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Numerical Analysis of the Electro-Magnetic Perfectly Matched Layer in a Discontinuous Galerkin Discretization

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Contents

Introduction

Abstract The work at hand is devoted to the numerical analysis of linear Maxwell's equations in time-domain and in a discontinuous Galerkin discretization with an upwind flux. We mostly consider the continuous or semi-discrete setting, where the time is not yet discretized. In especially we focus on Berenger's perfectly matched layer as an absorbing boundary layer and have a detailed look at construction, well-posedness, numerical simulation, and error analysis for a layer in one direction. We present an analytic solution to Maxwell's system with a layer, that can be utilized to efficiently determine optimal parameter values for the layer and state a way to avoid an exponential growth in time of the semi-discrete error bound.

Organization of this work The chapters are organized as follows. In Chapter 1 we briefly discuss physical basics about Maxwell's equations. In the next chapter, Chapter 2, we present a well-posedness result based on Lumer-Phillips' theorem, the detailed construction of the discrete operator, and a standard discretization error analysis result for Maxwell's equations without any layer. The analytical construction of the perfectly matched layer in the unsplit formulation is the topic of Chapter 3, followed by a transfer of the well-posedness theory stated in the second chapter and an upwind discretization in space. In Chapter 4 numerical tests are performed. Here, we focus on the test setting itself, i.e. we present a quite flexible exact solution to Maxwell's system with a layer and afterwards use it to run basic tests. The last chapter, Chapter 5, is dedicated to the long-time behaviour of the perfectly matched layer. We are interested in the growth of the solution in time and present a spatial discretization error estimate, which accounts for the analytical long-time behaviour of the layer. It only works in a special two dimensional case, though, since it is based on a dissipative functional introduced by Bécache and Joly, that so far only works in two dimensions in space.

Keywords Maxwell's equations, discontinuous Galerkin method, perfectly matched layer, absorbing boundary condition, parameter determination, error analysis

Mathematics Subject Classification (2010) 35L65, 35Q61, 65M15, 65M60, 78A25, 78M10

What I consider my own contribution to the topic There is already a large amount of literature available under the topic of discontinuous Galerkin discretizations for hyperbolic problems, as well as for the perfectly matched layer. Up to and including Chapter 3 the content of this work is mostly known nowadays, but serves as a basis for the results in the later chapters. In Chapter 2 the calculation of the upwind flux is done in detail and the upwind flux on boundary faces is explained. Though the procedure to construct the upwind flux is well known, it is hard to find any literature, where the calculations are actually performed. The error estimate in Chapter 2 for the spatial discretization of Maxwell's equations is not new, but the inclusion of non-homogeneous boundary values I did not see so far. In Chapter 3 there is a discussion of the boundary conditions behind the layer. Here, some problems occur with the unsplit layer formulation, that might be of importance in order to understand the analytical behaviour of the layer. I did not find any literature, where this was mentioned. Finally, I consider the test setting in Chapter 4, including an exact solution to Maxwell's equations with a layer and the error estimate in Chapter 5, adjusted to the long-time behaviour of the layer to be my main contributions to the topic.

1 A short introduction into Maxwell's equations

Origin of this chapter This chapter is based on a literature research in [Jac06].

Maxwell's equations are a linear first order PDE system that connects the electric and magnetic fields to the current- and charge density, which are considered the sources of electromagnetic phenomena. In a vacuum Maxwell's equations read [Jac06, Sec. I.1]

$$-\frac{1}{c^2}\partial_t \mathbf{E} + \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \qquad \text{(Maxwell-Ampère's law)} \tag{1.1a}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0}, \qquad \text{(Faraday's law of induction)} \tag{1.1b}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \text{(Gauss's law)} \tag{1.1c}$$

$$\nabla \cdot \mathbf{B} = 0. \qquad \text{(Gauss's magnetic law)} \tag{1.1d}$$

Here $\mathbf{E}$ denotes the electric field, $\mathbf{B}$ the magnetic induction, $\mathbf{J}$ the current density and ρ the charge density. In the stationary case Maxwell-Ampère's law states, that a current induces a magnetic field. Maxwell added the time derivative of the electric field to validate this law for time-dependent phenomena. It tells us, that also time-dependent electric fields induce magnetic fields. Vice versa, we see from Faraday's law, that time-dependent magnetic fields induce electric fields. Gauss's law states, that charges are the sources of the electric field, whereas Gauss's magnetic law reveals the non-existence of magnetic monopoles. The physical constants ε_0 and μ_0 are connected with the speed of light c in vacuum via the equation $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. In practice the current- and charge density are assumed to be known and the system (1.1) is used to determine the fields $\mathbf{E}$ and $\mathbf{B}$. A point charge q with velocity $\mathbf{v}$ in an electromagnetic field experiences the Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

Plane waves in a vacuum Plane waves are special solutions to Maxwell's equations in a vacuum. Consider the space $\mathbb{R}^3$ without current- and charge density ($\mathbf{J} = \mathbf{0}$, $\rho = 0$) and with the right-handed orthogonal basis $(\mathbf{k}, \mathbf{E}_0, \mathbf{B}_0)$, then a special solution to the system (1.1) is given by

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t), \tag{1.2a}$$

$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t), \tag{1.2b}$$

with the frequency $\omega = c|\mathbf{k}|$ and the relation on the amplitudes $|\mathbf{E}_0| = c|\mathbf{B}_0|$. The plane wave (1.2) travels with velocity c in the direction of $\mathbf{k}$.

Proof: To verify that (1.2) solves (1.1), we calculate the derivatives as follows

$$\begin{aligned}
\partial_t \mathbf{E} &= -i\omega \mathbf{E}, \\
\partial_t \mathbf{B} &= -i\omega \mathbf{B}, \\
\nabla \times \mathbf{E} &= \nabla \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t) \times \mathbf{E}_0 = \mathrm{i}\mathbf{k} \times \mathbf{E} = \mathrm{i}\omega \mathbf{B}, \\
\nabla \times \mathbf{B} &= \mathrm{i}\mathbf{k} \times \mathbf{B} = -\frac{\mathrm{i}\omega}{c^2} \mathbf{E}, \\
\nabla \cdot \mathbf{E} &= \nabla \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t) \cdot \mathbf{E}_0 = \mathrm{i}\mathbf{k} \cdot \mathbf{E} = 0, \\
\nabla \cdot \mathbf{B} &= 0.
\end{aligned}$$

This concludes the proof. □

In the presence of materials, where the electrons and protons contain a lot of charges, the current- and charge density cannot be considered to be known any more. In that case the effect of tied charges is approximated by their dipol moments. The polarization and magnetization fields

$$\mathbf{P} = \sum_j \frac{\mathbf{p}_j}{\Delta V}, \qquad \mathbf{M} = \sum_j \frac{\mathbf{m}_j}{\Delta V}$$

are the electric and magnetic dipol density, where $\mathbf{p}_j$ and $\mathbf{m}_j$ are the electric and magnetic dipol moments in the volume ΔV. The testvolume ΔV is assumed to be small from a macroscopic point of view, but still contains millions of atoms.

Magnetism The magnetic behaviour of materials can be devided into three groups.

- In a *diamagnetic* material, the magnetic dipols are generated by an external magnetic induction field. They will act against the generating field and reduce the total magnetic induction.
- In a *paramagnetic* material, there already exist uncorrelated magnetic dipols. In the presence of an external magnetic induction field, the dipols will adjust to the field and increase the total magnetic induction.
- In a *ferromagnetic* material, there exist correlated magnetic dipols. Here, even without external fields there may be a magnetization.

Introducing the fields

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M},$$

Maxwell's equations in the presence of materials now read [Jac06, Sec. 6.6]

$$-\partial_t \mathbf{D} + \nabla \times \mathbf{H} = \mathbf{J}, \tag{1.3a}$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = \mathbf{0}, \tag{1.3b}$$

$$\nabla \cdot \mathbf{D} = \rho, \tag{1.3c}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{1.3d}$$

Here, $\mathbf{D}$ is called electric displacement and $\mathbf{H}$ is called magnetic field. The charge density ρ contains free charges as well as absolute molecule charges and $\mathbf{J}$ is the corresponding current density of free charges. The effect of tied charges is compressed in the fields $\mathbf{D}$ and $\mathbf{H}$. All fields in (1.3) are averages in space over macroscopical small volumes ΔV (for details on that averaging see [Jac06, Sec. 6.6]).

Current- and charge density are connected by the equation of continuity

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0,$$

which expresses the conservation of charges and can be seen by application of $\nabla\cdot$ and ∂_t to equations (1.3a) and (1.3c). We split the fields into an external part, that is applied to the system and assumed to be known, and an internal part

$$\mathbf{J} = \mathbf{J}_{ext} + \mathbf{J}_{int}, \qquad \rho = \rho_{ext} + \rho_{int}.$$

On the internal current density, introducing the material dependent conductivity σ, we apply Ohm's law

$$\mathbf{J}_{int} = \sigma \mathbf{E}. \tag{1.4}$$

Constitutive relations We assume a linear isotropic dependence between polarization (magnetization) and electric (magnetic) field

$$\mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}, \qquad \mathbf{M} = \chi_m \mathbf{H}. \tag{1.5a}$$

Therefore, with $\varepsilon_r = 1 + \chi_e$ and $\mu_r = 1 + \chi_m$, we have

$$\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}, \qquad \mathbf{B} = \mu_r \mu_0 \mathbf{H} = \mu \mathbf{H}. \tag{1.5b}$$

The material parameters χ_e and χ_m are called electric and magnetic susceptibility, ε and μ denote the permittivity and permeability, whereas $\varepsilon_r = \varepsilon\varepsilon_0^{-1}$ and $\mu_r = \mu\mu_0^{-1}$ are the corresponding relative permittivity and permeability. The idea behind the linear dependencies in (1.5a) is a Taylor expansion up to the first order. So these constitutive relations are only reliable, as long as the electromagnetic fields are 'small' (here we will not specify the meaning of small). The assumption on isotropy states, that there is no special direction in the material. Therefore, the terms of zero order have to vanish and the susceptibilities are scalar.

Paramagnetic materials are characterized by $\mu_r > 1$ and diamagnetic materials by $\mu_r < 1$. In most cases the relative permeability does not differ very much from one, $|\mu_r - 1| \lesssim 10^{-6}$ [Jac06, Sec. 5.8]. Ferromagnetic materials cannot be described with the linear relations (1.5b).

Using (1.5b) and Ohm's law (1.4) on a system, where all materials are at rest, Maxwell's equations (1.3) gain the form

$$\varepsilon \partial_t \mathbf{E} + \sigma \mathbf{E} - \nabla \times \mathbf{H} = -\mathbf{J}_{ext}, \tag{1.6a}$$
$$\mu \partial_t \mathbf{H} + \nabla \times \mathbf{E} = \mathbf{0}, \tag{1.6b}$$
$$\nabla \cdot (\varepsilon \mathbf{E}) = \rho_{ext} + \rho_{int}, \tag{1.6c}$$
$$\nabla \cdot (\mu \mathbf{H}) = 0. \tag{1.6d}$$

Here we already used the time-independence of the material parameters ε and μ. In general these parameters depend on the frequency ω of the electromagnetic waves. To describe phenomena in a small frequency range, they can be considered independent of the frequency, though. Furthermore we assume these parameters to be piecewise constant throughout this work. This idealisation leads to jump conditions for the electromagnetic fields on the interfaces between different materials.

Jump conditions on material interfaces Let $\mathbf{n}$ be a unit normal vector on the interface between two materials, pointing from the first material to the second. Then we have the following jump conditions [Jac06, Sec. I.5]

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \rho_f, \tag{1.7a}$$
$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0, \tag{1.7b}$$
$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = \mathbf{0}, \tag{1.7c}$$
$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_f, \tag{1.7d}$$

where $\mathbf{J}_f$ and ρ_f are the surface current- and surface charge density.

Plane waves in a homogeneous medium Consider a homogeneous medium ($\varepsilon \neq \varepsilon(\mathbf{x})$, $\mu \neq \mu(\mathbf{x})$) with positive material parameters ($\varepsilon, \mu > 0$) in the space $\mathbb{R}^3$ without current- and charge density ($\mathbf{J}_{ext} = \mathbf{0}$, $\sigma = 0$, $\rho = 0$). Let $(\mathbf{k}, \mathbf{E}_0, \mathbf{H}_0)$ be a right-handed orthogonal basis of $\mathbb{R}^3$. Then a special solution to the system (1.6) is given by

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t), \tag{1.8a}$$
$$\mathbf{H}(\mathbf{x}, t) = \mathbf{H}_0 \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t), \tag{1.8b}$$

with the frequency $\omega = c_m|\mathbf{k}|$, the speed of light in a medium $c_m = (\varepsilon\mu)^{-\frac{1}{2}}$ and the relation on the amplitudes $|\mathbf{E}_0| = c_m\mu|\mathbf{H}_0|$. We see, that the medium changes the speed of light.

Proof: Same as in the vacuum case (see page 4). □

Boundary conditions If we consider Maxwell's equations on a bounded domain, we need boundary conditions to obtain a well-posed problem. There a two boundary conditions that are motivated by material interfaces. First consider a perfect electric conductor (PEC) outside the boundary. The free charges in the conductor are assumed to react instantly on incoming electromagnetic waves in the way, that the electric field is compensated. Following jump condition (1.7c), we see that $\mathbf{n} \times \mathbf{E} = \mathbf{0}$ on the interface of a perfect electric conductor. On the interface of a material of high permeability ($\mu = \infty$) the magnetic field will behave in an analogous way, i.e. $\mathbf{n} \times \mathbf{H} = \mathbf{0}$ [Jac06, Sec. 5.8 (in the end)]. We will call such a material perfect magnetic conductor (PMC). Both boundary conditions, PEC and PMC, reflect the full incoming energy, but with a phase difference of π to each other. A third boundary condition, that will be used in this work is the impedance boundary condition $\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{0}$, with the scalar impedance Z.

Reflections on a straight boundary Consider the half-space

$$\Omega = \{\mathbf{x} \in \mathbb{R}^3 : \; x_1 < 0\}$$

filled with a homogeneous material $\varepsilon, \mu > 0$ and a plane wave $(\mathbf{E}^{\text{inc}}(\mathbf{x}, t), \mathbf{H}^{\text{inc}}(\mathbf{x}, t))$ in the shape of (1.8). We denote $\mathbf{n} = \mathbf{e}_1$ the unit normal vector on $\partial\Omega$, $\boldsymbol{\tau}_1$ a tangential vector, such that $\mathbf{k} = k_n\mathbf{n} + k_\tau\boldsymbol{\tau}_1$, $k_\tau \geq 0$ and $\boldsymbol{\tau}_2 = \mathbf{n} \times \boldsymbol{\tau}_1$ another tangential vector. Let the wave be incoming on the boundary, i.e. $k_n > 0$ and let φ be the angle of incidence, i.e. $\cos\varphi = k_n(k_n^2 + k_\tau^2)^{-\frac{1}{2}}$. Now we seek an outgoing plane wave $(\mathbf{E}^*(\mathbf{x}, t), \mathbf{H}^*(\mathbf{x}, t))$, so that certain boundary conditions hold true on $\partial\Omega$. We define the reflected wave vector by $\mathbf{k}^* = -k_n\mathbf{n} + k_\tau\boldsymbol{\tau}_1$ and the reflected electromagnetic field by

$$\begin{aligned}
\mathbf{E}^*(\mathbf{x}, t) &= \mathbf{E}_0^* \exp(\mathrm{i}\mathbf{k}^* \cdot \mathbf{x} - \mathrm{i}\omega t), \\
\mathbf{H}^*(\mathbf{x}, t) &= \mathbf{H}_0^* \exp(\mathrm{i}\mathbf{k}^* \cdot \mathbf{x} - \mathrm{i}\omega t).
\end{aligned}$$

The total fields are thus $\mathbf{E} = \mathbf{E}^{\text{inc}} + \mathbf{E}^*$ and $\mathbf{H} = \mathbf{H}^{\text{inc}} + \mathbf{H}^*$. The vectors $\mathbf{E}_0^*$ and $\mathbf{H}_0^*$ depend on the actual boundary condition, we want to realize. In case we set $\mathbf{n} \times \mathbf{E} = \mathbf{0}$ on $\partial\Omega$, we obtain

$$\begin{aligned}
\mathbf{E}_0^* &= (\mathbf{E}_0 \cdot \mathbf{n})\mathbf{n} - (\mathbf{E}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 - (\mathbf{E}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2, \\
\mathbf{H}_0^* &= -(\mathbf{H}_0 \cdot \mathbf{n})\mathbf{n} + (\mathbf{H}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 + (\mathbf{H}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2.
\end{aligned}$$

In case of $\mathbf{n} \times \mathbf{H} = \mathbf{0}$, we obtain

$$\begin{aligned}
\mathbf{E}_0^* &= -(\mathbf{E}_0 \cdot \mathbf{n})\mathbf{n} + (\mathbf{E}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 + (\mathbf{E}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2, \\
\mathbf{H}_0^* &= (\mathbf{H}_0 \cdot \mathbf{n})\mathbf{n} - (\mathbf{H}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 - (\mathbf{H}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2.
\end{aligned}$$

For the impedance boundary condition $\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{0}$ with a constant impedance Z, we obtain

$$\begin{aligned}
\mathbf{E}_0^* &= -\alpha_1(\mathbf{E}_0 \cdot \mathbf{n})\mathbf{n} + \alpha_1(\mathbf{E}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 - \alpha_2(\mathbf{E}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2, \\
\mathbf{H}_0^* &= -\alpha_2(\mathbf{H}_0 \cdot \mathbf{n})\mathbf{n} + \alpha_2(\mathbf{H}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 - \alpha_1(\mathbf{H}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2,
\end{aligned}$$

with the coefficients

$$\alpha_1 = \frac{1 - Z\frac{\sqrt{\mu}}{\sqrt{\varepsilon}}\cos\varphi}{1 + Z\frac{\sqrt{\mu}}{\sqrt{\varepsilon}}\cos\varphi}, \qquad \alpha_2 = \frac{Z - \frac{\sqrt{\varepsilon}}{\sqrt{\mu}}\cos\varphi}{Z + \frac{\sqrt{\varepsilon}}{\sqrt{\mu}}\cos\varphi}. \tag{1.9}$$

Proof: We just want to check the impedance boundary condition. On the boundary $\mathbf{x} \in \partial\Omega$ we can calculate the boundary values

$$\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \left[\mathbf{n} \times (\mathbf{H}_0 + \mathbf{H}_0^*) - Z\mathbf{n} \times \left(\mathbf{n} \times (\mathbf{E}_0 + \mathbf{E}_0^*)\right)\right] \\ \exp\left(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t\right)$$

For the first term on the right hand side we obtain

$$\mathbf{n} \times (\mathbf{H}_0 + \mathbf{H}_0^*) = (1 + \alpha_2)(\mathbf{H}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_2 - (1 - \alpha_1)(\mathbf{H}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_1 \tag{1.10}$$

and for the second term

$$Z\mathbf{n} \times \big(\mathbf{n} \times (\mathbf{E}_0 + \mathbf{E}_0^*)\big) = -Z(1 + \alpha_1)(\mathbf{E}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 - Z(1 - \alpha_2)(\mathbf{E}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2. \tag{1.11}$$

With the orthogonality of $\mathbf{k}$, $\mathbf{E}_0$, $\mathbf{H}_0$ and with $\mathbf{e}_k = \frac{\mathbf{k}}{|\mathbf{k}|} = \cos\varphi\,\mathbf{n} + \sin\varphi\,\boldsymbol{\tau}_1$, we can state that

$$\cos\varphi\,\mathbf{E}_0 \cdot \mathbf{n} + \sin\varphi\,\mathbf{E}_0 \cdot \boldsymbol{\tau}_1 = \cos\varphi\,\mathbf{H}_0 \cdot \mathbf{n} + \sin\varphi\,\mathbf{H}_0 \cdot \boldsymbol{\tau}_1 = 0. \tag{1.12}$$

Since $\mathbf{k}$, $\mathbf{E}_0$ and $\mathbf{H}_0$ form a right-handed system, we obtain the relation

$$\begin{aligned} \mathbf{H}_0 &= \frac{\sqrt{\varepsilon}}{\sqrt{\mu}}\mathbf{e}_k \times \mathbf{E}_0 \\ &= \frac{\sqrt{\varepsilon}}{\sqrt{\mu}}\big(\sin\varphi(\mathbf{E}_0 \cdot \boldsymbol{\tau}_2)\mathbf{n} - \cos\varphi(\mathbf{E}_0 \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_1 \\ &\qquad + \cos\varphi(\mathbf{E}_0 \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_2 - \sin\varphi(\mathbf{E}_0 \cdot \mathbf{n})\boldsymbol{\tau}_2\big). \end{aligned} \tag{1.13}$$

Equations (1.12) and (1.13) lead to

$$\begin{aligned} \mathbf{E}_0 \cdot \boldsymbol{\tau}_1 &= \frac{\sqrt{\mu}}{\sqrt{\varepsilon}}\cos\varphi\,\mathbf{H}_0 \cdot \boldsymbol{\tau}_2, \\ \mathbf{H}_0 \cdot \boldsymbol{\tau}_1 &= -\frac{\sqrt{\varepsilon}}{\sqrt{\mu}}\cos\varphi\,\mathbf{E}_0 \cdot \boldsymbol{\tau}_2. \end{aligned}$$

We use this in (1.10) and (1.11) and see that

$$\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{0}.$$

□

Energy of the electromagnetic field The energy density of the electromagnetic field is given by [Jac06, Sec. 6.7]

$$e = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}) = \frac{1}{2}(\varepsilon\mathbf{E} \cdot \mathbf{E} + \mu\mathbf{H} \cdot \mathbf{H}),$$

as well as the corresponding energy flux density by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}.$$

2 Maxwell's linear time-dependent system

Content of this chapter For Maxwell's linear system we show well-posedness via Lumer-Phillips' theorem, calculate the upwind flux to discretize the system in space, and calculate an error bound for the semi-discrete solution.

Origin of this chapter We consider the content of this chapter to be general knowledge to a large extent. To gain access to the topic, I got supported by Christian Wieners. My discrete thinking starts with the Riemann problem in Section 2.3.4. The theory behind the Riemann problem is general knowledge, the elaboration of the calculations is my contribution. The proof of Theorem 2.4.1 is general knowledge as well. The addition of inhomogeneous boundary values and an impedance boundary is my contribution.

A related publication During the construction of this chapter, I contributed to a publication [HPS+14] regarding space discretization and upwind flux. There are paralles to the work at hand in the basic theory. Based on that, the publication advances in a different direction, as it focuses on time integration, whereas this work focuses on the perfectly matched layer.

2.1 The continuous Problem

In this work we consider Maxwell's linear system in the following form

$$\varepsilon\partial_t\mathbf{E} + \sigma\mathbf{E} - \nabla \times \mathbf{H} = \mathbf{f}_E \qquad \text{in } \Omega_\infty, \tag{2.1a}$$
$$\mu\partial_t\mathbf{H} + \nabla \times \mathbf{E} = \mathbf{f}_H \qquad \text{in } \Omega_\infty, \tag{2.1b}$$
$$\mathbf{n} \times \mathbf{E} = \mathbf{g}_E \qquad \text{on } \partial\Omega_{E,\infty}, \tag{2.1c}$$
$$\mathbf{n} \times \mathbf{H} = \mathbf{g}_H \qquad \text{on } \partial\Omega_{H,\infty}, \tag{2.1d}$$

$$\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{g}_I \qquad \text{on } \partial\Omega_{I,\infty}, \tag{2.1e}$$
$$\mathbf{E}(\cdot, 0) = \mathbf{E}_0 \qquad \text{in } \Omega, \tag{2.1f}$$
$$\mathbf{H}(\cdot, 0) = \mathbf{H}_0 \qquad \text{in } \Omega. \tag{2.1g}$$

Here, $\Omega \subset \mathbb{R}^3$ is a bounded domain with piecewise linear boundary. The index ∞ denotes the space-time cylinder, e.g. $\Omega_\infty = \Omega \times [0, \infty)$. The boundary $\partial\Omega$ is decomposed into three parts $\partial\Omega_E$, $\partial\Omega_H$ and $\partial\Omega_I$, which can be empty, but otherwise also have relative piecewise linear boundaries. The outer unit normal on $\partial\Omega$ is denoted by $\mathbf{n}$. The unknowns $\mathbf{E}, \mathbf{H} : \Omega_\infty \to \mathbb{R}^3$ denote the space- and time-dependent electric and magnetic field, respectively. The parameters $\varepsilon, \mu : \Omega \to (0, \infty)$ denote the electric permittivity and magnetic permeability of the material distribution in Ω, as well as $\sigma : \Omega \to [0, \infty)$ denotes the conductivity. The right-hand side $\mathbf{f}_E : \Omega_\infty \to \mathbb{R}^3$ is the outer current density $\mathbf{f}_E = -\mathbf{j}_{\text{ext}}$, whereas $\mathbf{f}_H : \Omega_\infty \to \mathbb{R}^3$ vanishes in physics. The tangential boundary fields $\mathbf{g}_E : \partial\Omega_{E,\infty} \to \mathbb{R}^3$, $\mathbf{g}_H : \partial\Omega_{H,\infty} \to \mathbb{R}^3$ and $\mathbf{g}_I : \partial\Omega_{I,\infty} \to \mathbb{R}^3$ are given boundary values. The initial values for $\mathbf{E}$ and $\mathbf{H}$ are denoted by $\mathbf{E}_0, \mathbf{H}_0 : \Omega \to \mathbb{R}^3$. The boundary parameter $Z : \partial\Omega_I \to (0, \infty)$ is called impedance. All material parameters are considered to be independent of the frequency ω of the electromagnetic field.

The equations (2.1a) and (2.1b) can be rewritten in the form

$$\partial_t \mathbf{u} + A\mathbf{u} = \mathbf{f}, \tag{2.2}$$

with $\mathbf{u} = (\mathbf{E}, \mathbf{H})$, $\mathbf{f} = (\varepsilon^{-1}\mathbf{f}_E, \mu^{-1}\mathbf{f}_H)$ and the differential operator A defined by

$$A\mathbf{u} = \begin{pmatrix} -\varepsilon^{-1}\nabla \times \mathbf{H} \\ \mu^{-1}\nabla \times \mathbf{E} \end{pmatrix} + \begin{pmatrix} \varepsilon^{-1}\sigma\mathbf{E} \\ \mathbf{0} \end{pmatrix} = A_0^{-1} \sum_{j=1}^{3} \partial_{x_j} A_j \mathbf{u} + A_{-1}\mathbf{u}. \tag{2.3}$$

The matrices A_j are given by

$$A_0 = \begin{pmatrix} \varepsilon\mathbb{1} & 0 \\ 0 & \mu\mathbb{1} \end{pmatrix}, \qquad A_j = \begin{pmatrix} 0 & R_j^T \\ R_j & 0 \end{pmatrix}, \text{ for } j = 1, 2, 3, \tag{2.4a}$$
$$R_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad R_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \tag{2.4b}$$
$$R_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_{-1} = \varepsilon^{-1}\sigma \operatorname{diag}(1, 1, 1, 0, 0, 0). \tag{2.4c}$$

Note, that a multiplication with the matrix R_j describes the crossproduct with the j-th unit vector, i.e. $R_j\mathbf{E} = \mathbf{e}_j \times \mathbf{E}$. Some theory on system (2.2) can be found for example in [DPE12] under the name 'Friedrichs' Systems'.

Maxwell's two-dimensional system In case the geometry is homogeneous in x_3-direction one may consider solutions to (2.1) that are homogeneous in x_3-direction as well. In this case we can set $\partial_{x_3} = 0$ and (2.1a) and (2.1b) decouple into two systems in the variables (H_1, H_2, E_3), called transversal magnetic (TM) mode, and (E_1, E_2, H_3), called transversal electric (TE) mode. In our numerical tests in chapter 4 we use the TM-mode. For $\Omega \subset \mathbb{R}^2$ the corresponding PDE reads

$$\begin{aligned}
\varepsilon\partial_t E_3 + \sigma E_3 - \partial_{x_1} H_2 + \partial_{x_2} H_1 &= f_{E,3} && \text{in } \Omega_\infty, && (2.5a)\\
\mu\partial_t H_1 + \partial_{x_2} E_3 &= f_{H,1} && \text{in } \Omega_\infty, && (2.5b)\\
\mu\partial_t H_2 - \partial_{x_1} E_3 &= f_{H,2} && \text{in } \Omega_\infty, && (2.5c)\\
(n_2 E_3, -n_1 E_3) &= (g_{E,1}, g_{E,2}) && \text{on } \partial\Omega_{E,\infty}, && (2.5d)\\
n_1 H_2 - n_2 H_1 &= g_{H,3} && \text{on } \partial\Omega_{H,\infty}, && (2.5e)\\
n_1 H_2 - n_2 H_1 + Z E_3 &= g_{I,3} && \text{on } \partial\Omega_{I,\infty}, && (2.5f)\\
(E_3, H_1, H_2)(\cdot, 0) &= (E_{3,0}, H_{1,0}, H_{2,0}) && \text{in } \Omega. && (2.5g)
\end{aligned}$$

2.1.1 Further assumptions and remarks on the system

Material parameters We assume the material parameters ε, μ, σ and Z to be piecewise constant, non-negative, and except for σ non-zero, i.e.

$$\begin{aligned}
&\varepsilon, \mu, \sigma, Z \text{ piecewise constant}, && \varepsilon, \mu, \sigma \in \mathrm{L}_\infty(\Omega), && Z \in \mathrm{L}_\infty(\partial\Omega_I),\\
&0 < \text{const.} \le \varepsilon, \mu, Z, && 0 \le \sigma.
\end{aligned}$$

The domain Ω is decomposed into a finite number of maximal domains Λ_j, where all parameters are constant

$$\begin{aligned}
&\overline{\Omega} = \bigcup_j \overline{\Lambda_j}, && \varepsilon|_{\Lambda_j} = \text{const.}, && \mu|_{\Lambda_j} = \text{const.},\\
&\sigma|_{\Lambda_j} = \text{const.}, && Z|_{\partial\Lambda_j \cap \partial\Omega_I} = \text{const.}.
\end{aligned}$$

The domains Λ_j are supposed to have a piecewise linear boundary as well. In case of a function with broken regularity, e.g. $u \in \mathrm{H}^1(\Lambda_j)$ for every j, we denote the corresponding broken norm by

$$\|u\|_{1,\Lambda}^2 = \sum_j \|u\|_{1,\Lambda_j}^2 .$$

With an index V we denote a weighted norm

$$\|\mathbf{u}\|_{1,\mathrm{V},\Lambda}^2 = \left\|\sqrt{\varepsilon}\mathbf{E}\right\|_{1,\Lambda}^2 + \left\|\sqrt{\mu}\mathbf{H}\right\|_{1,\Lambda}^2 .$$

Inhomogeneity The right-hand side $\mathbf{f}$ is supposed to be continuous in time throughout this work

$$\mathbf{f} \in \mathrm{C}\left([0,\infty), \mathrm{L}_2(\Omega)^6\right). \tag{2.6}$$

Boundary data We assume the boundary values $\mathbf{g}_j$ to have an extension into the space-time cylinder Ω_∞. Therefor we consider the linear space

$$\begin{aligned}\mathrm{H}(\mathrm{curl},\Omega,\partial\Omega) = \big\{\mathbf{u} \in \mathrm{H}(\mathrm{curl},\Omega)^2 :\ &\mathbf{n}\times\mathbf{E} \in \mathrm{L}_2(\partial\Omega_E)^3,\ \mathbf{n}\times\mathbf{H} \in \mathrm{L}_2(\partial\Omega_H)^3,\\ &\mathbf{n}\times\mathbf{E} \in \mathrm{L}_2(\partial\Omega_I)^3,\ \mathbf{n}\times\mathbf{H} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) \in \mathrm{L}_2(\partial\Omega_I)^3\big\},\end{aligned}$$

with norm

$$\begin{aligned}\|\mathbf{u}\|^2_{\mathrm{H}(\mathrm{curl},\Omega,\partial\Omega)} = \|\mathbf{u}\|^2_{0,\Omega} + \|(\nabla\times\mathbf{E},\nabla\times\mathbf{H})\|^2_{0,\Omega} + \|\mathbf{n}\times\mathbf{E}\|^2_{0,\partial\Omega_E} + \|\mathbf{n}\times\mathbf{H}\|^2_{0,\partial\Omega_H}\\ + \|\mathbf{n}\times\mathbf{E}\|^2_{0,\partial\Omega_I} + \|\mathbf{n}\times\mathbf{H} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E})\|^2_{0,\partial\Omega_I}\end{aligned}$$

and assume for the boundary values

$$\begin{aligned}\big(\mathbf{g}_E,\mathbf{g}_H,\mathbf{g}_I\big) \in \Big\{\ \big(\mathbf{n}\times\mathbf{E}|_{\partial\Omega_E}\,, \mathbf{n}\times\mathbf{H}|_{\partial\Omega_H}\,, \mathbf{n}\times\mathbf{H} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E})|_{\partial\Omega_I}\big) :\\ \mathbf{u} \in \mathrm{C}^1\big([0,\infty),\mathrm{L}_2(\Omega)^6\big) \cap \mathrm{C}\big([0,\infty),\mathrm{H}(\mathrm{curl},\Omega,\partial\Omega)\big)\Big\}.\end{aligned} \tag{2.7}$$

Hyperbolicity The system (2.2) is called hyperbolic (see [Eva98, Sec. 7.3]), if the matrix $A_0^{-1}\sum_{j=1}^3 n_j A_j$ is diagonizable with real eigenvalues for every $\mathbf{x}$, t and unit vector $\mathbf{n}$. The matrices in (2.4) fulfill this hyperbolicity condition. We will state the eigenvectors later on in (2.18).

