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Dynamic equilibrium of collective degrees of freedom in strongly correlated quantum matter

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Abstract. Strongly correlated quantum systems may assume states stabilized by dynamic equilibrium of competing collective degrees of freedom. As a first example we consider the Kondo effect, which may be viewed as governed by a subtle balance of an infinite growth of the exchange coupling constant controled by an increasing local spin relaxation rate, leading to a Fermi liquid stable fixed point at low temperatures. As a second example the quantum critical behavior in antiferromagnetic metals will be considered. It is often found experimentally that the quasiparticle effective mass appears to diverge at the quantum critical point in spite of the fact that the critical spin fluctuations seem to be in the weak coupling domain - a contradiction. It will be shown that the feedback of the diverging mass into the spin fluctuation spectrum allows to establish a balance of the fermionic and bosonic singular degrees of freedom, leading to strong-coupling type quantum critical behavior. The resulting self-consistent theory obeys hyper-scaling of the dominant fermionic degrees of freedom, while the bosonic interaction is still in the weak coupling domain. Excellent agreement is found with experiment for the two candidate cases $\text{CeCu}_{6-x}\text{Au}_x$ and YbRh_2Si_2 .

1. Introduction

Over the last four decades strongly correlated quantum systems have attracted considerable attention owing to the rich variety of novel behavior found in these systems. Starting with the discovery of the superfluid phases of ^3He (for a comprehensive treatise see [1]), the Kondo effect and the advent of heavy fermion metallic compounds [2], the fractional quantum Hall effect, the cuprate high temperature superconductors and most recently the pnictide superconductors [3] it has become clear that correlations play a dominant role in all of these systems. The theoretical advances in attempting to describe these systems have been impressive: renormalized perturbation theory, renormalization group theories, slave particle theories, dynamical mean field theory, to name just a few. Yet there are problems that appear to defy our theoretical capabilities, like the cuprate superconductor problem. What is lacking is a systematic approach to tackle many-body problems at strong coupling. A first simple but powerful approximation scheme working at weak and strong coupling has been established more than a century ago: mean field theory. First developed to describe the liquid-gas system or magnetic systems, later generalized to describe superconductivity, charge density waves, spin density waves, or indeed any type of order, mean field theory appears to work reasonably well if there is a mean field order parameter of some sort. Generally speaking the mean field is an expectation value of an operator, which involves averaging over a number of neighbors of a given entity (charge, spin, Cooper pair,..). The larger the number of neighbors the better defined the mean field is.



However, fluctuations may destroy the mean field. In this case the fluctuations are the object of interest, and one may ask whether it is possible to devise a kind of mean field theory for the fluctuations. This is the subject of the present paper. I will argue in the following that there are examples of mean field theories of dynamical quantities, which give at least a good qualitative description of a situation in which fluctuations dominate the behavior. A first example is the Kondo effect, which is the formation of a quantum many-body resonance state of conduction electrons at a magnetic impurity in a metal. The famous perturbative renormalization group treatment of this model, Anderson's "poor man's scaling" [4], suggests infinite growth of the spin exchange coupling. We will see how this growth is controlled by the increasing inelastic scattering rate of the impurity spin such that a stable Fermi liquid state is attained. The second example is the quantum critical behavior at an antiferromagnetic instability point of a metal. Here the conventional Hertz-Millis theory (a perturbative renormalization group theory of the bosonic degrees of freedom) is found to often be in conflict with experiment, whenever the fermionic quasiparticles are characterized by a diverging effective mass (for an overview see [5]). It will be shown that in this case the fermions rather than the bosons are the dominant critical entities. The critical fermions modify the (bosonic) spin fluctuation in such a way that a self-consistent theory of the fermionic mass emerges which has a strong coupling solution in addition to the usual weak coupling solution, provided that the initial effective mass enhancement is sufficiently large.

