Optimal Constrained Investment and Reinsurance in Lundberg insurance models

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Abstract.

An insurance company with start capital s is considered. This company can buy dynamically, in time, reinsurance as well as invest into risky or riskless assets. It is assumed that the insurance risk model is the Cramér-Lundberg model and the price of a risky asset is governed by geometric Brownian motion. The investment strategy is restricted by a set of constraints and a general form of reinsurance is considered.

The ultimate ruin probability, i.e. the probability that the surplus process drops below zero in infinite time, can be considered as the solvency measure for an insurance business. Now the question arises: is there any investment and reinsurance control process such that the ruin probability takes its minimum value? This thesis deals with this question.

The dynamic programming approach is used to characterize the optimal investment and reinsurance controls via the Hamilton-Jacobi-Bellman (HJB) equation. The optimal strategies are computed via a recursive finite difference solution to the corresponding discretized HJB equation. The concept of viscosity solution is used to derive convergence of the numerical method. The uniqueness of the viscosity solution is obtained through a comparison theorem.

A collection of examples with different analytical properties is presented which demonstrates the importance of the concept of viscosity solutions. With the help of adjustment coefficient, for some examples the asymptotic optimal investment and reinsurance strategy is calculated when s goes to infinity.

This thesis is supervised by Prof. Dr. Christian Hipp at the institute of insurance science and it is written in English.

Contents

CHAPTER 1

Introduction

Dynamic optimization is beginning to play an increasingly important technique in different fields of studies like operational research, physics and economics. For example, insurance companies face the problem of allocating their resources dynamically across different financial tools in order to achieve a particular goal. One of the most important goals of them is to be solvent over time, i.e. its capital and income exceed its costs. The ultimate ruin probability, the probability that the surplus process drops below zero in infinite time, can be considered as the solvency measure for an insurance business. An optimization problem is then the problem of finding the optimal strategy and the minimum ultimate ruin probability.

The classical stochastic model of ruin theory was introduced by Lundberg in his thesis in 1903. This model, known as Cramér-Lundberg model, describes the evolution of the insurer's surplus with incoming premiums and outgoing claims. Beside these two opposing cash flows, however, the insurer has a collection of possible actions such as investment, reinsurance, dividend payment, and the combination of all these actions. Controlling these actions in order to achieve insurance company's goal is a challenge of the insurance management.

In this script, we assume an insurance company whose goal is to minimize its ruin probability. To accomplish this objective, the insurer has the possibility to invest in a risky asset as well as to buy reinsurance. We model the risky asset price dynamics through geometric Brownian motion. The investment strategy is restricted by a constraint set $\mathcal{A}(s) \subset \mathbb{R}$ which must be taken into account as the insurer manages its portfolio. We use a general form of reinsurance, and assume that the insurance company can dynamically buy reinsurance as well as invest into a risky asset. By dynamic we mean that the actions are selected and changed at each point in time according to the risk position of the company.

The mathematical model used in this dissertation is the subject of chapter 2. We first introduce the classical Cramér-Lundberg model, state its most important concepts and review some well-known premium principles. The geometric Brownian motion model for the price of a risky asset is considered in section 2.2. Next, the possible reinsurance forms are introduced in a general setup as in [**42**]. Particularly, the two types of reinsurance: proportional reinsurance and excess of loss reinsurance, are discussed. In section 2.4, the risk process with dynamic investment and reinsurance strategy is formulated. Both the Cramér-Lundberg model and the diffusion model are Markovian and hence we may restrict our attention to the case where the strategies are Markovian, too. This chapter is ended by defining the optimization problem and presenting the value function.

In chapter 3, stochastic control theory is used to derive the optimal dynamic investment and reinsurance strategies. We start this chapter by describing the dynamic programming principle which was first introduced by Bellman (1957). In section 3.1, we use this principle to find the so-called Hamilton-Jacobi-Bellman (HJB) equation. Assuming that the value function is smooth enough, the corresponding HJB equation is a second-order non-linear integro-differential equation. If there exists a twice continuously differentiable solution to the HJB equation, then one needs to verify that this solution is indeed the value function. However, the value function is not always differentiable and one has to rely on a weak notion of solution called viscosity solution. We discuss the concept of viscosity solutions in section 3.2 and show that the value function is a solution of the associated HJB equation in a viscosity sense. We then derive the uniqueness of the viscosity solution through a comparison theorem. In section 3.3, we present a recursive numerical method and prove its convergence to the viscosity solution for the problem optimal investment. It should be pointed out that there are many excellent books on stochastic control and viscosity solution for the reader to learn more on the subject. Two books particularly worth mentioning here are [**15**] and [**34**].

Chapter 4 is dedicated to the numerical calculation of the value function and its associated optimal strategy. Most of the numerical examples provided in this script are for two well-known types of claim size distributions, namely the light- and heavy-tailed distributions. We begin this chapter with the optimal investment problem. A number of different operators have been proposed for dealing with this problem, each solving it for a certain set of constraints. In section 4.1, we review these operators and show how our numerical method can solve the optimal problem for a general set of constraints. Specifically, we present a collection of examples with different analytical properties which show the importance of the concept of viscosity solutions. In the next three sections, we assume that in addition to investment, the insurer can transfer part of its risk to a reinsurance company. The reinsurance contract types considered here are proportional, excess of loss and limited excess of loss reinsurance. In each section, the optimal strategy is studied through examples.

In chapter 5, we change the setup of the previous chapter by taking the risk free bond into account. The optimal investment problem is the topic of the first section. In particular, we consider the problem of optimal investment under borrowing and short selling constraints which was studied by Belkina et al. (2011). In section 5.2, the reinsurance is added to the set of possible actions. It is noticed that for sufficiently large start capital, the first insurer can have a positive rate of income from the riskfree bond without bearing any risks.

CHAPTER 2

Mathematical Model

In this chapter, we introduce the optimization problem to be solve. We model the insurance risk process via the classical Lundberg model which was firstly introduced by F. Lundberg in 1903. The insurance company has the possibility to invest in some risky asset (whose dynamics is modeled by a geometric Brownian motion) as well as buy reinsurance. The insurance risk model with investment and reinsurance is derived in section (2.4). We finish this chapter by presenting the value function.

2.1. Insurance risk model

Here we consider the development in time of the surplus R_t of an insurance company at time t with initial surplus $s := R_0$. One can interpret the initial surplus s as the amount of capital which is required to cover the costs of the claims higher than the premium. In this research we modeled the surplus process by classical Cramér-Lundberg model.

In this script we work on a complete probability space (Ω, F, P) with a filtration $\mathbb{F} = \{\mathcal{F}_t\}$ which is a family of increasing σ-fields, $\mathcal{F}_t \subset \mathcal{F}$. We assume that the filtration $\mathbb{F}$ is right continuous, that is, for all $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_{t+}$. A reference for stochastic calculus is [**30**].

2.1.1. Cramér-Lundberg model. An important question for insurance company is how to model the development in time of insurance capital R_t. This is a stochastic process which contains earning premium and paying claims. In this research we assume a classical Carmér-Lundberg model for surplus process as follow:

$$R_t = s + ct - X_t, \;\; t \geq 0,$$

where R_t, s and c are the insurer's capital at time t, initial capital $R_0 = s$ and constant premium income, respectively. Also $X_t = \sum_{i=1}^{N_t} Y_i$ is the aggregate claim amount which is assumed to be a compound Poisson process, that is,

(1) The claim arrival process $\{N_t\}$ is a Poisson process with rate λ, i.e.

$$\Pr\left[N_t = k\right] = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \;\; k = 0, 1, 2, ...,$$

(2) Individual claims Y_i are independent and identically distributed, and

(3) The two processes $\{N_t\}$ and $\{Y_i\}$ are independent.

It is obvious that $X_t = 0$ if $N_t = 0$.

Let T_i be the occurrence time of the ith claims. The random variables defined by $\xi_n := T_n - T_{n-1},\ , n \geq 1$, are called the **inter-occurrence times** in between successive claims which are independent, exponentially distributed random variables with mean $\lambda^{-1} > 0$.

The first time point when the risk reserve process becomes negative is called **ruin**, and the point in time at which this occurs is denoted by τ. So

$$\tau = \inf\left\{t \geq 0 : R_t < 0\right\}.$$

The probability of ruin for the initial capital s is then given by

$$\psi\left(s\right) = \Pr\left\{\tau < \infty \mid R_0 = s\right\}.$$

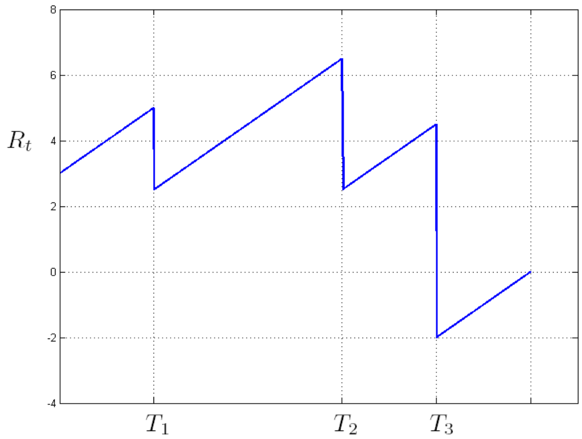
A typical realization of the risk process is depicted in Figure (1). At time point $t = 0$ due to initial capital $s = 3$ of insurer, $R_0 = 3$. The random variables $T_1, T_2, ...$ denote the occurrence times of claims. Insurance premiums are collected at the constant rate c. At times $t = T_i$ for some i, the ith claims with severity Y_i occurs, and there the capital drops. At time T_3 the risk reserve is less than 0 since the total of the incurred claims $Y_1 + Y_2 + Y_3$ is larger than the initial capital s plus the earned premium cT_3. So the ruin happens at time T_3 with severity -2.

We will assume that Y_i has a cumulative distribution function F_Y with expected value $\mu := E\left[Y\right]$.

Remark 1. When modeling the claims severity Y_i, insurers are usually concerned with the two following types of distributions:

1- **Light tails distributions**. This class of distributions consists of those distributions F with an exponentially bounded tail, i.e. $1 - F\left(y\right) \leq Ke^{-\alpha y} < \infty$, for some positive $\alpha > 0$ and K and all $y \geq 0$. The condition means that large claims are very

FIGURE 1. A realization of the Cramér-Lundberg process R_t.



unlikely. That is, the probability of their occurrence decreases exponentially fast to zero as the threshold y converges to ∞.

2- **Heavy tails distributions**. This class of distributions contains those distributions not having such exponential bound and huge claims are getting more likely. A well known class of heavy-tailed loss distributions is the class of subexponential distributions. △

2.1.2. Premiums. For an insurance company exposed to a liability X_t, a premium $\mathscr{P}(X_t)$ is the cost of risk transfer that the insurer must raise from the insured. For an insurance portfolio, the premiums are usually paid once at the beginning of insurance cover. But in this script we assume that the premiums are continuously paid to the insurance company. The premiums should be determined in such a way that the resolvability of the portfolio can be guaranteed. This means that the insurance premium $\mathscr{P}(X_t)$ should create an adequate insurance fund necessary to cover its liabilities at time $t > 0$. The reader must also note that a very high premium may result in lost customers, because other insurance companies might attract clients by offering lower premiums while covering the same risk.

A first reasonable estimate for a risk premium of policyholder would be the expected value of X_t. However, the insurer typically needs to charge premiums sufficient which cover the part of claims higher than expected value. In actuarial terminology, the positive amount $\mathscr{P}(X_t) - E[X_t]$ is called **safety loading** and usually denoted by safety loading factor $\eta > 0$.

Mathematically, a **premium principle** $\mathscr{P}$ is a map from the set $\mathtt{P}$ of all possible distribution function of risk X to $\mathbb{R}$, i.e. $\mathscr{P} : \mathtt{P} \to \mathbb{R}$. The well-known premium principles are:

(1) **Net premium principle**

$$\mathscr{P}(X_t) = E[X_t].$$

(2) **Expected Value Principle**

$$\mathscr{P}(X_t) = (1+\eta) E[X_t].$$

(3) **Variance Principle**

$$\mathscr{P}(X_t) = E[X_t] + \eta Var[X_t].$$

(4) **Standard Deviation Principle**

$$\mathscr{P}(X_t) = E[X_t] + \eta\sqrt{Var[X_t]}.$$

(5) **Exponential Principle**

$$\mathscr{P}(X_t) = \frac{1}{\eta}\log\left(E\left[e^{\eta X_t}\right]\right).$$

2.2. Investment

In practice, an insurer invests part of its capital in a financial market. We assume that the financial market consists of a risk-free asset like a bank account as well as a risky security, such as a stock or other risky asset.

We first consider the risky security and denote its price at time t by Z_t. Assume that $\{Z_t\}$ is an $\{\mathcal{F}_t\}$-adapted stochastic process which satisfies the stochastic differential equation

$$dZ_t = rZ_t dt + \sigma Z_t dW_t, \ Z_0 > 0,$$

where $\{W_t\}$ is a standard Brownian motion $W_t \sim N(0,t)$, and is adapted with respect to $\{\mathcal{F}_t\}$. The filtration $\{\mathcal{F}_t\}$ is generated by the two processes W_t and X_t, $t \geq 0$. We assume that X and W are independent. The drift r and diffusion σ are strictly positive and constant.

Moreover, the insurer has the possibility to invest some part of its capital into a risk-free bond with price B_t satisfying

$$dB_t = r_0 B_t dt,$$

for some constant $r_0 \geq 0$.

Let $\mathcal{A}$ denotes the set of all possible investment strategies at time t, then the insurer can invest amount $A_t \in \mathcal{A} \subset \mathbb{R}$ from its capital R_t into the risky asset and what is left is on the bank account earning (costing) interest r_0 if $R_t - A_t > 0$ (if $R_t - A_t < 0$). We shall neglect transaction costs and allow for shares of any (up to infinitesimal) size. We also assume $0 \in \mathcal{A}$, that is the insurer can always stop investing in risky asset.

The risk process of an insurance company with a constant investment strategy $A \in \mathcal{A}$ over time satisfies the stochastic differential equation

$$dR_t = r_0 R_t dt + cdt + (r - r_0) Adt + \sigma AdW_t - dX_t.$$

To simplify the setup and the notation we consider in chapters 3 and 4 that all the monetary quantities are discounted by inflation, so $r_0 = 0$. In the last chapter we consider the effect of risk-free bond in the optimization problem.

2.3. Reinsurance

An insurance company often transfers part of its risk to another insurance company (the reinsurance company). With the surplus, reinsurance is bought, and a premium has to be paid by the cedent insurer (the primary insurance company having issued the reinsurance contract) to the reinsurer. At any time the cedent can choose a reinsurance from a compact set $U \subset \mathbb{R}^n$. Here we assume that reinsurance contract acts on individual claims. Let $h(u)$ and $g(Y,u)$ denote reinsurance premium and part of the claim Y paid by the insurer. The function $g(Y,u)$ is called **risk sharing function** and must satisfy $0 \leq g(Y,u) \leq Y$ for each reinsurance strategy u. We assume that insurance company is not forced to buy reinsurance; that is, there exist $u_0 \in U$ which $h(u_0) = 0$ and $g(Y,u_0) = Y$.

Well-known reinsurance types are:

(1) **Proportional reinsurance**: $g(Y,\alpha) = \alpha Y$, $0 \le \alpha \le 1$ is called proportional reinsurance with proportion α. The reinsurer's share of claim is $(1-\alpha)Y$.
(2) **Excess of loss (XL) reinsurance**: In excess of loss reinsurance each claim of size Y is split between the first insurer and the reinsurer according to a priority $0 \le M \le \infty$: the insurer pays $g(Y,M) = \min\{Y,M\}$, and the reinsurer pays $(Y-M)^+ = \max\{Y-M,0\}$.
(3) **Limited Excess of loss reinsurance**: As special case of non-proportional reinsurance, limited excess of loss reinsurance will also charge the first insurer when the claims are larger than some barrier $L > 0$. The first insurer will pay $g(Y,(M,L)) = \min\{Y,M\} + (Y-M-L)^+$ of a claim of size Y and the reinsurer will pay $\min\left\{L,(Y-M)^+\right\}$. So in the limited XL reinsurance, the first insurer has two dimensional control process $u = (M,L)$ where $U = [0,\infty] \times (0,\infty]$.

We assume that the reinsurance premium function $h(u)$ is nonnegative, and reinsurance is expensive in the following sense: $g(Y,u) = 0$ implies $h(u) > c$. Otherwise the insurance company will get rid of all his risks by a full reinsurance and receive a positive return without any risk.

Depending on reinsurance strategy u chosen by the first insurer, reinsurance premium $h(u)$ can be calculated by the same premium principles presented in subsection (2.1.2). As an example, in case of proportional reinsurance with proportion level α, i.e. $g(Y,\alpha) = \alpha Y$, and reinsurer safety loading $\theta > \eta$, where $\eta \ge 0$ is first insurance safety loading, if an expected value principle is used, we have $h(\alpha) = (1+\theta)\lambda E[Y-\alpha Y]$. If the variance principle is used, then $h(\alpha) = \lambda E[Y-\alpha Y] + \theta\lambda Var[Y-\alpha Y]$.

Assume now that the insurer in addition to invest a constant amount of its capital A into risky asset, can buy reinsurance with constant strategy u. The risk process is then given by

$$R_t = s + (c - h(u) + rA)t + \sigma A \int_0^t dW_x - \sum_{i=1}^{N_t} g(Y_i,u).$$

2.4. The optimization problem

Insurers are always searching for opportunities to develop their business, increase revenues and improve profitability. The challenge is to maximize profitability and stability by achieving the optimal risk/reward relationship among reinsurance, investment and dividend payments. Portfolio optimization (for example, in a set of simultaneous reinsurance, investment and paying dividends) is a key to success in writing insurance, allowing a company to reduce its costs and solvency margin, and maximize profitability.

On the other hand, the solvency of an insurance company is one of the main concerns to the regulatory bodies. An insurer is solvent if its capital and expected income exceed its costs. In order to investigate solvency of an insurance company, regulators use some risk measures to determine the minimum capital that bears the risks and pays the claims. As a risk measure example for the risk process R_t, a regulator can uses ruin-consistent Value-at-risk (VaR), $\varrho_\epsilon[R]$, which is the capital required to ensure that the ruin probability is bounded by some constant $\epsilon > 0$, i.e. $\varrho_\epsilon[R] = \inf\{s : \psi(s) \leq \epsilon\}$ (see [**41**]). Hence, minimizing the ruin probability leads to a lower solvency capital requirement.

Here we consider the problem of minimizing the probability of ruin when an insurance company can dynamically invest into a risky asset as well as buy reinsurance. By dynamic we mean that the actions are selected and changed at each point in time according to the risk position of the company.

2.4.1. Dynamic investment and reinsurance. In section (2.3) we considered the risk reserve with the constant investment A and reinsurance u. We now assume that the insurance company can adjust his strategy (A, u) at each time point $t \geq 0$ based on revealed information before time t. We denote the dynamic investment and reinsurance strategy by $\{\pi_t\} = \{A_t, u_t\}$.

In the following considerations, we assume that Ω is the set of cadlag paths and $(\Omega, \mathcal{F}, (\mathcal{F}_t^{X,W})_{t\geq 0}, P)$ is a complete probability generated by the process $\{(X_t, W_t)\}$. Here $(\mathcal{F}_t^{X,W})_{t\geq 0}$ is the smallest right continuous filtration such that, $\{(X_t, W_t)\}$ is measurable. An adapted process π_t, $t \geq 0$, is called **predictable**, if it is the pointwise limit of left continuous processes. The control strategies are predictable processes π_t, $t \geq 0$, which take values in $\Pi(s) = (\mathcal{A}(s), U(s)) \subset \mathbb{R}^{n+1}$. The sets $\mathcal{A}(s) \subset \mathbb{R}$ and $U(s) \subset \mathbb{R}^n$ are closed. Moreover, due to fluctuations in the Brownian motion, we need to let $\mathcal{A}(0) = \{0\}$. We will come back to this restriction in section 3.1 when we derive the corresponding HJB equation and discuss properties of the optimization problem.

Because the process (X_t, W_t) is a Markov process, we may restrict ourselves to the set of strategies which are not path-dependent, that is, it depends just on the actual surplus R_{t-}^{π} where R_t^{π} is given by

$$R_t^{\pi} = s + ct - \int_0^t h(u_x)\,dx + \int_0^t rA_x dx + \int_0^t \sigma A_x dW_x - \sum_{i=1}^{N_t} g(Y_i, u_{T_i-})\,. \tag{2.1}$$

We suppose for feedback functions $\alpha_1, \alpha_2, ..., \alpha_{n+1}$, the set of strategies are given in feedback form:

$$\pi_t = (A_t, u_t) = \left(\alpha_1\left(R_{t-}^{\pi}\right), \alpha_2\left(R_{t-}^{\pi}\right), ..., \alpha_{n+1}\left(R_{t-}^{\pi}\right)\right),\ t \geq 0.$$

Let Γ be the set of all **piecewise left continuous** functions $\alpha : [0, \infty) \to \mathbb{R}$. A function $\alpha : [0, \infty) \to \mathbb{R}$ is said to be piecewise left continuous, if for a finite number of points $0 < s_1 \leq s_2 \leq ... \leq s_m$, the function $\alpha(s)$ is continuous on each subinterval (s_{i-1}, s_i) , and at the endpoints of each subinterval has left and right limits with $\alpha(s-) = \alpha(s)$. For $\alpha_i(s) \in \Gamma$, $i = 1, 2, ..., n+1$, we call $\pi(s) = (\alpha_1(s), \alpha_2(s), ..., \alpha_{n+1}(s))$, admissible if $\pi(s) \in \Pi(s)$, $s \geq 0$.

The constraint sets $\Pi(s)$ must be **time consistent** to allow for predictable strategies. For arbitrary $\pi = (\pi_1, \pi_2, ..., \pi_{n+1}) \in \Pi(s)$, and $s \geq 0$, the family $\Pi(s) = (\mathcal{A}(s), U(s))$, is called time consistent if for any π_i, $i = 1, 2, ..., n+1$, there exists a function $\alpha_i(x) \in \Gamma$, $x \geq 0$, such that $\pi(x) = (\alpha_1(x), \alpha_2(x), ..., \alpha_{n+1}(x)) \in \Pi(x)$, $x \geq 0$, and $\pi(s) = \pi$.

LEMMA 1. *Let $a_i(s)$, $b_i(s) \in \Gamma$, $i = 1, ..., n+1$, with finite values $a_i(s) \leq b_i(s)$, $s \geq 0$, and $a_1(0) = b_1(0) = 0$. Then the set of constraints*

$$\Pi(s) = ([a_1(s), b_1(s)], ..., [a_{n+1}(s), b_{n+1}(s)]),\ s \geq 0,$$

is time consistent.

PROOF. Fix s and choose $\pi = (\pi_1, \pi_2, ..., \pi_{n+1}) \in \Pi(s)$, then we must show that for any π_i, $i = 1, ..., n+1$, there exists $\alpha_i(x) \in \Gamma$ such that $\alpha_i(x) \in [a_i(x), b_i(x)]$, $x \geq 0$, and $\alpha_i(s) = \pi_i$. If $\pi_i = a_i(s)$, then we can choose the function $\alpha_i(x) = a_i(x)$, $x \geq 0$; a similar argument applies for $\pi_i = b_i(s)$.

Now assume that $a_i(s) < \pi_i < b_i(s)$. Left continuity of $a_i(x)$, $b_i(x)$ implies that for some $\varepsilon > 0$ we have $\pi_i \in [a_i(x), b_i(x)]$ for $x \in [s - \varepsilon, s]$. Let $\alpha_i(x) = \pi_i$ for these values x, and let the function $\alpha_i(x)$ jump to $a_i(s - \varepsilon)$ at $s-\varepsilon$. For $x > s$ define $\alpha_i(x) = a_i(x)$.

Then $\alpha_i(x)$ is a piecewise continuous function with $\alpha_i(x) \in [a_i(x), b_i(x)]$, $x \geq 0$, and $\alpha_i(s) = \pi_i$. □

As an example let's assume that at time $t \geq 0$ the insurer can invest A_t in risky asset with constraint set $\mathcal{A}(s) = [0, s]$ (no short-selling and no leverage), as well as proportional reinsurance with dynamic proportion $\alpha_t \in U = [0, 1]$. Furthermore, assume that the reinsurer calculates its premium using the Expected value principle with reinsurance safety loading factor θ. The set of constraint is then $\Pi(s) = [0, s] \times [0, 1]$. The corresponding risk reserve process is then

$$R_t^\pi = s + ct - (1+\theta)\lambda E[Y] \int_0^t (1-\alpha_x)\,dx + \int_0^t rA_x dx + \int_0^t \sigma A_x dW_x - \sum_{i=1}^{N_t} Y_i \alpha_{T_i -}.$$

2.4.2. The objective. The objective of this research is to find an optimal investment policy and reinsurance that minimize the probability of ruin. Indeed, if $\tau^\pi = \inf\{t \geq 0 : R_t^\pi < 0\}$ and $\psi^\pi(s) = \Pr\{\tau^\pi < \infty \mid R_0^\pi = s\}$ denotes the ruin time and ruin probability of risk reserve process (2.1), then our goal is to determine the minimal value $\psi(s) = \inf_{\pi \in \Pi} \psi^\pi(s)$ and the optimal control process $\{\pi_t^*\}$, i.e. the control process leading to the **value function** $\psi(s) = \psi^{\pi^*}(s)$.

It is possible that ruin never occurs and $\tau^\pi = \infty$, i.e. that (2.1) never becomes negative. The probability that this event happens known as **survival probability** and is

$$\delta^\pi(s) = 1 - \psi^\pi(s). \tag{2.2}$$

It is obvious that the problem of minimizing ruin probability $\psi^\pi(s)$ is equivalent to maximizing survival probability $\delta^\pi(s)$. Then our value function is

$$\delta(s) = \sup_{\pi \in \Pi} \delta^\pi(s). \tag{2.3}$$

We solve the optimality problem with help of the dynamic programming principle and the resulting Hamilton–Jacobi–Bellman (HJB) equation. In order to derive HJB equation for the risk process (2.1), we first assume that the value function is twice continuously differentiable. Under this assumption, the HJB equation of our optimization problem is a second-order integro-differential equation. Since we apply dynamic programming principle to our optimization problem and as the risk process (2.1) is a Markov process, we look for optimal investment and reinsurance strategies among markovian strategies, that is, the strategies depend just on the actual surplus and not on the history of the process.

Typically, after showing the existence of solution of this HJB equation, one uses the verification theorem to show that the solution to the HJB equation is the value function of the optimization problem (see for example [**23, 22, 5, 38**]). All this is done under the assumption that the value function (or the solution of the HJB equation) is twice differentiable. However this is not generally true and sometimes one has to rely on a weak (viscosity) solution concept that allows solution and its derivatives to be discontinuous. In the next chapter, we characterize the value function as a viscosity solution of the associated HJB equation.

CHAPTER 3

Stochastic Control

In this chapter we consider an insurance company with initial capital s whose risk model is Cramér-Lundberg process with mean number of claims λ and random claim size Y . At time t this company has two possibilities:

- Take reinsurance with strategy u_t. That is, the cedent pays $0 \leq g(Y, u_t) \leq Y$ and the premium $h(u_t)$ has to be paid by cedent insurer to reinsurer.
- Invest amount A_t into risky asset modeled as a Black–Scholes model with drift $r > 0$ and diffusion $\sigma > 0$.

We denote the set of all possible investment and reinsurance strategies respectively with $\mathcal{A} \in \mathbb{R}$ and $U \subset \mathbb{R}^n$. We then denote the combined set of all admissible investment and reinsurance controls by $\Pi = (\mathcal{A}, U) \subset \mathbb{R}^{n+1}$ and restrict ourselves to the set of strategies which are not path-dependent.

Furthermore, we assume that the insurer is not forced, neither to buy reinsurance, nor to invest in risky asset. For an arbitrary admissible strategy $\pi \in \Pi$ the surplus process R_t^π satisfies the stochastic equation below

$$R_t^\pi = s + ct - \int_0^t h(u_x)\,dx + \int_0^t rA_x dx + \int_0^t \sigma A_x dW_x - \sum_{i=1}^{N_t} g(Y_i, u_{T_i-}). \tag{3.1}$$

The corresponding ruin time is $\tau^\pi = \inf\{t \geq 0 : R_t^\pi < 0\}$, the ultimate ruin probability is $\psi^\pi(s) = \Pr\{\tau^\pi < \infty \mid R_0^\pi = s\}$, and the survival probability is $\delta^\pi(s) = 1 - \psi^\pi(s)$. We maximize $\delta^\pi(s)$ over all admissible strategy $\pi \in \Pi$ and let the value function be defined as $\delta(s) = \sup_{\pi\in\Pi} \delta^\pi(s)$.

An important approach dealing with finding optimal control is based on the dynamic programming principle. This principle relates the survival probability at time t to its expected value at time $t + \theta$ for $\theta > 0$. From the definition of survival probability we can write

$$\delta^\pi(s) = E_s\left[1_{\{\tau^\pi=\infty\}}\right] = E_s\left[1_{\{R_t^\pi \geq 0\,\forall\, t\geq 0\}}\right],$$

where $E_s[.] = E[.|\, R_0^\pi = s]$. For all stopping times θ,

$$E_s\left[1_{\{R_t^\pi \geq 0\,\forall\, t\geq 0\}}\right] = E_s\left[1_{\{R_t^\pi \geq 0\,\forall\, t\in[0,\theta]\}} 1_{\{R_t^\pi \geq 0\,\forall\, t\geq \theta\}}\right]$$

$$= E_s\left[1_{\{\tau^\pi > \theta\}} E\left[1_{\left\{R_{t+\theta}^\pi \geq 0\,\forall\, t\geq 0\right\}} \mid R_\theta^\pi\right]\right].$$

According to the **principle of dynamic programming**

$$\delta(s) = \sup_{\pi\in\Pi} E_s\left[1_{\{\tau^\pi>\theta\}}\delta\left(R_\theta^\pi\right)\right]. \tag{3.2}$$

This principle says that "*an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*" (see [**6**] Chap. III.3.). Applying the dynamic programming principle, we derive the so-called Hamilton-Jacobi-Bellman (HJB) equation in the next section. For the risk process (3.1), the HJB equation is a second-order non-linear integro-differential equation.

If there exists a twice differentiable solution for HJB equation, one can verify that this solution is indeed the value function. This part of the problem is called verification argument and often done by using martingale arguments (see for example [**22, 27, 39**]). However, sometimes, the value function is not twice continuously differentiable to satisfy the HJB equation in a classical sense. In fact, for the general set of admissible strategies Π, we can not even show the existence of a first continuously differentiable solution to the corresponding HJB equation and must use the concept of viscosity solutions. In section (3.2) we characterize $\delta(s)$ as the solution of (2.3) in the sense of viscosity solution. Then in section (3.2) we show that the value function is the unique viscosity solution of the associated HJB equation.

