Spectrally Localized Strichartz Estimates and Nonlinear Schrödinger Equations

Zur Erlangung des akademischen Grades eines

Doktors der Naturwissenschaften

bei der Fakultät für Mathematik
des Karlsruher Instituts für Technologie (KIT)

genehmigte

DISSERTATION

von

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aus Speyer

Datum der mündlichen Prüfung:	13.05.2015
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Abstract (English Version)

In this thesis we treat the nonlinear Schrödinger equation (NLS for short) formulated for a non-negative, selfadjoint linear operator $(A, D(A))$ on $L^2(\Omega)$, whereby Ω is an arbitrary measure space. In this setting we formulate a local existence result for mild solutions in $D(A^s)$ under the assumption of Strichartz estimates with loss of derivatives. We obtain an abstract framework as a generalization of methods given in [BGT04b] by a systematic study of spectrally localized Strichartz and dispersive estimates. These concepts are used successfully in the literature to prove Strichartz estimates with loss of derivatives. We show that the above methods can be applied in a unified way to the situations in [BGT04b, Ant08, BFHM12, YZ04].
Our approach leads to new results for the NLS formulated for the Laplace-Beltrami operator on $\mathbb{R}^n \times M$, whereby M is a connected, compact Riemannian C^∞-manifold without boundary and $\dim(M) = m$. We provide Strichartz estimates with loss for $(e^{it\Delta_{\mathbb{R}^n \times M}})_{t\in\mathbb{R}}$ exploiting the dispersive estimates for $(e^{it\Delta_{\mathbb{R}^n}})_{t\in\mathbb{R}}$ and the spectrally localized dispersive estimates for $(e^{it\Delta_M})_{t\in\mathbb{R}}$. For $n = m = 1$ we extend the known global existence result in [TTV14] with respect to the growth of the nonlinearity.

Abstract (German Version)

In dieser Dissertation behandeln wir die nichtlineare Schrödingergleichung (kurz NLS) für einen nicht-negativen, selbstadjungierten linearen Operator $(A, D(A))$ auf $L^2(\Omega)$ mit einem beliebigen Maßraum Ω. In diesem Rahmen formulieren wir ein lokales Existenzresultat für milde Lösungen in $D(A^s)$ unter der Annahme von Strichartzabschätzungen mit Verlust von Ableitungen. Wir abstrahieren Methoden aus [BGT04b] durch ein systematisches Studium von spektral lokalisierten Strichartz- und dispersiven Abschätzungen. Diese Konzepte werden in der Literatur mit großem Erfolg dazu verwendet Strichartzabschätzungen mit Verlust von Ableitungen herzuleiten. Wir zeigen, dass die obigen Methoden sich in einheitlicher Form auf die Situationen in [BGT04b, Ant08, BFHM12, YZ04] anwenden lassen.
Unser Zugang erlaubt uns die Herleitung neuer Resultate für NLS für den Laplace-Beltrami Operator auf $\mathbb{R}^n \times M$ mit einer zusammenhängenden, kompakten Riemannschen C^∞-Mannigfaltigkeit M ohne Rand und $\dim(M) = m$. Wir beweisen Strichartzabschätzungen mit Verlust für $(e^{it\Delta_{\mathbb{R}^n \times M}})_{t\in\mathbb{R}}$ unter Verwendung der dispersiven Abschätzungen für $(e^{it\Delta_{\mathbb{R}^n}})_{t\in\mathbb{R}}$ und der spektral lokalisierten dispersiven Abschätzungen für $(e^{it\Delta_M})_{t\in\mathbb{R}}$. Im Fall $n = m = 1$ erweitern wir das globale Existenzresultat aus [TTV14] im Hinblick auf das zulässige Wachstum der Nichtlinearitäten.

Acknowledgements

This thesis was supported by the German Research Foundation (DFG) through the Research Training Group 1294: Analysis, Simulation and Design of Nanotechnological Processes. I am very grateful for this funding and I enjoyed my time as a scholar very much. I additionally recieved the Kronmüller-Scholarship from the Department of Mathematics at KIT as financial support for the final stages of this thesis.

First of all, I want to address my deepest thanks to my advisor Prof. Dr. Lutz Weis for the possibility to work on such an interesting topic. I have greatly profited from his experience as a mathematician and his constructive criticism in particular. I also thank Prof. Dr. Roland Schnaubelt for co-examining this thesis and his availability for mathematical questions. In 2013 I enjoyed a two week stay with Prof. Dr. Lutz Weis at the University of South Carolina in Columbia, where the initial idea for this thesis emerged. Many thanks go to Prof. Maria Girardi for her help in organizing this stay and her great hospitality.

I want to thank the rest of my colleagues at the Institute of Analysis at KIT, in particular the functional analysis group, for creating a pleasant work environment. I enjoyed each discussion I had with them, may it have been mathematical or non-mathematical. Moreover, I want to express my thanks to Markus Antoni and Dominik Müller for many helpful comments and suggestions on a first draft of this thesis.

Also in my private life there are people who contributed to a successful completion of this thesis. First of all I want to thank all my friends and family, who unconditionally gave their support in times of need. Leo and Martin deserve many thanks for their professional support, but most of all for being good friends. Deepest thanks go to my sister Nicole for her efforts to put things into perspective, despite (or because) of being 1000 km away, and to my mother Ingrid, who always believed in me (and hopefully will never get tired to do so). Finally, I want to thank Natascha, who endured and comforted me through some rough times in the last couple of weeks and months. I can not overstate the warmth and joy she has brought into my life.

Andreas Bolleyer, 15th of March, 2015

"Et hätt noch imer jot jejange."
3. Artikel, Kölsches Grundgesetz.

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Introduction

Over the last three decades the nonlinear Schrödinger equation on $\mathbb{R}^d$ given by

$$\begin{aligned} \mathrm{i}\partial_t u(t,\boldsymbol{x}) &= -\Delta u(t,\boldsymbol{x}) \pm |u(t,\boldsymbol{x})|^\beta u(t,\boldsymbol{x}), \quad t \neq 0,\ \boldsymbol{x} \in \mathbb{R}^d, \\ u(0,\boldsymbol{x}) &= f(\boldsymbol{x}), \qquad\qquad\qquad\qquad\qquad\qquad \boldsymbol{x} \in \mathbb{R}^d, \end{aligned} \tag{NLS}$$

with $\beta \in (0,\infty)$ has been subject to extensive study regarding its local and global wellposedness as well as long time dynamics in general, which has led to a good understanding of these aspects (see e.g. [Tao06] for an overview). The results on global wellposedness highly depend on the sign of the nonlinearity. One therefore distinguishes the defocusing (+) and focusing (-) cases with respect to the nonlinearity and the corresponding equation. From a physicist's point of view this widespread interest in (NLS), in particular with $\beta = 2$, stems from its rather versatile applications. They include

- laser beam models in Kerr media, where several reductions of Maxwell's equations lead to (NLS) (e.g. Section 1.2 of [SS99]),
- models for Bose-Einstein condensates, where (NLS) with an additional potential describes the common state of the particles of a quantum gas at extremely low temperatures (e.g. [PS03]),
- the theory of water waves, where so-called breather solutions of (NLS) serve as a prototype for rogue waves in the ocean (e.g. [CKOP13]).

Although all these applications are interesting in their own right, from now on we take a mathematician's point of view and shed some light on the mathematical challenges we face when constructing solutions of (NLS). Firstly, we provide the proper context for our results by a description of the development of the existing theory for (NLS). After that we state our main results and describe the structure of this thesis.

Historical and methodological background

In the framework of nonlinear evolution equations we formulate the partial differential equation (NLS) as the nonlinear ordinary differential equation on $L^2(\mathbb{R}^d)$ given by

$$\begin{aligned} u'(t) &= \mathrm{i}\Delta u(t) \mp \mathrm{i}|u(t)|^\beta u(t), \quad t \neq 0, \\ u(0) &= f, \end{aligned} \tag{CP}$$

with $f \in L^2(\mathbb{R}^d)$ and $\mathrm{i}\Delta$ realized on $L^2(\mathbb{R}^d)$ with $D(\mathrm{i}\Delta) = H_2^2(\mathbb{R}^d)$. The method to construct a solution of (CP) provided by semigroup theory is to solve, in a suitable space, the fixed point equation

$$u(t) = e^{\mathrm{i}t\Delta} f \mp \mathrm{i} \int_0^t e^{\mathrm{i}(t-s)\Delta} |u(s)|^\beta u(s) \, \mathrm{d}s. \tag{FP}$$

Here $(e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$ denotes the C_0-group on $\mathcal{L}(L^2(\mathbb{R}^d))$ generated by $\mathrm{i}\Delta$. Also, $e^{\mathrm{i}t\Delta}$ is an isometry on $H_2^s(\mathbb{R}^d)$ for all $s \in [0,\infty)$ and $t \in \mathbb{R}$. The parabolic theory of nonlinear evolution equations is therefore not available for $(e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$, since it clearly lacks the crucial smoothing estimates for analytic semigroups $(e^{-tA})_{t\in[0,\infty)}$ on $\mathcal{L}(L^2(\mathbb{R}^d))$ of the form

$$\forall_{t\in(0,\infty),\, k\in\mathbb{N}} : \ \|e^{-tA}\|_{L^2(\mathbb{R}^d)\to D(A^k)} \leq C(t,k) t^{-k}, \tag{1}$$

(see e.g. [Lun95] for a good introduction). In 1977 Robert S. Strichartz took the first step in the direction of a suitable substitute for (1) (see Section 3 of [Str77]), by proving for $p = q = {}^{2(n+2)}/_{n}$ the estimates

$$\|e^{\mathrm{i}(\cdot)\Delta} f\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \lesssim \|f\|_{L^2(\mathbb{R}^d)}, \tag{HS}$$

$$\left\| \int_0^{\cdot} e^{\mathrm{i}(\cdot-s)\Delta} F(s) \, \mathrm{d}s \right\|_{L^p(\mathbb{R},L^q(\mathbb{R}^d))} \lesssim \|F\|_{L^{p^*}(\mathbb{R},L^{q^*}(\mathbb{R}^d))}. \tag{IS}$$

p^*, q^* denote the Hölder conjugates of p and q, respectively. At the heart of his argument is a restriction theorem for Fourier transformed functions on $\mathbb{R}^d$ to quadratic surfaces, which exposes a noteworthy connection between harmonic analysis and the theory of partial differential equations. In honor of his important contribution estimates of the form (HS) and (IS) are called Strichartz estimates.
In the subsequent development of Strichartz estimates several authors observe that (HS) and (IS) also hold for pairs (p,q) with $p \neq q$ and use them to construct solutions of (FP). In 1987 Kato provided in [Kat87] one of the most successful approaches via a contraction argument involving Banach's fixed point theorem. For initial data $f \in H_2^1(\mathbb{R}^d)$ and $\beta \in (0, {}^4/_{\max\{d-2,0\}})$ he first constructs a unique solution u of (FP) in

$$X_T := L^\infty([0,T], H^1(\mathbb{R}^d)) \cap L^p([0,T], H_q^1(\mathbb{R}^d)), \quad (p,q) = ({}^{4(\beta+2)}/_{\beta d}, \beta+2), \tag{2}$$

with T small enough. The key observation is the completeness of the metric space $(\overline{B}_{X_T}(0,R), d_T)$ with

$$d_T(v,w) := \|v-w\|_{L^\infty([0,T],L^2(\mathbb{R}^d))} + \|v-w\|_{L^p([0,T],L^q(\mathbb{R}^d))}, \tag{3}$$

where we stress that d_T contains no derivatives in space, in contrast to the norm on X_T. The local solution is a posteriori in $C([0,T], H_2^1(\mathbb{R}^d))$ and can be extended to a maximal existence interval $[0, T_+)$ such that

$$T_+ < \infty \implies \lim_{t\to T_+} \|u(t)\|_{H_2^1(\mathbb{R}^d)} = \infty. \tag{4}$$

This property is called the blow-up alternative. Combined with the energy conservation

$$\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \pm \frac{1}{\beta+2}\|u(t)\|^{\beta+2}_{L^{\beta+2}(\mathbb{R}^d)} = \frac{1}{2}\|\nabla f\|^2_{L^2(\Omega)} \pm \frac{1}{\beta+2}\|f\|^{\beta+2}_{L^{\beta+2}(\mathbb{R}^d)} \tag{5}$$

for all $t \in [0, T_+)$ he finally derives criteria for $u : [0, T_+) \to H_2^1(\mathbb{R}^d)$ to be bounded. With (4) this implies global existence in $H_2^1(\mathbb{R}^d)$, i.e. $T_+ = \infty$. Note that $\beta < 4/\max\{d-2,0\}$ provides $H_2^1(\mathbb{R}^d) \hookrightarrow L^{\beta+2}(\mathbb{R}^d)$ by Sobolev's embedding theorem and the expressions in (5) are therefore finite. In the literature this is known as the energy subcritical or $\dot{H}^1$-subcritical case.
Up to now, variants of Kato's scheme presented above have been extensively used in more general situations. As a rule of thumb the availability of Strichartz estimates always leads to a good local existence theory. It is therefore of interest to have as much flexibility for the pairs (p, q) in these estimates as possible.
In that context Ginibre and Velo showed in [GV92] by means of complex interpolation and the dispersive estimate

$$\|e^{\mathrm{i}t\Delta}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}, \quad t \neq 0, \tag{6}$$

that (HS) and the following extension of (IS)

$$\left\| \int_0^{\cdot} e^{\mathrm{i}(\cdot - s)\Delta} F(s)\, \mathrm{d}s \right\|_{L^p(\mathbb{R}, L^q(\mathbb{R}^d))} \lesssim \|F\|_{L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\mathbb{R}^d))} \tag{IS'}$$

hold for all sharp $d/2$-admissible $(p, q), (\tilde{p}, \tilde{q}) \in [2, \infty]^2$, i.e.

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \qquad (p, q, d) \neq (2, \infty, 2),$$

with the additional restriction $q, \tilde{q} < 2d/(d-2)$ if $d \geq 3$. The pair $(p, q) = (2, 2d/(d-2))$ for $d \geq 3$ is known as the endpoint of this admissibility scale. In 1998 Keel and Tao successfully prove in their groundbreaking paper the Strichartz estimates (HS) and (IS') in an operator theoretic setting for all (p, q), $(\tilde{p}, \tilde{q})$ sharp $d/2$-admissible *including* the endpoints for $d \geq 3$. Not only do they completely answer the question of necessary and sufficient criteria on (p, q) for (HS) to hold, but the striking generality of their result allows for very versatile applications. Note that the sharp $d/2$-admissibility of the pairs (p, q), $(\tilde{p}, \tilde{q})$ is not necessary for the inhomogeneous Strichartz estimate (IS') to hold (see e.g. [Vil07]).
Today there are countless results of dispersive and Strichartz estimates for more general operators on $\mathbb{R}^d$ than the Laplacian (see e.g. [Sch07, ST02, DFVV10]). In all of them the availability of euclidean Fourier analysis is of great importance.
In contrast to this rather favorable situation on $\mathbb{R}^d$ where the availability of Strichartz estimates has lead to a well developed local and global existence theory, there is considerably less development of (FP) for the Laplacian on either a compact C^∞-manifold or a bounded C^∞-domain with additional boundary conditions. The main problem here is on the one hand the failure of the dispersive estimate (6), due to the pure point spectrum of the respective Laplacian, and on the other hand the lack of euclidean Fourier analysis.
In his famous paper [Bou93b] Bourgain combines the theory of Fourier series and analytic number theory to produce a weaker form of the Strichartz estimate (HS), a so-called Strichartz estimate with "ϵ-loss of derivatives" for $(e^{\mathrm{i}t\Delta_{\mathbb{T}^d}})_{t\in\mathbb{R}}$ generated by the Laplacian $\Delta_{\mathbb{T}^d}$ on the flat torus. In these estimates the L^2-norm on the right-hand side of (HS) is substituted by the fractional $H_2^\epsilon(\mathbb{T}^d)$-norm with $\epsilon \in (0, \infty)$. Here $H_2^\epsilon(\mathbb{T}^d)$ denotes the fractional domain $D((-\Delta_{\mathbb{T}^d})^{\epsilon/2})$, which is the natural extension of the classical Sobolev spaces.

Inspired by this approach Burq, Gerard, and Tzvetkov produce in [BGT04b] Strichartz estimates with loss of derivatives for the Laplace-Beltrami operator $(-\Delta_M, D(-\Delta_M))$ on an arbitrary connected, compact Riemannian C^∞-manifold M without boundary and $\dim(M) = d \in \mathbb{N}$. Their key observation is the validity of the spectrally localized dispersive estimate

$$\|\psi(2^{-2k}\Delta_M)e^{it\Delta}\|_{L^1(M)\to L^\infty(M)} \lesssim |t|^{-d/2}, \quad 0 < |t| \lesssim 2^{-k}, \tag{7}$$

for $k \in \mathbb{N}_0$ and $\psi \in C_c^\infty(\mathbb{R})$. For particular $\psi_0, \psi \in C_c^\infty(\mathbb{R})$ the decomposition estimate of Littlewood-Paley type

$$\|e^{it\Delta_M} f\|_{L^q(M)} \lesssim \|\psi_0(\Delta_M)e^{it\Delta} f\|_{L^q(M)} + \left(\sum_{k=1}^{\infty} \|\psi(2^{-2k}\Delta_M)e^{it\Delta} f\|^2_{L^q(M)}\right)^{1/2}$$

with $q \in [2, \infty)$ and the abstract Keel-Tao result [KT98] applied to the operator families $(\psi(2^{-2k}\Delta_M)e^{it\Delta_M})_{t\in\mathbb{R}}$ for $k \in \mathbb{N}$ imply

$$\|e^{i(\cdot)\Delta_M} f\|_{L^p([0,T],L^q(M))} \le C(T)\|f\|_{H_2^{1/p}(M)}, \tag{LS}$$

with $(p, q) \in [2, \infty] \times [2, \infty)$ sharp $d/2$-admissible. As in the case of the flat torus the loss of derivatives is measured in terms of the fractional domains $H^{1/p}(M) = D((-\Delta_M)^{1/2p})$ of the Laplace-Beltrami operator. Based on (LS) they prove previously unknown global wellposedness results for (FP) with initial data in $H_2^1(M)$. They provide in particular a global wellposedness result in $H_2^1(M)$ for $d = 2$ with arbitrary growth of the nonlinearity and for $d = 3$ with cubic growth of the nonlinearity. One important step in their analysis is solving (FP) for initial data in $H_2^s(M)$ with arbitrary growth of the nonlinearity. This is done following a straightforward adaptation of the arguments in [Kat87] on the space

$$X_T := L^\infty([0, T], H_2^s(M)) \cap L^p([0, T], H_q^\sigma(M)), \quad s > \sigma + \frac{1}{p}, \ \sigma > \frac{d}{q}, \tag{8}$$

and (p, q) sharp $d/2$-admissible with p large enough. In contrast to (3), the loss of derivatives in (LS) forces the authors to work with the metric

$$d_T(w, v) := \|v - w\|_{L^\infty([0,T],H_2^s(M))} + \|v - w\|_{L^p([0,T],H_q^\sigma(M))},$$

which includes the fractional domain space $H_2^s(M)$. It is also noteworthy that in the case $d = 3$ with cubic growth of the nonlinearity they adapt the crucial logarithmic L^∞-estimate (2) used in [BG80] by Brezis and Gallouet.

They proceed to show that the loss of derivatives in (LS) is optimal on the sphere in the endpoint case for $d \ge 3$ and reduce the loss of derivatives under the geometric assumptions on M that all geodesics are closed with a common period. Up to this day the interplay of the geometry of the state space and the loss of derivatives phenomenon is only partially understood and remains an active area of research (see e.g. [Bou14] and the references therein).

The considerable success of [BGT04b] and the follow up paper [BGT04a] (treating (CP) on non-trapping exterior domains) unfolds in the impact their methods and ideas had in the subsequent developments. The spirit of their arguments is found for example in the papers [Ant08, BFHM12, Iva10, BSS12] and [YZ04], which deal with Strichartz

estimates with loss of derivatives and wellposedness of (CP) for the Laplacian on C^∞-, polygonal and exterior domains, compact manifolds with boundary, and on $\mathbb{R}^d$ with an added superquadratic potential.
It is natural to ask for local and global existence for (CP) formulated for the Laplacian Δ_Ω on the product space $\Omega = \mathbb{R}^n \times M$ with an arbitrary connected, compact Riemannian C^∞-manifold M without boundary. However, results in that area are rare. Recently, there has been considerable progress in the case $M = \mathbb{T}^m$ (see e.g. [TV14, GPT15] and the references therein). For a treatment of a general compact C^∞-manifold we are only aware of [TTV14] and [TV12], in which the respective initial data belongs to $H_2^1(\mathbb{R}^n \times M)$ or modified anisotropic Sobolev spaces. We are interested in the Cauchy problem (CP) with initial data in $H_2^s(\mathbb{R}^n \times M)$ and point out that the Strichartz estimates used in the local existence result in [TTV14] exclusively rely on the dispersive estimate (6) for the Laplacian on $\mathbb{R}^n$. The authors solve (FP) in the spirit of [Kat87] in a space exhibiting mixed $\mathbb{R}^n \times M$ integrability, which limits their argument with respect to the growth of the nonlinearity. However, the spectrally localized dispersive estimates (7) for the Laplacian on the compact manifold M provided by [BGT04b] are not exploited in this work. It is reasonable to expect that using (7) improvements are possible since $(e^{\mathrm{i}t\Delta_\Omega})_{t\in\mathbb{R}}$ decomposes as $e^{\mathrm{i}t\Delta_\Omega} = e^{\mathrm{i}t\Delta_{\mathbb{R}^n}} e^{\mathrm{i}t\Delta_M}$ for $t \in \mathbb{R}$.

Main results and organization of this thesis

We pursue the following goals:

(A) A systematic development of a functional analytic framework for (CP) based on generalizations of the methods introduced by [BGT04b] and [Kat87]. It covers many of the existing examples presented above and allows us to reproduce known local and global existence results in a unified way.

(B) The derivation of Strichartz estimates with loss of derivatives for $(e^{\mathrm{i}t\Delta_{\mathbb{R}^n\times M}})_{t\in\mathbb{R}}$ on $\mathbb{R}^n \times M$ with an arbitrary connected, compact Riemannian C^∞-manifold M without boundary, using the dispersive estimates (6) for $(e^{\mathrm{i}t\Delta_{\mathbb{R}^n}})_{t\in\mathbb{R}}$ *and* the spectrally localized dispersive estimate (7) for $(e^{\mathrm{i}t\Delta_M})_{t\in\mathbb{R}}$. As an application we prove a global existence result for (NLS) in $H_2^1(\mathbb{R} \times M)$ with $\dim(M) = 1$ for nonlinearities with larger growth than considered in [TTV14].

This thesis is organized as follows. In **Chapter I** we consider the Cauchy problem

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t) + F(u(t)), \quad t \neq t_0, \\ u(t_0) &= f, \end{aligned} \tag{ACP}$$

with a non-negative, selfadjoint linear operator $(A, D(A))$ on a Hilbert space H and aim to provide conditions for the existence of solutions of (ACP).
Section I.1 and I.2 give an accessible introduction into the necessary concepts involved in the subsequent analysis. First we recall important facts from the spectral theory for non-negative, selfadjoint operators on Hilbert spaces. Then we recall the notion of strong and weak solutions of (ACP) and present the usual criteria for them to be given

by mild solutions, i.e. solutions of the fixed-point equation

$$u(t) = e^{-\mathrm{i}(t-t_0)A} f - \mathrm{i} \int_{t_0}^{t} e^{\mathrm{i}(s-t)A} F(u(s))\, \mathrm{d}s. \tag{AFP}$$

In view of Kato's scheme we also recall the important concepts of H- and energy conservation and give several criteria for a solution to fulfill these conservation laws.
Section I.3 contains the central local existence theorem for maximal mild solutions of (AFP) in fractional domain spaces $D(A^s)$, which forms our first main result. The crucial assumptions that we make in this theorem are Strichartz estimates with loss of the form

$$\|e^{-\mathrm{i}(\cdot)A} f\|_{L^p(I,L^q(\Omega))} \leq C(|I|)\|f\|_{D(A^\ell)} \tag{ALS}$$

for bounded intervals I and mapping properties of the nonlinearity on $D(A^s)$. Criteria for the boundedness of strong solutions in $D(A^{1/2})$ are also provided. These results generalize the local existence result in [BGT04b] and the energy methods used in Kato's scheme. As there are many interesting examples where the established existence result for mild solutions cannot be applied, we provide a standard argument for the construction of weak solutions in $D(A^s)$ by means of an approximation with strong solutions.
The methods for the construction of mild and weak solutions are applied in Chapter III and IV. In particular, we provide there all the needed estimates for the nonlinearity, which we assumed in the existence result for mild solutions.
In the final Section I.4 we provide the necessary estimates for the model nonlinearity $F(u(t)) := \pm|u(t)|^\beta u(t)$ with $\beta \in (0, \infty)$ to fit into the framework of Section I.2. We additionally prove a criterion for energy conservation of a strong solution of the corresponding nonlinear Schrödinger equation (ACP), which will be used frequently in Chapter III and IV.

Having applied the Strichartz estimates with loss from (ALS) in Section I.3 to the construction of solutions of (AFP), we focus in the first part of **Chapter II** exclusively on the derivation of (ALS). The second part is devoted to the presentation of further methods abstracted from the "$d = 3$"-case in [BGT04b] with respect to solutions of (ACP).
Section II.1 provides a systematic introduction to the concepts of Strichartz and dispersive estimates. We prove a variant of the important Keel-Tao result with complex interpolation spaces instead of real interpolation spaces in the non-endpoint situation. We present in Section II.2 a precise formulation of the following hierarchy:

Spectrally localized dispersive estimates
$\Longrightarrow$ *Spectrally localized Strichartz estimates*
$\Longrightarrow$ *Strichartz estimates with loss* (ALS).

To underline the relevance of the previous approach we provide in Section II.3 several examples from the literature where the arguments and ideas of Section II.2 are used to provide Strichartz estimates with loss of derivatives. In Chapter III we will further develop some of these examples.
In the final Section II.4 we come back to the Cauchy problem (ACP). We provide criteria for uniqueness of weak solutions of (ACP) in $D(A^s)$ with $s \geq 1/2$ and a priori estimates for strong solutions of (ACP) in $D(A^s)$ with $s > 1/2$. In these results we highlight the

role of so-called Bernstein inequalities and the spectrally localized Strichartz estimates from Section II.2.

In **Chapter III** we provide several applications in order to underline the flexibility of the abstract framework developed so far. In Section III.1 we treat the nonlinear Schrödinger equation (ACP) for $(A, D(A))$ being the Laplace-Beltrami operator on a connected, complete Riemannian C^∞-manifold Ω without boundary, with bounded geometry, and $\dim(\Omega) = d$. We derive the crucial mapping properties for the model nonlinearity on Sobolev spaces for such manifolds. We formulate a local existence result in $H_2^s(\Omega)$ for $d \geq 2$ and a global existence result in dimensions $d \in \{2, 3\}$ in $H_2^1(\Omega)$. Doing so allows us to recover known results for $\Omega = \mathbb{R}^d$ from [Kat87] and $\Omega = M$ from [BGT04b], where M is a connected, compact Riemannian C^∞-manifold without boundary. We furthermore show that our framework can be applied to the following situations:

- $A = -\operatorname{div}(B(\cdot)\nabla) + V$ on $\mathbb{R}^d$ with diagonal $B \in C_b^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ and $V \in C_b^1(\mathbb{R}^d)$ (see the appendix of [BGT04b] for $V = 0$).
- $A = -\Delta + V$ on $\mathbb{R}^d$ with superquadratic potential (see [YZ04]).
- $A = -\Delta_D$ on a polygonal or C^∞-domain in $\mathbb{R}^2$ with homogeneous Dirichlet boundary conditions (see e.g. [BFHM12, Ant08])

We recover all the respective local existence results and almost all of the global existence results stated there. In some cases we obtain results beyond the statements in the respective papers.

In **Chapter IV** we turn to the product situation. We in particular construct local and global strong solutions of

$$\begin{aligned} iu'(t) &= -\Delta_{\mathbb{R}^n\times M} u(t) \pm |u(t)|^\beta u(t), \quad t \neq 0, \\ u(0) &= f, \end{aligned} \tag{9}$$

whereby M is a connected, compact Riemannian C^∞-manifold without boundary and $\dim(M) = m$. $(-\Delta_{\mathbb{R}^n\times M}, D(-\Delta_{\mathbb{R}^n\times M}))$ denotes the Laplace-Beltrami operator on the product manifold $\mathbb{R}^n \times M$.
In an abstract setting we consider in Section IV.1 an operator family U in $\mathcal{L}(L^2(X \times Y))$, which can be decomposed as $U(t) = U_x(t)U_y(t)$ for $t \in \mathbb{R}$, with commuting operator families U_x in $\mathcal{L}(L^2(X))$ and U_y in $\mathcal{L}(L^2(Y))$. We show that U satisfies Strichartz estimates with loss under the assumption that U_x and U_y satisfy spectrally localized dispersive estimates. This result provides in Section IV.2 Strichartz estimates with loss of $1/p$ derivatives for $(e^{it\Delta_{\mathbb{R}^n\times M}})_{t\in\mathbb{R}}$, whereby the loss is comparable to the loss in (LS). We then use the local existence result of Section III.1 to produce a local existence result for (9) in $H_2^s(\mathbb{R}^n \times M)$ with arbitrary $n, m \in \mathbb{N}$. Combined with the energy methods in Section I.3 we prove global existence in $H_2^1(\mathbb{R} \times M)$ for $m = 1$ and defocusing nonlinearity with $\beta \in [2, \infty)$. We are not aware of these results in the literature. The only comparable global existence result in $H_2^1(\mathbb{R} \times M)$ with $m = 1$ we know of is contained in [TTV14]. Their methods work under the restriction $\beta < 4$.

Notation

In this section let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed $\mathbb{K}$-vector spaces with $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$.

Normed spaces and linear operators

- For $x \in X$ and $R \in [0, \infty)$ we put $B_X(x, R) := \{x \in X \mid \|x\|_X < R\}$ and $\overline{B}_X(x, R) := \overline{B_X(x, R)}$. If $(X, \|\cdot\|_X) = (\mathbb{R}^d, |\cdot|)$ we let $B(x, R) := B_{\mathbb{R}^d}(x, R)$.
- Let S be a set. $T_1, T_2 : S \to \mathbb{R}$. Then $T_1 \lesssim T_2$ denotes the existence of a constant $C \in [0, \infty)$ such that $T_1(s) \leq CT_2(s)$ for all $s \in S$. Moreover, $T_1 \cong T_2$ denotes $T_1 \lesssim T_2$ and $T_2 \lesssim T_1$. We write $T_1 \leq C(r)T_2$ if we want to stress the dependence of the constant on the expression r.
- $(X, \|\cdot\|_X) \cong (Y, \|\cdot\|_Y)$ means $X = Y$ and $\|\cdot\|_X \cong \|\cdot\|_Y$. $(X, \|\cdot\|_X) \equiv (Y, \|\cdot\|_Y)$ denotes isomorphic equivalence, i.e. the existence of a bounded isomorphism $\mathscr{I} : X \to Y$.
- $X \hookrightarrow Y$ denotes that X is continuously embedded in Y, i.e. there is a continuous, injective mapping $e : X \to Y$. The embedding is said to be compact if e is compact. If $e(X)$ is dense in Y, then the embedding is said to be dense.
- By $(X^*, \|\cdot\|_{X^*})$ we denote the topological dual space of $(X, \|\cdot\|_X)$ and for $x^* \in X^*$ and $x \in X$ we write $\langle x^*, x\rangle = \langle x^*, x\rangle_{X^*,X} := x^*(x)$.
- Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $(X, \|\cdot\|_X)$ and $x \in X$. The expression $x_n \overset{n\to\infty}{\longrightarrow} x$ denotes strong convergence and $x_n \rightharpoonup x$ weak convergence of $(x_n)_{n\in\mathbb{N}}$ to x in X as $n \to \infty$.
- Let (X, Y) be a Banach interpolation couple and $\theta \in (0, 1)$. Then $[X, Y]_\theta$ denotes the complex interpolation space generated by the complex interpolation method explained in Section A.1. For $q \in [1, \infty]$ we denote by $(X_0, X_1)_{\theta,q}$ the real interpolation space generated in Section 1.3 of [Tri95].
- Let $D(A)$ be a subspace of X and $A : D(A) \to Y$ be $\mathbb{K}$-linear. The pair $(A, D(A))$ is called a linear operator from X to Y. We call $\|\cdot\|_{D(A)} := \|\cdot\|_X + \|A\cdot\|_Y$ the graph norm of A on $D(A)$. $\rho(A)$ and $\sigma(A)$ denote the resolvent set and the spectrum of A, respectively.
- Let $(A, D(A))$ and $(B, D(B))$ be linear operators from X to Y. $(A, D(A)) \subseteq (B, D(B))$ denotes the situation $D(A) \subseteq D(B)$ and $Af = Bf$ for $f \in D(A)$. $A = B$ means $A \subseteq B$ and $B \subseteq A$.
- $\mathcal{C}(X, Y)$ denotes the set of densely defined, closed and $\mathcal{L}(X, Y)$ the set of bounded linear operators from X to Y. For $A \in \mathcal{L}(X, Y)$ we denote by $\|A\|_{X\to Y}$ the operator norm of A. Moreover, $\mathcal{C}(X) := \mathcal{C}(X, Y)$ and $\mathcal{L}(X) := \mathcal{L}(X, X)$.
- Let $(A, D(A))$ be a linear operator on an inner product space $(H, (\cdot,\cdot)_H)$. Then we call $(\cdot,\cdot)_{D(A)} := (\cdot,\cdot)_H + (A\cdot, A\cdot)_H$ the graph inner product of A on $D(A)$. $(A, D(A))$ is called non-negative if $(Af, f)_H \geq 0$ for all $f \in D(A)$. It is called positive definite if there is a constant $C \in (0, \infty)$ such that $(Af, f)_H \geq C\|f\|_H^2$ for all $f \in D(A)$.

Intervals and special function spaces

- $\mathbb{N}_{even} := \{n \in \mathbb{N} \mid n \text{ is even}\}$. $N_{\leq k}$ and $N_{\geq k}$ are the sets of $n \in \mathbb{N}_0$ with $n \leq k$ and $n \geq k$, respectively.
- We let $\mathcal{I} := \{I \subseteq \mathbb{R} \mid I \text{ is an interval }\}$ and define the subclasses $\mathcal{I}_o$, $\mathcal{I}_b$ and $\mathcal{I}_c$ of open, bounded and compact elements of $\mathcal{I}$. For $I \in \mathcal{I}$ we let $|I| := \lambda(I)$, where λ is the Lebesgue measure on $\mathbb{R}$.
- Let $(\Omega, \mathscr{S}, \mu)$ be a measure space and $p \in [1, \infty]$. $M(\Omega)$ denotes the space of measurable complex valued functions and $M_b(\Omega)$ the set of bounded elements in $M(\Omega)$. $L^p(\Omega)$ denotes the usual Lebesgue space of equivalence classes of $f \in M(\Omega)$ with $|f|^p$ Lebesgue integrable if $p \in [1, \infty)$ or with $|f|$ bounded almost everywhere if $p = \infty$.
- By $\mathscr{B}(\mathbb{R}^d)$ we denote the Borel σ-algebra on $\mathbb{R}^d$ and by λ the Lebesgue measure on $\mathscr{B}(\mathbb{R}^d)$. For $\Omega \subseteq \mathbb{R}^d$ we denote the trace σ-algebra of $\mathscr{B}(\mathbb{R}^d)$ with respect to Ω by $\mathscr{B}(\Omega)$.
- Let Ω be either an open subset of $\mathbb{R}^d$ or a Riemannian C^∞-manifold and $s \in [0, \infty)$, $p \in [1, \infty], q \in (1, \infty)$.
  - $W_p^s(\Omega)$ denotes for $s \in \mathbb{N}_0$ the classical Sobolev space of $\mathbb{C}$-valued functions defined via weak or covariant derivatives, respectively.
  - We define the fractional Sobolev space $H_q^s(\Omega) := [W_q^{[s]}(\Omega), W_q^{[s]+1}(\Omega)]_{s-[s]}$ for $s \notin \mathbb{N}$ and we let $H_q^s(\Omega) := W_q^s(\Omega)$ for $s \in \mathbb{N}_0$.
  - $C_c^\infty(\Omega)$ denotes the set of complex valued C^∞ functions on Ω with bounded support and $H_{q,0}^s(\Omega)$ denotes the closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{H_q^s(\Omega)}$.
- For $I \in \mathcal{I}$ and $\alpha \in (0, 1]$ we define:

$$
\begin{aligned}
C_w(I, X) &:= \{u : I \to X \mid u \text{ is weakly continuous}\}, \\
C_b(I, X) &:= \{u \in C(I, X) \mid u \text{ is bounded}\}, \\
C_{b,u}(I, X) &:= \{u \in C_b(I, X) \mid u \text{ is uniformly continuous}\}, \\
C^{0,\alpha}(I, X) &:= \{u \in C_{b,u}(I, X) \mid \|u\|_{C^{0,\alpha}(I,X)} < \infty\}, \\
&\qquad \|u\|_{C^{0,\alpha}(I,X)} := \|u\|_{L^\infty(I,X)} + \sup_{t,s \in I,\, s \neq t} \frac{\|u(t) - u(s)\|_X}{|t - s|^\alpha}.
\end{aligned}
$$

- Let $I \in \mathcal{I}$, $p \in [1, \infty]$, and $(X, \|\cdot\|_X)$ be a Banach space. $L^p(I, X)$ and $W_p^1(I^o, X)$ denote the respective L^p and Sobolev space for functions $u : I \to X$ described in Section A.3.

I. A functional analytic framework for the nonlinear Schrödinger equation

This chapter contains the exposition of a functional analytic framework for the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t) + F(u(t)), \quad t \neq t_0, \\ u(t_0) &= f, \end{aligned} \tag{I.1}$$

with a non-negative selfadjoint linear operator $(A, D(A))$ on a Hilbert space H. It is organized as follows.
In Section I.1 we recall the functional calculus in Hilbert spaces and the extrapolation theory for the operator $(A, D(A))$. We furthermore review some useful properties of the fractional powers $(A^s, D(A^s))$ for $s \in [0, \infty)$.
In Section I.2 we use these concepts to give a precise formulation of the Cauchy problem (I.1) for $f \in D(A^s)$ and $F : D(A^s) \to D(A^{1/2})^*$ with $s \in [1/2, \infty)$. We recall the standard notions of solutions, namely strong, weak and mild solutions and discuss their relation depending on the mapping properties of F. In addition, we give criteria for solutions to have a conserved H-norm or energy.
In Section I.3 we present methods to construct mild and weak solutions of the Cauchy problem (I.1). Under the assumption of Strichartz estimates with loss and suitable estimates for the nonlinearity we prove the central existence result for maximal mild solutions of (I.1) with $f \in D(A^s)$. This existence theorem is our first main result and it will be applied frequently in Chapter III and IV in various situations. There we will also provide the needed estimates for the nonlinearity, which we have assumed here. For situations in which the latter result can not be applied, we provide a standard approximation scheme, which allows us to construct weak solutions of (I.1) from a sequence of strong solutions.
In the final Section I.4 we review the most commonly used nonlinearity in this thesis, namely the model nonlinearity $F(u(t)) := \pm|u(t)|^\beta u(t)$ with $\beta \in (0, \infty)$. We provide the needed mapping properties to fit this nonlinearity into the framework of Section I.2 and prove an additional criterion for strong solutions of the corresponding Cauchy problem to have energy conservation.

Throughout this chapter $(H, (\cdot, \cdot)_H)$ denotes a complex Hilbert space, which we equip with the real scalar product $\langle \cdot, \cdot \rangle_H := \mathrm{Re}(\cdot, \cdot)_H$. We always consider H and H^* identified via $f \mapsto \langle f, \cdot \rangle_H$. Oberserve that the complex scalar product $(\cdot, \cdot)_H$ and the real scalar product $\langle \cdot, \cdot \rangle_H$ induce an equivalent norm on H. We let $(A, D(A))$ be a non-negative, selfadjoint $\mathbb{C}$-linear operator on $(H, \langle \cdot, \cdot \rangle_H)$ if not stated otherwise.

I.1. Spectral calculus, extrapolation, and energy space

The spectral theorem for selfadjoint operators on Hilbert spaces in multiplication form in Theorem 1.7 of [Tay11] states the existence of a measure space $(\Omega_A, \Sigma_A, \mu_A)$, a unitary map $V_A : H \to L^2(\Omega_A)$, and a function $m_A \in M(\Omega_A)$ with $m_A(\Omega_A) \subseteq \mathbb{R}$ such that

$$D(A) = \{h \in H \mid m_A V_A h \in L^2(\Omega)\}, \qquad A = V_A^* m_A V_A \text{ on } D(A).$$

Theorem VII.3.1 in [Wer00] states that $(\Omega_A, \Sigma_A, \mu_A)$ is σ-finite if $(H, \langle\cdot,\cdot\rangle_H)$ is separable. It is easy to show that $m_A \geq 0$ almost everywhere on Ω_A since $(A, D(A))$ is non-negative. For $\varphi \in M(\mathbb{R})$ the spectral theorem allows us to define the linear operator $(\varphi(A), D(\varphi(A)))$ by

$$D(\varphi(A)) := \{h \in H \mid (\varphi \circ m_A) V_A h \in L^2(\Omega_A)\},$$
$$\varphi(A) := V_A^*(\varphi \circ m_A) V_A \text{ on } D(\varphi(A)).$$

This definition gives rise to the following map.

Theorem I.1.1
The map

$$\Phi_A : M(\mathbb{R}) \to \mathcal{C}(X), \qquad \varphi(A) := \Phi_A(\varphi)$$

has the following properties for $\varphi, \eta \in M(\mathbb{R})$*:*

(SC1) $\varphi(A) + \eta(A) \subseteq (\varphi + \eta)(A)$ *and* $\varphi(A)\eta(A) \subseteq (\varphi\eta)(A)$*, whereby*

$$D(\varphi(A)\eta(A)) = D((\varphi\eta)(A)) \cap D(\eta(A)),$$

(SC2) $\varphi|_{\sigma(A)} = 0 \implies \varphi(A) = 0$ *and* $\varphi|_{\sigma(A)} = 1 \implies \varphi(A) = \mathrm{id}$,

(SC3) $\varphi(A)^* = \overline{\varphi}(A)$,

(SC4) $\Phi_A : (M_b(\mathbb{R}), \|\cdot\|_{L^\infty([0,\infty))}) \to \mathcal{L}(H)$ *is a bounded algebra homomorphism.*

Remarks: The first assertion in (SC2) implies that exclusively the part of the function φ on $\sigma(A)$ is relevant for the definition of Φ_A. Recall that every non-negative, selfadjoint operator has its spectrum in $[0, \infty)$. The functions in use will therefore only be defined on $[0, \infty)$.

Proof. Let $\varphi \in M(\mathbb{R})$, $m := \varphi \circ m_A$, and the linear multiplication operator $(T_m, D(T_m))$ on $L^2(\Omega_A)$ be defined by

$$D(T_m) := \{f \in L^2(\Omega_A) \mid mf \in L^2(\Omega_A)\},$$
$$T_m f := mf \text{ on } D(T_m).$$

The unitary equivalence of $\varphi(A)$ and T_m implies

$$(\varphi(A), D(\varphi(A))) \in \mathcal{C}(H) \iff (T_m, D(T_m)) \in \mathcal{C}(L^2(\Omega_A)).$$

The latter follows from the fact that Ω_A admits the measurable partition

$$\Omega_A = \bigcup_{n\in\mathbb{N}} \{\omega \in \Omega_A \mid |m(\omega)| \le n\}$$

and

$$\operatorname{span}\{f \in L^2(\Omega_A) \mid \exists_{n\in\mathbb{N}} : \operatorname{supp}(f) \subseteq \{\omega \in \Omega_A \mid |m(\omega)| \le n\}\} \subseteq D(T_m).$$

For the density of the set on the left-hand side in $L^2(\Omega_A)$ we use Theorem 2.28 in [Els05] on each set $\{\omega \in \Omega_A \mid |m(\omega)| \le n\}$. The properties (SC1) - (SC3) are straightforward consequences of the definition of $\varphi(A)$ and its unitary equivalence to the multiplication operator T_m. Property (SC4) follows from (SC1) and $D(\varphi(A)) = H$ for $\varphi \in M_b(\mathbb{R})$. □

For further reference we fix the following notion for Φ_A and $\Phi_A|_{M_b(\mathbb{R})}$ to emphasize its origin.

Definition I.1.2
The maps Φ_A and $\Phi_A|_{M_b(\mathbb{R})}$ from Theorem I.1.1 are called the spectral calculus and bounded spectral calculus of $(A, D(A))$, respectively. We always equipp $D(\varphi(A))$ with the graph norm $\|\cdot\|_{\varphi(A)}$.

The spectral calculus introduced in Theorem I.1.1 can be used to construct important operators, including fractional powers and the unitary C_0-group of isometries generated by $(A, D(A))$.

Corollary I.1.3
Let $\theta \in (0,1)$ and $\alpha, \beta \in [0,\infty)$.

(a) Let $p_\alpha \in M([0,\infty))$ be defined by $p_\alpha(\lambda) := \lambda^\alpha$. The fractional powers $A^\alpha := p_\alpha(A)$ satisfy:

- *The embedding $(D(A^{\alpha+\beta}), \|\cdot\|_{D(A^{\alpha+\beta})}) \hookrightarrow (D(A^\beta), \|\cdot\|_{D(A^\beta)})$ is dense,*
- *$A^\alpha A^\beta = A^{\alpha+\beta}$,*
- *$[D(A^\alpha), D(A^\beta)]_\theta \cong D(A^{(1-\theta)\alpha+\theta\beta})$.*

(b) $U : \mathbb{R} \to \mathcal{L}(H)$ defined by $U(t) := e^{-\mathrm{i}tA}$ is a unitary C_0-group of isometries on $D(A^\alpha)$.

Remarks:

(1) Assertion *(b)* is known as Stone's theorem. We denote U also by $(e^{-\mathrm{i}tA})_{t\in\mathbb{R}}$ and call it the Schrödinger group of $(A, D(A))$.

(2) Since $D(A^{1/2}) \hookrightarrow H$ is dense and H is identified with its dual space H^*, we also have $D(A^{1/2}) \hookrightarrow D(A^{1/2})^*$ with $e : D(A^{1/2}) \to D(A^{1/2})^*$ defined by $e(f) := \langle f, \cdot\rangle_H$.

Proof. (a) Let us first prove the properties of the fractional powers. For $h \in D(A^{\alpha+\beta})$ holds

$$\begin{aligned}\|m_A^\beta V_A h\|_{L^2(\Omega_A)} &\le \|V_A h\|_{L^2(\Omega_A)} + \|\mathbb{1}_{\{|m_A|>1\}} m_A^{\alpha+\beta} V_A h\|_{L^2(\Omega_A)} \\ &\le \|h\|_H + \|A^{\alpha+\beta} h\|_H.\end{aligned}$$

Hence, $D(A^{\alpha+\beta}) \hookrightarrow D(A^{\beta})$ and with (SC1) follows $A^{\alpha+\beta} = A^{\alpha}A^{\beta}$. Concerning the density of $D(A^{\alpha+\beta})$ in $D(A^{\beta})$ it is enough to prove the density of $D(m_A^{\alpha+\beta})$ in $D(m_A^{\beta})$. Let $f \in D(m_A^{\beta})$. Since $D(m_A^{\alpha+\beta})$ is dense in $L^2(\Omega_A)$ by Theorem I.1.1, there are sequences $(f_{1,n})_{n\in\mathbb{N}}$ and $(f_{2,n})_{n\in\mathbb{N}}$ in $D(m_A^{\alpha+\beta})$ with

$$\|f_{1,n} - f\|_{L^2(\Omega_A)} + \|f_{2,n} - m_A^{\beta} f\|_{L^2(\Omega_A)} \overset{n\to\infty}{\longrightarrow} 0. \tag{I.2}$$

The sequence $(f_n)_{n\in\mathbb{N}}$ in $D(m_A^{\beta+\alpha})$ defined by

$$f_n : \Omega_A \to \mathbb{C}, \quad f_n := \mathbb{1}_{\{|m_A|\leq 1\}} f_{1,n} + \mathbb{1}_{\{|m_A|>1\}} \frac{f_{2,n}}{m_A^{\beta}}$$

satisfies $\|f_n - f\|_{D(m_A^{\beta})} \overset{n\to\infty}{\longrightarrow} 0$ and thus $D(A^{\alpha+\beta})$ is dense in $D(A^{\beta})$. It remains to prove the interpolation property. For $f \in D(A^{\alpha})$ follows

$$\begin{aligned}\|(m_A+1)^{\alpha} V_A f\|_{L^2(\Omega_A)} &\lesssim \|V_A f\|_{L^2(\Omega_A)} + \|\mathbb{1}_{\{|m_A|>1\}} m_A^{\alpha} V_A f\|_{L^2(\Omega_A)} \\ &\lesssim \|f\|_H + \|A^{\alpha} f\|_H.\end{aligned}$$

Hence, $D(A^{\alpha}) \hookrightarrow D((\mathrm{id}+A)^{\alpha})$ and the same argument yields $D((\mathrm{id}+A)^{\alpha}) \hookrightarrow D(A^{\alpha})$. Consequently, $D(A^{\alpha}) \cong D((\mathrm{id}+A)^{\alpha})$. Since $(\mathrm{id}+A, D(A))$ is positive definite the interpolation rule for fractional powers (A.4) implies

$$\begin{aligned}[D(A^{\alpha}), D(A^{\beta})]_{\theta} &\cong [D((\mathrm{id}+A)^{\alpha}), D((\mathrm{id}+A)^{\beta})]_{\theta} \\ &\cong D((\mathrm{id}+A)^{(1-\theta)\alpha+\theta\beta}) \cong D(A^{(1-\theta)\alpha+\theta\beta}).\end{aligned}$$

(b) Let $\varphi_t : [0,\infty) \to \mathbb{C}$ be defined by $\varphi_t(\lambda) := e^{-\mathrm{i}t\lambda}$. The properties of the family $(\varphi_t)_{t\in\mathbb{R}}$ and (SC2)-(SC4) immediately provide that U is a unitary group of isometries on H. For $h \in H$ additionally holds

$$\|e^{-\mathrm{i}tA}h - h\|_H \overset{t\to 0}{\longrightarrow} 0 \iff \|\varphi_t V_A h - V_A h\|_{L^2(\Omega_A)} \overset{t\to 0}{\longrightarrow} 0,$$

where the latter assertion follows from the dominated convergence theorem. The fact that $U(t)$ and A^{α} commute on $D(A^{\alpha})$ implies all the mentioned properties of U on $\mathcal{L}(H)$ additionally on $\mathcal{L}(D(A^{s}))$. □

In the next theorem we prove the existence of an extension of the linear operator $(A, D(A))$ on H to a linear operator $(\tilde{A}, D(A^{1/2}))$ on $D(A^{1/2})^*$. We additionally prove useful properties of $(\tilde{A}, D(A^{1/2}))$ for Section I.2.

Theorem I.1.4
There is a linear operator $(\tilde{A}, D(A^{1/2}))$ on $D(A^{1/2})^*$ such that:

(a) $\tilde{A} = A$ on $D(A)$.

(b) $(\tilde{A}, D(A^{1/2}))$ is non-negative and selfadjoint.

(c) The Schrödinger group $\tilde{U} : \mathbb{R} \to \mathcal{L}(D(A^{1/2})^*)$ generated by $(\tilde{A}, D(A^{1/2}))$ satisfies $\tilde{U}(t) = U(t)$ on H for $t \in \mathbb{R}$.

Remark: In *(a)* the assertion $A = \tilde{A}$ on $D(A)$ means $\langle \tilde{A}f, \cdot\rangle = \langle Af, \cdot\rangle_H$ for all $f \in D(A)$. The equality in *(c)* has to be understood in the same way.

Proof. First note that $A^{1/2} : D(A^{1/2}) \to H$ is bounded. By the Cauchy-Schwarz inequality and $D(A^{1/2}) \hookrightarrow H$ holds

$$\left|\langle A^{1/2}f, A^{1/2}g\rangle_H\right| \leq \|f\|_{D(A^{1/2})}\|g\|_{D(A^{1/2})}.$$

Then the $\mathbb{C}$-linear operator

$$\tilde{A} : D(A^{1/2}) \to D(A^{1/2})^*, \qquad \langle \tilde{A}f, \cdot\rangle := \langle A^{1/2}f, A^{1/2}\cdot\rangle_H$$

is well-defined and bounded. From now on we consider $(\tilde{A}, D(A^{1/2}))$ as an unbounded operator on $D(A^{1/2})^*$.
(a) For $f \in D(A)$ holds $\langle \tilde{A}f, g\rangle = \langle Af, g\rangle_H$ for all $g \in D(A^{1/2})$. Since $D(A^{1/2})$ is dense in H we have that $\tilde{A} = A$ on $D(A)$.
(b) We first show that

$$\mathrm{id} + \tilde{A} : D(A^{1/2}) \to D(A^{1/2})^*, \qquad \langle (\mathrm{id} + \tilde{A})f, \cdot\rangle := \langle f, \cdot\rangle_{D(A^{1/2})}$$

is a bijection. For $f \in D(A^{1/2})$ holds

$$\|(\mathrm{id} + \tilde{A})f\|_{D(A^{1/2})^*} = \sup_{\|g\|_{D(A^{1/2})}\leq 1} |\langle f, g\rangle_{D(A^{1/2})}| = \|f\|_{D(A^{1/2})}. \tag{I.3}$$

This implies that $\mathrm{id} + \tilde{A}$ is an isometry, hence injective. The bilinear form $\langle\cdot,\cdot\rangle_{D(A^{1/2})}$ is furthermore bounded and coercive. The Lax-Milgram Lemma (see [Eva10] Section 6.2.1) applied to $\langle\cdot,\cdot\rangle_{D(A^{1/2})}$ guarantees for each $f^* \in D(A^{1/2})^*$ a unique $f \in D(A^{1/2})$ with $\langle f^*, \cdot\rangle = \langle f, \cdot\rangle_{D(A^{1/2})}$. $\mathrm{id} + \tilde{A}$ is therefore surjective and in particular $-1 \in \rho(\tilde{A})$. With $-1 \in \rho(A) \cap \rho(\tilde{A})$ we have $(\mathrm{id} + A)^{-1} = (\mathrm{id} + \tilde{A})^{-1}$ on H. For $f \in D(A^{1/2})$ then holds

$$(\mathrm{id} + \tilde{A})^{-1}\tilde{A}f = f - (\mathrm{id} + A)^{-1}f. \tag{I.4}$$

We then define a scalar product on $D(A^{1/2})^*$ by

$$\langle f^*, g^*\rangle_{D(A^{1/2})^*}f := \langle (\mathrm{id} + \tilde{A})^{-1}f^*, (\mathrm{id} + \tilde{A})^{-1}g^*\rangle_{D(A^{1/2})}.$$

Note that the induced norm is equivalent to the prior $\|\cdot\|_{D(A^{1/2})^*}$-norm and we use the same symbol for both of them. The equalities (I.3) and (I.4) as well as the fact that $(I + A)^{-1}$ is positive definite allows for $f \in D(A^{1/2})$ the estimate

$$\begin{aligned}
&\|f\|_{D(A^{1/2})^*} + \|\tilde{A}f\|_{D(A^{1/2})^*} \\
&\quad\cong \|f\|_{D(A^{1/2})} + \left(\|f\|^2_{D(A^{1/2})} + \|(\mathrm{id} + A)^{-1}f\|^2_{D(A^{1/2})} - 2\langle f, (\mathrm{id} + A)^{-1}f\rangle_{D(A^{1/2})}\right)^{1/2} \\
&\quad\lesssim \|f\|_{D(A^{1/2})} \lesssim \|f\|_{D(A^{1/2})^*} + \|\tilde{A}f\|_{D(A^{1/2})^*}.
\end{aligned}$$

Hence, $(\tilde{A}, D(A^{1/2}))$ is closed. The equality (I.4) also gives for $f, g \in D(A^{1/2})$

$$\begin{aligned}\langle \tilde{A}f, g\rangle_{D(A^{1/2})^*} &= \langle (\mathrm{id}+\tilde{A})^{-1}\tilde{A}f, (\mathrm{id}+\tilde{A})^{-1}g\rangle_{D(A^{1/2})} \\ &= \langle f, (\mathrm{id}+A)^{-1}g\rangle_{D(A^{1/2})} - \langle (\mathrm{id}+A)^{-1}f, (\mathrm{id}+A)^{-1}g\rangle_{D(A^{1/2})} \\ &= \langle (\mathrm{id}+A)^{-1}f, g\rangle_{D(A^{1/2})} - \langle (\mathrm{id}+A)^{-1}f, (\mathrm{id}+A)^{-1}g\rangle_{D(A^{1/2})} \\ &= \langle f, \tilde{A}g\rangle_{D(A^{1/2})^*}.\end{aligned}$$

$(\tilde{A}, D(A^{1/2}))$ is therefore symmetric and clearly non-negative on $D(A^{1/2})$. Proposition 1.1.9 in [Roy07] implies that $(\tilde{A}, D(A^{1/2}))$ is selfadjoint since $-1 \in \rho(\tilde{A})$.
(c) $(\tilde{A}, D(A^{1/2}))$ also generates a Schrödinger group $\tilde{U} : \mathbb{R} \to \mathcal{L}(D(A^{1/2})^*)$ by the spectral calculus. For $f \in D(A)$ we define $u : \mathbb{R} \to D(A)$ by $u(t) := U(t)f$. Proposition 9.10 in [Tay11] yields that $u \in C(\mathbb{R}, D(A)) \cap C^1(\mathbb{R}, H)$ is the unique solution of

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t), \quad t \in \mathbb{R}, \\ u(0) &= f. \end{aligned}$$

We have $u(t) \in D(A)$ for $t \in \mathbb{R}$ and therefore $Au(t) = \tilde{A}u(t)$. Consequently, u is also the unique solution of

$$\begin{aligned} \mathrm{i}u'(t) &= \tilde{A}u(t), \quad t \in \mathbb{R}, \\ u(0) &= f. \end{aligned}$$

This implies $\tilde{U}(t) = U(t)$ for $t \in \mathbb{R}$ on $D(A)$. By means of the density of $D(A)$ in H we additionally have $\tilde{U}(t) = U(t)$ for all $t \in \mathbb{R}$ on H. □

Definition I.1.5
We call the Hilbert space

$$(E_A, \langle\cdot,\cdot\rangle_{E_A}) := (D(A^{1/2}), \langle\cdot,\cdot\rangle_{D(A^{1/2})})$$

the energy space and $\|\cdot\|_{E_A}$ *the energy norm associated to* $(A, D(A))$*. We furthermore call the non-negative, selfadjoint linear operator* $(\tilde{A}, E_A)$ *from Theorem I.1.4 the extrapolation operator of* $(A, D(A))$ *and* $\tilde{U}$ *its extrapolation group.*

Recall that $E_A \hookrightarrow E_A^*$ with the map $f \mapsto \langle f, \cdot\rangle_H$ as mentioned in the remark after Corollary I.1.3. The notion of the energy space will become clear in Section I.2 due to its relation to the energy functional.

Examples I.1.6
We end this section with the two main examples of differential operators considered in this thesis and their energy spaces. The example in (2) will be explained more detailed in Section III.1.

(1) **Elliptic operators in euclidean space:** Let $\Omega \subseteq \mathbb{R}^d$ be an open set, which equipped with the trace Borel σ-algebra $\mathscr{B}(\Omega)$ and the Lebesgue measure λ becomes a measure space. Let $B \in C^1(\Omega, \mathbb{R}^{d\times d})$ be uniformly elliptic, i.e. $\|\xi\|^2 \cong \xi^T B\xi$ for all $\xi \in \mathbb{R}^d$, and $V \in L^\infty(\Omega)$ with $V \geq 0$. We define the differential expression

$$A : C_c^\infty(\Omega) \to L^2(\Omega), \qquad Af := -\operatorname{div}(B\nabla f) + Vf.$$

By means of integration by parts we have for $f, g \in C_c^\infty(\Omega)$

$$\langle Af, g\rangle_{L^2(\Omega)} = \langle B\nabla f, \nabla g\rangle_{L^2(\Omega)} + \langle Vf, g\rangle_{L^2(\Omega)} =: a(f,g).$$

The sesquilinear form $a : H^1_{2,0}(\Omega) \times H^1_{2,0}(\Omega) \to \mathbb{C}$ is accretive, symmetric, continuous, closed and densely defined. Proposition 1.24 in Section 1.2.3 of [Ouh05] yields the existence of a non-negative, selfadjoint linear operator $(A_D, D(A_D))$ on $L^2(\Omega)$ such that $D(A_D) \subseteq H^1_{2,0}(\Omega)$ and $\langle A_D f, g\rangle_{L^2(\Omega)} = a(f,g)$ for all $f \in D(A_D)$ and $g \in H^1_{2,0}(\Omega)$. $(A_D, D(A_D))$ is called the realization of the differential expression A in $L^2(\Omega)$ with homogeneous Dirichlet boundary conditions. For $f \in D(A_D)$ integration by parts yields

$$\begin{aligned} \|A_D^{1/2} f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} &= \langle A_D f, f\rangle_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)} \\ &\cong \|\nabla f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}. \end{aligned} \tag{I.5}$$

Hence, $\|\cdot\|_{E_{A_D}} \cong \|\cdot\|_{H^1_2(\Omega)}$ on $D(A_D)$, which implies $H^1_{2,0}(\Omega) \hookrightarrow D(A_D^{1/2})$. The converse embedding follows from the density of $D(A_D)$ in E_{A_D} so that $E_{A_D} \cong H^1_{2,0}(\Omega)$. For further details consult Chapter 6 in [Tri92a] or Chapter 4 in [Ouh05]. For the characterization of the domain of the square root of a differential operator with complex coefficients see for example [AHL⁺02] or [AHMT01] for $\Omega = \mathbb{R}^d$ and [AT03] for certain Lipschitz domains and Dirichlet- or Neumann boundary conditions. The assertion, that in these situations the fractional domain of the square root of the operator is equivalent to the corresponding first order Sobolev space as above is part of the famous Square-Root Problem formulated by Tosio Kato.

(2) **The Laplace-Beltrami operator on manifolds with bounded geometry:** Let Ω be a connected, complete Riemannian C^∞-manifold with bounded geometry and without boundary and let $(-\Delta_\Omega, D(-\Delta_\Omega))$ denote the Laplace-Beltrami operator on Ω (for definitions and details see Section III.1). In Section III.1 we present a characterization of $D((-\Delta_\Omega)^{1/2}) \cong X$ with a Hilbert space X such that $X \cong W^1_2(\Omega)$. The defintion of X will not rely on covariant derivatives. However, using the latter, it would also be possible to prove $D((-\Delta_\Omega)^{1/2}) \cong W^1_2(\Omega)$ directly by establishing an analogon of the integration by parts formula for the Laplace-Beltrami operator. Then we can proceed as in the first example to get the norm equivalence from (I.5) and use the density of $C_c^\infty(\Omega)$ in $W^1_2(\Omega)$ shown in Theorem 3.1 in [Heb99]. For more details on this and on Sobolev spaces on Riemannian manifolds in general see the textbooks [Aub98, Heb99].

I.2. The nonlinear Cauchy problem and conservation laws

The abstract theory for the nonlinear Schrödinger equation in this section is derived from the standard theory of inhomogeneous Cauchy problems. This theory can be found in Section 4.1 of [CH98] or Section 4.2 of [Paz83]. For the readers convenience we will prove some assertions which are not explicitly proven in the given references, but which follow with the same methods used to prove the analogous result in the

inhomogeneous case.
Let $I \in \mathcal{I}$ with $t_0 \in I$ and $U := (e^{-\mathrm{i}tA})_{t\in\mathbb{R}}$. For the moment let $f \in D(A)$, $F \in C(I \times D(A), H)$ and consider the nonlinear Cauchy problem

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t) + F(t,u(t)), \quad t \neq t_0, \\ u(t_0) &= f. \end{aligned} \tag{I.6}$$

If $u \in C(I, D(A)) \cap C^1(I, H)$ solves (I.6), then the method of proof of Corollary 4.1.2 in [CH98] (with $f(t) := F(t, u(t))$) gives rise to the representation formula

$$\forall_{t\in I}: \; u(t) = U(t-t_0)f - \mathrm{i}\int_{t_0}^{t} U(t-s)F(s,u(s))\,\mathrm{d}s. \tag{I.7}$$

Observe that $u \in C(I, D(A))$ and (I.7) imply

$$\int_{t_0}^{t} U(t-s)F(s,u(s))\,\mathrm{d}s \in C(I, D(A)). \tag{I.8}$$

Conversely, a solution $u \in C(I, D(A))$ of (I.7) belongs to $C^1(I, H)$ and solves the Cauchy problem (I.6). This follows with the semigroup methods applied in the proof of Proposition I.2.4. Note also that if we drop the condition $u \in C(I, D(A))$, then this direction might fail even in the inhomogeneous situation. To see this let $F(\cdot, h) := U(\cdot)g$ for all $h \in D(A)$ with some $g \in H \setminus D(A)$ and $f := 0$. Then $F \in C(I \times D(A), H)$ but the integral expression in (I.8) is not in $D(A)$ for fixed $t \in I \setminus \{t_0\}$.
The formula in (I.7) allows us to establish a meaningful notion of a solution for the above Cauchy problem (I.6) for initial data in arbitrary fractional domain spaces $D(A^s)$ instead of $D(A)$. However, the most desirable space to consider initial values and solutions in is the energy space E_A due to its favorable relation to global existence results (see the discussion at the end of this section). In this thesis we aim at a global existence theory for initial data in E_A and subspaces of it. More precisely, we aim at a treatment of (I.6) with initial data in $D(A^s)$ with $s \in [1/2, \infty)$. We therefore "extrapolate" the nonlinear equation (I.6) from $D(A)$ to E_A using the extrapolation Theorem I.1.4.

Definition I.2.1
Let $I \in \mathcal{I}$, $t_0 \in I$, $s \in [1/2, \infty)$, $f \in D(A^s)$ and $F : I \times D(A^s) \to E_A^*$. We call the nonlinear Cauchy problem

$$\begin{aligned} \mathrm{i}u'(t) &= \tilde{A}u(t) + F(t,u(t)), \quad t \neq t_0, \\ u(t_0) &= f, \end{aligned} \tag{NLS}$$

the extrapolated nonlinear Schrödinger equation.

In our applications we consider only the cases $F(t, u(t)) = F(t)$ and $F(t, u(t)) = F(u(t))$. In the rest of the section we recall several concepts of solutions of (NLS) and provide some useful properties for them. We begin with a precise definition of strong and weak solutions.

Definition I.2.2
Let $s \in [1/2, \infty)$ and $u : I \to D(A^s)$.

(a) u is called a strong solution of (NLS) on I, if $u \in C(I, D(A^s)) \cap C^1(I, E_A^*)$ solves the Cauchy problem on E_A^* for all $t \in I$.

(b) *u is called a weak solution of (NLS) on I, if $u \in L^\infty(I, D(A^s)) \cap W^1_\infty(I^o, E_A^*)$ solves the Cauchy problem on E_A^* for almost all $t \in I$.*

A function $u : \mathbb{R} \to D(A^s)$ is called a global strong or weak solution, if for all $I \in \mathcal{I}_b$ the restriction $u|_I$ is a strong or weak solution, respectively.

Remarks:

(1) Let $u : I \to E_A$ and recall $E_A \hookrightarrow E_A^*$. Then $u \in C^1(I, E_A^*)$ means that $u : I \to E_A^*$ is continuously differentiable in the sense that $t \mapsto \langle u(t), \cdot \rangle_H$ is continuously differentiable. $u \in W^1_\infty(I^o, E_A^*)$ means that $u, u' \in L^\infty(I^o, E_A^*)$. We point out that the latter assertions mean that there is a function $f_{u'} \in L^\infty(I^o, E_A^*)$ such that

$$\forall_{\phi \in C_c^\infty(I^o)} : \int_I \phi'(t) u(t) \,\mathrm{d}t = - \int_I \phi(t) f_{u'}(t) \,\mathrm{d}t.$$

We're not going to distinguish $u : I \to E_A$ and $u : I \to E_A^*$ in our notation, since it will be clear from the context how u should be interpreted. If $u \in C^1(I, H)$, then $\frac{\mathrm{d}}{\mathrm{d}t} \langle u(t), g \rangle_H = \langle u'(t), g \rangle_H$ for all $g \in H$.

(2) Some care is needed when dealing with the imaginary unit on E_A^*. Let $H = L^2(\Omega)$. For all $g, h \in E_A$ holds $\langle \mathrm{i}g, h \rangle_{L^2(\Omega)} = \langle g, -\mathrm{i}h \rangle_{L^2(\Omega)}$. We therefore define $\langle \mathrm{i}g^*, h \rangle := \langle g^*, -\mathrm{i}h \rangle$ for $g^* \in E_A^*$.

(3) Let $f \in D(A)$ and $F \in C(I \times D(A), H)$. Then any strong solution $u : I \to D(A)$ of (NLS) satifies $u \in C^1(I, H)$ and solves (I.6).

By a straightforward generalization of the ideas presented at the beginning of this section, in particular formula (I.7), we introduce the following weaker concept of a mild solution.

Definition I.2.3
Let $s \in [1/2, \infty)$, $f \in D(A^s)$, and $F : I \times D(A^s) \to E_A^*$. We call a function $u : I \to D(A^s)$ a mild solution of (NLS) on I if:

$$\forall_{t \in I} : \; u(t) = U(t - t_0) f - \mathrm{i} \int_{t_0}^t \tilde{U}(t - s) F(s, u(s)) \,\mathrm{d}s \;\; \text{in } E_A^*. \tag{I.9}$$

From now on we refer to (I.9) as Duhamel's formula. If it only holds almost everywhere on I, then u is called an almost everywhere (or a.e.) mild solution on I.

The properties of the extrapolation group $\tilde{U}$ of $(A, D(A))$ reduce (I.9) to equation (I.7) if $F(s, u(s)) \in H$ for almost all $s \in I$.
Next we gather the equivalence of mild and strong (or a.e. mild and weak) solutions in the inhomogeneous and autonomous nonlinear case. For further reference we state these equivalences for initial data $f \in D(A^s)$ with $s \in [1/2, \infty)$.

Proposition I.2.4
Let $s \in [1/2, \infty)$ and $f \in D(A^s)$.

(a) *Let $u \in L^\infty(I, D(A^s))$ and $F : I \to E_A^*$. If u is a weak solution of (NLS) on I, then u is an a.e. mild solution. If u is an a.e. mild solution of (NLS) on I and $F \in L^\infty(I, E_A^*)$, then u is a weak solution.*

(b) *Let* $u \in L^\infty(I, D(A^s))$ *and* $F : D(A^s) \to E_A^*$. *If* u *is a weak solution of* (NLS) *on* I, *then* u *is an a.e. mild solution. If* u *is an a.e. mild solution of* (NLS) *on* I *with* $F(u) \in L^\infty(I, E_A^*)$, *then* u *is a weak solution.*

(c) *Let* $u \in C(I, D(A^s))$ *and* $F : D(A^s) \to E_A^*$. *If* u *is a strong solution of* (NLS) *on* I, *then* u *is a mild solution. If* u *is a mild solution of* (NLS) *on* I *and* $F(u) \in C(I, E_A^*)$, *then* u *is a strong solution.*

Proof. We make use of the following instance of the product rule: If $h : I \to E_A^*$ is differentiable and $h(I) \subseteq E_A$, then the map $\tilde{U}(\cdot)h : I \to E_A^*$ is differentiable with

$$[\tilde{U}(\cdot)h]' = -\mathrm{i}\tilde{A}\tilde{U}(\cdot)h + \tilde{U}(\cdot)h'. \tag{I.10}$$

For $\tilde{F} \in L^1_{loc}(I, E_A^*)$ the function

$$G : I \to E_A^*, \qquad G(t) := \int_{t_0}^{t} \tilde{U}(-s)\tilde{F}(s)\,\mathrm{d}s \tag{I.11}$$

is well-defined and $G \in C(I, E_A^*)$ by means of the dominated convergence theorem. Proposition 1.4.29 in [CH98] additionally states that G is almost everywhere differentiable on I with $G'(t) = \tilde{U}(-t)\tilde{F}(t)$ for almost all $t \in I$.

(a+b) Let $u \in L^\infty(I, D(A^s))$ and let either $\tilde{F} := F$ or $\tilde{F} := F \circ u$. Observe that $\tilde{F} \in L^\infty(I, E_A^*) \subseteq L^1_{loc}(I, E_A^*)$ is sufficient that the function G from (I.11) has the mentioned properties.

"$\Longrightarrow$": Let u be a weak solution of (NLS) on I. First, (NLS) directly implies $\tilde{F} \in L^\infty(I, E_A^*)$. Let $t \in I$ and

$$g : I \to E_A^*, \qquad g(s) := \tilde{U}(t-s)u(s). \tag{I.12}$$

By the product rule (I.10) g is differentiable on I such that for almost all $s \in I$ holds

$$g'(s) = i\tilde{A}\tilde{U}(t-s)u(s) + \tilde{U}(t-s)\big(-\mathrm{i}\tilde{A}u(s) - \mathrm{i}\tilde{F}(s)\big) = -\mathrm{i}\tilde{U}(t-s)\tilde{F}(s). \tag{I.13}$$

Hence, $g \in W^1_{1,loc}(I, E_A^*)$. Using A.3.5;(1) and (I.13) we compute for almost all $t, s \in I$

$$u(t) = U(t-s)f + \int_s^t g'(\tau)\,\mathrm{d}\tau = U(t-s)f - \mathrm{i}\int_s^t \tilde{U}(t-\tau)\tilde{F}(\tau)\,\mathrm{d}\tau, \tag{I.14}$$

what in particular implies the equality for $s = t_0$. In case t_0 is excluded by the null set, we approximate t_0 with a sequence which belongs to the null set and use the continuity of the above expressions in s.

