Positive Solutions for the Discrete Nonlinear Schrödinger Equation: A Priori Estimates and Convergence

Zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften

(Dr. rer. nat)

von der Fakultät für Mathematik des

Karlsruher Instituts für Technologie (KIT)

genehmigte

DISSERTATION

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Tag der mündlichen Prüfung: 25.06.2014

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Chapter 1

Introduction

1.1 Motivation

We begin with the (time-dependent) nonlinear Schrödinger equation in the form

$$i\frac{\partial \Psi}{\partial t} + \Delta\Psi + |\Psi|^2\Psi = 0, \tag{†}$$

used in nonlinear optics or with the (standard) one-dimensional discrete nonlinear Schrödinger equation

$$i\frac{d\Psi_n}{dz} + \alpha(\Psi_{n+1} + \Psi_{n-1} - 2\Psi_n) + \mu|\Psi_n|^2\Psi_n = 0, \tag{‡}$$

with $\alpha \in \mathbb{R}$, $\mu > 0$, used for modeling of nonlinear waveguide arrays.

The first equation describes the envelope Ψ of a traveling wave in a weakly nonlinear and dispersive medium (see [33], [39] and [29]). The character of the nonlinearity $|\Psi|^2\Psi$ is due to the physical properties of the medium (so called Kerr medium) and

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

denotes the standard Laplace operator on $\mathbb{R}^3$. Whereas this equation is usually considered on the whole space corresponding to propagation in bulk media, some phenomena can be modeled on a smooth bounded domain using the zero Dirichlet boundary condition (e.g. the propagation of intense laser beams in hollow-core fibers with a noble gas, see [21]).

By waveguide one understands a medium that allows wave propagation only along its axes, i.e. it "guides light" (see [28], where the connected concept of a photonic crystal is also described). The second equation (‡) describes the light propagation in a one-dimensional coupled waveguide array, i.e. the propagation in every waveguide is allowed only along the z-axis, the waveguides lie parallel and regularly spaced on a surface and for every waveguide we take into account only the interaction with the two neighboring waveguides in the array. The quantity Φ_n describes some transformation of the electric field E_n in n-th waveguide (for the transformation, see [15]). The array is usually assumed infinite (big enough) for modeling purposes (see [1] and [19]).

Using the standing-wave ansatz $\Psi(x,t) = e^{i\lambda t}u(x)$, $\lambda > 0$ for the first equation (†) we obtain the (time-independent) nonlinear Schrödinger equation (NLS),

$$-\Delta u + \lambda u = |u|^{p-1}u \tag{+}$$

with $p = 3$. Since the Laplacian Δ can be approximated by its standard discretization, the discrete Laplace operator Δ_h

$$\Delta u(x) = \lim_{h\to 0} \Delta_h u(x)$$
$$\Delta_h u(x) := \sum_{i=1}^{3} \frac{u(x+he_i) - 2u(x) + u(x-he_i)}{h^2}$$

for all sufficiently smooth $u \in C^3$, it is natural to consider the discrete (time-independent) nonlinear Schrödinger equation (DNLS)

$$-\Delta_h u + \lambda u = |u|^{p-1}u \tag{$*$}$$

on some (possibly unbounded) subset of the h-grid $\mathbb{R}^d_h := \{hz \mid z \in \mathbb{Z}^d\}$, where d is the space dimension and $p > 1$, and to investigate the behavior of its solutions as $h \to 0$.

Interestingly enough, using the same ansatz $\Psi_n(z) = e^{i\lambda z}u_n$, $\lambda > 0$ for our second motivational equation (‡), we obtain the same DNLS in one dimension

$$\frac{-u_{n+1} + 2u_n - u_{n-1}}{h^2} + \lambda u_n = \mu |u_n|^2 u_n,$$

with $h = 1/\sqrt{\alpha}$, i.e. h now describes some physical parameter of the model.

The NLS (+), $p > 1$ is often treated with variational methods (see [5], [41]) using the energy functional

$$V(u) = \int_\Omega \left[\frac{|\nabla u|^2}{2} + \lambda \frac{u^2}{2} - \frac{|u|^{p+1}}{p+1}. \right]$$

One common notion of solution is the one of *ground states*, i.e. minimizers of $V(u)$ w.r.t. a constraint (see [13] and [8] for details). The ground states usually have no zeros (see [18], [11]). We therefore restrict our attention to the non-negative solutions of DNLS.

It is known that the NLS type equations accompanied with the zero Dirichlet boundary condition on a bounded domain exhibits three different regimes depending on the parameter p (see [35], [6]). In the subcritical region $p \in (1, \frac{d}{d-2})$ all positive distributional solutions are smooth and bounded. For $p \geq \frac{d}{d-2}$ we begin to see so called *very weak solution*, basically regular distributions that also satisfy the equation in the distributional sense and may be unbounded. Moreover, for $p \geq \frac{d+2}{d-2}$ under some additional regularity assumptions on the solution and the domain the only non-negative solution is the trivial one (see [36], [20] Chapter 9 and [11], but also [16] for existence results on other domain types).

It is therefore of interest whether the DNLS in some sense reflects such behavior: do there exist a priori bounds as defined below or discrete solutions that blow up for $h \to 0$? See also [31], where the authors discuss how discrete solutions reflect the symmetry of the corresponding continuous solutions. Such question can also be important in numerical analysis to distinguish whether the unboundedness of a numerical approximation is due to the purely computational reasons or is an inherent property of the discretization (see [12]).

We consider the DNLS both on bounded box domains with the zero Dirichlet boundary condition (next 8 chapters) and on the whole space (last chapter). An essential feature of this thesis is the separation of a priori estimates and the convergence theory:

A Priori Estimates Let $h > 0$, $a_i, b_i \in \mathbb{R}_h$, $1 \leq i \leq d$ and define $\Omega := \prod_{i=1}^{d}(a_i, b_i)$ (so called admissible domains). We are interested in non-negative functions $u\colon \Omega_h \to \mathbb{R}$ defined on the bounded discrete set $\Omega_h := \overline{\Omega} \cap \mathbb{R}_h^d$ that satisfy the DNLS $(*)$

$$-\Delta_h u(x) + \lambda u(x) = |u(x)|^{p-1}u(x)$$

for $x \in \Omega \cap \mathbb{R}_h^d$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega \cap \mathbb{R}_h^d$. It is easy to show (see the proof of Theorem 10.9) that there exists a uniform (for all such solutions) bound that may be h-dependent. It has been proved in our joint paper [32] that the a priori estimate

$$\|u\|_{L^\infty(\Omega_h)} := \max_{x \in \Omega_h} |u(x)| \leq K$$

holds also for all $h > 0$ (for which Ω stays admissible) with some h-independent $K > 0$. The corresponding statement for $\Omega = \mathbb{R}^d$, Theorem 10.9, is one of two main results of the thesis. For the sake of completeness we also present the proofs from [32] for the one-dimensional case in Chapter 2.

Convergence theory In the rest of the work, we *assume* a priori estimates in L^∞-norm for non-negative solutions (as discussed above) of the (general) DNLS

$$-\Delta_h u = f(u) \qquad (**)$$

and derive the existence of a classical solution for the corresponding (general) NLS

$$-\Delta u = f(u) \tag{++}$$

by approximating it with solutions of DNLS (∗∗) (*convergence theory*), assuming only continuity of f. This is Theorem 9.10, the second main result of the thesis. The advantage of separating a priori bounds from the convergence theory consists in allowing the same convergence results for other types of nonlinearities than in (∗), provided the L^∞ bounds are already available.

1.2 Outline of the work

The following nine chapters of this thesis can be loosely divided in three parts.

The first part includes the next two chapters and provides motivation and some fundamental ideas for the rest of the work. In the second chapter a priori estimates for the *one-dimensional* NLS (++), Theorem 2.4, and DNLS (∗∗), Theorem 2.7, on bounded intervals for a class of nonlinearities with superlinear growth, that includes the case $f(u) = |u|^{p-1}u - \lambda u$ from (∗), are presented. These results are generalized to d-dimensional bounded boxes, $d \geq 1$, in our joint paper [32].

In the third chapter we deal with the convergence theory on one-dimensional bounded intervals. A uniform in space and grid spacing bound on the non-negative solutions of (∗∗) derived in the previous chapter yields the existence of a classical solution of (++) in Theorem 3.19; moreover, this solution $u \in C^2((a,b)) \cap C([a,b])$, $a < b$ can be approximated by solutions u_n, $u_n\colon [a,b] \cap \mathbb{R}_{h_n} \to \mathbb{R}$ of (++) uniformly on the h_n-grids of u_n as $h_n \to 0$, i.e.

$$\|u - u_n\|_{L^\infty([a,b]\cap\mathbb{R}_{h_n})} \xrightarrow{n\to\infty} 0.$$

The first two chapters provide the complete theory for bounded one-dimensional intervals and expose the main technical difficulty for the generalization to $d \geq 2$, namely the need for uniform convergence in Lemma 3.13. We used three ingredients to obtain it in $1d$ case. Firstly, we needed the discrete energy estimates from Lemma 3.15 that allows us to control the discrete $W^{1,2}$-norm, i.e. the quantity

$$\|u\|^2_{W_0^{1,2}([a,b]\cap\mathbb{R}_h)} := \sum_{x\in[a,b)\cap\mathbb{R}_h} \left|\frac{u(x+h)-u(x)}{h}\right|^2 h$$

of the solutions u, $u\colon [a,b] \cap \mathbb{R}_h \to \mathbb{R}$ of (∗∗) through the L^∞-norm of $f(u)$. Secondly, by Lemma 3.11 we have a way to extend a grid function to a continuous one that preserves $W^{1,2}$-norms, i.e. the continuous $W_0^{1,2}$-norm of the interpolant can be controlled by the discrete $W_0^{1,2}$-norm of the underlying grid function. Thirdly, we used that $W_0^{1,2}(I)$ embeds compactly into $C(I)$ on a bounded one-dimensional interval I, Lemma 3.13, allowing extraction of uniformly convergent subsequences from sequences which are bounded in $W_0^{1,2}$. Since this embedding is valid only for $d = 1$, in higher dimensions we have to use $W^{1,q}(\Omega)$ spaces which embed compactly into $C(\Omega)$ for $q > d$ and bounded $\Omega \subset \mathbb{R}^d$. This means we have to adapt the first two steps to the $W^{1,q}$ setting.

In the second part, Chapters 4 till 9 we carry out this program. In the forth chapter we show how to obtain a $W^{1,q}$ bound

$$\|u\|_{W_0^{1,q}(\Omega)} \leq C\|g\|_{L^r(\Omega)},$$

with $r > \frac{dq}{d+q}$ and $C > 0$ for the unique weak solution of the Poisson problem

$$\begin{aligned} -\Delta u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

on a bounded convex domain Ω. The main emphasis is to present a proof of this by no means original result that can be transferred into the discrete setting. At the heart of the proof lie Lemmas 4.9 and 4.13 that provide pointwise estimates for Green's function G of Ω and its first derivative:

$$\begin{aligned} |G(x,y)| &\leq K_1|x-y|^{2-d}, \quad \forall (x \neq y) \in \Omega. \\ |\partial_{x_i} G(x,y)| &\leq K_2|x-y|^{1-d}, \quad \forall (x \neq y) \in \Omega,\ 1 \leq i \leq d \end{aligned}$$

with some $K_1, K_2 > 0$. We spend two next chapters obtaining analogous estimates for the discrete Green's function.

We start in Chapter 5 with the basic definition of discrete objects (discrete derivatives, discrete Green's function etc) and the crucial Theorems 5.31 and 5.37 taken from [10]. Exactly as one can control the value of a harmonic function in the center of a ball through the values on the boundary, those theorems allow us to control discrete derivatives $D_i^+ u$ and $D_i^{\pm} u$ of a grid function u, $u\colon \Omega_h \to \mathbb{R}$ in the center of a discrete cube $\Omega_h := \overline{\Omega} \cap \mathbb{R}_h^d$, $\Omega = (-R, R)^d$, $R > h$ through the values of u on the boundary $\partial\Omega \cap \mathbb{R}_h^d$ and the values of $\Delta_h u$ in $\Omega \cap \mathbb{R}_h^d$.

We then proceed in Chapter 6 with pointwise estimates for the discrete Green's function, Lemma 6.3 and its discrete derivatives, Lemma 6.9, analogous to the ones in Chapter 4

$$\begin{aligned} G(x,y) &\leq K_1[|x-y|^2 + \gamma h^2]^{\frac{2-d}{2}} \quad \forall x \in \overline{\Omega}_h, \forall y \in \Omega_h \\ |D_{x_i}^+ G(x,y)| &\leq K_2[|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}}, \quad \forall x \in \Omega_h \cup \partial_i^- \Omega_h, \forall y \in \Omega_h, 1 \leq i \leq d, \end{aligned}$$

where $K_1, K_2, \gamma > 0$ are h-independent. This is the most technical part of the thesis.

We reap the fruits of our preparations in Chapter 7, obtaining the discrete analogue to the $W^{1,q}$-bound of Chapter 4 in Theorem 7.9. As a short intermezzo before the final step we present in Chapter 8 an interpolation method that provides control for the $W^{1,p}$ norm of the continuous interpolant through the discrete $W^{1,p}$ norm of the underlying grid function, Theorem 8.13. We conclude this part with Chapter 9 where the convergence theory from Chapter 3 is carried over to the higher dimensions, yielding Theorem 9.10:

Theorem
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, $h_n \overset{n\to\infty}{\to} 0$ and a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of discrete problems $(**)$ with $h := h_n$, $u_n|_{\partial\Omega_{h_n}} = 0$ and a positive $C > 0$ such that $\|u_n\|_{L^\infty(\overline{\Omega}_{h_n})} \le C$, $\forall n \in \mathbb{N}$. Then, there exists a (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ and a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of $(**)$, $u|_{\partial\Omega} = 0$ such that

$$\|u_n - u\|_{L^\infty(\overline{\Omega}_{h_n})} \to 0, \quad \text{as } n \to \infty.$$

The third and the last part of the thesis consists of only one chapter, Chapter 10. Here we consider the equation $(*)$ on the whole grid $\mathbb{R}^d_h$. We derive an uniform in space and grid spacing bound for non-negative solutions of $(*)$ in Theorem 10.9:

Theorem
There exists $C > 0$ such that $\|u\|_{L^\infty(\mathbb{R}^d_h)} \le C$ for all non-negative solutions of

$$-\Delta_h u(x) + \lambda u(x) = u^p(x), \quad x \in \mathbb{R}^d_h$$

uniformly in $h \in (0, 1]$ for fixed $\lambda > 0$, $p \in (1, \frac{d}{d-2})$.

By analogy with the continuous case (see [30]) we believe that $\frac{d}{d-2}$ is sharp in the sense that for every $p \in (\frac{d}{d-2}, \frac{d}{d-2} + \varepsilon)$ with sufficiently small $\varepsilon > 0$ there is a sequence $(u_n)_{n\in\mathbb{N}}$ of non-negative solutions with $\|u_n\|_{L^\infty(\mathbb{R}^d_{h_n})} \to \infty$ for $n \to \infty$.

The proof is based on a rescaling argument and yields a nontrivial solution either of the continuous equation

$$-\Delta u(x) = u^p(x), \quad x \in \mathbb{R}^d$$

or of the discrete equation

$$-\Delta_h u(x) = u^p(x), \quad x \in \mathbb{R}^d_h.$$

In the first case we obtain a contradiction to the nonlinear Liouville theorem 10.2 from [22] that prohibits any nontrivial solution. The corresponding *discrete* nonlinear Liouville theorem 10.8 was obtained in this thesis using a refinement of the discrete Hardy's Inequality 10.7 and the discrete version of Agmon-Allegretto-Piepenbrink principle (see the proof of 10.8).

Finally, using the convergence theory on bounded discrete domains from Chapter 9 we can convert the a priori estimates from Theorem 10.9 into the convergence result on bounded subsets of $\mathbb{R}^d$ in Corollary 10.11.

Chapter 2

A Priori Estimates for One-Dimensional Model Problem

2.1 Model Estimate for Continuous Case

2.1 Motivation (One-dimensional a priori estimate)**.**
The purpose of this chapter is to present the one-dimensional a priori estimates in the continuous, Theorem 2.4, and in the discrete setting, Theorem 2.7. In both cases we consider the class of nonlinearites with superlinear growth (see assumptions on f in the mentioned theorems) that include the classical Schrödinger nonlinearity $f(x,u) = |u|^{p-1}u - \lambda u$, $p > 1$, $\lambda > 0$ discussed in the Introduction. We begin with the elliptic comparison in Lemma 2.3 that extends the (elliptic) maximum principle (see [23] and Lemma 5.15 in the discrete setting) from $-u''$ to $-u'' - \lambda u$ with λ below the first eigenvalue.

2.2 Remark (Publication)**.**
The discrete a priori estimate from Theorem 2.7 together with Lemma 2.6 was already published in [32].

2.3 Lemma (Elliptic comparison)**.**
Let $w \in C^2((-L,L)) \cap C([-L,L])$ satisfy

$$\begin{aligned} -w''(x) &\leq \lambda_0 w(x), \quad x \in (-L,L), \\ w(-L) &= w(L) = 0 \end{aligned} \tag{$*$}$$

with $0 < \lambda_0 < \lambda_1$, where $\lambda_1 = \lambda_1([-L,L]) = \left(\frac{\pi}{2L}\right)^2$ is the first eigenvalue of

$$\begin{aligned} -u''(x) &= \lambda u(x), \quad x \in (-L,L), \\ u(-L) &= u(L) = 0. \end{aligned}$$

Then $w \leq 0$ on $[-L,L]$.

▶

We show that

$$w_+(x) := \max\{w(x), 0\} \in W_0^{1,2}((-L, L))$$

for $w \in C^2((-L, L)) \cap C([-L, L])$. To see this, define $v(x) := \chi_{w>0}\, w'(x)$. By monotonicity $v \in L^2((-L, L))$. Thus, it is sufficient to show

$$\int_{-L}^{L} w_+(x)\varphi'(x)\,\mathbf{d}\,x = -\int_{-L}^{L} v(x)\varphi(x)\,\mathbf{d}\,x$$

for all $\varphi \in C_0^\infty((-L, L))$.
Since $I_+ := \{x \in (-L, L) \mid w(x) > 0\}$ is open by continuity of w, we have $I_+ = \sqcup_{i=1}^\infty I_i$ with some open disjunct intervals I_i, $i \in \mathbb{N}$. We define $w_k := \chi_{\sqcup_{i=1}^k I_i} w$, $v_k := \chi_{\sqcup_{i=1}^k I_i} w'$ with $w_k \to w_+$, $v_k \to v$ pointwise, obtaining

$$\begin{aligned}\int_{-L}^{L} w_+(x)\varphi'(x)\,\mathbf{d}\,x &= \int_{I_+} w(x)\varphi'(x)\,\mathbf{d}\,x = \int_{I_+} \lim_{k\to\infty} w_k(x)\cdot\varphi'(x)\,\mathbf{d}\,x \\ &= \sum_{i=1}^{\infty}\int_{I_i} w(x)\varphi'(x)\,\mathbf{d}\,x = -\sum_{i=1}^{\infty}\int_{I_i} w'(x)\varphi(x)\,\mathbf{d}\,x \\ &= -\lim_{k\to\infty}\int_{-L}^{L} v_k(x)\varphi(x)\,\mathbf{d}\,x = -\int_{-L}^{L} v(x)\varphi(x)\,\mathbf{d}\,x\end{aligned}$$

by dominated convergence and $w(x) = 0$ for all $x \in \partial I_i$, $i \in \mathbb{N}$.
The Poincaré inequality with the optimal constant yields

$$\int_{-L}^{L} |u(x)|^2\,\mathbf{d}\,x \le \frac{1}{\lambda_1}\int_{-L}^{L} |u'(x)|^2\,\mathbf{d}\,x$$

for all $u \in W_0^{1,2}([-L, L])$.
Multiplying $(*)$ with $w_+(x)$, integrating over $(-L, L)$, we obtain

$$\int_{-L}^{L} -w''(x)w_+(x)\,\mathbf{d}\,x \le \lambda_0\int_{-L}^{L} w(x)w_+(x)\,\mathbf{d}\,x.$$

After integrating by parts using dominated convergence and applying the Poincaré inequality we get

$$\int_{-L}^{L} |w_+'(x)|^2\,\mathbf{d}\,x \le \lambda_0\int_{-L}^{L} |w_+(x)|^2\,\mathbf{d}\,x \le \frac{\lambda_0}{\lambda_1}\int_{-L}^{L} |w_+'(x)|^2\,\mathbf{d}\,x.$$

Since by assumption $\lambda_0 < \lambda_1$,

$$\int_{-L}^{L} |w_+'(x)|^2\,\mathbf{d}\,x = 0$$

and the Poincaré inequality yields

$$\int_{-L}^{L} |w_+(x)|^2 \,\mathbf{d}\, x = 0.$$

This implies $w_+ \equiv 0$, i.e. $w \leq 0$.
◀

2.4 Theorem (Continuous Model Estimate)**.**
Let $K > 0$, $L > 0$ and let

(1°) $g\colon [0, +\infty) \to \mathbb{R}$ be continuous, positive and strictly increasing with

(2°) $\lim_{s\to\infty} \frac{s}{\sqrt{G(s)}} = 0$ for $G(s) := \int_K^s g(t)\,\mathbf{d}\,t$;

(3°) $f\colon [-L, L] \times \mathbb{R} \to \mathbb{R}$ be continuous with

$$f(x, s) \geq g(s) > 0, \quad \forall s \geq K,\ \forall x \in [-L, L].$$

Then, there exists $\overline{M} = \overline{M}(g, L, K)$ with $\|u\|_\infty \leq \overline{M}$ uniformly for all classical non-negative solutions of

$$\begin{aligned} -u''(x) &= f(x, u(x)), \quad x \in (-L, L), \\ u(-L) &= u(L) = 0. \end{aligned} \tag{$*$}$$

▶
⟨*1*⟩ *Preliminary observation*
From (1°) we have $G(s) \leq (s - K)g(s)$, yielding

$$\frac{s^2}{G(s)} \geq \frac{s \cdot s}{(s-K)g(s)} \geq \frac{s}{g(s)}$$

for $s > K$. By (2°), the l.h.s of this inequality converges to zero and since the r.h.s. is positive, we have

$$\lim_{s\to\infty} \frac{s}{g(s)} = 0.$$

This implies that g has at least linear growth, i.e. there exists $K_1 > K$ such that

$$g(s) \geq s, \quad \forall s \geq K_1.$$

It follows that

$$g(s) \geq \lambda_0 s, \quad \forall s \geq K_1,$$

with every $0 < \lambda_0 \leq 1$. Let $\lambda_0 \in (0, \min\{1, \lambda_1/4\})$ be arbitrary, but fixed, where $\lambda_1 = \left(\frac{\pi}{2L}\right)^2$ is the first eigenvalue of

$$\begin{aligned} -u''(x) &= \lambda u(x), \quad x \in (-L, L) \\ u(-L) &= u(L) = 0. \end{aligned}$$

Since g is continuous, we can set

$$A_g := \max_{s \in [0, K_1]} (\lambda_0 s - g(s))_+ \geq 0,$$

obtaining

$$\begin{aligned} g(s) &\geq \lambda_0 s \geq \lambda_0 s - A_g, \quad && \forall s \geq K_1, \\ g(s) &\geq \lambda_0 s - A_g, && \forall s \in [0, K_1], \end{aligned}$$

i.e.

$$g(s) \geq \lambda_0 s - A_g, \quad \forall s \geq 0.$$

Since f is continuous on $[-L, L] \times [0, K_1]$, we repeat the argument, setting

$$A_f := \max_{\substack{s \in [0, K_1] \\ x \in [-L, L]}} (\lambda_0 s - A_g - f(x, s))_+ \geq 0.$$

Taking into account (3°), we now analogously have

$$f(x, s) \geq \lambda_0 s - A, \quad \forall s \geq 0, \ \forall x \in [-L, L] \tag{+}$$

with $A := A_f + A_g$.

⟨*2*⟩ *Proof idea*

We have at our disposal two estimates for f. Whereas the linear estimate (+) holds without any restriction on s, the superlinear estimate (3°) is valid only for sufficiently large values of s (correspondingly, for sufficiently large values of u). Let M be the maximum of a non-negative solution u. We can assume M to be sufficiently large, $M > K$, otherwise there is nothing to prove. By continuity of u we have a closed interval, a neighborhood of the maximum point, where $u \geq K$ and (3°) still holds. Using this condition it is possible to obtain an estimate from above for $u'(x_1)$ at the right end of this interval, point x_1. Then, using (+) and a comparison argument on $[x_1, L]$ we can obtain for $u'(x_1)$ an estimate from below. Comparing these two estimates we will obtain an upper bound for M.

⟨*3*⟩ *Auxiliary estimate*

We want to prove that

$$\int_{K/\mu}^{1} \frac{\mathbf{d}\, t}{\sqrt{1 - \frac{G(\mu t)}{G(\mu)}}} \leq 2 \tag{†}$$

for every $\mu \geq K \geq 0$. Since g is positive and strictly increasing, we have

$$G(\mu t) = \int_K^{\mu t} g(s)\,\mathbf{d}\,s = t\int_{K/t}^{\mu} g(\tau t)\,\mathbf{d}\,\tau \leq t\int_K^{\mu} g(\tau t)\,\mathbf{d}\,\tau$$
$$\leq t\int_K^{\mu} g(\tau)\,\mathbf{d}\,\tau = tG(\mu)$$

for every $t \in (0,1]$. We therefore obtain

$$\frac{G(\mu t)}{G(\mu)} \leq t,$$
$$\frac{1}{\sqrt{1-\frac{G(\mu t)}{G(\mu)}}} \leq \frac{1}{\sqrt{1-t}}, \quad \forall t \in (0,1).$$

Now the claim follows from monotonicity:

$$\int_{\frac{K}{\mu}}^{1} \frac{1}{\sqrt{1-\frac{G(\mu t)}{G(\mu)}}}\,\mathbf{d}\,t \leq \int_{\frac{K}{\mu}}^{1} \frac{1}{\sqrt{1-t}}\,\mathbf{d}\,t \leq \int_0^1 \frac{1}{\sqrt{1-t}}\,\mathbf{d}\,t = -2\sqrt{1-t}\Big|_0^1 = 2.$$

⟨*4*⟩ *Descending from maximum*

Let u be an arbitrary, but fixed non-negative classical solution of $(*)$ with $\|u\|_\infty =: M$. Without loss of generality we may assume $M > K$.

We also may assume that $M = u(x_0)$ with some $x_0 \in (-L, 0]$. If this is not the case, we replace u with v, $v(x) := u(-x)$ which is a solution to $(*)$ with r.h.s $\tilde{f}(x, v(x)) := f(-x, v(x))$. For this r.h.s. both $(3°)$ and $(+)$ stay valid due to uniformity in x and we then use $\|v\|_\infty = \|u\|_\infty$.

Furthermore, we denote $x_1 := \min\{x \in (x_0, L) \mid u(x) = K\}$, $x_1 > x_0$.
Now $u(x) \geq K$ for $x \in [x_0, x_1]$. Using $(3°)$ we first obtain that

$$-u''(x) = f(x, u(x)) \geq g(u(x)) > 0, \quad \forall x \in [x_0, x_1]$$

i.e., $u(x)$ is strictly concave and, since $u(x_0) = M$, strictly decreasing on $[x_0, x_1]$, implying that

$$u'(x) \leq 0, \quad \forall x \in [x_0, x_1].$$

Multiplying $(*)$ with $u'(x) \leq 0$, on $[x_0, x_1]$ we also get

$$-\frac{(u'(x)^2)'}{2} = -u''(x)u'(x) = f(x, u(x))u'(x) \leq g(u(x))u'(x).$$

Integrating this inequality over $[x_0, x]$, $x_0 < x \le x_1$ we obtain

$$-\int_{x_0}^{x} \frac{(u'(s)^2)'}{2} \, \mathbf{d}\, s \le \int_{x_0}^{x} g(u(s))u'(s) \, \mathbf{d}\, s,$$
$$-\frac{u'(x)^2}{2} + \underbrace{\frac{u'(x_0)^2}{2}}_{=0} \le G(u(x)) - G(u(x_0)),$$
$$u'(x)^2 \ge 2G(M) - 2G(u(x)).$$

Since $u'(x_1) \le 0$, in particular this implies

$$u'(x_1) \le -\sqrt{2G(M) - 2G(K)} = -\sqrt{2G(M)} \qquad (\ddagger)$$

and, more generally, from $u'(x) \le 0$ and $G(u(x)) < G(M)$ for $x \in (x_0, x_1]$ follows

$$u'(x) \le -\sqrt{2G(M) - 2G(u(x))}$$
$$\frac{u'(x)}{\sqrt{2G(M) - 2G(u(x))}} \le -1.$$

We now integrate over $[x_0, x_1]$

$$\int_{x_0}^{x_1} \frac{u'(s)}{\sqrt{2G(M) - 2G(u(s))}} \, \mathbf{d}\, s \le -\int_{x_0}^{x_1} 1 \, \mathbf{d}\, s,$$
$$\int_{u(x_0)}^{u(x_1)} \frac{1}{\sqrt{2G(M) - 2G(s)}} \, \mathbf{d}\, s \le (x_0 - x_1),$$
$$\int_{K}^{M} \frac{1}{\sqrt{2G(M) - 2G(s)}} \, \mathbf{d}\, s \ge x_1 - x_0,$$

obtaining

$$x_1 - x_0 \le \int_K^M \frac{1}{\sqrt{2G(M) - 2G(s)}} \, \mathbf{d}\, s = \frac{1}{\sqrt{2G(M)}} \int_K^M \frac{1}{\sqrt{1 - \frac{G(s)}{G(M)}}} \, \mathbf{d}\, s$$
$$= \frac{M}{\sqrt{2G(M)}} \int_{K/M}^{1} \frac{1}{\sqrt{1 - \frac{G(Mt)}{G(M)}}} \, \mathbf{d}\, t \le 2\frac{M}{\sqrt{2G(M)}},$$

where we have used the auxiliary estimate (†). Taking (2°) into account, we see that $x_1 \to x_0$ as $M \to \infty$ uniformly for all solutions of $(*)$. We denote

$$M_1 := \min \left\{ \mu > K \mid \frac{2\lambda}{\sqrt{2G(\lambda)}} \le \frac{L}{2}, \forall \lambda \ge \mu \right\}$$

and assume $M > M_1$, obtaining $x_1 - x_0 \leq \frac{L}{2}$. Since $x_0 \leq 0$, the length of the interval $[x_1, L]$ has a positive lower bound,

$$L - x_1 \geq \frac{L}{2}. \tag{\#}$$

⟨5⟩ *Comparison function*
We define $v\colon [x_1, L] \to \mathbb{R}$ as the solution of the following linear BVP

$$\begin{aligned} -v''(x) &= \lambda_0 v(x) - A, \\ v(x_1) &= K, \\ v(L) &= 0 \end{aligned}$$

with λ_0 as in (+). Since u satisfies

$$\begin{aligned} -u''(x) &= f(x, u(x)) \geq \lambda_0 u(x) - A, \\ u(x_1) &= K, \\ u(L) &= 0, \end{aligned}$$

we get for $w := v - u$

$$\begin{aligned} -w''(x) &\leq \lambda_0 w(x), \\ w(x_1) &= w(L) = 0, \end{aligned}$$

and since $0 < \lambda_0 < \lambda_1([-L, L]) \leq \lambda_1([x_1, L])$, Lemma 2.3 yields $u \geq v$ on $[x_1, L]$. From $u(x_1) = v(x_1)$ we get

$$\tfrac{1}{h}[u(x_1 + h) - u(x)] \geq \tfrac{1}{h}[v(x_1 + h) - v(x)], \quad \forall h \in (0, L - x_1)$$

and consequently, $u'(x_1) \geq v'(x_1)$.

We have the explicit representation for v

$$v(x) = \alpha \cos(\sqrt{\lambda_0}(x - x_1)) + \beta \sin(\sqrt{\lambda_0}(x - x_1)) + \frac{A}{\lambda_0}.$$

From $v(x_1) = K$ we get $\alpha + \frac{A}{\lambda_0} = K$, implying $\alpha = K - \frac{A}{\lambda_0}$. From $v(L) = 0$ we get

$$\alpha \cos(\sqrt{\lambda_0}(L - x_1)) + \beta \sin(\sqrt{\lambda_0}(L - x_1)) + \frac{A}{\lambda_0} = 0.$$

We have $\lambda_0 < \frac{\lambda_1}{4} = \left(\frac{\pi}{4L}\right)^2$, implying $\frac{\pi}{2} > 2\sqrt{\lambda_0}L > \sqrt{\lambda_0}(L - x_1)$; so, we obtain

$$\beta = \frac{-\frac{A}{\lambda_0} - (K - \frac{A}{\lambda_0}) \cos\left(\sqrt{\lambda_0}(L - x_1)\right)}{\sin\left(\sqrt{\lambda_0}(L - x_1)\right)}.$$

Since $v'(x_1) = \sqrt{\lambda_0}\beta$, taking into account (‡) we obtain

$$\sqrt{\lambda_0}\frac{-\frac{A}{\lambda_0} - (K - \frac{A}{\lambda_0})\cos\left(\sqrt{\lambda_0}(L - x_1)\right)}{\sin\left(\sqrt{\lambda_0}(L - x_1)\right)} = v'(x_1) \leq u'(x_1) \leq -\sqrt{2G(M)},$$
$$\sqrt{\lambda_0}\frac{\frac{A}{\lambda_0}(1 - \cos\left(\sqrt{\lambda_0}(L - x_1)\right)) + K\cos\left(\sqrt{\lambda_0}(L - x_1)\right)}{\sin\left(\sqrt{\lambda_0}(L - x_1)\right)} \geq \sqrt{2G(M)},$$

Using (#), from $\sqrt{\lambda_0}\frac{L}{2} \leq \sqrt{\lambda_0}(L - x_1) \leq 2\sqrt{\lambda_0}L \leq \frac{\pi}{2}$ we have

$$\sin\left(\sqrt{\lambda_0}(L - x_1)\right) \geq \sin\left(\sqrt{\lambda_0}\tfrac{L}{2}\right),$$
$$0 \leq \cos\left(\sqrt{\lambda_0}(L - x_1)\right) \leq \cos\left(\sqrt{\lambda_0}\tfrac{L}{2}\right),$$

implying that the l.h.s. of the previous inequality, which is an upper bound for $\sqrt{2G(M)}$, depends only on the assumption constants, and we have

$$G(M) \leq \frac{\lambda_0}{2}\left(\frac{\frac{A}{\lambda_0} + K\cos\left(\sqrt{\lambda_0}\frac{L}{2}\right)}{\sin\left(\sqrt{\lambda_0}\frac{L}{2}\right)}\right)^2,$$
$$M \leq M_2 := \max\left\{\mu \geq K \mid G(\mu) \leq \frac{\lambda_0}{2}\left(\frac{\frac{A}{\lambda_0} + K\cos\left(\sqrt{\lambda_0}\frac{L}{2}\right)}{\sin\left(\sqrt{\lambda_0}\frac{L}{2}\right)}\right)^2\right\}.$$

Putting everything together we get

$$M \leq \max\{M_1, M_2\} =: \overline{M}.$$

◀

2.2 Model Estimate for the Discrete Case

2.5 Notation.
For $h > 0$ denote

$$\mathcal{G}_h := \{zh \mid z \in \mathbb{Z}\},$$
$$(a, b)_h := (a, b) \cap \mathcal{G}_h,$$
$$[a, b]_h := [a, b] \cap \mathcal{G}_h$$

and

$$D_h^+ f(x) := \frac{f(x + h) - f(x)}{h},$$
$$D_h^- f(x) := \frac{f(x) - f(x - h)}{h},$$

as long as r.h.s. are well-defined for $f\colon \Omega_h \subset \mathcal{G}_h \to \mathbb{R}$.

2.6 Lemma (Discrete Elliptic Comparison)**.**
Let $h, L > 0$ with $\frac{L}{h} \in \mathbb{N}$ and let $w\colon [-L, L]_h \to \mathbb{R}$ satisfy

$$\begin{aligned} -D_h^+ D_h^- w(x) &\leq \lambda_0^h w(x), \quad x \in (-L, L)_h \\ w(-L) = w(L) &= 0 \end{aligned} \tag{$*$}$$

with $0 < \lambda_0^h < \lambda_1^h$, where $\lambda_1^h = \frac{4}{h^2}\sin^2\left(\frac{\pi h}{4L}\right)$ is the first eigenvalue of

$$\begin{aligned} -D_h^+ D_h^- u(x) &= \lambda u(x), \quad x \in (-L, L)_h, \\ w(-L) = w(L) &= 0. \end{aligned}$$

Then, $w(x) \leq 0$, $x \in [-L, L]_h$.

▶
We multiply $(*)$ with w_+, sum up over $(-L, L)_h$ and exploit partial summation, obtaining

$$\begin{aligned} I := \lambda_0^h \sum_{x\in(-L,L)_h} w_+^2(x) &\geq - \sum_{x\in(-L,L)_h} D_h^- D_h^+ w(x) \cdot w_+(x) \\ &= \sum_{x\in(-L,L)_h} D_h^+ w(x) \cdot D_h^+ w_+(x) + \tfrac{1}{h}[D_h^+ w(-L) w_+(h) - D_h^+ w(L-h) w_+(L)] \\ &= \sum_{x\in[-L,L)_h} D_h^+ w(x) \cdot D_h^+ w_+(x) \geq \sum_{x\in[-L,L)_h} D_h^+ w_+(x) \cdot D_h^+ w_+(x). \end{aligned}$$

To see the last inequality let $x \in [-L, L)_h$. If $w(x), w(x+h) \geq 0$ or $w(x), w(x+h) \leq 0$ then

$$D_h^+ w(x) D_h^+ w_+(x) = D_h^+ w_+(x) D_h^+ w_+(x).$$

If $w(x) \leq 0 \leq w(x+h)$ then $D_h^+ w_+(x) \geq 0$ and $D_h^+ w(x) \geq D_h^+ w_+(x)$, implying

$$D_h^+ w(x) D_h^+ w_+(x) \geq D_h^+ w_+(x) D_h^+ w_+(x).$$

Finally, if $w(x) \geq 0 \geq w(x+h)$ then $D_h^+ w_+(x) \leq 0$ and $D_h^+ w(x) \leq D_h^+ w_+(x)$, also implying

$$D_h^+ w(x) D_h^+ w_+(x) \geq D_h^+ w_+(x) D_h^+ w_+(x).$$

Using the discrete Poincaré inequality, we get

$$0 \leq \sum_{x\in[-L,L)_h} \left(D_h^+ w_+(x)\right)^2 \leq \lambda_0^h \sum_{x\in(-L,L)_h} w_+^2(x) \leq \frac{\lambda_0^h}{\lambda_1^h} \sum_{x\in[-L,L)_h} \left(D_h^+ w_+(x)\right)^2,$$

and since $w_+(-L) = 0$ consequently $w_+ \equiv 0$.
◀

2.7 Theorem (Discrete Model Estimate)**.**
Let $K, L > 0$ and let

(1°) $g\colon [0,+\infty) \to \mathbb{R}$ be continuous, positive and strictly increasing with

(2°) $\lim_{s\to\infty} \frac{s}{\sqrt{G(s)}} = 0$ for $G(s) := \int_K^s g(t)\,\mathbf{d}\,t$;

(3°) $f\colon [-L,L]\times\mathbb{R} \to \mathbb{R}$ be continuous with

$$f(x,s) \geq g(s) > 0, \quad \forall s \geq K,\ \forall x \in [-L,L].$$

Now let $h > 0$, $\frac{L}{h} \in \mathbb{N}$, $L \geq 4h$ and denote

$$\begin{aligned}\mathring{\Omega}_h &:= (a,b)_h,\\ \Omega_h &:= [a,b]_h.\end{aligned}$$

Then every non-negative solution $u\colon \Omega_h \to [0,+\infty)$ of

$$\begin{aligned}-\frac{u(x+h)-2u(x)+u(x-h)}{h^2} &= f(x,u(x)), \quad x \in \mathring{\Omega}_h\\ u(-L) = u(+L) &= 0\end{aligned} \tag{$*$}$$

satisfies

$$\|u\|_\infty \leq \overline{M}$$

with some $\overline{M} > 0$ independent of h.
▶
⟨*1*⟩ *Preliminary observation*
From (1°) we have $G(s) \leq (s-K)g(s)$, yielding

$$\frac{s^2}{G(s)} \geq \frac{s\cdot s}{(s-K)g(s)} \geq \frac{s}{g(s)}$$

for all $s > K$. By (2°), the lhs of this inequality converges to zero and since the rhs is positive, we have

$$\lim_{s\to\infty} \frac{s}{g(s)} = 0.$$

This implies that g has arbitrary linear growth, i.e. for every $\lambda > 0$ there exists $K_\lambda \geq K$ such that

$$g(s) \geq \lambda s, \quad \forall s \geq K_\lambda.$$

We choose $\lambda := \lambda_0 := \left(\frac{\pi}{L}\right)^2$, $K_1 := K_{\lambda_0}$, yielding

$$g(s) \geq \lambda_0 s, \quad \forall s \geq K_1.$$

Since g is continuous, we can set

$$A_g := \max_{s\in[0,K_1]} (\lambda_0 s - g(s))_+ \geq 0,$$

obtaining

$$g(s) \geq \lambda_0 s \geq \lambda_0 s - A_g, \quad \forall s \geq K_1,$$
$$g(s) \geq \lambda_0 s - A_g, \quad \forall s \in [0, K_1],$$

i.e.

$$g(s) \geq \lambda_0 s - A_g, \quad \forall s \geq 0.$$

Since f is continuous on $[-L, L] \times [0, K_1]$, we repeat the argument, setting

$$A_f := \max_{\substack{s\in[0,K_1]\\ x\in[-L,L]}} (\lambda_0 s - A_g - f(x,s))_+ \geq 0.$$

Taking into account (3°), we now analogously have

$$f(x,s) \geq \lambda_0 s - A, \quad \forall s \geq 0,\ \forall x \in [-L, L] \tag{+}$$

with $A := A_f + A_g$.

⟨*2*⟩ *Equivalent estimation value*
Let u be an arbitrary but fixed non-negative solution of $(*)$. We denote $M := \|u\|_\infty$ and assume without loss of generality that $M > K$. Since our problem is symmetrical w.r.t axis reflection, we can assume that $M = u(x_0)$ with $x_0 \leq 0$. Both $R := u(x_0 + h)$ and $u(x_0 + 2h)$ are well-defined due to $L \geq 4h$ and from (+) we obtain

$$\frac{2R - M + 0}{h^2} \geq \frac{2u(x_0+h) - u(x_0) - u(x_0+2h)}{h^2} = f(R) \geq \lambda_0 R - A \geq -A,$$
$$M \leq 2R + Ah^2 \leq 2R + A\frac{L^2}{16}$$

i.e. a bound for R implies a bound for M. It is therefore sufficient to find a bound for R and without loss of generality we assume $R > K$.