Divergence equations In our PDE (2.1) we neglected the two equations on the divergence (1.6c) and (1.6d) of the fields $\mathbf{E}$ and $\mathbf{H}$, respectively. The second equation (1.6d) is fulfilled as long as $\nabla\cdot(\mu\mathbf{H}_0) = 0$ and $\nabla\cdot\mathbf{f}_H = 0$ (in especially for $\mathbf{f}_H = \mathbf{0}$), which can be seen by application of the divergence on equation (2.1b), since $\nabla\cdot(\nabla\times\cdot) = 0$. So this equation is a condition on the initial values $\mathbf{H}_0$. The corresponding divergence equation for the electric field does not yield a condition on the initial values for the electric field, since we have the freedom to adjust the internal charge density ρ_{int}, so that (1.6c) is fulfilled.

Conservation of energy An electromagnetic field contains energy. This energy $\mathcal{E}$ can be expressed by a weighted inner product in $\mathrm{L}_2(\Omega)$

$$\mathcal{E} = \frac{1}{2}\int_\Omega \Big(\varepsilon\,\mathbf{E}\cdot\mathbf{E} + \mu\,\mathbf{H}\cdot\mathbf{H}\Big)\,\mathrm{dx}\,.$$

The total energy in Ω can change by outer forces, by a conductivity or through the boundary. Mathematically speaking, i.e.

$$\begin{aligned}
\partial_t \mathcal{E}(t) &= \int_\Omega \Big(\varepsilon\, \mathbf{E}(t)\cdot \partial_t \mathbf{E}(t) + \mu\, \mathbf{H}(t)\cdot \partial_t \mathbf{H}(t)\Big)\,\mathrm{dx} \\
&= \int_\Omega \Big(\mathbf{E}(t)\cdot \big(\mathbf{f}_E(t) - \sigma \mathbf{E}(t) + \nabla\times\mathbf{H}(t)\big) + \mathbf{H}(t)\cdot\big(\mathbf{f}_H(t) - \nabla\times\mathbf{E}(t)\big)\Big)\,\mathrm{dx} \\
&= \int_\Omega \Big(\mathbf{E}(t)\cdot\mathbf{f}_E(t) + \mathbf{H}(t)\cdot\mathbf{f}_H(t) - \sigma|\mathbf{E}(t)|^2\Big)\,\mathrm{dx} + \int_{\partial\Omega} \mathbf{E}(t)\cdot\mathbf{n}\times\mathbf{H}(t)\,\mathrm{da}\,.
\end{aligned}$$

For PEC and PMC boundary conditions and without conductivity and outer forces, i.e. $\sigma = 0$ and $(\mathbf{f}_E, \mathbf{f}_H) = \mathbf{0}$, we have conservation of energy.

Finite speed of propagation The PDE (2.1) describes the propagation of electromagnetic waves. These waves travel with a maximal speed $c = (\varepsilon\mu)^{-\frac{1}{2}}$, which can be found in special plane wave solutions (1.8), as well as in the system (2.1) in the following sense (see also [Eva98, Sec. 2.4.3]). For a time $0 \le t \le \frac{R}{c_{\max}}$ define the ball $B(t) = B(\mathbf{x}_0, R - c_{\max}t) \subset \mathbb{R}^3$ of decreasing radius $R - c_{\max}t$, where $c_{\max} = \sup_{\mathbf{x}\in\Omega}(\varepsilon(\mathbf{x})\mu(\mathbf{x}))^{-\frac{1}{2}}$. Additionally define the cone C to be the union of all these balls

$$C = \{(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, Rc_{\max}^{-1}) : \ \mathbf{x} \in B(t)\}.$$

Lemma 2.1.1:
Let $\mathbf{u}(t) = (\mathbf{E}(t), \mathbf{H}(t))$ *be a solution to* (2.1), *which is smooth in the cone* C. *If* $\mathbf{u}(0) = \mathbf{0}$ *on* $B(0)$, *then* $\mathbf{u}(t) = \mathbf{0}$ *on* C, *as long as there are no outer forces, i.e.* $(\mathbf{f}_E, \mathbf{f}_H) = \mathbf{0}$.

Proof: The electromagnetic energy inside the ball $B(t)$ at time t is given by

$$\mathcal{E}(t) = \frac{1}{2}\int_{B(t)} \Big(\varepsilon|\mathbf{E}(t)|^2 + \mu|\mathbf{H}(t)|^2\Big)\,\mathrm{dx}\,.$$

Now we calculate the time derivative of the energy

$$\begin{aligned}
\partial_t \mathcal{E}(t) = \partial_t \frac{1}{2}\int_{B(s)} \Big(\varepsilon|\mathbf{E}(t)|^2 + \mu|\mathbf{H}(t)|^2\Big)\,\mathrm{dx}\Big|_{s=t} \\
+ \partial_t \frac{1}{2}\int_0^{R-c_{\max}t}\int_{\partial B(\mathbf{x}_0, r)} \Big(\varepsilon|\mathbf{E}(s)|^2 + \mu|\mathbf{H}(s)|^2\Big)\,\mathrm{da}\,\mathrm{dr}\Big|_{s=t}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{B(t)} \Big(\varepsilon \mathbf{E}(t) \cdot \partial_t \mathbf{E}(t) + \mu \mathbf{H}(t) \cdot \partial_t \mathbf{H}(t) \Big) \, \mathrm{dx} \\
&\quad - \frac{c_{\max}}{2} \int_{\partial B(t)} \Big(\varepsilon |\mathbf{E}(t)|^2 + \mu |\mathbf{H}(t)|^2 \Big) \, \mathrm{da} \\
&= \int_{B(t)} \Big(\mathbf{E}(t) \cdot \big(\mathbf{f}_E(t) - \sigma \mathbf{E}(t) + \nabla \times \mathbf{H}(t) \big) + \mathbf{H}(t) \cdot \big(\mathbf{f}_H(t) - \nabla \times \mathbf{E}(t) \big) \Big) \, \mathrm{dx} \\
&\quad - \frac{c_{\max}}{2} \int_{\partial B(t)} \Big(\varepsilon |\mathbf{E}(t)|^2 + \mu |\mathbf{H}(t)|^2 \Big) \, \mathrm{da} \\
&= \int_{B(t)} \Big(\mathbf{E}(t) \cdot \mathbf{f}_E(t) + \mathbf{H}(t) \cdot \mathbf{f}_H(t) - \sigma |\mathbf{E}(t)|^2 \Big) \, \mathrm{dx} \\
&\quad + \int_{\partial B(t)} \mathbf{E} \cdot \mathbf{n} \times \mathbf{H} \, \mathrm{da} - \frac{c_{\max}}{2} \int_{\partial B(t)} \Big(\varepsilon |\mathbf{E}(t)|^2 + \mu |\mathbf{H}(t)|^2 \Big) \, \mathrm{da} \\
&\le \int_{B(t)} \Big(\mathbf{E}(t) \cdot \mathbf{f}_E(t) + \mathbf{H}(t) \cdot \mathbf{f}_H(t) - \sigma |\mathbf{E}(t)|^2 \Big) \, \mathrm{dx} \\
&\quad + \frac{c_{\max}}{2} \int_{\partial B(t)} \Big(2\sqrt{\varepsilon\mu} |\mathbf{H}(t)||\mathbf{E}(t)| - \varepsilon |\mathbf{E}(t)|^2 - \mu |\mathbf{H}(t)|^2 \Big) \, \mathrm{da} \\
&= \int_{B(t)} \Big(\mathbf{E}(t) \cdot \mathbf{f}_E(t) + \mathbf{H}(t) \cdot \mathbf{f}_H(t) - \sigma |\mathbf{E}(t)|^2 \Big) \, \mathrm{dx} \\
&\quad - \frac{c_{\max}}{2} \int_{\partial B(t)} \Big(\sqrt{\varepsilon} |\mathbf{E}(t)| - \sqrt{\mu} |\mathbf{H}(t)| \Big)^2 \, \mathrm{da} \, .
\end{aligned}
$$

For $(\mathbf{f}_E, \mathbf{f}_H) = \mathbf{0}$, we see that $\partial_t \mathcal{E}(t) \le 0$. So $\mathcal{E}(0) = 0 \Rightarrow \mathcal{E}(t) = 0 \ \forall t \in [0, Rc_{\max}^{-1}]$ and thereby the statement is proven. □

2.2 Existence and uniqueness of the solution to Maxwell's equations

We check the assumptions of Lumer-Phillips' theorem to show existence and uniqueness of the solution to (2.1). Details on semigroup theory and Lumer-Phillips' theorem can be found in [RR04, Chap. 12]. In 2011 Serge Nicaise had given two talks in Karlsruhe about Maxwell's equations including well-posedness. We gratefully like to mention this.

2.2.1 Homogeneous boundary conditions

Theorem 2.2.1 (Lumer-Phillips, [RR04, Theorem 12.22])**:**
Let H *be a Hilbert space and let* A *be a linear operator in* H *satisfying the following conditions:*

1. $\mathcal{D}(A)$ *is dense in* H.
2. $Re(x, Ax) \leq \omega(x, x)$ *for every* $x \in \mathcal{D}(A)$.
3. *There exists a* $\lambda_0 > \omega$ *such that* $A - \lambda_0 I$ *is onto.*

Then A *generates a quasicontraction semigroup and* $\|\exp(At)\| \leq \exp(\omega t)$.

We consider the Hilbert space $\mathrm{V} = \mathrm{L}_2(\Omega, \mathbb{R}^3 \times \mathbb{R}^3)$ equipped with the inner product

$$\big((\mathbf{E},\mathbf{H}),(\tilde{\mathbf{E}},\tilde{\mathbf{H}})\big)_{\mathrm{V}} = (\mathbf{E},\tilde{\mathbf{E}})_\varepsilon + (\mathbf{H},\tilde{\mathbf{H}})_\mu = \int_\Omega \Big(\varepsilon\mathbf{E}\cdot\tilde{\mathbf{E}} + \mu\mathbf{H}\cdot\tilde{\mathbf{H}}\Big)\,\mathrm{dx}\,.$$

The indices V, ε, and μ refer to the weights in the inner products. For the operator $A : \mathcal{D}(A) \subset \mathrm{V} \to \mathrm{V}$ as previously defined in (2.3), we choose the domain

$$\begin{aligned}\mathcal{D}(A) = \big\{(\mathbf{E},\mathbf{H}) \in \mathrm{H}(\mathrm{curl},\Omega)^2 :\ &\mathbf{n}\times\mathbf{E} = \mathbf{0} \text{ on } \partial\Omega_E,\ \mathbf{n}\times\mathbf{H} = \mathbf{0} \text{ on } \partial\Omega_H,\\ &\mathbf{n}\times\mathbf{E} \in \mathrm{L}_2(\partial\Omega_I)^3,\ \mathbf{n}\times\mathbf{H} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) = \mathbf{0} \text{ on } \partial\Omega_I\big\},\end{aligned}$$

equipped with the norm

$$\|\mathbf{u}\|^2_{\mathcal{D}(A)} = \|\mathbf{u}\|^2_{\mathrm{V}} + \|A\mathbf{u}\|^2_{\mathrm{V}} + \|\mathbf{n}\times\mathbf{E}\|^2_{\varepsilon,\partial\Omega_I}.$$

Lemma 2.2.2:
The operator $-A : \mathcal{D}(A) \subset \mathrm{V} \to \mathrm{V}$ *satisfies the assumptions of Theorem 2.2.1.*

Proof: We have to check three assumptions.

First assumption $\mathcal{D}(A)$ is dense in V, since $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{L}_2(\Omega)$.

Second assumption To check the second assumption, for $(\mathbf{E},\mathbf{H}),(\boldsymbol{\psi},\boldsymbol{\varphi}) \in \mathcal{D}(A)$ we calculate

$$\begin{aligned}
\big(A(\mathbf{E},\mathbf{H}),(\boldsymbol{\psi},\boldsymbol{\varphi})\big)_{\mathrm{V}} &= \int_\Omega (-\nabla\times\mathbf{H}+\sigma\mathbf{E})\cdot\boldsymbol{\psi}+\nabla\times\mathbf{E}\cdot\boldsymbol{\varphi}\,\mathrm{dx}\\
&= \int_\Omega \sigma\mathbf{E}\cdot\boldsymbol{\psi}-\mathbf{H}\cdot\nabla\times\boldsymbol{\psi}+\mathbf{E}\cdot\nabla\times\boldsymbol{\varphi}\,\mathrm{dx}\\
&\quad+\int_{\partial\Omega} -\mathbf{n}\times\mathbf{H}\cdot\boldsymbol{\psi}+\mathbf{n}\times\mathbf{E}\cdot\boldsymbol{\varphi}\,\mathrm{da}\\
&= -\big((\mathbf{E},\mathbf{H}),A(\boldsymbol{\psi},\boldsymbol{\varphi})\big)_{\mathrm{V}}+2\int_\Omega\sigma\mathbf{E}\cdot\boldsymbol{\psi}\,\mathrm{dx}\\
&\quad+\int_{\partial\Omega_I} -\mathbf{n}\times\mathbf{H}\cdot\boldsymbol{\psi}+\mathbf{n}\times\mathbf{E}\cdot\boldsymbol{\varphi}\,\mathrm{da}\\
&= -\big((\mathbf{E},\mathbf{H}),A(\boldsymbol{\psi},\boldsymbol{\varphi})\big)_{\mathrm{V}}+2\int_\Omega\sigma\mathbf{E}\cdot\boldsymbol{\psi}\,\mathrm{dx}\\
&\quad-\int_{\partial\Omega_I}\big(Z(\mathbf{n}\times(\mathbf{n}\times\mathbf{E}))\cdot\boldsymbol{\psi}+Z\mathbf{E}\cdot(\mathbf{n}\times(\mathbf{n}\times\boldsymbol{\psi}))\big)\,\mathrm{da}\\
&= -\big((\mathbf{E},\mathbf{H}),A(\boldsymbol{\psi},\boldsymbol{\varphi})\big)_{\mathrm{V}}+2\int_\Omega\sigma\mathbf{E}\cdot\boldsymbol{\psi}\,\mathrm{dx}\\
&\quad+2\int_{\partial\Omega_I} Z\,\mathbf{n}\times\mathbf{E}\cdot\mathbf{n}\times\boldsymbol{\psi}\,\mathrm{da}\\
\Rightarrow\ \big(A(\mathbf{E},\mathbf{H}),(\mathbf{E},\mathbf{H})\big)_{\mathrm{V}} &= \int_\Omega\sigma\mathbf{E}\cdot\mathbf{E}\,\mathrm{dx}+\int_{\partial\Omega_I} Z\,\mathbf{n}\times\mathbf{E}\cdot\mathbf{n}\times\mathbf{E}\,\mathrm{da}\geq 0.
\end{aligned}$$

With $\omega = 0$ we have the second assumption for the operator $-A$.

Third assumption We set $\lambda_0 = 1$ and check the third assumption. For a given $\mathbf{F} = (\mathbf{F}_E,\mathbf{F}_H) \in V$ find $(\mathbf{E},\mathbf{H}) \in \mathcal{D}(A)$ with

$$A(\mathbf{E},\mathbf{H}) + (\mathbf{E},\mathbf{H}) = \mathbf{F}$$

or equivalently

$$-\varepsilon^{-1}\nabla\times\mathbf{H}+\varepsilon^{-1}\sigma\mathbf{E}+\mathbf{E} = \mathbf{F}_E, \tag{2.8a}$$

$$\mu^{-1}\nabla\times\mathbf{E}+\mathbf{H} = \mathbf{F}_H. \tag{2.8b}$$

We take

$$\boldsymbol{\psi} \in \tilde{\mathrm{V}} = \{\mathbf{E}\in\mathrm{H}(\mathrm{curl},\Omega):\ \mathbf{n}\times\mathbf{E}=\mathbf{0}\text{ on }\partial\Omega_E,\ \mathbf{n}\times\mathbf{E}\in\mathrm{L}_2(\partial\Omega_I)^3\}$$

and test the second equation with $\nabla \times \boldsymbol{\psi}$

$$\begin{aligned}\int_\Omega \mathbf{F}_H \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} &= \int_\Omega \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} + \mathbf{H} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} \\ &= \int_\Omega \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} + \nabla \times \mathbf{H} \cdot \boldsymbol{\psi} \,\mathrm{dx} + \int_{\partial\Omega_I} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi} \,\mathrm{da}\,.\end{aligned}$$

Insertion of the first equation (2.8a) and the boundary conditions for $(\mathbf{E}, \mathbf{H})$ yield

$$\begin{aligned}\int_\Omega \mathbf{F}_H \cdot \nabla \times \boldsymbol{\psi} + \varepsilon\, \mathbf{F}_E \cdot \boldsymbol{\psi} \,\mathrm{dx} = \int_\Omega \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} + (\varepsilon + \sigma)\, \mathbf{E} \cdot \boldsymbol{\psi} \,\mathrm{dx} \\ + \int_{\partial\Omega_I} Z\, \mathbf{n} \times \mathbf{E} \cdot \mathbf{n} \times \boldsymbol{\psi} \,\mathrm{da}\,.\end{aligned} \tag{2.9}$$

The space $\tilde{\mathrm{V}}$ equipped with the inner product

$$(\mathbf{E}, \boldsymbol{\psi})_{\tilde{\mathrm{V}}} = \int_\Omega \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} + (\varepsilon + \sigma)\, \mathbf{E} \cdot \boldsymbol{\psi} \,\mathrm{dx} + \int_{\partial\Omega_I} Z\, \mathbf{n} \times \mathbf{E} \cdot \mathbf{n} \times \boldsymbol{\psi} \,\mathrm{da}$$

is a Hilbert space, so Riesz grants a unique solution $\mathbf{E} \in \tilde{\mathrm{V}}$ of (2.9). Define $\mathbf{H}$ by

$$\mathbf{H} = \mathbf{F}_H - \mu^{-1} \nabla \times \mathbf{E} \in \mathrm{L}_2(\Omega)^3.$$

Now, with the help of (2.9), we calculate the weak curl of $\mathbf{H}$. For $\boldsymbol{\psi} \in \mathrm{C}_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega \mathbf{H} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} &= \int_\Omega \mathbf{F}_H \cdot \nabla \times \boldsymbol{\psi} - \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} \\ &= \int_\Omega -\varepsilon\, \mathbf{F}_E \cdot \boldsymbol{\psi} + (\varepsilon + \sigma)\, \mathbf{E} \cdot \boldsymbol{\psi} \,\mathrm{dx}\,.\end{aligned}$$

So we have $\nabla \times \mathbf{H} = -\varepsilon \mathbf{F}_E + \varepsilon \mathbf{E} + \sigma \mathbf{E} \in \mathrm{L}_2(\Omega)$. Next we check the boundary conditions to assure $(\mathbf{E}, \mathbf{H}) \in \mathcal{D}(A)$. Again for $\boldsymbol{\psi} \in \tilde{\mathrm{V}} \cap \mathrm{H}^1(\Omega)$, we use (2.8a) and (2.9) to calculate

$$\begin{aligned}\int_\Omega \mathbf{H} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} &= \int_\Omega \nabla \times \mathbf{H} \cdot \boldsymbol{\psi} \,\mathrm{dx} + \int_{\partial\Omega} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi} \,\mathrm{da} \\ &= \int_\Omega -\varepsilon\, \mathbf{F}_E \cdot \boldsymbol{\psi} + (\varepsilon + \sigma)\, \mathbf{E} \cdot \boldsymbol{\psi} \,\mathrm{dx} + \int_{\partial\Omega} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi} \,\mathrm{da} \\ &= \int_\Omega \mathbf{F}_H \cdot \nabla \times \boldsymbol{\psi} - \mu^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} \\ &\quad - \int_{\partial\Omega_I} Z\, \mathbf{n} \times \mathbf{E} \cdot \mathbf{n} \times \boldsymbol{\psi} \,\mathrm{da} + \int_{\partial\Omega} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi} \,\mathrm{da}\,.\end{aligned}$$

Due to the definition of $\mathbf{H}$ the volume integrals sum up to zero and we obtain

$$\int_{\partial\Omega_I} Z\,\mathbf{n}\times\mathbf{E}\cdot\mathbf{n}\times\boldsymbol{\psi}\,\mathrm{da} = \int_{\partial\Omega_I} \mathbf{H}\cdot\mathbf{n}\times\boldsymbol{\psi}\,\mathrm{da} + \int_{\partial\Omega_H} \mathbf{H}\cdot\mathbf{n}\times\boldsymbol{\psi}\,\mathrm{da}$$

and thereby the boundary conditions $\mathbf{n}\times\mathbf{H}-Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) = \mathbf{0}$ on $\partial\Omega_I$ and $\mathbf{n}\times\mathbf{H} = \mathbf{0}$ on $\partial\Omega_H$. So the third assumption of Lumer-Phillips' theorem is fulfilled. □

We state an existence and uniqueness result for homogeneous boundary conditions.

Lemma 2.2.3 (Existence and uniqueness, [RR04, Sec. 12.1.3])**:**
Assume initial values $\mathbf{u}_0 = (\mathbf{E}_0, \mathbf{H}_0) \in \mathcal{D}(A)$, $\mathbf{f} \in \mathrm{C}([0,\infty), \mathrm{V})$ *and either* $\mathbf{f} \in \mathrm{W}^{1,1}_{\mathrm{loc}}([0,\infty), \mathrm{V})$ *or* $\mathbf{f} \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty), \mathcal{D}(A))$. *Then the PDE* $\partial_t \mathbf{u} + A\mathbf{u} = \mathbf{f}$ *has a unique classical solution* $\mathbf{u} \in \mathrm{C}^1([0,\infty), \mathrm{V}) \cap \mathrm{C}([0,\infty), \mathcal{D}(A))$ *given by*

$$\mathbf{u}(t) = \exp(-At)\mathbf{u}_0 + \int_0^t \exp\big(-A(t-s)\big)\mathbf{f}(s)\,\mathrm{ds}\,.$$

2.2.2 Non-homogeneous boundary conditions

In case of non-homogeneous boundary values $(\mathbf{g}_E, \mathbf{g}_H, \mathbf{g}_I) \neq \mathbf{0}$, we assume to have an extension $\mathbf{u}_B \in \mathrm{C}^1([0,\infty), \mathrm{V}) \cap \mathrm{C}([0,\infty), \mathrm{H}(\mathrm{curl}, \Omega, \partial\Omega))$ of the boundary values into the space-time cylinder Ω_∞ (see (2.7)) to state existence and uniqueness.

Lemma 2.2.4 (Existence and uniqueness, [RR04, Sec. 12.1.3])**:**
Assume initial values $\mathbf{u}_0 = (\mathbf{E}_0, \mathbf{H}_0) \in \mathrm{H}(\mathrm{curl}, \Omega)^2$, *which fit the boundary values, i.e.* $\mathbf{n}\times\mathbf{E}_0|_{\partial\Omega_E} = \mathbf{g}_E(\cdot, 0)$, $\mathbf{n}\times\mathbf{H}_0|_{\partial\Omega_H} = \mathbf{g}_H(\cdot, 0)$, $\mathbf{n}\times\mathbf{E}_0 \in \mathrm{L}_2(\partial\Omega_I)^3$, *and* $\mathbf{n}\times\mathbf{H}_0|_{\partial\Omega_I} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}_0)|_{\partial\Omega_I} = \mathbf{g}_I(\cdot, 0)$. *Further assume* $\mathbf{f} \in \mathrm{C}([0,\infty), \mathrm{V})$ *and either* $\mathbf{f} - \partial_t\mathbf{u}_B - A\mathbf{u}_B \in \mathrm{W}^{1,1}_{\mathrm{loc}}([0,\infty), \mathrm{V})$ *or* $\mathbf{f} - \partial_t\mathbf{u}_B - A\mathbf{u}_B \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty), \mathcal{D}(A))$. *Then the PDE* $\partial_t\mathbf{u} + A\mathbf{u} = \mathbf{f}$ *with boundary values* $\mathbf{g}_j$, $j \in \{E, H, I\}$ *has a unique classical solution* $\mathbf{u} \in \mathrm{C}^1([0,\infty), \mathrm{V}) \cap \mathrm{C}([0,\infty), \mathrm{H}(\mathrm{curl}, \Omega, \partial\Omega))$ *given by*

$$\begin{aligned} \mathbf{u}(t) &= \mathbf{u}_B(t) + \mathbf{u}_{\mathrm{hom}}(t), \\ \mathbf{u}_{\mathrm{hom}}(t) &= \exp(-At)\mathbf{u}_{\mathrm{hom},0} + \int_0^t \exp\big(-A(t-s)\big)(\mathbf{f} - \partial_t\mathbf{u}_B - A\mathbf{u}_B)(s)\,\mathrm{ds}\,. \end{aligned}$$

2.3 A discretization in space

2.3.1 The finite dimensional space of approximation

We decompose the domain Ω into a finite number of disjoint triangular ($\Omega \subset \mathbb{R}^2$) or tetrahedral ($\Omega \subset \mathbb{R}^3$) cells $K \in \mathcal{T}_h$ with maximal diameter $h = \max_{K\in\mathcal{T}_h} \operatorname{diam} K$ and $\overline{\Omega} = \cup_{K\in\mathcal{T}_h} \overline{K}$. For every cell $K \in \mathcal{T}_h$ we assume the diameter $h_K = \operatorname{diam} K$ to be bounded, up to a constant, by the radius r_K of the largest inner ball of K

$$h_K \leq c_{\text{mesh}}\, r_K, \quad r_K = \sup\{r \in \mathbb{R} : \ B(r, \mathbf{x}_0) \subset K, \mathbf{x}_0 \in K\}. \tag{2.10}$$

By $\mathcal{F}_K$ we denote the set of faces f of the cell K, as well as by $\mathcal{F}_K^\circ$ the set of inner faces and by $\mathcal{F}_K^\partial = \mathcal{F}_K^E \cup \mathcal{F}_K^H \cup \mathcal{F}_K^I$ the sets of boundary faces, where the indices E, H and I correspond to the three kind of boundary conditions in (2.1). Later on in Chapter 5 we will also need a bound on jumps in the size of the cells

$$\frac{h_K}{h_{K_f}} \leq c_{\text{h}}, \tag{2.11}$$

where K_f denotes the next neighbour cell of K across the face f. In the absence of hanging nodes, this bound is already implied by (2.10). On each cell K we assume the material parameters ε, μ, σ and Z to be constant, i.e. $K \subset \Lambda_j$ for some j. The inner products in $\mathrm{L}_2(K)$ and $\mathrm{L}_2(f)$ are denoted by $(\cdot,\cdot)_{0,K}$, $(\cdot,\cdot)_{0,f}$, and weighted with the material parameters by $(\cdot,\cdot)_{\mathrm{V},K}$. Our finite dimensional space for the approximate solution and the test functions is the piecewise polynomial space

$$\mathrm{V}_h^p = \{\mathbf{u}_h \in \mathrm{V} : \ \mathbf{u}_h|_K \in \mathbb{P}_p^6(K) \quad \forall K \in \mathcal{T}_h\},$$

with $\mathbb{P}_p$ the space of polynomial functions of degree less or equal p. In the literature the expression 'Finite Volume' refers to the case $p = 0$, where the approximate solution is supposed to be piecewise constant. Generalized methods with higher polynomial degrees are called 'Discontinuous Galerkin' methods. The L_2-orthogonal projection on V_h^p is denoted by Π_h^p

$$\Pi_h^p : \mathrm{V} \to \mathrm{V}_h^p, \ \mathbf{u} \mapsto \operatorname*{argmin}_{\mathbf{w}\in\mathrm{V}_h^p} \|\mathbf{u} - \mathbf{w}\|_{\mathrm{V}}.$$

In case of a vector with other than six components, we denote the projection on piecewise polynomial functions in the same way. The corresponding projection on polynomials on a single cell K is denoted by Π_K^p.

2.3.2 Projection error estimates

In our convergence analysis we will need some estimates on projection errors.

Lemma 2.3.1:
Let $K \subset \mathbb{R}^d$, $d = 2, 3$ be a triangle or tetrahedron with diameter $h_K = \operatorname{diam} K$ bounded as in (2.10) and let $s = \bar{s} - \hat{s}$, $\bar{s} \in \mathbb{N}$ the rounded up integer part of s, $\hat{s} \in [0, 1)$, $p = \bar{s} - 1$. Then there exist a constant c_{cell}, independent of K, such that

$$\|u - \Pi_K^p u\|_{0,K} \leq c_{\text{cell}}\, h_K^s |u|_{s,K} \quad \forall u \in \mathrm{H}^s(K).$$

Lemma 2.3.2:
Let $K \subset \mathbb{R}^d$, $d = 2, 3$ be a triangle or tetrahedron with diameter $h_K = \operatorname{diam} K$ bounded as in (2.10) and let $s = \bar{s} - \hat{s} \geq \frac{1}{2}$, $\bar{s} \in \mathbb{N}$ the rounded up integer part of s, $\hat{s} \in [0, 1)$, $p = \bar{s} - 1$. Then there exist a constant c_{boundary}, independent of K, such that

$$\|u - \Pi_K^p u\|_{0,\partial K} \leq c_{\text{boundary}}\, h_K^{s-\frac{1}{2}} |u|_{s,K} \quad \forall u \in \mathrm{H}^s(K).$$

These results are standard finite element analysis. We omit the proofs and refer to [Bra03, Sec. 2.6]. The tools for the estimates on fractional Sobolev spaces can be found in [DS80, Sec. 6]. The corresponding fractional semi-norm is definded by

$$|u|_{s,K}^2 = \sum_{|\boldsymbol{\alpha}| = \bar{s}-1} \int_K \int_K \frac{|\partial_{\boldsymbol{\alpha}} u(\mathbf{x}) - \partial_{\boldsymbol{\alpha}} u(\mathbf{y})|^2}{|\mathbf{x} - \mathbf{y}|^{d+2(1-\hat{s})}}\, \mathrm{dx}\, \mathrm{dy},$$

where $\boldsymbol{\alpha}$ is a multi-index. Another estimate on the trace of a polynomial function will be used later on in Chapter 5.