"$\Longleftarrow$": Let $u \in L^\infty(I, D(A^s))$ be an a.e. mild solution of (NLS) on I with $\tilde{F} \in L^\infty(I, E_A^*)$. Clearly $u(t_0) = f$ and

$$G(t) = i\tilde{U}(-t)u(t) - \mathrm{i}\tilde{U}(-t_0)f \qquad \text{a.e. on } I. \tag{I.15}$$

Since $\tilde{U}(t)(D(A^s)) = D(A^s)$ for all $t \in \mathbb{R}$ we have $G \in L^\infty(I, D(A^s))$ and $\tilde{A}G \in L^\infty(I, E_A^*)$. We already know that $G : I \to E_A^*$ is almost everywhere differentiable on I with $G'(t) = \tilde{U}(-t)F(t)$ for almost all $t \in I$. We therefore have $G' \in L^\infty(I, E_A^*)$ and $G \in W^1_\infty(I, E_A^*)$. The product rule in (I.10) implies that $\tilde{U}(\cdot)G(\cdot) : I \to E_A^*$ is almost everywhere differentiable with

$$[\tilde{U}(\cdot)G(\cdot)]' = -\mathrm{i}\tilde{U}(\cdot)\tilde{A}G(\cdot) + \tilde{F}(\cdot).$$

Hence, $U(\cdot)G(\cdot) \in W^1_\infty(I, E_A^*)$. Moreover, $U(\cdot - t_0)f \in C^1_b(I, E_A^*) \subseteq W^1_\infty(I, E_A^*)$ with

$$[i\tilde{U}(\cdot - t_0)f]' = \tilde{U}(\cdot - t_0)\tilde{A}f.$$

The previous two equations combined with (I.15) show $u \in W^1_\infty(I, E_A^*)$ and for almost all $t \in I$

$$\mathrm{i}u'(t) = \tilde{A}\big(\tilde{U}(t-t_0)f - \mathrm{i}\tilde{U}(t)G(t)\big) + \tilde{F}(t) = \tilde{A}u(t) + \tilde{F}(t).$$

Hence, u is a weak solution of (NLS) on I.
(c) Let $u \in C(I, D(A^s))$, $F : D(A^s) \to E_A^*$ and $\tilde{F} := F \circ u$.
"$\Longrightarrow$" Let u be a strong solution of (NLS) on I. Then the equation (NLS) again yields $\tilde{F} \in C(I, E_A^*)$. We define g as in (I.12) to generate the formulas (I.13) and (I.14) for all $t, s \in I$ by the product rule and the fundamental theorem.
"$\Longleftarrow$" Let $u \in C(I, D(A^s))$ be a mild solution of (NLS) on I with $\tilde{F} = F \circ u \in C(I, E_A^*)$. Then the equation (I.15) holds everywhere on I. With $\tilde{F} \in C(I, E_A^*)$ in (I.11), the continuity of the integrand implies $G \in C^1(I, E_A^*)$ with $G'(t) = \tilde{U}(-t)\tilde{F}(t)$ for all $t \in I$. The rest of the proof is similar to **(a+b)** "$\Longleftarrow$". □

The above Proposition yields the equivalence between strong and mild solutions of (NLS) under suitable assumptions on the nonlinearity. It therefore provides the possibility to construct strong solutions to the nonlinear Schrödinger equation (NLS) by solving the fixed-point equation given by the Duhamel formula on $C(I, D(A^s))$. This scheme is for example carried out in Theorem I.3.4 with initial data in $D(A^s)$ by means of Banach's fixed-point theorem. However, it is common, that this contraction argument does not work in $C(I, D(A^s))$ (for example for small s) and one rather considers $C(I, D(A^s)) \cap Y(I)$ with an auxiliary space $Y(I)$ as contraction space. With this in mind we formulate the following notion of uniqueness.

Definition I.2.5
Let $s \in [1/2, \infty)$, $I \in \mathcal{I}_b$ and $X_I \subseteq C(I, D(A^s))$ (or $L^\infty(I, D(A^s))$). A strong (or weak) solution $u \in X_I$ of (NLS) is called unconditionally unique , if it is unique in the space $C(I, D(A^s))$ (or $L^\infty(I, D(A^s))$ respectively). If it is unique in X_I, we use the expression of a (conditionally) unique solution $u \in X_I$.

In the rest of this section we discuss the two most relevant conservation laws admitted by the nonlinear Schrödinger equation (NLS) with an autonomous nonlinearity $F : E_A \to E_A^*$. We also provide basic conditions for their validity.
The first conservation law is the conservation of $\|\cdot\|_H$ along the graph of a solution u of the Cauchy problem (NLS).

Definition I.2.6
Any solution (weak, strong, mild, or a.e. mild) $u : I \to E_A$ of (NLS) on I is said to fulfill H-conservation, if $\|u(t)\|_H = \|f\|_H$ for all $t \in I$.

One frequently applied method to prove such a conservation law is to differentiate the quantity in question and show that its derivative vanishes on I. For this method it is of great interest to have as much regularity of a given solution as possible. In the next Proposition we show that the regularity of either a weak or a strong solution u is enough to have differentiability of $\|u(\cdot)\|_H^2$. We also provide some additional a-priori regularity of u.

Proposition I.2.7
Let $I \in \mathcal{I}$, $u : I \to E_A$ and $g : I \to \mathbb{R}$ defined by $g(t) := \|u(t)\|_H^2$.

(a) If $u \in C(I, E_A) \cap C^1(I, E_A^*)$, then $g \in C^1(I)$ with $g'(t) = 2\langle u'(t), u(t)\rangle$ for all $t \in I$.

(b) If $u \in L^\infty(I, E_A) \cap W^1_\infty(I^o, E_A^*)$, then $u \in C^{0,1/2}(I, H) \cap C^{0,1}(\overline{I}, E_A^*)$ and $g \in C_b(I) \cap W^1_\infty(I^o)$ with $g'(t) = 2\langle u'(t), u(t)\rangle$ for almost all $t \in I^o$.

Remarks:

(1) We do not need the specific structure of the Hilbert space E_A in the above assertions. Both of them stay valid if we substitute E_A with a Banach space X with $X \hookrightarrow H$.

(2) The proofs of *(a)* and *(b)* are essentially known. Let $E_A = H^1_{2,0}(\Omega)$ with a domain $\Omega \subseteq \mathbb{R}^d$. The assertion in *(a)* is contained in Lemma 3.3.6 of [Caz03]. If Ω is bounded and has C^2-boundary the assertion in *(b)* is contained in section 5.9 of [Eva10].

Proof. (a) Let $u \in C(I, E_A) \cap C^1(I, E_A^*)$ and $t \in I$. We put $I_0 := [t-1, t+1] \cap I$ and with $h \in \mathbb{R}$ such that $t + h \in I_0$ follows

$$\frac{\|u(t+h) - u(t)\|_H^2}{h} \leq \|u(t+h) - u(t)\|_{E_A} \frac{\|u(t+h) - u(t)\|_{E_A^*}}{h} \xrightarrow{h \to 0} 0,$$

and

$$\begin{aligned}&\left|\frac{g(t+h) - g(t)}{h} - 2\langle u'(t), u(t)\rangle\right| \\ &\qquad \leq 2\left|\left\langle \frac{u(t+h) - u(t)}{h} - u'(t), u(t)\right\rangle\right| + \frac{\|u(t+h) - u(t)\|_H^2}{h} \xrightarrow{h \to 0} 0.\end{aligned}$$

Hence, g is differentiable on I and the continuity of $g' = 2\langle u'(\cdot), u(\cdot)\rangle$ is obvious.
(b) Let $u \in L^\infty(I, E_A) \cap W^1_\infty(I, E_A^*)$. The embedding $W^1_\infty(I^o, E_A^*) \hookrightarrow C^{0,1}(\overline{I}, E_A^*)$ contained in A.3.5;(2) implies that there is a null set $N \subseteq I$ such that for all $t, s \in I \setminus N$ holds

$$\begin{aligned}\|u(t) - u(s)\|_H^2 &= |\langle u(t) - u(s), u(t) - u(s)\rangle_H| \\ &\leq 2\|u\|_{L^\infty(I, E_A)} \|u(t) - u(s)\|_{E_A^*} \leq C(u)|t - s|. \qquad \text{(I.16)}\end{aligned}$$

To construct a version of u which belongs to $C^{0,1/2}(I, H)$ we define $\tilde{u} : I \to H$ by $\tilde{u}(t) := u(t)$ for $t \in I \setminus N$ and $\tilde{u}(t) := \lim_{n\to\infty} u(t_n)$ for $t \in N$, whereby $(t_n)_{n\in\mathbb{N}} \subseteq I \setminus N$ satisfies $t_n \xrightarrow{n\to\infty} t$. With (I.16) it is straightforward to check that $\tilde{u}$ is well-defined and belongs to $C^{0,1/2}(I, H)$ with $\|\tilde{u}\|_{L^\infty(I,H)} = \|u\|_{L^\infty(I,H)}$. We can therefore choose a continuous version $g \in C_b(I)$.
The property $u \in L^\infty(I, E_A) \cap W^1_\infty(I^o, E_A^*)$ implies $\tilde{g} := 2\langle u'(\cdot), u(\cdot)\rangle \in L^\infty(I^o)$. Since the differentiability of g is a local property it is enough to show that g is almost everywhere differentiable with $g' = \tilde{g}$. It is furthermore enough to treat the case $I \in \mathcal{I}_b$. Then $u \in L^2(I, E_A) \cap W^1_2(I^o, E_A^*)$ and we define the extension

$$u_0 : \mathbb{R} \to E_A, \qquad u_0(t) := \begin{cases} u(t), & t \in I, \\ 0, & t \in I^c. \end{cases}$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ be an approximate identity and $(u_n)_{n\in\mathbb{N}}$ be defined by $u_n := \varphi_n * u_0$. Then $u_n \in C_c^1(\mathbb{R}, E_A)$ and the following properties are valid:

(i) $u_n \xrightarrow{n\to\infty} u$ in $L^2(I, E_A)$ and $u_n \xrightarrow{n\to\infty} u$ in E_A almost everywhere on I.

(ii) $u_n' \xrightarrow{n\to\infty} u'$ in $L^2(J, E_A^*)$ for all $J \in \mathcal{I}_o$ with $\overline{J} \subseteq I^o$.

The property (i) immediately follows from Lemma 1.3.3 in [ABHN11]. For (ii) we show that there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ holds $u_n'(t) = (\varphi_n * u)'(t)$ on E_A^* almost everywhere on J and can again use Lemma 1.3.3 in [ABHN11]. Indeed, if we let $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ holds $J - \operatorname{supp}(\varphi_n) \subseteq I^o$, then $\varphi_n * u_0 = \varphi_n * u$ on I. For $\phi \in C_c^\infty(J)$ and $g \in E_A$ then follows

$$\begin{aligned}\langle -\int_{I^o} \phi'(t) u_n(t)\, \mathrm{d}t, g\rangle &= \int_{I^o} \phi(t) \int_{\mathbb{R}} \varphi_n(s) \big(\langle u(t-r), g\rangle_H\big)'\, \mathrm{d}r\, \mathrm{d}t \\ &= \langle \int_{I^o} \phi(t)(\varphi_n * u')\, \mathrm{d}t, g\rangle.\end{aligned}$$

We then apply the sequence $(u_n)_{n\in\mathbb{N}}$ and its properties in the following way. From **(a)** we know $\|u_n(\cdot)\|_H^2 \in C^1(\mathbb{R})$ with $(\|u_n(\cdot)\|_H^2)' = 2\langle u_n'(\cdot), u_n(\cdot)\rangle_H$ since $u_n \in C^1(\mathbb{R}, E_A^*)$. The fundamental theorem of calculus and (i) then imply for almost all $s, t \in I$

$$\int_s^t 2\langle u_n'(r), u_n(r)\rangle_H\, \mathrm{d}r = \|u_n(t)\|_H^2 - \|u_n(s)\|_H^2 \xrightarrow{n\to\infty} \|u(t)\|_H^2 - \|u(s)\|_H^2. \tag{I.17}$$

Let $s, t \in I^o$ with $s < t$ satisfy the convergence property in (I.17) and let $J := (s, t)$. Property (i) implies the boundedness of $(u_n)_{n\in\mathbb{N}}$ in $L^2(J, E_A)$ and (ii) the boundedness of $(u_n')_{n\in\mathbb{N}}$ in $L^2(J, E_A^*)$. Furthermore

$$\begin{aligned}&\Big| \int_J \langle u_n'(r), u_n(r)\rangle_H - \langle u'(r), u(r)\rangle\, \mathrm{d}r \Big| \\ &\qquad\qquad \lesssim \|u_n - u\|_{L^2(J,E_A)} + \|u_n' - u'\|_{L^2(J,E_A^*)} \xrightarrow{n\to\infty} 0.\end{aligned}$$

The convergence in (I.17) therefore yields for almost all $s, t \in I^o$

$$\|u(t)\|_H^2 = \|u(s)\|_H^2 + \int_s^t \tilde{g}(r)\, \mathrm{d}r.$$

We recall that $\tilde{g} \in L^\infty(I)$ and use A.3.5;(1) to ensure that g is almost everywhere differentiable on I^o with $g' = \tilde{g}$. □

With the aid of Proposition I.2.7 we introduce the following simple structural condition on the nonlinearity in (NLS) to ensure that all strong and weak solutions have H-conservation.

Proposition I.2.8
Let $s \in [1/2, \infty)$ and $u : I \to D(A^s)$ be either a strong or weak solution of (NLS) on I. If $\langle F(t, u(t)), \mathrm{i}u(t)\rangle = 0$ for almost all $t \in I$, then u fulfills H-conservation.

Proof. Let $t \in I$ and $J := [\min\{t_0, t\}, \max\{t_0, t\}]$. Due to $J \in \mathcal{I}_b$, every strong solution on J is a weak solution on J and we only need to prove H-conservation in the case that $u : I \to E_A$ is a weak solution of (NLS).
From Proposition I.2.7 we know that $g : J \to \mathbb{R}$ with $g(t) := \|u(t)\|_H^2$ belongs to $C_b(J) \cap W^1_\infty(J^o)$ with $g'(t) = 2\langle u'(t), u(t)\rangle$ for almost all $t \in J^o$. Then differentiating g and using the equation (NLS) yields almost everywhere on J^o

$$\begin{aligned} g'(t) &= 2\langle u'(t), u(t)\rangle \\ &= -2\big(\langle \tilde{A}u(t), \mathrm{i}u(t)\rangle + \langle F(t, u(t)), \mathrm{i}u(t)\rangle\big) \\ &= 2\mathrm{i}\,\mathrm{Im}\langle A^{1/2}u(t), A^{1/2}u(t)\rangle_H = 0. \end{aligned}$$

Thus, there is a constant $C \in [0, \infty)$ such that $\|u(s)\|_H^2 = C$ for almost $s \in J^o$. Finally, $g \in C_b(J)$ extends this almost everywhere equality to all $t \in J$ and consequently $\|u(t)\|_H = g(t)^{1/2} = g(t_0)^{1/2} = \|u(t_0)\|_H$. □

The structural condition $\langle F(g), \mathrm{i}g\rangle = 0$ for all $g \in D(A^s)$ is proven in Proposition I.4.2 for $H := L^2(\Omega)$ and the model nonlinearity $F := F_{\beta,\pm}$ introduced in Section I.4. In our applications in Chapter III and IV we always use the model nonlinearity. The above proposition will therefore be sufficient to ensure $L^2(\Omega)$-conservation of solutions there.
The second important conservation law for the nonlinear Schrödinger equation (NLS) concerns the so-called energy functional. To define it in a meaningful way, we need additional assumptions on the nonlinearity F.

Definition I.2.9
Let $F : E_A \to E_A^*$ have a real antiderivative $\hat{F}$ on E_A, i.e. there is a differentiable $\hat{F} : E_A \to \mathbb{R}$ with $\hat{F}' = F$.

(a) The energy functional associated to (NLS) is defined as

$$\mathcal{E} : E_A \to \mathbb{R}, \qquad \mathcal{E}(f) := \frac{1}{2}\|A^{1/2}f\|_H^2 + \hat{F}(f).$$

(b) Any solution (weak, strong, mild, or a.e. mild) $u : I \to E_A$ to (NLS) on I is said to fulfill energy conservation, if $\mathcal{E}(u(t)) = \mathcal{E}(f)$ for all $t \in I$.

Remark: It is easy to show that the energy functional $\mathcal{E} : E_A \to \mathbb{R}$ is differentiable with $\mathcal{E}' : E_A \to E_A^*$ fulfilling $\mathcal{E}'(g) = \tilde{A}g + F(g)$. Consequently, the nonlinear Schrödinger equation (NLS) can be formulated as

$$\begin{aligned} \mathrm{i}u'(t) &= \mathcal{E}'(u(t)), \qquad t \neq t_0, \\ u(t_0) &= f. \end{aligned} \tag{I.18}$$

Energy conservation of a solution $u : I \to E_A$ usually requires more regularity than our notions of weak and strong solutions provide. One strategy to circumvent this problem is approximating the given solution with more regular strong solutions on which the following proposition applies.

Proposition I.2.10
Let $u \in C(I, E_A)$ be a weak solution of (NLS) on I which is almost everywhere differentiable . Then u fulfills energy conservation.

Proof. The chain rule and $u' \in E_A$ almost everywhere imply for almost all $t \in I$

$$\begin{aligned}[\mathcal{E}(u(t))]' &= \langle \mathcal{E}'(u(t)), u'(t)\rangle \\ &= \langle \mathrm{i}u'(t), u'(t)\rangle_H = -\mathrm{i}\,\mathrm{Im}\langle u'(t), u'(t)\rangle_H = 0.\end{aligned}$$

Hence, there is a constant $C \in \mathbb{R}$ such that $\mathcal{E}(u(\cdot)) = C$ almost everywhere on I. Finally, $u \in C(I, E_A)$ implies $\mathcal{E}(u(\cdot)) \in C(I)$ and consequently $\mathcal{E}(u(t)) = \mathcal{E}(u(t_0)) = \mathcal{E}(f)$ for all $t \in I$. □

Another strategy is approximating the energy functional $\mathcal{E}$ itself with a family $(\mathcal{E}_\epsilon)_{\epsilon\in(0,1)}$ of more regular ones. The family $(\mathcal{E}_\epsilon)_{\epsilon\in(0,1)}$ has to be chosen such that it can be handled in a similar fashion as in the above proposition and the convergence of $\mathcal{E}_\epsilon(u(\cdot))$ to $\mathcal{E}(u(\cdot))$ for $\epsilon \to 0$ is strong enough to carry over the differentiability properties of $\mathcal{E}_\epsilon(u(\cdot))$ to its limit. This is carried out in Lemma I.4.3 in the special case of (NLS) with the model nonlinearity $F := F_{\beta,\pm}$.

The introduced conservation laws are important quantities in the context of global existence of solutions to (NLS) in the energy space E_A, since they yield the possibility to control a solution in the $\|\cdot\|_{E_A}$-norm. Consider for example a strong solution $u : I \to E_A$ of (NLS) with $f \in E_A$ and $F : E_A \to E_A^*$. If u additionally satisfies H- and energy conservation, then for all $t \in I$ holds

$$\|u(t)\|_{E_A}^2 = 2\mathcal{E}(u(t)) + \|u(t)\|_H^2 - 2\hat{F}(u(t)) = \|f\|_H^2 + 2\mathcal{E}(f) - 2\hat{F}(u(t)).$$

If for example $\hat{F}(g) \geq 0$ for all $g \in E_A$, then

$$\|u\|_{L^\infty(I,E_A)} \lesssim \left(\mathcal{E}(f) + \|f\|_H^2\right)^{1/2}. \tag{I.19}$$

This a priori estimate clearly prevents $\|u(t)\|_{E_A} \longrightarrow \infty$ as $t \to \inf I$ and $t \to \sup I$. The relevance of this estimate lies in the fact that the local solutions we are going to construct in Theorem I.3.4 have the property that their existence time $T^\star$ depends on $\|f\|_{E_A}$ in a non-increasing manner. Then (I.19) allows us to restart (NLS) with initial value $u(t_0 + T^\star)$ and uniquely extend the solution u to the interval $[t_0 + T^\star, t_0 + 2T^\star]$, and so on. This is the reason that the energy space E_A is the most favorable one in view of global existence results. An estimate of the form (I.19) with E_A substituted by $D(A^s)$ with $s > 1/2$ is usually more difficult to prove. This will be the topic of Section II.4.

In the previous example the presence of H- and energy conservation induced (I.19) merely by assuming that $\hat{F}(g) \geq 0$ for all $g \in E_A$. For further reference we introduce such a condition on $\hat{F}$ in the following definition.

Definition I.2.11
Let $F : E_A \to E_A^*$ have a real antiderivative $\hat{F} : E_A \to \mathbb{R}$. Then F is said to be defocusing, if $\hat{F}(h) \geq 0$ for all $h \in E_A$ and focusing if $\hat{F}(h) \leq 0$ for all $h \in E_A$. The corresponding nonlinear Schrödinger equation is then also called defocusing or focusing, respectively.

I.3. Strichartz estimates with ℓ-loss and construction of solutions

In this section we provide construction schemes for mild and weak solutions of (NLS). We always assume $(A, D(A))$ to be a non-negative, selfadjoint linear operator on $(L^2(\Omega), \langle\cdot,\cdot\rangle_{L^2(\Omega)})$, whereby $(\Omega, \mathscr{S}, \mu)$ is an arbitrary measure space. Recall that we defined $\langle f, g\rangle_{L^2(\Omega)} := \mathrm{Re}(f, g)_{L^2(\Omega)}$. As in the previous section we denote by U the Schrödinger group generated by $(A, D(A))$. Moreover, $(\tilde{A}, E_A)$ denotes the extrapolation operator and $\tilde{U}$ the extrapolation group of $(A, D(A))$. If needed, we denote by $(A_q, D(A_q))$ with $q \in (1, \infty)$ the realization of $(A, D(A))$ on $L^q(\Omega)$. In this case we implicitly assume that this realization admits the definition of fractional powers $(A_q^\alpha, D(A_q^\alpha))$ as closed linear operators on $L^q(\Omega)$.
First, we turn to the construction of mild solutions of (NLS), i.e. we seek solutions $u \in C(I, D(A^s))$ of the fixed-point equation

$$u(t) = U(t - t_0)f - \mathrm{i}\int_{t_0}^{t} \tilde{U}(t - s)F(u(s))\,\mathrm{d}s =: [\Phi_f(u)](t). \tag{I.20}$$

Motivated by the structure of the right-hand side of (I.20) we introduce the following notation.

Notation I.3.1
We define the homogeneous flow $\mathcal{U}$ and the inhomogeneous flow Φ of U as follows:

$$\mathcal{U} : L^2(\Omega) \to L^\infty(\mathbb{R}, L^2(\Omega)), \qquad (\mathcal{U}f)(t) := U(t)f, \tag{I.21}$$

$$\Phi : L^1(\mathbb{R}, L^2(\Omega)) \to L^\infty(\mathbb{R}, L^2(\Omega)), \qquad (\Phi F)(t) := \int_{-\infty}^{t} U(t - \tau)F(\tau)\,\mathrm{d}\tau, \tag{I.22}$$

Our method to solve (I.20) is based on a contraction argument in $C(I, D(A^s))$. In many applications the nonlinearity F does not have the mapping property $F : D(A^s) \to D(A^s)$, in particular if s is small. We therefore need a suitable space $X(I) \subseteq C(I, D(A^s))$ for a contraction argument to work. A very successful tool when searching for such a space $X(I)$, are so-called Strichartz estimates for $\mathcal{U}$ and Φ (see Section II.1 for a systematic introduction). For $(A, D(A)) = (-\Delta, H_2^2(\mathbb{R}^d))$ such estimates are available and the existence theory for solutions of (I.20) is developed very well in this case. If we change the state space from $\mathbb{R}^d$ to an arbitrary domain or a Riemannian manifold such estimates for the corresponding Laplacian might fail. The consequence is that the existence theory in these cases is much less developed. However, there are several examples in which one can prove weaker versions compared to the Strichartz estimates for $(-\Delta, H_2^2(\mathbb{R}^d))$ and can still prove local and global existence results. We will formalize these weaker estimates in the following definition. For a list of examples in which such estimates are available we refer to Section II.3.

Definition I.3.2
Let $\ell \in [0, \infty)$ and $(p, q) \in [1, \infty]^2$. U satisfies a local (p, q) Strichartz estimate with ℓ-loss, if there is a non-decreasing $C_{\mathcal{U}} : [0, \infty) \to [0, \infty)$ such that for all $I \in \mathcal{I}_b$ holds

$$\|\mathcal{U}\|_{D(A^\ell) \to L^p(I, L^q(\Omega))} \le C_{\mathcal{U}}(|I|). \tag{I.23}$$

If $\ell = 0$, then we omit the dependency on ℓ. If $\mathcal{U} \in \mathcal{L}(L^2(\Omega), L^p(\mathbb{R}, L^q(\Omega)))$, then we say U satisfies a global (p,q) Strichartz estimate.

Remarks:

(1) Let $(p,q) \in [1,\infty] \times [2,\infty)$ and $(A, D(A)) := (-\Delta, H_2^2(\mathbb{R}^d))$. The Sobolev embedding A.2.1 implies $D((-\Delta)^{\ell^\star/2}) \hookrightarrow L^q(\mathbb{R}^d)$ for $\ell^\star := d(\frac{1}{2} - \frac{1}{q})$. For $I \in \mathcal{I}_b$ and $f \in D(A^{\ell^\star})$ therefore holds

$$\|\mathcal{U}f\|_{L^p(I,L^q(\mathbb{R}^d))} \lesssim \|\mathcal{U}f\|_{L^p(I,D((-\Delta)^{\ell^\star/2}))} \cong |I|^{1/p} \|f\|_{H_2^{\ell^\star}(\mathbb{R}^d)}. \tag{I.24}$$

Hence, (p,q) Strichartz estimates with $\ell^\star/2$-loss hold. The goal when proving estimates of the form (I.23) is therefore to generate a loss, which is below $\ell^\star/2$. Otherwise no results can be expected, which do not follow from methods involving the Sobolev embedding directly. $\ell^\star$ is known as the Sobolev loss in the literature.

(2) We stress, that we measure the loss in (I.23) in terms of the fractional domain $D(A^\ell)$. This has to be remembered when comparing our notion to the existing literature, in which the loss is usually measured in terms of Sobolev spaces. In the previous example the loss in terms of the fractional domain of the involved operator is $\ell^\star/2$ and the loss in terms of Sobolev spaces is $\ell^\star$.

(3) The phenomenon in (I.23) is often referred to as Strichartz estimates with "loss of derivatives". This notion explains itself in example (1) where the fractional domains of the involved operator are the Bessel potential spaces.

(4) We restrict our study to *local* Strichartz estimates with ℓ-loss, because in this thesis the case of global Strichartz estimates with loss will not occur. Such estimates can for example be proven for the classical Wave equation on $\mathbb{R}^d$, where the loss is measured in terms of the homogeneous Sobolev space $\dot{H}^\ell(\mathbb{R}^d)$ (see Corollary 8.27 in [BCD11]).

As a consequence of the notion of a local (p,q) Strichartz estimates with ℓ-loss we can collect the following estimates.

Corollary I.3.3
Let $I \in \mathcal{I}_b$, $\ell, \delta \in [0,\infty)$, $(p,q) \in [1,\infty]$ and U satisfy a local (p,q) Strichartz estimate with ℓ-loss. Let additionally $\theta \in [0,1]$ and

$$\tilde{\ell} = \ell\theta, \qquad \tilde{p} = \frac{p}{\theta}, \qquad \frac{1}{\tilde{q}} = \frac{1-\theta}{2} + \frac{\theta}{q}. \tag{I.25}$$

Then there is a non-decreasing $C_\mathcal{U} : [0,\infty) \to [0,\infty)$ such that for $f \in D(A^{\tilde{\ell}+\delta})$ and $F \in L^1(I, D(A^{\tilde{\ell}+\delta}))$ holds

$$\|\mathcal{U}f\|_{L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))} + \|A^\delta \mathcal{U}f\|_{L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))} \le C_\mathcal{U}(|I|)\|f\|_{D(A^{\tilde{\ell}+\delta})}, \tag{I.26}$$

$$\|\Phi F\|_{L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))} + \|A^\delta \Phi F\|_{L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))} \le C_\mathcal{U}(|I|)\|F\|_{L^1(I,D(A^{\tilde{\ell}+\delta}))}. \tag{I.27}$$

Remark: This is a consequence of the mapping properties of the linear flow $\mathcal{U}$ and complex interpolation. If $D(A^{\ell})$ is replaced by any Banach space $X \subseteq L^2(\Omega)$ an analogous result for $\delta = 0$ holds, in which the right hand side $D(A^{\tilde{\ell}})$ is substituted by $[L^2(\Omega), X]_{\theta}$.

Proof. Let $\delta = 0$. We have by assumption $\mathcal{U} \in \mathcal{L}(L^2(\Omega), L^{\infty}(\mathbb{R}, L^2(\Omega)))$ and $\mathcal{U} \in \mathcal{L}(D(A^{\ell}), L^p(I, L^q(\Omega)))$, which is our claim for $\theta \in \{0,1\}$. By means of complex interpolation (A.2) and Corollary I.1.3;(a) the operator $\mathcal{U}$ is then bounded from $D(A^{\tilde{\ell}})$ to $L^{\tilde{p}}(I, L^{\tilde{q}}(\Omega))$ with $\tilde{\ell}, \tilde{p}, \tilde{q}$ given in (I.25) and $\theta \in (0,1)$. This proves (I.26).
For $F \in L^1(I, D(A^{\tilde{\ell}}))$ holds $\Phi F \in L^{\infty}(I, L^2(\Omega))$ and Corollary I.1.3;(b) implies

$$\begin{aligned}\|\Phi F\|_{L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))} &\leq \int_I \|\mathcal{U}U(s)^* F(s)\|_{L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))}\,\mathrm{d}s \\ &\leq C_{\mathcal{U}}(|I|)\|F\|_{L^1(I,D(A^{\tilde{\ell}}))}.\end{aligned}$$

This shows (I.27). If $\delta \in (0,\infty)$, then we apply the previous estimates to $A^{\delta}f$ and $A^{\delta}F$, respectively. In this argument we also use the commutativity of A^{δ} and $U(s)^*$ on $D(A^{\delta})$ for all $s \in \mathbb{R}$ as well as the embedding $D(A^{\tilde{\ell}+\delta}) \hookrightarrow D(A^{\tilde{\ell}}) \cap D(A^{\delta})$ from Corollary I.1.3;(a). □

The estimates (I.26) and (I.27) will be useful for the following local existence result for the nonlinear Schrödinger equation (NLS). The proof relies on a contraction argument to generate mild solutions. Such an argument was carried out for the Laplacian on $\mathbb{R}^d$ in [Kat87] and for the Laplace-Beltrami operator on a compact manifold in [BGT04b]. The key assumptions are local (p,q) Strichartz estimates with ℓ-loss and mapping properties of the nonlinearity F on the fractional domains $D(A^s)$. Both are used in a purely functional analytic argument.

Theorem I.3.4
Let $\alpha, \beta_1, ..., \beta_5 \in [0,\infty)$, $\ell, s, \tilde{s} \in [0,\infty)$ with $s \leq \tilde{s}$, $(p,q) \in [1,\infty]^2$ and $(A, D(A))$ be a selfadjoint, non-negative linear operator on $L^2(\Omega)$. We furthermore assume:

(i) U satisfies a local (p,q) Strichartz estimate with ℓ-loss.

(ii) $F : D(A^s) \cap L^{\infty}(\Omega) \to D(A^s)$ satisfies $F(0) = 0$ and the following estimates:

$$\|F(g)\|_{D(A^s)} \lesssim (1 + \|g\|^{\beta_1}_{L^{\infty}(\Omega)})\|g\|_{D(A^s)}, \tag{I.28}$$

$$\begin{aligned}\|F(g) - F(h)\|_{D(A^s)} &\lesssim (1 + \|g\|^{\beta_2}_{L^{\infty}(\Omega)} + \|h\|^{\beta_2}_{L^{\infty}(\Omega)})\|g-h\|_{D(A^s)} \\ &+ (1 + \|g\|^{\beta_3}_{L^{\infty}(\Omega)} + \|h\|^{\beta_3}_{L^{\infty}(\Omega)})(1 + \|g\|^{\beta_4}_{D(A^s)} + \|h\|^{\beta_4}_{D(A^s)})\|g-h\|_{L^{\infty}(\Omega)},\end{aligned} \tag{I.29}$$

$$\|F(g) - F(h)\|_{L^2(\Omega)} \lesssim (1 + \|g\|^{\beta_5}_{L^{\infty}(\Omega)} + \|h\|^{\beta_5}_{L^{\infty}(\Omega)})\|g-h\|_{L^2(\Omega)}. \tag{I.30}$$

(iii) $p > \max\{\beta_1, \beta_2, \beta_3 + 1, \beta_4, \beta_5\}$ and either $(s,q) = (\ell,\infty)$ or $s > \ell$ and $D(A_q^{s-\ell}) \hookrightarrow L^{\infty}(\Omega)$.

Then for each $f \in D(A^s)$ there is a $T_f \in (0,\infty)$ such that with $I := [t_0 - T_f, t_0 + T_f]$ the equation

$$\forall_{t\in I}: \ u(t) = U(t-t_0)f - \mathrm{i}\int_{t_0}^{t} U(t-s)F(u(s))\, \mathrm{d}s \tag{I.31}$$

has a unique solution $u_f \in C(I, D(A^s)) \cap L^p(I, L^\infty(\Omega))$ with the following properties:

(a) $\|u_f\|_{L^\infty(I,D(A^s))\cap L^p(I,L^\infty(\Omega))} \lesssim \|f\|_{D(A^s)}$.

(b) $T_f = G(\|f\|_{D(A^s)})$ *for some decreasing, continuous function* $G : [0,\infty) \to (0,\infty)$.

(c) There are $T_\pm \in [T_f, \infty]$ and a unique solution $u \in C(I(f), D(A^s)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ of (I.31) on $I(f) := (t_0 - T_-, t_0 + T_+)$ with

$$T_\pm < \infty \implies \lim_{t\to t_0 \pm T_\pm} \|u(t)\|_{D(A^s)} = \infty. \tag{I.32}$$

(d) If additionally $f \in D(A^{\tilde{s}})$ and (I.28) holds with s substituted by $\tilde{s}$, then u from (b) belongs to $C(I(f), D(A^{\tilde{s}}))$ and satisfies (I.32) for s substituted by $\tilde{s}$.

(e) For each $I \in \mathcal{I}_c$ with $I \subseteq I(f)$ there is $\delta \in (0,\infty)$ such that the nonlinear flow

$$\mathcal{N} : \overline{B}_{D(A^s)}(f,\delta) \to C(I, D(A^s)) \cap L^p(I, L^\infty(\Omega)), \qquad g \mapsto v,$$

is well-defined and Lipschitz continuous. Here v denotes the solution from (c) of (I.31) with $v(t_0) = g$.

Remarks:

(1) We call the solution of (I.31) a mild solution of (NLS), although $s < 1/2$ is possible. Recall that we have introduced this notion of solution in Definition I.2.3 under the assumption $s \geq 1/2$. However, in the examples in Chapter III and IV we will always have that $F \in C(D(A^s), E_A^*)$ with $s \geq 1/2$. Every solution of (I.31) will therefore be a strong solution in these cases.

(2) The above theorem is meant for situations in which $D(A^s) \hookrightarrow L^\infty(\Omega)$ does not hold. If $D(A^s) \hookrightarrow L^\infty(\Omega)$ is true we only need to assume the nonlinear estimates (I.28) and (I.29) to construct a unique solution in $C(I, D(A^s))$ of (I.31).

(3) In the case $s > \ell$ and $q < \infty$ the Lipschitz continuity of the nonlinear flow $\mathcal{N}$ is valid in the topology of $C(I, D(A^s)) \cap L^p(I, D(A_q^{s-\ell}))$, which is stronger than the one given in the theorem. Also, the estimate (I.30) is only needed to show uniqueness in the space $C(I, D(A^s)) \cap L^p(I, L^\infty(\Omega))$. If we drop this assumption all the statements remain true if $L^\infty(\Omega)$ is substituted by $D(A_q^{s-\ell})$.

(4) For $f \in D(A^s)$ we call $I(f)$ the maximal existence interval and u the maximal mild solution of (NLS) with respect to f. A useful property of the maximal solution from *(c)* is the so-called blow-up alternative in (I.32). It states that either u is global or $\|u(t)\|_{D(A^s)}$ blows up if we approach $t_0 \pm T_\pm$ in time. The property that $u \in C(I(f), D(A^{\tilde{s}}))$ if $f \in D(A^{\tilde{s}})$ from *(d)* is called transport of $D(A^{\tilde{s}})$ regularity. Property *(e)* is called the local Lipschitz continuity of the nonlinear flow.

Proof. We restrict the proof to the forward in time problem as all assertions follow with the same arguments backwards in time. Let $I := [t_0, t_0 + T]$ with some $T \in (0, 1]$, $\beta := \max\{\beta_1, \beta_2, \beta_3 + 1, \beta_4, \beta_5\}$ and $Y(I) := L^\infty(I, D(A^s)) \cap L^p(I, L^\infty(\Omega))$ as well as $u, v \in Y(I)$. With estimate (I.29), Hölder's inequality, and $p > \beta$ follows

$$\begin{aligned}
&\|F(u) - F(v)\|_{L^1(I,D(A^s))} \\
&\quad \lesssim \Big(\int_I (1 + \|u(\tau)\|^{\beta_2}_{L^\infty(\Omega)} + \|v(\tau)\|^{\beta_2}_{L^\infty(\Omega)} \,\mathrm{d}\tau \Big) \|u - v\|_{L^\infty(I,D(A^s))} \\
&\qquad + \int_I (1 + \|u(\tau)\|^{\beta_3}_{L^\infty(\Omega)} + \|v(\tau)\|^{\beta_3}_{L^\infty(\Omega)}) \\
&\qquad\quad \times (1 + \|u(\tau)\|^{\beta_4}_{D(A^s)} + \|v(\tau)\|^{\beta_4}_{D(A^s)}) \|u(\tau) - v(\tau)\|_{L^\infty(\Omega)} \,\mathrm{d}\tau \\
&\quad \lesssim T^{1-\frac{\beta}{p}} \big(1 + \|u\|_{Y(I)} + \|v\|_{Y(I)}\big)^{2\beta} \|u - v\|_{Y(I)}.
\end{aligned} \tag{I.33}$$

Exploiting the estimates (I.28), (I.30), and again Hölder's inequality we obtain

$$\|F(u)\|_{L^1(I,D(A^s))} \lesssim T^{1-\frac{\beta}{p}} \big(1 + \|u\|_{L^p(I,L^\infty(\Omega))}\big)^\beta \|u\|_{L^\infty(I,D(A^s))}, \tag{I.34}$$

and

$$\begin{aligned}
&\|F(u) - F(v)\|_{L^1(I,L^2(\Omega))} \\
&\quad \lesssim T^{1-\frac{\beta}{p}} \big(1 + \|u\|_{L^p(I,L^\infty(\Omega))} + \|v\|_{L^p(I,L^\infty(\Omega))}\big)^\beta \|u - v\|_{L^\infty(I,L^2(\Omega))}.
\end{aligned} \tag{I.35}$$

We can choose the same constant in all previous three estimates. Then the map

$$\Phi_f : Y(I) \to C(I, D(A^s)), \qquad \Phi_f(u) := \tau_{t_0} \mathcal{U} f - \mathrm{i}\Phi(F(u))$$

is well-defined. Since $\mathcal{U} f \in C(I, D(A^s))$, we only need to prove the same for $\Phi(F(u))$. Indeed, $F(u) \in L^1(I, D(A^s))$ and the continuity of U in the strong operator topology imply for all $t, r \in I$ with $t \geq r$

$$\begin{aligned}
&\big\|[\Phi(F(u))](t) - [\Phi(F(u))](r)\big\|_{D(A^s)} \\
&\leq \Big\| \int_r^t U(t - \tau) F(u(\tau)) \,\mathrm{d}s \Big\|_{D(A^s)} + \Big\| \int_{t_0}^r \big(U(t - \tau) - U(r - \tau)\big) F(u(\tau)) \,\mathrm{d}\tau \Big\|_{D(A^s)} \\
&\leq \|F(u)\|_{L^1([r,t],D(A^s))} + \Big\| \big(U(t) - U(r)\big) \int_{t_0}^r U(-\tau) F(u(\tau)) \,\mathrm{d}s \Big\|_{D(A^s)} \xrightarrow{t \to r} 0.
\end{aligned}$$

For $f \in D(A^s)$ we write (I.31) as the fixed-point problem

$$\Phi_f(u) = u, \qquad u \in C(I, D(A^s)) \cap L^p(I, L^\infty(\Omega)). \tag{I.36}$$

We first show that this problem has at most one solution. Let $J \in \mathcal{I}$ with $J \subseteq I$ and $u, v \in C(I, D(A^s)) \cap L^p(I, L^\infty(\Omega))$ be two solutions of (I.36) on I. We put

$$M := \max\{\|u\|_{L^p(I,L^\infty(\Omega))}, \|v\|_{L^p(I,L^\infty(\Omega))}\}$$

and we use (I.35) to provide

$$\begin{aligned}\|u-v\|_{L^\infty(J,L^2(\Omega))} &= \|\Phi_f(u)-\Phi_f(v)\|_{L^\infty(J,L^2(\Omega))}\\ &\le \|F(u)-F(v)\|_{L^1(J,L^2(\Omega))}\\ &\le C_0|J|^{1-\frac{\beta}{p}}(1+M)^\beta\|u-v\|_{L^\infty(J,L^2(\Omega))}. \end{aligned} \tag{I.37}$$

This implies

$$\|u-v\|_{L^\infty(J,L^2(\Omega))}>0 \quad\Longrightarrow\quad 1\le C_0|J|^{1-\frac{\beta}{p}}(1+M)^\beta. \tag{I.38}$$

If $|J|\le \big(2C_0(1+M)^{2\beta}\big)^{\frac{p}{\beta-p}}$, then $C_0|J|^{1-\frac{\beta}{p}}(1+M)^\beta\le\frac12$ and therefore $u=v$ on J since (I.38) implies $\|u-v\|_{L^\infty(J,L^2(\Omega))}=0$. This allows us to conclude $u=v$ on I by covering I with finitely many overlapping intervals J with above length.
Let us turn to the construction of a solution of (I.36), which will involve a contraction argument exploiting the Strichartz estimates from Corollary I.3.3. With the a priori uniqueness of (I.36) it is enough to construct a solution in a suitable subspace of $C(I,D(A^s))\cap L^p(I,L^\infty(\Omega))$. For this we make the following minor case distinction.
If $q=\infty$ we choose

$$X(I):=\big(C(I,D(A^s))\cap L^p(I,L^\infty(\Omega)),\|\cdot\|_{Y(I)}\big)$$

as contraction space. If $q<\infty$ we still want our solution to be in $L^p(I,L^\infty(\Omega))$ and therefore we substitute $L^\infty(\Omega)$ by $D(A_q^{s-\ell})$ since we assumed $D(A_q^{s-\ell})\hookrightarrow L^\infty(\Omega)$. Then we choose

$$X(I):=\big(C(I,D(A^s))\cap L^p(I,D(A_q^{s-\ell})),\|\cdot\|_{L^\infty(I,D(A^s))\cap L^p(I,D(A_q^{s-\ell}))}\big).$$

In the following the only relevant information on $X(I)$ is the embedding $X(I)\hookrightarrow Y(I)$ as it is sufficient to exploit the nonlinear estimates (I.33)-(I.35). We therefore do not distinguish between the above cases.
Let $R\in(0,\infty)$ and $u,v\in X(I)$ with $\|u\|_{X(I)},\|v\|_{X(I)}\le R$. Using (I.37) and $X(I)\hookrightarrow Y(I)$ yields

$$\begin{aligned}\|\Phi_f(u)-\Phi_f(v)\|_{X(I)} &\le C\|F(u)-F(v)\|_{L^1(I,D(A^s))}\\ &\le CT^{1-\frac{\beta}{p}}\big(1+\|u\|_{Y(I)}+\|v\|_{Y(I)}\big)^{2\beta}\|u-v\|_{Y(I)}\\ &\le C_1T^{1-\frac{\beta}{p}}\big(1+R\big)^{2\beta}\|u-v\|_{X(I)}. \end{aligned}\tag{I.39}$$

Then

$$T=\big(2C_1(1+R)^{2\beta}\big)^{\frac{p}{\beta-p}}\implies C_1T^{1-\frac{\beta}{p}}(1+R)^{2\beta}=\frac12, \tag{I.40}$$

in which case the map Φ_f is Lipschitz continuous with Lipschitz constant $1/2$. In this situation we additionally have to find R such that $\|u\|_{X_T}\le R$ implies $\|\Phi_f(u)\|_{X_T}\le R$.

The boundedness of U, the Strichartz estimates from corollary I.3.3, and estimate (I.34) imply

$$\begin{aligned}\|\Phi_f(u)\|_{X(I)} &\le C\Big(\|f\|_{D(A^s)} + \|F(u)\|_{L^1(I,D(A^s))}\Big)\\ &\le C\Big(\|f\|_{D(A^s)} + T^{1-\frac{\beta}{p}}(1+\|u\|_{L^p(I,L^\infty(\Omega))})^\beta\|u\|_{L^\infty(I,D(A^s))}\Big)\\ &\le C\Big(\|f\|_{D(A^s)} + T^{1-\frac{\beta}{p}}(1+\|u\|^\beta_{X(I)})\|u\|_{X(I)}\Big)\\ &\le C_2\Big(\|f\|_{D(A^s)} + T^{1-\frac{\beta}{p}}(1+R)^\beta R\Big).\end{aligned}$$

Then

$$T = \big(2C_2(1+R)^{2\beta}\big)^{\frac{p}{\beta-p}} \implies C_2T^{1-\frac{\beta}{p}}(1+R)^\beta \le \frac{1}{2}, \tag{I.41}$$

in which case we have the estimate

$$\|\Phi_f(u)\|_{X(I)} \le C_2\|f\|_{D(A^s)} + \frac{R}{2}. \tag{I.42}$$

We put $C_3 := \max\{1, C_1, C_2\}$, $\tilde{R} \ge 2C_3$, as well as

$$R := \tilde{R}\|f\|_{D(A^s)} \qquad \text{and} \qquad T_f := \big(2C_3(1+R)^{2\beta}\big)^{\frac{p}{\beta-p}}. \tag{I.43}$$

Then T_f satisfies the left-hand sides in (I.40) and (I.41) and we fix $I := [t_0, t_0 + T_f]$. Since $X_R(I) := \overline{B}_{X(I)}(0, R)$ equipped with the metric $d(w,z) := \|w-z\|_{X(I)}$ is a complete metric space, we conclude that $\Phi_f(X_R(I)) \subseteq X_R(I)$ and Φ_f is a strict contraction on $X_R(I)$. Thus, Banach's fixed-point theorem guarantees the existence of a unique $u_f \in X_R(I)$, which satisfies (I.36).

Now we come to the additional properties *(a)-(e)*. Property *(a)* is clear since $u \in X_R(I) \hookrightarrow C(I, D(A^s)) \cap L^p(I, L^\infty(\Omega))$ and $R \cong \|f\|_{D(A^s)}$.

(b) The choices in (I.43) suggest that $T_f = G(\|f\|_{D(A^s)})$ with

$$G : [0,\infty) \to (0,\infty), \qquad G(x) := \big(2C_3(1+\tilde{R}x)^{2\beta}\big)^{\frac{p}{\beta-p}}, \tag{I.44}$$

which is clearly a decreasing, continuous function.

(c) Recall that we only consider the problem forward in time, hence we only prove the existence of T_+. We put

$$\mathcal{T}_+ := \big\{T \in (0,\infty) \mid \text{there is a unique solution } u \in X([t_0, t_0+T]) \text{ of (I.31)}\big\}.$$

Note that $T_f \in \mathcal{T}_+$ implies $\mathcal{T}_+ \neq \emptyset$. Let additionally $T_+ := \sup \mathcal{T}_+$ for which we check the blow-up alternative (I.32). We let $T_+ < \infty$ and assume

$$\exists_{(t_n)_{n\in\mathbb{N}}\subseteq[0,T_+),\ (t_n)\overset{n\to\infty}{\longrightarrow}T_+} :\ L := \sup_{n\in\mathbb{N}} \|u(t_0+t_n)\|_{D(A^s)} < \infty. \tag{I.45}$$

Then we choose $n \in \mathbb{N}$ such that $T_+ - t_n < G(L)$ with G from (I.44). Then $t_n \in \mathcal{T}_+$ with unique solution $u \in X(I)$ on $I := [t_0, t_0+t_n]$. Since $\|u(t_0+t_n)\|_{D(A^s)} \le L$ and G is decreasing, we have

$$T_+ - t_n < G(\|u(t_0+t_n)\|_{D(A^s)}) =: T_0. \tag{I.46}$$

With the above scheme we construct a unique solution $v \in X(J)$ of (I.31) on $J := [t_n, t_n + T_0]$ subject to the initial conditions $v(t_n) = u(t_n)$. Note that (I.46) implies that J covers $[t_n, t_n + T_+)$. With $K := I \cup J$ the function

$$w \in X(K), \qquad w(t) := \mathbb{1}_I(t)u(t) + \mathbb{1}_{J\setminus I}(t)v(t)$$

is a unique solution of (I.31) on K, what is a contradiction to the maximality of T_+. Hence the assumption (I.45) is false, which implies

$$\forall_{(t_n)_{n\in\mathbb{N}}\subseteq[t_0,t_0+T_+),\ (t_n)\overset{n\to\infty}{\longrightarrow}T_+} : \lim_{n\to\infty} \|u(t_0+t_n)\|_{D(A^s)} = \sup_{n\in\mathbb{N}} \|u(t_0+t_n)\|_{D(A^s)} = \infty.$$

The construction of the maximal solution u on $I(f) := [t_0, t_0 + T_+)$ with $T_+ \in (0, \infty]$ is straightforward. Let $(t_n)_{n\in\mathbb{N}}$ in $[0, T_+)$ with $t_n \overset{n\to\infty}{\longrightarrow} T_+$ and $t_n < t_{n+1}$. With $I_n := [t_0, t_0 + t_n]$ the corresponding unique solutions $u_n \in X(I_n)$ of (I.31) on I_n satisfy $u_{n+1} = u_n$ on I_n for all $n \in \mathbb{N}$ by uniqueness. Then the function

$$u : I(f) \to D(A^s), \qquad u(t) := \mathbb{1}_{I_1}(t)u_1(t) + \sum_{n\in\mathbb{N}} \mathbb{1}_{I_{n+1}\setminus I_n}(t)u_{n+1}(t)$$

is well-defined and belongs to $C(I(f), D(A^s)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$. By definition u is a unique solution of (I.31) on $I(f)$.

(d) Let $f \in D(A^{\tilde{s}})$ with $\tilde{s} \in (s, \infty)$ and $u \in C(I(f), D(A^s)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ be the maximal mild solution constructed in **(c)**, which exists since $D(A^{\tilde{s}}) \hookrightarrow D(A^s)$. We again put $I := [t_0 - T, t_0 + T]$ and define for $R \in (0, \infty)$

$$\tilde{X}_R(I) := \{u \in X(I) \cap C(I, D(A^{\tilde{s}})) \mid \|u\|_{X(I)\cap L^\infty(I,D(A^{\tilde{s}}))} \le R\}.$$

Since $D(A^{\tilde{s}})$ is reflexive and $D(A^{\tilde{s}}) \hookrightarrow D(A^s)$ Theorem 1.2.5 in [Caz03] provides that $(\tilde{X}_R(I), d)$ is a complete metric space. Then the validity of (I.28) for s substituted with $\tilde{s}$ allows us to repeat the above existence proof with $X_R(I)$ substituted by $\tilde{X}_R(I)$; note that we only need (I.28) for $\Phi_f(\tilde{X}_R(I)) \subseteq \tilde{X}_R(I)$, since we did not change the metric d. Thus there is a unique mild solution

$$\tilde{u} \in C(\tilde{I}(f), D(A^{\tilde{s}})) \cap L^p_{loc}(\tilde{I}(f), L^\infty(\Omega)) \subseteq C(\tilde{I}(f), D(A^s)) \cap L^p_{loc}(\tilde{I}(f), L^\infty(\Omega))$$

on $\tilde{I}(f) := [t_0, t_0 + \tilde{T}_+)$, where $\tilde{T}_+$ satisfies the blow-up alternative (I.32) with respect to $\|\cdot\|_{D(A^{\tilde{s}})}$. By means of the uniqueness of u follows $u = \tilde{u}$ on $I(f) \cap \tilde{I}(f)$ and consequently $u \in C(I(f) \cap \tilde{I}(f), D(A^{\tilde{s}}))$. We proceed by checking $\tilde{I}(f) = I(f)$.

First we assume $\tilde{T}_+ < T_+$. Then

$$\tilde{T}_+ < \infty, \tag{I.47}$$

and we define $K(T, \epsilon) := [t_0 + \tilde{T}_+ - T, t_0 + \tilde{T}_+ - \epsilon]$ with $\epsilon < T < \tilde{T}_+$ and $K := [t_0, t_0 + \tilde{T}_+]$. Duhamel's formula and (I.28) with $s = \tilde{s}$ provide the estimate

$$\begin{aligned}&\|\tilde{u}\|_{L^\infty(K(T,\epsilon),D(A^{\tilde{s}}))}\\ &\le C\Big(\|u(t_0 + T_+ - T)\|_{D(A^{\tilde{s}})} + T^{1-\frac{\beta}{p}}(1 + \|u\|_{L^p(K,L^\infty(\Omega))})^{2\beta}\|\tilde{u}\|_{C(K(T,\epsilon),D(A^{\tilde{s}}))}\Big).\end{aligned}$$

Consequently for $T < \big(2C(1 + \|u\|_{L^p(K,L^\infty(\Omega))})^{2\beta}\big)^{\frac{p}{\beta-p}}$ we have

$$\|\tilde{u}\|_{C([t_0+\tilde{T}_+-T,t_0+\tilde{T}_+),D(A^{\tilde{s}}))} \lesssim \|u(t_0 + T_+ - T)\|_{D(A^{\tilde{s}})} < \infty.$$

The last inequality implies with the blow-up alternative for $\tilde{u}$ that $\tilde{T}_+ = \infty$ and this is clearly a contradiction to (I.47). Hence, $\tilde{T}_+ \geq T_+$.
Now we assume $\tilde{T}_+ > T_+$. Then $T_+ < \infty$ and $D(A^{\tilde{s}}) \hookrightarrow D(A^s)$ immediately imply

$$\|u\|_{C([t_0,t_0+T_+),D(A^s))} \leq \|\tilde{u}\|_{C([t_0,t_0+T_+],D(A^s))} < \infty.$$

Thus, $T_+ = \infty$ by the blow-up alternative for u. This contradicts $T_+ < \tilde{T}_+$, so that $\tilde{T}_+ = T_+$.
(e) Without loss of generality we put $I_0 := [t_0, t_0 + T] \subseteq I(f)$ and first prove

$$\exists_{\delta \in (0,\infty)} \forall_{g \in \overline{B}_{D(A^s)}(f,\delta)} : \ I \subseteq I(g), \tag{I.48}$$

for the nonlinear flow $\mathcal{N}$ to make sense. Let $g \in D(A^s)$ and $v \in C(I(g), D(A^s)) \cap L^p_{loc}(I(g), L^\infty(\Omega))$ be the corresponding maximal solution of (I.31) on $I(g)$. We put $\|g - f\|_{D(A^s)} \leq \delta$ with $\delta \in (0,1)$, $r := 1 + \|u\|_{L^\infty(I_0,D(A^s))} < \infty$ and $R := 2C_3 r$. This implies

$$\|g\|_{D(A^s)} \leq \|g - f\|_{D(A^s)} + \|f\|_{D(A^s)} \leq \delta + \|u\|_{L^\infty(I_0,D(A^s))} < r.$$

With $\|f\|_{D(A^s)} < r$ property *(b)* yields $I_1 := [t_0, t_0 + G(r)] \cap I \subseteq I(g) \cap I(f)$ and *(a)* yields $\|v\|_{X(I_1)} \leq R$. For $i \in \{1,2\}$ we let $g_i \in D(A^s)$ with $\|g_i - f\|_{D(A^s)} \leq \delta$ and $v_i \in C(I(g_i), D(A^s)) \cap L^p(I(g_i), L^\infty(\Omega))$ be the corresponding maximal solution of (I.31). Then $I_1 \subseteq I(g_i)$ and $\|v_i\|_{X(I_1)} \leq R$. With the Strichartz estimates from Corollary I.3.3 and (I.39) we obtain

$$\begin{aligned} \|v_1 - v_2\|_{X(I_1)} &\leq C\|g_1 - g_2\|_{D(A^s)} + \|F(v_1) - F(v_2)\|_{L^1(I_1,D(A^s))} \\ &\leq C\|g_1 - g_2\|_{D(A^s)} + C_3 G(r)^{1-\frac{\beta}{p}} (1+R)^{2\beta} \|v_1 - v_2\|_{X(I_1)} \\ &= C\|g_1 - g_2\|_{D(A^s)} + \frac{1}{2}\|v_1 - v_2\|_{X(I_1)}, \end{aligned}$$

and therefore

$$\|v_1 - v_2\|_{X(I_1)} \lesssim \|g_1 - g_2\|_{D(A^s)}. \tag{I.49}$$

In case $t_1 < T$ let $I_2 := [t_0 + G(r), t_0 + 2G(r)] \cap I_0$. Then (I.49) yields for $i \in \{1,2\}$

$$\begin{aligned} \|v_i(t_0 + G(r))\|_{D(A^s)} &\leq \|u - v_i\|_{X(I_1)} + \|u(t_0 + G(r))\|_{D(A^s)} \\ &\leq C\delta + \|u\|_{L^\infty(I_0,D(A^s))}. \end{aligned}$$

Hence, we choose $\delta < \min\{1, 1/c\}$ so that $\|v_i(t_0 + G(r))\|_{D(A^s)} < r$ and repeat the previous argument on I_2 to generate

$$\begin{aligned} \|v_1 - v_2\|_{X(I_2)} &\lesssim \|v_1(t_0 + G(r)) - v_2(t_0 + G(r))\|_{D(A^s)} \\ &\lesssim \|v_1 - v_2\|_{X(I_1)} \lesssim \|g_1 - g_2\|_{D(A^s)}. \end{aligned} \tag{I.50}$$

Successively repeating this argument on $I_j := [t_0 + (j-1)G(r), t_0 + jG(r)]$ with $j \in \mathbb{N}$ finitely often until $jG(r) \geq T$ leads to a further reduction of δ. This implies (I.48) and analogous to (I.50) the estimate

$$\|v_1 - v_2\|_{X(I_{j+1})} \lesssim \|v_1 - v_2\|_{X(I_j)}. \tag{I.51}$$

With (I.48) for some $\delta \in (0,1)$ the map

$$\mathcal{N} : \overline{B}_{D(A^s)}(f,\delta) \to X(I), \qquad \mathcal{N}(g) := v$$

is well-defined and Lipschitz continuous since (I.51) implies

$$\|\mathcal{N}(g_1) - \mathcal{N}(g_2)\|_{X(I)} \lesssim \sup_{j\in\{1,\dots,N\}} \|v_1 - v_2\|_{X(I_j)} \lesssim \|g_1 - g_2\|_{D(A^s)}.$$

Finally, the embedding $X(I) \hookrightarrow C(I, D(A^s)) \cap L^p(I, L^\infty(\Omega))$ finishes the proof. □

The above constructed maximal solution satisfies the blow-up alternative (I.32). It is therefore of interest to have criteria for these solutions to be bounded on their maximal existence interval since this is sufficient to conclude $I(f) = \mathbb{R}$. If $s = 1/2$ such a criterion is provided in the next Lemma using $L^2(\Omega)$- and energy bounds, which are often available. The case $s > 1/2$ is considerably harder and will be dealt with in Section II.4 under much stronger assumptions.

Lemma I.3.5
Let $I \in \mathcal{I}$ with $t_0 \in I$, $\beta_1 \in [0,\infty)$, and $\beta_2 \in [0,2]$. Let additionally $f \in E_A$, $F \in C(E_A, E_A^*)$ with antiderivative $\hat{F} \in C^1(E_A, \mathbb{R})$, and $u \in C(I, E_A)$ be a strong solution of (NLS). We furthermore assume:

(i) $\|u(t)\|_{L^2(\Omega)} \lesssim \|f\|_{L^2(\Omega)}$ *and* $\mathcal{E}(u(t)) \lesssim \mathcal{E}(f)$ *for all* $t \in I$,

(ii) $\hat{F}(h) \gtrsim -\|h\|_{L^2(\Omega)}^{\beta_1} \|h\|_{E_A}^{\beta_2}$ *for* $h \in E_A$.

If either $\beta_2 < 2$ or $\|f\|_{L^2(\Omega)}$ is small enough, then $\|u\|_{L^\infty(I,E_A)} < \infty$.

Remark: A defocusing nonlinearity satisfies *(ii)* with $\beta_1 = \beta_2 = 0$ without any additional assumptions. A major limitation arises in the focusing case. Consider for example $\Omega = \mathbb{R}^d$ and $(A, D(A)) = (-\Delta, H_2^2(\mathbb{R}^d))$. Put furthermore $F(z) := -|z|^\beta z$ with $\beta \in (0,\infty)$ such that $\beta(d-2) < 4$. In Section I.4 we show that $\hat{F}(g) = -\frac{1}{\beta+2}\|g\|_{L^{\beta+2}(\mathbb{R}^d)}^{\beta+2}$ for $g \in H_2^1(\mathbb{R}^d)$. We have $H_2^s(\mathbb{R}^d) \hookrightarrow L^{\beta+2}(\mathbb{R}^d)$ with $s = d\beta/2(\beta+2) \in (0,1)$. By means of complex interpolation A.1.4;(2) and (A.1) follows

$$\|g\|_{L^{\beta+2}(\mathbb{R}^d)}^{\beta+2} \lesssim \|g\|_{H_2^s(\mathbb{R}^d)}^{\beta+2} \lesssim \|g\|_{L^2(\mathbb{R}^d)}^{(1-s)(\beta+2)} \|g\|_{H_2^1(\mathbb{R}^d)}^{s(\beta+2)}. \tag{I.52}$$

Then $s(\beta+2) = \beta d/2$ and $s(\beta+2) < 2$ if and only if $\beta < 4/d$. This gives a hint, that for $d \geq 2$ the focusing cubic nonlinear Schrödinger equation may not be treatable with the energy methods in the above lemma. In fact Theorem 6.5.10 in [Caz03] shows that there are blow-up solutions in finite time for the cubic focusing nonlinear Schrödinger equation with radial $H_2^1(\mathbb{R}^2)$-initial data with negative energy. However, initial data with small $L^2(\Omega)$-norm are still treatable.

Proof. For $t \in I$ follows with *(i)* and *(ii)* the estimate

$$\begin{aligned}
\|u(t)\|_{E_A}^2 &= \|u(t)\|_{L^2(\Omega)}^2 + \|A^{1/2}u(t)\|_{L^2(\Omega)}^2 \\
&\leq C_1\big(\|f\|_{L^2(\Omega)}^2 + \mathcal{E}(f)\big) - 2\hat{F}(u(t)) \\
&\leq C_1\big(\|f\|_{L^2(\Omega)}^2 + \mathcal{E}(f)\big) + C_2\|f\|_{L^2(\Omega)}^{\beta_1}\|u(t)\|_{E_A}^{\beta_2}.
\end{aligned} \tag{I.53}$$

If $\beta_2 < 2$ the last term of (I.53) can be estimated further by means of $ab \le a^{(2/\beta_2)^*} + b^{2/\beta_2}$ for $a, b \in [0, \infty)$. Then for $t \in I$ holds

$$\|u(t)\|_{E_A}^2 \le C(f) + \frac{1}{2}\|u(t)\|_{E_A}^2$$

and we can conclude $\|u\|_{L^\infty(I,E_A)} < \infty$.
Next assume $\|f\|_{L^2(\Omega)} < C_2^{-1/\beta_1}$. If $t \in I$ satisfies $\|u(t)\|_{E_A} \ge 1$, then $\|u(t)\|_{E_A}^{\beta_2-2} \le 1$. The estimate (I.53) additionally implies

$$\|u(t)\|_{E_A}^2 (1 - C_2\|f\|_{L^2(\Omega)}^{\beta_1}) \le \|u(t)\|_{E_A}^2 (1 - C_2\|f\|_{L^2(\Omega)}^{\beta_1}\|u(t)\|_{E_A}^{\beta_2-2}) \le C(f).$$

With $1 - C_2\|f\|_{L^2(\Omega)}^{\beta_1} > 0$ the previous estimate implies $\|u\|_{L^\infty(I,E_A)} < \infty$. □

This Lemma states more precisely what we discussed on a heuristic level after Proposition I.2.10 and again emphasizes the relevance of the energy functional $\mathcal{E}$ and energy conservation in particular.
A further situation where energy conservation is useful arises as follows. Unfortunately, there are situations where Theorem I.3.4 cannot be applied to construct mild solutions in the energy space E_A due to the magnitude of either the loss ℓ or of the exponent α from the embedding $D(A_q^\alpha) \hookrightarrow L^\infty(\Omega)$. In this situation one can try to construct a solution u of (NLS) by a weak limit argument applied to a sequence of solutions $(u_n)_{n\in\mathbb{N}}$ from a sequence of related equations, whereby either the nonlinearity F or the initial value f is approximated. In the next lemma we formulate such a scheme, in which we approximate $f \in D(A^s)$ by a sequence of initial data $(f_n)_{n\in\mathbb{N}}$ with corresponding strong solutions $(u_n)_{n\in\mathbb{N}}$. As we will see, energy conservation for the solutions $(u_n)_{n\in\mathbb{N}}$ can be transferred into an energy bound for the solution u. However, this is not necessary for the scheme itself to work but it provides additional information on the approximated solution.
Since we could not find a reference for the result below, we give a full proof. Note that we use a similar approximation argument as in Theorem 3.3.5 in [Caz03], in which the nonlinearity F is "smoothened" with the resolvent of the Laplacian.

Theorem I.3.6
Let $I \in \mathcal{I}_c$ with $t_0 \in I$, $s \in [1/2, \infty)$, $e \in (2, \infty]$, $p, q \in [2, e)$ and $(A, D(A))$ be a non-negative, selfadjoint linear operator on $L^2(\Omega)$. We furthermore assume:

(i) The embedding $E_A \hookrightarrow L^e(\Omega)$ is dense.

(ii) $F : E_A \to E_A^*$ satisfies $F(0) = 0$, $\langle F(g), ig\rangle = 0$ for all $g \in E_A$, and for all $L \in [0, \infty)$ holds

$$g, h \in \overline{B}_{E_A}(0, L) \implies \|F(g) - F(h)\|_{L^{p^*}(\Omega)} \le C(L)\|g - h\|_{L^q(\Omega)}. \tag{I.54}$$

(iii) There is an at most countable family of sets $\mathcal{S} \subseteq \mathscr{S}$ such that $\bigcup_{S\in\mathcal{S}} S = \Omega$ and any weakly convergent sequence in E_A is convergent in $L^p(S)$ for all $S \in \mathcal{S}$.

(iv) There is a sequence $(f_n)_{n\in\mathbb{N}} \subseteq D(A^s)$ such that $(f_n) \xrightarrow{n\to\infty} f$ in $D(A^s)$ and a bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $C(I, D(A^s))$ of strong solutions of (NLS) on I with $u_n(t_0) = f_n$.

Then the following holds:

(a) *There is a weak solution* $u \in C_w(I, D(A^s)) \cap C^{0,1}(I, E_A^*)$ *of* (NLS) *with initial value* $u(t_0) = f$ *and* $L^2(\Omega)$*-conservation. Moreover,* $u \in C(I, L^r(\Omega))$ *for all* $r \in [2, e)$.

(b) *In case F has a real antiderivative, the following assertions hold:*

- *If* u_n *has energy conservation for all* $n \in \mathbb{N}$*, then* $\mathcal{E}(u(t)) \leq \mathcal{E}(f)$ *for all* $t \in I$.
- *If u has energy conservation, then u is a strong solution of* (NLS) *on I. In particular,* $u \in C(I, D(A^s)) \cap C^1(I, E_A^*)$.

Remarks:

(1) The above weak solution u from *(a)* may not be unique, since it is constructed by choosing a subsequence of $(u_n)_{n\in\mathbb{N}}$, which converges weakly in $D(A^s)$ for all $t \in I$. The question of uniqueness of weak solutions will be picked up in Section II.4 in the next chapter. Note that the weak solution u from *(a)* belongs to $L^\infty(I, E_A) \cap W^1_\infty(I^o, E_A^*)$.

(2) The core of the proof is Theorem B.0.1, which is proven in Appendix B.