⟨*3*⟩ *Auxiliary inequality*
We show the following auxiliary estimate (cf. ⟨*3*⟩ in the proof of Theorem 2.4)

$$\int_0^R \frac{\mathbf{d}\, s}{\sqrt{G(R) - G(s)}} \leq \frac{2R}{\sqrt{G(R)}}. \tag{†}$$

Since g is positive and strictly increasing we get

$$G(Rt) = \int_K^{Rt} g(s)\,\mathbf{d}\, s = t\int_{K/t}^{R} g(t\tau)\,\mathbf{d}\,\tau \leq t\int_K^R g(\tau)\,\mathbf{d}\,\tau = tG(R) > 0$$

for every $t \in (0,1)$. This yields

$$\frac{G(Rt)}{G(R)} \leq t,$$
$$\frac{1}{\sqrt{1-\frac{G(Rt)}{G(R)}}} \leq \frac{1}{\sqrt{1-t}}$$

for the same values of t and we get

$$\int_0^R \frac{\mathbf{d}\, s}{\sqrt{G(R)-G(s)}} = R\int_0^1 \frac{\mathbf{d}\, t}{\sqrt{G(R)-G(Rt)}} = \frac{R}{\sqrt{G(R)}}\int_0^1 \frac{\mathbf{d}\, t}{\sqrt{1-\frac{G(Rt)}{G(R)}}}$$
$$\leq \frac{R}{\sqrt{G(R)}}\int_0^1 \frac{\mathbf{d}\, t}{\sqrt{1-t}} = \frac{2R}{\sqrt{G(R)}}.$$

⟨*4*⟩ *Descending from maximum*

Since $0 < K < M$, there exists $x_1 \in [x_0, L-h]_h$ such that $u \geq K$ on $[x_0, x_1]_h$ and $u(x_1+h) < K$. Since $R > K$, we have $x_1 \neq x_0$, i.e. $[x_0+h, x_1] \neq \emptyset$. Using (1°) and (3°) we see that

$$D_h^+ D_h^- u(x) = -f(u(x)) \leq -g(u(x)) < 0,$$
$$D_h^- u(x+h) - D^- u(x) < 0,$$
$$D_h^- u(x+h) < D_h^- u(x)$$

as long as $x \in [x_0, x_1]_h$. It follows inductively that

$$D_h^+ u(x) = D_h^- u(x+h) < D_h^- u(x) < \ldots < D_h^- u(x_0+h)$$
$$= D^+ u(x_0) = \frac{u(x_0+h)-M}{h} \leq 0$$
$$\Rightarrow u(x+h) < u(x)$$

as long as $x \in [x_0+h, x_1]_h$.

⟨*5*⟩ *Derivative bound at the end of the interval*

We now want to derive an upper bound for the forward difference of u at x_1. For $x \in [x_0+h, x_1]_h$ we have on one hand, using (3°)

$$-D_h^+\left((D_h^- u(x))^2\right) = -D_h^+(D_h^- u(x) \cdot D_h^- u(x))$$
$$= -D_h^- u(x+h) \cdot D_h^+ D_h^- u(x) - D_h^- u(x) \cdot D_h^+ D_h^- u(x)$$
$$= \underbrace{-D_h^+ D_h^- u(x)}_{\geq g(u(x))>0}(\underbrace{D_h^- u(x+h)}_{<0} + \underbrace{D_h^- u(x)}_{\leq 0})$$
$$\leq g(u(x)) D_h^- u(x+h) = g(u(x)) D_h^+ u(x)$$

and on the other hand with some $\bar{u} \in [u(x+h), u(x)]$

$$D_h^+ G(u(x)) = \frac{1}{h}\left[G(u(x+h)) - G(u(x))\right] = \frac{1}{h}\int_{u(x)}^{u(x+h)} g(t)\,\mathbf{d}\,t$$
$$= g(\bar{u})\frac{u(x+h)-u(x)}{h} \geq g(u(x))D_h^+ u(x),$$

since $\bar{u} \leq u(x)$ implies $g(\bar{u}) \leq g(u(x))$, and $D_h^+ u(x) < 0$. Together, we obtain

$$-D_h^+\left((D_h^- u(x))^2\right) \leq D_h^+ G(u(x))$$

on $x \in [x_0+h, x_1]_h$ and summing up over $[x_0+h, x]_h$, $x \leq x_1$ we also get

$$\begin{aligned}
-\left(D_h^- u(x+h)\right)^2 &\leq -\left(D_h^- u(x+h)\right)^2 + \left(D_h^- u(x_0+h)\right)^2 \\
&= -\sum_{z\in[x_0+h,x]_h} D_h^+\left[\left(D_h^- u(z)\right)^2\right] h \leq \sum_{z\in[x_0+h,x]_h} D_h^+ G(u(z))h \\
&= G(u(x+h)) - G(u(x_0+h)) \\
\Rightarrow \underbrace{(D_h^+ u(x))}_{<0}{}^2 &\geq \underbrace{G(R) - G(u(x+h))}_{>0}
\end{aligned}$$

and finally

$$D_h^+ u(x) \leq -\sqrt{G(R) - G(u(x+h))} < 0. \tag{‡}$$

In particular, this inequality holds for $x_1 \in [x_0+h, x_1]_h$.

⟨6⟩ *Shrinking interval*

We now want to show that $[x_0, x_1]$ shrinks as R goes to infinity. More precisely, we want to show that

$$x_1 - x_0 \leq \frac{2R}{\sqrt{G(R)}}.$$

Consider

$$\varkappa(t) := \int_t^R \frac{\mathbf{d}\,t}{\sqrt{G(R)-G(s)}}\,\mathbf{d}\,s, \quad t \in [0, R],$$
$$\varkappa'(t) := -\frac{1}{\sqrt{G(R)-G(t)}} < 0, \qquad t \in [0, R)$$

with $\varkappa$ being well-defined due to (†). Since $G(t)$ is growing on $[0, R)$, the derivative $\varkappa'(t)$ is decreasing on that interval. We gather

$$D_h^+\varkappa(u(x)) = \frac{1}{h}\int_{u(x)}^{u(x+h)} \varkappa'(t)\,\mathbf{d}\,t = \frac{1}{h}\int_{u(x+h)}^{u(x)} -\varkappa'(t)\,\mathbf{d}\,t$$
$$\geq -\varkappa(u(x+h))\cdot(-D_h^+ u(x)) = \frac{-1}{\sqrt{G(R)-G(u(x+h))}} D_h^+ u(x).$$

Rewriting (‡)

$$D_h^+ u(x) \le -\sqrt{G(R) - G(u(x+h))} < 0, \quad \forall x \in [x_0 + h, x_1]_h$$
$$\Rightarrow \frac{-D_h^+ u(x)}{\sqrt{G(R) - G(u(x+h))}} \ge 1,$$

we get

$$D_h^+ \varkappa(u(x)) \ge 1.$$

Using this and (†) we obtain

$$\begin{aligned} x_1 - x_0 &= \sum_{x \in [x_0+h, x_1]_h} h \le \sum_{x \in [x_0+h, x_1]_h} D_h^+ \varkappa(u(x)) h \\ &= \varkappa(u(x_1+h)) - \varkappa(u(x_0+h)) = \varkappa(u(x_1+h)) - \varkappa(R) \\ &= \int_{u(x_1+h)}^{R} \frac{1}{\sqrt{G(R) - G(s)}} \, \mathbf{d}\, s \le \int_0^R \frac{1}{\sqrt{G(R) - G(s)}} \, \mathbf{d}\, s \le \frac{2R}{\sqrt{G(R)}}. \end{aligned}$$

From (2°) we can define

$$R_1 := \min \left\{ r > K \mid \frac{2\rho}{\sqrt{G(\rho)}} \le \frac{L}{2}, \quad \forall \rho \ge r \right\}.$$

Assuming now $R \ge R_1$, we have

$$x_1 - x_0 \le \frac{L}{2} \Rightarrow L - x_1 \ge \frac{L}{2}.$$

⟨7⟩ *Preparations for comparison function*
Let $\tilde{\lambda}_1^h$ be the first eigenvalue of

$$-D_h^+ D_h^- v(x) = \lambda v(x), \quad x \in (x_1, L)_h,$$
$$v(x_1) = v(L) = 0.$$

Note that

$$L - x_1 \ge \frac{L}{2} \ge \frac{4h}{2} = 2h.$$

Since, exactly as in the continuous case, we have the representation

$$v(x) = \alpha \cos(\nu(x - x_1)) + \beta \sin(\nu(x - x_1)), \quad x \in [x_1, L]_h$$

for the general solution, plugging in the boundary conditions we find $\alpha = 0$ and setting $x - x_1 = kh$, $k \in \mathbb{N}$ we also get

$$\begin{aligned}
-D_h^+ D_h^- \sin(\nu(x-x_1)) &= -\frac{1}{h^2}(\sin(\nu kh + \nu h) - 2\sin(\nu kh) + \sin(\nu kh - \nu h)) \\
&= -\frac{1}{h^2}(2\sin(\nu kh)\cos(\nu h) - 2\sin(\nu kh)) \\
&= \sin(\nu kh)\frac{2 - 2\cos(\nu h)}{h^2} = \sin(\nu kh)\frac{4\sin^2(\frac{\nu h}{2})}{h^2} \\
&= \sin(\nu(x-x_1))\frac{4\sin^2(\frac{\nu h}{2})}{h^2}.
\end{aligned}$$

For the first (positive) eigenfunction we have

$$\nu_1(L - x_1) = \pi \Rightarrow \nu_1 = \frac{\pi}{L - x_1}, \quad \tilde{\lambda}_1^h = \lambda(\nu_1) = \frac{4\sin^2\left(\frac{\pi h}{2(L-x_1)}\right)}{h^2}.$$

We define $\nu_* = \frac{\nu_1}{2} = \frac{\pi}{2(L-x_1)}$ and

$$\lambda_*^h := \lambda(\nu_*) = \frac{4}{h^2}\sin^2\left(\frac{\pi h}{4(L-x_1)}\right).$$

We aim to show that the double-sided estimate

$$\frac{L}{2} \leq L - x_1 \leq 2L$$

results in a double-sided estimate λ_*^h that is independent of h.
We have

$$\frac{\pi}{4L} \leq \frac{\nu_1}{2} = \nu_* = \frac{\pi}{2(L-x_1)} \leq \frac{\pi}{L}$$

and from $L \geq 4h$ also

$$\frac{\pi h}{4(L-x_1)} = \frac{\nu_* h}{2} \leq \frac{\pi h}{2L} \leq \frac{\pi h}{8h} = \frac{\pi}{8}.$$

This implies, $\lambda_*^h < \tilde{\lambda}_1^h$ and, moreover, on one hand

$$\lambda_*^h = \frac{4\sin^2\left(\frac{\pi h}{4(L-x_1)}\right)}{h^2} \leq \frac{4\left(\frac{\pi h}{4(L-x_1)}\right)^2}{h^2} = \left(\frac{\pi}{2(L-x_1)}\right)^2 \leq \left(\frac{\pi}{L}\right)^2 = \lambda_0$$

and on the other hand

$$\begin{aligned}
\lambda_*^h &= \frac{4\sin^2\left(\frac{\pi h}{4(L-x_1)}\right)}{h^2} = \frac{4}{\left(\frac{4(L-x_1)}{\pi}\right)^2} \cdot \frac{\sin^2\left(\frac{\pi h}{4(L-x_1)}\right)}{\left(\frac{\pi h}{4(L-x_1)}\right)^2} \geq \frac{\pi^2}{4(L-x_1)^2} \cdot \frac{\sin^2\left(\frac{\pi}{8}\right)}{\left(\frac{\pi}{8}\right)^2} \\
&\geq \frac{\sin^2\left(\frac{\pi}{8}\right)}{\left(4L\frac{1}{8}\right)^2} = \frac{4\sin^2\left(\frac{\pi}{8}\right)}{L^2} =: \underline{\lambda}
\end{aligned}$$

since $\sin(x)/x$ is monotonically decreasing on $(0,\pi]$:

$$\begin{aligned} -t\sin t + \cos t &\le \cos t, && \forall t \in [0,\pi], \\ \int_0^x \mathbf{d}\,(t\cos t) &\le \int_0^x \mathbf{d}\,(\sin t), && \forall x \in [0,\pi], \\ x\cos x - \sin x &\le 0, && \forall x \in [0,\pi], \\ \left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2} &\le 0, && \forall x \in (0,\pi]. \end{aligned}$$

⟨*8*⟩ *Comparison function*

We define $v\colon [x_1, L] \to \mathbb{R}$ as the solution of

$$\begin{aligned} -D_h^+ D_h^- v(x) &= \lambda_*^h v(x) - A, \quad x \in (x_1, L)_h, \\ v(x_1) &= u(x_1), \\ v(L) &= 0. \end{aligned}$$

Since $u(x) \ge 0$ satisfies

$$\begin{aligned} -D_h^+ D_h^- u(x) &= f(u(x)) \ge \lambda_0 u(x) - A \ge \lambda_*^h u(x) - A, \quad \forall x \in (x_1, L)_h \\ u(x_1) &= u(x_1), \\ u(L) &= 0 \end{aligned}$$

for $w := v - u$ we have

$$\begin{aligned} -D_h^+ D_h^- w(x) &\le \lambda_*^h w(x), \quad \forall x \in (x_1, L)_h, \\ u(x_1) &= w(L) = 0 \end{aligned}$$

with $0 < \lambda_*^h < \tilde{\lambda}_1^h$ From discrete elliptic comparison 2.6 we obtain $v(x) \le u(x)$ on $[x_1, L]_h$ and due to $v(x_1) = u(x_1)$ also $D_h^+ v(x_1) \le D_h^+ u(x_1)$.

We want to estimate $D_h^+ v(x_1)$. To this end we use the explicit representation

$$v(x) = \alpha\cos(\nu_*(x - x_1)) + \beta\sin(\nu_*(x - x_1)) + \frac{A}{\lambda_*^h}$$

From $v(x_1) = u(x_1)$ we get $\alpha + \frac{A}{\lambda_*^h} = u(x_1)$, thus $\alpha = u(x_1) - \frac{A}{\lambda_*^h}$. From $v(L) = 0$ and $\nu_*(L - x_1) = \frac{\nu_1}{2}(L - x_1) = \frac{\pi}{2}$ we get

$$0 = \alpha\cos(\nu_*(L - x_1)) + \beta\sin(\nu_*(L - x_1)) + \frac{A}{\lambda_*^h} = \beta + \frac{A}{\lambda_*^h} \Rightarrow \beta \ = -\frac{A}{\lambda_*^h}.$$

We evaluate $D_h^+ v(x_1)$:

$$\begin{aligned} hD_h^+ v(x_1) &= v(x_1+h) - v(x_1) = \alpha\cos(\nu_* h) + \beta\sin(\nu_* h) + \frac{A}{\lambda_*^h} - \alpha - \frac{A}{\lambda_*^h} \\ &= \alpha(\cos(\nu_* h) - 1) + \beta\sin(\nu_* h) \end{aligned}$$

Since $D_h^+ v(x_1) \le D_h^+ u(x_1)$, using (‡) and $u(x_1) \le R$ we get

$$\begin{aligned} &\frac{1}{h}[\alpha(\cos(\nu_* h) - 1) + \beta\sin(\nu_* h)] \le -\sqrt{G(R) - G(u(x_1+h))}, \\ &\quad \Rightarrow I := \sqrt{G(R) - G(K)} \le \sqrt{G(R) - G(u(x_1+h))} \\ &\qquad \le \frac{1}{h}\left[\alpha(1 - \cos(\nu_* h)) - \beta\sin(\nu_* h)\right] \\ &\qquad = \left(u(x_1) - \frac{A}{\lambda_*^h}\right)\frac{1-\cos(\nu_* h)}{h} + \frac{A}{\lambda_*^h}\frac{\sin(\nu_* h)}{h} \\ &\qquad \le R\frac{1-\cos(\nu_* h)}{h} + \frac{A}{\lambda_*^h}\frac{\sin(\nu_* h) + \overbrace{\cos(\nu_* h) - 1}^{\le 0}}{h} \\ &\qquad \le R\nu_*\frac{1-\cos(\nu_* h)}{\nu_* h} + \frac{A\nu_*}{\lambda_*^h}\frac{\sin(\nu_* h)}{\nu_* h} \\ &\qquad \le R\nu_* + \frac{A\nu_*}{\lambda_*^h} \le R\frac{\pi}{L} + \frac{A\pi}{\underline{\lambda} L}, \end{aligned}$$

implying

$$\begin{aligned} \frac{G(R)}{R^2} &\le \frac{G(K)}{R^2} + \left(\frac{\pi}{L} + \frac{A\pi}{\underline{\lambda} LR}\right)^2 \\ &\le \frac{G(K)}{K^2} + \left(\frac{\pi}{L} + \frac{A\pi}{\underline{\lambda} LK}\right)^2 \end{aligned}$$

Since $G(t)/t^2 \to \infty$ as $t \to \infty$, we have the following bound

$$R \le \min\left\{ r > K \mid \frac{G(\rho)}{\rho^2} \ge \frac{G(K)}{K^2} + \left(\frac{\pi}{L} + \frac{A\pi}{\underline{\lambda} LK}\right)^2 \quad \forall \rho \ge r \right\}.$$

◀

Chapter 3

Convergence Result for One-Dimensional Model Problem

3.1 Discrete formulation

3.1 Motivation (Model Problem)**.**
In this chapter we consider the convergence problem in the simplest, one-dimensional setting. The aim is to present the essential ideas without the burden of technicalities and to point out the main difficulties which arise in higher dimensions.

3.2 Convention (Admissible grid spacings)**.**
The grid spacing h is called *admissible* for the interval $[0, L]$ if $\frac{L}{h} \in \mathbb{Z}_+$. We assume all grid spacings in this chapter to be admissible.

3.3 Notation (Grid)**.**
We denote by $M_h := \{x_i\}_{i=0}^{N_h}$, $N_h := L/h$ the *equidistant grid* on $[0, L]$ with grid spacing h, i.e. $h = x_i - x_{i-1}$, $1 \le i \le N_h$. We also denote $\mathring{M}_h := \{x_i\}_{i=1}^{N_h-1}$.

3.4 Definition (Finite Differences)**.**
For $u\colon M_h \to \mathbb{R}$ we define the *forward finite difference quotient* by

$$D_h^+ u(x_i) := \frac{u(x_{i+1}) - u(x_i)}{h}, \quad 0 \le i \le N_h - 1,$$

the *backward finite difference quotient* by

$$D_h^- u(x_i) := \frac{u(x_i) - u(x_{i-1})}{h}, \quad 1 \le i \le N_h,$$

and the *discrete Laplace operator* by

$$\begin{aligned} \Delta_h u(x_i) &:= D_h^+ D_h^- u(x_i) = D_h^- D_h^+ u(x_i) \\ &= \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}, \quad 1 \le i \le N_h - 1. \end{aligned}$$

3.5 Lemma (Summation by parts)**.**
Let $u, v\colon M_h \to \mathbb{R}$. Then

$$\sum_{i=1}^{N_h-1} D_h^+ u(x_i)v(x_i)h = -\sum_{i=1}^{N_h-1} u(x_i)D_h^- v(x_i)h + u(x_{N_h})v(x_{N_h-1}) - u(x_1)v(x_0),$$
$$\sum_{i=1}^{N_h-1} D_h^- u(x_i)v(x_i)h = -\sum_{i=1}^{N_h-1} u(x_i)D_h^+ v(x_i)h + u(x_{N_h-1})v(x_{N_h}) - u(x_0)v(x_1).$$

►
Direct calculation yields

$$\begin{aligned}\sum_{i=1}^{N_h-1} D_h^+ u(x_i)v(x_i)h &= \sum_{i=2}^{N_h} u(x_i)v(x_{i-1}) - \sum_{i=1}^{N_h-1} u(x_i)v(x_i)\\ &= -\sum_{i=1}^{N_h-1} u(x_i)(v(x_i) - v(x_{i-1})) - u(x_1)v(x_0) + u(x_{N_h})v(x_{N-1})\\ &= -\sum_{i=1}^{N_h-1} u(x_i)D_h^- v(x_i)h + u(x_{N_h})v(x_{N_h-1}) - u(x_1)v(x_0).\end{aligned}$$

Swapping u and v in the first equality we obtain the second one.
◄

3.6 Definition (Discrete Formulations)**.**
In the classical continuous setting we are looking for $u \in C^2((0,L)) \cap C([0,L])$ such that

$$\begin{aligned}-\Delta u(x) = -u''(x) &= f(u(x)), \quad x \in (0,L),\\ u(0) = u(L) &= 0\end{aligned} \tag{BVP}$$

with some $f \in C(\mathbb{R})$.
In the discrete case we are looking for $u\colon M_h \to \mathbb{R}$ that satisfy one of following three equivalent formulations (see the next lemma).
The *classical discrete formulation* for (BVP) is the problem

$$\begin{aligned}-\Delta_h u(x) &= f(u(x)), \quad x \in \mathring{M}_h,\\ u(0) = u(L) &= 0.\end{aligned} \tag{P_h}$$

The *weak discrete formulation* for (BVP) is the problem

$$\begin{aligned}\sum_{i=0}^{N_h-1} D_h^+ u(x_i) D_h^+ \varphi(x_i)h &= \sum_{i=0}^{N_h-1} f(u(x_i))\varphi(x_i)h,\\ u(0) = u(L) &= 0,\end{aligned} \tag{P_h'}$$

where equality has to hold for all $\varphi\colon M_h \to \mathbb{R}$ with $\varphi(0) = \varphi(L) = 0$.
The *very weak discrete formulation* for (BVP) is the problem

$$\begin{aligned} -\sum_{i=1}^{N_h-1} u(x_i)\Delta_h\varphi(x_i)h &= \sum_{i=1}^{N_h-1} f(u(x_i))\varphi(x_i)h, \\ u(0) = u(L) &= 0, \end{aligned} \tag{P_h''}$$

where equality has to hold for all $\varphi\colon M_h \to \mathbb{R}$ with $\varphi(0) = \varphi(L) = 0$.

3.7 Remark (Summation sets)**.**
The summation sets in the right hand sides of (P_h') and (P_h'') are chosen for the sake of symmetry with the corresponding left hand sides; condition $\varphi(0) = \varphi(L) = 0$ naturally implies

$$\sum_{i=0}^{N_h} f(u(x_i))\varphi(x_i) = \sum_{i=1}^{N_h-1} f(u(x_i))\varphi(x_i) = \sum_{i=1}^{N_h} f(u(x_i))\varphi(x_i) = \sum_{i=0}^{N_h-1} f(u(x_i))\varphi(x_i).$$

3.8 Lemma (Equivalence of the discrete formulations)**.**
The discrete formulations $(P_h), (P_h')$ and (P_h'') are equivalent.

▶
Let $\varphi\colon M_h \to \mathbb{R}$ be a grid function with $\varphi(0) = \varphi(L) = 0$. We multiply (P_h) with φ and sum up over $1 \le i \le N_h - 1$

$$\begin{aligned} \sum_{i=1}^{N_h-1} f(u(x_i))\varphi(x_i) &= -\sum_{i=1}^{N_h-1} \Delta_h u(x_i)\varphi(x_i) = \sum_{i=1}^{N_h-1} \frac{-D_h^+u(x_i) + D_h^-u(x_i)}{h}\varphi(x_i) \\ &= \sum_{i=0}^{N_h-1} \frac{-D_h^+u(x_i)}{h}\varphi(x_i) + \sum_{i=1}^{N_h} \frac{D_h^-u(x_i)}{h}\varphi(x_i) \\ &= \sum_{i=0}^{N_h-1} \frac{-D_h^+u(x_i)}{h}\varphi(x_i) + \sum_{i=0}^{N_h-1} \frac{D_h^+u(x_i)}{h}\varphi(x_{i+1}) \\ &= \sum_{i=0}^{N_h-1} D_h^+u(x_i)D_h^+\varphi(x_i). \end{aligned}$$

Adding $f(u(x_0))\varphi(x_0) = 0$ to the left hand side, we get $(P_h) \Rightarrow (P_h')$.
Since $u(0) = u(L) = 0$, we can expand $\sum D_h^+u(x_i)D_h^+\varphi(x_i)$ backwards, swapping u and φ. This yields $(P_h') \Leftrightarrow (P_h'')$.
Plugging $\varphi_j\colon M_h \to \mathbb{R}$, $1 \le j \le N_h - 1$, $\varphi_j(x_i) := \delta_{ij}$, $0 \le i \le N_h$ into (P_h') and

(P_h'') and in each case dividing by h we obtain (P_h):

$$f(u(x_j)) = -\frac{D_h^+ u(x_j)}{h} + \frac{D_h^+ u(x_{j-1})}{h} = -\Delta_h u(x_j), \quad 1 \le j \le N_h - 1,$$

$$f(u(x_j)) = -\sum_{i=j-1}^{j+1} u(x_i)\Delta_h \varphi(x_i) = -\Delta_h u(x_j), \qquad 1 \le j \le N_h - 1.$$

◀

3.2 Main result for the model problem

3.9 Announcement (Convergence Theorem)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, $h_n \overset{n\to\infty}{\to} 0$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon M_{h_n} \to \mathbb{R}$ of discrete problems (P_{h_n}) and a positive $C > 0$ such that

$$\|u_n\|_{L^\infty(M_{h_n})} := \max\{|u_n(x)| \mid x \in M_{h_n}\} \le C, \quad \forall n \in \mathbb{N}.$$

Then, there exists a (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ and a classical solution $u \in C^2((0,L)) \cap C([0,L])$ of (BVP) such that

$$\|u_n - u\|_{L^\infty(M_{h_n})} \to 0, \quad \text{as } n \to \infty.$$

3.10 Notation (Linear Interpolation)**.**
For $u\colon M_h \to \mathbb{R}$ we denote by $\hat{u}\colon [0,L] \to \mathbb{R}$ the *piecewise linear approximation* of u, i.e. $\hat{u}(x) = u(x)$, $\forall x \in M_h$ and $\hat{u}$ is linear on $[x_i, x_{i+1}]$, $0 \le i \le N_h - 1$.

3.11 Lemma (Regularity of linear interpolation)**.**
For $u\colon M_h \to \mathbb{R}$ we have $\hat{u} \in W_0^{1,2}((0,L))$ with

$$\int_0^L (\hat{u}'(x))^2 \,\mathbf{d}\, x = \sum_{i=0}^{N_h-1} |D_h^+ u(x_i)|^2 h.$$

▶

We have the following explicit representation

$$\hat{u}(x) = \frac{u(x_{i+1}) - u(x_i)}{h}(x - x_i) + u(x_i) \quad x_i \le x \le x_{i+1}, \quad 0 \le i \le N_h - 1$$

and pointwise

$$\hat{u}'(x) = D_h^+ u(x_i), \quad x_i < x < x_{i+1}, \quad 0 \le i \le N_h - 1.$$

Continuous piecewise differentiable function on the real line are weakly differentiable. We also have

$$\int_0^L (\hat{u}')^2 \,\mathbf{d}\, x = \sum_{i=0}^{N_h-1} \int_{x_i}^{x_{i+1}} (\hat{u}')^2 \,\mathbf{d}\, x = \sum_{i=0}^{N_h-1} |D_h^+ u(x_i)|^2 h.$$

◀

3.12 Motivation (Embedding results)**.**
The following classical embedding result is exceptional to $\mathbb{R}^1$. Since the proof is short and does not involve geometrical considerations, we find it suitable to present it here. This result can be extended to the compact embedding $W^{1,2} \hookrightarrow C^{0,\alpha}$ for all $\alpha \in (0, 1/2)$, see [2], Chapter 6.

3.13 Lemma (Embedding results)**.**
Let $u \in W_0^{1,2}((0,L))$. Then there exists $\bar{u} \in C([0,L])$, $\bar{u}(0) = \bar{u}(L) = 0$ with $\bar{u}|_{(0,L)} = u$ a.e. Now let $(u_n)_{n\in\mathbb{N}} \subset W_0^{1,2}((0,L))$ with $\|u_n\|_{W^{1,2}(0,L)} \le C$ for some $C > 0$. Then, there exists $\tilde{u} \in C([0,L])$ such that $\bar{u}_n \rightrightarrows \tilde{u}$ (i.e. uniformly on $[0,L]$) up to a subsequence as $n \to \infty$, where $u_n \mapsto \bar{u}_n$ is defined as above.

▶⟨*1*⟩ *Continuous embedding*

Let $u \in C_0^\infty((0,L))$. We can extend it to $\bar{u} \in C([0,L])$, $\bar{u}(0) = \bar{u}(L) = 0$. Identifying u and $\bar{u}$ we obtain

$$\begin{aligned} |u(x)| &\le \left| u(0) + \int_0^x u'(t) \,\mathbf{d}\, t \right| \le \int_0^L |u'(t)| \,\mathbf{d}\, t \\ &\le \sqrt{L} \sqrt{\int_0^L |u'(t)|^2 \,\mathbf{d}\, t} \le \sqrt{L} \|u\|_{W^{1,2}((0,L))}. \end{aligned}$$

Taking the maximum, we get

$$\|u\|_{C([0,L])} \le \sqrt{L} \|u\|_{W^{1,2}((0,L))}. \tag{+}$$

Now let, $(u_n)_{n\in\mathbb{N}} \in C_0^\infty((0,L))$ be such that $u_n \to u$ in $W_0^{1,2}((0,L))$. Since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_0^{1,2}((0,L))$ it is also (after the extension) a Cauchy sequence in $C([0,L])$. For limit $\tilde{u} \in C([0,L])$, $u_n \rightrightarrows \tilde{u}$, we also have $u_n \to \tilde{u}$ in $L^2((0,L))$. This yields $\tilde{u} = u$ a.e.

⟨*2*⟩ *Compact embedding*

From (+) follows that $(u_n)_{n\in\mathbb{N}}$ is also bounded in $C([0,L])$. In order to use Arzelà-Ascoli theorem we need only to show equicontinuity. Without loss of generality assume $x < y$, then

$$\begin{aligned} |u_n(x) - u_n(y)| &= \left| \int_x^y u_n'(t) \,\mathbf{d}\, t \right| \le \sqrt{\int_x^y |u_n'(t)|^2 \,\mathbf{d}\, t} \sqrt{\int_x^y 1 \,\mathbf{d}\, t} \\ &\le \|u_n\|_{W^{1,2}((0,L))} \sqrt{y-x} \le C\sqrt{y-x}. \end{aligned}$$

◀

3.14 Convention.
We will identify the function $u \in W_0^{1,p}((0,L))$ with its continuous representative $\bar{u} \in C([0,L])$ in the sense of the previous lemma.

3.15 Lemma (Boundedness)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon M_{h_n} \to \mathbb{R}$ of discrete problems (P_{h_n}) and a positive $C > 0$ such that

$$\|u_n\|_{L^\infty(M_{h_n})} \le C, \quad \forall n \in \mathbb{N}.$$

Then, there exists $\hat{C} > 0$ such that

$$\|\hat{u}_n\|_{W_0^{1,2}(0,L)} \le \hat{C}, \quad \forall n \in \mathbb{N}.$$

▶
From the uniform boundedness of $(u_n)_{n\in\mathbb{N}}$ and the continuity of f we get $\|f(u_n)\|_{L^\infty(M_{h_n})} \le \tilde{C}$ for some $\tilde{C} = \tilde{C}(C, f) > 0$. We denote $N_n := N_{h_n}$, obtaining

$$\int_0^L (\hat{u}_n')^2 \,\mathbf{d}\, x = \sum_{i=0}^{N_n-1} |D_h^+ u_n(x_i)|^2 h_n = \sum_{i=0}^{N_n-1} f(u_n(x_i)) u_n(x_i) h_n \le \tilde{C} C L =: \hat{C}.$$

◀

3.16 Remark (Higher Dimensions)**.**
The uniform convergence in the previous lemma is essential. In higher dimensions we have to use discrete L^p-estimates instead of the energy estimates in order to apply compact embeddings.

3.17 Lemma (Passing to the limit in the linear part)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon M_{h_n} \to \mathbb{R}$, $N_n := N_{h_n}$ of the discrete problems (P_{h_n}) and a positive $C > 0$ such that $\|u_n\|_{L^\infty(M_{h_n})} \le C$, $\forall n \in \mathbb{N}$. Then, there exists a (renamed) subsequence of $(u_n)_{n\in\mathbb{N}}$ such that

$$-\sum_{i=1}^{N_n-1} u_n(x_i) \Delta_{h_n} \varphi(x_i) h_n \xrightarrow{n\to\infty} -\int_{\mathbb{R}} u(x) \Delta\varphi(x) \,\mathbf{d}\, x,$$

for all $\varphi \in C^3([0,L])$ and

$$\hat{u}_n \rightrightarrows u$$

for some $u \in C([0,L])$, $u(0) = u(L) = 0$.
▶

⟨*1*⟩ *Subsequence in* $C^*([0,L])$
The functionals

$$\Phi_n \colon \begin{cases} C([0,L]) \to \mathbb{R} \\ \qquad \Psi \mapsto \sum_{i=1}^{N_n-1} u_n(x_i)\Psi(x_i)h_n \end{cases}$$

are uniformly bounded in n, because

$$|\Phi_n(\Psi)| \le \|u_n\|_{L^\infty(\mathring{M}_{h_n})}\|\Psi\|_{C([0,L])}L \le CL\|\Psi\|_{C([0,L])}.$$

Due to separability of $C([0,L])$ there exists $\Phi \in C^*([0,1])$ such that $\Phi_n \overset{*}{\rightharpoonup} \Phi$ up to a (renamed) subsequence.

⟨*2*⟩ *Extending the limit element to* $L^2([0,L])$
For any $\Psi\colon M_{h_n} \to \mathbb{R}$ we denote

$$\|\Psi\|_{L^2(M_{h_n})} := \sqrt{\sum_{i=1}^{N_n} \Psi^2(x_i)h_n}.$$

From

$$\begin{aligned} |\Phi_n(\Psi)| &= \left|\sum_{i=1}^{N_n-1} u_n(x_i)\Psi(x_i)h_n\right| \le \sqrt{\sum_{i=1}^{N_n} u_n^2(x_i)h_n}\sqrt{\sum_{i=1}^{N_n} \Psi^2(x_i)h_n} \\ &\le C\sqrt{L}\|\Psi\|_{L^2(M_{h_n})} \end{aligned}$$

letting $n \to \infty$ we get for a fixed $\Psi \in C([0,L])$

$$|\Phi(\Psi)| \le C\sqrt{L}\|\Psi\|_{L^2([0,L])}.$$

Since $C([0,L])$ is dense in the complete space $L^2([0,L])$ we can extend Φ by continuity onto $L^2([0,L])$. We denote by $\tilde{u} \in L^2([0,L])$ the Riesz representative of this extension.

⟨*3*⟩ *Convergence*
For $\Psi\colon M_{h_n} \to \mathbb{R}$ we set $\Phi_n(\Psi) := \Phi_n(\hat{\psi})$. We want to show that

$$\Phi_n(-\Delta_{h_n}\varphi) = -\sum_{i=1}^{N_n-1} u_n(x_i)\Delta_{h_n}\varphi(x_i)h_n \to -\int_0^L \Delta\varphi(x)\tilde{u}(x)\,\mathbf{d}\,x = \Phi(-\Delta\varphi)$$

for every $\varphi \in C^3([0,L])$ as $n \to \infty$. For $\varphi \in C^3([0,L])$ we have the consistency

$$\|\Delta_{h_n}\varphi - \Delta\varphi\|_{L^\infty(\mathring{M}_{h_n})} \to 0 \text{ as } n \to \infty,$$

implying

$$|\Phi_n(-\Delta_{h_n}\varphi + \Delta\varphi)| \le CL\|\Delta_{h_n}\varphi - \Delta\varphi\|_{L^\infty(\mathring{M}_{h_n})} \to 0 \text{ as } n \to \infty.$$

Hence, letting $n \to \infty$ we get

$$\Phi_n(-\Delta_{h_n}\varphi) = \Phi_n(-\Delta\varphi) + \Phi_n(\Delta\varphi - \Delta_{h_n}\varphi) \to \Phi(-\Delta\varphi) = -\int_0^L \tilde{u}\Delta\varphi \,\mathbf{d}\, x.$$

⟨*4*⟩ *Identification of* $\tilde{u}$
We now want to show that $\tilde{u}$ can also be obtained as an accumulation point of $(\hat{u}_n)_{n\in\mathbb{N}}$ in $C([0,L])$. Taking the (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ that corresponds to the extracted subsequence used in the first step and applying Lemma 3.15 and Lemma 3.13 we extract a (renamed) convergent subsequence $(\hat{u}_n)_{n\in\mathbb{N}}$ with a limit $u \in C([0,L])$, $\hat{u}_n \rightrightarrows u$ as $n \to \infty$. We now only need to show that $\tilde{u} = u$, yielding also uniqueness of u for fixed Φ from the first step. First we show

$$\Phi_n(-\Delta_{h_n}\varphi) \xrightarrow{n\to\infty} -\int_0^L u(x)\Delta\varphi(x)\,\mathbf{d}\, x, \quad \forall \varphi \in C^3([0,L]).$$

We get

$$\begin{aligned}
\Phi_n(-\Delta_{h_n}\varphi) &= -\sum_{i=1}^{N_n-1} u_n(x_i)\Delta_{h_n}\varphi(x_i)h_n = -\sum_{i=1}^{N_n-1} \hat{u}_n(x_i)\Delta_{h_n}\varphi(x_i)h_n \\
&= -\sum_{i=1}^{N_n-1} \hat{u}_n(x_i)\Delta\varphi(x_i)h_n + \sum_{i=1}^{N_n-1} \hat{u}_n(x_i)[\Delta\varphi(x_i) - \Delta_{h_n}\varphi(x_i)]h_n \\
&= -\sum_{i=1}^{N_n-1} u(x_i)\Delta\varphi(x_i)h_n + \sum_{i=1}^{N_n-1} [u(x_i) - \hat{u}_n(x_i)]\Delta\varphi(x_i)h_n \\
&\quad + \sum_{i=1}^{N_n-1} \hat{u}_n(x_i)[\Delta\varphi(x_i) - \Delta_{h_n}\varphi(x_i)]h_n \\
&\xrightarrow{n\to\infty} -\int_0^L u(x)\Delta\varphi(x)\,\mathbf{d}\, x,
\end{aligned}$$

since for $n \to \infty$

$$\left|\sum_{i=1}^{N_n-1} [u(x_i) - \hat{u}_n(x_i)]\Delta\varphi(x_i)h_n\right| \le \overbrace{\|u - \hat{u}_n\|_{C([0,L])}}^{\to 0} \overbrace{\sum_{i=1}^{N_n-1} |\Delta\varphi(x_i)|h_n}^{\to \int_0^L |\Delta\varphi(x)|\,\mathbf{d}\, x} \to 0$$

and

$$I := \left| \sum_{i=1}^{N_n-1} \hat{u}_n(x_i)[\Delta\varphi(x_i) - \Delta_{h_n}\varphi(x_i)]h_n \right|$$
$$\leq \underbrace{\sum_{i=1}^{N_n-1} |\hat{u}_n(x_i)|h_n}_{\leq CL} \underbrace{\|\Delta\varphi - \Delta_{h_n}\varphi\|_{L^\infty(M_{h_n})}}_{\to 0} \to 0.$$

We now have

$$\int_0^L [u(x) - \tilde{u}(x)]\varphi''(x)\,\mathbf{d}\,x = 0, \quad \forall \varphi \in C^3([0,L]),$$

implying

$$\int_0^L [u(x) - \tilde{u}(x)]\psi(x)\,\mathbf{d}\,x = 0, \quad \forall \psi \in C^1([0,L]).$$

Since $C^1([0,L]) \supset C_0^\infty((0,L))$ is dense in $L^2([0,L])$ we have $\tilde{u} = u$ a.e.
◀

3.18 Lemma (Passing to the limit in the nonlinear part)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, $h_n \overset{n\to\infty}{\to} 0$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon M_{h_n} \to \mathbb{R}$, $N_n := N_{h_n}$ of discrete variational problems (P_{h_n}) and a positive constant $C > 0$ such that $\|u_n\|_{L^\infty(M_{h_n})} \leq C$, $\forall n \in \mathbb{N}$. Then, there exists a (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\sum_{i=1}^{N_n-1} f(u_n(x_i))\varphi(x_i)h_n \xrightarrow{n\to\infty} \int_{\mathbb{R}} f(u(x))\varphi(x)\,\mathbf{d}\,x,$$

for all $\varphi \in C([0,L])$ and

$$\hat{u}_n \rightrightarrows u$$

for some $u \in C([0,L])$, $u(0) = u(L) = 0$.
▶
⟨*1*⟩ *Subsequence in* $C^*([0,L])$
The functionals

$$\Phi_n\colon \begin{cases} C([0,L]) \to \mathbb{R} \\ \qquad \Psi \mapsto \sum_{i=1}^{N_n-1} f(u_n(x_i))\Psi(x_i)h_n \end{cases}$$

are uniformly bounded, because

$$|\Phi_n(\Psi)| \leq \|f(u_n)\|_{L^\infty(M_{h_n})}\|\Psi\|_{C([0,L])}L \leq \|f\|_{L^\infty([-C,C])}\|\Psi\|_{C([0,L])}L.$$

Due to separability of $C([0,L])$ there exists $\Phi \in C^*([0,1])$ such that $\Phi_n \overset{*}{\rightharpoonup} \Phi$ up to a (renamed) subsequence.

⟨*2*⟩ *Extending the limit element to* $L^2([0,L])$
From

$$|\Phi_n(\Psi)| = \left|\sum_{i=1}^{N_n-1} f(u_n(x_i))\Psi(x_i)h_n\right| \le \sqrt{\sum_{i=1}^{N_n} f^2(u_n(x_i))h_n}\sqrt{\sum_{i=1}^{N_n} \Psi^2(x_i)h_n}$$
$$\le \|f\|_{L^\infty([-C,C])}\sqrt{L}\|\Psi\|_{L^2(M_{h_n})}$$

letting $n \to \infty$ we get for a fixed $\Psi \in C([0,L])$

$$|\Phi(\Psi)| \le \|f\|_{L^\infty([-C,C])}\sqrt{L}\|\Psi\|_{L^2([0,L])}.$$

Since $C([0,L])$ is dense in the complete space $L^2([0,L])$ we can extend Φ to $L^2([0,L])$. We denote by $F \in L^2([0,L])$ the Riesz representative of Φ.

⟨*3*⟩ *Identification of* F
We take the (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ that corresponds to the convergent subsequence in the first step. Using Lemma 3.15 and Lemma 3.13 we now extract from it a further (renamed) convergent subsequence $(u_n)_{n\in\mathbb{N}}$, this time with $\hat{u}_n \rightrightarrows u$ as $n \to \infty$ for some $u \in C([0,L])$. Our aim is to show that $F = f(u)$, also implying that u is unique once F is fixed. We have

$$\begin{aligned}
\Phi_n(\varphi) &= \sum_{i=1}^{N_n-1} f(u_n(x_i))\varphi(x_i)h_n = \sum_{i=1}^{N_n-1} f(\hat{u}_n(x_i))\varphi(x_i)h_n \\
&= \sum_{i=1}^{N_n-1} f(u(x_i))\varphi(x_i)h_n + \sum_{i=1}^{N_n-1} [f(\hat{u}_n(x_i)) - f(u(x_i))]\varphi h_n \\
&\to \int_0^L f(u(x))\varphi(x)\,\mathbf{d}\,x, \quad \text{as } n \to \infty, \text{ for all } \varphi \in C([0,L]),
\end{aligned}$$

provided

$$\left|\sum_{i=1}^{N_n-1} [f(\hat{u}_n(x_i)) - f(u(x_i))]\varphi(x_i)h_n\right| \to 0.$$

But this claim follows from uniform continuity of f on $[-C,C]$ and uniform convergence of $\hat{u}_n$ toward u. We have therefore

$$\int_0^L [f(u(x)) - F(x)]\varphi(x)\,\mathbf{d}\,x = 0 \quad \forall \varphi \in C([0,L]).$$

Since $C([0,L])$ is dense in $L^2([0,L])$ we finally obtain $f(u) = F$ a.e.