2. Kondo effect

The Kondo effect is arguably the best studied quantum many-body problem. It is defined by a local spin \mathbf{S} , representing a quantum impurity with internal degrees of freedom, coupled by isotropic spin exchange interaction J to a bath of conduction electrons [2]. At the heart of the Kondo problem is the excitation of low energy particle-hole pairs occurring after each spin-flip process, when the conduction electron Fermi sea has to adjust to the suddenly switched potential provided by the spin, and undergoes an *orthogonality catastrophe* [6]. The quantum coherent readjustment of the Fermi sea is interrupted by any phase breaking process, which may happen at a rate Γ , the local spin relaxation rate. A perturbative treatment of this problem requires a systematic resummation of infinitely many relevant contributions, which may be achieved by the renormalization group (RG) method. While an analytic formulation in the form of Anderson's *poor man's scaling* [4] provides a correct and simple description in the perturbative regime, i.e. for dimensionless coupling $g_0 = JN(0) \ll 1$, or equivalently, temperatures much larger than the Kondo temperature, $T \gg T_K = D \exp(-1/2g_0)$, the strong coupling regime has so far only been accessible by Wilson's Numerical RG (NRG) [7]. The latter method has established that the low energy behavior of a Kondo system ($T \ll T_K$) is governed by local Fermi liquid theory, with far-reaching consequences as noted by Nozieres [8]. These results, at least as far as thermodynamic properties are concerned, have been confirmed by the Bethe-Ansatz solution [9, 10].

2.1. Kondo model

We consider a local spin $S = 1/2$ exchange coupled to the local conduction electron spin density, as described by the Hamiltonian [12]

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + J \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma',i} \frac{1}{2} c_{\mathbf{k}\sigma}^\dagger \tau_{\sigma\sigma'}^i c_{\mathbf{k}'\sigma'} S^i \quad (1)$$

Here τ is the vector of Pauli matrices, and $c_{\mathbf{k}\sigma}^\dagger$ creates an electron of momentum \mathbf{k} and spin σ . In order to make use of standard many-body techniques we represent the spin operator by $\mathbf{S} = \frac{1}{2} \sum_{\sigma,\sigma'} f_\sigma^\dagger \boldsymbol{\tau}_{\sigma\sigma'} f_{\sigma'}$, where $f_\sigma^\dagger, f_\sigma$ are pseudo-fermion (pf) operators ($\sigma = \uparrow, \downarrow$). We

take the pf chemical potential to be $\lambda = 0$ so that the pf spectral function is particle-hole symmetric, $A(\omega) = A(-\omega) \simeq \frac{\Gamma}{\omega^2 + \Gamma^2}$. The projection onto the physical sector of Hilbert space, $Q = \sum_{\sigma} f_{\sigma}^{\dagger} f_{\sigma} = 1$, is done by omitting all higher pf-loop diagrams. In case of the pf-selfenergy this amounts to taking only diagrams without any pf-loop, whereas in case of the conductivity/conductance and of the local spin susceptibility only diagrams with a single loop are kept. In addition, the latter quantities are corrected by a factor [11]

$$Y = \frac{2}{2 - \frac{Z_0}{Z}}, \quad \frac{Z_0}{Z} = \frac{1}{4} \exp\left[\frac{1}{T} \int_0^{\infty} d\lambda \langle Q \rangle_{\lambda}\right], \quad (2)$$

where $Z_0 = \text{Tr}[e^{-H_0/T}]$ and $Z = \text{Tr}[e^{-H/T}]$ are the partition functions of, respectively, the noninteracting and interacting systems. A useful approximation is provided by $Z_0/Z = \frac{1}{4} \exp[2 \int d\omega \ln(1 + e^{-\omega/T}) A(\omega)]$.

2.2. RG-equation.

The coupling g as a function of the temperature T , three frequencies (which we put to zero) and the running UV-cutoff D obeys the RG-equation [13, 14]

$$\frac{dg}{dD} = \beta(g) \frac{D}{D^2 + \Gamma^2}. \quad (3)$$

At weak coupling ($\Gamma \ll D$), $\beta(g) = -c_2 g^2 + O(g^3)$, while at strong coupling ($D \ll \Gamma$) $\beta(g) = -\frac{1}{2} c_3 g^3 + O(g^2)$. Assuming the interpolation formula $\beta = -\frac{[g(c_2 + c_3 g)]^2}{c_2 + 2c_3 g}$, the RG-equation may be integrated to give $\frac{1}{g(c_2 + c_3 g)} - \frac{1}{g_0 c_2} = \ln(\sqrt{D^2 + \Gamma^2}/D_0)$ ($g_0 \ll 1$). The leading g^3 -dependence of $\beta(g)$ at $g \gg 1$, which is reached at scales $D \ll T_K$, follows from an analysis of the contributions to $\beta(g)$ in arbitrary order of perturbation theory. It turns out that a third order (in g) contribution is dominant, as all higher order contributions vanish as $(D/\Gamma)^n$, $n > 4$, in the limit $D \ll \Gamma$. The two-loop order vertex correction diagram at strong coupling, $D \ll \Gamma$, which is the dominant contribution since self-energy corrections are smaller by a factor $(D/\Gamma)^2$, gives $c_3 = 1$. We also use the weak coupling result $c_2 = 2$. The temperature dependence of g (at zero energy) is found by solving the RG equation, setting $D = T$ as [13, 14]