(3) Recall that for $\alpha \in (0, 1]$ the space $C^{0,\alpha}(I, X)$ is the space of bounded, uniformly continuous functions $v : I \to X$ such that

$$\|v\|_{C^{0,\alpha}(I,X)} := \|v\|_{L^\infty(I,X)} + \sup_{t,s\in I,\ t\neq s} \left(\frac{\|v(t) - v(s)\|_X}{|t - s|^\alpha} \right) < \infty.$$

Proof. First note that $E_A \hookrightarrow L^e(\Omega)$ yields $E_A \hookrightarrow L^r(\Omega)$ for all $r \in [2, e)$ by complex interpolation. This embedding is also dense and we therefore have $L^{r^*}(\Omega) \hookrightarrow E_A^*$. Let $\tilde{s} \in [1/2, \infty)$ and $L \in [0, \infty)$. The estimate (I.54) and $D(A^{\tilde{s}}) \hookrightarrow D(A^s)$ yield for $g, h \in \overline{B}_{D(A^{\tilde{s}})}(0, L)$ that

$$\begin{aligned} \|F(g) - F(h)\|_{E_A^*} &\leq \|F(g) - F(h)\|_{L^{p^*}(\Omega)} \\ &\leq C(L)\|g - h\|_{L^q(\Omega)} \leq C(L)\|g - h\|_{D(A^{\tilde{s}})}. \end{aligned} \tag{I.55}$$

With $F(0) = 0$ this implies that $F : D(A^{\tilde{s}}) \to E_A^*$ is Lipschitz continuous on bounded sets for all $\tilde{s} \in [1/2, \infty)$. We furthermore put

$$M := \sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(I,D(A^s))} < \infty.$$

(a) Let $f, (f_n)_{n\in\mathbb{N}}$ and $(u_n)_{n\in\mathbb{N}}$ be as in *(iv)* and let $n \in \mathbb{N}$. Then u_n has $L^2(\Omega)$-conservation on I by assumption *(ii)* and Proposition I.2.8. Moreover, $u_n \in L^\infty(I, D(A^s))$ and (I.55) imply

$$\begin{aligned} \|u_n'\|_{L^\infty(I^o,E_A^*)} &\leq \|\tilde{A}u_n\|_{L^\infty(I,E_A^*)} + \|F(u_n)\|_{L^\infty(I,E_A^*)} \\ &\leq C\|u_n\|_{L^\infty(I,D(A^s))} + C(M)\|u_n\|_{L^\infty(I,D(A^s))} \leq C(M). \end{aligned}$$

$(u_n)_{n\in\mathbb{N}}$ is therefore a bounded sequence in $L^\infty(I, D(A^s)) \cap W^1_\infty(I^o, E_A^*)$. We want to choose a sequence $(n(k))_{k\in\mathbb{N}}$ such that $(u_{n(k)}(t))_{k\in\mathbb{N}}$ converges weakly in $D(A^s)$ for all $t \in I$. We achieve this by checking the assumptions of Theorem B.0.1, where we take $X := D(A^s)$ and $Z := E_A^*$. Since $D(A^s)$ is a Hilbert space with $D(A^s) \hookrightarrow E_A^*$, the conditions in B.0.1;(i) are satisfied. The embedding $W_\infty^1(I^o, E_A^*) \hookrightarrow C_b^{0,1}(I, E_A^*)$ in A.3.5;(2) implies the boundedness of $(u_n)_{n\in\mathbb{N}} \subseteq C^{0,1}(I, E_A^*)$ and we define

$$N := \sup_{n\in\mathbb{N}} \|u_n\|_{C^{0,1}(I,E_A^*)} < \infty.$$

For all $s, t \in I$ then holds

$$\sup_{n\in\mathbb{N}} \|u_n(s) - u_n(t)\|_{E_A^*} \leq N|s-t|$$

and thus $(u_n)_{n\in\mathbb{N}}$ is uniformly equicontinuous (even uniformly Lipschitz continuous) on E_A^*. The boundedness of $(u_n)_{n\in\mathbb{N}}$ in $L^\infty(I, D(A^s))$ then completes the validation of the assumptions in B.0.1;(ii). Theorem B.0.1 guarantees the existence of $u \in C_w(I, D(A^s) \cap C^{0,1}(I, E_A^*)$ and a sequence $(n(k))_{k\in\mathbb{N}}$ such that

$$\forall_{t\in I}: \ (u_{n(k)}(t)) \rightharpoonup u(t) \ \text{ in } D(A^s). \tag{I.56}$$

Then the weak lower semicontinuity of $\|\cdot\|_{D(A^s)}$ (see e.g. (B.2)) yields

$$\|u\|_{L^\infty(I,D(A^s))} \leq \Big\| \liminf_{n\in\mathbb{N}} \|u_n(\cdot)\|_{D(A^s)} \Big\|_{L^\infty(I)} \leq M,$$

and $u \in C^{0,1}(I, E_A^*)$ implies $u \in W_\infty^1(I^o, E_A^*)$ by A.3.5;(2).
We proceed by analyzing $(F(u_{n(k)}))_{k\in\mathbb{N}}$. For $k \in \mathbb{N}$ Proposition I.2.7 yields $u_{n(k)} \in C^{0,1/2}(I, L^2(\Omega))$ which combined with (I.55) and complex interpolation (A.1) yields

$$\begin{aligned}
&\|F(u_{n(k)})\|_{C^{0,\alpha}(I,L^{p^*}(\Omega))} \\
&\quad \leq C(M)\left(\|u_{n(k)}\|_{L^\infty(I,D(A^s))} + \sup_{s,t\in I,\ s\neq t}\left(\frac{\|u_{n(k)}(s)-u_{n(k)}(t)\|_{L^q(\Omega)}}{|s-t|^\alpha}\right)\right) \\
&\quad \leq C(M)\left(1 + \sup_{s,t\in I,\ s\neq t}\left(\frac{\|u_{n(k)}(s)-u_{n(k)}(t)\|_{L^2(\Omega)}^\theta \|u_{n(k)}(s)-u_{n(k)}(t)\|_{L^e(\Omega)}^{1-\theta}}{|s-t|^\alpha}\right)\right) \\
&\quad \leq C(M)\left(1 + \|u_{n(k)}\|_{L^\infty(I,D(A^s))}^{1-\theta} \sup_{t,s\in I,\ s\neq t}\left(|t-s|^{\frac{\theta}{2}-\alpha}\right)\right) \leq C(M),
\end{aligned}$$

where $\frac{1}{q} = \frac{1-\theta}{e} + \frac{\theta}{2}$ and $\alpha := \frac{\theta}{2} = \frac{e-q}{q(e-2)} \in (0, 1/2)$. The sequence $(F(u_{n(k)}))_{k\in\mathbb{N}}$ is therefore bounded in $C^{0,\alpha}(I, L^{p^*}(\Omega))$. As above $(F(u_{n(k)}))_{k\in\mathbb{N}}$ is uniformly equicontinuous in $L^{p^*}(\Omega)$. The latter space is reflexive since $p^* \in (e^*, 2]$ and therefore we can apply Theorem B.0.1 with $X = Z = L^{p^*}(\Omega)$. This establishes the existence of $\tilde{F} \in C^{0,\alpha}(I, L^{p^*}(\Omega))$ and a subsequence of $(n(k))_{k\in\mathbb{N}}$, which we will still denote by $(n(k))_{k\in\mathbb{N}}$, such that

$$\forall_{t\in I}: \ F(u_{n(k)}(t)) \rightharpoonup \tilde{F}(t) \quad \text{in } L^{p^*}(\Omega). \tag{I.57}$$

Let $\mathcal{S}$ be the family of sets from assumption *(iii)* and let $S \in \mathcal{S}$ as well as $t \in I$. Then $F(0) = 0$ and $\langle F(g), \mathrm{i}g\rangle_{L^2(\Omega)} = 0$ for all $g \in D(A^s)$ implies

$$\begin{aligned}
&|\langle \tilde{F}(t), \mathrm{i}u(t)\rangle_{L^2(S)}| \\
&\quad = \left|\langle \tilde{F}(t), \mathrm{i}u(t)\rangle_{L^2(S)} - \langle F(u_{n(k)}(t)), \mathrm{i}u_{n(k)}(t)\rangle_{L^2(S)}\right| \\
&\quad \leq \left|\langle \mathrm{i}\mathbb{1}_S u(t), \tilde{F}(t) - F(u_{n(k)}(t))\rangle_{L^2(\Omega)}\right| + \left|\langle F(u_{n(k)}(t)), \mathrm{i}(u(t) - u_{n(k)}(t))\rangle_{L^2(S)}\right|.
\end{aligned}$$

Due to (I.57) and $\mathbb{1}_S u(t) \in L^p(\Omega)$ the first expression converges to 0 for $k \to \infty$. The same holds true for the second term, since with Hölder's inequality follows

$$\begin{aligned}\big|\langle F(u_{n(k)}(t)), \mathrm{i}(u(t) - u_{n(k)}(t))\rangle_{L^2(S)}\big| &\leq \|F(u_{n(k)}(t)\|_{L^{p^*}(S)}\|u(t) - u_{n(k)}(t)\|_{L^p(S)} \\ &\leq C(M)\|u(t) - u_{n(k)}(t)\|_{L^p(S)}.\end{aligned}$$

(I.56) and *(iii)* imply $u_{n(k)}(t) \overset{k\to\infty}{\longrightarrow} u(t)$ in $L^p(S)$. Consequently, $\langle \tilde{F}(t), \mathrm{i}u(t)\rangle_{L^2(S)} = 0$ and since $\tilde{F}(t)\overline{u(t)} \in L^1(\Omega)$ the dominated convergence theorem implies

$$\langle \tilde{F}(t), \mathrm{i}u(t)\rangle_{L^2(\Omega)} = \sum_{S\in\mathcal{S}} \langle \tilde{F}(t), \mathrm{i}u(t)\rangle_{L^2(S)} = 0. \tag{I.58}$$

We proceed by showing that u is a weak solution on I of the equation

$$\begin{aligned}\mathrm{i}u'(t) &= \tilde{A}u(t) + \tilde{F}(t), \quad t \neq t_0, \\ u(t_0) &= f.\end{aligned} \tag{I.59}$$

Let $g \in E_A$, $\eta \in C_c^\infty(I^o, \mathbb{R})$ and $t \in I$. Then $g \in L^p(\Omega)$ so that by (I.57) and (I.56) follows

$$\begin{aligned}\langle \mathrm{i}u'_{n(k)}(t), g\rangle &= \langle \tilde{A}u_{n(k)}(t) + F(u_{n(k)}(t)), g\rangle \\ &= \langle \tilde{A}g, u_{n(k)}(t)\rangle + \langle g, F(u_{n(k)}(t)\rangle_{L^2(\Omega)} \\ &\overset{k\to\infty}{\longrightarrow} \langle \tilde{A}g, u(t)\rangle + \langle g, \tilde{F}(t)\rangle_{L^2(\Omega)} = \langle \tilde{A}u(t) + \tilde{F}(t), g\rangle.\end{aligned} \tag{I.60}$$

Moreover, the weak convergence (I.56) combined with A.3.3;(4) and the dominated convergence theorem (note that the integrand and I are bounded) yields

$$\begin{aligned}\Big\langle \int_{I^o} (u'_{n(k)}(t) - u'(t))\eta(t)\,\mathrm{d}t, g\Big\rangle &= \Big\langle \int_{I^o} (u(t) - u_{n(k)}(t))\eta'(t)\,\mathrm{d}t, g\Big\rangle \\ &= \int_{I^o} \langle u(t) - u_{n(k)}(t), g\rangle_{L^2(\Omega)}\eta'(t)\,\mathrm{d}t \overset{k\to\infty}{\longrightarrow} 0.\end{aligned} \tag{I.61}$$

(I.60) and (I.61) imply $\mathrm{i}u_{n(k)}(t) \overset{k\to\infty}{\longrightarrow} \tilde{A}u(t) + \tilde{F}(t)$ and $\mathrm{i}u'_{n(k)}(t) \overset{k\to\infty}{\longrightarrow} \mathrm{i}u'(t)$ with respect to the weak* topology on E_A^* a.e. on I. Consequently, $\mathrm{i}u'(t) = \tilde{A}u(t) + \tilde{F}(t)$ on E_A^* a.e. on I and therefore u is a weak solution of (I.59). Observe that $u(t_0) = f$ follows from $f_{n(k)} \overset{k\to\infty}{\longrightarrow} f$ and $f_{n(k)} \rightharpoonup u(t_0)$ in $D(A^s)$.
It remains to show $\tilde{F}(t) = F(u(t))$ for $t \in I$. First a familiar argument involving (I.54), $E_A \hookrightarrow L^e(\Omega)$, and complex interpolation (A.1) implies

$$\begin{aligned}\|F(u_{n(k)}(t)) - F(u(t))\|_{L^{p^*}(\Omega)} &\leq C(M)\|u_{n(k)}(t) - u(t)\|_{L^q(\Omega)} \\ &\leq C(M)\|u_{n(k)}(t) - u(t)\|_{L^e(\Omega)}^{1-\theta}\|u_{n(k)}(t) - u(t)\|_{L^2(\Omega)}^{\theta} \\ &\leq C(M)\|u_{n(k)}(t) - u(t)\|_{L^2(\Omega)}^{\theta}.\end{aligned} \tag{I.62}$$

(I.58) implies $L^2(\Omega)$-conservation of u by Proposition I.2.8 and this also holds for all elements of $(u_{n(k)})_{k\in\mathbb{N}}$. Then $f_{n(k)} \overset{k\to\infty}{\longrightarrow} f$ in $L^2(\Omega)$ implies that $\|u_{n(k)}(\cdot)\|_{L^2(\Omega)} \overset{k\to\infty}{\longrightarrow} \|u(\cdot)\|_{L^2(\Omega)}$ uniformly on I. Theorem B.0.1;(c) yields $u_{n(k)} \overset{k\to\infty}{\longrightarrow} u$ in $C(I, L^2(\Omega))$ since $u_{n(k)}(t) \rightharpoonup u(t)$ in $L^2(\Omega)$ for all $t \in I$. This information and estimate (I.62) imply

$u_{n(k)} \xrightarrow{k\to\infty} u$ in $C(I, L^q(\Omega))$ and $F(u_{n(k)}) \xrightarrow{k\to\infty} F(u(t))$ in $C(I, L^{p^*}(\Omega))$. Consequently, $F(u(t)) = \tilde{F}(t)$ for all $t \in I$. Note that in the spirit of (I.62) we can easily prove $u_{n(k)} \xrightarrow{k\to\infty} u$ in $C(I, L^r(\Omega))$ for all $r \in [2, e)$ and thus $u \in C(I, L^r(\Omega))$.
(b) Now let F have a real antiderivative $\hat{F}$. The continuity of F implies $\hat{F} \in C^1(E_A, \mathbb{R})$ and consequently $\mathcal{E} \in C^1(E_A, \mathbb{R})$.
For the first statement we assume that u_n has energy conservation for all $n \in \mathbb{N}$. With $A^{1/2} \in \mathcal{L}(E_A, L^2(\Omega))$ and (I.56) we have $(A^{1/2} u_{n(k)}(t)) \rightharpoonup A^{1/2} u(t)$ in $L^2(\Omega)$ for all $t \in I$. The weak lower semicontinuity of $\|\cdot\|_{L^2(\Omega)}$ and $\mathcal{E} \in C^1(E_A, \mathbb{R})$ then imply

$$\begin{aligned}
\mathcal{E}(u(t)) &= \frac{1}{2}\|A^{1/2}u(t)\|^2_{L^2(\Omega)} + \hat{F}(u(t)) \\
&\le \liminf_{k\in\mathbb{N}} \Big(\mathcal{E}(f_{n(k)}) + \hat{F}(u(t)) - \hat{F}(u_{n(k)}(t)) \Big) \\
&= \lim_{k\to\infty} \mathcal{E}(f_{n(k)}) + \lim_{k\to\infty} \big(\hat{F}(u(t)) - \hat{F}(u_{n(k)}(t))\big) = \mathcal{E}(f).
\end{aligned}$$

The second limit is 0 since

$$\begin{aligned}
&\big|\hat{F}(u(t)) - \hat{F}(u_{n(k)}(t))\big| \\
&\quad = \Big| \int_0^1 [\frac{\mathrm{d}}{\mathrm{d}s}\hat{F}((\cdot)u(t) + (1-(\cdot))u_{n(k)}(t))](s)\,\mathrm{d}s \Big| \\
&\quad \le \int_0^1 \Big| \langle F\big(su(t) + (1-s)u_{n(k)}(t)\big), u(t) - u_{n(k)}(t)\rangle_{L^2(\Omega)} \Big| \,\mathrm{d}s \\
&\quad \le \int_0^1 \|F\big(su(t) + (1-s)u_{n(k)}(t)\big)\|_{L^{p^*}(\Omega)}\,\mathrm{d}s \|u(t) - u_{n(k)}(t)\|_{L^p(\Omega)} \\
&\quad \le C(M)\|u(t) - u_{n(k)}(t)\|_{L^p(\Omega)} \xrightarrow{k\to\infty} 0.
\end{aligned} \tag{I.63}$$

For the second statement let $s, t \in I$. The $L^2(\Omega)$- and energy conservation of u implies

$$\begin{aligned}
&\|u(s)\|^2_{E_A} - \|u(t)\|^2_{E_A} \\
&\quad = \|u(s)\|^2_{L^2(\Omega)} - \|u(t)\|^2_{L^2(\Omega)} + 2\big(\mathcal{E}(u(s)) - \mathcal{E}(u(t)) + \hat{F}(u(s)) - \hat{F}(u(t))\big) \\
&\quad = 2\big(\hat{F}(u(s)) - \hat{F}(u(t))\big).
\end{aligned}$$

Similar to (I.63) we have

$$\big|\hat{F}(u(t)) - \hat{F}(u(s))\big| \le C(M)\|u(t) - u(s)\|_{L^p(\Omega)}.$$

The fact that $u \in C(I, L^p(\Omega))$ implies the continuity of $\hat{F}(u(\cdot))$ on I and consequently the continuity of $\|u(\cdot)\|_{E_A}$. Since $u \in C_w(I, D(A^s))$ and thus $u \in C_w(I, E_A)$, the uniform convexity of E_A yields $u \in C(I, E_A)$ (see (B.4)). We then have $\tilde{A}u, F(u) \in C(I, E_A^*)$ and the function

$$v : I \to E_A^*, \qquad v(t) := -\mathrm{i}\int_{t_0}^t \tilde{A}u(\tau) + F(u(\tau))\,\mathrm{d}\tau + u(t_0),$$

belongs to $C^1(I, E_A^*)$. The equation (NLS) then implies $u \in W^1_\infty(I^o, E_A^*)$ with $u' = v'$ in E_A^* almost everywhere on I^o. Consequently, A.3.5;(1) yields $u = v$ in E_A^* almost everywhere on I^o. Finally, the continuity of u and v implies $u = v$ in E_A^* everywhere on I. We therefore have that $u \in C(I, D(A^s)) \cap C^1(I, E_A^*)$ is a strong solution of (NLS) on I. □

I.4. The model nonlinearity

In this section we introduce the model nonlinearity $F_{\beta,\pm}$, which is the most commonly used nonlinearity in this thesis. The structure of these nonlinearities naturally arises in physical applications of the nonlinear Schrödinger equation. From a mathematical point of view it is worth mentioning that these nonlinearities are only differentiable up to a certain degree, which depends on its growth. This will restrict the availability of important nonlinear estimates, which are needed for Theorem I.3.4. We will prove these estimates later in Chapter III.
In this section we provide the necessary material to fit $F_{\beta,\pm}$ in the functional analytic framework of Section I.2. We also introduce the notion of the energy subcritical nonlinear Schrödinger equation. All these concepts and results can be transferred to more general local nonlinearities $F : \mathbb{C} \to \mathbb{C}$ with minor or no modifications if suitable growth and structural assumptions are in place. However, we do not strive for maximal generality. The model nonlinearities are already a rich class which allows us to expose important underlying principles.
If not stated otherwise, in this section we always let $(A, D(A))$ be a non-negative, self-adjoint linear operator on $(L^2(\Omega), \langle\cdot,\cdot\rangle_{L^2(\Omega)})$, whereby $(\Omega, \mathscr{S}, \mu)$ is an arbitrary measure space.

Notation I.4.1
Let $\beta, \nu \in (0, \infty)$. We put

$$F_{\beta,\pm} : \mathbb{C} \to \mathbb{C}, \qquad F_{\beta,\pm}(z) := \pm\nu|z|^\beta z,$$

and call the induced Nemytskii map $g \mapsto F_{\beta,\pm} \circ g$ for $g : \mathbb{R}^d \to \mathbb{C}$ the model nonlinearity. We additionally put

$$\hat{F}_{\beta,\pm} : L^{\beta+2}(\Omega) \to \mathbb{R}, \qquad \hat{F}_{\beta,\pm}(g) := \frac{\pm\nu}{\beta+2}\|g\|^{\beta+2}_{L^{\beta+2}(\Omega)}.$$

Remarks:

(1) Note that by a change of sign we switch between the defocusing (+) and the focusing (-) case. We always assume $\nu = 1$ from now on, but every result holds for an arbitrary ν by changing the involved constants.

(2) We usually ignore the difference between a function $F : \mathbb{C} \to \mathbb{C}$ and its induced Nemytskii map $g \mapsto F \circ g$ in our notation.

(3) For $F : \mathbb{C} \to \mathbb{C}$ and $k \in \mathbb{N}_0$ the notation $F \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ means that the function

$$F_{\mathbb{R}} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad F_{\mathbb{R}}(\operatorname{Re} z, \operatorname{Im} z) := (\operatorname{Re} F(z), \operatorname{Im} F(z))$$

belongs to $C^k(\mathbb{R}^2, \mathbb{R}^2)$. This corresponds to the identification $\mathbb{C} \equiv \mathbb{R}^2$ given by $z \equiv (\operatorname{Re} z, \operatorname{Im} z)$ for $z \in \mathbb{C}$. Moreover, we define $\partial^\alpha F(z) := \partial^\alpha F_{\mathbb{R}}(\operatorname{Re} z, \operatorname{Im} z)$ for $\alpha \in \mathbb{N}_0^2$.

The next proposition contains important assertions for $F_{\beta,\pm}$ with respect to mapping properties on $L^p(\Omega$-spaces and the structural condition introduced in Section I.2 for $L^2(\Omega)$-conservation.

Proposition I.4.2
Let $\beta \in (0,\infty)$, $k \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ *and* $p \in [\beta+1,\infty]$.

(a) $F_{\beta,\pm} \in C^k(\mathbb{R}^2,\mathbb{R}^2)$ *for* $k < \beta+1$ *with* $|\partial^{\boldsymbol{\alpha}} F_{\beta,\pm}(z)| \lesssim |z|^{\beta+1-|\boldsymbol{\alpha}|}$ *for* $|\boldsymbol{\alpha}| < \beta+1$. *If* $\beta \in \mathbb{N}_{even}$, *then additionally* $F_{\beta,\pm} \in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ *with* $|\partial^{\boldsymbol{\alpha}} F(z)| \lesssim |z|^{\beta+1-|\boldsymbol{\alpha}|}$ *if* $|\boldsymbol{\alpha}| \le \beta+1$ *and* $\partial^{\boldsymbol{\alpha}} F = 0$ *for* $|\boldsymbol{\alpha}| > \beta+1$. *Moreover,*

$$|F_{\beta,\pm}(w) - F_{\beta,\pm}(z)| \lesssim (|w|^\beta + |z|^\beta)|w-z|. \tag{I.64}$$

(b) *Let* $g,h \in L^p(\Omega)$. *Then* $F_{\beta,\pm} : L^p(\Omega) \to L^{p/(\beta+1)}(\Omega)$ *satisfies*

$$\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{L^{p/(\beta+1)}(\Omega)} \lesssim \big(\|g\|_{L^p(\Omega)}^\beta + \|h\|_{L^p(\Omega)}^\beta\big)\|g-h\|_{L^p(\Omega)}. \tag{I.65}$$

(c) *Let* $E_A \hookrightarrow L^{\beta+2}(\Omega)$ *and* $g,h \in E_A$. *Then* $F_{\beta,\pm} : E_A \to E_A^*$ *satisfies*

$$\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{E_A^*} \lesssim \big(\|g\|_{E_A}^\beta + \|h\|_{E_A}^\beta\big)\|g-h\|_{E_A}, \tag{I.66}$$

and

- $F_{\beta,\pm}(g) \in L^{1+\frac{1}{\beta+1}}(\Omega)$ *and* $\langle F_{\beta,\pm}(g), \mathrm{i}g\rangle_{L^2(\Omega)} = 0$,
- $\hat{F}_{\beta,\pm} \in C^1(E_A,\mathbb{R})$ *with* $\langle \hat{F}'_{\beta,\pm}(g), h\rangle = \langle F_{\beta,\pm}(g), h\rangle_{L^2(\Omega)}$.

Remarks:

(1) The expression $F_{\beta,\pm} : E_A \to E_A^*$ means that

$$F_{\beta,\pm}^* : E_A \to E_A^*, \qquad \langle F_{\beta,\pm}^*(g), \cdot\rangle := \langle F_{\beta,\pm}(g), \cdot\rangle_{L^2(\Omega)}$$

is well-defined. Any property of $F_{\beta,\pm} : E_A \to E_A^*$ has to be understood as a property of $F_{\beta,\pm}^*$. We usually do not distinguish between $F_{\beta,\pm}$ and $F_{\beta,\pm}^*$ in our notation.

(2) In *(c)* we actually prove more than is stated, namely that all statements are valid for E_A substituted by $L^{\beta+2}(\Omega)$. The embedding $E_A \hookrightarrow L^{\beta+2}(\Omega)$ then yields the corresponding assertions on E_A.

Proof. Surely, it is enough to consider the defocusing case. We fix the function

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \qquad F(\boldsymbol{x}) := |\boldsymbol{x}|^\beta \boldsymbol{x},$$

and denote by F_1, F_2 the component functions. In the proof below always let $\boldsymbol{\alpha} \in \mathbb{N}_0^2$.
(a) First, we assume that $\beta \in \mathbb{N}_{even}$. Then $\boldsymbol{x} \mapsto |\boldsymbol{x}|^\beta$ belongs to $C^\infty(\mathbb{R}^2,\mathbb{R})$ and so $F \in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$. For $|\boldsymbol{\alpha}| \le \beta+1$ furthermore holds

$$|\partial^{\boldsymbol{\alpha}} F(\boldsymbol{x})| \lesssim \sum_{i=1}^2 |\partial^{\boldsymbol{\alpha}} F_i(\boldsymbol{x})| = \sum_{i=1}^2 \Bigg| \sum_{|\gamma|=\beta+1-|\alpha|} C_\gamma \boldsymbol{x}^\gamma \Bigg| \lesssim |\boldsymbol{x}|^{\beta+1-|\boldsymbol{\alpha}|}.$$

We now let $\beta \in (0,\infty)$ and $|\boldsymbol{\alpha}| < \beta+1$. Let $i \in \{1,2\}$. On $\mathbb{R}^2 \setminus \{\boldsymbol{0}\}$, F_i is infinitely often differentiable as a composition of C^∞-functions with

$$\partial^{\boldsymbol{\alpha}} F_i(\boldsymbol{x}) = \|\boldsymbol{x}\|^{\beta-2|\alpha|} \sum_{|\gamma|=|\boldsymbol{\alpha}|+1} C_\gamma \boldsymbol{x}^\gamma.$$

With $\partial^\alpha F_i(\mathbf{0}) := \mathbf{0}$ the previous equality implies that $\partial^\alpha F_i$ is differentiable in $\boldsymbol{x} = \mathbf{0}$ with $\partial^{e_j}\partial^\alpha F_i(\mathbf{0}) = \mathbf{0}$. Consequently, $F \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ and

$$|\partial^\alpha F(\boldsymbol{x})| \lesssim |\boldsymbol{x}|^{\beta+1-|\alpha|}.$$

This estimate and a simple application of the mean value theorem yield the Lipschitz estimate (I.64).
(b) Let $p \in [\beta+1, \infty]$ and $g, h \in L^p(\Omega)$. Then $\|F_{\beta,+}(g)\|_{L^{p/(\beta+1)}(\Omega)} = \|g\|^{\beta+1}_{L^p(\Omega)}$. The Lipschitz estimate (I.64) for $F_{\beta,+} : \mathbb{C} \to \mathbb{C}$ yields

$$\|F_{\beta,+}(g) - F_{\beta,+}(h)\|_{L^\infty(\Omega)} \lesssim (\|g\|^\beta_{L^\infty(\Omega)} + \|h\|^\beta_{L^\infty(\Omega)})\|g-h\|_{L^\infty(\Omega)}.$$

For $p < \infty$ the Lipschitz estimate (I.64) and Hölder's inequality with $\frac{\beta}{\beta+1} + \frac{1}{\beta+1} = 1$ imply

$$\begin{aligned}\|F_{\beta,+}(g) - F_{\beta,+}(h)\|_{L^{p/(\beta+1)}(\Omega)} &\lesssim \left(\int_\Omega (|g|^{\frac{p\beta}{\beta+1}} + |h|^{\frac{p\beta}{\beta+1}})|g-h|^{\frac{p}{\beta+1}}\,\mathrm{d}\mu\right)^{(\beta+1)/p} \\ &\lesssim (\|g\|^\beta_{L^p(\Omega)} + \|h\|^\beta_{L^p(\Omega)})\|g-h\|_{L^p(\Omega)}.\end{aligned}$$

(c) Let $g, h \in E_A$ and $p := \beta + 2$. Then $p^* = 1 + \frac{1}{\beta+1}$. All the assertions on mapping properties of $F_{\beta,+}$, in particular the Lipschitz estimate (I.66), follow with Hölder's inequalitity, (I.65), and $E_A \hookrightarrow L^{\beta+2}(\Omega)$. We additionally have

$$\langle F_{\beta,+}(g), \mathrm{i}g\rangle_{L^2(\Omega)} = \operatorname{Re}\int_\Omega \mathrm{i}|g(\omega)|^{\beta+2}\,\mathrm{d}\omega = 0.$$

It remains to prove the continuous differentiability of $\hat{F}_{\beta,+} : E_A \to \mathbb{R}$. For this let $\hat{F} : \mathbb{R}^2 \to \mathbb{R}$ with $\hat{F}(\boldsymbol{x}) := \frac{1}{\beta+2}\|\boldsymbol{x}\|^{\beta+2}$ and $\boldsymbol{g} := (\operatorname{Re} g, \operatorname{Im} g)$, $\boldsymbol{h} := (\operatorname{Re} h, \operatorname{Im} h)$. First, observe that

$$\begin{aligned}\langle F_{\beta,+}(g), h\rangle_{L^2(\Omega)} &= \operatorname{Re}\int_\Omega |g|^\beta g\bar{h}\,\mathrm{d}\mu \\ &= \int_\Omega F(\boldsymbol{g})\cdot\boldsymbol{h}\,\mathrm{d}\mu = \int_\Omega (\nabla\hat{F})(\boldsymbol{g})\cdot\boldsymbol{h}\,\mathrm{d}\mu.\end{aligned}$$

The fundamental theorem, Fubini's theorem, Hölder's inequality, and the Lipschitz estimate (I.65) then provide

$$\begin{aligned}&\left|\hat{F}_{\beta,+}(g+h) - \hat{F}_{\beta,+}(g) - \langle F_{\beta,+}(g), h\rangle_{L^2(\Omega)}\right| \\ &\quad= \left|\int_\Omega \hat{F}(\boldsymbol{g}+\boldsymbol{h}) - \hat{F}(\boldsymbol{g})\,\mathrm{d}\mu - \int_\Omega (\nabla\hat{F})(\boldsymbol{g})\cdot\boldsymbol{h}\,\mathrm{d}\mu\right| \\ &\quad= \left|\int_\Omega\int_0^1 \big((\nabla\hat{F})(\boldsymbol{g}+s\boldsymbol{h}) - (\nabla\hat{F})(\boldsymbol{g})\big)\,\mathrm{d}s\cdot\boldsymbol{h}\,\mathrm{d}\mu\right| \\ &\quad\le \left(\int_0^1 \|F_{\beta,+}(g+sh) - F_{\beta,+}(g)\|_{L^{(\beta+2)^*}(\Omega)}\,\mathrm{d}s\right)\|h\|_{L^{\beta+2}(\Omega)} \\ &\quad\lesssim (\|g\|^\beta_{L^{\beta+2}(\Omega)} + \|h\|^\beta_{L^{\beta+2}(\Omega)})\|h\|^2_{L^{\beta+2}(\Omega)} \lesssim (\|g\|^\beta_{E_A} + \|h\|^\beta_{E_A})\|h\|^2_{E_A}.\end{aligned}$$

This implies that $\hat{F}_{\beta,+} : E_A \to \mathbb{R}$ is differentiable with $\hat{F}'_{\beta,+}(g) = \langle F_{\beta,+}(g), \cdot \rangle_{L^2(\Omega)}$. Furthermore, (I.66) and $E_A \hookrightarrow L^p(\Omega)$ yield

$$\|\hat{F}'_{\beta,+}(g) - \hat{F}'_{\beta,+}(h)\|_{E_A^*} \lesssim \|F_{\beta,+}(g) - F_{\beta,+}(h)\|_{L^{p^*}(\Omega)} \lesssim (\|g\|^{\beta}_{E_A} + \|h\|^{\beta}_{E_A})\|g - h\|_{E_A}.$$

Hence, $\hat{F}'_{\beta,+} \in C(E_A, E_A^*)$ and $\hat{F}_{\beta,+} \in C^1(E_A, \mathbb{R})$. □

The condition $E_A \hookrightarrow L^{\beta+2}(\Omega)$ in the above Lemma plays an important role for the following reasons. Firstly, it ensures that the corresponding model nonlinearity $F_{\beta,\pm} : E_A \to E_A^*$ is Lipschitz continuous on bounded sets, hence continuous and bounded on bounded sets. This establishes equivalence of the different notions of solutions which were introduced in the previous section (see Proposition I.2.4). Secondly, we have shown that $\hat{F}_{\beta,\pm} \in C^1(E_A, \mathbb{R})$ is the antiderivative of $F_{\beta,\pm}$. The energy functional $\mathcal{E} : E_A \to \mathbb{R}$ is therefore defined and belongs to $C^1(E_A, \mathbb{R})$.
Recall from the examples in I.1.6 that in many cases the energy space E_A is the corresponding Sobolev space on Ω of order 1. Let us for a moment discuss the case $E_A \cong W^1_2(\mathbb{R}^d)$. Then $W^1_2(\mathbb{R}^d) \hookrightarrow L^{\beta+2}(\mathbb{R}^d)$ if and only if $\beta(d-2) \leq 4$. This can be seen via the Sobolev embedding A.2.1. Since the Sobolev embedding will be available in our applications, we also refer to $\beta(d-2) < 4$ as the energy subcritical condition. If $d \geq 3$ we call the case $\beta = 4/(d-2)$ energy critical. Moreover, there is an important heuristic which leads to these notions, namely scaling invariance. Let $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ be a sufficiently regular solution of the equation

$$\begin{aligned} i\partial_t u(t,x) &= -\Delta u(t,x) + F_{\beta,\pm}(u(t,x)), \quad && (t,x) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}^d, \\ u(0,x) &= f(x), && x \in \mathbb{R}^d. \end{aligned} \tag{I.67}$$

For all $\lambda \in (0,\infty)$ the function $u(\lambda,\cdot,\cdot) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$, $u(\lambda,t,x) := \lambda^{2/\beta} u(\lambda^2 t, \lambda x)$ then satisfies

$$\begin{aligned} i\partial_t u(\lambda,t,x) &= -\Delta u(\lambda,t,x) + F_{\beta,\pm}(u(\lambda,t,x)), \quad && (t,x) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}^d, \\ u(0,x) &= \lambda^{2/\beta} f(\lambda x), && x \in \mathbb{R}^d. \end{aligned}$$

We define for $\lambda \in (0,\infty)$ the space scaling $(S_\lambda u)(t,x) := \lambda^{2/\beta} u(t, \lambda x)$. It is straightforward to check for all $t \in \mathbb{R}$

$$\begin{aligned} \|(S_\lambda u)(t,\cdot)\|_{L^2(\mathbb{R}^d)} &= \lambda^{\frac{2}{\beta} - \frac{d}{2}} \|u(t,\cdot)\|_{L^2(\mathbb{R}^d)}, \\ \|\nabla (S_\lambda u)(t,\cdot)\|_{L^2(\mathbb{R}^d)} &= \lambda^{\frac{2}{\beta} - \frac{d-2}{2}} \|\nabla u(t,\cdot)\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

The invariance of $\|\cdot\|_{L^2(\mathbb{R}^d)}$ under the space scaling S_λ is therefore valid if and only if $\beta = 4/d$, which is called the mass or L^2-critical case. The invariance of $\|\nabla \cdot\|_{L^2(\mathbb{R}^d)}$ is valid if and only if $\beta = 4/(d-2)$, which is called the energy critical or $\dot{H}^1_2$-critical case. Compared to the case $\beta < 4/(d-2)$ the global existence theory in $\dot{H}^1_2(\mathbb{R}^d)$ for $\beta = 4/(d-2)$ is much harder. Here, the existence time in local existence results depends on the $\dot{H}^1_2(\mathbb{R}^d)$-norm and on the profile of the initial data. See for example Chapter 6 in

[Caz03] for a further discussion of the blow-up phenomenon for (I.67) in the focusing and defocusing case.
We already mentioned that the embedding $E_A \hookrightarrow L^{\beta+2}(\Omega)$ allows the meaningful definition of the energy functional $\mathcal{E} \in C^1(E_A, \mathbb{R})$ and we turn back to the question of energy conservation. In Propsition I.2.10 we have proven a first basic criterion for energy conservation. It relied on the differentiablility of the solution $u : I \to E_A$ of (NLS). Next, we give a more elaborate criterion, in which we assume that $u : I \to D(A)$ is continuously differentiable as map from I to $L^2(\Omega)$.

Lemma I.4.3
Let $\beta \in (0,\infty)$ and $f \in D(A)$. Let furthermore $D(A) \hookrightarrow L^{2(\beta+1)}(\Omega)$ and $u \in C(I, D(A))$ be a strong solution of the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t) + F_{\beta,\pm}(u(t)), \quad t \neq t_0, \\ u(t_0) &= f. \end{aligned}$$

If $u \in C^1(I, L^2(\Omega))$, then u has energy conservation on I.

Remark: The embedding $D(A) \hookrightarrow L^{2(\beta+1)}(\Omega)$ in particular yields $D(A) \hookrightarrow L^{\beta+2}(\Omega)$ by means of the complex interpolation results in Theorem A.1.3 and (A.1) and the trivial embedding $D(A) \hookrightarrow L^2(\Omega)$.

Proof. We argue as in Theorem 13.2 of [HMMS13] with an approximative energy functional and lift the given proof to our situation of an arbitrary measure space $(\Omega, \mathscr{S}, \mu)$. Let $J \in \mathcal{I}_c$ with $t_0 \in J \subseteq I^o$ be arbitrary. It is surely enough to prove that $\mathcal{E} \circ u$ is constant almost everywhere on J since $\mathcal{E} \circ u \in C(I, \mathbb{R})$.
For $\epsilon \in (0,1)$ we put

$$\hat{F}_\epsilon : D(A) \to \mathbb{R}, \qquad \hat{F}_\epsilon(g) := \int_\Omega \eta_\epsilon(g) \, \mathrm{d}\mu,$$

whereby $\eta_\epsilon(\cdot) := \epsilon^{-2}\eta(\epsilon|\cdot|)$ with $\eta \in C_b^1(\mathbb{R})$ defined by

$$\eta(x) := \frac{1}{\beta+2}\Big(\mathbb{1}_{[0,1]}(x)x^{\beta+2} + \mathbb{1}_{(1,\infty)}(x)\big(1 + \arctan\big(\frac{(\beta+2)(x^2-1)}{2}\big)\big)\Big).$$

An elementary calculation shows

$$\begin{aligned} |\eta_\epsilon(\boldsymbol{x})| &\lesssim \mathbb{1}_{\overline{B}(\mathbf{0},1/\epsilon)}(\boldsymbol{x})\epsilon^\beta|\boldsymbol{x}|^{\beta+2} + \mathbb{1}_{\overline{B}(\mathbf{0},1/\epsilon)^c}(\boldsymbol{x})|\boldsymbol{x}|^2, \\ |\nabla\eta_\epsilon(\boldsymbol{x})| &\lesssim \mathbb{1}_{\overline{B}(\mathbf{0},1/\epsilon)}(\boldsymbol{x})\epsilon^\beta|\boldsymbol{x}|^{\beta+1} + \mathbb{1}_{\overline{B}(\mathbf{0},1/\epsilon)^c}(\boldsymbol{x})|\boldsymbol{x}|. \end{aligned}$$

We therefore have

$$|(\nabla\eta_\epsilon)(\boldsymbol{x})| \lesssim |\boldsymbol{x}|, \qquad |\eta_\epsilon(\boldsymbol{x})| \lesssim |\boldsymbol{x}|^{\beta+2}, \qquad |(\nabla\eta_\epsilon)(\boldsymbol{x})| \lesssim |\boldsymbol{x}|^{\beta+1}. \tag{I.68}$$

Next, we define the "approximative energy functional" as

$$\mathcal{E}_\epsilon : D(A) \to \mathbb{R}, \qquad \mathcal{E}_\epsilon(g) := \frac{1}{2}\|A^{1/2}g\|^2_{L^2(\Omega)} + \hat{F}_\epsilon(g).$$

First we show that the functions

$$e_1, e_2 : J \to \mathbb{R}, \qquad e_1(t) := \frac{1}{2}\|A^{1/2}u(t)\|^2_{L^2(\Omega)}, \qquad e_2(t) := \hat{F}_\epsilon(u(t))$$

are differentiable. Let $t \in J$ and $h \in \mathbb{R} \setminus \{0\}$ with $t + h \in J$. Then $u \in C(I, D(A)) \cap C^1(I, L^2(\Omega))$ implies

$$\left|\frac{e_1(t+h) - e_1(t)}{h} - \langle Au(t), u'(t)\rangle_{L^2(\Omega)}\right|$$
$$\lesssim \|Au(t+h) - Au(t)\|_{L^2(\Omega)} + \left\|\frac{u(t+h) - u(t)}{h} - u'(t)\right\|_{L^2(\Omega)} \xrightarrow{h\to 0} 0.$$

e_1 is therefore continuously differentiable with

$$e_1' : J \to \mathbb{R}, \qquad e_1'(t) = \langle Au(t), u'(t)\rangle_{L^2(\Omega)}. \tag{I.69}$$

For e_2 we show that $\hat{F}_\epsilon : L^2(\Omega) \to \mathbb{C}$ is differentiable. Let $(h_n)_{n\in\mathbb{N}}$ be a null sequence in $L^2(\Omega)$. Then $(h_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$ which contains a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ such that $h_{n_k} \xrightarrow{k\to\infty} 0$ almost everywhere and

$$H : \Omega \to \mathbb{R}, \qquad H := |h_{n_1}| + \sum_{l=1}^{\infty} |h_{n_{l+1}} - h_{n_l}|$$

belongs to $L^2(\Omega)$. For all $k \in \mathbb{N}$ holds $|h_{n_k}| \leq H$ almost everywhere. Then (I.68) yields

$$|(\nabla\eta_\epsilon)(\boldsymbol{g} + s\boldsymbol{h}_{n_k}) - (\nabla\eta_\epsilon)(\boldsymbol{g})| \lesssim |g| + |H| \quad \text{a.e. on } \Omega.$$

We furthermore have

$$|(\nabla\eta_\epsilon)(\boldsymbol{g} + s\boldsymbol{h}_{n_k}) - (\nabla\eta_\epsilon)(\boldsymbol{g})| \xrightarrow{k\to\infty} 0 \quad \text{a.e. on } \Omega$$

and the dominated convergence theorem yields

$$\int_0^1 \|(\nabla\eta_\epsilon)(\boldsymbol{g} + s\boldsymbol{h}) - (\nabla\eta_\epsilon)(\boldsymbol{g})\|_{L^2(\Omega)}\, \mathrm{d}s \to 0 \quad \text{for } h \to 0 \text{ in } L^2(\Omega). \tag{I.70}$$

With this information we assume $g, h \in L^2(\Omega)$ with $h \neq 0$. The usual identification $\mathbb{C} \cong \mathbb{R}^2$ and the same arguments as in the proof of Proposition I.4.2;(b) yield

$$\left|\hat{F}_\epsilon(g+h) - \hat{F}_\epsilon(g) - \int_\Omega (\nabla\eta_\epsilon)(\boldsymbol{g})\boldsymbol{h}\, \mathrm{d}\mu\right|$$
$$= \left|\int_\Omega \int_0^1 \big((\nabla\eta_\epsilon)(\boldsymbol{g} + s\boldsymbol{h}) - (\nabla\eta_\epsilon)(\boldsymbol{g})\big)\, \mathrm{d}s \cdot \boldsymbol{h}\, \mathrm{d}\mu\right|$$
$$\lesssim \|h\|_{L^2(\Omega)} \int_0^1 \big\|(\nabla\eta_\epsilon)(\boldsymbol{g} + s\boldsymbol{h}) - (\nabla\eta_\epsilon)(\boldsymbol{g})\big\|_{L^2(\Omega)}\, \mathrm{d}s.$$

Then (I.70) implies the differentiability of $\hat{F}_\epsilon : L^2(\Omega) \to \mathbb{C}$ and the chain rule implies the differentiability of e_2 with

$$e_2' : J \to \mathbb{R}, \qquad e_2'(t) := \int_\Omega (\nabla\eta_\epsilon)(\boldsymbol{u}(t))\boldsymbol{u}'(t)\, \mathrm{d}\mu.$$

The latter equation combined with (I.69) implies for $t \in J$

$$(\mathcal{E}_\epsilon \circ u)'(t) = \langle Au(t), u'(t) \rangle_{L^2(\Omega)} + \int_\Omega (\nabla \eta_\epsilon)(\boldsymbol{u}(t))\boldsymbol{u}'(t) \, \mathrm{d}\mu. \tag{I.71}$$

Let $t \in J$. The growth estimates (I.68) imply

$$|\eta_\epsilon(u(t))| \lesssim |u(t)|^{\beta+2}, \qquad |(\nabla \eta_\epsilon)(\boldsymbol{u}(t))\boldsymbol{u}'(t)| \lesssim |u(t)|^{\beta+1}|u'(t)| \quad \text{a.e. on } \Omega, \tag{I.72}$$

and $u(t) \in D(A)$, $u'(t) \in L^2(\Omega)$ as well as $D(A) \hookrightarrow L^{\beta+2}(\Omega) \cap L^{2(\beta+1)}(\Omega)$ yield

$$|u(t)|^{\beta+2}, |u(t)|^{\beta+1}|u'(t)| \in L^1(\Omega). \tag{I.73}$$

Moreover, we have

$$\eta_\epsilon(u(t)) \xrightarrow{\epsilon \to 0} \frac{1}{\beta+2}|u(t)|^{\beta+2} \text{ a.e. on } \Omega,$$

$$(\nabla \eta_\epsilon)(\boldsymbol{u}(t))\boldsymbol{u}'(t) \xrightarrow{\epsilon \to 0} (\nabla \hat{F}_{\beta,\pm}(\boldsymbol{u}(t))\boldsymbol{u}'(t) = \frac{1}{\beta+2}|u(t)|^\beta \operatorname{Re}(u(t)\overline{u'(t)}) \text{ a.e. on } \Omega.$$

With (I.72) and (I.73) the dominated convergence theorem and the equation (NLS) provide

$$(\mathcal{E}_\epsilon \circ u)(t) \xrightarrow{\epsilon \to 0} \mathcal{E}(u(t)), \qquad (\mathcal{E}_\epsilon \circ u)'(t) \xrightarrow{\epsilon \to 0} 0.$$

Since $u \in C(J, D(A)) \cap C^1(J, L^2(\Omega))$ it is easy to prove that

$$\sup_{\epsilon \in (0,1)} \left(\|\mathcal{E}_\epsilon \circ u\|_{L^\infty(J)} + \|(\mathcal{E}_\epsilon \circ u)'\|_{L^\infty(J)} \right) < \infty,$$

so that once again the dominated convergence theorem yields

$$\mathcal{E}_\epsilon \circ u \xrightarrow{\epsilon \to 0} \mathcal{E} \circ u, \; (\mathcal{E}_\epsilon \circ u)' \xrightarrow{\epsilon \to 0} 0 \quad \text{in } L^1(J).$$

Consequently, $\mathcal{E} \circ u \in W_1^1(J^o)$ with $(\mathcal{E} \circ u)' = 0$ and therefore $\mathcal{E} \circ u$ is constant almost everywhere on J. □

II. Strichartz and spectrally localized estimates

In Section I.3 we have derived the central existence result for maximal mild solutions of the Cauchy problem

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t) + F(u(t)), \quad t \neq t_0, \\ u(t_0) &= f, \end{aligned} \tag{II.1}$$

where $(A, D(A))$ was a non-negative, selfadjoint linear operator on $L^2(\Omega)$ with Schrödinger group U. The crucial assumptions in this result were suitable estimates for F on $D(A^s)$ and a local (p,q) Strichartz estimate with ℓ-loss. In the first part of this chapter we will provide criteria and examples for local (p,q) Strichartz estimates with ℓ-loss. In the second part we further exploit the tools developed in the first part to provide a priori information on solutions of (II.1). We proceed as follows.
In Section II.1 we recall the notions of dispersive and Strichartz estimates. We prove a slight variation of the important result of Keel-Tao from [KT98] with complex interpolation spaces instead of real interpolation spaces in the non-endpoint situation.
In Section II.2 we turn to a method to prove local (p,q) Strichartz estimates with ℓ-loss. It was initially used in [BGT04b] in a special case and is based on the following idea: Let $(p,q) \in [2,\infty]^2$ and $(\psi_k)_{k\in\mathbb{N}_0}$ be a dyadic partition of unity, which in particular satisfies $\sum_{k=0}^{\infty} \psi_k = 1$ and $|\operatorname{supp}(\psi_k)| \cong 2^k$ for $k \in \mathbb{N}_0$ (see Definition II.2.4). If the estimate

$$\|\mathcal{U}f\|_{L^p(I,L^q(\Omega))} \lesssim \left(\sum_{k=0}^{\infty} \|\psi_k(A)\mathcal{U}f\|^2_{L^p(I,L^q(\Omega))} \right)^{1/2} \tag{II.2}$$

holds, then Strichartz estimates for $(\psi_k(A)U(t))_{t\in\mathbb{R}}$ imply Strichartz estimates for U. In order to formalize this approach, we highlight the following crucial ingredients:

(1) Spectrally localized Strichartz estimates on intervals J with $|J| \cong 2^{-k/2}$:

$$\|\psi_k(A)\mathcal{U}f\|_{L^p(J,L^q(\Omega))} \lesssim \|\psi_k(A)f\|_{L^2(\Omega)}. \tag{II.3}$$

(2) Inequalities derived from Littlewood-Paley decompositions:

$$\forall_{f\in L^2(\Omega)}: \ \|f\|_{L^q(\Omega)} \lesssim \left(\sum_{k=0}^{\infty} \|\psi_k(A)f\|^2_{L^q(\Omega)} \right)^{1/2}. \tag{II.4}$$

We show that the spectrally localized dispersive estimates

$$\|\psi_k(A)U(t)\|_{L^1(\Omega)\to L^\infty(\Omega)} \lesssim |t|^{-\sigma}, \qquad 0 < |t| \lesssim 2^{-k\gamma}, \tag{II.5}$$

are sufficient for spectrally localized Strichartz estimates of the form (II.3) to hold with $|J| \cong 2^{-k\gamma}$. Such estimates are frequently used in the literature and we call them (p,q) Strichartz estimates of SL-type (γ,ν) (SL for spectrally localized). We then discuss the availability of the decomposition estimates of the form (II.4) for $q=2$ and $q \neq 2$. We give the proof that (p,q) Strichartz estimates of SL-type (γ,ν) combined with (II.4) imply a local (p,q)-Strichartz estimate with ℓ-loss. Section II.3 contains an extensive list of examples where the arguments in Section II.2 are applied in the literature.
In Section II.4 we return to the Cauchy problem (II.1). We combine (p,q) Strichartz estimates of SL-type γ with so-called Bernstein inequalities to provide further abstract versions of the arguments used in [BGT04b]. In particular, we provide criteria for weak solutions of (II.1) with sub-cubic nonlinearity in $D(A^s)$ with $s \geq 1/2$ to be unique, and a priori estimates for strong solutions in $D(A^s)$ with $s > 1/2$. Both of these results will be applied in the global existence Theorem III.1.6 in Chapter III.

II.1. Strichartz estimates and the Keel-Tao result

In Section I.3 we proved a local existence theorem for the fixed point equation

$$u(t) = U(t-t_0)f - \mathrm{i}\int_{t_0}^{t} U(t-s)F(u(s))\,\mathrm{d}s \tag{II.6}$$

in $C(I, D(A^s))$ under the assumption of estimates for F on $D(A^s)$ and local (p,q) Strichartz estimates with ℓ-loss. However, we did not specify how to prove the latter estimates. In order to do so we review the abstract Keel-Tao result from [KT98].
First, let us introduce a rather elementary notation, which is used throughout this thesis. It arises by separating the two terms in (II.6) and abstract the unitary group $U \in \mathcal{L}(L^2(\Omega))$ to a bounded family $T \in \mathcal{L}(H, X^*)$ (compare to I.3.1).

Notation II.1.1
Let $I \in \mathcal{I}$, H a Hilbert space, X a Banach space, $T : I \to \mathcal{L}(H, X^*)$ be bounded. We put

$$\mathcal{T} : H \to L^\infty(I, X^*), \qquad (\mathcal{T}f)(t) := T(t)f, \tag{II.7}$$

$$\Phi : L^1(I, X) \to L^\infty(I, X^*), \qquad (\Phi F)(t) := \int_{-\infty}^{t} T(t)T(s)^* F_0(s)\,\mathrm{d}s, \tag{II.8}$$

whereby $F_0(s) := F(s)$ for $s \in I$ and $F_0(s) := 0$ for $s \in I^c$. $\mathcal{T}$ is called the homogeneous and Φ the inhomogeneous flow of T.

Remarks:

(1) The operator $\mathcal{T}$ is bounded and for $F \in L^1(I, X)$ holds

$$\mathcal{T}^* F = \int_I T(s)F(s)\,\mathrm{d}s.$$

Indeed, an application of Hille's Theorem stated in A.3.3;(4) implies for all $h \in H$

$$\langle F, \mathcal{T}h\rangle_{L^\infty(I,X)^*, L^\infty(I,X)} = \int_I \langle F(t), T(t)h\rangle_{X^*,X}\,\mathrm{d}t = \Big\langle \int_I T(t)^* F(t)\,\mathrm{d}t, h\Big\rangle_H.$$

Then $\mathcal{T}\mathcal{T}^* \in \mathcal{L}(L^1(I,X), L^\infty(I,X^*))$ is bounded and by means of A.3.4;(4), so is Φ. Note that we usually consider $\mathcal{T}^* : L^\infty(I,X^*)^* \to H$ to be restricted onto $L^1(I,X)$.

(2) The homogeneous flow of the Schrödinger group U is denoted by $\mathcal{U}$.

In the following we always consider T given as in II.1.1. The mapping properties of $\mathcal{T}$ and Φ are generated without any additional assumptions on T. The most prominent estimate to improve these properties is the so-called dispersive estimate. In our abstract setting it takes the following form.

Definition II.1.2
Let $\sigma \in (0,\infty)$, $I \in \mathcal{I}$ and X_1 a Banach space. We call T (σ, X_1)-dispersive on I, if for all $t,s \in I$ with $t \neq s$ and $f \in X \cap X_1$ holds

$$\|T(t)T(s)^* f\|_{X_1^*} \lesssim |t-s|^{-\sigma} \|f\|_{X_1}. \tag{II.9}$$

In the case $I = \mathbb{R}$ we omit the reference to the interval. An estimate of the form (II.9) is generally referred to as a dispersive estimate for T.

Remark: Let $\sigma \in (0,\infty)$, $I = \mathbb{R}$, $X = H = L^2(\Omega)$, $X_1 = L^1(\Omega)$ and U be a unitary group on $\mathcal{L}(L^2(\Omega))$. We then have $U(t)U(s)^* = U(t-s)$ on $L^2(\Omega)$ and for the estimate (II.9) to hold, it is sufficient that for all $f \in L^2(\Omega) \cap L^1(\Omega)$ and $t \in \mathbb{R} \setminus \{0\}$ holds

$$\|U(t)f\|_{L^\infty(\Omega)} \lesssim |t|^{-\sigma} \|f\|_{L^1(\Omega)}. \tag{II.10}$$

For $(e^{\mathrm{i}t\Delta})_{t\in\mathbb{R}}$ with the Laplacian $(-\Delta, H_2^2(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ the latter is obtained by the kernel representation

$$(e^{\mathrm{i}t\Delta} f)(x) = (4\pi \mathrm{i} t)^{-d/2} \int_{\mathbb{R}^d} e^{\frac{\mathrm{i}|x-y|^2}{4t}} f(y)\,\mathrm{d}y, \tag{II.11}$$

which holds for all $f \in S(\mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$. We refer to Section 2.2 in [Tao06] for a proof of this formula. If we either change the state space $\mathbb{R}^d$ or consider other differential operators, then a kernel representation such as (II.11) is not available in general. However, there is a vast literature concerning the question whether for Schrödinger operators $-\Delta + V$ the generated unitary group $(e^{\mathrm{i}t(\Delta - V)})_{t\in\mathbb{R}}$ still admits the dispersive estimate (II.10). For a good overview of this topic consult [Sch07]. We like to point out that if V is bounded and periodic, dispersive estimates for $(e^{\mathrm{i}t(\Delta-V)})_{t\in\mathbb{R}}$ have been established for $d = 1$ in [Fir96, Cai06, Cuc08]. For $d > 1$ this is still an open problem.

Next, we state a result along the lines of Theorem 10.1 in [KT98] (see [Tag08] for a proof), where we use the complex interpolation scale instead of the real interpolation scale in the non-endpoint situation. This enables us to review the proof of that particular case. We furthermore include the case of a local in time dispersive estimate, which leads to a corresponding local in time result.

Theorem II.1.3
Let $\sigma \in (0,\infty)$, $I \in \mathcal{I}$, (X_0, X_1) be a Banach interpolation couple as well as $T : I \to \mathcal{L}(H, X_0^*)$ and $X_\nu \in \{(X_0, X_1)_{\nu,2}, [X_0, X_1]_\nu\}$ for $\nu \in (0,1)$. Let additionally

$$N_\sigma := \begin{cases} [0,1], & \sigma < 1, \\ [0, 1/\sigma), & \sigma \geq 1, \end{cases}$$

and $p := 2/\sigma\theta$, $\tilde{p} := 2/\sigma\tilde{\theta}$ *with* $\theta, \tilde{\theta} \in N_\sigma$. *If* T *is bounded and* (σ, X_1)*-dispersive on* I, *then*

$$\mathcal{T} \in \mathcal{L}(H, L^p(I, X_\theta^*)), \tag{II.12}$$

$$\Phi \in \mathcal{L}(L^{\tilde{p}^*}(I, X_{\tilde{\theta}}), L^p(I, X_\theta^*)). \tag{II.13}$$

If $\sigma \in (1, \infty)$ *and either* $\theta = 1/\sigma$ *and/or* $\tilde{\theta} = 1/\sigma$, *the above estimates hold for* $X_\theta = (X_0, X_1)_{\theta,2}$ *and* $X_{\tilde{\theta}} = (X_0, X_1)_{\tilde{\theta},2}$, *respectively.*

Remarks:

(1) $\|\mathcal{T}\|_{H \to L^p(I, X_\theta^*)}$ and $\|\Phi\|_{L^{\tilde{p}^*}(I, X_{\tilde{\theta}}) \to L^p(I, X_\theta)}$ do not depend on I.

(2) We are not aware of any result concerning the question whether the endpoint case in the complex interpolation scale holds. However, the proof of the endpoint case in the real interpolation scale heavily relies on subtleties in the context of real interpolation of certain weighted sequence spaces and bilinear real interpolation. These methods do not carry over to the complex case.

Proof. We proceed as in [Tag08] and [KT98], where in the prior reference the case $X_\theta = (X_0, X_1)_{\theta,2}$ is proven including the endpoint case. Hence, we only consider the complex interpolation case $X_\theta = [X_0, X_1]_\theta$ in the non-endpoint situation. In (a) we assume $I = \mathbb{R}$, which in (b) easily implies the same result for an arbitrary I.
Before we start, let us gather some helpful notations and results. Let X be a Banach space, $p \in [1, \infty]$ and $F \in L^{p^*}(I, X^*)$, $G \in L^p(I, X)$. We identify F as an element in $L^p(I, X)^*$ in the fashion of A.3.4;(1) and for abbreviation we let

$$\langle F, G \rangle_{\mathscr{I}} := \langle F, G \rangle_{L^p(I,X)^*, L^p(I,X)} = \int_I \langle F(t), G(t) \rangle_{X^*, X} \, dt.$$

We will constantly use $X \hookrightarrow X^{**}$ and $L^p(I, X) \hookrightarrow L^p(I, X^{**})$ as well as restrict bounded linear operators acting on X^{**} and $L^p(I, X^{**})$ to X and $L^p(I, X)$, respectively.
As we have seen in (II.8) the operator Φ has an explicit integral representation on $L^1(I, X_0)$. We are going to use it by exploiting the fact, that $L^1(I, X_0) \cap L^p(I, X_\theta)$ is dense in $L^p(I, X_\theta)$ for $p < \infty$ and $\theta \in [0, 1)$. For $\theta = 0$ this is obvious and for $\theta \in (0, 1)$ we use the density of $X_0 \cap X_1$ in X_θ stated in A.1.2;(1) to approximate an element $F \in L^p(I, X_\theta)$ by a sequence $(F_n)_{n \in \mathbb{N}}$ of step functions with values in $X_0 \cap X_1$ and $\lambda(\operatorname{supp}(F_n)) < \infty$ for $n \in \mathbb{N}$. This sequence clearly belongs to $L^1(I, X_0) \cap L^p(I, X_\theta)$.
(a) Let $I = \mathbb{R}$. We first prove (II.12) and let $\theta \in N_\sigma$ and $p := 2/\sigma\theta$. We already know $\mathcal{T} \in \mathcal{L}(H, L^\infty(\mathbb{R}, X_0^*))$ and therefore we assume $\theta > 0$. Since $p < \infty$ we have $L^p(\mathbb{R}, X_\theta^*) \hookrightarrow L^{p^*}(\mathbb{R}, X_\theta)^*$ from A.3.4;(1) and for $f \in H$ with $\|f\|_H = 1$ therefore holds

$$\begin{aligned} \|\mathcal{T} f\|_{L^p(\mathbb{R}, X_\theta^*)} &= \sup_{\|G\|_{L^{p^*}(\mathbb{R}, X_\theta)} \le 1} |\langle \mathcal{T} f, G \rangle_{L^{p^*}(\mathbb{R}, X_\theta)^*, L^{p^*}(\mathbb{R}, X_\theta)}| \\ &= \sup_{\|G\|_{L^{p^*}(\mathbb{R}, X_\theta)} \le 1} |\langle f, \mathcal{T}^* G \rangle_H| \le \sup_{\|G\|_{L^{p^*}(\mathbb{R}, X_\theta)} \le 1} \|\mathcal{T}^* G\|_H. \end{aligned}$$

By the density of $L^{p^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)$ in $L^{p^*}(\mathbb{R}, X_\theta)$ it is enough for (II.12) to prove for $G \in L^{p^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)$ the estimate

$$\|\mathcal{T}^* G\|_H \lesssim \|G\|_{L^{p^*}(\mathbb{R}, X_\theta)}. \tag{II.14}$$

We fix $G \in L^{p^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)$. Then Fubini's and Hille's Theorem A.3.3;(4)+(5) yield

$$\begin{aligned}
\|\mathcal{T}^* G\|_H^2 &= \Big\langle \int_{\mathbb{R}} T(s)^* G(s)\,\mathrm{d}s, \int_{\mathbb{R}} T(t)^* G(t)\,\mathrm{d}t \Big\rangle_H \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \langle T(s)^* G(s), T(t)^* G(t)\rangle_H \,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{\mathbb{R}} \int_{-\infty}^{t} \langle T(s)^* G(s), T(t)^* G(t)\rangle_H \,\mathrm{d}s\,\mathrm{d}t \\
&\quad + \int_{\mathbb{R}} \int_{t}^{\infty} \langle T(s)^* G(s), T(t)^* G(t)\rangle_H \,\mathrm{d}s\,\mathrm{d}t. \qquad \text{(II.15)}
\end{aligned}$$

To estimate the latter integrals we consider for $t, s \in \mathbb{R}$ with $t \neq s$ the sesquilinear form

$$\tau_{s,t} : (X_0 \cap X_1)^2 \to \mathbb{C}, \qquad \tau_{s,t}(f, g) := \langle T(s)^* f, T(t)^* g\rangle_H.$$

By means of the Cauchy-Schwarz inequality for $f, g \in X_0 \cap X_1$ holds

$$|\tau_{s,t}(f, g)| \leq \|T(s)^* f\|_H \|T(t)^* g\|_H \lesssim \|f\|_{X_0} \|g\|_{X_0},$$

and the dispersive estimate (II.9) yields

$$\begin{aligned}
|\tau_{s,t}(f, g)| &= |\langle T(t) T(s)^* f, g\rangle_{X_1}| \\
&\leq \|T(t) T(s)^* f\|_{X_1^*} \|g\|_{X_1} \lesssim |s-t|^{-\sigma} \|f\|_{X_1} \|g\|_{X_1}.
\end{aligned}$$

Thus, the bilinear complex interpolation result from Theorem A.1.3;(a) provides for $f, g \in X_\theta$

$$|\tau_{s,t}(f, g)| \leq C |s-t|^{-\sigma\theta} \|f\|_{X_\theta} \|g\|_{X_\theta}. \qquad \text{(II.16)}$$

For further reference we define the sesquilinear form

$$\tau : L^1(\mathbb{R}, X_0) \times L^1(\mathbb{R}, X_0) \to \mathbb{C}, \quad \tau(F, G) := \int_{\mathbb{R}} \int_{-\infty}^{t} \tau_{s,t}(F(s), G(t))\,\mathrm{d}s\,\mathrm{d}t. \quad \text{(II.17)}$$

Estimate (II.16) then yields for all $F \in L^{p^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)$ with Hölder's inequality

$$\begin{aligned}
|\tau(F, G)| &\lesssim \int_{\mathbb{R}} \int_{-\infty}^{t} |s-t|^{-\sigma\theta} \|F(s)\|_{X_\theta} \|G(t)\|_{X_\theta} \,\mathrm{d}s\,\mathrm{d}t \\
&\lesssim \Big\| \int_0^\infty \frac{1}{|r|^{1-(1-\sigma\theta)}} \|F(\cdot - r)\|_{X_\theta} \,\mathrm{d}r \Big\|_{L^p(\mathbb{R})} \|G\|_{L^{p^*}(\mathbb{R}, X_\theta)} \\
&\lesssim \|F\|_{L^{p^*}(\mathbb{R}, X_\theta)} \|G\|_{L^{p^*}(\mathbb{R}, X_\theta)}. \qquad \text{(II.18)}
\end{aligned}$$

In the last line we have applied the Hardy-Littlewood-Sobolev Inequality of Theorem A.2.2. Its requirements are met since

$$1 < p^* < 2 < p < \infty, \qquad \frac{1}{p^*} - \frac{1}{p} = 1 - \sigma\theta \in (0, 1). \qquad \text{(II.19)}$$

Estimate (II.18) applied in (II.15) clearly implies (II.14). A closer look on (II.19) reveals that the Hardy-Littlewood-Sobolev Inequality is not applicable for $\theta = {}^1/_\sigma$ in the case

$\sigma \in [1, \infty)$ since $p^* < p$ is not satisfied.
We turn to (II.13) and define $p(\theta) := {}^{2}/_{\sigma\theta}$ for $\theta \in N_\sigma$. Recall that the operator

$$\mathcal{T}\mathcal{T}^* : L^1(\mathbb{R}, X_0) \to L^\infty(\mathbb{R}, X_0^*), \qquad (\mathcal{T}\mathcal{T}^* F)(t) := \int_{\mathbb{R}} T(t)T(s)^* F(s) \, \mathrm{d}s,$$

is bounded as mentioned after Definition II.1.1. This allows us to exclude the case $\theta_1 = \theta_2 = 0$ in the following consideration.
We first prove several estimates for the sesquilinear form τ from (II.17). For this reason let $\theta \in N_\sigma$ and $F, G \in L^{p(\theta)^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)$. Estimate (II.18) implies

$$|\tau(F, G)| \lesssim \|F\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)} \|G\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)}.$$

With the Cauchy-Schwarz inequality, $\mathcal{T}^* \in \mathcal{L}(L^1(\mathbb{R}, X_0), H)$ and the Christ-Kiselev Lemma A.3.4;(4) additionally follows

$$\begin{aligned} |\tau(F, G)| &= \left| \int_{\mathbb{R}} \left\langle \int_{-\infty}^{t} T(s)^* F(s) \, \mathrm{d}s, T(t)^* G(t) \right\rangle_H \mathrm{d}t \right| \\ &\lesssim \left\| \int_{-\infty}^{\cdot} T(s)^* F(s) \, \mathrm{d}s \right\|_{L^\infty(\mathbb{R}, H)} \left(\int_{\mathbb{R}} \|T(t)^* G(t)\|_H \, \mathrm{d}t \right) \\ &\lesssim \|F\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)} \|G\|_{L^1(\mathbb{R}, X_0)}. \end{aligned}$$

By means of the Cauchy-Schwarz inequality, (II.12), and the previous estimate we get

$$|\tau(F, G)| \leq \|\mathcal{T}^* F\|_H \|\mathcal{T}^* G\|_H + |\tau(G, F)| \leq \|F\|_{L^1(\mathbb{R}, X_0)} \|G\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)}.$$

Gathering the previous estimates yields that for all $\theta \in N_\sigma$ the sesquilinear form

$$\tau : \left(L^{p(\theta)^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)\right) \times \left(L^{p(\theta)^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)\right) \to \mathbb{C}$$

satisfies for $F, G \in L^{p(\theta)^*}(\mathbb{R}, X_\theta) \cap L^1(\mathbb{R}, X_0)$ the estimates

$$|\tau(F, G)| \lesssim \|F\|_{L^1(\mathbb{R}, X_0)} \|G\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)}, \tag{II.20}$$

$$|\tau(F, G)| \lesssim \|F\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)} \|G\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)}, \tag{II.21}$$

$$|\tau(F, G)| \lesssim \|F\|_{L^{p(\theta)^*}(\mathbb{R}, X_\theta)} \|G\|_{L^1(\mathbb{R}, X_0)}. \tag{II.22}$$

Now, let $\theta_1, \theta_2 \in (0, 1)$ with $\theta_1 \neq \theta_2$ and $p_i := p(\theta_i)$ for $i \in \{1, 2\}$. We first assume $\theta_1 \in (0, \theta_2)$. With (II.20) and (II.21) follows for $F, G \in L^{p_2^*}(\mathbb{R}, X_{\theta_2}) \cap L^1(\mathbb{R}, X_0)$

$$|\tau(F, G)| \lesssim \|F\|_{L^1(\mathbb{R}, X_0)} \|G\|_{L^{p_2^*}(\mathbb{R}, X_{\theta_2})},$$

$$|\tau(F, G)| \lesssim \|F\|_{L^{p_2^*}(\mathbb{R}, X_{\theta_2})} \|G\|_{L^{p_2^*}(\mathbb{R}, X_{\theta_2})}.$$

For $\theta := {}^{\theta_1}/_{\theta_2} \in (0, 1)$ the complex interpolation results A.1.2;(3) and (A.1) imply

$$L^{p_1^*}(\mathbb{R}, X_{\theta_1}) \hookrightarrow L^{p_1^*}(\mathbb{R}, [X_0, X_{\theta_2}]_{\theta_1/\theta_2}) \cong [L^1(\mathbb{R}, X_0), L^{p_2^*}(\mathbb{R}, X_{\theta_2})]_\theta.$$

Theorem A.1.3 then provides for $F \in L^{p_1^*}(\mathbb{R}, X_{\theta_1}) \cap L^1(\mathbb{R}, X_0)$ the estimate

$$|\tau(F,G)| \lesssim \|F\|_{L^{p_1^*}(\mathbb{R},X_{\theta_1})} \|G\|_{L^{p_2^*}(\mathbb{R},X_{\theta_2})}.$$

If $\theta_2 < \theta_1$ we just switch the roles of θ_1 and θ_2 in the above argument and use inequality (II.22) instead of (II.20).
Consequently, we have for all $\theta_1, \theta_2 \in N_\sigma$, $F \in L^{p_1^*}(\mathbb{R}, X_{\theta_1}) \cap L^1(\mathbb{R}, X_0)$ and $G \in L^{p_2^*}(\mathbb{R}, X_{\theta_2}) \cap L^1(\mathbb{R}, X_0)$ the estimate

$$|\tau(F,G)| \lesssim \|F\|_{L^{p_1^*}(\mathbb{R},X_{\theta_1})} \|G\|_{L^{p_2^*}(\mathbb{R},X_{\theta_2})}. \tag{II.23}$$

We now let $\tilde{\theta} \in N_\sigma$, $\tilde{p} := p(\tilde{\theta})$ and $F \in L^{\tilde{p}^*}(\mathbb{R}, X_{\theta_1}) \cap L^1(\mathbb{R}, X_0)$. $\mathcal{T}\mathcal{T}^*F \in L^\infty(\mathbb{R}, X_0^*)$ and duality (see A.3.4;(1)) imply

$$\|\mathcal{T}\mathcal{T}F\|_{L^p(\mathbb{R},X_{\tilde{\theta}}^*)} = \sup_{\substack{G \in L^{\tilde{p}^*}(\mathbb{R},X_{\tilde{\theta}}) \cap L^1(\mathbb{R},X_0) \\ \|G\|_{L^{\tilde{p}^*}(\mathbb{R},X_{\tilde{\theta}})} \leq 1}} |\langle \mathcal{T}\mathcal{T}^*F, G\rangle_{\mathscr{I}}|.$$

For $G \in L^{\tilde{p}^*}(\mathbb{R}, X_{\tilde{\theta}}) \cap L^1(\mathbb{R}, X_0)$ holds

$$\begin{aligned} \langle \mathcal{T}\mathcal{T}^*F, G\rangle_{\mathscr{I}} &= \int_{\mathbb{R}} \langle (\mathcal{T}\mathcal{T}^*F)(t), G(t)\rangle_{X_0^*, X_0} \, \mathrm{d}t \\ &= \int_{\mathbb{R}} \Big\langle \int_{\mathbb{R}} T(s)^*F(s) \, \mathrm{d}s, T(t)^*G(t)\Big\rangle_H \, \mathrm{d}t \\ &= \tau(F,G) + \overline{\tau(G,F)}. \end{aligned} \tag{II.24}$$

(II.24) combined with estimate (II.23) provides

$$\|\mathcal{T}\mathcal{T}^*F\|_{L^p(\mathbb{R},X_{\tilde{\theta}}^*)} \lesssim \|F\|_{L^{\tilde{p}^*}(\mathbb{R},X_{\tilde{\theta}})}.$$

The last estimate implies (II.13) by the density of $L^{\tilde{p}^*}(\mathbb{R}, X_{\tilde{\theta}}) \cap L^1(\mathbb{R}, X_0)$ in $L^{\tilde{p}^*}(\mathbb{R}, X_{\tilde{\theta}})$. Finally, $\tilde{p}^* < 2 < p$ allows us to apply the Christ-Kiselev Lemma A.3.4;(4) to obtain (II.12).
(b) Let $I \in \mathcal{I}$, T be (σ, X_1) dispersive on I and (p, θ), $(\tilde{p}, \tilde{\theta})$ chosen as in **(a)**. We define $S : \mathbb{R} \to \mathcal{L}(H, X_0^*)$ by $S(t) := T(t)$ for $t \in I$ and $S(t) := 0$ for $t \in I^c$. Then S is bounded and for $t, s \in \mathbb{R}$ with $t \neq s$ and $f \in X_0 \cap X_1$ we have the estimate

$$\|S(t)S(s)^*f\|_{X_1^*} = \|\mathbb{1}_I(t)\mathbb{1}_I(s)T(t)T(s)^*f\|_{X_1^*} \lesssim |t-s|^{-\sigma}\|f\|_{X_1}.$$

Let $f \in H$, $F \in L^{\tilde{p}^*}(I, X_{\tilde{\theta}}) \cap L^1(I, X_0)$ and F_0 be the continuation of F to $\mathbb{R}$ by 0. The assertions in **(a)** applied to S implies the estimates

$$\begin{aligned} \|\mathcal{T}f\|_{L^{p^*}(I,X_\theta)} &= \|\mathcal{S}f\|_{L^{p^*}(\mathbb{R},X_\theta)} \lesssim \|f\|_H \\ \|\Phi F\|_{L^{p^*}(I,X_\theta)} &= \left\| \int_{-\infty}^{\cdot} S(\cdot)S(s)^*F_0(s) \, \mathrm{d}s \right\|_{L^{p^*}(\mathbb{R},X_\theta)} \lesssim \|F\|_{L^{\tilde{p}^*}(I,X_{\tilde{\theta}})}. \end{aligned}$$

□

In the example considered after the Definition II.1.2 we showed that the Schrödinger group $(e^{it\Delta})_{t\in\mathbb{R}}$ on $L^2(\mathbb{R}^d)$ is $(d/2, L^1(\mathbb{R}^d))$-dispersive. The above theorem therefore implies the estimates (II.12) and (II.13) with $X_\theta = [L^2(\mathbb{R}^d), L^1(\mathbb{R}^d)]_\theta \cong L^{2-\theta}(\mathbb{R}^d)$ (see (A.1)). In our applications we are always interested in the L^p scale, which enables us to introduce the following notion.