3.1. Hamilton-Jacobi-Bellman equation

The HJB equation can be derived heuristically by considering dynamic programming principle (3.2). Extensive discussion of HJB equation can be found in [**39**]. Let $\Pi = (\mathcal{A}, U)$ be the set of all admissible investment and reinsurance strategies and consider a short time interval $[0,\theta]$, in which a constant strategy $(A,u)\in\Pi$ is used. Then there will be no claim in $[0,\theta]$ with probability $1-\lambda\theta + o(\theta)$ and if this happens, the reserve of the company at time θ is given by

$$R_\theta = s + (c - h(u) + rA)\,\theta + \sigma A W_\theta.$$

If a claim occurs with claim size $Y \sim F_Y$ in the time interval, then the reserve can be written as:

$$R_\theta = s + (c - h(u) + rA)\theta + \sigma A W_\theta - g(Y, u),$$

and this happens with probability $\lambda\theta + o(\theta)$.

Taking expectations and averaging over all possible claim sizes, we arrive at the equation

$$\delta^\pi(s) = (1 - \lambda\theta + o(\theta)) E[\delta^\pi(s + c\theta - h(u)\theta + rA\theta + \sigma A W_\theta)]$$

$$+ (\lambda\theta + o(\theta)) E[\delta^\pi(s + c\theta - h(u)\theta + rA\theta + \sigma A W_\theta - g(Y, u))]. \tag{3.3}$$

The first term on the right-hand side represents the expected survival probability, if there is no claim. The second term gives the expected survival probability, if there is a claim. For $\theta > 0$ we have

$$E\left[\frac{\delta^\pi(s + c\theta - h(u)\theta + rA\theta + \sigma A W_\theta) - \delta^\pi(s)}{\theta}\right]$$
$$-\left(\lambda - \frac{o(\theta)}{\theta}\right) E[\delta^\pi(s + c\theta - h(u)\theta + rA\theta + \sigma A W_\theta)]$$
$$+\left(\lambda + \frac{o(\theta)}{\theta}\right) E[\delta^\pi(s + c\theta - h(u)\theta + rA\theta + \sigma A W_\theta - g(Y, u))] = 0.$$

Let $C_2(0, \infty)$ be the set of continuous functions on $[0, \infty)$, which are twice continuously differentiable on $(0, \infty)$ and assume that $\delta^\pi(s) \in C_2(0, \infty)$. Letting $\theta \to 0$, we obtain with Itô's lemma

$$\frac{1}{2}\sigma^2 A^2 \delta''^\pi(s) + (c - h(u) + rA)\delta'^\pi(s) + \lambda E[\delta^\pi(s - g(Y, u)) - \delta^\pi(s)] = 0.$$

Finally by maximizing over all possible values $\pi \in \Pi$, the Hamilton-Jacobi-Bellman equation for our optimization problem is

$$\sup_{\pi \in \Pi}\left\{\frac{1}{2}\sigma^2 A^2 \delta''(s) + (c - h(u) + rA)\delta'(s) + \lambda E[\delta(s - g(Y, u)) - \delta(s)]\right\} = 0, \tag{3.4}$$

where $\delta(s) = \sup_{\pi \in \Pi} \delta^\pi(s)$. The term in the bracket is related to the **infinitesimal generator**. For smooth enough function $\phi(s)$ and Markov process R_t, the infinitesimal generator $\mathcal{L}$ is defined as the following operator:

$$\mathcal{L}_t\phi(s) = \lim_{\theta \searrow 0} \frac{1}{\theta} E[\phi(R_{t+\theta}) - \phi(s) \mid R_t = s],$$

where the function ϕ must be in the domain of $\mathcal{L}$ for which this limit exists.

For the state dependent claims intensity, the probability that no claim reported in small time interval $[0,\theta]$ is $\lambda(R_\theta)\theta + o(\theta)$ and the probability that one claim occurs in $[0,\theta]$ is $1-\lambda(R_\theta)\theta + o(\theta)$. Replacing these probabilities in (3.3), with the same argument like above the corresponding HJB equation can be written as

$$\sup_{\pi\in\Pi}\left\{\frac{1}{2}\sigma^2 A^2 \delta''(s) + (c - h(u) + rA)\,\delta'(s) + \lambda(s)\,E\left[\delta(s - g(Y,u)) - \delta(s)\right]\right\} = 0.$$

The proof of the following lemma is similar to the proof of Lemma 2.3 in [**39**], p. 35.

LEMMA 2. *The objective function $\delta(s) = \sup_{\pi\in\Pi}\delta^\pi(s)$ is an increasing function of s for all $s \geq 0$.*

PROOF. Let u_0 denote the no-reinsurance strategy and choose two initial capitals $0 \leq x \leq y$. For an arbitrary strategy $\pi = \{\pi_t\}$ let τ_x^π be the ruin time of the risk process (3.1) with initial capital x. Since $\pi_0 = (0, u_0) \in \Pi$, we may define the strategy $\tilde{\pi}_t$ as follow: $\tilde{\pi}_t = \pi_t$ for $t \leq \tau_x^\pi$ and $\tilde{\pi}_t = \pi_0$ for $t > \tau_x^\pi$. If τ_y^π denotes the ruin time for the risk process with initial capital y, then

$$\delta^{\tilde{\pi}}(y) = E\left[1_{\{\tau_y^{\tilde{\pi}}=\infty\}}\right]$$

$$= E\left[1_{\{\tau_y^{\tilde{\pi}}=\infty\}}\middle| 1_{\{\tau_x^\pi=\infty\}}\right] E\left[1_{\{\tau_x^\pi=\infty\}}\right] + E\left[1_{\{\tau_y^{\tilde{\pi}}=\infty\}}\middle| 1_{\{\tau_x^\pi<\infty\}}\right] E\left[1_{\{\tau_x^\pi<\infty\}}\right]$$

$$= \delta^\pi(x) + E\left[1_{\{\tau_y^{\tilde{\pi}}=\infty\}}\middle| 1_{\{\tau_x^\pi<\infty\}}\right](1 - \delta^\pi(x))$$

$$\geq \delta^\pi(x).$$

Since π was chosen arbitrary, by taking suprimum over all possible strategies π, we thus have the desirable result, i.e. $\delta(x) \leq \delta(y)$. □

Since

$$\delta(s) = 1 - \int_s^\infty \delta'(x)\,dx, \tag{3.5}$$

we can maximize survival probability by minimizing $\delta'(s)$ for all $s \geq 0$.

REMARK 2. Assume $c > \lambda E[Y]$ and let $\delta_0(s)$ be the survival probability without investment and reinsurance. Then $\delta_0(s)$ satisfies

$$\delta_0'(s) = \frac{\lambda}{c} E\left[\delta_0(s) - \delta_0(s - Y)\right] \tag{3.6}$$

(see for example [**35, 9**]). Integrating (3.6) over the interval $(0,x]$ yields

$$\frac{c}{\lambda}\left(\delta_0(x)-\delta_0(0)\right)=\int_0^x \delta_0(s)\,ds-\int_0^x\int_0^s \delta_0(s-y)\,dF(y)\,dx$$

$$=\int_0^x \delta_0(x-s)\left(1-F(x)\right)dx. \tag{3.7}$$

Letting $x\to\infty$, (3.7) implies

$$\delta_0(0)=\frac{c-\lambda E[Y]}{c}. \tag{3.8}$$

△

At point $s=0$, $A(0)=0$, otherwise the fluctuation of the Wiener process would lead to immediate ruin, i.e. $\delta(0)=0$, which can not be optimal, since without investment we have $\delta_0(0)=1-{}^{\lambda E[Y]}/_{c}$ if $\lambda E[Y]<c$. Thus, the natural boundary conditions for value function $\delta(s)$ are

$$\delta(s)=0,\ s<0,\ \delta(\infty)=1\ \text{and}\ \delta'(0)=\lambda\delta(0)\inf_{u\in U}\left\{\frac{1-\Pr\left(g(Y,u)=0\right)}{c-h(u)}\right\}. \tag{3.9}$$

Note that for arbitrary constant $\alpha>0$, the function $V(s)=\alpha\delta(s)$ is also a solution to (3.4). Hence, we set $V(0)=1$ and look for a solution of the following equation

$$\sup_{\pi\in\Pi}\left\{\frac{1}{2}\sigma^2A^2V''(s)+\left(c-h(u)+rA\right)V'(s)+\lambda E\left[V\left(s-g(Y,u)\right)-V(s)\right]\right\}=0. \tag{3.10}$$

3.2. Viscosity solution

Generally, not only a twice continuously differentiable solution of (3.4) cannot be always expected but also we do not even know whether the value function is once continuously differentiable. An appropriate notion of solution in this case is that of a viscosity solution which is introduced by Crandall and Lions. In section 4.1, we present examples with jumps in the functions $\delta''(s)$ and $\delta'(s)$. For an extensive discussion about viscosity solution see [**10**] and [**15**]. To define this concept, consider the following second-order non-linear integro-differential equation:

$$\mathcal{H}\left(s,w,w',w''\right)=$$

$$\sup_{\pi\in\Pi}\left\{\frac{1}{2}\sigma^2A^2w''(s)+\left(c-h(u)+rA\right)w'(s)+\lambda E\left[w\left(s-g(Y,u)\right)-w(s)\right]\right\}=0,$$

in which $\mathcal{H}$ is a continuous function of $\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Furthermore, let $C^m[0,\infty)$ be the set of monotone non-decreasing continuous functions satisfying the boundary conditions (3.9).

DEFINITION 3. A function $w(s) \in C^m[0,\infty)$ is said to be a **viscosity subsolution** of (3.4) at $s \in (0,\infty)$ if for any function $\varphi(s) \in C_2(0,\infty)$ with $\varphi(s) = w(s)$ for which $w(x) - \varphi(x)$ reaches the local maximum at s, satisfies

$$\mathcal{H}\left(s, w, \varphi', \varphi''\right) \geq 0, \tag{3.11}$$

and a function $w(s) \in C^m[0,\infty)$ is said to be a **viscosity supersolution** of (3.4) at $s \in (0,\infty)$, if for any function $\varphi(s) \in C_2(0,\infty)$ with $\varphi(s) = w(s)$ for which $w(x) - \varphi(x)$ reaches the local minimum at s, satisfies

$$\mathcal{H}\left(s, w, \varphi', \varphi''\right) \leq 0. \tag{3.12}$$

A **viscosity solution** to (3.4) is a function $w(s) \in C^m[0,\infty)$, if it is both a viscosity subsolution and a viscosity supersolution at any $s \in (0,\infty)$.

Proposition (4) gives an equivalent formulation for viscosity solutions which is needed to prove the theorem (6) later.

PROPOSITION 4. *Let $w(s) \in C^m[0,\infty)$, then*

i) w is a viscosity subsolution of (3.4) at $s \in (0,\infty)$, if and only if

$$\mathcal{H}\left(s, \varphi, \varphi', \varphi''\right) \geq 0,$$

whenever $\varphi(s) \in C_2(0,\infty)$ with $\varphi(s) = w(s)$ and $w(x) \leq \varphi(x)$, for $x > 0$.

ii) w is a viscosity subsolution of (3.4) at $s \in (0,\infty)$, if and only if

$$\mathcal{H}\left(s, \varphi, \varphi', \varphi''\right) \leq 0,$$

whenever $\varphi(s) \in C_2(0,\infty)$ with $\varphi(s) = w(s)$ and $w(x) \geq \varphi(x)$, for $x > 0$.

For the proof that the value function $\delta(s)$ is a viscosity solution of (3.4), we need the following lemma.

LEMMA 5. *Let $\pi = (A, u)$, an admissible strategy where $A = \alpha(s)$, $\alpha(s) \in \Gamma$,* with *$\alpha(0) = 0$. For start capital $s > 0$, denote R_t^π, $t \geq 0$ the corresponding risk process with* *investment A and reinsurance u and let τ be the ruin time of R_t^π. Then for $0<\theta\to 0$*

$$\Pr\{\tau \le \theta \,\&\, R_\theta^\pi \ge 0\} = o(\theta). \tag{3.13}$$

PROOF. First we consider that there is no claim in $[0,\theta]$. The risk process without claim is

$$R_\theta^\pi = s + c\theta - \int_0^\theta h(u_x)\,dx + \int_0^\theta rA_x dx + \int_0^\theta \sigma A_x dW_x.$$

Since the dynamics

$$dM_t := (c - h(u_t) + rA_t)\,dt,\; M_0 = 0,$$

has bounded variation we can just look at the process $N_t := s - R_t^\pi + M_t$. Note that the process N_t starts at zero and is a martingale. The function $\alpha(x)$ is bounded with $\alpha(0)=0$. From martingale inequality 7.31 in [**30**], p. 200, we have

$$\Pr\left\{\max_{0\le t\le\theta} |N_t| > s\right\} \le s^2 E\left[\int_0^\theta \alpha^2(N_t)\,dt\right]$$

which has order $o(\theta)$.

Next, consider that a single claim Y at time $\tau\in[0,\theta]$ occurs and let D be the set on which this is true. We need to study the case where this single claim causes ruin at time τ. If $R_{\tau-}^\pi$ denotes the insurance capital before ruin, then we have $R_{\tau-}^\pi < g(Y, u_{\tau-})$. For $s\le 0$ we let $A(s)=0$ and define

$$t_1 = \inf\{\tau < t < \theta :\, R_t^\pi = 0\}.$$

Therefore, on $t\in[\tau,t_1]$ we have no investment and the process R_t^π can grow at most by premium income rate c. Hence at t_1, $R_{\tau-}^\pi - g(Y,u_{\tau-}) + c(t_1-\tau) \ge 0$ and for $\tau<\theta$ we obtain $R_\theta^\pi \ge 0$ if

$$R_{\tau-}^\pi < g(Y,u_{\tau-}) \le R_{\tau-}^\pi + c\theta.$$

Letting $\theta\to 0$ we get $\Pr\left(R_{\tau-}^\pi < g(Y,u_{\tau-}) \le R_{\tau-}^\pi + c\theta\right)\to 0$. Since a single claim occurs with probability $\lambda\theta + o(\theta)$, we have

$$\Pr\{\tau\le\theta \,\&\, R_\theta^\pi \ge 0 \,\&\, D\} \le \Pr\left\{R_{\tau-}^\pi < g(Y,u_{\tau-}) \le R_{\tau-}^\pi + c\theta \,\&\, D\right\}$$

$$\le (\lambda\theta + o(\theta))\Pr\left(R_{\tau-}^\pi < g(Y,u_{\tau-}) \le R_{\tau-}^\pi + c\theta\right) = o(\theta).$$

For more than one claim in $[0,\theta]$, we obtain a set with probability $o(\theta)$ and therefore we come to the desired result (3.13). □

In the following theorem 6 on the viscosity subsolution part of the proof, due to technical difficulties, we need to assume that the set of constraints is of the form

$$\Pi(s) = ([a_1(s), b_1(s)], ..., [a_{n+1}(s), b_{n+1}(s)]), \; s \geq 0,$$

where $a_i(s)$ and $b_i(s)$, $i = 1, ..., n+1$ are continuous functions on $s \in (0, \infty)$. We also assume that the reinsurance premium $h(u)$ is a continuous function. Throughout the proof of this Theorem, for $s > 0$ and functions $w(s) \in C^m[0, \infty)$ and $\varphi(s) \in C_2[0, \infty)$, and $(A, u) \in \Pi(s)$, we use the following notation:

$$\begin{aligned} &K(s, \varphi, A, u) \\ &= \frac{1}{2}\sigma^2 A^2 \varphi''(s) + (c - h(u) + rA)\varphi'(s) + \lambda \int_0^\infty \varphi(s - g(Y, u))\, dF_Y - \lambda\varphi(s). \end{aligned}$$

THEOREM 6. *Let $a_i(s)$ and $b_i(s)$, $i = 1, ..., n+1$ be continuous functions on $s \in (0, \infty)$ and $\Pi(s) = (\mathcal{A}(s), U(s)) = ([a_1(s), b_1(s)], ..., [a_{n+1}(s), b_{n+1}(s)])$ be the constraint set of investment and reinsurance. Furthermore assume that the reinsurance premium $h(u)$ is a continuous function of $u \in U(s)$. Then the value function $\delta(s) = \sup_{\pi \in \Pi} \delta^\pi(s)$, is a viscosity solution of (3.4).*

PROOF. Let us first show that $\delta(s)$ is a viscosity supersolution of (3.4). Let $\varphi : (0, \infty) \to \mathbb{R}$ be any twice continuously differentiable function with $\varphi(s) = \delta(s)$ such that $\delta - \varphi$ reaches the minimum at s. We need to show that

$$\mathcal{H}\left(s, \varphi, \varphi', \varphi''\right) = \sup_{\pi \in \Pi(s)} \{K(s, \varphi, A, u)\} \leq 0. \tag{3.14}$$

We use the same argument in [**12**]. Choose an arbitrary admissible strategy $\pi := \pi_t = (A_t, u_t) \in \Pi(R_t^\pi)$, with the risk process R_t^π and let $\tau = \inf\{t \geq 0 : R_t^\pi < 0\}$. For stopping time θ, the dynamic programming principle (3.2), together with Lemma (5), yields

$$\begin{aligned} \delta(s) &\geq E_s\left[1_{\{\tau > \theta\}} \varphi(R_\theta^\pi)\right] \\ &\geq E_s[\varphi(R_\theta^\pi)] - \Pr\{\tau \leq \theta \,\&\, R_\theta^\pi \geq 0\} \\ &\geq \varphi(s) + \theta K(s, \varphi, A_0, u_0) + o(\theta). \end{aligned} \tag{3.15}$$

The last inequality holds because $K(s, \varphi, A_t, u_t)$ is the infinitesimal generator of the stochastic process R_t^π. Because $\delta(s) = \varphi(s)$, dividing the (3.15) by θ and letting $\theta \to 0$, we obtain

$$K(s,\varphi,A_0,u_0)\le 0.$$

Since $\pi\in\Pi(R_t^\pi)$ was arbitrary , we conclude the desired inequality (3.14).

We now show that $\delta(s)$ is a viscosity subsolution of (3.4). To this end, let $\varphi_0\in C_2(0,\infty)$ be a test function such that

$$0=(\delta-\varphi_0)(s)=\max_x(\delta-\varphi_0)(x).$$

Similar to the contradiction argument used in the proof of proposition 4.3.2 in [**34**], assume $\mathcal{H}\left(s,\varphi_0,\varphi_0^{'},\varphi_0^{''}\right)<0$. That is for some $\varepsilon>0$

$$K(x,\varphi_0,A,u)<-\varepsilon,$$

for all $(A,u)\in\Pi(s)$. We may replace $\varphi_0(x)$ by a function $\varphi(x)\in C_2(0,\infty)$ satisfying $\varphi(x)\ge\delta(x)$, $\varphi(s)=\delta(s)$, $\varphi^{'}(s)=\varphi_0^{'}(s)$, $\varphi_0^{''}(s)=\varphi_0^{''}(s)$ and

$$|E[\delta(s-g(Y,u))-\varphi(s-g(Y,u))]|<\frac{\varepsilon}{2}.$$

Therefore $K(x,\varphi,A,u)<-\frac{\varepsilon}{2}$ for all $(A,u)\in\Pi(s)$. Since the function $\mathcal{H}$, is a continuous function of s, A and u, and since $a_i(s)$ and $b_i(s)$, $i=1,...,n+1$ are continuous for some $\eta>0$ we have

$$K(x,\varphi,A,u)<-\frac{\varepsilon}{2}\,\text{for all}\,s-\eta<x<s+\eta\,\text{and all }\ (A,u)\in\Pi(s).$$

Let $0<\gamma_m\to 0$ and $\pi_m(x)=(A_m(x),u_m(x))$ be feedback function such that the strategy $\pi_{mt}=\pi\left(R_{t-}^\pi\right)$ is admissible and satisfies

$$\Pr\{\tau^{\pi_m}=\infty|\,R_0^{\pi_m}=s\}\ge\delta(s)-\frac{\varepsilon\gamma_m}{4},$$

where τ^{π_m} is the ruin time of $R_t^{\pi_m}$ with $R_0^{\pi_m}=s$. Then for any stopping time $\theta\le\tau^{\pi_m}$ we have

$$\begin{aligned}\Pr\{\tau^{\pi_m}=\infty|\,R_0^{\pi_m}=s\}&=E\left[\Pr\{\tau^{\pi_m}=\infty|\,R_\theta^{\pi_m}\}\right]\\&\le E\left[\delta\left(R_\theta^{\pi_m}\right)\right]\le E\left[\varphi\left(R_\theta^{\pi_m}\right)\right].\end{aligned}$$

Let τ_m be the first exit time from $[s-\eta,s+\eta]$ and $\theta_m=\min(\tau_m,\gamma_m)$. Notice that for sufficiently small η we have $\tau_m\le\tau^{\pi_m}$ which yields $\theta_m\le\tau^{\pi_m}$. Thus

$$\varphi(s)=\delta(s)\le\Pr\{\tau^{\pi_m}=\infty|\,R_0^{\pi_m}=s\}+\frac{\varepsilon\gamma_m}{4}\le$$

$$E\left[\varphi\left(R_{\theta_m}^{\pi_m}\right)\right]+\frac{\varepsilon\gamma_m}{4}=\varphi\left(s\right)+E\left[\int_0^{\theta_m}K\left(R_t^{\pi_m},\varphi,A_{mt},u_{mt}\right)dt\right]+\frac{\varepsilon\gamma_m}{4}$$
$$\leq\varphi\left(s\right)+\frac{\varepsilon\gamma_m}{4}-\frac{\varepsilon E\left[\theta_m\right]}{2}.$$

Since the probability of one or more claims in $[0,\gamma_m]$ goes to zero for $m\to\infty$, and the without claims process is continuous, we have $\Pr\left\{\tau_m\leq\gamma_m\right\}\to0$ for $m\to\infty$. On the othe hand, Tchebyshev's inequality gives us

$$\Pr\left\{\tau_m\geq\gamma_m\right\}\leq\frac{1}{\gamma_m}E\left[\theta_m\right]\leq1.$$

By letting m goes to infinity we come to the contradiction

$$0\leq\varepsilon\left(\frac{1}{4}-\frac{1}{2}\right).$$

□

We have shown, that $\delta\left(s\right)$ is a viscosity solution to (3.4). Next we are going to characterize the optimal survival probability as the unique viscosity solution of the HJB equation (3.4). The uniqueness of the value function can be derived through a comparison principle, which we present in the next proposition. The proof technique used here is as the same as that developed in [**10**]. Throughout the next Proposition, for two symmetric matrices $M,N\in\mathbb{R}^{n\times n}$, we write $M\geq N$ if for any vector $z\in\mathbb{R}^n$, $z^tMz\geq z^tNz$.

Proposition 7. *Let $v\left(s\right)$, $w\left(s\right)$ be continuous, uniformly Lipschitz, increasing functions, $w\left(s\right)$ a super- and $v\left(s\right)$ a subsolution to (3.4). Assume that the constraint set, $\Pi\left(s\right)=\left(\mathcal{A}\left(s\right),U\left(s\right)\right)$, satisfies*

$$\Pi\left(x\right)\subset\Pi\left(y\right),\ \textit{for } x<y. \tag{3.16}$$

Moreover, assume that for all reinsurance strategy $u\in U\left(s\right)$ either $\Pr\left\{g\left(Y,u\right)<s\right\}<1$ for all s, or $g\left(Y,u\right)$ has a positive density on an interval (a,b).

If it holds $v\left(0\right)\leq w\left(0\right)$ and $v\left(\infty\right)\leq w\left(\infty\right)$, then $v\left(s\right)\leq w\left(s\right)$ on $[0,\infty)$.

Proof. We use contradiction argument. Assume there is $s_0\in(0,\infty)$ such that $v\left(s_0\right)-w\left(s_0\right)>0$. Let m be a common Lipschitz constant for $v\left(s\right)$ and $w\left(s\right)$. For $k>1$ the function

$$w_k\left(s\right)=kw\left(s\right),\ s\geq0,$$

is also a supersolution which is also increasing and uniformly Lipschitz with constant km. Fix $k > 1$ such that $v(s_0) - w_k(s_0) > 0$. Because v and w_k are Lipschitz continuous, there exists an interval $D := [\varepsilon, L]$, $\varepsilon, L > 0$ and $\delta > 0$ such that for $s \notin D$,

$$v(s) - w_k(s) \leq -\delta.$$

Define furthermore

$$M := \sup_{s \geq 0} (v(s) - w_k(s)),$$

which is positive. Continuity of $v(s)$, $w_k(s)$ and $v(\infty) - w_k(\infty) \leq 0$ imply that for some $0 < s^* < \infty$ we have $M = v(s^*) - w_k(s^*)$.

For $\xi > 0$ and $x, y > 0$, define

$$f_\xi(x,y) := v(x) - w_k(y) - \frac{\xi}{2}(y-x)^2 + mk\frac{y-x}{\xi(y-x)^2+1}.$$

Because f_ξ is continuous, there exists $(x_\xi, y_\xi) \geq 0$ maximizing $f_\xi(x,y)$. It is easy to see that

$$\frac{|x-y|}{\xi(x-y)^2+1} \leq \xi^{-1/2}.$$

Therefore, we obtain

$$M \leq f_\xi(x_\xi, y_\xi) \leq M - \frac{\xi}{2}(x_\xi - y_\xi)^2 + \xi^{-1/2},$$

and so $\xi(x_\xi - y_\xi)^2 \to 0$, as $\xi \to \infty$. Defining

$$\phi(x,y) = \frac{\xi}{2}(x-y)^2 - mk\frac{y-x}{\xi(y-x)^2+1},$$

we have

$$f_\xi(x,y) = v(x) - w_k(y) - \phi(x,y).$$

Note that

$$\phi_x(x,y) := \frac{\partial}{\partial x}\phi(x,y) = \xi(x-y) + \frac{mk}{\xi(x-y)^2+1} - 2mk\frac{\xi(x-y)^2}{\left(\xi(x-y)^2+1\right)^2},$$

and $\phi_x(x,y) = -\phi_y(x,y)$. Thus for large ξ we have

$$m \geq \limsup_{\theta \searrow 0} \frac{v(x_\xi) - v(x_\xi - \theta)}{\theta} \geq \phi_x(x_\xi, y_\xi) \geq \xi(x_\xi - y_\xi) + m,$$

which yields $x_\xi \le y_\xi$ and $\phi_x(x_\xi, y_\xi) < 0$. Furthermore, $y_\xi \le L$ and $x_\xi \ge \varepsilon/2$.

We also have $\phi_{xx}(x_\xi, y_\xi) := \frac{\partial}{\partial x \partial x}\phi(x,y) = \phi_{yy}(x_\xi, y_\xi) =: Q_\xi$, where

$$Q_\xi = \xi\left[1 + \frac{6mk(x_\xi - y_\xi)}{\left(\xi(x_\xi - y_\xi)^2 + 1\right)^2} - \frac{8mk\xi(x_\xi - y_\xi)^3}{\left(\xi(x_\xi - y_\xi)^2 + 1\right)^3}\right].$$

The bracket converges to 1 for $\xi \to \infty$, hence for sufficiently large ξ we have $0 \le Q_\xi \le 2\xi$. Since $\phi_{xy}(x_\xi, y_\xi) = -\phi_{xx}(x_\xi, y_\xi)$, the Hessian matrix $\mathcal{Q}$ of $\phi(x,y)$ is

$$\mathcal{Q} = Q_\xi \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

which satisfies $\mathcal{Q}^2 = 2Q_\xi \mathcal{Q}$.

For the continuous function $f(x)$, the set $\mathcal{J}^+ f(x_0)$ is called the superjet of the function $f(x)$ at point x_0, if for all numbers $(p,d) \in \mathcal{J}^+ f(x_0)$,

$$f(x) \le f(x_0) + p(x - x_0) + \frac{1}{2}d(x - x_0)^2 + o(x - x_0).$$

Similarly, the set $\mathcal{J}^- f(x_0)$ is called the subjet of the function $f(x)$ at point x_0, if for all numbers $(p,d) \in \mathcal{J}^- f(x_0)$,

$$f(x) \ge f(x_0) + p(x - x_0) + \frac{1}{2}d(x - x_0)^2 + o(x - x_0).$$

If $\mathcal{J}^+ f(x_0) \cap \mathcal{J}^- f(x_0) = \emptyset$, then $f'(x_0)$ and $f''(x_0)$ exist and

$$\mathcal{J}^+ f(x_0) \cap \mathcal{J}^- f(x_0) = \left\{\left(f'(x_0), f''(x_0)\right)\right\}.$$

From Crandall and Ishii maximum principle there exists d_1, d_2 in the closure of $\mathcal{J}^+ v(x_\xi)$ and $\mathcal{J}^- w_k(y_\xi)$, respectively, such that

$$\begin{pmatrix} d_1 & 0 \\ 0 & -d_2 \end{pmatrix} \le 3Q_\xi \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{3.17}$$

Hence we have $d_1 \le d_2$.