Lemma 2.3.3:
Let $K \subset \mathbb{R}^d$, $d = 2, 3$ be a triangle or tetrahedron with diameter $h_K = \operatorname{diam} K$ bounded as in (2.10) and let $p \in \mathbb{N}_0$. Then there exist a constant c_{trace}, independent of K, such that

$$\|u_K\|_{0,\partial K} \leq c_{\text{trace}}\, h_K^{-\frac{1}{2}} \|u_K\|_{0,K} \quad \forall u_K \in \mathbb{P}_p(K).$$

Again, this is standard finite element analysis (see e.g. [DPE12, Lemma 1.46]).

2.3.3 Construction of the discrete operator A_h

We want to approximate our problem (2.1) with an ODE in the finite dimensional space V_h^p. Therefor we need to define an operator $A_h : \mathrm{V}_h^p \to \mathrm{V}_h^p$. So we test our continuous operator A with argument $\mathbf{u} = (\mathbf{E}, \mathbf{H}) \in \mathrm{H}(\mathrm{curl}, \Omega)^2$ on a cell K with a function $(\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h) \in \mathrm{V}_h^p$ and integrate by parts

$$\begin{aligned}\left(A\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\psi}_h\\ \boldsymbol{\varphi}_h\end{pmatrix}\right)_{\mathrm{V},K} &= -(\nabla\times\mathbf{H}, \boldsymbol{\psi}_h)_{0,K} + (\nabla\times\mathbf{E}, \boldsymbol{\varphi}_h)_{0,K} + (\sigma\mathbf{E}, \boldsymbol{\psi}_h)_{0,K}\\ &= -(\mathbf{H}, \nabla\times\boldsymbol{\psi}_h)_{0,K} + (\mathbf{E}, \nabla\times\boldsymbol{\varphi}_h)_{0,K} + (\sigma\mathbf{E}, \boldsymbol{\psi}_h)_{0,K}\\ &\quad + \sum_{f\in\mathcal{F}_K}\left(F_{K,f}(\mathbf{u}), \begin{pmatrix}\boldsymbol{\psi}_K\\ \boldsymbol{\varphi}_K\end{pmatrix}\right)_{0,f}.\end{aligned} \tag{2.12}$$

The boundary term

$$F_{K,f}(\mathbf{u}) = \sum_{j=1}^{3} n_j A_j \mathbf{u} = \begin{pmatrix}-\mathbf{n}\times\mathbf{H}\\ \mathbf{n}\times\mathbf{E}\end{pmatrix} \tag{2.13}$$

is called flux. In the volume integrals we can replace the fields $\mathbf{E}$ and $\mathbf{H}$ by discrete fields $\mathbf{E}_h$ and $\mathbf{H}_h$ to define the discrete operator A_h, but in the boundary integral we have to decide in what way to replace the flux $F_{K,f}(\mathbf{u})$ by a discrete version $F_{K,f}^*(\mathbf{u}_h)$. For hyperbolic PDEs the continuous flux can be replaced by a so called upwind flux. How to obtain the upwind flux by consideration of a Riemann problem will be explained next.

2.3.4 Upwind flux and Riemann problem

To construct the upwind flux, we have to solve a Riemann problem. Therefor we consider a two dimensional subspace f of $\mathbb{R}^3$, so to say an infinitely extended face, with unit normal $\mathbf{n}$, i.e. $f = \{\mathbf{x} \in \mathbb{R}^3 : \ \mathbf{x}\cdot\mathbf{n} = 0\}$. On each half-space defined by f we choose constant initial values and material parameters

$$\mathbf{u}_0(\mathbf{x}) = \begin{cases}\mathbf{u}^I, & \mathbf{x}\cdot\mathbf{n} < 0,\\ \mathbf{u}^{IV}, & \mathbf{x}\cdot\mathbf{n} > 0,\end{cases} \qquad (\mu, \varepsilon)(\mathbf{x}) = \begin{cases}(\mu^I, \varepsilon^I), & \mathbf{x}\cdot\mathbf{n} < 0,\\ (\mu^{IV}, \varepsilon^{IV}), & \mathbf{x}\cdot\mathbf{n} > 0.\end{cases}$$

The piecewise constant initial values $\mathbf{u}_0(\mathbf{x})$ correspond to the local behaviour near a face of a piecewise constant approximate solution in a Finite Volume scheme. Now we consider the time evolution of that initial values under Maxwell's system

$$\varepsilon\partial_t\mathbf{E} - \nabla\times\mathbf{H} = \mathbf{0} \qquad \text{in } \mathbb{R}^3\times[0,\infty), \tag{2.14a}$$

$$\mu\partial_t\mathbf{H} + \nabla\times\mathbf{E} = \mathbf{0} \qquad \text{in } \mathbb{R}^3\times[0,\infty). \tag{2.14b}$$

So to say, we neglect the right-hand side $(\mathbf{f}_E, \mathbf{f}_H)$ and the terms of order zero, i.e. $\sigma = 0$, in the PDE (2.1). Since we work on the whole of $\mathbb{R}^3$, we do not need any boundary conditions. The described problem is called Riemann problem.

The classical formulation (2.14) of the PDE is not appropriate to handle the jump in the initial values, so in the usual way we multiply (2.14) with a test function, integrate over space and time, and then integrate by parts to obtain the weak formulation of the problem.

Weak formulation Find $\mathbf{u} \in \mathrm{L}^1_{loc}(\mathbb{R}^3 \times [0,\infty))^6$ such that $\forall \boldsymbol{\phi} \in \mathrm{C}_0^\infty(\mathbb{R}^3 \times [0,\infty))^6$

$$\int_{[0,\infty)} \int_{\mathbb{R}^3} \mathbf{u} \cdot A_0 \partial_t \boldsymbol{\phi} + \mathbf{u} \cdot A_0 A \boldsymbol{\phi} \,\mathrm{dx}\,\mathrm{dt} + \int_{\mathbb{R}^3} \mathbf{u}_0(\mathbf{x}) \cdot A_0 \boldsymbol{\phi}(\mathbf{x}, 0)\,\mathrm{dx} = 0. \tag{2.15}$$

Since the initial values are constant in two directions, we actually have a problem in one space dimension. To understand the qualitative behaviour of such problems, we refer to [LeV92]. Here, we will just state a solution of (2.15) and show that it really is a solution. For our purposes this will be sufficient.

Our solution consists of four constant values and three travelling discontinuities

$$\mathbf{u}(\mathbf{x}, t) = \begin{cases} \mathbf{u}^I, & (\mathbf{x}, t) \in \Omega^I_\infty, \\ \mathbf{u}^{II}, & (\mathbf{x}, t) \in \Omega^{II}_\infty, \\ \mathbf{u}^{III}, & (\mathbf{x}, t) \in \Omega^{III}_\infty, \\ \mathbf{u}^{IV}, & (\mathbf{x}, t) \in \Omega^{IV}_\infty. \end{cases} \tag{2.16}$$

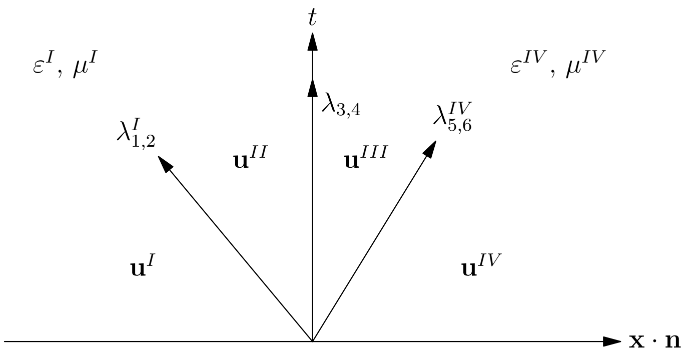


Figure 2.1: Solution to the Riemann problem.

The domains of constant values $\Omega^j_\infty \subset \mathbb{R}^3 \times [0,\infty)$ are given by

$$
\begin{aligned}
\Omega^I_\infty &= \{(\mathbf{x},t) \in \mathbb{R}^3 \times [0,\infty):\ \mathbf{x}\cdot\mathbf{n} < \lambda^I_{1,2} t\},\\
\Omega^{II}_\infty &= \{(\mathbf{x},t) \in \mathbb{R}^3 \times [0,\infty):\ \lambda^I_{1,2} t < \mathbf{x}\cdot\mathbf{n} < 0\},\\
\Omega^{III}_\infty &= \{(\mathbf{x},t) \in \mathbb{R}^3 \times [0,\infty):\ 0 < \mathbf{x}\cdot\mathbf{n} < \lambda^{IV}_{5,6} t\},\\
\Omega^{IV}_\infty &= \{(\mathbf{x},t) \in \mathbb{R}^3 \times [0,\infty):\ \lambda^{IV}_{5,6} t < \mathbf{x}\cdot\mathbf{n}\}.
\end{aligned}
$$

In order for $\mathbf{u}(\mathbf{x},t)$ to fulfill the weak formulation (2.15), we have to impose jump conditions on the discontinuities. They are known by the name Rankine-Hugoniot's jump conditions and read

$$
\begin{aligned}
F_{K,f}(\mathbf{u}^{II} - \mathbf{u}^{I}) &= \lambda^I_{1,2} A^I_0 (\mathbf{u}^{II} - \mathbf{u}^{I}),\\
F_{K,f}(\mathbf{u}^{III} - \mathbf{u}^{II}) &= \mathbf{0},\\
F_{K,f}(\mathbf{u}^{IV} - \mathbf{u}^{III}) &= \lambda^{IV}_{5,6} A^{IV}_0 (\mathbf{u}^{IV} - \mathbf{u}^{III}).
\end{aligned}
\tag{2.17}
$$

The flux $F_{K,f}$ was defined in (2.13), the matrix A_0 in (2.4a). Since A_0 is material dependent, the upper indices I, IV refer to the material parameters on the left- and right-hand side of f, respectively. The jump conditions (2.17) are eigenvalue equations for the matrix $A_0^{-1}F_{K,f}$. The eigenvalues denoted by λ^k_j are also the travelling speeds of the discontinuities of $\mathbf{u}(\mathbf{x},t)$. We have the following eigenvalues and eigenvectors of $A_0^{-1}F_{K,f}(\mathbf{u}) = (-\varepsilon^{-1}\mathbf{n}\times\mathbf{H}, \mu^{-1}\mathbf{n}\times\mathbf{E})$ written in unit normal and unit tangential vectors $\mathbf{n}$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ on f, with $\mathbf{n}\times\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$ and $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 = 0$

eigenvalue	eigenvector
$\lambda_1 = -\frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_1 = \begin{pmatrix}\sqrt{\mu}\boldsymbol{\tau}_2\\ \sqrt{\varepsilon}\boldsymbol{\tau}_1\end{pmatrix}$
$\lambda_2 = -\frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_2 = \begin{pmatrix}\sqrt{\mu}\boldsymbol{\tau}_1\\ -\sqrt{\varepsilon}\boldsymbol{\tau}_2\end{pmatrix}$
$\lambda_3 = 0$	$\mathbf{w}_3 = \begin{pmatrix}\mathbf{n}\\ \mathbf{0}\end{pmatrix}$
$\lambda_4 = 0$	$\mathbf{w}_4 = \begin{pmatrix}\mathbf{0}\\ \mathbf{n}\end{pmatrix}$
$\lambda_5 = \frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_5 = \begin{pmatrix}\sqrt{\mu}\boldsymbol{\tau}_1\\ \sqrt{\varepsilon}\boldsymbol{\tau}_2\end{pmatrix}$
$\lambda_6 = \frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_6 = \begin{pmatrix}\sqrt{\mu}\boldsymbol{\tau}_2\\ -\sqrt{\varepsilon}\boldsymbol{\tau}_1\end{pmatrix}.$

(2.18)

Since the eigenvalues as well as the eigenvectors depend on the material parameters, we again use the upper indices I and IV to distinguish between the two materials divided by f. The upper index 0 denotes the material parameters $\varepsilon = \mu = 1$.

Next, we show that the piecewise constant function $\mathbf{u}(\mathbf{x},t)$ given by (2.16) with the jump conditions (2.17) is a solution to the weak formulation (2.15).

Lemma 2.3.4:
The piecewise constant function $\mathbf{u}(\mathbf{x},t)$ *defined in* (2.16) *solves the weak formulation* (2.15), *as long as the jump conditions* (2.17) *are fulfilled.*

Proof: We denote the boundaries of Ω^j_∞ as follows

$$\begin{aligned}
&\Gamma_1 = \partial\Omega^I_\infty \cap \mathbb{R}^3 \times \{0\}, && \Gamma_2 = \partial\Omega^{II}_\infty \cap \partial\Omega^I_\infty, && \Gamma_3 = \partial\Omega^{III}_\infty \cap \partial\Omega^{II}_\infty,\\
&\Gamma_4 = \partial\Omega^{IV}_\infty \cap \partial\Omega^{III}_\infty, && \Gamma_5 = \mathbb{R}^3 \times \{0\} \cap \partial\Omega^{IV}_\infty .
\end{aligned}$$

The unit normal in space-time will be denoted $\mathbf{n}_T$. Since we have to clarify the direction of the normal, we introduce an upper index $\mathbf{n}^j_T$ for the outward unit normal on Ω^j_∞. We obtain the following values for the unit normal vector

$$\begin{aligned}
&\mathbf{n}^I_T\big|_{\Gamma_1} = \begin{pmatrix}\mathbf{0} & -1\end{pmatrix}, && \mathbf{n}^I_T\big|_{\Gamma_2} = -\,\mathbf{n}^{II}_T\big|_{\Gamma_2} = \frac{1}{\sqrt{1+(\lambda^I_{1,2})^2}}\begin{pmatrix}\mathbf{n} & -\lambda^I_{1,2}\end{pmatrix},\\
&\mathbf{n}^{II}_T\big|_{\Gamma_3} = -\,\mathbf{n}^{III}_T\big|_{\Gamma_3} = \begin{pmatrix}\mathbf{n} & 0\end{pmatrix}, && \mathbf{n}^{III}_T\big|_{\Gamma_4} = -\,\mathbf{n}^{IV}_T\big|_{\Gamma_4} = \frac{1}{\sqrt{1+(\lambda^{IV}_{5,6})^2}}\begin{pmatrix}\mathbf{n} & -\lambda^{IV}_{5,6}\end{pmatrix},\\
&\mathbf{n}^{IV}_T\big|_{\Gamma_5} = \begin{pmatrix}\mathbf{0} & -1\end{pmatrix}.
\end{aligned}$$

For $\boldsymbol{\phi} \in C^\infty_0(\mathbb{R}^3 \times [0,\infty))$ we have

$$\int_{[0,\infty)}\int_{\mathbb{R}^3} \mathbf{u}\cdot A_0\partial_t\boldsymbol{\phi} + \mathbf{u}\cdot A_0 A\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} = \sum_{j=I}^{IV}\int_{\Omega^j_\infty} \mathbf{u}\cdot A_0\partial_t\boldsymbol{\phi} + \mathbf{u}\cdot A_0 A\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt}\,.$$

With an integration by parts we obtain

$$\begin{aligned}
\sum_{j=I}^{IV}\int_{\Omega^j_\infty} \mathbf{u}\cdot A_0\partial_t\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} = &-\int_{\Gamma_1}\boldsymbol{\phi}\cdot A^I_0\mathbf{u}^I\,\mathrm{da} + \int_{\Gamma_2}\frac{\lambda^I_{1,2}}{\sqrt{1+(\lambda^I_{1,2})^2}}\boldsymbol{\phi}\cdot A^I_0(\mathbf{u}^{II}-\mathbf{u}^I)\,\mathrm{da}\\
&+\int_{\Gamma_4}\frac{\lambda^{IV}_{5,6}}{\sqrt{1+(\lambda^{IV}_{5,6})^2}}\boldsymbol{\phi}\cdot A^{IV}_0(\mathbf{u}^{IV}-\mathbf{u}^{III})\,\mathrm{da}\\
&-\int_{\Gamma_5}\boldsymbol{\phi}\cdot A^{IV}_0\mathbf{u}^{IV}\,\mathrm{da}
\end{aligned}$$

and also

$$\sum_{j=I}^{IV} \int_{\Omega_\infty^j} \mathbf{u} \cdot A_0 A \boldsymbol{\phi} \,\mathrm{dx}\,\mathrm{dt} = -\int_{\Gamma_2} \frac{1}{\sqrt{1+(\lambda_{1,2}^I)^2}} \boldsymbol{\phi} \cdot F_{K,f}(\mathbf{u}^{II} - \mathbf{u}^{I}) \,\mathrm{da}$$
$$- \int_{\Gamma_3} \boldsymbol{\phi} \cdot F_{K,f}(\mathbf{u}^{III} - \mathbf{u}^{II}) \,\mathrm{da}$$
$$- \int_{\Gamma_4} \frac{1}{\sqrt{1+(\lambda_{5,6}^{IV})^2}} \boldsymbol{\phi} \cdot F_{K,f}(\mathbf{u}^{IV} - \mathbf{u}^{III}) \,\mathrm{da}\,.$$

Using the jump conditions (2.17), we see that $\mathbf{u}(\mathbf{x}, t)$ fulfills the weak formulation (2.15). □

Now we want to calculate $\mathbf{u}^{II}$ and $\mathbf{u}^{III}$. Therefore we define the two bases of $\mathbb{R}^6$

$$\begin{aligned}
\mathcal{B}_0 &= \left(\frac{1}{\sqrt{2}}\mathbf{w}_1^0, \frac{1}{\sqrt{2}}\mathbf{w}_2^0, \mathbf{w}_3, \mathbf{w}_4, \frac{1}{\sqrt{2}}\mathbf{w}_5^0, \frac{1}{\sqrt{2}}\mathbf{w}_6^0\right) \\
&= \left(\frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\tau}_2\\ \boldsymbol{\tau}_1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\tau}_1\\ -\boldsymbol{\tau}_2\end{pmatrix}, \begin{pmatrix}\mathbf{n}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{n}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\tau}_1\\ \boldsymbol{\tau}_2\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}\boldsymbol{\tau}_2\\ -\boldsymbol{\tau}_1\end{pmatrix}\right), \\
\mathcal{B} &= \left(\mathbf{w}_1^I, \mathbf{w}_2^I, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5^{IV}, \mathbf{w}_6^{IV}\right) \\
&= \left(\begin{pmatrix}\sqrt{\mu^I}\boldsymbol{\tau}_2\\ \sqrt{\varepsilon^I}\boldsymbol{\tau}_1\end{pmatrix}, \begin{pmatrix}\sqrt{\mu^I}\boldsymbol{\tau}_1\\ -\sqrt{\varepsilon^I}\boldsymbol{\tau}_2\end{pmatrix}, \begin{pmatrix}\mathbf{n}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\ \mathbf{n}\end{pmatrix}, \begin{pmatrix}\sqrt{\mu^{IV}}\boldsymbol{\tau}_1\\ \sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_2\end{pmatrix}, \begin{pmatrix}\sqrt{\mu^{IV}}\boldsymbol{\tau}_2\\ -\sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_1\end{pmatrix}\right).
\end{aligned}$$

In the case of $\mathcal{B}_0$ we use the material parameters $\varepsilon = \mu = 1$, so we have an orthonormal eigenbasis of the symmetric matrix $F_{K,f}$. Right now it is not quite clear that also $\mathcal{B}$ is a basis of $\mathbb{R}^6$, but we will see this later on. If we now decompose the initial jump $\mathbf{u}^{IV} - \mathbf{u}^I$ into the basis $\mathcal{B}$

$$\mathbf{u}^{IV} - \mathbf{u}^I = \alpha_1\mathbf{w}_1^I + \alpha_2\mathbf{w}_2^I + \alpha_3\mathbf{w}_3 + \alpha_4\mathbf{w}_4 + \alpha_5\mathbf{w}_5^{IV} + \alpha_6\mathbf{w}_6^{IV}, \tag{2.19}$$

we see that, in order to fulfill the jump conditions (2.17), the intermediate values of $\mathbf{u}(\mathbf{x}, t)$ have to be

$$\mathbf{u}^{II} = \mathbf{u}^I + \alpha_1\mathbf{w}_1^I + \alpha_2\mathbf{w}_2^I, \tag{2.20a}$$
$$\mathbf{u}^{III} = \mathbf{u}^{IV} - \alpha_6\mathbf{w}_6^{IV} - \alpha_5\mathbf{w}_5^{IV}. \tag{2.20b}$$

Since $\mathcal{B}$ is not orthonormal, we calculate the decomposition of $\mathbf{u}^{IV} - \mathbf{u}^I$ via projections on the basis $\mathcal{B}_0$. To eventually exchange the basis to $\mathcal{B}$, we need the following identities

$$\mathbf{w}_1^0 = \begin{pmatrix} \boldsymbol{\tau}_2 \\ \boldsymbol{\tau}_1 \end{pmatrix} = \beta_1 \begin{pmatrix} \sqrt{\mu^I}\boldsymbol{\tau}_2 \\ \sqrt{\varepsilon^I}\boldsymbol{\tau}_1 \end{pmatrix} + \beta_6 \begin{pmatrix} \sqrt{\mu^{IV}}\boldsymbol{\tau}_2 \\ -\sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_1 \end{pmatrix} = \beta_1 \mathbf{w}_1^I + \beta_6 \mathbf{w}_6^{IV},$$
$$\mathbf{w}_2^0 = \begin{pmatrix} \boldsymbol{\tau}_1 \\ -\boldsymbol{\tau}_2 \end{pmatrix} = \beta_2 \begin{pmatrix} \sqrt{\mu^I}\boldsymbol{\tau}_1 \\ -\sqrt{\varepsilon^I}\boldsymbol{\tau}_2 \end{pmatrix} + \beta_5 \begin{pmatrix} \sqrt{\mu^{IV}}\boldsymbol{\tau}_1 \\ \sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_2 \end{pmatrix} = \beta_2 \mathbf{w}_2^I + \beta_5 \mathbf{w}_5^{IV},$$
$$\mathbf{w}_5^0 = \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{pmatrix} = \gamma_2 \begin{pmatrix} \sqrt{\mu^I}\boldsymbol{\tau}_1 \\ -\sqrt{\varepsilon^I}\boldsymbol{\tau}_2 \end{pmatrix} + \gamma_5 \begin{pmatrix} \sqrt{\mu^{IV}}\boldsymbol{\tau}_1 \\ \sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_2 \end{pmatrix} = \gamma_2 \mathbf{w}_2^I + \gamma_5 \mathbf{w}_5^{IV},$$
$$\mathbf{w}_6^0 = \begin{pmatrix} \boldsymbol{\tau}_2 \\ -\boldsymbol{\tau}_1 \end{pmatrix} = \gamma_1 \begin{pmatrix} \sqrt{\mu^I}\boldsymbol{\tau}_2 \\ \sqrt{\varepsilon^I}\boldsymbol{\tau}_1 \end{pmatrix} + \gamma_6 \begin{pmatrix} \sqrt{\mu^{IV}}\boldsymbol{\tau}_2 \\ -\sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_1 \end{pmatrix} = \gamma_1 \mathbf{w}_1^I + \gamma_6 \mathbf{w}_6^{IV},$$

with

$$\beta_1 = \beta_2 = \frac{\sqrt{\varepsilon^{IV}} + \sqrt{\mu^{IV}}}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}},$$
$$\beta_5 = \beta_6 = \frac{\sqrt{\varepsilon^I} - \sqrt{\mu^I}}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}},$$
$$\gamma_1 = \gamma_2 = \frac{\sqrt{\varepsilon^{IV}} - \sqrt{\mu^{IV}}}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}},$$
$$\gamma_5 = \gamma_6 = \frac{\sqrt{\varepsilon^I} + \sqrt{\mu^I}}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}}.$$

From here on we know that $\mathcal{B}$ indeed is a basis of $\mathbb{R}^6$, since we can decompose the basisvectors of $\mathcal{B}_0$ into the vectors of $\mathcal{B}$. The next step is to calculate the projections of the initial jump $\mathbf{u}^{IV} - \mathbf{u}^I$ onto the basis $\mathcal{B}_0$. To shorten the notation we introduce the jump $[\mathbf{E}] = \mathbf{E}^{IV} - \mathbf{E}^I$, so $\mathbf{u}^{IV} - \mathbf{u}^I = ([\mathbf{E}], [\mathbf{H}])$. The projections are

$$\big((\mathbf{u}^{IV} - \mathbf{u}^I) \cdot \mathbf{w}_1^0\big)\mathbf{w}_1^0 = \frac{1}{2}\big([\mathbf{E}] \cdot \boldsymbol{\tau}_2 + [\mathbf{H}] \cdot \boldsymbol{\tau}_1\big) \begin{pmatrix} \boldsymbol{\tau}_2 \\ \boldsymbol{\tau}_1 \end{pmatrix},$$
$$\big((\mathbf{u}^{IV} - \mathbf{u}^I) \cdot \mathbf{w}_2^0\big)\mathbf{w}_2^0 = \frac{1}{2}\big([\mathbf{E}] \cdot \boldsymbol{\tau}_1 - [\mathbf{H}] \cdot \boldsymbol{\tau}_2\big) \begin{pmatrix} \boldsymbol{\tau}_1 \\ -\boldsymbol{\tau}_2 \end{pmatrix},$$
$$\big((\mathbf{u}^{IV} - \mathbf{u}^I) \cdot \mathbf{w}_3\big)\mathbf{w}_3 = [\mathbf{E}] \cdot \mathbf{n} \begin{pmatrix} \mathbf{n} \\ \mathbf{0} \end{pmatrix},$$
$$\big((\mathbf{u}^{IV} - \mathbf{u}^I) \cdot \mathbf{w}_4\big)\mathbf{w}_4 = [\mathbf{H}] \cdot \mathbf{n} \begin{pmatrix} \mathbf{0} \\ \mathbf{n} \end{pmatrix},$$
$$\big((\mathbf{u}^{IV} - \mathbf{u}^I) \cdot \mathbf{w}_5^0\big)\mathbf{w}_5^0 = \frac{1}{2}\big([\mathbf{E}] \cdot \boldsymbol{\tau}_1 + [\mathbf{H}] \cdot \boldsymbol{\tau}_2\big) \begin{pmatrix} \boldsymbol{\tau}_1 \\ \boldsymbol{\tau}_2 \end{pmatrix},$$
$$\big((\mathbf{u}^{IV} - \mathbf{u}^I) \cdot \mathbf{w}_6^0\big)\mathbf{w}_6^0 = \frac{1}{2}\big([\mathbf{E}] \cdot \boldsymbol{\tau}_2 - [\mathbf{H}] \cdot \boldsymbol{\tau}_1\big) \begin{pmatrix} \boldsymbol{\tau}_2 \\ -\boldsymbol{\tau}_1 \end{pmatrix}.$$

Now we can decompose the initial jump into the basis $\mathcal{B}$

$$\begin{aligned}
\mathbf{u}^{IV} - \mathbf{u}^{I} =& \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_2 + [\mathbf{H}]\cdot\boldsymbol{\tau}_1\big)\left(\beta_1\mathbf{w}_1^I + \beta_6\mathbf{w}_6^{IV}\right)\\
&+ \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_1 - [\mathbf{H}]\cdot\boldsymbol{\tau}_2\big)\left(\beta_2\mathbf{w}_2^I + \beta_5\mathbf{w}_5^{IV}\right)\\
&+ \big([\mathbf{E}]\cdot\mathbf{n}\big)\mathbf{w}_3 + \big([\mathbf{H}]\cdot\mathbf{n}\big)\mathbf{w}_4\\
&+ \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_1 + [\mathbf{H}]\cdot\boldsymbol{\tau}_2\big)\left(\gamma_2\mathbf{w}_2^I + \gamma_5\mathbf{w}_5^{IV}\right)\\
&+ \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_2 - [\mathbf{H}]\cdot\boldsymbol{\tau}_1\big)\left(\gamma_1\mathbf{w}_1^I + \gamma_6\mathbf{w}_6^{IV}\right).
\end{aligned}$$

For the intermediate values $\mathbf{u}^{II}$ and $\mathbf{u}^{III}$ we obtain the expressions

$$\begin{aligned}
\mathbf{u}^{II} &= \mathbf{u}^I + \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_2(\beta_1+\gamma_1) + [\mathbf{H}]\cdot\boldsymbol{\tau}_1(\beta_1-\gamma_1)\big)\mathbf{w}_1^I\\
&\quad + \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_1(\gamma_2+\beta_2) + [\mathbf{H}]\cdot\boldsymbol{\tau}_2(\gamma_2-\beta_2)\big)\mathbf{w}_2^I,\\
\mathbf{u}^{III} &= \mathbf{u}^{IV} - \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_2(\beta_6+\gamma_6) + [\mathbf{H}]\cdot\boldsymbol{\tau}_1(\beta_6-\gamma_6)\big)\mathbf{w}_6^{IV}\\
&\quad - \frac{1}{2}\big([\mathbf{E}]\cdot\boldsymbol{\tau}_1(\gamma_5+\beta_5) + [\mathbf{H}]\cdot\boldsymbol{\tau}_2(\gamma_5-\beta_5)\big)\mathbf{w}_5^{IV}.
\end{aligned}$$

The numerical upwind flux is defined via the mean value of $\mathbf{u}^{II}$ and $\mathbf{u}^{III}$

$$F^*_{K,f}(\mathbf{u}) = F_{K,f}\frac{\mathbf{u}^{II}+\mathbf{u}^{III}}{2} = F_{K,f}\mathbf{u}^{II} = F_{K,f}\mathbf{u}^{III}.$$

In comparison to the upwind flux the so called central flux is defined via the mean value of the initial values $F^*_{K,f,central}(\mathbf{u}) = \frac{1}{2}F_{K,f}(\mathbf{u}^I + \mathbf{u}^{IV})$. We proceed with our evaluation of the upwind flux and state that

$$\begin{aligned}
F_{K,f}\mathbf{w}_1^I &= -\begin{pmatrix} -\sqrt{\varepsilon^I}\boldsymbol{\tau}_2\\ -\sqrt{\mu^I}\boldsymbol{\tau}_1\end{pmatrix}, & F_{K,f}\mathbf{w}_2^I &= -\begin{pmatrix} -\sqrt{\varepsilon^I}\boldsymbol{\tau}_1\\ \sqrt{\mu^I}\boldsymbol{\tau}_2\end{pmatrix},\\
F_{K,f}\mathbf{w}_5^{IV} &= \begin{pmatrix} \sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_1\\ \sqrt{\mu^{IV}}\boldsymbol{\tau}_2\end{pmatrix}, & F_{K,f}\mathbf{w}_6^{IV} &= \begin{pmatrix} \sqrt{\varepsilon^{IV}}\boldsymbol{\tau}_2\\ -\sqrt{\mu^{IV}}\boldsymbol{\tau}_1\end{pmatrix},
\end{aligned}$$

as well as

$$\begin{aligned}
\beta_1 - \gamma_1 &= \frac{2\sqrt{\mu^{IV}}}{\sqrt{\varepsilon^I\mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}}, & \beta_1 + \gamma_1 &= \frac{2\sqrt{\varepsilon^{IV}}}{\sqrt{\varepsilon^I\mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}},\\
\beta_6 - \gamma_6 &= -\frac{2\sqrt{\mu^{I}}}{\sqrt{\varepsilon^I\mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}}, & \beta_6 + \gamma_6 &= \frac{2\sqrt{\varepsilon^{I}}}{\sqrt{\varepsilon^I\mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}}.
\end{aligned}$$

To express the upwind flux in terms of the normal $\mathbf{n}$, we also need the following identities

$$\mathbf{n} \times \mathbf{E} = (\mathbf{E} \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_2 - (\mathbf{E} \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_1, \tag{2.21a}$$

$$\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = -(\mathbf{E} \cdot \boldsymbol{\tau}_1)\boldsymbol{\tau}_1 - (\mathbf{E} \cdot \boldsymbol{\tau}_2)\boldsymbol{\tau}_2. \tag{2.21b}$$

Finally, we obtain the result

$$\begin{aligned}
F^*_{K,f}(\mathbf{u}) = {} & F_{K,f}\mathbf{u}^I + F_{K,f}\frac{\mathbf{u}^{IV} - \mathbf{u}^I}{2} \\
& + \frac{1}{2}[\mathbf{E}] \cdot \boldsymbol{\tau}_2 \frac{1}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}} \begin{pmatrix} -2\sqrt{\varepsilon^I \varepsilon^{IV}}\boldsymbol{\tau}_2 \\ \left(\sqrt{\varepsilon^I \mu^{IV}} - \sqrt{\varepsilon^{IV}\mu^I}\right)\boldsymbol{\tau}_1 \end{pmatrix} \\
& + \frac{1}{2}[\mathbf{H}] \cdot \boldsymbol{\tau}_1 \frac{1}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}} \begin{pmatrix} \left(-\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}\right)\boldsymbol{\tau}_2 \\ -2\sqrt{\mu^I \mu^{IV}}\boldsymbol{\tau}_1 \end{pmatrix} \\
& + \frac{1}{2}[\mathbf{E}] \cdot \boldsymbol{\tau}_1 \frac{1}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}} \begin{pmatrix} -2\sqrt{\varepsilon^I \varepsilon^{IV}}\boldsymbol{\tau}_1 \\ \left(-\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}\right)\boldsymbol{\tau}_2 \end{pmatrix} \\
& + \frac{1}{2}[\mathbf{H}] \cdot \boldsymbol{\tau}_2 \frac{1}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}} \begin{pmatrix} \left(\sqrt{\varepsilon^I \mu^{IV}} - \sqrt{\varepsilon^{IV}\mu^I}\right)\boldsymbol{\tau}_1 \\ -2\sqrt{\mu^I \mu^{IV}}\boldsymbol{\tau}_2 \end{pmatrix} \\
= {} & F_{K,f}\mathbf{u}^I + \frac{1}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}} \begin{pmatrix} \sqrt{\varepsilon^I \varepsilon^{IV}}\mathbf{n} \times (\mathbf{n} \times [\mathbf{E}]) \\ \sqrt{\mu^I \mu^{IV}}\mathbf{n} \times (\mathbf{n} \times [\mathbf{H}]) \end{pmatrix} \\
& + \frac{1}{\sqrt{\varepsilon^I \mu^{IV}} + \sqrt{\varepsilon^{IV}\mu^I}} \begin{pmatrix} -\sqrt{\varepsilon^I \mu^{IV}}\mathbf{n} \times [\mathbf{H}] \\ \sqrt{\varepsilon^{IV}\mu^I}\mathbf{n} \times [\mathbf{E}] \end{pmatrix}.
\end{aligned}$$

We formulate this in a lemma.