Definition II.1.4
Let $\sigma \in (0,\infty)$ and $(p,q) \in [1,\infty]^2$. We call (p,q) sharp σ-admissible, if p,q satisfy

$$p,q \geq 2, \qquad \frac{1}{p} + \frac{\sigma}{q} = \frac{\sigma}{2}, \qquad (p,q,\sigma) \neq (2,\infty,1).$$

By $(e_1(\sigma), e_2(\sigma))$ we denote the endpoint of the sharp σ-admissibility scale, which is given by

$$(e_1(\sigma), e_2(\sigma)) := \begin{cases} (\frac{2}{\sigma}, \infty), & \sigma \in (0,1], \\ (2, \frac{2\sigma}{\sigma-1}), & \sigma \in (1,\infty). \end{cases}$$

Remarks:

(1) Observe that

$$(p,q) \text{ is sharp } \sigma\text{-admissible} \qquad \Longleftrightarrow \qquad \frac{1}{p^*} + \frac{\sigma}{q^*} = \frac{\sigma}{2} + 1.$$

For $\theta \in (0,1)$ and (p_0,q_0), (p_1,q_1) sharp σ-admissible, the pair (p,q) defined by

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

is sharp σ-admissible, too.

(2) We have to exclude the case $(p,q,\sigma) = (2,\infty,1)$ in the above definition since there are counterexamples for $\Omega = \mathbb{R}^d$. For the wave equation this is due to Klainerman and Machedon [KM93] and for the Schrödinger equation this was shown by Montgomery-Smith in [MS98].

Using the notion of sharp σ-admissible pairs, Theorem II.1.3 takes the following form.

Corollary II.1.5
Let $\sigma \in (0,\infty)$, $(p,q), (\tilde{p},\tilde{q})$ be sharp σ-admissible, and $T : \mathbb{R} \to \mathcal{L}(L^2(\Omega))$. If T is bounded and $(\sigma, L^1(\Omega))$-dispersive, then for $f \in L^2(\Omega)$ holds

$$\|\mathcal{T} f\|_{L^p(\mathbb{R},L^q(\Omega))} \lesssim \|f\|_{L^2(\Omega)}, \tag{II.25}$$

and for $F \in L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega)) \cap L^1(\mathbb{R}, L^2(\Omega))$ holds

$$\|\Phi F\|_{L^p(\mathbb{R},L^q(\Omega))} \lesssim \|F\|_{L^{\tilde{p}^*}(\mathbb{R},L^{\tilde{q}^*}(\Omega))}. \tag{II.26}$$

Remarks:

(1) The estimate (II.25) is referred to as homogeneous Strichartz estimate and (II.26) as inhomogeneous Strichartz estimate for T.

(2) Since we want to reproduce the Strichartz estimates including the endpoint cases, we are forced to use the real interpolation spaces $(L^2(\Omega), L^1(\Omega))_{\theta,2}$ for $\theta \in (0,1)$. By means of Theorem 1.18.6;2 in [Tri95] we have $(L^2(\Omega), L^1(\Omega))_{\theta,2} \cong L^{2/(1+\theta),2}(\Omega)$, where $L^{r,s}(\Omega)$ denotes the classical Lorentz spaces. These are defined in Section 1.18.6 of [Tri95] or Section 1.4 of [Gra08]. The latter reference contains all the needed properties for the proof.

Proof. In Theorem II.1.3 we take $H = X_0 = L^2(\Omega)$ and $X_1 = L^1(\Omega)$ and we let (p,q) sharp σ-admissible. $(p,q) = (\infty, 2)$ is the trivial case and we can assume $q \in (2,\infty]$. First, we assume $\sigma \geq 1$. We then have $q < \infty$ and

$$\frac{1}{p} = \sigma\left(\frac{1}{2} - \frac{1}{q}\right) = \frac{\sigma}{2}\left(1 - \frac{2}{q}\right).$$

For fixed $\theta := 1 - \frac{2}{q} \in (0,1)$ holds $p = {}^{2}/_{\sigma\theta}$. As mentioned in the above remark we have

$$X_\theta := (L^2(\Omega), L^1(\Omega))_{\theta,2} \cong L^{2/(1+\theta),2}(\Omega) = L^{q^*,2}(\Omega).$$

Proposition 1.4.10 and Theorem 1.4.13 in [Gra08] imply the density of the embedding $L^{q^*}(\Omega) \hookrightarrow L^{q^*,2}(\Omega)$. Following the proof of Theorem 1.4.16;(iv) we conclude $L^{q,2}(\Omega) \hookrightarrow L^{q^*,2}(\Omega)^*$. Hence, $L^{q,2}(\Omega) \hookrightarrow L^q(\Omega)$ and

$$L^p(\mathbb{R}, X_\theta^*) \hookrightarrow L^p(\mathbb{R}, L^q(\Omega)),$$
$$L^{p^*}(\mathbb{R}, L^{q^*}(\Omega)) \hookrightarrow L^{p^*}(\mathbb{R}, X_\theta).$$

Combined with the real interpolation version of Theorem II.1.3, these embeddings imply the claimed Strichartz estimates (II.25) and (II.26).
For $\sigma < 1$ the complex interpolation version of Theorem II.1.3 immediately proves our claim. Note that if $q < \infty$, then complex interpolation A.1.4;(1) yields $X_\theta \cong L^{q^*}(\Omega)$ with $\theta = 1 - \frac{2}{q}$. Hence, $X_\theta^* \equiv L^q(\Omega)$. □

For the sake of completeness we close this section with a brief discussion of necessary conditions for the pairs $(\tilde{p}, \tilde{q})$, (p,q) such that (II.25) and (II.26) hold. The methods used below are closely related to the scaling arguments given after Proposition I.4.2. We start with an examination of the homogeneous estimate (II.25) for T and formulate the necessity of the sharp σ-admissibility of the pair (p,q). This result can be applied to $T = (e^{it\Delta})_{t\in\mathbb{R}}$ with the Laplacian $(-\Delta, H_2^2(\mathbb{R}^d))$.

Lemma II.1.6
Let $\sigma \in (0,\infty)$, $(p,q) \in [1,\infty]^2$ such that $(p,q,\sigma) \neq (2,\infty,1)$ and $T : \mathbb{R} \to \mathcal{L}(L^2(\Omega))$. Let additionally $T \neq 0$ be bounded, $(\sigma, L^1(\Omega))$-dispersive and fulfill $T(t)T(s)^* = T(t+r)T(s+r)^*$ for all $r,s,t \in \mathbb{R}$. If T satisfies (II.25), then (p,q) is sharp σ-admissible.

Proof. Let T satisfy (II.25) with $(p,q) \in [1,\infty]^2$. For $\lambda \in (0,\infty)$ we let $S : \mathbb{R} \to \mathcal{L}(L^2_\sigma(\Omega))$ given by $S_\lambda(t) := T({}^{t}/_{\lambda})$ where $L^p_\sigma(\Omega) := L^p(\Omega, \lambda^\sigma \mu)$ and $L^\infty_\sigma(\Omega) = L^\infty(\Omega)$.

S then is bounded and for $t, s \in \mathbb{R}$ with $t \neq s$ and $f \in L^2_\sigma(\Omega) \cap L^1_\sigma(\Omega)$ holds

$$\begin{aligned}\|S_\lambda(t)S_\lambda(s)^* f\|_{L^\infty(\Omega)} &= \|T(t/\lambda)T(s/\lambda)f\|_{L^\infty(\Omega)} \\ &\lesssim |t-s|^{-\sigma}\lambda^\sigma \|f\|_{L^1(\Omega)} \cong |t-s|^{-\sigma}\|f\|_{L^1_\sigma(\Omega)}.\end{aligned}$$

We then apply Corollary II.1.5 to S and T to provide (II.25) with

$$\|\mathcal{S}_\lambda\|_{L^2_\sigma(\Omega)\to L^p(\mathbb{R},L^q_\sigma(\Omega))} \cong \|\mathcal{T}\|_{L^2(\Omega)\to L^p(\mathbb{R},L^q(\Omega))} > 0. \tag{II.27}$$

If we assume $\frac{1}{p} + \frac{\sigma}{q} \neq \frac{\sigma}{2}$, then we have for all $f \in L^2_\sigma(\Omega)$ with $\|f\|_{L^2_\sigma(\Omega)} \leq 1$

$$\begin{aligned}\|\mathcal{S}_\lambda f\|_{L^p(\mathbb{R},L^q_\sigma(\Omega))} &= \lambda^{\frac{1}{p}+\frac{\sigma}{q}} \|\mathcal{T} f\|_{L^p(\mathbb{R},L^q(\Omega))} \\ &\lesssim \lambda^{\frac{1}{p}+\frac{\sigma}{q}} \|f\|_{L^2(\Omega)} \cong \lambda^{\frac{1}{p}+\frac{\sigma}{q}-\frac{\sigma}{2}} \to 0,\end{aligned}$$

for $\lambda \to 0$ or $\lambda \to \infty$, respectively. Hence, $\|S_\lambda\|_{L^2_\sigma(\Omega)\to L^p(\mathbb{R},L^q_\sigma(\Omega))} \to 0$ in both these cases. Combined with (II.27) this implies $\|\mathcal{T}\|_{L^2(\Omega)\to L^p(\mathbb{R},L^q(\Omega))} = 0$, which is absurd.
$\frac{1}{p} + \frac{\sigma}{q} = \frac{\sigma}{2}$ then implies $q \in [2, \infty]$ since otherwise $p < 0$. Moreover, (II.25) for T implies $\mathcal{T}\mathcal{T}^* \in \mathcal{L}(L^{p^*}(\mathbb{R}, L^{q^*}(\Omega)), L^p(\mathbb{R}, L^q(\Omega)))$. For $r \in \mathbb{R}$ and $F \in L^{p^*}(\mathbb{R}, L^{q^*}(\Omega)) \cap L^1(\mathbb{R}, L^2(\Omega))$ we additionally have

$$\begin{aligned}\tau_r(\mathcal{T}\mathcal{T}^* F) &= \int_{\mathbb{R}} T(t-r)T(s)^* F(s)\,\mathrm{d}s \\ &= \int_{\mathbb{R}} T(t)T(s)^* F(s-r)\,\mathrm{d}s = \mathcal{T}\mathcal{T}^*(\tau_r F).\end{aligned}$$

Using the density of $L^{p^*}(\mathbb{R}, L^{q^*}(\Omega)) \cap L^1(\mathbb{R}, L^2(\Omega))$ in $L^{p^*}(\mathbb{R}, L^{q^*}(\Omega))$ the operator $\mathcal{T}\mathcal{T}^*$ is translation invariant. Theorem 1.1 in [Hör60] then states $p^* \leq p$ and thus $p \in [2, \infty]$. □

For the inhomogeneous estimate (II.26) the situation is not so clear. Even for the Schrödinger group $(e^{it\Delta})_{t\in\mathbb{R}}$ with the Laplacian $(-\Delta, H^2_2(\mathbb{R}^d))$ the sharp σ-admissibility of (p, q) and $(\tilde{p}, \tilde{q})$ is not necessary for (II.26) to hold. For results in that direction see for example [Caz03, Fos05, Tag08, Vil07] where more general inhomogeneous Strichartz estimates are proven for $(e^{it\Delta})_{t\in\mathbb{R}}$. However, in the generality of Lemma II.1.6 we at least have the following.

Lemma II.1.7
Let $\sigma \in (0, \infty)$, $(p, q), (\tilde{p}, \tilde{q}) \in [1, \infty]^2$ and $T : \mathbb{R} \to \mathcal{L}(L^2(\Omega))$. Let additionally T be bounded, $(\sigma, L^1(\Omega))$-dispersive and $T(t)T(s)^* = T(t+r)T(s+r)^*$ for all $r, s, t \in \mathbb{R}$. If T satisfies (II.26), then $\frac{1}{p} + \frac{\sigma}{q} + \frac{1}{\tilde{p}} + \frac{\sigma}{\tilde{q}} = \sigma$ and $\tilde{p}^* \leq p$.

Proof. The first condition again arises from scaling. Hence, we consider $S : \mathbb{R} \to \mathcal{L}(L^2_\sigma(\Omega))$ given by $S(t) := T(t/\lambda)$ and its induced inhomogeneous flow Φ_S. Then we establish the translation invariance of $\Phi_S \in \mathcal{L}(L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega)), L^p(\mathbb{R}, L^q(\Omega)))$ and again use Theorem 1.1 of [Hör60] to conclude $\tilde{p}^* \leq p$. □

II.2. Spectrally localized dispersive and Strichartz estimates

In this section we recall the concepts of spectrally localized dispersive and Strichartz estimates. These estimates have far reaching consequences with respect to local (p, q) Strichartz estimates with ℓ-loss and solutions of the Cauchy problem (II.1). The latter will be explored in Section II.4. We provide in this section an abstract scheme for the proof of local (p, q) Strichartz estimates with ℓ-loss. All the tools which will be introduced in this section are justified in Section II.3 in various situations.
In this section we always let $(\Omega, \mathscr{S}, \mu)$ be a measure space, $T : \mathbb{R} \to \mathcal{L}(L^2(\Omega))$ be bounded and $(P, D(P))$ a selfadjoint linear operator on $(L^2(\Omega), \langle \cdot, \cdot \rangle_{L^2(\Omega)})$.
Similar to Definition II.1.2 we introduce the following notion.

Definition II.2.1
Let $\sigma \in (0, \infty)$ and $\gamma \in [0, \infty)$. T is called σ-dispersive of SL-type (γ, P), if for all $\varphi \in C_c^\infty(\mathbb{R})$, $h \in (0, 1]$ and $t, s \in \mathbb{R}$ with $0 < |t - s| \leq h^\gamma$ holds

$$\|\varphi(hP)T(t)T(s)^*\varphi(hP)\|_{L^1(\Omega)\to L^\infty(\Omega)} \lesssim |t - s|^{-\sigma}. \tag{II.28}$$

Remark: For $T = (e^{-\mathrm{i}tP})_{t\in\mathbb{R}}$ to be σ-dispersive of SL-type (γ, P), it is sufficient that for all $\varphi \in C_c^\infty(\mathbb{R})$, $h \in (0, 1]$ and $t \in \mathbb{R}$ with $0 < |t| \leq h^\gamma$ holds

$$\|\varphi(hP)T(t)\|_{L^1(\Omega)\to L^\infty(\Omega)} \lesssim |t|^{-\sigma}. \tag{II.29}$$

In this case we say T is σ-dispersive of SL-type γ and omit the dependency on $(P, D(P))$.

The next proposition shows how σ-dispersive estimates of SL-type imply (p, q) Strichartz estimates of SL-type by means of Corollary II.1.5.

Proposition II.2.2
Let $\sigma \in (0, \infty)$, $\gamma \in [0, \infty)$, $(p, q), (\tilde{p}, \tilde{q})$ be sharp σ-admissible, and $T : \mathbb{R} \to \mathcal{L}(L^2(\Omega)$ be bounded. We furthermore let $\varphi \in C_c^\infty(\mathbb{R})$, $h \in (0, 1]$, and $I \in \mathcal{I}$ with $|I| \leq h^\gamma$. If T is σ-dispersive of SL-type (γ, P) and commutes with $\varphi(hP)$ on $L^2(\Omega)$, then for all $f \in L^2(\Omega)$ and $F \in L^1(I, L^2(\Omega))$ with $\varphi(hP)F \in L^{\tilde{p}}(I, L^{\tilde{q}}(\Omega))$ holds

$$\|\varphi(hP)\mathcal{T}f\|_{L^p(I,L^q(\Omega))} \lesssim \|\varphi(hP)f\|_{L^2(\Omega)},$$
$$\|\varphi(hP)\Phi F\|_{L^p(I,L^q(\Omega))} \lesssim \|\varphi(hP)F\|_{L^{\tilde{p}^*}(I,L^{\tilde{q}^*}(\Omega))}.$$

Remark: The assumption that T commutes with $\varphi(hA)$ is for example satisfied if $T := (e^{-\mathrm{i}tP})_{t\in\mathbb{R}}$ by the properties of the spectral calculus on $L^2(\Omega)$.

Proof. Let $h \in (0, 1]$, $I \in \mathcal{I}$ with $|I| \leq h^\gamma$ and (p, q), $(\tilde{p}, \tilde{q})$ sharp σ-admissible as well as $\varphi, \tilde{\varphi} \in C_c^\infty(\mathbb{R})$ such that $\tilde{\varphi}(\mathrm{supp}(\varphi)) = \{1\}$. Let furthermore $f \in L^2(\Omega)$ and $F \in L^1(I, L^2(\Omega))$ satisfy $\varphi(hP)F \in L^{\tilde{p}^*}(I, L^{\tilde{q}^*}(\Omega))$. The operator family

$$\tilde{T}_h : \mathbb{R} \to \mathcal{L}(L^2(\Omega)), \qquad \tilde{T}_h(t) := \mathbb{1}_I(t)\tilde{\varphi}(hP)T(t),$$

is bounded by $\|T\|_{L^\infty(\mathbb{R},\mathcal{L}(L^2(\Omega)))}$ due to the properties of the spectral calculus on $L^2(\Omega)$. For $t,s \in \mathbb{R}$ with $t \neq s$ additionally holds with the σ-dispersivity of SL-type (γ, P) of T that

$$\begin{aligned}\|\tilde{T}_h(t)\tilde{T}_h(s)\|_{L^1(\Omega)\to L^\infty(\Omega)} &= \|\mathbb{1}_I(t)\mathbb{1}_I(s)\tilde{\varphi}(hP)T(t)T(s)^*\tilde{\varphi}(hP)\|_{L^1(\Omega)\to L^\infty(\Omega)}\\ &\lesssim |t-s|^{-\sigma}.\end{aligned}$$

Corollary II.1.5 and the commutativity of T and $\varphi(hP)$ on $L^2(\Omega)$ then yield

$$\|\varphi(hP)\mathcal{T}f\|_{L^p(I,L^q(\Omega))} = \|\tilde{\mathcal{T}}_h\varphi(hP)f\|_{L^p(\mathbb{R},L^q(\Omega))} \lesssim \|\varphi(hP)f\|_{L^2(\Omega)}.$$

With $F_0 := F$ on I and $F_0 := 0$ on I^c we also have

$$\begin{aligned}\|\varphi(hP)\Phi F\|_{L^p(I,L^q(\Omega))} &= \left\| \int_{-\infty}^{\cdot} \tilde{T}_h(\cdot)\tilde{T}_h(s)\varphi(hP)F_0(s)\,\mathrm{d}s\right\|_{L^p(\mathbb{R},L^q(\Omega))}\\ &\lesssim \|\varphi(hP)F\|_{L^{\tilde{p}^*}(J,L^{\tilde{q}^*}(\Omega))}.\end{aligned}$$

□

We formalize the estimates from the previous proposition into a notion of its own, which will be the basis of our study.

Definition II.2.3
Let $(p,q),(\tilde{p},\tilde{q}) \in [1,\infty]^2$, and $\gamma,\nu \in [0,\infty)$.

(a) T fulfills (p,q)-Strichartz estimates of SL-type (γ,ν,P) if for all $\varphi \in C_c^\infty(\mathbb{R})$, $h \in (0,1]$, $I \in \mathcal{I}$ with $|I| \le h^\gamma$ and $f \in L^2(\Omega)$ holds

$$\|\varphi(hP)\mathcal{T}f\|_{L^p(I,L^q(\Omega))} \lesssim h^{-\nu}\|\varphi(hP)f\|_{L^2(\Omega)}. \tag{II.30}$$

(b) T fulfills a $(\tilde{p},\tilde{q})$-(p,q) Strichartz estimate of SL-type (γ,ν,P) if for all $\varphi \in C_c^\infty(\mathbb{R})$, $h \in (0,1]$, $I \in \mathcal{I}$ with $|I| \le h^\gamma$ and $F \in L^1(I,L^2(\Omega))$ with $\varphi(hP)F \in L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))$ holds

$$\|\varphi(hP)\Phi F\|_{L^p(I,L^q(\Omega))} \lesssim h^{-\nu}\|\varphi(hP)F\|_{L^{\tilde{p}}(I,L^{\tilde{q}}(\Omega))}. \tag{II.31}$$

Remarks:

(1) If $T = (e^{-\mathrm{i}tP})_{t\in\mathbb{R}}$ we omit the dependency on P in the notation above. Moreover, we omit the dependency of ν if $\nu = 0$.

(2) We stress that the constants in (II.30) and (II.31) are not allowed to depend on h and I. They may however depend on φ.

(3) Estimate (II.30) can be established in many examples, which will be listed in Section II.3. We stress that there other methods to prove (II.30) than by means of spectrally localized dispersive estimates, which was shown in Lemma II.2.2.

We now aim to present the important argument how to deduce a local (p,q) Strichartz estimates with ℓ-loss for an operator family T from (p,q) Strichartz estimates of SL-type (γ,ν,P).
In order to utilize the (p,q) Strichartz estimates of SL-type we first need a decomposition estimate of the form (II.3). To that end we first fix the dyadic partition of unity, which is frequently used throughout this thesis.

Definition II.2.4
Let $\psi_c \in C_c^\infty(\mathbb{R},[0,1])$ such that

$$\operatorname{supp}(\psi_c) \subseteq \{\lambda \in \mathbb{R} \mid \frac{1}{2} < \lambda < 2\}, \qquad \forall_{\lambda\in(0,\infty)} : \sum_{k=-\infty}^{\infty} \psi_c(2^{1-k}\lambda) = 1. \tag{II.32}$$

We call the sequence $(\psi_k)_{k\in\mathbb{N}_0}$ defined by

$$\psi_k := \psi_c(2^{1-k}|\cdot|),\ k \in \mathbb{N}, \qquad \psi_0 := 1 - \sum_{l=1}^{\infty} \psi_l, \tag{II.33}$$

a dyadic partition of unity. We additionally define the sequence $(\tilde{\psi}_k)_{k\in\mathbb{N}_0} \subseteq C_c^\infty(\mathbb{R},[0,1])$, $\operatorname{supp}(\tilde{\psi}_0) \subsetneq (-2,2)$, $\operatorname{supp}(\tilde{\psi}_1) \subsetneq (-4,-\frac{1}{4}) \cup (\frac{1}{4},4)$ and $\tilde{\psi}_k(\operatorname{supp}(\psi_k)) = \{1\}$ for $k \in \{0,1\}$ as well as $\tilde{\psi}_k := \sum_{l=k-1}^{k+1} \psi_l$ for $k \geq 2$. Note that for $k \in \mathbb{N}_0$ holds

$$\tilde{\psi}_k \psi_k = \psi_k. \tag{II.34}$$

The existence of the function ψ_c and the partition $(\psi_k)_{k\in\mathbb{N}_0}$ is proven in Lemma 6.1.7 of [BL76]. If not otherwise stated ψ_c, $(\psi_k)_{k\in\mathbb{N}_0}$, and $(\tilde{\psi}_k)_{k\in\mathbb{N}_0}$ will always be taken from Definition II.2.4.
The spectral calculus on $L^2(\Omega)$ allows us to define the following operators.

Notation II.2.5
Let $(P,D(P))$ be selfadjoint linear operator on $L^2(\Omega)$ and $k \in \mathbb{N}_0$. We put $P_k := \psi_k(P)$ and $\tilde{P}_k := \tilde{\psi}_k(P)$ and call P_k and $\tilde{P}_k$ a spectral localization. The sequence $(P_k)_{k\in\mathbb{N}_0}$ is called the spectral decomposition of $(P,D(P))$.

Let us recall some important properties of spectral decompositions on $L^2(\Omega)$ first.

Lemma II.2.6
Let $\alpha \in (0,\infty)$ and $(P,D(P))$ a selfadjoint linear operator on $L^2(\Omega)$.

(a) For all $k \in \mathbb{N}_0$ holds $P_k = \tilde{P}_k P_k = P_k \tilde{P}_k$ on $L^2(\Omega)$.

(b) Let $f \in L^2(\Omega)$. Then $f = \sum_{k=0}^{\infty} P_k f$, whereby the sum converges unconditionally, and

$$\|f\|_{L^2(\Omega)} \cong \left(\sum_{k=0}^{\infty} \|P_k f\|_{L^2(\Omega)}^2\right)^{1/2}. \tag{II.35}$$

(c) Let $(P,D(P))$ be additionally non-negative and $f \in D(P^\alpha)$. Then holds:

$$\|P_k f\|_{L^2(\Omega)} \cong 2^{-k\alpha} \|P_k P^\alpha f\|_{L^2(\Omega)}, \qquad k \in \mathbb{N}, \tag{II.36}$$

$$\|f\|_{D(P^\alpha)} \cong \left(\sum_{k=0}^{\infty} 2^{2k\alpha} \|P_k f\|_{L^2(\Omega)}^2\right)^{1/2}. \tag{II.37}$$

Proof. Our claims are consequences of the properties (SC1)-(SC4) of the spectral calculus in Theorem I.1.1 and well known. We present these short proofs since similar arguments are used frequently in this thesis.
(a) This is a consequence of $\psi_k = \tilde{\psi}_k \psi_k$ and (SC2).
(b) Let $f \in L^2(\Omega)$. The first assertion immediately follows from the almost orthogonality lemma by Cotlar and Stein formulated in Lemma 18.6.5 of [Hör07]. The equivalence (II.35) immediately follows with the Cauchy-Schwarz inequality and *(a)* via

$$\begin{aligned}\|f\|_{L^2(\Omega)}^2 &= \sum_{k=0}^{\infty} (P_k f, \tilde{P}_k f)_{L^2(\Omega)} \\ &\lesssim \sum_{k=0}^{\infty} \|P_k f\|_{L^2(\Omega)}^2 + \sum_{k=0}^{\infty} |(P_k f, P_{k+1} f)_{L^2(\Omega)}| \\ &\lesssim \sum_{k=0}^{\infty} \|P_k f\|_{L^2(\Omega)}^2 \lesssim \Big\| \sum_{k=0}^{\infty} \psi_k^2 \Big\|_{L^\infty(\mathbb{R})} \|f\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Omega)}^2.\end{aligned}$$

(c) Let $\alpha \in (0, \infty)$ and $f \in D(P^\alpha)$. We first prove (II.36) and let $k \in \mathbb{N}$. For $\beta \in \mathbb{R}$ we define the functions $\psi_{k,\beta}, \tilde{\psi}_{k,\beta} \in C_c^\infty(\mathbb{R})$ by

$$\psi_{k,\beta}(\lambda) := \lambda^\beta \psi_k(\lambda), \qquad \tilde{\psi}_{k,\beta}(\lambda) := \lambda^\beta \tilde{\psi}_k(\lambda),$$

so that $\|\psi_{k,\beta}\|_{L^\infty(\mathbb{R})} \lesssim 2^{k\beta}$. Then *(a)*, (SC1), and (SC4) yield

$$\begin{aligned}\|P_k f\|_{L^2(\Omega)} &= \|\tilde{\psi}_{k,-\alpha}(P) \psi_{k,\alpha}(P) f\|_{L^2(\Omega)} \\ &\lesssim 2^{-k\alpha} \|P_k P^\alpha f\|_{L^2(\Omega)} \lesssim 2^{-k\alpha} \|\tilde{\psi}_{k,\alpha}\|_{L^\infty(\mathbb{R})} \|P_k f\|_{L^2(\Omega)} \lesssim \|P_k f\|_{L^2(\Omega)}.\end{aligned}$$

We now turn to (II.37) and let $\alpha \in (0, \infty)$ and $f \in D(P^\alpha)$. By means of (II.35), (SC4) and (II.36) we have

$$\begin{aligned}\|P^\alpha f\|_{L^2(\Omega)}^2 &\cong \sum_{k=0}^{\infty} \|P_k P^\alpha f\|_{L^2(\Omega)}^2 \\ &\lesssim \sum_{k=0}^{\infty} 2^{2k\alpha} \|P_k f\|_{L^2(\Omega)}^2 \lesssim \|f\|_{L^2(\Omega)}^2 + \sum_{k=1}^{\infty} \|P_k P^\alpha f\|_{L^2(\Omega)}^2 \lesssim \|f\|_{D(P^\alpha)}^2.\end{aligned}$$

Combined with (II.35) the previous estimate yields (II.37). □

Lemma II.2.6 yields a stronger assertion than (II.3) for $q = 2$, which based exclusively on the spectral calculus of $(P, D(P))$ on $L^2(\Omega)$. For $q \neq 2$ such a functional calculus can not be provided without making stronger assumptions on the operator $(P, D(P))$. We therefore introduce the following notion, which captures (II.3). For an exensive list of examples where such estimates are valid we refer to Section II.3.

Definition II.2.7
Let $(P, D(P))$ be selfadjoint linear operator on $L^2(\Omega)$.

(a) $(P_k)_{k\in\mathbb{N}_0}$ *has the* (LP) *property, if for each $q \in [2, \infty)$ and $f \in L^2(\Omega)$ holds: If $(P_k f)_{k\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, L^q(\Omega))$, then $f \in L^q(\Omega)$ and*

$$\|f\|_{L^q(\Omega)} \leq C(q) \Big(\sum_{k=0}^{\infty} \|P_k f\|_{L^q(\Omega)}^2 \Big)^{1/2}. \tag{II.38}$$

(b) $(P_k)_{k\in\mathbb{N}_0}$ *has the reversed* (LP) *property, if for each* $q \in (1,2]$ *and* $f \in L^2(\Omega)$ *holds: If* $f \in L^q(\Omega)$*, then* $(P_k f)_{k\in\mathbb{N}_0} \in \ell^2(\mathbb{N}_0, L^q(\Omega))$ *and*

$$\|f\|_{L^q(\Omega)} \geq C(q)\bigg(\sum_{k=0}^{\infty} \|P_k f\|^2_{L^q(\Omega)}\bigg)^{1/2}. \tag{II.39}$$

Next, we show how the above notions result in local (p,q) Strichartz estimates with ℓ-loss. Our method of proof is abstracted from an argument initially used in [BGT04b]. It is one of the most important tools in this thesis and we give a detailed proof.

Lemma II.2.8
Let $I \in \mathcal{I}_b$*,* $(p,q),(\tilde{p},\tilde{q}) \in [2,\infty)^2$*,* $\gamma,\nu \in [0,\infty)$*, and* $T : \mathbb{R} \to \mathcal{L}(L^2(\Omega))$ *be bounded. Let* $(P, D(P))$ *be a non-negative, selfadjoint operator on* $L^2(\Omega)$ *and* $(P_k)_{k\in\mathbb{N}_0}$ *satisfy the* (LP) *property.*

(a) Let $\ell := \frac{\gamma}{p} + \nu$ *and* T *admit a* (p,q) *Strichartz estimate of SL-type* (γ,ν,P)*. Then there is a non-decreasing* $C_{\mathcal{T}} \in C([0,\infty),[0,\infty))$ *such that for all* $f \in D(P^\ell)$ *holds*

$$\|\mathcal{T} f\|_{L^p(I,L^q(\Omega))} \leq C_{\mathcal{T}}(|I|)\|f\|_{D(P^\ell)}. \tag{II.40}$$

(b) Let additionally $(P, D(P))$ *fulfill the reversed* (LP) *property and* $\varphi(P)$ *can be extended to a bounded operator on* $L^{\tilde{q}^*}(\Omega)$ *for all* $\varphi \in C_c^\infty(\mathbb{R})$*. If* T *admits a* $(\tilde{p}^*,\tilde{q}^*)$*-*$(p,q)$ *Strichartz estimate of SL-type* (γ,ν,P)*, then for all* $F \in L^1(I, D(P^\nu))$ *with* $P^\nu F \in L^{\tilde{p}^*}(I, L^{\tilde{q}^*}(\Omega))$ *holds*

$$\|\Phi F\|_{L^p(I,L^q(\Omega))} \lesssim \|F\|_{L^{\tilde{p}^*}(I,L^{\tilde{q}^*}(\Omega))} + \|P^\nu F\|_{L^{\tilde{p}^*}(I,L^{\tilde{q}^*}(\Omega))}. \tag{II.41}$$

Remark: We exclude $q,\tilde{q} = \infty$ since the (LP) property is usually not valid in these cases.

Since a similar time splitting argument as in the proof below is used frequently in the subsequent sections, we define the following partition of a bounded interval.

Notation II.2.9
Let $I \in \mathcal{I}_b$ *and* $\rho \in (0,\infty)$ *and* $N := [|I|/\rho]$*. The family* $(I_i)_{i=0}^N$ *defined by*

$$I_i := \big[\inf I + i\rho, \inf I + (i+1)\rho\big] \cap I, \qquad (i \in \{0,\dots,N-1\}),$$
$$I_N := \big[\inf I + N\rho, b\big] \cap I$$

is called the ρ*-partition of* I*. Observe that* $\lambda(I_j) \leq \rho$ *for* $j \in \{0,\dots,N\}$*,* $\bigcup_{j=0}^N I_j = I$ *and* $I_j^o \cap I_i^o = \emptyset$ *for* $j \neq i$*.*

Proof of Lemma II.2.8. We let $I \in \mathcal{I}_b$ with $2^{(1-k)\gamma}$-partition $(I_i)_{i=0}^{N_k}$ for $k \in \mathbb{N}_0$. Then $N_k \cong 2^{k\gamma}|I|$.
(a). Let $f \in D(P^\ell)$. First note that there is $l \in \mathbb{N}$ such that $N_k \geq 1$ for $k \geq l$. We

use the (LP) property (II.38), Minkowski inequality and the (p,q) Strichartz estimate of SL-type (γ,ν,P) to produce

$$\begin{aligned}\|\mathcal{T}f\|_{L^p(I,L^q(\Omega))} &\lesssim \left\| \left(\sum_{k=0}^{\infty} \|P_k\mathcal{T}f\|^2_{L^q(\Omega)} \right)^{1/2} \right\|_{L^p(I)} \\ &\lesssim \left(\sum_{k=0}^{\infty} \left(\sum_{i=0}^{N_k} \|P_k\mathcal{T}f\|^p_{L^p(I_i,L^q(\Omega))} \right)^{2/p} \right)^{1/2} \\ &\lesssim \left(\sum_{k=0}^{\infty} (N_k+1)^{\frac{2}{p}} 2^{2k\nu} \|P_k f\|^2_{L^2(\Omega)} \right)^{1/2} \\ &\lesssim (1+|I|^{2/p}) \left(\|f\|_{L^2(\Omega)} + \left(\sum_{k=l}^{\infty} 2^{2k\ell} \|P_k f\|^2_{L^2(\Omega)} \right)^{1/2} \right) \lesssim C(|I|)\|f\|_{D(P^\ell)}.\end{aligned}$$

(b) Let $F \in L^1(I, D(P^\nu))$ with $P^\nu F \in L^{\tilde{p}^*}(I, L^{\tilde{q}^*}(\Omega))$. The idea of the proof remains the same since we use the same splitting procedure of the time interval as in **(a)**. Minkowski's integral inequality (A.8), the (LP) property, the $(\tilde{p}^*,\tilde{q}^*)$-$(p,q)$ Strichartz estimate of SL-type (γ,ν,P), and $\ell^{\tilde{p}^*}(\mathbb{N}_0) \hookrightarrow \ell^p(\mathbb{N}_0)$ yield

$$\begin{aligned}\|\Phi F\|_{L^p(I,L^q(\Omega))} &\lesssim \left(\sum_{k=0}^{\infty} \left(\sum_{i=0}^{N_k} \|P_k\Phi F\|^p_{L^p(I_i,L^q(\Omega))} \right)^{2/p} \right)^{1/2} \\ &\lesssim \left(\sum_{k=0}^{\infty} 2^{2k\nu} \|P_k F\|^2_{L^{\tilde{p}^*}(I,L^{\tilde{q}^*}(\Omega))} \right)^{1/2}.\end{aligned}$$

Continuing this estimate with Minkowski's integral inequality and the reversed (LP) property yield

$$\begin{aligned}\|\Phi F\|_{L^p(I,L^q(\Omega))} &\lesssim \left\| \left(\sum_{k=0}^{\infty} 2^{2k\nu} \|P_k F(\cdot)\|^2_{L^{\tilde{q}^*}(\Omega)} \right)^{1/2} \right\|_{L^{\tilde{p}^*}(I)} \\ &\lesssim \left\| \|F(\cdot)\|_{L^{\tilde{q}^*}(\Omega)} + \left(\sum_{k=1}^{\infty} \|P_k P^\nu F(\cdot)\|^2_{L^{\tilde{q}^*}(\Omega)} \right)^{1/2} \right\|_{L^{\tilde{p}^*}(I)} \\ &\lesssim \|F\|_{L^{\tilde{p}^*}(I,L^{\tilde{q}^*}(\Omega))} + \|P^\nu F\|_{L^{\tilde{p}^*}(I,L^{\tilde{q}^*}(\Omega))}.\end{aligned}$$

In particular, we used the estimate

$$\|P_k g\|_{L^{\tilde{q}^*}(\Omega)} \lesssim 2^{-k\nu} \|P_k P^\nu g\|_{L^{\tilde{q}^*}(\Omega)}$$

for $k \in \mathbb{N}$ and $g \in D(P^\nu)$ with $P^\nu g \in L^{\tilde{q}^*}(\Omega)$. It follows similarly to (II.36), whereby we use the boundedness of $\psi_k(P)P^{-\nu}$ as an operator on $L^{\tilde{p}^*}(\Omega)$ instead of $L^2(\Omega)$. □

Combining Proposition II.2.2 and Lemma II.2.8 allows us to formulate the following important special case of the results developed so far. It will be used in some of the coming applications.

Corollary II.2.10
Let $\sigma \in (0,\infty)$, $\gamma \in [0,\infty)$, and $(p,q),(\tilde{p},\tilde{q}) \in [2,\infty)^2$ be sharp σ-admissible. Let furthermore $(A,D(A))$ be a non-negative, selfadjoint linear operator on $L^2(\Omega)$ such that $(A_k)_{k\in\mathbb{N}_0}$ has the (LP) *property and its generated Schrödinger group U be σ-dispersive of SL-type γ.*

(a) *U fulfills a local (p, q) Strichartz estimate with γ/p-loss.*

(b) *If $(A_k)_{k\in\mathbb{N}_0}$ satisfies the reversed* (LP) *property and $\varphi(A)$ may be extended to a bounded operator on $L^{\tilde{q}^*}(\Omega)$ for all $\varphi \in C_c^\infty(\mathbb{R})$, then $\Phi \in \mathcal{L}(L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega)), L^p(\mathbb{R}, L^q(\Omega)))$.*

Proof. Let $(p, q), (\tilde{p}, \tilde{q}) \in [2, \infty)^2$ be sharp σ-admissible as well as $f \in D(A^{\gamma/p})$ and $F \in L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega))$. Proposition II.2.2 and Lemma II.2.8 yield for all $I, \tilde{I} \in \mathcal{I}_b$ with $\tilde{I} \subseteq I$ the estimates

$$\|\mathcal{U}f\|_{L^p(I,L^q(\Omega))} \lesssim C_{\mathcal{U}}(|I|)\|f\|_{D(A^{\gamma/p})},$$

$$\|\Phi(\mathbb{1}_{\tilde{I}}F)\|_{L^p(I,L^q(\Omega))} \lesssim \|\mathbb{1}_{\tilde{I}}F\|_{L^{\tilde{p}^*}(\mathbb{R},L^{\tilde{q}^*}(\Omega))}.$$

Since $I \in \mathcal{I}_b$ is arbitrary in the latter estimate, we have for all $k \in \mathbb{N}$ that $\Phi(\mathbb{1}_{[-k,k]}F) \in L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega))$ by the monotone convergence theorem. Finally, $\mathbb{1}_{[-k,k]}F \overset{k\to\infty}{\longrightarrow} F$ in $L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega))$ lets us conclude $\Phi \in \mathcal{L}(L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega)), L^p(\mathbb{R}, L^q(\Omega)))$. □

II.3. Examples for Strichartz estimates with ℓ-loss

In this section we present several situations in which the above scheme can be applied. The examples II.3.1-II.3.3 as well as the abstract example II.3.7 are based on dispersive estimates of SL-type.

II.3.1. The Laplacian on $\mathbb{R}^d$

Let $\Omega = \mathbb{R}^d$ with $d \in \mathbb{N}$ and $(A, D(A)) := (-\Delta, H_2^2(\mathbb{R}^d))$ as well as $U = (e^{it\Delta})_{t\in\mathbb{R}}$. Since U is $(d/2, L^1(\mathbb{R}^d))$-dispersive Corollary II.1.5 provides the full set of homogeneous and inhomogeneous Strichartz estimates. It is interesting to compare these estimates to the estimates provided by means of the results of Section II.2. For $k \in \mathbb{N}_0$ and $f \in S(\mathbb{R}^d)$ we have $A_k f = \psi_k(-\Delta)f = (\mathcal{F}^{-1}\psi_k(|\cdot|^2)) * f$. By means of the Mikhlin multiplier theorem we extend A_k to a bounded operator on $L^q(\mathbb{R}^d)$ for all $q \in (1, \infty)$. Exercise 5.1.11 in [Gra08] or Section 6.7.14 in [Ste93] show furthermore for $q \in (1, \infty)$

$$\|f\|_{L^q(\mathbb{R}^d)} \cong \|A_0 f\|_{L^q(\mathbb{R}^d)} + \Big\|\Big(\sum_{k=1}^{\infty} |A_k f|^2\Big)^{1/2}\Big\|_{L^q(\mathbb{R}^d)}. \tag{II.42}$$

Minkowski's integral inequality yields for $q \in [2, \infty)$ the estimate

$$\|f\|_{L^q(\Omega)} \lesssim \left(\sum_{k=0}^{\infty} \|A_k f\|^2_{L^q(\mathbb{R}^d)}\right)^{1/2},$$

which is the (LP) property for $(A_k)_{k\in\mathbb{N}_0}$. For $q \in (1, 2]$ the same argument provides the reversed (LP) property (II.39). Note that we use $2^{-k/2}$ as frequency dilation instead of 2^{-k}. For $\varphi \in C_c^\infty(\mathbb{R})$ holds $\mathcal{F}^{-1}[\varphi(|\cdot|^2)] \in L^1(\mathbb{R}^d)$ and Young's inequality implies

$$\|\varphi(-h\Delta)\|_{L^\infty(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \le \|\mathcal{F}^{-1}\varphi(h|\cdot|^2)\|_{L^1(\mathbb{R}^d)} = \|\mathcal{F}^{-1}\varphi(|\cdot|^2)\|_{L^1(\mathbb{R}^d)}. \tag{II.43}$$

For $h \in (0,1]$ and $t \in \mathbb{R} \setminus \{0\}$ therefore holds

$$\|\varphi(-h\Delta)U(t)\|_{L^1(\Omega)\to L^\infty(\Omega)} \lesssim |t|^{-\sigma}.$$

Hence, U is σ-dispersive of SL-type 0. For $(p,q),(\tilde{p},\tilde{q}) \in [2,\infty)^2$ sharp $d/2$-admissible Corollary II.2.10 provides $\Phi \in \mathcal{L}(L^{\tilde{p}^*}(\mathbb{R}, L^{\tilde{q}^*}(\Omega)), L^p(\mathbb{R}, L^q(\Omega)))$. We additionally have $\|\mathcal{U}\|_{L^2(\Omega)\to L^p(I,L^q(\Omega))} \lesssim 1$ for all $I \in \mathcal{I}_b$. The last argument in the proof of Corollary II.2.10 even provides $\mathcal{U} \in \mathcal{L}(L^2(\Omega), L^p(\mathbb{R}, L^q(\Omega)))$. Hence, we recover the full set of Strichartz estimates for U provided by Corollary II.1.5, except the cases where $q = \infty$ or $\tilde{q} = \infty$.

II.3.2. Divergence form operators on $\mathbb{R}^d$

Let $\Omega = \mathbb{R}^d$ and $d \in \mathbb{N}$. Section A.1. of [BGT04b] contains the proof of local (p,q) Strichartz estimates with $1/2p$-loss for $U = (e^{-itA})_{t\in\mathbb{R}}$ for certain divergence form operators $(A, H_2^2(\mathbb{R}^d))$ (see Section III.2 for details). This is an interesting situation, since the authors use the spectral decomposition $(P_k)_{k\in\mathbb{N}_0}$ with $(P, D(P)) = (-\Delta, H_2^2(\mathbb{R}^d))$ and U and $(P_k)_{k\in\mathbb{N}_0}$ may not commute. To bypass this problem they show that there is $\varphi_1 \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}(\varphi_1) \subseteq (0,\infty)$ and $\varphi_1(\operatorname{supp}(\psi_1)) = \{1\}$ such that with $\varphi_k(\lambda) := \varphi_1(2^{1-k}\lambda)$ holds

$$\forall_{k\in\mathbb{N},\, s\in(0,\infty)} : \ \|P_k(\mathrm{id} - \varphi_k(A))\|_{L^2(\mathbb{R}^d)\to H_2^s(\mathbb{R}^d)} \lesssim 2^{-k}.$$

Let $I \in \mathcal{I}_b$ and $(p,q) \in [2,\infty] \times [2,\infty)$ be sharp $d/2$-admissible. With $s > \frac{d}{2} - \frac{d}{q}$ the Sobolev embedding A.2.1 yields

$$\begin{aligned}
\|\mathcal{U}f\|_{L^p(I,L^q(\mathbb{R}^d))} &\lesssim \Big(\sum_{k=0}^{\infty} \|P_k\mathcal{U}f\|^2_{L^p(I,L^q(\Omega))}\Big)^{1/2} \\
&\lesssim C(|I|)\|f\|_{L^2(\mathbb{R}^d)} + \Big(\sum_{k=1}^{\infty} \|P_k(\mathrm{id} - \varphi_k(A))\mathcal{U}f\|^2_{L^p(I,H_2^s(\mathbb{R}^d))}\Big)^{1/2} \\
&+ \Big(\sum_{k=0}^{\infty} \|P_k\mathcal{U}\varphi_k(A)f\|^2_{L^p(I,L^q(\mathbb{R}^d))}\Big)^{1/2} \\
&\lesssim C(|I|)\Big(\|f\|_{L^2(\mathbb{R}^d)} + \Big(\sum_{k=1}^{\infty} \|P_k\mathcal{U}\varphi_k(A)f\|^2_{L^p(I,L^q(\mathbb{R}^d))}\Big)^{1/2}\Big).
\end{aligned}$$

Since U is shown in Lemma A.3 of [BGT04b] to be $d/2$-dispersive of SL-type $(1/2, -\Delta)$ we apply the time splitting procedure from the proof of Lemma II.2.8;(a) to the sum on the right-hand side to obtain

$$\Big(\sum_{k=1}^{\infty} \|P_k\mathcal{U}\varphi_k(A)f\|^2_{L^p(I,L^q(\mathbb{R}^d))}\Big)^{1/2} \lesssim \Big(\sum_{k=1}^{\infty} 2^{k/p}\|\varphi_k(A)f\|^2_{L^2(\mathbb{R}^d)}\Big)^{1/2} \lesssim \|A^{1/2p}f\|_{L^2(\mathbb{R}^d)}.$$

Hence, $(A, H_2^2(\mathbb{R}^d))$ satisfies local (p,q) Strichartz estimates with $1/2p$-loss.

II.3.3. The Laplace-Beltrami on compact manifolds without boundary

Let Ω be a connected, compact Riemannian C^∞-manifold with $\partial\Omega = \emptyset$ and $\dim(\Omega) = d \in \mathbb{N}$. For the Laplace-Beltrami operator $(-\Delta_\Omega, H_2^2(\Omega))$ on Ω the (LP) property (II.38) is shown in Corollary 2.3 of [BGT04b]. Lemma 2.5 there furthermore contains estimate (II.29) with $\gamma = 1/2$ and $\sigma = d/2$, i.e. $(e^{it\Delta_\Omega})_{t\in\mathbb{R}}$ is $d/2$-dispersive of SL-type $1/2$. Hence, Corollary II.2.10;(a) provides local (p,q) Strichartz estimates with $1/2p$-loss for all $(p,q) \in [2,\infty] \times [2,\infty)$ sharp $d/2$-admissible. This corresponds to Theorem 1 in [BGT04b].

II.3.4. The Dirichlet and Neumann Laplacian on manifolds with boundary

Let Ω be a connected, compact Riemannian C^∞-manifold with boundary and $\dim(\Omega) = d \geq 2$. In [BSS12] local (p,q) Strichartz estimates with ℓ-loss for $U = (e^{it\Delta_\Omega})_{t\in\mathbb{R}}$ with the Dirichlet and Neumann Laplacian $(-\Delta_\Omega, D(-\Delta_\Omega))$ on Ω are proven. The (LP) property for $((-\Delta_\Omega)_k)_{k\in\mathbb{N}_0}$ is provided by heat kernel methods (see also example II.3.7). In our notation (put $\lambda = 2^{k/2}$ in [BSS12]) Theorem 2.1 there contains the following estimates for $I \in \mathcal{I}$ with $|I| \lesssim 2^{-k/2}$ with $k \in \mathbb{N}_0$: If $(p,q) \in (2,\infty] \times [2,\infty)$ and $s \in [0,\infty)$ satisfy

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{2} - s, \qquad \text{and} \qquad \begin{cases} \frac{3}{q} + \frac{2}{q} \leq 1, & d = 2, \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, & d \geq 3, \end{cases} \tag{II.44}$$

then

$$\|(-\Delta_\Omega)_k \mathcal{U} f\|_{L^p(I,L^q(\Omega))} \lesssim 2^{\frac{ks}{2}} \|(-\Delta_\Omega)_k f\|_{L^2(\Omega)}.$$

Consequently, $\mathcal{U}$ satisfies (p,q) Strichartz estimates of SL-type $(1/2, s/2)$ for all pairs (p,q) and s, which satisfy (II.44). Lemma II.2.8;(a) once more yields a local (p,q) Strichartz estimates with $\frac{s}{2} + \frac{1}{2p}$-loss. This corresponds to Theorem 1.2 in [BSS12]. See also [BSS08] for local (p,q) Strichartz estimates with $2/3p$-loss on compact C^∞-manifolds with boundary for sharp $d/2$-admissible pairs (p,q).

II.3.5. The Dirichlet and Neumann Laplacian on bounded domains

Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain. Let $(A, D(A))$ be either the Dirichlet or Neumann Laplacian on Ω and $U = (e^{itA})_{t\in\mathbb{R}}$. For details on the definitions in the following examples see the provided references or Section III.4 for the Dirichlet case.
Let $d = 2$ and $\partial\Omega$ be polygonal. In [BFHM12] local (p,q) Strichartz estimates with $1/2p$-loss for U are proven. The method used is a doubling procedure of the domain which transfers the problem to the consideration of the linear flow on an euclidean surface (S,g) with conical singularities. For the Laplacian $(-\Delta_S, D(-\Delta_S))$ on such surfaces the (LP) property is provided by Gaussian upper bounds. Estimate (3.7) in [BFHM12] gives a (p,q)-Strichartz estimate of SL-type $(1/2, 0)$ for $(e^{it\Delta_S})_{t\in\mathbb{R}}$ for all sharp 1-admissible pairs (p,q). Applying Theorem II.2.8;(a) for such pairs consequently yields a local (p,q) Strichartz estimate with $1/2p$-loss for $(e^{it\Delta_S})_{t\in\mathbb{R}}$. This corresponds to Theorem 1.5

in [BFHM12] since $D((-\Delta_S)^{1/2p}) \cong H_2^{1/p}(S)$.
Let $d \in \{2,3\}$ and Ω have a C^∞ boundary. In [Ant08] a similar reflection argument as above transfers the problem to the Laplace-Beltrami operator $(-\Delta_{\tilde{\Omega}}, W_2^2(\tilde{\Omega}))$ on a compact, connected, Riemannian C^∞-manifold $\tilde{\Omega}$ whose metric g is only Lipschitz continuous. Let $\mathscr{A} = (O_j, \kappa_j)_{j\in J}$ be an atlas of $\tilde{\Omega}$ with subordinate C^∞-partition of unity $(\chi_j)_{j\in J}$ and $(\tilde{\chi}_j)_{j\in J} \subseteq C_c^\infty(M)$ such that $\operatorname{supp}(\tilde{\chi}_j) \subseteq O_j$ and $\tilde{\chi}_j(\operatorname{supp}(\chi_j)) = \{1\}$. For $k \in \mathbb{N}_0$ we put

$$(P_k f)(\cdot) := \sum_{j\in J} \tilde{\chi}_j(\cdot)\Big(\psi_k((-\Delta)^{1/2})[(\chi_j f)\circ \kappa_j^{-1}]\Big)(\kappa_j(\cdot)).$$

Proposition 4.17 in [Ant08] contains for $k \in \mathbb{N}$, $f \in H^1(\tilde{\Omega})$, $I \in \mathcal{I}_b$ and $(p,q) \in [2,\infty]^2$ sharp $d/2$-admissible the estimate

$$\|P_k e^{\mathrm{i}(\cdot)\Delta_{\tilde{\Omega}}} f\|_{L^p(I,L^q(\tilde{\Omega}))} \leq C(|I|)2^{k(\frac{3}{2p}-1)}\|f\|_{H^1(\tilde{\Omega})}.$$

A very similiar approach as in the proof of Lemma II.2.8, involving the triangle inequality instead of the (LP) property, then provides (p,q) Strichartz estimates for U with ℓ-loss for all $\ell > 3/2p$.

II.3.6. The Dirichlet Laplacian on exterior domains

Let $K \subseteq \mathbb{R}^d$ with $d \in \mathbb{N}_{\geq 2}$ be a compact, strictly convex C^∞-domain and $\Omega = \mathbb{R}^d \setminus K$. Let $(-\Delta_D, H_2^2(\Omega) \cap H_{2,0}^1(\Omega))$ be the Dirichlet Laplacian and $U = (e^{\mathrm{i}t\Delta_D})_{t\in\mathbb{R}}$. The (LP) property for $((-\Delta_D)_k)_{k\in\mathbb{N}_0}$ is proven in [IP08] for general C^∞-domains. In [Iva10] global (p,q) Strichartz estimates for U are proven. Proposition 3.1 in [Iva10] provides for $k \in \mathbb{N}_0$ and $f \in L^2(\Omega)$ the estimate

$$\|(-\Delta_D)_k \mathcal{U} f\|_{L^p(\mathbb{R},L^q(\Omega))} \lesssim \|(-\Delta_D)_k f\|_{L^2(\Omega)}$$

for $(p,q) \in (2,\infty] \times [2,\infty)$ sharp $d/2$ admissible. Hence, (p,q) Strichartz estimates of SL-type 0 hold and we derive a global (p,q) Strichartz estimate by means of the (LP) property.

II.3.7. Operators with Gaussian upper bounds on metric measures spaces

Let $(\Omega, \mathscr{S}, \mu, d_\Omega)$ be a σ-finite metric measure space, whose measure μ has the doubling property (see the beginning of chapter 7 in [Ouh05]). Let $(A, D(A))$ be a non-negative, selfadjoint linear operator on $L^2(\Omega)$ and $U = (e^{\mathrm{i}tA})_{t\in\mathbb{R}}$. We additionally assume that the generated C_0 semigroup $(e^{-tA})_{t\in[0,\infty)}$ has an integral kernel $p_t : \Omega\times\Omega \to \mathbb{R}$ for $t \in (0,\infty)$, which satisfies the following Gaussian upper bound:

$$\forall_{t\in(0,\infty)} : \ |p_t(x,y)| \lesssim t^{-d/2}\exp\big(-Ct^{-1}d_\Omega(x,y)^2\big) \quad \text{a.e. on } \Omega\times\Omega. \tag{II.45}$$

For $\Omega = \mathbb{R}^d$ and $(A, D(A)) = (-\Delta, H_2^2(\mathbb{R}^d))$ a standard argument involving Fourier analysis gives (II.45). However, such an estimate (or generalized versions of it) are valid for more general differential operators of second order on $\mathbb{R}^d$, domains and manifolds (see e.g. [Ouh05, SC10]).

The spectral multiplier result in Theorem 7.23 of [Ouh05], provides that A_k with $k \in \mathbb{N}_0$ can be extended to a bounded operator on $L^q(\Omega)$ with $q \in (1,\infty)$. Let $(r_k)_{k\in\mathbb{N}_0}$ be sequence of Rademacher functions on $[0,1]$ and $q \in [2,\infty)$. The properties of $(r_k)_{k\in\mathbb{N}_0}$ and the fact that $L^q(\Omega)$ is a type 2 Banach space, imply for $f \in L^q(\Omega)$ the estimate

$$\|f\|_{L^q(\Omega)} \lesssim \int_0^1 \Big\| \sum_{k=0}^{\infty} r_k(t)\psi_k(A)f \Big\|_{L^q(\Omega)} \, dt \lesssim \Big(\sum_{k=0}^{\infty} \|\psi_k(A)f\|^2_{L^q(\Omega)} \Big)^{1/2}. \tag{II.46}$$

Hence, $(A, D(A))$ has the (LP) property. Furthermore, (II.45) implies

$$\|e^{-tA}\|_{L^2(\Omega)\to L^\infty(\Omega)} \lesssim t^{-d/4}. \tag{II.47}$$

Let $\varphi \in C_c^\infty(\mathbb{R},[0,1])$ and $h \in (0,1]$. Then (II.47) and the spectral calculus of $(A, D(A))$ show

$$\|\varphi(hA)f\|_{L^\infty(\Omega)} \lesssim \|e^{-hA}\|_{L^2(\Omega)\to L^\infty(\Omega)} \|e^{hA}\varphi(hA)f\|_{L^2(\Omega)} \lesssim h^{-d/4}\|f\|_{L^2(\Omega)}. \tag{II.48}$$

Hence, $\varphi(hA) \in \mathcal{L}(L^2(\Omega), L^\infty(\Omega))$. Then $\varphi(hA)^*|_{L^1(\Omega)} \in \mathcal{L}(L^1(\Omega), L^2(\Omega))$ satisfies $\varphi(hA)^* = \varphi(hA)$ on $L^1(\Omega)\cap L^2(\Omega)$, whereby the latter space is dense in $L^1(\Omega)$. Consequently, $\varphi(hA)$ can be extended from $L^1(\Omega)\cap L^2(\Omega)$ to a bounded operator in $\mathcal{L}(L^1(\Omega), L^2(\Omega))$ which coincides with $\varphi(hA)^*|_{L^1(\Omega)}$ and

$$\|\varphi(hA)\|_{L^1(\Omega)\to L^2(\Omega)} = \|\varphi(hA)\|_{L^2(\Omega)\to L^\infty(\Omega)}.$$

For $t \in \mathbb{R}$ with $0 < |t| \le h$ then holds by means of (II.48)

$$\|\varphi(hA)U(t)\varphi(hA)\|_{L^1(\Omega)\to L^\infty(\Omega)} \le \|\varphi(hA)\|^2_{L^2(\Omega)\to L^\infty(\Omega)} \|U(t)\|_{L^2(\Omega)\to L^2(\Omega)} \lesssim |t|^{-d/2}.$$

Consequently, U is $d/2$-dispersive of SL-type 1 and Corollary II.2.10 provides local (p,q) Strichartz estimates with $1/p$-loss for U for all $(p,q) \in [2,\infty)^2$ sharp $d/2$-admissible. For $(A, D(A)) = (-\Delta, H_2^2(\mathbb{R}^d))$ we can compare this to Remark (2) after Definition I.3.2 where we called $\ell^\star = d(\frac{1}{2} - \frac{1}{q}) = 2/p$ Sobolev-type loss. $\gamma = 1$ therefore corresponds to the loss which can be deduced by the Sobolev embedding. There is also a deep connection between the estimate (II.47) and the Sobolev embedding itself (see e.g. [SC02, CSCV92]). The approach for spectral decompositions involving Rademacher functions as in (II.46) is applied in more general situations (see e.g. [KW14]).

II.4. Criteria for uniqueness of weak solutions and a priori estimates

In this section we want to further exploit the spectrally localized Strichartz estimates from Definition II.2.3. In Section II.2 we used them to deduce local (p,q) Strichartz estimates with ℓ-loss. In this section we come back to the Cauchy problem from Chapter I for the nonlinear Schrödinger equation given by

$$\begin{aligned} iu'(t) &= \tilde{A}u(t) + F(u(t)), \quad t \neq t_0, \\ u(t_0) &= f. \end{aligned} \tag{CPA}$$

Recall that $(A, D(A))$ is a non-negative, selfadjoint operator on $(L^2(\Omega), \langle\cdot,\cdot\rangle_{L^2(\Omega)})$, $f \in D(A^s)$, and $F : D(A^s) \to E_A^*$, whereby $s \in [1/2, \infty)$. The results we will present are generalizations of the methods used in Section 3.3 of [BGT04b] to (CPA). To formulate them we need to recall an additional type of estimate, for which we give the following motivation.

Example: Let $\Omega = \mathbb{R}^d$, $q \in (2,\infty)$ and $s = \frac{d}{2}(\frac{1}{2} - \frac{1}{q})$. Assume for the moment that $D(A^s) \cong H_2^{2s}(\mathbb{R}^d)$. Then the Sobolev embedding A.2.1 yields $D(A^s) \hookrightarrow L^q(\mathbb{R}^d)$. For $\varphi \in C_c^\infty(\mathbb{R})$, $h \in (0,1]$ and $f \in L^2(\mathbb{R}^d)$ the property (SC4) of the spectral calculus furthermore implies

$$\begin{aligned}\|\varphi(hA)f\|_{L^q(\mathbb{R}^d)} &\lesssim \|\varphi(hA)f\|_{L^2(\mathbb{R}^d)} + \|\varphi(hA)A^s f\|_{L^2(\mathbb{R}^d)} \\ &\lesssim h^{\frac{d}{2}(\frac{1}{q}-\frac{1}{2})}\|f\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{II.49}$$

On the other hand, we now assume that (II.49) holds for all $\varphi \in C_c^\infty(\mathbb{R})$ and $h \in (0,1]$. Then Hölder's inequality and (II.36) imply for $f \in D(A^{s+\epsilon})$ with $\epsilon \in (0,\infty)$ the estimate

$$\begin{aligned}\|f\|_{L^q(\mathbb{R}^d)} &\lesssim \|A_0 f\|_{L^q(\mathbb{R}^d)} + \sum_{k=1}^{\infty}\|A_k f\|_{L^q(\mathbb{R}^d)} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^d)} + \sum_{k=1}^{\infty} 2^{-\epsilon s}\|A_k A^{s+\epsilon} f\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{D(A^{s+\epsilon})}.\end{aligned}$$

Hence, $D(A^{s+\epsilon}) \hookrightarrow L^q(\mathbb{R}^d)$ for all $\epsilon \in (0,\infty)$. If $(A_k)_{k\in\mathbb{N}_0}$ additionally has the (LP) property then for $f \in D(A^s)$ we use (II.37) to produce

$$\begin{aligned}\|f\|_{L^q(\mathbb{R}^d)} &\lesssim \|A_0 f\|_{L^q(\mathbb{R}^d)} + \left(\sum_{k=1}^{\infty}\|A_k f\|_{L^q(\mathbb{R}^d)}^2\right)^{1/2} \\ &\lesssim \|f\|_{L^2(\mathbb{R}^d)} + \left(\sum_{k=1}^{\infty} 2^{2ks}\|A_k f\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2} \cong \|f\|_{D(A^s)}.\end{aligned}$$

Hence, $D(A^s) \hookrightarrow L^q(\mathbb{R}^d)$.

The previous example illustrates the intimate relationship between the Sobolev embedding and estimates of the form (II.49). In the literature these are called Bernstein or Nikol'skij inequalities. We abstract (II.49) in the following notion and provide several examples afterward.

Definition II.4.1
Let $p, q \in [1,\infty]$ with $p \le q$ and $\alpha \in (0,\infty)$. $(A, D(A))$ is said to fulfill (p,q,α)-Bernstein inequalities, if for all $\varphi \in C_c^\infty(\mathbb{R})$, $h \in (0,1]$ and $f \in L^p(\Omega) \cap L^2(\Omega)$ holds

$$\|\varphi(hA)f\|_{L^q(\Omega)} \lesssim h^{\alpha(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^p(\Omega)}. \tag{II.50}$$

$(A, D(A))$ is said to fulfill the full range of (p,q,α)-Bernstein inequalities, if $(A, D(A))$ fulfills (p,q,α)-Bernstein inequalities for all $p, q \in [1,\infty]$ with $p \le q$.

Remarks:

(1) For $(A, D(A)) = (-\Delta, H_2^2(\mathbb{R}^d))$ the operators $\varphi(-h\Delta)$ are convolution operators which are translation invariant and not the zero map. Theorem 1.1 in [Hör60] provides $p \leq q$ if $\varphi(-h\Delta) \in \mathcal{L}(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))$. The restriction of p, q in the above definition is therefore justified.

(2) For $p < \infty$ the estimate (II.50) implies that $\varphi(hA)$ can be uniquely extended to an element of $\mathcal{L}(L^p(\Omega), L^q(\Omega))$.

(3) Let $p_0, q_0, p_1, q_1 \in [1, \infty]$ and $\alpha \in (0, \infty)$. If $(A, D(A))$ fulfills (p_0, q_0, α) and (p_1, q_1, α) Bernstein inequalities. Then $(A, D(A))$ fulfills (p, q, α)-Bernstein inequalities if there is $\theta \in (0, 1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

This immediately follows with complex interpolation (see Theorem A.1.3 and (A.1)).

(4) If $(A, D(A))$ fulfills (p, q, α)-Bernstein inequalities, then for $k \in \mathbb{N}_0$ holds

$$\|A_k\|_{L^p(\Omega) \to L^q(\Omega)} \lesssim 2^{k\alpha(\frac{1}{p} - \frac{1}{q})}. \tag{II.51}$$

The previous estimate will be used frequently in the following proofs.

Examples II.4.2
Here we list some (classes of) operators for which Bernstein inequalities are available.

(1) Let $(A, D(A)) = (-\Delta, H_2^2(\mathbb{R}^d))$ and $\varphi \in C_c^\infty(\mathbb{R})$ with $|\max \operatorname{supp}(\varphi)| < C$ as well as $p, q \in [1, \infty]$ with $p \leq q$. For $f \in S(\mathbb{R}^d)$ and $h \in (0, 1]$ holds

$$\operatorname{supp}(\mathcal{F}[\varphi(-h\Delta)f]) = \operatorname{supp}(\varphi(h|\cdot|^2)\mathcal{F}f) \subseteq B(0, \sqrt{C/h}).$$

Lemma 2.1 in Section 2.1 of [BCD11] applied to $u = \varphi(-h\Delta)f$ yields

$$\|\varphi(-h\Delta)\|_{L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)} \lesssim h^{\frac{d}{2}(\frac{1}{q} - \frac{1}{p})}.$$

Hence, $(-\Delta, H_2^2(\mathbb{R}^d))$ fulfills $(p, q, d/2)$-Bernstein inequalities. Moreover, the cited lemma implies for all $l \in \mathbb{N}$ that

$$\sum_{|\alpha|=l} \|\partial^\alpha \varphi(-h\Delta)f\|_{L^p(\mathbb{R}^d)} \lesssim h^{-l} \|\varphi(-h\Delta)f\|_{L^p(\mathbb{R}^d)}. \tag{II.52}$$

If additionally $\operatorname{supp}(\varphi) \subseteq \mathbb{R} \setminus \{0\}$, then

$$\sum_{|\alpha|=l} \|\partial^\alpha \varphi(-h\Delta)f\|_{L^p(\mathbb{R}^d)} \cong h^{-l} \|\varphi(-h\Delta)f\|_{L^p(\mathbb{R}^d)}. \tag{II.53}$$

(2) Let $(A, D(A)) = (-\Delta_\Omega, D(-\Delta_\Omega))$ be the Laplace-Beltrami operator on a connected, compact Riemannian C^∞-manifold Ω without boundary and $\dim(\Omega) = d \in \mathbb{N}$. Corollary 2.2 of [BGT04b] provides the full range of $(p, q, d/2)$-Bernstein inequalities.

(3) Recall that in Example II.3.7 we have shown the $(2,\infty,d/2)$-Bernstein inequalities (II.48) by means of gaussian upper bounds of the heat kernel.

Having recalled the notion of Bernstein inequalities we come back to the Cauchy problem (CPA). The results of this section require rather strong assumptions on the involved operator $(A, D(A))$ and the nonlinearity F. It makes sense to fix these assumptions since they are needed frequently in the coming proofs.

Convention
In the rest of this section we fix $I \in \mathcal{I}_b$ with $t_0 \in I$, $\alpha \in (0,\infty)$, $\gamma \in [0,\infty)$, $s \in [1/2,\infty)$ as well as $(p,q) \in [2,\infty) \times (2,\infty)$ and $r \in [q,\infty)$ with the following properties:

(A1) *$A_k \in \mathcal{L}(L^{r^*}(\Omega), L^{q^*}(\Omega))$ for $k \in \mathbb{N}$ and the embedding $E_A \hookrightarrow L^r(\Omega)$ holds.*

(A2) *U satisfies (p,q) Strichartz estimates of SL-type γ.*

(A3) *$F : D(A^s) \to L^{r^*}(\Omega)$ satisfies $A_k F : L^\infty(I, D(A^s)) \to L^\infty(I, L^{q^*}(\Omega))$ for $k \in \mathbb{N}$.*

Remarks:

(1) Recall that $(A_k)_{k\in\mathbb{N}_0}$ denotes the spectral decomposition of $(A, D(A))$.

(2) (A1) implies in particular $E_A \hookrightarrow L^q(\Omega)$ by complex interpolation of $E_A \hookrightarrow L^2(\Omega)$ and $E_A \hookrightarrow L^r(\Omega)$.

(3) If U is σ-dispersive of SL-type γ, then Lemma II.2.2;(a) provides (A2).

With this convention in mind we state the following central result of this section.

Lemma II.4.3
Besides (A1)-(A3) let $\beta \in [1,p]$ and $\rho(x) := x - \frac{\gamma}{p}$. There is a non-decreasing $C : [0,\infty) \to [0,\infty)$ such that for all weak solutions $u \in L^\infty(I, D(A^s))$ of (CPA) and $k \in \mathbb{N}$ holds

$$\|A_k u\|_{L^\beta(I,L^q(\Omega))} \leq C(|I|)\left(2^{-k\rho(s)}\|A_k u\|_{L^{\max\{\beta,2\}}(I,D(A^s))} + 2^{-k\rho(\frac{\gamma}{p^*})}\|A_k F(u)\|_{L^\infty(I,L^{q^*}(\Omega))}\right). \quad \text{(II.54)}$$

Remarks:

(1) The estimate (II.54) is used in this section to bound $\|A_k u\|_{L^\beta(I,L^q(\Omega))}$ for a solution u of (CPA) in terms of norms involving the fractional domains $D(A^s)$. This will be the key tool in the subsequent results.

(2) Let $I \in \mathcal{I}$ and $|I| \leq 2^{(1-k)\gamma}$. Then (A2) implies for $F \in L^1(I, L^2(\Omega)) \cap L^{\tilde{p}^*}(I, L^{\tilde{q}^*}(\Omega))$ the estimates

$$\|A_k \mathcal{U}\mathcal{U}^* A_k F\|_{L^p(I,L^q(\Omega))} \lesssim \|A_k \mathcal{U}\|_{L^2(\Omega)\to L^p(I,L^q(\Omega))}\|(A_k\mathcal{U})^* F\|_{L^2(\Omega)} \lesssim \|F\|_{L^1(I,L^2(\Omega))},$$

$$\|A_k \mathcal{U}\mathcal{U}^* A_k F\|_{L^p(I,L^q(\Omega))} \lesssim \|A_k \mathcal{U}\|_{L^2(\Omega)\to L^p(I,L^q(\Omega))}\|(A_k\mathcal{U})^* F\|_{L^2(\Omega)} \lesssim \|F\|_{L^{p^*}(I,L^{q^*}(\Omega))}.$$

Then the Christ-Kiselev Lemma in A.3.4;(4) yields $(1,2)$-(p,q) and (p^*,q^*)-(p,q) Strichartz estimates of SL-type γ for $\mathcal{U}$. In the following proof we use exclusively this consequence of (A2), not (A2) itself.