The viscosity sub- and super-solution properties imply that for $(\phi_x(x_\xi, y_\xi), d_1) \in \mathcal{J}^+ v(x_\xi)$ and $(-\phi_y(x_\xi, y_\xi), d_2) \in \mathcal{J}^- w_k(y_\xi)$, there exists $\left(A_{x_\xi}, u_{x_\xi}\right) \in \Pi(x_\xi)$ with

(3.18)

$$\lambda E\left[v\left(x_\xi - g\left(Y, u_{x_\xi}\right)\right) - v(x_\xi)\right] + \left(c - h\left(u_{x_\xi}\right) + rA_{x_\xi}\right)\phi_x(x_\xi, y_\xi) + \frac{1}{2}d_1\sigma^2 A_{x_\xi}^2 \ge 0,$$

and for all $\left(A_{y_\xi}, u_{y_\xi}\right) \in \Pi\left(y_\xi\right)$ we have

(3.19)

$$\lambda E\left[w_k\left(y_\xi - g\left(Y, u_{y_\xi}\right)\right) - w_k\left(y_\xi\right)\right] + \left(c - h\left(u_{y_\xi}\right) + rA_{y_\xi}\right)\phi_x\left(x_\xi, y_\xi\right) + \frac{1}{2}d_2\sigma^2 A_{y_\xi}^2 \leq 0.$$

Since $x_\xi \leq y_\xi$, we have $\Pi\left(x_\xi\right) \subset \Pi\left(y_\xi\right)$ and so we can choose $\left(A_{y_\xi}, u_{y_\xi}\right) = \left(A_{x_\xi}, u_{x_\xi}\right)$. The assumption $g\left(Y, u_{x_\xi}\right) > 0$, together with the inequalities (3.18) and (3.19), yield

$$E\left[v\left(x_\xi - g\left(Y, u_{x_\xi}\right)\right) - v\left(x_\xi\right)\right] - E\left[w_k\left(y_\xi - g\left(Y, u_{x_\xi}\right)\right) - w_k\left(y_\xi\right)\right] \geq 0.$$

For a sequence $\xi_n \to \infty$ as $n \to \infty$, since $\xi_n\left(x_{\xi_n} - y_{\xi_n}\right)^2 \to 0$, we get that $\left|x_{\xi_n} - y_{\xi_n}\right| \to 0$, that is for some $\bar{x} > 0$, $\left(x_{\xi_n} - y_{\xi_n}\right) \to \left(\bar{x}, \bar{x}\right)$. Choose a sequence $\xi_n \to \infty$, such that $u_{x_{\xi_n}}$ converges to some $\bar{u} \in U\left(\bar{x}\right)$. Then with

$$M \geq E\left[v\left(\bar{x} - g\left(Y, \bar{u}\right)\right) - w_k\left(\bar{x} - g\left(Y, \bar{u}\right)\right)\right] \geq v\left(\bar{x}\right) - w_k\left(\bar{x}\right) = M$$

we have a contradiction in the case $\Pr\left\{g\left(Y, \bar{u}\right) \leq \bar{x}\right\} < 1$. In case $\Pr\left\{g\left(Y, \bar{u}\right) \leq \bar{x}\right\} = 1$, since $g\left(Y, \bar{u}\right)$ has a positive density on an interval (a, b), we can find some intervals $\left(z_1, z_2\right) \subset \left(0, \bar{x}\right)$ with positive length on which the $v\left(x\right) - w_k\left(x\right)$ is constant and equal to M. Since the set of possible values for $k > 1$ is uncountable, these intervals cannot be disjoint. Thus, there exists a non void interval $\left(s_1, s_2\right)$ and constants $k_1 < k_2$, such that $v\left(x\right) - w_{k_i}\left(x\right)$ is constant and equal to some constant $M_i > 0$ on $\left(s_1, s_2\right)$ for $i = 1, 2$. Therefore, $v\left(s_1\right) = v\left(s_2\right)$ and $w\left(s_1\right) = w\left(s_2\right)$, which contradict the fact that the $v\left(x\right)$ and $w\left(x\right)$ are increasing functions. □

All examples, presented in the next chapter, satisfy the constraints condition 3.16, except example 19. For the case of example 19, one can repeat the above argument with

$$\phi\left(x, y\right) = \frac{\xi}{2}\left(x - y\right)^2 - mk\frac{x - y}{\xi\left(y - x\right)^2 + 1},$$

for which the maximizers satisfy $x_\xi \geq y_\xi$.

In the next section we will present a stable recursive numerical method which can solve the HJB equation (3.10) in general sense, i.e. it works even when there is no smooth solution to the equation (3.4).

3.3. Existence of solution and numerical algorithm

Different operators have been proposed to prove the existence of solutions of integro-differential equation (3.10). See for example [**3, 27, 23, 22, 26, 36, 38**]. In this section we shall present a numerical method which can be used to solve the problem of optimal reinsurance and investment with or without constrained.

In order to obtain recursive numerical algorithm we discretize the state space with some small step size Δ and recursively define a family of function $V(s) = V_\Delta(s)$, starting with

$$V(0) = 1 \text{ and } V^{'}(0) = \inf_u \left\{ \lambda \frac{1 - \Pr(g(y,u) = 0)}{c - h(u)} \right\}. \tag{3.20}$$

For $s = i\Delta > 0$, we use the approximations and notations

$$V_\Delta(s) = V_\Delta(s - \Delta) + \Delta V^{'}_\Delta(s), \tag{3.21}$$

$$V^{'}_\Delta(s) = \frac{V_\Delta(s) - V_\Delta(s - \Delta)}{\Delta}, \tag{3.22}$$

$$V^{''}_\Delta(s) = \frac{V^{'}(s) - V^{'}(s - \Delta)}{\Delta}. \tag{3.23}$$

We approximate $E[V(s - g(Y,u))]$ by

$$G_{\Delta,u}(s)$$

$$= \sum_{\{g(j\Delta,u) \leq (i-1)\Delta\}} V_\Delta\left((i-1)\Delta - \left\lfloor \frac{g(j\Delta,u)}{\Delta} \right\rfloor \Delta\right) \Pr\{(j-1)\Delta < Y \leq j\Delta\},$$

where $j = 0, 1, 2, ...$ and $\lfloor . \rfloor$ maps a real number to the largest previous integer.

Starting from the initial values for $V_\Delta(0)$ and $V^{'}_\Delta(0)$, we define for $s = i\Delta$ the functions $V_\Delta(s)$, $i = 1, 2, ...$, by

$$V^{'}_\Delta(s) = \inf_{\pi \in \Pi} \frac{\lambda\Delta(V_\Delta(s - \Delta) - G_u(s)) + \frac{1}{2}\sigma^2 A^2 V^{'}_\Delta(s - \Delta)}{\Delta(c - h(u) + rA - \lambda\Delta) + \frac{1}{2}\sigma^2 A^2}. \tag{3.24}$$

This recursion is equivalent to the following equation

$$V^{''}_\Delta(s) = \inf_{\pi \in \Pi} \frac{\lambda(V_\Delta(s - \Delta) - G_u(s)) - (c - h(u) + rA - \lambda\Delta) V^{'}_\Delta(s - \Delta)}{\Delta(c - h(u) + rA - \lambda\Delta) + \frac{1}{2}\sigma^2 A^2}. \tag{3.25}$$

This implies in particular that the minimizer in (3.24) is equal to the minimizer in (3.25). Since we are interested to the positive values of (3.24), we can restrict our admissible strategy to the set $\bar{\Pi}_\Delta = \left\{(A,u) \in \Pi : \Delta\left(c - h(u) + A - \lambda\Delta\right) + \frac{1}{2}A^2 > 0\right\}$. It is obvious that $(0, u_0) \in \bar{\Pi}_\Delta$. The function $G_{\Delta,u}(s)$ takes its maximum $V_\Delta(s-\Delta)$, when $g(Y,u) = 0$, i.e. full reinsurance.

For notational convenience we denote the discretizations again by $V(s)$ and $G_u(s)$ whenever this causes no confusion.

Remark 3. Let $\delta_0(s)$ be the survival probability without investment and reinsurance. Then the function

$$V(s) = \frac{c}{c - \lambda E[Y]}\delta_0(s),$$

satisfies $V(0) = 1$, $V'(0) = \lambda/c$. Using the approximation method in (3.22), we have

$$V'_\Delta(s) = \frac{\lambda}{c}\left(V_\Delta(s-\Delta) - G_{\Delta,u_0}(s)\right),$$

where u_0 denotes the no-reinsurance strategy, i.e. $g(Y,u_0) = Y$. Note that from (3.21), for $s = i\Delta$, we have

$$V_\Delta(s-\Delta) = V_\Delta(0) + \Delta\sum_{j=1}^{i} V'_\Delta(j\Delta),$$

and

$$G_{u_0}(s) = \sum_{j=1}^{i}\left(V(0) + \Delta\sum_{k=1}^{i-k} V'(k\Delta)\right)\Pr\left\{(j-1)\Delta < Y \leq j\Delta\right\}.$$

We obtain then

$$V(s-\Delta) - G_{u_0}(s) = V(0)\Pr\left\{Y \geq j\Delta\right\} + \Delta\sum_{j=1}^{i-1} V'(j\Delta)\Pr\left\{Y \geq (i-j)\Delta\right\}. \tag{3.26}$$

$\triangle$

Lemma 8. *Let* $0 < \Delta < c/(\lambda+1/2)$ *be arbitrary and let* $D = \{0, \Delta, 2\Delta, ...\}$. *Define* $V(s)$, $s \in D$ *with* $V(0) = 1$, $V'(0) = \lambda/c$ *together with recursion (3.24), for* $0 < s \in D$. *Then* $V'(s) \geq 0$, *for all* $s \in D$.

PROOF. Assume that i is a positive integer with $V^{'}(k\Delta) \geq 0$, $k = 1, ..., i-1$. Then the numbers $V(0) \leq V(\Delta) \leq ... \leq V((i-1)\Delta)$ are non-decreasing and thus

$$G_u(s) = \sum_{\{g(j\Delta,u) \leq (i-1)\Delta\}} V_\Delta \left(i\Delta - \left\lfloor \frac{g(j\Delta, u)}{\Delta} \right\rfloor \Delta \right) \Pr\{(j-1)\Delta < Y \leq j\Delta\}$$

$$\leq V_\Delta((i-1)\Delta), \quad j = 0, 1, 2, ...$$

So for $s = i\Delta$ the numerator of

$$\frac{\lambda\Delta\left(V_\Delta(s-\Delta) - G_u(s)\right) + \frac{1}{2}\sigma^2 A^2 V_\Delta^{'}(s-\Delta)}{\Delta\left(c - h(u) + rA - \lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2} \tag{3.27}$$

is nonnegative for all A. Since the denominator of (3.22) is positive for all A, the infimum of (3.22) over $\pi \in \bar{\Pi}$ must be non-negative, so $V^{'}(s) \geq 0$. □

LEMMA 9. *Assume* $0 < \Delta < c/(\lambda + 1/2)$ *and that for all* $s \in D$ *there exist a strategy* $(A, u) \in \Pi$ *with* $A \geq 0$ *and* $g(Y, u_0) = Y$. *Then for all* $k \geq 0$

$$V(k\Delta) \leq \left(1 - \frac{\lambda}{c}\Delta\right)^{-k} \leq e^{\frac{\lambda}{c}k\Delta}, \tag{3.28}$$

and

$$V^{'}(k\Delta) \leq \frac{\lambda}{c} V(k\Delta). \tag{3.29}$$

PROOF. For $k = 0$ the two assertions holds. Assume now that $s = k\Delta > 0$ and the assertion (3.29) is true for $s - \Delta$. If $V^{''}(s) \leq 0$, then

$$V^{'}(s) \leq V^{'}(s-\Delta) \leq \frac{\lambda}{c} V(s-\Delta) \leq \frac{\lambda}{c} V(s).$$

If $V^{''}(s) \geq 0$, then for $0 \leq A$ and no-reinsurance strategy u_0, we obtain from (3.4)

$$0 \geq (c + A) V^{'}(s) + \lambda E\left[V(s-Y) - V(s)\right]$$

$$\geq -\lambda V(s) + cV^{'}(s)$$

which gives us (3.29), for s. This implies

$$V(s) \leq \left(1 - \frac{\lambda}{c}\Delta\right)^{-1} V(s-\Delta),$$

and thus we obtain (3.28) for s from (3.29) for $s - \Delta$. □

Beside inequality (3.29) we can show that $V^{'}(s) \leq V_0^{'}(s)$ where $V_0^{'}(s)$ is the result of our recursion for admissible strategy restricted to no-reinsurance without investment, i.e. $\Pi_0(s_0) = (0, u_0)$. To this end let $\Pi(s_0) \supset \Pi_0(s_0)$ and consider their corresponding schemes $V_0(s)$, $V(s)$, $s \in D$, with a common $\Delta > 0$ and norming $V_0(0) = V(0) = 1$. Set $V_0^{'}(0) = \frac{\lambda}{c}$, and

$$V^{'}(0) = \inf_{u \in \Pi(0)} \left\{ \lambda \frac{1 - \Pr(g(Y,u) = 0)}{c - h(u)} \right\}.$$

We can show by induction that for $s \in D$, $V_0^{'}(s) \geq V^{'}(s)$. It is clear that $V_0^{'}(0) \geq V^{'}(0)$. Assume that $s > 0$ is such that for all $s_0 \in D$, $s_0 \leq s - \Delta$,

$$V_0^{'}(s_0) \geq V^{'}(s_0).$$

Let u_0 denotes the no-reinsurance strategy, then from (3.22) and (3.26) we have

$$V^{'}(s) = \inf_{\pi \in \bar{\Pi}} \frac{\lambda\Delta\left(V(s-\Delta) - G_u(s)\right) + \frac{1}{2}\sigma^2 A^2 V^{'}(s-\Delta)}{\Delta\left(c - h(u) + rA - \lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2}. \tag{3.30}$$

$$\leq \frac{\lambda\left(V(s-\Delta) - G_{u_0}(s)\right)}{c}$$

$$= \frac{\lambda}{c}\left(V(0)\Pr\{Y \geq j\Delta\} + \Delta\sum_{j=1}^{i-1} V^{'}(j\Delta)\Pr\{Y \geq (i-j)\Delta\}\right)$$

$$\leq \frac{\lambda}{c}\left(V_0(0)\Pr\{Y \geq j\Delta\} + \Delta\sum_{j=1}^{i-1} V_0^{'}(j\Delta)\Pr\{Y \geq (i-j)\Delta\}\right) \leq V_0^{'}(s),$$

which completes the induction.

In the next Theorem we use the same argument as in [**15**], Chapter IX, section 4, to show that the function $V(s)$ is the unique viscosity solution to (3.10). To this end, however, we need to assume that $\Pi(s) \subset \Pi_+$ where

$$\Pi_+ = \{(A,u) : c - h(u) + A \geq 0\}.$$

THEOREM 10. *Let $V_\Delta(s)$ be the solution to (3.24) and define*

$$V^*(s) = \limsup_{\Delta \to 0, i\Delta \to s} V_\Delta(i\Delta),$$

and

$$V_*(s) = \liminf_{\Delta \to 0, i\Delta \to s} V_\Delta(i\Delta).$$

If $\Pi \subset \Pi_+$, *then the functions* $V^*(s)$ *and* $V_*(s)$ *are respectively, sub- and supersolution of (3.10).*

Moreover, if $V^*(\infty) \leq V_*(\infty)$ *or if* $V^*(s) - V_*(s)$ *has a local maximum* $s \in (0,\infty)$ *where* $V^*(s) > V_*(s)$, *then the sequence* $V_\Delta(s)$ *converges to the unique viscosity solution* V *of (3.10) which is continuous on* $[0,\infty)$.

Proof. We start by showing that $V^*(s)$ is a viscosity subsolution of (3.10), while $V_*(s)$ is a viscosity supersolution of (3.10). We only prove the subsolution case $V^*(s)$. The proof for the supersolution case is analogous. Let $s > 0$ and $\varphi(s) \in C_2(0,\infty)$ with $\varphi(s) = V^*(s)$ for which $V^*(x) - \varphi(x)$ has strict local maximum at $x = s$. We want to show that

$$\mathcal{H}\left(s, V, \varphi', \varphi''\right) \geq 0. \tag{3.31}$$

To show this, note that for Δ sufficiently small we can find $\Delta < s_\Delta \in D_\Delta$ such that

$$V_\Delta(s_\Delta) - \varphi_\Delta(s_\Delta) \geq V_\Delta(x) - \varphi_\Delta(x), \; x \in \{s_\Delta - 2\Delta, s_\Delta\}.$$

This implies that

$$V_\Delta'(s_\Delta) \leq \varphi_\Delta'(s_\Delta), \; V_\Delta''(s_\Delta) \leq \varphi_\Delta''(s_\Delta).$$

Take a sequence $\Delta_n \to 0$ as $n \to \infty$ for which $V_{\Delta_n}(s_{\Delta_n}) \to V^*(s)$. Then by Fatou's lemma

$$\limsup_{n\to\infty} G_{\Delta_n,u}(s_{\Delta_n}) \leq E\left[V^*(s - g(Y,u))\right], \; \lim_{n\to\infty} V_{\Delta_n}(s_{\Delta_n}) = V^*(s),$$

$$\limsup_{n\to\infty} V_{\Delta_n}'(s_{\Delta_n}) \leq \varphi'(s), \text{ and} \limsup_{n\to\infty} V_{\Delta_n}''(s_{\Delta_n}) \leq \varphi''(s),$$

which imply (3.31). Now from the comparison results Proposition (7) if $V^*(\infty) \leq V_*(\infty)$, we have that $V^* \leq V_*$. Since $V^* \geq V_*$ by definition, we have convergence. If $V^*(s) - V_*(s)$ has a local maximum $s^* \in (0,\infty)$ where $V^*(s^*) > V_*(s^*)$, then with the same argument used in the proof of Proposition (7), we have $V^* \leq V_*$ and therefore we conclude the convergence.

It remains to prove that $V(s) = V^*(s)$. Since $V(s)$ is the viscosity supersolution of (3.10), from Proposition (7) we have $V^*(s) \leq V(s)$. Because $V(s)/V(\infty)$ is the value function of (2.3) with the boundary values (3.9), for all $\varepsilon > 0$ there exists a predictable strategy π such that $V^\pi(s) > V(s) - \varepsilon$. The function $V^\pi(s)$ is also a viscosity solution for HJB equation (3.10) with the strategy π. Because $V^*(s)$ is a viscosity supersolution too, we obtain from Proposition 7, $V^\pi(s) \le V^*(s)$ and therefore $V^*(s) > V(s) - \varepsilon$. Since ε can be made arbitrarily small, we come to the desired results, i.e., $V = V_*$. □

For the general admissible set $\Pi(s) \subset \mathbb{R}^{n+1}$, we define the following notations

$$V_+^{'}(s) = \frac{V(s+\Delta) - V(s)}{\Delta},\ V_-^{'}(s) = \frac{V(s) - V(s-\Delta)}{\Delta},$$

$$V_\Delta^{''}(s) = \frac{V(s-\Delta) + V(s+\Delta) - 2V(s)}{\Delta^2}.$$

These are respectively called the forward and backward first order difference quotient approximations, and the second order difference quotient approximation in s. Using these notations, we can approximate the HJB equation (3.10) as

$$\mathcal{H}_\Delta\left(s, V, V_+^{'}, V_-^{'}, V_\Delta^{''}\right) = \sup_{(A,u)\in\Pi} \{\lambda\left(G_{\Delta,u}(s) - V(s)\right)$$

$$+ (c - h(u) + rA)^+ V_+^{'}(s) + (c - h(u) + rA)^- V_-^{'}(s) + \frac{1}{2}\sigma^2 A^2 V_\Delta^{''}(s)\} = 0. \tag{3.32}$$

With a similar method used in the proof of Theorem 10, we can show the above HJB-scheme also converges to the unique viscosity solution of HJB (3.10). For instance, in the subsolution case,

$$V_\Delta(s_\Delta - \Delta) - \varphi_\Delta(s_\Delta - \Delta) \ge V_\Delta(x) - \varphi(x),\ x \in \{s_\Delta - 2\Delta, s_\Delta\},$$

implies

$$\varphi_+^{'}(s_\Delta) \ge V_+^{'}(s_\Delta),\ \varphi_-^{'}(s_\Delta) \le V_-^{'}(s_\Delta),\ \varphi^{''}(s_\Delta) \ge V^{''}(s_\Delta).$$

Note that the HJB-scheme (3.32) does not produce a simple recursion like (3.24) and one must apply iterative method for numerical purposes. In numerical Examples presented in the next two chapters, we did not observe a difference between the solutions to (3.24) and (3.32) after a few iterations.

Note that the lemma 9 implies indeed that the two functions $V^*(s) = \limsup V_\Delta(k\Delta)$ and $V_*(s) = \liminf V_\Delta(k\Delta)$ are continuous and locally Lipschitz on $[0,\infty)$. To show this let $0 \le s_2 < s_1$ be arbitrary and $k_n^{(1)}$, $k_n^{(2)}$, Δ_n be sequences such that

$$k_n^{(i)}\Delta_n \to s_i,\ V_{\Delta_n}\left(k_n^{(1)}\Delta_n\right) \to V^*(s_1).$$

From

$$\limsup V_{\Delta_n}\left(k_n^{(2)}\Delta_n\right) \leq V^*(s_2)$$

we obtain

$$V^*(s_1) - V^*(s_2) \leq \lim V_{\Delta_n}\left(k_n^{(1)}\Delta_n\right) - \limsup V_{\Delta_n}\left(k_n^{(2)}\Delta_N\right)$$

$$\leq \limsup\left(V_{\Delta_n}\left(k_n^{(1)}\Delta_n\right) - V_{\Delta_n}\left(k_n^{(2)}\Delta_n\right)\right)$$

$$\leq \limsup K\left|k_n^{(1)}\Delta_n - k_n^{(1)}\Delta_n\right| = K(s_1 - s_2),$$

where K is a common Lipschitz constant for the function $V_{\Delta_n}(x)$, $0 \leq x \leq s_1$.

In the Theorem 10 we used comparison results Proposition (7), without studying the necessary condition $V^*(\infty) \leq V_*(\infty)$. We now verify this condition for the problem of optimal investment without reinsurance, that is $\Pi = (\mathcal{A}, u_0)$, where $g(Y, u_0) = Y$. The proof method used here is as the same as the one in [**21**].

Theorem 11. *Let $V_\Delta(s)$ be the solution to (3.24) and define*

$$V^*(s) = \limsup_{\Delta\to 0, i\Delta\to s} V_\Delta(i\Delta),$$

and

$$V_*(s) = \liminf_{\Delta\to 0, i\Delta\to s} V_\Delta(i\Delta).$$

If $\Pi = (\mathcal{A}, U) = (\mathcal{A}, u_0)$, then $V^*(s) = V_*(s)$ for all $s \in [0, \infty)$.

Proof. If $V^*(\infty) \leq V_*(\infty)$ or if $V^*(s) - V_*(s)$ has a local maximum $s^* \in (0, \infty)$ where $V^*(s^*) > V_*(s^*)$, then with the same argument used in the proof of Proposition (7), we have $V^* \leq V_*$. Because $V^* \geq V_*$ by definition, we have $V^*(s) = V_*(s)$.

It remains to show that $V^*(s) = V_*(s)$, $0 \leq s < \infty$ for the case that $V^*(s) - V_*(s)$ is increasing on $[s_0, \infty)$, where

$$s_0 = \max\{s : V^*(x) = V_*(x), 0 \leq x \leq s\}.$$

With the help of Proposition (7), we first show that the functions $V^*(s)$ and $V_*(s)$ are pointwise limits, i.e. for some $s_1 > 0$ there exists sequences $\Delta_n^* \to 0$ and $\Delta_*^n \to 0$ such that for $0 \leq s \leq s_1$

$$\lim V_{\Delta_n^*}(k_n\Delta_n^*) \to V^*(s) \text{ as } k_n\Delta_n^* \to s,$$

and

$$\lim V_{\Delta_*^n}(k_n\Delta_*^n) \to V_*(s) \text{ as } k_n\Delta_*^n \to s.$$

Select k_n and Δ_n^* such that $V_{\Delta_n^*}(k_n\Delta_n^*) \to V^*(s_1)$ when $k_n\Delta_n^* \to s_1$. From lemma 9, for arbitrary $k_n^{'}$ satisfying $k_n^{'}\Delta_n^* \to s_1$, we have $V_{\Delta_n^*}\left(k_n^{'}\Delta_n^*\right) \to V^*(s_1)$. The function $V_0(s) = \liminf V_{\Delta_n^*}(s)$, is a viscosity super-solution of our HJB equation and Lipschitz which $V_0(0) = V^*(0)$, $V_0(s_1) = V^*(s_1)$. With the Propostion (7) we have $V^*(x) \leq V_0(x)$ for $0 \leq x \leq s_1$. Since by definition $V^*(x) \geq V_0(x)$, we have $V^*(x) = V_0(x)$ for $0 \leq x \leq s_1$. Moreover,

$$\liminf V_{\Delta_n^*}(x) = \limsup V_{\Delta_n^*}(x), \; 0 \leq x \leq s_1.$$

Since any pointwise limit of a sequence of discretizations is both lim sup and lim inf, the functions $V_0(x)$ and $V^*(x)$ for $0 \leq x \leq s_1$ are viscosity solutions of the HJB equation (3.10). The proof for the function $V_*(s)$ can be done with the same argument with choosing a sequence of $k_n\Delta_*^n$ converging to s_1 and derive $V_*(x) = \limsup V_{\Delta_*^n}(x) = \lim V_{\Delta_*^n}(x)$.

Second, we let $s_1 > s_0$ and with the contradiction argument show that $V^*(x) = V_*(x)$, $x \in [0, s_1)$. To this end, let $V^*(s_1) > V_*(s_1)$. Choose an arbitrary $s \in [s_0, s_1]$ and define two discretization schemes;

Δ_{n*}^*, with step-size Δ_n^* for $k \leq k^*(s)$ and step-size Δ_*^n for $k > k^*(s)$, where $k^*(s) = \max\{k : k\Delta_n^* \leq s\}$, and

Δ_*^{n*}, with step-size Δ_*^n for $k \leq k_*(s)$ and step-size Δ_n^* for $k > k_*(s)$, where $k_*(s) = \max\{k : k\Delta_*^n \leq s\}$.

For $0 \leq x \leq s_1$, set

$$\bar{V}^*(x) = \limsup V_{\Delta_{n*}^*}(x),$$

and

$$\bar{V}_*(x) = \liminf V_{\Delta_*^{n*}}(x).$$

At the change points $s^* = k^*(s)\Delta_n^*$ and $s_* = k_*(s)\Delta_*^n$ we modify the discretization scheme (3.23) respectively by

$$V_{\Delta_{n*}^*}^{''}(s^*) = 2\frac{V_{\Delta_*^n}^{'}(s^*) - V_{\Delta_n^*}^{'}(s^* - \Delta_n^*)}{\Delta_n^* + \Delta_*^n},$$

and

$$V''_{\Delta_*^{n*}}(s_*) = 2\frac{V'_{\Delta_n^*}(s_*) - V'_{\Delta_*^n}(s_* - \Delta_*^n)}{\Delta_n^* + \Delta_*^n}.$$

With this modification and the same argument used at the Theorem 10, we can show that the functions $\bar{V}^*(x)$ and $\bar{V}_*(x)$ are the viscosity sub- and supersolution of HJB equation (3.10) for $0 \leq x \leq s_1$, respectively. We now find an upper bound for $\bar{V}^*(s_1)$. Define $\tilde{V}(x)$ as $\bar{V}^*(x)$ with the constraint $(\mathcal{A}, U) = (0, u_0)$ for $s_0 \leq x \leq s$. Notice that the discretization step-sizes has no influence in the limit for the range $0 \leq x \leq s$. Therefore, we can use the discretizations $\tilde{V}_{\Delta_n^*}(x)$ for $0 \leq x \leq s$ which are defined as $V_{\Delta_n^*}(x)$ but with $(\mathcal{A}, U) = (0, u_0)$ for $s_0 \leq x \leq s$. Using (3.29), for $s_0 \leq x \leq s$ we have

$$\tilde{V}_{\Delta_*^n}(x) \leq V_{\Delta_*^n}(s_0)\left(1 + \Delta\frac{\lambda}{c}\right)^{K_*^n(s_0,x)} \leq V_{\Delta_*^n}(s_0)\left(1 + \Delta\frac{\lambda}{c}\right)^{K_*^n(s_0,s)},$$

where

$$K_*^n(x,y) = \#\left\{k \geq 0 : x < k\Delta_*^n \leq y\right\}.$$

Recalling recursion (3.24) and using (3.26) for constraint set $(\mathcal{A}, U) = (A, u_0)$ and $s = k\Delta$ we obtain

(3.33)

$$V'_{\Delta}(s) = \inf_{A \in \mathcal{A}} \frac{\lambda\Delta\left(V(0)\Pr\{Y \geq j\Delta\} + \Delta\sum_{j=1}^{k-1} V'(j\Delta)\Pr\{Y \geq (k-j)\Delta\}\right) + \frac{1}{2}\sigma^2 A^2 V'_{\Delta}(s-\Delta)}{\Delta(c + rA - \lambda\Delta) + \frac{1}{2}\sigma^2 A^2}.$$

Now by induction, it is easy to see that for $x \geq s$

$$\tilde{V}'_{\Delta_*^n}(x) \leq V'_{\Delta_*^n}(x)\left(1 + \Delta\frac{\lambda}{c}\right)^{K_*^n(s_0,s)}.$$

The above inequality with $\bar{V}^*(x) \leq \tilde{V}(x)$, for $n \to \infty$ yields

$$\bar{V}^*(s_1) \leq V_*(s_1)\exp\left(\frac{\lambda}{c}(s - s_0)\right).$$

A lower bound for $\bar{V}_*(s_1)$ can be found as well. Define $\hat{V}(x)$ as $\bar{V}_*(x)$ with the investment constraint $\mathcal{A}(x) = (-\infty, \infty)$ for $s_0 \leq x \leq s$. Notice that the discretization step-sizes has no influence in the limit for the range $0 \leq x \leq s$. Therefore, we can use the discretizations $\hat{V}_{\Delta_*^n}(x)$ for $0 \leq x \leq s$ which are defined as $V_{\Delta_*^n}(x)$ but with the investment constraint $\mathcal{A}(x) = (-\infty, \infty)$ for $s_0 \leq x \leq s$, i.e. $(\mathcal{A}, U) = ((-\infty, \infty), u_0)$.

From (3.29), for $s_0 \leq x \leq s$ we obtain

$$V_{\Delta_n^*}(x) \leq V_{\Delta_n^*}(s_0)\left(1+\Delta\frac{\lambda}{c}\right)^{K_n^*(s_0,s)},$$

where

$$K_n^*(x,y) = \#\{k \geq 0 : x < k\Delta_n^* \leq y\}.$$

So

$$\hat{V}_{\Delta_n^*}(x) \geq V_{\Delta_n^*}(s_0) \geq V_{\Delta_n^*}(x)\left(1+\Delta\frac{\lambda}{c}\right)^{-K_n^*(s_0,s)}.$$

Using (3.33), by induction we can show that

$$\hat{V}'_{\Delta_n^*}(x) \leq V'_{\Delta_n^*}(x)\left(1+\Delta\frac{\lambda}{c}\right)^{-K_n^*(s_0,s)}.$$

This inequality with $\bar{V}_*(x) \geq \hat{V}(x)$ yields

$$\bar{V}_*(s_1) \geq V^*(s_1)\exp\left(-\frac{\lambda}{c}(s-s_0)\right),$$

when $n \to \infty$.

Notice that the functions $\bar{V}_*(x)$ and $\bar{V}^*(x)$ are Lipschitz and increasing viscosity super- and subsolutions of HJB equation (3.10), respectively, and for value s close to s_0 the conditions $\bar{V}_*(0) = \bar{V}^*(0)$ and $\bar{V}^*(s_1) \leq \bar{V}_*(s_1)$ are satisfied. Hence from the Proposition 7, $\bar{V}^*(x) \leq \bar{V}_*(x)$ for $0 \leq x \leq s_1$ which contradicts $\bar{V}_*(x) = V_*(x) \geq V^*(x) = \bar{V}^*(x)$ for $s_0 \leq x \leq s$. □

We finish this chapter by showing that the limit of discretization schemes $V(s)$ for the optimal investment problem without investment is equal to its value function. The proof is as the same as that in [**21**].