Lemma 2.3.5:
Let $K \in \mathcal{T}_h$ be a cell with face $f \in \mathcal{F}_K$ and material parameters ε_K and μ_K and let $K_f \in \mathcal{T}_h$ be the next neighbour cell of K in direction of f with material parameters ε_{K_f} and μ_{K_f}. Let $\mathbf{u}_h = (\mathbf{E}_h, \mathbf{H}_h) \in \mathrm{V}_h^p$ be a piecewise polynomial function. Then the upwind flux $F^*_{K,f}(\mathbf{u}_h)$ of $\mathbf{u}_h$ on the face f is given by

$$\begin{aligned}
F^*_{K,f}(\mathbf{u}_h) = {} & F_{K,f}\mathbf{u}_K + \begin{pmatrix} -\alpha_{K,f}\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f} \\ \beta_{K,f}\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f} \end{pmatrix} \\
& + \begin{pmatrix} \gamma_f \mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}) \\ \delta_f \mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}) \end{pmatrix},
\end{aligned} \tag{2.22}$$

where

$$\alpha_{K,f} = \frac{\sqrt{\varepsilon_K \mu_{K_f}}}{\sqrt{\varepsilon_K \mu_{K_f}} + \sqrt{\varepsilon_{K_f} \mu_K}}, \qquad \beta_{K,f} = \frac{\sqrt{\varepsilon_{K_f} \mu_K}}{\sqrt{\varepsilon_K \mu_{K_f}} + \sqrt{\varepsilon_{K_f} \mu_K}},$$
$$\gamma_f = \frac{\sqrt{\varepsilon_K \varepsilon_{K_f}}}{\sqrt{\varepsilon_K \mu_{K_f}} + \sqrt{\varepsilon_{K_f} \mu_K}}, \qquad \delta_f = \frac{\sqrt{\mu_K \mu_{K_f}}}{\sqrt{\varepsilon_K \mu_{K_f}} + \sqrt{\varepsilon_{K_f} \mu_K}}.$$

Here, e.g. $[\mathbf{E}_h]_{K,f} = \mathbf{E}_{K_f} - \mathbf{E}_K$ *denotes the jump of* $\mathbf{E}_h$ *on the face* f *and* $\mathbf{E}_K$ *and* $\mathbf{E}_{K_f}$ *denote the left- and right-hand side value of* $\mathbf{E}_h$ *on* f*, respectively. The flux* $F_{K,f}$ *was defined in* (2.13)*.*

Proof: See calculations above. □

The parameters $\alpha_{K,f}$ and $\beta_{K,f}$ fulfill the properties

$$\alpha_{K,f} = \beta_{K_f,f}, \qquad \alpha_{K,f} + \alpha_{K_f,f} = 1, \tag{2.23a}$$
$$\alpha_{K_f,f} = \beta_{K,f}, \qquad \beta_{K,f} + \beta_{K_f,f} = 1. \tag{2.23b}$$

In the special case of constant materials, i.e. $\varepsilon_K = \varepsilon_{K_f}$ and $\mu_K = \mu_{K_f}$, we have $\alpha_{K,f} = \beta_{K,f} = \frac{1}{2}$. In especially on boundary faces this is the case, as we will see later on in Lemma 2.3.6. Although, we do not focus on the central flux, we like to mention that we obtain the central flux from (2.22), if we set $\alpha_{K,f} = \beta_{K,f} = \frac{1}{2}$ and $\gamma_f = \delta_f = 0$. Later on in the error calculations we will divide by γ_f and δ_f and therefore take an advantage of the upwind flux over the central flux.

2.3.5 Upwind flux on the boundary

The previous lemma contains a formula for the upwind flux on inner faces. To construct the discrete operator A_h, we also need to make a choice for the discrete flux on boundary faces $f \subset \partial\Omega$. The idea behind that choice will be quite analogous to the construction via Riemann problem on inner faces.

$\mathbf{n} \times \mathbf{E} = \mathbf{g}_E$**:** First, we consider the boundary condition $\mathbf{n} \times \mathbf{E} = \mathbf{g}_E$. We restrict the solution $\mathbf{u}(\mathbf{x}, t)$ in (2.16) to the region $\{(\mathbf{x}, t) \in \mathbb{R}^3 \times [0, \infty) : \ \mathbf{x} \cdot \mathbf{n} < 0\}$, use the same jump behaviour as before

$$\mathbf{u}^{II} = \mathbf{u}^I + \alpha_1 \mathbf{w}_1^I + \alpha_2 \mathbf{w}_2^I \tag{2.24}$$

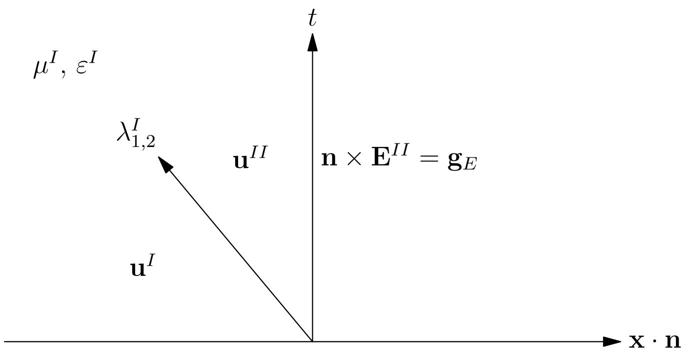


Figure 2.2: Construction of bounary fluxes.

and enforce the condition

$$\mathbf{n} \times \mathbf{E}^{II} = \mathbf{g}_E \tag{2.25}$$

to determine the parameters α_j. Here, $\mathbf{g}_E$ is an arbitrary tangential vector to the subspace f. Equations (2.24) and (2.25) together yield

$$\mathbf{g}_E = \mathbf{n} \times \mathbf{E}^I - \alpha_1 \sqrt{\mu^I} \boldsymbol{\tau}_1 + \alpha_2 \sqrt{\mu^I} \boldsymbol{\tau}_2.$$

A multiplication with the tangential vectors $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and the identity (2.21a) lead to

$$\alpha_1 = -\frac{1}{\sqrt{\mu^I}}(\mathbf{g}_E \cdot \boldsymbol{\tau}_1 + \mathbf{E}^I \cdot \boldsymbol{\tau}_2), \qquad \alpha_2 = \frac{1}{\sqrt{\mu^I}}(\mathbf{g}_E \cdot \boldsymbol{\tau}_2 - \mathbf{E}^I \cdot \boldsymbol{\tau}_1).$$

For the value of $\mathbf{u}^{II}$ and the discrete flux $F^*_{K,f}(\mathbf{u}) + G^*_{E,f}(\mathbf{g}_E) = F_{K,f}\mathbf{u}^{II}$ we obtain

$$\mathbf{u}^{II} = \mathbf{u}^I + \begin{pmatrix} \mathbf{n} \times (\mathbf{n} \times \mathbf{E}^I) \\ \sqrt{\frac{\varepsilon^I}{\mu^I}} \mathbf{n} \times \mathbf{E}^I \end{pmatrix} - \begin{pmatrix} \mathbf{n} \times \mathbf{g}_E \\ \sqrt{\frac{\varepsilon^I}{\mu^I}} \mathbf{g}_E \end{pmatrix},$$

$$F^*_{K,f}(\mathbf{u}) = F_{K,f}\mathbf{u}^I - \begin{pmatrix} \sqrt{\frac{\varepsilon^I}{\mu^I}} \mathbf{n} \times (\mathbf{n} \times \mathbf{E}^I) \\ \mathbf{n} \times \mathbf{E}^I \end{pmatrix}, \tag{2.26a}$$

$$G^*_{E,f}(\mathbf{g}_E) = \begin{pmatrix} \sqrt{\frac{\varepsilon^I}{\mu^I}} \mathbf{n} \times \mathbf{g}_E \\ \mathbf{g}_E \end{pmatrix}. \tag{2.26b}$$

Sometimes it is more convenient to work with the same definition of the upwind flux for inner and for boundary faces. Introducing a virtual cell K_f next to the boundary cell K, we can do this. With the definitions

$$\mathbf{n} \times \mathbf{E}_{K_f} = -\mathbf{n} \times \mathbf{E}_K, \qquad \mathbf{n} \times \mathbf{H}_{K_f} = \mathbf{n} \times \mathbf{H}_K,$$
$$\varepsilon_{K_f} = \varepsilon_K, \qquad \mu_{K_f} = \mu_K$$

on this virtual cell the inner upwind flux (2.22) and the boundary flux (2.26a) will coincide. Note, that these virtual definitions are not unique. Other virtual definitions might produce the same flux.

$\mathbf{n}\times\mathbf{H} = \mathbf{g}_H$**:** We do the same construction for the boundary condition $\mathbf{n}\times\mathbf{H} = \mathbf{g}_H$. The parameters α_j are determined by the two equations

$$\mathbf{u}^{II} = \mathbf{u}^I + \alpha_1 \mathbf{w}_1^I + \alpha_2 \mathbf{w}_2^I, \qquad \mathbf{n}\times\mathbf{H}^{II} = \mathbf{g}_H,$$

where we obtain the explicit values

$$\alpha_1 = \frac{1}{\sqrt{\varepsilon^I}}(\mathbf{g}_H\cdot\boldsymbol{\tau}_2 - \mathbf{H}^I\cdot\boldsymbol{\tau}_1), \qquad \alpha_2 = \frac{1}{\sqrt{\varepsilon^I}}(\mathbf{g}_H\cdot\boldsymbol{\tau}_1 + \mathbf{H}^I\cdot\boldsymbol{\tau}_2).$$

For the values of $\mathbf{u}^{II}$ and the discrete flux $F^*_{K,f}(\mathbf{u}) + G^*_{H,f}(\mathbf{g}_H) = F_{K,f}\mathbf{u}^{II}$ we obtain

$$\begin{aligned}
\mathbf{u}^{II} &= \mathbf{u}^I + \begin{pmatrix} -\sqrt{\frac{\mu^I}{\varepsilon^I}}\mathbf{n}\times\mathbf{H}^I \\ \mathbf{n}\times(\mathbf{n}\times\mathbf{H}^I)\end{pmatrix} + \begin{pmatrix}\sqrt{\frac{\mu^I}{\varepsilon^I}}\mathbf{g}_H \\ -\mathbf{n}\times\mathbf{g}_H\end{pmatrix},\\
F^*_{K,f}(\mathbf{u}) &= F_{K,f}\mathbf{u}^I + \begin{pmatrix}\mathbf{n}\times\mathbf{H}^I \\ -\sqrt{\frac{\mu^I}{\varepsilon^I}}\mathbf{n}\times(\mathbf{n}\times\mathbf{H}^I)\end{pmatrix},\\
G^*_{H,f}(\mathbf{g}_H) &= \begin{pmatrix}-\mathbf{g}_H \\ \sqrt{\frac{\mu^I}{\varepsilon^I}}\mathbf{n}\times\mathbf{g}_H\end{pmatrix},
\end{aligned}$$

as well as the virtual definitions

$$\begin{aligned}
\mathbf{n}\times\mathbf{E}_{K_f} &= \mathbf{n}\times\mathbf{E}_K, & \mathbf{n}\times\mathbf{H}_{K_f} &= -\mathbf{n}\times\mathbf{H}_K,\\
\varepsilon_{K_f} &= \varepsilon_K, & \mu_{K_f} &= \mu_K.
\end{aligned}$$

$\mathbf{n}\times\mathbf{H} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) = \mathbf{g}_I$**:** For the impedance boundary condition we use the two equations

$$\mathbf{u}^{II} = \mathbf{u}^I + \alpha_1\mathbf{w}_1^I + \alpha_2\mathbf{w}_2^I, \qquad \mathbf{n}\times\mathbf{H}^{II} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}^{II}) = \mathbf{g}_I$$

to obtain the explicit parameter values

$$\begin{aligned}
\alpha_1 &= \frac{1}{\sqrt{\varepsilon^I} + Z\sqrt{\mu^I}}(\mathbf{g}_I\cdot\boldsymbol{\tau}_2 - \mathbf{H}^I\cdot\boldsymbol{\tau}_1 - Z\mathbf{E}^I\cdot\boldsymbol{\tau}_2),\\
\alpha_2 &= \frac{1}{\sqrt{\varepsilon^I} + Z\sqrt{\mu^I}}(\mathbf{g}_I\cdot\boldsymbol{\tau}_1 + \mathbf{H}^I\cdot\boldsymbol{\tau}_2 - Z\mathbf{E}^I\cdot\boldsymbol{\tau}_1).
\end{aligned}$$

They lead to the values of $\mathbf{u}^{II}$ and the discrete flux $F^*_{K,f}(\mathbf{u}) + G^*_{I,f}(\mathbf{g}_I) = F_{K,f}\mathbf{u}^{II}$ given by

$$\begin{aligned}
\mathbf{u}^{II} = \mathbf{u}^I &+ \begin{pmatrix} \frac{Z\sqrt{\mu^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times(\mathbf{n}\times\mathbf{E}^I) \\ \frac{Z\sqrt{\varepsilon^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times\mathbf{E}^I \end{pmatrix} + \begin{pmatrix} -\frac{\sqrt{\mu^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times\mathbf{H}^I \\ \frac{\sqrt{\varepsilon^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times(\mathbf{n}\times\mathbf{H}^I) \end{pmatrix} \\
&+ \begin{pmatrix} \frac{\sqrt{\mu^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{g}_I \\ -\frac{\sqrt{\varepsilon^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times\mathbf{g}_I \end{pmatrix}, \\
F^*_{K,f}(\mathbf{u}) = F_{K,f}\mathbf{u}^I &- \begin{pmatrix} \frac{Z\sqrt{\varepsilon^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times(\mathbf{n}\times\mathbf{E}^I) \\ \frac{Z\sqrt{\mu^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times\mathbf{E}^I \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{\varepsilon^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times\mathbf{H}^I \\ -\frac{\sqrt{\mu^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times(\mathbf{n}\times\mathbf{H}^I) \end{pmatrix}, \\
G^*_{I,f}(\mathbf{g}_I) &= \begin{pmatrix} -\frac{\sqrt{\varepsilon^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{g}_I \\ \frac{\sqrt{\mu^I}}{\sqrt{\varepsilon^I}+Z\sqrt{\mu^I}}\mathbf{n}\times\mathbf{g}_I \end{pmatrix},
\end{aligned}$$

as well as the virtual definitions

$$\begin{aligned}
\mathbf{n}\times\mathbf{E}_{K_f} &= -\frac{\sqrt{\mu_K}Z - \sqrt{\varepsilon_K}}{\sqrt{\mu_K}Z + \sqrt{\varepsilon_K}}\mathbf{n}\times\mathbf{E}_K, & \mathbf{n}\times\mathbf{H}_{K_f} &= \frac{\sqrt{\mu_K}Z - \sqrt{\varepsilon_K}}{\sqrt{\mu_K}Z + \sqrt{\varepsilon_K}}\mathbf{n}\times\mathbf{H}_K, \\
\varepsilon_{K_f} &= \varepsilon_K, & \mu_{K_f} &= \mu_K.
\end{aligned}$$

Again, we summarize these results in a lemma.

Lemma 2.3.6:
Let $K \in \mathcal{T}_h$ be a cell with boundary face $f \in \mathcal{F}_K$, $f \subset \partial\Omega$ and material parameters ε_K, μ_K, and optionally Z_f. Let $\mathbf{u}_h = (\mathbf{E}_h, \mathbf{H}_h) \in \mathrm{V}_h^p$ be a piecewise polynomial function and let $\mathbf{g}_j \in \mathrm{L}_2(f)$, $j \in \{E, H, I\}$ be a tangential field on f. Then the boundary upwind flux $F^*_{K,f}(\mathbf{u}_h) + G^*_{j,f}(\mathbf{g}_j)$ of $\mathbf{u}_h$ on the face f is given by (2.22) together with the definitions

$$\begin{aligned}
\mathbf{n}_{K,f}\times\mathbf{E}_{K_f} &= -\mathbf{n}_{K,f}\times\mathbf{E}_K, & \mathbf{n}_{K,f}\times\mathbf{H}_{K_f} &= \mathbf{n}_{K,f}\times\mathbf{H}_K, \\
\varepsilon_{K_f} &= \varepsilon_K, & \mu_{K_f} &= \mu_K, \\
G^*_{E,f}(\mathbf{g}_E) &= \begin{pmatrix} 2\gamma_f \mathbf{n}_{K,f}\times\mathbf{g}_E \\ \mathbf{g}_E \end{pmatrix}
\end{aligned}$$

in case of the boundary condition $\mathbf{n} \times \mathbf{E} = \mathbf{g}_E$ *on* f,

$$\begin{aligned} \mathbf{n}_{K,f} \times \mathbf{E}_{K_f} &= \mathbf{n}_{K,f} \times \mathbf{E}_K, & \mathbf{n}_{K,f} \times \mathbf{H}_{K_f} &= -\mathbf{n}_{K,f} \times \mathbf{H}_K, \\ \varepsilon_{K_f} &= \varepsilon_K, & \mu_{K_f} &= \mu_K, \\ G^*_{H,f}(\mathbf{g}_H) &= \begin{pmatrix} -\mathbf{g}_H \\ 2\delta_f \mathbf{n}_{K,f} \times \mathbf{g}_H \end{pmatrix} \end{aligned}$$

in case of $\mathbf{n} \times \mathbf{H} = \mathbf{g}_H$, *and*

$$\begin{aligned} \mathbf{n}_{K,f} \times \mathbf{E}_{K_f} &= (\chi_f - Z_f\kappa_f)\mathbf{n}_{K,f} \times \mathbf{E}_K, & \mathbf{n}_{K,f} \times \mathbf{H}_{K_f} &= (Z_f\kappa_f - \chi_f)\mathbf{n}_{K,f} \times \mathbf{H}_K, \\ \varepsilon_{K_f} &= \varepsilon_K, & \mu_{K_f} &= \mu_K, \\ G^*_{I,f}(\mathbf{g}_I) &= \begin{pmatrix} -\chi_f \mathbf{g}_I \\ \kappa_f \mathbf{n}_{K,f} \times \mathbf{g}_I \end{pmatrix} \end{aligned}$$

in case of $\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{g}_I$, *where*

$$\chi_f = \frac{\sqrt{\varepsilon_K}}{\sqrt{\varepsilon_K} + Z_f\sqrt{\mu_K}}, \qquad \kappa_f = \frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K} + Z_f\sqrt{\mu_K}}.$$

Later on, we also will use the notation h_{K_f} for the diameter of a virtual cell K_f at the boundary face f. This is to be understood in the natural way $h_{K_f} = h_K = \operatorname{diam} K$.

Proof: See calculations above. □

We note, that we have $\alpha_{K,f} = \beta_{K,f} = \frac{1}{2}$ on boundary faces, as well as

$$\begin{aligned} 2\chi_f\delta_f &= \kappa_f, \\ 2\kappa_f\gamma_f &= \chi_f, \\ \chi_f + Z_f\kappa_f &= 1 \end{aligned} \tag{2.27}$$

on faces $f \in \partial\Omega_I$ of the impedance boundary. The jumps of the discontinuous fields $\mathbf{E}_h$ and $\mathbf{H}_h$ on $\partial\Omega_I$ can be calculated to be $\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f} = -2Z_f\kappa_f\mathbf{n}_{K,f} \times \mathbf{E}_K$ and $\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f} = -2\chi_f\mathbf{n}_{K,f} \times \mathbf{H}_K$.

2.3.6 Construction of the discrete operator A_h (continued)

Now, we are able to handle the boundary integrals in (2.12) by the replacement $F_{K,f}(\mathbf{u}) \to F^*_{K,f}(\mathbf{u}_h)$ (see Lemma 2.3.5) on inner faces and $F_{K,f}(\mathbf{u}) \to F^*_{K,f}(\mathbf{u}_h) + G^*_{j,f}(\mathbf{g}_j)$ (see Lemma 2.3.6) on boundary faces. For $(\mathbf{E}_h, \mathbf{H}_h), (\boldsymbol{\psi}_h, \boldsymbol{\varphi}_h) \in \mathrm{V}_h^p$ we define

$$\begin{aligned}
\left(A_h \begin{pmatrix} \mathbf{E}_h \\ \mathbf{H}_h \end{pmatrix}, \begin{pmatrix} \boldsymbol{\psi}_h \\ \boldsymbol{\varphi}_h \end{pmatrix}\right)_{\mathrm{V}} &= \sum_{K \in \mathcal{T}_h} -(\mathbf{H}_h, \nabla \times \boldsymbol{\psi}_h)_{0,K} + (\mathbf{E}_h, \nabla \times \boldsymbol{\varphi}_h)_{0,K} \\
&\qquad + (\sigma_K \mathbf{E}_h, \boldsymbol{\psi}_h)_{0,K} \\
&\quad + \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} \left(F^*_{K,f}(\mathbf{u}_h), \begin{pmatrix} \boldsymbol{\psi}_K \\ \boldsymbol{\varphi}_K \end{pmatrix}\right)_{0,f} \\
&= \sum_{K \in \mathcal{T}_h} -(\nabla \times \mathbf{H}_h, \boldsymbol{\psi}_h)_{0,K} + (\nabla \times \mathbf{E}_h, \boldsymbol{\varphi}_h)_{0,K} \\
&\qquad + (\sigma_K \mathbf{E}_h, \boldsymbol{\psi}_h)_{0,K} \\
&\quad + \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} -\alpha_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \boldsymbol{\psi}_K)_{0,f} \\
&\qquad\qquad + \beta_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \boldsymbol{\varphi}_K)_{0,f} \\
&\qquad\qquad + \gamma_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \boldsymbol{\psi}_K\big)_{0,f} \\
&\qquad\qquad + \delta_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \boldsymbol{\varphi}_K\big)_{0,f}
\end{aligned} \tag{2.28}$$

and thereby the linear operator $A_h : \mathrm{V}_h^p \to \mathrm{V}_h^p$. The boundary values are handled in a vector $\mathbf{G}_h \in \mathrm{V}_h^p$ defined by

$$\begin{aligned}
\left(\mathbf{G}_h, \begin{pmatrix} \boldsymbol{\psi}_h \\ \boldsymbol{\varphi}_h \end{pmatrix}\right)_{\mathrm{V}} &= \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^E} \left(G^*_{E,f}(\mathbf{g}_E), \begin{pmatrix} \boldsymbol{\psi}_K \\ \boldsymbol{\varphi}_K \end{pmatrix}\right)_{0,f} \\
&\quad + \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^H} \left(G^*_{H,f}(\mathbf{g}_H), \begin{pmatrix} \boldsymbol{\psi}_K \\ \boldsymbol{\varphi}_K \end{pmatrix}\right)_{0,f} \\
&\quad + \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^I} \left(G^*_{I,f}(\mathbf{g}_I), \begin{pmatrix} \boldsymbol{\psi}_K \\ \boldsymbol{\varphi}_K \end{pmatrix}\right)_{0,f}.
\end{aligned} \tag{2.29}$$

With A_h and $\mathbf{G}_h$ defined, we now can formulate the semi-discrete (discrete in space) ODE for the approximate solution $\mathbf{u}_h(t) = (\mathbf{E}_h(t), \mathbf{H}_h(t))$.

The semi-discrete problem *Find* $\mathbf{u}_h \in \mathrm{C}^1([0,\infty), \mathrm{V}_h^p)$, *such that*

$$\partial_t \mathbf{u}_h(t) + A_h \mathbf{u}_h(t) = \mathbf{f}_h(t) - \mathbf{G}_h(t), \tag{2.30a}$$

$$\mathbf{u}_h(0) = \Pi_h^p \mathbf{u}_0. \tag{2.30b}$$

Here Π_h^p *denotes the* $\mathrm{L}_2(\Omega)$*-projection on the space* V_h^p. *The right-hand side term* $\mathbf{f}_h(t) = \Pi_h^p \mathbf{f}(t)$ *is also defined via* $\mathrm{L}_2(\Omega)$*-projection.*

The system (2.30) has a well known unique solution (see [RR04, Theorem 12.14], [Wal86, Kap. III, §16, Satz III])

$$\mathbf{u}_h(t) = \exp(-A_h t)\Pi_h^p \mathbf{u}_0 + \int_0^t \exp\big(-A_h(t-s)\big)\big(\mathbf{f}_h(s) - \mathbf{G}_h(s)\big)\,\mathrm{d}s\,. \tag{2.31}$$

Note, that $\mathbf{f}_h(t)$ and $\mathbf{G}_h(t)$ are continuous in time due to our assumptions on $\mathbf{f}$ and $\mathbf{g}_j$ in (2.6) and (2.7), respectively.

Remark 2.3.7:
Apart from the numerical tests in Chapter 4, we will not discretize the time variable. In this matter we like to refer to Pažur's doctoral thesis [Paž13] devoted to time integration of linear Maxwell's equations.

2.4 Convergence of the semi-discrete solution

Under an additional regularity assumption on the solution to Maxwell's system (2.1), namely $\mathbf{u}(t)|_{\Lambda_j} \in \mathrm{H}^s(\Lambda_j)^6$, $s > \frac{1}{2}$, we will show convergence of the semi-discrete solution to the continuous solution as the mesh parameter h tends to zero. Our aim is the following result on a finite time interval $(0,T)$.

Theorem 2.4.1:
Let $\mathbf{u}(t)$ *be the solution to Maxwell's system* (2.1), *with an additional regularity* $\mathbf{u}|_{\Lambda_j \times [0,T]} \in \mathrm{C}([0,T], \mathrm{H}^s(\Lambda_j)^6)$ *for every* j *and a* $s > \frac{1}{2}$ *and let* $\mathbf{u}_h(t)$ *be the approximate solution given by* (2.31), *where* $p = \bar{s} - 1$ *and* $\bar{s}$ *denotes the rounded up integer part of* s *as in Lemma 2.3.1. Further assume on the regularity of the right-hand side* $\mathbf{f}|_{\Lambda_j \times [0,T]} \in \mathrm{C}([0,T], \mathrm{H}^{s-\frac{1}{2}}(\Lambda_j)^6)$ *for every* j. *Then we have the following estimate*

$$\|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{\mathrm{L}_2((0,T),\mathrm{V})} \leq C h^{s-\frac{1}{2}}.$$

The constant C *is independent of* h *and will be specified in* (2.35).

In [VV03] Vila and Villedieu presented an error estimate for an explicit in time finite volume scheme for first order symmetric systems without boundary conditions. We used their approach as a basis for the following proof of Theorem 2.4.1.

Remark 2.4.2 (on the regularity assumption in Theorem 2.4.1)**:**
Regarding the existence and uniqueness result in Lemma 2.2.4, we have a regularity $\mathbf{u} \in \mathrm{C}^1([0,\infty),\mathrm{V}) \cap \mathrm{C}([0,\infty),\mathrm{H}(\mathrm{curl},\Omega,\partial\Omega))$ of the solution $\mathbf{u}$ to Maxwell's system (2.1). For the convergence analysis, we need traces $\mathbf{u}|_f$ on faces f to be in $\mathrm{L}_2(f)^6$ and we need a positive power of h_K in the projection error estimate in Lemma 2.3.2. Therefore, we need local H^s-regularity of $\mathbf{u}(t)$ with $s > \frac{1}{2}$. Here in this work, we will not answer the question, under which assumptions the solution $\mathbf{u}$ actually has the desired regularity. We refer to [Mon03, Sec. 3.8] as a possible starting point, though.

Since the proof of Theorem 2.4.1 is long and technical, we split it into several lemmata, beginning with the following.

Lemma 2.4.3:
Under the assumptions of Theorem 2.4.1 we have the following error estimate

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathrm{L}_2((0,T),\mathrm{V})} \\ &\le 4\int_0^T \big(\mathbf{G}_h(t) + A_h\mathbf{u}_h(t) - A\mathbf{u}(t), \mathbf{u}(t)-\mathbf{u}_h(t)\big)_{\mathrm{V}}\eta_T(t)\,\mathrm{dt} \\ &\quad + 4\int_0^T \eta_T^2(t)\|\mathbf{f}(t)-\mathbf{f}_h(t)\|^2_{\mathrm{V}}\,\mathrm{dt} + 2T\|\mathbf{u}_0 - \Pi_h^p\mathbf{u}_0\|^2_{\mathrm{V}},\end{aligned} \tag{2.32}$$

where the time-dependent function η_T is defined by $\eta_T(t) = T - t$.