Proof. Let $k \in \mathbb{N}$ be fixed and for convenience put $h := 2^{1-k}$. Let furthermore $(I_j)_{j=0}^N$ be the $h^\gamma/4$-partition of I from Definition II.2.9 with $N = [4|I|h^{-\gamma}]$. We prove our claim by estimating $A_k u$ on each I_j and in order to do so we introduce the following cover of $(I_j)_{j=0}^N$ and a smooth partition of unity subordinate to it. For $j \in \{0, ..., N\}$ we let

$$I_j' := \left(I_j + \left[-\frac{h^\gamma}{8}, \frac{h^\gamma}{8}\right]\right) \cap I, \qquad m_j := \inf I + \frac{jh^\gamma}{4} + \frac{h^\gamma}{8}.$$

We additionally choose $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ with $\operatorname{supp}(\chi) \subseteq [-\frac{1}{4}, \frac{1}{4}]$ and $\chi([-\frac{1}{8}, \frac{1}{8}]) = \{1\}$ and define the function $\chi_j := \chi(h^{-\gamma}(\cdot - m_j))$ for $j \in \{0, ..., N\}$. For $j \in \{0, ..., N\}$ then holds

$$\chi_j(I_j) = \{1\}, \qquad \operatorname{supp}(\chi_j) \cap I \subseteq I_j', \qquad \|\chi_j'\|_{L^\infty(\mathbb{R})} \leq h^{-\gamma}\|\chi'\|_{L^\infty(\mathbb{R})}. \tag{II.55}$$

For $j \in \{0, ..., N\}$ we put $u_j := \chi_j A_k u \in L^\infty(I, D(A^s))$ and let $g \in E_A$. Then for all $t \in I_j'$ holds

$$\langle u_j(t), g\rangle_{L^2(\Omega)} = \langle u(t), \chi_j(t) A_k g\rangle_{L^2(\Omega)}. \tag{II.56}$$

Since $\chi_j A_k g \in C^1(I, E_A)$ with derivative $\chi_j' A_k g$ and $u \in W^1_\infty(I^o, E_A^*)$, equation (II.56) and the product rule yields that $u_j \in W^1_\infty(I^o, E_A^*)$ with

$$\langle u_j', g\rangle = \langle u', \chi_j A_k g\rangle + \langle \chi_j' A_k u, g\rangle_{L^2(\Omega)} \ \text{ a.e. on } I.$$

It is easy to check that the right-hand side belongs to $L^\infty(I^o, E_A^*)$. With (CPA) and the commutativity of A_k and $A^{1/2}$ on E_A we furthermore have

$$\begin{aligned}
\langle \mathrm{i}u_j', g\rangle &= \langle \mathrm{i}u', \chi_j A_k g\rangle + \langle \mathrm{i}\chi_j' A_k u, g\rangle_{L^2(\Omega)} \\
&= \langle \tilde{A}u, \chi_j A_k g\rangle + \langle F(u), \chi_j A_k g\rangle_{L^2(\Omega)} + \langle \mathrm{i}\chi_j' A_k u, g\rangle_{L^2(\Omega)} \\
&= \langle \tilde{A}u_j, g\rangle + \langle \mathrm{i}\chi_j' A_k u + \chi_j A_k F(u), g\rangle_{L^2(\Omega)}.
\end{aligned} \tag{II.57}$$

In the last line we used that $\langle g_1, A_k g_2\rangle_{L^2(\Omega)} = \langle A_k g_1, g_2\rangle_{L^2(\Omega)}$ for $g_1 \in L^{r^*}(\Omega)$ and $g_2 \in E_A$. This is trivial for $g_1 \in L^{r^*}(\Omega) \cap L^2(\Omega)$ and the rest follows by density and $A_k \in \mathcal{L}(L^{r^*}(\Omega), L^{q^*}(\Omega))$. The latter also implies that the function

$$F_j : I \to E_A^*, \qquad F_j(t) := \mathrm{i}\chi_j'(t) A_k u(t) + \chi_j A_k F(u(t)),$$

satisfies $F_j \in L^\infty(I_j', E_A^*)$. Indeed, $A_k \in \mathcal{L}(L^{r^*}(\Omega), L^{q^*}(\Omega))$ and (A3) yield

$$\begin{aligned}
\|F_j\|_{L^\infty(I_j', E_A^*)} &\lesssim h^{-\gamma}\|A_k u\|_{L^\infty(I_j', E_A)} + \|A_k F(u)\|_{L^\infty(I_j', L^{q^*}(\Omega))} \\
&\lesssim h^{-\gamma}\|u\|_{L^\infty(I, E_A)} + \|A_k F(u)\|_{L^\infty(I, L^{q^*}(\Omega))}.
\end{aligned}$$

Equation (II.57) additionally shows that u_j with $j \in \{1, ..., N\}$ is a weak solution of

$$\begin{aligned}
\mathrm{i}v'(t) &= \tilde{A}v(t) + F_j(t), \quad t \in I_j', \\
v(\min I_j') &= 0,
\end{aligned}$$

and u_0 is a weak solution of

$$\begin{aligned} \mathrm{i}v'(t) &= \tilde{A}v(t) + F_0(t), \quad t \in I_0', \\ v(\max I_0') &= 0. \end{aligned}$$

With Proposition I.2.4 the functions u_j are in fact a.e. mild solutions on I_j' and for almost all $t \in I_j'$ holds

$$u_j(t) = -\int_{\min I_j'}^t U(t-\tau)\chi_j'(\tau)A_k u(\tau)\,\mathrm{d}\tau + \mathrm{i}\int_{\min I_j'}^t \tilde{U}(t-\tau)\chi_j(\tau)A_k F(u(\tau))\,\mathrm{d}\tau$$

if $j \in \{1, ..., N\}$. For almost all $t \in I_0'$ holds

$$u_0(t) = -\int_t^{\max I_0'} U(t-\tau)\chi_j'(\tau)A_k u(\tau)\,\mathrm{d}\tau + \mathrm{i}\int_t^{\max I_0'} \tilde{U}(t-\tau)\chi_0(\tau)A_k F(u(\tau))\,\mathrm{d}\tau.$$

Both integrals are treated in a uniform fashion. Let $j \in \{0, ..., N\}$. Hölder's inequality and the (p^*, q^*)-(p, q) and $(1, 2)$-(p, q) Strichartz estimates of SL-type γ derived from (A2) yield

$$\begin{aligned} \|A_k u\|_{L^\beta(I_j, L^q(\Omega))} &\lesssim h^{\gamma(\frac{1}{\beta}-\frac{1}{p})}\|u_j\|_{L^p(I_j', L^q(\Omega))} \\ &\lesssim h^{\gamma(\frac{1}{\beta}-\frac{1}{p})}\Big(\|\chi_j' A_k u\|_{L^1(I_j', L^2(\Omega))} + \|A_k F(u)\|_{L^{p^*}(I_j', L^{q^*}(\Omega))}\Big). \end{aligned} \tag{II.58}$$

Let $\beta' := \max\{\beta, 2\}$. In the first term we apply (II.55) and Lemma II.2.6;(c) to get

$$\|\chi_j' A_k u\|_{L^1(I_j', L^2(\Omega))} \lesssim h^{s-\gamma}\|A_k A^s u\|_{L^1(I_j', L^2(\Omega))} \lesssim h^{s-\frac{\gamma}{\beta'}}\|A_k u\|_{L^{\beta'}(I_j', D(A^s))}.$$

We estimate the second term as before with (A1) and Hölder's inequality to get

$$\|A_k F(u)\|_{L^{p^*}(I_j', L^{q^*}(\Omega))} \lesssim h^{\frac{\gamma}{p^*}}\|A_k F(u)\|_{L^\infty(I, L^{q^*}(\Omega))}.$$

Applying the previous two estimates in (II.58) yields

$$\begin{aligned} &\|A_k u\|_{L^\beta(I_j, L^q(\Omega))} \\ &\quad\lesssim h^{\rho(s)+\gamma(\frac{1}{\beta}-\frac{1}{\beta'})}\|A_k u\|_{L^{\beta'}(I_j', D(A^s))} + h^{\rho(\frac{\gamma}{p^*})+\frac{\gamma}{\beta}}\|A_k F(u)\|_{L^\infty(I, L^{q^*}(\Omega))}, \end{aligned} \tag{II.59}$$

with $\rho(s) = s - \frac{\gamma}{p}$. Recall that $N \lesssim |I|h^{-\gamma}$ so that for $j \in \{0, ..., N\}$ we can apply (II.59) and Hölder's inequality to obtain

$$\begin{aligned} \|A_k u\|_{L^\beta(I, L^q(\Omega))} &= \Big(\sum_{j=0}^N \|A_k u\|^\beta_{L^\beta(I_j, L^q(\Omega))}\Big)^{1/\beta} \\ &\lesssim h^{\rho(s)+\gamma(\frac{1}{\beta}-\frac{1}{\beta'})}\Big(\sum_{j=0}^N \|A_k u\|^\beta_{L^{\beta'}(I_j', D(A^s))}\Big)^{1/\beta} \\ &\quad + (1+|I|)^{\frac{1}{\beta}} h^{\rho(\frac{\gamma}{p^*})}\|A_k F(u)\|_{L^\infty(I, L^{q^*}(\Omega))} \\ &\lesssim (1+|I|)^{\frac{1}{\beta}}\Big(h^{\rho(s)}\|A_k u\|_{L^{\beta'}(I, D(A^s))} + h^{\rho(\frac{\gamma}{p^*})}\|A_k F(u)\|_{L^\infty(I, L^{q^*}(\Omega))}\Big). \end{aligned}$$

Here we used that by the definition of $(I_j')_{j=0}^N$ we have

$$\left(\sum_{j=0}^{N}\|A_k u\|_{L^{\beta'}(I_j',D(A^s))}^{\beta'}\right)^{1/\beta'} \lesssim \left(\sum_{j=0}^{N}\|A_k u\|_{L^{\beta'}(I_j,D(A^s))}^{\beta'}\right)^{1/\beta'} \cong \|A_k u\|_{L^{\beta'}(I,D(A^s))}.$$

□

In the following two results we are going to exploit the a priori estimate (II.54) from Lemma II.4.3. It should be no surprise that both of them require a summation over all $k \in \mathbb{N}_0$. In order to deal with the expression in (II.54) involving the nonlinearity F, it is convenient to introduce the following function:

$$S_F : L^\infty(I, D(A^s)) \to [0,\infty], \quad S_F(u) := \sum_{k=1}^{\infty} 2^{k(\frac{\alpha}{q}+\frac{\gamma}{p}-\frac{\gamma}{p^*})}\|A_k F(u)\|_{L^\infty(I,L^{q^*}(\Omega))}. \tag{II.60}$$

In the applications of the results of this section in Chapter III we will discuss this expression and prove suitable bounds for it. For now we just accept it as one building block of the proofs below.
Let us formulate the first result, which gives a criterion for two weak solutions of (CPA) to be equal.

Theorem II.4.4
Besides (A1)-(A3) *let* $s \geq \frac{\alpha}{q} + \frac{\gamma}{p}$ *and* $\beta \in [1,2]$. *We additionally assume:*

(i) $(A, D(A))$ *satisfies* (q, b, α)*-Bernstein inequalities for all* $b \in [q, \infty)$.

(ii) $F : \mathbb{C} \to \mathbb{C}$ *satisfies* $F(0) = 0$ *and* $|F(z) - F(w)| \lesssim (|z|^\beta + |w|^\beta)|z - w|$.

If $u, v \in L^\infty(I, D(A^s))$ *are weak solutions of* (CPA) *with* $\max\{S_F(u), S_F(v)\} < \infty$, *then* $u = v$ *on* I.

Remarks:

(1) Recall that in (A1) we assumed $E_A \hookrightarrow L^r(\Omega)$ with $r > q \geq 2$. Then $D(A^s) \hookrightarrow L^r(\Omega)$ and by complex interpolation we have $D(A^s) \hookrightarrow L^{\tilde r}(\Omega)$ for all $\tilde r \in [2, r]$.

(2) The restriction $\beta \leq 2$ is essential for the method of proof and we do not know how to get rid of this restriction.

(3) If all weak solutions $u \in L^\infty(I, D(A^s))$ of (CPA) satisfy $S_F(u) < \infty$, then each weak solution $u : I \to D(A^s)$ of (CPA) is unconditionally unique.

Proof. Let $u, v \in L^\infty(I, D(A^s)) \cap W^1_\infty(I^o, E_A^*)$ be two weak solutions of (CPA) on I with $\max\{S_F(u), S_F(v)\} < \infty$. Then $u, v \in C(I, L^2(\Omega))$ by Proposition I.2.7;(b) and $u(t_0) = v(t_0)$. Throughout the proof we fix $R \in [0,\infty)$ such that

$$\|u\|_{L^\infty(I,D(A^s))} + \|v\|_{L^\infty(I,D(A^s))} + S_F(u) + S_F(v) \leq R. \tag{II.61}$$

The proof is divided into two parts. In (a) we provide suitable a priori estimates for the weak solutions u, v in spaces of the form $L^p(J, L^q(\Omega))$ with $J \subseteq I$. In (b) we use these estimates for a contradiction argument to prove $\|u(t) - v(t)\|_{L^2(\Omega)} = 0$ on I.

(a) Let $J \in \mathcal{I}_b$ with $J \subseteq I$ and $b \in [q,\infty)$. Minkowski's integral inequality and the (q,b,α)-Bernstein inequalities from *(i)* yield

$$\begin{aligned} \|u\|_{L^\beta(J,L^b(\Omega))} &\le \|A_0 u\|_{L^\beta(J,L^b(\Omega))} + \sum_{k=1}^{\infty} \|A_k u\|_{L^\beta(J,L^b(\Omega))} \\ &\le C(|I|)\Big(\|u\|_{L^\infty(I,D(A^s))} + \sum_{k=1}^{\infty} 2^{k\alpha(\frac{1}{q}-\frac{1}{b})} \|A_k u\|_{L^\beta(J,L^q(\Omega))}\Big). \end{aligned} \tag{II.62}$$

Since $\beta \le 2 \le p$ we can apply estimate (II.54) for $k \in \mathbb{N}$ to get

$$\|A_k u\|_{L^\beta(J,L^q(\Omega))} \le C(|I|,R)\Big(2^{-k\rho(s)}\|A_k u\|_{L^2(J,D(A^s))} + 2^{-k\rho(\frac{\gamma}{p^*})}\|A_k F(u)\|_{L^\infty(I,L^{q^*}(\Omega))}\Big),$$

with $\rho(x) = x - \frac{\gamma}{p}$. The condition $s \ge \frac{\gamma}{p} + \frac{\alpha}{q}$ implies $\rho(s) - \frac{\alpha}{q} \ge 0$. The previous estimate and (II.62) then yield

$$\begin{aligned} \|u\|_{L^\beta(J,L^b(\Omega))} &\le C(|I|,R)\Big(\sum_{k=1}^{\infty} 2^{-k(\frac{\alpha}{b}+\rho(s)-\frac{\alpha}{q})}\|A_k u\|_{L^2(J,D(A^s))} + S_F(u) + 1\Big) \\ &\le C(|I|,R)\Big(\sum_{k=1}^{\infty} 2^{-\frac{k\alpha}{b}}\|A_k u\|_{L^2(J,D(A^s))} + 1\Big). \end{aligned} \tag{II.63}$$

We observe that

$$\frac{1}{b}\sum_{k=1}^{\infty} 2^{-\frac{2k\alpha}{b}} = \frac{1}{b(1-2^{-2\alpha/b})} - \frac{1}{b} \overset{b\to\infty}{\longrightarrow} \frac{1}{2\alpha\log(2)}$$

implies the boundedness of the first expression uniformly in $b \in [q,\infty)$. Continuing the estimation in (II.63) with Hölder's inequality yields

$$\begin{aligned} \|u\|_{L^\beta(J,L^b(\Omega))} &\le C(|I|,R)\Big(b^{\frac{1}{2}}\Big(\frac{1}{b}\sum_{k=1}^{\infty} 2^{-\frac{2k\alpha}{b}}\Big)^{1/2}\Big(\sum_{k=1}^{\infty}\|A_k u\|^2_{L^2(J,D(A^s))}\Big)^{1/2} + 1\Big) \\ &\le C(|I|,R)\Big(b^{\frac{1}{2}}\|u\|_{L^2(J,D(A^s))} + 1\Big) \\ &\le C(|I|,R)\big((|J|b)^{\frac{1}{2}} + 1\big). \end{aligned}$$

We have chosen R such that (II.61) holds, so that the previous estimate also holds for v instead of u. We conclude for $b \in [q,\infty)$

$$\|u\|^\beta_{L^\beta(J,L^b(\Omega))} + \|v\|^\beta_{L^\beta(J,L^b(\Omega))} \le C(|I|,R)\big(|J|^{\beta/2}b + 1\big). \tag{II.64}$$

With this estimate at our disposal we proceed to the next step.
(b) Let $g : I \to \mathbb{R}$ be given by $g(t) := \|u(t)-v(t)\|^2_{L^2(\Omega)}$. Then $g(t_0) = 0$ and Proposition I.2.7;(b) provides $g \in C_b(I) \cap W^1_\infty(I^o)$ with $g' = 2\langle u'-v', u-v\rangle$. We assume the following:

$$\exists_{t_2 \in I^o\setminus\{t_0\}} : \; g(t_2) > 0. \tag{II.65}$$

Without loss of generality we assume $t_2 > t_0$ as the following argument also works in the case $t_2 < t_0$ without any change. $g \in C(I)$ and $g(t_0) = 0$ imply

$$\exists_{t_1 \in [t_0,t_2)} : \; g(t_1) = 0 \quad \wedge \quad \forall_{t\in(t_1,t_2)} : \; g(t) > 0. \tag{II.66}$$

With the observation $\beta \le 2 < q$ we let $\nu \in (1, \min\{\frac{r}{2}, 1 + \frac{\beta}{q-\beta}\})$ be arbitrary. We use (CPA) and *(ii)* to generate for almost all $t \in (t_1, t_2)$ the inequality

$$\begin{aligned} g'(t) &= 2\langle \mathrm{i}u'(t) - \mathrm{i}v'(t), \mathrm{i}u(t) - \mathrm{i}v(t)\rangle \\ &= 2\Big(\langle A^{1/2}[u(t) - v(t)], \mathrm{i}A^{1/2}[u(t) - v(t)]\rangle_{L^2(\Omega)} \\ &\qquad + \langle F(u(t)) - F(v(t)), \mathrm{i}[u(t) - v(t)]\rangle_{L^2(\Omega)}\Big) \\ &\lesssim \int_\Omega |F(u(t)) - F(v(t))||u(t) - v(t)| \,\mathrm{d}\mu \\ &\lesssim \int_\Omega (|u(t)|^\beta + |v(t)|^\beta)|u(t) - v(t)|^2 \,\mathrm{d}\mu \\ &\lesssim (\|u(t)\|^\beta_{L^{\beta\nu^*}(\Omega)} + \|v(t)\|^\beta_{L^{\beta\nu^*}(\Omega)})\|u(t) - v(t)\|^2_{L^{2\nu}(\Omega)}. \end{aligned}$$

Then we choose $\theta \in (0, 1)$ such that $\frac{1}{2\nu} = \frac{1-\theta}{2} + \frac{\theta}{r}$ so that complex interpolation yields for all $t \in (t_1, t_2)$

$$\begin{aligned} \|u(t) - v(t)\|^2_{L^{2\nu}(\Omega)} &\lesssim \|u(t) - v(t)\|^{2(1-\theta)}_{L^2(\Omega)} \|u(t) - v(t)\|^{2\theta}_{L^r(\Omega)} \\ &\lesssim g(t)^{1-\theta} \|u(t) - v(t)\|^{2\theta}_{D(A^s)} \\ &\le C(R) g(t)^{1-\theta}, \end{aligned}$$

with $C(R) \ge 1$. Consequently, for almost all $t \in (t_1, t_2)$ holds

$$g'(t) g(t)^{\theta-1} \le C(R)\big(\|u(t)\|^\beta_{L^{\beta\nu^*}(\Omega)} + \|v(t)\|^\beta_{L^{\beta\nu^*}(\Omega)}\big). \tag{II.67}$$

Now we put $J_\epsilon := (t_1, t_1 + \epsilon)$ with some $\epsilon \in (0, t_2 - t_1)$. The weak chain rule in Theorem 7.8 of [GT01] yields that $g^\theta \in W^1_{\infty,loc}(J_\epsilon)$ with $(g^\theta)' = \theta g^{\theta-1} g'$ almost everywhere on J_ϵ. By means of A.3.5;(1) then follows that for almost all $t, s \in J_\epsilon$ holds

$$g(t)^\theta - g(s)^\theta = \theta \int_s^t g'(\tau) g(\tau)^{\theta-1} \,\mathrm{d}\tau. \tag{II.68}$$

Then there is $t \in J_\epsilon$ and a sequence $(s_n)_{n\in\mathbb{N}}$ in J_ϵ with $s_n \overset{n\to\infty}{\longrightarrow} t_1$ such that (II.67) and (II.64) (note $\beta\nu^* \ge q$ since $\nu < 1 + \frac{\beta}{q-\beta}$) imply

$$\begin{aligned} g(t)^\theta - g(s_n)^\theta &= \theta \int_{s_n}^t g'(\tau) g(\tau)^{\theta-1} \,\mathrm{d}\tau \\ &\le \frac{C(|I|, R)}{\nu^*}\left(\|u\|^\beta_{L^\beta(J_\epsilon, L^{\beta\nu^*}(\Omega))} + \|v\|^\beta_{L^\beta(J_\epsilon, L^{\beta\nu^*}(\Omega))}\right) \le C(|I|, R)\left(\epsilon^{\beta/2} + \frac{1}{\nu^*}\right). \end{aligned}$$

Since $g(s_n) \overset{n\to\infty}{\longrightarrow} 0$ we also have

$$g(t)^\theta \le C(|I|, R)\left(\epsilon^{\beta/2} + \frac{1}{\nu^*}\right). \tag{II.69}$$

Now let $\epsilon \in (0, \min\{t_2 - t_1, C(|I|, R)^{-2/\beta}\})$. Then (II.69) and $\frac{1}{\theta\nu^*} = 1 - \frac{2}{r}$ yields

$$g(t)^{\frac{r}{r-2}} \le \left(C(|I|, R)(\epsilon^{\beta/2} + \frac{1}{\nu^*})\right)^{\nu^*} = (C(|I|, R)\epsilon^{\beta/2})^{\nu^*}\left(1 + \frac{1}{\epsilon^{\beta/2}\nu^*}\right)^{\nu^*} \overset{\nu^*\to\infty}{\longrightarrow} 0.$$

Hence, $g(t) = 0$ with $t \in (t_1, t_2)$, what clearly contradicts (II.66). (II.65) must therefore be false and consequently $u = v$ on I. □

The next result deals with a priori estimates for the strong solutions of (CPA) constructed by means of Theorem I.3.4 in $L^\infty(I, D(A^s))$ with $s > 1/2$. For $s = 1/2$ we can deduce such bounds with $L^2(\Omega)$- and energy conservation. Indeed, Lemma I.3.5 provides a criterion for a strong solution u to satisfy $\|u\|_{L^\infty(I,E_A)} < \infty$. Such an estimate is one of the crucial ingredients of the following proof.

Theorem II.4.5
Besides (A1)-(A3) *let* $s \in (1/2, \infty)$, $R \in [0, \infty)$, *and* $\beta \in [1, p]$. *We additionally assume:*

(i) $(A, D(A))$ *satisfies* (q, ∞, α)*-Bernstein inequalities.*

(ii) $\|F(g)\|_{D(A^s)} \lesssim (1 + \|g\|^\beta_{L^\infty(\Omega)})\|g\|_{D(A^s)}$ *for all* $g \in D(A^s) \cap L^\infty(\Omega)$.

(iii) $\frac{1}{2} \geq \frac{\gamma}{p} + \frac{\alpha}{q}$ *if* $\beta \leq 2$ *and "$>$" in the previous inequality if* $\beta > 2$.

Then there is an increasing $C : [0, \infty) \to [0, \infty)$ *such that for all strong solutions* $u \in C(I, D(A^s)) \cap L^\beta_{loc}(I, L^\infty(\Omega))$ *of* (CPA) *on* I *with* $\|u\|_{L^\infty(I,E_A)} + S_F(u) \leq R$ *holds:*

(a) There is a decreasing $T \in C([0, \infty), (0, 1])$ *such that for all* $t \in I$ *holds*

$$\|u\|_{L^\infty([t-T(R),t+T(R)]\cap I, D(A^s))} \leq C(R)\big(\|u(t)\|_{D(A^s)} + \|u(t)\|^2_{D(A^s)}\big). \tag{II.70}$$

(b) $u \in L^\infty(I, D(A^s))$ *and* $\|u\|_{L^\beta(I,L^\infty(\Omega))} \leq C(R, |I|) \log(\exp(1) + \|u\|_{L^\infty(I,D(A^s))})$.

Proof. **(a;1)** Let $t_1 \in I$, $T \in (0, 1]$ and define $J := [t_1 - T, t_1 + T] \cap I$. We first prove a logarithmic estimate for $\|u\|_{L^\beta(J,L^\infty(\Omega))}$ similar to the estimate in *(b)*, which will enable us in (a;2) to use Gronwall's lemma for the desired bound on $\|u\|_{L^\infty(J,D(A^s))}$ if T is suitably small.
The (q, ∞, α)-Bernstein inequalities from *(i)* and (II.54) imply for all $k \in \mathbb{N}$ and $\tilde{s} \in [1/2, \infty)$ the estimate

$$\begin{aligned}\|A_k u\|_{L^\beta(J,L^\infty(\Omega))} &\lesssim 2^{\frac{k\alpha}{e}} \|A_k u\|_{L^\beta(J,L^e(\Omega))} \\ &\lesssim 2^{2k\epsilon(\tilde{s})}\|A_k u\|_{L^{\max\{\beta,2\}}(J,D(A^{\tilde{s}}))} + 2^{2k\epsilon(\frac{\gamma}{p^*})}\|A_k F(u)\|_{L^\infty(I,L^{q^*}(\Omega))},\end{aligned} \tag{II.71}$$

with $\epsilon(x) := \frac{1}{2}(\frac{\alpha}{q} + \frac{\gamma}{p} - x)$. Then $\epsilon(s) < \epsilon(1/2) \leq 0$ by *(iii)*. Let $l \in \mathbb{N}$. If $\beta \leq 2$, we use the triangle inequality, (II.71), and $E_A \hookrightarrow L^q(\Omega)$ from (A1) to produce

$$\begin{aligned}&\|u\|_{L^\beta(J,L^\infty(\Omega))} \\ &\leq \|A_0 u\|_{L^\beta(J,L^\infty(\Omega))} + \sum_{k=1}^\infty \|A_k u\|_{L^\beta(J,L^\infty(\Omega))} \\ &\lesssim \|u\|_{L^\infty(I,E_A)} + \sum_{k=1}^{l} 2^{2k\epsilon(\frac{1}{2})}\|A_k u\|_{L^2(J,E_A)} + \sum_{k=l}^\infty 2^{2k\epsilon(s)}\|A_k u\|_{L^2(J,D(A^s))} + S_F(u).\end{aligned}$$

By means of $\|u\|_{L^\infty(I,E_A)} + S_F(u) \leq R$, $\epsilon(s) < \epsilon(1/2) \leq 0$, and Hölder's inequality we continue the previous estimate as follows:

$$\begin{aligned}
\|u\|_{L^\beta(J,L^\infty(\Omega))} &\leq C(R)\Big(\sum_{k=1}^{l}\|A_k u\|_{L^2(J,E_A)} + 2^{l\epsilon(s)}\sum_{k=l}^{\infty}2^{k\epsilon(s)}\|A_k u\|_{L^2(J,D(A^s))} + 1\Big)\\
&\leq C(R)\Big(l^{\frac{1}{2}}\Big(\sum_{k=1}^{l}\|A_k u\|^2_{L^2(J,E_A)}\Big)^{1/2} + 2^{l\epsilon(s)}\Big(\sum_{k=l}^{\infty}\|A_k u\|^2_{L^2(J,D(A^s))}\Big)^{1/2} + 1\Big)\\
&\leq C(R)\Big((lT)^{\frac{1}{2}}\|u\|_{L^\infty(J,E_A)} + 2^{l\epsilon(s)}\|u\|_{L^2(J,D(A^s))}\Big)\\
&\leq C(R)\Big(l^{\frac{1}{\beta}}T^{\frac{1}{2}} + 2^{l\epsilon(s)}\|u\|_{L^\infty(J,D(A^s))}\Big). \qquad \text{(II.72)}
\end{aligned}$$

If $\beta > 2$ we have $\epsilon(1/2) < 0$ by *(iii)* so that similarly

$$\begin{aligned}
\|u\|_{L^\beta(J,L^\infty(\Omega))} &\leq C(|I|,R)\Big(\sum_{k=1}^{\infty}2^{2k\epsilon(\frac{1}{2})}\|A_k u\|_{L^\beta(J,E_A)} + 1\Big)\\
&\leq C(R)\Big(\Big(\sum_{k=1}^{\infty}2^{2k\epsilon(\frac{1}{2})\beta^*}\Big)^{1/\beta^*}\Big(\sum_{k=1}^{\infty}\|A_k u\|^\beta_{L^\beta(J,E_A)}\Big)^{1/\beta} + 1\Big)\\
&\leq C(R)\big(\|u\|_{L^\beta(J,E_A)} + 1\big) \leq C(R)\big((lT)^{\frac{1}{\beta}} + 1\big). \qquad \text{(II.73)}
\end{aligned}$$

With $K := [\frac{1}{-\epsilon(s)\log(2)}] + 1$ it is straightforward to check that the function

$$H : [0,\infty) \to [0,\infty), \qquad H(x) := 2^{K\epsilon(s)\log(\exp(1)+x)}x$$

is bounded. In the estimates (II.72) and (II.73) we fix $l := [K\log(\exp(1) + \|u\|_{L^\infty(J,D(A^s))})]$ to ensure

$$2^{K\epsilon(s)\log(\exp(1)+\|u\|_{L^\infty(J,D(A^s))})}\|u\|_{L^\infty(J,D(A^s))} \leq \|H\|_{L^\infty([0,\infty))}.$$

We therefore have

$$\|u\|^\beta_{L^\beta(J,L^\infty(\Omega))} \leq C_1(R)\Big(\log(\exp(1) + \|u\|_{L^\infty(J,D(A^s))})T^{\frac{\beta}{2}} + 1\Big), \qquad \beta \leq 2, \qquad \text{(II.74)}$$

$$\|u\|^\beta_{L^\beta(J,L^\infty(\Omega))} \leq C_1(R)\Big(\log(\exp(1) + \|u\|_{L^\infty(J,D(A^s))})T + 1\Big), \qquad \beta > 2. \qquad \text{(II.75)}$$

We can choose $C_1 \in C([0,\infty))$ increasing with $C_1(x) \geq 1$ for all $x \in [0,\infty)$.
(a;2) Since u is a strong solution of (CPA) on I, it satisfies Duhamel's formula

$$\forall_{t\in I}: \ u(t) = U(t-t_1)u(t_1) - \mathrm{i}\int_{t_1}^{t} U(t-\tau)F(u(\tau))\,\mathrm{d}\tau.$$

Combined with the estimate for F in *(ii)* this implies for all $t \in I$

$$\begin{aligned}
\|u(t)\|_{D(A^s)} &\leq \|u(t_1)\|_{D(A^s)} + \int_{t_1}^{t}\|F(u(\tau)\|_{D(A^s)}\,\mathrm{d}\tau\\
&\leq \|u(t_1)\|_{D(A^s)} + C_2\int_{t_1}^{t}(1 + \|u(\tau)\|^\beta_{L^\infty(\Omega)})\|u(\tau)\|_{D(A^s)}\,\mathrm{d}\tau,
\end{aligned}$$

with some $C_2 \geq 1$. The Lemma of Gronwall formulated in Lemma 4.2.1 in [CH98] implies

$$\begin{aligned}\|u(t)\|_{D(A^s)} &\leq \|u(t_1)\|_{D(A^s)} \exp\left(C_2 \int_{t_1}^{t} (1 + \|u(\tau)\|^{\beta}_{L^\infty(\Omega)})\, d\tau\right) \\ &\lesssim \|u(t_1)\|_{D(A^s)} \exp\left(C_2 \|u\|^{\beta}_{L^\beta(J,L^\infty(\Omega))}\right). \end{aligned} \tag{II.76}$$

We put $T(R) := (2C_1(R)C_2)^{-2/\beta}$. Then $T(\cdot)$ is a continuous decreasing function. If $\beta \leq 2$ and $T := T(R)$, then (II.74) and (II.76) yield

$$\begin{aligned}\|u\|_{L^\infty(J,D(A^s))} &\leq C(R)\|u(t_1)\|_{D(A^s)} \exp\left(\log(\exp(1) + \|u\|_{L^\infty(J,D(A^s))}) T^{\frac{\beta}{2}} C_1(R) C_2 \right) \\ &= C(R)\|u(t_1)\|_{D(A^s)} \left(1 + \|u\|_{L^\infty(J,D(A^s))}\right)^{1/2}.\end{aligned}$$

If $\beta > 2$ and $T := T(R)^{\beta/2}$, then (II.75) and (II.76) similarly yield

$$\|u\|_{L^\infty(J,D(A^s))} \leq C(R)\|u(t_1)\|_{D(A^s)} \left(1 + \|u\|_{L^\infty(J,D(A^s))}\right)^{1/2}.$$

In the previous two estimates $C(R) \cong \exp(C_1(R)C_2)$. Hence, with the above choices for T we have in case $\|u\|_{L^\infty(J,D(A^s))} > 0$ that

$$\begin{aligned}\|u\|_{L^\infty(J,D(A^s))} &\leq C(R)\|u(t_1)\|^2_{D(A^s)} \left(\frac{1}{\|u\|_{L^\infty(J,D(A^s))}} + 1 \right) \\ &\leq C(R) \left(\|u(t_1)\|_{D(A^s)} + \|u(t_1)\|^2_{D(A^s)} \right). \end{aligned} \tag{II.77}$$

This is trivial if $\|u\|_{L^\infty(J,D(A^s))} = 0$, and therefore the proof of (II.70) is finished.
(b) We take T from **(a;2)** and let $(I_j)_{j=0}^N$ be the T-partition of I where $t_j \in I_j^o$ denotes the center of I_j for $j \in \{0, ..., N\}$. Then (II.77) yields for $j \in \{0, ..., N\}$

$$\|u\|_{L^\infty(I_j,D(A^s))} \leq C(R) \left(\|u(t_j)\|_{D(A^s)} + \|u(t_j)\|^2_{D(A^s)} \right),$$

and we therefore have

$$\|u\|_{L^\infty(I,D(A^s))} \leq C(R) \max_{j\in\{0,...,N\}} \left(\|u(t_j)\|_{D(A^s)} + \|u(t_j)\|^2_{D(A^s)} \right) < \infty.$$

Finally, (II.74), (II.75) and $T \leq 1$ imply the estimate in *(b)* by

$$\begin{aligned}\|u\|_{L^\beta(I,L^\infty(\Omega))} &= \left(\sum_{j=0}^{N} \|u\|^{\beta}_{L^\beta(I_j,L^\infty(\Omega))} \right)^{1/\beta} \\ &\leq C(|I|, R) \left(\log(\exp(1) + \|u\|_{L^\infty(I,D(A^s))})^{\frac{1}{\beta}} (T^{\frac{1}{2}} + T^{\frac{1}{\beta}}) + 1 \right) \\ &\leq C(|I|, R) \log(\exp(1) + \|u\|_{L^\infty(I,D(A^s))}).\end{aligned}$$

□

III. Local and global existence results for the nonlinear Schrödinger equation

This chapter is devoted to the application of the existence results provided in Section I.3 and the refined results of Section II.4 to the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t) \pm |u(t)|^{\beta} u(t), \quad t \neq 0, \\ u(0) &= f, \end{aligned} \tag{III.1}$$

in various situations. It is organized as follows.

In Section III.1 we first provide some background material on Riemannian manifolds (Ω, g) with bounded geometry. We recall a useful characterization of Sobolev spaces defined on such manifolds, which will allow us to prove the crucial nonlinear estimates for an application of Theorem I.3.4. We first deduce a *local* existence result in $H_2^s(\Omega)$ for strong solutions of (III.1) where $(A, D(A))$ is the Laplace-Beltrami operator on Ω. This result will also be applied in Chapter IV.

On the basis of the local existence result in $H_2^s(\Omega)$ we prove a *global* existence result in $H_2^1(\Omega)$ for the defocusing nonlinear Schrödinger equation in dimension $d \in \{2, 3\}$. One of the the crucial assumptions in this result are (p, q) Strichartz estimates of SL-type so that the results of Section II.4 are available. This global existence result shows how the global existence theory relies on the quality of the (p, q) Strichartz estimates of SL-type. It contains the global existence results of [BGT04b] as a special case.

In the remaining Sections III.2-III.4 we show that Theorem I.3.4 contains known local and global existence results from the literature. We consider the following situations:

III.2: $A = -\operatorname{div}(B(\cdot)\nabla) + V$ on $\mathbb{R}^d$ with diagonal $B \in C_b^{\infty}(\mathbb{R}^d, \mathbb{R}^{d\times d})$ and $V \in C_b^1(\mathbb{R}^d)$ (see the appendix of [BGT04b] for $V = 0$).

III.3: $A = -\Delta + V$ on $\mathbb{R}^d$ with superquadratic potential (see [YZ04]).

III.4: $A = -\Delta_D$ on a polygonal or C^{∞}-domain in $\mathbb{R}^2$ with homogeneous Dirichlet boundary conditions (see e.g. [BFHM12, Ant08])

In the corresponding section, each of the existence theorems given in the references above will be compared to the results that we deduce by means of Theorem I.3.4. In some cases we can even slightly extend the existing results.

III.1. The Laplace-Beltrami operator on C^{∞}-manifolds with bounded geometry

In this section let (Ω, g) always be a connected, Riemannian C^{∞}-manifold without boundary and $\dim(\Omega) = d \in \mathbb{N}$. We always consider (Ω, g) to be equipped with the

Levi-Civita connection. For details and some standard notation used throughout this thesis we refer to Appendix A.4. We aim at a treatment of the nonlinear Schrödinger equation formulated on manifolds with bounded geometry. We therefore introduce the following property and discuss some useful consequences of it afterward.

(M) The injectivity radius satisfies $\operatorname{inj}(\Omega) \in (0, \infty]$ and (Ω, g) has bounded geometry, i.e. for all $\alpha \in \mathbb{N}_0^d$ and $k, l \in \{1, ..., d\}$ there is $C \in (0, \infty)$ such that $|\partial^\alpha g_{k,l}| \leq C$

Let us gather some properties of manifolds (Ω, g) which satisfy (M). We refer to Section 7.2.1 of [Tri92b] for the proof of the assertions in (M1) and (M2) below and further details.

(M1) There is $r_0 \in (0, \operatorname{inj}(\Omega))$ such that for $r \in (0, {r_0}/{4})$ there is an at most countable and uniformly locally finite geodesic atlas $\mathscr{A}(r) = \{(O_i(r), \kappa_i) \mid i \in I\}$, i.e.

$$\sup_{i \in I} \left(\#\{j \in I \mid O_j(r) \cap O_i(r)\}\right) < \infty. \tag{III.2}$$

Recall that κ_i is defined via the inverse of the exponential map and $O_i(r)$ is a geodesic ball (for details see A.12).

(M2) There is a smooth partition of unity $(\psi_{i,r})_{i \in I}$ with the following properties:

- $\psi_{i,r} \in C^\infty(\Omega, [0,1])$ with $\operatorname{supp}(\psi_{i,r}) \subseteq O_i(r)$ and $\sum_{i \in I} \psi_{i,r} = \operatorname{id}$.
- For all $\beta \in \mathbb{N}_0^d$ there is $C \in (0, \infty)$ such that $\|\partial^\beta \psi_{i,r}\|_{L^\infty(O_i(r))} \leq C$.

We say that $(\psi_{i,r})_{i \in I}$ is subordinate to $\mathscr{A}(r)$.

(M3) For $i \in I$ we define

$$J(i) := \{j \in \mathbb{N} \mid \operatorname{supp}(\psi_{r,i}) \cap \operatorname{supp}(\psi_{r,j}) \neq \emptyset\}.$$

By means of (III.2) we have $\sup_{i \in I}(\#J(i)) < \infty$. By means of $r < {r_0}/{4} < \operatorname{inj}(\Omega)$ for all $j \in J(i)$ follows $\operatorname{supp}(\psi_{j,r}) \subseteq O_j(r) \subseteq O_i(r_0)$. This implies that we can treat all localizations to a chart $O_j(r)$ with $j \in J(i)$ in the local chart $O_i(r_0)$.

Convention
From now on we always let (Ω, g) be a connected, (geodesically) complete Riemannian C^∞-manifold without boundary and $\dim(\Omega) = d \in \mathbb{N}$. If (Ω, g) satisfies (M), *then we consider (Ω, g) to be equipped with a geodesic atlas $\mathscr{A}(r)$ in* (M1) *with $r \in (0, {r_0}/{4})$. In that case $(\psi_{i,r})_{i \in I}$ denotes the smooth partition of unity subordinate to $\mathscr{A}(r)$ from* (M2).

Examples III.1.1
The following examples are the most relevant ones in view of our applications.

(1) $(\mathbb{R}^d, \operatorname{id})$ is complete and satisfies (M).

(2) Let (Ω, g) be a connected Riemmanian C^∞-manifold, which is additionally compact. Then (Ω, g) is complete and (M) is satisfied. Indeed, the Hopf-Rinow Theorem formulated in Theorem 1.37 in Section 1.4 in [Aub98] implies that Ω is complete. Theorem 1.36 in [Aub98] shows that $\operatorname{inj}(\Omega) > 0$. Since (Ω, g) is compact we can always choose a finite atlas for (Ω, g) and therefore the geometry is bounded.

(3) In Lemma IV.2.1 of Section IV.2 we will show that $\mathbb{R}^n \times M$ with a connected, compact Riemannian C^∞-manifold (M, g) without boundary is complete and satisfies (M).

We continue with the definition of $L^p(\Omega)$ and the introduction of the Laplace-Beltrami operator. First, we need an integral and for $S \in \mathscr{B}(\Omega)$ we put

$$\mu(S) := \sum_{i \in I} \int_{\kappa(S \cap O_i)} \left((\psi_{r,i} \det(G)^{1/2}) \circ \kappa^{-1} \right)(z) \, \mathrm{d}z.$$

This definition is well known to be independent of the atlas and the partition of unity. μ is furthermore a Radon measure and therefore induces an integral which satisfies all the properties of the Lebesgue integral on euclidean spaces (for details see Section 3.4 in [Aub98]). Hence, the $L^p(\Omega)$ spaces with respect to μ for $p \in [1, \infty]$ are Banach spaces. $L^2(\Omega)$ equipped with the inner product

$$(f, g)_{L^2(\Omega)} := \int_\Omega f \overline{g} \, \mathrm{d}\mu$$

is a Hilbert space.
The operator we want to consider is the canonical generalization of the Laplacian on euclidean space to Riemannian manifolds. For $f \in C_c^\infty(\Omega)$ and a local chart (O, κ) we define the differential expression Δ by

$$\Delta f := \sum_{k,l=1}^{d} g^{k,l} \left(\partial_{\omega_k} \partial_{\omega_l} - \sum_{m=1}^{d} \Gamma_{k,l}^m \partial_{\omega_m} \right) f. \tag{III.3}$$

Here $(g^{k,l})_{k,l=1,\dots,d} = (g_{k,l})_{k,l=1,\dots,d}^{-1}$ and $\Gamma_{k,l}^m$ denotes the Christoffel symbols from (A.10). In [Str83] it was shown that $(-\Delta, C_c^\infty(\Omega))$ is a non-negative, essentially selfadjoint linear operator on $L^2(\Omega)$. Then there is a unique non-negative selfadjoint extension

$$(-\Delta_\Omega, D(-\Delta_\Omega)), \qquad D(-\Delta_\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{D(-\Delta)}}. \tag{III.4}$$

Theorem 3.5 in [Str83] additionally states that the heat semigroup $(e^{t\Delta_\Omega})_{t \in [0,\infty)}$ given by the spectral calculus on $L^2(\Omega)$, can be extended to a contraction semigroup on $L^q(\Omega)$ for all $q \in [1, \infty]$. With this extended semigroup the fractional powers $(-\Delta_\Omega)_q^\alpha$ on $L^q(\Omega)$ for $\alpha \in [0, \infty) \setminus \mathbb{N}_0$ and $q \in (1, \infty)$ can be defined via

$$D((-\Delta_\Omega)_q^\alpha) := \left\{ f \in L^q(\Omega) \mid \lim_{\epsilon \to 0} \int_\epsilon^\infty t^{-\alpha-1} e^{t\Delta_\Omega} f \, \mathrm{d}t \text{ exists} \right\},$$

$$(-\Delta_\Omega)_q^\alpha f := \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} e^{t\Delta_\Omega} f \, \mathrm{d}t \quad \text{on } D((-\Delta_\Omega)_q^\alpha).$$

This defines a closed linear operator on $L^q(\Omega)$. For $q = 2$ this definition coincidces with the fractional powers defined by the spectral calculus of $(-\Delta_\Omega, D(-\Delta_\Omega))$. In the following theorem we gather some important properties of the introduced operators. We in particular provide a convenient characterization of the fractional domains of the Laplace-Beltrami operator via pullback of the corresponding fractional domains of the Laplacian on $\mathbb{R}^d$.

Theorem III.1.2
Let $s, s_0, s_1 \in [0,\infty)$, $p, p_0, p_1 \in (1,\infty)$, and $k \in \mathbb{N}$. If (Ω, g) satisfies (M)*, then the space*

$$H_p^s(\Omega) := \left\{ f \in L^p(\Omega) \mid \|f\|_{H_p^s(\Omega)} := \left(\sum_{i\in I} \|(\psi_{i,r} f) \circ \kappa_i^{-1}\|_{H_p^s(\mathbb{R}^d)}^p \right)^{1/p} < \infty \right\}.$$

has the following properties:

(a) $H_p^s(\Omega)$ *is a Banach space and independent of the geodesic atlas* $\mathscr{A}(r)$ *and the smooth partition of unity* $(\psi_{i,r})_{i\in I}$.

(b) $H_p^s(\Omega) \cong D((-\Delta_\Omega)_p^{s/2})$ *and* $C_c^\infty(\Omega)$ *is dense in* $H_p^s(\Omega)$.

(c) $[H_{p_1}^{s_1}(\Omega), H_{p_2}^{s_2}(\Omega)]_\theta \cong H_p^s(\Omega)$ *with* $s = (1-\theta)s_1 + \theta s_2$ *and* $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *for all* $\theta \in (0,1)$.

(d) Let $e(d,s) := 2d/(\max\{d-2s,0\})$ *with* $2d/0 := \infty$*. The following embeddings hold:*

(d1) $H_p^s(\Omega) \hookrightarrow L^\infty(\Omega)$ *if* $s > \frac{d}{p}$.

(d2) $H_2^s(\Omega) \hookrightarrow L^q(\Omega)$ *if either* $q \in [2, e(d,s))$ *or* $d > 2s$ *and* $q = e(d,s)$*. These embeddings are dense.*

(e) If $q \in [2, e(d,1))$*, then any weakly convergent sequence in* $H_2^1(\Omega)$ *is convergent in* $L^q(O_i(r))$ *for all* $i \in I$.

Remarks:

(1) In Theorem 7.4.5 in [Tri92b] it is shown that for $k \in \mathbb{N}_0$ and $p \in (1,\infty)$ holds $H_p^k(\Omega) \cong W_p^k(\Omega)$. $W_p^k(\Omega)$ is the classical Sobolev space defined via covariant derivatives.

(2) *(b)* includes the assertion $D(-\Delta_\Omega) \cong H_2^2(\Omega)$ and $(-\Delta_\Omega, H_2^2(\Omega))$ is therefore non-negative and selfadjoint on $L^2(\Omega)$.

(3) The assertion in *(e)* is needed for an application of Theorem I.3.6 (see I.3.6;(iii)). Recall that the geodesic atlas $\mathscr{A}(r)$ with $r \in (0, r_0/4)$ is at most countable.

Proof. Chapter 7 in [Tri92b] contains the assertions *(a)*, *(c)* and the density claim in *(b)*. Combined with the results of [Str83] also $D((-\Delta_\Omega)_p^{s/2}) \cong H_p^s(\Omega)$ follows. It remains to show *(d)* and *(e)*.
(d1) We let $\delta \in (0, 1/2)$ such that $s - \frac{d}{p} \geq 2\delta > 0$ and $f \in H_p^s(\Omega)$. Then Theorem 7.4.2;(ii)+(iv) in [Tri92b] implies $H_p^s(\Omega) = F_{p,2}^s(\Omega) \hookrightarrow F_{p,\infty}^s(\Omega) \hookrightarrow B_{p,\infty}^{s-\delta}(\Omega) \hookrightarrow C^{0,\delta}(\Omega)$ and

$$\|f\|_{C^{0,\delta}(\Omega)} = \sup_{i\in I} \|(\psi_{r,i} f) \circ \kappa_i^{-1}\|_{C^{0,\delta}(\mathbb{R}^d)}.$$

For a definition of the Besov- and Triebel-Lizorkin spaces on manifolds with bounded geometry we refer to Section 7.2.1 and 7.3.1 in [Tri92b]. By means of $C^{0,\delta}(\mathbb{R}^d) \hookrightarrow$

$L^\infty(\mathbb{R}^d)$, $H^s_p(\Omega) \hookrightarrow C^{0,\delta}(\Omega)$ and $\sup_{i\in I}(\#J(i)) < \infty$ follows

$$\begin{aligned}
\|f\|_{L^\infty(\Omega)} &= \sup_{i\in I} \|f \circ \kappa_i^{-1}\|_{L^\infty(\kappa_i(O_i(r)))} \\
&\leq \sup_{i\in I} \sum_{j\in J(i)} \|(\psi_{r,j} f) \circ \kappa_j^{-1} \circ \kappa_j \circ \kappa_i^{-1}\|_{L^\infty(\kappa_i(O_i(r)\cap O_j(r)))} \\
&\leq \sup_{i\in I} (\#J(i)) \sup_{j\in I} \|[\psi_{j,r} f] \circ \kappa_j^{-1}\|_{L^\infty(\kappa_j(O_j(r)))} \\
&\lesssim \sup_{j\in I} \|(\psi_{j,r} f) \circ \kappa_j^{-1}\|_{C^{0,\delta}(\mathbb{R}^d)} \lesssim \|f\|_{H^s_p(\Omega)}.
\end{aligned}$$

(d2) Let either $q \in [2, e(d,s))$ or $d > 2s$ and $q = e(d,s)$. The Sobolev embedding A.2.1 implies $H^s_2(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ and we have $\ell^2 \hookrightarrow \ell^q$. For $f \in H^s_2(\Omega)$ then follows

$$\begin{aligned}
\|f\|_{L^q(\Omega)} &= \left(\sum_{i\in I} \|(\psi_{r,i} f)\kappa_i^{-1}\|^q_{L^q(\mathbb{R}^d)} \right)^{1/q} \\
&\lesssim \left(\sum_{i\in I} \|(\psi_{r,i} f)\kappa_i^{-1}\|^2_{H^s_2(\mathbb{R}^d)} \right)^{1/2} = \|f\|_{H^s_2(\mathbb{R}^d)}.
\end{aligned}$$

The density of this embedding follows from the density of $C_c^\infty(\Omega)$ in $L^q(\Omega)$.
(e) Let $i \in I$, $j \in J(i)$, $q \in [2, e(d,1))$, and $f \in H^1_2(\Omega)$. Let additionally $(f_n)_{n\in\mathbb{N}}$ be a sequence in $H^1_2(\Omega)$ with $f_n \rightharpoonup f$ in $H^1_2(\Omega)$. From *(b)* we know that $H^1_2(\Omega) \cong H^1_{2,0}(\Omega)$ so that $\psi_{r,j} f_n, \psi_{r,j} f \in H^1_{2,0}(O_i(r_0))$ for all $n \in \mathbb{N}$ and

$$\psi_{r,j} f_n \rightharpoonup \psi_{r,j} f \quad \text{in } H^1_{2,0}(O_i(r_0)). \tag{III.5}$$

Since $O_i(r_0) = \exp(B(0,r_0))$ and exp is a diffeomorphism we can equipp $\overline{O_i(r)}$ with the atlas $\mathscr{A}_i := \{(O_j(r), \kappa_j) \mid j \in J(i)\}$. Then $(\overline{O_i(r)}, g)$ is a connected, compact Riemannian C^∞-manifold with boundary. Theorem 10.1 in [Heb99] states the compactness of the embedding $H^1_{2,0}(O_i(r)) \hookrightarrow L^q(O_i(r))$. The weak convergence in (III.5) therefore implies $\psi_{r,j} f_n \overset{n\to\infty}{\longrightarrow} \psi_{r,j} f$ in $L^q(O_i(r_0))$. Finally,

$$\|f_n - f\|_{L^q(O_i(r))} \leq \sum_{j\in J(i)} \|\psi_{j,r}(f_n - f)\|_{L^q(O_i(r_0))} \overset{n\to\infty}{\longrightarrow} 0.$$

□

By means of a localization argument we can transfer nonlinear estimates for $H^s_2(\mathbb{R}^d)$ to $H^s_2(\Omega)$. This is done in the following Lemma.

Lemma III.1.3
Let $s \in (0,\infty)$, $k, m \in \mathbb{N}$ and (Ω, g) satisfy (M). *We put $S(\Omega)^k := S(\Omega, \mathbb{R}^k)$ for $S \in \{H^s_2, L^\infty\}$ and let $g, h \in H^s_2(\Omega)^m \cap L^\infty(\Omega)^m$. We then have:*

(a) $\|g \cdot h\|_{H^s_2(\Omega)} \lesssim (\|g\|_{L^\infty(\Omega)^m} \|h\|_{H^s_2(\Omega)^m} + \|g\|_{H^s_2(\Omega)^m} \|h\|_{L^\infty(\Omega)^m})$.

(b) For $F \in C^{[s]+1}(\mathbb{R}^m, \mathbb{R}^k)$ with $F(\mathbf{0}) = \mathbf{0}$ holds $F : L^\infty(\Omega)^m \cap H^s_2(\Omega)^m \to H^s_2(\Omega)^k$ and there is a non-decreasing $C_{1,F} : [0,\infty) \to [0,\infty)$ such that

$$\|F(g)\|_{H^s_2(\Omega)^k} \leq C_{1,F}(\|g\|_{L^\infty(\Omega)^m}) \|g\|_{H^s_2(\Omega)^m}. \tag{III.6}$$

(c) For $F \in C^{[s]+2}(\mathbb{R}^m, \mathbb{R}^k)$ there is a non-decreasing $C_{2,F} : [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned}
&\|F(g) - F(h)\|_{H_2^s(\Omega)^k} \\
&\quad \le C_{2,F}\big(\|g\|_{L^\infty(\Omega)^m} + \|h\|_{L^\infty(\Omega)^m}\big)\|g-h\|_{H_2^s(\Omega)^m} \qquad \text{(III.7)}\\
&\quad\quad + C_{1,F'}\big(\|g\|_{L^\infty(\Omega)^m} + \|h\|_{L^\infty(\Omega)^m}\big)\big(\|g\|_{H_2^s(\Omega)^m} + \|h\|_{H_2^s(\Omega)^m}\big)\|g-h\|_{L^\infty(\Omega)^m}.
\end{aligned}$$

More precisely, there is $C \in [2,\infty)$ such that

$$C_{1,F}(x) \cong (1+x)^{[s]} \sup_{1\le|\alpha|\le[s]+1} \|\partial^\alpha F\|_{L^\infty(\overline{B}(0,Cx))^k}, \qquad \text{(III.8)}$$

$$C_{2,F}(x) \cong \sup_{|\alpha|=1} \|\partial^\alpha F\|_{L^\infty(\overline{B}(0,Cx))^k}. \qquad \text{(III.9)}$$

Remarks:

(1) Let $\Omega = \mathbb{R}^d$. The estimates stated above can be found in Lemma A.8, A.9, and Exercise A.12 of [Tao06]. The estimates in *(a)* and *(b)* with $F \in C^\infty(\mathbb{R},\mathbb{R})$ can also be found in Chapter II.A of [AG07]. However, all these result do not state a precise enough information on how the functions $C_{1,F}$ and $C_{2,F}$ depend on $\|g\|_{L^\infty(\mathbb{R}^d)^m}$ and $\|h\|_{L^\infty(\mathbb{R}^d)^m}$. This is an important information for our local and global existence theory and we are forced to repeat the necessary proofs. In contrast to that, the dependencies on s and d in the constants are irrelevant and will be omitted.

(2) If F is a polynomial of order $k \in \mathbb{N}$ with $F(\mathbf{0}) = \mathbf{0}$, then the product estimates in *(a)* easily yields the estimates (III.6) and (III.7) with

$$C_{1,F}(x) \cong (1+x)^{k-1}, \qquad C_{2,F} \cong C_{1,F}. \qquad \text{(III.10)}$$

Proof. It is surely enough to treat the case $k = 1$. In (a+b;1) we show all the estimates in (a) and (b) for $(\Omega, g) = (\mathbb{R}^d, \mathrm{id})$. We use these estimates in (a+b;2) for the general case. In (c) we show (III.7). In all parts of the proof we let $s \in (0,\infty)$, $F \in C^{[s]+1}(\mathbb{R}^m)$, and $g, h \in L^\infty(\Omega)^m \cap H_2^s(\Omega)^m$. We put

$$g_\infty := \|g\|_{L^\infty(\Omega)^m}, \quad h_\infty := \|h\|_{L^\infty(\Omega)^m}, \quad r_\infty := g_\infty + h_\infty.$$

(a+b;1) Let $\Omega = \mathbb{R}^d$ and $F(\mathbf{0}) = \mathbf{0}$. Lemma A.8 in [Tao06] contains the product estimate in *(a)*, what leaves (III.6). Since $F \in C^{[s]+1}(\mathbb{R}^m)$ we have $F|_{\overline{B}(0,r_\infty)} \in C^{[s]+1}(\overline{B}(0,r_\infty))$ and the mean value theorem yields

$$\|F(g) - F(h)\|_{L^2(\mathbb{R}^d)} \le \|\nabla F\|_{L^\infty(\overline{B}(0,r_\infty))^m}\|g-h\|_{L^2(\mathbb{R}^d)^m}. \qquad \text{(III.11)}$$

We recall the following Littlewood-Paley projections with $(\psi_k)_{k\in\mathbb{N}_0}$ from Definition II.2.4. For $k \in \mathbb{N}_0$ we define

$$\Delta_k := \psi_k((-\Delta)^{1/2}) = \mathcal{F}^{-1}\psi_k(|\cdot|)\mathcal{F}, \qquad \Delta_{\le k} := \sum_{l=0}^{k} \Delta_k, \qquad \Delta_{>k} := \sum_{l=k+1}^{\infty} \Delta_k.$$

We use the characterization of $H_2^s(\mathbb{R}^d)$ given by Theorem 6.2.6 in [Gra09]. It states that for $f \in L^2(\Omega)$ holds

$$\|f\|_{H_2^s(\mathbb{R}^d)^m} \cong \|\Delta_0 f\|_{L^2(\mathbb{R}^d)^m} + \left(\sum_{k=1}^{\infty} 2^{2ks}\|\Delta_k f\|^2_{L^2(\mathbb{R}^d)^m}\right)^{1/2}. \tag{III.12}$$

This has to be interpreted as follows: If the right-hand side is finite, then $f \in H_2^s(\mathbb{R}^d)$ and (III.12) holds, and vice versa. Recall that the Bernstein inequalities in II.4.2;(1) yield for $k, N \in \mathbb{N}$ and $p, q \in [1, \infty]$ with $p \leq q$ the estimates

$$\|\Delta_{<k}\|_{L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \lesssim 2^{\frac{kd}{2}(\frac{1}{p}-\frac{1}{q})}, \qquad \sum_{|\alpha|=N} \|\partial^\alpha \Delta_k\|_{L^p(\mathbb{R}^d)\to L^q(\mathbb{R}^d)} \cong 2^{k(\frac{d}{2}(\frac{1}{p}-\frac{1}{q})-N)}. \tag{III.13}$$

$F(\mathbf{0}) = \mathbf{0}$ and the Lipschitz estimate (III.11) yield $F(g) \in L^2(\mathbb{R}^d)$. Then the isometry properties of the Fourier transform provide

$$\|\Delta_0 F(g)\|_{L^2(\mathbb{R}^d)} \leq \|F(g)\|_{L^2(\mathbb{R}^d)} \leq \|\nabla F\|_{L^\infty(\overline{B}(0,g_\infty))^m}\|g\|_{L^2(\mathbb{R}^d)^m}.$$

For $k \in \mathbb{N}$ additionally holds

$$\|\Delta_k F(g)\|_{L^2(\mathbb{R}^d)} \leq \|F(g) - F(\Delta_{<k} g)\|_{L^2(\mathbb{R}^d)} + \|\Delta_k F(\Delta_{<k} g)\|_{L^2(\mathbb{R}^d)}. \tag{III.14}$$

In view of (III.12) we have to estimate the ℓ^2-norms of both expressions on the right-hand side of (III.14). We treat these expressions seperately.
For the first term observe that (III.13) implies $\|\Delta_{<k} g\|_{L^\infty(\mathbb{R}^d)^m} + \|g\|_{L^\infty(\mathbb{R}^d)^m} \leq C g_\infty$. Then the Lipschitz estimate (III.11), (II.36), and Hölder's inequality imply

$$\begin{aligned}
&\left(\sum_{k=1}^{\infty} 2^{2ks}\|F(g) - F(\Delta_{<k} g)\|^2_{L^2(\mathbb{R}^d)}\right)^{1/2} \\
&\quad\leq \|\nabla F\|_{L^\infty(\overline{B}(0,Cg_\infty))^m}\left(\sum_{k=1}^{\infty} 2^{2ks}\|\Delta_{\geq k} g\|^2_{L^2(\mathbb{R}^d)^m}\right)^{1/2} \\
&\quad\leq C(F, g_\infty)\left(\sum_{k=1}^{\infty} 2^{2ks}\left(\sum_{l=k}^{\infty} 2^{-ls}\|\Delta_l(-\Delta)^{s/2} g\|_{L^2(\mathbb{R}^d)^m}\right)^2\right)^{1/2} \\
&\quad\leq C(F, g_\infty)\left(\sum_{k=1}^{\infty} 2^{2ks}\left(\sum_{l=k}^{\infty} 2^{-ls}\right)\left(\sum_{l=k}^{\infty} 2^{-ls}\|\Delta_l(-\Delta)^{s/2} g\|^2_{L^2(\mathbb{R}^d)^m}\right)\right)^{1/2} \\
&\quad\leq C(F, g_\infty)\left(\sum_{k=1}^{\infty}\sum_{l=k}^{\infty} 2^{(k-l)s}\|\Delta_l(-\Delta)^{s/2} g\|^2_{L^2(\mathbb{R}^d)^m}\right)^{1/2} \\
&\quad= C(F, g_\infty)\left(\sum_{l=1}^{\infty}\left(\sum_{k=1}^{l} 2^{ks}\right)2^{-ls}\|\Delta_l(-\Delta)^{s/2} g\|^2_{L^2(\mathbb{R}^d)^m}\right)^{1/2} \\
&\quad\leq C(F, g_\infty)\left(\sum_{l=1}^{\infty} 2^{2ls}\|\Delta_l g\|^2_{L^2(\mathbb{R}^d)^m}\right)^{1/2} \leq C_1(F, g_\infty)\|g\|_{H_2^s(\mathbb{R}^d)^m}.
\end{aligned} \tag{III.15}$$

The ℓ^2-norm of the second term in (III.14) is a bit more delicate to handle. We put $N := [s] + 1$. Recall $\Delta_{<k} g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Theorem 1.7.5 in [Hör76] provides the analyticity of $\Delta_{<k} g$ as it is a Fourier transformed L^2-function with compact support.

Since $F \in C^N(\mathbb{R}^m)$ we deduce $F(\Delta_k g) \in C^N(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \subseteq W_2^N(\mathbb{R}^d)$. (III.13) then yields

$$\begin{aligned}\|\Delta_k F(\Delta_{<k} g)\|_{L^2(\mathbb{R}^d)} &\cong 2^{-kN} \sum_{|\alpha|=N} \|\partial^\alpha \Delta_k F(\Delta_{<k} g)\|_{L^2(\mathbb{R}^d)} \\ &\cong 2^{-kN} \sum_{|\alpha|=N} \|\partial^\alpha F(\Delta_{<k} g)\|_{L^2(\mathbb{R}^d)}. \end{aligned} \tag{III.16}$$

The given regularity properties of F and $\Delta_{<k} g$ allow us to apply the chain and Leibniz rule on $\partial^\alpha F(\Delta_{<k} g)$. With the set

$$M_n := \left\{ \boldsymbol{B} = (\boldsymbol{\beta^1}, ..., \boldsymbol{\beta^n}) \in \left(\mathbb{N}_0^d \setminus \{\boldsymbol{0}\}\right)^n \;\middle|\; \sum_{i=1}^n |\boldsymbol{\beta^i}| = N \right\}$$

we have

$$\partial^\alpha F(\Delta_{<k} g) = \sum_{n=1}^N \sum_{\boldsymbol{B} \in M_n} C(\boldsymbol{B}) F^{(n)}(\Delta_{<k} g)(\partial^{\boldsymbol{\beta^1}} \Delta_{<k} g, ..., \partial^{\boldsymbol{\beta^n}} \Delta_{<k} g).$$

Since (III.13) yields $\|\Delta_{<k} g\|_{L^\infty(\mathbb{R}^d)} \le C\|g\|_{L^\infty(\mathbb{R}^d)}$ we additionally have

$$\begin{aligned} &\|\partial^\alpha F(\Delta_{<k} g)\|_{L^2(\mathbb{R}^d)} \\ &\lesssim \sup_{1 \le |\gamma| \le N} \left(\|\partial^\gamma F\|_{L^\infty(\overline{B}(0, C g_\infty))} \right) \sup_{1 \le n \le N,\ \boldsymbol{B} \in M_n} \left(\prod_{i=1}^{n-1} \|\partial^{\boldsymbol{\beta^i}} \Delta_{<k} g\|_{L^\infty(\mathbb{R}^d)^m} \|\partial^{\boldsymbol{\beta^n}} \Delta_{<k} g\|_{L^2(\mathbb{R}^d)^m} \right). \end{aligned} \tag{III.17}$$

An application of the previous estimate, (II.52), and $|\boldsymbol{\beta^i}| > 0$ yields

$$\begin{aligned} &\|\partial^\alpha F(\Delta_{<k} g)\|_{L^2(\mathbb{R}^d)} \\ &\le C(F, g_\infty) \sup_{1 \le n \le N,\ \boldsymbol{B} \in M_n} \left(\sum_{0 \le k_1, ..., k_n < k} \prod_{i=1}^{n-1} \|\partial^{\boldsymbol{\beta^i}} \Delta_{k_i} g\|_{L^\infty(\mathbb{R}^d)^m} \|\partial^{\boldsymbol{\beta^n}} \Delta_{k_n} g\|_{L^2(\mathbb{R}^d)^m} \right) \\ &\le C(F, g_\infty)(1 + g_\infty)^{N-1} \sup_{1 \le n \le N,\ \boldsymbol{B} \in M_n} \left(\sum_{0 \le k_1 \le ... \le k_n < k} \prod_{i=1}^{n-1} 2^{|\boldsymbol{\beta^i}| k_i} \|\Delta_{k_n} g\|_{L^2(\mathbb{R}^d)^m} \right) \\ &\le C(F, g_\infty) \sup_{1 \le n \le N,\ \boldsymbol{B} \in M_n} \left(\sum_{0 \le k_3 \le ... \le k_n < k} \prod_{i=3}^{n-1} 2^{|\boldsymbol{\beta^i}| k_i} \left(\sum_{0 \le k_2 \le k_3} 2^{(|\boldsymbol{\beta^1}| + |\boldsymbol{\beta^2}|) k_2} \right) \|\Delta_{k_n} g\|_{L^2(\mathbb{R}^d)^m} \right) \\ &\le C(F, g_\infty) \sum_{0 \le l < k} 2^{lN} \|\Delta_l g\|_{L^2(\mathbb{R}^d)^m}. \end{aligned} \tag{III.18}$$

Observe that the Cauchy Schwarz and Young inequality yield for any real, non-negative sequence $(a_l)_{l \in \mathbb{N}_0}$ that

$$\begin{aligned} \sum_{k=1}^\infty 2^{2k(s-N)} \left(\sum_{0 \le l < k} 2^{(N-s)l} a_l \right)^2 &= \sum_{l,m=0}^\infty \left(\sum_{k=\max\{l,m\}+1}^\infty 2^{(s-N)2k} \right) 2^{(N-s)(l+m)} a_l a_m \\ &\cong \sum_{l=0}^\infty \left(\sum_{m=0}^\infty 2^{(s-N)|l-m|} a_m \right) a_l \\ &\lesssim \|(2^{(s-N)l})\|_{\ell^1(\mathbb{N}_0)} \|(a_l)\|_{\ell^2(\mathbb{N}_0)}^2. \end{aligned}$$

Then (III.16), (III.18), the previous estimate with $a_l := 2^{ls}\|\Delta_l g\|_{L^2(\mathbb{R}^d)^m}$, and (III.12) imply

$$\left(\sum_{k=1}^{\infty} 2^{2ks}\|\Delta_k F(\Delta_{<k}g)\|_{L^2(\mathbb{R}^d)}^2\right)^{1/2} \leq C(F,g_\infty)\left(\sum_{k=1}^{\infty} 2^{2k(s-N)}\left(\sum_{0\leq l<k} 2^{lN}\|\Delta_l g\|_{L^2(\mathbb{R}^d)^m}\right)^2\right)^{1/2}$$
$$\leq C(F,g_\infty)\left(\sum_{l=0}^{\infty} 2^{2ls}\|\Delta_l g\|_{L^2(\mathbb{R}^d)^m}^2\right)^{1/2}$$
$$\leq C_2(F,g_\infty)\|g\|_{H_2^s(\mathbb{R}^d)^m}. \tag{III.19}$$

Applying the estimates (III.15) and (III.19) in (III.12) consequently yields

$$\|F(g)\|_{H^s(\mathbb{R}^d)} \leq \big(C_1(F,g_\infty)+C_2(F,g_\infty)\big)\|g\|_{H_2^s(\mathbb{R}^d)^m} \leq C_{1,F}(g_\infty)\|g\|_{H_2^s(\mathbb{R}^d)^m}. \tag{III.20}$$

We have chosen $C \in [2,\infty)$ large enough and

$$C_{1,F}(x) := C(1+x)^{[s]} \sup_{1\leq|\alpha|\leq[s]+1} \|\partial^\alpha F\|_{L^\infty(\overline{B}(0,Cx))}. \tag{III.21}$$

(a+b;2) Let (Ω, g) satisfy (M) and $F(\mathbf{0}) = \mathbf{0}$. Recall that the smooth partition of unity $(\psi_{r,i})_{i\in\mathbb{N}}$ satisfies $\sum_{j\in J(i)} \psi_{r,j}(\operatorname{supp}(\psi_{r,i})) = \{1\}$ and $\sup_{i\in I} \#J(i) < \infty$. For any real, non-negative sequence $(a_j)_{j\in\mathbb{N}}$ then holds

$$\sum_{i=1}^{\infty}\sum_{j\in J(i)} a_j = \sum_{j=1}^{\infty}(\#J(j))a_j \lesssim \sum_{j=1}^{\infty} a_j. \tag{III.22}$$

The product estimate for $\Omega = \mathbb{R}^d$ in **(a+b;1)** and Theorem 4.3.2 in [Tri92b] yields for all $i \in I$

$$\begin{aligned}
&\|(\psi_{r,i}g\cdot h)\circ\kappa^{-1}\|_{H_2^s(\mathbb{R}^d)} \\
&\quad\leq \sum_{j\in J(i)} \|((\psi_{r,i}g)\cdot(\psi_{r,j}h))\circ\kappa_i^{-1}\|_{H_2^s(\mathbb{R}^d)} \\
&\quad\lesssim \sum_{j\in J(i)} \|\psi_{r,i}g\|_{L^\infty(\Omega)^m}\|(\psi_{r,j}h)\circ\kappa_j^{-1}\|_{H_2^s(\mathbb{R}^d)^m} + \|(\psi_{r,i}g)\circ\kappa_i^{-1}\|_{H_2^s(\mathbb{R}^d)^m}\|\psi_{r,j}h\|_{L^\infty(\Omega)^m} \\
&\quad\lesssim g_\infty\left(\sum_{j\in J(i)} \|(\psi_{r,j}h)\circ\kappa_i^{-1}\|_{H_2^s(\mathbb{R}^d)}^2\right)^{1/2} + h_\infty\|(\psi_{r,i}g)\circ\kappa_i^{-1}\|_{H_2^s(\mathbb{R}^d)}.
\end{aligned}$$

With (III.22) we therefore obtain

$$\begin{aligned}
\|g\cdot h\|_{H_2^s(\Omega)} &\lesssim g_\infty\left(\sum_{i\in I}\sum_{j\in J(i)} \|(\psi_{j,r}h)\circ\kappa_j^{-1}\|_{H_2^s(\mathbb{R}^d)}^2\right)^{1/2} + h_\infty\left(\sum_{i\in I}\|(\psi_{r,i}g)\circ\kappa_i^{-1}\|_{H_2^s(\mathbb{R}^d)}^2\right)^{1/2} \\
&\lesssim g_\infty\|h\|_{H_2^s(\Omega)^m} + h_\infty\|g\|_{H_2^s(\Omega)^m}.
\end{aligned}$$

The same approach also works for the nonlinear estimate (III.6) due to $F(\mathbf{0}) = \mathbf{0}$. More precisely, we additionally apply Theorem 4.2.2 in [Tri92b] and the uniform boundedness of the derivatives of $(\psi_{r,i})_{i\in I}$ from (M2) to provide

$$\begin{aligned}
\|F(g)\|_{H_2^s(\Omega)} &= \Big(\sum_{i\in I}\big\|\big(\psi_{r,i}F(\sum_{j\in J(i)}\psi_{r,j}g)\big)\circ\kappa_i^{-1}\big\|^2_{H_2^s(\mathbb{R}^d)}\Big)^{1/2}\\
&\lesssim \Big(\sum_{i\in I}\big\|\big(F(\sum_{j\in J(i)}\psi_{r,j}g)\big)\circ\kappa_j^{-1}\big\|^2_{H_2^s(\mathbb{R}^d)}\Big)^{1/2}\\
&\lesssim C_{1,F}(g_\infty)\Big(\sum_{i\in I}\sum_{j\in J(i)}\big\|(\psi_{r,j}g)\circ\kappa_j^{-1}\big\|^2_{H_2^s(\mathbb{R}^d)^m}\Big)^{1/2} \lesssim C_{1,F}(g_\infty)\|g\|_{H_2^s(\Omega)^m}.
\end{aligned}$$

(c) Let (Ω, g) satisfy (M) and additionally $F\in C^{[s]+2}(\mathbb{R}^m)$. We define $G\in C^{[s]+2}(\mathbb{R}^m)$ by $G(\boldsymbol{x}) := F(\boldsymbol{x}) - (\nabla F)(\mathbf{0})\boldsymbol{x}$. Then $\nabla G\in C^{[s]+1}(\mathbb{R}^m,\mathbb{R}^m)$ with $(\nabla G)(\mathbf{0}) = \mathbf{0}$. We use the fundamental theorem, the product estimate in *(a)* and (III.6) to prove

$$\begin{aligned}
\|G(g)-G(h)\|_{H_2^s(\Omega)} &\le \int_0^1 \|(\nabla G)(h+t(g-h))\cdot(g-h)\|_{H_2^s(\Omega)}\,\mathrm{d}t\\
&\lesssim \Big(\sup_{t\in[0,1]}\|(\nabla G)(h+t(g-h))\|_{L^\infty(\Omega)^m}\|g-h\|_{H_2^s(\Omega)^m}\\
&\qquad + \sup_{t\in[0,1]}\|(\nabla G)(h+t(g-h))\|_{H_2^s(\Omega)^m}\|g-h\|_{L^\infty(\Omega)^m}\Big)\\
&\lesssim \Big(\big(|\nabla F(\mathbf{0})| + \|\nabla F\|_{L^\infty(\overline{B}(0,2r_\infty))^m}\big)\|g-h\|_{H_2^s(\Omega)^m}\\
&\qquad + C_{1,\nabla G}(r_\infty)(\|g\|_{H_2^s(\Omega)^m}+\|h\|_{H_2^s(\Omega)^m})\|g-h\|_{L^\infty(\Omega)^m}\Big)\\
&\lesssim \Big(\sup_{|\boldsymbol{\alpha}|=1}\|\partial^{\boldsymbol{\alpha}}F\|_{L^\infty(\overline{B}(0,2r_\infty))}\|g-h\|_{H_2^s(\Omega)^m}\\
&\qquad + C_{1,F'}(r_\infty)(\|g\|_{H_2^s(\Omega)^m}+\|h\|_{H_2^s(\Omega)^m})\|g-h\|_{L^\infty(\Omega)^m}\Big).
\end{aligned}$$

If we take $C\in[2,\infty)$ large enough and put

$$C_{2,F}(x) := \sup_{|\boldsymbol{\alpha}|=1}\|\partial^{\boldsymbol{\alpha}}F\|_{L^\infty(\overline{B}(0,Cr))},$$

then we have

$$\begin{aligned}
&\|F(g)-F(h)\|_{H_2^s(\Omega)}\\
&\quad\le \|G(g)-G(h)\|_{H_2^s(\Omega)} + |(\nabla F)(\mathbf{0})|\|g-h\|_{H_2^s(\Omega)^m}\\
&\quad\lesssim C_{2,F}(r_\infty)\|g-h\|_{H_2^s(\Omega)^m} + C_{1,F'}(r_\infty)(\|g\|_{H_2^s(\Omega)^m}+\|h\|_{H_2^s(\Omega)^m})\|g-h\|_{L^\infty(\Omega)^m}.
\end{aligned}$$

□

From the previous lemma we can extract the nonlinear estimates for the model nonlinearity $F_{\beta,\pm}$ on $H_2^s(\Omega)$, which are needed for an application of Theorem I.3.4. However, for an application of the results in Section II.4 the estimates in $H_2^s(\Omega)$ are not sufficient.