Theorem 12. *Let $V(s)$ be the value function of the HJB equation (3.10) for the optimal investment problem without investment, i.e. $\Pi = (\mathcal{A}, u_0)$ and $W(s)$ be the limit solution of the discretization scheme (3.24) for this problem. Then $V(s) = W(s)$.*

Proof. Note that if $V(\infty) = W(\infty)$, then one can use the comparison results Proposition (7). For $V(\infty) \neq W(\infty)$ one can apply an extended version of functions used in previous theorem. Here we only consider the case $W(\infty) > V(\infty)$. The case $W(\infty) < V(\infty)$ can be proved with the same argument. Assuming $W(\infty) > V(\infty)$ we have $W(x) \geq V(x)$ for all $x \geq 0$ and we need to show that $W(x) \leq V(x)$ for all $x \geq 0$.

First note that both functions are semi-concave and thus almost everywhere twice differentiable on $(0,\infty)$ (see [**4**] Remark 3.4, p.54, and Proposition 3.3, p. 55). Assume that for some $s>0$ we have $W(s)>V(s)$. For some $\varepsilon>0$, choose $s_0<s_1<s_0+\varepsilon$ such that $W(x)$ and $V(x)$ are twice differentiable at s_1 and $W(s_1)>V(s_1)$, where

$$s_0=\max\{s\geq 0: V(x)=W(x)\,,\,0\leq x\leq s\}\,.$$

Consider the following HJB equation:

$$0=\sup_{A\in\mathcal{A}(s)}\{\frac{1}{2}\sigma^2A^2V^{''}(s)+(c+rA)\,V^{'}(s)+\lambda E\left[V(s-Y)-V(s)\right]$$

$$+\lambda E\left[V(s-Y)\,1_{\{Y\leq s-s_1\}}+p(s-Y)\,1_{\{Y>s-s_1\}}-V(s)\right]\},\ s\geq s_1, \tag{3.34}$$

where $p(x)$ is a continuous monotone function on $-\infty<x<s_1$ and is left differentiable at s_1. We first construct a sub-solution $\bar{W}(x)$ for the HJB equation (3.10) satisfying $\bar{W}(x)=W(x)$, $0\leq x\leq s_1$, and $\bar{W}(\infty)\leq V(\infty)\exp\left(\frac{\lambda}{c}(s_1-s_0)\right)$. To this end let $W_\Delta(x)$, $x\geq s_1$be the discretization scheme for the equation (3.34) with

$$W_\Delta(s_1)=p(s_1)\,,\,V^{'}(s_1)=p^{'}(s_1)\,,$$

and denote its limsup by $V^p(x)$. Notice that $V^p(x)$ is a sub-solution of equation (3.34) and for $p(x)=W(x)$, $0\leq x\leq s_1$, we have

$$V^p(x)=V(x)\,,\,x\geq s_1,$$

and for $\alpha>1$ and $p(x)\leq p_1(x)$, it holds

$$V^p(x)\leq V^{p_1}(x)\text{ and }V^{\alpha p}(x)\leq\alpha V^p(x)\,,\,x\geq s_1. \tag{3.35}$$

If $p(x)=V(x)$, $x\leq s_1$, define

$$\bar{W}(x)=\begin{cases}W(x) & x\leq s_1\\ V^p(x) & x>s_1,\end{cases}$$

which is a subsolution of equation (3.10) for $s>0$. Proving this is easy in the case $0\leq s<s_1$. For $s>s_1$ one must notice that for any $s\geq s_1$ and with $p(x)=W(x)$ it holds

$$E\left[\bar{W}(s-Y)\right]=E\left[W(s-Y)1_{\{Y>s-s_1\}}\right]+E\left[V^p(s-Y)1_{\{Y\leq s-s_1\}}\right].$$

At the point $s=s_1$ one can use the initial values for the discretization $W_\Delta(s)$. Let $\varphi(x)\in C_2(0,\infty)$ be a test function such that $\bar{W}(x)-\varphi(x)$ has a strict maximum at $x=s_1$. We want to show

$$\mathcal{H}\left(s_1,\bar{W},\varphi',\varphi''\right)\geq 0.$$

From [**15**], p. 334, there is a sequence $\Delta\to 0$ such that $W_\Delta(x)-\varphi(x)$ has its maximum on $\{x=k\Delta: x\geq s_1\}$ in some x_Δ with $x_\Delta\to s_1$ and $W_\Delta(x_\Delta)\to W(s_1)$. If $x_\Delta=s_1$ infinitely often then for these Δ's we have

$$W_\Delta(s_1+\Delta)-\varphi(s_1+\Delta)\leq W_\Delta(s_1)-\varphi(s_1)$$

and so $\varphi'(s_1)\geq W'(s_1)$. From $\varphi(x)>W(x)$ for $x<s_1$ we obtain $\varphi'(s_1)\leq W'(s_1)$which yields $\varphi'(s_1)=W'(s_1)$. Thus for $0<\gamma\to 0$ we get

$$\varphi(s_1-\gamma)\geq W(s_1)-\gamma W'(s_1)+\frac{\gamma^2}{2}W''(s_1)+o\left(\gamma^2\right),$$

and this leads to $\varphi''(s_1)\geq W''(s_1)$. This gives us to the desired result, i.e. $\mathcal{H}\left(s_1,\bar{W},\varphi',\varphi''\right)\geq 0$.

If $x_\Delta>0$ for a subsequence of Δ's, then

$$W_\Delta'(x_\Delta)\leq\frac{\varphi(x_\Delta+\Delta)-\varphi(x_\Delta)}{\Delta}\to\varphi'(s_1),\ \Delta\to 0,$$

$$W_\Delta''(x_\Delta)\leq\frac{\varphi(x_\Delta+\Delta)-2\varphi(x_\Delta)+\varphi(x_\Delta-\Delta)}{\Delta^2}\to\varphi''(s_1),\ \Delta\to 0,$$

and therefore $\mathcal{H}\left(s_1,\bar{W},\varphi',\varphi''\right)\geq 0$.
Inequalities (3.35) and $W(x)\leq V(x)\exp\left(\frac{\lambda}{c}(s_1-s_0)\right)$, $x\leq s_1$, yields

$$\bar{W}(x)\leq V(x)\exp\left(\frac{\lambda}{c}(s_1-s_0)\right),\ x\geq 0.$$

Now we built a supersolution $\bar{V}(x)$ for HJB equation (3.10) such that $\bar{V}(x)=V(x)$, for $0\leq x\leq s_1$ and $\bar{V}(\infty)\geq\bar{W}(\infty)$. Since for $x\geq 0$, $0\in\mathcal{A}(x)$ it holds for all $\gamma>0$

$$W(x+\gamma)-W(x)\leq\frac{\lambda}{c}W(x).$$

Choose an admissible strategy A_t, $t \geq 0$ with the value function

$$p(x) = \Pr\left\{ \tau^A = \infty \middle| R_0^A = x \right\}, \, x \geq 0,$$

such that for $\eta > 0$, we have $p(x) \geq \frac{V(x)}{V(\infty)} - \eta$ for $x \geq 0$. Let τ_1^A be the first exit time of the stochastic process R_t^A from the set $[s, \infty)$ and

$$\kappa = V(s_1) - E\left[V\left(R^A_{\tau_1^A} \right) 1_{\left\{ \tau_1^A < \infty \right\}} \middle| R_0^A = s_1 \right].$$

Let $p_1(x) = \Pr\left\{ \tau_1^A = \infty \middle| R_0^A = x \right\}$ and define

$$q(x) = E\left[V\left(R^A_{\tau_1^A} \right) 1_{\left\{ \tau_1^A < \infty \right\}} \middle| R_0^A = s_1 \right] + \kappa \frac{p_1(x)}{p_1(s_1)}, \, x \geq 0.$$

We obtain $q(x) = V(x)$, $0 \leq x \leq s_1$, $q(s_1+) = q(s_1)$, $q(\infty) = \kappa/p_1(s_1)$, and $q(x)$ is a supersolution of (3.10) for $0 < x < s_1$ and$x > s_1$. At point s_1 we change the constant premium c with a continuous function $c(s_1) = \lambda V(s_1)/V'(s_1)$ in the range $[s_1, s_1 + \varepsilon]$ and $c(x) = c$ for $x \geq s_1 + \varepsilon$. Thus

$$\limsup_{l \to 0} \frac{q(s_1 + l) - q(s_1)}{l} \leq V'(s_1),$$

which gives us the supersolution property. Note that the $p_1(s_1)$ will change a bit by choosing ε small enough. The inequality

$$W(s_1) - V(s_1) \leq V(s_1)\left(1 - \exp\left(\frac{\lambda}{c}(s_1 - s_0) \right) \right) = I_1,$$

leads to

$$\begin{aligned}
\kappa &\geq W(s_1) - E\left[W\left(R^A_{\tau_1} \right) 1_{\left\{ \tau_1^A < \infty \right\}} \middle| R_0^A = s_1 \right] - I_1 \\
&\geq p(s_1) W(\infty) - E\left[p\left(R^A_{\tau_1} \right) 1_{\left\{ \tau_1^A < \infty \right\}} \right] W(\infty) - 2\eta - I_1 \\
&= p_1(s_1) W(\infty) - 2\eta - I_1.
\end{aligned}$$

Because $p_1(s_1) \geq p(s_1) - \eta \geq (1 - \lambda\mu/c) - \eta$, by choosing ε and η small enough, we obtain $q(\infty) = \kappa/p_1(s_1)$ larger than $\bar{W}(\infty)$.

Now by applying the Proposition (7) on the two functions $\bar{W}(x)$ and $\bar{V}(x)$ we come to the contradiction. □

CHAPTER 4

Optimal Dynamic reinsurance and investment with constraints

In the previous chapter we showed that the HJB equation has a continuous solution $\delta(s)$. Because this solution is not always twice continuously differentiable or even once differentiable, we consider it as a weak solution to HJB equation within the framework of viscosity solutions. We showed under some assumptions that the value function is the viscosity solution of our HJB equation (3.4). We developed a numerical algorithm in section 3.3 and proved that for the optimal investment problem, our numerical algorithm converges to the value function.

In this chapter, we apply our numerical method in section 3.3 and present a number of numerical examples showing different analytical properties. Here must be mentioned that there are still some technical gaps to prove the convergence of our numerical method to the value function of HJB equation (3.4) in the optimal investment and reinsurance problem.

In the first section, we consider the problem of optimal investment without reinsurance. Using our numerical method for the case optimal investment without constraint $\mathcal{A}(s) = (-\infty, \infty)$, in subsection 4.1.1, we can represent the corresponding HJB equation in the form of a quadratic equation. In subsection 4.1.2, we see that the proposed numerical method is able to solve the problem of optimal investment even when the solution $\delta(s)$ is not smooth.

The problem of optimal reinsurance has been extensively considered in [**42**]. Therefore, in this chapter we skip the problem of optimal reinsurance without investment and just consider the combined optimization of investment and reinsurance. In sections 4.2, 4.3 and 4.4 we solve the problem optimal investment and reinsurance for three kinds of reinsurance defined in section 2.3. Here we assume that the reinsurance company calculates its premium via the expected value principle with safety loading $\theta > 0$.

During this chapter, we calculate the optimal strategies and survival probabilities for the following two claim size distributions:

(1) An exponential claim size with parameter m and distribution function

$$F(y) = 1 - e^{-my}, \;\; m > 0, \;\; y \in (0, \infty).$$

This distribution is a typical case of light tails distributions.

(2) A Pareto claim size with parameter p and distribution function

$$F(y) = 1 - (1+y)^{-p}, \;\; p > 1, \;\; y \in (0, \infty).$$

The Pareto distribution is a example from the family of subexponential distributions.

For the case of an exponential claim size distribution, the survival probability without reinsurance and investment $\delta_0(s)$, can be given explicitly:

$$\delta_0(s) = 1 - \frac{\lambda}{mc} \exp\left(-\left(m - \frac{\lambda}{c}\right)s\right). \tag{4.1}$$

In fact, the analytical solution for $\delta_0(s)$ exists only for claims distributions that are mixtures and combinations of exponential distributions (see [**1**]). For other distributions, the formula (3.6) is so complicated and it should rather be viewed as basis for numerical algorithms. However, it is common that instead of looking for survival probability $\delta_0(s)$, often one just look at its **Cramér-Lundberg lower bound** $1 - \delta_0(s) = \psi_0(s) \geq e^{-ls}$, where $l > 0$ is the so-called **adjustment coefficient** of risk process without investment and reinsurance R_t^0, $t \geq 0$. Let ξ be the exponentially distributed random variable of the inter-occurrence times between successive claims with mean $\lambda^{-1} > 0$. The adjustment coefficient of risk process R_t^0 is the positive solution of

$$\mathcal{M}_Z(l) = 1, \tag{4.2}$$

where $\mathcal{M}_Z(l) = E[\exp(lZ)]$ is the moment generating function of random variable $Z = Y - c\xi$. The equation (4.2) has just one positive solution $l > 0$ (see [**35**]). Since ξ has an exponential distribution, the equation (4.2) writes

$$\lambda + lc = \lambda \mathcal{M}_Z(l).$$

The details of calculation formula (4.1) as well as Cramér-Lundberg inequality can be found for example in [**35**] and [**1**]. With the help of adjustment coefficient, for some examples we calculate the asymptotic optimal investment and reinsurance strategy when $s \to \infty$.

THEOREM 13. *Let $\psi_0(s)$ be the ruin probability of the risk process without investment and reinsurance. If the corresponding adjustment coefficient l exists, then there exists a constant $k > 0$, such that*

$$\lim_{s\to\infty} \psi_0(s)\, e^{ls} = k. \tag{4.3}$$

PROOF. The proof can be found, for instance, in [**35**], p.172. □

REMARK 4. The adjustment coefficient does not exist for all claim size distribution F_Y. In order that the adjustment coefficient exists, it is needed that the claim size distribution F_Y is light tailed in the sense of Remark 1. △

4.1. Optimal investment

In this section we consider two cases: optimal unconstrained investment i.e. $\mathcal{A}(s) = (-\infty, \infty)$ and optimal constrained investment without reinsurance. Different iterative operators have been used by different authors [**3, 23, 22**] to solve certain special cases of the problem of optimal investment. Here we briefly review each of these operators and compare them with the numerical method presented in section (3.3). In the sequel of this section we denote the no-reinsurance strategy by u_0, that is $g(u_0, Y) = Y$ and $h(u_0) = 0$, and use the norming $V(0) = 1$ and $V'(0) = \lambda/c$.

4.1.1. Optimal investment without constraint. Let us first consider the unconstrained case with $\mathcal{A}(s) = (-\infty, \infty)$. The HJB equation (3.10) is then simplified to

$$\sup_A \left\{ \frac{1}{2}\sigma^2 A^2 V''(s) + (c + rA)\, V'(s) + \lambda E\left[V(s - Y) - V(s)\right] \right\} = 0. \tag{4.4}$$

A maximizing A exists only if $V''(s) \le 0$ and is

$$A(s) = -\frac{rV'(s)}{\sigma^2 V''(s)}. \tag{4.5}$$

Since at point $s=0$, $A(0)=0$, we have $V''(0)=-\infty$. If we plug in the optimal investment, then equation (4.4) reads

$$0=\lambda\int_0^\infty [V(s-Y)-V(s)]\,dF(y)+cV'(s)-\frac{1}{2}\frac{r^2V'(s)^2}{\sigma^2V''(s)}.$$

For the reason of simplicity we shall substitute λ and c by $\lambda\frac{\sigma^2}{r^2}$ and $c\frac{\sigma^2}{r^2}$, respectively, and rewrite the HJB with this new notations as

$$0=\lambda\int_0^\infty [V(s-Y)-V(s)]\,dF(y)+cV'(s)-\frac{1}{2}\frac{V'(s)^2}{V''(s)}. \tag{4.6}$$

With integration by parts

$$\lambda\int_0^\infty [V(s-Y)-V(s)]\,dF(y)=-V(0)\bar{F}(s)-\int_0^s V'(s-y)\bar{F}(y)\,dy. \tag{4.7}$$

Plugging (4.7) into (4.6) and using transformation $v(s)=V'\left(s^2\right)$, we have

$$v'(s)\left(\frac{1}{s}\left(cv(s)-\lambda\bar{F}\left(s^2\right)\right)-2\lambda s\int_0^1 xv(sx)\bar{F}\left(s^2\left(1-x^2\right)\right)dx\right)=v(s)^2.$$

Using a contraction argument, Hipp and Plum (2000) showed that there exists a unique solution $v(s)$ for the above equation with

$$v(s)=\frac{\lambda}{c}-\frac{\lambda}{c}\frac{s}{\sqrt{c}}+o\left(\sqrt{s}\right)\quad as\;\; s\to 0.$$

But this contraction method does not generate a stable numerical algorithm for the computation of optimal survival probability, since for the small amount of capital the singularity of $V''(0)$ is disturbing. For computational purpose, Hipp and Plum (2003) introduced the function $U(s)=A(s)^2$ and rewrite (4.6) as

$$\lambda\int_0^\infty [V(s-Y)-V(s)]\,dF(y)+cV'(s)=-\frac{1}{2}V'(s)\sqrt{U(s)},\;\; s\geq 0, \tag{4.8}$$

where $\sqrt{U(s)}$ denotes the positive root of $U(s)$. If we assume that the distribution function $F(y)$ is smooth, we are able to differentiate equation (4.8) and get for $s\geq 0$,

$$\lambda\left(V(0)f(s)+\int_0^s V'(s-y)f(y)\,dy-V'(s)\right)+cV'(s)$$
$$=-\frac{1}{2}V''(s)A(s)-\frac{1}{2}V'(s)A'(s).$$

Using $V^{''}(s)\,A(s) = -V^{'}(s)$ we arrive at

$$\sqrt{U(s)}\left(\left(\lambda + \frac{1}{2}\right)V^{'}(s) - \lambda V(0)\,f(s) - \lambda\int_0^s V^{'}(s-y)\,f(y)\,dy\right) + cV^{'}(s)$$

$$= \frac{1}{4}U^{'}(s)\,V^{'}(s)\,,\quad s \geq 0. \tag{4.9}$$

The two interaction differential equation (4.8) and (4.9) are equivalent to the equation (4.6). Using this system of equation, we get rid of the problem of singularity at zero and the result gives a stable numerical method. This approach can only be used in the problem of optimal unconstrained investment, where one can easily insert the optimal investment (4.5) into HJB equation (4.4).

Let us now return to the numerical approach introduced in section 3.3. Using the approximations (3.23), (3.22) and (3.21), the equation (4.6) can be simplified into the following quadratic equation

$$\alpha_1 V^{'}(s)^2 + a_1(s)\,V^{'}(s) + b_1(s) = 0, \tag{4.10}$$

where

$$\alpha_1 = c - \frac{1}{2}(\lambda+1)\,\Delta,$$

$$a_1(s) = \lambda\left(G_{u_0}(s) - V(s-\Delta)\right) - (c - \lambda\Delta)\,V^{'}(s-\Delta)\,,$$

$$b_1(s) = -\lambda V^{'}(s-\Delta)\left(G_{u_0}(s) - V(s-\Delta)\right),$$

and

$$G_{u_0}(s) = \sum_{j=1}^{i} V((i-j)\,\Delta)\Pr\{(j-1)\,\Delta < Y \leq j\Delta\}\,.$$

Notice that (for small Δ) $\alpha_1,\ b_1(s),\ -a_1(s) > 0$, and since $V^{'}(s)$ is minimized we obtain

$$V^{'}(s) = -\frac{a_1(s)}{2\alpha_1} - \sqrt{\frac{a_1(s)^2}{4\alpha_1^2} - \frac{b_1(s)}{\alpha_1}}. \tag{4.11}$$

If in (4.6) we approximate $V^{'}(s)$ by $V^{'}(s-\Delta) + \Delta V^{''}(s)$ and solve for $V^{''}(s)$ we derive the following quadratic equation

$$\alpha_2 V^{''}(s)^2 + a_2(s)\,V^{''}(s) + b_2(s) = 0,\ 0 < s \in D, \tag{4.12}$$

where

$$\alpha_2 = c\Delta - \frac{1}{2}(\lambda+1)\Delta^2,$$
$$a_2(s) = \lambda(G_{u_0}(s) - V(s-\Delta)) + cV'(s-\Delta) - (\lambda+1)\Delta V'(s-\Delta),$$
$$b_2(s) = -\frac{1}{2}V'(s-\Delta)^2.$$

For small Δ, we have $\alpha_2, -b_2(s) > 0$, and from (4.6) $a_2(s) \le 0$, we get

$$V''(s) = -\frac{a_2(s)}{2\alpha_2} - \sqrt{\frac{a_2(s)^2}{4\alpha_2^2} - \frac{b_2(s)}{\alpha_2}}. \tag{4.13}$$

At $s = \Delta$, the equation (4.12) reads

$$0 = \alpha V''(\Delta)^2 + V''(\Delta)\left(\lambda F(\Delta) - (\lambda+1)\Delta\frac{\lambda}{c}\right) - \frac{\lambda^2}{2c^2}.$$

Letting $\Delta \to 0$, we have

$$\frac{a_2(\Delta)}{2\alpha_2} \to \frac{\lambda}{c}\left(f(0) - \frac{\lambda}{c} - \frac{1}{c}\right).$$

Hence from (4.13)

$$\lim_{\Delta\to 0}\sqrt{\Delta}V''_{\Delta}(\Delta) = \frac{\lambda}{c\sqrt{2c}}, \tag{4.14}$$

which is the same results as in Hipp and Plum 2000.

Remark 5. For $s = \Delta$, the relation (3.25) writes

$$V''(\Delta) = \inf_A \frac{-\lambda F(\Delta) - A\frac{\lambda}{c} + \frac{\lambda^2}{c}\Delta}{\Delta(c + A - \lambda\Delta) + \frac{1}{2}A^2}.$$

Differentiating with respect to A, the optimal A must satisfy

$$\frac{1}{2}A^2 + A\left(-F(\Delta) - \frac{\lambda}{c}\Delta\right) - \Delta(1 - F(\Delta)) = 0.$$

This yields $A \to \sqrt{2c\Delta}$ when $\Delta \to 0$ which corresponds to (4.14). △

Hipp and Schmidli [**24**] studied the asymptotic behavior of optimal survival probability for the light tail claim distribution. They first considered a constant investment strategy A, and defined random variable $Z := Y - (c + rA)\xi - \sigma AW(\xi)$ where the random variables ξ and $W(\xi)$ are, respectively, exponentially distributed with mean $\lambda^{-1} > 0$ and normally distributed with mean 0 and variance ξ. The adjustment coefficient of risk process with constant investment strategy, A, is the positive solution of equation

$$\mathcal{M}_Z(l) = 1,$$

for l. Since $\xi \sim \exp(\lambda)$ and $W(\xi) \sim N(0, \xi)$, the above equation reads

$$\frac{1}{2}\sigma^2 A^2 l^2 - (c + rA)\, l + \lambda \mathcal{M}_Y(l) - \lambda = 0. \tag{4.15}$$

Note that the above equation corresponds to the HJB equation (4.4) with $V(s) = 1 - e^{-ls}$ and a constant strategy A. Let $l(A)$ be the solution of equation (4.15). In order to obtain an asymptotically optimal constant strategy, A^*, among all constant strategies, we need to find $l^* = \sup_{A \geq 0} l(A)$. Since at l^*, the left hand side of equation (4.15) is nonnegative, the equation (4.15) gets its minimum at the optimal constant strategy. Therefore, $A^* = \frac{r}{l^* \sigma^2}$ which is the minimizer of

$$\inf_A \left\{ \frac{1}{2}\sigma^2 A^2 l^2 - (c + rA)\, l + \lambda \mathcal{M}_Y(l) - \lambda \right\} = 0.$$

By inserting the optimal constant strategy, A^*, into the above equation we have

$$\frac{1}{2}\frac{r^2}{\sigma^2} + cl^* + \lambda - \lambda \mathcal{M}_Y(l^*) = 0. \tag{4.16}$$

Comparing l^* with the corresponding adjustment coefficient $l(0)$ without investment, we have $l^* = \sup_{A \geq 0} l(A) \geq l(0)$. Moreover, the solution of (4.16) exists even if the adjustment coefficient $l(0)$ does not exist. Similar to the theorem 13, Hipp and Schmidli (2004) showed that, if the claim distribution is light tailed, then there exists a constant $k \in (0, \infty)$ such that $\psi(s)\, e^{ls} \to k$ as $s \to \infty$. Furthermore, they proved that the optimal strategy $A(s)$ converges to the optimal constant strategy $A^* = \frac{r}{l^* \sigma^2}$ as $s \to \infty$.

In the following examples we choose $\Delta = 0.0001$, $\lambda = 1$, $c = 2$.

EXAMPLE 14. For the first numerical example we consider the exponential claim size with mean 1. The premium income is $c = 2$ and the survival probability without investment at $s = 0$ is then $\delta_0(0) = 0.5$. In figure (1) the optimal survival probability as well as survival probability without investment is depicted. For start capital zero the survival probability increases by about one third and reaches 0.64 due to optimization. Since the exponential distribution is a light tail distribution, the optimal ruin probability goes to zero exponentially fast for $s \to \infty$.

FIGURE 1. $\delta(s)$ for exponential distribution claim size Y.

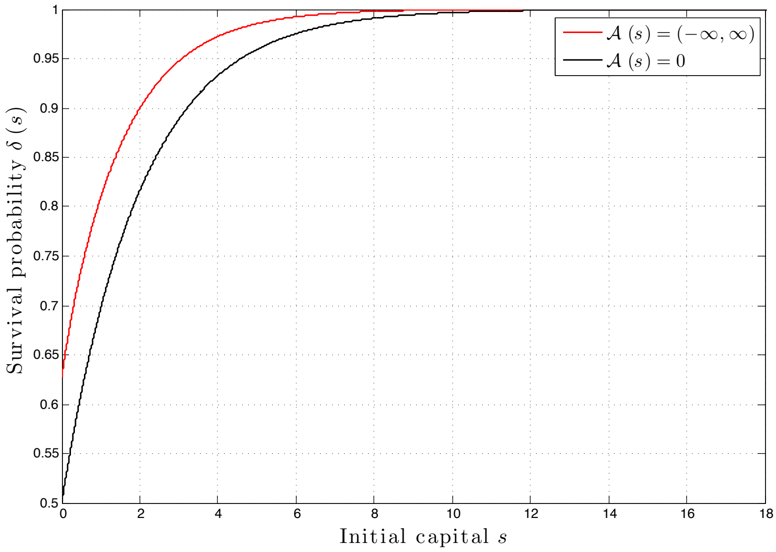


Figure (2) gives the optimal amount of investment A. For small s the optimal investment strategy is highly leveraged. For $s \to \infty$, the asymptotic optimal investment can be computed by solving (4.15). For the exponential distribution with mean 1, we obtain

$$\mathcal{M}_Y(l) = \frac{1}{1-l}.$$

Thus from (4.15), we have $\lim\limits_{s\to\infty} A(s) \approx 1.562$.

EXAMPLE 15. Consider Pareto distributed claim sizes with parameter $p = 2$. The premium income is $c = 2$ and from (3.8), the survival probability without investment at point $s = 0$ is $\delta_0(0) = 0.5$. Figure (3) depicts the optimal survival probability as well as survival probability without investment for $s \in [0, 30]$. It is obvious that the optimal investment gives a considerably higher survival probability; specifically for start capital zero the optimal survival probability growth to 0.65.

Figure (4) shows the optimal investment strategy $A(s)$ divided by s for $s \in [0, 30]$; for $s \leq 1.395$ the optimal strategy is to invest more than the surplus, that is $A(s)/s > 1$. For $s \to \infty$ we have $A(s)/s \to 0.342$. Gaier and Grandits (2002) showed that if the claim size Y with the distribution function $F(y)$ satisfies

$$\lim_{x\to\infty} \frac{F(tx)}{F(x)} = t^p,$$

FIGURE 2. $A(s)$ for exponential distribution claim size Y.

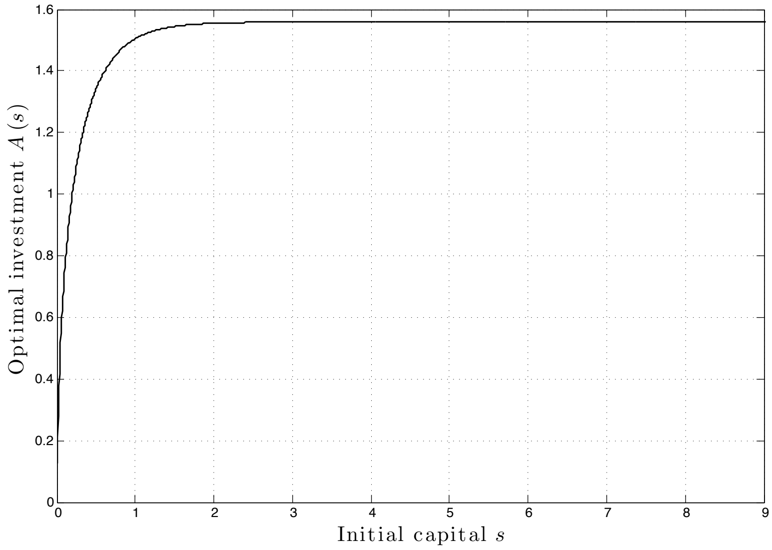
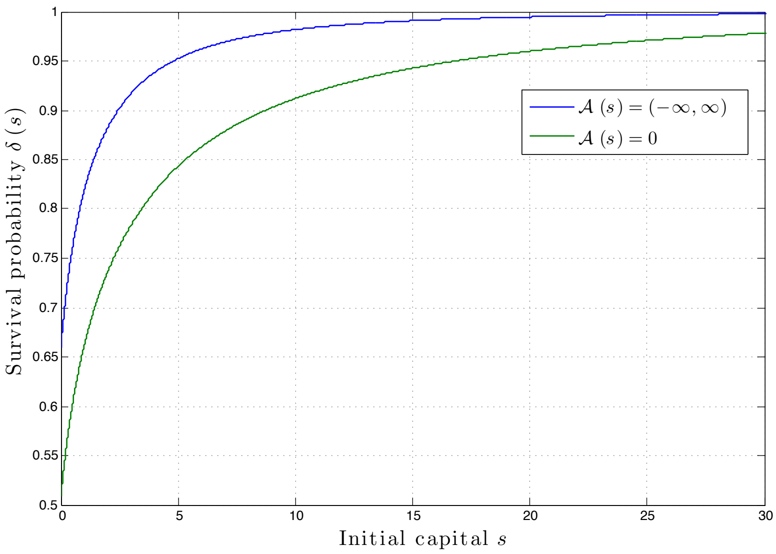


FIGURE 3. $\delta(s)$ for Pareto distributed claim size Y.