◀

3.19 Theorem (Convergence Theorem)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, $h_n \overset{n\to\infty}{\to} 0$ and a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of discrete problems (P_{h_n}) and a positive $C > 0$ such that $\|u_n\|_{L^\infty(M_{h_n})} \leq C$, $\forall n \in \mathbb{N}$. Then, there exists a (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ and a classical solution $u \in C^2((0, L)) \cap C([0, L])$ of (BVP) such that

$$\|u_n - u\|_{L^\infty(M_{h_n})} \to 0, \quad \text{as } n \to \infty.$$

►

Using the convergent subsequence obtained after applying Lemma 3.17 as the initial sequence for Lemma 3.18, we obtain a subsequence of $(u_n)_{n\in\mathbb{N}}$ for which the claims of both those results hold with the same uniform limit, so letting $n \to \infty$ we have

$$\int_0^L -u(x)\varphi''(x)\,\mathbf{d}\,x = \int_0^L f(u(x))\varphi(x)\,\mathbf{d}\,x,$$

for all $\varphi \in C^3([0, L])$ with $\varphi(0) = \varphi(L) = 0$ with some $u \in C([0, L])$, $u(0) = u(L) = 0$. Let $v \in C^2((0, L)) \cap C([0, L])$ be the classical solution of

$$\begin{aligned} -v''(x) &= f(u(x)), \quad x \in (0, L) \\ v(0) &= v(L) = 0. \end{aligned}$$

Taking the weak formulation of this BVP and subtracting it from the weak formulation of our original problem we get

$$\int_0^L (u - v)(x)\varphi''(x)\,\mathbf{d}\,x = 0,$$

for all $\varphi \in C^3([0, L])$ with $\varphi(0) = \varphi(L) = 0$. Repeating the final argument from Lemma 3.17 we get $u = v$ a.e., i.e. $u \in C^2((0, L))$.

◄

Chapter 4

A Priori Estimates for Convex Domains in the Continuous Case

4.1 Estimates for Green's function

4.1 Motivation (Continuous case).
The results in this chapter are either known, often in a much stronger form ([4],[40]) or classical ([20],[23]). They and especially their proofs are presented here in order to provide an guideline for and to motivate the corresponding discrete results in the following three chapters. As in the previous chapter this has the advantage of separating the essential ideas from the technical difficulties arising from discretization.

4.2 Convention (Dimension).
We will always assume $d \geq 3$.

4.3 Announcement (Regularity estimate [4]).
Let $\Omega \subset \mathbb{R}^n$ be a bounded and convex domain. Let $f \in L^q(\Omega)$, $q > \frac{dp}{d+p}$ for some $1 < p < \infty$. Then, there exists a unique weak solution $u \in W_0^{1,p}(\Omega)$ of the boundary-value problem

$$\begin{aligned} -\Delta u(x) &= f(x), \quad x \in \Omega, \\ u(x) &= 0, \qquad x \in \partial\Omega \end{aligned}$$

and it holds

$$\|u\|_{1,p} \leq C(d,p,q,\Omega)\|f\|_q,$$

with some $C(d,p,q,\Omega) > 0$.

4.4 Notation (Generic constants).
We reserve the letters K and K^* to denote generic constants, i.e. expressions like

$$A(x) \leq KB(x)$$

mean that

$$A(x) \leq CB(x)$$

for some particular real number $C > 0$. We explicitly allow the case $B(x) \equiv 1$.
The same generic constant within a chain of inequalities can stand for different particular constants; the change of particular constants is indicated by adding or removing of asterisk, i.e. in

$$A(x) \overset{(1)}{\leq} KB(x) \overset{(2)}{\leq} K^*C(x) \overset{(3)}{\leq} K^*D(x) \overset{(4)}{\leq} KP(x)$$

the particular constants may need to be changed in the second and fourth inequalities, one can always choose the same particular constant in the third inequality, and K in the first and in the last inequalities can stand for different particular constants.
Using the notation

$$A(x) \leq KB(x) \leq: CB(x)$$

we pick some particular constant C for which the previous inequality holds. If we have some (fixed within our consideration) quantities $s_1, \dots, s_n$, $n \in \mathbb{N}$, we write $K(s_1, \dots, s_n)$ to indicate that the corresponding particular constants can be chosen depending only on those quantities.

4.5 Theorem (Classical existence result)**.**
Let Ω be a bounded convex domain. If f is a bounded, locally Hölder continuous function in Ω, then the classical Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

is uniquely solvable in $C^2(\Omega) \cap C(\overline{\Omega})$ for any continuous boundary value $\varphi \in C(\partial\Omega)$.

▶

The proof follows from Theorem 4.3 in [23] and the fact that convex domains have regular boundary by the exterior cone condition, see Chapter 2 in the same book.

◀

4.6 Definition (Green's Function)**.**
We denote by $\Phi\colon \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) := \frac{1}{d(d-2)\omega_d} |x|^{2-d}$$

the *fundamental solution* to Laplace's operator $-\Delta$ in $\mathbb{R}^d$, where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$. We have $-\Delta u = f$ for

$$u(x) := \int_{\mathbb{R}^d} \Phi(x-y) f(y) \, \mathbf{d}\, y, \quad x \in \mathbb{R}^d$$

as long as $f \in C_0^\infty(\mathbb{R}^d)$ (see [23]).

For $\Omega \subset \mathbb{R}^d$ open and bounded we define the *corrector function* $\varphi\colon \Omega \times \overline{\Omega} \to \mathbb{R}$ as the classical solution (if it exists) of

$$\begin{aligned} -\Delta_y \varphi(x,y) &= 0, & y \in \Omega, \\ \varphi(x,y) &= \Phi(y-x), & y \in \partial\Omega \end{aligned}$$

for all $x \in \Omega$ fixed.

Green's function (if it exists) is defined by

$$G(x,y) := \Phi(y-x) - \varphi(x,y), \quad \forall (x \neq y) \in \Omega.$$

4.7 Remark (Representation and smoothness).
It is a classical result ([20], [23]) that if the corrector function is sufficiently smooth, $\varphi(x,\cdot) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ for all $x \in \Omega$, then the solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

$$\begin{aligned} -\Delta u(x) &= f(x), & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \tag{$*$}$$

has the representation

$$u = G * f :\Leftrightarrow u(x) = \int_\Omega G(x,y) f(y) \,\mathbf{d}\, x, \quad x \in \Omega, \tag{$+$}$$

provided $f \in C^{0,\alpha}(\overline{\Omega})$ with some $\alpha \in (0,1]$ and the boundary of Ω is C^2.

The existence of G satisfying $(+)$ for the weak solutions of the Poisson problem $(*)$ can be obtained on general bounded domains (see [37] and [24]).

The following theorem states that in the case of convex domains the representation via the corrector function also holds.

4.8 Theorem (Green's representation on bounded convex domains).
Green's function $G\colon \Omega \times \Omega \to \mathbb{R}$ exists for every bounded convex domain Ω. Moreover,

$$u(x) := \int_\Omega G(x,y) f(y) \,\mathbf{d}\, y, \quad x \in \Omega$$

satisfies the equation

$$\begin{aligned} -\Delta u(x) &= f(x), & x \in \Omega, \\ u(x) &= 0, & x \in \partial\Omega, \end{aligned} \tag{$*$}$$

for $f \in C_0^\infty(\Omega)$.

▶

The existence and uniqueness of G follows from Theorem 4.5. Using Definition 3.2.3 from [27] we construct $\tilde{G}\colon \Omega \times \Omega \to \mathbb{R}$ such that $\tilde{G}(x,y) = \Phi(x-y) - h(x,y)$

with $\Delta_y h(x,y) = 0$ for all $x, y \in \Omega$. Theorem 3.3.15 from the same book yields $\lim_{z\to y} \tilde{G}(x,z) = 0$ for all $x \in \Omega$, $y \in \partial\Omega$. This implies $\varphi = h$ and $G = \tilde{G}$. We now use three other theorems from [27]. From Theorem 3.4.10 we obtain $-\Delta u = f$ for

$$u(x) = \int_\Omega G(x,y) f(y) \, \mathbf{d}\, y$$

with $f \in C_0^\infty(\Omega)$ and from Theorems 3.3.9 and 3.4.17 we also have $\lim_{z\to x} u(z) = 0$ for all $x \in \partial\Omega$, i.e. u is the classical solution to $(*)$.

◀

4.9 Lemma (Rough estimate for Green's function).
Let $G(x,y)$ be Green's function for a bounded domain $\Omega \subset \mathbb{R}^d$. Then it holds

$$0 \le G(x,y) \le K|x-y|^{2-d}, \quad \forall (x \ne y) \in \Omega.$$

▶

⟨*1*⟩ *Estimate from below*
Let $x \in \Omega$ be arbitrary but fixed. The corrector function $\varphi(x,\cdot)$ is harmonic and therefore bounded. Since $\Phi(x,\cdot)$ is positive and unbounded in every pricked neighborhood of x, there exists $\varepsilon > 0$ such that $G(x,y) > 0$ for all $y \in B(x,\varepsilon) \setminus \{x\}$. We now apply the minimum principle to $G(x,\cdot)$ in $\Omega \setminus B(x,\varepsilon)$ to obtain the claim.

⟨*2*⟩ *Estimate from above*
From minimum principle for the corrector function we have $\varphi \ge 0$. The representation formula now implies

$$G(x,y) \le \Phi(y-x) \le K|x-y|^{2-d}, \quad \forall (x \ne y) \in \Omega.$$

◀

4.10 Motivation.
From the structure of Green's function one sees that $G(x,y_n) \to 0$ for $x \in \Omega$, $x \ne y_n \to y \in \partial\Omega$, as $n \to \infty$. The symmetry of Green's function implies also $G(x_n,y) \to 0$ for $y \in \Omega$, $y \ne x_n \to x \in \partial\Omega$, as $n \to \infty$. The following inequality reflects this behavior. Another possible way of looking at this result is to consider a sequence $(x_n,y_n)_{n\in\mathbb{N}}$ with $x_n \to x \in \partial\Omega$, $0 \ne |y_n - x_n| \to 0$ as $n \to \infty$. The following estimates states that $G(x_n,y_n)$ will stay uniformly bounded provided

$$\operatorname{dist}(x_n,\partial\Omega) \le K|y_n - x_n|^{d-1}.$$

4.11 Lemma (Combined estimate for Green's Function, [40]).
Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and convex set. Let G be Green's function for Ω. Then, it holds

$$G(x,y) \le K|x-y|^{1-d}\delta(x), \quad \forall (x \ne y) \in \Omega,$$

where $\delta(x) := \operatorname{dist}(x,\partial\Omega)$.

▶

⟨*1*⟩ *Auxiliary function*

We construct a harmonic function to be used later in a comparison argument. Denote $A := B_{\frac{11}{5}}(0) \setminus \overline{B_{\frac{1}{5}}(0)}$ and let $h\colon A \to \mathbb{R}$ be the classical solution of

$$\begin{aligned} -\Delta h &= 0, \quad x \in A, \\ h(x) &= 0, \quad x \in \partial B_{\frac{1}{5}}(0), \\ h(x) &= 1, \quad x \in \partial B_{\frac{11}{5}}(0). \end{aligned}$$

We show that

$$h(tz) \leq \varkappa\left(t - \frac{1}{5}\right), \quad \forall t \in \left[\frac{1}{5}, \frac{11}{5}\right], \quad \forall z \in S_1 := \{x \in \mathbb{R}^d \mid |x| = 1\}$$

holds for some $\varkappa > 0$. Since the limit

$$\lim_{t \to \frac{1}{5}+0} \frac{h(tz)}{t - \frac{1}{5}} = \lim_{\tau \to 0+0} \frac{h(z\tau + \frac{z}{5}) - h(\frac{z}{5})}{\tau} = \frac{\partial}{\partial z} h\left(\frac{z}{5}\right) = \operatorname{grad} h\left(\frac{z}{5}\right) \cdot z$$

is uniformly bounded on S_1 ([23], Ch. 6), the claim follows. We also define $A_\varepsilon := \varepsilon A = \{\varepsilon x \mid x \in A\}$ for all $\varepsilon > 0$ and the corresponding functions $h_\varepsilon\colon A_\varepsilon \to \mathbb{R}$ by $h_\varepsilon(x) := h(\frac{x}{\varepsilon})$, obtaining

$$h_\varepsilon(tz) = h\left(\frac{tz}{\varepsilon}\right) \leq \varkappa\left(\frac{t - \frac{\varepsilon}{5}}{\varepsilon}\right), \quad \forall t \in \left[\frac{\varepsilon}{5}, \frac{11\varepsilon}{5}\right], \quad \forall z \in S_1.$$

⟨*2*⟩ *Estimate*

To prove the estimate we fix a point $y \in \Omega$ and consider two cases

(i) $|x - y| \leq 2\delta(x)$. In this case we have

$$G(x, y) \leq K|x - y|^{2-d} \leq K^*|x - y|^{1-d}\delta(x).$$

(ii) $|x - y| > 2\delta(x)$ $[\Leftrightarrow \delta(x) < \frac{|x-y|}{2}]$.

Setting $\varepsilon := \frac{1}{4}|x - y|$ we obtain $2\varepsilon = \frac{|x-y|}{2} > \delta(x)$. Let $x' \in \partial\Omega$ be a point with $|x - x'| = \delta(x)$ and denote $x_0 := x' + \frac{\varepsilon}{5}\frac{(x'-x)}{|(x'-x)|}$. Since $|x - x'| = \delta(x)$ and Ω is convex, the open half-space $\{p \in \mathbb{R}^d \mid \langle p - x', x_0 - x'\rangle > 0\}$ does not intersect $\overline{\Omega}$. Therefore, $\overline{B(x_0, \frac{\varepsilon}{5})} \cap \overline{\Omega} = \{x'\}$.

We show that the estimate $|x - z| \leq 3\varepsilon$ holds for $z \in \overline{\Gamma}$, $\overline{\Gamma} := \overline{B(x_0, \frac{11}{5}\varepsilon)} \cap \overline{\Omega}$. Denote by z' the projection of z onto the line joining x and x', i.e. $z - x = (z - z') + (z' - x)$, $\langle z - z', z' - x\rangle = 0$ (see Figure 4.1). Now

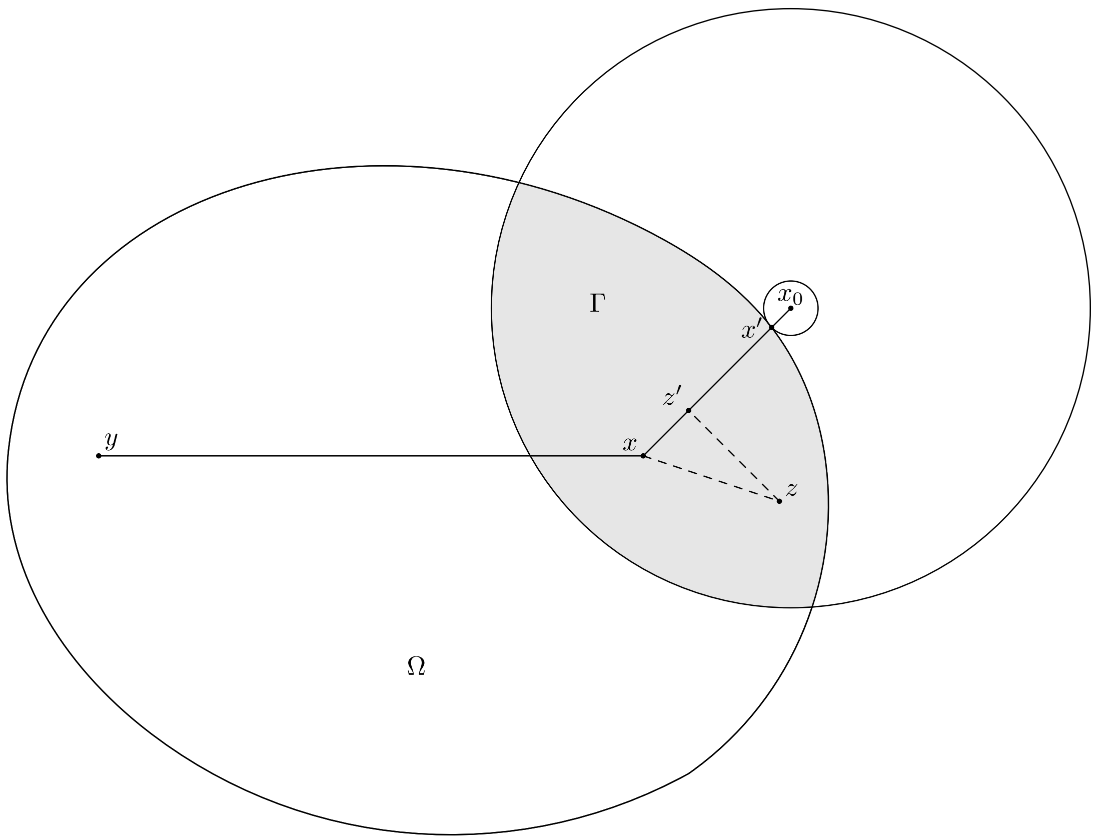

Figure 4.1

$$|z-x| = \sqrt{|z'-x|^2 + |z-z'|^2} \le \sqrt{(2\varepsilon)^2 + \left(\frac{11}{5}\varepsilon\right)^2} = \sqrt{4\varepsilon^2 + \frac{121}{25}\varepsilon^2}$$
$$< 3\varepsilon.$$

Since $|x-y| = 4\varepsilon$, this implies

$$\frac{1}{4}|x-y| = \varepsilon < |x-y| - |x-z| \le |z-y|, \quad \forall z \in \overline{\Gamma}.$$

So, the function $G(\cdot, y)$ is harmonic in Γ for our fixed y and satisfies the following boundary conditions

$$\begin{aligned} G(z,y) &= 0, \quad \forall z \in \partial\Gamma \cap \partial\Omega, \\ G(z,y) &\le K|z-y|^{2-d} \le K^*|x-y|^{2-d} \\ &\le: \tilde{K}|x-y|^{2-d}, \quad \forall z \in \partial\Gamma \setminus \partial\Omega. \end{aligned}$$

We construct the following comparison function

$$H(z) := h_\varepsilon(z-x_0)\tilde{K}|x-y|^{2-d}, \quad z \in \overline{\Gamma}.$$

By construction $H(z)$ is harmonic in Γ with

$$H(z) \equiv \tilde{K}|x-y|^{2-d} \quad \forall z \in \partial\Gamma \setminus \partial\Omega,$$
$$H(z) \geq 0, \quad \forall z \in \partial\Gamma \cap \partial\Omega.$$

So, by comparison principle we have

$$G(z,y) \leq H(z) = h_\varepsilon(z-x_0)\tilde{K}|x-y|^{2-d}, \quad z \in \overline{\Gamma}.$$

Particularly, since $x \in \Gamma$ we also have

$$G(x,y) \leq h_\varepsilon(x-x_0)\tilde{K}|x-y|^{2-d} \leq \varkappa\tilde{K}\frac{|x-x_0|-\varepsilon/5}{\varepsilon}|x-y|^{2-d}$$
$$= \varkappa\tilde{K}\frac{\delta(x)}{\varepsilon}|x-y|^{2-d} \leq K^*\delta(x)|x-y|^{1-d}.$$

◀

4.12 Motivation.
We now prove the previous result in a fashion which is more suitable for the discrete case.

▶
⟨*1*⟩ *Auxiliary function*
We construct a harmonic function to be used later in a comparison argument. Denote $Q_{r_1,r_2} := \{x \in \mathbb{R}^d \mid r_1 \leq |x|_\infty, |x|_1 \leq r_1 + r_2\}$, $r_2 := \frac{1}{2\sqrt{d}}$, $r_1 := \frac{1}{2\sqrt{d}d}$ and let $h\colon Q_{r_1,r_2} \to \mathbb{R}$ be the solution of

$$-\Delta h = 0, \quad x \in Q_{r_1,r_2},$$
$$h(x) = 0, \quad |x|_\infty = r_1,$$
$$h(x) = 1, \quad |x|_1 = r_1 + r_2.$$

The choice of r_1 and r_2 implies that Q_{r_1,r_2} is a domain. Really, since from $|x|_\infty = r_1$ follows $|x|_1 \leq dr_1$, it is sufficient to have

$$dr_1 < r_1 + r_2 \Leftrightarrow r_1 < \frac{r_2}{d-1} :\Leftrightarrow \frac{1}{2\sqrt{d}d} < \frac{1}{2\sqrt{d}(d-1)}.$$

From the classical Schauder ([23], Ch. 6) estimates we have

$$h(te_1) \leq \varkappa(t-r_1), \quad \forall t \in [r_1, r_1+r_2],$$

with some $\varkappa > 0$. By scaling

$$Q^\varepsilon_{r_1,r_2} := \varepsilon Q_{r_1,r_2} = \{\varepsilon x \mid x \in Q_{r_1,r_2}\}$$

the corresponding function $h_\varepsilon\colon Q^\varepsilon_{r_1,r_2} \to \mathbb{R}$, $h_\varepsilon(x) := h\left(\frac{x}{\varepsilon}\right)$ satisfy

$$h_\varepsilon(te_1) = h\left(\frac{te_1}{\varepsilon}\right) \leq \varkappa\left(\frac{t}{\varepsilon} - r_1\right) = \varkappa\left(\frac{t-r_1\varepsilon}{\varepsilon}\right), \quad \forall t \in [\varepsilon r_1, \varepsilon(r_1+r_2)].$$

⟨*2*⟩ *Estimate*

To prove the estimate we fix a point $y \in \Omega$ and consider two cases

(i) $|x-y|_1 \le 4\sqrt{d}\delta(x)$. In this case we have

$$\begin{aligned} G(x,y) &\le K|x-y|^{2-d} = K|x-y|^{1-d}|x-y| \\ &\le K^*|x-y|^{1-d}|x-y|_1 \le K|x-y|^{1-d}\delta(x). \end{aligned}$$

(ii) $|x-y|_1 > 4\sqrt{d}\delta(x)$ $[\Leftrightarrow \delta(x) < \frac{|x-y|_1}{4\sqrt{d}}]$.
Setting

$$\varepsilon := |x-y|_1 > 4\sqrt{d}\delta(x) \Rightarrow r_2\varepsilon > \frac{4\sqrt{d}}{2\sqrt{d}}\delta(x) = 2\delta(x).$$

Let $x' \in \partial\Omega$ be a point with $|x-x'| = \delta(x)$ and denote $x_0 := x' + r_1\varepsilon\frac{(x'-x)}{|x-x'|}$. Let $O \in SO(d)$ be a rigid rotation that transforms the vector $x_0 - x'$ to $r_1\varepsilon e_1$. We now define

$$\begin{aligned} C_r(z) &:= \{y \in \mathbb{R}^d \mid |O(y-z)|_1 < r\} = \{y \in \mathbb{R}^d \mid |O(y)|_1 < r\} + z, \\ Q_r(z) &:= \{y \in \mathbb{R}^d \mid |O(y-z)|_\infty < r\} = \{y \in \mathbb{R}^d \mid |O(y)|_\infty < r\} + z. \end{aligned}$$

By construction and convexity we have (see Figure 4.2).

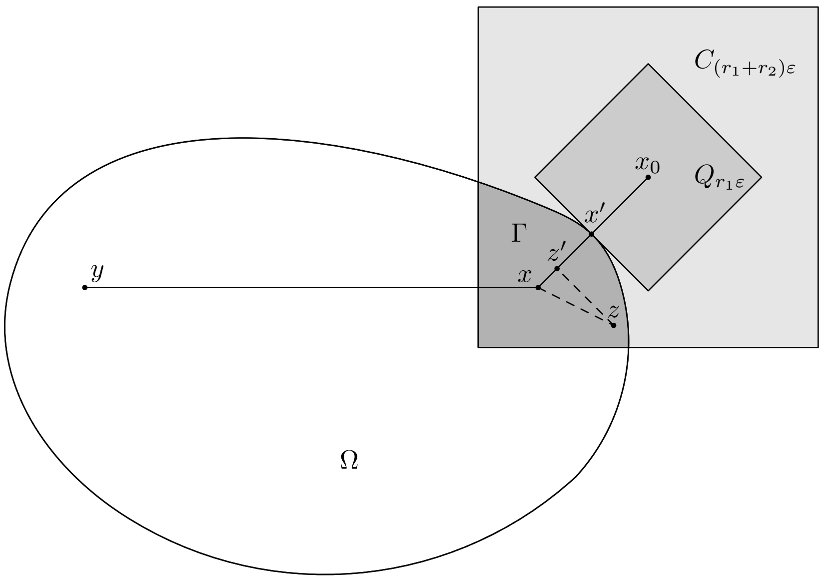

Figure 4.2

$$\begin{aligned} &\overline{Q_{r_1\varepsilon}} \cap \Omega = \emptyset, \\ &\overline{Q_{r_1\varepsilon}} \cap \overline{\Omega} \supset \{x'\}. \end{aligned}$$

We now show that the estimate $|x-z| \le \varepsilon\sqrt{2}r_2$ holds for $z \in \overline{\Gamma}$, $\Gamma := C_{(r_1+r_2)\varepsilon}(x_0) \cap \Omega$. Denote by z' the projection of z onto the line joining x and x', i.e.

$$\begin{cases} (z-z')+(z'-x) = z-x, \\ \langle z-z', z'-x\rangle = 0. \end{cases}$$

By geometrical properties of $C_{(r_1+r_2)\varepsilon}$ we obtain

$$|z-x| = \sqrt{|z'-x|^2+|z-z'|^2} \le \sqrt{(r_2\varepsilon)^2+(r_2\varepsilon)^2} = \sqrt{2}r_2\varepsilon.$$

We now have

$$\begin{aligned} |z-y| &\ge |x-y|-|x-z| \ge \frac{1}{\sqrt{d}}|x-y|_1 - |x-z| \ge \frac{1}{\sqrt{d}}\varepsilon - \sqrt{2}r_2\varepsilon \\ &= \varepsilon \underbrace{\left(\frac{1}{\sqrt{d}} - \frac{\sqrt{2}}{2\sqrt{d}}\right)}_{:=\gamma>0} = \gamma\varepsilon = \gamma|x-y|_1 \ge \gamma|x-y| \end{aligned}$$

for all $z \in \overline{\Gamma}$. So, the function $G(\cdot, y)$ is harmonic in Γ for our fixed y and satisfies the following boundary condition

$$\begin{aligned} &G(z,y) = 0, && \forall z \in \partial\Gamma \cap \partial\Omega, \\ &G(z,y) \le K|z-y|^{2-d} \le: \tilde{K}|x-y|^{2-d}, && \forall z \in \partial\Gamma \setminus \partial\Omega. \end{aligned}$$

We construct the following comparison function

$$H(z) := h_\varepsilon(O(z-x_0))\tilde{K}|x-y|^{2-d}, \quad z \in \overline{\Gamma}.$$

By construction $H(z)$ is harmonic in Γ with

$$\begin{aligned} &H(z) \ge 0, && z \in \partial\Gamma \cap \partial\Omega, \\ &H(z) \equiv \tilde{K}|x-y|^{2-d}, && z \in \partial\Gamma \setminus \partial\Omega. \end{aligned}$$

So, by comparison principle we have

$$G(z,y) \le H(z) = h_\varepsilon(z-x_0)\tilde{K}|x-y|^{2-d}, \quad z \in \Gamma.$$

Particularly, since $x \in \Gamma$ and moreover $O(x-x_0) \in \operatorname{span}\{e_1\}$, we also have

$$\begin{aligned} G(x,y) &\le h_\varepsilon(O(x-x_0))\tilde{K}|x-y|^{2-d} \le \varkappa\tilde{K}\left[\frac{|x-x_0|-r_1\varepsilon}{\varepsilon}\right]|x-y|^{2-d} \\ &\le \varkappa K\frac{\delta(x)}{\varepsilon}|x-y|^{2-d} \le \varkappa K\delta(x)|x-y|^{1-d}\underbrace{\frac{|x-y|}{|x-y|_1}}_{\le 1} \\ &\le K^*\delta(x)|x-y|^{1-d}. \end{aligned}$$

◀

4.13 Lemma (Estimates for the derivatives of Green's function, [40])**.**
Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and convex set. Let G be the Green's function for Ω. Then the estimate holds

$$|\partial_{x_i} G(x,y)| \leq K|x-y|^{1-d}, \quad \forall (x \neq y) \in \Omega,\ 1 \leq i \leq d.$$

▶

⟨*1*⟩ *Case distinction*
Let y be arbitrary but fixed. As above we denote $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ and consider two cases

⟨*2*⟩ $|x-y| \geq \delta(x)$
Denote $R := \frac{\delta(x)}{2}$. The function $G(\cdot, y)$ is harmonic in $B(x,R)$ since $B(x,R) \Subset \Omega$ and $|x-y| \geq 2R$. We can therefore write

$$\begin{aligned} |\partial_{x_i} G(x,y)| &= \frac{1}{\omega_d R^d} \left| \int_{B(x,R)} \partial_{\xi_i} G(\xi, y) \, \mathbf{d}\,\xi \right| \\ &\leq \frac{1}{\omega_d R^d} \int_{\partial B(x,R)} G(\xi,y) \frac{|\xi_i - x_i|}{R} \, \mathbf{d}\,\sigma_\xi \qquad (*) \\ &\leq \frac{1}{\omega_d R^d} \int_{\partial B(x,R)} G(\xi,y) \, \mathbf{d}\,\sigma_\xi. \end{aligned}$$

Since for $\xi \in \partial B(x,R)$

$$\begin{aligned} &\delta(\xi) \leq \delta(x) + |x - \xi| = 3R, \\ &|\xi - y| \geq |x-y| - |x-\xi| = |x-y| - R \geq |x-y| - \frac{|x-y|}{2} = \frac{|x-y|}{2}, \end{aligned}$$

using Lemma 4.11 we obtain

$$\begin{aligned} |\partial_{x_i} G(x,y)| &\leq \frac{K}{\omega_d R^d} \int_{\partial B(x,R)} |\xi - y|^{1-d} \delta(\xi) \, \mathbf{d}\,\sigma_\xi \\ &\leq \frac{K^*}{\omega_d R^d} |x-y|^{1-d} R \int_{\partial B(x,R)} \mathbf{d}\,\sigma_\xi \\ &\leq K|x-y|^{1-d}. \end{aligned}$$

⟨*3*⟩ $|x-y| < \delta(x)$
Denote $R := \frac{|x-y|}{2}$. Once again $G(\cdot, y)$ is harmonic in $B(x,R) \Subset \Omega$. Since

$|\xi - y| \geq |x-y| - |x-\xi| = \frac{|x-y|}{2}$ for $\xi \in \partial B(x,R)$, using $(*)$ together with Lemma 4.9 we obtain

$$\begin{aligned}
|\partial_{x_i} G(x,y)| &\leq \frac{1}{\omega_d R^d} \int_{\partial B(x,R)} G(\xi,y)\,\mathbf{d}\,\sigma_\xi \\
&\leq \frac{K}{\omega_d R^d} \int_{\partial B(x,R)} |\xi - y|^{2-d}\,\mathbf{d}\,\sigma_\xi \\
&\leq \frac{K^*}{\omega_d R^d} |x-y|^{2-d} d\omega_d R^{d-1} \\
&\leq K\frac{|x-y|^{2-d}}{R} = K^*|x-y|^{1-d}.
\end{aligned}$$

◀

4.2 Riesz potentials

4.14 Motivation.
In order to obtain the L^p estimates for Green's function we introduce the following potential operator. Lemma 4.17, Theorem 4.18 and Theorem 4.5 are taken from [23], Ch. 7. The proofs of the first two results will be translated into discrete setting in Chapter 7.

4.15 Definition (Riesz Potential Operator)**.**
The Riesz Potential Operator is defined by

$$(V_\mu f)(x) = \int_\Omega |x-y|^{d(\mu-1)} f(y)\,\mathbf{d}\,y$$

with $\mu \in (0,1]$ and $\Omega \subset \mathbb{R}^d$, $|\Omega| := \mathrm{meas}(\Omega) < \infty$.

4.16 Motivation (Well-Posedness and Continuity)**.**
Our next aim will be to show that V_μ is well-posed and continuous as a mapping from $L^p(\Omega)$ into $L^q(\Omega)$ for some $p, q \geq 1$. In order to apply the Young inequality for convolution we need the following lemma.

4.17 Lemma (L^p estimate for the Riesz Potential)**.**
Let $\Omega \subset \mathbb{R}^d$ be bounded and $x \in \Omega$ be fixed. Then $h(y) := |x-y|^{d(\mu-1)} \in L^p(\Omega)$ for all $1 \leq p < \frac{1}{1-\mu}$, $\mu < 1$ and all $1 \leq p \leq \infty$, $\mu = 1$, with

$$\|h\|_p \leq \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_d^{1-\mu} |\Omega|^{\mu-\delta}$$

where $\delta := 1 - p^{-1}$.
▶

⟨*1*⟩ L^1 *estimate*
Since $h(y)$ is radial-symmetric around x and decreasing along axes of symmetry, changing Ω to a ball $B(x,R)$ with $w_d R^d = |B(x,R)| = |\Omega|$ will not decrease the integral:

$$\int_\Omega |x-y|^{d(\mu-1)}\,\mathbf{d}\,y \le \int_{B(x,R)} |x-y|^{d(\mu-1)}\,\mathbf{d}\,y = d\omega_d \int_0^R r^{d(\mu-1)} r^{d-1}\,\mathbf{d}\,r$$
$$= \frac{\omega_d R^{d\mu}}{\mu} = \frac{\omega_d^{1-\mu}|\Omega|^\mu}{\mu}.$$

⟨*2*⟩ L^p *estimate*
The assumption $1 \le p < \frac{1}{1-\mu}$, $\mu < 1$ implies that $d(\mu-1)p = d(\mu'-1)$ for some $\mu' \in (0,1)$, since

$$\mu - 1 \ge p(\mu-1) > -1 \Rightarrow 1 > \mu \ge \mu' = p(\mu-1)+1 > 0.$$

We therefore have

$$\|h\|_p \le \left(\frac{\omega_d^{1-\mu'}|\Omega|^{\mu'}}{\mu'}\right)^{\frac{1}{p}} = (p(\mu-1)+1)^{-\frac{1}{p}}\omega_d^{1-\mu}|\Omega|^{\mu-1+\frac{1}{p}}$$
$$= \left(\frac{\mu-1+1-\delta}{1-\delta}\right)^{\delta-1}\omega_d^{1-\mu}|\Omega|^{\mu-\delta} = \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta}\omega_d^{1-\mu}|\Omega|^{\mu-\delta}$$

for $\mu \in (0,1)$ and

$$\|h\|_p = |\Omega|^{1/p} = |\Omega|^{1-(1-1/p)}, \quad 1 \le p < \infty,$$
$$\|h\|_\infty = 1 = |\Omega|^0$$

for $\mu = 1$.

◀

4.18 Theorem (L^p estimates for the Riesz potential operator)**.**
The operator V_μ maps $L^p(\Omega)$, $1 \le p \le \infty$ continuously into $L^q(\Omega)$ for any q, $1 \le q \le \infty$ satisfying

$$0 \le \delta := \frac{1}{p} - \frac{1}{q} < \mu.$$

Under this assumption we have

$$\|V_\mu f\|_q \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta}\omega_d^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_p, \quad \forall f \in L^p(\Omega).$$

▶

⟨*1*⟩ $\mu = 1$
For $1 \le q < \infty$ we get

$$\|V_1 f\|_q = \left(\int_\Omega \left|\int_\Omega f(y)\,\mathbf{d}\,y\right|^q \mathbf{d}\,x\right)^{\frac{1}{q}} \le \left(\int_\Omega \|f\|_1^q \,\mathbf{d}\,x\right)^{\frac{1}{q}} = \|f\|_1 |\Omega|^{\frac{1}{q}}$$
$$\overset{p>1}{\le} |\Omega|^{\frac{1}{q}+\frac{1}{p'}} \|f\|_p = |\Omega|^{1-\delta}\|f\|_p, \forall\forall f \in L^p(\Omega)$$

for all $1 \le p \le \infty$ and for $q = \infty$ we get

$$\|V_1 f\|_\infty \le \|f\|_1 \overset{p>1}{\le} |\Omega|^{1/p'}\|f\|_p, \quad \forall f \in L^p(\Omega)$$

also for all $1 \le p \le \infty$.

⟨*2*⟩ $\mu \in (0,1)$
Denote $r := \frac{1}{1-\delta}$. From $1 \ge 1-\delta > 1-\mu > 0$ we obtain $1 \le r < \frac{1}{1-\mu}$, implying

$$h(y) := |x-y|^{d(\mu-1)} \in L^r(\Omega), \quad x \in \Omega$$

with uniform in x bound

$$\|h\|_r \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta} \omega_d^{1-\mu} |\Omega|^{\mu-\delta} =: C.$$

Writing

$$h|f| = h^{r/q}|f|^{p/q} \cdot h^{r(1-1/p)} \cdot |f|^{p\delta}$$

where

$$\frac{r}{q} + r\left(1 - \frac{1}{p}\right) = r(1-\delta) = 1,$$
$$\frac{p}{q} + p\delta = p\left(\frac{1}{q} + \frac{1}{p} - \frac{1}{q}\right) = 1$$

we can apply the Hölder inequality with three multipliers (assuming $q < \infty$ and w.l.o.g $\|f\|_p \ne 0$)

$$I := |(V_\mu f)(x)| \le \int_\Omega h(x-y)|f(y)|\,\mathbf{d}\,y$$
$$\le \left\{\int_\Omega h^r(x-y)|f(y)|^p \,\mathbf{d}\,y\right\}^{1/q} \left\{\int_\Omega h^r(x-y)\,\mathbf{d}\,y\right\}^{1-1/p} \left\{\int_\Omega |f(y)|^p \,\mathbf{d}\,y\right\}^{\delta}.$$

Since the last two multipliers can be estimated independently of x we obtain

$$\begin{aligned}
\|V_\mu f\|_q &\le \left\{\int_\Omega\int_\Omega h^r(x-y)|f(y)|^p \,\mathbf{d}\,y\,\mathbf{d}\,x\right\}^{1/q} C^{r-r/p}\|f\|_p^{p\delta} \\
&= \left\{\int_\Omega\int_\Omega h^r(y-x)\,\mathbf{d}\,x|f(y)|^p\,\mathbf{d}\,y\right\}^{1/q} C^{r-r/p}\|f\|_p^{p\delta} \\
&\le \sup_{y\in\Omega}\left\{\int_\Omega h^r(y-x)\,\mathbf{d}\,x\right\}^{1/q} \|f\|_p^{p/q} C^{r-r/p}\|f\|_p^{p\delta} \\
&= C^{r(1-1/p+1/q)}\|f\|_p^{p\delta+p/q} = \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta}\omega_d^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_p.
\end{aligned}$$

We have used $h(x-y)=h(y-x)$ and the Fubini-Tonelli theorem.
For $q=\infty$ we have $r=p'$ and therefore

$$\|V_\mu f\|_\infty \le \|h\|_r\|f\|_p \le \left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta}\omega_d^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_p.$$

◀

4.19 Remark (L^p estimates for the Riesz potential operator)**.**
The previous result still holds in the case $\delta=\mu$ for $p>1$ (the proof requires the classical integral Hardy-Littlewood inequality [26]). For the corresponding result with $\Omega=\mathbb{R}^d$ see Chapter 5 in [38]. The same remark applies to the case $\frac{dp}{d+p}=q$ in the following theorem. The questions concerning the regularity of the boundary are discussed in [14] and [24].

4.3 Linear a priori estimate

4.20 Theorem (Regularity Estimates)**.**
Let $\Omega\subset\mathbb{R}^n$ be a bounded and convex domain. Let $f\in L^q(\Omega)$, $q>\frac{dp}{d+p}$ for some $1<p<\infty$. Then, there exists a unique weak solution $u\in W_0^{1,p}(\Omega)$ of the boundary-value problem

$$\begin{aligned}
-\Delta u(x) &= f(x), \quad x\in\Omega,\\
u(x) &= 0, \qquad x\in\partial\Omega
\end{aligned}$$

and it holds

$$\|u\|_{1,p}\le C(d,p,q,\Omega)\|f\|_q,$$

with some $C(d,p,q,\Omega)>0$.

▶

⟨*1*⟩ *Restricting q*

Without loss of generality we can assume $q \in (\frac{dp}{d+p}, p] \neq \emptyset$, since

$$\|f\|_q \leq K(q,p,\Omega)\|f\|_p \leq K^*(q,p,\tilde{q},\Omega)\|f\|_{\tilde{q}}$$

for all $q \leq p < \tilde{q}$.

⟨*2*⟩ *Estimates for Green's function*

We want to show that under our assumption $f \mapsto G * f$ is a bounded operator from $L^q(\Omega)$ to $W^{1,p}(\Omega)$. Since all finite-dimensional norms are equivalent, it is sufficient to show that

$$\begin{aligned} \|G * f\|_p &\leq K\|f\|_q, \quad \forall f \in L^q(\Omega), \\ \|\partial_i (G * f)\|_p &\leq K\|f\|_q, \quad \forall f \in L^q(\Omega), \quad 1 \leq i \leq d. \end{aligned}$$

Furthermore, Lemmata 4.9 and 4.13 yield

$$\begin{aligned} \left| \int_\Omega G(x,y) f(y) \, \mathbf{d}\, y \right| &\leq \int_\Omega G(x,y)|f(y)| \, \mathbf{d}\, y \leq K \int_\Omega \frac{|f(y)|}{|x-y|^{d-2}} \, \mathbf{d}\, y, \\ \left| \int_\Omega \partial_{x_i} G(x,y) f(y) \, \mathbf{d}\, y \right| &\leq \int_\Omega |\partial_{x_i} G(x,y)||f(y)| \, \mathbf{d}\, y \leq K \int_\Omega \frac{|f(y)|}{|x-y|^{d-1}} \, \mathbf{d}\, y, \end{aligned}$$

i.e. it is enough to show that $V_{1/d}$ and $V_{2/d}$ have the desired properties. By Lemma 4.18 this is the case, if

$$0 \overset{!}{\leq} \frac{1}{q} - \frac{1}{p} \overset{!}{<} \frac{1}{d} < \frac{2}{d}.$$

From our assumption we have

$$0 \leq \frac{1}{q} - \frac{1}{p} < \frac{d+p}{dp} - \frac{1}{p} = \frac{1}{d}.$$

⟨*3*⟩ *Approximating solution*

We approximate f with $(f_k)_{k\in\mathbb{N}} \subseteq C_0^\infty(\Omega)$ in L^q,

$$\|f - f_k\|_q \to 0 \quad \text{as } k \to \infty.$$

By the classical existence result 4.5 and the representation theorem 4.8 there exist $(u_k)_{k\in\mathbb{N}} \subseteq C^2(\Omega) \cap C(\overline{\Omega})$, solutions of

$$\begin{aligned} -\Delta u_k(x) &= f_k(x), \quad x \in \Omega, \\ u_k(x) &= 0, \qquad\quad x \in \partial\Omega. \end{aligned}$$

with $u_k = G * f_k$. The a priori estimate

$$\|u_k\|_{1,p} \leq C(d,p,q,\Omega)\|f_k\|_q$$

yields the uniform boundedness of u_k,$k \in \mathbb{N}$, implying the existence of a weak limit $u_k \rightharpoonup u$ in $W^{1,p}(\Omega)$ as $k \to \infty$. Passing to the limit in the weak formulation of our problem, we see that u is indeed a weak solution.