$$g(T) = -1 + \sqrt{1 + 1/\ln(\sqrt{(T^2 + \Gamma^2)}/T_K)}. \quad (4)$$

, where $T_K = D_0 e^{-1/2g_0}$ is the Kondo temperature and $g_0 = N(0)J$ is the bare coupling. We observe that the growth of $g(T)$ as $T \rightarrow 0$ is controlled by the relaxation rate Γ , which is again a function of temperature and depends on $g(T)$ in turn. In the limit of $T \gg T_K$ we recover the weak coupling result $g(T) = 1/[2 \ln(\sqrt{(T^2 + \Gamma^2)}/T_K)]$, and $\Gamma \ll T_K$. In the limit $T \ll T_K$, considering that $\Gamma \rightarrow T_K + O(T^2)$, and consequently $\ln(\sqrt{(T^2 + \Gamma^2)}/T_K) \propto T^2$, as will be shown below, we find $g(T) \propto T_K/T$, diverging in the limit $T \rightarrow 0$.

It is worthwhile to note that the coefficient of the g^3 -term of $\beta(g)$ at $g \ll 1$ has the opposite sign, $c_3 = -4$. We may improve the approximate form of the β -function by requiring that the g^3 -term at weak coupling be reproduced, in the form $\beta(g) = -2g^2 \frac{1+g^2/4}{1+g}$.

2.3. Selfenergy

The spin relaxation rate Γ is given by the imaginary part of the pf self-energy $\text{Im}\Sigma$ plus vertex corrections, which we omit at present, as we do not expect those to have a major effect. It is

best to consider the diagrams of the components $\Sigma^{>,<}$, which may be classified according to the number n of particle-hole (ph) excitations in the intermediate state:

$$\text{Im}\Sigma(\omega) = \Gamma(\omega) = \frac{i}{2} \sum_{n=1}^{\infty} [\Sigma_n^>(\omega) - \Sigma_n^<(\omega)] \quad (5)$$

$$\begin{aligned} \Sigma_n^>(\omega) &= \sum_{n=1}^{\infty} \int d\omega' A(\omega') f(-\omega') \int d\epsilon_1 d\bar{\epsilon}_1 \dots d\epsilon_n \times \\ &\times d\bar{\epsilon}_n f(\epsilon_1) f(-\bar{\epsilon}_1) \dots f(\epsilon_n) f(-\bar{\epsilon}_n) \times |U_n|^2 \times \\ &\times \delta(\omega' - \omega + \bar{\epsilon}_1 + \epsilon_1 + \dots + \bar{\epsilon}_n + \epsilon_n) \end{aligned} \quad (6)$$

and a similar expression for $\Sigma_n^<(\omega)$. Here $f(\omega) = 1/[e^{\omega/T} + 1]$, the pf spectral function is given by $A(\omega) = \frac{1}{\pi} \Gamma(\omega)/[(\omega - \text{Re}\Sigma)^2 + \Gamma^2(\omega)]$, U_n is the n -particle-hole vertex function and the local conduction electron spectral functions $N(\epsilon) = N(0)\Theta(D - |\epsilon|)$ can be taken as constant and are combined with the (renormalized) exchange coupling J to give g . We will first analyze the single particle-hole contribution ($n = 1$), for which $U_1(\epsilon_1) = g(\epsilon_1)$. Since the Fermi functions confine the integrations to $|\omega'|, |\epsilon_1| \lesssim T$, we may take $g(\epsilon_1) \approx g(T)$ out of the integral. Further we replace $\Gamma(\omega')$ by $\Gamma(T)$ under the integral, resulting in the self-consistency equation

$$\Gamma(T) = 3g^2(T) \int d\omega \omega n(\omega) f(-\omega) \frac{\Gamma(T)}{\omega^2 + \Gamma^2(T)}$$