Proof: A straight forward calculation yields

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|^2_{\mathrm{L}_2((0,T),\mathrm{V})} \\ &= -\int_0^T \|\mathbf{u}(t)-\mathbf{u}_h(t)\|^2_{\mathrm{V}}\partial_t\eta_T(t)\,\mathrm{dt} \\ &= \int_0^T \partial_t\|\mathbf{u}(t)-\mathbf{u}_h(t)\|^2_{\mathrm{V}}\eta_T(t)\,\mathrm{dt} + T\|\mathbf{u}(0)-\mathbf{u}_h(0)\|^2_{\mathrm{V}} \\ &= 2\int_0^T \big(\partial_t\mathbf{u}(t) - \partial_t\mathbf{u}_h(t), \mathbf{u}(t)-\mathbf{u}_h(t)\big)_{\mathrm{V}}\eta_T(t)\,\mathrm{dt} + T\|\mathbf{u}_0 - \Pi_h^p\mathbf{u}_0\|^2_{\mathrm{V}}\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^T \big(\mathbf{f}(t) - \mathbf{f}_h(t) + \mathbf{G}_h(t) + A_h\mathbf{u}_h(t) - A\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_h(t)\big)_{\mathrm{V}}\eta_T(t)\,\mathrm{dt} \\
&\quad + T\|\mathbf{u}_0 - \Pi_h^p\mathbf{u}_0\|_{\mathrm{V}}^2.
\end{aligned}
$$

Now we use Cauchy-Schwarz' inequality and

$$
2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon} \text{ for } a, b \in \mathbb{R} \text{ and } \epsilon > 0 \tag{2.33}
$$

to obtain

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{L}_2((0,T),\mathrm{V})}^2 \leq{}& 2\int_0^T \big(\mathbf{G}_h(t) + A_h\mathbf{u}_h(t) - A\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_h(t)\big)_{\mathrm{V}}\eta_T(t)\,\mathrm{dt} \\
&+ \frac{1}{2}\int_0^T \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{\mathrm{V}}^2\,\mathrm{dt} + 2\int_0^T \eta_T^2(t)\|\mathbf{f}(t) - \mathbf{f}_h(t)\|_{\mathrm{V}}^2\,\mathrm{dt} \\
&+ T\|\mathbf{u}_0 - \Pi_h^p\mathbf{u}_0\|_{\mathrm{V}}^2.
\end{aligned}
$$

Collecting the error on the left-hand side, the desired estimate (2.32) is obtained. □

The last two summands in (2.32) contain projection errors, which can be handled with the projection error estimate stated in Lemma 2.3.1. Applied to the terms in (2.32) it yields

$$
\|\mathbf{f}(t) - \mathbf{f}_h(t)\|_{\mathrm{V}}^2 \leq c_{\mathrm{cell}}^2 h^{2s-1}|\mathbf{f}(t)|_{s-\frac{1}{2},\mathrm{V},\Lambda}^2, \tag{2.34a}
$$

$$
\|\mathbf{u}_0 - \Pi_h^p\mathbf{u}_0\|_{\mathrm{V}}^2 \leq c_{\mathrm{cell}}^2 h^{2s}|\mathbf{u}_0|_{s,\mathrm{V},\Lambda}^2. \tag{2.34b}
$$

Next, we focus on the term $\big(\mathbf{G}_h(t) + A_h\mathbf{u}_h(t) - A\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_h(t)\big)_{\mathrm{V}}$ in (2.32). For the discrete operator A_h we have the following positivity result.

Lemma 2.4.4:
The operator A_h defined in (2.28) is positive semi-definite, i.e. for all $\mathbf{u}_h \in \mathrm{V}_h^p$

$$
\begin{aligned}
(A_h\mathbf{u}_h, \mathbf{u}_h)_{\mathrm{V}} ={}& \sum_{K\in\mathcal{T}_h} \sigma_K(\mathbf{E}_h, \mathbf{E}_h)_{0,K} \\
&+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\circ} \frac{\gamma_f}{2}\|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{\delta_f}{2}\|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2 \\
&+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^E} 2\gamma_f\|\mathbf{n}_{K,f} \times \mathbf{E}_K\|_{0,f}^2 \\
&+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^H} 2\delta_f\|\mathbf{n}_{K,f} \times \mathbf{H}_K\|_{0,f}^2
\end{aligned}
$$

$$+ \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^I} Z_f \chi_f \|\mathbf{n}_{K,f} \times \mathbf{E}_K\|_{0,f}^2 + \kappa_f \|\mathbf{n}_{K,f} \times \mathbf{H}_K\|_{0,f}^2$$
$$\geq 0.$$

In case we have no outer forces, i.e. $\mathbf{f}_h = \mathbf{0}$, and homogeneous boundary conditions, i.e. $\mathbf{G}_h = \mathbf{0}$, we have a discrete loss of energy for the solution of (2.30), since

$$\frac{1}{2}\partial_t \|\mathbf{u}_h\|_{\mathrm{V}}^2 = (\mathbf{u}_h, \partial_t \mathbf{u}_h)_{\mathrm{V}} = -(\mathbf{u}_h, A_h \mathbf{u}_h)_{\mathrm{V}} \leq 0.$$

This loss of energy even exists for full reflecting boundary conditions and without a conductivity σ and is caused by the upwind specific parameters γ_f and δ_f.

Proof: We use the properties (2.23) of the parameters $\alpha_{K,f}$ and $\beta_{K,f}$ to calculate the following identities on inner faces $f \in \mathcal{F}_K^\circ$

$$\begin{aligned}
&- \alpha_{K,f}(\mathbf{n}_{K,f} \times \mathbf{H}_{K_f}, \mathbf{E}_K)_{0,f} + \beta_{K_f,f}(\mathbf{n}_{K_f,f} \times \mathbf{E}_K, \mathbf{H}_{K_f})_{0,f} = 0, \\
&- \alpha_{K_f,f}(\mathbf{n}_{K_f,f} \times \mathbf{H}_K, \mathbf{E}_{K_f})_{0,f} + \beta_{K,f}(\mathbf{n}_{K,f} \times \mathbf{E}_{K_f}, \mathbf{H}_K)_{0,f} = 0, \\
&(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \mathbf{E}_K)_{0,f} + (\mathbf{n}_{K_f,f} \times (\mathbf{n}_{K_f,f} \times [\mathbf{E}_h]_{K_f,f}), \mathbf{E}_{K_f})_{0,f} \\
&\qquad = \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2, \\
&(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \mathbf{H}_K)_{0,f} + (\mathbf{n}_{K_f,f} \times (\mathbf{n}_{K_f,f} \times [\mathbf{H}_h]_{K_f,f}), \mathbf{H}_{K_f})_{0,f} \\
&\qquad = \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2.
\end{aligned}$$

Additionally, on every cell K we obtain with an integration by parts

$$-(\nabla \times \mathbf{H}_h, \mathbf{E}_h)_{0,K} + (\nabla \times \mathbf{E}_h, \mathbf{H}_h)_{0,K} = \sum_{f \in \mathcal{F}_K} (\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{H}_K)_{0,f}.$$

Regarding the virtual definitions in Lemma 2.3.6, we now use straight forward calculations to show our statement

$$\begin{aligned}
(A_h \mathbf{u}_h, \mathbf{u}_h)_{\mathrm{V}} = &\sum_{K \in \mathcal{T}_h} -(\nabla \times \mathbf{H}_h, \mathbf{E}_h)_{0,K} + (\nabla \times \mathbf{E}_h, \mathbf{H}_h)_{0,K} + \sigma_K(\mathbf{E}_h, \mathbf{E}_h)_{0,K} \\
&+ \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} -\alpha_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \mathbf{E}_K)_{0,f} \\
&\qquad + \beta_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \mathbf{H}_K)_{0,f} \\
&\qquad + \gamma_f(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \mathbf{E}_K)_{0,f} \\
&\qquad + \delta_f(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \mathbf{H}_K)_{0,f}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}_h} \sigma_K(\mathbf{E}_h,\mathbf{E}_h)_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\circ} \frac{\gamma_f}{2}\|\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{\delta_f}{2}\|\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}\|_{0,f}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\partial} -\alpha_{K,f}(\mathbf{n}_{K,f}\times\mathbf{H}_{K_f},\mathbf{E}_K)_{0,f} + \beta_{K,f}(\mathbf{n}_{K,f}\times\mathbf{E}_{K_f},\mathbf{H}_K)_{0,f} \\
&\qquad\qquad + \gamma_f(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}),\mathbf{E}_K)_{0,f} \\
&\qquad\qquad + \delta_f(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}),\mathbf{H}_K)_{0,f} \\
&= \sum_{K\in\mathcal{T}_h} \sigma_K(\mathbf{E}_h,\mathbf{E}_h)_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\circ} \frac{\gamma_f}{2}\|\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{\delta_f}{2}\|\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}\|_{0,f}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^E} 2\gamma_f\|\mathbf{n}_{K,f}\times\mathbf{E}_K\|_{0,f}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^H} 2\delta_f\|\mathbf{n}_{K,f}\times\mathbf{H}_K\|_{0,f}^2 \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^I} Z_f\chi_f\|\mathbf{n}_{K,f}\times\mathbf{E}_K\|_{0,f}^2 + \kappa_f\|\mathbf{n}_{K,f}\times\mathbf{H}_K\|_{0,f}^2 \\
&\geq 0.
\end{aligned}
$$

□

Due to the boundary values $\mathbf{g}_j$ of $\mathbf{u}$, the analogous term $(A\mathbf{u},\mathbf{u})_{\mathrm{V}}$ for the continuous operator does not need to be non-negative. With an integration by parts we get the following result for all $\mathbf{u}\in\mathrm{H}(\mathrm{curl},\Omega)^2$ with $(\mathbf{n}\times\mathbf{E},\mathbf{n}\times\mathbf{H})\in\mathrm{L}_2(\partial\Omega)^6$

$$
\begin{aligned}
(A\mathbf{u},\mathbf{u})_{\mathrm{V}} &= \int_\Omega -\nabla\times\mathbf{H}\cdot\mathbf{E} + \nabla\times\mathbf{E}\cdot\mathbf{H} + \sigma\mathbf{E}\cdot\mathbf{E}\,\mathrm{dx} \\
&= \int_\Omega \sigma\mathbf{E}\cdot\mathbf{E}\,\mathrm{dx} + \int_{\partial\Omega}\mathbf{n}\times\mathbf{E}\cdot\mathbf{H}\,\mathrm{da} \\
&= \sum_{K\in\mathcal{T}_h}\sigma_K(\mathbf{E},\mathbf{E})_{0,K} + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\partial}(\mathbf{n}_{K,f}\times\mathbf{E},\mathbf{H})_{0,f}.
\end{aligned}
$$

Now, we calculate the mixed terms, namely $(A\mathbf{u},\mathbf{u}_h)_{\mathrm{V}}$ and $(A_h\mathbf{u}_h,\Pi_h^p\mathbf{u})_{\mathrm{V}}$. We have the following result.

Lemma 2.4.5:
All $\mathbf{u} \in \mathrm{H}(\mathrm{curl}, \Omega)^2$, *with* $\mathbf{u}|_{\Lambda_j} \in \mathrm{H}^{\frac{1}{2}}(\Lambda_j)^6$ *for all* j *and all* $\mathbf{u}_h \in \mathrm{V}_h^p$ *fulfill the following equation*

$$\begin{aligned}
(A\mathbf{u}, \mathbf{u}_h)_{\mathrm{V}} + (A_h \mathbf{u}_h, \Pi_h^p \mathbf{u})_{\mathrm{V}} \\
&= \sum_{K \in \mathcal{T}_h} \sigma_K (\mathbf{E}_K, \Pi_K^p \mathbf{E} + \mathbf{E})_{0,K} \\
&\quad + \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^\circ} -\alpha_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \Pi_K^p \mathbf{E} - \mathbf{E})_{0,f} \\
&\qquad + \beta_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \Pi_K^p \mathbf{H} - \mathbf{H})_{0,f} \\
&\qquad + \gamma_f (\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \Pi_K^p \mathbf{E} - \mathbf{E})_{0,f} \\
&\qquad + \delta_f (\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \Pi_K^p \mathbf{H} - \mathbf{H})_{0,f} \\
&\quad + \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^\partial} (\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{H})_{0,f} - (\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{E})_{0,f} \\
&\qquad - \alpha_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \Pi_K^p \mathbf{E})_{0,f} \\
&\qquad + \beta_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \Pi_K^p \mathbf{H})_{0,f} \\
&\qquad + \gamma_f (\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \Pi_K^p \mathbf{E})_{0,f} \\
&\qquad + \delta_f (\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \Pi_K^p \mathbf{H})_{0,f}.
\end{aligned}$$

Proof: On inner faces $f \in \mathcal{F}_K^\circ$ we have the identities

$$\begin{aligned}
&\gamma_f (\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \mathbf{E})_{0,f} + \gamma_f (\mathbf{n}_{K_f,f} \times (\mathbf{n}_{K_f,f} \times [\mathbf{E}_h]_{K_f,f}), \mathbf{E})_{0,f} = 0, \\
&\delta_f (\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \mathbf{H})_{0,f} + \delta_f (\mathbf{n}_{K_f,f} \times (\mathbf{n}_{K_f,f} \times [\mathbf{H}_h]_{K_f,f}), \mathbf{H})_{0,f} = 0, \\
&- (\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{E})_{0,f} - (\mathbf{n}_{K_f,f} \times \mathbf{H}_{K_f}, \mathbf{E})_{0,f} \\
&\qquad = \alpha_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \mathbf{E})_{0,f} + \alpha_{K_f,f} (\mathbf{n}_{K_f,f} \times [\mathbf{H}_h]_{K_f,f}, \mathbf{E})_{0,f}, \\
&(\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{H})_{0,f} + (\mathbf{n}_{K_f,f} \times \mathbf{E}_{K_f}, \mathbf{H})_{0,f} \\
&\qquad = -\beta_{K,f} (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \mathbf{H})_{0,f} - \beta_{K_f,f} (\mathbf{n}_{K_f,f} \times [\mathbf{E}_h]_{K_f,f}, \mathbf{H})_{0,f}
\end{aligned}$$

and therefore

$$\begin{aligned}
(A\mathbf{u}, \mathbf{u}_h)_{\mathrm{V}} &= \sum_{K \in \mathcal{T}_h} -(\nabla \times \mathbf{H}, \mathbf{E}_K)_{0,K} + (\nabla \times \mathbf{E}, \mathbf{H}_K)_{0,K} + \sigma_K (\mathbf{E}, \mathbf{E}_K)_{0,K} \\
&= \sum_{K \in \mathcal{T}_h} -(\mathbf{H}, \nabla \times \mathbf{E}_K)_{0,K} + (\mathbf{E}, \nabla \times \mathbf{H}_K)_{0,K} + \sigma_K (\mathbf{E}, \mathbf{E}_K)_{0,K} \\
&\quad + \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} -(\mathbf{n}_{K,f} \times \mathbf{H}, \mathbf{E}_K)_{0,f} + (\mathbf{n}_{K,f} \times \mathbf{E}, \mathbf{H}_K)_{0,f}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}_h} -(\mathbf{H}, \nabla\times\mathbf{E}_K)_{0,K} + (\mathbf{E}, \nabla\times\mathbf{H}_K)_{0,K} + \sigma_K(\mathbf{E},\mathbf{E}_K)_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} (\mathbf{n}_{K,f}\times\mathbf{E}_K, \mathbf{H})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{H}_K, \mathbf{E})_{0,f} \\
&= \sum_{K\in\mathcal{T}_h} -(\mathbf{H}, \nabla\times\mathbf{E}_K)_{0,K} + (\mathbf{E}, \nabla\times\mathbf{H}_K)_{0,K} + \sigma_K(\mathbf{E},\mathbf{E}_K)_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\circ} -\beta_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}, \mathbf{H})_{0,f} + \alpha_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}, \mathbf{E})_{0,f} \\
&\qquad\qquad - \gamma_f(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}), \mathbf{E})_{0,f} \\
&\qquad\qquad - \delta_f(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}), \mathbf{H})_{0,f} \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\partial} (\mathbf{n}_{K,f}\times\mathbf{E}_K, \mathbf{H})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{H}_K, \mathbf{E})_{0,f}.
\end{aligned}
$$

Per definition of A_h in (2.28), the other term reads

$$
\begin{aligned}
(A_h\mathbf{u}_h, \Pi_h^p\mathbf{u})_{\mathrm{V}} = \sum_{K\in\mathcal{T}_h} & -(\nabla\times\mathbf{H}_K, \Pi_K^p\mathbf{E})_{0,K} + (\nabla\times\mathbf{E}_K, \Pi_K^p\mathbf{H})_{0,K} \\
& + \sigma_K(\mathbf{E}_K, \Pi_K^p\mathbf{E})_{0,K} \\
+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} & -\alpha_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}, \Pi_K^p\mathbf{E})_{0,f} \\
& + \beta_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}, \Pi_K^p\mathbf{H})_{0,f} \\
& + \gamma_f(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}), \Pi_K^p\mathbf{E})_{0,f} \\
& + \delta_f(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}), \Pi_K^p\mathbf{H})_{0,f}.
\end{aligned}
$$

We sum up both terms and use the orthogonality relation

$$(\nabla\times\mathbf{H}_K, \Pi_K^p\mathbf{E}-\mathbf{E})_{0,K} = (\nabla\times\mathbf{E}_K, \Pi_K^p\mathbf{H}-\mathbf{H})_{0,K} = 0$$

to obtain the desired result. □

The last term we need to calculate is the one related to $\mathbf{G}_h$. Per definition of $\mathbf{G}_h$ in (2.29) and with the boundary flux values in Lemma 2.3.6, we obtain

$$
\begin{aligned}
&(\mathbf{G}_h, \Pi_h^p\mathbf{u}-\mathbf{u}_h)_{\mathrm{V}} \\
&\quad = \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^E} \left(G_{E,f}^*(\mathbf{g}_E), \begin{pmatrix} \Pi_K^p\mathbf{E}-\mathbf{E}_K \\ \Pi_K^p\mathbf{H}-\mathbf{H}_K \end{pmatrix} \right)_{0,f} \\
&\quad\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^H} \left(G_{H,f}^*(\mathbf{g}_H), \begin{pmatrix} \Pi_K^p\mathbf{E}-\mathbf{E}_K \\ \Pi_K^p\mathbf{H}-\mathbf{H}_K \end{pmatrix} \right)_{0,f}
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^I}\left(G_{I,f}^*(\mathbf{g}_I), \begin{pmatrix}\Pi_K^p\mathbf{E}-\mathbf{E}_K\\ \Pi_K^p\mathbf{H}-\mathbf{H}_K\end{pmatrix}\right)_{0,f}\\
&= \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^E} 2\gamma_f(\mathbf{n}_{K,f}\times\mathbf{g}_E, \Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f} + (\mathbf{g}_E, \Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}\\
&\quad+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^H} -(\mathbf{g}_H, \Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f} + 2\delta_f(\mathbf{n}_{K,f}\times\mathbf{g}_H, \Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}\\
&\quad+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^I} -\chi_f(\mathbf{g}_I, \Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f} + \kappa_f(\mathbf{n}_{K,f}\times\mathbf{g}_I, \Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}.
\end{aligned}
$$

In total, we can formulate the following lemma for the desired term in (2.32).

Lemma 2.4.6:
Let $\mathbf{u}\in\mathrm{H}(\mathrm{curl},\Omega)^2$ *with regularity* $\mathbf{u}|_{\Lambda_j}\in\mathrm{H}^{\frac{1}{2}}(\Lambda_j)^6$ *for all* j *and with boundary values* $\mathbf{g}_E$*,* $\mathbf{g}_H$*, and* $\mathbf{g}_I$ *as in* (2.1) *and let* $\mathbf{u}_h\in\mathrm{V}_h^p$*. Then we have the following estimate*

$$
\begin{aligned}
&(\mathbf{G}_h + A_h\mathbf{u}_h - A\mathbf{u}, \mathbf{u}-\mathbf{u}_h)_{\mathrm{V}}\\
&\qquad\le \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\circ}\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}}\|\mathbf{E}-\Pi_K^p\mathbf{E}\|_{0,f}^2 + \frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}}\|\mathbf{H}-\Pi_K^p\mathbf{H}\|_{0,f}^2\\
&\qquad\quad+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\partial}\frac{1}{2}\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}}\|\mathbf{E}-\Pi_K^p\mathbf{E}\|_{0,f}^2 + \frac{1}{2}\frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}}\|\mathbf{H}-\Pi_K^p\mathbf{H}\|_{0,f}^2.
\end{aligned}
$$

Proof: We just calculated the individual terms of $(\mathbf{G}_h + A_h\mathbf{u}_h - A\mathbf{u}, \mathbf{u}-\mathbf{u}_h)_{\mathrm{V}}$. Summed up, we obtain

$$
\begin{aligned}
&(\mathbf{G}_h + A_h\mathbf{u}_h - A\mathbf{u}, \mathbf{u}-\mathbf{u}_h)_{\mathrm{V}}\\
&= \sum_{K\in\mathcal{T}_h}\sigma_K(\mathbf{E}_K, \Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,K} + \sigma_K(\mathbf{E}_K-\mathbf{E}, \mathbf{E})_{0,K}\\
&\quad+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^\circ} -\alpha_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}, \Pi_K^p\mathbf{E}-\mathbf{E})_{0,f}\\
&\qquad\qquad+ \beta_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}, \Pi_K^p\mathbf{H}-\mathbf{H})_{0,f}\\
&\qquad\qquad+ \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}), \Pi_K^p\mathbf{E}-\mathbf{E}\big)_{0,f}\\
&\qquad\qquad+ \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}), \Pi_K^p\mathbf{H}-\mathbf{H}\big)_{0,f}\\
&\qquad\qquad- \frac{\gamma_f}{2}\|\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}\|_{0,f}^2 - \frac{\delta_f}{2}\|\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}\|_{0,f}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^\partial} (\mathbf{n}_{K,f}\times\mathbf{E}_K,\mathbf{H})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{H}_K,\mathbf{E})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{E},\mathbf{H})_{0,f}\\
&\qquad - \alpha_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f},\Pi_K^p\mathbf{E})_{0,f}\\
&\qquad + \beta_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f},\Pi_K^p\mathbf{H})_{0,f}\\
&\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}),\Pi_K^p\mathbf{E}\big)_{0,f}\\
&\qquad + \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}),\Pi_K^p\mathbf{H}\big)_{0,f}\\
&+ \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^E} 2\gamma_f(\mathbf{n}_{K,f}\times\mathbf{g}_E,\Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f} + (\mathbf{g}_E,\Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}\\
&\qquad - 2\gamma_f\|\mathbf{n}_{K,f}\times\mathbf{E}_K\|_{0,f}^2\\
&+ \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^H} -(\mathbf{g}_H,\Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f} + 2\delta_f(\mathbf{n}_{K,f}\times\mathbf{g}_H,\Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}\\
&\qquad - 2\delta_f\|\mathbf{n}_{K,f}\times\mathbf{H}_K\|_{0,f}^2\\
&+ \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^I} -\chi_f(\mathbf{g}_I,\Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f} + \kappa_f(\mathbf{n}_{K,f}\times\mathbf{g}_I,\Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}\\
&\qquad - Z_f\chi_f\|\mathbf{n}_{K,f}\times\mathbf{E}_K\|_{0,f}^2 - \kappa_f\|\mathbf{n}_{K,f}\times\mathbf{H}_K\|_{0,f}^2\\
&= S_K + S_\circ + S_E + S_H + S_I.
\end{aligned}
$$

Here, we denoted by S_K the sum over the volume integrals, by $S_\circ$ the sum over inner faces and by S_E, S_H, S_I the sums over boundary faces. Now, we look at the sums individually.

Volume term The first term can be estimated as follows

$$
\begin{aligned}
S_K &= \sum_{K\in\mathcal{T}_h} \sigma_K(\mathbf{E}_K,\Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,K} + \sigma_K(\mathbf{E}_K-\mathbf{E},\mathbf{E})_{0,K}\\
&= \sum_{K\in\mathcal{T}_h} \sigma_K(\mathbf{E}_K,\Pi_K^p\mathbf{E}-\mathbf{E})_{0,K} - \sigma_K(\mathbf{E}-\mathbf{E}_K,\mathbf{E}-\mathbf{E}_K)_{0,K}\\
&= -\sum_{K\in\mathcal{T}_h} \sigma_K(\mathbf{E}-\mathbf{E}_K,\mathbf{E}-\mathbf{E}_K)_{0,K}\\
&\le 0.
\end{aligned}
$$

Inner faces To estimate the sum over inner faces, we use Cauchy-Schwarz' inequality and (2.33) to obtain

$$
S_\circ = \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^\circ} -\alpha_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f},\Pi_K^p\mathbf{E}-\mathbf{E})_{0,f}
$$

$$
\begin{aligned}
&+ \beta_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \Pi_K^p \mathbf{H} - \mathbf{H})_{0,f} \\
&+ \gamma_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \Pi_K^p \mathbf{E} - \mathbf{E}\big)_{0,f} \\
&+ \delta_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \Pi_K^p \mathbf{H} - \mathbf{H}\big)_{0,f} \\
&- \frac{\gamma_f}{2} \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2 - \frac{\delta_f}{2} \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2 \\
\leq \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^\circ} &\alpha_{K,f} \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f} \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f} \\
&+ \beta_{K,f} \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f} \|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f} \\
&+ \gamma_f \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f} \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f} \\
&+ \delta_f \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f} \|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f} \\
&- \frac{\gamma_f}{2} \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2 - \frac{\delta_f}{2} \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2 \\
\leq \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^\circ} &\frac{\delta_f}{4} \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{0,f}\|_{0,f}^2 + \frac{\alpha_{K,f}^2}{\delta_f} \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2 \\
&+ \frac{\gamma_f}{4} \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{\beta_{K,f}^2}{\gamma_f} \|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2 \\
&+ \frac{\gamma_f}{4} \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2 + \gamma_f \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2 \\
&+ \frac{\delta_f}{4} \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2 + \delta_f \|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2 \\
&- \frac{\gamma_f}{2} \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2 - \frac{\delta_f}{2} \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2 \\
= \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^\circ} &\left(\frac{\alpha_{K,f}^2}{\delta_f} + \gamma_f\right) \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2 + \left(\frac{\beta_{K,f}^2}{\gamma_f} + \delta_f\right) \|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2 \\
= \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^\circ} &\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}} \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2 + \frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}} \|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2.
\end{aligned}
$$

E-**Boundary faces** To handle the three kind of boundary faces, we recall that $\alpha_{K,f} = \beta_{K,f} = \frac{1}{2}$ on boundary faces, as well as the virtual definitions for the next neighbour cell $\mathbf{n}_{K,f} \times \mathbf{E}_{K_f} = -\mathbf{n}_{K,f} \times \mathbf{E}_K$ and $\mathbf{n}_{K,f} \times \mathbf{H}_{K_f} = \mathbf{n}_{K,f} \times \mathbf{H}_K$ on faces $f \subset \partial\Omega_E$. We obtain

$$
\begin{aligned}
S_E = \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^E} &(\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{H})_{0,f} - \underbrace{(\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{E})_{0,f}}_{(*)} - (\mathbf{n}_{K,f} \times \mathbf{E}, \mathbf{H})_{0,f} \\
&- \underbrace{\alpha_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \Pi_K^p \mathbf{E})_{0,f}}_{0} + \beta_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \Pi_K^p \mathbf{H})_{0,f}
\end{aligned}
$$

$$
\begin{aligned}
&+ \underbrace{\gamma_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \Pi_K^p \mathbf{E}\big)_{0,f} + \delta_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \Pi_K^p \mathbf{H}\big)_{0,f}}_{0} \\
&+ 2\gamma_f(\mathbf{n}_{K,f} \times \mathbf{g}_E, \Pi_K^p \mathbf{E} - \mathbf{E}_K)_{0,f} + (\mathbf{g}_E, \Pi_K^p \mathbf{H})_{0,f} - \underbrace{(\mathbf{g}_E, \mathbf{H}_K)_{0,f}}_{-(*)} \\
&- 2\gamma_f(\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{n}_{K,f} \times \mathbf{E}_K)_{0,f} \\
= &\sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^E} (\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{H})_{0,f} - (\mathbf{n}_{K,f} \times \mathbf{E}, \mathbf{H})_{0,f} \\
&\qquad - (\mathbf{n}_{K,f} \times \mathbf{E}_K, \Pi_K^p \mathbf{H})_{0,f} + (\mathbf{n}_{K,f} \times \mathbf{E}, \Pi_K^p \mathbf{H})_{0,f} \\
&\qquad - 2\gamma_f\big(\mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E}), \mathbf{n}_{K,f} \times (\mathbf{E}_K - \Pi_K^p \mathbf{E})\big)_{0,f} \\
= &\sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^E} \big(\mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E}), \mathbf{H} - \Pi_K^p \mathbf{H}\big)_{0,f} \\
&\qquad - 2\gamma_f\big(\mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E}), \mathbf{n}_{K,f} \times (\mathbf{E} - \Pi_K^p \mathbf{E})\big)_{0,f} \\
&\qquad - 2\gamma_f\big(\mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E}), \mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E})\big)_{0,f} \\
\leq &\sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^E} \gamma_f \|\mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E})\|_{0,f}^2 + \frac{1}{4\gamma_f}\|\mathbf{H} - \Pi_K^p \mathbf{H}\|_{0,f}^2 \\
&\qquad + \gamma_f \|\mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E})\|_{0,f}^2 + \gamma_f \|\mathbf{E} - \Pi_K^p \mathbf{E}\|_{0,f}^2 \\
&\qquad - 2\gamma_f \|\mathbf{n}_{K,f} \times (\mathbf{E}_K - \mathbf{E})\|_{0,f}^2 \\
= &\sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^E} \frac{1}{4\gamma_f}\|\mathbf{H} - \Pi_K^p \mathbf{H}\|_{0,f}^2 + \gamma_f \|\mathbf{E} - \Pi_K^p \mathbf{E}\|_{0,f}^2 \\
= &\sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^E} \frac{1}{2}\frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}}\|\mathbf{H} - \Pi_K^p \mathbf{H}\|_{0,f}^2 + \frac{1}{2}\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}}\|\mathbf{E} - \Pi_K^p \mathbf{E}\|_{0,f}^2.
\end{aligned}
$$