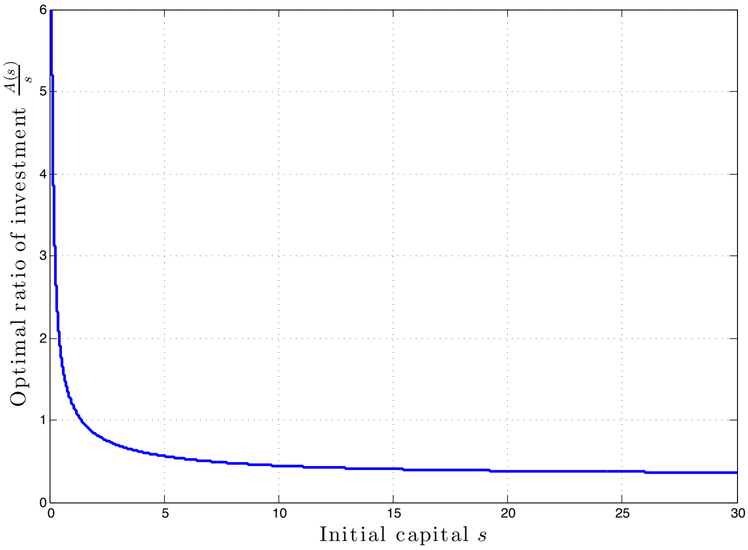


then for $s \to \infty$,

$$\frac{A(s)}{s} \to \frac{r}{\sigma^2 (1-p)}.$$

Using the parameters in this example we obtain $p = -2$ and for $s \to \infty$, $\frac{A(s)}{s} \to \frac{1}{3}$.

FIGURE 4. Optimal ratio of investment to initial capital, $A(s)/s$, for Pareto distributed claim size Y.



4.1.2. Optimal investment with constraint. Azcue and Muler 2009 solved the problem of optimal investment under the investment constraint set $\mathcal{A}(s) = [0, as]$ for $a > 0$. The maximizer of HJB equation

$$\sup_{A \in \mathcal{A}(s)} \left\{ \frac{1}{2}\sigma^2 A^2 V''(s) + (c + rA) V'(s) + \lambda E[V(s-Y) - V(s)] \right\} = 0, \tag{4.17}$$

is either $A(s) = -\frac{rV'(s)}{\sigma^2 V''(s)}$ or $A(s) = as$ or $A(s) = 0$. The HJB equation (4.17) can be rewritten as

$$0 = \sup_{A \in \mathcal{A}_+} \left(V'(s) A^2(s) H_A(s) + 2\lambda \int_0^s H_A(x) E[V(x-Y) - V(x)] dx \right),$$

where $\mathcal{A}_+$ is the set of all piecewise continuous functions satisfying

$$0 \leq A(s) \leq as \ \ and \ \inf_s \frac{A(s)}{s} > 0,$$

and the function $H_A(s)$ is specified by

$$H_A(s) = \frac{1}{A^2(s)} \exp\left(\int_1^s \frac{2(c + rA(x))}{\sigma^2 A^2(x)} dx \right).$$

Define operator T in $[0,\infty)$ as

$$Tw(s) = \inf_{A\in\mathcal{A}_+} \frac{2\lambda\int_0^s H_A(x)\,E\left[W(x-Y)-W(x)\right]dx}{\sigma^2 A^2(s)\,H_A(s)}, \tag{4.18}$$

where $W(x) = 1 + \int_0^x w(t)\,dt$. Azcue and Muller proved that there exists a unique twice differentiable solution to (4.17). They also showed that

$$\lim_{s\to 0} V''(s) = \frac{\lambda}{c}\left(\frac{\lambda}{c} - \frac{ar}{c} - f(0)\right), \tag{4.19}$$

and found an interval $[0,\varepsilon)$, $\varepsilon > 0$ where $A(s) = as$, $s\in[0,\varepsilon)$.

For the numerical purpose, one can rewrite the HJB equation (4.17) as

$$V''(s) = \inf_{A\in\mathcal{A}(s)} \frac{2\lambda E\left[V(s)-V(s-Y)\right] - 2(c+rA)V'(s)}{\sigma^2 A^2},\ s>0, \tag{4.20}$$

and use it for $s>0$ (see [**5**]). The equation (4.20) is formally true for $s\le\varepsilon$, but it results to $V''(0) = -\infty$, also in the case with constraint which from (4.19) can not be true. The operator (4.20) also fails in case without smooth value functions.

Consider now a general constraint set $\mathcal{A}(s)$ which is time consistent in the sense of subsection 2.4.1 and return to the numerical method defined in section 3.3, that is

$$V'_\Delta(s) = \inf_{A\in\mathcal{A}(s)} \frac{\lambda\Delta\left(V_\Delta(s-\Delta) - G_{u_0}(s)\right) + \frac{1}{2}\sigma^2A^2V'_\Delta(s-\Delta)}{\Delta(c+rA-\lambda\Delta)+\frac{1}{2}\sigma^2A^2},$$

with norming $V(0) = 1$ and $V'(0) = \lambda/c$. For $s=\Delta$, from (3.25) we have

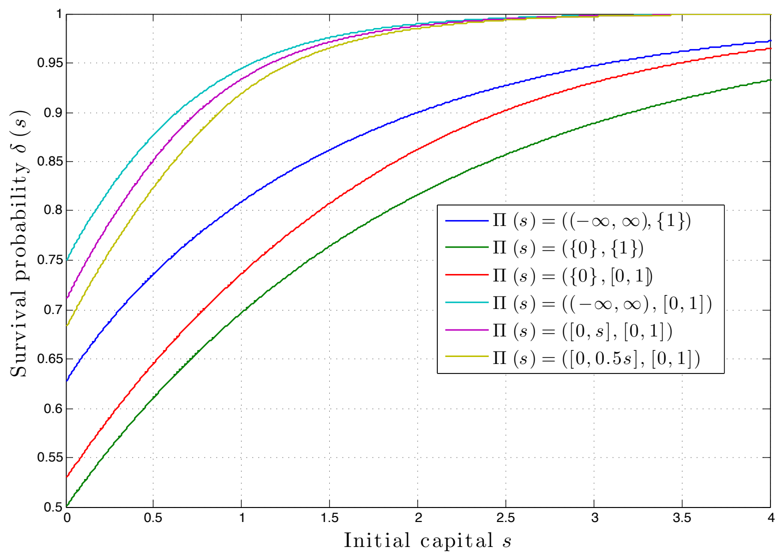
$$V''(\Delta) = \inf_{A\in[0,a\Delta]} \frac{-\lambda F(\Delta) - rA\frac{\lambda}{c} + \frac{\lambda^2}{c}\Delta}{\Delta(c+rA-\lambda\Delta)+\frac{1}{2}\sigma^2A^2}. \tag{4.21}$$

From the Remark (5), the optimal investment is $A = as$. Inserting the optimal investment into (4.21) and letting $\Delta\to 0$ we obtain (4.19).

Example 16. As a numerical example, we first calculate the optimal survival probabilities for constraints $a = 0.2,\ 1$ when the claim sizes are exponentially distributed with mean 1. We choose $\Delta = 0.0001$, $\lambda = 1$, $c = 2$ and for the sake of comparison, we plot in figure 5 the corresponding two optimal survival probabilities as well as the optimal survival probabilities in the unconstrained case ($a=\infty$) and survival probability without investment ($a = 0$).

The optimal amount of investments for $a = 0.2, 1\,\text{and}\,\infty$ are depicted in Figure 6. As it is shown in [**3**], for small $\varepsilon>0$, the optimal investment strategy with constraint

FIGURE 5. $\delta(s)$ for exponential claim size distribution Y.



is to invest $A(s) = as$, $s \in [0, \varepsilon)$. It can be seen that the smaller the a, the larger the ε. Here $\varepsilon = 7.8$ for $a = 0.2$ and $\varepsilon = 1.5$ for $a = 1$. Similar to the Example 14, for $s \to \infty$ the optimal strategies for all values of a tends to be constant at $A(s) = 1.561$. Similar to the theorem 6 in [**24**], we can deduce that for large s, the asymptotic behavior of $\delta(s)$ in the constrained cases coincides with the one in the unconstrained case (see Figure 5).

EXAMPLE 17. We assume that Y is Pareto distributed with parameter $p = 2$. We choose the same parameters as in Example 16. We calculate the optimal amount of investment with two different constraints $a = 0.2$ and 1 and compare it with the case investment without constraint (see Figure 7). In the case $\mathcal{A}(s) = [0, s]$, the optimal ratio of investment $A(s)/s$ converges to $1/3$ as $s \to \infty$, and so the survival function for this constrained case coincides with the one for the unconstrained case for s large enough (see Figure 8). For the constraint set $\mathcal{A}(s) = [0, 0.2s]$, however, the optimal ratio of investment is $A(s)/s = 0.2$ for $s > 0$.

In the next example we see that $\delta''(s)$ is not continuous.

EXAMPLE 18. Assume that Y is exponentially distributed with mean 1 and let $\mathcal{A}(s) = [0, 0.2s]$, for $0 < s < 0.5$ and $\mathcal{A}(s) = (-\infty, \infty)$, for $s \geq 0.5$. Note that this

FIGURE 6. $A(s)$ for exponential claim size distribution Y.

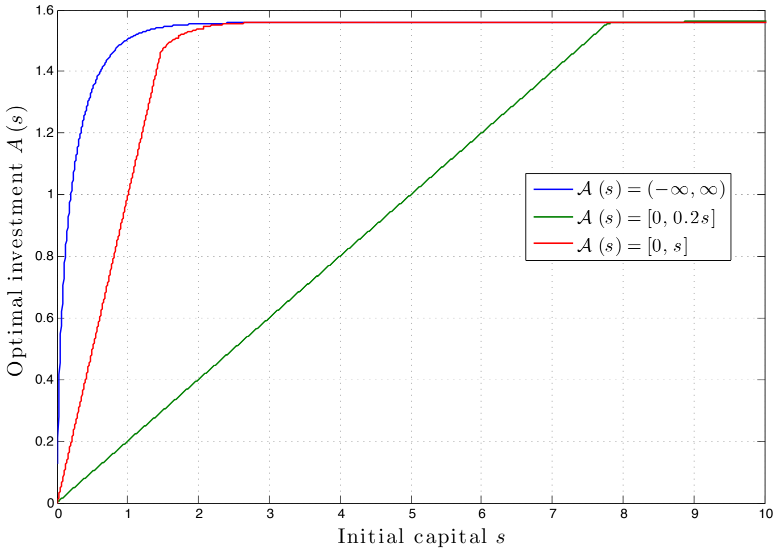
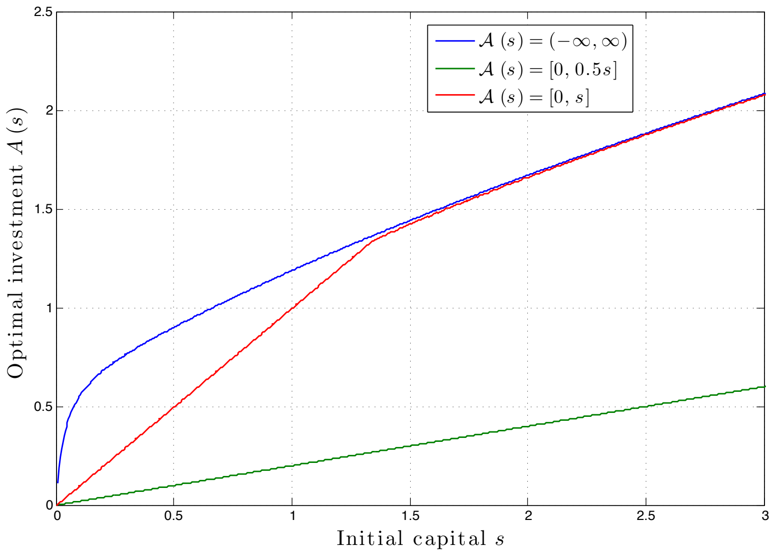
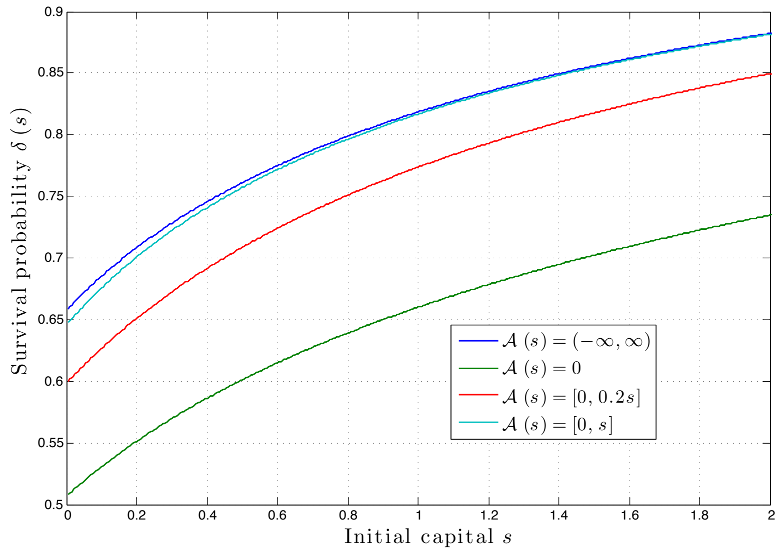


FIGURE 7. $A(s)$ for Pareto distributed claim size Y.



set of constraint is not satisfying the assumption in the theorem 6. Set the parameters $\lambda = 1$, $c = 0.5$, $r = \sigma^2 = 1$ and choose $\Delta = 0.0001$. There is a jump in the function $\delta''(s)$ as well as $A(s)$, at the point where the feedback function jumps (see Figure 9

FIGURE 8. $\delta(s)$ for Pareto distributed claim size Y.



and 10). Note that the equation (4.19) yields

$$\lim_{s\to 0}\delta''(s) = \lim_{s\to 0}\frac{V''(s)}{V(\infty)} = \frac{1}{V(\infty)}\frac{1}{0.5}\left(\frac{1}{0.5} - \frac{0.2}{0.5} - 1\right) = \frac{1.2}{V(\infty)} > 0.$$

Here we have $\delta''(s) = \frac{V''(s)}{V(\infty)} = 0.1749V''(s)$.

The solution of HJB equation under the constraint set $\mathcal{A}(s) = [0, 0.2s]$, $s \geq 0$, is twice continuously differentiable (see [**3**]). Thus we can find an interval $[0, \varepsilon)$, $\varepsilon > 0$, on which $\delta''(s) \geq 0$. On the other hand, for the constraint set $\mathcal{A}(s) = (-\infty, \infty)$, it holds $\delta''(s) < 0$, for $s \geq 0$. Hence, for the constraint sets of the form

$$\mathcal{A}(s) = \begin{cases} [0, 0.2s] & 0 < s < \varepsilon \\ (-\infty, \infty) & s \geq \varepsilon, \end{cases}$$

by choosing small enough $\varepsilon > 0$, we have $\delta''(\varepsilon-) \neq \delta''(\varepsilon+)$.

In the following example we construct the admissible set of strategy $\mathcal{A}(s)$ in such a manner that $\delta'(s)$ is not continuous. Notice that the constraint set below does not fulfill the assumptions in the theorem 6 and Proposition 7.

EXAMPLE 19. We use the same parameters as in Example 16, but we redesign the set of admissible strategy as $\mathcal{A}(s) = (-\infty, \infty)$, $0 < s < 0.5$, $\mathcal{A}(s) = \{0\}$, $s \geq 0.5$. The

FIGURE 9. $\delta''(s)$ for $\mathcal{A}(s) = [0, 0.2s]$, for $0 < s < 0.5$ and $\mathcal{A}(s) = (-\infty, \infty)$, for $s > 0.5$.

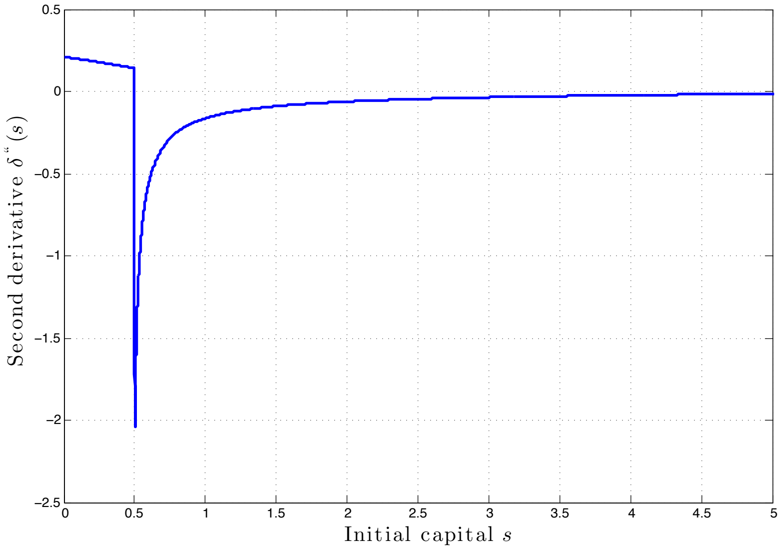
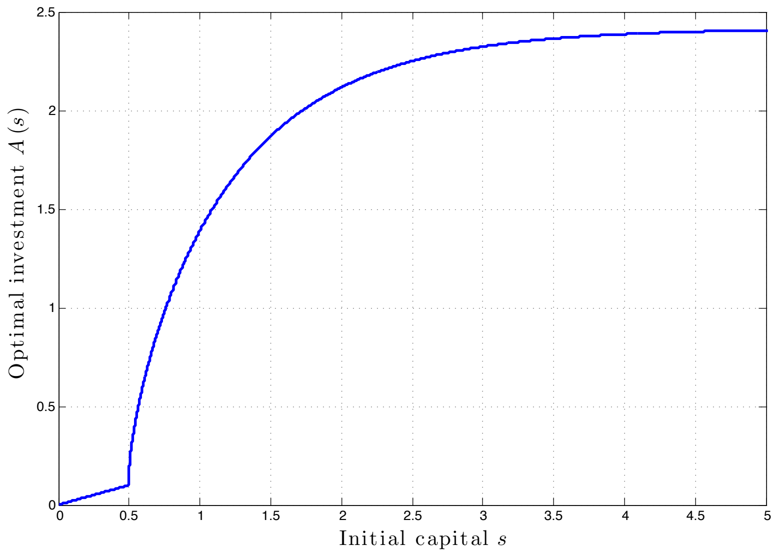
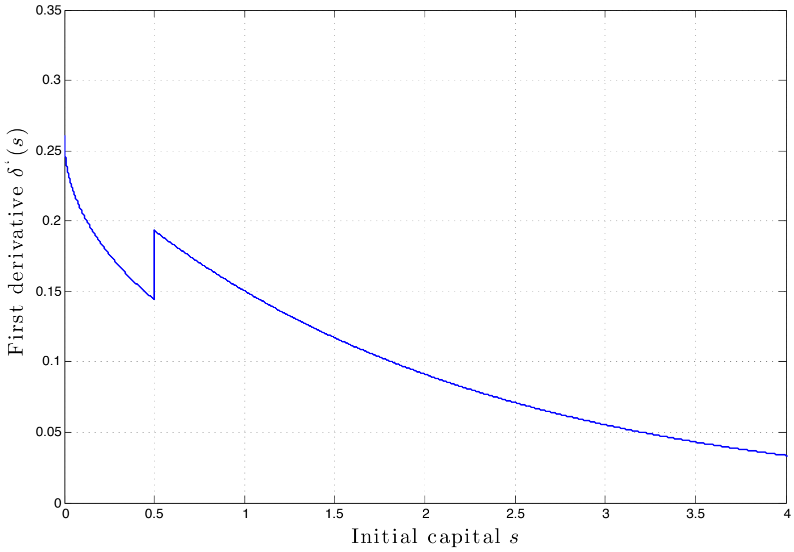


FIGURE 10. $A(s)$ for $\mathcal{A}(s) = [0, 0.2s]$, for $0 < s < 0.5$ and $\mathcal{A}(s) = (-\infty, \infty)$, for $s > 0.5$.



function $\delta'(s)$ for $s \in [0, 4]$ is drawn in figure 11. There is a jump at $s = 0.5$ where the feedback function jumps. In fact, from (4.6) for $s < s_0$ where $\mathcal{A}(s) = (-\infty, \infty)$ we can

FIGURE 11. $\delta^{'}(s)$ for $\mathcal{A}(s) = (-\infty, \infty)$, $0 < s < 0.5$, $\mathcal{A}(s) = \{0\}$, $s \geq 0.5$.



write

$$\delta^{'}(s_0-) = \lambda \frac{\delta(s_0) - G_{u_0}(s_0)}{c + 1/2 A(s_0)},$$

while

$$\delta^{'}(s_0+) = \frac{\lambda}{c}(\delta(s_0) - G_{u_0}(s_0)) \neq \delta^{'}(s_0-),$$

if for $s \geq s_0$, $\mathcal{A}(s) = \{0\}$.

Next, we present an example that does not have optimal strategy for no $s \geq 0$. The reader must notice again that the constraint set below does not satisfy the assumptions in the theorem 6 and Proposition 7.

EXAMPLE 20. We let $\mathcal{A}(s) = (-\infty, s]$, $s > 0$, $\mathcal{A}(0) = \{0\}$ and assume that the risky asset governed by geometric Brownian motion with drift $r = 0.02$ and volatility $\sigma = 0.1$. Moreover, we assume that Y is exponentially distributed with mean 1 and choose $c = 0.05$, $\lambda = 0.09$ and $\Delta = 0.001$. For small value of s say $0 < s \leq s_0$, we have $\delta^{''}(s) = 0$ and optimal strategy in this interval is $A(s) = -\infty$. For $s \leq s_0$ we have $\delta^{'}(s) = \delta^{'}(0) = \delta(0)\frac{\lambda}{c}$ and $\delta(s) = \delta(0)\left(1 + s\frac{\lambda}{c}\right)$, and therefore s_0 can be computed by

$$s_0 = \inf\left\{s > 0 : \lambda E[\delta(s) - \delta(s - Y)] - (c + rs)\delta(0)\frac{\lambda}{c} < 0\right\} = 1.593.$$

The function $\delta^{''}(s)$ and $A(s)$ are given in figure 12 and figure 13, respectively.

FIGURE 12. $\delta''(s)$ for exponential claims with $\mathcal{A}(s) = (-\infty, s]$.

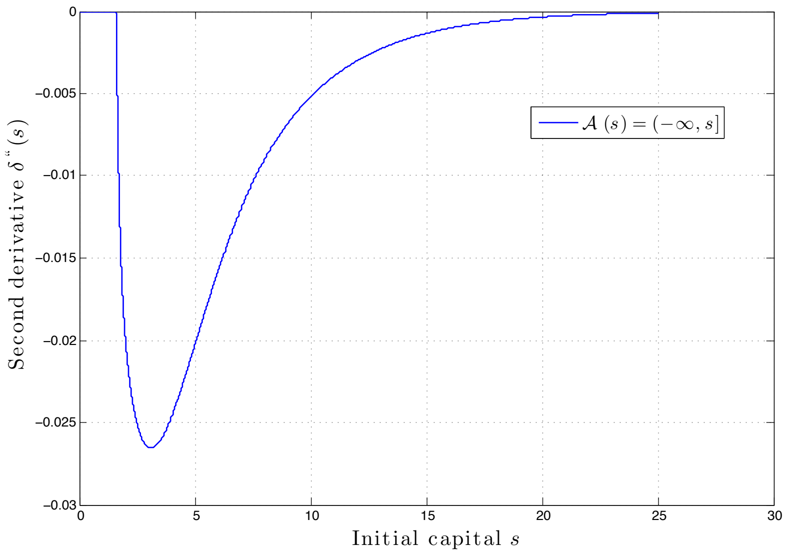
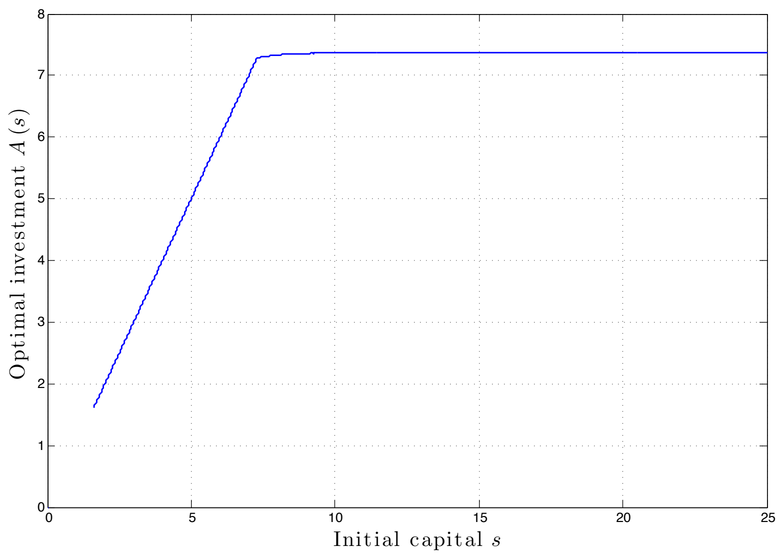


FIGURE 13. $A(s)$ for exponential claims with $\mathcal{A}(s) = (-\infty, s]$.



In the next example we solve the optimal investment problem for distribution having isolated point with positive mass.

EXERCISE 21. We let the claim size probability mass function be $\Pr\{Y = 1\} = 1$ and set the parameters as follows: $\lambda = 1$, $c = 2$, $r = 1$, $\sigma = 1$. We select $\Delta = 0.0005$ and use our numerical method to solve the optimal investment problem for two constraint sets $\mathcal{A}_1(s) = (-\infty, \infty)$ and $\mathcal{A}_2(s) = [0, s]$.

The optimal investment $A(s)$, and the function $\delta''(s)$, for the two constraint sets $\mathcal{A}_1$ and $\mathcal{A}_2$, are plotted in figure 14 and 15, respectively. In the interval $[0, 1)$, the optimal investment is highly leveraged for the unconstrained case $\mathcal{A}_1$, and $A(s) = s$ for the constrained set $\mathcal{A}_2$. This is because, in this interval the insurer gets ruined with probability one if a claim occurs, that is $E[\delta(s - Y)] = 0$. Thus, for $s \in [0, 1)$ the insurer's goal is to achieve capital one as fast as possible.

In the constrained set $\mathcal{A}_2(s) = [0, s]$, for $s \in [0, 1)$, $\delta''(s) = 0$ and therefore $\delta(s) = \delta(0)(1 + s/c)$. This can be seen by inserting $A(s) = s$ and applying the operator 4.18. Note that in the operator 4.18 for the parameters $\lambda = r = \sigma = 1$ and $A(s) = s$, we obtain

$$H_A(s) = \exp\left(2c - \frac{2c}{s}\right), \; s \in [0, 1),$$

and hence

$$w(s) = \frac{1}{c}.$$

At point $s = 1$, $\delta''(1)$ tends to minus infinity and therefore the optimal investment is $A(1) = 0$, for both constraint sets $\mathcal{A}_1$ and $\mathcal{A}_2$. Note that for $A(1) \neq 0$, the ruin probability is one if a claim happens because of the fluctuation of the Wiener process. So, if c is considerably larger than λ, then the optimal is to be risk averse and not to invest in risky asset.

As $s \to \infty$ the optimal investments for the two constrained sets tends to be constant. Again, with the same argument used in Example 14 by solving equation (4.16), we can find the asymptotic optimal investment. The adjustment coefficient l^* for this example is 1.507 and the asymptotic optimal strategy is $\lim_{s\to\infty} A(s) = 0.663$.

4.2. Optimal proportional reinsurance with investment

In this section we assume that the part of claim paid by insurer is $g(Y, \alpha) = \alpha Y$, $0 \leq \alpha \leq 1$ and the rest of it, i.e. $(1 - \alpha) Y$, will be paid by reinsurance. For this, insurer pays reinsurance premium based on expected value principle, i.e. $h(\alpha) = (1 - \alpha)(1 + \theta)\lambda E[Y]$ where θ is the reinsurance safety loading. Notice that the reinsurance safety loading θ must be always more than the first insurance safely loading

FIGURE 14. $A(s)$ for claim size probability mass function $\Pr\{Y=1\}=1$.

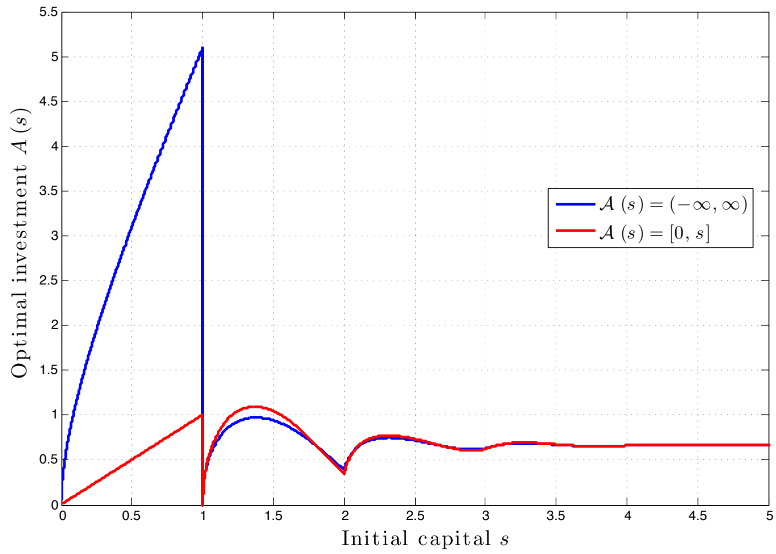
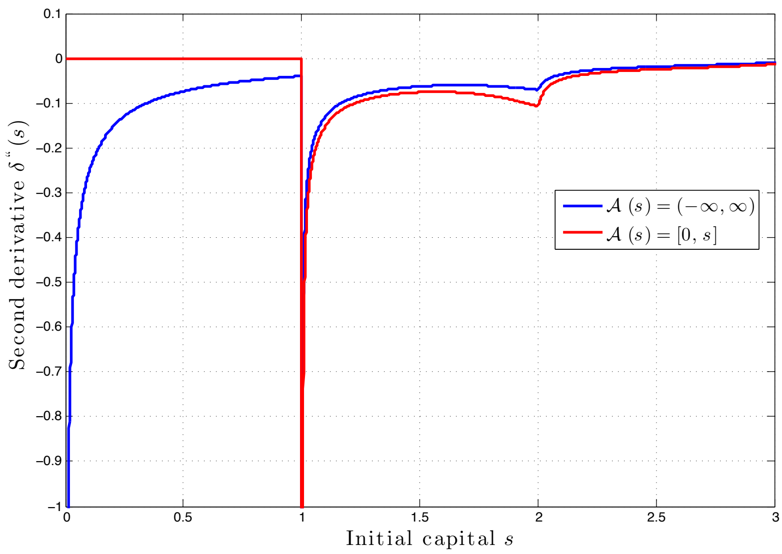


FIGURE 15. $\delta''(s)$ for claim size probability mass function $\Pr\{Y=1\}=1$.