with $n(\omega)$ being the Bose function. Here we neglected $\text{Re}\Sigma(\omega')$, which is justified at all T . At $T \ll T_K$, we may neglect ω^2 in the denominator of $A(\omega)$. Then one finds $\Gamma^2(T) = c_{T,2} T^2 g^2(T)$, and further, substituting g , we find

$$\Gamma/T_K = 1 + c_{\Gamma} (T/T_K)^2 + \mathcal{O}\left((T/T_K)^3\right), \quad g = c_g(T_K/T) \quad (7)$$

with $c_{T,2} = \frac{3}{4}\pi^2$, $c_{\Gamma} = (c_{T,2} - \frac{1}{2})$ and $c_g = 1/\sqrt{c_{T,2}}$. We observe that g is indeed scaling to infinite strength as $T \rightarrow 0$, while the rate Γ assumes a finite value T_K with Fermi liquid corrections $\propto T^2$.

Let us now take a brief look at the higher order contributions. Analyzing the diagrams of U_n we note that only the n vertices g with external legs grow large. The internal g 's are integrated over and enter only the prefactor. We therefore get as generalized self-consistent equation $\Gamma^2 = \sum_{n=1}^{\infty} c_{T,n} g^{2n} (T^{2n}/\Gamma^{2n-2})$, which has the low T solution $\Gamma = T_K + c_{\Gamma} T^2$, and $g = c_g(T_K/T)$, with modified coefficients c_{Γ}, c_g , where $\sum_{n=1}^{\infty} c_{T,n} c_g^{2n} = 1$.

The limiting value of $\Gamma = T_K$ has been obtained before by Mattuck and collaborators [15], using the reasoning applied above. However, these authors employed the weak coupling form of the RG equation, which leads to finite temperature corrections linear in T , in disagreement with Fermi liquid theory. It is the form of the RG β -function $\propto g^3$, which is responsible for recovering the correct T^2 -dependence, the hallmark of Fermi liquid theory.

2.4. Conductance and spin susceptibility

With the aid of the above results one may now proceed to calculate the conductance G of a Kondo quantum dot (in units of $2e^2/\hbar$)

$$\frac{G}{G_0} = 3\pi^2 g^2 Y \int d\epsilon d\epsilon' f(\epsilon) f(-\epsilon') f(\epsilon' - \epsilon) A(\epsilon) A(\epsilon').$$

Here Y is the normalization factor introduced above. We may calculate Y analytically at high and low temperatures to find $Y = 2$ and $Y = 1$, respectively. The conductance at $T \gg T_K$ is found as $G/G_0 = 3\pi^2/[16\ln^2(T/T_K)]$ in agreement with perturbation theory. At low temperatures $T \ll T_K$ we may drop ϵ^2 and $(\epsilon')^2$ in the denominators of A compared to Γ^2 . The remaining integral is exactly the same as in the selfconsistency equation for Γ , and therefore $G/G_0 = c_{T,2}g^2\frac{T^2}{\Gamma^2} = 1$ at $T = 0$. In other words, the requirement of unitarity of the scattering amplitude is met. The finite temperature corrections are $O(T^2)$ and are obtained, e.g. by keeping the ϵ^2 in the denominator, as well as similar terms in the equation for the selfenergy and results in

$$G/G_0 = 1 - c_G \left(\frac{T}{T_K} \right)^2 \quad \text{for} \quad \frac{T}{T_K} \ll 1 \quad (8)$$

The spin susceptibility is given by

$$\chi_{imp}^R(T) = (g\mu_B)^2 [\chi_{SS}^R(T) + 2\chi_{sS}^R(T) + \chi_{ss}^R(T) - \chi_{ss,0}^R(T)], \quad (9)$$

where $\chi_{\alpha\beta}^R(T)$, $\{\alpha, \beta\} = s, S$ are the spin-spin-response functions of impurity spin and conduction electron spins, respectively. Neglecting vertex corrections, $\chi_{SS}^R(T)$ is given by the pf-bubble diagram,

$$\chi_{SS}^R(T) = \pi Y \int d\omega \tanh\left(\frac{\omega}{2T}\right) A^2(\omega) \frac{\omega}{\Gamma} \quad (10)$$