We in particular need to control the expression $S_{F_{\beta,\pm}}$ defined in (II.60). For this we prove that $F_{\beta,\pm} : H_2^1(\Omega) \to H_{r^*}^1(\Omega)$ for some $r^* < 2$ is bounded on bounded sets. In the next Lemma we gather all the nonlinear estimates for the model nonlinearity, which will be used frequently throughout this section.

Lemma III.1.4
Let $s, \beta \in (0, \infty)$ and (Ω, g) satisfy (M). *We then have:*

(a) $F_{\beta,\pm} : L^2(\Omega) \cap L^\infty(\Omega) \to L^2(\Omega)$ *and*

$$\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{L^2(\Omega)} \lesssim \left(\|g\|_{L^\infty(\Omega)}^\beta + \|h\|_{L^\infty(\Omega)}^\beta\right)\|g - h\|_{L^2(\Omega)}. \tag{III.23}$$

(b) If $\beta > [s]$, then $F_{\beta,\pm} : H_2^s(\Omega) \cap L^\infty(\Omega) \to H_2^s(\Omega)$ and

$$\|F_{\beta,\pm}(g)\|_{H_2^s(\Omega)} \lesssim \|g\|_{L^\infty(\Omega)}^\beta \|g\|_{H_2^s(\Omega)}. \tag{III.24}$$

If additionally $\beta > [s] + 1$, then

$$\begin{aligned} &\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{H_2^s(\Omega)} \\ &\quad\lesssim \left(\|g\|_{L^\infty(\Omega)}^\beta + \|h\|_{L^\infty(\Omega)}^\beta\right)\|g - h\|_{H_2^s(\Omega)} \\ &\qquad + \left(\|g\|_{L^\infty(\Omega)}^{\beta-1} + \|h\|_{L^\infty(\Omega)}^{\beta-1}\right)\left(\|g\|_{H_2^s(\Omega)} + \|h\|_{H_2^s(\Omega)}\right)\|g - h\|_{L^\infty(\Omega)}. \end{aligned} \tag{III.25}$$

The estimates (III.24) *and* (III.25) *also hold under the assumption $\beta \in \mathbb{N}_{even}$.*

(c) If $s = 1$, then (III.24) *holds for $\beta > 0$ and* (III.25) *holds for $\beta > 1$.*

(d) Let $q \in [2, \infty)$ such that $H_2^1(\Omega) \hookrightarrow L^q(\Omega)$ and $r \in (1, \infty)$ with $\frac{1}{r^*} = \frac{1}{2} + \frac{\beta}{q}$. Then holds $F_{\beta,\pm} : H_2^1(\Omega) \to H_{r^*}^1(\Omega)$ and

$$\|F_{\beta,\pm}(g)\|_{H_{r^*}^1(\Omega)} \lesssim \|g\|_{H_2^1(\Omega)}^{\beta+1}. \tag{III.26}$$

Proof. In all the parts of the proof we let $s, \beta \in (0, \infty)$ and $g, h \in H_2^s(\Omega) \cap L^\infty(\Omega)$ with $\|g\|_{L^\infty(\Omega)} > 0$ if not stated otherwise. The nonlinear estimates of Lemma III.1.3 are available for $F_{\beta,\pm}$ through the identification $H_2^s(\Omega, \mathbb{C})$ and $H_2^s(\Omega, \mathbb{R}^2)$. Similar to the proof of Proposition I.4.2 we define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $F(\boldsymbol{x}) := \pm|\boldsymbol{x}|^\beta \boldsymbol{x}$, which satisfies with $\boldsymbol{g} := (\operatorname{Re} g, \operatorname{Im} g)$

$$\|g\|_{H_2^s(\Omega)} \cong \|\boldsymbol{g}\|_{H_2^s(\Omega,\mathbb{R}^2)}, \qquad \|F_{\beta,\pm}(g)\|_{H_2^s(\Omega)} \cong \|F(\boldsymbol{g})\|_{H_2^s(\Omega,\mathbb{R}^2)}.$$

We will use these identifications in the proof below without mentioning.
(a) The Lipschitz estimate (I.64) in Proposition I.4.2 directly implies for $g, h \in L^2(\Omega) \cap L^\infty(\Omega)$

$$\begin{aligned} \|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{L^2(\Omega)} &= \left(\int_\Omega |F_{\beta,\pm}(g) - F_{\beta,\nu}(h)|^2 \, \mathrm{d}\mu\right)^{1/2} \\ &\lesssim \left(\|g\|_{L^\infty(\Omega)}^\beta + \|h\|_{L^\infty(\Omega)}^\beta\right)\|g - h\|_{L^2(\Omega)}. \end{aligned}$$

(b) We assume $\beta > [s]$ and put $\tilde{g} := g/\|g\|_{L^\infty(\Omega)}$. Proposition I.4.2;(a) yields $F_{\beta,\pm} \in C^{[s]+1}(\mathbb{R}^2, \mathbb{R}^2)$ and $|\partial^{\boldsymbol{\alpha}} F_{\beta,\pm}(\boldsymbol{x})| \lesssim |\boldsymbol{x}|^{\beta+1-|\boldsymbol{\alpha}|}$. Then the homogeneity of $F_{\beta,\pm}$ of order $\beta+1$ and (III.6) from Lemma III.1.3 imply

$$\begin{aligned}\|F_{\beta,\pm}(g)\|_{H_2^s(\Omega)} &\lesssim \|g\|_{L^\infty(\Omega)}^{\beta+1}(1+\|\tilde{g}\|_{L^\infty(\Omega)})^{[s]} \sup_{1\le|\boldsymbol{\alpha}|\le[s]+1} \|\partial^{\boldsymbol{\alpha}} F_{\beta,\pm}\|_{L^\infty(\overline{B}(0,C\|\tilde{g}\|_{L^\infty(\Omega)}))}\|\tilde{g}\|_{H_2^s(\Omega)} \\ &\lesssim \|g\|_{L^\infty(\Omega)}^{\beta} \sup_{1\le|\boldsymbol{\alpha}|\le[s]+1} \||\cdot|^{\beta+1-|\boldsymbol{\alpha}|}\|_{L^\infty(\overline{B}(0,C))}\|g\|_{H_2^s(\Omega)} \\ &\lesssim \|g\|_{L^\infty(\Omega)}^{\beta}\|g\|_{H_2^s(\Omega)}.\end{aligned}$$

Now we assume $\beta > [s]+1$. We put $r_\infty := \|g\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)}$ and $\tilde{g} := g/r_\infty$, $\tilde{h} := h/r_\infty$. Proposition I.4.2;(a) provides $F_{\beta,\pm} \in C^{[s]+2}(\mathbb{R}^2, \mathbb{R}^2)$ and again the homogeneity of $F_{\beta,\pm}$ of order $\beta+1$ and (III.7) from Lemma III.1.3 yield

$$\begin{aligned}&\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{H_2^s(\Omega)} \\ &\quad\lesssim r_\infty^{\beta+1}\Big((1+\|\tilde{g}\|_{L^\infty(\Omega)} + \|\tilde{h}\|_{L^\infty(\Omega)})^{\beta}\|\tilde{g}-\tilde{h}\|_{H_2^s(\Omega)} \\ &\qquad+(1+\|\tilde{g}\|_{L^\infty(\Omega)} + \|\tilde{h}\|_{L^\infty(\Omega)})^{\beta+[s]-1}(\|\tilde{g}\|_{H^s(\Omega)} + \|\tilde{h}\|_{H_2^s(\Omega)})\|\tilde{g}-\tilde{h}\|_{L^\infty(\Omega)}\Big) \\ &\quad\lesssim (\|g\|_{L^\infty(\Omega)}^{\beta} + \|h\|_{L^\infty(\Omega)}^{\beta})\|g-h\|_{H_2^s(\Omega)} \\ &\qquad+(\|g\|_{L^\infty(\Omega)}^{\beta-1} + \|h\|_{L^\infty(\Omega)}^{\beta-1})(\|g\|_{H_2^s(\Omega)} + \|h\|_{H_2^s(\Omega)})\|g-h\|_{L^\infty(\Omega)}.\end{aligned}$$

If we assume $\beta \in \mathbb{N}_{even}$, then Proposition I.4.2;(a) yields $F_{\beta,\pm} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ with $|\partial^{\boldsymbol{\alpha}} F_{\beta,\pm}(\boldsymbol{x})| \lesssim |\boldsymbol{x}|^{\max\{\beta+1-|\boldsymbol{\alpha}|,0\}}$. The estimates (III.24) and (III.25) follow as above.
(c+d) Let $s = 1$. We show the assertions with the same localization argument, which was used in the proof of Lemma III.1.3;(a+b). In (c+d;1) we show (III.24) and (III.26) for the case $\Omega = \mathbb{R}^d$ and use this in (c+d;2) for the general case and (III.25).
(c+d;1) Let $\Omega = \mathbb{R}^d$. From Theorem 2.1.6 in [Zie89] and its proof, we can derive the following useful criterion for a function $h \in L^p(\mathbb{R}^d)$ with $p \in (1,\infty)$ to belong to the Sobolev space $H_p^1(\mathbb{R}^d)$:

$$h \in H_p^1(\mathbb{R}^d) \iff \sup_{\boldsymbol{y}\neq\boldsymbol{0}} \frac{\|h(\cdot+\boldsymbol{y}) - h\|_{L^p(\mathbb{R}^d)}}{|\boldsymbol{y}|} < \infty. \tag{III.27}$$

In particular, $h \in H_p^1(\mathbb{R}^d)$ implies

$$\sup_{\boldsymbol{y}\neq\boldsymbol{0}} \frac{\|h(\cdot+\boldsymbol{y}) - h\|_{L^p(\mathbb{R}^d)}}{|\boldsymbol{y}|} \le \|\nabla h\|_{L^p(\mathbb{R}^d)}. \tag{III.28}$$

Let $g \in H_2^1(\mathbb{R}^d) \cap L^\infty(\Omega)$. Then $F_{\beta,\pm} \circ g \in L^2(\mathbb{R}^d)$ and for $\boldsymbol{y} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ the Lipschitz estimate in *(a)* yields

$$\|F_{\beta,\pm}(g(\cdot+\boldsymbol{y})) - F_{\beta,\pm}(g)\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|_{L^\infty(\mathbb{R}^d)}^{\beta}\|g(\cdot+\boldsymbol{y}) - g\|_{L^2(\mathbb{R}^d)}.$$

By means of (III.27) and (III.28) we have $F_{\beta,\pm} \circ g \in H^1_2(\mathbb{R}^d)$ and for $i \in \{1, \dots, d\}$

$$\begin{aligned}\|\partial_{x_i} F_{\beta,\pm}(g)\|_{L^2(\mathbb{R}^d)} &\leq \sup_{y \neq 0} \frac{\|F_{\beta,\pm}(g(\cdot + y)) - F_{\beta,\pm}(g)\|_{L^2(\mathbb{R}^d)}}{|y|} \\ &\lesssim \|g\|^\beta_{L^\infty(\mathbb{R}^d)} \sup_{y \neq 0} \frac{\|g(\cdot + y) - g\|_{L^2(\mathbb{R}^d)}}{|y|} \lesssim \|g\|^\beta_{L^\infty(\mathbb{R}^d)} \|\nabla g\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

The previous estimate implies

$$\|F_{\beta,\pm}(g)\|_{H^1_2(\mathbb{R}^d)} \lesssim \|g\|^\beta_{L^\infty(\mathbb{R}^d)} \|g\|_{H^1_2(\mathbb{R}^d)}. \tag{III.29}$$

Now let $g \in H^1_2(\mathbb{R}^d)$ and choose q, r as in *(d)*. Then we have $\frac{1}{r^*} = \frac{1}{2} + \frac{\beta}{q} < 1$ and $2 \leq (\beta+1)r^* < e(d,1)$. By means of the Sobolev embedding A.2.1 we have $H^1_2(\mathbb{R}^d) \hookrightarrow L^{(\beta+1)r^*}(\mathbb{R}^d)$ and therefore $F_{\beta,\pm} \circ g \in L^{r^*}(\mathbb{R}^d)$ with

$$\|F_{\beta,\pm}(g)\|_{L^{r^*}(\mathbb{R}^d)} \lesssim \|g\|^{\beta+1}_{H^1_2(\mathbb{R}^d)}. \tag{III.30}$$

Hölder's inequality yields for $y \neq 0$ as above

$$\|F_{\beta,\pm}(g(\cdot + y)) - F_{\beta,\pm}(g)\|_{L^{r^*}(\mathbb{R}^d)} \lesssim \|g\|^\beta_{L^q(\mathbb{R}^d)} \|g(\cdot + y) - g\|_{L^2(\mathbb{R}^d)}.$$

Then (III.27), (III.28), and $H^1_2(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ imply $F_{\beta,\pm} \circ g \in H^1_{r^*}(\mathbb{R}^d)$ and

$$\|\nabla F_{\beta,\pm}(g)\|_{L^{r^*}(\mathbb{R}^d)} \lesssim \|g\|^\beta_{L^q(\mathbb{R}^d)} \|\nabla g\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|^{\beta+1}_{H^1_2(\mathbb{R}^d)}. \tag{III.31}$$

Consequently, (III.30) and (III.31) provide

$$\|F_{\beta,\pm}(g)\|_{H^1_{r^*}(\mathbb{R}^d)} \lesssim \|g\|^{\beta+1}_{H^1_2(\mathbb{R}^d)}. \tag{III.32}$$

(c+d;2) Let (Ω, g) satisfy (M). The estimate (III.29), $F_{\beta,\pm}(0) = 0$, and Theorem 4.2.2 and 4.3.2 in [Tri92b] yield

$$\begin{aligned}\|F_{\beta,\pm}(g)\|_{H^1_2(\Omega)} &\lesssim \Big(\sum_{i \in I} \|(F_{\beta,\pm}(\sum_{j \in J(i)} \psi_{r,j} g)) \circ \kappa_j^{-1}\|^2_{H^1_2(\mathbb{R}^d)} \Big)^{1/2} \\ &\lesssim \|g\|^\beta_{L^\infty(\Omega)} \Big(\sum_{i \in I} \sum_{j \in J(i)} \|(\psi_{r,i} g) \circ \kappa_j^{-1}\|^2_{H^1_2(\mathbb{R}^d)} \Big)^{1/2} \lesssim \|g\|^\beta_{L^\infty(\Omega)} \|g\|_{H^1_2(\Omega)}.\end{aligned}$$

We apply the same results of [Tri92b] combined with (III.32) and the embedding $\ell^2 \hookrightarrow \ell^{(\beta+1)r^*}$ to prove

$$\begin{aligned}\|F_{\beta,\pm}(g)\|_{H^1_{r^*}(\Omega)} &\lesssim \Big(\sum_{i \in I} \|(F_{\beta,\pm}(\sum_{j \in J(i)} \psi_{r,j} g)) \circ \kappa_j^{-1}\|^{r^*}_{H^1_{r^*}(\mathbb{R}^d)} \Big)^{1/r^*} \\ &\lesssim \Big(\sum_{i \in I} \Big(\sum_{j \in J(i)} \|(\psi_{r,j} g) \circ \kappa_j^{-1}\|_{H^1_2(\mathbb{R}^d)} \Big)^{(\beta+1)r^*} \Big)^{1/r^*} \\ &\lesssim \Big(\sum_{i \in I} \sum_{j \in J(i)} \|(\psi_{r,j} g) \circ \kappa_j^{-1}\|^2_{H^1_2(\mathbb{R}^d)} \Big)^{(\beta+1)/2} \lesssim \|g\|^{\beta+1}_{H^1_2(\Omega)}.\end{aligned}$$

This leaves the proof of estimate (III.25). We use the exact same argument as in the proof of the estimate (III.7) in Lemma III.1.3;(c). It relies on the product estimate in Lemma III.1.3;(a) and estimate (III.24). We omit the details. ☐

Having gathered all the necessary background material and nonlinear estimates we now come back to the nonlinear Schrödinger equation. We aim at local and global existence results of strong solutions for

$$\begin{aligned} \mathrm{i}u'(t) &= -\tilde{\Delta}_\Omega u(t) + F_{\beta,\pm}(u(t)), \quad t \neq t_0, \\ u(t_0) &= f, \end{aligned} \tag{CPM}$$

whereby (Ω, g) satisfies (M) and $f \in H_2^s(\Omega)$ with $s \in [1, \infty)$. Recall that $(-\tilde{\Delta}_\Omega, H_2^1(\Omega))$ is the extrapolation operator of $(-\Delta_\Omega, H_2^2(\Omega))$.
We first derive the following local existence result as a corollary of Theorem I.3.4. We state this as a base result from which we develop the existence theory for (CPM) further. Moreover, the next result can be applied in Chapter IV to construct maximal strong solutions for a nonlinear Schrödinger equation on certain product manifolds. It therefore plays a major role in one of the main results of this thesis.

Theorem III.1.5
Let $d \in \mathbb{N}_{\geq 2}$, $\ell \in [0,\infty)$, $s, \tilde{s} \in [1,\infty)$ with $s \leq \tilde{s}$, $\beta \in (1,\infty)$ and $(p,q) \in [2,\infty) \times [2,\infty]$ with $p > \max\{\beta, 2\}$. We additionally assume:

(i) $(e^{it\Delta_\Omega})_{t\in\mathbb{R}}$ satisfies a local (p,q) Strichartz estimate with ℓ-loss.

(ii) Either $\beta \in \mathbb{N}_{even}$ or $\beta > [s]+1$ or $s = 1$.

(iii) $\beta(d-2s) \leq 2(s+1)$ and $s > \frac{d}{q} + 2\ell$.

Then for each $f \in H_2^s(\Omega)$ the nonlinear Schrödinger equation (CPM) has a conditionally unique maximal strong solution $u \in C(I(f), H_2^s(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ with the following properties:

(a) u has $L^2(\Omega)$-conservation and the induced nonlinear flow is locally Lipschitz continuous.

(b) The nonlinear flow transports $H_2^{\tilde{s}}(\Omega)$ regularity if either $\beta \in \mathbb{N}_{even}$ or $\beta > [\tilde{s}]$. In that case u satisfies the blow-up alternative with respect to $H_2^{\tilde{s}}(\Omega)$.

(c) If $\beta(d-4) \leq 4$ and the nonlinear flow transports $H_2^{\tilde{s}}(\Omega)$ regularity with some $\tilde{s} \geq 2$, then u has energy conservation.

Remark:

(1) The key result of the proof is Theorem I.3.4. Let $(A, D(A))$ be a non-negative, selfadjoint linear operator on $L^2(\Omega)$ with $D(A^{s/2}) \cong H_2^s(\Omega)$ for all $s \in [1,\infty)$ and $D(A_q^\alpha) \hookrightarrow L^\infty(\Omega)$ for $\alpha > d/2q$. Then the assertions of the above theorem remain true if we substitute $(-\Delta_\Omega, H_2^2(\Omega))$ with $(A, D(A^{s/2}))$. See for example Theorem III.2.2 for a similar result.

(2) The case $d = 1$ is not interesting, since $H_2^1(\Omega) \hookrightarrow L^\infty(\Omega)$. In this case the nonlinear estimates (III.24) and (III.25) itself are sufficient for the construction of strong solutions of (CPM) in $C(I, H_2^1(\Omega))$. No Strichartz estimates are required.

Proof. Recall that by Theorem III.1.2;(b+d) holds $H_2^s(\Omega) \cong D((-\Delta_\Omega)^{s/2})$ and the usual Sobolev embeddings, in particular $D((-\Delta_\Omega)_q^\alpha) \hookrightarrow L^\infty(\Omega)$ for $\alpha > d/2q$. Since $H_2^s(\Omega) \cong D((-\Delta_\Omega)^{s/2})$ we want to stress that we apply Theorem I.3.4 with $s/2$, not s.

Let $f \in H^s_2(\Omega)$. We first check that the conditions in Theorem I.3.4 are satisfied. *(i)* clearly matches I.3.4;(i) and *(iii)* implies I.3.4;(iii). The condition in *(ii)* and Lemma III.1.4 imply the nonlinear estimates (III.23)-(III.25). Hence, the nonlinear estimates in I.3.4;(ii) are satisfied. Theorem I.3.4 provides a conditionally unique maximal mild solution $u \in C(I(f), H^s_2(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ of (CPM) with locally Lipschitz continuous nonlinear flow. Let us show that u is in fact a strong solution of (CPM). The condition $\beta(d-2s) \leq 2(s+1)$ implies $H^s_2(\Omega) \hookrightarrow L^q(\Omega)$ with $q := {}^{2d(\beta+1)}/_{(d+2)}$ by Theorem III.1.2;(d). The Lipschitz estimate (I.65) from Proposition I.4.2 implies for $g, h \in H^s_2(\Omega)$

$$\begin{aligned}\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{H^1_2(\Omega)^*} &\lesssim \|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{L^{\frac{2d}{d+2}}(\Omega)} \\ &\lesssim (\|g\|^\beta_{L^q(\Omega)} + \|h\|^\beta_{L^q(\Omega)})\|g-h\|_{L^q(\Omega)} \\ &\lesssim (\|g\|^\beta_{H^s_2(\Omega)} + \|h\|^\beta_{H^s_2(\Omega)})\|g-h\|_{H^s_2(\Omega)}.\end{aligned}$$

$F_{\beta,\pm} : H^s_2(\Omega) \to H^1_2(\Omega)^*$ is therefore Lipschitz continuous on bounded sets. Proposition I.2.4;(c) then provides that u is a strong solution of (CPM). $L^2(\Omega)$-conservation of u follows with Lemma I.4.2;(c) and Proposition I.2.8. This ends the proof of our existence claim and *(a)*. It remains to show *(b)* and *(c)*.

(b) This property follows from the nonlinear estimate (III.24) for $\tilde{s}$ and Theorem I.3.4;(d).

(c) We assume $\beta(d-4) \leq 4$ and let $\tilde{s} \geq 2$ for which the nonlinear flow transports $H^{\tilde{s}}_2(\Omega)$ regularity. It is enough to show energy conservation on an arbitrary $I \in \mathcal{I}_c$ with $t_0 \in I \subseteq I(f)$. We fix such an interval I and let $(f_n)_{n\in\mathbb{N}} \subseteq H^{\tilde{s}}_2(\Omega)$ with $f_n \xrightarrow{n\to\infty} f$ in $H^s_2(\Omega)$ with corresponding strong solutions $(u_n)_{n\in\mathbb{N}}$ of (NLS) with $u_n(t_0) = f_n$. Since $H^{\tilde{s}}_2(\Omega)$ regularity is transported by the nonlinear flow, we have $u_n \in C(I(f_n), H^{\tilde{s}}_2(\Omega)) \cap L^p_{loc}(I(f_n), L^\infty(\Omega))$ for all $n \in \mathbb{N}$. The local Lipschitz continuity of the nonlinear flow in $H^s_2(\Omega)$ implies the existence of $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ holds $I \subseteq I(f_n)$ and $u_n(t) \xrightarrow{n\to\infty} u(t)$ in $H^s_2(\Omega)$ for all $t \in I$. From now on let $n \geq n_0$. Note that $\beta(d-4) \leq 4$ and Theorem III.1.2;(d) imply the embedding $H^{\tilde{s}}_2(\Omega) \hookrightarrow H^2_2(\Omega) \hookrightarrow L^{2(\beta+1)}(Omega)$. The Lipschitz estimate (I.65) shows for $n \geq n_0$ and $t, s \in I$

$$\begin{aligned}&\|F_{\beta,\pm}(u_n(t)) - F_{\beta,\pm}(u_n(s))\|_{L^2(\Omega)} \\ &\qquad\lesssim (\|u_n(t)\|^\beta_{L^{2(\beta+1)}(\Omega)} + \|u_n(s)\|^\beta_{L^{2(\beta+1)}(\Omega)})\|u_n(t) - u_n(s)\|_{L^{2(\beta+1)}(\Omega)} \\ &\qquad\lesssim \|u_n\|^\beta_{L^\infty(I, H^{\tilde{s}}_2(\Omega))}\|u_n(t) - u_n(s)\|_{H^{\tilde{s}}_2(\Omega)}.\end{aligned}$$

Hence, $F_{\beta,\pm}(u_n) \in C(I, L^2(\Omega))$. The equation $\mathrm{i}u_n = -\Delta_\Omega u_n + F_{\beta,\pm}(u_n)$ in $H^1_2(\Omega)^*$ implies $u_n \in C^1(I, L^2(\Omega))$. Lemma I.4.3 then provides energy conservation for u_n. Let $t \in I$. $u_n(t) \xrightarrow{n\to\infty} u(t)$ in $H^s_2(\Omega)$ and $\mathcal{E} \in C(H^s_2(\Omega), \mathbb{R})$ then shows

$$\mathcal{E}(u(t)) = \lim_{n\to\infty} \mathcal{E}(u_n(t)) = \lim_{n\to\infty} \mathcal{E}(f_n) = \mathcal{E}(f).$$

□

The remaining section is devoted to the application of the results of Section II.4 in the context of the nonlinear Schrödinger equation (CPM). For these results to be applicable, we need much stronger assumptions on the properties of the Laplace-Beltrami operator. We need in particular that $(e^{it\Delta_\Omega})_{t\in\mathbb{R}}$ satisfies the spectrally localized Strichartz estimates and $(-\Delta_\Omega, H_2^2(\Omega))$ satisfies Bernstein inequalities and the (LP) property. All of these properties have been introduced in Section II.2.

Convention
In the rest of this section we let $\gamma \in [0,1]$ and assume the following:

(O1) *$(-\Delta_\Omega, H_2^2(\Omega))$ satisfies $(p, q, {}^{d}/_{2})$-Bernstein inequalities for all $1 \le p \le q \le \infty$.*

(O2) *$(e^{it\Delta_\Omega})_{t\in\mathbb{R}}$ satisfies (p,q) Strichartz estimates of SL-type γ if (p,q) is sharp ${}^{d}/_{2}$-admissible.*

(O3) *The spectral decomposition $((-\Delta_\Omega)_k)_{k\in\mathbb{N}_0}$ has the* (LP) *property.*

Remarks:

(1) In the examples of Section II.3 and II.4.2 we already gathered two operators for which the conditions (O1)-(O3) are satisfied. The first one is $(-\Delta, H_2^2(\mathbb{R}^d))$ with $\gamma = 0$. The second one is $(-\Delta_\Omega, H_2^2(\Omega))$ on a connected, compact Riemannian C^∞-manifold (Ω, g) without boundary and $\gamma = {}^{1}/_{2}$.

(2) We restrict our study to the case $\gamma \in [0,1]$. We already mentioned in Section II.3.7 that $\gamma = 1$ induces the Sobolev-type loss. The case $\gamma > 1$ is therefore not relevant.

Let us bring the conditions (O1)-(O3) into the context of Section II.2 and II.4, in particular with respect to (A1)-(A3). Let (p,q) be a sharp ${}^{d}/_{2}$-admissible pair.
The Bernstein inequalities in (O1) and the Sobolev embedding $H_2^1(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [2, e(d,1))$ do compare to the assumption in (A1). (O2) provides (A2) for the pair (p,q). Condition (A3) involves the nonlinearity and will be checked separately for $F_{\beta,\pm}$ in the proof of Theorem III.1.6. There we also analyze the expression $S_{F_{\beta,\pm}}$ from (II.60), which played a major role in Section II.4. By means of (O2) and (O3) we can also apply Lemma II.2.8;(a) to produce local (p,q) Strichartz estimate with ${}^{\gamma}/_{p}$-loss.
Now we state the central global existence result for $d \in \{2,3\}$ for the defocusing nonlinear Schrödiner equation (CPM). The proof combines the existence results in Theorem III.1.5, Theorem I.3.6 and the results of Section II.4.

Theorem III.1.6
Let $d \in \{2,3\}$, $\gamma \in [0, \frac{4-d}{2}]$, and $\beta \in [2,\infty)$. Besides (O1)-(O3) *we assume one of the following:*

(i) $\gamma = \frac{4-d}{2}$ and $\beta = 2$,

(ii) $\gamma < \frac{4-d}{2}$ and $\beta(d-2) < 4(1-\gamma)$.

Then for each $f \in H_2^1(\Omega)$ the defocusing nonlinear Schrödinger equation (CPM) *has a global strong solution $u \in C_b(\mathbb{R}, H_2^1(\Omega))$ with the following properties:*

(a) u has $L^2(\Omega)$- and energy conservation.

(b) If $\beta = 2$, then u is unconditionally unique. If $\beta > 2$, then there is $p \in (\beta, \infty)$ such that u is conditionally unique in $C_b(\mathbb{R}, H_2^1(\Omega)) \cap L^p_{loc}(\mathbb{R}, L^\infty(\Omega))$.

(c) The induced nonlinear flow transports $H_2^2(\Omega)$ regularity. If $\gamma < \frac{4-d}{2}$, then the nonlinear flow is locally Lipschitz continuous.

Remarks:

(1) The cases $d \in \{2,3\}$ and $\gamma = 1/2$ in the above theorem recover Theorem 2 and 3 in [BGT04b].

(2) For $\Omega = \mathbb{R}^3$ and $\gamma = 0$, the above theorem provides global solutions for (CPM) for $\beta \in [2,4)$ in $H_2^1(\mathbb{R}^3)$. This compares well to [Kat87], which establishes global existence in $H_2^1(\mathbb{R}^3)$ for $\beta < 4$.

(3) For $d = 4$ and $\beta = 2$ our method of proof fails. With $\Omega = \mathbb{R}^4$ and $\gamma = 0$ this corresponds to the energy-critical nonlinear Schrödinger equation. In this case global existence in $H_2^1(\mathbb{R}^4)$ was established in [RV07] by means of a much more refined analysis than presented below.

Proof. Some differences in the Sobolev embeddings and the sharp $d/2$-admissibility scales for $d = 2$ and $d = 3$ force us to consider these cases seperately. We start in (1) with the harder one $d = 3$ and use the same arguments in (2) for $d = 2$.
(1) Let $d = 3$. In (1.1) and (1.2) we consider $\beta = 2$. (1.1) contains the unconditional uniqueness for $\gamma \leq 1/2$ and the global existence for $\gamma < 1/2$. (1.2) contains global existence for $\gamma = 1/2$. (1.3) contains our claims for $\beta > 2$.
(1.1) Let $\beta = 2$ and $\gamma \leq 1/2$. We fix $I \in \mathcal{I}_c$ with $t_0 \in I$. Since $H_2^1(\Omega) \cong D((-\Delta_\Omega)^{1/2})$ we want to apply Theorem II.4.4 with $s := 1/2$ and the sharp $3/2$-admissible pair $(p,q) = (2,6)$. Then $\frac{1}{2} \geq \frac{3}{2q} + \frac{\gamma}{p}$. Due to (O1) and the Sobolev embedding $H_2^1(\Omega) \hookrightarrow L^6(\Omega)$ condition II.4.4;(i) is fulfilled. Also $F_{2,+}$ satisfies II.4.4;(ii) by (I.64). Now let $u \in L^\infty(I, H_2^1(\Omega))$ be a weak solution of (CPM). Lemma III.1.4;(d) provides for $g \in H_2^1(\Omega)$ the estimate

$$\|F_{2,+}(g)\|_{H^1_{6/5}(\Omega)} \lesssim \|g\|^3_{H^1_2(\Omega)}. \tag{III.33}$$

With $S_{F_{2,+}}(u)$ from (II.60) the Bernstein inequalities for $(-\Delta_\Omega, H_2^2(\Omega))$ from (O1) and (II.36) imply

$$\begin{aligned}
S_{F_{2,+}}(u) &= \sum_{k=1}^{\infty} 2^{k(\frac{3}{2q}+\gamma(\frac{1}{p}-\frac{1}{p^*}))} \|(-\Delta_\Omega)_k F_{2,+}(u)\|_{L^\infty(I,L^{q^*}(\Omega))} \\
&\lesssim \sum_{k=1}^{\infty} 2^{-\frac{k}{4}} \|(-\Delta_\Omega)_k (-\Delta_\Omega)^{1/2} F_{2,+}(u)\|_{L^\infty(I,L^{6/5}(\Omega))} \\
&\lesssim \|F_{2,+}(u)\|_{L^\infty(I,L^{6/5}(\Omega))} \lesssim \|u\|^3_{L^\infty(I,H^1_2(\Omega))}.
\end{aligned} \tag{III.34}$$

Hence, every weak solution u satisfies $S_{F_{2,+}}(u) < \infty$. Theorem II.4.4 provides that there is at most one weak solution to (CPM). The same then holds true for strong solutions of (CPM).
Let either $\gamma < 1/2$ and $s = 1$ or $\gamma = 1/2$ and $s \in (1,2)$. We fix $f \in H_2^s(\Omega)$. In both cases we choose a sharp $3/2$-admissible pair (p,q) with $p > 2$ such that $s > \frac{3}{2} + \frac{2(\gamma-1)}{p}$ (put $p \in (2, 4(1-\gamma))$ if $\gamma < 1/2$). The sharp $3/2$-admissibility of (p,q) implies $s > \frac{3}{q} + \frac{2\gamma}{p}$. By means of (O2), (O3), and Lemma II.2.8 we have a local (p,q) Strichartz estimate with γ/p-loss. Since $\beta = 2$ we also have the nonlinear estimates (III.24) and (III.25) at our disposal. Theorem III.1.5 then provides a unique maximal strong solution $u \in C(I(f), H_2^s(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ with blow-up alternative, $L^2(\Omega)$-conservation

and locally Lipschitz continuous nonlinear flow. (III.24) additionally holds for $s = 2$ and Theorem III.1.5;(b+c) therefore yields transport of $H_2^2(\Omega)$ regularity and energy conservation, respectively. The conservation laws and the defocusing nature of $F_{2,+}$ imply

$$\|u\|_{L^\infty(I(f),H_2^1(\Omega))} = \sup_{t\in I(f)} \left(\|u(t)\|^2_{L^2(\Omega)} + 2\mathcal{E}(u(t)) - \frac{1}{2}\|u(t)\|^4_{L^4(\Omega)} \right)^{1/2}$$
$$\lesssim \left(\|f\|^2_{L^2(\Omega)} + \mathcal{E}(f)\right)^{1/2} =: R < \infty. \tag{III.35}$$

If $\gamma < 1/2$ and $s = 1$, then estimate (III.35) proves $I(f) = \mathbb{R}$ with the blow-up alternative.

For the case $\gamma = 1/2$ and $s \in (1, 2)$ we assume $T_+ < \infty$ and put $I_+(f) := [t_0, t_0 + T_+)$. We want to use the a priori estimate from Theorem II.4.5 in $D((-\Delta_\Omega)^{s/2}) \cong H_2^s(\Omega)$. We again put $(p, q) = (2, 6)$ so that $\frac{1}{2} \geq \frac{3}{2q} + \frac{\gamma}{p}$. II.4.5;(iii) is therefore satisfied. By means of $\beta = 2$ and (O1) the conditions in II.4.5;(i)+(ii) are also satisfied. We combine (III.34) and (III.35) to prove with $C(x) := x + x^3$ the estimate

$$\|u\|_{L^\infty(I_+(f),H_2^1(\Omega))} + S_{F_{2,+}}(u) \lesssim C(R).$$

Theorem II.4.5;(b) then yields $u \in L^\infty(I_+(f), H_2^s(\Omega))$ and therefore $T_+ = \infty$ by the blow-up alternative. $T_- = \infty$ follows with the exact same argument.

(1.2) Let $\beta = 2$ and $\gamma = 1/2$. We fix $f \in H_2^1(\Omega)$ and choose $(f_n)_{n\in\mathbb{N}} \subseteq H_2^2(\Omega)$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in $H_2^1(\Omega)$. Then (1.1) provides a sequence $(u_n)_{n\in\mathbb{N}} \subseteq C(\mathbb{R}, H_2^2(\Omega)) \cap L^p_{loc}(\mathbb{R}, L^\infty(\Omega))$ of global strong solution of (CPM) with $u_n(t_0) = f_n$, which satisfy $L^2(\Omega)$- and energy conservation. With $\mathcal{E} \in C^1(H_2^1(\Omega), \mathbb{R})$ we provide similar to (III.35) a $n_0 \in \mathbb{N}$ such that

$$\sup_{n\geq n_0} \|u_n\|_{L^\infty(\mathbb{R},H_2^1(\Omega))} \lesssim \left(1 + \|f\|^2_{L^2(\Omega)} + \mathcal{E}(f)\right)^{1/2} < \infty. \tag{III.36}$$

Let us check the assumptions of Theorem I.3.6. The density of the embedding $H_2^1(\Omega) \hookrightarrow L^4(\Omega)$ and Proposition I.4.2;(b) show that I.3.6;(i)+(ii) is satisfied (with $(p, q) = (4, 4)$). Moreover, I.3.6;(iii) has been checked in Theorem III.1.2;(e). Theorem I.3.6 then provides on $I_T := [t_0 - T, t_0 + T]$ with an arbitrary $T \in (0, \infty)$ a weak solution $u_T \in C_w(I_T, H_2^1(\Omega))$ of (CPM) on I_T with $u_T(t_0) = f$, $L^2(\Omega)$-conservation and $\mathcal{E}(u(t)) \leq \mathcal{E}(f)$ for all $t \in I_T$. We in particular have

$$\|u_T\|_{L^\infty(I_T,H_2^1(\Omega))} \lesssim \left(\|f\|^2_{L^2(\Omega)} + \mathcal{E}(f)\right)^{1/2} \tag{III.37}$$

and $u_T \in C^{0,1/2}(I_T, L^2(\Omega))$ by Proposition I.2.7;(a). In (1.1) the weak solution u_T was shown to be unqiue . For $t_1 \in I \setminus \{t_0\}$ we can then construct a unique $v_T \in C_w(I_T, H_2^1(\Omega))$ with $v(t_1) = u(t_1)$, which satisfies $\mathcal{E}(v(s)) \leq \mathcal{E}(v(t_1))$ for all $s \in I_T$. The uniqueness implies $u(r) = v(r)$ for all $r \in I_T$. This implies $\mathcal{E}(f) \leq \mathcal{E}(u(t_1) \leq \mathcal{E}(f)$ and u_T therefore has energy conservation. Then Theorem I.3.6;(b) provides $u_T \in C(I_T, H_2^1(\Omega))$ and that u_T is a strong solution of (CPM) on I_T. As T was arbitrary and u_T unique we can construct a well-defined global strong solution $u \in C(\mathbb{R}, H_2^1(\Omega))$ of (CPM) by

$$u : \mathbb{R} \to H_2^1(\Omega), \qquad u(t) := u_1(t) + \sum_{n=2}^{\infty} u_n|_{I_n \setminus I_{n-1}}(t),$$

which also satisfies $L^2(\Omega)$- and energy conservation. Finally, (III.37) yields that $u \in L^\infty(\mathbb{R}, H_2^1(\Omega))$.

(1.3) Let $\beta \in (2, 4(1-\gamma))$ and $\gamma < 1/2$. We fix $f \in H_2^1(\Omega)$. We choose (p,q) sharp $3/2$-admissible such that $p \in (\beta, 4(1-\gamma))$. Then the sharp $3/2$-admissibility implies the inequality $1 > \frac{3}{2} - \frac{2(\gamma-1)}{p} = \frac{3}{q} + \frac{2\gamma}{p}$. As in (1.1) Theorem III.1.5 provides a maximal strong solution $u \in C(I(f), H_2^1(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ of (CPM) with $L^2(\Omega)$- and energy conservation, blow-up alternative and locally Lipschitz continuous nonlinear flow. We additionally have transport of $H_2^2(\Omega)$ regularity, since $\beta > 2$ and therefore (III.24) holds with $s = 2$. $I(f) = \mathbb{R}$ follows via the energy bound (III.35). This follows from the fact that $F_{\beta,+}$ is defocusing.

(2) Let $d = 2$. We proceed as for $d = 3$ where in the case $\gamma = 1$ instead of $\gamma = 1/2$ Theorem III.1.5 fails.

(2.1) Let $\beta = 2$ and $\gamma \leq 1$. We fix $I \in \mathcal{I}_c$ with $t_0 \in I$. We first show that (CPM) has at most one strong solution. We use Theorem II.4.4 for $s = 1/2$ and the sharp 1-admissible pairs $(p,q) = (4,4)$. The sharp 1-admissibility condition yields $\frac{1}{2} \geq \frac{1}{q} + \frac{\gamma}{p}$. $H_2^s(\Omega) \hookrightarrow L^4(\Omega)$ and (O1) imply II.4.4;(i). Also $F_{\beta,+}$ satisfies II.4.4;(ii) by (I.64). Let $u \in L^\infty(I, H_2^1(\Omega))$ be a weak solution of (CPM) on I. Lemma III.1.4;(d) provides for $g \in H_2^1(\Omega)$ the estimate

$$\|F_{2,+}(g)\|_{H^1_{4/3}(\Omega)} \lesssim \|g\|^3_{H_2^1(\Omega)}. \tag{III.38}$$

With the Bernstein inequalities for $(-\Delta_\Omega, H_2^2(\Omega))$, (II.36) and (III.38) follows

$$\begin{aligned} S_{F_{2,+}}(u) &= \sum_{k=1}^{\infty} 2^{k(\frac{1}{q}+\gamma(\frac{1}{p}-\frac{1}{p^*}))} \|(-\Delta_\Omega)_k F_{2,+}(u)\|_{L^\infty(I, L^{q^*}(\Omega))} \\ &\lesssim \sum_{k=1}^{\infty} 2^{-\frac{k(1+2\gamma)}{4}} \|(-\Delta_\Omega)^{1/2}(-\Delta_\Omega)_k F_{2,+}(u)\|_{L^\infty(I, L^{4/3}(\Omega))} \\ &\lesssim \|F_{2,+}(u)\|_{L^\infty(I, H^1_{4/3}(\Omega))} \lesssim \|u\|^3_{L^\infty(I, H_2^1(\Omega))}. \end{aligned} \tag{III.39}$$

Hence, every weak solution u satisfies $S_{F_{2,+}}(u) < \infty$. Theorem II.4.4 provides that there is at most one strong solution of (CPM) as before.

Let either $\gamma < 1$ and $s = 1$ or $\gamma = 1$ and $s \in (1,2)$. We fix $f \in H_2^s(\Omega)$. In both cases there is (p,q) sharp 1-admissible with $p > \beta$ and $s > 1 + \frac{2(\gamma-1)}{p} = \frac{2}{q} + \frac{2\gamma}{p}$. Theorem III.1.5 provides a unique maximal strong solution $u \in C(I(f), H_2^s(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ with blow-up alternative, $L^2(\Omega)$-conservation and Lipschitz continuous nonlinear flow. By means of $\beta = 2$ and (III.24) for $s = 2$ we have transport of $H_2^2(\Omega)$ regularity and therefore energy conservation. With the conservation laws and the defocusing nonlinearity $F_{\beta,+}$ we have

$$\|u\|_{L^\infty(I(f), H_2^1(\Omega))} \lesssim \left(\|f\|^2_{L^2(\Omega)} + 2\mathcal{E}(f)\right)^{1/2} =: R < \infty. \tag{III.40}$$

If $\gamma < 1$ and $s = 1$ the estimate (III.40) proves $I(f) = \mathbb{R}$ by the blow-up alternative.

For the case $\gamma = 1$ and $s \in (1,2)$ we assume $T_+ < \infty$ and put $I_+(f) := [t_0, t_0 + T_+)$. We want to use the a priori estimate from Theorem II.4.5 in $D((-\Delta_\Omega)^{s/2}) \cong H_2^s(\Omega)$. We put $(p,q) = (4,4)$. Then $\frac{s}{2} > \frac{1}{2} \geq \frac{1}{q} + \frac{1}{p}$. II.4.5;(iii) is therefore satisfied and so are II.4.5;(i)+(ii). With (III.39) and (III.40) follows with $C(x) := x + x^3$ the estimate

$$\|u\|_{L^\infty(I_+(f), H_2^1(\Omega))} + S_{F_{2,+}}(u) \lesssim C(R).$$

Theorem II.4.5 then implies $u \in L^\infty(I_+(f), H_2^s(\Omega))$ and therefore $T_+ = \infty$ by the blow-up alternative. The same argument provides $T_- = \infty$, hence $I(f) = \mathbb{R}$.
(2.2) In the limit case $\beta = 2$ and $\gamma = 1$ we repeat the exact same argument as in (1.2).
(2.3) Let $\beta \in (2, 4(1-\gamma))$ and $\gamma < 1/2$. We fix $f \in H_2^1(\Omega)$. We choose (p,q) sharp 1-admissible such that $p \in (\beta, 4(1-\gamma))$. Then the sharp 1-admissibility implies the inequality $1 > 1 - \frac{2(\gamma-1)}{p} = \frac{2}{q} + \frac{2\gamma}{p}$. As in (2.1) Theorem III.1.5 provides a maximal strong solution $u \in C(I(f), H_2^1(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ of (CPM) with $L^2(\Omega)$- and energy conservation, blow-up alternative and locally Lipschitz continuous nonlinear flow. We additionally have transport of $H_2^2(\Omega)$ regularity, since $\beta > 2$ and (III.24) holds with $s = 2$. $I(f) = \mathbb{R}$ follows via the energy bound (III.40). This follows from the fact that $F_{\beta,+}$ is defocusing.

□

III.2. Divergence form operators with potential on $\mathbb{R}^d$

The appendix of [BGT04b] contains the proof for local (p,q) Strichartz estimates with $1/2p$-loss for all sharp $d/2$-admissible pairs (p,q) for the Schrödinger group $(e^{-\mathrm{i}tP})_{t\in\mathbb{R}}$ generated by a certain divergence form operator $(P, H_2^2(\mathbb{R}^d))$ on $\mathbb{R}^d$. It is indicated in [BGT04b] that the global existence results of the paper should carry over to this situation but no proof is given.
In this section we want to consider operators of the form $P + V$ on $\mathbb{R}^d$, whereby V is a bounded potential. By means of a perturbation argument we transfer the local (p,q) Strichartz estimates $1/2p$-loss for $(e^{-\mathrm{i}tP})_{t\in\mathbb{R}}$ to the Schrödinger group $(e^{-\mathrm{i}t(P+V)})_{t\in\mathbb{R}}$. As a consequence, we are able to prove the existence of maximal strong solutions in $H_2^s(\mathbb{R}^d)$ for the corresponding nonlinear Schrödinger equation. We furthermore derive a global existence result in $H_2^1(\mathbb{R}^2)$.
Let $b \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $B := \mathrm{diag}(b_1, ..., b_d)$, and $V \in C^1(\mathbb{R}^d, \mathbb{R})$ satisfy the following:

(C1) $b_- := \inf_{i\in\{1,...,d\}, x\in\mathbb{R}^d} b_i(x) > 0$ and $V_- := \inf_{i\in\{1,...,d\}, x\in\mathbb{R}^d} V_i(x) \geq 0$,

(C2) For all $\alpha \in \mathbb{N}_0^d$ holds $\|\partial^\alpha b\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^d)} < \infty$ and $\sup_{\alpha\in\mathbb{N}_0^d,\ |\alpha|\leq 1} \|\partial^\alpha V\|_{L^\infty(\mathbb{R}^d)} < \infty$.

We define the linear operators

$$\begin{aligned} P &: C_c^\infty(\mathbb{R}^d) \to L^q(\mathbb{R}^d), \qquad Pf := -\det(B)^{-1/2}\,\mathrm{div}(\det(B)^{1/2} B^{-1} \cdot \nabla f), \\ A &: C_c^\infty(\mathbb{R}^d) \to L^q(\mathbb{R}^d), \qquad Af := Pf + Vf. \end{aligned} \tag{III.41}$$

Then we realize $(A, C_c^\infty(\mathbb{R}^d))$ on the space $L^q(\mathbb{R}^d)$ for $q \in (1,\infty)$ by means of the operator $(A_q, D(A_q))$ defined by

$$D(A_q) := \{f \in H_q^1(\mathbb{R}^d) \mid Af \in L^q(\mathbb{R}^d) \text{ in distributional sense}\},$$

$$A_q f := Af \quad \text{on } D(A_q).$$

We gather the needed properties of $(A_q, D(A_q))$ and its fractional powers in the next Lemma.

Lemma III.2.1
Let $d \in \mathbb{N}$, $q \in (1,\infty)$, and $\theta \in (0,1)$. Then $(A_q, D(A_q))$ is a closed linear operator with the following properties:

(a) $D(A_q) \cong H^2_q(\mathbb{R}^d)$ and $(A_q, D(A_q))$ has fractional powers $(A_q^\theta, D(A_q^\theta))$ with $D(A_q^\theta) \hookrightarrow L^\infty(\mathbb{R}^d)$ if $\theta > d/2q$.

(b) $(A_2, H^2_2(\mathbb{R}^d))$ is non-negative and selfadjoint on $L^2(\mathbb{R}^d)$ with respect to the weighted scalar product $\langle \cdot, \cdot \det(B)^{1/2} \rangle_{L^2(\mathbb{R}^d)}$ and $D(A_2^\theta) \cong H^{2\theta}_2(\mathbb{R}^d)$.

(c) $(e^{-\mathrm{i}tA_2})_{t\in\mathbb{R}}$ satisfies (p,r) Strichartz estimates with $1/2p$-loss for all sharp $d/2$-admissible $(p,r) \in [2,\infty] \times [2,\infty)$.

Remarks:

(1) In *(b)* the induced norm of the weighted scalar product $\langle \cdot, \cdot \det(B)^{1/2} \rangle_{L^2(\mathbb{R}^d)}$ is equivalent to $\|\cdot\|_{L^2(\mathbb{R}^d)}$ on $L^2(\mathbb{R}^d)$ by means of the properties of b.

(2) The perturbation argument we use in *(c)* can be applied for all sharp $d/2$-admissible pairs (p,q), to prove local (p,q) Strichartz estimates for $(e^{it(\Delta - V)})_{t\in\mathbb{R}}$ with $V \in L^\infty(\mathbb{R}^d)$.

Proof. Let $q \in (1,\infty)$. Note that the conditions on b imply that there are $C_1, C_2 \in (0,\infty)$ such that $C_1 \le \det(B(\boldsymbol{x})) \le C_2$ for all $\boldsymbol{x} \in \mathbb{R}^d$. We will use this fact below without mentioning.
(a) Since

$$Af = -\sum_{i=1}^d b_i \partial_{x_i}^2 f - \sum_{i=1}^d \left(\det(B)^{-1/2} b_i (\partial_{x_i} \det(B)^{1/2}) + \partial_{x_i} b_i \right) \partial_{x_i} f + Vf$$

the conditions (C1) and (C2) imply that $-A$ satisfies the assumptions in Theorem 3.1.1 in [Lun95]. Consequently, we have $D(A_q) \cong H^2_q(\mathbb{R}^d)$ and $(A_q, H^2_q(\mathbb{R}^d))$ is closed. Section 3.1.1 in [Lun95] furthermore provides that $(-A_q, H^2_q(\mathbb{R}^d))$ is sectorial. Then we can define fractional powers $(A_q^\theta, D(A_q^\theta))$ such that Proposition 2.2.15 in [Lun95] and Theorems 2.4.2.2 and 2.8.1;(d) in [Tri95] provide for $\theta \in (d/2q, 1)$ the embedding

$$D(A_q^\theta) \hookrightarrow (L^q(\mathbb{R}^d), H^2_q(\mathbb{R}^d))_{\theta,\infty} \cong B^{2\theta}_{q,\infty}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d).$$

For a definition of the Besov space $B^s_{p,q}(\mathbb{R}^d)$ and further details we refer to Section 2.3 in [Tri95].
(b) Let $q = 2$. The sesquilinear form

$$a : H^1_2(\mathbb{R}^d) \times H^1_2(\mathbb{R}^d) \to \mathbb{C},$$
$$a(f,g) := \sum_{i=1}^d \int_{\mathbb{R}^d} b_i \partial_{x_i} f \cdot \overline{\partial_{x_i} g} \det(B)^{1/2} \, \mathrm{d}\lambda + \int_{\mathbb{R}^d} V f \overline{g} \det(B)^{1/2} \, \mathrm{d}\lambda$$

is densely defined, continuous, accretive, closed and symmetric. Proposition 1.24 in Section 1.2.3 of [Ouh05] implies that $(\tilde{A}, D(\tilde{A}))$ given by

$$D(\tilde{A}) := \{ f \in L^2(\mathbb{R}^d) \mid \exists_{g \in L^2(\mathbb{R}^d)} \forall_{\varphi \in H^1_2(\mathbb{R}^d)} : \ a(f,\varphi) = \langle g, \varphi \det(B)^{1/2} \rangle_{L^2(\mathbb{R}^d)} \},$$
$$\tilde{A} f := g \quad \text{on } D(\tilde{A}),$$

is a well-defined, non-negative, and selfadjoint linear operator on $L^2(\Omega)$ with respect to the inner product $\langle \cdot, \cdot \ \det(B)^{1/2}\rangle_{L^2(\mathbb{R}^d)}$. We have $H_2^2(\mathbb{R}^d) \subseteq D(\tilde{A})$ and $D(\tilde{A}) \subseteq D(A_2) \cong H_2^2(\mathbb{R}^d)$. Hence, $D(\tilde{A}) = H_2^2(\mathbb{R}^d)$. Integration by parts and $\det(B)(H_2^1(\mathbb{R}^d)) = H_2^1(\mathbb{R}^d)$ then imply

$$\forall_{f \in H_2^2(\mathbb{R}^d), g \in H_2^1(\mathbb{R}^d)} : \ \langle \tilde{A} f, g\rangle_{L^2(\mathbb{R}^d)} = \langle A f, g\rangle_{L^2(\mathbb{R}^d)}.$$

Thus, $(\tilde{A}, D(\tilde{A})) = (A_2, D(A_2))$ and Corollary I.1.3;(c) yields for $\theta \in (0,1)$

$$D(A_2^\theta) \cong [L^2(\mathbb{R}^d), D(A_2)]_\theta \cong [L^2(\mathbb{R}^d), H_2^2(\mathbb{R}^d)]_\theta \cong H_2^{2\theta}(\mathbb{R}^d).$$

(c) Let $(p,q) \in [2,\infty] \times [2,\infty)$ be sharp $d/2$-admissible. The realizations of $(P, C_c^\infty(\mathbb{R}^d))$ and $(A, C_c^\infty(\mathbb{R}^d))$ given in (III.41) on $L^2(\mathbb{R}^d)$ are non-negative and selfadjoint operators on $L^2(\mathbb{R}^d)$. We denote them by $(P_2, H_2^2(\mathbb{R}^d))$ and $(A_2, H_2^2(\mathbb{R}^d))$, respectively. We furthermore have $D(A_2^{\theta/2}) \cong D(P_2^{\theta/2}) \cong H_2^\theta(\mathbb{R}^d)$ for $\theta \in (0,2)$, which implies

$$\|e^{-\mathrm{i}(\cdot)A_2}\|_{L^\infty(\mathbb{R}, \mathcal{L}(H_2^\theta(\mathbb{R}^d)))} \cong \|e^{-\mathrm{i}(\cdot)P_2}\|_{L^\infty(\mathbb{R}, \mathcal{L}(H_2^\theta(\Omega)))} \cong 1. \tag{III.42}$$

Local (p,q) Strichartz estimates are translation invariant in I. It is therefore enough to prove such an estimate for $I := [0,T]$ with an arbitrary $T \in (0,\infty)$. Moreover, we can assume $f \in H_2^2(\mathbb{R}^d)$ since $H_2^2(\mathbb{R}^d)$ is dense in $H_2^{1/p}(\mathbb{R}^d)$. Theorem 5 in [BGT04b] provides a non-decreasing $C : [0,\infty) \to [0,\infty)$ such that

$$\|e^{-\mathrm{i}(\cdot)P_2} f\|_{L^p(I, L^q(\mathbb{R}^d))} \leq C(T)\|f\|_{H_2^{1/p}(\mathbb{R}^d)}. \tag{III.43}$$

We define the function $u \in C^1(\mathbb{R}, H_2^2(\mathbb{R}^d))$ by $u(t) := e^{-itA_2} f$. u is the unique solution of the Cauchy problem

$$\begin{aligned} \mathrm{i}u'(t) &= P_2 u(t) + V u(t), \quad t \neq 0, \\ u(0) &= f. \end{aligned} \tag{III.44}$$

Observe that (C2) yields $V \in W_\infty^1(\mathbb{R}^d)$ and therefore $V \in \mathcal{L}(H_2^1(\mathbb{R}^d))$. By means of complex interpolation with $V \in \mathcal{L}(L^2(\mathbb{R}^d))$ we have $V \in \mathcal{L}(H_2^\theta(\mathbb{R}^d))$ for all $\theta \in (0,1)$. We then have $Vu \in C(\mathbb{R}, H_2^1(\mathbb{R}^d))$ and we can consider (III.44) as an inhomogeneous equation with respect to $(P_2, H_2^2(\mathbb{R}^d))$. Duhamel's formula yields for all $t \in \mathbb{R}$

$$u(t) = e^{-\mathrm{i}tA_2} f = e^{-\mathrm{i}tP_2} f - \mathrm{i} \int_0^t e^{-\mathrm{i}(t-s)P_2} V u(s) \, \mathrm{d}s.$$

The previous formula combined with Minkowski's integral inequality, (III.43), $V \in \mathcal{L}(H_2^{1/p}(\mathbb{R}^d))$, and (III.42) provides

$$\begin{aligned} \|e^{-\mathrm{i}(\cdot)A_2} f\|_{L^p(I, L^q(\mathbb{R}^d))} &\leq \|e^{-\mathrm{i}(\cdot)P_2} f\|_{L^p(I, L^q(\mathbb{R}^d))} + \|e^{-\mathrm{i}(\cdot)P_2} V e^{-\mathrm{i}(\cdot)A_2} f\|_{L^1(I, H_2^{1/p}(\mathbb{R}^d))} \\ &\leq C(T)\|f\|_{H_2^{1/p}(\mathbb{R}^d)} \cong C(T)\|f\|_{D(A_2^{1/2p})}. \end{aligned}$$

□

We now turn to the construction of solutions for the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}u'(t) &= \tilde{A}_2 u(t) + F_{\beta,\pm}(u(t)), \quad t \neq t_0, \\ u(t_0) &= f. \end{aligned} \tag{III.45}$$

Recall that $(\tilde{A}_2, H_2^1(\mathbb{R}^d))$ is the extrapolation operator of $(A_2, H_2^2(\mathbb{R}^d))$. Compared to Theorem III.1.5 in the previous section we could say that we are in a "$\gamma = 1/2$" situation and we expect analogous results to hold.
Indeed, we first apply Theorem I.3.4 to deduce an existence result for maximal strong solutions of (III.45) in $H_2^s(\mathbb{R}^d)$.

Theorem III.2.2
Let $d \in \mathbb{N}_{\geq 2}$, $s, \tilde{s} \in [1,2]$ with $s \leq \tilde{s}$, and $\beta \in (1,\infty)$, which are assumed to satisfy:

(i) $\beta(d-2s) \leq 2(s+1)$ *and* $s > \frac{d}{2} - \frac{1}{\beta}$.

(ii) Either $\beta \in \mathbb{N}_{even}$ *or* $\beta > [s]+1$ *or* $s = 1$.

Then there is $p \in (\max\{\beta,2\},\infty)$ such that for each $f \in H_2^s(\mathbb{R}^d)$ the nonlinear Schrödinger equation (III.45) has a conditionally unique maximal strong solution $u \in C(I(f), H_2^s(\mathbb{R}^d)) \cap L^p_{loc}(I(f), L^\infty(\mathbb{R}^d))$ with the following properties:

(a) u has $L^2(\mathbb{R}^d)$-conservation and the induced nonlinear flow is locally Lipschitz continuous.

(b) The nonlinear flow transports $H_2^{\tilde{s}}(\Omega)$ regularity if either $\beta \in \mathbb{N}_{even}$ of $\beta > [\tilde{s}]$. In that case u satisfies the blow-up alternative with respect to $H_2^{\tilde{s}}(\Omega)$.

(c) If $\beta(d-4) \leq 4$ and u transports $H_2^{\tilde{s}}(\mathbb{R}^d)$ regularity for some $\tilde{s} \geq 2$, then u has energy conservation.

Proof. We fix $f \in H_2^s(\mathbb{R}^d)$ and let (p,q) be a sharp $d/2$-admissible pair such that $p \in (\max\{\beta,2\},\infty)$ and $\frac{s}{2} > \frac{d}{4} - \frac{1}{2\beta} > \frac{d}{2q} + \frac{1}{2p}$. Condition *(ii)* and Lemma III.1.4 yield the nonlinear estimates (III.23), (III.24), and (III.25). Lemma III.2.1;(c) provides a local (p,q) Strichartz estimate with $1/2p$-loss. Theorem I.3.4 then implies the existence of a unique maximal mild solution $u \in C(I(f), H_2^s(\mathbb{R}^d)) \cap L^p_{loc}(I(f), L^\infty(\mathbb{R}^d))$ of (III.45) with a locally Lipschitz continuous nonlinear flow. The condition $\beta(d-2s) \leq 2(s+1)$ and the Sobolev embedding A.2.1 imply that $H_2^s(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$ with $q = 2d(\beta+1)/(d+2)$. The Lipschitz estimate (I.65) from Proposition I.4.2;(b) then implies for $g, h \in H_2^s(\mathbb{R}^d)$

$$\begin{aligned} \|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{H_2^1(\mathbb{R}^d)^*} &\lesssim \|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{L^{2d/(d+2)}(\mathbb{R}^d)} \\ &\lesssim \left(\|g\|^\beta_{L^q(\mathbb{R}^d)} + \|h\|^\beta_{L^q(\mathbb{R}^d)}\right)\|g-h\|_{L^q(\mathbb{R}^d)} \\ &\lesssim \left(\|g\|^\beta_{H_2^s(\mathbb{R}^d)} + \|h\|^\beta_{H_2^s(\mathbb{R}^d)}\right)\|g-h\|_{H_2^s(\mathbb{R}^d)}. \end{aligned}$$

Hence, $F_{\beta,\pm}(u) \in C(I(f), E_A^*)$ and Proposition I.2.4 yields that u is in fact a strong solution of (III.45). Moreover, Lemma I.2.8 provides $L^2(\mathbb{R}^d)$-conservation. This finishes the proof of *(a)*.
(b) This property follows from the nonlinear estimate (III.24) for $\tilde{s}$ and Theorem I.3.4;(d).
(c) We repeat the exact same argument from the proof of Theorem III.1.5;(c) and we provide all the details. We assume $\beta(d-4) \leq 4$ and let $\tilde{s} \geq 2$ for which the nonlinear

flow transports $H_2^{\tilde{s}}(\mathbb{R}^d)$ regularity. It is enough to show energy conservation on an arbitrary $I \in \mathcal{I}_c$ with $t_0 \in I \subseteq I(f)$. We fix such an interval I and let $(f_n)_{n\in\mathbb{N}} \subseteq H_2^{\tilde{s}}(\mathbb{R}^d)$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in $H_2^s(\mathbb{R}^d)$ with corresponding strong solutions $(u_n)_{n\in\mathbb{N}}$ of (III.45) with $u_n(t_0) = f_n$. Since $H_2^{\tilde{s}}(\mathbb{R}^d)$ regularity is transported by the nonlinear flow, we have $u_n \in C(I(f_n), H_2^{\tilde{s}}(\mathbb{R}^d)) \cap L^p_{loc}(I(f_n), L^\infty(\mathbb{R}^d))$ for all $n \in \mathbb{N}$. The local Lipschitz continuity of the nonlinear flow in $H_2^s(\mathbb{R}^d)$ implies the existence of $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ holds $I \subseteq I(f_n)$ and $u_n(t) \overset{n\to\infty}{\longrightarrow} u(t)$ in $H_2^s(\mathbb{R}^d)$ for all $t \in I$. From now on let $n \geq n_0$. Note that $\beta(d-4) \leq 4$ and the Sobolev embedding A.2.1 imply the embedding $H_2^{\tilde{s}}(\mathbb{R}^d) \hookrightarrow H_2^2(\mathbb{R}^d) \hookrightarrow L^{2(\beta+1)}(\mathbb{R}^d)$. The Lipschitz estimate (I.65) shows for $n \geq n_0$ and $t, s \in I$

$$\begin{aligned}
&\|F_{\beta,\pm}(u_n(t)) - F_{\beta,\pm}(u_n(s))\|_{L^2(\Omega)} \\
&\qquad\lesssim (\|u_n(t)\|^\beta_{L^{2(\beta+1)}(\Omega)} + \|u_n(s)\|^\beta_{L^{2(\beta+1)}(\Omega)})\|u_n(t) - u_n(s)\|_{L^{2(\beta+1)}(\Omega)} \\
&\qquad\lesssim \|u_n\|^\beta_{L^\infty(I, H_2^{\tilde{s}}(\Omega))}\|u_n(t) - u_n(s)\|_{H_2^{\tilde{s}}(\Omega)}.
\end{aligned}$$

Hence, $F_{\beta,\pm}(u_n) \in C(I, L^2(\mathbb{R}^d))$. The equation $\mathrm{i}u_n = \tilde{A}_2 u_n + F_{\beta,\pm}(u_n)$ in $H_2^1(\mathbb{R}^d)^*$ implies $u_n \in C^1(I, L^2(\mathbb{R}^d))$. Lemma I.4.3 then provides energy conservation for u_n. Let $t \in I$. $u_n(t) \overset{n\to\infty}{\longrightarrow} u(t)$ in $H_2^s(\mathbb{R}^d)$ and $\mathcal{E} \in C(H_2^s(\mathbb{R}^d), \mathbb{R})$ then shows

$$\mathcal{E}(u(t)) = \lim_{n\to\infty} \mathcal{E}(u_n(t)) = \lim_{n\to\infty} \mathcal{E}(f_n) = \mathcal{E}(f).$$

□

The energy methods from Lemma I.3.5 allow us to deduce the following global existence result for (III.45) in $H_2^1(\mathbb{R}^2)$.

Corollary III.2.3
Let $\beta \in [2, \infty)$, $p \in (\beta, \infty)$, and $f \in H_2^1(\mathbb{R}^2)$. Then the nonlinear Schrödinger equation (III.45) has a conditionally unique maximal strong solution $u \in C(I(f), H_2^1(\mathbb{R}^2)) \cap L^p_{loc}(I(f), L^\infty(\mathbb{R}^2))$ with the following properties:

(a) u has $L^2(\mathbb{R}^2)$- and energy conservation.

(b) If either the equation is defocusing or $\beta = 2$ and $\|f\|_{L^2(\mathbb{R}^2)}$ is small enough, then $I(f) = \mathbb{R}$ and $u \in C_b(\mathbb{R}, H_2^1(\mathbb{R}^2))$.

(c) The induced nonlinear flow is locally Lipschitz continuous. It furthermore transports $H_2^2(\mathbb{R}^2)$-regularity.

Proof. We fix $f \in H_2^1(\mathbb{R}^2)$. Since $\beta \geq 2$ the conditions III.2.2;(i)+(ii) are satisfied. Theorem III.2.2 provides a unique maximal strong solution $u \in C(I(f), H_2^1(\mathbb{R}^2)) \cap L^p_{loc}(I(f), L^\infty(\mathbb{R}^2))$ of (III.45). Theorem III.2.2;(a) provides $L^2(\mathbb{R}^2)$-conservation and the local Lipschitz continuity of the nonlinear flow. Moreover, Theorem III.2.2;(b)+(c) provides transport of $H_2^2(\mathbb{R}^2)$ regularity (recall $\beta \geq 2$) and energy conservation of u. It remains to show *(b)*. We derive the criteria for $I(f) = \mathbb{R}$ from Lemma I.3.5, whose assumptions we check now. We have already established $L^2(\mathbb{R}^2)$- and energy conservation and therefore I.3.5;(i) is satisfied. Since $(\beta+2)\hat{F}_{\beta,\pm}(g) = \pm\|g\|^{\beta+2}_{L^{\beta+2}(\mathbb{R}^2)}$ the condition I.3.5;(ii) is satisfied in the defocusing case. For the focusing case and $\beta = 2$ the estimate (I.52) shows for $g \in H_2^1(\mathbb{R}^2)$ that

$$-\|g\|_{L^4(\mathbb{R}^2)}^4 \gtrsim -\|g\|_{L^2(\mathbb{R}^2)}^2 \|g\|_{H_2^1(\mathbb{R}^2)}^2.$$

Hence, I.3.5;(ii) is satisfied with $\beta_2 = 2$. In both cases Lemma I.3.5 provides $u \in L^\infty(I(f), H_2^1(\mathbb{R}^2))$ under the given assumptions in *(b)* above. This shows $I(f) = \mathbb{R}$ by the blow-up alternative. □

III.3. Schrödinger operators with superquadratic potentials on $\mathbb{R}^d$

The remark after Lemma III.2.1 states that local (p,q) Strichartz estimates without loss hold for the Schrödinger group generated by Schrödinger operators $-\Delta + V$ with bounded potential. In [YZ04] local (p,q) Strichartz estimates with ℓ-loss are shown for the Schrödinger group generated by Schrödinger operators with certain unbounded potentials. Moreover, a local existence result in the fashion of Theorem I.3.4 is proven. In this section we briefly want to discuss their setting and how it fits into the framework of Section I.3. We additionally state a slightly extended version of Theorem 1.5 in [YZ04].
Let $q \in (1,\infty)$ and $V \in C^\infty(\mathbb{R}^d, \mathbb{R})$ which satisfies $\inf_{x\in\mathbb{R}^d} V(x) \geq 1$. We assume that there is $m \in (2,\infty)$ with the following properties:

(V1) There is $R > 0$ such that $V(x) \cong (1+|x|^2)^{\frac{m}{2}}$ for $x \in B(\mathbf{0}, R)^c$,

(V2) For all $\alpha \in \mathbb{N}_0^d$ holds $|(\partial^\alpha V)(x)| \leq C_\alpha (1+|x|^2)^{\frac{m-|\alpha|}{2}}$.