η, otherwise the insurance company can transfer the whole risk to the reinsurance company and still receive a positive return without any risk.

The set of admissible strategy is $\Pi(s) = (\mathcal{A}(s), [0,1]) \in \mathbb{R}^2$ and the HJB equation to be solve for $s \geq 0$ is

$$\sup_{(A,\alpha)\in(\mathcal{A}(s),[0,1])} \{\frac{1}{2}\sigma^2 A^2 V''(s) + (c - (1-\alpha)\rho\lambda E[Y] + rA) V'(s) + \lambda E[V(s-\alpha Y) - V(s)]\} = 0, \tag{4.22}$$

where $\rho = (1+\theta)$. We use again the norming $V(0) = 1$ and from (3.20), $V'(0) = \lambda/c$.

Consider now the unconstrained investment, i.e. $\Pi(s) = ((-\infty,\infty), [0,1])$. If we plug in the optimal investment $A(s) = -\frac{rV'(s)}{\sigma^2 V''(s)}$ into (4.22), we get for $s \geq 0$,

$$\sup_{\alpha\in[0,1]} \left\{ -\frac{1}{2}\frac{r^2 V'(s)^2}{\sigma^2 V''(s)} + (c - (1-\alpha)\rho\lambda E[Y]) V'(s) + \lambda E[V(s-\alpha Y) - V(s)] \right\} = 0. \tag{4.23}$$

Rearranging above equation we have

$$\frac{V''(s)}{V'(s)^2} = \frac{r^2}{2\sigma^2} \frac{1}{\sup\limits_{\alpha} \{(c - (1-\alpha)\rho\lambda E[Y]) V'(s) + \lambda E[V(s-\alpha Y) - V(s)]\}}. \tag{4.24}$$

Integration yields

$$V'(x) = \left(\frac{r^2}{2\sigma^2} \int_0^x \frac{1}{\inf\limits_{\alpha} \{\lambda E[V(s) - V(s-\alpha Y)] - (c - h(\alpha)) V'(s)\}} ds + \frac{c}{\lambda} \right)^{-1}. \tag{4.25}$$

Schmidli (2002) applied iterative operator theory in order to prove that there exists a smooth solution to (4.25). He also showed that there is always an interval $[0,\varepsilon)$, $\varepsilon > 0$, on which the optimal is not buying reinsurance. So, on this interval, one can use the system of equation (4.8) and (4.9) and numerically calculate the optimal solution. For $s \geq \varepsilon$, the numerical solution can be iterated using (4.25). This numerical method can only be applied if $\mathcal{A}(s) = (-\infty,\infty)$.

Let A and α be respectively, an arbitrary constant investment and reinsurance strategy. Define $Z := \alpha Y - (c - h(\alpha) + rA)\xi - \sigma A W(\xi)$, where the random variables ξ and $W(\xi)$ are, respectively, exponentially distributed with mean $\lambda^{-1} > 0$ and normally distributed with mean 0 and variance ξ. If $\mathcal{M}_{\alpha Y}(l)$ exists, then the adjustment coefficient $l = l(A,\alpha) > 0$ satisfies

$$\frac{1}{2}\sigma^2 A^2 l^2 - (c - h(\alpha) + rA) l + \lambda \mathcal{M}_{\alpha Y}(l) - \lambda = 0. \tag{4.26}$$

For the problem of optimal constant investment and reinsurance strategies, the adjustment coefficient l^* must satisfy $l^* = \sup_{A\geq 0, 1\geq\alpha\geq 0} l(A,\alpha)$. Since the left hand side of equation (4.26) is nonnegative for l, finding l is equivalent to solving the following equation

$$\inf_{A\geq 0, 1\geq\alpha>0}\left\{\frac{1}{2}\sigma^2 A^2 l^2 - (c-h(\alpha)+rA)\,l + \lambda\mathcal{M}_{\alpha Y}(l) - \lambda\right\} = 0.$$

By inserting the optimal constant strategy, $A^* = \frac{r}{l^*\sigma^2}$, into the above equation we have

$$\inf_{1\geq\alpha>0}\left\{\frac{1}{2}\frac{r^2}{\sigma^2} + (c-h(\alpha))\,l^* + \lambda - \lambda\mathcal{M}_{\alpha Y}(l^*)\right\} = 0. \tag{4.27}$$

Differentiating above relation with respect to α, for $h(\alpha) = (1-\alpha)\rho\lambda E[Y]$ we have

$$l^*\rho\lambda E[Y] = \lambda\int_0^\infty l^* y e^{l^*\alpha y} dF(y). \tag{4.28}$$

For $\int_0^\infty l^* y e^{rl^*\sigma^{-2}\alpha y} dF_Y < \infty$, the above equation has a solution α^*. Comparing l^* with the corresponding adjustment coefficient $l(0)$ without investment, we have $l^* = \sup_{A\geq 0} l(A,\alpha) \geq l(0,1)$.

Let $\psi(s)$ be the optimal ruin probability. With the exact same argument used in [**24**], one can show that there exists a constant $k\in(0,\infty)$ such that $\lim_{s\to\infty}\psi(s)e^{l^*s} = k$. A direct result of this is that the optimal investment and reinsurance strategies converge to the optimal constant investment and reinsurance strategies, i.e. $\lim_{s\to\infty} A(s) = A^*$ and $\lim_{s\to\infty}\alpha(s) = \alpha^*$.

Recalling the approximation defined in section (3.3), for $s = i\Delta$, $i = 1,2,...$, we have

$$V_\Delta'(s) = \inf_{\pi\in\Pi}\frac{\lambda\Delta\left(V_\Delta(s-\Delta) - G_\alpha(s)\right) + \frac{1}{2}\sigma^2 A^2 V_\Delta'(s-\Delta)}{\Delta\left(c-h(\alpha)+rA-\lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2}, \tag{4.29}$$

where

$$G_\alpha(s) = \sum_{\{\alpha j\leq i\}} V_\Delta\left((i-1)\Delta - \left\lfloor\frac{\alpha j}{\Delta}\right\rfloor\Delta\right)\Pr\left\{(j-1)\Delta < Y \leq j\Delta\right\},\ \ j=0,1,2,....$$

Note that $G_\alpha(s)\to 0$ as $s\to 0$. So, for small s the optimal reinsurance strategy is buying no-reinsurance which maximizes the denominator in (4.29).

We now illustrate the result of using our numerical method (4.29). In the following examples we choose $\Delta = 0.001$, $\lambda = 1$, $c = 2$, $\rho = (1+\theta) = 2.5$, $r = \sigma = 1$ and calculate the survival probabilities for the following scenarios:

(1) $\Pi_1(s) = (\{0\}, [0,1])$: optimal reinsurance without investment,
(2) $\Pi_2(s) = ((-\infty,\infty), \{1\})$: optimal investment without constraint,
(3) $\Pi_3(s) = ((-\infty,\infty), [0,1])$: optimal reinsurance and investment without constraint,
(4) $\Pi_4(s) = ([0,s], [0,1])$: optimal reinsurance and investment with constraint $\mathcal{A}(s) = [0,s]$, and
(5) $\Pi_5(s) = ([0,0.5s], [0,1])$: optimal reinsurance and investment with constraint $\mathcal{A}(s) = [0,0.5s]$.

We also denote the optimal investment and reinsurance for each above scenarios with $A_i(s)$ and $\alpha_i(s)$, $i = 1, ..., 5$, respectively.

Example 22. Consider exponentially distributed claim size with mean 1. The optimal strategies for different above sets of constraints are given in Figures 16 and 17. As we expected, there always exists an interval $[0,\varepsilon)$, where the optimal reinsurance strategy is no-reinsurance. In this interval the optimal investment strategies are as the same as Example 16.

In the case reinsurance and investment (scenarios $i = 3, 4$ and 5), from certain point $s_{0,i} > 0$, the optimal is to buy reinsurance. In this example $s_{0,3} = 0.19$, $s_{0,4} = 0.485$, $s_{0,5} = 0.725$. From point $s_{0,i}$, $i = 3,4,5$, the insurer must reduce its investment in risky asset in order to finance its reinsurance costs (see Figure 16).

For exponential distribution with mean $1/m$, the equation (4.28) reads

$$l^*\rho\frac{1}{m} = \int_0^\infty l^* m y e^{(l^*\alpha - m)y} dy.$$

This equation, for $\alpha < \frac{m}{l}$, has the solution

$$\alpha^* = m\frac{\rho - \sqrt{\rho}}{l^*\rho}. \tag{4.30}$$

Hence, the equation (4.27) at α^* reads

$$\frac{1}{2}\frac{r^2}{\sigma^2} + \left(c - (1-\alpha^*)\rho\lambda\frac{1}{m}\right)l^* + \lambda = \frac{\lambda m}{m - l^*\alpha^*}. \tag{4.31}$$

FIGURE 16. $A(s)$ for exponential claim size distribution Y.

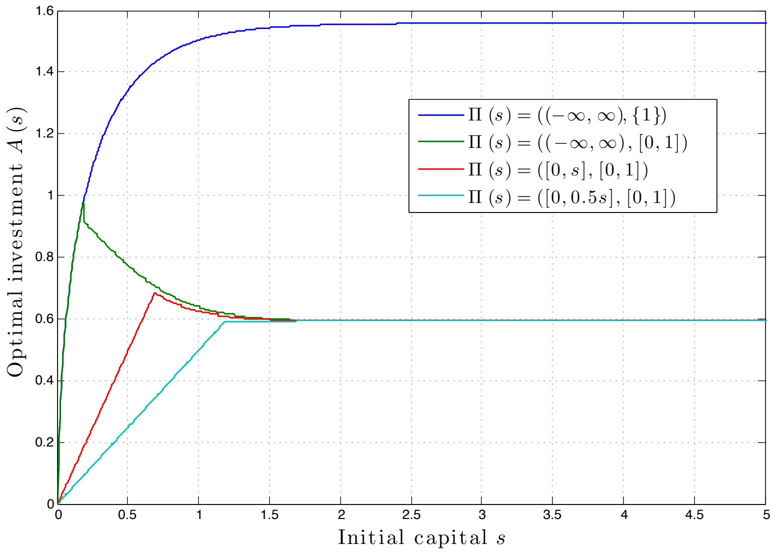
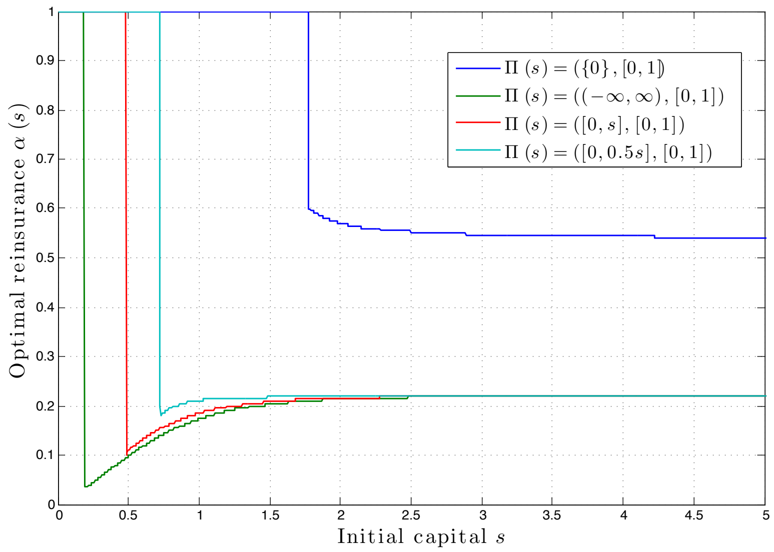
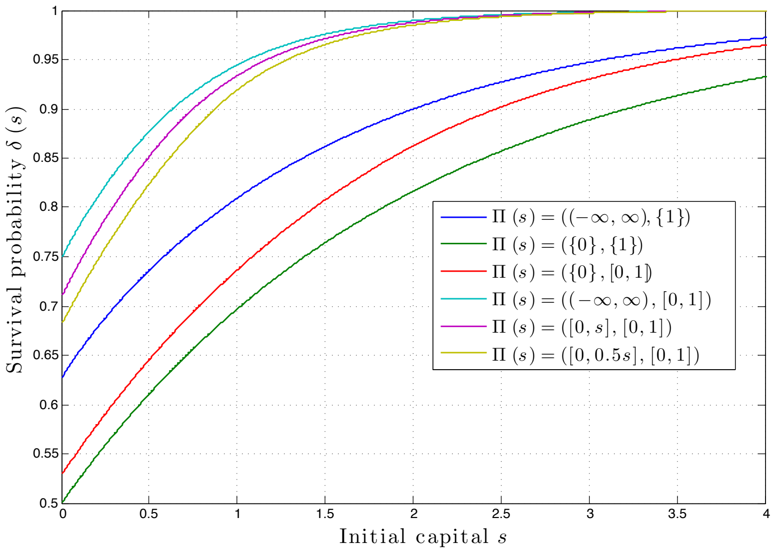


FIGURE 17. $\alpha(s)$ for exponential claim size distribution Y.



Using the parameters in this example, by solving the system of two equations (4.30) and (4.31), we obtain for $i = 3, 4, 5$, $A_i(s) \to 0.6$ and $\alpha_i(s) \to 0.22$ as $s \to \infty$. For the optimal constant reinsurance without investment α_1^*, instead of equation (4.31), we

FIGURE 18. $\delta(s)$ for exponential claim size distribution Y.



have

$$\left(c - (1-\alpha_1^*)\,\rho\lambda\frac{1}{m}\right) l^* + \lambda = \frac{\lambda m}{m - l^*\alpha_1^*}.$$

Thus we can deduce from the above equation and (4.30), $\alpha_1(s) \to 0.55$ as $s \to \infty$.

Figure 18 depicts the survival probabilities for $s \in [0,5]$. It is clear that the combination of investment and reinsurance for $s \geq 0$ significantly increase the survival probability.

EXAMPLE 23. Consider Pareto distributed claim size distribution with parameter $p = 2$, so $E[Y] = 1$. Figure 21 shows the optimal reinsurance strategies $\alpha_i(s)$ for the scenarios $i = 1,3,4,5$. As we expected, for given constraint set $\Pi_i(s)$, there exists an interval $[0, s_{0,i})$ where the optimal reinsurance strategy is no reinsurance. Here $s_{0,1} = 1.77$, $s_{0,3} = 0.53$, $s_{0,4} = 0.68$ and $s_{0,5} = 1$. For given scenario i, from the point $s_{0,i}$ the insurer can afford reinsurance and, to compensate the reinsurance cost, the insurer must invest less than $A_2(s)$, the optimal investment without reinsurance (see figure 19). The interesting fact for the combined set of investment and reinsurance, is that the proportion after a slight increase, gradually decreases and then jumps to zero (see figure 20). Thus, from some point $s_{1,i} > s_{0,i}$, $i = 3,4,5$, the whole insurance risk is transferred to the reinsurer and the insurance premium rate left to the insurer is then $-1/2$. Since for $s > s_{1,i}$ the insurer is left with the investment risk, from the HJB equation (4.22) as $s \to \infty$ we get

$$c - \lambda\rho E[Y] + \frac{1}{2}A = 0, \tag{4.32}$$

and thus for $i = 3, 4, 5$, $A_i(s) \to 1$ as $s \to \infty$. In this example $s_{1,3} = 7.18$, $s_{1,4} = 7.21$ and $s_{1,5} = 7.34$. We can also conclude that under the optimal investment and reinsurance the survival probabilities goes to one exponentially fast (see [**38**], Example 5.2, where a different numerical method has been used).

For the case optimal reinsurance without investment, since $E\left[e^{l\alpha Y}\right]$ is not bounded for $l, \alpha > 0$, the equation

$$\left(c - (1-\alpha_1^*)\,\rho\lambda E[Y]\right) l^* + \lambda = \mathcal{M}_{\alpha_1^* Y}(l^*)$$

does not have a solution. However, we can still compute the asymptotic optimal reinsurance strategy for subexponential distribution by applying the following approximation for ruin probability $\psi(s)$ (see [**14**] and [**42**]):

$$\lim_{s\to\infty} \frac{\psi(s)}{\bar{H}(s)} = \frac{q}{1-q}, \tag{4.33}$$

where $q = \frac{\lambda E[Y]}{c}$ and $\bar{H}(s) = \frac{1}{E[Y]}\int_s^\infty \bar{F}(y)\,dy$.

Let $\alpha_1 \in [0,1]$ be a constant proportional reinsurance strategy, then the ruin happens if $Y > s/\alpha_1$. Therefore, from (4.33) we have

$$\lim_{s\to\infty} \frac{1-\delta_1(s)}{\frac{1}{E[Y]}\int_{s/\alpha_1}^\infty \bar{F}(y)\,dy} = \frac{\lambda\alpha_1 E[Y]}{c - \rho\lambda(1-\alpha_1)E[Y]}.$$

For the Pareto distributed claim size with parameter $p = 2$, holds $1/E[Y]\int_{s/\alpha_1}^\infty \bar{F}(y)\,dy = \alpha_1/(\alpha_1+s)$. Hence for the large values of s, we yield

$$\delta_1(s) \approx 1 - \frac{\alpha_1}{(c-\rho(1-\alpha_1))\left(1+\frac{s}{\alpha_1}\right)}.$$

The survival probability is maximum when $\frac{\alpha_1}{(c-\rho(1-\alpha_1))\left(1+\frac{s}{\alpha_1}\right)}$ is minimum, so

$$\alpha_1 = \frac{2(\theta-\eta)s}{\theta s - \theta - \eta},$$

FIGURE 19. $A(s)$ for Pareto distributed claim size Y.

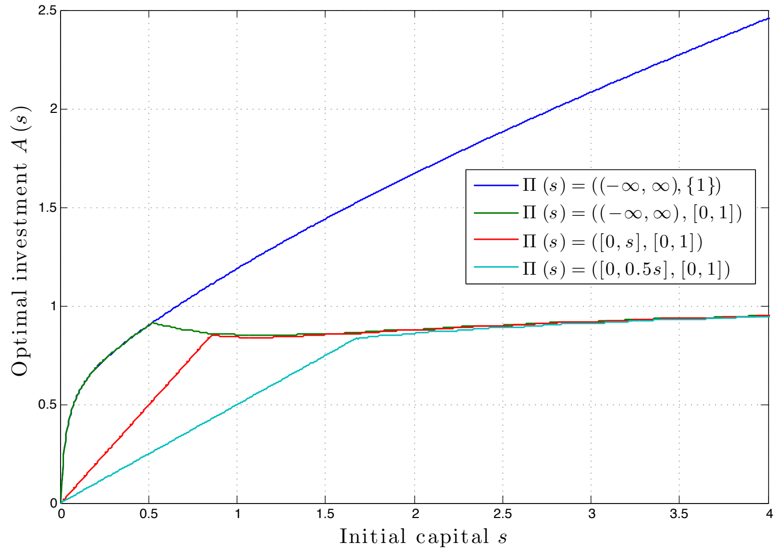
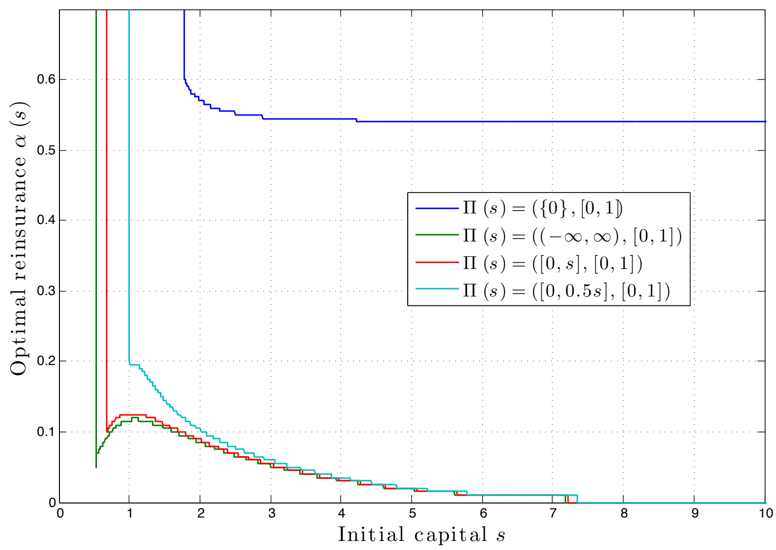


FIGURE 20. $\alpha(s)$ for Pareto distributed claim size Y.

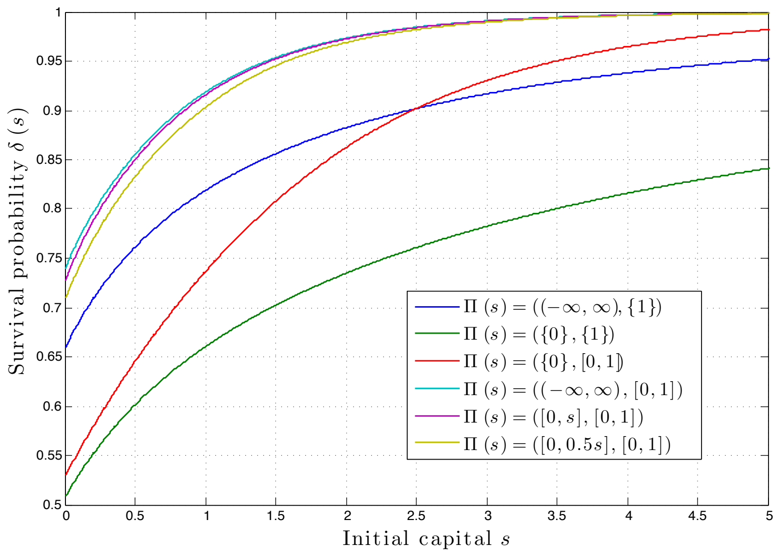


and for $s \to \infty$ we find the asymptotic optimal reinsurance

$$\alpha_1 = \min\left\{\frac{2(\theta-\eta)}{\theta}, 1\right\}.$$

Hence, in this example, $\alpha_1(s) \to 0.55$ as $s \to \infty$.

FIGURE 21. $\delta(s)$ for Pareto distributed claim size Y.



4.3. Optimal XL reinsurance with investment

Recall that in the XL reinsurance with priority $0 \leq M \leq \infty$, if a claim Y happens, then the first insurer pays $g(Y,M) = \min\{Y,M\}$, and the rest i.e. $(Y-M)^+ = \max\{Y-M,0\}$, will be payed by the reinsurer. The set of admissible strategy is then $\Pi(s) = (\mathcal{A}(s), \mathbb{R}^+) \in \mathbb{R}^2$ and the HJB equation to be solve for $s \geq 0$ is

$$\sup_{(A,M)\in(\mathcal{A}(s),\mathbb{R}^+)} \{\frac{1}{2}\sigma^2 A^2 V''(s) + (c - h(M) + rA) V'(s) + \lambda E[V(s - \min\{Y,M\}) - V(s)]\} = 0. \tag{4.34}$$

We consider that the reinsurance premium is calculated using the expected value principle, i.e. $h(M) = \lambda\rho E\left[(Y-M)^+\right]$, $\rho = (1+\theta) > 1$. With integration by part

$$h(M) = \lambda\rho \int_M^\infty (1 - F(y))\, dy.$$

Consider now the three following cases:

(1) $M > s$: Since the term $E[V(s - \min\{Y,M\})]$ does not depend on M, the optimal is no-reinsurance, i.e. $M = \infty$.

(2) $M = s$: This strategy can be interpreted as the cheapest reasonable reinsurance, since for $M > s$ optimal is buy no-reinsurance.

(3) $M \leq s$: Here with integration by parts we have

$$E\left[V\left(s-\min\left\{Y,M\right\}\right)-V\left(s\right)\right]=-\int_0^M V^{'}\left(s-y\right)\left(1-F\left(y\right)\right)dy. \tag{4.35}$$

Differentiating argument (4.34) respect to M, we also have the following normal equation

$$\rho V^{'}\left(s\right)=V^{'}\left(s-M\right). \tag{4.36}$$

Therefore, we can restrict ourselves to the set $\Pi\left(s\right)=\left(\mathcal{A}\left(s\right),\left[0,s\right]\cup+\infty\right)$. For small s, since for $M \leq s$ the term in (4.35) varies slowly whereas $h\left(M\right)$ varies quickly, there exists a neighborhood of zero such that optimal is no-reinsurance, i.e. $M=\infty$.

We now study the optimal constant investment and reinsurance strategies. Choose arbitrary constant investment and reinsurance strategies A and M. Let $Z:=\min\left\{Y,M\right\}-\left(c-h\left(M\right)+rA\right)\xi-\sigma AW\left(\xi\right)$, where the random variables ξ and $W\left(\xi\right)$ are, respectively, exponentially distributed with mean $\lambda^{-1}>0$ and normally distributed with mean 0 and variance ξ. If $\mathcal{M}_{\min\{Y,M\}}\left(l\right)$ exists then the adjustment coefficient $l=l\left(A,M\right)>0$ satisfies

$$\frac{1}{2}\sigma^2A^2l^2-\left(c-h\left(M\right)+rA\right)l+\lambda\mathcal{M}_{\min\{Y,M\}}\left(l\right)-\lambda=0. \tag{4.37}$$

This equation corresponds to the HJB equation (4.34) with the solution $V\left(s\right)=1-e^{-ls}$ and constant strategies A and M. The adjustment coefficient $l^*=\sup_{A,M} l\left(A,M\right)$ is then the solution of our optimal problem. The left hand side of equation (4.37) is nonnegative for l^*, thus finding l^* is equivalent to solving the following equation

$$\inf_{A,M}\left\{\frac{1}{2}\sigma^2A^2l^2-\left(c-h\left(M\right)+rA\right)l+\lambda\mathcal{M}_{\min\{Y,M\}}\left(l\right)-\lambda\right\}=0.$$

Plugging in the optimal constant strategy, $A^*=\frac{r}{l^*\sigma^2}$, into the above equation, it holds

$$\inf_{M}\left\{\frac{1}{2}\frac{r^2}{\sigma^2}-\left(c-h\left(M\right)\right)l^*+\lambda-\lambda\int_0^M e^{l^*y}dF\left(y\right)-\lambda\int_M^\infty e^{l^*M}dF\left(y\right)\right\}=0. \tag{4.38}$$

Differentiating above relation with respect to M, for $h\left(M\right)=\lambda\rho\int_M^\infty\left(1-F\left(y\right)\right)dy$, we have

$$M^*=\frac{\ln\left(\rho\right)}{l^*}. \tag{4.39}$$

With a similar argument used in [**24**] (see also [**42**]), for the optimal ruin probability $\psi(s) = 1 - \delta(s)$, one can show that there exists a constant $k \in (0,\infty)$ such that $\lim_{s\to\infty} \psi(s) e^{l^* s} = k$. Moreover, $\lim_{s\to\infty} A(s) = A^*$ and $\lim_{s\to\infty} M(s) = M^*$.

Letting $V(0) = 1$, (3.20) yields $V'(0) = \lambda/c$. The numerical method in section (3.3) reads

$$V'_\Delta(s) = \inf_{\pi\in\Pi} \frac{\lambda\Delta\left(V_\Delta(s-\Delta) - G_M(s)\right) + \frac{1}{2}\sigma^2 A^2 V'_\Delta(s-\Delta)}{\Delta\left(c - h(M) + rA - \lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2},$$

where for $M = k\Delta \le s$, we have

$$G_M(s) = \sum_{j=1}^{k} V_\Delta((i-j)\Delta) \Pr\{(j-1)\Delta < Y \le j\Delta\} + V((i-k)\Delta)\bar{F}(M).$$

We now present some numerical examples with the similar scenarios provided in Example 22, which are

(1) $\Pi_1(s) = (\{0\}, [0,\infty])$: optimal reinsurance without investment,

(2) $\Pi_2(s) = ((-\infty,\infty), \{\infty\})$: optimal investment without constraint,

(3) $\Pi_3(s) = ((-\infty,\infty), [0,\infty])$: optimal reinsurance and investment without constraint,

(4) $\Pi_4(s) = ([0,s], [0,\infty])$: optimal reinsurance and investment with constraint $\mathcal{A}(s) = [0,s]$ and

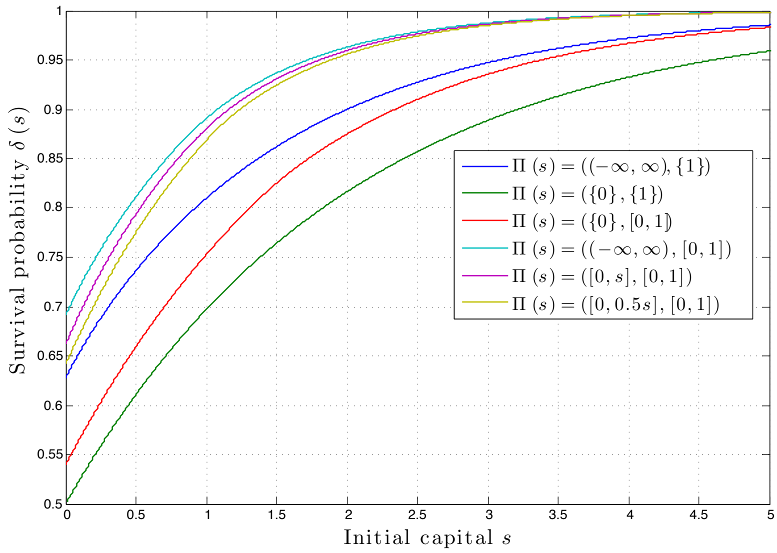
(5) $\Pi_5(s) = ([0,0.5s], [0,\infty])$: optimal reinsurance and investment with constraint $\mathcal{A}(s) = [0,0.5s]$.

In the following examples we choose $\Delta = 0.001$, $\lambda = 1$, $c = 2$. We denote the optimal investment and reinsurance for each above scenarios by $A_i(s)$ and $M_i(s)$, $i = 1,...,5$, respectively.

Example 24. We let Y be exponentially distributed with mean 1 and set $\rho = 4$. The survival probabilities for the above mentioned sets of constraints are given in figure 22. Because of the light tail property of exponential distribution, the ruin probabilities for all above scenarios go to zero exponentially fast.