The mixed response function $\chi_{sS}^R(T)$ is given by a pf-bubble and a local conduction electron bubble, joined by an interaction vertex g as $\chi_{sS}^R(T) = -8\Pi_g^R \chi_{SS}^R(T)$ with $\Pi_g^R(T) = \frac{1}{N(0)} \int d\omega g(\omega) \omega \tanh\left(\frac{\omega}{2T}\right) N^2(\omega)$. The impurity induced c-electron contribution is given by two c-bubbles at the ends and a pf-bubble in the middle $\chi_{ss}^R(T) - \chi_{ss,0}^R(T) = c_{ss}[\Pi_g^R]^2 \chi_{SS}^R(T)$. In the limit $T \rightarrow 0$ the susceptibility is found to tend to a finite value, with quadratic temperature correction, as required by Fermi liquid theory, $\chi^R(T) = \alpha \frac{1}{T_K} - O(T^2)$. For high temperatures we find the perturbative result $\chi^R(T) = 1/(4T) (1 - 1/\ln(T/T_K) \pm \dots)$.

2.5. Summary of the Kondo problem part

In the above we demonstrated that the Kondo problem may be solved in the framework of renormalized perturbation theory, employing a self-consistent relation for the central quantity, the spin relaxation rate Γ . As to be expected, it is necessary to sum infinite classes of contributions in perturbation theory, which is done with the aid of the renormalization group method. The relaxation rate Γ of the local spin acts to separate the weak coupling (scale parameter $D \gg \Gamma$) from the strong coupling regime ($D \ll \Gamma$) of the renormalization group flow. Surprisingly, the renormalization group β -function in the strong coupling regime is dominated by a two-loop (cubic in g) contribution, all higher contributions being small in the parameter D/Γ . The coupling function acquires a strong dependence on energy or temperature, $g(T) \propto T_K/T$, and thus diverges in the limit $T \rightarrow 0$. In this work we determine Γ from renormalized perturbation theory, keeping all orders, in principle. As noticed in the 1970s by Mattuck and collaborators [15], the divergence of the coupling function $g(T)$ is exactly cancelled by the phase space factors of the energy integrations, so that Γ tends to the finite value T_K as $T \rightarrow 0$. This statement is correct even in the case of multiple particle-hole excitations in the intermediate state (in this case the numerical coefficients have not yet been calculated). The finite temperature corrections are $\propto T^2$, as required by Fermi liquid theory. Given the relatively simple structure of the theory, one may calculate thermodynamic, transport or dynamic properties. We presented results on the linear conductance through a Kondo quantum-dot. We observe that the conductance reaches the unitarity limit at $T \rightarrow 0$ and decreases $\propto T^2$ in the regime $T \ll T_K$. We further calculated the temperature dependent spin susceptibility, again in good agreement with known results.

3. Critical quasiparticles at an antiferromagnetic quantum critical point

The theory of quantum criticality in metals goes back to the 1960s, when the first RPA-type theories of metals with strong spin fluctuations were formulated. A field-theoretical formulation was pioneered by J. Hertz [16] and later completed by A. Millis [17]. The latter theories are based on the assumption that the critical excitations are the bosonic spin fluctuations, while the fermionic quasiparticle excitations may be integrated out. The resulting ϕ^4 - field theory has an upper critical dimension $d = 4$. Observing that the temporal dimension is given by the dynamical exponent z , the bare value of which is $z = 3$ for ferromagnets and $z = 2$ for antiferromagnets, one finds that the effective dimension $d_{eff} = d + z$ is frequently above the upper critical dimension, meaning that the theory is Gaussian. Nonetheless the results of the theory are often in conflict with experimental observation. The reason is that in those cases the fermionic quasiparticles show critical behavior, as deduced from a specific heat coefficient $\gamma(T) = C(T)/T$ diverging as $T \rightarrow 0$. Provided that the single-particle self-energy $\Sigma(\omega, k)$ is a weak function of momentum k , which appears to be the case in these systems, it follows that the quasiparticle effective mass ratio $m^*/m \propto \gamma(T)$ diverges, implying that the quasiparticles are critical.