⟨*4*⟩ *Uniqueness*
Suppose that $u_1, u_2 \in W_0^{1,p}(\Omega)$ are two weak solutions for our problem. Then $u = u_1 - u_2$ satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi \, \mathbf{d}\, x = 0, \quad \forall \varphi \in C_0^\infty(\Omega)$$

By Hölder inequality this holds also for all $\varphi \in W_0^{1,p'}(\Omega)$. For $u \in L^p(\Omega)$ we have $|u|^{p-2}u \in L^{p'}(\Omega)$. From ⟨*2*⟩ we can choose $\varphi \in W_0^{1,p'}(\Omega)$ as a weak solution of

$$\begin{aligned} -\Delta\varphi &= |u|^{p-2}u, && \text{in } \Omega \\ \varphi &= 0, && \text{on } \partial\Omega. \end{aligned}$$

Testing the weak formulation of this equation against $u \in W_0^{1,p}(\Omega)$ we obtain

$$0 = \int_\Omega \nabla u \cdot \nabla \varphi \, \mathbf{d}\, x = \int_\Omega |u|^p \, \mathbf{d}\, x \Rightarrow u = 0.$$

◀

Chapter 5

Discrete Laplace Operator

5.1 Discretizations

5.1 Convention.
We will always assume that the dimension of the underlying space is $d \geq 3$.

5.2 Notation (Grid, Discrete Neighbors)**.**
For $0 < h \leq 1$ we denote by $\mathcal{G}_h := \{hk \mid k \in \mathbb{Z}^d\}$ the *equidistant grid* on $\mathbb{R}^d$ with *grid spacing* h. We call subsets of $\mathcal{G}_h$ *discrete sets.* For $x \in \mathcal{G}_h$ we define

$$N_h(x) := \{x \pm he_i \mid 1 \leq i \leq d\} \subset \mathcal{G}_h$$

the *set of discrete neighbors.* Note that $x \notin N_h(x)$.

5.3 Remark (Discrete Laplace)**.**
We want to use the classical discretization of the Laplace operator which determines the way we define $N_h(x)$ and, successively, other "discrete topological" concepts. For other discretizations and operators we may need to use different definitions of the discrete neighborhood.

5.4 Motivation (Discretization, Recovery)**.**
We will now consider the two following questions concerning discretization: firstly, how general sets in $\mathbb{R}^d$ are discretized, and, secondly, which sets can be uniquely recovered from their discretizations.

5.5 Notation (Domain Discretization)**.**
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $h > 0$. For $x \in \Omega \cap \mathcal{G}_h$ and $y \in N_h(x)$ we set

$$n_\Omega(x, y) := x + \alpha(y - x),$$

where $\alpha := \sup\{\gamma \in [0, h] \mid x + s\gamma(y - x) \in \Omega \;\; \forall s \in [0, 1)\}$. We denote for $x \in \Omega \cap \mathcal{G}_h$

$$N_h^\Omega(x) := \{n_\Omega(x, y) \mid y \in N_h(x)\}$$

the *set of discrete neighbors w.r.t.* Ω.
We define the *discretized interior* by

$$(\Omega)_h^\circ := \{x \in \Omega \cap \mathcal{G}_h \mid N_h(x) = N_h^\Omega(x)\}.$$

Further, we define the *discretized boundary layer* by

$$(\Omega)_h' := (\Omega \cap \mathcal{G}_h) \setminus (\Omega)_h^\circ.$$

In other words, $\Omega \cap \mathcal{G}_h$ consists of two types of points: those which can be connected with all their discrete neighbors by open line segments lying fully in Ω and those which cannot. We also define the *discretized boundary* by

$$\partial_h \Omega := \cup_{x \in \Omega \cap \mathcal{G}_h} \cup_{y \in N_h(x)} \{n_\Omega(x,y) \mid n_\Omega(x,y) \in \partial\Omega\},$$

i.e. in general $\partial_h \Omega \not\subset \mathcal{G}_h$ and we set

$$\mathrm{cl}_h\, \Omega := (\Omega \cap \mathcal{G}_h) \cup \partial_h \Omega \subset \overline{\Omega}.$$

for the *discretized closure*.

5.6 Remark (Discrete closure)**.**
Take note that in general $\overline{\Omega} \cap \mathcal{G}_h \not\subset \mathrm{cl}_h\, \Omega \not\subset \mathcal{G}_h$. First, consider $\Omega = (-h,h)^2 \subset \mathbb{R}^2$. Then $(\Omega)_h^\circ = \{(0,0)\}$, $(\Omega)_h' = \emptyset$ and $\partial_h \Omega = \{(0, \pm h), (\pm h, 0)\}$. This shows that

$$\{(h,h)\} \subset \left(\overline{\Omega} \cap \mathcal{G}_h\right) \setminus (\mathrm{cl}_h\, \Omega) \neq \emptyset.$$

Now, let $\Omega := \{x \in \mathbb{R}^2 \mid |x| < 2h\}$. We have $(\Omega)_h^\circ = \{(0,0), (\pm h, 0), (0, \pm h)\}$, $(\Omega)_h' = \{(\pm h, \pm h)\}$ (4 points) and

$$\partial_h \Omega = \{(0, \pm 2h), (\pm 2h, 0), (\pm h, \pm\sqrt{3}h), (\pm\sqrt{3}h, \pm h)\} \not\subset \mathcal{G}_h.$$

5.7 Notation (Discrete Sets)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be bounded. We denote

$$\overline{\Omega}_h := \cup_{x \in \Omega_h} (N_h(x) \cup \{x\}),$$
$$\partial \Omega_h := \overline{\Omega}_h \setminus \Omega_h.$$

the *discrete closure* and the *discrete boundary*, correspondingly.

5.8 Remark (Analogy with open sets)**.**
Take note that for bounded discrete sets one always has $\partial\Omega_h \cap \Omega_h = \emptyset$ and $\Omega_h \subsetneq \overline{\Omega}_h$.

5.9 Convention.
To avoid confusion we use the following convention: discrete sets will always be denoted with the grid spacing in the subscript.

5.10 Definition (Admissible Domains, Grid Spacing and Discrete Sets)**.**
If for $\Omega := \prod_{i=1}^{d}(a_i, b_i)$, $a_i, b_i \in \mathbb{R}$, $a_i < b_i$, $1 \le i \le d$ holds

$$\{a_1, b_1\} \times \cdots \times \{a_d, b_d\} \subset \mathcal{G}_h,$$

i.e. if the corners of Ω belong to $\mathcal{G}_h$, then the domain Ω is called *admissible* w.r.t. h (if h is considered to be fixed) or the grid spacing h is called *admissible* for Ω (if Ω is considered to be fixed).

Take note that not every rectangular domain possesses an admissible grid spacing (e.g. $(0,1) \times (0, \sqrt{3}) \subset \mathbb{R}^2$). If Ω is admissible w.r.t. h, then $(\Omega)'_h = \emptyset$ and defining

$$\Omega_h := (\Omega)^{\circ}_h$$

we get

$$\partial\Omega_h = \partial_h\Omega,$$
$$\overline{\Omega}_h = \operatorname{cl}_h \Omega.$$

Moreover, as long as $\Omega_h \neq \emptyset$ we can recover

$$\Omega = \prod_{i=1}^{d} \left(\min_{x \in \overline{\Omega}_h} x_i, \max_{x \in \overline{\Omega}_h} x_i \right).$$

In general, we call a non-void discrete set $\Omega_h \subset \mathcal{G}_h$ *admissible* if $\Omega_h = (\Omega)^{\circ}_h$ or, equivalently $\overline{\Omega}_h = \operatorname{cl}_h \Omega$ for some admissible (w.r.t. h) domain Ω; in this case we define the *full discretized closure* and the *full discrete boundary* by

$$\operatorname{cl}_h^+ \Omega := \overline{\Omega} \cap \mathcal{G}_h,$$
$$\hat{\partial}\Omega_h := \partial\Omega \cap \mathcal{G}_h.$$

5.11 Notation (Faces)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible. We denote by

$$\partial^+\Omega_h := \cup_{i=1}^{d} \partial_i^+\Omega_h$$
$$\partial_i^+\Omega_h := \partial\Omega_h \cap \{x_i = \max_{x \in \overline{\Omega}_h} x_i\}$$

discrete front face (in the direction i) and by

$$\partial^-\Omega_h = \cup_{i=1}^{d} \partial_i^-\Omega_h$$
$$\partial_i^-\Omega_h = \partial\Omega_h \cap \{x_i = \min_{x \in \overline{\Omega}_h} x_i\}$$

discrete back face (in the direction i).
Take note that

$$\partial_i^+\Omega_h \cap \partial_j^+\Omega_h = \partial_i^-\Omega_h \cap \partial_j^-\Omega_h = \partial_i^+\Omega_h \cap \partial_j^-\Omega_h = \emptyset$$

for $1 \le i \ne j \le d$.
Analogously are defined

$$\begin{aligned}
\hat{\partial}^+\Omega_h &:= \cup_{i=1}^d \hat{\partial}_i^+\Omega_h, \\
\hat{\partial}^-\Omega_h &:= \cup_{i=1}^d \hat{\partial}_i^-\Omega_h, \\
\hat{\partial}_i^+\Omega_h &:= \hat{\partial}\Omega_h \cap \{x_i = \max_{x\in\overline{\Omega}_h} x_i\}, \\
\hat{\partial}_i^-\Omega_h &:= \hat{\partial}\Omega_h \cap \{x_i = \min_{x\in\overline{\Omega}_h} x_i\}.
\end{aligned}$$

5.2 Discrete Green's function

5.12 Definition (Finite difference quotients)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible and let $u\colon \overline{\Omega}_h \to \mathbb{R}$. The *forward* respectively *backward finite difference quotient* are defined as

$$\begin{aligned}
D_i^+u(x) &:= \frac{u(x+e_ih)-u(x)}{h}, \quad x \in \overline{\Omega}_h \setminus \partial_i^+\Omega_h, \\
D_i^-u(x) &:= \frac{u(x)-u(x-e_ih)}{h}, \quad x \in \overline{\Omega}_h \setminus \partial_i^-\Omega_h.
\end{aligned}$$

In the case of several variables we write $D_{x_i}^+u(x,y)$.

5.13 Definition (Discrete Laplace Operator)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $x \in \Omega_h$ and $u\colon \overline{\Omega}_h \to \mathbb{R}$. The *discrete Laplace operator* is defined by

$$\Delta_h u(x) := \frac{1}{h^2}\left(-2du(x) + \sum_{y\in N_h(x)} u(y)\right).$$

5.14 Definition (Discretely connected component)**.**
Two points x, y in $\Omega_h \subset \mathcal{G}_h$ are *discretely path-connected* if there is a sequence $x = x_1, x_2, \dots, x_n = y$, $n \in \mathbb{N}$ such that $x_2, \dots, x_{n-1} \in \Omega_h$ and $x_{i+1} \in N_h(x_i)$, $1 \le i \le n-1$. This defines an equivalence relation that divides Ω_h into equivalence classes which we call *discretely connected components*. A discrete set consisting of a single discretely connected component is called *discrete-connected*.

5.15 Lemma (Discrete Maximum Principle)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $u\colon \overline{\Omega}_h \to \mathbb{R}$ and $\lambda \ge 0$ be such that

$$-\Delta_h u(x) + \lambda u(x) \le 0, \quad x \in \Omega_h.$$

If $\max_{x\in\overline{\Omega}_h} u(x) =: u(\bar{x}) \geq 0$ with some $\bar{x} \in \Omega$, then $u \equiv u(\bar{x})$ on the discretely connected component of $\overline{\Omega}_h$ containing $\bar{x}$. If Ω_h is bounded with

$$u(x) \leq 0, \quad x \in \partial\Omega_h,$$

then $u(x) \leq 0$, $x \in \overline{\Omega}_h$.

▶

For the first part, assume that $\max_{x\in\overline{\Omega}_h} u(x) =: u(\bar{x}) \geq 0$ for some $\bar{x} \in \Omega_h$. Since

$$(-\Delta_h + \lambda)u(\bar{x}) = \frac{1}{h^2}\left((2d + \lambda h^2)u(\bar{x}) - \sum_{y\in N_h(\bar{x})} u(y)\right) \geq \frac{1}{h^2}\sum_{y\in N_h(\bar{x})} [u(\bar{x}) - u(y)]\,,$$

we obtain

$$\sum_{y\in N_h(\bar{x})} [u(\bar{x}) - u(y)] \leq 0.$$

On the other hand, since $u(\bar{x}) \geq u(y)$ for all $y \in \overline{\Omega}_h$, we obtain $u(y) = u(\bar{x})$ for $y \in N_h(\bar{x})$. Repeating this argument, we see that u is constant (and non-negative) on the discretely connected component of $\overline{\Omega}_h$ which contains $\bar{x}$.
For the second part, assume $u(\bar{x}) > 0$ and let $z \in \partial\Omega_h$ belong to the discretely connected component containing $\bar{x}$. The first part yields $u(z) > 0$, a contradiction.

◀

5.16 Theorem (Solvability for the discrete Poisson equation)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be bounded with $f\colon \Omega_h \to \mathbb{R}$, $g\colon \partial\Omega_h \to \mathbb{R}$. Then there is a unique solution $u\colon \overline{\Omega}_h \to \mathbb{R}$ to discrete Poisson equation

$$\begin{aligned} -\Delta_h u(x) &= f(x), \quad x \in \Omega_h,\\ u(x) &= g(x), \quad x \in \partial\Omega_h. \end{aligned}$$

▶

This is a linear system with equal number of variables and equations. The solvability is therefore equivalent to the uniqueness of the trivial solution for

$$\begin{aligned} -\Delta_h u(x) &= 0, \quad x \in \Omega_h,\\ u(x) &= 0, \quad x \in \partial\Omega_h. \end{aligned}$$

This follows from the Discrete Maximum Principle 5.15. See Ch. 4 from [25]) for numerical treatment.

◀

5.17 Notation (Discrete δ-function)**.**
The function $\delta_h\colon \mathcal{G}_h \to \mathbb{R}$ is defined by

$$\delta_h(x) := \begin{cases} \frac{1}{h^d}, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

We set $\delta(x) := \delta_1(x)$.

5.18 Definition (Discrete Green's function)**.**
Let Ω_h be a bounded subset of $\mathcal{G}_h$. We define the *discrete Green's function* (for zero boundary conditions) $G\colon \overline{\Omega}_h \times \Omega_h \to \mathbb{R}$ as the solution of the following DBVP with fixed $y \in \Omega_h$

$$\begin{aligned} -\Delta_{h,x} G(x,y) &= \delta_h(x-y), \quad x \in \Omega_h, \\ G(x,y) &= 0, \qquad\qquad\quad x \in \partial\Omega_h. \end{aligned}$$

5.19 Lemma (Representation formula)**.**
Let Ω_h be a bounded subset of $\mathcal{G}_h$ and let $G\colon \overline{\Omega}_h \times \Omega_h \to \mathbb{R}$ be the corresponding Green's function. Let $u\colon \overline{\Omega}_h \to \mathbb{R}$ be an arbitrary grid function with $u|_{\partial\Omega_h} = 0$. Then the following representation holds

$$u(x) = \sum_{y \in \Omega_h} G(x,y)[-\Delta_h u(y)] h^d, \quad x \in \overline{\Omega}_h.$$

▶
We denote the right hand side of the representation by v. By direct calculations we obtain

$$\begin{aligned} -\Delta_h v(x) &= \sum_{y \in \Omega_h} [-\Delta_{h,x} G(x,y)][-\Delta_{h,y} u(y)] h^d = \sum_{y \in \Omega_h} \delta_h(x-y)[-\Delta_{y,h} u(y)] h^d \\ &= -\Delta_h u(x), \quad x \in \Omega_h, \\ v(x) &= \sum_{y \in \Omega_h} 0 \cdot [-\Delta_{h,y} u(y)] h^d = 0, \quad x \in \partial\Omega_h. \end{aligned}$$

Since $-\Delta_h[v(x) - u(x)] = 0$ for $x \in \Omega_h$ and $v(x) = u(x) = 0$ for $x \in \partial\Omega_h$, the discrete maximum principle 5.15 yields $v(x) \equiv u(x)$ in $\overline{\Omega}_h$.
◀

5.3 Comparison principle

5.20 Motivation.
We want to obtain a result which represents a discrete generalization of the mean value property for harmonic functions (see [23]), namely an estimate for a discrete

difference quotient of a grid function in the center of a $|\cdot|_\infty$ ball given the boundary values of the function, the values of the discrete Laplacian in the ball and the radius of the ball.

Such a result was obtained by Brandt in ([10]). This paper deals rigorously only with the case of the central finite difference quotients with a concluding remark that analogous estimates for the forward f.d.q. can be obtained in a similar manner.

We therefore include full proofs for the forward f.d.q. case and, for the sake of completeness, also the proofs for the central f.d.q. case. The proofs in the rest of this chapter are essentially due to Brandt.

5.21 Definition (Nearest neighbor to a boundary point)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible. For $x \in \partial\Omega_h$ we denote $\check{x} \in \Omega_h$ with $|x - \check{x}| = h$. Since Ω_h is admissible, $\check{x}$ is unique.

5.22 Notation (Disjoint union)**.**
We write $C = A \sqcup B$ for *disjoint union*, i.e. for $C = A \cup B$ with $A \cap B = \emptyset$.

5.23 Lemma (Discrete Maximum Principle for admissible set)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible, with $u\colon \overline{\Omega}_h \to \mathbb{R}$ such that

$$\begin{aligned} -\Delta_h u(x) &\leq 0, \quad x \in \Omega_h, \\ u(x) &\leq 0, \quad x \in \Gamma'_h, \\ u(x) + u(\check{x}) &\leq 0, \quad x \in \Gamma''_h, \end{aligned}$$

where $\Gamma'_h \sqcup \Gamma''_h = \partial\Omega_h$. Then

$$u(x) \leq 0, \quad x \in \Omega_h.$$

▶

Let, by contradiction, $u(\bar{x}) = \max_{x\in\Omega_h} u(x) > 0$ with some $\bar{x} \in \Omega_h$. Exactly as in the proof of the Discrete Maximum Principle 5.15, this implies $u(y) > 0$ in some boundary point $y \in \partial\Omega_h$. If $y \in \Gamma'_h$ we get a direct contradiction. Otherwise, $y \in \Gamma''_h$ and by the construction also $u(\check{y}) > 0$, i.e.

$$u(y) + u(\check{y}) > 0$$

again a contradiction.

◀

5.24 Remark (Generalization)**.**
In the previous result the assumption of admissibility can be replaced with the assumption that $x \mapsto \check{x}$ is uniquely defined only on Γ''_h, since the proof does not change in this case.

5.25 Theorem (Solvability for generalized boundary condition)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible with $\partial\Omega_h = \Gamma'_h \sqcup \Gamma''_h$. Further, let $f\colon \Omega_h \to \mathbb{R}$ and $g\colon \partial\Omega_h \to \mathbb{R}$. Then there exists a unique solution to

$$\begin{aligned} -\Delta_h u(x) &= f(x), \quad x \in \Omega_h, \\ u(x) &= g(x), \quad x \in \Gamma'_h, \\ u(x) + u(\check{x}) &= g(x), \quad x \in \Gamma''_h. \end{aligned}$$

►
Replace Lemma 5.15 with Lemma 5.23 in the proof of Theorem 5.16.
◄

5.26 Lemma (Comparison principle)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible with $u\colon \overline{\Omega}_h \to \mathbb{R}$ such that

$$\begin{aligned} \nu_1 \le -\Delta_h u(x) \le \nu_2, &\qquad x \in \Omega_h \\ \mu_1 \le u(x) \le \mu_2, &\qquad x \in \Gamma'_h \\ 2\mu_1 \le u(x) + u(\check{x}) \le 2\mu_2, &\quad x \in \Gamma''_h, \end{aligned}$$

where $\Gamma'_h \sqcup \Gamma''_h = \partial\Omega_h$ and $\nu_{1,2}, \mu_{1,2} \in \mathbb{R}$. Then if $\overline{\Omega}_h = \mathrm{cl}_h\left((-R,R)^d\right)$ we have

$$\mu_1 + \frac{\nu_1}{2}(R^2 - x_i^2) \le u(x) \le \mu_2 + \frac{\nu_2}{2}(R^2 - x_i^2), \quad \forall x \in \Omega_h,\ 1 \le i \le d$$

and if $\overline{\Omega}_h = \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\}\right)$ we have

$$\mu_1 + \frac{\nu_1}{2} x_i(R - x_i) \le u(x) \le \mu_2 + \frac{\nu_2}{2} x_i(R - x_i), \quad \forall x \in \Omega_h,\ 1 \le i \le d.$$

►
⟨*1*⟩ *First reduction*
Let $i \in \{1, \ldots, d\}$ be arbitrary, but fixed. We denote

$$v(x) := \begin{cases} \dfrac{\nu_2}{2}(R^2 - x_i^2) & \text{if } \overline{\Omega}_h = \mathrm{cl}_h\left((-R,R)^d\right), \\ \dfrac{\nu_2}{2} x_i(R - x_i) & \text{if } \overline{\Omega}_h = \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\}\right). \end{cases}$$

It is only necessary to prove the estimates from above using the upper bounds from the assumptions. Considering $-u$ we then obtain the corresponding estimates from below.

⟨*2*⟩ *Second reduction*
It is also sufficient to show that

$$\begin{aligned} -\Delta_h w(x) &\le \nu_2, \quad x \in \Omega_h, \\ w(x) &\le 0, \quad x \in \Gamma'_h, \\ w(x) + w(\check{x}) &\le 0, \quad x \in \Gamma''_h \end{aligned}$$

for $w\colon \overline{\Omega}_h \to \mathbb{R}$ implies

$$w(x) \le v(x),$$

since plugging in $w := u - \mu_2$ yields the previous claim.

⟨3⟩ *Comparison function*
Direct calculation yields

$$-\Delta_h v(x) = \frac{\nu_2}{2}\Delta_h\left(x_i^2\right) = \frac{\nu_2}{2}\frac{(x_i+h)^2 - 2x_i^2 + (x_i-h)^2}{h^2} = \nu_2, \quad x \in \Omega_h,$$

since constant and linear functions are discrete harmonic. This implies

$$-\Delta_h[w(x) - v(x)] \le 0, \quad x \in \Omega_h$$

and since $v(x) \ge 0$, $x \in \overline{\Omega}_h$ also

$$\begin{aligned} w(x) - v(x) &\le 0, \quad x \in \Gamma'_h, \\ [w(x) - v(x)] + [w(\check{x}) - v(\check{x})] &\le 0, \quad x \in \Gamma''_h. \end{aligned}$$

Applying Lemma 5.23 we get the claim.

◀

5.4 Inner estimates for forward differences

5.27 Notation.
We introduce the following notation

$$(x_1, x_i = a, x_d) := (x_1, \dots, x_{i-1}, a, x_{i+1}, \dots, x_d).$$

5.28 Lemma (Vanishing boundary values estimate, Forward Differences)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h((-R,R)^d \cap \{x_i > -R+h\})$ be admissible with $R > h$ and $i \in \{1,\dots,d\}$. Let $\nu > 0$ and let $u\colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} |\Delta_h u(x)| &\le \nu, \quad x \in \Omega_h, \\ u(x) &= 0, \quad x \in \partial\Omega_h. \end{aligned}$$

Then

$$|D_i^+ u(0)| \le \nu R.$$

▶
We define $\overline{G}_h := \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\}\right)$ and $U\colon \overline{G}_h \to \mathbb{R}$

$$U(x) := \tfrac{1}{2}\left[u(x) - u(x_1, x_i = -x_i + h, x_d)\right],$$

implying

$$U(x_1, x_i = h, x_d) = \frac{h}{2} D_i^+ u(x_1, x_i = 0, x_d), \quad x \in \Omega_h.$$

Moreover, we have for $x \in \partial_i^- G_h$

$$\begin{aligned} I &:= 2\left[U(x) + U(\check{x})\right] = 2\left[U(x_1, x_i = 0, x_d) + U(x_1, x_i = h, x_d)\right] \\ &= u(x_1, x_i = 0, x_d) - u(x_1, x_i = h, x_d) + u(x_1, x_i = h, x_d) - u(x_1, x_i = 0, x_d) = 0 \end{aligned}$$

and for $x \in \partial_i^+ G_h$

$$U(x) = U(x_1, x_i = R, x_d) = \frac{1}{2}[u(x_1, x_i = R, x_d) - u(x_1, x_i = -R + h, x_d)] = 0.$$

Finally, for $x \in \partial G_h \setminus \partial_i G_h$ holds

$$U(x) = 0,$$

since $x \in \partial_j G_h \subset \partial_j \Omega_h$ implies $(x_1, x_i = -x_i + h, x_d) \in \partial_j \Omega_h$, $1 \leq i, j \leq d$, $i \neq j$. Since $|\Delta_h U| \leq \nu$ on G_h, we can apply Lemma 5.26 on this set, yielding

$$\left|D_i^+ u(x_1, x_i = 0, x_d)\right| = \frac{2}{h} \left|U(x_1, x_i = h, x_d)\right| \leq \frac{2}{h} \frac{\nu}{2} h(R - h) \leq \nu R,$$

for $x \in G_h \subset \Omega_h$. Since $he_i \in G_h$ this estimate also holds for $D_i^+ u(0)$.
◀

5.29 Lemma (Simple harmonic estimate, Forward Differences)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h := \mathrm{cl}_h\left((-R, R)^d \cap \{x_i > -R + h\}\right)$ be admissible with $R > h$. Let $\mu > 0$ and let $u \colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} -\Delta_h u(x) &= 0, \quad x \in \Omega_h, \\ |u(x)| &\leq \mu, \quad x \in \partial_i \Omega_h, \\ u(x) &= 0, \quad x \in \partial \Omega_h \setminus \partial_i \Omega_h. \end{aligned}$$

Then

$$\left|D_i^+ u(0)\right| \leq 2\frac{\mu}{R}.$$

▶
Define $\overline{G}_h := \mathrm{cl}_h\left((-R, R)^d \cap \{x_i > 0\}\right)$, and $U \colon \overline{G}_h \to \mathbb{R}$

$$U(x) := \tfrac{1}{2}\left[u(x) - u(x_1, x_i = -x_i + h, x_d)\right],$$

then, for $x \in G_h$ we have

$$\begin{aligned} \Delta_h U(x) &= 0, \\ U(x_1, x_i = h, x_d) &= \tfrac{h}{2} D_i^+ u(x_1, x_i = 0, x_d). \end{aligned}$$

Moreover, we have for $x \in \partial_i^- G_h$

$$I := 2\left[U(x) + U(\check{x})\right] = 2\left[U(x_1, x_i = 0, x_d) + U(x_1, x_i = h, x_d)\right]$$
$$= u(x_1, x_i = 0, x_d) - u(x_1, x_i = h, x_d) + u(x_1, x_i = h, x_d) - u(x_1, x_i = 0, x_d) = 0 \tag{1}$$

and for $x \in \partial_i^+ G_h$

$$|U(x)| = |U(x_1, x_i = R, x_d)| = \tfrac{1}{2}\left|u(x_1, x_i = R, x_d) - u(x_1, x_i = -R + h, x_d)\right| \leq \mu. \tag{2}$$

Finally, for $\partial G_h \setminus \partial_i G_h$ holds

$$U(x) = 0, \tag{3}$$

since $x \in \partial_j G_h \subset \partial_j \Omega_h$ implies $(x_1, x_i = -x_i + h, x_d) \in \partial_j \Omega_h$, $1 \leq i, j \leq d$, $i \neq j$. We introduce $V(x) := \frac{\mu}{R} x_i$, $x \in \overline{G}_h$. Subtraction of a linear function preserves harmonicity, i.e.

$$-\Delta_h \left(U(x) - V(x)\right) = 0, \quad x \in G_h.$$

From $V(x) \geq 0$ in $\overline{G}_h$ we get on $\partial_i^- G_h$

$$U(x) - V(x) + U(\check{x}) - V(\check{x}) \leq U(x) + U(\check{x}) = 0$$

and on $\partial G_h \setminus \partial_i G_h$

$$U(x) - V(x) \leq U(x) = 0.$$

Since $V(x) = V(x_1, x_i = R, x_d) \equiv \mu$, we obtain

$$U(x) - V(x) \leq 0$$

on $\partial_i^+ G_h$. Applying Lemma 5.23 we get

$$U(x) \leq V(x), \quad x \in G_h.$$

Analogously, since (1) – (3) hold also for $-U(x)$ we can apply the same argument to $-U(x) - V(x)$, obtaining

$$U(x) \geq -V(x), \quad x \in G_h,$$

i.e.

$$|U(x)| \leq V(x), \quad x \in G_h$$

Since $(x_1, x_i = h, x_d) \in G_h$ for $x \in G_h$ we get

$$|D_i^+ u(x_1, x_i = 0, x_d)| = \frac{2}{h}|U(x_1, x_i = h, x_d)| \leq \frac{2}{h}\mu\frac{h}{R} = 2\frac{\mu}{R},$$

and setting $x = 0 \in G_h$ we get the claim.

◀

5.30 Lemma (Difficult harmonic estimate, Forward Differences)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h := \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > -R+h\}\right)$ be admissible with $R > h$ and $i \in \{1,\dots,d\}$.

Let $j \in \{1,\dots,d\} \setminus \{i\}$, $\mu > 0$ and let $u\colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} -\Delta_h u(x) &= 0, \quad x \in \Omega_h \\ |u(x)| &\le \mu, \quad x \in \partial_j \Omega_h \\ u(x) &= 0, \quad x \in \partial\Omega_h \setminus \partial_j\Omega_h. \end{aligned}$$

Then

$$\left|D_i^+ u(0)\right| \le 2\frac{\mu}{R}.$$

▶

⟨*1*⟩ *Auxiliary function*

We introduce the following notation $P_i, P_j \colon \overline{\Omega}_h \to \overline{\Omega}_h$

$$\begin{aligned} P_i(x) &:= (x_1, x_i = -x_i + h, x_d), \\ P_j(x) &:= (x_1, x_j = -x_j, x_d) \end{aligned}$$

with $P_iP_j = P_jP_i$ due to orthogonality. We define

$$U(x) := \tfrac{1}{4}\left[u(x) - u(P_ix) + u(P_jx) - u(P_iP_jx)\right],$$

obtaining

$$U(he_i) = \tfrac{1}{4}\left[u(he_i) - u(0) + u(he_i) - u(0)\right] = \tfrac{h}{2} D_i^+ u(0).$$

Define $G_h \subset \mathcal{G}_h$, $\overline{G}_h := \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\}\right)$ (see Figure 5.1). We observe that

$$x \in \partial_j G_h \subset \partial_j\Omega_h \Rightarrow P_ix, P_jx, P_iP_j \in \partial_j\Omega_h,$$

implying

$$|U(x)| \le \mu, \quad x \in \partial_j G_h.$$

Considering analogous properties of P_i,P_j, P_iP_j on the other parts of the boundary, we get

$$U(x) = 0, \quad x \in \partial G_h \setminus \left(\partial_j G_h \cup \partial_i^- G_h\right)$$

For $x = (x_1, x_i = 0, x_d) \in \partial_i^- G_h$ we have $\check{x} = x + he_i$, i.e. $P_i\check{x} = x$, $P_ix = \check{x}$, $P_iP_j\check{x} = P_jx$, $P_iP_jx = P_j\check{x}$. This yields

$$\begin{aligned} U(x) + U(\check{x}) = {}& [u(x) - u(P_i\check{x})] - [u(P_ix) - u(\check{x})] \\ &+ [u(P_jx) - u(P_iP_j\check{x})] - [u(P_iP_jx) - u(P_j\check{x})] = 0 \end{aligned}$$

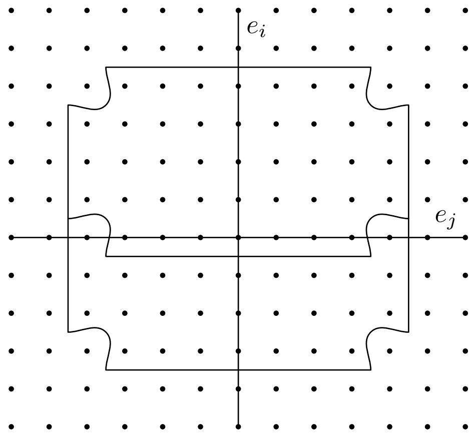

Figure 5.1: Sets $\overline{\Omega}_h$ and $\overline{G}_h$

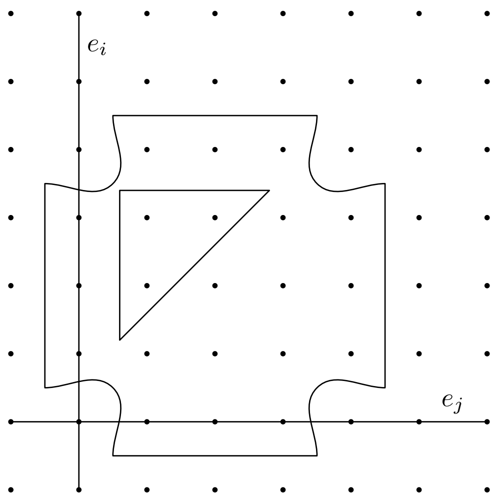

Figure 5.2: Sets $\overline{D}_h$ and $D_h \cap \{x_i > x_j\}$

for $x \in \partial_i^- G_h$. In other words, U satisfies

$$\begin{aligned} -\Delta_h U(x) &= 0, \quad x \in G_h, \\ U(x) &= 0, \quad x \in \partial G_h \setminus \left(\partial_j G_h \cup \partial_i^- G_h\right), \\ |U(x)| &\leq \mu, \quad x \in \partial_j G_h, \\ U(x) + U(\check{x}) &= 0, \quad x \in \partial_i^- G_h. \end{aligned}$$

⟨*2*⟩ *First comparison function*
We define $V \colon \overline{G}_h \to \mathbb{R}$ by

$$\begin{aligned} -\Delta_h V(x) &= 0, \quad x \in G_h \\ V(x) &= 0, \quad x \in \partial G_h \setminus \left(\partial_j G_h \cup \partial_i^- G_h\right) \\ V(x) &= \mu, \quad x \in \partial_j G_h, \\ V(x) + V(\check{x}) &= 0, \quad x \in \partial_i^- G_h. \end{aligned}$$

Here and further in this proof such definitions are well-defined due to Theorem 5.25. Applying Lemma 5.23 to $U - V$ and $-U - V$ we get

$$|U| \leq V \qquad \text{on } G_h.$$

It is therefore sufficient to estimate V.

⟨*3*⟩ *Second comparison function*
Define $W \colon \overline{G}_h \to \mathbb{R}$ by

$$\begin{aligned} -\Delta_h W(x) &= 0, \qquad x \in G_h, \\ W(x) &= \mu, \qquad x \in \partial_i^+ G_h, \\ W(x) &= 0, \qquad x \in \partial G_h \setminus (\partial_i G_h \cup \partial_j G_h), \\ W(x) &= -\mu, \quad x \in \partial_j G_h, \\ W(x) + W(\check{x}) &= 0, \qquad x \in \partial_i^- G_h. \end{aligned}$$

The sum $V + W$ is discrete harmonic with

$$\begin{aligned} (V+W)(x) + (V+W)(\check{x}) &= 0, \quad x \in \partial_i^- G_h, \\ (V+W)(x) &= \mu, \quad x \in \partial_i^+ G_h, \\ (V+W)(x) &= 0, \quad x \in \partial G_h \setminus \partial_i G_h. \end{aligned}$$

Since $X(x) := \frac{\mu}{R} x_i$ is non-negative on $\overline{G}_h$ and discrete harmonic in G_h with

$$X(x) = \mu, \quad x \in \partial_i^+ G_h,$$

we can apply Lemma 5.23 to $V + W - X$, obtaining

$$V(x) + W(x) \leq \frac{\mu x_i}{R}, \quad x \in G_h.$$

We will see later that $W(the_i) \geq 0$ for $t \in \{t \in \mathbb{N} \mid the_i \in G_h\}$. In particular, for $t = 1$ this implies

$$V(he_i) \leq \frac{\mu h}{R}$$

and consequently

$$|D_i^+ u(0)| = \frac{2}{h}|U(he_i)| \leq \frac{2}{h}V(he_i) \leq \frac{2}{h}\frac{\mu h}{R} = 2\frac{\mu}{R}.$$

⟨*4*⟩ *Third comparison function*
We want to prove that W is non-negative along e_i. To this end, we define $\Psi^+ \colon \overline{D}_h \to \mathbb{R}$, $\overline{D}_h := \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\} \cap \{x_j > 0\}\right)$ (see Figure 5.2) by

$$\begin{aligned} \Delta_h \Psi^+(x) &= 0, & x &\in D_h, \\ \Psi^+(x) + \Psi^+(\check{x}) &= 0, & x &\in \partial_j^- D_h \cup \partial_i^- D_h, \\ \Psi^+(x) &= -\mu, & x &\in \partial_j^+ D_h, \\ \Psi^+(x) &= \mu, & x &\in \partial_i^+ D_h, \\ \Psi^+(x) &= 0, & x &\in \partial D_h \setminus (\partial_j D_h \cup \partial_i D_h) \end{aligned}$$

Taking the symmetric part of Ψ^+ w.r.t. hyperplane $\{x_j = x_i\}$,

$$\Psi^+_{sym} := \Psi^+(x) + \Psi^+(x_1, x_i = x_j, x_j = x_i, x_d)$$

we observe that

$$\begin{aligned} -\Delta_h \Psi^+_{sym} &= 0, & x &\in D_h \\ \Psi^+_{sym} &= 0, & x &\in \partial D_h \setminus \left(\partial_j^- D_h \cup \partial_i^- D_h\right) \\ \Psi^+_{sym}(x) + \Psi^+_{sym}(\check{x}) &= 0, & x &\in \partial_j^- D_h \cup \partial_i^- D_h, \end{aligned}$$

implying $\Psi^+_{sym} = 0$ in D_h by Lemma 5.23. In particular, we have

$$\Psi^+_{sym}(x) = 2\Psi^+(x) = 0, \quad x \in D_h \cap \{x_i = x_j\}.$$

Applying now Lemma 5.23 and the Remark thereafter to Ψ^+ on $D_h \cap \{x_i > x_j\}$ (not an admissible set!) we get $\Psi^+ \geq 0$ on this set, in particular

$$\Psi^+(x) \geq 0, x \in \overline{D}_h \cap \{x_j = h\},$$

implying from $\Psi^+(x) + \Psi^+(\check{x}) = 0$ on $\partial_j^- D_h$

$$\Psi^+(x) \le 0, x \in \overline{D}_h \cap \{x_j = 0\}.$$

We now construct $\Psi \colon \overline{G}_h \to \mathbb{R}$ by

$$\Psi(x) := \begin{cases} \Psi^+(x), & \text{if } x_j > 0, \\ \Psi^+(x_1, x_j = -x_j, x_d), & \text{if } x_j < 0, \\ 0, & \text{if } x_j = 0 \text{ and } x_i < R, \\ \mu, & \text{if } x_j = 0 \text{ and } x_i = R. \end{cases}$$

This function is discrete subharmonic in G_h, since it is harmonic in $G_h \cap (\{x_j > h\} \cup \{x_j < -h\})$ by construction, for $x \in G_h \cap \{x_j = h\}$ (analogously, for $x \in G_h \cap \{x_j = -h\}$)

$$-\Delta_h \Psi = \frac{1}{h^2}[-\underbrace{\Psi(x - he_j)}_{=0} + \underbrace{\Psi^+(x - he_j)}_{\le 0}] - \underbrace{\Delta_h \Psi^+}_{=0} \le 0.$$

and for $x \in G_h \cap \{x_j = 0\}$ we have $\Psi(x) = 0$, $\Psi(y) \ge 0$ for $y \in N_h(x)$, so $-\Delta_h \Psi(x) \le 0$ by definition.

⟨5⟩ *Final comparison*
Now consider the difference $\Psi - W$. We want to show that $\Psi - W \le 0$ in G_h. We have

$$\begin{aligned} -\Delta_h(\Psi(x) - W(x)) &\le 0, \quad x \in G_h, \\ (\Psi - W)(x) &= 0, \quad x \in \partial G_h \setminus \partial_i^- G_h, \\ (\Psi - W)(x) + (\Psi - W)(\check{x}) &= 0, \quad x \in \partial_i^- G_h. \end{aligned}$$

Once again, using Lemma 5.23, we get claim. This implies $0 \le \Psi(x) \le W(x)$ for $x \in \{the_i \in G_h \mid t \in \mathbb{N}\} \subset G_h \cap \{x_j = 0\}$.

◀

5.31 Theorem (Combined estimate for the forward finite difference quotient)**.** Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h := \mathrm{cl}_h((-R, R)^d \cap \{x_i > -R + h\})$ be admissible with $R > h$ and with some $i \in \{1, \ldots, d\}$. Let $\nu, \mu > 0$ and let $u \colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} |\Delta_h u(x)| &\le \nu, \quad x \in \Omega_h, \\ |u(x)| &\le \mu, \quad x \in \partial\Omega_h. \end{aligned}$$

Then

$$|D_i^+ u(0)| \le \frac{2d\mu}{R} + \nu R.$$

►

By Theorem 5.16, we can decompose $u = \sum_{j=0}^{d} u_j$, where

$$\begin{aligned} -\Delta_h u_0(x) &= -\Delta_h u(x), & x &\in \Omega_h, \\ u_0(x) &= 0, & x &\in \partial\Omega_h \end{aligned}$$

and, for $1 \leq j \leq d$,

$$\begin{aligned} -\Delta_h u_j(x) &= 0, & x &\in \Omega_h, \\ u_j(x) &= u(x), & x &\in \partial_j\Omega_h, \\ u_j(x) &= 0, & x &\in \partial\Omega_h \setminus \partial_j\Omega_h. \end{aligned}$$

Using Lemmas 5.28, 5.29 and 5.30, we get

$$|D_i^+ u(0)| \leq \sum_{j=0}^{d} |D^+ u_j(0)| = |D^+ u_0(0)| + \sum_{j=1}^{d} |D^+ u_j(0)| \leq \nu R + 2\frac{d\mu}{R}.$$

◄

5.32 Remark (Special case)**.**
We conclude with the proof of the special case $R = h$ that is not covered by the previous theorem.