H-Boundary faces The boundary terms on $\partial\Omega_H$ can be handled analogously. For faces $f \subset \partial\Omega_H$ we recall, that $\mathbf{n}_{K,f} \times \mathbf{E}_{K_f} = \mathbf{n}_{K,f} \times \mathbf{E}_K$ and $\mathbf{n}_{K,f} \times \mathbf{H}_{K_f} = -\mathbf{n}_{K,f} \times \mathbf{H}_K$, to obtain

$$
\begin{aligned}
S_H = &\sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K^H} \underbrace{(\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{H})_{0,f}}_{(*)} - (\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{E})_{0,f} - (\mathbf{n}_{K,f} \times \mathbf{E}, \mathbf{H})_{0,f} \\
&- \alpha_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \Pi_K^p \mathbf{E})_{0,f} + \underbrace{\beta_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \Pi_K^p \mathbf{H})_{0,f}}_{0} \\
&+ \underbrace{\gamma_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \Pi_K^p \mathbf{E}\big)_{0,f}}_{0}
\end{aligned}
$$

$$
\begin{aligned}
&+ \delta_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \Pi_K^p \mathbf{H}\big)_{0,f} \\
&- (\mathbf{g}_H, \Pi_K^p \mathbf{E})_{0,f} + \underbrace{(\mathbf{g}_H, \mathbf{E}_K)_{0,f}}_{-(*)} + 2\delta_f(\mathbf{n}_{K,f} \times \mathbf{g}_H, \Pi_K^p \mathbf{H} - \mathbf{H}_K)_{0,f} \\
&- 2\delta_f(\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{n}_{K,f} \times \mathbf{H}_K)_{0,f} \\
= \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^H} &-(\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{E})_{0,f} + (\mathbf{n}_{K,f} \times \mathbf{H}_K, \Pi_K^p \mathbf{E})_{0,f} \\
&- (\mathbf{n}_{K,f} \times \mathbf{H}, \Pi_K^p \mathbf{E})_{0,f} - (\mathbf{n}_{K,f} \times \mathbf{E}, \mathbf{H})_{0,f} \\
&- 2\delta_f\big(\mathbf{n}_{K,f} \times (\mathbf{H}_K - \mathbf{H}), \mathbf{n}_{K,f} \times (\mathbf{H}_K - \Pi_K^p \mathbf{H})\big)_{0,f} \\
= \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^H} &\big(\mathbf{n}_{K,f} \times (\mathbf{H} - \mathbf{H}_K), \mathbf{E} - \Pi_K^p \mathbf{E}\big)_{0,f} \\
&- 2\delta_f\big(\mathbf{n}_{K,f} \times (\mathbf{H}_K - \mathbf{H}), \mathbf{n}_{K,f} \times (\mathbf{H} - \Pi_K^p \mathbf{H})\big)_{0,f} \\
&- 2\delta_f\big(\mathbf{n}_{K,f} \times (\mathbf{H}_K - \mathbf{H}), \mathbf{n}_{K,f} \times (\mathbf{H}_K - \mathbf{H})\big)_{0,f} \\
\leq \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^H} &\delta_f \|\mathbf{n}_{K,f} \times (\mathbf{H}_K - \mathbf{H})\|_{0,f}^2 + \frac{1}{4\delta_f} \|\mathbf{E} - \Pi_K^p \mathbf{E}\|_{0,f}^2 \\
&+ \delta_f \|\mathbf{n}_{K,f} \times (\mathbf{H}_K - \mathbf{H})\|_{0,f}^2 + \delta_f \|\mathbf{H} - \Pi_K^p \mathbf{H}\|_{0,f}^2 \\
&- 2\delta_f \|\mathbf{n}_{K,f} \times (\mathbf{H}_K - \mathbf{H})\|_{0,f}^2 \\
= \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^H} &\frac{1}{4\delta_f} \|\mathbf{E} - \Pi_K^p \mathbf{E}\|_{0,f}^2 + \delta_f \|\mathbf{H} - \Pi_K^p \mathbf{H}\|_{0,f}^2 \\
= \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^H} &\frac{1}{2} \frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}} \|\mathbf{E} - \Pi_K^p \mathbf{E}\|_{0,f}^2 + \frac{1}{2} \frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}} \|\mathbf{H} - \Pi_K^p \mathbf{H}\|_{0,f}^2.
\end{aligned}
$$

I-Boundary faces The last kind of boundary $\partial\Omega_I$ will be treated in the same way, though here, the terms are more complex. We use the parameter identities (2.27), as well as the jumps for the discrete fields $\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f} = -2Z_f\kappa_f \mathbf{n}_{K,f} \times \mathbf{E}_K$ and $\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f} = -2\chi_f \mathbf{n}_{K,f} \times \mathbf{H}_K$ to obtain the estimate

$$
\begin{aligned}
S_I = \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^I} &(\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{H})_{0,f} - (\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{E})_{0,f} - (\mathbf{n}_{K,f} \times \mathbf{E}, \mathbf{H})_{0,f} \\
&- \alpha_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}, \Pi_K^p \mathbf{E})_{0,f} + \beta_{K,f}(\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}, \Pi_K^p \mathbf{H})_{0,f} \\
&+ \gamma_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}), \Pi_K^p \mathbf{E}\big)_{0,f} \\
&+ \delta_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}), \Pi_K^p \mathbf{H}\big)_{0,f} \\
&- \chi_f(\mathbf{g}_I, \Pi_K^p \mathbf{E} - \mathbf{E}_K)_{0,f} + \kappa_f(\mathbf{n}_{K,f} \times \mathbf{g}_I, \Pi_K^p \mathbf{H} - \mathbf{H}_K)_{0,f} \\
&- Z_f\chi_f(\mathbf{n}_{K,f} \times \mathbf{E}_K, \mathbf{n}_{K,f} \times \mathbf{E}_K)_{0,f} - \kappa_f(\mathbf{n}_{K,f} \times \mathbf{H}_K, \mathbf{n}_{K,f} \times \mathbf{H}_K)_{0,f}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^I} (\mathbf{n}_{K,f}\times\mathbf{E}_K,\mathbf{H})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{H}_K,\mathbf{E})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{E},\mathbf{H})_{0,f}\\
&\qquad + \chi_f(\mathbf{n}_{K,f}\times\mathbf{H}_K,\Pi_K^p\mathbf{E})_{0,f} - Z_f\kappa_f(\mathbf{n}_{K,f}\times\mathbf{E}_K,\Pi_K^p\mathbf{H})_{0,f}\\
&\qquad - 2Z_f\kappa_f\gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{E}_K),\Pi_K^p\mathbf{E}\big)_{0,f}\\
&\qquad - 2\chi_f\delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{H}_K),\Pi_K^p\mathbf{H}\big)_{0,f}\\
&\qquad - \chi_f(\mathbf{n}_{K,f}\times\mathbf{H},\Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f}\\
&\qquad + Z_f\chi_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{E}),\Pi_K^p\mathbf{E}-\mathbf{E}_K\big)_{0,f}\\
&\qquad + \kappa_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{H}),\Pi_K^p\mathbf{H}-\mathbf{H}_K\big)_{0,f}\\
&\qquad + Z_f\kappa_f(\mathbf{n}_{K,f}\times\mathbf{E},\Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}\\
&\qquad - Z_f\chi_f(\mathbf{n}_{K,f}\times\mathbf{E}_K,\mathbf{n}_{K,f}\times\mathbf{E}_K)_{0,f}\\
&\qquad - \kappa_f(\mathbf{n}_{K,f}\times\mathbf{H}_K,\mathbf{n}_{K,f}\times\mathbf{H}_K)_{0,f}\\
&= \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^I} \chi_f\big((\mathbf{n}_{K,f}\times\mathbf{E}_K,\mathbf{H})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{H}_K,\mathbf{E})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{E},\mathbf{H})_{0,f}\\
&\qquad\quad + (\mathbf{n}_{K,f}\times\mathbf{H}_K,\Pi_K^p\mathbf{E})_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{H},\Pi_K^p\mathbf{E}-\mathbf{E}_K)_{0,f}\big)\\
&\qquad + Z_f\chi_f\big(-\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{E}_K),\Pi_K^p\mathbf{E}-\mathbf{E}_K\big)_{0,f}\\
&\qquad\qquad + \big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{E}),\Pi_K^p\mathbf{E}-\mathbf{E}_K\big)_{0,f}\big)\\
&\qquad + \kappa_f\big(-\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{H}_K),\Pi_K^p\mathbf{H}-\mathbf{H}_K\big)_{0,f}\\
&\qquad\qquad + \big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times\mathbf{H}),\Pi_K^p\mathbf{H}-\mathbf{H}_K\big)_{0,f}\big)\\
&\qquad + Z_f\kappa_f\big(\big(\mathbf{n}_{K,f}\times(\mathbf{E}_K-\mathbf{E}),\mathbf{H}\big)_{0,f} - (\mathbf{n}_{K,f}\times\mathbf{H}_K,\mathbf{E})_{0,f}\\
&\qquad\qquad - (\mathbf{n}_{K,f}\times\mathbf{E}_K,\Pi_K^p\mathbf{H})_{0,f} + (\mathbf{n}_{K,f}\times\mathbf{E},\Pi_K^p\mathbf{H}-\mathbf{H}_K)_{0,f}\big)\\
&= \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^I} \chi_f\big(\mathbf{n}_{K,f}\times(\mathbf{H}_K-\mathbf{H}),\Pi_K^p\mathbf{E}-\mathbf{E}\big)_{0,f}\\
&\qquad + Z_f\chi_f\big(\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times(\mathbf{E}-\mathbf{E}_K)),\Pi_K^p\mathbf{E}-\mathbf{E}\big)_{0,f}\\
&\qquad\qquad - \big(\mathbf{n}_{K,f}\times(\mathbf{E}-\mathbf{E}_K),\mathbf{n}_{K,f}\times(\mathbf{E}-\mathbf{E}_K)\big)_{0,f}\big)\\
&\qquad + \kappa_f\big(\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times(\mathbf{H}-\mathbf{H}_K)),\Pi_K^p\mathbf{H}-\mathbf{H}\big)_{0,f}\\
&\qquad\qquad - \big(\mathbf{n}_{K,f}\times(\mathbf{H}-\mathbf{H}_K),\mathbf{n}_{K,f}\times(\mathbf{H}-\mathbf{H}_K)\big)_{0,f}\big)\\
&\qquad + Z_f\kappa_f\big(\mathbf{n}_{K,f}\times(\mathbf{E}-\mathbf{E}_K),\Pi_K^p\mathbf{H}-\mathbf{H}\big)_{0,f}\\
&\le \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K^I} \frac{\kappa_f}{2}\|\mathbf{n}_{K,f}\times(\mathbf{H}-\mathbf{H}_K)\|_{0,f}^2 + \frac{\chi_f^2}{2\kappa_f}\|\Pi_K^p\mathbf{E}-\mathbf{E}\|_{0,f}^2\\
&\qquad + \frac{Z_f\chi_f}{2}\|\mathbf{n}_{K,f}\times(\mathbf{E}-\mathbf{E}_K)\|_{0,f}^2 + \frac{Z_f\chi_f}{2}\|\Pi_K^p\mathbf{E}-\mathbf{E}\|_{0,f}^2\\
&\qquad - Z_f\chi_f\|\mathbf{n}_{K,f}\times(\mathbf{E}-\mathbf{E}_K)\|_{0,f}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\kappa_f}{2}\|\mathbf{n}_{K,f} \times (\mathbf{H} - \mathbf{H}_K)\|_{0,f}^2 + \frac{\kappa_f}{2}\|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2 \\
&- \kappa_f \|\mathbf{n}_{K,f} \times (\mathbf{H} - \mathbf{H}_K)\|_{0,f}^2 \\
&+ \frac{Z_f \chi_f}{2}\|\mathbf{n}_{K,f} \times (\mathbf{E} - \mathbf{E}_K)\|_{0,f}^2 + \frac{Z_f \kappa_f^2}{2\chi_f}\|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2 \\
= &\sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^I} \gamma_f \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2 + \delta_f \|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2 \\
= &\sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^I} \frac{1}{2}\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}}\|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2 + \frac{1}{2}\frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}}\|\Pi_K^p \mathbf{H} - \mathbf{H}\|_{0,f}^2.
\end{aligned}
$$

This finishes the proof. □

Proof of Theorem 2.4.1: For the remaining part of the proof, we just have to put the pieces together. With (2.32), (2.34), Lemma 2.3.2, and Lemma 2.4.6 we get the desired estimate

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{L}_2((0,T),\mathrm{V})}^2 \leq\ & 4 c_{\text{light}} c_{\text{boundary}}^2 h^{2s-1} \int_0^T |\mathbf{u}(t)|_{s,\mathrm{V},\Lambda}^2 \eta_T(t)\,\mathrm{dt} \\
&+ 4 c_{\text{cell}}^2 h^{2s-1} \int_0^T \eta_T^2(t) |\mathbf{f}(t)|_{s-\frac{1}{2},\mathrm{V},\Lambda}^2\,\mathrm{dt} + 2T c_{\text{cell}}^2 h^{2s} |\mathbf{u}_0|_{s,\mathrm{V},\Lambda}^2 \\
\leq\ & 2 c_{\text{light}} c_{\text{boundary}}^2 T^2 h^{2s-1} \max_{t \in [0,T]} |\mathbf{u}(t)|_{s,\mathrm{V},\Lambda}^2 \\
&+ \frac{4}{3} c_{\text{cell}}^2 T^3 h^{2s-1} \max_{t \in [0,T]} |\mathbf{f}(t)|_{s-\frac{1}{2},\mathrm{V},\Lambda}^2 + 2 c_{\text{cell}}^2 T h^{2s} |\mathbf{u}_0|_{s,\mathrm{V},\Lambda}^2,
\end{aligned}
\tag{2.35}
$$

where

$$
c_{\text{light}} = \max_{K \in \mathcal{T}_h} \frac{1}{\sqrt{\varepsilon_K \mu_K}}. \tag{2.36}
$$

□

For the convergence proof to work, we just needed positivity and the identities (2.23) and (2.27) on the parameters $\alpha_{K,f}$, $\beta_{K,f}$, γ_f, δ_f, χ_f, and κ_f. From that point of view, we have some freedom to alter the upwind flux obtained by the Riemann problem and still maintain convergence. So a natural question to ask is about the specialty of the upwind flux defined via the Riemann problem. Looking at plane waves in a material (1.8), there is a factor $\sqrt{\mu\varepsilon^{-1}}$ between the magnitudes of electric and magnetic field, which is compensated by the weights in the norm $\|\cdot\|_{\mathrm{V}}$. Because of the special parameters obtained via Riemann problem, the estimate in (2.35) can be formulated in weighted semi-norms in a natural way.

3 The perfectly matched layer

Content of this chapter We recap the construction of the PML by complex coordinate stretching, have a look at the boundary conditions at the outer end of the layer, investigate well-posedness and afterwards discretize the PDE in a DG upwind scheme.

Origin of this chapter Idea and construction of the perfectly matched layer are general knowledge. The discussion of the boundary values behind the layer and the transfer of the well-posedness theory in Chapter 2 to the situation with a layer are my contributions. The approach to the error estimate in Section 3.3.4 again is general knowledge.

3.1 The idea behind the PML

In the previous chapter we considered Maxwell's equations in a bounded domain with appropriate boundary conditions. In practical situations there often is a bounded domain of interest surrounded by a vast vacuum or a homogeneous medium. When a wave hits the boundary of the domain of interest it is supposed to leave the domain without reflections. The boundary conditions used in (2.1) are not able to handle such a situation (unless we know the exact values of $\mathbf{g}_j$, $j = E, H, I$ - which we do not). Therefore, a common technique to absorb outgoing waves is to introduce a perfectly matched layer (PML) around the domain of interest, which damps down outgoing waves. In 1994 Berenger [Ber94] introduced the PML with a split field formulation. Another convenient way to introduce the layer is a method called complex coordinate stretching. Some notes of Johnson [Joh10] explain this very nicely. In the following we give a shortened explanation of this method.

3.1.1 The PML by complex coordinate stretching

Consider a plane wave (1.8) that solves Maxwell's equations in a homogeneous medium. We focus on the exponential part $f(\mathbf{x}, t) = \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x} - \mathrm{i}\omega t)$, where the direction of $\mathbf{k} \in \mathbb{R}^3$ is the direction of propagation. We assume $k_1 > 0$. Since the argument of the exponential is purely imaginary, we have an oscillating function. Now we add an imaginary part to x_1

$$y_1 = x_1 + \mathrm{i}\frac{\theta}{\omega}x_1, \qquad \theta = \theta(x_1) = \begin{cases} 0, & x_1 \leq 0, \\ \theta_0, & x_1 > 0, \end{cases}$$

where $\theta_0 > 0$ in the simplest case is a positive constant. Meanwhile the other space variables remain untouched, i.e. $y_2 = x_2$, $y_3 = x_3$. Now the plane wave $\mathbf{E}(\mathbf{y}, t) = \mathbf{E}_0 f(\mathbf{y}, t)$, $\mathbf{H}(\mathbf{y}, t) = \mathbf{H}_0 f(\mathbf{y}, t)$ as a function of $(\mathbf{y}, t)$ solves Maxwell's equations, whereas as a function of $(\mathbf{x}, t)$ is damped exponentially by the factor $\exp(-\frac{\theta_0}{\omega}k_1 x_1)$ (see Fig. 3.1). Note that for $k_1 < 0$ we have an exponential amplification of the plane wave towards $x_1 \to \infty$. If we add a reflecting boundary inside the layer, the reflected wave therefore will be damped again travelling towards the layer free region.

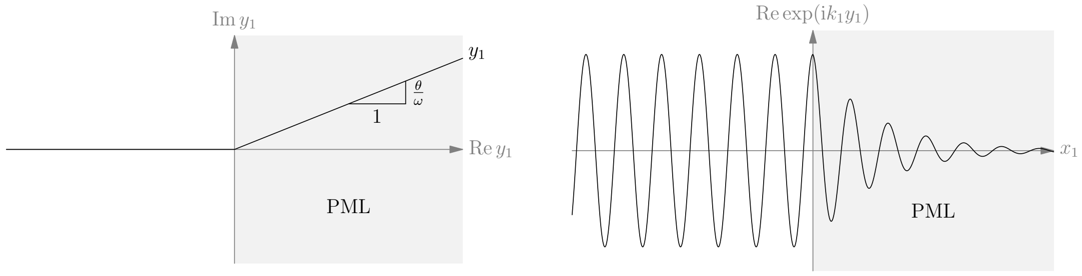


Figure 3.1: A plane wave under complex coordinate stretching.

In order to express Maxwell's equations in the real space variables $\mathbf{x}$, we have to transform the derivatives in y_1 by the chain rule

$$\partial_{x_1} f(\mathbf{y}(\mathbf{x}), t) = \left(1 + \mathrm{i}\frac{\theta}{\omega}\right) \partial_{y_1} f(\mathbf{y}, t).$$

That is, in our PDE we have to replace

$$\partial_{x_1} \to \frac{1}{1 + \mathrm{i}\frac{\theta}{\omega}} \partial_{x_1} \tag{3.1}$$

to obtain a layer in x_1-direction.

Remark 3.1.1:
The layer is called perfectly matched, because it does not produce any reflections at the interface between layer and domain of interest. In Fig. 3.1 this is expressed in the fact, that the plane wave in the region $x_1 < 0$ is not altered during the construction of the PML.

3.1.2 Derivation of the PML PDE system

The geometric situation we want to handle now is displayed in Fig. 3.2. We have a layer Ω_{PML} of thickness d in x_1-direction next to a cuboidal (rectangular) domain of interest Ω_c

$$\Omega_c = (0, a_1) \times (0, a_2) \times (0, a_3),$$
$$\Omega_{\mathrm{PML}} = (a_1, a_1 + d) \times (0, a_2) \times (0, a_3).$$

Our initial values are supported in Ω_c. In the layer we assume to have no current density and no conductivity, i.e.

$$\mathbf{f}_E|_{\Omega_{\mathrm{PML}}} = \mathbf{f}_H|_{\Omega_{\mathrm{PML}}} = \mathbf{0}, \qquad \sigma|_{\Omega_{\mathrm{PML}}} = 0.$$

The PML parameter $\theta(\mathbf{x})$ vanishes in Ω_c, is non-negative and bounded in Ω_{PML} and only depends on x_1

$$\theta(\mathbf{x}) = \chi_{\Omega_{\mathrm{PML}}} \theta(x_1) \geq 0,$$

where χ denotes the characteristic function. The parameter θ may vary inside the layer.

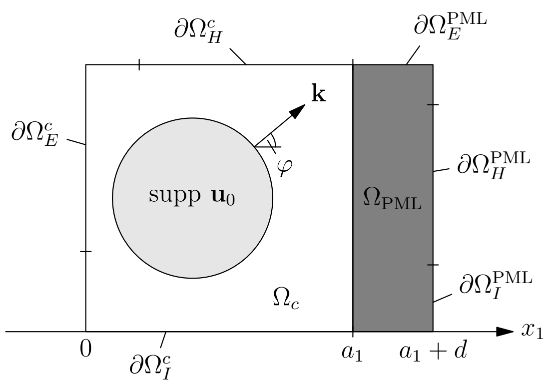


Figure 3.2: Geometry of a PML.

Remark 3.1.2:
The PML can be utilized with non-homogeneous materials, but in order to make a reasonable use of the layer, the materials ε and μ have to be homogeneous in x_1-direction for $x_1 > a_1$. Otherwise they may produce reflections, which re-enter the domain of interest Ω_c. These reflections will be damped by the layer, though they are not supposed to. In [OZAJ08, Fig. 1] there are some pictures on that topic.

Remark 3.1.3:
Sometimes in applications the parameter θ is chosen to be continuously increasing from $x_1 = a_1$ to $x_1 = a_1 + d$. The intention is to surpress reflections from parameter jumps, that occur because the discretization scheme does not suit discontinuous parameters. Discontinuous Galerkin methods, nevertheless, are supposed to handle discontinuities in the parameters quite well, so we will only work with a piecewise constant θ. In his doctoral thesis [Nie09, Sec. 5.5.3] Niegemann tested several shapes of the parameter function θ for a DG method with the result, that a piecewise constant parameter performs the best. Though, it is still recommended to increase the value of θ towards the outer boundary (see Chapter 4.3.5).

First equation of the PDE To derive the PDE system for this situation, we start with the first equation of (2.1a)

$$\varepsilon\partial_t E_1 + \sigma E_1 - \partial_{x_2} H_3 + \partial_{x_3} H_2 = f_{E,1}. \tag{3.2}$$

Now we do a Fourier transform in time, that reads for a function $g(t)$

$$\hat{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t) \exp(\mathrm{i}\omega t)\,\mathrm{d}t,$$
$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(\omega) \exp(-\mathrm{i}\omega t)\,\mathrm{d}\omega\,.$$

To transform (3.2) into frequency space we have to do the substitution $\partial_t \to -\mathrm{i}\omega$ to obtain

$$-\mathrm{i}\omega\varepsilon\hat{E}_1 + \sigma\hat{E}_1 - \partial_{x_2}\hat{H}_3 + \partial_{x_3}\hat{H}_2 = \hat{f}_{E,1}. \tag{3.3}$$

Since there is no derivative in x_1-direction, we can neglect the substitution (3.1), but we redefine the first components of the electromagnetic field

$$\tilde{\mathbf{E}} = \left(\left(1 + \mathrm{i}\frac{\theta}{\omega}\right)\hat{E}_1, \hat{E}_2, \hat{E}_3\right), \tag{3.4a}$$

$$\tilde{\mathbf{H}} = \left(\left(1 + \mathrm{i}\frac{\theta}{\omega}\right)\hat{H}_1, \hat{H}_2, \hat{H}_3\right). \tag{3.4b}$$

Hereby the fields do not change in the domain of interest Ω_c and equation (3.3) after a multiplication with the factor $1 + \mathrm{i}\theta\omega^{-1}$ reads

$$-\mathrm{i}\omega\varepsilon\tilde{E}_1 + \sigma\tilde{E}_1 - \partial_{x_2}\tilde{H}_3 + \partial_{x_3}\tilde{H}_2 - \mathrm{i}\frac{\theta}{\omega}\partial_{x_2}\tilde{H}_3 + \mathrm{i}\frac{\theta}{\omega}\partial_{x_3}\tilde{H}_2 = \hat{f}_{E,1}. \tag{3.5}$$

Note, since the right-hand side $\hat{f}_{E,1}$ vanishes in the layer, we have $\theta\hat{f}_{E,1} = 0$. We define an auxiliary function

$$\tilde{\xi}_1 = \frac{\varepsilon^{-1}}{\mathrm{i}\omega}\big(\sigma\tilde{E}_1 - \hat{f}_{E,1} - (\nabla \times \tilde{\mathbf{H}})_1\big) - \tilde{E}_1 \tag{3.6}$$

and do a reverse Fourier transform to obtain the equation in time domain

$$-\varepsilon\partial_t(\xi_1 + E_1) = \sigma E_1 - f_{E,1} - (\nabla \times \mathbf{H})_1.$$

With (3.6), the fact that $\theta\hat{f}_{E,1} = 0$ and $\theta\sigma = 0$ and a transformation of (3.5) into time domain we finally obtain

$$\begin{aligned} \varepsilon\partial_t E_1 + \sigma E_1 - (\nabla \times \mathbf{H})_1 - \varepsilon\theta\xi_1 - \varepsilon\theta E_1 &= f_{E,1}, \\ \partial_t\xi_1 + \theta\xi_1 + \theta E_1 &= 0. \end{aligned} \tag{3.7}$$

So the steps we did here were a Fourier transform in time, a substitution of the x_1-derivative (3.1), a redefinition of the fields (3.4), a definition of the auxiliary function (3.6) and finally a reverse Fourier transform

$$\mathbf{u} \overset{\text{FT}}{\longrightarrow} \hat{\mathbf{u}} \overset{(3.1)}{\longrightarrow} \hat{\mathbf{u}} \overset{(3.4)}{\longrightarrow} \tilde{\mathbf{u}} \overset{(3.6)}{\longrightarrow} (\tilde{\mathbf{u}}, \tilde{\boldsymbol{\xi}}) \overset{\text{FT}}{\longrightarrow} (\mathbf{u}, \boldsymbol{\xi}).$$

Second equation of the PDE In the second equation of (2.1a)

$$\varepsilon\partial_t E_2 + \sigma E_2 - \partial_{x_3}H_1 + \partial_{x_1}H_3 = f_{E,2}$$

we have a derivative in x_1-direction and therefore do the substitution (3.1) after a Fourier transform

$$-\mathrm{i}\omega\varepsilon\hat{E}_2 + \sigma\hat{E}_2 - \partial_{x_3}\hat{H}_1 + \frac{1}{1+\mathrm{i}\frac{\theta}{\omega}}\partial_{x_1}\hat{H}_3 = \hat{f}_{E,2}.$$

A multiplication with the denominator and the fact that the conductivity, as well as the right-hand side vanish inside the layer lead to

$$-\mathrm{i}\omega\varepsilon\tilde{E}_2 + \varepsilon\theta\tilde{E}_2 + \sigma\tilde{E}_2 - \partial_{x_3}\tilde{H}_1 + \partial_{x_1}\tilde{H}_3 = \hat{f}_{E,2}. \tag{3.8}$$

Here we do not need an auxiliary function, so we set $\tilde{\xi}_2 = 0$ and do the reverse Fourier transform

$$\varepsilon\partial_t E_2 + \varepsilon\theta E_2 + \sigma E_2 - (\nabla \times \mathbf{H})_2 = f_{E,2}.$$

Third equation of the PDE The third equation of (2.1a)

$$\varepsilon\partial_t E_3 + \sigma E_3 - \partial_{x_1} H_2 + \partial_{x_2} H_1 = f_{E,3}$$

is similar to the second one. We obtain $\tilde{\xi}_3 = 0$ and

$$\varepsilon\partial_t E_3 + \varepsilon\theta E_3 + \sigma E_3 - (\nabla \times \mathbf{H})_3 = f_{E,3}.$$

Fourth to sixth equation of the PDE For the equations (2.1b) the procedure remains the same. Here, the auxiliary functions are defined as

$$\begin{aligned}
\tilde{\xi}_4 &= \frac{\mu^{-1}}{\mathrm{i}\omega}\big((\nabla \times \tilde{\mathbf{E}})_1 - \hat{f}_{H,1}\big) - \tilde{H}_1, \\
\tilde{\xi}_5 &= 0, \\
\tilde{\xi}_6 &= 0.
\end{aligned}$$

In total our system with a layer in x_1-direction reads

$$\varepsilon\partial_t \mathbf{E} + \sigma\mathbf{E} - \nabla \times \mathbf{H} + \varepsilon(2\Theta - \theta\mathbb{1})\mathbf{E} + \varepsilon(\Theta - \theta\mathbb{1})\boldsymbol{\xi}_E = \mathbf{f}_E, \tag{3.9a}$$

$$\mu\partial_t \mathbf{H} + \nabla \times \mathbf{E} + \mu(2\Theta - \theta\mathbb{1})\mathbf{H} + \mu(\Theta - \theta\mathbb{1})\boldsymbol{\xi}_H = \mathbf{f}_H, \tag{3.9b}$$

$$\partial_t \boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\mathbf{E} = \mathbf{0}, \tag{3.9c}$$

$$\partial_t \boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\mathbf{H} = \mathbf{0}, \tag{3.9d}$$

where we have $\boldsymbol{\xi} = (\boldsymbol{\xi}_E, \boldsymbol{\xi}_H)$, a unit matrix $\mathbb{1}$ and a diagonal Matrix Θ defined by

$$\Theta = \mathrm{diag}(0, \theta, \theta).$$

Equations (3.9a) and (3.9b) describe Maxwell's system with a perturbation of zeroth order. For the auxiliary function we have an ordinary differential equation without any spatial derivatives. The function lives inside the layer and though we only have two non-vanishing components of $\boldsymbol{\xi}$, we write it as a six component vector. It is not a topic of this work, but for a layer in every space direction the additional components are needed.

Remark 3.1.4:
The redefinition (3.4) of the Fourier fields does not look straightforward. Looking at equation (3.8), we can define another auxiliary function $\hat{\xi}_2^{\mathrm{nh}} = \mathrm{i}\frac{\varepsilon^{-1}\theta}{\omega}\partial_{x_3}\hat{H}_1$ and do the reverse Fourier transform without the redefinition (3.4) to obtain the equations

$$\begin{aligned}
\varepsilon\partial_t E_2 + \varepsilon\theta E_2 + \sigma E_2 - (\nabla \times \mathbf{H})_2 - \varepsilon\xi_2^{\mathrm{nh}} &= f_{E,2}, \\
\varepsilon\partial_t \xi_2^{\mathrm{nh}} &= \theta\partial_{x_3} H_1.
\end{aligned}$$

Proceeding like this, we obtain the system

$$\begin{aligned}\varepsilon\partial_t\mathbf{E} + \varepsilon\Theta\mathbf{E} + \sigma\mathbf{E} - \nabla\times\mathbf{H} - \varepsilon\boldsymbol{\xi}_E^{\mathrm{nh}} &= \mathbf{f}_E,\\ \mu\partial_t\mathbf{H} + \mu\Theta\mathbf{H} + \nabla\times\mathbf{E} - \mu\boldsymbol{\xi}_H^{\mathrm{nh}} &= \mathbf{f}_H,\\ \varepsilon\partial_t\boldsymbol{\xi}_E^{\mathrm{nh}} - \Theta\nabla\times\mathbf{H}_1 &= \mathbf{0},\\ \mu\partial_t\boldsymbol{\xi}_H^{\mathrm{nh}} + \Theta\nabla\times\mathbf{E}_1 &= \mathbf{0},\end{aligned}$$

where $\mathbf{H}_1 = \mathbf{e}_1 \otimes \mathbf{e}_1\mathbf{H} = (H_1, 0, 0)$. This is a more intuitive way to define the auxiliary function, but the resulting system is not hyperbolic, so we cannot define an upwind flux here. For example Bonnet and Poupaud worked with such a system in [BP97].

3.1.3 Initial values and boundary conditions

Initial values For the electromagnetic field we use initial values $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ compactly supported in the domain of interest, i.e. $\operatorname{supp}\mathbf{u}_0 \subset\subset \Omega_c$, whereas for the auxiliary function we use vanishing initial values, i.e. $\boldsymbol{\xi}(\mathbf{x}, 0) = \mathbf{0}$. That way, the auxiliary function evolves as the electromagnetic field penetrates the layer. Choosing non-vanishing initial values inside the layer may produce unwanted non-physical effects.

The dilemma with the boundary conditions In the construction of the PML we used a redefinition of the electromagnetic field (3.4) to obtain a hyperbolic PDE system (3.9), but this advantage has some negative effect on the boundary conditions. As a result of 3.4 the first components of $\mathbf{E}$ and $\mathbf{H}$ do not decay exponentially in the layer any more. Let us denote the Fourier transform of $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ by $\mathbf{E}_{\exp}$ and $\mathbf{H}_{\exp}$. These are actually the fields with exponential decay. Now we want to express them in terms of $\mathbf{u}$ and $\boldsymbol{\xi}$. A transformation of the first equation in (3.4a) into time domain leads to

$$\varepsilon\partial_t E_1 = \varepsilon\partial_t E_{\exp,1} + \varepsilon\theta E_{\exp,1}.$$

Summed up with (3.3) transformed back into time domain

$$\varepsilon\partial_t E_{\exp,1} + \sigma E_{\exp,1} - (\nabla\times\mathbf{H})_1 = f_{E,1}$$

and (3.7), we obtain

$$\varepsilon\theta\xi_1 + \varepsilon\theta E_1 = \varepsilon\theta E_{\exp,1}.$$

Here again, we used that σ vanishes in the layer, therefore $\sigma E_1 = \sigma E_{\text{exp},1}$. In total we obtain the following results

$$\mathbf{E}_{\text{exp}} = \mathbf{E} + \boldsymbol{\xi}_E, \tag{3.10a}$$
$$\mathbf{H}_{\text{exp}} = \mathbf{H} + \boldsymbol{\xi}_H \tag{3.10b}$$

for the fields of exponential decay.