The optimal retention level $M_i(s)$, $i = 1,3,4,5$, are given in figure 23. As we expected, there is always an interval $[0, s_{0,i})$, for which the optimal is to keep the whole risk, i.e. $M_i(s) = \infty$ for given scenarios $i = 1,...,5$. This interval is longer for those scenarios with more restricted constraint. In some interval $[s_{0,i}, s_{1,i})$, $s_{1,i} > s_{0,i}$, the optimal reinsurance strategy is $M_i(s) = s$. For the constraint sets Π_1, Π_3, Π_4

FIGURE 22. $\delta(s)$ for exponentially distributed claim size Y.



and Π_5, these intervals are respectively $[1.389, 2.3)$, $[0.644, 1.298)$, $[0.799, 1.405)$ and $[0.945, 1.508)$. From the point $s_{1,i}$, the insurer can afford more expensive reinsurance, that is $M_i(s) < s$, and for $s \to \infty$ the optimal $M_i(s)$ tends to be constant. For exponential distribution with mean $1/m$, the reinsurance premium is $h(M) = \frac{\lambda\rho}{m}e^{-Mm}$ and for $r \neq m$,

$$\mathcal{M}_{\min\{Y,M\}}(l) = \frac{l}{l-m}e^{(l-m)M} - \frac{m}{l-m}.$$

The equation (4.38) at M^* reads

$$\frac{1}{2}\frac{r^2}{\sigma^2} - (c - h(M^*))\,l^* + \lambda = \lambda\int_0^{M^*} e^{l^*y}dF(y) + \lambda\int_{M^*}^{\infty} e^{l^*M^*}dF(y). \tag{4.40}$$

Note that for optimal reinsurance without investment (scenario 1) the above equation turns to

$$c - \lambda\rho\int_{M^*}^{\infty}\bar{F}(y)\,dy + \lambda = \lambda\int_0^{M^*} e^{l^*y}dF(y) + \lambda\int_{M^*}^{\infty} e^{l^*M^*}dF(y). \tag{4.41}$$

Solving the system of two equations (4.40) and (4.39) under the optimal investment and reinsurance (scenarios 3, 4 and 5), we can calculate the asymptotic optimal investment and reinsurance when $s \to \infty$. So the asymptotic optimal investment and reinsurance in this example for $i = 3, 4, 5$, are $A_i(s) \to 0.924$ and $M_i(s) \to 1.28$ (see

FIGURE 23. $M(s)$ for exponentially distributed claim size Y.

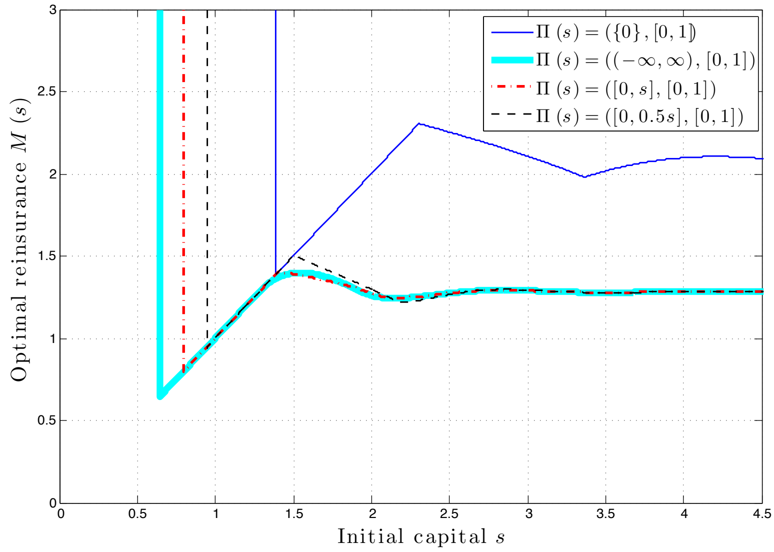
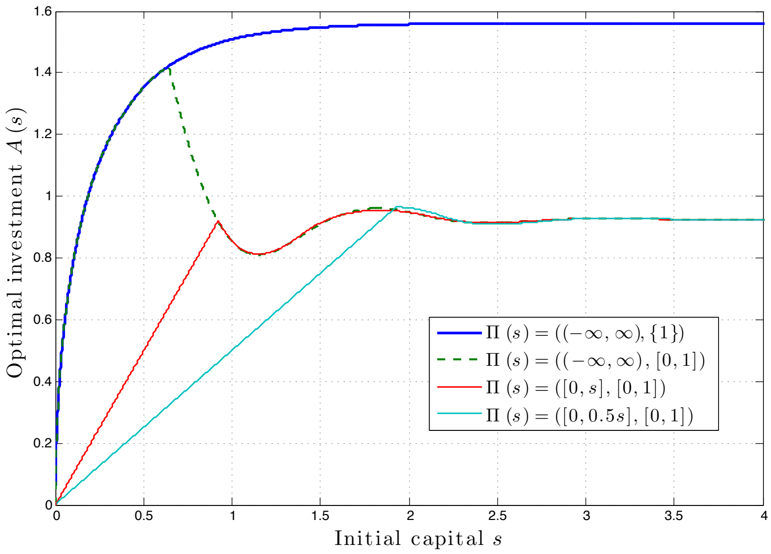


FIGURE 24. $A(s)$ for exponentially distributed claim size Y.



also figure 24). For the case optimal reinsurance without investment, Π_1, from the equations (4.41) and (4.36), we get $M_1(s) \to 2.07$ as $s \to \infty$.

EXAMPLE 25. We let Y be Pareto distributed with the parameter $p = 2$ and set $\rho = 3$ and use our numerical method to calculate the optimal survival probabilities for the five given scenarios. Figure 25 gives the optimal reinsurance strategies for different scenarios. As expected, there is an interval $[0, s_{0,i})$ for the scenario i, $i = 1, ..., 5$, on which optimal is buying no-reinsurance. Under the combined set of investment and reinsurance, the constraint sets Π_i, $i = 3, 4, 5$, for $s \in [s_{0,i}, s_{1,i})$, $s_{1,i} > s_{0,i}$, we have $M_i(s) = s$ and for $s \to \infty$ the optimal reinsurance tends to be constant. In this example, in the constraint sets Π_3, Π_4, and Π_5, we obtain $[0.515, 0.755)$, $[0.625, 0.79)$ and $[0.815, 0.9)$, respectively.

Figure 26 gives the optimal survival probabilities. Having the possibility to invest and reinsurance at the same time (scenarios 3-5) causes that the optimal investment strategies $A_i(s)$, $i = 3, 4, 5$, converge to a constant for $s \to \infty$. Hence, the optimal survival probabilities for these scenarios goes to one exponentially fast (See Figure 27). For these scenarios by solving the system of two equations (4.40) and (4.39) we can calculate the optimal asymptotic investment and reinsurance strategies. For the Pareto distribution with parameter p we have

$$\mathcal{M}_{\min\{Y,M\}}(l) = \frac{e^{lM}}{(1+M)^p} + \int_0^M e^{ly} p (1+y)^{-(p+1)} dy$$

and

$$E[\max\{Y - M, 0\}] = \frac{1}{(1+M)^{p-1}(p-1)}.$$

The given parameters in this example for $i = 3, 4, 5$, yield $A_i(s) \to 0.65$ and $M_i(s) \to 0.88$. For the optimal reinsurance, scenario 2, from equations (4.41) and (4.39) we get $M_1(s) \to 2$, as $s \to \infty$.

4.4. Optimal limited XL reinsurance with investment

The XL reinsurance may become too expensive for those portfolio with heavy tail severity distributions. Limited XL reinsurance as a special case of the XL reinsurance, will retain the tail part of claim severity by some barrier L. A claim of size Y is divided in the first insurer's payment with $g(Y, (M, L)) = \min\{Y, M\} + (Y - M - L)^+$ and the reinsurer's payment $\min\{L, (Y - M)^+\}$. As a return, we assume that the insurance company pays reinsurance premium $h(M, L)$ according to the expected value principle, i.e.

$$h(M, L) = \lambda\rho E\left[\min\left\{L, (Y - M)^+\right\}\right], \quad \rho = (1 + \theta) > 1.$$

FIGURE 25. $M(s)$ for Pareto distributed claim size Y.

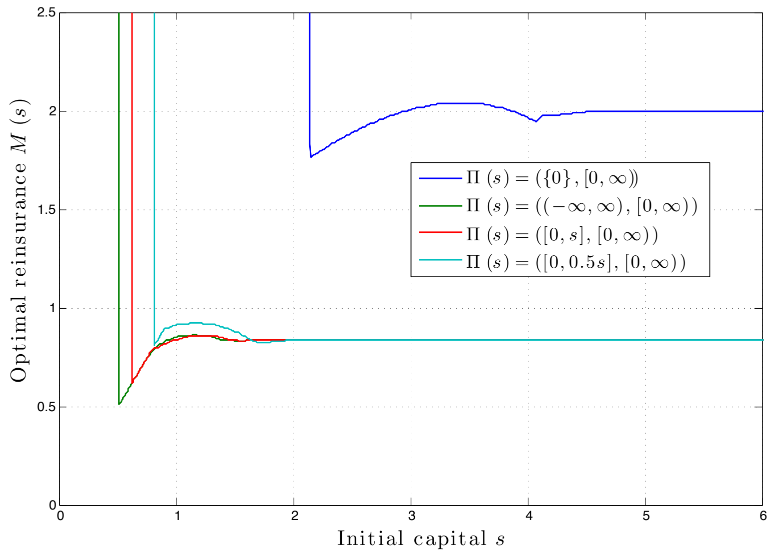
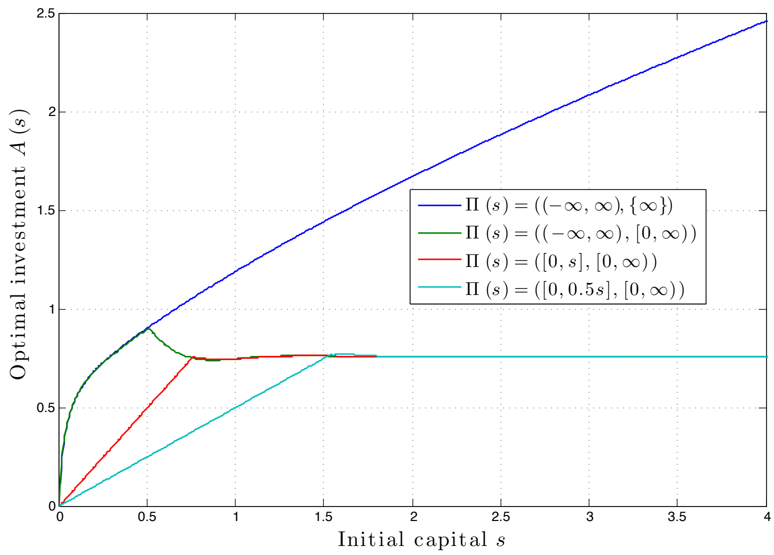


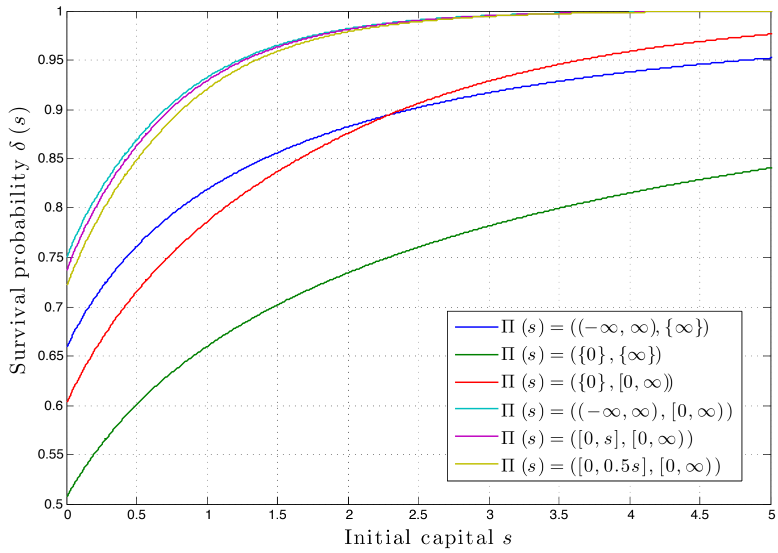
FIGURE 26. $A(s)$ for Pareto distributed claim size Y.



Applying integration by parts we have

$$h(M,L) = \lambda\rho \int_{M}^{M+L} 1 - F(y)\, dy.$$

FIGURE 27. $\delta(s)$ for Pareto distributed claim size Y.



So in the limited XL reinsurance, the first insurer has two dimensional reinsurance control process $u = (M, L)$ where $u \in U = [0, \infty] \times (0, \infty]$. The case no-reinsurance is described by $M = L = \infty$. The HJB equation (3.10) is now for $s \geq 0$

$$\sup_{(A,M,L)\in(\mathcal{A}(s),U)} \{\frac{1}{2}\sigma^2 A^2 V''(s) + (c - h(M, L) + rA) V'(s) + \lambda E[V(s - g(Y,(M,L))) - V(s)]\} = 0. \tag{4.42}$$

The optimal dynamic limited XL reinsurance was studied by Vogt (2003). He showed that having extra control variable L in limited XL reinsurance could bring higher survival probability just in some cases.

For $s > 0$ we divide initial capital into three cases as follows:

(1) $M(s) > s$: In this case we have

$$E\left[V\left(s - \min\{Y, M\} + (Y - M - L)^+\right)\right] = \int_0^s V(s - y)\, dF(y).$$

So the optimal strategy is no-reinsurance i.e. $M = L = \infty$ and $h(\infty, \infty) = 0$.

(2) $M(s) = s$: We can rewrite $E[V(s - g(Y,(M,L)))]$ as

$$\int_0^s V(s - y)\, dF(y) + (F(s + L) - F(s))\, V(0).$$

and from normal equation for $L < \infty$ we have

$$\rho V^{'}(s) = V(0)\frac{f(s+L)}{1-F(s+L)}. \tag{4.43}$$

(3) $M(s) < s$: we can rewrite $E[V(s-g(Y,(M,L)))]$ as

$$\int_0^M V(s-y)\,dF(y) + (F(M+L)-F(M))\,V(s-M) + \int_{M+L}^{s+L} V(s-y+L)\,dF(y).$$

From normal equation we have

$$\rho V^{'}(s) = V^{'}(s-M), \tag{4.44}$$

and for $L < \infty$

$$\rho\overline{F}(M+L)\,V^{'}(s) = V(0)\,f(s+L) + \int_{M+L}^{s+L} V^{'}(s-y+L)\,dF(y). \tag{4.45}$$

Remark 6. At $s = 0$, the HJB equation (4.42) becomes

$$0 = \sup_L\left\{\lambda V(0)\,F(L) - \lambda V(0) + cV^{'}(0) - V^{'}(0)\,\rho\left(\int_0^L yf(y)\,dy + L(1-F(L))\right)\right\}. \tag{4.46}$$

The optimal reinsurance strategy is then either $M^*(0) = L^*(0) = \infty$ (no-reinsurance), which gives

$$V^{'}(0) = V(0)\frac{\lambda}{c},$$

or $M^*(0) = 0$ and $L^*(0) < \infty$, which gives

$$V^{'}(0) = V(0)\inf_L\left\{\lambda\frac{1-F(L)}{c-h(0,L)}\right\} < V(0)\frac{\lambda}{c}.$$

From (4.43) and (4.46), for $L^* := L^*(0) < \infty$ we obtain

$$0 = F(L^*) - 1 + \frac{cf(L^*)}{\rho(1-F(L^*))} - \frac{f(L^*)\int_0^{L^*} yf(y)\,dy}{(1-F(L^*))} - f(L^*)\,L^*. \tag{4.47}$$

$\triangle$

Lemma 26. *Assume that the HJB equation (4.42) has a solution $V(s)$ which is continuously differentiable on $(0,x)$, $x > 0$. If the reinsurance premium calculate via expected value principle with fixed safety loading $\rho > 1$, then there exists an interval $[0,\varepsilon)$, $\varepsilon > 0$, on which the optimal retention is either $M^*(s) = s$ or $M^*(s) = \infty$.*

PROOF. We already know that for $M(s) > s$, optimal is buying no-reinsurance that is $M^*(s) = \infty$. Assume by way of contradiction that for all $\varepsilon > 0$ there exists $s < \varepsilon$ such that $M^*(s) < s$. Then from normal equation we have for $\rho = (1+\theta) > 1$,

$$\rho V'(s) = V'(s - M^*).$$

The term $\frac{V'(s-M^*)}{V'(s)}$ can be made arbitrary close to one by choosing s small enough which is a contradiction to the fact that $\rho > 1$. □

Similar to the Theorem 6.1.8 in [**42**], we can show that for exponentially distributed claim sizes Y, the optimal barrier is $L^*(s) = \infty$, $s \geq 0$ and consequently the solution of the HJB equation (4.42) corresponds to the solution of the HJB equation (4.34).

THEOREM 27. *Consider the HJB equation (4.42) and assume that the claim size distribution F_Y is exponential with mean $1/m$. If the reinsurance premium is given by expected value principle with safety loading $\theta > 0$, then for all $s \geq 0$ the solution of the HJB equation (4.42) corresponds to the solution of the HJB equation (4.34).*

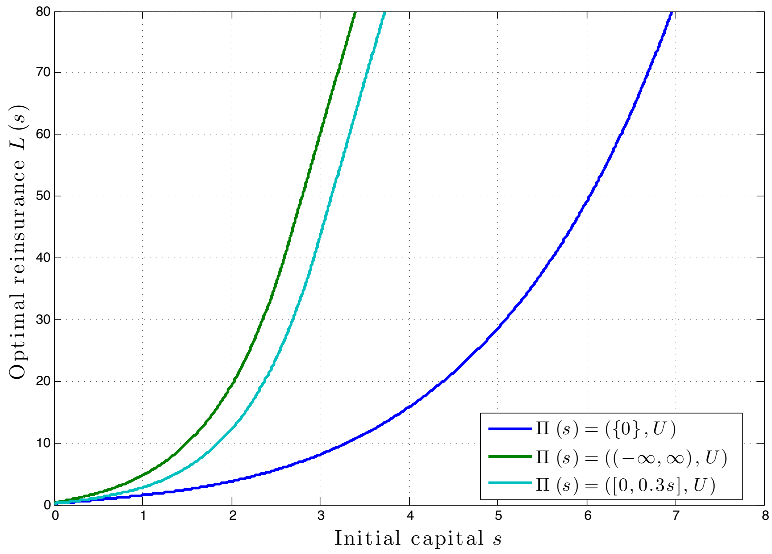
PROOF. We first show that for an arbitrary start capital $s \geq 0$, the optimal barrier is $L^*(s) = \infty$. Let $M^*(s) \in [0, \infty)$ be the optimal priority strategy and choose an arbitrary investment strategy $A \in \mathcal{A}(s)$. Then, for $M^*(s) > s$, it is obvious that $M^*(s) = L^*(s) = \infty$. If $M^*(s) = s$, then for the exponential claim size F_Y, from the normal equation (4.43), if $L < \infty$, then we must have $\theta = 0$, which is contradiction. If $M^*(s) < s$, then from (4.45), for the exponential claim size distribution with mean $1/m$ we get

$$V'(s) = \frac{mV(s - M^*(s)) - \int_{M^*(s)}^{s} V(s-y)\, m^2 e^{-m(y-M^*(s))} dy}{1+\theta}.$$

The right hand side of the above equation does not depend on L, so the optimal $L^*(s)$ is ∞. Because s and $A(s)$ were chosen arbitrarily, we conclude that for the optimal values $A^*(s)$ and $M^*(s)$ it holds $L^*(s) = \infty$ and the solution of HJB equations (4.42) and (4.34) correspond to each other. □

EXAMPLE 28. Let the claim size Y be Pareto distributed with parameter $p = 2$ and assume that the reinsurance premium calculated via the expected value principle with safety loading $\theta = 2.5$, so $\rho = (1+\theta) = 3.5$. Choose $\lambda = 1$, $c = 2$ and set the step size $\Delta = 0.005$. We calculate numerically the optimal investment and reinsurance strategies

FIGURE 28. $L(s)$ for Pareto distributed claim size Y.



for investment constraints sets $\mathcal{A}(s) = (-\infty, \infty)$, $\mathcal{A}(s) = [0, 0.3s]$ and $\mathcal{A}(s) = \{0\}$. The optimal barrier $L^*(s)$ for given scenarios is plotted in figure 28. Since for the start capital zero the optimal investment strategies for all set of investment constraints are equal to zero, with the same argument in [**42**] by solving (4.47) the optimal barrier is

$$L^*(0) = \exp\left(\frac{\ln\left(-\frac{1+\theta}{p(\eta-\theta)}\right)}{p-1}\right) - 1,$$

if $\theta < \frac{1+p\eta}{p-1}$. For $\theta \geq \frac{1+p\eta}{p-1}$ the optimal is to buy no-reinsurance, that is $M^*(0) = L^*(0) = \infty$. In this example, we have $\eta = 1$, and so $L^*(0) = 0.1667$. For $s > 0$, the optimal barriers are increasing and tends to infinity.

Figure 29 shows the optimal priority strategies $M^*(s)$ for different scenarios. As expected from the theory, there is an interval $[0, s_0)$, $s_0 > 0$, on which $M^*(s) = s$. For $s > s_0$, the optimal priority increases slowly and converges to a constant. Letting $L^*(s) \to \infty$ for $s \to \infty$, similar to the example 24, by solving the equations (4.40) and (4.39) for the combined set of investment and reinsurance we get $M^*(s) \to 1.24$ and $A^*(s) \to 1$ (see figure 30). We can therefore conclude that the optimal survival probability goes exponentially fast to one as $s \to \infty$ (see figure 31).

In the problem of optimal reinsurance without investment, from (4.41) and (4.39), we find the asymptotic optimal priority $M^*(s) \to 3$ for $s \to \infty$. So, in this case

FIGURE 29. $M(s)$ for Pareto distributed claim size Y.

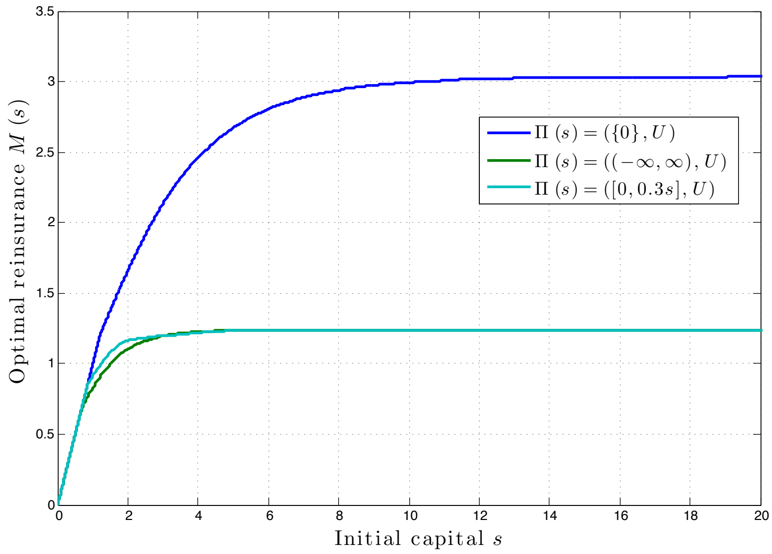
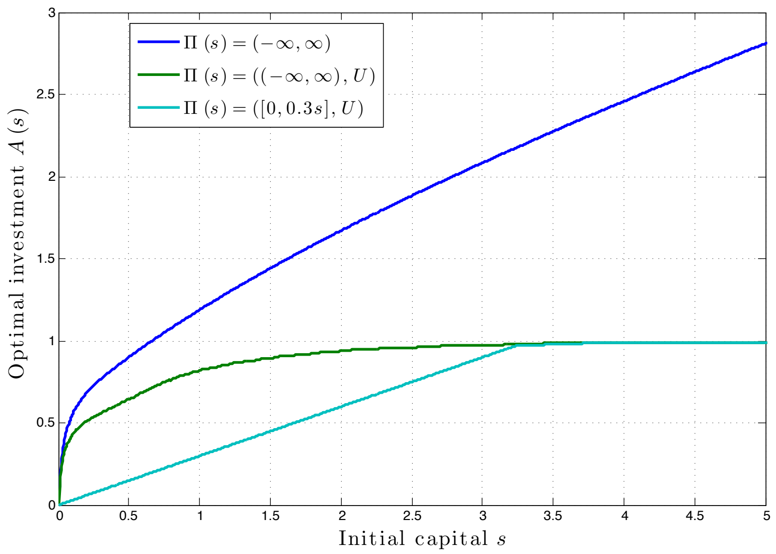


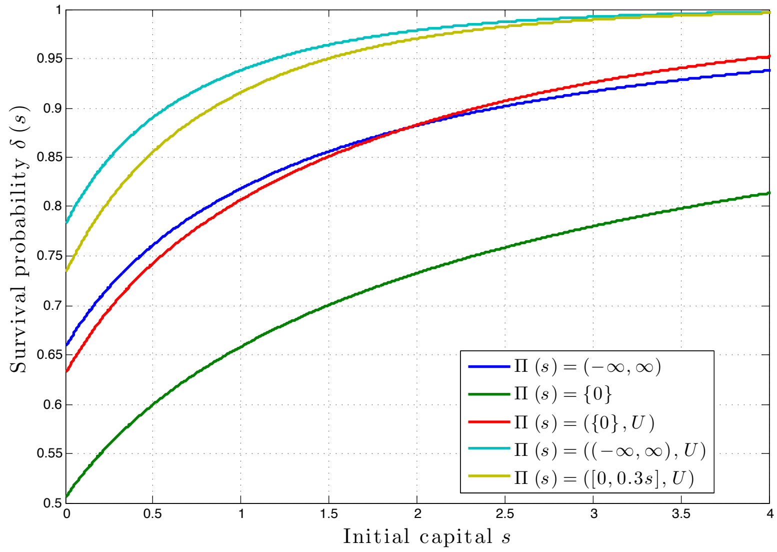
FIGURE 30. $A(s)$ for Pareto distributed claim size Y.



the optimal survival probability goes also exponentially fast to one for $s \to \infty$. This result was expected, because from the previous section , example 25, we know that the survival probability for the optimal XL reinsurance goes also exponentially fast to one.

The survival probabilities for the given scenarios as well as $\delta_0(s)$, the survival probability without investment and reinsurance, are given in figure 31. It is obvious

FIGURE 31. $\delta(s)$ for Pareto distributed claim size Y.



that the combination of investment and reinsurance significantly increases the survival probabilities.

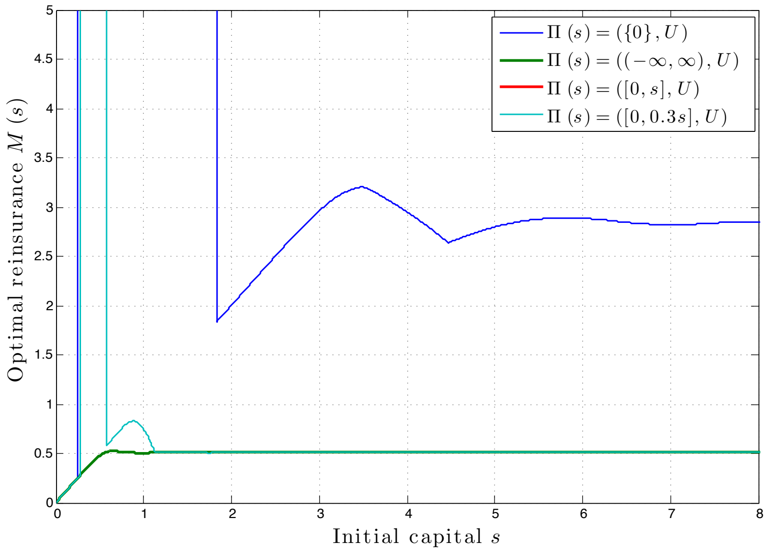
EXAMPLE 29. We let the claim size Y follows the mixture of exponential and Pareto distributions with distribution function

$$F_Y(y) = 1 - 0.1e^{-\frac{1}{2}y} - 0.9(1+y)^{-30},\ y \geq 0.$$

We choose $c = 0.4621$, $\rho = 3.5$, $\lambda = 1$ and $\Delta = 0.001$. The optimal strategies are calculated for the following constraint sets: $\Pi_1(s) = (\{0\}, U)$, $\Pi_2(s) = ((-\infty,\infty), u_0)$, $\Pi_3(s) = ((-\infty,\infty), U)$ and $\Pi_4(s) = ([0, 0.3s], U)$.

The optimal priorities, $M(s)$, are plotted in figure 32. As expected from the lemma 26, for small initial capital, $M^*(s) = s$. Then, the optimal priority jumps to infinity at some point s_0. Hence for small values of s, insurer tries to eliminate the occurrence of ruin due to the small claims. For the constraint sets Π_1 and Π_4, in which investment is not allowed or, respectively, restricted by $\mathcal{A}(s) = [0, 0.3s]$, from s_0 to some point $s_1 > s_0$, the optimal is taking no-reinsurance, that is $M(s) = L(s) = \infty$. This is because, the insurer has enough capital to bear the small claims by itself and can save money by not buying expensive reinsurance. The optimal priority then drops to some $M(s_1) < \infty$ and tends to be constant as $s \to \infty$. In this example, for constraint

FIGURE 32. $M(s)$ for mixture of exponential and Pareto distributed claim size Y.

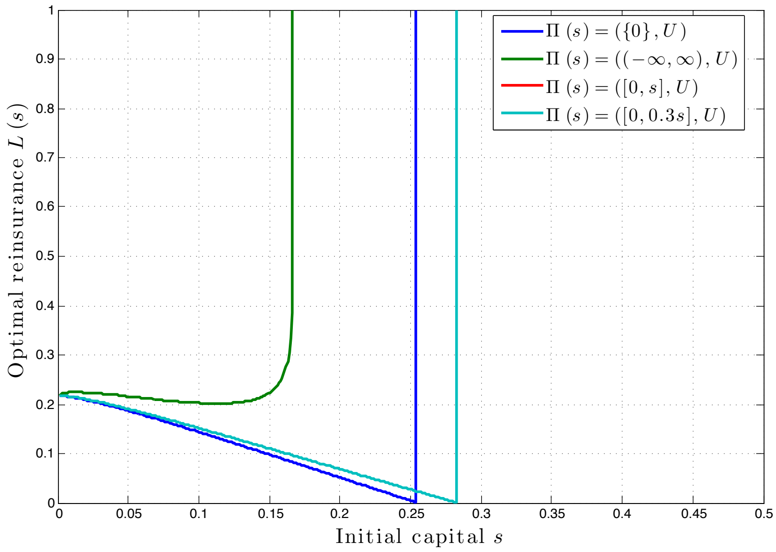


set Π_1 (respectively, Π_4), $s_0 = 0.25$, $s_1 = 1.83$, $M(s_1) = 1.83$ and $\lim_{s\to\infty} M(s) = 2.84$ (respectively, $s_0 = 0.28$, $s_1 = 0.57$, $M(s_1) = 0.57$ and $\lim_{s\to\infty} M(s) = 0.5$).
In the unconstraint investment and reinsurance case, Π_3, the optimal is always to take reinsurance into account for $s > 0$, since the insurer can fully gain from investing in risky asset and can afford the reinsurance costs. As s goes to infinity, the optimal priority converges to 0.5. It must be mentioned that the convergence values can be obtained with the same method as in the previous example.