We define the differential expression

$$A : C_c^\infty(\mathbb{R}^d) \to L^q(\mathbb{R}^d), \qquad Af := -\Delta f + Vf.$$

In [HS96] Theorem 8.14 it is shown that $(A, C_c^\infty(\mathbb{R}^d))$ is essentially selfadjoint on $L^2(\mathbb{R}^d)$ with unique selfadjoint and positive definite extension $(A_2, D(A_2))$. Lemma 2.4 in [YZ04] shows for $s \in [0,\infty)$ that $(A_2^s, D(A_2^s))$ can be extended to a closed operator on $(A_q^s, D(A_q^s))$ on $L^q(\mathbb{R}^d)$. They additionally show

$$D(A_q^{s/2}) = \overline{C_c^\infty(\mathbb{R}^d)}^{\|\cdot\|_{D(A_q^{s/2})}}, \qquad \|f\|_{D(A_q^{s/2})} := \|f\|_{H_q^s(\mathbb{R}^d)} + \|(1+|\cdot|^2)^{\frac{ms}{4}} f\|_{L^q(\mathbb{R}^d)}.$$

Consequently, the Sobolev embedding A.2.1 implies $D(A_q^\alpha) \hookrightarrow L^\infty(\mathbb{R}^d)$ for $\alpha > d/2q$ and $D(A_2^{s/2}) \hookrightarrow L^q(\mathbb{R}^d)$ for either $q \in [2, e(d,s))$ or $q = e(d,s)$ if $d > 2s$. Theorem 1.3 in [YZ04] states for the Schrödinger group $(e^{-itA_2})_{t\in\mathbb{R}}$ local (p,q) Strichartz estimates with ℓ-loss for all sharp $d/2$-admissible pairs (p,q) and $\ell > \frac{1}{p}(\frac{1}{2} - \frac{1}{m})$. The Lipschitz estimate (I.64) implies for $g, h \in L^2(\mathbb{R}^d, \phi \mathrm{d}\lambda) \cap L^\infty(\mathbb{R}^d)$ with some $\phi \in C(\mathbb{R}^d)$ the estimates

$$\|F_{\beta,\pm}(g)\|_{L^2(\mathbb{R}^d,\phi\mathrm{d}\lambda)} \lesssim \|g\|_{L^\infty(\mathbb{R}^d)}^\beta \|g\|_{L^2(\mathbb{R}^d,\phi\mathrm{d}\lambda)},$$

$$\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{L^2(\mathbb{R}^d,\phi\mathrm{d}\lambda)} \lesssim \left(\|g\|_{L^\infty(\mathbb{R}^d)}^\beta + \|h\|_{L^\infty(\mathbb{R}^d)}^\beta\right) \|g-h\|_{L^2(\mathbb{R}^d,\phi\mathrm{d}\lambda)}.$$

Combined with the nonlinear estimates of Lemma III.1.4 we have all the nonlinear estimates from I.3.4;(ii) for $F_{\beta,\pm}$ on $D(A_2^{s/2})$. With $\alpha > d/2q$ and $\ell > \frac{1}{p}(\frac{1}{2} - \frac{1}{m})$ follows

$$\alpha + \ell > \frac{d}{2q} + \frac{1}{p}\left(\frac{1}{2} - \frac{1}{m}\right) = \frac{d}{4} - \frac{1}{p}\left(\frac{1}{2} + \frac{1}{m}\right).$$

As a consequence of Theorem I.3.4 we then recover the following existence result for the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}u'(t) &= \tilde{A}_2 u(t) + F_{\beta,\pm}(u(t)), \quad t \neq t_0 \\ u(t_0) &= f. \end{aligned} \tag{III.46}$$

Theorem III.3.1
Let $t_0 \in \mathbb{R}$, $d \in \mathbb{N}$, $s \in [0,\infty)$, $\beta \in (1,\infty)$ and $(p,q) \in [2,\infty)^2$ sharp $d/2$-admissible with $p > \beta$. We furthermore assume:

(i) $s > \frac{d}{2} - \frac{1}{p}\left(1 + \frac{2}{m}\right)$.

(ii) Either $\beta \in \mathbb{N}_{even}$ or $\beta > [s] + 1$ or $s = 1$.

Then for each $f \in D(A_2^{s/2})$ the nonlinear Schrödinger equation (III.46) has a conditionally unique maximal mild solution $u \in C(I(f), D(A_2^{s/2})) \cap L^p_{loc}(I(f), L^\infty(\mathbb{R}^d))$ with the following properties:

(i) The induced nonlinear flow is locally Lipschitz continuous.

(ii) If $\beta(d - 2s) \leq 2(s+1)$, then u is a strong solution of (III.46) with $L^2(\mathbb{R}^d)$-conservation.

Remark: Compared to Theorem 1.5 in [YZ04] we state in the above theorem maximal solutions with uniqueness in $C(I(f), D(A_2^s)) \cap L^p_{loc}(I(f), L^\infty(\mathbb{R}^d))$ instead in the smaller space $C(I(f), D(A_2^s)) \cap L^p_{loc}(I(f), D(A_q^{s-\ell}))$. We additionally obtain local Lipschitz continuity of the nonlinear flow instead of continuity.

III.4. The Dirichlet Laplacian on bounded domains

In this section we treat the nonlinear Schrödinger equation for the Dirichlet Laplacian on a bounded domain with initial data in the energy space and the model nonlinearity $F_{\beta,\pm}$. In order to avoid some inessential technical difficulties we choose $\beta \in \mathbb{N}_{even}$. Proposition I.4.2;(a) then ensures $F_{\beta,\pm} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$.
We first fix the notion of a Lipschitz and C^∞-domain.

Definition III.4.1
Let $d \in \mathbb{N}_{\geq 2}$ and $\Omega \subseteq \mathbb{R}^d$ be a bounded domain.

(a) Ω is called a Lipschitz domain, if there is $N \in \mathbb{N}$ and an open cover $(\Omega_i)_{i=0}^N$ of Ω with the following properties:

- $\overline{\Omega_0} \subseteq \Omega$ *and* $\Omega_i \cap \partial\Omega \neq \emptyset$ *for* $i \in \{1, ..., N\}$.

- *There is a family of open sets $(O_i)_{i=1}^N$ in $\mathbb{R}^{d-1}$ and functions $(b_i)_{i=1}^N$ in $C^{0,1}(O_i, \mathbb{R})$ such that for $i \in \{1, ..., N\}$ holds (possibly after permutation of coordinates)*

$$\partial\Omega \cap \Omega_i = \{x \in \mathbb{R}^d \mid (x_1, ..., x_{d-1}) \in O_i,\ x_d = b_i(x_1, ..., x_{d-1})\},$$
$$\Omega \cap \Omega_i \subseteq \{x \in \mathbb{R}^d \mid (x_1, ..., x_{d-1}) \in O_i,\ x_d > b_i(x_1, ..., x_{d-1})\}.$$

(b) Ω is called a C^∞-domain if it is a Lipschitz domain such that $b_i \in C^\infty(O_i, \mathbb{R})$ for all $i \in \{1, ..., N\}$.

Remarks III.4.2
Let us give the most important examples and comment on possible extensions.

(1) Clearly all bounded C^∞-domains are bounded Lipschitz domains.

(2) We are interested in the following special Lipschitz domains. Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. We call $\Omega \subseteq \mathbb{R}^2$ polygonal, if there is $P \in \mathbb{N}$ and $x_1, ..., x_P$ such that with $x_{P+1} := x_1$ holds $\partial\Omega = \bigcup_{i=1}^P L_i$ with

$$L_i := \{x \in \mathbb{R}^2 \mid \exists_{t\in[0,1]} :\ x = x_i + t(x_{i+1} - x_i)\}.$$

(3) The notion of a bounded Lipschitz domain has a straightforward generalization to unbounded domains. The notion of a strong local Lipschitz domain (not necessarily bounded) introduced in Definition 4.9 of [AF03] is such a generalization. For such domains Lemma III.4.3 below would still hold true. This is relevant since local (p, q) Strichartz estimates with and without loss have been proven on certain unbounded exterior domains in [BSS12] and [Iva10]. We therefore could study consequences of Theorem I.3.4 for the nonlinear Schrödinger equation on such domains. However, we will not pursue this here and restrict our study to the case of bounded domains.

(4) We will need a bounded extension operator in Sobolev spaces. Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain (or an unbounded strong local Lipschitz domain). Then Stein's extension theorem formulated in Theorem 5.24 of [AF03] states the existence of ext_Ω such that for all $k \in \mathbb{N}_0$ and $p \in [1, \infty)$ we have $\mathrm{ext}_\Omega \in \mathcal{L}(H_p^k(\Omega), H_p^k(\mathbb{R}^d))$ and $(\mathrm{ext}_\Omega f)|_\Omega = f$ for $f \in H_p^k(\Omega)$. The construction of ext_Ω furthermore provides for $f \in H_p^k(\Omega) \cap L^\infty(\Omega)$ the inequality $\|\mathrm{ext}_\Omega f\|_{L^\infty(\mathbb{R}^d)} \leq \|f\|_{L^\infty(\Omega)}$. The complex interpolation Theorem A.1.3 then implies $\mathrm{ext}_\Omega \in \mathcal{L}(H_p^s(\Omega), H_p^s(\mathbb{R}^d))$ for all $s \in [0, \infty)$ and $p \in (1, \infty)$.

We produce next the needed nonlinear estimates for $F_{\beta,\pm}$. We use the extension operator ext_Ω to transfer estimates known on $H_2^k(\mathbb{R}^d)$ to $H_2^k(\Omega)$. The prior have been proved in Lemma III.1.3.

Lemma III.4.3
Let $d \in \mathbb{N}_{\geq 2}$, $k \in \mathbb{N}$, $\beta \in \mathbb{N}_{even}$, and $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Then holds

$F_{\beta,\pm}(H_2^k(\Omega)\cap L^\infty(\Omega)) \subseteq H_2^k(\Omega)\cap L^\infty(\Omega)$ and

$$\|F_{\beta,\pm}(g)\|_{H_2^k(\Omega)} \lesssim \|g\|_{L^\infty(\Omega)}^{\beta}\|g\|_{H_2^k(\Omega)}, \tag{III.47}$$

$$\begin{aligned}\|F_{\beta,\pm}(g)-F_{\beta,\pm}(h)\|_{H_2^k(\Omega)} &\lesssim \big(\|g\|_{L^\infty(\Omega)}^{\beta}+\|h\|_{L^\infty(\Omega)}^{\beta}\big)\|g-h\|_{H_2^k(\Omega)} \\ &\quad+\big(\|g\|_{L^\infty(\Omega)}^{\beta-1}+\|h\|_{L^\infty(\Omega)}^{\beta-1}\big)\big(\|g\|_{H_2^k(\Omega)}+\|h\|_{H_2^k(\Omega)}\big)\|g-h\|_{L^\infty(\Omega)}.\end{aligned} \tag{III.48}$$

Moreover, $F_{\beta,\pm}(H_{2,0}^1(\Omega)\cap L^\infty(\Omega)) \subseteq H_{2,0}^1(\Omega)\cap L^\infty(\Omega)$.

Proof. Since $\beta\in\mathbb{N}_{even}$ the nonlinearity $F_{\beta,\pm}\in C^\infty(\mathbb{R}^2,\mathbb{R}^2)$ is a polynomial in z and $\bar z$. Then the properties of the extension operator ext_Ω from Remark III.4.2;(4) and the product estimate from Lemma III.1.3;(a) with $\Omega=\mathbb{R}^d$ provides for $g,h\in H_2^1(\Omega)\cap L^\infty(\Omega)$ the estimate

$$\begin{aligned}\|gh\|_{H_2^k(\Omega)} &= \|(\mathrm{ext}_\Omega g)|_\Omega\cdot(\mathrm{ext}_\Omega h)|_\Omega\|_{H_2^k(\Omega)} \\ &\lesssim \|\mathrm{ext}_\Omega g\|_{L^\infty(\mathbb{R}^d)}\|\mathrm{ext}_\Omega h\|_{H_2^k(\mathbb{R}^d)}+\|\mathrm{ext}_\Omega g\|_{H_2^k(\mathbb{R}^d)}\|\mathrm{ext}_\Omega h\|_{L^\infty(\mathbb{R}^d)} \\ &\lesssim \|g\|_{L^\infty(\Omega)}\|h\|_{H_2^k(\Omega)}+\|g\|_{H_2^k(\Omega)}\|h\|_{L^\infty(\Omega)}.\end{aligned} \tag{III.49}$$

A successive application of (III.49) implies (III.47). The estimate (III.48) then follows via the fundamental theorem as in the proof of Lemma III.1.3;(c). For the remaining assertion it is sufficient to prove that $H_{2,0}^1(\Omega)\cap L^\infty(\Omega)$ is closed under multiplication. We fix $g,h\in H_{2,0}^1(\Omega)\cap L^\infty(\Omega)$. By means of the estimate (III.49) we have $gh\in H_2^1(\Omega)\cap L^\infty(\Omega)$, which leaves us to prove $gh\in H_{2,0}^1(\Omega)$. To this end we first construct a sequence $(h_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ such that

$$h_n \overset{n\to\infty}{\longrightarrow} h \quad \text{in } H_2^1(\Omega), \qquad \sup_{n\in\mathbb{N}}\|h_n\|_{L^\infty(\Omega)}<\infty. \tag{III.50}$$

Following the proof of Theorem 3.17 in [AF03] we define

$$\begin{aligned}\Omega_m &:= \{x\in\Omega \mid |x|<m \ \wedge\ d(\{x\},\partial\Omega)>1/m\}, \qquad (m\in\mathbb{N})\\ \Omega_0=\Omega_{-1} &:= \emptyset,\\ O_m &:= \Omega_{m+1}\cap(\overline{\Omega_{m-1}})^c,\\ U_m &:= \Omega_{m+2}\cap(\overline{\Omega_{m-2}})^c.\end{aligned}$$

$\mathscr{O}:=\{O_m \mid m\in\mathbb{N}\}$ is an open cover of Ω which admits a C^∞-partition of unity $(\chi_m)_{m\in\mathbb{N}}$ subordinate to $\mathscr{O}$ such that for all $m\in\mathbb{N}$ only χ_m satisfies $\mathrm{supp}(\chi_m)\subseteq O_m$. Let furthermore $(\varphi_\epsilon)_{\epsilon\in(0,1)}$ be an approximate identity and $(\epsilon_m)_{m\in\mathbb{N}}$ a sequence in $(0,1)$ with $\epsilon_m<1/(m+1)(m+2)$. This implies $\mathrm{supp}(\varphi_{\epsilon_m}*\chi_m h)\subseteq O_m$. Moreover, $(\epsilon_m)_{m\in\mathbb{N}}$ can be chosen such that $(h_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\Omega)$ defined by

$$h_n:\Omega\to\mathbb{C}, \qquad h_n:=\sum_{m=1}^{n}\varphi_{\epsilon_m}*(\chi_m h), \tag{III.51}$$

satisfies $h_n \overset{n\to\infty}{\longrightarrow} h$ in $H_2^1(\Omega)$. Since $O_k \cap O_m = \emptyset$ for $|k-m| > 4$ we have

$$\|h_n\|_{L^\infty(\Omega)} \leq \sup_{k\in\mathbb{N}} \sum_{\substack{1\leq m\leq n \\ |k-m|\leq 4}} \|\varphi_{\epsilon_m} * (\chi_m h)\|_{L^\infty(U_k)} \lesssim \|h\|_{L^\infty(\Omega)}.$$

Consequently, $(h_n)_{n\in\mathbb{N}}$ from (III.51) satisfies (III.50).
We now define $(p_n)_{n\in\mathbb{N}}$ by $p_n := gh_n$. Then $p_n \in H_{2,0}^1(\Omega)$ for all $n \in \mathbb{N}$. The dominated convergence theorem yields

$$\|(\nabla g)(h-h_n)\|_{L^2(\Omega)} \overset{n\to\infty}{\longrightarrow} 0. \tag{III.52}$$

Indeed, $(\partial^\alpha g)(h-h_n) \overset{n\to\infty}{\longrightarrow} 0$ almost everywhere on Ω and

$$D \in L^2(\Omega), \qquad D(\omega) := |\nabla g(\omega)|\big(\|h\|_{L^\infty(\Omega)} + \sup_{n\in\mathbb{N}} \|h_n\|_{L^\infty(\Omega)}\big) \text{ a.e.}$$

is a majorant of $((\nabla g)(h-h_n))_{n\in\mathbb{N}}$ almost everywhere on Ω. With (III.52) and the product rule we obtain

$$\begin{aligned}\|gh - p_n\|_{H_2^1(\Omega)} &\leq \big(\|g\|_{L^\infty(\Omega)} + \|h\|_{L^\infty(\Omega)}\big)\|h-h_n\|_{H_2^1(\Omega)} + \|(\nabla g)(h-h_n)\|_{L^2(\Omega)} \\ &\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Consequently, we have $gh \in H_{2,0}^1(\Omega)$. □

We now turn to the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}u'(t) &= -\tilde{\Delta}_D u(t) + F_{\beta,\pm}(u(t)), \quad t \neq t_0, \\ u(t_0) &= f, \end{aligned} \tag{III.53}$$

with $f \in H_{2,0}^1(\Omega)$ and a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$. $(-\Delta_D, D(-\Delta_D))$ denotes the Dirichlet Laplacian on Ω given by

$$\begin{aligned} D(-\Delta_D) &:= \{f \in H_{2,0}^1(\Omega) \mid \Delta f \in L^2(\Omega) \text{ in distributional sense}\}, \\ -\Delta_D f &:= -\Delta f \quad \text{ on } D(-\Delta_D). \end{aligned}$$

In Example I.1.6;(1) we defined $(-\Delta_D, D(-\Delta_D))$ via a sesquilinear form and showed that its energy space is given by $E_{-\Delta_D} \cong H_{2,0}^1(\Omega)$.

Theorem III.4.4
Let $\beta \in \mathbb{N}_{even}$, $p \in (\max\{\beta, 4\}, \infty)$ and $\Omega \subseteq \mathbb{R}^2$ be a bounded domain, which is either C^∞ or polygonal. Then for each $f \in H_{2,0}^1(\Omega)$ there is a conditionally unique maximal strong solution $u \in C(I(f), H_{2,0}^1(\Omega)) \cap L_{loc}^p(I(f), L^\infty(\Omega))$ of the nonlinear Schrödinger equation (III.53) with the following properties:

(a) u has $L^2(\Omega)$-conservation and the induced nonlinear flow is locally Lipschitz continuous.

(b) If Ω is C^∞, then u has energy conservation. Then $I(f) = \mathbb{R}$ if either the equation is defocusing or $\beta = 2$ and $\|f\|_{L^2(\Omega)}$ is small enough.

Remarks:

(1) The statements of the above theorem for a bounded C^∞-domain with $\beta \in (1, \infty)$ are contained in [Ant08]. Since the local existence result there does not rely on the sign of the nonlinearity, Lemma I.3.5 provides global existence for the focusing equation if $\beta \in (1, 2)$.

(2) In [BFHM12] the needed Strichartz estimate is proven for polygonal domains. However, the authors do not state any existence results for the corresponding nonlinear Schrödinger equation.

(3) If $\beta = 2$ and Ω is polygonal, then Theorem 3.6.1 in [Caz03] provides that the above solution is unconditionally unique and satisfies energy conservation. This again results in $I(f) = \mathbb{R}$ if either the equation is defocusing or $\|f\|_{L^2(\Omega)}$ is small enough by Lemma I.3.5. Compared to Theorem 3.6.1 in [Caz03] we gained the information that $u \in L^p_{loc}(\mathbb{R}, L^\infty(\Omega))$ and the nonlinear flow is locally Lipschitz continuous with respect to the norm of $C(I(f), H^1_{2,0}(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$.

Proof. Let us first gather the necessary function spaces, in particular the fractional domains of $-\Delta_D$. Theorem 1 in Section 5.7 of [Eva10] provides that $H^1_{2,0}(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Then there is a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in $(0, \infty)$ of eigenvalues associated to the eigenfunctions $(\varphi_k)_{k\in\mathbb{N}}$ of $(-\Delta_D, D(-\Delta_D))$ such that $\lambda_k \xrightarrow{k\to\infty} \infty$. $(\varphi_k)_{k\in\mathbb{N}}$ additionally forms an orthonormal basis of $L^2(\Omega$ and $\varphi_k \in H^1_{2,0}(\Omega) \cap C^\infty_b(\Omega)$ for all $k \in \mathbb{N}$ (see Section 9.8 in [Bre11]). For $s \in [0, \infty)$ the fractional powers $((-\Delta_D)^{s/2}, H^s_{2,D}(\Omega))$ of the Dirichlet Laplacian are given by

$$H^s_{2,D}(\Omega) := \left\{ f = \sum_{k=1}^\infty \langle f, \varphi_k\rangle_{L^2(\Omega)} \varphi_k \in L^2(\Omega) \;\middle|\; \sum_{k=1}^\infty \lambda_k^s |\langle f, \varphi_k\rangle_{L^2(\Omega)}|^2 < \infty \right\},$$
$$(-\Delta_D)^{s/2} f := \sum_{k=1}^\infty \lambda_k^{s/2} \langle f, \varphi_k\rangle_{L^2(\Omega)} \varphi_k \quad \text{on } H^s_{2,D}(\Omega).$$

We equip $H^s_{2,D}(\Omega)$ with the norm

$$\|\cdot\|_{H^s_{2,D}(\Omega)} : H^s_{2,D}(\Omega) \to [0, \infty), \qquad \|f\|_{H^s_{2,D}(\Omega)} := \left(\sum_{k=1}^\infty \lambda_k^s |\langle f, \varphi_k\rangle_{L^2(\Omega)}|^2 \right)^{1/2}.$$

Then $(H^s_{2,D}(\Omega), \|\cdot\|_{H^s_{2,D}(\Omega)})$ is a Banach space. Recall Weyl's law for the asymptotic behavior of the eigenvalues $(\lambda_k)_{k\in\mathbb{N}}$ for $k \to \infty$ from Theorem 1 in Section 11.6 of [Str08]. It provides $\lim_{k\to\infty} \frac{\lambda_k}{k} = \frac{4\pi}{\lambda(\Omega)}$. Then there is $k_0 \in \mathbb{N}$ such that $\lambda_k \cong k$ for all $k > k_0$.
Let $f \in C^\infty_c(\Omega)$ with $K := \operatorname{supp}(f)$, $s \in [0, \infty)$, and $\tilde{s} \in \mathbb{N}$ such that $s - 2\tilde{s} > 1$. We use the asymptotics of $(\lambda_k)_{k\in\mathbb{N}}$ and the orthonormality of $(\varphi_k)_{k\in\mathbb{N}}$ in $L^2(\Omega)$ to provide

$$\begin{aligned}\sum_{k=1}^\infty \lambda_k^s |\langle f, \varphi_k\rangle_{L^2(\Omega)}|^2 &= \sum_{k=1}^\infty \lambda_k^{s-2\tilde{s}} |\langle f, (-\Delta_D)^{\tilde{s}} \varphi_k\rangle_{L^2(K)}|^2 \\ &\le \|(-\Delta_D)^{\tilde{s}} f\|^2_{L^\infty(\Omega)} \left(\sum_{k=1}^{k_0} \lambda_k + \sum_{k=k_0+1}^\infty k^{2\tilde{s}-s} \right) < \infty.\end{aligned}$$

Hence, $f \in H^s_{2,D}(\Omega)$ and therefore $C^\infty_c(\Omega) \subseteq H^s_{2,D}(\Omega)$ for all $s \in [0,\infty)$. We stress that $H^1_{2,0}(\Omega) \cong H^1_{2,D}(\Omega)$ and $H^s_{2,D}(\Omega)$ is dense in $H^{\tilde{s}}_{2,D}(\Omega)$ for $0 \le \tilde{s} \le s$.
(a) We already showed in Lemma III.4.3 the nonlinear estimates (III.47) and (III.48) for $k = 1$ and $F_{\beta,\pm}(H^1_{2,0}(\Omega) \cap L^\infty(\Omega)) \subseteq H^1_{2,0}(\Omega)$ in Lemma III.4.3. Also the mean value theorem yields

$$\|F_{\beta,\pm}(g) - F_{\beta,\pm}(h)\|_{L^2(\Omega)} \lesssim \big(\|g\|^\beta_{L^\infty(\Omega)} + \|h\|^\beta_{L^\infty(\Omega)}\big)\|g-h\|_{L^2(\Omega)}.$$

We now collect the following local (p,q) Strichartz estimates with ℓ-loss from Theorem 1.1 in [BFHM12] and Theorem 1.2 in [Ant08]. Let $f \in C^\infty_c(\Omega)$ and (p,q) be sharp 1-admissible such that $p \in (\max\{\beta,4\},\infty)$ (note that $q \in (2,4)$ by the admissibility condition). There is $C : [0,\infty) \to [0,\infty)$ such that for $I \in \mathcal{I}_b$ holds

$$\|e^{\mathrm{i}(\cdot)\Delta_D} f\|_{L^p(I,L^q(\Omega))} \le C(|I|)\|f\|_{H^{\ell(\Omega)}_{2,D}(\Omega)},$$

where we can choose $\ell(\Omega)$ according to

$$\ell(\Omega) \begin{cases} = \frac{1}{p}, & \Omega \text{ is polygonal,} \\ > \frac{3}{2p}, & \Omega \text{ is } C^\infty. \end{cases}$$

We additionally have

$$\|(-\Delta_D)^{1/2} e^{\mathrm{i}(\cdot)\Delta_D} f\|_{L^p(I,L^q(\Omega))} \le C(|I|)\|(-\Delta_D)^{1/2} f\|_{H^{\ell(\Omega)}_{2,D}(\Omega)} \quad \le C(|I|)\|f\|_{H^{\ell(\Omega)+1}_{2,D}(\Omega)}.$$

Since $q \in (2,4)$ and $(-\Delta_D)^{1/2} e^{-i(\cdot)\Delta_D} f \in L^q(\Omega)$ almost everywhere on I, Theorem 7.5;(a) in [JK95] provides the estimate

$$\|\nabla e^{\mathrm{i}t\Delta_D} f\|_{L^q(\Omega)} \lesssim \|(-\Delta_D)^{1/2} e^{\mathrm{i}t\Delta_D} f\|_{L^q(\Omega)} \qquad \text{a.e on } I.$$

We then have

$$\|e^{\mathrm{i}(\cdot)\Delta_D} f\|_{L^p(I,H^1_q(\Omega))} \le C(|I|)\|f\|_{H^{\ell(\Omega)+1}_{2,D}(\Omega)}.$$

The complex interpolation results (A.1) and Theorem A.1.3 as well as Corollary I.1.3 imply for $\theta \in (2/q, 1)$ the estimate

$$\begin{aligned} \|e^{\mathrm{i}(\cdot)\Delta_D} f\|_{L^p(I,H^\theta_q(\Omega))} &\cong \|e^{\mathrm{i}(\cdot)\Delta_D} f\|_{[L^p(I,L^q(\Omega)),L^p(I,H^1_q(\Omega))]_\theta} \\ &\le C(|I|)\|f\|_{[H^{\ell(\Omega)}_{2,D}(\Omega),H^{\ell(\Omega)+1}_{2,D}(\Omega)]_\theta} \le C(|I|)\|f\|_{H^{\ell(\Omega)+\theta}_{2,D}(\Omega)}. \end{aligned} \tag{III.54}$$

The embedding $H^\theta_q(\Omega) \hookrightarrow L^\infty(\Omega)$ from A.2.1;(a) and (III.54) then provide

$$\|e^{\mathrm{i}(\cdot)\Delta_D} f\|_{L^p(I,L^\infty(\Omega))} \lesssim \|e^{\mathrm{i}(\cdot)\Delta_D} f\|_{L^p(I,H^\theta_q(\Omega))} \le C(|I|)\|f\|_{H^{\ell(\Omega)+\theta}_{2,D}(\Omega)}.$$

The sharp 1-admissibility condition for the pair (p,q) implies that there is $\theta \in (2/q, 1)$ such that $\ell(\Omega) + \theta \le 1$. This implies the following local (p,∞) Strichartz estimate with $1/2$-loss

$$\|e^{\mathrm{i}(\cdot)\Delta_D} f\|_{L^p(I,L^\infty(\Omega))} \le C(|I|)\|f\|_{H^1_2(\Omega)} \cong C(|I|)\|f\|_{D((-\Delta_D)^{1/2})}.$$

By density of $C_c^\infty(\Omega)$ in $H^1_{2,0}(\Omega)$ the previous estimate holds for all $f \in H^1_{2,0}(\Omega)$. This concludes the establishment of the conditions to apply Theorem I.3.4 with $s = \ell = 1/2$. Hence, for all $f \in H^1_{2,0}(\Omega)$ there is a conditionally unique maximal mild solution $u \in C(I(f), H^1_{2,0}(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ of (III.53) with all the properties stated in *(a)*. Since $H^1_{2,0}(\Omega) \hookrightarrow L^{\beta+2}(\Omega$ Proposition I.4.2 implies that $F_{\beta,\pm} \in C(H^1_{2,0}(\Omega), H^1_{2,0}(\Omega)^*)$. Proposition I.2.4;(c) now implies that u is in fact a strong solution of (III.53). $L^2(\Omega)$-conservation of u then follows from Proposition I.2.8.

(b) Let Ω be a bounded C^∞-domain. We first prove energy conservation. By density of $C_c^\infty(\Omega)$ in $H^1_{2,0}(\Omega)$, the continuity of the nonlinear flow, and $\mathcal{E} \in C(H^1_{2,0}(\Omega), \mathbb{R})$ it is enough to prove energy conservation for u with intial data in $C_c^\infty(\Omega)$. Let $f \in C_c^\infty(\Omega)$ and $u \in C(I(f), H^1_{2,0}(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ be the maximal solution of (III.53) from *(a)* with $u(t_0) = f$. The regularity of the boundary ensures $g \in H^2_2(\Omega)$ for all $g \in D(-\Delta_D)$. Hence, $D(-\Delta_D) = H^1_{2,0}(\Omega) \cap H^2_2(\Omega)$ and $\|g\|_{H^2_2(\Omega)} \lesssim \|\Delta_D g\|_{L^2(\Omega)}$ for $g \in H^1_{2,0}(\Omega) \cap H^2_2(\Omega)$ (see Theorem 9.15 and Lemma 9.17 in [GT01]). We consequently have

$$(D(-\Delta_D), \|\cdot\|_{D(-\Delta_D)}) \cong (H^1_{2,0}(\Omega) \cap H^2_2(\Omega), \|\cdot\|_{H^2_2(\Omega)}).$$

Lemma III.4.3 implies $F(D(-\Delta_D) \cap L^\infty(\Omega)) \subseteq D(-\Delta_D) \cap L^\infty(\Omega)$ as well as the nonlinear estimates (III.47) and (III.48) for $k = 2$. Theorem I.3.4;(d) then implies $u \in C(I(f), D(-\Delta_D)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$. The Sobolev embedding A.2.1 provides the embedding $D(-\Delta_D) \hookrightarrow L^{2(\beta+1)}(\Omega)$ and the Lipschitz estimate (I.65) implies for $t, s \in I(f)$

$$\begin{aligned}
&\|F_{\beta,\pm}(u(t)) - F_{\beta,\pm}(u(s))\|_{L^2(\Omega)} \\
&\qquad \lesssim (\|u(t)\|^\beta_{L^{2(\beta+1)}(\Omega)} + \|u(s)\|^\beta_{L^{2(\beta+1)}(\Omega)})\|u(t) - u(s)\|_{L^{2(\beta+1)}(\Omega)} \\
&\qquad \lesssim \|u\|^\beta_{L^\infty(I, H^2_2(\Omega))}\|u(t) - u(s)\|_{H^2_2(\Omega)}.
\end{aligned}$$

Hence, $F_{\beta,\pm}(u) \in C(I(f), L^2(\Omega))$ and the equation $\mathrm{i}u' = -\Delta_D u + F_{\beta,\pm}(u)$ yields $u \in C^1(I(f), L^2(\Omega))$. Lemma I.4.3 then provides energy conservation of u.

We derive the criteria for $I(f) = \mathbb{R}$ from Lemma I.3.5. The condition I.3.5;(i) is satisfied since we already have provided $L^2(\Omega)$- and energy conservation. Recall that $(\beta + 2)\hat{F}_{\beta,\pm}(g) = \pm\|g\|^{\beta+2}_{L^{\beta+2}(\Omega)}$. For the defocusing equation I.3.5;(ii) is satisfied. For the focusing case with $\beta = 2$ we use Sobolev embedding A.2.1 and complex interpolation (A.1) to prove for all $g \in H^1_2(\Omega)$ the estimate

$$-\|g\|^4_{L^4(\Omega)} \gtrsim -\|g\|^4_{H^{1/2}_2(\Omega)} \gtrsim -\|g\|^2_{L^2(\Omega)}\|g\|^2_{H^1_2(\Omega)}.$$

This provides I.3.5;(ii) with $\beta_2 = 2$. Hence, Lemma I.3.5 implies $u \in L^\infty(I(f), H^1_{2,0}(\Omega))$ under the conditions in *(b)*. This shows $I(f) = \mathbb{R}$ by the blow-up alternative. □

IV. The nonlinear Schrödinger equation on product spaces

In this chapter we consider the nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i}u'(t) &= Au(t) + F_{\beta,\pm}(u(t)), \quad t \neq t_0, \\ u(t_0) &= f, \end{aligned} \tag{IV.1}$$

where $(A, D(A))$ is a non-negative, selfadjoint linear operator on $L^2(\Omega)$ and $\Omega := X \times Y$ with σ-finite measure spaces $(X, \mathscr{X}, \mu_x)$ and $(Y, \mathscr{Y}, \mu_y)$. We always consider Ω to be equipped with the product σ-algebra $\mathscr{S} := \mathscr{X} \otimes \mathscr{Y}$ and the product measure $\mu := \mu_x \otimes \mu_y$. We furthermore always assume that $L^2(X)$ and $L^2(Y)$ are seperable. This will always be the case in our applications.
We are interested in situations in which the Schrödinger group U generated by A can be decomposed into a product of two Schrödinger groups U_x and U_y acting on $L^2(X)$ and $L^2(Y)$, respectively.
In Section IV.1 we provide local (p, q) Strichartz estimates with ℓ-loss for U under the assumption that U_x and U_y satisfy either dispersive estimates or dispersive estimates of SL-type. For the proof we adapt the methods of Section II.2 to the product situation. As a corollary of the abstract result we deduce in Section IV.2 local (p, q) Strichartz estimates with $1/2p$-loss for $(e^{\mathrm{i}t\Delta_{\mathbb{R}^n\times M}})_{t\in\mathbb{R}}$, where $(-\Delta_{\mathbb{R}^n\times M}, H_2^2(\mathbb{R}^n \times M))$ denotes the Laplace-Beltrami operator on the product manifold $\mathbb{R}^n \times M$. Here (M, g) is an arbitrary connected, compact Riemannian C^∞-manifold without boundary and $\dim(M) = m$.
We will show that $\mathbb{R}^n \times M$ meets the requirements on the manifolds in Section III.1. We apply Theorem III.1.5 to deduce a local existence result in $H_2^s(\mathbb{R}^n \times M)$ for (IV.1) with $A = -\Delta_{\mathbb{R}^n\times M}$ and $n, m \in \mathbb{N}$. We additionally prove a global existence result in $H_2^1(\mathbb{R} \times M)$ with $m = 1$ for $\beta \in [2, \infty)$. The latter is one of the main results of this thesis.
The best result in this direction known to us is provided in [TTV14] under the restriction $\beta < 4$. We will present the method of [TTV14], which relies on Strichartz estimates with mixed space integrability. These are interesting in their own right, because comparable estimates are available on flat and distorted waveguides (see [DR12]).

We constantly use the fact that $L^p(X \times Y) \cong L^p(X, L^p(Y)) \cong L^p(Y, L^p(X))$ for $p \in [1, \infty)$ by A.3.4;(2) without mentioning. To shorten the notation we sometimes denote spaces of the form $R(X, S(Y))$ by $R_x S_y$ and $R(X \times Y)$ by $R_{x,y}$.

IV.1. Strichartz estimates with directional loss

In this section we discuss an adaptation of the methods developed in Section II.2 to the product situation $X \times Y$. We first present the main idea. We assume for the moment that $U \in L^\infty(\mathbb{R}, \mathcal{L}(L^2(\Omega))$ is given by $U = U_x U_y$ with $U_x \in L^\infty(\mathbb{R}, L^2(X))$ and $U_y \in L^\infty(\mathbb{R}, L^2(Y))$. If U_x and U_y are $(\sigma_x, L^1(X))$- and $(\sigma_y, L^1(Y))$-dispersive, respectively, then for $f \in L^1(X \times Y) \cap L^2(X \times Y)$ we have the estimate

$$\begin{aligned} \|U(t)U(s)^* f\|_{L^\infty_{x,y}} &\leq \big\| \|U_x(t-s)U_y(t-s)f\|_{L^\infty_x} \big\|_{L^\infty_y} \\ &\lesssim |t-s|^{-\sigma_x} \big\| \|U_y(t-s)f\|_{L^1_x} \big\|_{L^\infty_y} \\ &\lesssim |t-s|^{-\sigma_x} \big\| \|U_y(t-s)f\|_{L^\infty_y} \big\|_{L^1_x} \lesssim |t-s|^{-(\sigma_x+\sigma_y)} \|f\|_{L^1_{x,y}}. \end{aligned} \tag{IV.2}$$

U therefore is $(\sigma_x + \sigma_y, L^1(X \times Y))$-dispersive. Corollary II.1.5 yields (p, q) Strichartz estimates for U for all sharp $\sigma_x + \sigma_y$-admissible pairs (p, q). Since we are interested in situations where U_x and/or U_y may only satisfy dispersive estimates of SL-type we use the idea of (IV.2) on the level of a spectral decomposition. We first produce (p, q) Strichartz estimates with "directional loss", which will become clear in the central Theorem IV.1.1. We also deduce local (p, q) Strichartz estimates with ℓ-loss from these (p, q) Strichartz estimates with directional loss.
We now describe the functional analytic setup for Theorem IV.1.1. Let $(A_x, D(A_x))$ and $(A_y, D(A_y))$ be non-negative, selfadjoint linear operators on $L^2(X)$ and $L^2(Y)$, respectively. For the action of A_x and A_y on a function in $L^2(X \times Y)$ to make sense we extend them canonically to the operators on $L^2(\Omega)$ given by

$$\begin{aligned} D(\tilde{A}_x) &:= L^2(Y, D(A_x)), & \tilde{A}_x f(x,y) &:= (A_x f(\cdot, y))(x) \quad \text{a.e. on } Y, \\ \tilde{D}(\tilde{A}_y) &:= L^2(X, D(A_y)), & \tilde{A}_y f(x,y) &:= (A_y f(x, \cdot))(y) \quad \text{a.e. on } X. \end{aligned}$$

It is straightforward to check that both operators are closed, non-negative, and symmetric on $L^2(\Omega)$. The separability of $L^2(Y)$ implies that

$$\mathcal{P} := \operatorname{span}\{f \in M(X \times Y) \mid \exists_{f_x \in L^2(X),\, f_y \in L^2(Y)} : \ f = f_x f_y\} \tag{IV.3}$$

is dense in $L^2(\Omega)$. Proposition 4.8 in Chapter X.4 in [AE09] furthermore implies for all $\alpha \in [0, \infty)$ that

$$\mathcal{E}_{Y,\alpha} := \operatorname{span}\{f \in M(X \times Y) \mid \exists_{S_x \in \mathscr{S}_x \text{ s.t. } \mu_x(S_x) < \infty,\, f_y \in D(A_y^\alpha)} : f = \mathbb{1}_{S_x} f_y\} \tag{IV.4}$$

is dense in $L^2(X, D(A_y^\alpha))$. The density of $D(A_y^\alpha) \hookrightarrow D(A_y^\beta)$ given by Corollary I.1.3;(a) additionally provides the density of $L^2(X, D(A_y^\alpha))$ in $L^2(X, D(A_y^\beta))$ for $\alpha \geq \beta \geq 0$. The analogous versions of these results for $L^2(Y, D(A_x^\alpha))$ are also valid. In particular, $(\tilde{A}_x, D(\tilde{A}_x))$ and $(\tilde{A}_y, D(\tilde{A}_y))$ are densely defined on $L^2(\Omega)$. Moreover, the sesquilinear forms

$$\begin{aligned} a_x &: L^2(Y, D(A_x^{1/2})) \times L^2(Y, D(A_x^{1/2})) \to \mathbb{C}, & a_x(f, g) &:= \langle A_x^{1/2} f, A_x^{1/2} g \rangle_{L^2(X \times Y)}, \\ a_y &: L^2(X, D(A_y^{1/2})) \times L^2(X, D(A_y^{1/2})) \to \mathbb{C}, & a_y(f, g) &:= \langle A_y^{1/2} f, A_y^{1/2} g \rangle_{L^2(X \times Y)}, \end{aligned}$$

are densely defined, continuous, accretive, closed, and symmetric. Then a_x and a_y generate by means of Proposition 1.24 in Section 1.2.3 of [Ouh05] uniquely determined non-negative, selfadjoint linear operators which coincide with $\tilde{A}_x$ and $\tilde{A}_y$, respectively.

This can be checked using the density results for $D(\tilde{A}_x)$ and $D(\tilde{A}_y)$. $(\tilde{A}_x, D(\tilde{A}_x))$ and $(\tilde{A}_y, D(\tilde{A}_y))$ are therefore selfadjoint. We can easily check via uniqueness of the corresponding Cauchy problem and the density properties of $\mathcal{E}_{Y,1}$ and $\mathcal{E}_{X,1}$ that the generated Schrödinger groups $\tilde{U}_x$ and $\tilde{U}_y$ satisfy for all $t \in \mathbb{R}$ and $f \in L^2(X \times Y)$

$$(\tilde{U}_x f)(x,y) = (U_x f(\cdot, y))(x), \qquad (\tilde{U}_y f)(x,y) = (U_x f(x, \cdot))(y) \quad \text{a.e. on } X \times Y.$$

We then have $\tilde{U}_x \tilde{U}_y = \tilde{U}_y \tilde{U}_x$ on $\mathcal{P}$ and the density of $\mathcal{P}$ in $L^2(X \times Y)$ shows that $\tilde{U}_x$ and $\tilde{U}_y$ commute on $L^2(X \times Y)$. The same argument provides that $(\tilde{A}_x, D(\tilde{A}_x))$ and $(\tilde{A}_y, D(\tilde{A}_y))$ have commuting resovents. Then by [Sz.67] (see also Section 5.3 and 5.5 in [Sch12]) we can define the following multivariate version of the spectral calculus of Theorem I.1.1 for the pair $\boldsymbol{A} = (A_x, A_y)$. There is a map

$$\Phi_{\boldsymbol{A}} : M(\mathbb{R}^2) \to \mathcal{C}(L^2(X \times Y)), \qquad \varphi(\boldsymbol{A}) := \Phi_{\boldsymbol{A}}(\varphi)$$

such that for $\varphi, \eta \in M(\mathbb{R}^2)$ holds:

(MC1) $\varphi(\boldsymbol{A}) + \eta(\boldsymbol{A}) \subseteq (\varphi + \eta)(\boldsymbol{A})$ and $\varphi(\boldsymbol{A})\eta(\boldsymbol{A}) \subseteq (\varphi\eta)(\boldsymbol{A})$, whereby

$$D(\varphi(\boldsymbol{A})\eta(\boldsymbol{A})) = D((\varphi\eta)(\boldsymbol{A})) \cap D(\eta(\boldsymbol{A})),$$

(MC2) $\varphi|_{\sigma(A_x) \times \sigma(A_y)} = 0 \implies \varphi(\boldsymbol{A}) = 0$ and $\varphi|_{\sigma(A_x) \times \sigma(A_y)} = 1 \implies \varphi(\boldsymbol{A}) = \mathrm{id}$,

(MC3) $\varphi(\boldsymbol{A})^* = \overline{\varphi}(\boldsymbol{A})$,

(MC4) $\Phi_{\boldsymbol{A}} : (M_b(\mathbb{R}^2), \|\cdot\|_{L^2([0,\infty)^2)}) \to \mathcal{L}(L^2(X \times Y))$ is a bounded algebra homomorphism.

The operator $(A, D(A))$ defined by

$$D(A) := L^2(Y, D(A_x)) \cap L^2(X, D(A_y)),$$
$$Af := A_x f + A_y f \quad \text{on } D(A),$$

is symmetric and non-negative. (MC4) implies $(\mathrm{id} + A)^{-1} \in \mathcal{L}(L^2(X \times Y))$. Hence, $-1 \in \rho(A)$ and $(A, D(A))$ is closed. In fact Proposition 1.1.9 in [Roy07] provides that $(A, D(A))$ is selfadjoint on $L^2(\Omega)$. The properties of the multivariate spectral calculus also imply that the Schrödinger group U of $(A, D(A))$ is given by $U = \tilde{U}_x \tilde{U}_y = \tilde{U}_y \tilde{U}_x$. From now on we are not going to distinguish the operators $(\tilde{A}_x, D(\tilde{A}_x))$, $(\tilde{A}_y, D(\tilde{A}_y))$ and $(A_x, D(A_x))$, $(A_y, D(A_y))$ in our notation. It will be clear from the context what we mean. Recall the Definition of the dyadic partition of unity $(\psi_k)_{k \in \mathbb{N}_0}$ from II.2.4 and the definition of the homogeneous and inhomogeneous flows $\mathcal{U}$ and Φ for an operator family $U \in L^\infty(\mathbb{R}, \mathcal{L}(L^2(\Omega))$ from I.3.1.

In the above setting we can prove the following main result.

Theorem IV.1.1
Let $I \in \mathcal{I}_b$, $\sigma_x, \sigma_y \in (0,\infty)$, $\gamma, \nu \in [0,1]$, and (p,q) be sharp $\sigma_x + \sigma_y$-admissible with $q < \infty$.

(a) Let U_x be $(\sigma_x, L^1(X))$-dispersive and U_y be $(\sigma_y, L^1(Y))$-dispersive of SL-type γ as well as $(\psi_k(A_y))_{k\in\mathbb{N}_0}$ have the (LP) *property. Then there is a non-decreasing $C : [0,\infty) \to [0,\infty)$ such that for all $f \in L^2(X, D(A_y^{\gamma/p}))$ holds*

$$\|\mathcal{U}f\|_{L^p(I,L^q_{x,y})} \leq C(|I|)\|f\|_{L^2(X,D(A_y^{\gamma/p}))}. \tag{IV.5}$$

(b) Let U_x in (a) be only $(\sigma_x, L^1(X))$-dispersive of SL-type ν and $(\psi_k(A_x))_{k\in\mathbb{N}_0}$ satisfy the (LP) *property. Then there is a non-decreasing $C : [0,\infty) \to [0,\infty)$ such that for all $f \in L^2(X, D(A_y^{\gamma/p})) \cap L^2(Y, D(A_x^{\nu/p}))$ holds*

$$\|\mathcal{U}f\|_{L^p(I,L^q_{x,y})} \leq C(|I|)\big(\|f\|_{L^2(X,D(A_y^{\gamma/p}))} + \|f\|_{L^2(Y,D(A_x^{\nu/p}))}\big). \tag{IV.6}$$

(c) In (a) a local (p,q) Strichartz estimate with γ/p-loss and in (b) a local (p,q) Strichartz estimate with $\max\{\gamma/p, \nu/p\}$-loss holds.

Remark: The estimates (IV.5) and (IV.6) are the analogue to our notion of local (p,q) Strichartz estimates with loss. In a rather natural fashion the loss only occurs in the direction, in which the corresponding Schrödinger group satisfies dispersive estimates of SL-type.

Proof. We fix $I \in \mathcal{I}_b$, $\sigma := \sigma_x + \sigma_y$, and (p,q) sharp σ-admissible with $q < \infty$. The claims in *(a)* and *(b)* are trivial in the case $(p,q) = (\infty,2)$ since $U \in L^\infty(\mathbb{R}, \mathcal{L}(L^2_{x,y}))$. We therefore assume $p < \infty$ in the rest of the proof. We use the same approach as in the proof of Theorem II.2.8, but this time pointwise in the X or Y direction.
(a) For $k \in \mathbb{N}_0$ we put $h_k := \min\{2^{(1-k)\gamma}, 1\}$ and let $(I_{j,k})_{j=0}^{N_k}$ be the h_k-partition of I with $N_k = [|I|h_k^{-1}]$ from Defintion II.2.9. For $k \in \mathbb{N}_0$ and $j \in \{0,...,N_k\}$ we define

$$\tilde{U}_{j,k} : \mathbb{R} \to \mathcal{L}(L^2_{x,y}), \qquad \tilde{U}_{j,k}(t) := \mathbb{1}_{I_{j,k}}(t)\tilde{\psi}_k(A_y)U(t).$$

Then $\tilde{U}_{j,k} \in L^\infty(\mathbb{R}, \mathcal{L}(L^2_{x,y}))$ with $\|\tilde{U}_{j,k}\|_{L^\infty(\mathbb{R},\mathcal{L}(L^2_{x,y}))} \leq 1$ by means off (MC4). We now let $f \in L^1(X\times Y) \cap L^2(X \times Y)$ and $t,s \in \mathbb{R}$ with $t \neq s$. If either $t \notin I_{j,k}$ or $s \notin I_{j,k}$, then $\tilde{U}_{j,k}(t)\tilde{U}_{j,k}(s)^* = 0$. If $t,s \in I_{j,k}$, then $|t-s| \leq h_k$. The σ_y-dispersivity of SL-type γ of U_y and Minkowski's integral inequality imply the estimate

$$\begin{aligned}\|\tilde{U}_{j,k}(t)\tilde{U}_{j,k}(s)^*f\|_{L^\infty_{x,y}} &\leq \big\|\|\mathbb{1}_{I_{j,k}}(t)\mathbb{1}_{I_{j,k}}(s)\tilde{\psi}_k^2(A_y)U_y(t-s)U_x(t-s)f\|_{L^\infty_y}\big\|_{L^\infty_x} \\ &\lesssim |t-s|^{-\sigma_y}\big\|\|U_x(t-s)f\|_{L^\infty_x}\big\|_{L^1_y} \lesssim |t-s|^{-\sigma}\|f\|_{L^1_{x,y}}.\end{aligned} \tag{IV.7}$$

We stress that the constant in this estimate is independent of j and k. Corollary II.1.5 then provides global (p,q)-Strichartz estimates for $\tilde{U}_{j,k}$ with a constant independent of j and k. We fix $f \in L^2(X, D(A_y^{\gamma/p}))$. By means of the (LP) property of $(\psi_k(A_y))_{k\in\mathbb{N}_0}$ and Minkowski's integral inequality we have

$$\begin{aligned}\|\mathcal{U}f\|^2_{L^p(I,L^q_{x,y})} &\lesssim \Big\|\Big(\sum_{k=0}^\infty \|\psi_k(A_y)\mathcal{U}f\|^2_{L^q_y}\Big)^{1/2}\Big\|^2_{L^p(I,L^q_x)} \\ &\lesssim \sum_{k=0}^\infty \Big(\sum_{j=0}^{N_k} \|\tilde{\mathcal{U}}_{j,k}\psi_k(A_y)f\|^p_{L^p(\mathbb{R},L^q_{x,y})}\Big)^{2/p} \lesssim \sum_{k=0}^\infty (1+N_k)^{\frac{2}{p}}\|\psi_k(A_y)f\|^2_{L^2_{x,y}}.\end{aligned}$$

With $N_k \cong |I|2^{k\gamma}$ for k large enough and (II.36) we continue the previous estimate to conclude

$$\begin{aligned}\|\mathcal{U}f\|_{L^p(I,L^q_{x,y})} &\lesssim C(|I|)\Big(\|f\|^2_{L^2_{x,y}} + \sum_{k=0}^{\infty}\|\psi_k(A_y)A_y^{\gamma/p}f\|^2_{L^2_{x,y}}\Big)^{1/2}\\ &\lesssim C(|I|)\Big(\|f\|_{L^2_{x,y}} + \Big\|\Big(\sum_{k=0}^{\infty}\|\psi_k(A_y)A_y^{\gamma/p}f\|^2_{L^2_y}\Big)^{1/2}\Big\|_{L^2_x}\Big)\\ &\lesssim C(|I|)\|f\|_{L^2(X,D(A_y^{\gamma/p}))}. \end{aligned} \tag{IV.8}$$

(b) We slightly modify the above procedure but the guiding idea remains the same. For $k,l \in \mathbb{N}_0$ we put $h_{k,l} := \min\{2^{(1-k)\gamma}, 2^{(1-l)\nu}, 1\}$ and let $(I_{j,k,l})_{j=0}^{N_{k,l}}$ be the $h_{k,l}$-partition of I . Note that as above we have $N_{k,l} = [|I|h_{k,l}^{-1}]$. For $k,l \in \mathbb{N}_0$ and $j \in \{0,...,N_{k,l}\}$ we define

$$\tilde{U}_{j,k,l} : \mathbb{R} \to \mathcal{L}(L^2(\Omega)), \qquad \tilde{U}_{j,k,l}(t) := \mathbb{1}_{I_{j,k,l}}(t)\tilde{\psi}_k(A_y)\tilde{\psi}_l(A_x)U(t).$$

$\tilde{U}_{j,k,l}$ is bounded with uniform constant in j,k,l by means of (MC4). For $f \in L^1(X \times Y) \cap L^2(X \times Y)$ and $t,s \in \mathbb{R}$ with $t \neq s$ the same argument as in (IV.7) yields

$$\begin{aligned}\|\tilde{U}_{j,k,l}(t)\tilde{U}_{j,k,l}(s)^*f\|_{L^\infty_{x,y}} &\leq \Big\|\|\mathbb{1}^2_{I_{j,k,l}}(t)\mathbb{1}^2_{I_{j,k,l}}(s)\tilde{\psi}^2_k(A_y)U_y(t-s)\tilde{\psi}^2_l(A_x)U_x(t-s)f\|_{L^\infty_y}\Big\|_{L^\infty_x}\\ &\lesssim |t-s|^{-\sigma_y}\Big\|\|\mathbb{1}_{I_{j,k,l}}(t)\mathbb{1}_{I_{j,k,l}}(s)\tilde{\psi}^2_k(A_x)U_x(t-s)f\|_{L^\infty_x}\Big\|_{L^1_y}\\ &\lesssim |t-s|^{-\sigma}\|f\|_{L^1_{x,y}},\end{aligned}$$

with a uniform constant in j,k,l. Then Corollary II.1.5 again provides global (p,q) Strichartz estimates for $\tilde{U}_{j,k,l}$ with a uniform constant in j,k,l. We therefore have for $f \in L^2(X, D(A_y^{\gamma/p})) \cap L^2(Y, D(A_x^{\nu/p}))$ the estimate

$$\begin{aligned}\|\psi_k(A_y)\psi_l(A_x)\mathcal{U}f\|_{L^p(I,L^q_{x,y})} &= \Big(\sum_{j=0}^{N_{k,l}}\|\tilde{\mathcal{U}}_{j,k,l}\psi_k(A_y)\psi_l(A_x)f\|^p_{L^p(\mathbb{R},L^q_{x,y})}\Big)^{1/p}\\ &\lesssim (1+N_{k,l})^{1/p}\|\psi_k(A_y)\psi_l(A_x)f\|_{L^2_{x,y}}.\end{aligned} \tag{IV.9}$$

The dependency of $N_{k,l}$ in estimate (IV.9) calls for a case distinction for which we define

$$M := \{(k,l) \in \mathbb{N}_0^2 \mid 2^{(1-k)\gamma} \leq 2^{(1-l)\nu}\}.$$

Then the respective (LP) properties of $(\psi_k(A_y))_{k\in\mathbb{N}_0}$ and $(\psi_l(A_x))_{l\in\mathbb{N}_0}$ and Minkowski's integral inequality yield

$$\begin{aligned}&\|\mathcal{U}f\|^2_{L^p(I,L^q_{x,y})}\\ &\lesssim \Big\|\Big(\sum_{l=0}^{\infty}\|\psi_l(A_x)\mathcal{U}f\|^2_{L^q_{x,y}}\Big)^{1/2}\Big\|^2_{L^p(I)}\\ &\lesssim \Big\|\Big(\sum_{l=0}^{\infty}\|\Big(\sum_{k=0}^{\infty}\|\psi_k(A_y)\psi_l(A_x)\mathcal{U}f\|^2_{L^q_x}\Big)^{1/2}\|^2_{L^q_y}\Big)^{1/2}\Big\|^2_{L^p(I)}\\ &\lesssim \sum_{(k,l)\in M}\|\psi_k(A_y)\psi_l(A_x)\mathcal{U}f\|^2_{L^p(I,L^q_{x,y})} + \sum_{(k,l)\in\mathbb{N}_0^2\setminus M}\|\psi_k(A_y)\psi_l(A_x)\mathcal{U}f\|^2_{L^p(I,L^q_{x,y})}.\end{aligned} \tag{IV.10}$$

It is enough to handle the first sum since the analogous result is going to hold for the second one with $Y \leftrightarrow X$, $k \leftrightarrow l$, and $\gamma \leftrightarrow \nu$. Similar to the argument in (IV.8) the estimates (IV.9) and (II.36) provide

$$\begin{aligned} &\sum_{(k,l)\in M} \|\psi_k(A_y)\psi_l(A_x)\mathcal{U}f\|^2_{L^p(I,L^q_{x,y})} \\ &\quad = \sum_{(k,l)\in M} \left(\sum_{j=0}^{N_{k,l}} \|\tilde{\mathcal{U}}_{j,k,l}\psi_k(A_y)\psi_l(A_x)f\|^p_{L^p(\mathbb{R},L^q_{x,y})} \right)^{2/p} \\ &\quad \leq C(|I|) \sum_{k,l=0}^{\infty} (1+2^{k\gamma})^{\frac{2}{p}} \|\psi_k(A_y)\psi_l(A_x)f\|^2_{L^2_{x,y}} \\ &\quad \leq C(|I|) \sum_{l=0}^{\infty} \left(\sum_{k=0}^{\infty} \|\psi_k(A_y)\psi_l(A_x)f\|^2_{L^2_{x,y}} + \sum_{k=1}^{\infty} \|\psi_l(A_x)\psi_k(A_y)A_y^{\gamma/p}f\|^2_{L^2_{x,y}} \right) \\ &\quad \lesssim C(|I|)\|f\|^2_{L^2(X,D(A_y^{\gamma/p}))}. \end{aligned}$$

The previous estimate and its corresponding result for the second sum in (IV.10) immediately provide (IV.6).

(c) The embedding $D(A) \hookrightarrow L^2(X, D(A_y))$, (A.1), and Corollary I.1.3;(a) imply for all $\theta \in (0,1)$

$$D(A^\theta) \cong [L^2(X\times Y), D(A)]_\theta \hookrightarrow [L^2(X, L^2(Y)), L^2(X, D(A_y))]_\theta \cong L^2(X, D(A_y^\theta)).$$

Choosing $\theta = \gamma/p$ in *(a)* and $\theta = \max\{\gamma/p, \nu/p\}$ in *(b)* implies our claims regarding the local (p,q) Strichartz estimate with ℓ-loss. In the latter case we have also used $L^2(Y, D(A_x^\theta)) \hookrightarrow L^2(Y, D(A_x^{\tilde{\theta}}))$ for $\tilde{\theta} \leq \theta$. □

IV.2. Existence results for the nonlinear Schrödinger equation on $\mathbb{R}^n \times M$

In this section we treat the nonlinear Schrödinger equation for the Laplace-Beltrami operator on a cylindrical product manifold. We fix $m, n \in \mathbb{N}$ and define $\Omega := \mathbb{R}^n \times M$, whereby (M, g_m) is a connected, compact Riemannian C^∞-manifold without boundary and $\dim(M) = m$. We first review some of the properties of $\mathbb{R}^n$ and (M, g_m) in order to bring Ω into the framework of Section III.1.

We consider $\mathbb{R}^n$ as a connected Riemannian C^∞-manifold equipped with the trivial atlas $\{(\mathbb{R}^n, \mathrm{id})\}$. We choose the Riemannian metric $g_n : \mathbb{R}^n \to T^2\mathbb{R}^n$ given by $g_n(\boldsymbol{x}) := \langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ for $\boldsymbol{x} \in \mathbb{R}^n$ and identify g_n with the identity matrix $\mathrm{id}_n \in \mathbb{R}^{n\times n}$. We then have $\Gamma^p_{k,l} = 0$ on $\mathbb{R}^n$ for all $k, l, p \in \mathbb{N}_{\leq n}$ by (A.10). The geodesic differential equations (A.11) yield that all geodesics are given by line segments. Moreover, $(\mathbb{R}^n, \mathrm{id}_n)$ is geodesically complete and we have $\mathrm{inj}(\mathbb{R}^n) = \infty$.

We already noted in Remark III.1.1;(2) that (M, g_m) is complete, $\mathrm{inj}(M) > 0$, and has bounded geometry. We equip (M, g_m) with the finite geodesic atlas $\{(\tilde{O}_i(r), \tilde{\kappa}_i) \mid i \in I\}$ and denote by G_m the coefficient matrix of the Riemannian metric g_m. We fix r and the smooth partition of unity $(\tilde{\psi}_i)_{i\in I}$ subordinate to $\mathscr{A}(r)$ are chosen as in (M1) and (M2) at the beginning of Section III.1. We usually denote $\omega \in \Omega$ by $\omega = (\boldsymbol{x}, y)$ with $\boldsymbol{x} \in \mathbb{R}^n$ and $y \in M$. We now equipp Ω with the finite atlas

$$\mathscr{A}(r) := \{(O_i(r), \kappa_i) \mid i \in I\}, \qquad O_i(r) := \mathbb{R}^n \times \tilde{O}^i(r), \ \ \kappa_i(\boldsymbol{x}, y) := (\boldsymbol{x}, \tilde{\kappa}_i(y)).$$

Then $(\Omega, \mathscr{A}(r))$ is a connected C^∞-manifold without boundary and $\dim(\Omega) = n + m$. We choose the smooth partition of unity $(\psi_i)_{i \in I}$ subordinate to $\mathscr{A}(r)$ defined by $\psi_i(\omega) := \tilde{\psi}_i(y)$. As Riemannian metric for $(\Omega, \mathscr{A}(r))$ we choose

$$g : \Omega \to T^2\Omega, \qquad g(\boldsymbol{x}, y) = \langle \cdot, \cdot \rangle_\omega := \langle \pi|_{\mathbb{R}^n} \cdot, \pi|_{\mathbb{R}^n} \cdot \rangle_{\mathbb{R}^n} + g_m(y)\big(\pi|_M \cdot, \pi|_M \cdot\big).$$

$\pi|_{\mathbb{R}^n}$ and $\pi|_M$ denote the respective canonical projections onto $\mathbb{R}^n$ and M. We consequently have

$$G = \begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & \tilde{G}(y) \end{pmatrix}, \qquad G^{-1} = \begin{pmatrix} \mathrm{id}_n & 0 \\ 0 & \tilde{G}(y)^{-1} \end{pmatrix}. \tag{IV.11}$$

In the next Lemma we check that (Ω, g) satisfies the condition (M) at the beginning of Section III.1.

Lemma IV.2.1
Let $\Omega := \mathbb{R}^n \times M$ as above. Then (Ω, g) is complete, has bounded geometry, and $\mathrm{inj}(\Omega) > 0$. *Moreover, $\Delta_\Omega = \Delta_{\mathbb{R}^n} + \Delta_M$.*

Proof. Let $d := n + m$. In the proof we denote $\tilde{g} := g_m$ with respective Christoffel symbols $\tilde{\Gamma}^p_{k,l}$ for $k, l, p \in \mathbb{N}_{\leq m}$. Equation (IV.11) yields that (Ω, g) has bounded geometry, since $\mathbb{R}^n$ and M have bounded geometry and the atlas $\mathscr{A}(r)$ is finite. We next calculate the Christoffel symbols $\Gamma^p_{k,l}$ of (Ω, g) in an arbitrary local chart $(O, \kappa) \in \mathscr{A}(r)$. (A.10) and (IV.11) imply for all $k, l, p \in \mathbb{N}_{\leq d}$

$$\Gamma^p_{k,l} = \frac{1}{2} \sum_{j=n+1}^{n+m} \big(\partial_{\omega_k} g_{j,l} + \partial_{\omega_l} g_{j,k} - \partial_{\omega_j} g_{k,l}\big) g^{j,p}.$$

This expression is 0 if either $k \leq n$ or $l \leq n$ or $p \leq n$. For $k, l, p > n$ and $\omega = (\boldsymbol{x}, y) \in \Omega$ additionally holds

$$\begin{aligned}
\Gamma^p_{k,l}(\omega) &= \frac{1}{2} \sum_{j=n+1}^{n+m} \big(\partial_{\omega_k} \tilde{g}_{j-n,l-n}(\omega) + \partial_{\omega_l} \tilde{g}_{j-n,k-n}(\omega) - \partial_{\omega_j} \tilde{g}_{k-n,l-n}(\omega)\big) g^{j-n,p-n}(\omega) \\
&= \frac{1}{2} \sum_{j=1}^{m} \big(\partial_{y_{k-n}} \tilde{g}_{j,l-n}(\omega) + \partial_{y_{l-n}} \tilde{g}_{j,k-n}(\omega) - \partial_{y_j} \tilde{g}_{k-n,l-n}(\omega)\big) g^{j,p-n}(\omega) \\
&= \tilde{\Gamma}^{p-n}_{k-n,l-n}(y).
\end{aligned}$$

We consequently have

$$\Gamma^p_{k,l}(\omega) = \begin{cases} \tilde{\Gamma}^{p-n}_{k-n,l-n}(y), & n+1 \leq k, l, p \leq n+m, \\ 0, & \text{otherwise.} \end{cases} \tag{IV.12}$$

With (IV.12) the geodesic differential equations (A.11) for (Ω, g) are

$$\gamma_p''(t) = 0, \qquad 1 \le p \le n,$$
$$\gamma_p''(t) = -\sum_{k,l=n+1}^{n+m} \tilde{\Gamma}_{k-n,l-n}^{p-n}(\gamma_{n+1}(t), ..., \gamma_{n+m}(t))\gamma_k'(t)\gamma_l'(t), \qquad n+1 \le p \le n+m.$$

Hence, the geodesic differential equations of (Ω, g) decouple into the geodesic differential equations for $\mathbb{R}^n$ and $(M, \tilde{g})$. γ is therefore a geodesic of Ω if and only if $\gamma = (\gamma_{\mathbb{R}^n}, \tilde{\gamma})$, whereby $\gamma_{\mathbb{R}^n}$ and $\tilde{\gamma}$ are geodesics of $\mathbb{R}^n$ and $(M, \tilde{g})$, respectively. For $\omega = (x, y) \in \Omega$ and $v \in T_\omega\Omega = \mathbb{R}^n \times T_y M$ then holds

$$\exp_\omega(v) = (\exp_x(v_1, ..., v_n), \exp_y(v_{n+1}, ..., v_{n+m})).$$

If $\|v\|_\omega < r$, then $(v_1, ..., v_n) \in B_x(0, r)$ and $(v_{n+1}, ..., v_{n+m}) \in B_y(0, r)$. $\exp_x$ is a diffeomorphism on $B_x(0, r)$ and $\exp_y$ is a diffeomorphism on $B_y(0, r)$ since $r < \mathrm{inj}(M)$. $\exp_\omega$ is therefore a diffeomorphism on $B_\omega(0, r)$ and consequently

$$\forall_{\omega\in\Omega}: \ r \in \left\{\epsilon \in (0, \infty) \mid \exp_\omega : B_\omega(0, \epsilon) \to \Omega \text{ is a diffeomorphism } \right\}.$$

This implies $\mathrm{inj}(\Omega) \ge r > 0$. The decoupling of the geodesics also shows that (Ω, g) is geodesically complete since $\mathbb{R}^n$ and $(M, \tilde{g})$ are geodesically complete.
(b) We already know that $(-\Delta_+, D(-\Delta_+))$ defined by

$$D(-\Delta_+) := L^2(\mathbb{R}^n, H_2^2(M)) \cap L^2(M, H_2^2(\mathbb{R}^n)),$$
$$-\Delta_+ f := -\Delta_{\mathbb{R}^n} f - \Delta_M f \quad \text{on } D(-\Delta_+),$$

is non-negative and selfadjoint on $L^2(\Omega)$. Recall that Theorem III.1.2;(b) states that $C_c^\infty(\Omega)$ is dense in $H_2^2(\Omega)$. Since $C_c^\infty(\Omega) \subseteq D(-\Delta_+) \subseteq H_2^2(\Omega)$ we have

$$\overline{D(-\Delta_+)}^{H_2^2(\Omega)} \cong H_2^2(\Omega).$$

It is therefore enough to show that $\Delta_+ = \Delta_\Omega$ on $C_c^\infty(\Omega)$. Then (III.3) and (IV.12) imply for $f \in C_c^\infty(\Omega)$

$$\Delta_\Omega f = \sum_{k,l=1}^{n} \partial_{\omega_k}\partial_{\omega_l} f + \sum_{k>n \,\vee\, l>n}^{n+m} g^{kl}\partial_{\omega_k}\partial_{\omega_l} f - \sum_{k,l,p=1}^{n+m} g^{kl}\Gamma_{k,l}^p \partial_{\omega_p} f$$
$$= \Delta_{\mathbb{R}^n} f + \sum_{k,l=1}^{m} \tilde{g}^{kl}\left(\partial_{y_k}\partial_{y_l} - \sum_{p=1}^{m} \tilde{\Gamma}_{k,l}^p \partial_{y_p}\right) f = \Delta_+ f.$$

□

Having checked the condition (M) for (Ω, g) in the previous lemma, we can now apply Theorem III.1.5 in Section III.1 to the nonlinear Schrödinger equation

$$\begin{aligned} iu'(t) &= -\tilde{\Delta}_\Omega u(t) + F_{\beta,\pm}(u(t)), \quad t \ne t_0, \\ u(t_0) &= f. \end{aligned} \qquad \text{(CPP)}$$

But before we exploit this, let us discuss some known results for (CPP). The most studied case is $M = \mathbb{T}^m$, where $\mathbb{T}^m$ is the m-dimensional flat torus. There has been considerable progress in recent years with respect to global existence in $H_2^1(\mathbb{R}^n \times \mathbb{T}^m)$ for which we refer to [HTT14], [HPTV14] and the references therein. The proofs given there heavily rely on the availability of the theory of Fourier series and methods adapted from the initial groundbreaking work of Bourgain in [Bou93b] and [Bou93a] on the nonlinear Schrödinger equation on the torus. These methods unfortunately do not transfer to the case where (M, g_m) is an arbitrary connected, compact C^∞-manifold without boundary. In this general situation there are much less results. The only two references we are aware of are [TV12] and [TTV14]. The prior paper deals with intial data in mixed-norm Sobolev spaces. The latter paper deals with (CPP) with initial data in $H_2^1(\mathbb{R}^n \times M)$. We therefore compare the results below with the one given in Theorem 1.4 of [TTV14]. For a convenient comparison we state a slight extension of their existence theorem and provide a sketch of the proof.

Theorem IV.2.2 ([TTV14])
Let $n \in \mathbb{N}$, $m = 1$, $\beta \in (0, 4/n)$, $(p, q) := (4(\beta+2)/\beta n, \beta + 2)$ and $f \in H_2^1(\Omega)$.

(a) The defocusing nonlinear Schrödinger equation (CPP) has a conditionally unique global strong solution $u \in C_b(\mathbb{R}, H_2^1(\Omega))$.

(b) The focusing nonlinear Schrödinger equation (CPP) has a conditionally unique global strong solution $u \in C_b(\mathbb{R}, H_2^1(\Omega))$ if either $\beta < 4/(n+1)$ or $\|f\|_{L^2(\Omega)}$ is small enough.

In (a) and (b) holds $L^2(\Omega)$- and energy conservation and the conditional uniqueness holds for all $I \in \mathcal{I}_b$ with respect to

$$
\begin{aligned}
X(I) &:= \left\{u \in L^p(I, L^q(\mathbb{R}^n, L^2(M))) \mid \partial_y u, \nabla_x u \in L^p(I, L^q(\mathbb{R}^n, L^2(M)))\right\}, \\
\|u\|_{X(I)} &:= \|u\|_{L^p(I, L^q(\mathbb{R}^n, H_2^1(M)))} + \|\nabla_x u\|_{L^p(I, L^q(\mathbb{R}^n, L^2(M)))}.
\end{aligned}
\tag{IV.13}
$$

Remarks:

(1) The authors of [TTV14] are exclusively concerned with long-term dynamics of the focusing equation. They therefore prove *(b)* with $\beta < 4/(n+1)$, since the latter condition ensures global existence with the energy method from Lemma I.3.5. We sketch below how to construct a maximal strong solution of (CPP) in $H_2^1(\Omega)$ with blow-up alternative if $\beta < 4/n$. Lemma I.3.5 then provides global existence for the defocusing equation. For $n = 1$ this implies the restriction $\beta < 4$ on the nonlinearity.

(2) In the next proposition we provide the needed Strichartz estimates for the proof of Theorem IV.2.2. In [DR12] the same estimates are provided on flat and distorted waveguides. The authors do not prove any results regarding local and global existence and it would be interesting to explore the possibilities of the method below in this context. However, we will not pursue this.

The proof is based on the following observation in the abstract setting of the previous Section IV.1.