Now, if we want to prescribe boundary conditions on the outer boundary of Ω_{PML}, we are supposed to do that in terms of $\mathbf{E}_{\text{exp}}$ and $\mathbf{H}_{\text{exp}}$, e.g. $\mathbf{n} \times \mathbf{E}_{\text{exp}} = \mathbf{0}$ for a perfect electric conductor. With these sort of boundary conditions the well-posedness theory based on Lumer-Phillips' theorem does not work any more in the way it is presented in Section 3.2. For that reason we will continue the theory with boundary conditions e.g. $\mathbf{n} \times \mathbf{E} = \mathbf{0}$. In some special cases we might not even notice any difference. Since our boundary conditions are posed on the tangential components of the electromagnetic field, we do not obtain any problems on the boundary Γ_2^{PML}, see Fig. 3.3, but only on Γ_1^{PML} and Γ_3^{PML}. And in the generic test example on a

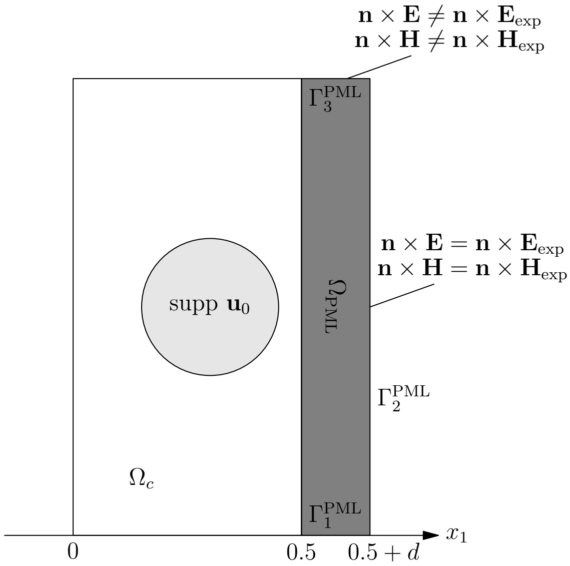


Figure 3.3: On the right boundary of the PML there are no problems with the boundary conditions. On the upper and lower boundary of the layer, it is not obvious, which boundary condition to use.

rectangular or cuboidal domain we automatically have $\mathbf{n} \times \mathbf{E}_{\text{exp}} = \mathbf{0}$ in case we use the boundary condtion $\mathbf{n} \times \mathbf{E} = \mathbf{0}$. To understand that, we recall the first equation of (3.9c)

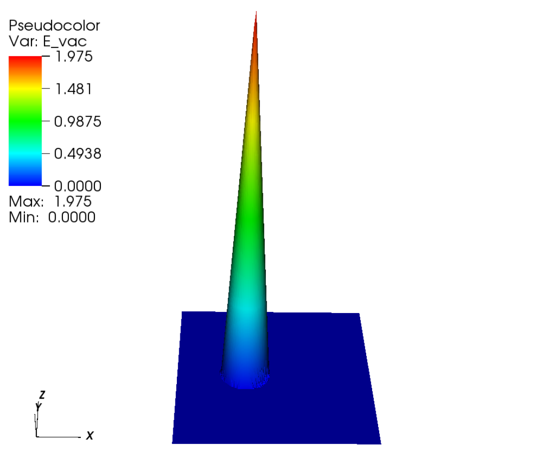
$$\partial_t \xi_{E,1} + \theta \xi_{E,1} + \theta E_1 = 0.$$

Using homogeneous initial values, an integral form of $\xi_{E,1}$ can be calculated to be

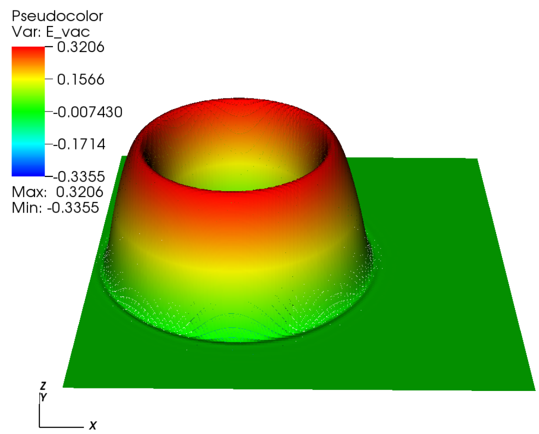
$$\xi_{E,1}(t) = -\int_0^t \theta E_1(\tau) \exp\big(\theta(\tau - t)\big)\,\mathrm{d}\tau\,. \tag{3.11}$$

So with $\mathbf{n} \times \mathbf{E} = \mathbf{0}$ we can see that $\mathbf{n} \times \boldsymbol{\xi}_E = \mathbf{0}$ as well. This argument also works fine for the magnetic field. It does not work, though, for the impedance boundary condition, since different components of $\mathbf{E}$ and $\mathbf{H}$ are coupled in that condition. In the numerical simulation we therefore observe reflections out of corner regions.

To show that, we use two-dimensional calculations with unknowns (E_3, H_1, H_2) in the TM mode in a quadratic domain $\Omega = (0,1) \times (0,1)$. The domain is divided into $8 \cdot 4^6 = 32768$ triangles with a cellwidth of $h = 2^{-\frac{13}{2}}$. In space, we use a first order polynomial approximation, with a second order explicit time stepping scheme and a timestepwidth of $\tau = 0.0005$. On the boundary, we use an impedance boundary condition. We start with non-vanishing initial values only for the electric field E_3 (see Fig. 3.4(a)) in shape of a hat function. As time evolves the electric



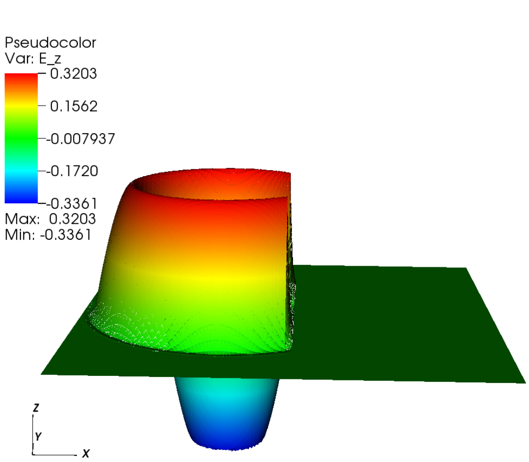
(a) The initial values $E_3(0)$ of the electric field are a hat-shaped function.



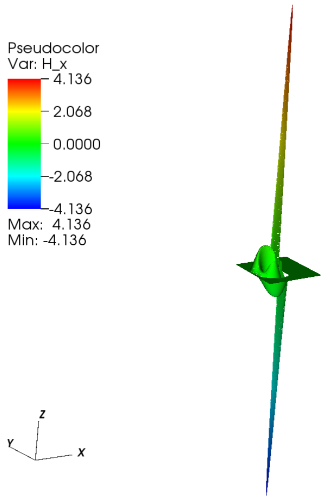
(b) As time evolves the electric field E_3 stays radially symmetric, as long as it does not hit the boundary or a PML. Here, the time $t = 0.2$ is illustrated.

Figure 3.4: The pictures show the electric field E_3 of our test example at times $t = 0$ and $t > 0$.

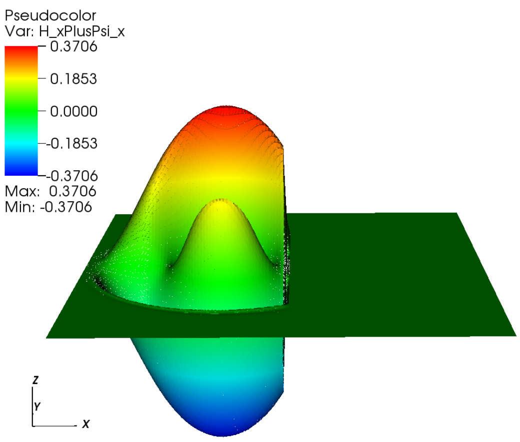
field remains radial symmetric (Fig. 3.4(b)), as long as it does not hit any obstacle. In Fig. 3.5(a) we see the effect of a layer with a constant parameter $\theta = 200$ in the region $0.5 < x_1 < 0.5 + d$, $d = 64^{-1}$ on the electric field E_3. We actually see that



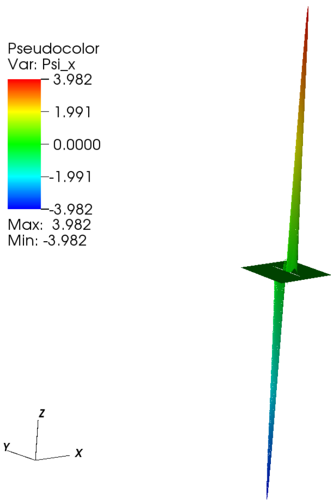
(a) Due to the layer, we see an exponential decay in the electric field E_3. The right-side boundary is programmed to be almost in the middle of the picture. On the right-hand side of the graph the field is set to zero.



(b) The magnetic field H_1 in- and outside the layer is shown. We can see that H_1 does not decay exponentially inside the layer.



(c) The field $H_1 + \xi_{H,1}$ is shown, where an exponential damping inside the layer can be seen.

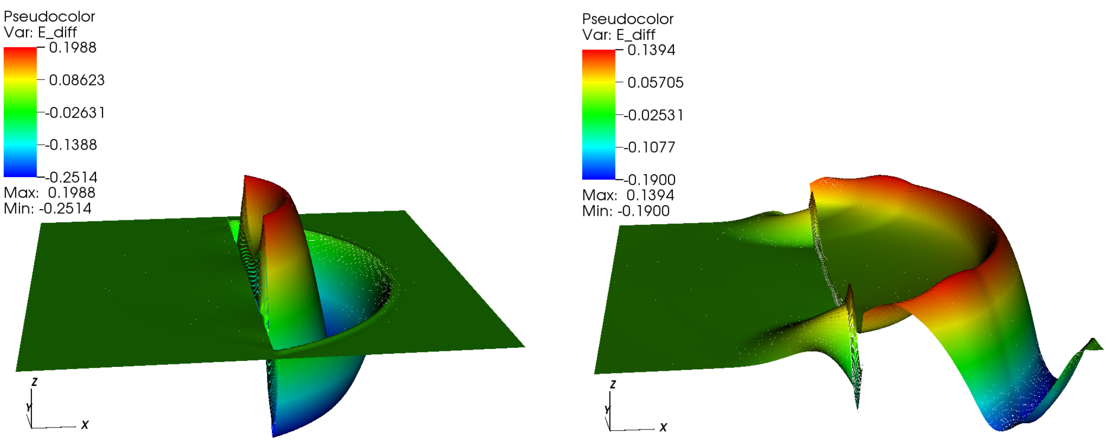


(d) Except for a sign, the field $\xi_{H,1}$ shows the spikes inside the layer of Fig. 3.5(b).

Figure 3.5: Here, we see the influence of a layer, that is positioned almost in the center of the domain, on the fields E_3 and H_1.

the field is damped. For the magnetic field H_1 the situation is different. Since we redefined the first components of the electromagnetic fields in (3.4), we observe a rather large magnetic field H_1 in the layer (Fig. 3.5(b)). In Fig. 3.5(c) and 3.5(d) we see the fields $H_1 + \xi_{H,1}$ and $\xi_{H,1}$. The former shows an exponential damping behaviour inside the layer, the latter only lives inside the layer and produces the non-damping behaviour of H_1. A discussion about the long-time growth of the auxiliary function $\boldsymbol{\xi}$ can be found in Chapter 5. Regarding (3.9c) and (3.9d), we already know $\boldsymbol{\xi}$ to be proportional to the parameter θ at the layer vacuum interface.

Taking a look at the difference between the electric field that evolves freely in Ω and the one with a PML, we see the reflections caused by the layer. In the left-hand half of Fig. 3.6(a) we see the reflections by the layer before the initial wave hits the corners of the PML. There are hardly any reflections to see. After the wave hits the corners of the layer, we notice reflections (see Fig. 3.6(b)) caused by the impedance boundary condition $\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{0}$ on the boundaries Γ_1^{PML} and Γ_3^{PML}.



(a) In the left half of the graph the reflections of the electric field by a PML with impedance boundary condition at time $t = 0.35$ are shown. The right half shows (except for a minus) the electric field as it evolves in free space.

(b) Due to the discrepancy between $\mathbf{u}$ and $\mathbf{u}_{\text{exp}}$, we obtain reflections out of corner regions, here at time $t = 0.65$, of a PML with an impedance boundary condition.

Figure 3.6: We observe undesired reflections out of corner regions, if we use the impedance boundary condition in the shape of (3.12c) behind the layer.

Boundary conditions for well-posedness theory In this work we utilize a well-posedness theory based on Lumer-Phillips' theorem. In order to obtain a well-posed problem, our choice of boundary conditions is one of the following

$$\mathbf{n} \times \mathbf{E} = \mathbf{0}, \tag{3.12a}$$
$$\mathbf{n} \times \mathbf{H} = \mathbf{0}, \tag{3.12b}$$
$$\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{0}. \tag{3.12c}$$

They can be combined, so that on several different parts of the boundary we use different boundary conditions. They also can be inhomogeneous, but that is only advisable in special cases, e.g. when we have exact knowledge of the electromagnetic field. In the numerical tests in Chapter 4 this will be the case.

Boundary conditions for theoretical understanding of the PML In order to fully understand the theoretical behaviour of the layer, we deem it necessary - as explained above - to use the boundary conditions

$$\mathbf{n} \times (\mathbf{E} + \boldsymbol{\xi}_E) = \mathbf{0},$$
$$\mathbf{n} \times (\mathbf{H} + \boldsymbol{\xi}_H) = \mathbf{0},$$
$$\mathbf{n} \times (\mathbf{H} + \boldsymbol{\xi}_H) - Z\mathbf{n} \times \big(\mathbf{n} \times (\mathbf{E} + \boldsymbol{\xi}_E)\big) = \mathbf{0},$$

since they have a well-known reflection and absorption behaviour.

Boundary conditions for practical use In applications the absorbing boundary condition

$$\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = -\mathbf{n} \times \boldsymbol{\xi}_H + Z\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\xi}_E),$$

with $Z = \varepsilon^{\frac{1}{2}}\mu^{-\frac{1}{2}}$, seems to be the most useful. How to treat the right-hand side in terms of the upwind flux was explained in Section 2.3.5 and will be continued in Section 3.3.2. In Lemma 2.3.6 we just have to choose $\mathbf{g}_I = -\mathbf{n} \times \boldsymbol{\xi}_H + Z\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\xi}_E)$.

3.1.4 Maxwell's system with a PML in x_1-direction

Now we are able to state the PDE we are interested in. The boundary conditions behind the layer will be chosen as in (3.12)

$$\varepsilon\partial_t\mathbf{E} + \sigma\mathbf{E} - \nabla \times \mathbf{H} + \varepsilon(2\Theta - \theta\mathbb{1})\mathbf{E} + \varepsilon(\Theta - \theta\mathbb{1})\boldsymbol{\xi}_E = \mathbf{f}_E \quad \text{in } \Omega_\infty, \tag{3.13a}$$
$$\mu\partial_t\mathbf{H} + \nabla \times \mathbf{E} + \mu(2\Theta - \theta\mathbb{1})\mathbf{H} + \mu(\Theta - \theta\mathbb{1})\boldsymbol{\xi}_H = \mathbf{f}_H \quad \text{in } \Omega_\infty, \tag{3.13b}$$

$$\partial_t \boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\mathbf{E} = \mathbf{0} \quad \text{in } \Omega_\infty, \tag{3.13c}$$
$$\partial_t \boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\mathbf{H} = \mathbf{0} \quad \text{in } \Omega_\infty, \tag{3.13d}$$
$$\mathbf{n} \times \mathbf{E} = \mathbf{g}_E \quad \text{on } \partial\Omega_{E,\infty}, \tag{3.13e}$$
$$\mathbf{n} \times \mathbf{H} = \mathbf{g}_H \quad \text{on } \partial\Omega_{H,\infty}, \tag{3.13f}$$
$$\mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times \big(\mathbf{n} \times \mathbf{E}\big) = \mathbf{g}_I \quad \text{on } \partial\Omega_{I,\infty}, \tag{3.13g}$$
$$\mathbf{E}(\cdot, 0) = \mathbf{E}_0 \quad \text{in } \Omega, \tag{3.13h}$$
$$\mathbf{H}(\cdot, 0) = \mathbf{H}_0 \quad \text{in } \Omega, \tag{3.13i}$$
$$\boldsymbol{\xi}_E(\cdot, 0) = \mathbf{0} \quad \text{in } \Omega, \tag{3.13j}$$
$$\boldsymbol{\xi}_H(\cdot, 0) = \mathbf{0} \quad \text{in } \Omega. \tag{3.13k}$$

Again, the index ∞ denotes e.g. $\Omega_\infty = \Omega \times [0, \infty)$. The domain Ω has a layer and a non-layer part, i.e. $\overline{\Omega} = \overline{\Omega}_c \cup \overline{\Omega}_{\text{PML}}$. The boundary parts are split in the same way $\overline{\partial\Omega_j^c} \cup \overline{\partial\Omega_j^{\text{PML}}} = \overline{\partial\Omega_j}$, $j = E, H, I$. The conductivity σ and the right-hand sides $\mathbf{f}_E$, $\mathbf{f}_H$ are supposed to vanish in Ω_{PML}. The Matrix Θ was defined by $\Theta = \text{diag}(0, \theta, \theta)$, where θ is the non-negative parameter of the layer, that vanishes in Ω_c. Therefore we can see by (3.13c) and (3.13d) that the auxiliary function $\boldsymbol{\xi}$ also vanishes in Ω_c. In the layer only the first and fourth component of $\boldsymbol{\xi}$ are non-vanishing. The initial values $\mathbf{E}_0$ and $\mathbf{H}_0$ are supposed to be supported in Ω_c.

Remark 3.1.5:
The PML in the present hyperbolic formulation (3.13) is often referred to as PML in Zhao-Cangellaris' formulation or unsplit PML compared to Berenger's version, who introduced the PML by a splitting of the electromagnetic fields.

3.2 Existence and Uniqueness for the PML setting

The next step will be to verify the assumptions of Lumer-Phillips' theorem (Theorem 2.2.1). Here again, we will work with homogeneous boudary values $\mathbf{g}_j = \mathbf{0}$ and rewrite the system (3.13a) to (3.13d) in terms of $\mathbf{v} = (\mathbf{u}, \boldsymbol{\xi})$. We consider the Hilbert space $\text{V}_{\text{PML}} = \text{V} \times \text{V}$ equipped with the inner product

$$(\mathbf{v}, \tilde{\mathbf{v}})_{\text{V}_{\text{PML}}} := (\mathbf{u}, \tilde{\mathbf{u}})_{\text{V}} + (\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}})_{\text{V}}.$$

Let the operator $B : \mathcal{D}(B) \subset \text{V}_{\text{PML}} \to \text{V}_{\text{PML}}$ be defined by

$$B\mathbf{v} = \begin{pmatrix} (2\boldsymbol{\Theta} - \theta\mathbb{1})\mathbf{u} + (\boldsymbol{\Theta} - \theta\mathbb{1})\boldsymbol{\xi} + A\mathbf{u} \\ (\theta\mathbb{1} - \boldsymbol{\Theta})(\mathbf{u} + \boldsymbol{\xi}) \end{pmatrix}, \tag{3.14}$$

with domain

$$\begin{aligned}\mathcal{D}(B) := \big\{(\mathbf{u},\boldsymbol{\xi}) \in \mathrm{H}(\mathrm{curl},\Omega)^2 \times \mathrm{V}:\ &\mathbf{n}\times\mathbf{E} = \mathbf{0} \text{ on } \partial\Omega_E,\\ &\mathbf{n}\times\mathbf{H} = \mathbf{0} \text{ on } \partial\Omega_H,\ \mathbf{n}\times\mathbf{E} \in \mathrm{L}_2\big(\partial\Omega_I\big)^3,\\ &\mathbf{n}\times\mathbf{H} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) = \mathbf{0} \text{ on } \partial\Omega_I\big\},\end{aligned}$$

corresponding norm

$$\|\mathbf{v}\|^2_{\mathcal{D}(B)} = \|\mathbf{v}\|^2_{\mathrm{V}_{\mathrm{PML}}} + \|B\mathbf{v}\|^2_{\mathrm{V}_{\mathrm{PML}}} + \|\mathbf{n}\times\mathbf{E}\|^2_{\varepsilon,\partial\Omega_I},$$

and $A : \mathcal{D}(A) \subset \mathrm{V} \to \mathrm{V}$ defined in (2.3). The 6×6-analogon of the 3×3-matrix Θ is denoted by $\boldsymbol{\Theta} = \mathrm{diag}(\Theta,\Theta)$. The coefficient matrices in B are

$$\begin{aligned}2\boldsymbol{\Theta} - \theta\mathbb{1} &= \mathrm{diag}(-\theta,\theta,\theta,-\theta,\theta,\theta),\\ \boldsymbol{\Theta} - \theta\mathbb{1} &= \mathrm{diag}(-\theta,0,0,-\theta,0,0).\end{aligned}$$

We want to investigate the problem $\partial_t\mathbf{v} + B\mathbf{v} = (\mathbf{f},\mathbf{0})$ and therefor check Lumer-Phillip's assumptions on the operator $-B$.

Lemma 3.2.1:
The operator $-B : \mathcal{D}(B) \subset \mathrm{V}_{\mathrm{PML}} \to \mathrm{V}_{\mathrm{PML}}$ satisfies the assumptions of Theorem 2.2.1.

Proof: There are three assumptions to check.

First assumption $\mathcal{D}(B)$ is dense in $\mathrm{V}_{\mathrm{PML}}$ with the same argument as before, namely $\mathrm{C}_0^\infty(\Omega)$ is dense in $\mathrm{L}_2(\Omega)$.

Second assumption With an appropriate ω and for $\mathbf{v} \in \mathcal{D}(B)$ we have to show the estimate

$$(\mathbf{v}, B\mathbf{v})_{\mathrm{V}_{\mathrm{PML}}} \overset{!}{\geq} -\omega(\mathbf{v},\mathbf{v})_{\mathrm{V}_{\mathrm{PML}}}. \tag{3.15}$$

Therefor we calculate

$$\begin{aligned}(\mathbf{v}, B\mathbf{v})_{\mathrm{V}_{\mathrm{PML}}} &= \big((2\boldsymbol{\Theta} - \theta\mathbb{1})\mathbf{u},\mathbf{u}\big)_{\mathrm{V}} + \big((\boldsymbol{\Theta} - \theta\mathbb{1})\boldsymbol{\xi},\mathbf{u}\big)_{\mathrm{V}} + (A\mathbf{u},\mathbf{u})_V\\ &\quad + \big((\theta\mathbb{1} - \boldsymbol{\Theta})\mathbf{u},\boldsymbol{\xi}\big)_{\mathrm{V}} + \big((\theta\mathbb{1} - \boldsymbol{\Theta})\boldsymbol{\xi},\boldsymbol{\xi}\big)_{\mathrm{V}}\\ &= \big((2\boldsymbol{\Theta} - \theta\mathbb{1})\mathbf{u},\mathbf{u}\big)_{\mathrm{V}} + (A\mathbf{u},\mathbf{u})_V + \big((\theta\mathbb{1} - \boldsymbol{\Theta})\boldsymbol{\xi},\boldsymbol{\xi}\big)_{\mathrm{V}}.\end{aligned}$$

With an integration by parts we have

$$(A\mathbf{u},\mathbf{u})_V = \int_{\Omega_c} \sigma\,\mathbf{E}\cdot\mathbf{E}\,\mathrm{dx} - \int_{\partial\Omega} \mathbf{E}\cdot\mathbf{n}\times\mathbf{H}\,\mathrm{da}$$
$$= \int_{\Omega_c} \sigma\,\mathbf{E}\cdot\mathbf{E}\,\mathrm{dx} + \int_{\partial\Omega_I} Z\,\mathbf{n}\times\mathbf{E}\cdot\mathbf{n}\times\mathbf{E}\,\mathrm{da} \geq 0$$

and as a result the desired estimate

$$(\mathbf{v}, B\mathbf{v})_{\mathrm{V_{PML}}} \geq -\theta_{\sup}\,(\mathbf{v},\mathbf{v})_{\mathrm{V_{PML}}}\,.$$

We have $\omega = \theta_{\sup} = \sup_{\mathbf{x}\in\Omega}\theta$.

Third assumption For $\lambda_0 > \theta_{\sup}$ and $\mathbf{F} = (\mathbf{F}_u, \mathbf{F}_\xi) = (\mathbf{F}_E, \mathbf{F}_H, \mathbf{F}_{\xi_E}, \mathbf{F}_{\xi_H}) \in \mathrm{V_{PML}}$ we seek a $\mathbf{v} \in \mathcal{D}(B)$ such that

$$B\mathbf{v} + \lambda_0\mathbf{v} = \mathbf{F}.$$

Rewritten in four equations i.e.

$$(2\Theta - \theta\mathbb{1})\mathbf{E} + (\Theta - \theta\mathbb{1})\boldsymbol{\xi}_E - \varepsilon^{-1}\nabla\times\mathbf{H} + \varepsilon^{-1}\sigma\mathbf{E} + \lambda_0\mathbf{E} = \mathbf{F}_E, \tag{3.16a}$$
$$(2\Theta - \theta\mathbb{1})\mathbf{H} + (\Theta - \theta\mathbb{1})\boldsymbol{\xi}_H + \mu^{-1}\nabla\times\mathbf{E} + \lambda_0\mathbf{H} = \mathbf{F}_H, \tag{3.16b}$$
$$(\theta\mathbb{1} - \Theta)\mathbf{E} + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_E + \lambda_0\boldsymbol{\xi}_E = \mathbf{F}_{\xi_E}, \tag{3.16c}$$
$$(\theta\mathbb{1} - \Theta)\mathbf{H} + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_H + \lambda_0\boldsymbol{\xi}_H = \mathbf{F}_{\xi_H}. \tag{3.16d}$$

With the definition $T = (\theta + \lambda_0)\mathbb{1} - \Theta = \mathrm{diag}(\theta + \lambda_0, \lambda_0, \lambda_0)$ we can express $\boldsymbol{\xi}$ by $\mathbf{F}_\xi$ and $\mathbf{u}$ as follows

$$\boldsymbol{\xi}_E = T^{-1}(\mathbf{F}_{\xi_E} - (\theta\mathbb{1} - \Theta)\mathbf{E}), \tag{3.17a}$$
$$\boldsymbol{\xi}_H = T^{-1}(\mathbf{F}_{\xi_H} - (\theta\mathbb{1} - \Theta)\mathbf{H}). \tag{3.17b}$$

Now we replace $\boldsymbol{\xi}$ in (3.16a) and (3.16b)

$$(2\Theta - \theta\mathbb{1})\mathbf{E} + (\Theta - \theta\mathbb{1})^2 T^{-1}\mathbf{E} - \varepsilon^{-1}\nabla\times\mathbf{H} + \varepsilon^{-1}\sigma\mathbf{E} + \lambda_0\mathbf{E} = \mathbf{F}_E + (\theta\mathbb{1} - \Theta)T^{-1}\mathbf{F}_{\xi_E}, \tag{3.18a}$$
$$(2\Theta - \theta\mathbb{1})\mathbf{H} + (\Theta - \theta\mathbb{1})^2 T^{-1}\mathbf{H} + \mu^{-1}\nabla\times\mathbf{E} + \lambda_0\mathbf{H} = \mathbf{F}_H + (\theta\mathbb{1} - \Theta)T^{-1}\mathbf{F}_{\xi_H}. \tag{3.18b}$$

To shorten the notation, we define

$$\begin{aligned}
P_E &= (2\Theta - \theta\mathbb{1}) + (\Theta - \theta\mathbb{1})^2 T^{-1} + (\lambda_0 + \varepsilon^{-1}\sigma)\mathbb{1} \\
&= \operatorname{diag}\left(\frac{\lambda_0^2}{\theta + \lambda_0} + \varepsilon^{-1}\sigma, \theta + \lambda_0 + \varepsilon^{-1}\sigma, \theta + \lambda_0 + \varepsilon^{-1}\sigma\right), \\
P_H &= (2\Theta - \theta\mathbb{1}) + (\Theta - \theta\mathbb{1})^2 T^{-1} + \lambda_0\mathbb{1} = \operatorname{diag}\left(\frac{\lambda_0^2}{\theta + \lambda_0}, \theta + \lambda_0, \theta + \lambda_0\right), \\
\tilde{\mathbf{F}}_E &= \mathbf{F}_E + (\theta\mathbb{1} - \Theta)T^{-1}\mathbf{F}_{\xi_E}, \\
\tilde{\mathbf{F}}_H &= \mathbf{F}_H + (\theta\mathbb{1} - \Theta)T^{-1}\mathbf{F}_{\xi_H}
\end{aligned}$$

and obtain

$$\varepsilon P_E \mathbf{E} - \nabla \times \mathbf{H} = \varepsilon \tilde{\mathbf{F}}_E, \tag{3.19a}$$

$$\mathbf{H} + \mu^{-1} P_H^{-1} \nabla \times \mathbf{E} = P_H^{-1} \tilde{\mathbf{F}}_H. \tag{3.19b}$$

From here on the procedure will be the same as in the proof of Lemma 2.2.2. We test (3.19b) with $\nabla \times \boldsymbol{\psi}$, where

$$\boldsymbol{\psi} \in \tilde{\mathrm{V}} = \{\mathbf{E} \in \mathrm{H}(\mathrm{curl}, \Omega) : \ \mathbf{n} \times \mathbf{E} = \mathbf{0} \text{ on } \partial\Omega_E, \ \mathbf{n} \times \mathbf{E} \in \mathrm{L}_2(\partial\Omega_I)^3\},$$

and do an integration by parts

$$\begin{aligned}
\int_\Omega P_H^{-1} \tilde{\mathbf{F}}_H \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} = &\int_\Omega \boldsymbol{\psi} \cdot \nabla \times \mathbf{H} + \mu^{-1} P_H^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} \\
&+ \int_{\partial\Omega} \mathbf{n} \times \boldsymbol{\psi} \cdot \mathbf{H} \,\mathrm{da}\,.
\end{aligned}$$

Now we insert (3.19a) and use the boundary conditions of $\mathbf{E}$, $\mathbf{H}$, and $\boldsymbol{\psi}$

$$\begin{aligned}
\int_\Omega P_H^{-1} \tilde{\mathbf{F}}_H \cdot \nabla \times \boldsymbol{\psi} + \varepsilon \tilde{\mathbf{F}}_E \cdot \boldsymbol{\psi} \,\mathrm{dx} = &\int_\Omega \varepsilon P_E \mathbf{E} \cdot \boldsymbol{\psi} \,\mathrm{dx} \\
&+ \int_\Omega \mu^{-1} P_H^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} \\
&+ \int_{\partial\Omega_I} Z\mathbf{n} \times \boldsymbol{\psi} \cdot \mathbf{n} \times \mathbf{E} \,\mathrm{da}\,.
\end{aligned} \tag{3.20}$$