Figure 33 depicts the optimal barriers $L(s)$. At $s = 0$ the optimal barrier is $L(0) = 0.218$ for all sets of constraint with reinsurance. For the constraint set Π_1 (respectively, Π_4), as the start capital s grows, $L(s)$ shrinks towards zero. Then at point s_0, in which the optimal priority $M(s_0)$ jumps to infinity, the optimal barrier $L(s_0)$ jumps to infinity and never comes back.
In the unconstraint investment and reinsurance case, Π_3, the optimal barrier $L(s)$, for the small initial capital stays at about 0.21. It, but, then suddenly increases and goes to infinity and never returns back.

The optimal amount of investments, $A(s)$, are shown in figure 34 for different constraint sets. For s close to zero, it appears that the optimal investment in the case Π_3, is more than the optimal investment in the case Π_2. Thus for small initial capital,

FIGURE 33. $L(s)$ for mixture of exponential and Pareto distributed claim size Y.



insurer should take more investment risk in order to get away from zero as fast as possible. As $s \to \infty$, in the two cases Π_3 and Π_4, the optimizer function $A(s) \to 0.4$, while, in the case Π_2, the optimizer function $A(s) \to 2.3$. We can, therefore, deduce that the the optimal survival probabilities go to one exponentially fast (See figure 35). It must be mentioned that the convergence values can be obtained with the same method as in the previous example.

FIGURE 34. $A(s)$ for mixture of exponential and Pareto distributed claim size Y.

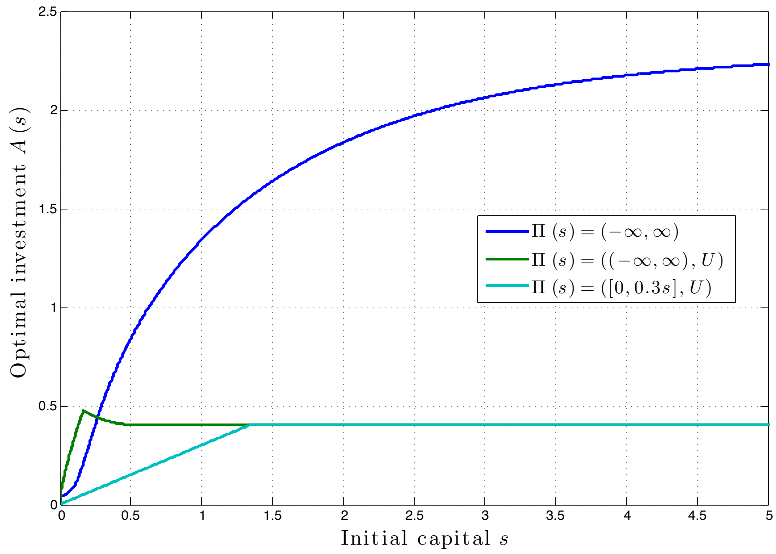
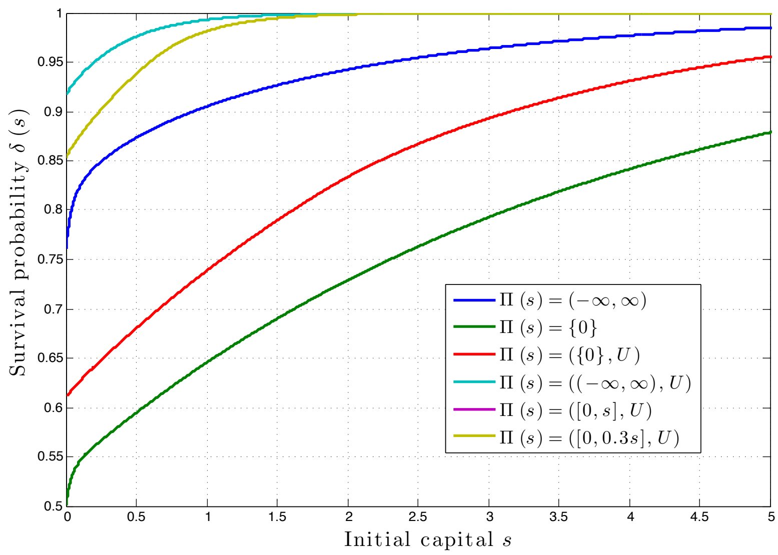


FIGURE 35. $\delta(s)$ for mixture of exponential and Pareto distributed claim size Y.



CHAPTER 5

The model with risk-free bond

In this chapter we take the risk free-bond, $r_0 > 0$, into account. The risk process (2.1) turns to

$$R_t^\pi = s + ct + \int_0^t r_0 R_x^\pi dx - \int_0^t h(u_x)\, dx + \int_0^t (r - r_0) A_x dx$$
$$+ \int_0^t \sigma A_x dW_x - \sum_{i=1}^{N_t} g\left(Y_i, u_{T_i-}\right).$$

Considering the general constraint set Π, the optimal dynamic survival probability, $\delta(s)$, satisfies the following HJB equation

$$\sup_{\pi\in\Pi}\{\frac{1}{2}\sigma^2 A^2 \delta^{''}(s) + (c - h(u) + r_0 s + (r - r_0) A)\, \delta^{'}(s)$$
$$+\lambda E\left[\delta\left(s - g(Y, u)\right) - \delta(s)\right]\} = 0.$$

We first study the problem of optimal investment without reinsurance that is $\Pi(s) = (\mathcal{A}(s), u_0)$, where $g(Y, u_0) = Y$ and $h(u_0) = 0$. W begin with the optimal unconstraint investment problem, i.e. $\mathcal{A}(s) = (-\infty, \infty)$. In this case, the HJB equation can be solved numerically through a quadratic equation. Next, we consider constraint set $\mathcal{A}(s) = [-bs, as]$, $a, b \geq 0$, and see that although optimal investment strategy has jumps for some cases, the value function is surprisingly always continuous.

In section 5.2, we bring the reinsurance into the set of possible action and deal with the optimal investment and reinsurance problem. The risk free bond and reinsurance give the first insurer this chance to get rid of all his risks for sufficiently large initial capital, s_0, and achieve $\delta(s) = 1$, $s > s_0$.

5.1. Optimal constrained investment

Consider a time consistent constraint set of investment $\mathcal{A}(s) \subset \mathbb{R}$. The HJB equation to be solved is

(5.1)
$$\sup_{A \in \mathcal{A}(s)} \left\{ \frac{1}{2}\sigma^2 A^2 V''(s) + (c + r_0 s + (r - r_0) A) V'(s) + \lambda E[V(s-Y) - V(s)] \right\} = 0.$$

We use the norming $V(0) = 1$ and $V'(0) = \frac{\lambda}{c}$ and solve the above equation for two special cases $\mathcal{A}(s) = (-\infty, \infty)$ and $\mathcal{A}(s) = [-bs, as]$, $a, b > 0$.

If $\mathcal{A}(s) = (-\infty, \infty)$, then the optimizer function is $A(s) = -\frac{(r-r_0)V'(s)}{\sigma^2 V''(s)}$. Inserting the optimal investment into (5.1), we get

$$(c + r_0 s) V'(s) + \lambda E[V(s-Y) - V(s)] = \frac{1}{2} \frac{(r-r_0)^2 V'(s)^2}{\sigma^2 V''(s)}$$

Applying the approximations (3.23), (3.22) and (3.21) to the above equation, one obtains the following quadratic equation

$$\alpha_1(s) V'(s)^2 + a_1(s) V'(s) + b_1(s) = 0, \tag{5.2}$$

where

$$\alpha_1 = (c - \Delta\lambda + r_0 s)\sigma^2 - \frac{1}{2}(r - r_0)^2 \Delta,$$
$$a_1(s) = \sigma^2 \Big(\lambda G_{u_0}(s) - \lambda V(s - \Delta) - (c + r_0 s - \lambda\Delta) V'(s - \Delta) \Big),$$
$$b_1(s) = -\lambda\sigma^2 V'(s-\Delta)\left(G_{u_0}(s) - V(s-\Delta)\right),$$

and

$$G_{u_0}(s) = \sum_{j=1}^{i} V((i-j)\Delta) \Pr\{(j-1)\Delta < Y \le j\Delta\}.$$

For Δ small enough we have α_1, $b_1(s)$, $-a_1(s) > 0$. Since we are looking for minimum $V'(s)$, we obtain

$$V'(s) = -\frac{a_1(s)}{2\alpha_1} - \sqrt{\frac{a_1(s)^2}{4\alpha_1^2} - \frac{b_1(s)}{\alpha_1}}. \tag{5.3}$$

Now let $\mathcal{A}(s) = [-bs, as]$. This means that the insurance company can not borrow more than bs, $b > 0$ and his investment in risky asset should be smaller than as, $a > 0$. Using Proposition 4.2 in [**3**], Belkina et al. [**5**] proved that there exists a unique solution $V(s)$, $s > 0$, to the HJB equation (5.1) which is twice continuously differentiable on

$(0, \infty)$ with $V'(0) = \frac{\lambda}{c}$. They showed that for small enough start capital, the optimal investment is either a if $r > r_0$, or $-b$ if $r < r_0$. Using this fact we obtain

$$V''(0) = \frac{\lambda}{c}\left(\frac{\lambda}{c} - f(0) - \frac{r_0 + \gamma(r - r_0)}{c}\right), \tag{5.4}$$

where $\gamma \in \{a, -b\}$.

Recall the numerical method with the initial value $V_\Delta(0) = 1$. For $s = i\Delta$ the function $V_\Delta(s)$, $i = 1, 2, ...$, is defined via

$$V'_\Delta(s) = \inf_{A \in [-bs, as]} \frac{\lambda\Delta\left(V_\Delta(s - \Delta) - G_{u_0}(s)\right) + \frac{1}{2}\sigma^2 A^2 V'_\Delta(s - \Delta)}{\Delta\left(c + r_0 s + (r - r_0)A - \lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2}, \tag{5.5}$$

which is similar to

$$V''_\Delta(s) = \inf_{A \in [-bs, as]} \frac{\lambda\left(V_\Delta(s - \Delta) - G_{u_0}(s)\right) - \left(c + r_0 s + (r - r_0)A - \lambda\Delta\right)V'_\Delta(s - \Delta)}{\Delta\left(c + r_0 s + (r - r_0)A - \lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2}.$$

At $s = \Delta$, we have

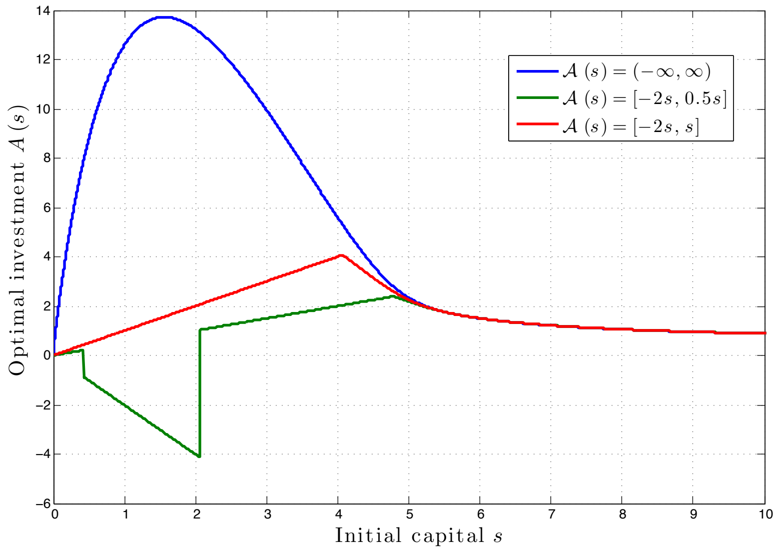
$$V''(\Delta) = \inf_{A \in [-b\Delta, a\Delta]} \frac{-\lambda F(\Delta) - r_0\frac{\lambda}{c}\Delta - (r - r_0)A\frac{\lambda}{c} + \frac{\lambda^2}{c}\Delta}{\Delta\left(c + r_0\Delta + (r - r_0)A - \lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2}. \tag{5.6}$$

If $r > r_0$, then the coefficient of A is positive and consequently the optimal investment is $A = as$. The contrary holds if $r < r_0$, which makes $A = -bs$ the optimal investment. Note that by letting $\Delta \to 0$ in (5.6), we obtain again (5.4).

Example 30. Consider the exponential distributed claim size Y with mean 1. Choose parameters as follow: $\sigma^2 = 0.01$, $r = 0.02$, $r_0 = 0.015$, $\lambda = 0.09$, $c = 0.02$. The numerical method is applied with $\Delta = 0.001$ in order to find the optimal investment for the constraint sets $\mathcal{A}_1(s) = [-2s, 0.5s]$, $\mathcal{A}_2(s) = [-2s, s]$ and $\mathcal{A}_3(s) = (-\infty, \infty)$.

Figure 1 denotes the optimal investments for different sets of constraints. Since $r > r_0$, the optimal investment for small initial capital is $0.5s$ for constraint set $\mathcal{A}_1(s)$ and is s for constraint set $\mathcal{A}_2(s)$. For the constraint set $\mathcal{A}_1(s) = [-2s, 0.5s]$, it then at $s = 0.431$ drops to $-2s$ and then at $s = 2.057$ jumps to $0.5s$ again. This strange behavior is because the insurer tries to get quickly away from zero where a small claim can causes ruin. To achieve this, since b is sufficiently larger than a, from some point s_1, the insurer must take the largest possible risk of gambling on the effect of volatility and switch from the maximal long position to the maximal short position in the risky asset. Surprisingly, despite the fact that the optimal investment has two jumps, the

FIGURE 1. $A(s)$ for exponential distributed claim size Y.



$\delta''(s)$ is still continuous on $(0,\infty)$ (see figure 2). A simple argument for the continuity of $\delta''(s)$ is as follows: Let $\delta''(s)$ piecewise continuous, and assume at s_0 the optimal investment, $A(s)$, jumps from as to $-bs$. Then

$$\frac{1}{2}a^2s_0^2\sigma^2\delta''(s_0-) + (c + r_0s_0 + ars_0)\,\delta'(s_0) + \lambda E\left[\delta(s_0 - Y) - \delta(s_0)\right] = 0,$$

$$\frac{1}{2}b^2s_0^2\sigma^2\delta''(s_0+) + (c + r_0s_0 - brs_0)\,\delta'(s_0) + \lambda E\left[\delta(s_0 - Y) - \delta(s_0)\right] = 0.$$

Since as_0- and $-bs_0+$ are the maximizer of HJB equation, we obtain

$$-brs_0\delta'(s_0) + \frac{1}{2}b^2\sigma^2s_0^2\delta''(s_0-) \le ars_0\delta'(s_0) + \frac{1}{2}a^2\sigma^2s_0^2\delta''(s_0-),$$

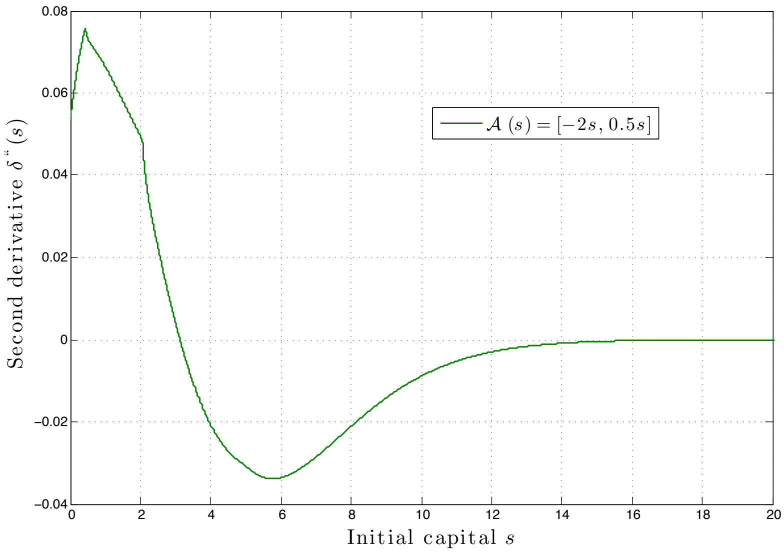
$$ars_0\delta'(s_0) + \frac{1}{2}a^2\sigma^2s_0^2\delta''(s_0+) \le -brs_0\delta'(s_0) + \frac{1}{2}b^2\sigma^2s_0^2\delta''(s_0+).$$

Hence $\delta''(s_0-) = \delta''(s_0+)$. For more details about this proof, we refer the interested reader to [**5**].

The function $\delta''(s)$, for the constraint $\mathcal{A}_1(s) = [-2s, 0.5s]$, is shown in figure 2. We remind the reader that using the norming $V(0) = 1$, from (5.4) one obtains $V''(s) = 11.81$. Here we have $\delta''(s) = \frac{V''(s)}{V(\infty)} = 0.0046V''(s)$.

In the constraint set $\mathcal{A}_2(s) = [-2s, s]$, as b is not enough larger than a, the insurer has to relies on the highest possible rate of return and stays on the long position for

FIGURE 2. $\delta''(s)$ for exponential distributed claim size Y.



the small initial capital. As $s \to \infty$, the optimal investment for all set of constraints converges to a constant 0.6.

The optimal survival probabilities, $\delta(s)$, are shown in figure 3. Note that without possibility of investment in the risky asset, the ruin probability is one, since $c \leq \lambda E[Y]$. In fact, the investment gives us the possibility to reduce premium even less than net premium. The optimal survival probability in the unconstraint investment case is significantly more than the optimal survival probabilities in the two constraint investment cases $\mathcal{A}_1(s)$ and $\mathcal{A}_2(s)$.

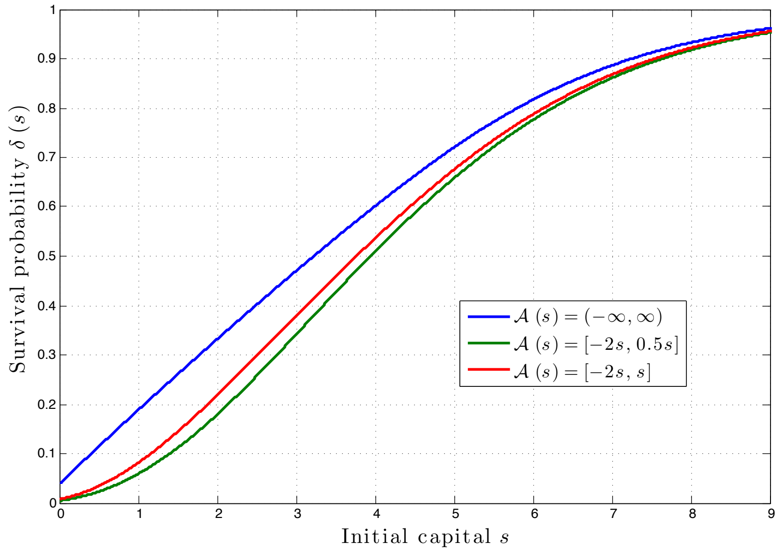
5.2. Optimal constrained investment and reinsurance

Here we consider that the insurer, beside investment, can dynamically transfer part of its claim Y to the reinsurer by choosing a reinsurance strategy $u \in U$. The risk sharing function is then $g(Y,u)$ and the reinsurance premium of this risk transformation is $h(u)$. The HJB equation to be solved is

$$\sup_{\pi \in \Pi} \{ \frac{1}{2} \sigma^2 A^2 \delta''(s) + (c - h(u) + r_0 s + (r - r_0) A) \delta'(s)$$
$$+ \lambda E[\delta(s - g(Y,u)) - \delta(s)] \} = 0,$$

where $\Pi = (\mathcal{A}, U)$ is the set of possible investment and reinsurance strategies.

FIGURE 3. $\delta(s)$ for exponential distributed claim size Y.



With the risk free bond $r_0 > 0$, from some initial capital $\bar{s} > 0$, the first insurer has the possibility to get rid of its whole insurance and investment risks and still earns positive income rate. Let $\bar{u}$ be the full-reinsurance strategy, that is $g(Y, \bar{u}) = 0$. Setting $u = \bar{u}$ and $A = 0$, the risk process of insurer with initial capital s, increases constantly over time with factor $c - h(\bar{u}) + r_0 s$ which is positive for $s > \bar{s}$, where

$$\bar{s} = \frac{1}{r_0}\left(h(\bar{u}) - c\right).$$

Hence $\delta(s) = 1$ for $s > \bar{s}$.

If we use the norming $V(0) = 1$, then

$$V'(0) = \lambda \inf_{u \in U}\left\{\frac{1 - \Pr(g(y, u) = 0)}{c - h(u)}\right\}.$$

Return to the numerical method in section (3.3), we have a family of function $V_\Delta(s)$ which is defined via

$$V'_\Delta(s) = \inf_{(A,u)\in\Pi} \frac{\lambda\Delta\left(V_\Delta(s-\Delta) - G_u(s)\right) + \frac{1}{2}\sigma^2 A^2 V'_\Delta(s-\Delta)}{\Delta\left(c - h(u) + r_0 s + (r - r_0)A - \lambda\Delta\right) + \frac{1}{2}\sigma^2 A^2}, \tag{5.7}$$

where

$$G_u(s)$$

$$= \sum_{\{g(j\Delta,u)\leq(i-1)\Delta\}} V_\Delta\left((i-1)\Delta - \left\lfloor \frac{g(j\Delta,u)}{\Delta}\right\rfloor \Delta\right)\Pr\{(j-1)\Delta < Y \leq j\Delta\}.$$

Let $\bar{s} = \frac{1}{r_0}(h(\bar{u}) - c + \lambda\Delta)$, then for $s > \bar{s}$ by setting the strategies $u = \bar{u}$ and $A = 0$, the denominator of (5.7) is positive while the numerator of (5.7) equals to zero. So $V'(s) = 0$ for $s > \bar{s}$.

Example 31. We assume that that the claims Y are exponentially distributed with mean 1 and let $\lambda = 1$ and $c = 1.5$. The insurer can invest in risky asset with drift and volatility equal to 1 as well as risk free bond with $r_0 = 0.2$. At the same time, the insurer can buy XL reinsurance whose premium is calculated using the Expected value principle with reinsurance safety loading $\theta = 2.5$. The risk sharing function is $g(Y,M) = \min\{Y,M\}$ and the full reinsurance is achieved by letting $M = 0$. The full reinsurance premium is $h(0) = (1+\theta)\lambda E[Y] = 3.5$. We choose $\Delta = 0.001$ and solve the optimal problem for the constraint sets $\Pi_1(s) = (\{0\}, [0,\infty))$, $\Pi_2(s) = ((-\infty,\infty), \{\infty\})$, $\Pi_3(s) = ([0, 0.5s], [0,\infty))$ and $\Pi_4(s) = ((-\infty,\infty), [0,\infty))$.

The optimal investment functions, $A(s)$, for the different cases, are shown in Figure 4. For small s, the optimal investment is highly leveraged for the constraint sets Π_2 and Π_4. In the case Π_3, there is an interval $[0, s_0)$ on which the optimal investment is $A(s) = 0.5s$. In this example, $s_0 = 1.555$. If the reinsurance is allowed, the cases Π_3 and Π_4, then the optimal investment is gradually decreases and for $s > 10$, we have $A(s) = 0$.

In figure 5, we show the optimal reinsurance strategies for the constraint sets Π_1, Π_3 and Π_4. As we expected, for small initial capital s, since the reinsurance is expensive, optimal is no-reinsurance, i.e. $M(s) = \infty$. From some point s_1, the insurer can afford the reinsurance cost and we have an interval $[s_1, s_2)$ on which the optimal reinsurance is $M(s) = s$. In this example for the constraint sets Π_1, Π_3 and Π_4, those intervals are respectively $[1.64, 2.143)$, $[1.176, 1.581)$ and $[0.948, 1.415)$. For $s > s_2$, the optimal priority, $M(s)$, slowly decrease and for $s > 10$, $M(s) = 0$. In fact, for $s > 10$, the insurer can get rid of the investment and insurance risks, i.e. $A(s) = 0$ and $M(s) = 0$, and still have a positive income from riskless asset. So for the constraint sets Π_1, Π_3 and Π_4 , we can deduce that the optimal survival probability is $\delta(s) = 1$, $s > 10$ (see figure 6).

FIGURE 4. $A(s)$ for exponential distributed claim size Y.

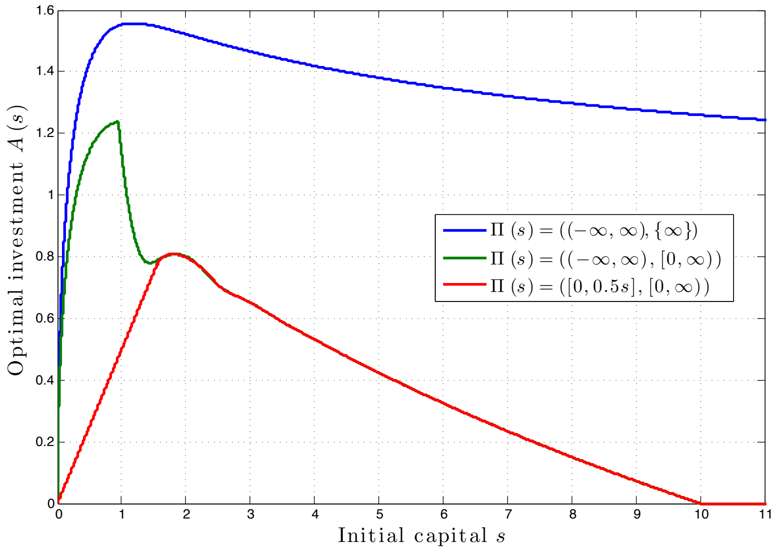


FIGURE 5. $M(s)$ for exponential distributed claim size Y.

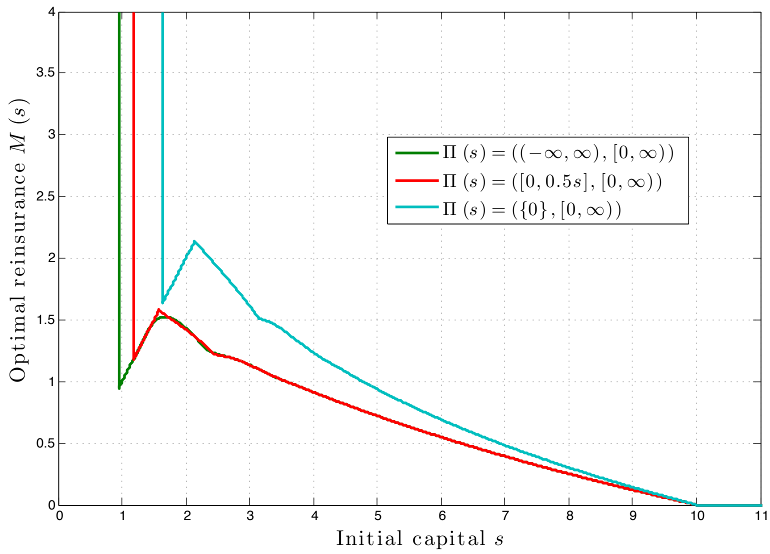
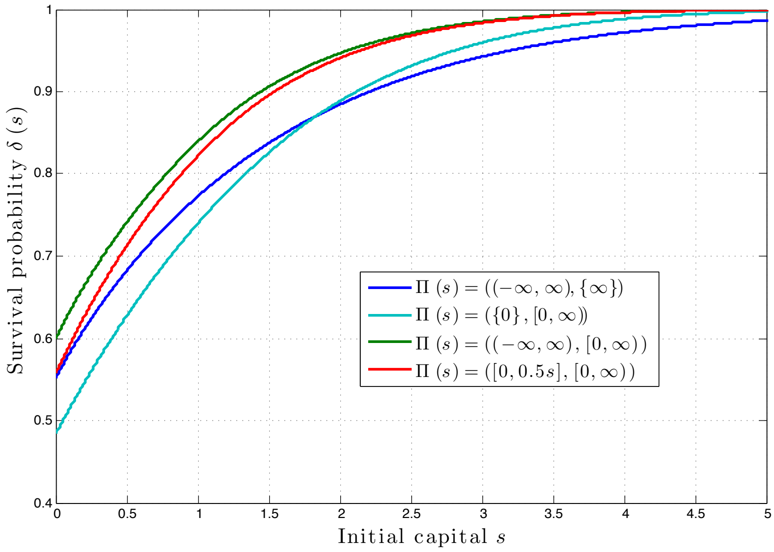


FIGURE 6. $\delta(s)$ for exponential distributed claim size Y.



Summary.

The stochastic dynamic control of the ruin probability is studied in a variant of the Cramer Lundberg model with different sets of possible actions. Different authors have presented different iteration operators to deal with the optimal problem. See for example [**3, 27, 23, 22, 26, 36, 38**]. In this thesis we studied the optimal dynamic investment and reinsurance problem. We considered a general set of constraints on investment which gave us a variety of optimal policies and value functions.

We presented a numerical method based on Euler type discretization to solve the optimal problem. This numerical method has the advantages of being universal, fast and stable. It is universal because it solves the optimal problem with or without constraints. It also works on different types of claim distributions, either continuous or discrete or mixed, and can be applied even when the value function is not smooth. See examples 16, 19 and 21. It is fast because it is recursive. Moreover, since we use the Euler approximation for the second derivative of the value function, we get rid of the singularity problem at the point $s = 0$ in the case of optimal investment without constraint and reinsurance. This makes our numerical method stable.

An important result of this thesis is showing the importance of the viscosity solution concept through some examples in section 4.1.2. In chapter 3, we showed that the value function is the viscosity solution of the obtained HJB equation 3.4 and then proved the comparison Principle 7. These two proofs are done with a few assumptions and one can try to prove these theorems under a more general setting. In section 3.3 we have presented our numerical method and shown that it converges to the value function for the optimal investment problem. It remains, however, to show that our numerical algorithm for the optimal investment and reinsurance problem converges to the value function of the HJB equation 3.4. This is still an important open question and requires further research.

Along with these questions one can consider a more general set of possible actions, for example optimal investment, reinsurance and new business. See [**25**]. Another interesting research area is the problem of optimal dividend under constraints on the ruin probability (see e.g. Hipp [**19**]). This problem appears to be difficult for a risk reserve process with jumps.

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