3.1. Collective excitations in antiferromagnetic metals

The clue to resolving this puzzle lies in a careful study of vertex corrections [18, 21]. The (three-point) vertices coupling to spin density, $\Lambda(\mathbf{k}, \omega; \mathbf{q}, \Omega)$, and spin current density, $\mathbf{\Lambda}(\mathbf{k}, \omega; \mathbf{q}, \Omega)$ satisfy a Ward identity based on the spin conservation law: $\Omega\Lambda - \mathbf{q}\mathbf{\Lambda} = G^{-1}(\mathbf{k} + \mathbf{q}/2, \omega + \Omega/2) - G^{-1}(\mathbf{k} - \mathbf{q}/2, \omega - \Omega/2)$. It follows that in the case that the effective mass ratio $m^*/m = 1 - \partial Re\Sigma/\partial\omega$ is singularly enhanced in the limit $\omega \rightarrow 0$, the vertices Λ will be enhanced, too, in particular for momenta at the antiferromagnetic ordering wave vector $\mathbf{q} = \mathbf{Q}$, $\Lambda(\mathbf{k}_F, 0; \mathbf{Q}, 0) = \lambda_Q \propto m^*/m$. The latter vertex corrections enter the evaluation of diagrammatic contributions in several places. First in the Landau damping term, accounting for the decay of collective spin excitations into particle-hole pairs, and second at the vertices coupling the spin fluctuations to quasiparticles. The imaginary part of the spin susceptibility (the spectrum of spin fluctuations) for wave vectors near the AFM ordering wave vector \mathbf{Q} thus takes the form

$$\text{Im}\chi(\mathbf{q}, \omega) = \frac{N_0(\omega\lambda_Q^2/v_F Q)}{[(r + (\mathbf{q} - \mathbf{Q})^2\xi_0^2)^2 + (\omega\lambda_Q^2/v_F Q)^2]}, \quad (11)$$

Here, N_0 is the bare density of states at the Fermi surface, v_F is the bare Fermi velocity and $\xi_0 = k_F^{-1}$ is the microscopic AFM correlation length. The control parameter r is a function of both, the tuning field and the temperature. Outside the critical regime we have $r = r_0^{2\nu}$, where $\nu = 1/(1 + z/2)$, whereas in the critical regime $r \propto T^{2/z}$.

The above spin fluctuation propagator may be viewed as mediating a retarded interaction transferring momentum $\mathbf{q} \approx \mathbf{Q}$. It follows that a quasiparticle with momentum \mathbf{k} on the Fermi surface scattering off a spin fluctuation will end up in a state of momentum $\mathbf{k} + \mathbf{Q}$, in general far from the Fermi surface, except if \mathbf{k} and $\mathbf{k} + \mathbf{Q}$ are located in so-called "hot spots". There are, however, ways to get singular scattering even on the cold parts of the Fermi surface. One possibility is that impurity scattering helps to distribute the singular scattering at the hot spots all over the Fermi surface [19, 20]. Another mechanism operative even in clean systems is provided by the simultaneous scattering off two spin fluctuations carrying opposite momenta, such that the net transferred momentum is small [18, 22]. An excitation combined of two spin fluctuations may be considered as an (exchange) energy fluctuation χ_E . Its spectral weight is given by the expression

$$\text{Im}\chi_E(q, \omega) \approx N_0^3 \lambda_Q^{-2} \frac{(\omega \lambda_Q^2 / v_F Q)^{d-1/2}}{[\frac{|\omega| \lambda_Q^2}{v_F Q} + q^2 \xi_0^2 + r]^{(d+1)/2}} \quad (12)$$

3.2. Selfconsistent theory of fermion effective mass

The effective mass m^* is calculated from the fermion selfenergy Σ . The leading scale-dependent contribution to the latter is obtained from the single boson exchange diagram of the energy fluctuation propagator, the imaginary part of which is given by

$$\begin{aligned} \text{Im}\Sigma(\mathbf{k}, \omega) &= -\lambda_E^2 \int \frac{d\nu}{\pi} \sum_{\mathbf{q}} \text{Im}G(\mathbf{k} + \mathbf{q}, \omega + \nu) \text{Im}\chi_E(\mathbf{q}, \nu) [b(\nu) + f(\omega - \nu)] \\ &\approx v_F Q (m^*/m)^2 (\omega \lambda_Q^2 / v_F Q)^{d-1/2} \\ &\propto |\omega|^{d-1/2-\eta(2d+1)}, \end{aligned} \quad (13)$$

