_n]}(.)$ is supermodular. Because $\{\sigma_i\}$ is independent of $\{X_i\}$ we obtain $Z \geq_{st} Z'$ i.e. the greater the dependence in the intensity process, the longer is the time it takes for the first claim arrival.

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