5.33 Lemma (Forward differences, special case)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h := \mathrm{cl}_h((-h,h)^d)$ be admissible. Let $\nu, \mu > 0$ and let $u \colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} |\Delta_h u(x)| &\leq \nu, & x &\in \Omega_h, \\ |u(x)| &\leq \mu, & x &\in \partial\Omega_h. \end{aligned}$$

Then

$$|D_i^+ u(0)| \leq \frac{2\mu}{R} + \frac{\nu}{2d} R, \quad 1 \leq i \leq d.$$

►

From

$$|\Delta_h u(0)| \leq \nu,$$

we get

$$\frac{-\nu h^2 - 2d\mu}{2d} \leq u(0) \leq \frac{\nu h^2 + 2d\mu}{2d},$$

$$|u(0)| \leq \frac{\nu h^2}{2d} + \mu.$$

Direct calculation yields

$$|D_i^+ u(0)| = \frac{|u(e_i h) - u(0)|}{h} \leq \frac{|u(e_i h)| + |u(0)|}{h} \leq 2\frac{\mu}{h} + \frac{\nu h}{2d}.$$

◄

5.5 Inner estimates for central differences

5.34 Lemma (Estimate for vanishing boundary value, Central Differences)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h((-R,R)^d)$ be admissible. Let $\nu > 0$ and $u\colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} |\Delta_h u(x)| &\le \nu, \quad x \in \Omega_h, \\ u(x) &= 0, \quad x \in \partial\Omega_h. \end{aligned}$$

Then

$$|D_i^{\pm} u(0)| \le \frac{\nu}{2} R, \quad 1 \le i \le d.$$

►
We introduce $G_h \subset \mathcal{G}_h$, $\overline{G}_h = \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\}\right)$ and define $U\colon \overline{G}_h \to \mathbb{R}$ by

$$U(x) := \frac{1}{2}\left[u(x) - u(x_1, x_i = -x_i, x_d)\right],$$

obtaining

$$U(x_1, x_i = h, x_d) = h D_i^{\pm} u(x_1, x_i = 0, x_d), \quad x \in G_h.$$

We now have for $x \in \partial_i^- G_h$

$$U(x) = U(x_1, x_i = 0, x_d) = \tfrac{1}{2}[u(x_1, x_i = 0, x_d) - u(x_1, x_i = 0, x_d)] = 0,$$

respectively for $x \in \partial_i^+ G_h$

$$U(x) = U(x_1, x_i = R, x_d) = \tfrac{1}{2}[u(x_1, x_i = R, x_d) - u(x_1, x_i = -R, x_d)] = 0,$$

and finally, $U(x) = 0$ for $x \in \partial G_h \setminus \partial_i G_h$.
This means that U satisfies the assumptions of Lemma 5.26 on G_h, yielding

$$\left|D_i^{\pm} u(x_1, x_i = 0, x_d)\right| = \frac{1}{h}\left|U(x_1, x_i = h, x_d)\right| \le \frac{1}{h}\frac{\nu}{2} h(R-h) \le \frac{\nu}{2} R,$$

for $x \in \Omega_h$, in particular for $x = 0$, and all $1 \le i \le d$.
◄

5.35 Lemma (Simple harmonic estimate, Central Differences)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h := \mathrm{cl}_h\left((-R,R)^d\right)$ be admissible. Let $\mu > 0$ and $u\colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} -\Delta_h u(x) &= 0, \quad x \in \Omega_h, \\ |u(x)| &\le \mu, \quad x \in \partial_i \Omega_h, \\ u(x) &= 0, \quad x \in \partial\Omega_h \setminus \partial_i \Omega_h. \end{aligned}$$

Then

$$\left|D_i^{\pm} u(0)\right| \le \frac{\mu}{R}, \quad 1 \le i \le d.$$

►

We introduce $G_h \subset \mathcal{G}_h$, $\overline{G}_h = \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\}\right)$ and define $U\colon \overline{G}_h \to \mathbb{R}$ by

$$U(x) := \frac{1}{2}\left[u(x) - u(x_1, x_i = -x_i, x_d)\right],$$

obtaining

$$U(x_1, x_i = h, x_d) = hD_i^+ u(x_1, x_i = 0, x_d), \quad x \in G_h.$$

We now have

$$U(x) = \frac{1}{2}[u(x_1, x_i = 0, x_d) - u(x_1, x_i = 0, x_d)] = 0, \quad x \in \partial_i^- G_h \tag{1}$$

and

$$|U(x)| = \tfrac{1}{2}\left|u(x_1, x_i = R, x_d) - u(x_1, x_i = -R, x_d)\right| \le \mu, \quad x \in \partial_i^+ G_h. \tag{2}$$

Moreover,

$$U(x) = 0, \quad \partial G_h \setminus \partial_i G_h. \tag{3}$$

We define $V(x) := \frac{\mu}{R} x_i$, $x \in \overline{G}_h$. Subtraction of a linear function preserves discrete harmonicity, i.e.

$$-\Delta_h\left(U(x) - V(x)\right) = 0, \quad x \in G_h.$$

From $V \ge 0$ on $\overline{G}_h$ we get

$$U(x) - V(x) \le U(x) = 0, \quad x \in \partial G_h \setminus \partial_i^+ G_h.$$

Finally, since $V(x) \equiv \mu$, $x \in \partial_i^+ G_h$ we obtain

$$U(x) - V(x) \le 0, \quad x \in \partial_i^+ G_h.$$

Applying Lemma 5.15 we get

$$U(x) \le V(x), \quad x \in \overline{G}_h.$$

Analogously, since (1)-(3) hold also for $-U$ we can apply the same argument to $-U - V$, obtaining

$$U(x) \ge -V(x), \quad x \in \overline{G}_h,$$

i.e.

$$|U(x)| \le V(x), \quad x \in \overline{G}_h.$$

This yields

$$|D_i^+ u(0)| = \frac{1}{h}|U(he_i)| \le \frac{1}{h} V(he_i) = \frac{\mu}{h}\frac{h}{R} = \frac{\mu}{R}.$$

◄

5.36 Lemma (Difficult harmonic estimate, Central Difference)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h\left((-R,R)^d\right)$ be admissible.
Let $i, j \in \{1,\dots,d\}$, $i \neq j$ be arbitrary but fixed, $\mu > 0$ and let $u\colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} -\Delta_h u(x) &= 0, \quad x \in \Omega_h, \\ |u(x)| &\leq \mu, \quad x \in \partial_j \Omega_h, \\ u(x) &= 0, \quad x \in \partial\Omega_h \setminus \partial_j\Omega_h. \end{aligned}$$

Then

$$\left|D_i^{\pm} u(0)\right| \leq \frac{\mu}{R}.$$

▶
⟨*1*⟩ *Auxiliary function*
We denote $P_i, P_j\colon \overline{\Omega}_h \to \overline{\Omega}_h$

$$\begin{aligned} P_i x &:= (x_1, x_i = -x_i, x_d), \\ P_j x &:= (x_1, x_j = -x_j, x_d) \end{aligned}$$

with $P_iP_j = P_jP_i$ due to orthogonality. We now define

$$U(x) := \tfrac{1}{4}\left[u(x) - u(P_i x) + u(P_j x) - u(P_iP_j x)\right],$$

obtaining

$$U(he_i) = \tfrac{1}{4}[u(he_i) - u(-he_i) + u(he_i) - u(-he_i)] = hD_i^{\pm}u(0).$$

Introducing $G_h \subset \mathcal{G}_h$, $\overline{G}_h = \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\}\right)$ we get

$$\begin{aligned} -\Delta_h U(x) &= 0, \quad x \in G_h, \\ U(x) &= 0, \quad x \in \partial G_h \setminus \partial_j G_h, \\ |U(x)| &\leq \mu, \quad x \in \partial_j G_h, \end{aligned}$$

since

$$\begin{aligned} &x \in \partial G_h \setminus \left(\partial_j G_h \cup \partial_i^- G_h\right) \subset \partial\Omega_h \setminus \partial_j \Omega_h \Rightarrow P_i x, P_j x, P_iP_j x \in \partial\Omega_h \setminus \partial_j\Omega_h \\ &x \in \partial_j G_h \subset \partial_j \Omega_h \Rightarrow P_i x, P_j x, P_iP_j x \in \partial_j\Omega_h, \\ &x \in \partial_i^- G_h \Rightarrow P_i x = x, P_j x = P_iP_j x. \end{aligned}$$

⟨*2*⟩ *First comparison function*
We define $V\colon \overline{G}_h \to \mathbb{R}$ by

$$\begin{aligned} -\Delta_h V(x) &= 0, \quad x \in G_h, \\ V(x) &= 0, \quad x \in \partial G_h \setminus \partial_j G_h, \\ V(x) &= \mu, \quad x \in \partial_j G_h. \end{aligned}$$

Applying Lemma 5.15 to $U - V$ and $-U - V$ we get

$$|U(x)| \le V(x) \quad x \in \overline{G}_h.$$

It is therefore sufficient to estimate V.

⟨*3*⟩ *Second comparison function*
Define $W \colon \overline{G}_h \to \mathbb{R}$ by

$$\begin{aligned}
-\Delta_h W(x) &= 0, && x \in G_h, \\
W(x) &= 0, && x \in \partial_i^- G_h, \\
W(x) &= \mu, && x \in \partial_i^+ G_h, \\
W(x) &= -\mu, && x \in \partial_j G_h, \\
W(x) &= 0, && x \in \partial G_h \setminus (\partial_i G_h \cup \partial_j G_h).
\end{aligned}$$

The sum $V + W$ is discrete harmonic on G_h with

$$\begin{aligned}
(V+W)(x) &= 0, \quad x \in \partial G_h \setminus \partial_i^+ G_h \\
(V+W)(x) &= \mu, \quad x \in \partial_i^+ G_h.
\end{aligned}$$

Since $X(x) := \frac{\mu}{R} x_i$ is discrete harmonic on G_h and non-negative on $\overline{G}_h$ with

$$X(x) = \mu, \quad x \in \partial_i^+ G_h,$$

we can apply Lemma 5.15 again, obtaining

$$V(x) + W(x) \le X(x), \quad x \in \overline{G}_h.$$

We will see later that $W(the_i) \ge 0$ for $t \in \{t \in \mathbb{Z}_+ \mid the_i \in G_h\}$. In particular, for $t = 1$ this implies

$$V(he_i) \le X(he_i) = \frac{\mu h}{R}$$

and consequently

$$|D_i^{\pm} u(0)| = \frac{1}{h}|U(he_i)| \le \frac{1}{h} V(he_i) \le \frac{1}{h}\frac{\mu h}{R} = \frac{\mu}{R}.$$

⟨*4*⟩ *Third comparison function*
We want to prove that W is non-negative along e_i.

To this end, we denote $D_h \subset \mathcal{G}_h$, $\overline{D}_h := \mathrm{cl}_h\left((-R,R)^d \cap \{x_i > 0\} \cap \{x_j > 0\}\right)$ (in general not admissible) and define $\Psi^+ \colon \overline{D}_h \to \mathbb{R}$ by

$$\begin{aligned}
\Delta_h \Psi^+(x) &= 0, && x \in D_h, \\
\Psi^+(x) &= -\mu, && x \in \partial_j^+ D_h, \\
\Psi^+(x) &= \mu, && x \in \partial_i^+ D_h, \\
\Psi^+(x) &= 0, && x \in \partial D_h \setminus \left(\partial_j^+ D_h \cup \partial_i^+ D_h\right)
\end{aligned}$$

Taking the symmetric part of Ψ^+ w.r.t. hyperplane $\{x_j = x_i\}$,

$$\Psi^+_{sym} := \Psi^+(x) + \Psi^+(x_1, x_i = x_j, x_j = x_i, x_d)$$

we observe that

$$\begin{aligned} -\Delta_h \Psi^+_{sym} &= 0, \quad x \in D_h, \\ \Psi^+_{sym} &= 0, \quad x \in \partial D_h, \end{aligned}$$

implying $\Psi^+_{sym} = 0$ in D_h by Lemma 5.16. In particular, we have

$$\Psi^+_{sym}(x) = 2\Psi^+(x) = 0, \quad x \in D_h \cap \{x_i = x_j\}.$$

Applying now Lemma 5.15 to Ψ^+ on $D \cap \{x_i > x_j\}$ we get $\Psi^+ \geq 0$ on this set, in particular

$$\begin{aligned} \Psi^+(x) &\geq 0, \quad x \in \overline{D}_h \cap \{x_j = h\}, \\ \Psi^+(x) &= 0, \quad x \in \overline{D}_h \cap \{x_j = 0\}. \end{aligned}$$

We now construct $\Psi \colon \overline{G}_h \to \mathbb{R}$ by

$$\Psi(x) := \begin{cases} \Psi^+(x), & \text{if } x_j > 0, \\ \Psi^+(x_1, x_j = -x_j, x_d), & \text{if } x_j < 0, \\ 0, & \text{if } x_j = 0 \text{ and } x_i < R, \\ \mu, & \text{if } x_j = 0 \text{ and } x_i = R. \end{cases}$$

This function is discrete subharmonic in G_h, since it is harmonic in $G_h \cap (\{x_j > h\} \cup \{x_j < -h\})$ by construction, for $x \in G_h \cap \{x_j = h\}$ (analogously, for $x \in G_h \cap \{x_j = -h\}$)

$$-\Delta_h \Psi(x) = \frac{1}{h^2}[-\underbrace{\Psi(x - he_j)}_{=0} + \underbrace{\Psi^+(x - he_j)}_{=0}] - \underbrace{\Delta_h \Psi^+(x)}_{=0} = 0.$$

and for $x \in G_h \cap \{x_j = 0\}$ we have $\Psi(x) = 0$, $\Psi(y) \geq 0$ for $y \in N_h(x)$, so $-\Delta_h \Psi(x) \leq 0$ by definition.

⟨5⟩ *Final comparison*

Now consider the difference $\Psi - W$. We want to show that $\Psi - W \leq 0$ in $\overline{G}_h$. We have

$$\begin{aligned} -\Delta_h(\Psi - W)(x) &\leq 0, \quad x \in G_h, \\ (\Psi - W)(x) &= 0, \quad x \in \partial G_h. \end{aligned}$$

Once again, using Lemma 5.15, we get claim. This implies $0 \leq \Psi(x) \leq W(x)$ for $x \in \{the_i \in G_h \mid t \in \mathbb{N}\} \subset G_h \cap \{x_j = 0\}$.

◀

5.37 Theorem (Combined estimate for the central finite difference quotient). Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h((-R,R)^d)$ be admissible. Let $\nu, \mu > 0$ and let $u\colon \overline{\Omega}_h \to \mathbb{R}$ satisfy

$$\begin{aligned} |\Delta_h u(x)| &\le \nu, \quad x \in \Omega_h, \\ |u(x)| &\le \mu, \quad x \in \partial\Omega_h. \end{aligned}$$

Then

$$|D_i^{\pm} u(0)| \le \frac{d\mu}{R} + \frac{\nu}{2}R,$$

for all $1 \le i \le d$.

▶

By Theorem 5.16, we can decompose $u = \sum_{j=0}^{d} u_j$, where u_j, $0 \le j \le d$ are defined as the unique solutions of

$$\begin{aligned} -\Delta_h u_0(x) &= -\Delta_h u(x), & x &\in \Omega_h, \\ u_0(x) &= 0, & x &\in \partial\Omega_h \end{aligned}$$

and

$$\begin{aligned} -\Delta_h u_j(x) &= 0, & x &\in \Omega_h, \\ u_j(x) &= u(x), & x &\in \partial_j\Omega_h, \\ u_j(x) &= 0, & x &\in \partial\Omega_h \setminus \partial_j\Omega_h. \end{aligned}$$

Using Lemmas 5.34, 5.35 and 5.36, we get

$$|D_i^{\pm} u(0)| \le \sum_{j=0}^{d} |D^{\pm} u_j(0)| = |D^{\pm} u_0(0)| + \sum_{j=1}^{d} |D^{\pm} u_j(0)| \le \frac{\nu}{2}R + \frac{d\mu}{R},$$

for all $1 \le i \le d$.

◀

Chapter 6

Pointwise estimates for discrete Green's function

6.1 Estimates for discrete Green's function

6.1 Motivation.
The following two lemmas are due to Bramble, Hubbard and Zlámal [9]. Since we need the extension of the first lemma, we present the complete proof. The rather short proof of the second lemma is given for the sake of completeness.

6.2 Lemma (First estimate for discrete potentials, [9])**.**
Define

$$\sigma(x) := \sigma_\gamma(x) := \sqrt{|x|^2 + \gamma h^2}, \quad x \in \mathcal{G}_h.$$

Then, on the one hand

$$-\beta(d-2+\beta)\sigma^{\beta-2}(x) \leq -\Delta_h[\sigma(x)^\beta], \quad x \in \mathcal{G}_h \setminus \{0\},$$

and on the other hand

$$\begin{aligned} -\Delta_h[\sigma(x)^\beta] &\leq -\beta(d-2+\beta+\varepsilon)\sigma^{\beta-2}(x), && x \in \mathcal{G}_h, |x| \geq Rh, \\ -\Delta_h[\sigma(x)^\beta] &\leq -d\beta\sigma(x)^{\beta-2}, && x \in \mathcal{G}_h, \end{aligned}$$

for every $\beta < 0$, $\varepsilon \in (0, 2-\beta)$, and $\gamma > \tilde{\gamma} > 0$, $R > \tilde{R}$ with some $\tilde{\gamma} := \tilde{\gamma}(d,\beta) > 0$, $\tilde{R} := \tilde{R}(\varepsilon, d, \beta, \gamma)$.
▶
⟨*1*⟩ *Discretization error*
From the mean value theorem we obtain

$$-\Delta_h[\sigma(x)^\beta] = -\Delta[\sigma(x)^\beta] - \frac{h^2}{24}\sum_{i=1}^{d}\left\{\frac{\partial^4}{\partial x_i^4}\sigma^\beta(\xi_i) + \frac{\partial^4}{\partial x_i^4}\sigma^\beta(\eta_i)\right\}$$

with $\xi_i \in [x_i, x_i + he_i]$, $\eta \in [x_i - he_i, x_i]$, $1 \le i \le d$.
For the continuous Laplacian we have

$$\begin{aligned}
-\Delta[\sigma(x)^\beta] &= -\sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}\left[(|x|^2+\gamma h^2)^{\frac{\beta}{2}}\right] = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[\beta x_i (|x|^2+\gamma h^2)^{\frac{\beta-2}{2}}\right] \\
&= -\sum_{i=1}^{d}\left[\beta(|x|^2+\gamma h^2)^{\frac{\beta-2}{2}} + \beta(\beta-2)x_i^2(|x|^2+\gamma h^2)^{\frac{\beta-4}{2}}\right] \\
&= -d\beta\sigma(x)^{\beta-2} - \beta(\beta-2)|x|^2\sigma(x)^{\beta-4} \\
&= -d\beta\sigma(x)^{\beta-2} - \beta(\beta-2)(\sigma^2(x)-\gamma h^2)\sigma(x)^{\beta-4} \\
&= -\beta(\beta+d-2)\sigma(x)^{\beta-2} + \beta(\beta-2)\gamma h^2\sigma(x)^{\beta-4}.
\end{aligned}$$

⟨*2*⟩ *Estimates for the derivatives*
We calculate the derivatives needed for the error estimate:

$$\begin{aligned}
\frac{\partial}{\partial x_i}\sigma(x)^\beta &= \frac{\partial}{\partial x_i}(|x|^2+\gamma h^2)^{\frac{\beta}{2}} = \beta x_i \sigma(x)^{\beta-2}, \\
\frac{\partial^2}{\partial x_i^2}\sigma(x)^\beta &= \beta\sigma(x)^{\beta-2} + \beta(\beta-2)x_i^2\sigma(x)^{\beta-4}, \\
\frac{\partial^3}{\partial x_i^3}\sigma(x)^\beta &= \beta(\beta-2)x_i\sigma(x)^{\beta-4} + \beta(\beta-2)\left[2x_i\sigma(x)^{\beta-4} + (\beta-4)x_i^3\sigma(x)^{\beta-6}\right] \\
&= 3\beta(\beta-2)x_i\sigma(x)^{\beta-4} + \beta(\beta-2)(\beta-4)x_i^3\sigma(x)^{\beta-6}, \\
\frac{\partial^4}{\partial x_i^4}\sigma(x)^\beta &= 3\beta(\beta-2)\sigma(x)^{\beta-4} + 3\beta(\beta-2)(\beta-4)x_i^2\sigma(x)^{\beta-6} \\
&\quad + 3\beta(\beta-2)(\beta-4)x_i^2\sigma(x)^{\beta-6} + \beta(\beta-2)(\beta-4)(\beta-6)x_i^4\sigma(x)^{\beta-8} \\
&= \beta(\beta-2)\sigma(x)^{\beta-4}\left[3 + 6(\beta-4)\frac{x_i^2}{\sigma(x)^2} + (\beta-4)(\beta-6)\frac{x_i^4}{\sigma(x)^4}\right].
\end{aligned}$$

We observe that

$$0 \le x_i^2\sigma(x)^{-2} = \frac{x_i^2}{|x|^2+\gamma h^2} \le \frac{|x|^2}{|x|^2+\gamma h^2} \le 1.$$

With

$$\begin{aligned}
p(s) &:= 3 + 6(\beta-4)s + (\beta-4)(\beta-6)s^2, \\
p(x_i^2\sigma(x)^{-2}) &= 3 + 6(\beta-4)x_i^2\sigma(x)^{-2} + (\beta-4)(\beta-6)x_i^4\sigma(x)^{-4},
\end{aligned}$$

we can estimate

$$\min_{s\in[0,1]} p(s) \le p(x_i^2\sigma(x)^{-2}) \le \max_{s\in[0,1]} p(s).$$

We have $(\beta-4)(\beta-6)>0$ and the minimum is attained in

$$s_{min} = \frac{-6(\beta-4)}{2(\beta-4)(\beta-6)} = \frac{3}{6-\beta} \le \frac{1}{2},$$

taking into consideration the symmetry of $p(s)$ w.r.t. s_{min} this yields $s_{max}=1$. We therefore obtain

$$\begin{aligned}
\frac{\partial^4}{\partial x_i^4}\sigma(x)^\beta &\le \beta(\beta-2)\sigma(x)^{\beta-4}[3+6(\beta-4)+(\beta-4)(\beta-6)] \\
&= \beta(\beta-2)\sigma(x)^{\beta-4}[3+\beta(\beta-4)] \\
&= \beta(\beta-1)(\beta-2)(\beta-3)\sigma(x)^{\beta-4}, \\
\frac{\partial^4}{\partial x_i^4}\sigma(x)^\beta &\ge \beta(\beta-2)\sigma(x)^{\beta-4}\left[3+6(\beta-4)\frac{3}{6-\beta}+(\beta-4)(\beta-6)\frac{9}{(6-\beta)^2}\right] \\
&= \frac{3}{6-\beta}\beta(\beta-2)\sigma(x)^{\beta-4}[(6-\beta)+6(\beta-4)-3(\beta-4)] \\
&= \frac{3\beta(\beta-2)}{6-\beta}\sigma(x)^{\beta-4}[6-\beta+6\beta-24-3\beta+12] \\
&= -\frac{6\beta(\beta-2)(\beta-3)}{\beta-6}\sigma(x)^{\beta-4}.
\end{aligned}$$

⟨*3*⟩ *Neighbors comparison*

Let $x \in \mathcal{G}_h \setminus \{0\}$ and $\xi \in [x-he_i, x+he_i]$ for some $i \in \{1,\ldots,d\}$. We now show than for every $k \in (0,1)$ it is possible to determine $\tilde{\gamma}$ (independent of h and x) such that

$$k^{-2}\sigma^2(x) \ge \sigma^2(\xi) \ge k^2\sigma^2(x)$$

for all $\gamma \ge \tilde{\gamma}$. For $\xi = x+\tau h e_i$ with $|\tau| \le 1$ we obtain

$$\sigma^2(\xi) = |x+\tau h e_i|^2 + \gamma h^2 = |x|^2 + 2\tau h x_i + \tau^2 h^2 + \gamma h^2.$$

and consequently

$$\begin{aligned}
\sigma^2(\xi) - k^2\sigma^2(x) &= (1-k^2)|x|^2 + 2\tau h x_i + \tau^2 h^2 + (1-k^2)\gamma h^2 \\
&\ge (1-k^2)|x|^2 - 2h|x| + (1-k^2)\gamma h^2 \\
&\ge [(1-k^2)\gamma - (1-k^2)^{-1}]h^2 \\
&\ge 0, \quad \text{for } \gamma \ge (1-k^2)^{-2},
\end{aligned}$$

since

$$(1-k^2)|x|^2 - 2|x|h \ge -(1-k^2)^{-1}h^2$$

by the arithmetic-geometric mean inequality.
Now, using $h|x| \geq h^2$ we also obtain

$$\begin{aligned} k^{-2}\sigma^2(x) - \sigma^2(\xi) &= (k^{-2}-1)|x|^2 - 2\tau h x_i - \tau^2 h^2 + (k^{-2}-1)\gamma h^2 \\ &\geq (k^{-2}-1)|x|^2 - 2h|x| - h^2 + (k^{-2}-1)\gamma h^2 \\ &\geq (k^{-2}-1)|x|^2 - 4h|x| + (k^{-2}-1)\gamma h^2 \\ &\geq [(k^{-2}-1)\gamma - 4(k^{-2}-1)^{-1}]h^2 \\ &\geq 0, \quad \text{for } \gamma \geq 4(k^{-2}-1)^{-2}, \end{aligned}$$

since

$$(k^{-2}-1)|x|^2 - 4|x|h \geq -4(k^{-2}-1)^{-1}h^2$$

by the arithmetic-geometric mean inequality.

⟨*4*⟩ *Estimate from below*
We now can estimate

$$\begin{aligned} -\Delta_h[\sigma(x)^\beta] &= -\Delta[\sigma(x)^\beta] - \frac{h^2}{24}\sum_{i=1}^{d}\left\{\frac{\partial^4}{\partial x_i^4}\sigma^\beta(\xi) + \frac{\partial^4}{\partial x_i^4}\sigma^\beta(\eta)\right\} \\ &\geq -\beta(\beta+d-2)\sigma(x)^{\beta-2} + \beta(\beta-2)\gamma h^2\sigma(x)^{\beta-4} \\ &\quad - \frac{h^2}{24}\beta(\beta-1)(\beta-2)(\beta-3)\sum_{i=1}^{d}\left\{\sigma(\xi_i)^{\beta-4} + \sigma(\eta_i)^{\beta-4}\right\} \\ &\geq -\beta(\beta+d-2)\sigma(x)^{\beta-2}, \end{aligned}$$

provided γ can be chosen to satisfy

$$\beta(\beta-2)h^2\left[\gamma\sigma(x)^{\beta-4} - \frac{(\beta-1)(\beta-3)}{24}\sum_{i=1}^{d}\left\{\sigma(\xi_i)^{\beta-4} + \sigma(\eta_i)^{\beta-4}\right\}\right] \geq 0.$$

This is possible due to neighbors comparison

$$\begin{aligned} S &:= \gamma\sigma(x)^{\beta-4} - \frac{(\beta-1)(\beta-3)}{24}\sum_{i=1}^{d}\left\{\sigma(\xi_i)^{\beta-4} + \sigma(\eta_i)^{\beta-4}\right\} \\ &\geq \gamma\sigma(x)^{\beta-4} - \frac{(\beta-1)(\beta-3)}{24}\sum_{i=1}^{d} 2k^{-\beta+4}\sigma(x)^{\beta-4} \\ &= \sigma(x)^{\beta-4}\left[\gamma - \frac{d(\beta-1)(\beta-3)k^{-\beta+4}}{12}\right] \geq 0, \end{aligned}$$

for sufficiently large

$$\gamma > \tilde{\gamma}(\beta, d) := \inf_{k\in(0,1)} \max\left\{4(k^{-2}-1)^2, (1-k^2)^{-2}, \frac{d(\beta-1)(\beta-3)k^{-\beta+4}}{12}\right\}.$$

⟨5⟩ *Restricted estimate from above*
We now can estimate

$$
\begin{aligned}
-\Delta_h[\sigma(x)^\beta] &= -\Delta[\sigma(x)^\beta] - \frac{h^2}{24}\sum_{i=1}^{d}\left\{\frac{\partial^4}{\partial x_i^4}\sigma^\beta(\xi) + \frac{\partial^4}{\partial x_i^4}\sigma^\beta(\eta)\right\} \\
&\leq -\beta(\beta+d-2)\sigma(x)^{\beta-2} + \beta(\beta-2)\gamma h^2\sigma(x)^{\beta-4} \\
&+ \frac{h^2}{4}\frac{\beta(\beta-2)(\beta-3)}{\beta-6}\sum_{i=1}^{d}\left\{\sigma(\xi_i)^{\beta-4} + \sigma(\eta_i)^{\beta-4}\right\} \\
&\leq -\beta(\beta+d-2+\varepsilon)\sigma(x)^{\beta-2} \\
&+ [\beta\varepsilon\sigma(x)^{\beta-2} + \beta(\beta-2)\gamma h^2\sigma(x)^{\beta-4} \\
&+ \frac{h^2}{4}\frac{\beta(\beta-2)(\beta-3)}{\beta-6}\sum_{i=1}^{d}\left\{\sigma(\xi_i)^{\beta-4} + \sigma(\eta_i)^{\beta-4}\right\}] \\
&\leq -\beta(\beta+d-2+\varepsilon)\sigma(x)^{\beta-2},
\end{aligned}
$$

provided the sum in the square brackets can be made non-positive for $|x|$ sufficiently large. By neighborhood comparison we have

$$
\sum_{i=1}^{d}\left\{\sigma(\xi_i)^{\beta-4} + \sigma(\eta_i)^{\beta-4}\right\} \leq 2dk^{\beta-4}\sigma(x)^{\beta-4},
$$

for all $k \in (0,1)$ with corresponding $\gamma > \tilde{\gamma}_k$. We see that

$$
-\beta\sigma(x)^{\beta-4}h^2\left[-\varepsilon\left(\frac{|x|^2}{h^2}+\gamma\right) + (2-\beta)\gamma + \frac{d}{2}\frac{(2-\beta)(\beta-3)}{\beta-6}k^{\beta-4}\right] \leq 0
$$

for sufficiently large γ and sufficiently large $|x| \geq Rh$ for all $R \geq \tilde{R}(\varepsilon, d, \beta, \gamma)$.

⟨6⟩ *Unrestricted estimate from above*
For $\beta < 0$ and $x > -1$ it holds

$$
(1+x)^\beta \geq 1 + \beta x.
$$

This follows directly from Taylor's expansion for $(1+x)^\beta$ at zero

$$
(1+x)^\beta = 1 + \beta(1+0)^{\beta-1}x + \beta(\beta-1)(1+\xi)^{\beta-2}x^2 \geq 1 + \beta x
$$

with some intermediate value ξ. Using this we finally obtain

$$\begin{aligned}
\Delta_h \sigma^\beta(x) &= \sum_{i=1}^d \frac{\sigma(x+e_ih)^\beta - 2\sigma(x)^\beta + \sigma(x-e_ih)^\beta}{h^2} \\
&= \sum_{i=1}^d \frac{\sqrt{|x|^2+2x_ih+h^2+\gamma h^2}^\beta - \sigma(x)^\beta}{h^2} \\
&+ \sum_{i=1}^d \frac{\sqrt{|x|^2-2x_ih+h^2+\gamma h^2}^\beta - \sigma(x)^\beta}{h^2} \\
&= h^{-2}\sigma(x)^\beta \sum_{i=1}^d \left[\left(1+\frac{2x_ih+h^2}{\sigma^2(x)}\right)^{\frac{\beta}{2}} - 1\right] \\
&+ h^{-2}\sigma(x)^\beta \sum_{i=1}^d \left[\left(1+\frac{-2x_ih+h^2}{\sigma^2(x)}\right)^{\frac{\beta}{2}} - 1\right] \\
&\geq h^{-2}\sigma(x)^\beta \sum_{i=1}^d \left(\frac{\beta}{2}\frac{2x_ih+h^2}{\sigma^2(x)} + \frac{\beta}{2}\frac{h^2-2x_ih}{\sigma^2(x)}\right) \\
&= \beta d \sigma(x)^{\beta-2},
\end{aligned}$$

since

$$\begin{aligned}
\frac{\pm 2x_ih+h^2}{\sigma^2(x)} &\geq \frac{-2|x|h+h^2}{|x|^2+\gamma h^2} > -1 \\
\Leftarrow -2|x|h+h^2 &> -|x|^2-\gamma h^2 \\
\Leftarrow |x|^2-2|x|h+h^2 &> -\gamma h^2.
\end{aligned}$$

◀

6.3 Lemma (Basic estimate for discrete Green's function, [9])**.**
Let $\Omega_h \subset \mathcal{G}_h$ be bounded and let $G\colon \overline{\Omega}_h \times \Omega_h \to \mathbb{R}$ be its discrete Green function. Define

$$v_\gamma(x) := [|x|^2+\gamma h^2]^{\frac{2-d}{2}}.$$

Then for every $\gamma > \tilde{\gamma}(d, 2-d)$ with $\tilde{\gamma}$ as in the previous lemma holds

$$G(x,y) \leq K(d,\gamma)v_\gamma(x-y), \quad \forall x \in \overline{\Omega}_h,\ \forall y \in \Omega_h.$$

For fixed $\gamma := 2\tilde{\gamma}(d, 2-d)$ we correspondingly have

$$G(x,y) \leq K(d)v_\gamma(x-y) \quad \forall x \in \overline{\Omega}_h,\ \forall y \in \Omega_h.$$

▶
⟨*1*⟩ *Maximum principle*
Let $y \in \Omega_h$ be arbitrary but fixed. We want to apply the discrete maximum

principle to $G(\cdot, y) - Kv(\cdot - y)$ on G_h. Since the condition on the boundary follows from the positivity of v and K, we only need to show

$$-\Delta_{h,x}[G(x,y) - Kv(x-y)] \overset{!}{\leq} 0, \quad \forall x \in \Omega_h.$$

We consider two cases:

⟨*2*⟩ $x \neq y$
In this case we have

$$-\Delta_{h,x}[G(x,y) - Kv(x-y)] = K\Delta_{h,x}v(x-y).$$

From the previous Lemma with $\beta := 2 - d < 0$ and sufficiently large $\gamma > \tilde{\gamma}(d, 2-d)$ one obtains

$$-\Delta_{h,x}[v(x-y)] \geq 0.$$

⟨*3*⟩ $x = y$
In this case the symmetry of v, $v(he_i) = v(he_j)$, $1 \leq i,j \leq d$ yields

$$\begin{aligned}\Delta_h \, v(x)|_{x=0} &= \frac{2d}{h^2}\left[((1+\gamma)h^2)^{\frac{2-d}{2}} - (\gamma h^2)^{\frac{2-d}{2}}\right] = \frac{2d}{h^d}\left[(1+\gamma)^{\frac{2-d}{2}} - \gamma^{\frac{2-d}{2}}\right] \\ &=: \frac{1}{h^d}\Theta(d,\gamma) < 0\end{aligned}$$

and consequently

$$-\Delta_h\left[G(x,y) - Kv(x-y)\right] = \frac{1}{h^d}(1 + K\Theta(d,\gamma)) \leq 0$$

for all $K \geq -\frac{1}{\Theta(d,\gamma)}$.

◀

6.4 Lemma (Stairs estimate)**.**
Let $s_h\colon [h, +\infty) \to \mathbb{N}h$, $s_h(x) := \left\lfloor \frac{x}{h} \right\rfloor h$ for some $h > 0$. Then

$$s_h(x) > \frac{m}{m+2}(x+h), \quad \forall x \geq mh$$

holds for all $m \in \mathbb{N}$. In particular,

$$s_h(x) > \frac{1}{3}(x+h), \quad \forall x \geq h.$$

▶

Let $s_h(x) = kh$ for some $k \in \mathbb{N}$ and $x \geq mh$ implying $k \geq m$. Then $kh \leq x < (k+1)h$ yields

$$x + h < (k+2)h.$$

So, $k \geq m$ implies

$$s_h(x) = kh \geq \frac{m}{m+2}(k+2)h > \frac{m}{m+2}(x+h).$$

◀

6.5 Lemma (Discrete Schwarz's reflexion principle)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\Omega \subset \mathbb{R}^d$, $\overline{\Omega}_h = \mathrm{cl}_h\,\Omega$ be admissible with $\partial_1^- \Omega_h \subset \{x \in \mathbb{R}^d \mid x_1 = 0\}$ and let $u\colon \overline{\Omega}_h \to \mathbb{R}$ be discrete harmonic with $u|_{\partial_1^- \Omega_h} = 0$. Further, let $G_h \subset \mathcal{G}_h$, $\overline{G}_h = \mathrm{cl}_h\, G$ with $G := \Omega \cup \{(-x_1, x_2, \ldots, x_d) \mid x \in \Omega\}$, and define $v\colon \overline{G}_h \to \mathbb{R}$ by

$$v(x) := \begin{cases} u(x_1, x_2, \ldots, x_d), & \text{if } x_1 > 0, x \in \overline{G}_h \\ 0, & \text{if } x_1 = 0, x \in \overline{G}_h \\ -u(-x_1, x_2, \ldots, x_d), & \text{if } x_1 < 0, x \in \overline{G}_h. \end{cases}$$

Then v is discrete harmonic in G_h.

▶

By direct computation.

◀

6.6 Lemma (Second lemma on harmonic estimate)**.**
Let $\sigma_2 \in \mathbb{N}$ and $\sigma_1 := \left\lfloor \frac{\sigma_2}{d} \right\rfloor \in \mathbb{N} \cup \{0\}$. We define

$$U_h := \{x \in \mathcal{G}_h \mid \sigma_1 h < |x|_\infty, |x|_1 < (\sigma_1 + \sigma_2)h\}$$

for some $h > 0$. Furthermore, let $\eta\colon \overline{U}_h \to \mathbb{R}$ be discrete harmonic in U with

$$\begin{aligned} \eta(x) &= 0, \quad \text{for } |x|_\infty = \sigma_1 h, \\ \eta(x) &= 1, \quad \text{for } |x|_1 = (\sigma_1 + \sigma_2)h \end{aligned}$$

Let $\bar{s} := \left\lfloor \frac{\sigma_2}{d+1} \right\rfloor$. Then, the following inequality holds

$$\eta(x) \leq K(d)\frac{(t - \sigma_1)}{\sigma_2} = K(d)\frac{|x - \sigma_1 e_i h|}{\sigma_2 h} \tag{$*$}$$

for all $x = hte_i$, $\sigma_1 \leq t \leq \sigma_1 + \bar{s}$, $1 \leq i \leq d$.

▶

⟨*1*⟩ *Generality*

From symmetry it sufficient to show the claim for e_1. Both the left and the right side of the inequality are independent of h, and we therefore can assume h to be fixed. Now, for every fixed σ_2 the claim follows from the finiteness of U_h. Inductively, the claim also holds for finite set $\{1 \leq \sigma_2 \leq 2(d+2)\}$. We can therefore assume for the rest of the proof that

$$\sigma_2 \geq 2(d+2).$$

⟨*2*⟩ *Admissible geometry*

We want to show that

$$\{x \in \mathcal{G}_h \mid |x|_\infty = \sigma_1 h\} \subset \{x \in \mathcal{G}_h \mid |x|_1 < (\sigma_1 + \sigma_2)h\},$$

i.e. that the boundaries do not intersect.
First, we note that $\sigma_1 < \frac{\sigma_2}{d-1}$ by observing that

$$\sigma_1 = \left\lfloor \frac{\sigma_2}{d} \right\rfloor \leq \frac{\sigma_2}{d} < \frac{\sigma_2}{d-1}.$$

This implies $d\sigma_1 < \sigma_1 + \sigma_2$ and consequently

$$|x|_1 \leq d|x|_\infty = d\sigma_1 h < (\sigma_1 + \sigma_2)h$$

for $|x|_\infty = \sigma_1 h$.

⟨*3*⟩ *Admissible shifting depth*

To prove our estimate, we first want to obtain an estimate for $D_1^{\pm}\eta$ along $\{e_1 th \mid \sigma_1 \leq t \leq \sigma_1 + s\}$ using the combined harmonic estimate from Theorem 5.37. To use this theorem near the inner boundary of U_h, we need to be able to harmonically continue η onto some parts of $\{x \in \mathcal{G}_h \mid |x|_\infty < \sigma_1 h\}$ using Schwarz's reflection principle 6.5. This imposes the additional constraint $s < \sigma_1$.
To this end, we are looking for $s \in \mathbb{N}$, $s < \sigma_1$ such that

$$\{y \in \mathcal{G}_h \mid |x - y|_\infty \leq sh\} \subset \{y \in \mathcal{G}_h \mid |y|_1 \leq (\sigma_1 + \sigma_2)h\}$$

for all $x = e_1 th$, $\sigma_1 \leq t \leq \sigma_1 + s$ (see Figure 6.1 where $d = 2$, $\sigma_1 = 3$, $\sigma_2 = 7$, $s = 2$, $t = 4$). Since $\|\cdot\|_1$-balls are invariant in the sense

$$\{z \in \mathcal{G}_h \mid |z|_1 \leq Rh\} \cap \{z \in \mathcal{G}_h \mid z_1 \geq h\} = \{z \in \mathcal{G}_h \mid z_1 \geq h\} \cap \{z \in \mathcal{G}_h \mid |z - e_1 h|_1 \leq (R-1)h\}$$

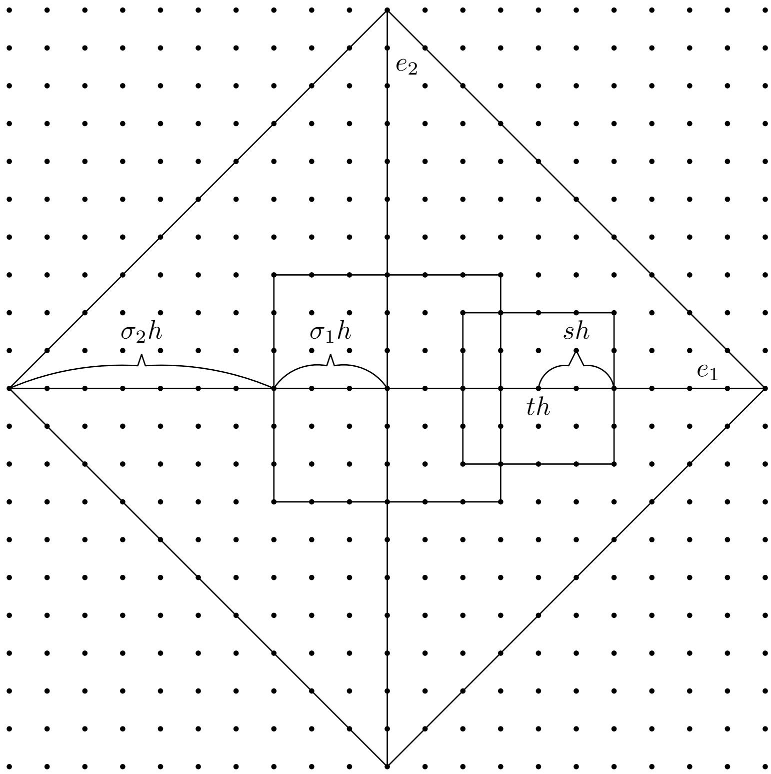

Figure 6.1

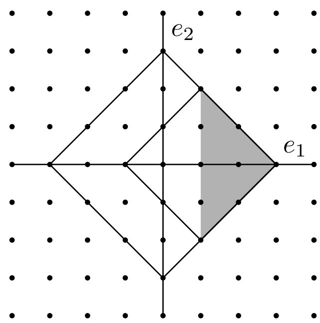

Figure 6.2

for all $R \in \mathbb{N}$ (see Figure 6.2 with $d = 2$, $R = 3$), it is sufficient to check

$$\{y \in \mathcal{G}_h \mid |x-y|_\infty \le sh, y_1 \ge x_1\} \\ \subset \{y \in \mathcal{G}_h \mid y_1 \ge x_1, |x-y|_1 \le (\sigma_2 - (t-\sigma_1))h\}$$

for $x = e_1 th$, $\sigma_1 \le t \le \sigma_1 + s$ (see Figure 6.3 where the left and right set in the situation of Figure 6.1 are marked by correspondingly darker and lighter filling). Using monotonicity of inclusion (if inclusion holds for $t = \sigma_1 + s$ it also holds for all $\sigma_1 \le t < \sigma_1 + s$), symmetry (dropping $y_1 \ge x_1$ condition) and translation invariance w.r.t. shift $|\cdot - x|$, we can reduce this condition even further to

$$\{y \in \mathcal{G}_h \mid |y|_\infty \le sh\} \subset \{y \in \mathcal{G}_h \mid |y|_1 \le (\sigma_2 - s)h\}.$$

From our previous geometrical considerations in ⟨*2*⟩, this holds if

$$ds \le \sigma_2 - s \Leftrightarrow s \le \frac{\sigma_2}{d+1},$$

i.e. we need to demand $s \le \left\lfloor \frac{\sigma_2}{d+1} \right\rfloor$. We also need to ensure $s < \sigma_1$ for Schwarz's reflection principle 6.5. This can be achieved by setting

$$s := \min\left\{ \left\lfloor \frac{\sigma_2}{d+1} \right\rfloor, \left\lfloor \frac{\sigma_2}{d} \right\rfloor - 1 \right\}.$$

We now have $s \ge 1$ due to the assumptions in ⟨*1*⟩ and $s + 1 \ge \left\lfloor \frac{\sigma_2}{d+1} \right\rfloor = \bar{s}$.