The interaction vertex $\lambda_E = \lambda_Q^2 \lambda_v$, where λ_v is $\propto (m^*/m)$, as it arises through a Ward identity connected to energy conservation. We used $\lambda_Q \propto (m^*/m)$, as discussed above. The Fermi and Bose functions $f(\omega), b(\omega)$ confine the ν -integration at low T to the interval $[0, \omega]$. In Eq. (4), we used the power law $m^*(\omega) \propto |\omega|^{-\eta}$. The scale dependent contribution to $\text{Re}\Sigma(\omega)$ follows from analyticity as $\text{Re}\Sigma(\omega) \propto (\omega/v_F Q)^{d-1/2} (m^*/m)^{2d+1}$. This leads to the self-consistency relation for $m^*(\omega)$ [18]

$$\begin{aligned} m^*(\omega)/m &= 1 - \partial \text{Re}\Sigma(\omega) / \partial \omega \\ &\approx 1 + (m^*/m)^{(2d+1)} (\omega/v_F Q)^{d-3/2} \end{aligned} \quad (14)$$

In dimensions $d > 3/2$ the scale dependent correction term $\propto (\omega/v_F Q)^{d-3/2}$ is seen to vanish in the limit $\omega \rightarrow 0$. This would suggest that m^* does not diverge in this limit, in agreement with the result of Hertz-Millis theory in that case. If, however, the effective mass is already sufficiently enhanced, e.g. by AFM fluctuations around the hot spots or by additional fluctuations, the scale-dependent term on the right-hand-side will dominate and a second solution becomes available, which is of a strong coupling type.

Before solving the self-consistent equation for $m^*(\omega)$ in the strong coupling regime we first note that in general, the ω and T dependence of m^* is obtained by substituting $\sqrt{\omega^2 + a^2 T^2}$ for ω , where a is a constant of order unity. For frequencies less than the temperature, we may replace ω by T . We then find

$$m^*(T) \propto (T/v_F Q)^\eta, \quad (15)$$

where the exponent η characterizing the scaling of m^* is found to be

$$\eta = (2d - 3)/4d \quad (16)$$

which gives $\eta = 1/4$ in the case of threedimensional AFM fluctuations and $\eta = 1/8$ in the case of quasi-twodimensional spin fluctuations ($d = 2$) in a threedimensional metal.

3.3. Critical exponents and dynamical scaling

The critical behavior of the spin-excitation spectrum as obtained above is given by power laws in frequency and in the bare control parameter r_0 (magnetic field H , pressure P , ..)

$$\omega \propto q^z, \quad z = \frac{2}{1 - 2\eta} = \frac{4d}{3} \quad (17)$$

$$\xi \propto r_0^{-\nu}, \quad \nu = \frac{3}{3 + d} \quad (18)$$

The spin correlation length is a function of T inside the critical regime, and a function of H (for example) outside

$$1/\xi(H, T) \propto T^{1/z}, \quad T > |H - H_c|^{z\nu} \quad (19)$$

$$1/\xi(H, T) \propto |H - H_c|^\nu, \quad |H - H_c|^{z\nu} > T \quad (20)$$

The scaling properties of the fermions may be derived from the single particle Green's function: $\Sigma(\omega) \propto v_F |k - k_F|$, i.e. the dynamical critical exponent of the fermions is given by $z_f = 1/(1-\eta)$ and the effective dimension of the fermions is $d_f = 1$ (the dimension perpendicular to the Fermi surface). The correlation length of the critical quasiparticles ξ_f follows power laws in temperature and the control parameter, respectively, $\xi_f \propto T^{-1/z_f}$ and $\xi_f \propto |H - H_c|^{-1/\nu_f}$. The exponent ν_f follows from the relation of the momentum to the (spindependent) shift in chemical potential $v_F |k - k_F| \propto \delta\mu_\sigma$ as $\nu_f = 1$. It is interesting to note that the Harris criterion $\nu_f > 2/3$ is satisfied so that the solution found here is stable with respect to spatial fluctuations induced by impurities. The correlation lengths of fermions and bosons are therefore related by $\xi \propto \xi_f^{z_f/z}$ inside the critical regime and $\xi \propto \xi_f^{\nu_f/\nu}$ outside.

Inside the critical regime and at $\mathbf{q} = \mathbf{Q}$ one finds E/T -scaling of the dynamical structure factor [18]

$$S(\mathbf{Q}, E) \sim T^{-2/z} \frac{\phi(E/T)}{1 + (\phi(E/T))^2}, \quad (21)$$

where $\phi(x) = x(x^2 + a^2)^{-\eta}$ and a is a constant.