Proposition IV.2.3
Let $\sigma \in (0,\infty)$ and $(p,q),(\tilde{p},\tilde{q})$ be sharp σ-admissible. If $\sigma \geq 1$ let $p,\tilde{p} > 2$. If U_x is $(\sigma, L^1(X))$-dispersive, then

$$\|\mathcal{U}f\|_{L^p(\mathbb{R},L^q_xL^2_y)} \lesssim \|f\|_{L^2_{x,y}}, \qquad \|\Phi F\|_{L^p(\mathbb{R},L^q_xL^2_y)} \lesssim \|F\|_{L^{\tilde{p}^*}(\mathbb{R},L^{\tilde{q}^*}_x L^2_y)}. \tag{IV.14}$$

Proof. By A.3.4;(1) we have $(L^1_xL^2_y)^* \equiv L^\infty_xL^2_y$. With Minkowski's integral inequality holds for $f \in L^1(X,L^2(Y)) \cap L^2(X\times Y)$ and $t,s \in I$ with $t \neq s$

$$\begin{aligned}\|U(t)U(s)^*f\|_{L^\infty_xL^2_y} &\lesssim \|U_x(t-s)f\|_{L^2_yL^\infty_x}\\ &\lesssim |t-s|^{-\sigma}\|f\|_{L^2_yL^1_x} \lesssim |t-s|^{-\sigma}\|f\|_{L^1_xL^2_y}.\end{aligned}$$

Theorem II.1.3 then implies (IV.14), since for $\theta \in (0,1)$ with $q = 2/(1+\theta)$ holds

$$[L^2(X\times Y), L^1(X,L^2(Y))]_\theta \cong L^q(X,L^2(Y))$$

by means of complex interpolation and (A.2). □

Proof of Theorem IV.2.2: We only sketch the local existence result. The rest of the proof is analogous to the one given in [TTV14] with the additional criteria for global solutions from Lemma I.3.5.
(1) Before we start the proof, the space $X(I)$ deserves some comments. It contains the mixed norm Sobolev space

$$\begin{aligned}H^1_{(r,2)}(\Omega) &:= \{g \in L^r(\mathbb{R}^n, L^2(M)) \mid \partial_y g, \nabla_x g \in L^r(\mathbb{R}^d, L^2(M))\},\\ \|g\|_{H^1_{(r,2)}(\Omega)} &:= \|f\|_{L^r(\mathbb{R}^n,H^1_2(M))} + \|\nabla_x g\|_{L^r(\mathbb{R}^n,L^2(M))}.\end{aligned}$$

with $r \in (1,\infty)$. Unfortunately we can not point to a suitable reference for the properties of these spaces. However, if (M, g_m) is substituted by $\mathbb{R}$, then Chapter 3 of [BIN78] provides a systematic treatment of such spaces. In particular, Theorems 9.1 and 14.14 there assert that $(H^1_{(r,2)}(\mathbb{R}^{n+1}), \|\cdot\|_{H^1_{(r,2)}(\mathbb{R}^{n+1})})$ is a Banach space which contains $C^\infty_c(\mathbb{R}^{n+1})$ as a dense subset. We can check that

$$\|g\|_{H^1_{(r,2)}(\Omega)} := \left(\sum_{i\in I}\|(\psi_i g)\circ\kappa_i^{-1}\|^2_{H^1_{(r,2)}(\mathbb{R}^{n+1})}\right)^{1/2},$$

with the smooth partition of unity $(\psi_i)_{i\in I}$ given at the beginning of the section, is an equivalent norm on $H^1_{(r,2)}(\Omega)$. Then a straightforward localization argument shows that the latter is also a Banach space with $C^\infty_c(\Omega)$ as a dense subset. $H^1_{(r,2)}(\Omega)$ is isometrically isomorphic to a closed subspace of the reflexive product space $L^r(\mathbb{R}^n, L^2(M))^{n+2}$, and therefore reflexive itself. It is useful for the nonlinear estimate (IV.17) below, to state the following variant of the characterization (III.27) for $g \in L^r(\mathbb{R}^n, L^2(M))$ and $i \in \{1,\dots,n\}$:

$$\partial_{x_i} g \in L^r(\mathbb{R}^n, L^2(M)) \iff \sup_{h\neq 0}\frac{\|g(\cdot + he_i) - g\|_{L^r_xL^2_y}}{|h|} < \infty. \tag{IV.15}$$

As in (III.28) the right-hand side can be bounded by $\|\partial_{x_i} g\|_{L^r_x L^2_y}$ if it is finite. To check these statements we can repeat the proof of Theorem 2.1.6 in [Zie89]. The essential ingredient for "$\Longrightarrow$" is the density of $C_c^\infty(\Omega)$ in $H^1_{(r,2)}(\Omega)$. For "$\Longleftarrow$" it is the reflexivity of $L^r(\mathbb{R}^n, L^2(M))$.

(2) With these preliminary remarks we start the contraction argument. We can use the Strichartz estimates in (IV.14), since $(e^{it\Delta_{\mathbb{R}^n}})_{t\in\mathbb{R}}$ is $({}^n/_2, L^1(\mathbb{R}^n))$-dispersive. Note that $H^1_2(M) \hookrightarrow L^\infty(M)$ by Theorem III.1.2;(d). We fix $f \in H^1_2(\Omega)$, $(p,q) := ({}^{4(\beta+2)}/_{\beta n}, \beta+2)$ and $X(I)$ as in (IV.13). We let $T \in (0,1]$ and put $I := [t_0 - T, t_0 + T]$. As before $\Phi_f(u)$ denotes the right-hand side of Duhamel's formula.

We put $\tilde{p} := {}^p/_{(p-2)}$, $\tilde{q} := {}^q/_\beta$ and observe that $\frac{1}{p^*} = \frac{1}{\tilde{p}} + \frac{1}{p}$ and $\frac{1}{q^*} = \frac{1}{\tilde{q}} + \frac{1}{q}$. The sharp ${}^n/_2$-admissibility of (p,q) and $\beta < {}^4/_n$ imply

$$\alpha := \frac{1}{\tilde{p}} - \frac{\beta}{p} = 1 - \frac{\beta+2}{p} = 1 - \frac{\beta n}{4} \in (0,1).$$

We in particular have $\beta\tilde{p} < p$. Let $u \in X(I)$. Then the nonlinear estimates from [TTV14], (III.24) and Hölder's inequality yield

$$\begin{aligned} \|\partial_y F_{\beta,\pm}(u)\|_{L^{p^*}(I,L_x^{q^*} L_y^2)} &\lesssim \big\| \|u\|^\beta_{L^\infty_y} \|u\|_{H^1_{2,y}} \big\|_{L^{p^*}(I,L_x^{q^*})} \\ &\lesssim \|u\|^\beta_{L^{\beta\tilde{p}}(I,L_x^{\beta\tilde{q}} L_y^\infty)} \|u\|_{L^p(I,L^q_x H^1_{2,y})} \\ &\lesssim T^\alpha \|u\|^{\beta+1}_{L^p(I,L^q_x H^1_{2,y})} \lesssim T^\alpha \|u\|^{\beta+1}_{X(I)}. \end{aligned} \tag{IV.16}$$

An application of (IV.15) as in the proof of Lemma III.1.4;(c+d) also implies

$$\|F_{\beta,\pm}(u)\|_{L^{p^*}(I,L_x^{q^*} L_y^2)} + \|\nabla_x F_{\beta,\pm}(u)\|_{L^{p^*}(I,L_x^{q^*} L_y^2)} \lesssim T^\alpha \|u\|^{\beta+1}_{X(I)}. \tag{IV.17}$$

Since (p,q) is sharp ${}^n/_2$-admissible we use (IV.14), (IV.16) and (IV.17) to provide the estimate

$$\|\Phi_f(u)\|_{X(I)} \le C_1\big(\|f\|_{H^1_2(\Omega)} + T^\alpha \|u\|^{\beta+1}_{X(I)}\big). \tag{IV.18}$$

Let $u, v \in X(I)$. Similar as in (IV.16) the estimates (IV.14) and Hölder's inequality yield

$$\begin{aligned} \|\Phi_f(u) - \Phi_f(v)\|_{L^p(I,L^q_x L^2_y)} &\le C \|F_{\beta,\pm}(u) - F_{\beta,\pm}(v)\|_{L^{p^*}(I,L_x^{q^*} L_y^2)} \\ &\le C \big\| \big(\|u\|^\beta_{L^\infty_y} + \|v\|^\beta_{L^\infty_y}\big) \|u - v\|_{L^2_y} \big\|_{L^{p^*}(I,L_x^{q^*})} \\ &\le C_2 T^\alpha \big(\|u\|^\beta_{X(I)} + \|v\|^\beta_{X(I)}\big) \|u - v\|_{L^p(I,L^q_x L^2_y)}. \end{aligned} \tag{IV.19}$$

Moreover, (IV.14) provides for $t, s \in I$ with $s < t$

$$\|\Phi_f(u)(t) - \Phi_f(u)(s)\|_{H^1_2(\Omega)} \lesssim \|\partial_y F_{\beta,\pm}(u)\|_{L^{p^*}([s,t],L_x^{q^*} L_y^2)} + \|\nabla_x F_{\beta,\pm}(u)\|_{L^{p^*}([s,t],L_x^{q^*} L_y^2)}.$$

Then (IV.16) and (IV.17) yield $\Phi_f \in C(I, H^1_2(\Omega))$. Hence, every fixed point of Φ_f in $X(I)$ belongs to $C(I, H^1_2(\Omega))$ and we are left to construct one. To this end we put $C_3 := \max\{C_1, 2C_2\}$ and

$$R := 2C_3 \|f\|_{H^1_2(\Omega)}, \qquad T \le \big(2C_3 R^\beta\big)^{-1/\alpha}. \tag{IV.20}$$

We define the metric space $(X(I), d_I)$ by

$$X(I,R) := \overline{B}_{X(I)}(0,R), \qquad d_I(u,v) := \|u-v\|_{L^p(I,L^q_x L^2_y)}.$$

The reflexivity of $H^1_{(q,2)}(\Omega)$ and Theorem 1.2.5 of [Caz03] imply that $(X(I,R), d_I)$ is complete. The choices in (IV.20) plugged into (IV.18) and (IV.19) imply that $\Phi_f : X(I,R) \to X(I,R)$ is a strict contraction. Φ_f has therefore a unique fixed point $u \in X(I,R)$. For the uniqueness of this fixed point in the larger space $X(I)$ we assume $v \in X(I)$ to be a second fixed point. Then (IV.19) shows that $u = v$ for some $\tilde{T} \leq T$. We successively repeat this argument to the intervals $[t_0 + (k-1)\tilde{T}, t_0 + (k+1)\tilde{T}]$, $[t_0 - (k+1)\tilde{T}, t_0 - (k-1)\tilde{T}]$ with $k \in \mathbb{N}$ until the whole interval I is covered. This shows $u = v$ on I.
Since $\beta < 4/n$ we have $\beta + 2 < e(n+1,1)$ and Theorem III.1.2;(d) provides the embedding $H^1_2(\Omega) \hookrightarrow L^{\beta+2}(\Omega)$. Then Proposition I.4.2;(c) provides $F_{\beta,\pm} \in C(H^1_2(\Omega), H^1_2(\Omega)^*)$ and Proposition I.2.4 yields that u is a strong solution of (CPP) □

Having reviewed the arguments of the global existence result in [TTV14], let us now state our existence results for (CPP). We first state a version of Theorem III.1.5 adapted to the situation $\Omega = \mathbb{R}^n \times M$.

Theorem IV.2.4
Let $m, n \in \mathbb{N}$, $s \in [1,\infty)$, and $\beta \in (1,\infty)$. We additionally assume:

(i) $\beta(d-2s) \leq 2(s+1)$ and $s > \frac{n+m}{2} - \frac{1}{\beta}$.

(ii) Either $\beta \in \mathbb{N}_{even}$ or $\beta > [s]+1$ or $s = 1$.

Then there is $p \in (\max\{\beta, 2\}, \infty)$ such that for each $f \in H^s_2(\Omega)$ the nonlinear Schrödinger equation (CPP) has a unique maximal strong solution $u \in C(I(f), H^s(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ with the following properties:

(a) u has $L^2(\Omega)$-conservation and the induced nonlinear flow is locally Lipschitz continuous.

(b) The nonlinear flow transports $H^{\tilde{s}}_2(\Omega)$ regularity if either $\beta \in \mathbb{N}_{even}$ or $\beta > [\tilde{s}]$. In this case u satisfies the blow-up alternative with respect to $H^{\tilde{s}}_2(\Omega)$.

(c) If $\beta(d-4) \leq 4$ and the nonlinear flow transports $H^{\tilde{s}}_2(\Omega)$ regularity with some $\tilde{s} \geq 2$, then u has energy conservation.

Proof. We showed in Proposition IV.2.1 that the cylindrical manifold (Ω, g) fits the framework of Section III.1 and we want to apply Theorem III.1.5. Note that the condition *(ii)* matches III.1.5;(ii). We are therefore left to check III.1.5;(i)+(iii).
We first prove local (p,q) Strichartz estimates with $1/2p$-loss for $(e^{it(\Delta_\Omega)})_{t\in\mathbb{R}}$. Indeed, $(e^{it\Delta_{\mathbb{R}^n}})_{t\in\mathbb{R}}$ is $(n/2, L^1(\mathbb{R}^n))$-dispersive and $(e^{it\Delta_M})_{t\in\mathbb{R}}$ is $(m/2, L^1(M))$-dispersive of SL-type $1/2$. Proposition IV.2.1 also provides that $e^{it\Delta_\Omega} = e^{it\Delta_{\mathbb{R}^n}} e^{it\Delta_M}$ for $t \in \mathbb{R}$. Theorem IV.1.1;(c) then implies local (p,q) Strichartz estimates with $1/2p$-loss for $(e^{it\Delta_\Omega})_{t\in\mathbb{R}}$ for all sharp $(n+m)/2$-admissible pairs (p,q) with $q < \infty$. Hence, III.1.5;(i) will be fulfilled as long as we choose the pair (p,q) accordingly.
Let (p,q) be sharp $(n+m)/2$-admissible pair such that $p \in (\max\{\beta,2\},\infty)$ and $s > \frac{n+m}{2} - \frac{1}{\beta} > \frac{n+m}{q} + \frac{1}{p}$. Combined with $\beta(d-2s) \leq 2(s+1)$ the latter provides III.1.5;(iii). Consequently Theorem III.1.5 provides all our claims. □

The standard energy methods from Lemma I.3.5 allow us to deduce the following global existence result for $n = m = 1$. This extends the corresponding result in Theorem IV.2.2 with respect to the growth β of the nonlinearity. Although this is one of our main results, the proof is rather short. We already provided in Section III.1 all the necessary estimates and tools that we need here.

Corollary IV.2.5
Let $m = 1$ and $\Omega = \mathbb{R} \times M$. We furthermore let $\beta \in [2, \infty)$, $p \in (\beta, \infty)$, and $f \in H_2^1(\Omega)$. If either the nonlinearity is defocusing or $\beta = 2$ and $\|f\|_{L^2(\Omega)}$ is small, then the nonlinear Schrödinger equation (CPP) has a conditionally unique global strong solution $u \in C_b(\mathbb{R}, H_2^1(\Omega)) \cap L^p_{loc}(\mathbb{R}, L^\infty(\Omega))$ with the following properties:

(a) u has $L^2(\Omega)$- and energy conservation.

(b) The induced nonlinear flow is locally Lipschitz continuous and transports $H_2^2(\Omega)$ regularity.

Remark: We stress that the comparable result in Theorem IV.2.2 requires $\beta < 4$. Here, we are able to prove global existence for the defocusing equation for an arbitrary $\beta \in [2, \infty)$.

Proof. We follow the lines of the proof of Corollary III.2.3. We fix $f \in H_2^1(\Omega)$. Note that IV.2.4;(i)+(ii) are satisfied. Then Theorem IV.2.4 provides a unique maximal strong solution $u \in C(I(f), H_2^1(\Omega)) \cap L^p_{loc}(I(f), L^\infty(\Omega))$ of (CPP). u has $L^2(\Omega)$-conservation and due to $\beta \in [2, \infty)$ there is transport of $H_2^2(\Omega)$ regularity by IV.2.4;(b). Then Theorem IV.2.4;(c) also yields energy conservation. The criteria for $I(f) = \mathbb{R}$ follow from Lemma I.3.5, whose assumptions we check now. We already established $L^2(\Omega)$- and energy conservation and therefore I.3.5;(i) is satisfied. Recall that $(\beta + 2)\hat{F}_{\beta,\pm}(g) = \pm\|g\|^{\beta+2}_{L^{\beta+2}(\Omega)}$. Then I.3.5;(ii) is clearly satisfied in the defocusing case. For the focusing case and $\beta = 2$ we follow the argument of (I.52). Theorem III.1.2 gives $H_2^{1/2}(\Omega) \hookrightarrow L^4(\Omega)$ and $[L^2(\Omega), H_2^1(\Omega)]_{1/2} \cong H_2^{1/2}(\Omega)$. For $g \in H_2^1(\Omega)$ follows the estimate

$$-\|g\|^4_{L^4(\Omega)} \gtrsim -\|g\|^4_{H_2^{1/2}(\Omega)} \gtrsim -\|g\|^2_{L^2(\Omega)}\|g\|^2_{H_2^1(\Omega)}.$$

Hence, I.3.5;(ii) is satisfied with $\beta_2 = 2$. Hence, if either the equation is defocusing or $\beta = 2$ and $\|f\|_{L^2(\Omega)}$ small enough, then Lemma I.3.5 provides $u \in L^\infty(I(f), H_2^1(\mathbb{R}^2))$. This shows $I(f) = \mathbb{R}$ by the blow-up alternative. □

As a final remark let us briefly comment on the problems we face when dealing with the defocusing equation (CPP) in the case $n = 1$, $m = 2$ and $\beta = 2$. This corresponds to the case $d = 3$ in [BGT04b]. For a more convenient notation let $(P_k^x)_{k\in\mathbb{N}_0}$, $(P_l^y)_{l\in\mathbb{N}_0}$ be defined by $P_k^x := \psi_k(-\Delta_\mathbb{R})$ and $P_l^y := \psi_l(-\Delta_M)$.
We fix $f \in H_2^1(\Omega)$. Theorem IV.2.4 is not applicable directly. However, for a sequence $(f_n)_{n\in\mathbb{N}}$ in $H_2^2(\Omega)$ with $f_n \overset{n\to\infty}{\longrightarrow} f$ in $H_2^1(\Omega)$, Theorem IV.2.4 provides a sequence of strong maximal solutions $(u_n)_{n\in\mathbb{N}}$ in $C(I(f_n), H_2^2(\Omega)) \cap L^p_{loc}(I(f_n), L^\infty(\Omega))$ to the cubic (CPP) with $u_n(0) = f$. Now we would like to approximate a weak solution in $H_2^1(\Omega)$ of (CPP) by these strong solutions in $H_2^2(\Omega)$ by means of Theorem I.3.6. A huge problem here is that we do not have control of the maximal existence intervals $I(f_n)$. The major

input of the a priori estimate in Theorem II.4.5 was the deduction of $I(f_n) = \mathbb{R}$ for all $n \in \mathbb{N}$ by means of the blow-up alternative. Then Theorem I.3.6 can be applied on each compact subinterval in $\mathbb{R}$.
Hence, one of the crucial ingredients for global existence in $H^1_2(\Omega)$ is the validity of Theorem II.4.5. The proof relied on estimates for $\|u\|_{L^2(I,L^\infty(\Omega))}$, which were obtain by Strichartz estimates of SL-type. A straightforward adaptation of the proof of II.4.3 to the product situation and the Bernstein inequalities for P^x_k and P^y_l yield for $k, l \in \mathbb{N}_0$ and $I \in \mathcal{I}_b$

$$\begin{aligned} \|P^x_k P^y_l u\|_{L^2(I,L^\infty_{x,y})} &\lesssim 2^{\frac{k}{12}} 2^{\frac{l}{6}} \|P^x_k P^y_l u\|_{L^2(I,L^6_{x,y})} \\ &\lesssim 2^{\frac{k}{12}} 2^{\frac{5l}{12}} \|P^x_k P^y_l u\|_{L^2(I,L^2_{x,y})} + 2^{\frac{k}{12}} 2^{\frac{l}{6}} \|P^x_k P^y_l F_{2,+}(u)\|_{L^\infty(I,L^{6/5}_{x,y})}. \end{aligned}$$

Then (II.36), (MC4), and (III.26) imply

$$\begin{aligned} &\|u\|_{L^2(I,L^\infty_{x,y})} \\ &\le \|P^x_0 P^y_0 u\| + \sum_{k=1}^\infty \|P^x_k P^y_0 u\|_{L^2(I,L^\infty_{x,y})} + \sum_{l=1}^\infty \|P^x_0 P^y_l u\|_{L^2(I,L^\infty_{x,y})} + \sum_{k,l=1}^\infty \|P^x_k P^y_l u\|_{L^2(I,L^\infty_{x,y})} \\ &\lesssim \|u\|_{L^\infty(I,H^1_2(\Omega))} + \sum_{k=1}^\infty \Big(2^{-\frac{5k}{12}} \|(-\Delta_M)^{\frac12} P^x_k u\|_{L^2(I,L^2_{x,y})} + 2^{\frac{k}{12}} \|P^x_k F_{2,+}(u)\|_{L^\infty(I,L^{6/5}_{x,y})} \Big) \\ &\quad + \sum_{l=1}^\infty \Big(2^{-\frac{l}{12}} \|(-\Delta_M)^{\frac12} P^y_l u\|_{L^2(I,L^2_{x,y})} + 2^{\frac{l}{6}} \|P^y_l F_{2,+}(u)\|_{L^\infty(I,L^{6/5}_{x,y})} \Big) \\ &\quad + \sum_{k,l=1}^\infty \Big(2^{\frac{k}{12}} 2^{\frac{5l}{12}} \|P^x_k P^y_l u\|_{L^2(I,L^2_{x,y})} + 2^{\frac{k}{12}} 2^{\frac{l}{6}} \|P^x_k P^y_l F_{2,+}(u)\|_{L^\infty(I,L^{6/5}_{x,y})} \Big) \\ &\lesssim \|u\|_{L^\infty(I,H^1_2(\Omega))} + \sum_{k,l=1}^\infty 2^{\frac{k}{12}} 2^{\frac{5l}{12}} \|P^x_k P^y_l u\|_{L^2(I,L^2_{x,y})} + \|(-\Delta_{\mathbb{R}})^{\frac{1+\epsilon}{12}} (-\Delta_M)^{\frac{1+\epsilon}{6}} F_{2,+}(u)\|_{L^\infty(I,L^{6/5}_{x,y})}, \end{aligned}$$

for all $\epsilon \in (0, 1)$. In view of the procedure in (II.72) the first two terms in the last line are managable. However, we still need to bound the term involving the nonlinearity by $C\|u\|_{L^\infty(I,H^1_2(\Omega))}$. The straightforward way to achieve this is to show the boundedness of $\varphi(-\Delta_{\mathbb{R}}, -\Delta_M) \in \mathcal{L}(L^{6/5}(\Omega))$, whereby

$$\varphi : [0,\infty)^2 \to \mathbb{R}, \qquad \varphi(\lambda_x, \lambda_y) := \frac{\lambda_x^{\frac{1+\epsilon}{6}} \lambda^{\frac{1}{12}}}{(\lambda_x + \lambda_y)^{\frac12}}. \tag{IV.21}$$

Such boundedness results are often obtained by multivariate spectral multiplier results (see for example [Sik09]). However, compared to the bounded spectral calculus on $L^2(\Omega)$, which only requires a function to be bounded, Theorem 2.1 of [Sik09] also requires the partial derivatives of a function to be bounded. The partial derivatives of φ in (IV.21) develop singularities for $\lambda_x = 0$ and $\lambda_y = 0$, respectively. The multivariate spectral multiplier result in [Sik09] is therefore not applicable. Unfortunately, we are not able to bypass this problem at this point.

Appendix

A. Supplementary Material

The following appendices are meant to supply additional material, which is used frequently throughout this thesis. We focus on mentioning the necessary results and adequate references for them so that a working basis is established. The details of proofs can be found in the provided references.

A.1. The complex interpolation method

There are several textbooks dealing with abstract interpolation theory, see for example [Tri95] or [BL76]. Both these textbooks contain the most commonly used results and also the real interpolation method. For the complex interpolation method the original article [Cal64] by A. Calderon is also worth a look. Applications to partial differential equations can be found in the textbook [Lun95].
It is not intended to give a precise introduction into this topic here, we merely want to present the central ideas and some relevant results used throughout this thesis.
The first instance of a complex interpolation result is the well-known Riesz-Thorin Interpolation Theorem. The proof heavily relies on complex function theory, more precisely, on Hadamard's three lines theorem.

Theorem A.1.1 ([Gra08] Theorem 1.3.4)
Let $(\Omega, \mathscr{S}_\Omega, \mu)$ *and* $(\Sigma, \mathscr{S}_\Sigma, \nu)$ *be measure spaces,* $p_0, p_1, q_0, q_1 \in [1, \infty]$*, and*

$$A : L^{p_0}(\Omega) \cap L^{p_1}(\Omega) \to L^{q_0}(\Sigma) + L^{q_1}(\Sigma)$$

be a linear operator. Let furthermore $\theta \in (0,1)$ *and* $p, q \in [1, \infty]$ *with*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If there are $C_0, C_1 \in [0, \infty)$ *such that for all* $f \in L^{p_0}(\Omega) \cap L^{p_1}(\Omega)$ *holds*

$$\|Af\|_{L^{q_0}(\Sigma)} \leq C_0 \|f\|_{L^{p_0}(\Omega)},$$

$$\|Af\|_{L^{q_1}(\Sigma)} \leq C_1 \|f\|_{L^{p_1}(\Omega)},$$

then A *can be extended uniquely to* $\overline{A} \in \mathcal{L}(L^p(\Omega), L^q(\Sigma))$ *with* $\|\overline{A}\|_{L^p(\Omega) \to L^q(\Sigma)} \leq C_0^{1-\theta} C_1^{\theta}$.

As it turns out, this is only a special case of a more general method called the complex interpolation method, which is defined as follows. Let $\theta \in (0,1)$ and the Banach spaces $(X_0, \|\cdot\|_{X_0})$ and $(X_1, \|\cdot\|_{X_1})$ be continuously embedded into a topological Hausdorff space $(X, \mathcal{O})$. In this situation we call (X_0, X_1) a Banach interpolation couple. With the strip $S := \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < 1\}$ we define the class of functions $\mathscr{F}(X_0, X_1)$ as the set of all functions $F : \overline{S} \to X_0 + X_1$ with the following properties

(1) $F \in C(\overline{S}, X_0 + X_1)$ and $F|_S$ is holomorphic,

(2) For $k \in \{0,1\}$ holds $F(k+is) \in X_k$ for $s \in \mathbb{R}$ and $F(k+is)) : \mathbb{R} \to X_k$ is continuous,

(3) $\sup_{z \in \overline{S}} \left(e^{-|\operatorname{Im}(z)|} \|F(z)\|_{X_1+X_2}\right) < \infty$.

For $F \in \mathscr{F}(X_0, X_1)$ we define

$$\|F\|_{\mathcal{F}(X_0,X_1)} := \max_{k \in \{0,1\}} \sup_{s \in \mathbb{R}} \left(e^{-s} \|F(k+is)\|_{X_k}\right).$$

Then we define the complex interpolation space

$$[X_0, X_1]_\theta := \left\{x \in X_0 + X_1 \mid \exists_{F \in \mathcal{F}(X_0,X_1)} : \; F(\theta) = x\right\}$$

and equipp it with the norm

$$\|x\|_{[X_0,X_1]_\theta} := \inf_{F \in \mathscr{F}(X_0,X_1), F(\theta)=x} \|F(\theta)\|_{\mathcal{F}(X_0,X_1)}.$$

As in Section 9.3 in [Cal64] we define $[X_0, X_1]_0 := \overline{X_0 \cap X_1}^{\|\cdot\|_{X_0}}$.

Facts A.1.2
Let (X_0, X_1) be a Banach interpolation couple and $\theta \in (0,1)$.

(1) **Density and Interpolation inequality (see [Tri95] Theorem 1.9.3):** The complex interpolation space $([X_0, X_1]_\theta, \|\cdot\|_{[X_0,X_1]_\theta})$ is a Banach space and we have

$$X_0 \cap X_1 \subseteq [X_0, X_1]_\theta \subseteq X_0 + X_1.$$

$X_0 \cap X_1$ is dense in $[X_0, X_1]_\theta$. Moreover, there is $C(\theta) \in (0, \infty)$ such that for all $x \in X_0 \cap X_1$ holds

$$\|x\|_{[X_0,X_1]_\theta} \leq C(\theta) \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta}.$$

(2) **Duality (see [Tri95] Theorem 1.9.2):** If either X_0 or X_1 is reflexive and $X_0 \cap X_1$ is dense in both, then $[X_0, X_1]_\theta^* = [X_1^*, X_0^*]_\theta$.

(3) **The Reiteration Theorem (see [Cal64] 12.3):** Let $\eta_0, \eta_1 \in (0,1)$ and either $X_0 \hookrightarrow X_1$ or $X_1 \hookrightarrow X_0$. Then holds

$$[[X_0, X_1]_{\eta_0}, [X_0, X_1]_{\eta_1}]_\theta \cong [X_0, X_1]_{(1-\theta)\eta_0 + \theta\eta_1}.$$

The proof of this fact in [Cal64] reveals that without the given additional assumptions on the Banach interpolation couple (X_0, X_1) the embedding

$$[X_0, X_1]_{\eta\theta} \hookrightarrow [\overline{X_0 \cap X_1}^{\|\cdot\|_{X_0}}, [X_0, X_1]_\eta]_\theta \cong [X_0, [X_0, X_1]_\eta]_\theta$$

is still valid. The last equivalence can be deduced from Section 9.3 there.

With these concepts we present the central interpolation result for linear and bilinear operators.

Theorem A.1.3 ([Cal64], Section 4 and 10.2)
Let (X_0, X_1), (Y_0, Y_1), and (Z_0, Z_1) be Banach interpolation couples and $\theta \in (0, 1)$.

(a) *If a linear operator $A : X_0 \cap X_1 \to Z_0 + Z_1$ satisfies for all $x \in X_0 \cap X_1$*

$$\|Ax\|_{Z_0} \leq C_0 \|x\|_{X_0},$$
$$\|Ax\|_{Z_1} \leq C_1 \|x\|_{X_1},$$

then there is a uniquely determined extension $\overline{A} \in \mathcal{L}([X_0, X_1]_\theta, [Z_0, Z_1]_\theta)$ with $\overline{A} = A$ on $X_0 \cap X_1$ and

$$\|\overline{A}\|_{[X_0,X_1]_\theta \to [Z_0,Z_1]_\theta} \leq C_0^{1-\theta} C_1^\theta.$$

(b) *If a bilinear operator $B : (X_0 \cap X_1) \times (Y_0 \cap Y_1) \to Z_0 + Z_1$ satisfies for all $(x, y) \in (X_0 \cap X_1) \times (Y_0 \cap Y_1)$*

$$\|B(x, y)\|_{Z_0} \leq C_0 \|x\|_{X_0} \|y\|_{Y_0},$$
$$\|B(x, y)\|_{Z_1} \leq C_1 \|x\|_{X_1} \|y\|_{Y_1},$$

then there is a uniquely determined continuation $\overline{B} : [X_0, X_1]_\theta \times [Y_0, Y_1]_\theta \to [Z_0, Z_1]_\theta$ with $\overline{B} = B$ on $(X_0 \cap X_1) \times (Y_0 \cap Y_1)$ and

$$\|\overline{B}\|_{[X_0,X_1]_\theta \times [Y_0,Y_1]_\theta \to [Z_0,Z_1]_\theta} \leq C_0^{1-\theta} C_1^\theta.$$

Examples A.1.4
We close this section with some relevant examples on how complex interpolation spaces can be characterized in particular situations.

(1) **L^p-spaces (see [Tri95] Theorem 1.18.4):** Let (X_0, X_1) be a Banach interpolation couple, $(\Omega, \mathscr{S}_\Omega, \mu)$, $(\Sigma, \mathscr{S}_\Sigma, \nu)$ be measure spaces, and $\theta \in (0, 1)$. Let furthermore $p, p_0, p_1 \in [1, \infty)$ with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then holds

$$[L^{p_0}(\Omega, X_0), L^{p_1}(\Omega, X_1)]_\theta \cong L^p(\Omega, [X_0, X_1]_\theta). \tag{A.1}$$

Consult Remark 3 in Section 1.18.4 of [Tri95] for a discussion of the case $p_1 = \infty$. Moreover, if $X_0 = X_1 = \mathbb{C}$ then (A.1) is true if either $p_0 = 1$ and/or $p_1 = \infty$ (see [Tri95] Theorem 1.18.6.2). A combination of this fact with Theorem A.1.3;(a) reproduces the Riesz-Thorin Interpolation Theorem A.1.1. In particular, if $X = L^{q_0}(\Sigma)$, $Y = L^{q_1}(\Sigma)$ and $q, q_1, q_2 \in [1, \infty]$ with $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, then

$$[L^{p_0}(\Omega, L^{q_0}(\Sigma)), L^{p_1}(\Omega, L^{q_1}(\Sigma))]_\theta \cong L^p(\Omega, L^q(\Sigma)). \tag{A.2}$$

(2) **Bessel potential spaces (see [Tri95] Theorem 2.4.2.1;(d) and Remark 2):** Let $s_0, s_1 \in [0, \infty)$ and $p_0, p_1 \in (1, \infty)$ and $s := (1-\theta)s_0 + \theta s_1$ and $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then holds

$$[H^{s_0}_{p_0}(\mathbb{R}^d), H^{s_1}_{p_1}(\mathbb{R}^d)]_\theta \cong H^s_p(\mathbb{R}^d). \tag{A.3}$$

(3) Let $(H, (\cdot, \cdot)_H)$ be a Hilbert space and $(A, D(A))$ be a positive definite, selfadjoint linear operator on H. Theorem 1.18.10 in [Tri95] then ensures

$$[D(A^\alpha), D(A^\beta)]_\theta \cong D(A^{(1-\theta)\alpha + \theta\beta}). \tag{A.4}$$

A.2. Sobolev's embedding theorem

In this short section we recall one of the most important tools in the study of partial differential equation, namely the Sobolev embeddings. We also formulate the Hardy-Littlewood-Sobolev inequality. Recall that for $p \in (1,\infty)$ and an open set $\Omega \subseteq \mathbb{R}^d$ we defined

$$H^s_p(\Omega) := \begin{cases} W^s_p(\Omega), & s \in \mathbb{N}_0, \\ [W^{[s]}_p(\Omega), W^{[s]+1}_p(\Omega)]_{s-[s]}, & s \in [0,\infty) \setminus \mathbb{N}_0. \end{cases}$$

Facts A.2.1
Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $s \in [0,\infty)$, $p \in (1,\infty)$, $q \in [1,\infty]$ and $e(d,p,s) := {}^{pd}/_{\max\{d-ps,0\}}$ with ${}^{pd}/_{0} := \infty$.

(1) The following embeddings hold:

$$\frac{1}{p} < \frac{s}{d} \quad \Longrightarrow \quad H^s_{p,0}(\Omega) \hookrightarrow L^\infty(\Omega), \tag{A.5}$$

$$\frac{1}{p} \leq \frac{s}{d} \quad \Longrightarrow \quad H^s_{p,0}(\Omega) \hookrightarrow L^q(\Omega), \qquad q \in [p, e(d,p,s)), \tag{A.6}$$

$$\frac{1}{p} > \frac{s}{d} \quad \Longrightarrow \quad H^s_{p,0}(\Omega) \hookrightarrow L^q(\Omega), \qquad q = e(d,p,s). \tag{A.7}$$

(2) If Ω is a bounded Lipschitz domain (or satisfies the strong local Lipschitz condition in Definition 4.9 of [AF03]), then (A.5)-(A.7) hold with $H^s_{p,0}(\Omega)$ replaced by $H^s_p(\Omega)$.

In this form the above Sobolev embeddings deserve some comments. For $\Omega = \mathbb{R}^d$ Theorem 2.8.1 in [Tri95] provides the embeddings in (1) since

$$H^s_{p,0}(\mathbb{R}^d) = H^s_p(\mathbb{R}^d) = F^s_{p,2}(\mathbb{R}^d).$$

The Triebel-Lizorkin space $F^s_{p,2}(\mathbb{R}^d)$ is defined in Section 2.3 of [Tri95]. For an arbitrary open set $\Omega \subseteq \mathbb{R}^d$ the density of $C^\infty_c(\Omega)$ in $H^s_{p,0}(\Omega)$ yields all the $\mathbb{R}^d$-embeddings in this case, what settles (1). For (2) we can use the extension operator ext_Ω in Remark III.4.2;(3) to carry over the $\mathbb{R}^d$-embeddings.
The central tools to prove the Sobolev embeddings in the $\mathbb{R}^d$-case is the following inequality. This inequality is interesting in its own right in the proof of Theorem II.1.3.

Theorem A.2.2 (Hardy-Littlewood-Sobolev, [Gra09] Theorem 6.1.3)
Let $d \in \mathbb{N}$, $s \in (0,d)$ and $1 < p < q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \frac{s}{d}$. Then the linear operator

$$A : S(\mathbb{R}^d) \to L^q(\mathbb{R}^d), \qquad (Af)(x) := \text{p.v.} \int_{\mathbb{R}^d} \frac{f(x-y)}{|y|^{d-s}} \,\mathrm{d}y,$$

extends uniquely to $\overline{A} \in \mathcal{L}(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))$.

A.3. Vector-valued L^p- and Sobolev spaces

In this thesis we extensively use integrals over functions with values in L^p-spaces defined on an interval. The necessary background material presented here is covered in many textbooks in the more general framework of functions with values in a Banach space defined on a σ-finite measure space; see for example Chapter X in [AE09], Chapter VI in [Lan93], and Section 2.2 in[DU77]. These textbooks contain all the proofs of the below assertions. However, the definition of these spaces follows the lines of the scalar-valued case. Moreover, assertions from the scalar-valued theory which do not rely on non-negativity usually carry over with almost no change of proof.
In this section always let $(X, \|\cdot\|_X)$ be a Banach space equipped with the Borel σ-algebra $\mathscr{B}(X)$ and $(\Omega, \mathscr{S}, \mu)$ be a σ-finite measure space.

Definition A.3.1
Let $f : \Omega \to X$ be a function.

(a) f is called simple, if there are $N \in \mathbb{N}$, $\Omega_1, ..., \Omega_n \in \mathscr{M}$ with $\mu(\Omega_i) < \infty$ for $i \in \{1, ..., N\}$, and $x_1, ..., x_N \in X$ such that for all $\omega \in \Omega$ holds $f(\omega) = \sum_{k=1}^{N} \mathbb{1}_{\Omega_i}(\omega) x_i$.

(b) f is called strongly measurable if there is a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions on X such that $f_n \overset{n\to\infty}{\longrightarrow} f$ almost everywhere.

Consequently, every simple function is strongly measurable and every strongly measurable function is measurable. Note that the function

$$f : (0,1) \to L^\infty((0,1), \mathbb{R}), \qquad [f(t)](\cdot) := \mathbb{1}_{(0,t)}(\cdot),$$

is measurable but not strongly measurable. Since $\|\cdot\|_X : X \to [0,\infty)$ is continuous, additionally $\|f(\cdot)\|_X : \Omega \to [0,\infty)$ is measurable if $f : \Omega \to X$ is strongly measurable and we can formulate the following definition.

Definition A.3.2
Let $f : \Omega \to X$ be strongly measurable and $(f_n)_{n\in\mathbb{N}}$ a sequence of simple functions with $f_n \overset{n\to\infty}{\longrightarrow} f$ alomst everywhere.

(a) If f is a simple function, then $\displaystyle\int_\Omega f \,\mathrm{d}\lambda := \sum_{i=1}^{N} \mu(\Omega_i) x_i$.

(b) f is called Bochner integrable, if $\displaystyle\lim_{n\to\infty} \int_\Omega \|f - f_n\|_X \,\mathrm{d}\lambda = 0$. In this case the Bochner integral of f is defined as

$$\int_\Omega f \,\mathrm{d}\lambda := \lim_{n\to\infty} \int_\Omega f_n \,\mathrm{d}\lambda.$$

Facts A.3.3
All the results presented here, except Fubini's theorem, can be found in [DU77] Section 2.2.

(1) **Alteration on null sets:** If $f, g : \Omega \to X$ are Bochner integrable with $f = g$ almost everywhere, then the Bochner integrals of f and g are equal. The equivalence class of a function f induced by almost everywhere equality is denoted by $f_\sim$.

(2) **Bochner's Theorem:** Let $f : \Omega \to X$ be strongly measurable. Then f is Bochner integrable if and only if $\int_\Omega \|f\|_X \, \mathrm{d}\mu < \infty$ and in this case

$$\left\| \int_\Omega f \, \mathrm{d}\mu \right\|_X \le \int_\Omega \|f\|_X \, \mathrm{d}\mu.$$

(3) **The dominated convergence theorem:** Let $(f_n)_{n\in\mathbb{N}}$ with $f_n : \Omega \to X$ Bochner integrable be almost everywhere convergent with limit function $f : \Omega \to X$. If there is a Bochner integrable function $g : \Omega \to \mathbb{C}$ with $\|f_n\|_X \le |g|$ almost everywhere, then f is Bochner integrable and

$$\lim_{n\to\infty} \int_\Omega \|f_n - f\|_X \, \mathrm{d}\mu = 0.$$

By means of (2) this implies

$$\lim_{n\to\infty} \int_\Omega f_n \, \mathrm{d}\mu = \int_\Omega f \, \mathrm{d}\mu.$$

(4) **Hille's Theorem:** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces, $f : \Omega \to X$ be Bochner integrable, and $(A, D(A))$ be a closed linear operator from X to Y. If $f \in D(A)$ almost everywhere and $Af : \Omega \to Y$ (which is defined almost everywhere) is Bochner integrable, then we have

$$\int_\Omega f \, \mathrm{d}\mu \in D(A), \qquad A \int_\Omega f \, \mathrm{d}\mu = \int_\Omega Af \, \mathrm{d}\mu.$$

(5) **Fubini's Theorem (see [Lan93] Theorem 8.4):** Let $(\Omega_1, \mathscr{S}_1, \mu_1)$, $(\Omega_2, \mathscr{S}_2, \mu_2)$ be σ-finite measure spaces and $f : \Omega_1 \times \Omega_2 \to X$ be Bochner integrable with respect to $\mu_1 \otimes \mu_2$. Then

$$\int_{\Omega_1\times\Omega_2} f \, \mathrm{d}(\mu_1 \otimes \mu_2) = \int_{\Omega_1} \left(\int_{\Omega_2} f \, \mathrm{d}\mu_2 \right) \mathrm{d}\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f \, \mathrm{d}\mu_1 \right) \mathrm{d}\mu_2.$$

Finally, we define the Banach space valued L^p-spaces by

$$L^p(\Omega, X) := \left\{ f_\sim \mid f : \Omega \to X \text{ Bochner integrable}, \int_\Omega \|f\|_X^p \, \mathrm{d}\mu < \infty \right\}, \quad p \in [1, \infty)$$

$$\|f_\sim\|_{L^p(\Omega,X)} := \left(\int_\Omega \|f\|_X^p \, \mathrm{d}\mu \right)^{1/p}$$

and

$$L^\infty(\Omega, X) := \left\{ f_\sim \mid f : \Omega \to X \text{ strongly measurable, bounded almost everywhere} \right\}$$

$$\|f_\sim\|_{L^\infty(\Omega,X)} := \operatorname{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_X.$$

As usual we completely ignore the difference of f and $f_\sim$ in our notation.

Facts A.3.4
As before we gather some important assertions.

(1) **Elementary properties (see [DU77] Section 4.1):** For $p \in [1, \infty]$ the space $(L^p(\Omega, X), \|\cdot\|_{L^p(\Omega,X)})$ is a Banach spaces. For $p < \infty$ it contains the simple functions as a dense subset and for $p = \infty$ it contains the countably valued functions as a dense subset. Moreover, the map

$$\mathscr{I} : L^{p^*}(\Omega, X^*) \to L^p(\Omega, X)^*, \qquad \langle \mathscr{I}F, \cdot\rangle := \int_\Omega \langle F, \cdot\rangle_{X^*,X} \, \mathrm{d}\mu$$

is an isometry. If $p \in [1, \infty)$ and $(X, \|\cdot\|_X)$ is reflexive, then $\mathscr{I}$ is surjective, hence $L^p(\Omega, X)^* \equiv L^{p^*}(\Omega, X^*)$ with isometric isomorphism $\mathscr{I}$. The latter assertion has been proven in [DU77], Theorem 1 in Section 4.1 for $\mu(\Omega) < \infty$ and in [Edw95], Theorem 8.18.3 and 8.20.4 for $(\Omega, \mathscr{S}, \mu)$ σ-finite.

(2) **Characterization of $L^p(\Omega_1 \times \Omega_2, X)$ (see [AE09] Theorem 6.22):** Let $(\Omega_1, \mathscr{S}_1, \mu_1)$ and $(\Omega_2, \mathscr{S}_2, \mu_2)$ be σ-finite measure spaces. For $p \in [1, \infty)$ Fubini's Theorem eventually implies that

$$L^p(\Omega_1 \times \Omega_2, X) \equiv L^p(\Omega_1, L^p(\Omega_2)),$$

with the isometric isomorphism

$$\mathscr{I} : L^p(\Omega_1 \times \Omega_2, X) \to L^p(\Omega_1, L^p(\Omega_2, X)), \qquad [\tau(f)(\omega_1)](\cdot) := f(\omega_1, \cdot).$$

This statement is false if $p = \infty$, since the strong measurability of $\mathscr{I}f : \Omega_1 \to L^\infty(\Omega_2, X)$ may fail as for example shown in the counterexample in Remark 6.23 of [AE09].

(3) **Minkowski's integral inequality (see [Gra08] 1.1.6):** Let $(\Omega_i, \mathscr{S}_i, \mu_i)$ with $i \in \{1, 2\}$ be σ-finite measure spaces, $p, q \in [1, \infty)$ with $1 \leq q \leq p$, and $F : \Omega_1 \times \Omega_2 \to \mathbb{C}$ be measurable with respect to $\mathscr{S}_1 \otimes \mathscr{S}_2$. Then

$$\|F\|_{L^p(\Omega_1, L^q(\Omega_2))} \leq \|F\|_{L^q(\Omega_2, L^p(\Omega_1))}. \tag{A.8}$$

(4) **The Lemma of Christ-Kiselev (see [CK01] Theorem 1.1):** Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be Banach spaces, $I \in \mathcal{I}$, $p, q \in [1, \infty]$, and $K : I \times I \to \mathcal{L}(X, Y)$ be locally integrable such that

$$T : L^p(I, X) \to L^q(I, Y), \qquad (TF)(t) := \int_I K(t, s)F(s) \, \mathrm{d}s$$

is bounded. If $p < q$, then

$$T|_r : L^p(I, X) \to L^q(I, Y), \qquad (T|_r F)(t) := T(\mathbb{1}_{(\inf I, t]} F)(t),$$

is bounded with $\|T|_r\|_{L^p(I,X) \to L^q(I,Y)} \lesssim \|T\|_{L^p(\mathbb{R},X) \to L^q(\mathbb{R},Y)}$.

Now we want to present the necessary material on the Sobolev space $W^1_p(I, X)$. For details and more results see for example Chapter 1 of [CH98].

From now on let $I \in \mathcal{I}_o$, $p \in [1,\infty]$, and $u \in L^p(I,X)$. We define the distributional derivative of u via

$$\forall_{\eta \in C_c^\infty(I)}:\; u'(\eta) := -\int_I u(t)\eta'(t)\,\mathrm{d}t.$$

$u' \in L^p(I,X)$ means that there is $v \in L^p(I,X)$ such that

$$\forall_{\eta \in C_c^\infty(I)}:\; u'(\eta) = \int_I v(t)\eta(t)\,\mathrm{d}t.$$

We then define the Sobolev space $W^1_p(I,X)$ by

$$W^{1,p}(I,X) := \{u \in L^p(I,X) \mid u' \in L^p(I,X)\},$$

$$\|u\|_{W^1_p(I,X)} := \|u\|_{L^p(I,X)} + \|u'\|_{L^p(I,X)}.$$

Facts A.3.5
Let $p \in [1,\infty]$. We gather an important characterization for functions in $W^1_p(I,X)$ and some useful embeddings.

(1) **Characterization of $W^1_p(I,X)$ (see [CH98] Theorem 1.4.35):** $(W^1_p(I,X), \|\cdot\|_{W^1_p(I,X)})$ is a Banach space and for $u \in L^p(I,X)$ the following assertions are equivalent.

(i) $u \in W^1_p(I,X)$.

(ii) There is $v \in L^p(I,X)$ such that for almost all $s,t \in I$ holds

$$u(t) - u(s) = \int_s^t v(\tau)\,\mathrm{d}\tau.$$

(iii) There are $v \in L^p(I,X)$, $x \in X$, and $s \in I$, such that for almost all $t \in I$ holds

$$u(t) - x = \int_s^t v(\tau)\,\mathrm{d}\tau.$$

(iv) u is absolutely continuous, almost everywhere differentiable on I, and $u' \in L^p(I,X)$.

As in the scalar-valued case the function v from (ii) and (iii) coincides with the distributional derivative u'. Moreover, if $(X, \|\cdot\|_X)$ is reflexive then a fifth characterization can be added, namely:

(v) There is $h \in L^p(I,\mathbb{R})$ such that for almost all $s,t \in I$ holds

$$\|u(s) - u(t)\|_X \leq \left| \int_s^t h(\tau)\,\mathrm{d}\tau \right|.$$

(2) **Embeddings for $W^1_p(I,X)$:** From (1) immediately follows:

(i) $W^1_p(I,X) \hookrightarrow C_{b,u}(\overline{I},X)$.

(ii) If $p > 1$, then $W^1_p(I,X) \hookrightarrow C^{0,1/p^*}(\overline{I},X)$.

(iii) If X is reflexive, then for $u \in C^{0,1}(\overline{I},X)$ holds $u|_I \in W^1_\infty(I,X)$.

Remark A.3.6
One important consequence of (2) is that, roughly speaking, for reflexive Banach spaces $(X, \|\cdot\|_X)$ the Sobolev space $W^1_\infty(I,X)$ coincides with $C^{0,1}(\overline{I},X)$.

A.4. Some Riemannian geometry

In this section we give a brief review of standard notions and concepts with respect to Riemannian manifolds. For more information we refer to the textbooks [Aub98, Heb99, Gri09].

Let $(\Omega, \tau, \mathscr{A})$ be a connected C^∞-manifold without boundary, with topology τ, smooth atlas $\mathscr{A}$, and $\dim(\Omega) = d$. We usually omit the topology τ and the atlas $\mathscr{A}$ and simply write Ω.

For a local chart $(O, \kappa) \in \mathscr{A}$ we associate a local coordinate system and distorted partial derivatives in the following manner: For $\omega \in O$ we call $\boldsymbol{z} = (z_1, ..., z_d) = \kappa(\omega) \in \mathbb{R}^d$ the local coordinates of ω in (O, κ). If $f : \Omega \to \mathbb{R}$ is differentiable in $\omega = \kappa^{-1}(\boldsymbol{z})$ then

$$\partial_{\omega_k}|_\omega f := (\partial_{\omega_k} f)(\omega) := (\partial_{z_k}(f \circ \kappa^{-1}))(\boldsymbol{z}).$$

Here ∂_{z_k} denotes the standard partial derivative on $\mathbb{R}^d$. We put $\partial_\omega^{\boldsymbol{\alpha}} = \partial_{\omega_1}^{\alpha_1} \ldots \partial_{\omega_d}^{\alpha_d}$ for $\boldsymbol{\alpha} \in \mathbb{N}_0^d$.

Let $\omega \in \Omega$. Then the tangential space of Ω in ω is given by the d-dimensional real vector space

$$T_\omega\Omega := \operatorname{span}\left\{\partial_{\omega_1}|_\omega, ..., \partial_{\omega_d}|_\omega\right\}.$$

We additionally denote by $T\Omega := \bigcup_{\omega \in \Omega} T_\omega\Omega$ the tangent bundle. For $k \in \mathbb{N}$ the space $T_\omega^k\Omega$ is given by all k-linear forms on $\bigotimes_{l=1}^k T_\omega\Omega$ and $T^k\Omega := \bigcup_{\omega \in \Omega} T_\omega^k\Omega$. All these structures allow the definition of a C^∞-atlas and we thus consider them as C^∞-manifolds themselves. Hence, we have a notion of differentiability on these sets.

A map $v : \Omega \to T\Omega$ with $v(\omega) \in T_\omega\Omega$ is called a vector field on Ω. $V\Omega$ denotes the space of differentiable vector fields on Ω. Note that $\partial_{\omega_k} : \Omega \to T\Omega$ given by $\partial_{\omega_k}(\omega) := \partial_{\omega_k}|_\omega$ is such a vector field.

Let $\tilde{\Omega}$ be second C^∞-manifold and $f : \Omega \to \tilde{\Omega}$ be differentiable. By $Df : T\Omega \to T\tilde{\Omega}$ we denote the differential map, given by

$$[Df(v)](h) := v(h \circ f)$$

for $v \in T_\omega\Omega$ and $h : \tilde{\Omega} \to \mathbb{R}$ differentiable.

A map $g : \Omega \to T^2\Omega$ is said to be a Riemannian metric if it is smooth and $g(\omega)$ is a scalar product on $T_\omega\Omega$ for all $\omega \in \Omega$. We denote this scalar product by $\langle \cdot, \cdot \rangle_\omega$ and its induced norm by $\| \cdot \|_\omega$ as well as $B_\omega(0, R) := \{v \in T_\omega\Omega \mid \|v\|_\omega < R\}$. For all $\omega \in \Omega$ in local coordinates the metric is uniquely determined by the matrix

$$G(\omega) := (g_{k,l}(\omega))_{k,l=1}^d, \qquad g_{k,l}(\omega) := g(\omega)\left(\partial_{\omega_k}|_\omega, \partial_{\omega_l}|_\omega\right)$$

and its inverse is denoted by $G(\omega)^{-1} = (g^{k,l}(\omega))_{k,l=1}^d$. We say that Ω has bounded geometry if for all $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ there is $C_{\boldsymbol{\alpha}} \in (0, \infty)$ such that $|\partial^{\boldsymbol{\alpha}} g_{k,l}| \leq C_{\boldsymbol{\alpha}}$.

As usual for $\omega_1, \omega_2 \in \Omega$ let

$$C_p^1(\omega_1, \omega_2) := \left\{\gamma \in C([a, b], \Omega) \mid \gamma(a) = \omega_1,\ \gamma(b) = \omega_2 \text{ and } \gamma \text{ is piecewise } C^1\right\}.$$

One defines the distance between ω_1 and ω_2 by

$$d_g(\omega_1, \omega_2) := \inf_{\gamma \in C_p^1(\omega_1, \omega_2)} L_g(\gamma) := \inf_{\gamma \in C_p^1(\omega_1, \omega_2)} \int_a^b \langle D\gamma(t), D\gamma(t)\rangle_{\gamma(t)}^{\frac{1}{2}}\, \mathrm{d}t.$$

This defines a metric on Ω since Ω is connected. If $L_g(\gamma) = d_g(\omega_1, \omega_2)$ the curve γ is called minimizing. In case a C^∞-manifold can be equipped with a Riemannian metric g, then we call $(\Omega, \tau, \mathscr{A}, g)$ a Riemannian C^∞-manifold. These manifolds admit a uniquely determined linear connection $L : T\Omega \times V\Omega \to T\Omega$ called the Levi-Civita connection, which does not depend on the chosen atlas. In a local chart (O, κ) we define $\nabla_k := L(\cdot, \partial_{\omega_k})$. Then for $k, l \in \mathbb{N}_{\leq d}$ one has

$$\nabla_k \partial_{\omega_l}|_\omega = \Gamma^m_{k,l}(\omega)\partial_{\omega_m}|_\omega \tag{A.9}$$

where the Christoffel symbols $\Gamma^m_{k,l}$ are given by

$$\Gamma^m_{k,l}(\omega) := \frac{1}{2}\sum_{n=1}^{d}\big((\partial_{\omega_k} g_{n,l})(\omega) + (\partial_{\omega_l} g_{n,k})(\omega) - (\partial_{\omega_n} g_{k,l})(\omega)\big) g^{n,m}(\omega). \tag{A.10}$$

The equality (A.10) holds for the Christoffel symbols generated by the Levi-Civita connection. The relation in (A.9) can be seen as the definition of these quantities for an arbitrary linear connection.

Let $I \in \mathcal{I}_c$, $t \in I$ and $\gamma : I \to \Omega$ be a curve. Then $(\gamma_m(t))_{m=1}^d$ denote the local coordinates of $\gamma(t)$ in (O, κ). The curve γ is called geodesic if for $m \in \mathbb{N}_{\leq d}$ and $t \in I$ holds

$$\gamma_m''(t) + \sum_{k,l=1}^{d} \Gamma^m_{k,l}(\gamma(t))\gamma_k'(t)\gamma_l'(t) = 0. \tag{A.11}$$

Let $\omega \in \Omega$ and $B \subseteq T_\omega\Omega$ such that for all $v \in B$ with coefficients $(v_1, ..., v_d)$, there is a unique geodesic $\gamma : [0,1] \to \Omega$ such that $\gamma_m'(t) = v_m$ for $m \in \mathbb{N}_{\leq d}$. Then one can define the exponential map $\exp_\omega : B \to \Omega$ by $\exp_\omega(v) := \gamma(1)$. We say $\exp_\omega$ is global if it is defined on $T_\omega\Omega$, i.e., the differential equation (A.11) equipped with the above initial conditions has a unique global solution for all initial values for the first derivatives of γ. The latter assertion is equivalent to the completeness of the metric space (Ω, d_g) by the famous Hopf-Rinow Theorem 1.37 in Section 1.4 of [Aub98]. In this case one can join any two points of Ω by a minimizing geodesic curve.

We continue with the important injectivity radius. There is $\epsilon \in (0, \infty)$ such that $\exp_\omega : B_\omega(0, \epsilon) \to \exp_\omega(B_\omega(0, \epsilon))$ is injective. Then one calls

$$\mathrm{inj}(\Omega, g) := \inf_{\omega \in \Omega}\Big(\sup\big\{\epsilon \in (0, \infty) \mid \exp_\omega : B_\omega(0, \epsilon) \to \Omega \text{ is injective }\big\}\Big)$$

the injectivity radius of (Ω, g).

We can use the exponential map to generate special local coordinates. Let $\omega \in \Omega$ and ϵ be given as above. With $O^\omega(\epsilon) := \exp_\omega(B_\omega(0, \epsilon))$ the map $\exp_\omega : B_\omega(0, \epsilon) \to O^\omega(\epsilon)$ is a diffeomorphism and we let $\kappa^\omega := \exp_\omega^{-1}$. We call $O^\omega(\epsilon)$ geodesic ball with center ω and ϵ its geodesic radius. Then $(O^\omega(\epsilon), \kappa^\omega)$ is a chart around ω and the induced local coordinates are called geodesic normal coordinates of ω. If $\mathrm{inj}(\Omega) > 0$ and $r \in (0, \mathrm{inj}(\Omega))$, then there is a family $(\omega_i)_{i \in I}$ in Ω with an arbitrary index set I, which satisfies $\Omega = \bigcup_{i \in I} O^{\omega_i}(r)$. We then define the geodesic atlas $\mathscr{A}(r)$ with geodesic radius r by

$$\mathscr{A}(r) := \big\{(O_i(r), \kappa_i) \mid i \in I\big\}, \qquad O_i(r) := O^{\omega_i}(r),\ \kappa_i := \kappa^{\omega_i}. \tag{A.12}$$

B. Proof of the weak limit argument in Theorem I.3.6

This appendix is devoted exclusively to the proof of the crucial ingredient of the proof of Theorem I.3.6. Because of its relevance and to keep our exposition reasonably self-contained, we want to give a detailed proof of this result. It can be found for example in Proposition 1.1.2 in [Caz03], where assertions (a) and (c) are proven. However, the exposition of [Caz03] is rather concise and we want to fill in some technical details concerning uniformly convex Banach spaces.

Theorem B.0.1
Let $I \in \mathcal{I}_c$ and $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be Banach spaces, as well as $(u_n)_{n\in\mathbb{N}}$ with $u_n : I \to X$. We furthermore assume:

(i) $(X, \|\cdot\|_X)$ *is reflexive and* $X \hookrightarrow Z$,

(ii) $\sup_{(n,t)\in\mathbb{N}\times I} \|u_n(t)\|_X < \infty$ *and* $(u_n)_{n\in\mathbb{N}} \subseteq C(I,Z)$ *is uniformly equicontinuous, i.e.,*

$$\forall_{\epsilon\in(0,\infty)} \exists_{\delta\in(0,\infty)} \forall_{(n,s,t)\in\mathbb{N}\times I\times I} : \ |t-s| < \delta \implies \|u_n(s) - u_n(t)\|_Z < \epsilon.$$

We then have:

(a) There is $u \in C_w(I,X) \cap C(I,Z)$ and a sequence $(n(k))_{k\in\mathbb{N}}$ in $\mathbb{N}$ with $u_{n(k)}(t) \rightharpoonup u(t)$ in X for all $t \in I$.

(b) If in addition $(u_n)_{n\in\mathbb{N}} \subseteq C^{0,\alpha}(I,Z)$ is bounded for some $\alpha \in (0,1]$, then $u \in C^{0,\alpha}(I,Z)$.

(c) Let in addition $(Y, \cdot\|_Y)$ be uniformly convex with $X \hookrightarrow Y \hookrightarrow Z$. If $(u_n)_{n\in\mathbb{N}} \subseteq C(I,Y)$ and $\|u_{n(k)}(\cdot)\|_Y \overset{k\to\infty}{\longrightarrow} \|u(\cdot)\|_Y$ uniformly on I, then $u \in C(I,Y)$ and $u_{n(k)} \overset{k\to\infty}{\longrightarrow} u$ in $C(I,Y)$.

Remarks:

(1) Examples for uniformly convex Banach spaces are Hilbert spaces, $L^p(\Omega)$ and $\ell^p(\mathbb{Z})$ for $p \in (1,\infty)$, which is enough for our purposes. For proofs see [Cla36].

(2) Note that the boundedness of $(u_n)_{n\in\mathbb{N}} \subseteq C^{0,\alpha}(I,Z)$ in *(b)* directly implies the uniform equicontinuity in Z from condition *(ii)*.

Proof. Without loss of generality we can assume that $X \subseteq Y \subseteq Z$. First we recall two helpful facts from functional analysis.
(1) For $z \in Z$ and a bounded sequence $(x_l)_{l\in\mathbb{N}}$ in X holds

$$x_l \rightharpoonup z \ \text{ in } Z \quad \implies \quad z \in X \quad \wedge \quad x_l \rightharpoonup z \ \text{ in } X. \tag{B.1}$$

Indeed, the reflexivity of X yields the existence of a sequence $(l(k))_{k\in\mathbb{N}} \subseteq \mathbb{N}$ and $x \in X$ such that $x_{l(k)} \rightharpoonup x$ in X. Due to $X \subseteq Z$ holds $Z^* \subseteq X^*$ and $x_{l(k)} \rightharpoonup x$ in Z and therefore $x = z$ by the uniqueness of weak limits. For the weak convergence in X we only need to show that Z^* is dense in X^*. Let $\ell \in X^{**}$ suffice $\ell|_{Z^*} = 0$. The reflexivity of X provides a unique $x \in X$ with $\ell(x^*) = x^*(x)$ for all $x^* \in X^*$. Then for all $z^* \in Z^*$ holds $z^*(x) = \ell(z^*) = 0$. Thus, $x = 0$ and therefore $\ell = 0$. This implies the claimed density by the Hahn-Banach Theorem.

(2) The second important tool is the weak lower semicontinuity of a norm, i.e. for a normed space $(S, \|\cdot\|_S)$ holds

$$x_n \rightharpoonup x \;\text{ in } S \qquad \Longrightarrow \qquad \|x\|_S \leq \liminf_{n\in\mathbb{N}} \|x_n\|_S. \tag{B.2}$$

The case $x = 0$ is trivial, so let $x \neq 0$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in S with $x_n \rightharpoonup x$ in S. We assume the opposite of the right-hand side. Then there is a sequence $(n(k))_{k\in\mathbb{N}}$ in $\mathbb{N}$ with $n(k) \xrightarrow{k\to\infty} \infty$ and $\|x_{n(k)}\|_S < \|x\|_S$ for all $k \in \mathbb{N}$. As $x \neq 0$ the Hahn-Banach Theorem yields the existence of $x^* \in S^*$ with $\|x^*\|_{S^*} = 1$ and $\|x\|_S = x^*(x)$. Consequently, $\|x\|_S = \lim_{k\to\infty} x^*(x_{n(k)}) \leq \|x_{n(k)}\|_S < \|x\|_S$, which is absurd.

(a) Let $Q := I \cap \mathbb{Q} = \{q_j \mid j \in \mathbb{N}\}$. By assumption *(ii)* the sequence $(u_n(q_1))_{n\in\mathbb{N}}$ is bounded in X. This implies by reflexivity the existence of a sequence $(n_1(k))_{k\in\mathbb{N}}$ such that $(u_{n_1(k)}(q_1))_{k\in\mathbb{N}}$ converges weakly in X. The same argument yields that $(u_{n_1(k)}(q_2))_{k\in\mathbb{N}}$ admits a subsequence $(n_2(k))_{k\in\mathbb{N}}$ of $(n_1(k))_{k\in\mathbb{N}}$ such that $(u_{n_2(k)}(q_2))_{k\in\mathbb{N}}$ converges weakly in X. By induction, for $i \in \mathbb{N}\setminus\{1\}$, there is a sequence $(n_i(k))_{k\in\mathbb{N}}$ such that $(u_{n_i(k)}(q_i))_{k\in\mathbb{N}}$ converges weakly in X and $(n_i(k))_{k\in\mathbb{N}} \subseteq (n_{i-1}(k))_{k\in\mathbb{N}}$. This implies that for all $i \in \mathbb{N}$ the weak convergence of $(u_{n_i(k)}(q_j))_{k\in\mathbb{N}}$ in X for all $j \in \{1,...,i\}$. Consequently, the sequence $(n(k))_{k\in\mathbb{N}}$ with $n(k) := n_k(k)$ admits the weak convergence of $(u_{n(k)})(q_j))_{k\in\mathbb{N}}$ in X for all $j \in \mathbb{N}$. Note that the weak convergence also holds in Z, since $Z^* \subseteq X^*$. For all $t \in I$, $q \in Q$ and $z^* \in Z^*$ holds

$$\begin{aligned} &\left|z^*\left(u_{n(k)}(t) - u_{n(l)}(t)\right)\right| \\ &\quad\leq\; \left|z^*\left(u_{n(k)}(t) - u_{n(k)}(q)\right) + z^*\left(u_{n(l)}(q) - u_{n(l)}(t)\right)\right| + \left|z^*\left(u_{n(k)}(q) - u_{n(l)}(q)\right)\right|. \end{aligned}$$

Let $\epsilon \in (0,\infty)$. If we choose $k, l \in \mathbb{N}$ large enough, then the last term is smaller than $\epsilon/3$. The uniform equicontinuity of $(u_n)_{n\in\mathbb{N}}$ from *(ii)* yields for $|t-q| < \delta$ with $\delta \in (0,\infty)$ small enough that for all $k, l \in \mathbb{N}$

$$\begin{aligned} &\left|z^*\left(u_{n(k)}(t) - u_{n(k)}(q)\right) + z^*\left(u_{n(l)}(q) - u_{n(l)}(t)\right)\right| \\ &\qquad\leq \|z^*\|_{Z^*}\left(\|u_{n(k)}(t) - u_{n(k)}(q)\|_Z + \|u_{n(l)}(q) - u_{n(l)}(t)\|_Z\right) < \frac{2\epsilon}{3}. \end{aligned}$$

This implies that $(u_{n(k)}(t))_{k\in\mathbb{N}}$ converges weakly in Z for all $t \in I$. Consequently, there is $u : I \to Z$ with $u_{n(k)}(t) \rightharpoonup u(t)$ in Z for all $t \in I$. Then for $s, t \in I$ with $|s-t| < \delta$ follows by (B.2) and *(ii)*

$$\|u(t) - u(s)\|_Z \leq \liminf_{k\in\mathbb{N}} \|u_{n(k)}(t) - u_{n(k)}(s)\|_Z < \epsilon. \tag{B.3}$$

Hence, $u \in C(I,Z)$. Let $t \in I$ and $(t_l)_{l\in\mathbb{N}}$ in I with $t_l \xrightarrow{l\to\infty} t$. Then *(ii)* and $u \in C_b(I,Z)$ combined with (B.1) shows that $u_{n(k)}(t) \rightharpoonup u(t)$ in X and $u(t_l) \rightharpoonup u(t)$ in X. We

therefore have $u \in C_w(I,X)$.
(b) Let additionally $(u_n)_{n\in\mathbb{N}} \subseteq C^{0,\alpha}(I,Z)$ be bounded, i.e,

$$\sup_{n\in\mathbb{N}} \sup_{s,t\in I, s\neq t} \left(\frac{\|u_n(s)-u_n(t)\|_Z}{|s-t|^\alpha} \right) \leq \sup_{n\in\mathbb{N}} \|u_n\|_{C^{0,\alpha}(I,Z)} =: L < \infty.$$

As in (B.3) the weak lower semicontinuity (B.2) of $\|\cdot\|_Z$ yields for $s,t \in I$

$$\|u(s)-u(t)\|_Z \leq \liminf_{k\in\mathbb{N}} \|u_{n(k)}(s)-u_{n(k)}(t)\|_Z \leq L|s-t|^\alpha.$$

Hence, $u \in C^{0,\alpha}(I,Z)$.
(c) We first prove a useful consequence of the uniform convexity of Y, namely

$$y_n \rightharpoonup y \ \text{ in } Y \quad \wedge \quad \|y_n\|_Y \xrightarrow{n\to\infty} \|y\|_Y \qquad \Longrightarrow \qquad y_n \xrightarrow{n\to\infty} y \ \text{ in } Y. \tag{B.4}$$

The uniform convexity provides per definition

$$\forall_{\epsilon\in(0,\infty)} \exists_{\delta\in(0,1]} \forall_{x,y\in S_Y(0,1)} : \frac{\|x+y\|}{2} > 1-\delta \implies \|x-y\| < \epsilon. \tag{B.5}$$

In the case $y=0$ there is nothing to show. Thus we can consider $y \neq 0$ and $(y_n)_{n\in\mathbb{N}}$ in $Y \setminus \{0\}$. We additionally define $(x_n)_{n\in\mathbb{N}}$ by $x_n := y_n/\|y_n\|_Y$ and $x := y/\|y\|_Y$. Then we have

$$x_n \rightharpoonup x \ \text{ in } Y, \qquad \frac{x_n+x_k}{2} \rightharpoonup x \ \text{ in } Y,$$

and (B.2) implies

$$\begin{aligned} 1 = \|x\|_Y &\leq \frac{1}{2} \liminf_{(n,k)\in\mathbb{N}^2} \|x_n+x_k\|_Y \\ &\leq \frac{1}{2} \limsup_{(n,k)\in\mathbb{N}^2} \|x_n+x_k\|_Y \leq \frac{1}{2} \sup_{(n,k)\in\mathbb{N}^2} \|x_n+x_k\|_Y \leq 1. \end{aligned}$$

Consequently, $\|x_n+x_k\|_Y/2 \xrightarrow{n,k\to\infty} 1$ and (B.5) yields that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y with strong limit x, since weak limits are unique. We then have

$$\begin{aligned} \|y_n-y\|_Y &= \big\| \|y_n\|_Y x_n - \|y\|_Y x \big\|_Y \\ &\leq \|y_n\|_Y \|x_n-x\|_Y + \big| \|y_n\|_Y - \|y\|_Y \big| \|x\|_Y \xrightarrow{n\to\infty} 0. \end{aligned}$$

Having proven (B.4) we proceed with the proof of *(c)*. Recall that $u \in C_w(I,X)$ and let $t \in I$. Then the embedding $X \hookrightarrow Y$ one the one hand yields $u \in C_w(I,Y)$ and on the other hand $u_{n(k)}(t) \rightharpoonup u(t)$ in Y. The latter fact combined with $\|u_{n(k)}(t)\|_Y \xrightarrow{k\to\infty} \|u(t)\|_Y$ yields $u_{n(k)}(t) \xrightarrow{k\to\infty} u(t)$ in Y by (B.4). For $s,t \in I$ we have

$$\begin{aligned} &\big| \|u(s)\|_Y - \|u(t)\|_Y \big| \\ &\qquad \leq \big| \|u(s)\|_Y - \|u_{n(k)}(s)\|_Y \big| + \big| \|u_{n(k)}(t)\|_Y - \|u(t)\|_Y \big| + \|u_{n(k)}(s)-u_{n(k)}(t)\|_Y. \end{aligned}$$

The first two terms converge to 0 for $k\to\infty$ by assumption. $u_{n(k)} \in C(I,Y)$ and the previous estimate provide $\|u\|_Y \in C(I)$. Since we already know $u \in C_w(I,Y)$ the

assertion in (B.4) provides $u \in C(I, Y)$.
It is left to show that $(u_{n(k)})_{k\in\mathbb{N}}$ converges in $C(I, Y)$ to u. Let us assume the opposite. Then a well known characterization of uniform convergence yields

$$\exists_{\epsilon\in(0,\infty),\ (t_k)_{k\in\mathbb{N}}\subseteq I}\forall_{k\in\mathbb{N}}:\ \|u_{n(k)}(t_k) - u(t_k)\|_Y \geq \epsilon. \tag{B.6}$$

By choosing a subsequence we can assume that $t_k \overset{k\to\infty}{\longrightarrow} t \in I$. Then the uniform equicontinuity of $(u_n)_{n\in\mathbb{N}}$ on Z and the weak convergence of $(u_{n(k)}(t))_{k\in\mathbb{N}}$ in Z yield for $z^* \in Z^*$

$$z^*\big(u_{n(k)}(t_k) - u(t)\big) = z^*\big(u_{n(k)}(t_k) - u_{n(k)}(t)\big) + z^*\big(u_{n(k)}(t) - u(t)\big) \overset{k\to\infty}{\longrightarrow} 0.$$

Assumption *(ii)* and $X \hookrightarrow Y$ additionally imply that $(u_{n(k)}(t_k))_{k\in\mathbb{N}}$ is bounded in Y. Then (B.1) yields $u_{n(k)}(t_k) \rightharpoonup u(t)$ in Y. Moreover, the uniform convergence of $(\|u_{n(k)}(\cdot)\|_Y)_{k\in\mathbb{N}}$ and $u \in C(I, Y)$ imply

$$\begin{aligned}&\big|\|u_{n(k)}(t_k)\|_Y - \|u(t)\|_Y\big|\\ &\qquad \leq \big|\|u_{n(k)}(t_k)\|_Y - \|u(t_k)\|_Y\big| + \|u(t_k) - u(t)\|_Y \overset{k\to\infty}{\longrightarrow} 0.\end{aligned}$$

(B.4) therefore provides $u_{n(k)}(t_k) \overset{k\to\infty}{\longrightarrow} u(t)$ in Y. Finally, $u \in C(I, Y)$ implies

$$\|u_{n(k)}(t_k) - u(t_k)\|_Y \leq \|u_{n(k)}(t_k) - u(t)\|_Y + \|u(t) - u(t_k)\|_Y \overset{k\to\infty}{\longrightarrow} 0.$$

This clearly contradicts (B.6). □

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