⟨*4*⟩ *Linear estimate*

Let $t \in \{\sigma_1, \dots, \sigma_1 + s\}$ be arbitrary, but fixed. By discrete maximum principle 5.15 we have $0 \le \eta \le 1$. Using discrete Schwarz reflection principle 6.5 if necessary, we can apply the combined harmonic estimate 5.37 to the discrete domain $\{y \in \mathcal{G}_h \mid |y - the_1|_\infty \le sh\}$, obtaining

$$\frac{\eta((t+1)he_1) - \eta((t-1)he_1)}{2h} \le \left| D_1^\pm \eta(the_1) \right| \le \frac{d}{sh}.$$

We now prove by induction that

$$\eta(te_1h) \overset{!}{\le} d\frac{t-\sigma_1}{s}$$

holds for $\sigma_1 \le t \le \sigma_1 + s + 1$. For $t = \sigma_1$ the claim follows by assumption, since $|\sigma_1 e_1 h|_\infty = \sigma_1 h$. For $t = \sigma_1 + 1$ Schwarz's reflection principle gives $\eta((\sigma_1 - 1)he_1) = -\eta((\sigma_1 + 1)he_1)$, implying

$$\frac{2\eta((\sigma_1+1)he_1)}{2h} = D_1^\pm \eta(\sigma_1 he_1) \le \frac{d}{sh},$$
$$\eta((\sigma_1+1)he_1) \le \frac{d}{s}.$$

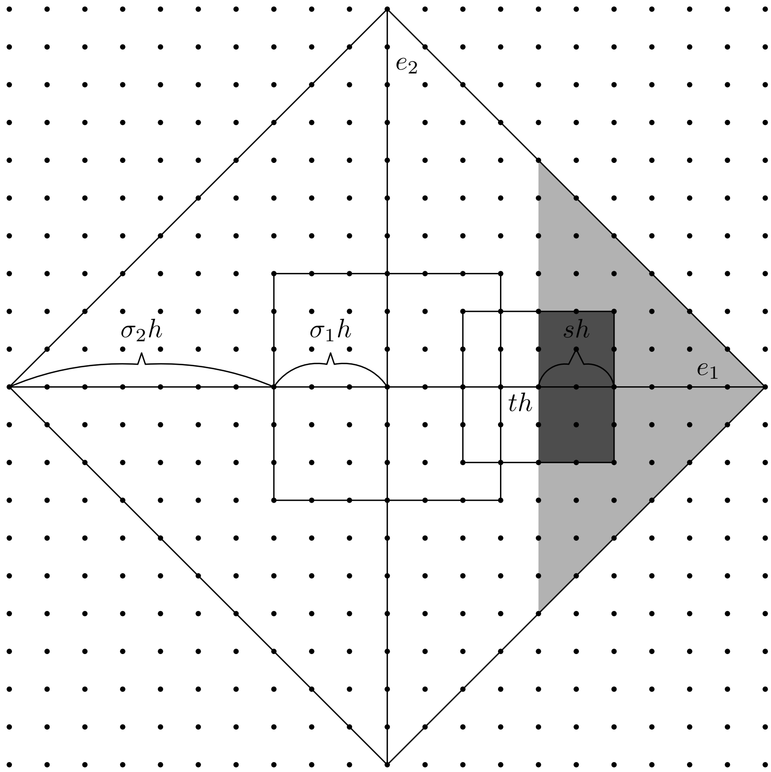

Figure 6.3

Now let the claim hold for all $t \in \mathbb{N}$, $\sigma_1 \le t \le \hat{t}$, $\hat{t} \le s$. Using Theorem 5.37 again and the induction step, we obtain

$$\begin{aligned} \frac{\eta((\hat{t}+1)he_1) - \eta((\hat{t}-1)he_1)}{2h} &\le \frac{d}{sh}, \\ \eta((\hat{t}+1)he_1) &\le \frac{2d}{s} + \frac{d}{s}[(\hat{t}-1) - \sigma_1], \\ \eta((\hat{t}+1)he_1) &\le \frac{d}{s}[(\hat{t}+1) - \sigma_1]. \end{aligned}$$

⟨*5*⟩ *Inequality*

Using the linear estimate from the previous step we only need to show that

$$\frac{d}{s}(t-\sigma_1) \overset{!}{\le} K(d)d\frac{t-\sigma_1}{\sigma_2} \Leftrightarrow s \overset{!}{\ge} \frac{\sigma_2}{K(d)}.$$

Applying the stairs estimate from Lemma 6.4 and using the assumption $-2 \ge -\frac{\sigma_2}{d+2}$ we obtain

$$\begin{aligned} s &\ge \left\lfloor \frac{\sigma_2}{d+1} \right\rfloor - 1 > \frac{1}{3}\left(\frac{\sigma_2}{d+1} + 1\right) - 1 = \frac{1}{3}\left(\frac{\sigma_2}{d+1} - 2\right) \\ &\ge \frac{1}{3}\left(\frac{\sigma_2}{d+1} - \frac{\sigma_2}{d+2}\right) = \frac{1}{3(d+1)(d+2)}\sigma_2. \end{aligned}$$

◀

6.7 Lemma (Combined estimate for discrete Green's function)**.**
Let Ω_h be an admissible subset of $\mathcal{G}_h$ and let $G\colon \overline{\Omega}_h \times \Omega_h \to \mathbb{R}$ be its discrete Green's function. Then it holds

$$G(x,y) \le K(d)[|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}}\rho(x), \quad x \in \overline{\Omega}_h, y \in \Omega_h,$$

where $\rho(x) = \operatorname{dist}(x, \partial\Omega_h)$ and γ is like in the Basic Estimate 6.3.

▶

⟨*1*⟩ *Proof idea*

Let $y \in \Omega_h$ be arbitrary but fixed and we can always assume $x \in \Omega_h$, i.e. $\rho(x) \ge h$. We prove the estimate by distinguishing two cases: we use a direct proof in the first case and a comparison argument in the second.

⟨*2*⟩ $|x-y| \le 2(d+1)\rho(x)$

Since $h \le \rho(x)$ we have

$$|x-y|^2 + \gamma h^2 \le 4(d+1)^2\rho^2(x) + \gamma\rho^2(x) = [\gamma + 4(d+1)^2]\rho^2(x),$$

and the basic estimate 6.3 yields

$$G(x,y) \le K[|x-y|^2 + \gamma h^2]^{\frac{2-d}{2}} \le K^*[|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}}\rho(x).$$

⟨*3*⟩ $|x-y| > 2(d+1)\rho(x)$

Here we proceed in two stages: firstly, we choose a suitable set and secondly, we construct an appropriate comparison function on this set.

(a) We set

$$\sigma_2 := \left\lceil \frac{|x-y|}{2h} \right\rceil, \quad \sigma_1 := \left\lfloor \frac{\sigma_2}{d} \right\rfloor, \quad s := \left\lfloor \frac{\sigma_2}{d+1} \right\rfloor.$$

We see that

$$\frac{|x-y|}{2h} \le \sigma_2,$$
$$\frac{|x-y|}{2(d+1)h} \le \frac{\sigma_2}{d+1} \le \left\lceil \frac{\sigma_2}{d+1} \right\rceil \le s+1,$$

and also

$$\left\lceil \frac{\sigma_2}{d+1} \right\rceil \le \frac{\sigma_2}{d+1} + 1 \le \sigma_2$$

since $\sigma_2 \ge 2 \ge \frac{d+1}{d}$.

(b) Let $x' \in \partial\Omega_h$ be such that $|x-x'| = \rho(x)$ and denote $x_0 := x' + \sigma_1 h \frac{(x'-x)}{|x'-x|}$ We introduce the following discrete sets

$$A_h := \{x \in \mathcal{G}_h \mid |x|_1 < (\sigma_1+\sigma_2)h, |x|_\infty > \sigma_1 h\},$$
$$U_h := (A_h + x_0) \cap \Omega_h,$$

with the usual notation $S + z_0 := \{z + z_0 \mid z \in S\}$. (see Figure 6.4 where $d = 2$, $\rho(x) = 2$, $|x-y| = \sqrt{146} > 12 = 2(2+1)\rho(x)$, $\sigma_2 = 7$, $\sigma_1 = 3$, $s = 2$, U_h is filled). Since Ω_h is admissible, we have $|x-x'|_1 = |x-x'|$ and

$$|x-x'| = \rho(x) < \frac{|x-y|}{2(d+1)} \le \frac{h}{d+1}\sigma_2 \le \left\lceil \frac{\sigma_2}{d+1} \right\rceil h \begin{matrix} \le (s+1)h \\ \le \sigma_2 h \end{matrix},$$

implying

$$|x-x_0| \le |x-x'| + |x'-x_0| \le (\sigma_1+s)h \qquad (*)$$

and analogously $|x-x_0|_1 = |x-x_0| < (\sigma_2+\sigma_1)h$, i.e. $x \in U_h$.

Now let $z \in U_h$. We decompose $z - x = (z-z') + (z'-x)$, where $\langle z-z', z'-x\rangle = \langle z-z', x-x'\rangle = 0$. We now have

$$\begin{aligned} |z-x| &= \sqrt{|z'-x|^2 + |z-z'|^2} \le \sqrt{\sigma_2^2 h^2 + \sigma_2^2 h^2} = \sqrt{2}\sigma_2 h \\ &= \sqrt{2}\left\lceil \frac{|x-y|}{2h} \right\rceil h < \sqrt{2}\left(\frac{|x-y|}{2} + h\right) \\ &< \sqrt{2}\left(\frac{|x-y|}{2} + \frac{|x-y|}{2(d+1)}\right) = \underbrace{\frac{\sqrt{2}}{2}\frac{d+2}{d+1}}_{=:\varkappa(d)=:\varkappa}|x-y|, \end{aligned}$$

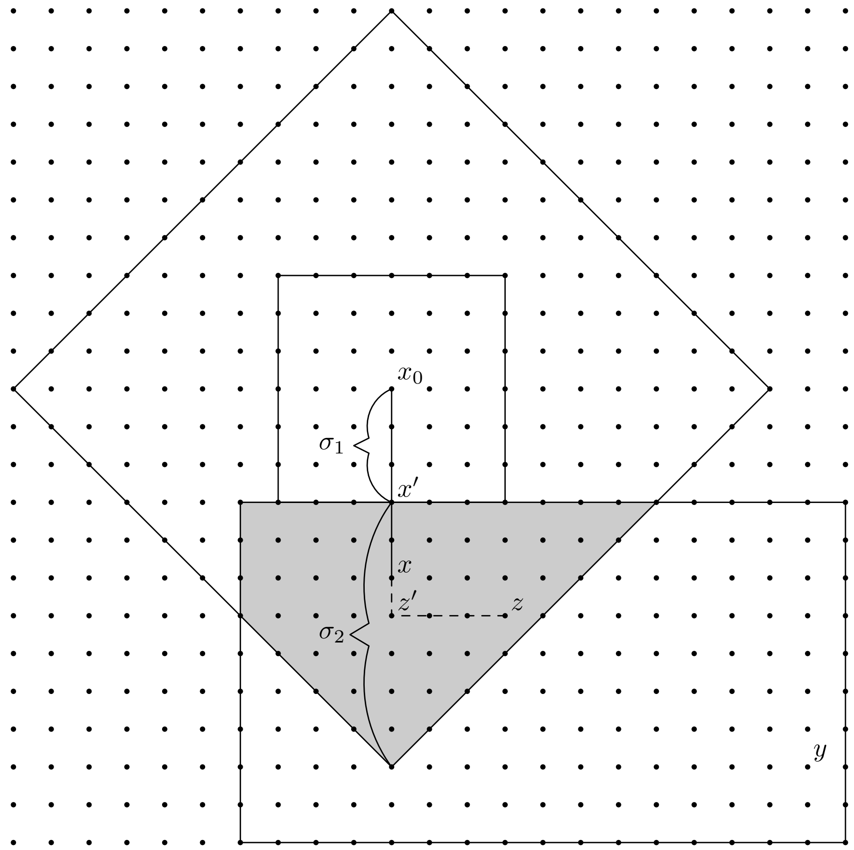

Figure 6.4

since $h \leq \rho(x) < \frac{|x-y|}{2(d+1)}$. From $d \geq 3$ we infer $\varkappa \in (0,1)$. This implies

$$|z-y| \geq |x-y| - |x-z| > (1-\varkappa)\,|x-y| > 0,$$

i.e. $y \notin \overline{U}_h$.
We therefore obtain that $G(\cdot, y)$ is discrete harmonic in U_h with

$$\begin{aligned} G(z,y) &= 0, \quad \forall z \in \partial\Omega_h, \\ G(z,y) &\leq K(d)[|z-y|^2 + \gamma h^2]^{\frac{2-d}{2}} \leq K^*(d)[|x-y|^2 + \gamma h^2]^{\frac{2-d}{2}} \\ &=: K_1(d)[|x-y|^2 + \gamma h^2]^{\frac{2-d}{2}}, \quad \forall z \in \partial U_h \cap \Omega_h, \end{aligned}$$

since for all $z \in \overline{U}_h$ holds

$$|z-y|^2 + \gamma h^2 \geq (1-\varkappa)^2|x-y|^2 + (1-\varkappa)^2\gamma h^2 = (1-\varkappa)^2(|x-y|^2 + \gamma h^2).$$

(c) Choosing a comparison function
Let $\eta\colon \overline{A}_h \to \mathbb{R}$ be defined as in Lemma 6.6. We now define $H\colon \overline{U}_h \to \mathbb{R}$

by

$$H(z) := \eta(z - x_0) K_1(d) [|x - y|^2 + \gamma h^2]^{\frac{2-d}{2}}.$$

This function is discrete harmonic by construction and

$$\begin{aligned} G(z, y) &\leq H(z), \quad \forall z \in \partial U_h \cap \Omega_h, \\ G(z, y) = 0 &\leq H(z), \quad \forall z \in \partial U_h \cap \partial\Omega_h. \end{aligned}$$

By discrete maximum principle we obtain

$$G(z, y) \leq H(z) = \eta(z - x_0) K_1(d) [|x - y|^2 + \gamma h^2]^{\frac{2-d}{2}}, \quad \forall z \in U_h.$$

Taking into account that $x \in U_h$ satisfies $(*)$ we can apply Lemma 6.6 with $e_i = \frac{x - x_0}{|x - x_0|}$, $i \in \{1, \ldots, d\}$, getting

$$\begin{aligned} G(x, y) &\leq \eta(x - x_0) K_1(d) [|x - y|^2 + \gamma h^2]^{\frac{2-d}{2}} \\ &\leq K(d) \frac{|x - x_0 - \sigma_1 e_i h|}{\sigma_2 h} [|x - y|^2 + \gamma h^2]^{\frac{2-d}{2}} \\ &= K(d) \frac{|x - x'|}{\sigma_2 h} [|x - y|^2 + \gamma h^2]^{\frac{2-d}{2}} \\ &\leq K^*(d) \frac{\rho(x)}{[|x - y|^2 + \gamma h^2]^{\frac{1}{2}}} [|x - y|^2 + \gamma h^2]^{\frac{2-d}{2}} \\ &= K^*(d) [|x - y|^2 + \gamma h^2]^{\frac{1-d}{2}} \rho(x), \end{aligned}$$

since

$$\begin{aligned} &\sigma_2 h \geq \frac{|x - y|}{2} > \frac{|x - y|}{2(d+1)} > \rho(x) \geq h, \\ \Rightarrow \quad &(1 + \tfrac{\gamma}{4}) \sigma_2^2 h^2 > \frac{|x - y|^2}{4} + \frac{\gamma}{4} h^2, \\ \Rightarrow \quad &\sigma_2 h \geq \left(1 + \tfrac{\gamma}{4}\right)^{-1/2} \tfrac{1}{2} (|x - y|^2 + \gamma h^2)^{\frac{1}{2}} = \frac{1}{\sqrt{\gamma + 4}} (|x - y|^2 + \gamma h^2)^{\frac{1}{2}}. \end{aligned}$$

◀

6.8 Remark (Norm equivalence)**.**
It holds

$$\begin{aligned} |x - y| &\leq \sqrt{d} |x - y|_\infty, \\ |x - y|_\infty &\leq |x - y| \end{aligned}$$

for all $x, y \in \mathbb{R}^d$. Moreover, for any $a_1, a_2, b_1, b_2 > 0$ we have

$$a_1 |x_1| + a_2 |x_2| \geq K(a_1, a_2, b_1, b_2) \sqrt{b_1 x_1^2 + b_2 x_2^2}, \quad \forall x_1, x_2 \in \mathbb{R}$$

due to the theorem of Riesz about equivalent norms.

6.9 Lemma (Estimates for quotients of discrete Green's function)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible, and let $G\colon \overline{\Omega}_h \times \Omega_h \to \mathbb{R}$ be its discrete Green's function. Then

$$|D^+_{x_i} G(x,y)| \le K(d)[|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}}, \quad x \in \Omega_h \cup \partial_i^- \Omega_h, y \in \Omega_h, 1 \le i \le d$$

with γ like in the Basic Estimate 6.3.
▶
⟨*1*⟩ *Dealing with the boundary*
Let $y \in \Omega_h$ be arbitrary but fixed. Again we denote $\rho(x) := \operatorname{dist}(x, \partial\Omega_h)$. For $x \in \partial_i^- \Omega_h$ we have

$$\begin{aligned} \left|D^+_{x_i} G(x,y)\right| &= \left|\frac{G(x+he_i,y) - G(x,y)}{h}\right| = \left|\frac{G(x+he_i,y)}{h}\right| \\ &\le \frac{K(d)[|x-y+he_i|^2 + \gamma h^2]^{\frac{1-d}{2}} \rho(x+he_i)}{h} \\ &\le K^*(d)[|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}} \end{aligned}$$

due to Lemma 6.7, the identity $\rho(x+he_i) = h$ and the third step in the proof of Lemma 6.2 (note $x \ne y$). We can now assume that $\rho(x) \ge h$.

⟨*2*⟩ *Case distinction*
We introduce $d^+ := \lfloor \sqrt{d} \rfloor + 1 > \sqrt{d}$, $\varkappa := \frac{\sqrt{d}}{d^+} < 1$ and consider the following three cases

⟨*3*⟩ $\frac{|x-y|_\infty}{d^+} \ge \rho(x)$
Choose $R := \rho(x)$. Since for $\xi \in \overline{B_h(x,R,|\cdot|_\infty)} := \{z \in \mathcal{G}_h \mid |z-x|_\infty \le R\}$

$$\begin{aligned} \rho(\xi) &\le \rho(x) + |x-\xi|_\infty \le R + R = 2R, \\ |\xi - y| &\ge |x-y| - |x-\xi| \ge |x-y| - \sqrt{d}|x-\xi|_\infty \\ &\ge |x-y| - \sqrt{d}R = |x-y| - \sqrt{d}\rho(x) \\ &\ge |x-y| - \sqrt{d}\frac{|x-y|_\infty}{d^+} \\ &\ge |x-y| - \varkappa|x-y| \\ &= \underbrace{(1-\varkappa)}_{=:\tilde{\varkappa} \in (0,1)} |x-y| > 0, \end{aligned}$$

we have $y \notin \overline{B_h(x,R,|\cdot|_\infty)}$, i.e. $G(\cdot,y)$ is discrete harmonic in this set. We denote

$$B_h^i(x,R,|\cdot|_\infty) := \{z \in \mathcal{G}_h \mid |z-x|_\infty \le R, z_i - x_i > -R + h\}.$$

Using Lemma 6.7 and Theorem 5.31 together with the previous estimates we obtain

$$\begin{aligned}|D^+_{x_i}G(x,y)| &\le 2\frac{d}{R}\max_{\xi\in\partial B^i_h(x,R,|\cdot|_\infty)}|G(\xi,y)|\\ &\le 2\frac{d}{R}\max_{\xi\in\partial B^i_h(x,R,|\cdot|_\infty)}K(d)[|\xi-y|^2+\gamma h^2]^{\frac{1-d}{2}}\rho(\xi)\\ &\le 4K(d)\max_{\xi\in\partial B^i_h(x,R,|\cdot|_\infty)}\bar{\varkappa}^{1-d}[|x-y|^2+\gamma h^2]^{\frac{1-d}{2}}\\ &\le K^*(d)[|x-y|^2+\gamma h^2]^{\frac{1-d}{2}},\end{aligned}$$

if $\rho(x)=R>h$ (see assumptions in 5.31). Using Lemma 5.33 instead of Theorem 5.31 one obtains the case $\rho(x)=h$.

⟨*4*⟩ $h\le\frac{|x-y|_\infty}{d^+}<\rho(x)$
Choose $R:=\lfloor\frac{|x-y|_\infty}{d^+h}\rfloor h$. This leads to

$$h\le R\le\frac{|x-y|_\infty}{d^+}\le\frac{|x-y|}{d^+}$$

and also to $y\notin\overline{B_h(x,R,|\cdot|_\infty)}\subset\Omega_h$, since for $\xi\in\overline{B_h(x,R,|\cdot|_\infty)}$ we have

$$\begin{aligned}|\xi-y| &\ge |x-y|-|x-\xi|\ge|x-y|-\sqrt{d}|x-\xi|_\infty\ge|x-y|-\sqrt{d}R\\ &\ge|x-y|-\frac{\sqrt{d}}{d^+}|x-y|=\underbrace{(1-\varkappa)}_{=:\bar{\varkappa}\in(0,1)}|x-y|>0.\end{aligned}$$

The stairs estimate 6.4 and the norm equivalence gives

$$R>\frac{1}{3}\left(\frac{|x-y|_\infty}{d^+}+h\right)\ge\frac{1}{3}\left(\frac{|x-y|}{\sqrt{d}d^+}+h\right)\ge K(d)[|x-y|^2+\gamma h^2]^{\frac{1}{2}},$$

implying

$$\begin{aligned}|D^+_{x_i}G(x,y)| &\le 2\frac{d}{R}\max_{\xi\in\partial B^i_h(x,R,|\cdot|_\infty)}|G(\xi,y)|\\ &\le 2\frac{d}{R}\max_{\xi\in\partial B^i_h(x,R,|\cdot|_\infty)}K(d)[|\xi-y|^2+\gamma h^2]^{\frac{2-d}{2}}\\ &\le 2\frac{d}{R}\max_{\xi\in\partial B^i_h(x,R,|\cdot|_\infty)}\bar{\varkappa}^{2-d}K(d)[|x-y|^2+\gamma h^2]^{\frac{2-d}{2}}\\ &\le K^*(d)[|x-y|^2+\gamma h^2]^{\frac{1-d}{2}},\end{aligned}$$

for $R>h$ and analogously with Lemma 5.33 for $R=h$.

$\langle 5 \rangle$ $\frac{|x-y|_\infty}{d^+} < h$

Choose $R := \lceil \frac{|x-y|_\infty}{d^+ h} \rceil h = h$. Then for $\xi \in \partial \overline{B_h(x, R, |\cdot|_\infty)}$, we can estimate

$$\begin{aligned} h &\leq [|\xi - y| + h] \leq K[|\xi - y|^2 + \gamma h^2]^{\frac{1}{2}}, \\ h &\geq \frac{h}{2} + \frac{|x-y|_\infty}{2d^+} \geq \frac{h}{2} + \frac{|x-y|}{2d^+\sqrt{d}} \geq K(d)[|x-y|^2 + \gamma h^2]^{\frac{1}{2}}. \end{aligned}$$

From the representation of Green's function and Lemma 5.33 we obtain (recall that $-\Delta_{h,x} G(x,y)|_{y=x} = h^{-d}$)

$$\begin{aligned} |D^+_{x_i} G(x,y)| &\leq \frac{2}{R} \max_{\xi \in \partial B_h(x,R,|\cdot|_\infty)} |G(\xi, y)| + \frac{1}{2d} h \frac{1}{h^d} \\ &\leq \frac{2}{h} \max_{\xi \in \partial B_h(x,R,|\cdot|_\infty)} K(d)[|\xi - y|^2 + \gamma h^2]^{\frac{2-d}{2}} + \frac{1}{2dh^{d-1}} \\ &\leq K^*(d) \frac{2}{h} h^{2-d} + \frac{1}{2dh^{d-1}} \leq K(d)[|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}}. \end{aligned}$$

◀

Chapter 7

Discrete A Priori Estimates for the Linear Case

7.1 Discrete Riesz potentials

7.1 Motivation (Discrete linear a priori estimates)**.**
The results of this chapter are essentially discrete pendants to the results of Chapter 4. For this section see also Chapter 7 from [23].

7.2 Definition (Discrete L^q Norms)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be bounded and let $u\colon \Omega_h \to \mathbb{R}$. We define by

$$\|u\|^q_{L^q(\Omega_h)} = \sum_{x\in\Omega_h} |u(x)|^q h^d$$

the *discrete L^q norm.*

7.3 Definition (Discrete Riesz potential operators)**.**
Let $\alpha \in (0,1]$, $\gamma > 0$ be fixed. The *discrete Riesz potential* is defined on $\mathcal{G}_h$ by

$$v_{\alpha,\gamma}(x) := [|x|^2 + \gamma h^2]^{\frac{-d(1-\alpha)}{2}}.$$

For bounded $\Omega_h \subset \mathcal{G}_h$ it generates the *discrete Riesz potential operator* defined by

$$(V_{\alpha,\gamma} f)(x) = \sum_{y\in\Omega_h} v_{\alpha,\gamma}(x-y) f(y) h^d, \quad x \in \overline{\Omega}_h$$

which maps $f\colon \Omega_h \to \mathbb{R}$ to $V_{\alpha,\gamma} f\colon \overline{\Omega}_h \to \mathbb{R}$.

7.4 Notation (Diameter of an admissible set)**.**
Let $\Omega \subset \mathbb{R}^d$ be admissible. We denote

$$\operatorname{diam}\Omega := \max_{1\le i\le d} \sup_{x,y\in\Omega} |x_i - y_i|.$$

7.5 Lemma (Uniform estimate for the discrete Riesz potential)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h\,\Omega$ be admissible with admissible $\Omega = \prod_{i=1}^d (a_i, b_i)$, $a_i < b_i$, $1 \le i \le d$. Then

$$\|v_{\alpha,\gamma}(x - \cdot)\|_{L^q(\overline{\Omega}_h)} \le K(\alpha, \gamma, d, q, \operatorname{diam}\Omega), \quad \forall x \in \overline{\Omega}_h$$

for

$$1 \le q < \begin{cases} \dfrac{1}{1-\alpha}, & \alpha \in (0,1), \\ \infty, & \alpha = 1. \end{cases}$$

▶

⟨*1*⟩ *Symmetrization*

We will sometimes suppress the (α,γ)-dependence, writing $v := v_{\alpha,\gamma}$. Let $x \in \overline{\Omega}_h$ be arbitrary. Since v is positive and radially symmetric we have

$$\|v(x - \cdot)\|_{L^q(\Omega_h)} \le \|v\|_{L^q(\Omega_h^{\pm})} \le 2^d \|v\|_{L^q(\Omega_h^{+})},$$

where $\overline{\Omega}_h^{\pm} = \mathrm{cl}_h((-c,c)^d)$, $\Omega_h^{+} = \mathrm{cl}_h^{+}((0,c)^d)$, $c = \max_{1\le i\le d}(b_i - a_i) = \operatorname{diam}\Omega$ (see Figure 7.1 with $d = 2$, $\Omega = (5h, 10h) \times (5h, 9h)$). It is therefore sufficient

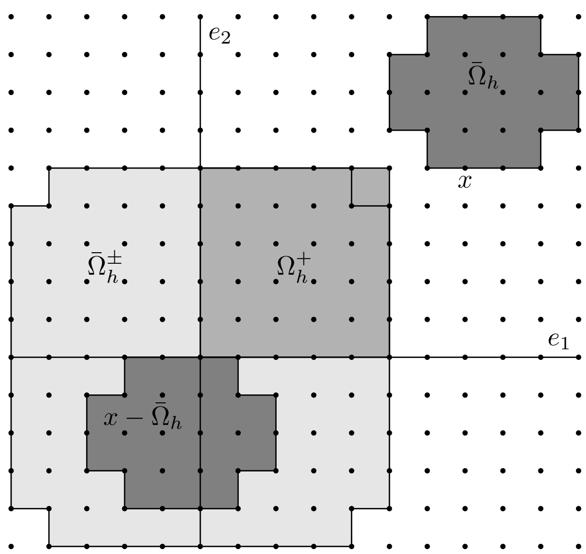

Figure 7.1

to establish an upper bound on $L^q(\Omega_h^{+})$. We also can assume $\alpha \in (0,1)$, since for $\alpha = 1$ we have

$$\|v_{1,\gamma}\|_{L^q(\Omega_h^{+})}^q = \sum_{x \in \Omega_h^{+}} h^d \le (\operatorname{diam}\Omega + 1)^d.$$

⟨*2*⟩ *Isolating the origin*

Using radial symmetry of v we can write

$$\|v\|^q_{L^q(\Omega_h^+)} = \sum_{x\in\Omega_h^+} |v(x)|^q h^d = \sum_{\substack{x\in\Omega_h^+\\ |x|_\infty\le h}} |v(x)|^q h^d + \sum_{\substack{x\in\Omega_h^+\\ |x|_\infty> h}} |v(x)|^q h^d$$
$$\le 2^d|v(0)|^q h^d + \sum_{\substack{x\in\Omega_h^+\\ |x|_\infty> h}} |v(x)|^q h^d,$$

since $2^d = \#\{x \in \mathcal{G}_h \mid x_i \ge 0, 1 \le i \le d, |x|_\infty \le h\}$. The term

$$|v_{\alpha,\gamma}(0)|^q h^d = \gamma^{-\frac{d(1-\alpha)q}{2}} h^{d(1-(1-\alpha)q)}$$

is bounded uniformly in h due to $q < \frac{1}{1-\alpha}$. We denote

$$G_h := \{x \in \Omega_h^+ \mid |x|_\infty > h\}.$$

and prove the boundedness of $\|v\|^q_{L^q(G_h)}$ in two steps.

⟨*3*⟩ *Interior estimate by comparison integral*

The main idea of the estimate is to interpret $\|v\|^q_{L^q(G_h)}$ as a lower Riemann sum for $\|v\|^q_{L^q(G)}$ with some $G \subset \mathbb{R}^d$. If $x \in G_h$ with $x_i > 0$, $1 \le i \le d$ then it follows by direct calculations that

$$v(x) = \min\{v(y) \mid y \in \prod_{i=1}^d (x_i - h, x_i)\}.$$

Denoting $G_h^+ := \{x \in G_h \mid x_i > 0, 1 \le i \le d\}$ and $G := [0,c]^d \setminus [0,h]^d$ we obtain

$$\sum_{x\in G_h^+} |v(x)|^q h^d \le \int_G |v(x)|^q \,\mathbf{d}\, x.$$

Replacing G with the larger set (see Figure 7.2, where the dots represents G_h^+)

$$G_{rad} := \{x \in \mathbb{R}^d \mid h < |x|_2 \le \sqrt{d}c, x_i > 0, 1 \le i \le d\}$$

and using polar coordinates we finally get

$$\sum_{G_h^+} |v(x)|^q h \le \int_G |v(x)|^q \,\mathbf{d}\, x \le \int_{G_{rad}} |v(x)|^q \,\mathbf{d}\, x \le \int_{G_{rad}} |x|^{-d(1-\alpha)q} \,\mathbf{d}\, x$$
$$= \frac{\omega_d}{2^d} \int_h^{\sqrt{d}c} r^{-d(1-\alpha)q} r^{d-1} \,\mathbf{d}\, r = \frac{\omega_d}{2^d} \left. \frac{r^{d(1-(1-\alpha)q)}}{d(1-(1-\alpha)q)} \right|_h^{\sqrt{d}c} \mathbf{d}\, r$$
$$= \frac{\omega_d}{2^d d} \left(\frac{(\sqrt{d}c)^{d(1-(1-\alpha)q)} - h^{d(1-(1-\alpha)q)}}{1-(1-\alpha)q} \right),$$

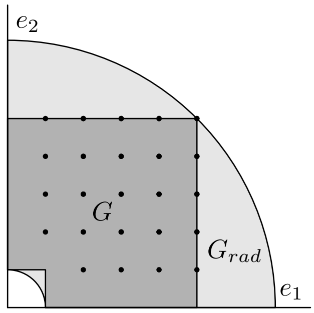

Figure 7.2

where $\omega_d/2^d$ is the surface area of $\{|x|=1\}\cap\mathbb{R}^d_+$. Since $(1-\alpha)q<1$, we avoid the logarithmic singularity and have uniform in h boundedness.

⟨*4*⟩ *Dimensional reduction*
Denoting $G_h^0 := G_h \setminus G_h^+$ we now have to show the boundedness of $\|v\|^q_{L^q(G_h^0)}$. The set G_h^0 represents points from G_h, for which at least one component x_i, $i\in\{1,\dots,d\}$ is zero. We can partition it in sets where exactly $1,2,\dots,d-1$ components are non-zero. For instance, the subset with exactly one non-zero component corresponds to the intersection of G_h with axes. Since v is radially symmetric, the sum over every such subset with exactly k non-zero components can be calculated via

$$\binom{d}{k}\sum_{x\in R_h^k}|v(x)|^q h^d,$$

with

$$\begin{aligned}
&R_h^k := R^k\cap G_h,\\
&R^k := \{x\in[0,c]^k\setminus[0,h]^k \mid x_i>0, 1\le i\le k\}\times\{0\}^{d-k}.
\end{aligned}$$

We now have to show boundedness for every such sum. Using the approach from the previous step (observe $R_h^d = G_h^+$) and introducing

$$R_{rad}^k := \{x\in\mathbb{R}^d \mid h<|x|_2\le\sqrt{k}c, x_i>0, 1\le i\le k\}\times\{0\}^{d-k}.$$

we estimate

$$\begin{aligned}
\sum_{x\in R_h^k}|v(x)|^q h^d &= h^{d-k}\sum_{x\in R_h^k}|v(x)|^q h^k \le h^{d-k}\int_{R^k}|v(x)|^q\,\mathbf{d}\,x\\
&\le h^{d-k}\int_{R_{rad}^k}|v(x)|^q\,\mathbf{d}\,(x_1,\dots,x_k)\le h^{d-k}\int_h^{\sqrt{k}c} r^{k-1-d(1-\alpha)q}\,\mathbf{d}\,r\\
&= h^{d-k}\cdot\frac{(\sqrt{k}c)^{k-d(1-\alpha)q}}{k-d(1-\alpha)q}-\frac{h^{d(1-(1-\alpha)q)}}{k-d(1-\alpha)q},
\end{aligned}$$

for $k \neq d(1-\alpha)q$ and

$$\sum_{x \in R_h^k} |v(x)|^q h^d \leq h^{d-k} \int_h^{\sqrt{k}c} r^{k-1-d(1-\alpha)q} \, \mathbf{d}\, r = h^{d-k} \left(\log(\sqrt{k}c) - \log(h) \right)$$

otherwise. In both cases $d > k$ and $1-(1-\alpha)q > 0$ ensure uniform in h boundedness.

◀

7.6 Lemma (Discrete Hardy-Littlewood-Sobolev inequality)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h \, \Omega$ be admissible with admissible $\Omega = \prod_{i=1}^d (a_i, b_i)$, $a_i < b_i$, $1 \leq i \leq d$. Then

$$\|V_{\alpha,\gamma} f\|_{L^q(\overline{\Omega}_h)} \leq K(\alpha, d, \operatorname{diam}\Omega, p, q, \gamma) \|f\|_{L^p(\Omega_h)}$$

for every $f \colon \Omega_h \to \mathbb{R}$, provided

$$0 \leq \delta := \delta(p,q) := \frac{1}{p} - \frac{1}{q} < \alpha$$

▶

⟨*1*⟩ *Case* $\alpha = 1$
After setting $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p' \leq \infty$ direct calculations yield

$$\begin{aligned}
\|V_{1,\gamma} f\|_{L^q(\overline{\Omega}_h)} &= \left(\sum_{x \in \overline{\Omega}_h} |(V_{1,\gamma} f)(x)|^q h^d \right)^{1/q} = \left(\sum_{x \in \overline{\Omega}_h} \left| \sum_{y \in \Omega_h} f(y) h^d \right|^q h^d \right)^{1/q} \\
&\leq \sum_{y \in \Omega_h} |f(y)| h^d \cdot \left(\sum_{x \in \overline{\Omega}} h^d \right)^{1/q} \leq \sum_{y \in \Omega_h} |f(y)| h^d \cdot (\operatorname{diam}\Omega + 1)^{d/q} \\
&\leq \left(\sum_{y \in \Omega_h} |f(y)|^p h^d \right)^{1/p} (\operatorname{diam}\Omega + 1)^{d/q + d/p'} \\
&= \|f\|_{L^p(\Omega_h)} (\operatorname{diam}\Omega + 1)^{d/q + d/p'}
\end{aligned}$$

for all $1 \leq p \leq q < +\infty$. We are therefore left with the

⟨*2*⟩ *Case* $\alpha \in (0,1)$
Denote $r := \frac{1}{1-\delta}$. From $1 \geq 1-\delta > 1-\alpha > 0$ we obtain

$$1 \leq r < \frac{1}{1-\alpha}.$$

Using Lemma 7.5 this implies

$$v_{\alpha,\gamma}(x-\cdot) \in L^r(\overline{\Omega}_h), \quad x \in \overline{\Omega}_h$$

with a uniform bound

$$\|v_{\alpha,\gamma}(x-\cdot)\|_{L^r(\Omega_h)} \le C(\alpha,\gamma,d,r,\operatorname{diam}\Omega) =: C.$$

We can write

$$v_{\alpha,\gamma}|f| = v_{\alpha,\gamma}^{r/q}|f|^{p/q} \cdot v_{\alpha,\gamma}^{r(1-1/p)} \cdot |f|^{p\delta}$$

and apply the Hölder inequality with three multipliers

$$\begin{aligned}
I &:= |(V_{\alpha,\gamma}f)(x)| \le \sum_{y\in\Omega_h} v_{\alpha,\gamma}(x-y)|f(y)|h^d \\
&\le \left(\sum_{y\in\Omega_h} v_{\alpha,\gamma}^r(x-y)|f(y)|^p h^d\right)^{\frac{1}{q}} \left(\sum_{y\in\Omega_h} v_{\alpha,\gamma}^r(x-y)h^d\right)^{1-\frac{1}{p}} \left(\sum_{y\in\Omega_h} |f(y)|^p h^d\right)^{\delta} \\
&\le \left(\sum_{y\in\Omega_h} v_{\alpha,\gamma}^r(x-y)|f(y)|^p h^d\right)^{\frac{1}{q}} C^{r-r/p}\|f\|_{L^p(\Omega_h)}^{p\delta}.
\end{aligned}$$

Lemma 7.5 finally yields

$$\begin{aligned}
\|V_{\alpha,\gamma}f\|_{L^q(\overline{\Omega}_h)} &\le \left(\sum_{x\in\overline{\Omega}_h}\sum_{y\in\Omega_h} v_{\alpha,\gamma}^r(x-y)|f(y)|^p h^d h^d\right)^{1/q} C^{r-r/p}\|f\|_{L^p(\Omega_h)}^{p\delta} \\
&= \left(\sum_{y\in\Omega_h}\left(\sum_{x\in\overline{\Omega}_h} v_{\alpha,\gamma}^r(x-y)h^d\right)|f(y)|^p h^d\right)^{1/q} C^{r-r/p}\|f\|_{L^p(\Omega_h)}^{p\delta} \\
&\le C^{r/q}\|f\|_{L^p(\Omega_h)}^{p/q} C^{r-r/p}\|f\|_{L^p(\Omega_h)}^{p\delta} \\
&= C^{r(1-\delta)}\|f\|_{L^p(\Omega_h)}.
\end{aligned}$$

◀

7.7 Remark (Extension)**.**
We did not consider the case $\delta = \alpha$, $p > 1$, since it is not necessary for our purposes (cf. Remark 4.19).

7.2 Discrete Linear A Priori Estimate

7.8 Definition (Discrete Sobolev Norms)**.**
Let $\Omega_h \subset \mathcal{G}_h$ be admissible and let $u\colon \overline{\Omega}_h \to \mathbb{R}$ with $u|_{\partial\Omega_h} = 0$. We define by

$$\|u\|^p_{W_0^{1,p}(\Omega_h)} := \sum_{i=1}^{d} \sum_{x \in \Omega_h \cup \partial_i^- \Omega_h} |D_i^+ u(x)|^p h^d.$$

the *discrete* $W_0^{1,p}$ *norm.*

7.9 Theorem (Discrete Linear A Priori Estimate)**.**
Let $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h\, \Omega$ be admissible with some admissible $\Omega \subset \mathbb{R}^d$. Let $f\colon \Omega_h \to \mathbb{R}$ be given and $u\colon \overline{\Omega}_h \to \mathbb{R}$ be defined as the solution of

$$\begin{aligned} -\Delta_h u(x) &= f(x), \quad x \in \Omega_h, \\ u(x) &= 0, \qquad x \in \partial\Omega_h. \end{aligned}$$

Then, for $p \in (1, \infty)$, $q > \frac{dp}{d+p}$ it holds

$$\|u\|_{W_0^{1,p}(\Omega_h)} \le K(d, p, q, \operatorname{diam}\Omega)\|f\|_{L^q(\Omega_h)}.$$

▶
Without loss of generality we can assume $q \in (\frac{dp}{d+p}, p] \neq \emptyset$, since

$$\|f\|_{L^q(\Omega_h)} \le K(q, p, d, \operatorname{diam}\Omega)\|f\|_{L^p(\Omega_h)} \le K^*(q, p, \tilde{q}, d, \operatorname{diam}\Omega)\|f\|_{L^{\tilde{q}}(\Omega_h)}$$

for $q \le p < \tilde{q}$. Due to norm equivalence on $\mathbb{R}^d$ we obtain

$$\begin{aligned} \|u\|_{W_0^{1,p}(\Omega_h)} &= \left(\sum_{i=1}^{d} \sum_{x \in \Omega_h \cup \partial_i^- \Omega_h} |D_i^+ u(x)|^p h^d \right)^{1/p} = \left(\sum_{i=1}^{d} \|D_i^+ u(x)\|^p_{L^p(\Omega_h \cup \partial_i^- \Omega_h)} \right)^{1/p} \\ &\le K(d) \max_{1 \le i \le d} \|D_i^+ u(x)\|_{L^p(\Omega_h \cup \partial_i^- \Omega_h)}. \end{aligned}$$

It is therefore sufficient to have an estimate for $\|D_i^+ u\|_{L^p(\Omega_h)}$, $1 \le i \le d$.
From the Representation Lemma 5.19 we have

$$u(x) = \sum_{y \in \Omega_h} G(x, y) f(y) h^d, \quad x \in \overline{\Omega}_h,$$

implying

$$D_i^+ u(x) = \sum_{y \in \Omega_h} D_{x_i}^+ G(x, y) f(y) h^d, \quad x \in \Omega_h \cup \partial_i^- \Omega_h$$

for $1 \le i \le d$. From Lemma 6.9 we know that

$$|D^+_{x_i} G(x,y)| \le K(d)[|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}}, \quad x \in \Omega_h \cup \partial_i^- \Omega_h,\ y \in \Omega_h$$

for fixed $\gamma := \gamma(d)$, yielding

$$|D^+_i u(x)| \le K(d) \sum_{y \in \Omega_h} [|x-y|^2 + \gamma h^2]^{\frac{1-d}{2}} |f(y)| h^d = K(d)(V_{1/d,\gamma}|f|)(x)$$

for $x \in \Omega_h \cup \partial_i^- \Omega_h$, $1 \le i \le d$. We now have

$$\begin{aligned} \|D^+_i u\|_{L^p(\Omega_h \cup \partial_i^- \Omega_h)} &\le K(d) \|V_{1/d,\gamma}|f|\|_{L^p(\Omega_h \cup \partial_i^- \Omega_h)} \le K(d) \|V_{1/d,\gamma}|f|\|_{L^p(\overline{\Omega}_h)} \\ &= K^*(d, \operatorname{diam}\Omega, p, q) \|f\|_{L^q(\Omega_h)}, \end{aligned}$$

for $1 \le i \le d$, since

$$0 \le \frac{1}{q} - \frac{1}{p} < \frac{d+p}{dp} - \frac{1}{p} = \frac{1}{d}$$

as Lemma 7.6 demands.