3.4. Free energy

The true critical excitations of the above strong coupling scenario are the fermionic quasiparticles. Their contribution to the free energy may be derived from the expression for the entropy density in terms of the self-energy:

$$\frac{S}{V} = \frac{1}{2\pi} N(0) \int \frac{d\omega \omega}{T^2 \cosh^2 \frac{\omega}{2T}} [\omega - \text{Re}\Sigma(\omega)] \quad (22)$$

Substituting the critical self energy found above and integrating over temperature we find the scaling form of the free energy density

$$f(H, T) = \xi_f^{-(d_f+z_f)} \Phi_f(r_0 \xi_f^{1/\nu_f}, T \xi_f^{z_f}) \quad (23)$$

which obeys hyperscaling. This expression translates into

$$f(H, T) = \xi^{-(2d+1)} \Phi_f(r_0 \xi^{1/\nu}, T \xi^z) \quad (24)$$

in terms of the bosonic variables. This is not of the hyperscaling form. Indeed, in the cases considered the effective dimension of the bosons is $d_{eff} = d + 4d/3 = 7d/3 > 4$ for spatial dimension $d > 12/7$, implying that the boson-boson interaction scales to zero.

The consequences of the free energy expression have been analyzed in detail in [19, 20, 18]. The power laws in temperature (inside the critical regime) and in the control parameter (in the quantum disordered regime) have been determined. The exponents found agree very well with the available extensive experimental data. There is not a single observable where a distinct discrepancy of theory and experiment is found.

3.5. Summary of the self-consistent theory of quantum criticality

The theory of quantum criticality in metallic systems has traditionally been formulated in terms of bosonic incoherent excitations such as spin fluctuations, charge fluctuations, pair fluctuations. While it is true that these fluctuations are harbingers of the incipient order, one should keep in mind that the more fundamental excitations of an itinerant interacting Fermi system are the fermionic quasiparticles. The concept of quasiparticles is not confined to the realm of Fermi liquids, where the Landau quasiparticles are stable in the sense that their line width is $\propto \omega^2$, which is much less than the quasiparticle energy ω in the limit $\omega \rightarrow 0$. In fact quasiparticles may be well defined even in so-called non-Fermi liquid states, with power law self-energy $\Sigma(\omega) \propto \omega^{1-\eta}$, provided the exponent $\eta < 1/2$. In that case the quasiparticle decay rate is still less than the energy and the quasiparticle weight is non-zero at any finite energy, a necessary condition for the existence of a well-defined quasiparticle peak in the single particle spectral function. Consequently, one may still use the familiar tools of renormalized perturbation theory, namely the expansion of response functions in terms of quasiparticle-quasihole pairs and the infinite summation in the spirit of RPA. There is one important new element here: if the quasiparticles turn critical, i.e. if their effective mass diverges at the QCP, this has fundamental consequences for the interaction vertices. Namely, the Ward identities deriving from conservation laws or from additional symmetries cause the vertices (e.g. the spin vertex at finite momentum \mathbf{Q} in the antiferromagnetic case) to scale with the divergent effective mass. As a result the usual single boson exchange diagram for the self-energy, which may be proportional to a positive power of energy ω^s and thus irrelevant in the limit $\omega \rightarrow 0$, will be multiplied by a factor originating from vertex corrections, which may diverge and may offset the vanishing of ω^s . This is the scenario considered in the above. It leads to a self-consistent relation for the effective mass, which allows for two entirely different solutions, depending on the initial condition, i.e. the system parameters at the far end of the critical regime. One of the solutions is the familiar weak coupling solution, which is experimentally found to be realized in a number of systems. The second and new solution is activated at strong coupling, if the effective mass is already enhanced by additional critical fluctuations. This case is apparently met in at least two of the most intensively studied systems. It turns out that the theory agrees in detail with experimental observation.

4. Conclusion

Dynamic equilibrium of collective degrees of freedom in strongly correlated quantum matter may characterize certain well-defined states at strong coupling. In particular, the dynamic equilibrium of renormalized single particle quantities such as the fermionic quasiparticle relaxation rate or/and the effective mass and of two-particle quantities such as dynamical interaction vertices has been shown here to lead to novel equilibrium states. In the cases considered here there are reasons to expect that the correction terms to the self-consistent problems describing the dynamic equilibrium do not change the character of the solution. It is hoped that the approach sketched here will be useful in solving more strong-coupling many-body problems of this type in the future.

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