◀

Chapter 8

Interpolation

8.1 Tensor product interpolation

8.1 Motivation (Interpolation).
Exactly as in the model problem 3 we need a way to interpolate a grid function in such a way that

1. the interpolated function coincides with the grid function in the grid points;
2. the interpolated function belongs to $W^{1,p}$ and is continuous;
3. the $W^{1,p}$-norm of the interpolated function can be estimated by the discrete $W^{1,p}$-norm of the grid function up to a grid-independent factor.

8.2 Motivation (Reference box).
We first construct an interpolation process on the simplest discrete set – reference box – and then we extend it to a general discrete admissible domain.

8.3 Notation (Reference box, tensor element, natural numeration).
Let $(i_1, \dots, i_d) \in \{0, 1\}^d$. We denote

$$x^{(i_1,\dots,i_d)} := x_1^{i_1} \cdots x_d^{i_d}$$

with the convention $x^0 \equiv 1$.

Using binary representation we can map the set $\{0, 1\}^d$ one-to-one onto the set $\{0, 1, \dots, 2^d - 1\}$. We denote by $\pi\colon \{0, \dots, 2^d - 1\} \to \{0, 1\}^d$ the corresponding binary decomposition *written in reverse order*, i.e. π is additive with

$$\begin{aligned} \pi(0) &= (0, \dots, 0), \\ \pi(2^k) &= (\underbrace{0, \dots, 0}_{k \text{ times}}, 1, 0, \dots, 0), \quad 0 \le k \le d - 1. \end{aligned}$$

We also define

$$\sigma(k) := |\pi(k)|_1, \quad k \in \{0, 1, \dots, 2^d - 1\},$$

i.e. if $\pi(k) = (i_1, \dots, i_d)$, then

$$\sigma(k) = \sum_{j=1}^{d} i_j.$$

We now define by

$$Q := [0, h]^d$$

the *reference box* with

$$Q_h := Q \cap \mathcal{G}_h = \{h\pi(k) \mid 0 \le k \le 2^d - 1\}.$$

Moreover, we define $\tau_k \colon Q \to \mathbb{R}$ by

$$\tau_k(x) = x^{\pi(k)} = x_1^{\pi_1(k)} \cdots x_d^{\pi_d(k)}$$

for $0 \le k \le 2^d - 1$; for instance,

$$\begin{aligned} \tau_0(x) &= 1, \\ \tau_3(x) &= x_1 x_2, \\ \tau_5(x) &= x_1 x_3. \end{aligned}$$

8.4 Definition (Interpolation on the reference box)**.**
Let $f \colon Q_h \to \mathbb{R}$ be given. Define $\hat{f} \colon Q \to \mathbb{R}$ by

$$\hat{f}(x) := \sum_{k=0}^{2^d-1} \alpha_k \tau_k(x),$$

where $(\alpha_k)_{k=0}^{2^d-1}$ is the solution of the linear system

$$\begin{aligned} &\sum_{k=0}^{2^d-1} \alpha_k \tau_k(x) = f(x), && x \in Q_h, \\ \Leftrightarrow &\sum_{k=0}^{2^d-1} \alpha_k \tau_k(h\pi(j)) = f(h\pi(j)), && 0 \le j \le 2^d - 1. \end{aligned}$$

This system is lower triangular with a positive diagonal, i.e. always solvable. Really, let $k \in \{0, \dots, 2^d - 1\}$, $\pi(k) = (i_1, \dots, i_d) \in \{0, 1\}^d$. Then,

$$\tau_k(h\pi(k)) = x_1^{i_1} \cdots x_d^{i_d}\big|_{x = h\pi(k)} = (i_1 h)^{i_1} \cdots (i_d h)^{i_d} = h^{\sum_{j=1}^{d} i_j} = h^{\sigma(k)} > 0.$$

Now let $k > j$, $\pi(j) = (i'_1, \dots, i'_d)$. This implies $1 = i_s > i'_s = 0$ for some $s \in \{1, \dots, d\}$ and consequently

$$\tau_k(h\pi(j)) = (i'_1 h)^{i_1} \cdots \underbrace{(i'_s h)^{i_s}}_{=0} \cdots (i'_d h)^{i_d} = 0.$$

8.5 Example (Interpolation in 2d)**.**
For instance, for $d = 2$ we have

$$\begin{aligned} \tau_0(x_1, x_2) &= 1, & h\pi(0) &= (0, 0), \\ \tau_1(x_1, x_2) &= x_1, & h\pi(1) &= (h, 0), \\ \tau_2(x_1, x_2) &= x_2, & h\pi(2) &= (0, h), \\ \tau_3(x_1, x_2) &= x_1 x_2, & h\pi(3) &= (h, h). \end{aligned}$$

i.e. the system for determining $(\alpha_k)_{k=0}^3$ is given by

$$\begin{cases} \alpha_0 & +0 & +0 & +0 = f_0 := f(0,0), \\ \alpha_0 & +\alpha_1 h & +0 & +0 = f_1 := f(h,0), \\ \alpha_0 & +0 & +\alpha_2 h & +0 = f_2 := f(0,h), \\ \alpha_0 & +\alpha_1 h & +\alpha_2 h & +\alpha_3 h^2 = f_3 := f(h,h). \end{cases}$$

8.6 Motivation (Representation lemma)**.**
We want to estimate the $W_0^{1,p}$-norm of the interpolant against the discrete $W_0^{1,p}$-norm of the corresponding grid function. As the first step towards this aim we want to obtain a representation of the coefficients α_i through the finite differences of the grid function. From Example 8.5 we see

$$\begin{aligned} \alpha_0 &= f_0, \\ \alpha_1 h &= f_1 - f_0, \\ \alpha_2 h &= f_2 - f_0, \\ \alpha_3 h^2 &= (f_3 - f_0) - (f_1 - f_0) - (f_2 - f_0). \end{aligned}$$

The following lemma generalizes this observation.

8.7 Lemma (Representation lemma)**.**
Let $f\colon Q_h \to \mathbb{R}$, $f_k := f(h\pi(k))$, $0 \le k \le 2^d - 1$ be given and let $\hat{f}\colon Q \to \mathbb{R}$

$$\hat{f}(x) = \sum_{k=0}^{2^d-1} \alpha_k \tau_k(x), \quad x \in Q$$

be the corresponding interpolant. Then it holds

$$\alpha_k h^{\sigma(k)} = \sum_{j=1}^{k} (f_j - f_0) \beta_j^{(k)}, \quad 1 \le k \le 2^d - 1,$$

where $\beta_j^{(k)}$, $1 \le j \le k$, $1 \le k \le 2^d - 1$ are some dimension-dependent constants. In particular this implies

$$|\alpha_k| \le C(d) \frac{1}{h^{\sigma(k)}} \sum_{j=1}^{k} |f_j - f_0|,$$

for $1 \le k \le 2^d - 1$ with some $C(d) > 0$.

▶

We prove this statement by induction. Let the statement hold for $(\alpha_k)_{k=1}^n$ with some $n \in [1, 2^d - 1) \cap \mathbb{N}$. Since the matrix for determining $(\alpha_k)_{k=1}^{n+1}$ is lower diagonal with $\alpha_0 = f_0$ and $\tau_k(h\pi(n+1)) = h^{\sigma(k)} t_k(\pi(n+1))$, we have

$$\alpha_{n+1} h^{\sigma(n+1)} = (f_{n+1} - f_0) - \sum_{k=1}^{n} \alpha_k h^{\sigma(k)} \gamma_k^{(n+1)},$$

with $\gamma_k^{(n+1)} := t_k(\pi(n+1))$, $1 \le k \le n$. Using the induction assumption we get the claim:

$$\begin{aligned}
\alpha_{n+1} h^{\sigma(n+1)} &= (f_{n+1} - f_0) - \sum_{k=1}^{n} \gamma_k^{(n+1)} \sum_{j=1}^{k} (f_j - f_0) \beta_j^{(k)} \\
&= (f_{n+1} - f_0) \underbrace{1}_{=:\beta_{n+1}^{(n+1)}} + \sum_{j=1}^{n} (f_j - f_0) \underbrace{\left\{ - \sum_{k=j}^{n} \gamma_k^{(n+1)} \beta_j^{(k)} \right\}}_{=:\beta_j^{(n+1)}}.
\end{aligned}$$

◀

8.8 Corollary.
Let $f \colon Q_h \to \mathbb{R}$, $f_k := f(h\pi(k))$, $0 \le k \le 2^d - 1$ be given and let $\hat{f} \colon Q \to \mathbb{R}$

$$\hat{f}(x) = \sum_{k=0}^{2^d - 1} \alpha_k \tau_k(x), \quad x \in Q$$

be the corresponding interpolant. Then it holds

$$\|\hat{f}\|_{L^\infty(Q)} \le K(d) \|f\|_{L^\infty(Q_h)}.$$

▶

By direct calculation and the previous lemma

$$\begin{aligned}
\|\hat{f}\|_{L^\infty(\Omega)} &\le \sum_{k=0}^{2^d - 1} |\alpha_k| \|\tau_k\|_{L^\infty(Q)} = |f_0| + \sum_{k=1}^{2^d - 1} |\alpha_k| h^{\sigma(k)} \\
&\le |f_0| + \sum_{k=1}^{2^d - 1} C(d) |f_k - f_0| \le 2^{d+1} C(d) \|f\|_{L^\infty(Q_h)}.
\end{aligned}$$

◀

8.9 Lemma (Estimate for a tensor product element)**.**
The following estimate

$$\int_Q \left(\frac{\partial}{\partial x_i}\tau_k(x)\right)^p \mathbf{d}\,x \leq h^{p(\sigma(k)-1)+d}$$

holds for $1 \leq i \leq d$, $0 \leq k \leq 2^d - 1$ and $p \in [1, \infty)$.

▶

By definition

$$\tau_k(x) = x^{\pi(k)} = x_1^{\pi_1(k)} \cdots x_d^{\pi_d(k)}$$

and therefore

$$\frac{\partial}{\partial x_i}\tau_k(x) = \begin{cases} 0, & \text{if } \pi_i(k) = 0 \\ x_1^{\pi_1(k)} \cdots x_{i-1}^{\pi_{i-1}(k)} \cdot x_{i+1}^{\pi_{i+1}(k)} \cdots x_d^{\pi_d(k)}, & \text{if } \pi_i(k) = 1 \end{cases}$$

for all $1 \leq i \leq d$, $0 \leq k \leq 2^d - 1$. For $\pi_i(k) = 1$ this yields

$$\begin{aligned} I := \int_Q \left(\frac{\partial}{\partial x_i}\tau_k(x)\right)^p \mathbf{d}\,x &= h \prod_{\substack{j=1\\ j\neq i}}^{d} \int_0^h x_j^{\pi_j(k)p}\,\mathbf{d}\,x_j \\ &= h \left(\prod_{\substack{j=1, j\neq i\\ \pi_j(k)=1}}^{d} \int_0^h x_j^{\pi_j(k)p}\,\mathbf{d}\,x_j\right)\left(\prod_{\substack{j=1, j\neq i\\ \pi_j(k)=0}}^{d} \int_0^h x_j^{\pi_j(k)p}\,\mathbf{d}\,x_j\right) \\ &= h \left(\prod_{\substack{j=1, j\neq i\\ \pi_j(k)=1}}^{d} \frac{h^{p+1}}{p+1}\right)\left(\prod_{\substack{j=1, j\neq i\\ \pi_j(k)=0}}^{d} h\right) \\ &\leq \underbrace{\left(\frac{1}{p+1}\right)^{\sigma(k)-1}}_{\leq 1} h^{(p+1)(\sigma(k)-1)} h^{d-\sigma(k)+1} \leq h^{p(\sigma(k)-1)+d}. \end{aligned}$$

and for $\pi_i(k) = 0$ we immediately get $I = 0$.

◀

8.10 Lemma ($W_0^{1,p}$ norm on reference box)**.**
Let $f\colon Q_h \to \mathbb{R}$, $f_k := f(h\pi(k))$, $0 \leq k \leq 2^d - 1$ be given and let $\hat{f}\colon Q \to \mathbb{R}$

$$\hat{f}(x) = \sum_{k=0}^{2^d-1} \alpha_k \tau_k(x)$$

be the corresponding interpolant. Then for $p \in [1, \infty)$ holds

$$\int_Q \sum_{i=1}^d \left|\frac{\partial}{\partial x_i}\hat{f}(x)\right|^p \mathbf{d}\, x \le K(p,d)h^d \sum_{x \in Q_h} \sum_{\substack{1 \le i \le d: \\ x + he_i \in Q_h}} |D_i^+ f(x)|^p.$$

►
We estimate

$$\begin{aligned} I &:= \int_Q \sum_{i=1}^d \left|\frac{\partial}{\partial x_i}\hat{f}(x)\right|^p \mathbf{d}\, x = \int_Q \sum_{i=1}^d \left|\frac{\partial}{\partial x_i} \sum_{k=0}^{2^d-1} \alpha_k \tau_k(x)\right|^p \mathbf{d}\, x \\ &= \int_Q \sum_{i=1}^d \left|\sum_{k=1}^{2^d-1} \alpha_k \frac{\partial}{\partial x_i}\tau_k(x)\right|^p \mathbf{d}\, x \le \int_Q \sum_{i=1}^d \left(\sum_{k=1}^{2^d-1} |\alpha_k| \frac{\partial}{\partial x_i}\tau_k(x)\right)^p \mathbf{d}\, x \end{aligned}$$

By norm equivalence on $\mathbb{R}^{2^d}$ we have

$$\sum_{k=1}^{2^d-1} |\alpha_k| \frac{\partial}{\partial x_i}\tau_k(x) \le K(p,d)^{1/p} \left(\sum_{k=1}^{2^d-1} |\alpha_k|^p \left(\frac{\partial}{\partial x_i}\tau_k(x)\right)^p\right)^{1/p},$$

for all $1 \le i \le d$, $x \in Q$, yielding

$$\begin{aligned} I &\le K(p,d) \int_Q \sum_{i=1}^d \sum_{k=1}^{2^d-1} |\alpha_k|^p \left(\frac{\partial}{\partial x_i}\tau_k(x)\right)^p \mathbf{d}\, x \\ &= K(p,d) \sum_{k=1}^{2^d-1} \left\{ |\alpha_k|^p \sum_{i=1}^d \int_Q \left(\frac{\partial}{\partial x_i}\tau_k(x)\right)^p \mathbf{d}\, x \right\} \end{aligned}$$

Invoking Lemmas 8.9 and 8.7 we obtain

$$\begin{aligned} I &\le K(p,d) \sum_{k=1}^{2^d-1} |\alpha_k|^p \sum_{i=1}^d h^{p(\sigma(k)-1)+d} \\ &\le K'(p,d) \sum_{k=1}^{2^d-1} \frac{h^{p(\sigma(k)-1)+d}}{h^{\sigma(k)p}} \left(\sum_{j=1}^k |f_j - f_0|\right)^p \\ &\le K(p,d) h^{d-p} \sum_{j=1}^{2^d-1} |f_j - f_0|^p \\ &\le K(p,d) h^d \sum_{j=1}^{2^d-1} \left|\frac{f_j - f_0}{h}\right|^p. \end{aligned}$$

Geometrically, for every $1 \le j \le 2^d - 1$ we can represent the difference $(f_j - f_0)$ as a telescope sum of at most d terms of the form $f_p - f_q$, where $h\pi(p)$ and $h\pi(q)$ have a common edge in Q. Really, let $j \in \{1, \ldots, 2^d - 1\}$, $\pi(j) = (i_1, \ldots, i_d)$. Choose $k_n^{(j)}$, $0 \le n \le d$ like this

$$\begin{aligned}
\pi(k_0^{(j)}) &= (0, 0, 0, \ldots, 0),\\
\pi(k_1^{(j)}) &= (i_1, 0, 0, \ldots, 0),\\
\pi(k_2^{(j)}) &= (i_1, i_2, 0, \ldots, 0),\\
&\cdots\\
\pi(k_d^{(j)}) &= (i_1, i_2, i_3, \ldots, i_d),
\end{aligned}$$

where some $k_n^{(j)}$'s can possibly coincide. By construction

$$f_j - f_0 = \sum_{n=1}^{d} f_{k_n^{(j)}} - f_{k_{n-1}^{(j)}}$$

and $h\pi(k_{n-1}^{(j)})$, $h\pi(k_n^{(j)})$, $1 \le n \le d$, unless they coincide, have a common edge $h\pi(k_n^{(j)}) = h\pi(k_{n-1}^{(j)}) + he_n$ with

$$\left| \frac{f_{k_n^{(j)}} - f_{k_{n-1}^{(j)}}}{h} \right| = \left| D_n^+ f_{k_{n-1}^{(j)}} \right|.$$

So finally, we write

$$\begin{aligned}
I &\le K'(p,d) h^d \sum_{j=1}^{2^d-1} \sum_{\substack{n=1\\ k_{n-1}^{(j)} \ne k_n^{(j)}}}^{d} \left| \frac{f_{k_n^{(j)}} - f_{k_{n-1}^{(j)}}}{h} \right|^p = K'(p,d) h^d \sum_{j=1}^{2^d-1} \sum_{\substack{n=1\\ k_{n-1}^{(j)} \ne k_n^{(j)}}}^{d} \left| D_n^+ f_{k_{n-1}^{(j)}} \right|^p\\
&\le K(p,d) h^d \sum_{x \in Q_h} \sum_{\substack{1 \le i \le d:\\ x + he_i \in Q_h}} |D_i^+ f(x)|^p,
\end{aligned}$$

since every term in the penultimate sum appears in the last sum with multiplicity at most 2^d.

◀

8.11 Lemma (Lower dimensional facets).
Let $f\colon Q_h \to \mathbb{R}$, $f_k := f(h\pi(k))$, $0 \le k \le 2^d - 1$ be given and let $\hat{f}\colon Q \to \mathbb{R}$

$$\hat{f}(x) = \sum_{k=0}^{2^d-1} \alpha_k \tau_k(x), \quad x \in Q$$

be the corresponding interpolant. Further, let $S := S_1 \times \cdots \times S_d$ with $S_i \in \{\{0\}, \{h\}, [0,h]\}$, $1 \le i \le d$. Then $\hat{f}|_S$ depends only on $\{f(x) \mid x \in Q_h \cap S\}$.

►

Define $s := \#\{i \mid S_i = [0,h]\}$, in other words S is an s-dimensional facet of Q. Since the other $d-s$ variables are fixed, $\hat{f}|_S$ possesses exactly 2^s degrees of freedom. The set $Q_h \cap S$ consists of exactly 2^s points. Repeating the argument from the Definition 8.4 we get the claim.

◄

8.12 Definition (Interpolation on discrete admissible set)**.**
Let $\Omega \subset \mathbb{R}^d$, $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h\,\Omega$ be admissible with $\hat{\Omega}_h := \Omega_h \cup \hat{\partial}\Omega_h$. Let $u\colon \hat{\Omega}_h \to \mathbb{R}$ be given. For every $\bar{x} \in \hat{\Omega}_h \setminus \hat{\partial}^+\Omega_h$ define

$$Q^{\bar{x}} := \bar{x} + Q = \prod_{i=1}^{d} [\bar{x}_i, \bar{x}_i + h].$$

We construct $\hat{u}\colon Q^{\bar{x}} \to \mathbb{R}$ in the following way. First, define $f\colon Q_h \to \mathbb{R}$ by

$$f(x) := u(x+\bar{x}), \quad x \in Q_h.$$

Now, using $\hat{f}\colon Q \to \mathbb{R}$ from Definition 8.4 define

$$\hat{u}(x) := \hat{f}(x - \bar{x}), \quad x \in Q^{\bar{x}}.$$

Since

$$\overline{\Omega} = \cup_{\bar{x} \in \hat{\Omega}_h \setminus \hat{\partial}^+\Omega_h} Q^{\bar{x}}$$

we have defined $\hat{u}$ on $\overline{\Omega}$, provided the values $\hat{u}$ on $\partial Q^{\bar{x}}$ are well-defined for all $\bar{x} \in \hat{\Omega}_h \setminus \hat{\partial}^+\Omega_h$, i.e. the values on the common facets of two different boxes coincide. This follows from Lemma 8.11.

8.13 Theorem (Norm estimate)**.**
Let $\Omega \subset \mathbb{R}^d$, $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h\,\Omega$ be admissible with $\hat{\Omega}_h := \Omega_h \cup \hat{\partial}\Omega_h$. Further, let $u\colon \hat{\Omega}_h \to \mathbb{R}$, $u|_{\hat{\partial}\Omega_h} = 0$ be given with its tensor product interpolant $\hat{u}\colon \overline{\Omega} \to \mathbb{R}$. Then it holds

$$\|\hat{u}\|_{W_0^{1,p}(\Omega)} \le K(d,p) \|u\|_{W_0^{1,p}(\Omega_h)}$$

for $p \in [1, \infty)$.

►

As in the previous Definition we write $Q^{\bar{x}} := Q + \bar{x}$ for all $\bar{x} \in \hat{\Omega}_h \setminus \hat{\partial}^+\Omega_h =: G_h$ Using Lemma 8.10 we estimate

$$\|\hat{u}\|^p_{W_0^{1,p}(\Omega)} = \sum_{\bar{x} \in G_h} \int_{Q^{\bar{x}}} \sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \hat{u}(x) \right|^p \mathbf{d}\, x \le K(p,d) h^p \sum_{\bar{x} \in G_h} \sum_{x \in Q^{\bar{x}}} \sum_{\substack{1 \le i \le d: \\ x + he_i \in Q^{\bar{x}}}} |D_i^+ u(x)|^p$$

$$\le K^*(p,d) h^p \sum_{i=1}^{d} \sum_{x \in \overline{\Omega}_h \setminus \partial_i^+ \Omega_h} |D_i^+ u(x)|^p = K^*(p,d) \|u\|^p_{W_0^{1,p}(\Omega_h)}.$$

Chapter 9

Convergence Result for bounded domains

9.1 Discrete formulation

9.1 Motivation (Main Result).
After the preparations in Chapters 5, 6 and 7 we are finally in the position to carry out the program outlined in Chapter 3.

9.2 Definition (Discrete Formulations).
Let $\Omega \subset \mathbb{R}^d$ and $\Omega_h \subset \mathcal{G}_h$, $\overline{\Omega}_h = \mathrm{cl}_h(\Omega)$ be admissible.
In the classical continuous setting we are looking for $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta u(x) &= f(u(x)), \quad && x \in \Omega, \\ u(x) &= 0, && x \in \partial\Omega \end{aligned} \tag{BVP}$$

with some $f \in C(\mathbb{R})$.
In the discrete case we are looking for $u\colon \overline{\Omega}_h \to \mathbb{R}$ that satisfy one of following three equivalent formulations (see the next lemma).
The *classical discrete formulation* for (BVP) is the problem

$$\begin{aligned} -\Delta_h u(x) &= f(u(x)), \quad && x \in \Omega_h, \\ u(x) &= 0, && x \in \partial\Omega_h. \end{aligned} \tag{P_h}$$

The *weak discrete formulation* for (BVP) is the problem

$$\begin{aligned} \sum_{i=1}^{d} \sum_{x \in \Omega_h \cup \partial_i^- \Omega_h} D^+ u(x) D^+ \varphi(x) h^d &= \sum_{x \in \Omega_h} f(u(x))\varphi(x) h^d, \\ u(x) = 0, \quad x &\in \partial\Omega_h, \end{aligned} \tag{P_h'}$$

where equality has to hold for all $\varphi\colon \overline{\Omega}_h \to \mathbb{R}$ with $\varphi|_{\partial\Omega_h} = 0$.
The *very weak discrete formulation* for (BVP) is the problem

$$-\sum_{x\in\Omega_h} u(x)\Delta_h\varphi(x)h^d = \sum_{x\in\Omega_h} f(u(x))\varphi(x)h^d, \qquad u(x) = 0, \quad x \in \partial\Omega_h, \tag{P_h''}$$

where equality has to hold for all $\varphi\colon \overline{\Omega}_h \to \mathbb{R}$ with $\varphi|_{\partial\Omega_h} = 0$.
Analogously to Lemma 3.8 we obtain

9.3 Lemma (Equivalence of the discrete formulations)**.**
The discrete formulations $(P_h), (P_h')$ and (P_h'') are equivalent.

▶
Let $\varphi\colon \overline{\Omega}_h \to \mathbb{R}$ be a grid function with $\varphi|_{\partial\Omega_h} = 0$. We multiply (P_h) with φ and sum up over $x \in \Omega$

$$\begin{aligned}
\sum_{x\in\Omega_h} f(u(x))\varphi(x) = -\sum_{x\in\Omega_h} \Delta_h u(x)\varphi(x) &= \sum_{i=1}^{d}\sum_{x\in\Omega_h} \frac{-D_i^+u(x) + D_i^-u(x)}{h}\varphi(x)\\
&= \sum_{i=0}^{d}\left(\sum_{x\in\Omega_h\cup\partial_i^-\Omega_h} \frac{-D_i^+u(x)}{h}\varphi(x) + \sum_{x\in\Omega_h\cup\partial_i^+\Omega_h} \frac{D_i^-u(x)}{h}\varphi(x)\right)\\
&= \sum_{i=0}^{d}\sum_{x\in\Omega_h\cup\partial_i^-\Omega_h}\left(\frac{-D_i^+u(x)}{h}\varphi(x) + \frac{D_i^+u(x)}{h}\varphi(x+he_i)\right)\\
&= \sum_{i=0}^{d}\sum_{x\in\Omega_h\cup\partial_i^-\Omega_h} D_i^+u(x)D_i^+\varphi(x),
\end{aligned}$$

obtaining $(P_h) \Rightarrow (P_h')$.
Since $u|_{\partial\Omega_h} = 0$, we can expand $\sum_{i=1}^d \sum_{x\in\Omega_h\cup\partial_i^-\Omega_h} D_i^+u(x)D_i^+\varphi(x)$ backwards, swapping u and φ. This yields $(P_h') \Leftrightarrow (P_h'')$.
Define $\varphi_y\colon \overline{\Omega}_h \to \mathbb{R}$ by

$$\varphi_y(x) := \begin{cases} 1, & \text{if } x = y,\\ 0, & \text{otherwise.}\end{cases}$$

with an arbitrary but fixed $y \in \Omega_h$. Plugging φ_y into (P_h')

$$f(u(y)) = \sum_{i=1}^{d}\left[-\frac{D_i^+u(y)}{h} + \frac{D_i^+u(y-he_i)}{h}\right] = -\Delta_h u(y),$$

and (P_h'')

$$f(u(y)) = \frac{1}{h^2}\sum_{i=1}^{d}[-u(y+he_i)+2u(y)-u(y-he_i)] = -\Delta_h u(y).$$

and in both cases dividing by h^d, we obtain (P_h).
◀

9.2 Main result for bounded domains

9.4 Announcement (Convergence Theorem)**.**
Let $f\colon \mathbb{R}\to\mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, $h_n \overset{n\to\infty}{\to} 0$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon \overline{\Omega}_{h_n}\to\mathbb{R}$ of discrete problems (P_{h_n}) and a positive $C>0$ such that

$$\|u_n\|_{L^\infty(\overline{\Omega}_{h_n})} := \max\{|u_n(x)| \mid x\in\overline{\Omega}_{h_n}\} \le C, \quad \forall n\in\mathbb{N}.$$

Then, there exists a (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ and a classical solution $u\in C^2(\Omega)\cap C(\overline{\Omega})$ of (BVP) such that

$$\|u_n - u\|_{L^\infty(\overline{\Omega}_{h_n})} \to 0, \quad \text{as } n\to\infty.$$

9.5 Lemma (Embedding results)**.**
Let $u\in W_0^{1,p}(\Omega)$ with Ω admissible and $p>d$. Then there exists a continuous representative $\bar{u}\in C(\overline{\Omega})$ with $\bar{u}|_{\partial\Omega}=0$ and $\bar{u}|_{\Omega}=u$ a.e. Now let $(u_n)_{n\in\mathbb{N}}\subset W_0^{1,p}(\Omega)$ with $\|u_n\|_{W^{1,p}(\Omega)}\le C$ for some $C>0$. Then, there exists $\tilde{u}\in C(\overline{\Omega})$ such that $\bar{u}_n \rightrightarrows \tilde{u}$ (i.e. uniformly on $\overline{\Omega}$) up to a subsequence as $n\to\infty$, where $\bar{u}_n$ is the continuous representative of u_n, $n\in\mathbb{N}$, as defined previously.

▶
The continuous embedding follows from Theorem 4.12 in [2]. From Theorem 6.3 in [2] we have the compact embedding $W^{1,p}(\Omega)\hookrightarrow C_B(\Omega)$. Extracting if needed a subsequence we obtain $\bar{u}_n\rightrightarrows\tilde{u}$ with some $\tilde{u}\in C(\overline{\Omega})$.
◀

9.6 Convention.
We will identify the function $u\in W_0^{1,p}(\Omega)$ with its continuous representative $\bar{u}\in C(\overline{\Omega})$ in the sense of the previous lemma.

9.7 Lemma (Boundedness)**.**
Let $f\colon\mathbb{R}\to\mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings

$(h_n)_{n\in\mathbb{N}}$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon \overline{\Omega}_{h_n} \to \mathbb{R}$ of the discrete problems (P_{h_n}) and a positive $C > 0$ such that

$$\|u_n\|_{L^\infty(\overline{\Omega}_{h_n})} \leq C, \quad \forall n \in \mathbb{N}.$$

Then, there exists $\hat{C} > 0$ such that

$$\|\hat{u}_n\|_{W_0^{1,p}(\Omega)} \leq \hat{C}, \quad \forall n \in \mathbb{N},$$

where $\hat{u}_n$ is the tensor product interpolant (as defined in 8.12) of u_n, $n \in \mathbb{N}$ and $p \in (1,\infty)$.

▶

From the uniform boundedness of $(u_n)_{n\in\mathbb{N}}$ and the continuity of f we get $\|f(u_n)\|_{L^p(\Omega_{h_n})} \leq \tilde{C}$ with some $\tilde{C} = \tilde{C}(C,f,p) > 0$ for all $p \in [1,\infty)$. Using Theorem 8.13 and Theorem 7.9, we obtain

$$\begin{aligned}\|\hat{u}_n\|_{W_0^{1,p}(\Omega)} &\leq K(d,p)\|u_n\|_{W_0^{1,p}(\Omega_{h_n})} \leq K^*(d,p,\operatorname{diam}\Omega)\|f\|_{L^q(\Omega_{h_n})} \\ &\leq K(d,p,\operatorname{diam}\Omega,C,f), \quad \forall n \in \mathbb{N}.\end{aligned}$$

with $q = \frac{2dp}{d+p} > \frac{dp}{d+p}$.

◀

9.8 Lemma (Passing to the limit in the linear part)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}} \subset (0,1]$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon \overline{\Omega}_{h_n} \to \mathbb{R}$ of the discrete problems (P_{h_n}) and a positive $C > 0$ such that $\|u_n\|_{L^\infty(\overline{\Omega}_{h_n})} \leq C$, $\forall n \in \mathbb{N}$. Then, there exists a (renamed) subsequence of $(u_n)_{n\in\mathbb{N}}$ such that

$$-\sum_{x\in\Omega_{h_n}} u_n(x)\Delta_{h_n}\varphi(x)h_n^d \xrightarrow{n\to\infty} -\int_\Omega u(x)\Delta\varphi(x)\,\mathbf{d}\,x,$$

for all $\varphi \in C^3(\overline{\Omega})$ and

$$\hat{u}_n \rightrightarrows u$$

for some $u \in C(\overline{\Omega})$, $u|_{\partial\Omega_h} = 0$.

▶

⟨*1*⟩ *Subsequence in* $C^*(\overline{\Omega})$

The functionals

$$\Phi_n\colon \begin{cases} C(\overline{\Omega}) \to \mathbb{R} \\ \quad \Psi \mapsto \displaystyle\sum_{x\in\Omega_{h_n}} u_n(x)\Psi(x)h_n^d \end{cases}$$

are uniformly bounded in n with $\|\Psi\|_{C(\overline{\Omega})} := \max_{x\in\overline{\Omega}} |\Psi(x)|$, because

$$|\Phi_n(\Psi)| \leq \|u_n\|_{L^\infty(\Omega_{h_n})} \|\Psi\|_{C(\overline{\Omega})} |\Omega| \leq C|\Omega| \|\Psi\|_{C(\overline{\Omega})}.$$

Due to separability of $C(\overline{\Omega})$ there exists $\Phi \in C^*(\overline{\Omega})$ such that $\Phi_n \overset{*}{\rightharpoonup} \Phi$ up to a (renamed) subsequence.

⟨2⟩ *Extending the limit element to* $L^2(\overline{\Omega})$
For any $\Psi \in C(\overline{\Omega})$ we denote

$$\|\Psi\|_{L^2(\overline{\Omega}_{h_n})} := \sqrt{\sum_{x\in\Omega_{h_n}} \Psi^2(x) h_n^d}.$$

From

$$\begin{aligned}|\Phi_n(\Psi)| &= \left|\sum_{x\in\Omega_{h_n}} u_n(x)\Psi(x)h_n^d\right| \leq \sqrt{\sum_{x\in\Omega_{h_n}} u_n^2(x)h_n^d}\sqrt{\sum_{x\in\Omega_{h_n}} \Psi^2(x)h_n^d} \\ &\leq C\sqrt{|\Omega|}\|\Psi\|_{L^2(\overline{\Omega}_{h_n})}\end{aligned}$$

letting $n \to \infty$ we get for a fixed $\Psi \in C(\overline{\Omega})$

$$|\Phi(\Psi)| \leq C\sqrt{|\Omega|}\|\Psi\|_{L^2(\overline{\Omega})}.$$

Since $C(\overline{\Omega})$ is dense in the complete space $L^2(\overline{\Omega})$ we can extend Φ by continuity onto $L^2(\overline{\Omega})$. We denote by $\tilde{u} \in L^2(\overline{\Omega})$ the Riesz representative of this extension.

⟨3⟩ *Convergence*
We formally extend Φ to all grid functions $\Psi\colon \overline{\Omega}_{h_n} \to \mathbb{R}$ and implicitly restrict all $\varphi \in C(\overline{\Omega})$ to the corresponding grids if necessary. We want to show that

$$\Phi_n(-\Delta_{h_n}\varphi) = -\sum_{x\in\Omega_{h_n}} u_n(x)\Delta_{h_n}\varphi(x)h_n^d \xrightarrow{n\to\infty} -\int_\Omega \Delta\varphi(x)\tilde{u}(x)\,\mathbf{d}\,x = \Phi(-\Delta\varphi)$$

for $\varphi \in C^3(\overline{\Omega})$. For $\varphi \in C^3(\overline{\Omega})$ we have the consistency

$$\|\Delta_{h_n}\varphi - \Delta\varphi\|_{L^\infty(\Omega_{h_n})} \to 0 \text{ as } n \to \infty,$$

implying

$$|\Phi_n(-\Delta_{h_n}\varphi + \Delta\varphi)| \leq C|\Omega|\|\Delta_{h_n}\varphi - \Delta\varphi\|_{L^\infty(\Omega_{h_n})} \to 0 \text{ as } n \to \infty.$$

Hence,

$$\Phi_n(-\Delta_{h_n}\varphi) = \Phi_n(-\Delta\varphi) + \Phi_n(-\Delta_{h_n}\varphi + \Delta\varphi) \to \Phi(-\Delta\varphi) = -\int_\Omega \tilde{u}\Delta\varphi\,\mathbf{d}\,x,$$

for all $\varphi \in C^3(\overline{\Omega})$ as $n \to \infty$.

⟨*4*⟩ *Identification of* $\tilde{u}$

We now want to show that $\tilde{u}$ can also be obtained as an accumulation point of $(\hat{u}_n)_{n\in\mathbb{N}}$ in $C(\overline{\Omega})$. Taking the (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ that corresponds to the extracted subsequence used in the first step and applying Lemma 9.5 and Lemma 9.7 we extract a (renamed) convergent subsequence $(\hat{u}_n)_{n\in\mathbb{N}}$ with a limit $u \in C(\overline{\Omega})$, $\hat{u}_n \rightrightarrows u$ as $n \to \infty$. We now only need to show that $\tilde{u} = u$, yielding also that the limit u is independent from the choice of the convergent subsequence from the sequence fixed in the first step. First we show

$$\Phi_n(-\Delta_{h_n}\varphi) \xrightarrow{n\to\infty} -\int_\Omega u(x)\Delta\varphi(x)\,\mathbf{d}\,x, \quad \forall \varphi \in C^3(\overline{\Omega}).$$

We get

$$\begin{aligned}\Phi_n(-\Delta_{h_n}\varphi) &= -\sum_{x\in\Omega_{h_n}} u_n(x)\Delta_{h_n}\varphi(x)h_n^d = -\sum_{x\in\Omega_{h_n}} \hat{u}_n(x)\Delta_{h_n}\varphi(x)h_n^d \\ &= -\sum_{x\in\Omega_{h_n}} \hat{u}_n(x)\Delta\varphi(x)h_n^d + \sum_{x\in\Omega_{h_n}} \hat{u}_n(x)[\Delta\varphi(x) - \Delta_{h_n}\varphi(x)]h_n^d \\ &= -\sum_{x\in\Omega_{h_n}} u(x)\Delta\varphi(x)h_n^d + \sum_{x\in\Omega_{h_n}} [u(x) - \hat{u}_n(x)]\Delta\varphi(x)h_n^d \\ &\quad + \sum_{x\in\Omega_{h_n}} \hat{u}_n(x)[\Delta\varphi(x) - \Delta_{h_n}\varphi(x)]h_n^d \\ &\xrightarrow{n\to\infty} -\int_\Omega u(x)\Delta\varphi(x)\,\mathbf{d}\,x,\end{aligned}$$

since for $n \to \infty$

$$\left|\sum_{x\in\Omega_{h_n}} [u(x) - \hat{u}_n(x)]\Delta\varphi(x)h_n^d\right| \le \overbrace{\|u - \hat{u}_n\|_{C(\overline{\Omega})}}^{\to 0} \overbrace{\sum_{x\in\Omega_{h_n}} |\Delta\varphi(x)|h_n}^{\to\int_\Omega |\Delta\varphi(x)|\,\mathbf{d}\,x} \to 0$$

and

$$\begin{aligned} I &:= \left|\sum_{x\in\Omega_{h_n}} \hat{u}_n(x)[\Delta\varphi(x) - \Delta_{h_n}\varphi(x)]h_n^d\right| \\ &\le \underbrace{\sum_{x\in\Omega_{h_n}} |\hat{u}_n(x)|h_n^d}_{\le C|\Omega|} \underbrace{\|\Delta\varphi - \Delta_{h_n}\varphi\|_{L^\infty(\overline{\Omega}_{h_n})}}_{\to 0} \to 0.\end{aligned}$$

We now have

$$\int_\Omega [u(x) - \tilde{u}(x)]\Delta\varphi(x)\,\mathbf{d}\,x = 0, \quad \forall \varphi \in C^3(\overline{\Omega}).$$

Consider the problem

$$\begin{aligned} \Delta\varphi(x) &= \psi(x) \quad x \in \Omega \\ \varphi(x) &= 0, \qquad x \in \partial\Omega \end{aligned} \tag{†}$$

with a prescribed $\psi \in C_0^\infty(\Omega)$. By the classical existence and regularity theory (see Chapter 4 in [23]) we have a unique solution in $C^2(\Omega)\cap C(\overline{\Omega})$. But since Ω is admissible with zero boundary condition, we can exploit Schwarz reflection principle (see Chapter 2 in [23]). The Schwarz reflection $\tilde{\varphi}$ of φ, i.e. after $2d$ reflections over the boundaries with sign change, satisfies (†) on the reflected domain $\tilde{\Omega}$, $\overline{\Omega} \subset \tilde{\Omega}$. Due to the compactness of the support of ψ the reflected $\tilde{\psi}$ is in $C_0^\infty(\tilde{\Omega})$. Applying the classical regularity theory (see Chapter 8 in [23]), we get $\varphi \in C^\infty(\overline{\Omega})$, and in particular $\varphi \in C^3(\overline{\Omega})$. This yields

$$\int_\Omega [u(x) - \tilde{u}(x)]\psi(x)\,\mathbf{d}\,x = 0, \quad \forall\psi \in C_0^\infty(\Omega).$$

Since $C_0^\infty(\Omega)$ is dense in $L^2(\overline{\Omega})$ we have $\tilde{u} = u$ a.e.

◀

9.9 Lemma (Passing to the limit in the nonlinear part)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, $h_h \overset{n\to\infty}{\to} 0$, a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$, $u_n\colon \overline{\Omega}_{h_n} \to \mathbb{R}$ of the discrete variational problems (P_{h_n}) and a positive constant $C > 0$ such that $\|u_n\|_{L^\infty(\overline{\Omega}_{h_n})} \le C$, $\forall n \in \mathbb{N}$. Then, there exists a (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\sum_{x\in\Omega_{h_n}} f(u_n(x))\varphi(x)h_n^d \xrightarrow{n\to\infty} \int_\Omega f(u(x))\varphi(x)\,\mathbf{d}\,x,$$

for all $\varphi \in C(\overline{\Omega})$ and

$$\hat{u}_n \rightrightarrows u$$

for some $u \in C(\overline{\Omega})$, $u|_{\partial\Omega} = 0$.

▶

⟨*1*⟩ *Subsequence in* $C^*(\overline{\Omega})$

The functionals

$$\Phi_n\colon \begin{cases} C(\overline{\Omega}) \to \mathbb{R} \\ \quad \Psi \mapsto \displaystyle\sum_{x\in\Omega_{h_n}} f(u_n(x))\Psi(x)h_n^d \end{cases}$$

are uniformly bounded, because

$$|\Phi_n(\Psi)| \le \|f(u_n)\|_{L^\infty(\overline{\Omega}_{h_n})}\|\Psi\|_{C(\overline{\Omega})}|\Omega| \le \|f\|_{L^\infty([-C,C])}\|\Psi\|_{C(\overline{\Omega})}|\Omega|.$$

Due to separability of $C(\overline{\Omega})$ there exists $\Phi \in C^*(\overline{\Omega})$ such that $\Phi_n \overset{*}{\rightharpoonup} \Phi$ up to a (renamed) subsequence.

⟨*2*⟩ *Extending the limit element to* $L^2(\overline{\Omega})$
From

$$|\Phi_n(\Psi)| = \left|\sum_{x\in\Omega_{h_n}} f(u_n(x))\Psi(x)h_n\right| \le \sqrt{\sum_{x\in\Omega_{h_n}} f^2(u_n(x))h_n}\sqrt{\sum_{x\in\Omega_{h_n}} \Psi^2(x)h_n}$$
$$\le \|f\|_{L^\infty([-C,C])}\sqrt{|\Omega|}\|\Psi\|_{L^2(\overline{\Omega}_{h_n})}$$

letting $n\to\infty$ we get for a fixed $\Psi \in C(\overline{\Omega})$

$$|\Phi(\Psi)| \le \|f\|_{L^\infty([-C,C])}\sqrt{|\Omega|}\|\Psi\|_{L^2(\overline{\Omega})}.$$

Since $C(\overline{\Omega})$ is dense in the complete space $L^2(\overline{\Omega})$ we can extend Φ to $L^2(\overline{\Omega})$. We denote by $F\in L^2(\overline{\Omega})$ the Riesz representative of Φ.

⟨*3*⟩ *Identification of F*
We take the (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ that corresponds to the convergent subsequence in the first step. Using Lemma 9.5 and Lemma 9.7 we now extract from it a further (renamed) convergent subsequence $(u_n)_{n\in\mathbb{N}}$, this time with $\hat{u}_n \rightrightarrows u$ as $n\to\infty$ for some $u\in C(\overline{\Omega})$. Our aim is to show that $F = f(u)$. We have

$$\begin{aligned}\Phi_n(\varphi) &= \sum_{x\in\Omega_{h_n}} f(u_n(x))\varphi(x)h_n^d = \sum_{x\in\Omega_{h_n}} f(\hat{u}_n(x))\varphi(x)h_n^d \\ &= \sum_{x\in\Omega_{h_n}} f(u(x))\varphi(x)h_n^d + \sum_{x\in\Omega_{h_n}} [f(\hat{u}_n(x)) - f(u(x))]\varphi(x)h_n^d \\ &\to \int_\Omega f(u(x))\varphi(x)\,\mathbf{d}\,x, \quad \text{as } n\to\infty, \text{ for all } \varphi\in C(\overline{\Omega}),\end{aligned}$$

provided

$$\left|\sum_{x\in\Omega_{h_n}} [f(\hat{u}_n(x)) - f(u(x))]\varphi(x)h_n^d\right| \to 0.$$

But this claim follows from uniform continuity of f on $[-C,C]$ and uniform convergence of $\hat{u}_n$ toward u. We have therefore

$$\int_\Omega [f(u(x)) - F(x)]\varphi(x)\,\mathbf{d}\,x = 0 \quad \forall\varphi\in C(\overline{\Omega}).$$

Since $C(\overline{\Omega})$ is dense in $L^2(\overline{\Omega})$ we finally obtain $f(u) = F$ a.e.

◀

9.10 Theorem (Convergence Theorem)**.**
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous and assume that there exists a sequence of grid spacings $(h_n)_{n\in\mathbb{N}}$, $h_n \overset{n\to\infty}{\to} 0$ and a corresponding sequence of solutions $(u_n)_{n\in\mathbb{N}}$ of the discrete problems (P_{h_n}) and a positive $C > 0$ such that $\|u_n\|_{L^\infty(\overline{\Omega}_{h_n})} \leq C$, $\forall n \in \mathbb{N}$. Then, there exists a (renamed) subsequence $(u_n)_{n\in\mathbb{N}}$ and a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of (BVP) such that

$$\|u_n - u\|_{L^\infty(\overline{\Omega}_{h_n})} \to 0, \quad \text{as } n \to \infty.$$

▶

Using the convergent subsequence obtained after applying Lemma 9.8 as the initial sequence for Lemma 9.9, we obtain a subsequence of $(u_n)_{n\in\mathbb{N}}$ for which the claims of both those results hold with the same uniform limit, so letting $n \to \infty$ we have

$$\int_\Omega -u(x)\Delta\varphi(x)\,\mathbf{d}\,x = \int_\Omega f(u(x))\varphi(x)\,\mathbf{d}\,x,$$

for all $\varphi \in C^3(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$ with some $u \in C(\overline{\Omega})$, $u|_{\partial\Omega} = 0$. Let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be the classical solution of

$$\begin{aligned} -\Delta v(x) &= f(u(x)), & x &\in \Omega \\ v(x) &= 0, & x &\in \partial\Omega. \end{aligned}$$

Taking the weak formulation of this BVP and subtracting it from the weak formulation of our original problem we get

$$\int_\Omega (u - v)(x)\Delta\varphi(x)\,\mathbf{d}\,x = 0,$$

for all $\varphi \in C^3(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$. Repeating the final argument from Lemma 9.8 we get $u = v$ a.e., i.e. $u \in C^2(\Omega)$.
◀

Chapter 10

A Priori Estimates on $\mathcal{G}_h$

10.1 Discrete Nonlinear Liouville Theorem

10.1 Motivation (Discrete Nonlinear Liouville Theorem).
We want to generalize the following nonlinear Liouville Theorem from [22] to the discrete setting. This is needed for a contradiction argument in the proof of the final result of this chapter, Theorem 10.9. We begin with the discrete version of Hardy's Inequality (see [7]) and one useful refinement thereof.

10.2 Theorem (Nonlinear Liouville Theorem, [22]).
Let $u(x)$ be a non-negative C^2 solution of

$$\Delta u + u^p = 0$$

in $\mathbb{R}^d$, $d \geq 3$ with

$$1 \leq p < \frac{d+2}{d-2}.$$

Then

$$u(x) \equiv 0.$$

10.3 Lemma (Discrete Regularized Hardy Inequality).
Let $d \geq 3$. There exists $C(\alpha, d) > 0$ such that

$$\sum_{x \in \mathcal{G}_h} \frac{|u(x)|^2}{|x|^2 + \alpha h^2} \leq C \sum_{x \in \mathcal{G}_h} \sum_{i=1}^{d} \left(D_i^+ u(x)\right)^2$$

for all $u \colon \mathcal{G}_h \to \mathbb{R}$ with finite support and for all sufficiently large $\alpha > 0$.
▶

⟨*1*⟩ *Vector field method*

Let $u\colon \mathcal{G}_h \to \mathbb{R}$ with finite support and $F_i\colon \mathcal{G}_h \to \mathbb{R}$, $1 \le i \le d$ be arbitrary. Then

$$
\begin{aligned}
D_i^+ \left(u^2(x)F_i(x)\right) &= \frac{u^2(x+he_i)F_i(x+he_i) - u^2(x)F_i(x)}{h} \\
&= u^2(x+he_i)D_i^+F_i(x) + \frac{u^2(x+he_i) - u^2(x)}{h}F_i(x) \\
&= u^2(x+he_i)D_i^+F_i(x) + D_i^+u(x)\left[u(x+he_i) + u(x)\right]F_i(x) \\
&\le u^2(x+he_i)D_i^+F_i(x) + \left(D_i^+u(x)\right)^2 \\
&\qquad\qquad + \left(\frac{u(x+he_i)+u(x)}{2}\right)^2 F_i^2(x) \\
&\le u^2(x+he_i)D_i^+F_i(x) + \left(D_i^+u(x)\right)^2 \\
&\qquad\qquad + \left(\frac{1}{2}u^2(x+he_i) + \frac{1}{2}u^2(x)\right) F_i^2(x)
\end{aligned}
$$

holds for all $x \in \mathcal{G}_h$ and $1 \le i \le d$. Since

$$
\sum_{i=1}^{d} \sum_{x\in\mathcal{G}_h} D_i^+ \left(u^2(x)F_i(x)\right) = 0
$$

due to the finiteness of $\operatorname{supp} u$, we obtain

$$
\begin{aligned}
\sum_{i=1}^{d} \sum_{x\in\mathcal{G}_h} \left(D_i^+u(x)\right)^2 &\ge -\sum_{i=1}^{d} \sum_{x\in\mathcal{G}_h} u^2(x+he_i)D_i^+F_i(x) \\
&\quad -\sum_{i=1}^{d} \sum_{x\in\mathcal{G}_h} \left(\frac{1}{2}u^2(x+he_i) + \frac{1}{2}u^2(x)\right) F_i^2(x) \\
&= -\sum_{i=1}^{d} \sum_{x\in\mathcal{G}_h} u^2(x)\left[D_i^-F_i(x) + \frac{1}{2}F_i^2(x-he_i) + \frac{1}{2}F_i^2(x)\right].
\end{aligned}
$$

⟨*2*⟩ *Comparison estimate*

We show that the estimates

$$
\begin{aligned}
(1-\varepsilon)(|x|^2 + \alpha h^2) \le& |x - \tau e_i h|^2 + \alpha h^2 \\
&|x - \tau e_i h|^2 + \frac{\alpha}{d}h^2 \le (1+\varepsilon)\left(|x|^2 + \frac{\alpha}{d}h^2\right)
\end{aligned}
$$

hold for all $x \in \mathcal{G}_h$, $\tau \in [0,1]$, $i \in \{1,\dots,d\}$ and every $\varepsilon \in (0,1)$ with sufficiently large $\alpha(\varepsilon, d)$. Denote $\beta_0 = 1-\varepsilon$, $\beta = 1+\varepsilon$ and let $x \in \mathcal{G}_h$,

$i \in \{1, \dots, d\}$ and $\tau \in [0,1]$ be arbitrary but fixed. We estimate, on the one hand

$$
\begin{aligned}
|x - \tau e_i h|^2 + \frac{\alpha}{d} h^2 &= |x|^2 - 2\tau h x_i + \tau^2 h^2 + \frac{\alpha}{d} h^2 \le |x|^2 + 2h|x| + \left(\frac{\alpha}{d} + 1\right) h^2 \\
&\le |x|^2 (1+\varepsilon) + \frac{d(\alpha/d + 1 + 1/\varepsilon)}{\alpha(1+\varepsilon)} (1+\varepsilon) \frac{\alpha}{d} h^2 \\
&\le \beta \left(|x|^2 + \frac{\alpha}{d} h^2 \right),
\end{aligned}
$$

provided

$$
\frac{\alpha + d + d/\varepsilon}{\alpha(1+\varepsilon)} \le 1 \Leftrightarrow \alpha \ge \frac{d}{\varepsilon} + \frac{d}{\varepsilon^2}.
$$

On the other hand

$$
\begin{aligned}
|x - \tau e_i h|^2 + \alpha h^2 &= |x|^2 - 2\tau h x_i + \tau^2 h^2 + \alpha h^2 \\
&\ge |x|^2 - 2h|x| + \alpha h^2 \ge |x|^2 (1-\varepsilon) + \left(\alpha - \frac{1}{\varepsilon}\right) h^2 \\
&= (1-\varepsilon) \left[|x|^2 + \frac{\alpha - 1/\varepsilon}{\alpha(1-\varepsilon)} \alpha h^2 \right] \ge \beta_0 (|x|^2 + \alpha h^2),
\end{aligned}
$$

provided

$$
\frac{\alpha - 1/\varepsilon}{(1-\varepsilon)\alpha} \ge 1 \Leftrightarrow \alpha \ge \frac{1}{\varepsilon^2}.
$$

⟨*3*⟩ *Choice of the vector field*

We set

$$
F_i(x) = -\frac{t x_i}{|x|^2 + \alpha h^2}, \quad x \in \mathcal{G}_h
$$

for $1 \le i \le d$, where $t, \alpha > 0$ are parameters yet to be determined. We now can estimate

$$
\begin{aligned}
F_i^2(x) &= \frac{t^2 x_i^2}{(|x|^2 + \alpha h^2)^2}, & 1 \le i \le d, \\
F_i^2(x - e_i h) &= \frac{t^2 (x_i - h)^2}{(|x - e_i h|^2 + \alpha h^2)^2} \le \frac{\beta t^2 (x_i^2 + \frac{\alpha}{d} h^2)}{\beta_0^2 (|x|^2 + \alpha h^2)^2}, & 1 \le i \le d,
\end{aligned}
$$

obtaining

$$
\begin{aligned}
\sum_{i=1}^d F_i^2(x) &\le \frac{t^2}{|x|^2 + \alpha h^2}, \\
\sum_{i=1}^d F_i^2(x - e_i h) &\le \frac{\beta}{\beta_0^2} \frac{t^2}{|x|^2 + \alpha h^2}.
\end{aligned}
$$

Moreover, from the mean value theorem

$$D_i^- F_i(x) = \frac{F_i(x) - F_i(x - e_i h)}{h} = \frac{\partial}{\partial x_i} F_i(x - \tau e_i h)$$

with some $\tau \in [0,1]$, $1 \le i \le d$. We calculate

$$\frac{\partial}{\partial x_i} F_i(x) = \frac{-t}{|x|^2 + \alpha h^2} + \frac{2t x_i^2}{(|x|^2 + \alpha h^2)^2}$$

and obtain

$$\begin{aligned} D_i^- F_i(x) &= \frac{-t}{|x - \tau e_i h|^2 + \alpha h^2} + \frac{2t(x_i - \tau h)^2}{(|x - \tau e_i h|^2 + \alpha h^2)^2} \\ &\le \frac{-t}{\beta(|x|^2 + \alpha h^2)} + \frac{2t\beta(x_i^2 + \frac{\alpha}{d} h^2)}{\beta_0^2(|x|^2 + \alpha h^2)^2}, \end{aligned}$$

for all $1 \le i \le d$. Summing up we also see that

$$\sum_{i=1}^{d} D_i^- F_i(x) \le t \left(\frac{2\beta}{\beta_0^2} - \frac{d}{\beta} \right) \frac{1}{|x|^2 + \alpha h^2}.$$

Plugging this estimates into the first step

$$\begin{aligned} \sum_{i=1}^{d} \sum_{x \in \mathcal{G}_h} \left(D_i^+ u(x) \right)^2 &\ge - \sum_{x \in \mathcal{G}_h} u^2(x) \sum_{i=1}^{d} \left[D_i^- F_i(x) + \frac{1}{2} F_i^2(x - h e_i) + \frac{1}{2} F_i^2(x) \right] \\ &\ge \sum_{x \in \mathcal{G}_h} \frac{u^2(x)}{|x|^2 + \alpha h^2} \left[t \left(\frac{d}{\beta} - \frac{2\beta}{\beta_0^2} \right) - t^2 \left(\frac{1}{2} + \frac{\beta}{2\beta_0^2} \right) \right] \end{aligned}$$

For $d \ge 3$ we can achieve

$$\frac{d}{\beta} - \frac{2\beta}{\beta_0^2} = \frac{d}{1+\varepsilon} - 2\frac{1+\varepsilon}{(1-\varepsilon)^2} > 0$$

by making $\varepsilon > 0$ sufficiently small, i.e. for all sufficiently large α. Since now

$$\max_{t \in [0, +\infty)} t \left(\frac{d}{\beta} - \frac{2\beta}{\beta_0^2} \right) - t^2 \left(\frac{1}{2} + \frac{\beta}{2\beta_0^2} \right)$$

is positive, we get the claim by setting $t := \bar{t}$, where $\bar{t}$ is the corresponding maximizer.

◀

10.4 Corollary (Discrete Hardy Inequality).
Let $d \geq 3$. There exists $C > 0$ such that

$$\sum_{x \in \mathcal{G}_h \setminus \{0\}} \frac{|u(x)|^2}{|x|^2} \leq C \sum_{x \in \mathcal{G}_h} \sum_{i=1}^{d} \left(D_i^+ u(x)\right)^2$$

for all $u \colon \mathcal{G}_h \to \mathbb{R}$ with finite support.

▶

For $x \neq 0$ we have

$$|x|^2 \geq h^2 \Rightarrow |x|^2 + \alpha h^2 \leq (\alpha + 1)|x|^2,$$

where α is sufficiently large for the previous lemma to hold, yielding

$$\frac{1}{\alpha+1} \sum_{x \in \mathcal{G}_h \setminus \{0\}} \frac{|u(x)|^2}{|x|^2} \leq \sum_{x \in \mathcal{G}_h \setminus \{0\}} \frac{|u(x)|^2}{|x|^2 + \alpha h^2} \leq \sum_{x \in \mathcal{G}_h} \frac{|u(x)|^2}{|x|^2 + \alpha h^2}.$$

◀

10.5 Remark (Dimension).
Compare the condition $d \geq 3$ with the best possible constant $C = \frac{4}{(d-2)^2}$, $d \geq 3$ in the continuous case, see [7].

10.6 Motivation (Shifted Hardy Inequality).
One possible question connected with the previous result is the following: can the constant C be made arbitrary small if we restrict the validity of the inequality to the functions whose support is far away from zero. The answer is negativ as the following lemma shows.

10.7 Lemma (Shifted Hardy Inequality).
There exists a sequence $(u_n)_{n \in \mathbb{N}}$, $u_n \colon \mathcal{G}_h \to \mathbb{R}$ with finite support, $u_n \not\equiv 0$, $\operatorname{supp}(u_n) \subset \{x \in \mathcal{G}_h \mid x_1 > nh\}$, $n \in \mathbb{N}$ such that

$$\sum_{i=1}^{d} \sum_{x \in \mathcal{G}_h} \left|D_i^+ u_n(x)\right|^2 h^d \leq C \sum_{x \in \mathcal{G}_h \setminus \{0\}} \frac{|u_n(x)|^2}{|x|^2} h^d,$$

with $C > 0$ independent of h and n.

▶

⟨*1*⟩ *Reduction*

Exploiting the scaling invariance w.r.t. h

$$\frac{\sum_{i=1}^{d} \sum_{x \in \mathcal{G}_h} \left|D_i^+ u_n(x)\right|^2 h^d}{\sum_{x \in \mathcal{G}_h \setminus \{0\}} \frac{|u_n(x)|^2}{|x|^2} h^d} = \frac{\sum_{i=1}^{d} \sum_{x \in \mathcal{G}_h} |u_n(x + e_i h) - u(x)|^2}{\sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{|u_n(kh)|^2}{|k|^2}}$$

and to norm equivalence on $\mathbb{R}^d$ it is sufficient to show

$$\frac{\sum_{i=1}^{d}\sum_{x\in\mathcal{G}_h}|u_n(x+e_ih)-u(x)|^2}{\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\frac{|u_n(kh)|^2}{|k|_1^2}} \overset{!}{\leq} C. \tag{$*$}$$

⟨*2*⟩ *Construction of* u_n
We set

$$u_n := \sum_{i=0}^{n}\frac{n-i}{n}\chi_{\{x\in\mathcal{G}_h \mid |x-2nhe_1|_1=ih\}},$$

where χ denotes the characteristic function of a set (see Figure 10.1 with $n=3$, $d=2$, where the values of u_3 in the corresponding points are labeled). By construction we have $\operatorname{supp}(u_n)\subset\{x\in\mathcal{G}_h \mid x_1>nh\}$.

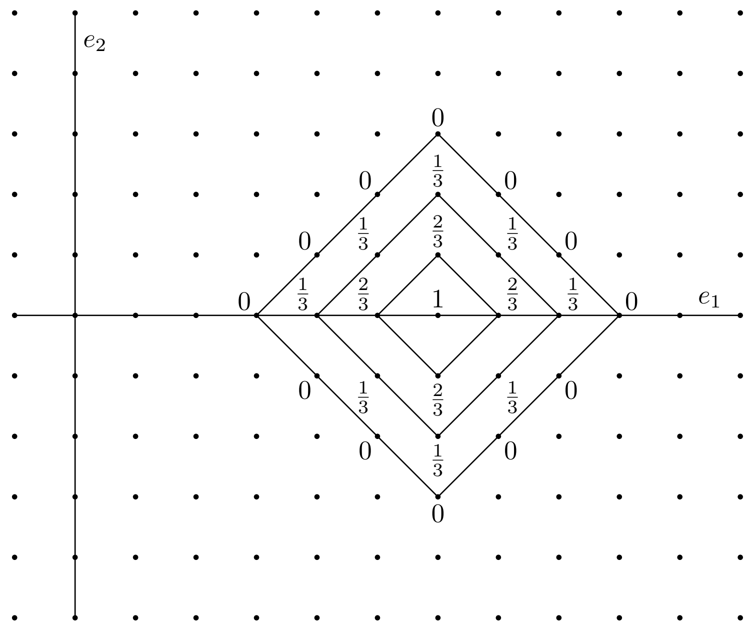

Figure 10.1

⟨*3*⟩ *Estimate for the numerator*
We estimate the numerator in $(*)$. By construction of u_n we have

$$|u_n(x+e_ih)-u_n(x)|^2 = \begin{cases} 0, & \text{if } \{x,x+e_ih\}\cap\operatorname{supp}(u_n)=\emptyset,\\ \frac{1}{n^2}, & \text{otherwise.}\end{cases}$$

for $n\in\mathbb{N}$, $1\leq i\leq d$, $x\in\mathcal{G}_h$. We should therefore estimate the number of elements in

$$\bigcup_{i=1}^{d}\{x\in\mathcal{G}_h \mid \{x,x+e_ih\}\cap\operatorname{supp}(u_n)\neq\emptyset\}.$$

Since

$$\{x \pm e_i h \mid 1 \le i \le d, x \in \operatorname{supp}(u_n)\} = \{x \in \mathcal{G}_h \mid |x - 2nhe_1|_1 \le n\},$$

we obtain

$$\sum_{i=1}^{d} \sum_{x \in \mathcal{G}_h} |u_n(x + e_i h) - u_n(x)|^2 \le d(2n+1)^d \frac{1}{n^2} \le K(d) n^{d-2}.$$

⟨*4*⟩ *Estimate for the denominator*
We can estimate

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^d} \frac{|u_n(kh)|^2}{|k|_1^2} &= \sum_{k \in \mathbb{Z}^d} \sum_{i=0}^{n} \frac{(n-i)^2}{n^2} \frac{\chi_{\{x \in \mathcal{G} \,|\, |x-2hne_1|_1 = ih\}}(kh)}{|k|_1^2} \\
&\ge \sum_{k \in \mathbb{Z}^d} \sum_{i=0}^{n} \frac{(n-i)^2}{n^2} \frac{\chi_{\{x \in \mathcal{G} \,|\, |x-2hne_1|_1 = ih\}}(kh)}{(2n+i)^2} \\
&\ge \sum_{i=0}^{n} \frac{(n-i)^2}{n^2(2n+i)^2} \sum_{k \in \mathbb{Z}^d} \chi_{\{x \in \mathcal{G} \,|\, |x-2hne_1|_1 = ih\}}(kh) \\
&\ge K'(d) \frac{1}{(3n)^2} \sum_{i=0}^{n} \frac{(n-i)^2}{n^2} (i+1)^{d-1} \ge K(d) n^{d-2},
\end{aligned}$$

provided

$$\sum_{i=0}^{n} \frac{(n-i)^2}{n^2} (i+1)^{d-1} \ge \sum_{i=1}^{n} \frac{(n-i)^2}{n^2} i^{d-1} \overset{!}{\ge} K(d) n^d.$$

To see this just observe that

$$\frac{1}{n^d} \sum_{i=1}^{n} \frac{(n-i)^2}{n^2} i^{d-1} = \sum_{i=1}^{n} \left(1 - \frac{i}{n}\right)^2 \left(\frac{i}{n}\right)^{d-1} \frac{1}{n}$$

is a Riemann sum for

$$\int_0^1 (1-x)^2 x^{d-1} \, \mathbf{d}\, x = \frac{2}{d(d+1)(d+2)} > 0.$$

◀

10.8 Theorem (Discrete Nonlinear Liouville Theorem)**.**
Let $d \ge 3$ and $1 < p < \frac{d}{d-2}$. Then $u \equiv 0$ is the only non-negative solution of

$$-\Delta_h u(x) = u^p(x), \quad x \in \mathcal{G}_h.$$

►⟨1⟩ *Comparison argument*

Let there be a further non-negative solution, $u \geq 0$, $u \not\equiv 0$. Suppose $u(x_0) = \min_{x\in\mathcal{G}_h} u(x) = 0$ for some $x_0 \in \mathcal{G}_h$. By Lemma 5.15, we obtain $u \equiv 0$, hence we may assume $u > 0$ on $\mathcal{G}_h$. From $1 < p < \frac{d}{d-2}$ we get

$$\delta := 2 - (p-1)(d-2) > 0$$

and setting $\varepsilon := \frac{\delta}{2(p-1)}$ we also get

$$\begin{aligned} \varepsilon(p-1) &< 2 - (p-1)(d-2) \\ (p-1)(2-d-\varepsilon) &> -2 \end{aligned} \tag{$*$}$$

Setting $\beta := 2 - d - \varepsilon$ and using Lemma 6.2 we obtain

$$-\Delta_h \sigma_\gamma^\beta(x) \leq -\beta(d-2+\beta+\varepsilon)\sigma^{\beta-2}(x) = 0$$

for $x \in \mathcal{G}_h$, $|x|_\infty \geq Rh$ with sufficiently large γ and R.
The set $\{x \in \mathcal{G}_h \mid |x|_\infty = Rh\}$ is finite, i.e. we can always find $C > 0$ such that

$$C\sigma(x)^\beta \leq u(x), \quad x \in \mathcal{G}_h, |x|_\infty = Rh.$$

Since σ^β is radially falling to zero, for every $w > 0$ there exists $\bar{R}_w \in \mathbb{N}$ such that

$$C\sigma(x)^\beta - w \leq 0 \quad x \in \mathcal{G}_h, |x|_\infty = \bar{R}_w h$$

with $\bar{R}_w \to \infty$ as $w \to 0$. This implies

$$\begin{aligned} C\sigma^\beta(x) - w &\leq u(x), & |x|_\infty &= \bar{R}_w h \\ C\sigma^\beta(x) - w \leq C\sigma^\beta(x) &\leq u(x), & |x|_\infty &= Rh \\ -\Delta_h\left[C\sigma^\beta(x) - w - u(x)\right] &\leq 0, & |x|_\infty &> Rh \end{aligned}$$

with $\bar{R}_w > R$ for all $w > 0$ sufficiently small. Applying the discrete maximum principle on the set $\{x \in \mathcal{G}_h \mid Rh \leq |x|_\infty \leq \bar{R}_w h\}$ and letting $w \to 0$, we obtain

$$u(x) \geq C\sigma(x)^\beta \geq \underbrace{C\left(1+\frac{\gamma}{R}\right)^{\frac{\beta}{2}}}_{\tilde{C}} |x|^\beta$$

for all $x \in \mathcal{G}_h$, $|x| \geq Rh$. This implies

$$-\Delta_h u(x) = u^{p-1}(x)u(x) \geq \tilde{C}^{p-1}|x|^{2+\beta(p-1)}|x|^{-2}u(x)$$

for $x \in \mathcal{G}_h$, $|x|_\infty \geq \max\{Rh, 1\}$. From $(*)$ we get $2 + \beta(p-1) > 0$, i.e. for all sufficiently large K there exists $R_K \in \mathbb{N}$ (depending on h in general) such that

$$-\Delta_h u(x) \geq K|x|^{-2}u(x)$$

for all $|x|_\infty > R_K h$.

⟨*2*⟩ *Allegretto-Piepenbrinck-Agmon-Trick*

We now apply the discrete version of the so called Allegretto-Piepenbrinck-Agmon-Trick (see [3], Theorem 3.1 or [17], Theorem 1.5.12). Let

$$-\Delta_h v(x) - \frac{K}{|x|^2} v(x) \geq 0, \quad \forall x \in H_s \tag{+}$$

hold for a fixed $v\colon \mathcal{G}_h \to \mathbb{R}$, $v > 0$ and a fixed $s \in \mathbb{Z}_+$, $H_s := \{x \in \mathcal{G}_h \mid x_1 \geq sh\}$. Let $\varphi\colon \mathcal{G}_h \to \mathbb{R}$, $\operatorname{supp}(\varphi) \subset H_s$ be arbitrary. Multiplying (+) with $\varphi^2/v \geq 0$ and partially summing over H_s we obtain

$$\sum_{x \in H_s} \left\{ \sum_{i=1}^{d} D_i^+ v(x) D_i^+ \frac{\varphi^2}{v}(x) - \frac{K}{|x|^2} \varphi^2(x) \right\} \geq 0.$$

By direct calculation for every $1 \leq i \leq d$ and $x \in \mathcal{G}_h$ we obtain

$$\begin{aligned}
I := & D_i^+ v(x) D_i^+ \frac{\varphi^2}{v}(x) = \frac{1}{h^2} \left(v(x+he_i) - v(x) \right) \left(\frac{\varphi^2(x+he_i)}{v(x+he_i)} - \frac{\varphi^2(x)}{v(x)} \right) \\
= & \frac{1}{h^2} \left[\varphi^2(x+he_i) - 2\varphi(x+he_i)\varphi(x) + \varphi^2(x) \right] \\
& - \frac{v(x+he_i)v(x)}{h^2} \left[\frac{\varphi^2}{v^2}(x) - 2\frac{\varphi(x+he_i)\varphi(x)}{v(x+he_i)v(x)} + \frac{\varphi^2}{v^2}(x+he_i) \right] \\
= & \left(D_i^+ \varphi(x) \right)^2 - v(x+he_i)v(x) \left(D_i^+ \frac{\varphi}{v}(x) \right)^2,
\end{aligned}$$

yielding

$$\begin{aligned}
J := & \sum_{x \in H_s} \left\{ \sum_{i=1}^{d} \left(D_i^+ \varphi(x) \right)^2 - \frac{K}{|x|^2} \varphi^2(x) \right\} \\
\geq & \sum_{x \in H_s} \sum_{i=1}^{d} v(x+he_i)v(x) \left(D_i^+ \frac{\varphi}{v}(x) \right)^2 \geq 0.
\end{aligned}$$

Together with the first step this implies that for all sufficiently large $C > 0$ there exists $s \in \mathbb{Z}_+$ such that

$$\sum_{x \in H_s} \sum_{i=1}^{d} \left(D_i^+ \varphi(x) \right)^2 \geq C \sum_{x \in H_s} \frac{\varphi^2(x)}{|x|^2}$$

for *all* $\varphi\colon \mathcal{G}_h \to \mathbb{R}$ with $\operatorname{supp}(\varphi) \subset H_s$. This contradicts Lemma 10.7, i.e. our original equation admits only the trivial solution.

◀

10.2 A Priori Estimate and Convergence Theory on $\mathcal{G}_h$

10.9 Theorem (A Priori Estimate on $\mathcal{G}_h$)**.**
There exists $C > 0$ such that $\|u\|_{L^\infty(\mathcal{G}_h)} \leq C$ for all non-negative solutions of

$$-\Delta_h u(x) + \lambda u(x) = u^p(x), \quad x \in \mathcal{G}_h \qquad (*)_h$$

uniformly in $h \in (0,1]$ for fixed $\lambda > 0$, $p \in (1, \frac{d}{d-2})$.
▶

⟨*1*⟩ *Boundedness*
Let $u \colon \mathcal{G}_h \to \mathbb{R}$ be a non-negative solution for $(*)_h$. This implies

$$u^p(x) = -\Delta_h u(x) + \lambda u(x) \leq \frac{2du(x)}{h^2} + \lambda u(x)$$
$$\Rightarrow u^{p-1}(x) \leq \lambda + \frac{2d}{h^2}, \quad \forall x \in \mathcal{G}_h.$$

This yields the uniform boundedness for every fixed $h \in (0,1]$. Moreover, we have

$$(hu^{\frac{p-1}{2}}(x))^2 \leq h^2\lambda + 2d \leq \lambda + 2d$$

for all $x \in \mathcal{G}_h$ and $h \in (0,1]$.

⟨*2*⟩ *Rescaling*
Arguing by contradiction, assume there exists a sequence $(h_n)_{n\in\mathbb{N}} \subset (0,1]$ and a corresponding sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \colon \mathcal{G}_{h_n} \to \mathbb{R}$ of non-negative (and w.l.o.g. non-trivial) solutions for $(*)_h$ such that

$$\|u_n\|_{L_\infty(\mathcal{G}_{h_n})} \overset{n\to\infty}{\to} \infty.$$

This implies the existence of $(x_n)_{n\in\mathbb{N}}$, $x_n \in \mathcal{G}_{h_n}$ such that $M_n := u_n(x_n) \to \infty$ as $n \to \infty$ and $u_n(x) \leq 2u_n(x_n)$ for all $x \in \mathcal{G}_{h_n}$ and $n \in \mathbb{N}$. We set $\mu_n := M_n^{\frac{1-p}{2}}$ and define $v_n \colon \mathcal{G}_{\tau_n} \to \mathbb{R}$ by

$$\tau_n := \frac{h_n}{\mu_n} = M_n^{\frac{p-1}{2}} h_n,$$
$$v_n(x) := \frac{1}{M_n} u_n(x_n + \mu_n x), \quad x \in \mathcal{G}_{\tau_n},$$

implying $v_n(0) = 1$ and $v_n(x) \le 2$ for all $x \in \mathcal{G}_{\tau_n}$, $n \in \mathbb{N}$. We obtain

$$\begin{aligned} I = -\Delta_{\tau_n} v_n(x) &= \sum_{i=1}^{d} \frac{2v_n(x) - v_n(x + e_i\tau_n) - v_n(x - e_i\tau_n)}{\tau_n^2} \\ &= \frac{\mu_n^2}{M_n} \sum_{i=1}^{d} \left(\frac{2u_n(y) - u_n(y + e_i\mu_n\tau_n) - u_n(y - e_i\mu_n\tau_n)}{h_n^2} \right)\Bigg|_{y=x_n+\mu_n x} \\ &= \frac{\mu_n^2}{M_n} \left(-\Delta_{h_n} u_n(y)\right)|_{y=x_n+\mu_n x} = \frac{\mu_n^2}{M_n} \left(-\lambda u_n(y) + u_n^p(y)\right)|_{y=x_n+\mu_n x} \\ &= \frac{\mu_n^2}{M_n} \left(-M_h \lambda v_n(x) + M_n^p v_n^p(x)\right), \end{aligned}$$

i.e. v_n satisfies the equation

$$-\Delta_{\tau_n} v_n(x) = -\mu_n^2 \lambda v_n(x) + \underbrace{\mu_n^2 M_n^{p-1}}_{=1} v_n^p(x) = -\mu_n^2 \lambda v_n(x) + v_n^p(x). \tag{$\dagger$}$$

The first step assures the boundedness of $(\tau_n)_{n\in\mathbb{N}}$. Two cases are therefore possible: either $\tau_n \to \tau > 0$ for $n \to \infty$ up to a subsequence or $\tau_n \to 0$ as $n \to \infty$. We consider those cases separately. Take note that $\mu_n \to 0$ as $n \to \infty$.

⟨3⟩ *Positive accumulation point*
We construct $v_\tau \colon \mathcal{G}_\tau \to [0, \infty)$ as follows. The sequence $(v_n(e_1\tau_n))_{n\in\mathbb{N}} \subset (0, 2]$ possesses by Bolzano-Weierstrass a convergent subsequence $(v_{n_k}(e_1\tau_{n_k}))_{k\in\mathbb{N}}$ with some limit in $[0, 2]$. We assign this limit to $v_\tau(e_1\tau)$. Analogously, we extract a convergent subsequence from $(v_{n_k}(e_2\tau_{n_k}))_{k\in\mathbb{N}}$ and assign its limit to $v_\tau(e_2\tau)$, and so on for every $v_\tau(x)$, $x \in \mathcal{G}_\tau$. Since the discrete Laplacian depends only on a finite number of points, taking the limit in (†) after having extracted and renamed all involved subsequences, we obtain

$$-\Delta_\tau v_\tau(x) = v_\tau^p(x), \quad x \in \mathcal{G}_\tau$$

with $v_\tau(0) = 1$, resulting in a contradiction to Theorem 10.8.

⟨4⟩ *Continuous limit*
Let $(R_n)_{n\in\mathbb{N}}$, $R_n > 0$, $n \in \mathbb{N}$ be a strictly increasing sequence with $R_n \to \infty$ as $n \to \infty$, i.e. $([-R_n, R_n]^d)_{n\in\mathbb{N}}$ monotonically exhausts $\mathbb{R}^d$. We define

$$R_n^{(k)} := \left\lceil \frac{R_n}{\tau_k} \right\rceil \tau_k$$

for $n, k \in \mathbb{N}$, implying that $(-R_n^{(k)}, R_n^{(k)})^d$ is admissible for $\mathcal{G}_{\tau_k}$ with $R_n \le R_n^{(k)} \le R_n + 1$ for sufficiently large $k \in \mathbb{N}$.

From $\|v_k\|_{L^\infty(\mathcal{G}_{\tau_k})} \leq 2$ we obtain

$$\|\Delta_{\tau_k} v_k\|_{L^\infty(\mathcal{G}_{\tau_k})} \leq \lambda\mu_k \|v_n\|_{L^\infty(\mathcal{G}_{\tau_k})} + \|v_k^p\|_{L^\infty(\mathcal{G}_{\tau_k})} \leq 2 + 2^p.$$

for all sufficiently large $k \in \mathbb{N}$. Since for every $\tau_k \in (0,1]$ exists a natural number $s_k \in \mathbb{N}$ with $s_k\tau_k \in (1/2, 1]$ we use Theorem 5.31 and obtain

$$\begin{aligned} \|D_i^+ v_k\|_{L^\infty([-R_1^{(k)}, R_1^{(k)}]^d \cap \mathcal{G}_{\tau_k})} \leq \frac{2d}{s_k\tau_k} \|v_k\|_{L^\infty([-R_1^{(k)}, R_1^{(k)}]^d \cap \mathcal{G}_{\tau_k})} \\ + s_k\tau_k \|\Delta_{\tau_k} v_k\|_{L^\infty([-R_1^{(k)}, R_1^{(k)}]^d \cap \mathcal{G}_{\tau_k})} \leq K \end{aligned} \tag{+}$$

for all sufficiently large $k \in \mathbb{N}$ and all $1 \leq i \leq d$.

Using the interpolation operator from Definition 8.12 we now construct $\hat{v}_k \in C([-R_1^{(k)}, R_1^{(k)}])$, $k \in \mathbb{N}$.

Since by Corollary 8.8

$$\|\hat{v}_k\|_{L^q([-R_1^{(k)}, R_1^{(k)}]^d)} \leq (2R_1^{(k)})^{d/q} \max_{x \in [-R_1^{(k)}, R_1^{(k)}]^d} |\hat{v}_k(x)| \leq 2^{1+d/q}(R_1 + 1)^{d/q} K.$$

for sufficiently large $k \in \mathbb{N}$ and by Theorem 8.13 with (+) (we write the seminorm $|\cdot|_{W^{1,p}}$ instead of $\|\cdot\|_{W_0^{1,p}}$, because we in general do not have zero on the boundary)

$$|\hat{v}_k|_{W^{1,q}([-R_1^{(k)}, R_1^{(k)}]^d)} \leq K,$$

we get

$$\|\hat{v}_k\|_{W^{1,q}([-R_1^{(k)}, R_1^{(k)}]^d)} \leq K$$

with $q > d$. Extracting from $\left(\hat{v}_k|_{[-R_1, R_1]^d}\right)_{k \in \mathbb{N}}$ a uniformly convergent subsequence we obtain $v \in C([-R_1, R_1]^d) \cap W^{1,q}([-R_1, R_1]^d)$, $v \geq 0$, $v(0) = 1$. Using the extracted subsequence as a starting point we repeat this argument on $[-R_2, R_2]^d$ and so on. The resulting function $v \colon \mathbb{R}^d \to [0, \infty)$ lies in $C(\mathbb{R}^d) \cap W^{1,q}_{loc}(\mathbb{R}^d)$ with $v(0) = 1$. Moreover, taking the diagonal subsequence, i.e. the first element of the convergent subsequence on $[-R_1, R_1]^d$, the second element of the convergent subsequence on $[-R_2, R_2]^d$ etc., we obtain subsequence $(v_{k_i})_{i \in \mathbb{N}}$ of $(v_k)_{k \in \mathbb{N}}$ such that $\hat{v}_{k_i} \to v$ uniformly on every bounded subset of $\mathbb{R}^d$ as $i \to \infty$. We want to show that v is a classical solution of the continuous nonlinear Liouville equation 10.2, thus yielding a contradiction.

⟨5⟩ *Passing to the limit in the equation*
Repeating the arguments in Theorems 9.10 and noting that the $\mu_n^2 \lambda v_n(x)$ term

converges to zero due to the uniform boundedness of $v_n(x)$ in $x \in \mathcal{G}_{h_n}$ and $n \in \mathbb{N}$, we get that v satisfies

$$\int_{\mathbb{R}^d} v(-\Delta\varphi) = \int_{\mathbb{R}^d} v^p \varphi$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. Here the boundedness of $\operatorname{supp}\varphi$ for $\varphi \in C_0^\infty(\mathbb{R}^d)$ and uniform convergence on bounded subsets derived above replaces the boundedness of the domain in the original Theorem 9.10, but we obtain only a very weak solution. Nonetheless, it is well-known (see [35], [34]) that if $1 < p < \frac{d}{d-2}$, $v \in L^p_{loc}(\mathbb{R}^d)$, then $v \in C^2$ and Theorem 10.2 applies.

◀

10.10 Remark (Convergence)**.**
Now we know that every sequence $(u_n)_{n\in\mathbb{N}}$, $u_n \colon \mathcal{G}_{h_n} \to \mathbb{R}$ of non-negative solutions of $(*)_{h_n}$ is uniformly bounded in $L^\infty(\mathcal{G}_{h_n})$. Assuming $h_n \to 0$ as $n \to \infty$ we can repeat the last two steps in the proof with $v_n := u_n$, $n \in \mathbb{N}$. The crucial difference is that whereas v_n satisfies

$$-\Delta_{\tau_n} v_n(x) = -\mu_n^2 \lambda v_n(x) + v_n^p(x), \quad x \in \mathcal{G}_{\tau_n}$$

with $\mu_n \to 0$ as $n \to \infty$, we now have

$$-\Delta_{h_n} u_n(x) = -\lambda u_n(x) + u_n^p(x), \qquad x \in \mathcal{G}_{h_n}$$

and the linear part of r.h.s. does not vanish in the limit.

10.11 Corollary (Convergence on $\mathcal{G}_h$)**.**
Let $\lambda > 0$ and $p \in (1, \frac{d}{d-2})$ be fixed. Further, let $(u_n)_{n\in\mathbb{N}}$, $u_n \colon \mathcal{G}_{h_n} \to [0,\infty)$ satisfy

$$-\Delta_{h_n} u_n(x) + \lambda u_n(x) = u_n^p(x), \quad x \in \mathcal{G}_{h_n}$$

with $h_n \in (0,1]$, $n \in \mathbb{N}$, $h_n \xrightarrow{n\to\infty} 0$.

Then, there exists a (renamed) subsequence of $(u_n)_{n\in\mathbb{N}}$ and a non-negative solution $u \in C^2(\mathbb{R}^d)$ of

$$-\Delta u(x) + \lambda u(x) = u^p(x), \quad x \in \mathbb{R}^d$$

such that

$$\|u - u_n\|_{L^\infty(\Omega\cap\mathcal{G}_{h_n})} \to 0$$

as $n \to \infty$ for every bounded Ω.

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Hiermit erkläre ich an Eides Statt, dass ich die vorliegende Dissertation selbstständig und nur unter Verwendung der angegebenen Hilfsmittel verfasst habe. Alle Stellen, die im Wortlaut oder dem Sinn nach schriftlich oder im Internet veröffentlichten Quellen entnommen sind, habe ich durch genaue Quellenangabe gekennzeichnet.

Karlsruhe, den 10. Juni 2014 Anton Verbitsky

Acknowledgments

This thesis would not have been possible without the invaluable help of my advisers. I would like to express my deepest gratitude to

Prof. Dr. Wolfgang Reichel for believing in me at the beginning, supporting me on the way and being patient with me at the end. His steady encouragement and sincere interest guided me all the time.

Prof. Dr. Willy Dörfler for helping when nothing else did by letting me draw on his unprecedented mathematical expertise.

The time I spent in the Research Training Group 1294 has brought me together with many beautiful people who made these four years joyful and enriching. I'm grateful to

Hans-Jürgen and Hannes for all our friendly discussions,

Tomislav for conversations and a new name,

Abdullah and Bilal for always keeping the door open.

I finally want to mention and thank people whom I know and love since a long time

Michael and Roman for being there when I needed them,

my mother for not forgetting to remind.