The space $\tilde{\mathrm{V}}$ equipped with the inner product

$$\begin{aligned}
(\mathbf{E}, \boldsymbol{\psi})_{\tilde{\mathrm{V}},\mathrm{PML}} = &\int_\Omega \varepsilon P_E \mathbf{E} \cdot \boldsymbol{\psi} + \mu^{-1} P_H^{-1} \nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi} \,\mathrm{dx} \\
&+ \int_{\partial\Omega_I} Z\mathbf{n} \times \boldsymbol{\psi} \cdot \mathbf{n} \times \mathbf{E} \,\mathrm{da}
\end{aligned}$$

is a Hilbert space, so Riesz grants a unique solution $\mathbf{E} \in \tilde{\mathrm{V}}$ of (3.20). We define the corresponding magnetic field $\mathbf{H}$ by (3.19b)

$$\mathbf{H} = P_H^{-1}\tilde{\mathbf{F}}_H - \mu^{-1}P_H^{-1}\nabla \times \mathbf{E} \in \mathrm{L}_2(\Omega)^3$$

and the auxiliary function $\boldsymbol{\xi} \in \mathrm{L}_2(\Omega)^6$ by (3.17). With the aid of (3.20), we calculate the weak curl of $\mathbf{H}$. Therefor let $\boldsymbol{\psi} \in \mathrm{C}_0^\infty(\Omega)^3$

$$\begin{aligned}\int_\Omega \mathbf{H} \cdot \nabla \times \boldsymbol{\psi}\,\mathrm{dx} &= \int_\Omega P_H^{-1}\tilde{\mathbf{F}}_H \cdot \nabla \times \boldsymbol{\psi} - \mu^{-1}P_H^{-1}\nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi}\,\mathrm{dx}\\ &= \int_\Omega \varepsilon P_E\mathbf{E} \cdot \boldsymbol{\psi} - \varepsilon\tilde{\mathbf{F}}_E \cdot \boldsymbol{\psi}\,\mathrm{dx}\,.\end{aligned}$$

By the definition of the weak curl we have $\nabla \times \mathbf{H} = \varepsilon P_E\mathbf{E} - \varepsilon\tilde{\mathbf{F}}_E \in \mathrm{L}_2(\Omega)^3$. To assure $(\mathbf{E}, \mathbf{H}, \boldsymbol{\xi}_E, \boldsymbol{\xi}_H) \in \mathcal{D}(B)$ we still have to assure the correct boundary conditions. Again for $\boldsymbol{\psi} \in \tilde{\mathrm{V}} \cap \mathrm{H}^1(\Omega)$, using (3.19a) and (3.20) we have

$$\begin{aligned}\int_\Omega \mathbf{H} \cdot \nabla \times \boldsymbol{\psi}\,\mathrm{dx} &= \int_\Omega \nabla \times \mathbf{H} \cdot \boldsymbol{\psi}\,\mathrm{dx} + \int_{\partial\Omega} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi}\,\mathrm{da}\\ &= \int_\Omega -\varepsilon\tilde{\mathbf{F}}_E \cdot \boldsymbol{\psi} + \varepsilon P_E\mathbf{E} \cdot \boldsymbol{\psi}\,\mathrm{dx} + \int_{\partial\Omega} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi}\,\mathrm{da}\\ &= \int_\Omega P_H^{-1}\tilde{\mathbf{F}}_H \cdot \nabla \times \boldsymbol{\psi} - \mu^{-1}P_H^{-1}\nabla \times \mathbf{E} \cdot \nabla \times \boldsymbol{\psi}\,\mathrm{dx}\\ &\quad - \int_{\partial\Omega_I} Z\mathbf{n} \times \boldsymbol{\psi} \cdot \mathbf{n} \times \mathbf{E}\,\mathrm{da} + \int_{\partial\Omega} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi}\,\mathrm{da}\,.\end{aligned}$$

With the definition of the magnetic field $\mathbf{H}$ by (3.19b), the volume integrals sum up to zero

$$\begin{aligned}\int_{\partial\Omega_I} Z\mathbf{n} \times \mathbf{E} \cdot \mathbf{n} \times \boldsymbol{\psi}\,\mathrm{da} &= \int_{\partial\Omega_I} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi}\,\mathrm{da}\\ &\quad + \int_{\partial\Omega_H} \mathbf{H} \cdot \mathbf{n} \times \boldsymbol{\psi}\,\mathrm{da}\end{aligned}$$

and this yields the boundary conditions $\mathbf{n}\times\mathbf{H} - Z\mathbf{n}\times(\mathbf{n}\times\mathbf{E}) = \mathbf{0}$ on $\partial\Omega_I$ and $\mathbf{n}\times\mathbf{H} = \mathbf{0}$ on $\partial\Omega_H$. So the third assumption of Lumer-Phillips' theorem is fulfilled. □

In the situation with a layer, we have $\omega > 0$ in (3.15). This allows for an exponential growth of the solution in time, compared to the situation of Lemma 2.2.2, where we have $\omega = 0$. In case of homogeneous boundary conditions we now can state existence and uniqueness.

Lemma 3.2.2 (Existence and uniqueness, [RR04, Sec. 12.1.3])**:**
Assume $\mathbf{v}_0 = (\mathbf{E}_0, \mathbf{H}_0, \mathbf{0}, \mathbf{0}) \in \mathcal{D}(B)$*, with support* $\operatorname{supp} \mathbf{u}_0 \subset \Omega_c$*,* $\mathbf{f} \in \mathrm{C}([0,\infty), \mathrm{V})$ *and either* $\mathbf{f} \in \mathrm{W}^{1,1}_{\mathrm{loc}}([0,\infty), \mathrm{V})$ *or* $(\mathbf{f}, \mathbf{0}) \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty), \mathcal{D}(B))$*. Then there exists a unique classical solution* $\mathbf{v} \in \mathrm{C}^1([0,\infty), \mathrm{V}_{\mathrm{PML}}) \cap \mathrm{C}([0,\infty), \mathcal{D}(B))$ *of the PDE* $\partial_t \mathbf{v} + B\mathbf{v} = (\mathbf{f}, \mathbf{0})$*, with initial values* $\mathbf{v}_0$ *given by*

$$\mathbf{v}(t) = \exp(-Bt) \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{0} \end{pmatrix} + \int_0^t \exp\big(-B(t-s)\big) \begin{pmatrix} \mathbf{f}(s) \\ \mathbf{0} \end{pmatrix} \mathrm{d}s\,.$$

In case of non-homogeneous boundary values $(\mathbf{g}_E, \mathbf{g}_H, \mathbf{g}_I) \neq \mathbf{0}$, we again assume to have an extension $\mathbf{u}_B \in \mathrm{C}^1([0,\infty), \mathrm{V}) \cap \mathrm{C}([0,\infty), \mathrm{H}(\mathrm{curl}, \Omega, \partial\Omega))$ (see (2.7)) of the boundary values into the space-time cylinder Ω_∞.

Lemma 3.2.3 (Existence and uniqueness, [RR04, Sec. 12.1.3])**:**
Assume initial values $\mathbf{v}_0 = (\mathbf{E}_0, \mathbf{H}_0, \mathbf{0}, \mathbf{0}) \in \mathrm{H}(\mathrm{curl}, \Omega)^2 \times \mathrm{V}$*, which fit the boundary values, i.e.* $\mathbf{n} \times \mathbf{E}_0|_{\partial\Omega_E} = \mathbf{g}_E(\cdot, 0)$*,* $\mathbf{n} \times \mathbf{H}_0|_{\partial\Omega_H} = \mathbf{g}_H(\cdot, 0)$*,* $\mathbf{n} \times \mathbf{E}_0 \in \mathrm{L}_2(\partial\Omega_I)^3$*, and* $(\mathbf{n} \times \mathbf{H}_0 - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}_0))|_{\partial\Omega_I} = \mathbf{g}_I(\cdot, 0)$*. Further assume the regularity of the right-hand side* $\mathbf{f} \in \mathrm{C}([0,\infty), \mathrm{V})$ *and either* $\mathbf{f} - \partial_t \mathbf{u}_B - A\mathbf{u}_B \in \mathrm{W}^{1,1}_{\mathrm{loc}}([0,\infty), \mathrm{V})$ *or* $(\mathbf{f}, \mathbf{0}) - \partial_t(\mathbf{u}_B, \mathbf{0}) - B(\mathbf{u}_B, \mathbf{0}) \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty), \mathcal{D}(B))$*. Then there exists a unique classical solution* $\mathbf{v} \in \mathrm{C}^1([0,\infty), \mathrm{V}_{\mathrm{PML}}) \cap \mathrm{C}([0,\infty), \mathrm{H}(\mathrm{curl}, \Omega, \partial\Omega) \times \mathrm{V})$ *of the system* $\partial_t \mathbf{v} + B\mathbf{v} = (\mathbf{f}, \mathbf{0})$*, with boundary values* $\mathbf{g}_j$*,* $j = E, H, I$ *given by*

$$\begin{aligned} \mathbf{v}(t) &= \begin{pmatrix} \mathbf{u}_B(t) \\ \mathbf{0} \end{pmatrix} + \mathbf{v}_{\mathrm{hom}}(t), \\ \mathbf{v}_{\mathrm{hom}}(t) &= \exp(-Bt) \begin{pmatrix} \mathbf{u}_{\mathrm{hom},0} \\ \mathbf{0} \end{pmatrix} \\ &\quad + \int_0^t \exp\big(-B(t-s)\big) \left(\begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix} - \partial_t \begin{pmatrix} \mathbf{u}_B \\ \mathbf{0} \end{pmatrix} - B \begin{pmatrix} \mathbf{u}_B \\ \mathbf{0} \end{pmatrix} \right)(s)\, \mathrm{d}s\,. \end{aligned}$$

3.3 A discretization in space with a PML

3.3.1 The finite dimensional space of approximation

There are a lot of similarities to the situation without a layer. Therefore the notation can remain quite similar, despite of some indices B or PML to meet the higher space dimension due to the auxiliary function. For the setting with a layer we use the piecewise polynomial space of approximation

$$\mathrm{V}_{h,\mathrm{PML}}^{p} = \{\mathbf{v}_h \in \mathrm{V}_{\mathrm{PML}} :\ \mathbf{v}_h|_K \in \mathbb{P}_p^{12}(K) \quad \forall K \in \mathcal{T}_h\},$$

together with the L_2-orthogonal projection on $\mathrm{V}_{h,\mathrm{PML}}^{p}$

$$\Pi_h^p : \mathrm{V}_{\mathrm{PML}} \to \mathrm{V}_{h,\mathrm{PML}}^{p},\ \mathbf{v} \mapsto \operatorname*{argmin}_{\mathbf{w}\in \mathrm{V}_{h,\mathrm{PML}}^{p}} \|\mathbf{v} - \mathbf{w}\|_{\mathrm{V}_{\mathrm{PML}}}.$$

3.3.2 Upwind flux with a PML

Since the system $\partial_t \mathbf{v} + B\mathbf{v} = (\mathbf{f}, \mathbf{0})$ is hyperbolic (see page 14 for the definition), we already know the procedure to calculate the corresponding upwind flux and to discretize the system in an upwind scheme. We already did detailed calculations regarding Riemann problem and upwind flux in Sections 2.3.4 and 2.3.5 for Maxwell's equations without a PML. Here, we will transfer these results to the problem with a layer. The operator B can be rewritten similarly to (2.3) as

$$B\mathbf{v} = B_0^{-1} \sum_{j=1}^{3} \partial_{x_j} B_j \mathbf{v} + B_{-1}\mathbf{v}. \tag{3.21}$$

Expressed in terms of A_j (see (2.4)), the matrices B_j are given by

$$B_0 = \begin{pmatrix} A_0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \qquad B_j = \begin{pmatrix} A_j & 0 \\ 0 & 0 \end{pmatrix}, \text{ for } j = 1,2,3, \tag{3.22a}$$

$$B_{-1} = \begin{pmatrix} 2\boldsymbol{\Theta} - \theta\mathbb{1} + A_{-1} & \boldsymbol{\Theta} - \theta\mathbb{1} \\ \theta\mathbb{1} - \boldsymbol{\Theta} & \theta\mathbb{1} - \boldsymbol{\Theta} \end{pmatrix}. \tag{3.22b}$$

Analogously to (2.13), we define the flux $F_{K,f}^{B}$ through a face f for the operator B

$$F_{K,f}^{B}\mathbf{v} = \sum_{j=1}^{3} n_j B_j \mathbf{v} = \begin{pmatrix} F_{K,f}(\mathbf{u}) \\ \mathbf{0} \end{pmatrix} \tag{3.23}$$

and lead our interest to an eigendecomposition of the linear mapping

$$B_0^{-1} F_{K,f}^{B}\mathbf{v} = B_0^{-1} \sum_{j=1}^{3} n_j B_j \mathbf{v} = \begin{pmatrix} A_0^{-1} F_{K,f}(\mathbf{u}) \\ \mathbf{0} \end{pmatrix}.$$

Since we already know the eigendecomposition of $A_0^{-1}F_{K,f}(\mathbf{u})$ (see (2.18)), we can identify the desired eigendecomposition to be

eigenvalue	eigenvector	eigenvalue	eigenvector
$\lambda_1^B = -\frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_1^B = \begin{pmatrix}\mathbf{w}_1\\ \mathbf{0}\end{pmatrix}$	$\lambda_7^B = 0$	$\mathbf{w}_7^B = \begin{pmatrix}\mathbf{0}\\ \mathbf{e}_3\end{pmatrix}$
$\lambda_2^B = -\frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_2^B = \begin{pmatrix}\mathbf{w}_2\\ \mathbf{0}\end{pmatrix}$	$\lambda_8^B = 0$	$\mathbf{w}_8^B = \begin{pmatrix}\mathbf{0}\\ \mathbf{e}_4\end{pmatrix}$
$\lambda_3^B = 0$	$\mathbf{w}_3^B = \begin{pmatrix}\mathbf{w}_3\\ \mathbf{0}\end{pmatrix}$	$\lambda_9^B = 0$	$\mathbf{w}_9^B = \begin{pmatrix}\mathbf{0}\\ \mathbf{e}_5\end{pmatrix}$
$\lambda_4^B = 0$	$\mathbf{w}_4^B = \begin{pmatrix}\mathbf{w}_4\\ \mathbf{0}\end{pmatrix}$	$\lambda_{10}^B = 0$	$\mathbf{w}_{10}^B = \begin{pmatrix}\mathbf{0}\\ \mathbf{e}_6\end{pmatrix}$
$\lambda_5^B = 0$	$\mathbf{w}_5^B = \begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix}$	$\lambda_{11}^B = \frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_{11}^B = \begin{pmatrix}\mathbf{w}_5\\ \mathbf{0}\end{pmatrix}$
$\lambda_6^B = 0$	$\mathbf{w}_6^B = \begin{pmatrix}\mathbf{0}\\ \mathbf{e}_2\end{pmatrix}$	$\lambda_{12}^B = \frac{1}{\sqrt{\mu\varepsilon}}$	$\mathbf{w}_{12}^B = \begin{pmatrix}\mathbf{w}_6\\ \mathbf{0}\end{pmatrix}$.

(3.24)

Similar to Section 2.3.4 we start with two-valued initial values

$$\mathbf{v}_0(\mathbf{x}) = \begin{cases}\mathbf{v}^I, & \mathbf{x}\cdot\mathbf{n} < 0,\\ \mathbf{v}^{IV}, & \mathbf{x}\cdot\mathbf{n} > 0,\end{cases} \qquad (\mu,\varepsilon)(\mathbf{x}) = \begin{cases}(\mu^I,\varepsilon^I), & \mathbf{x}\cdot\mathbf{n} < 0,\\ (\mu^{IV},\varepsilon^{IV}), & \mathbf{x}\cdot\mathbf{n} > 0\end{cases}$$

in the Riemann problem to determine the upwind flux. As time evolves, the initial discontinuity will split into three discontinuities

$$\mathbf{v}(\mathbf{x},t) = \begin{cases}\mathbf{v}^I, & (\mathbf{x},t)\in\Omega_\infty^I,\\ \mathbf{v}^{II}, & (\mathbf{x},t)\in\Omega_\infty^{II},\\ \mathbf{v}^{III}, & (\mathbf{x},t)\in\Omega_\infty^{III},\\ \mathbf{v}^{IV}, & (\mathbf{x},t)\in\Omega_\infty^{IV}.\end{cases}$$

We do a decomposition of the initial jump similar to (2.19)

$$\mathbf{v}^{IV} - \mathbf{v}^I = \alpha_1\mathbf{w}_1^{B,I} + \alpha_2\mathbf{w}_2^{B,I} + \sum_{j=3}^{10}\alpha_j\mathbf{w}_j^B + \alpha_{11}\mathbf{w}_{11}^{B,IV} + \alpha_{12}\mathbf{w}_{12}^{B,IV}$$

and obtain for the intermediate values

$$\begin{aligned}\mathbf{v}^{II} &= \mathbf{v}^I + \alpha_1\mathbf{w}_1^{B,I} + \alpha_2\mathbf{w}_2^{B,I},\\ \mathbf{v}^{III} &= \mathbf{v}^{IV} - \alpha_{11}\mathbf{w}_{11}^{B,IV} - \alpha_{12}\mathbf{w}_{12}^{B,IV}.\end{aligned}$$

In the variable $\mathbf{u}$ this is completely analogous to (2.20), so we can skip the calculations and state the upwind flux for the system with a PML

$$F^{B*}_{K,f}\mathbf{v} = \frac{1}{2}F^{B}_{K,f}\left(\mathbf{v}^{II} + \mathbf{v}^{III}\right) = \begin{pmatrix} F_{K,f}\frac{\mathbf{u}^{II}+\mathbf{u}^{III}}{2} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} F^{*}_{K,f}(\mathbf{u}) \\ \mathbf{0} \end{pmatrix}. \tag{3.25}$$

We formulate this result in a Lemma.

Lemma 3.3.1:
Let $K \in \mathcal{T}_h$ be a cell with face $f \in \mathcal{F}_K$ and material parameters ε_K and μ_K and let $K_f \in \mathcal{T}_h$ be the next neighbour cell of K in direction of f with material parameters ε_{K_f} and μ_{K_f}. Let $\mathbf{v}_h = (\mathbf{u}_h, \boldsymbol{\xi}_h) = (\mathbf{E}_h, \mathbf{H}_h, \boldsymbol{\xi}_{E,h}, \boldsymbol{\xi}_{H,h}) \in \mathrm{V}^p_{h,\mathrm{PML}}$ be a piecewise polynomial function. Then the upwind flux $F^{B*}_{K,f}\mathbf{v}_h$ of $\mathbf{v}_h$ on the face f is given by

$$F^{B*}_{K,f}\mathbf{v}_h = \begin{pmatrix} F^{*}_{K,f}(\mathbf{u}_h) \\ \mathbf{0} \end{pmatrix}, \tag{3.26}$$

where $F^{*}_{K,f}(\mathbf{u}_h)$ was defined in (2.22).

In other words, to define the discrete operator $B_h : \mathrm{V}^p_{h,\mathrm{PML}} \to \mathrm{V}^p_{h,\mathrm{PML}}$ in an upwind scheme, we have to do the same substitution $F_{K,f}(\mathbf{u}) \to F^{*}_{K,f}(\mathbf{u}_h)$, that we already did to define the operator A_h. On the boundary this is not any different, but we have to remember, that we need an extra term in case of non-homogeneous boundary values. Here, we do the substitution $F_{K,f}(\mathbf{u}) \to F^{*}_{K,f}(\mathbf{u}_h)+G^{*}_{j,f}(\mathbf{g}_j)$, where a definition of $G^{*}_{j,f}(\mathbf{g}_j)$, $j = E, H, I$ and the virtual definitions used in $F^{*}_{K,f}(\mathbf{u}_h)$ can be found in Lemma 2.3.6.

3.3.3 The discrete operator B_h

Since for the system with and without PML the upwind fluxes are quite similar, see (3.26), the discretization of the operator B defined in (3.14) is straightforward. We define the operator $B_h : \mathrm{V}^p_{h,\mathrm{PML}} \to \mathrm{V}^p_{h,\mathrm{PML}}$ by

$$B_h\mathbf{v}_h = \begin{pmatrix} (2\boldsymbol{\Theta} - \theta\mathbb{1})\mathbf{u}_h + (\boldsymbol{\Theta} - \theta\mathbb{1})\boldsymbol{\xi}_h + A_h\mathbf{u}_h \\ (\theta\mathbb{1} - \boldsymbol{\Theta})(\mathbf{u}_h + \boldsymbol{\xi}_h) \end{pmatrix}. \tag{3.27}$$

With the vector $\mathbf{G}_h \in \mathrm{V}^p_h$ given by (2.29), we investigate the semi-discrete problem described by the following ordinary differential equation.

The semi-discrete problem *Find* $\mathbf{v}_h \in \mathrm{C}^1([0,\infty), \mathrm{V}^p_{h,\mathrm{PML}})$*, such that*

$$\partial_t \mathbf{v}_h(t) + B_h \mathbf{v}_h(t) = \begin{pmatrix} \mathbf{f}_h(t) \\ \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{G}_h(t) \\ \mathbf{0} \end{pmatrix}, \tag{3.28a}$$

$$\mathbf{v}_h(0) = \Pi^p_h \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{0} \end{pmatrix}. \tag{3.28b}$$

The right-hand side term $\mathbf{f}_h(t) = \Pi^p_h \mathbf{f}(t)$ *again is defined via* $\mathrm{L}_2(\Omega)$*-projection.*

3.3.4 A first rough error estimate

Now that we have a solution $\mathbf{v}$ to the continuous problem (3.13) and a solution $\mathbf{v}_h$ to the semi-discrete problem (3.28) in the shape of (2.31), we would like to show an error estimate similar to Theorem 2.4.1 for the present situation with a PML in x_1-direction. Since the terms of zeroth order in the PDE (3.13a) to (3.13d) allow for an exponential growth of the solution in time, we can reproduce the error estimate in Theorem 2.4.1, if we handle this exponential growth appropriately and repeat the calculations of Lemma 2.4.3. The presented approach is quite standard and can be found e.g. in [VV03]. Nevertheless, the resulting error estimate is very rough. In the context of this work, it serves as a motivation to find another approach to the problem. So we will not perform the calculations in every detail

$$\begin{aligned}
&\|\exp(-\theta_{\sup} t)(\mathbf{v} - \mathbf{v}_h)\|^2_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})} \\
&\quad= -\int_0^T \exp(-2\theta_{\sup} t)\|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{\mathrm{V}_{\mathrm{PML}}} \partial_t \eta_T(t)\,\mathrm{dt} \\
&\quad= -2\theta_{\sup} \int_0^T \exp(-2\theta_{\sup} t)\|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{dt} + T\|\mathbf{v}(0) - \mathbf{v}_h(0)\|^2_{\mathrm{V}_{\mathrm{PML}}} \\
&\qquad+ \int_0^T \exp(-2\theta_{\sup} t)\partial_t \|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{dt} \\
&\quad= -2\theta_{\sup} \int_0^T \exp(-2\theta_{\sup} t)\|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{dt} + T\|\mathbf{v}_0 - \Pi^p_h \mathbf{v}_0\|^2_{\mathrm{V}_{\mathrm{PML}}} \\
&\qquad+ 2\int_0^T \exp(-2\theta_{\sup} t)\big(\partial_t \mathbf{v}(t) - \partial_t \mathbf{v}_h(t), \mathbf{v}(t) - \mathbf{v}_h(t)\big)_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{dt} \\
&\quad= -2\theta_{\sup} \int_0^T \exp(-2\theta_{\sup} t)\|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{dt} + T\|\mathbf{v}_0 - \Pi^p_h \mathbf{v}_0\|^2_{\mathrm{V}_{\mathrm{PML}}} \\
&\qquad+ 2\int_0^T \exp(-2\theta_{\sup} t)\big(B_h \mathbf{v}_h(t) - B\mathbf{v}(t), \mathbf{v}(t) - \mathbf{v}_h(t)\big)_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{dt} \\
&\qquad+ 2\int_0^T \exp(-2\theta_{\sup} t)\big(\mathbf{f}(t) - \mathbf{f}_h(t) + \mathbf{G}_h(t), \mathbf{u}(t) - \mathbf{u}_h(t)\big)_{\mathrm{V}} \eta_T(t)\,\mathrm{dt}
\end{aligned}$$

$$
\begin{aligned}
&= -2\theta_{\text{sup}} \int_0^T \exp(-2\theta_{\text{sup}} t)\|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{d}t + T\|\mathbf{v}_0 - \Pi_h^p \mathbf{v}_0\|^2_{\mathrm{V}_{\mathrm{PML}}} \\
&\quad - 2\int_0^T \exp(-2\theta_{\text{sup}} t)\Big((2\boldsymbol{\Theta} - \theta\mathbb{1})\big(\mathbf{u}(t) - \mathbf{u}_h(t)\big) + (\boldsymbol{\Theta} - \theta\mathbb{1})\big(\boldsymbol{\xi}(t) - \boldsymbol{\xi}_h(t)\big), \\
&\qquad\qquad \mathbf{u}(t) - \mathbf{u}_h(t)\Big)_{\mathrm{V}} \eta_T(t)\,\mathrm{d}t \\
&\quad - 2\int_0^T \exp(-2\theta_{\text{sup}} t)\Big((\theta\mathbb{1} - \boldsymbol{\Theta})\big(\mathbf{u}(t) - \mathbf{u}_h(t) + \boldsymbol{\xi}(t) - \boldsymbol{\xi}_h(t)\big), \\
&\qquad\qquad \boldsymbol{\xi}(t) - \boldsymbol{\xi}_h(t)\Big)_{\mathrm{V}} \eta_T(t)\,\mathrm{d}t \\
&\quad + 2\int_0^T \exp(-2\theta_{\text{sup}} t)\big(\mathbf{f}(t) - \mathbf{f}_h(t) + \mathbf{G}_h(t) + A_h\mathbf{u}_h(t) - A\mathbf{u}(t), \\
&\qquad\qquad \mathbf{u}(t) - \mathbf{u}_h(t)\big)_{\mathrm{V}} \eta_T(t)\,\mathrm{d}t\,.
\end{aligned}
$$

Here, the first, third and fourth summand are bounded from above by zero

$$
\begin{aligned}
0 \geq &-2\theta_{\text{sup}} \int_0^T \exp(-2\theta_{\text{sup}} t)\|\mathbf{v}(t) - \mathbf{v}_h(t)\|^2_{\mathrm{V}_{\mathrm{PML}}} \eta_T(t)\,\mathrm{d}t \\
&- 2\int_0^T \exp(-2\theta_{\text{sup}} t)\Big((2\boldsymbol{\Theta} - \theta\mathbb{1})\big(\mathbf{u}(t) - \mathbf{u}_h(t)\big) + (\boldsymbol{\Theta} - \theta\mathbb{1})\big(\boldsymbol{\xi}(t) - \boldsymbol{\xi}_h(t)\big), \\
&\qquad\qquad \mathbf{u}(t) - \mathbf{u}_h(t)\Big)_{\mathrm{V}} \eta_T(t)\,\mathrm{d}t \\
&- 2\int_0^T \exp(-2\theta_{\text{sup}} t)\Big((\theta\mathbb{1} - \boldsymbol{\Theta})\big(\mathbf{u}(t) - \mathbf{u}_h(t) + \boldsymbol{\xi}(t) - \boldsymbol{\xi}_h(t)\big), \\
&\qquad\qquad \boldsymbol{\xi}(t) - \boldsymbol{\xi}_h(t)\Big)_{\mathrm{V}} \eta_T(t)\,\mathrm{d}t\,.
\end{aligned}
$$

With (2.33) we obtain the estimate

$$
\begin{aligned}
&\|\exp(-\theta_{\text{sup}} t)(\mathbf{v} - \mathbf{v}_h)\|^2_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})} \\
&\qquad \leq 2\int_0^T \exp(-2\theta_{\text{sup}} t)\big(\mathbf{f}(t) - \mathbf{f}_h(t) + \mathbf{G}_h(t) + A_h\mathbf{u}_h(t) - A\mathbf{u}(t), \\
&\qquad\qquad\qquad \mathbf{u}(t) - \mathbf{u}_h(t)\big)_{\mathrm{V}} \eta_T(t)\,\mathrm{d}t \\
&\qquad\quad + T\|\mathbf{v}_0 - \Pi_h^p \mathbf{v}_0\|^2_{\mathrm{V}_{\mathrm{PML}}} \\
&\qquad \leq 2\int_0^T \exp(-2\theta_{\text{sup}} t)\big(\mathbf{G}_h(t) + A_h\mathbf{u}_h(t) - A\mathbf{u}(t), \mathbf{u}(t) - \mathbf{u}_h(t)\big)_{\mathrm{V}} \eta_T(t)\,\mathrm{d}t \\
&\qquad\quad + 2\int_0^T \exp(-2\theta_{\text{sup}} t)\|\mathbf{f}(t) - \mathbf{f}_h(t)\|^2_{\mathrm{V}} \eta_T^2(t)\,\mathrm{d}t \\
&\qquad\quad + \frac{1}{2}\int_0^T \exp(-2\theta_{\text{sup}} t)\|\mathbf{u}(t) - \mathbf{u}_h(t)\|^2_{\mathrm{V}}\,\mathrm{d}t \\
&\qquad\quad + T\|\mathbf{v}_0 - \Pi_h^p \mathbf{v}_0\|^2_{\mathrm{V}_{\mathrm{PML}}}.
\end{aligned}
$$

Now, we substract the third summand on the right-hand side to obtain an inequality similar to the one of Lemma 2.4.3

$$\begin{aligned}\|\exp(-\theta_{\sup} t)(\mathbf{v}-\mathbf{v}_h)\|^2_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})} \\ \leq 4\int_0^T \exp(-2\theta_{\sup} t)\big(\mathbf{G}_h(t)+A_h\mathbf{u}_h(t)-A\mathbf{u}(t),\mathbf{u}(t)-\mathbf{u}_h(t)\big)_{\mathrm{V}}\eta_T(t)\,\mathrm{dt} \\ +4\int_0^T \exp(-2\theta_{\sup} t)\|\mathbf{f}(t)-\mathbf{f}_h(t)\|^2_{\mathrm{V}}\eta_T^2(t)\,\mathrm{dt}+2T\|\mathbf{v}_0-\Pi_h^p\mathbf{v}_0\|^2_{\mathrm{V}_{\mathrm{PML}}}.\end{aligned}$$

Proceeding with Lemma 2.3.1 and 2.3.2 and the estimates in (2.34) and Lemma 2.4.6 we obtain the estimate

$$\begin{aligned}\|\exp(-\theta_{\sup} t)(\mathbf{v}-\mathbf{v}_h)\|^2_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})} \\ \leq 4c_{\mathrm{light}}c^2_{\mathrm{boundary}}h^{2s-1}\int_0^T \exp(-2\theta_{\sup} t)|\mathbf{u}(t)|^2_{s,\mathrm{V},\Lambda}\eta_T(t)\,\mathrm{dt} \\ +4c^2_{\mathrm{cell}}h^{2s-1}\int_0^T \exp(-2\theta_{\sup} t)|\mathbf{f}(t)|^2_{s-\frac{1}{2},\mathrm{V},\Lambda}\eta_T^2(t)\,\mathrm{dt} \\ +2Tc^2_{\mathrm{cell}}h^{2s}|\mathbf{v}_0|^2_{s,\mathrm{V}_{\mathrm{PML}},\Lambda},\end{aligned}$$

where c_{light} was given in (2.36). The reason why we consider this estimate to rough is the exponential term on the left-hand side. Brought to the right-hand side it turns into a factor $\exp(\theta_{\sup} T)$. For a typical calculation over a time $T = 1$ this factor reaches values of exp(100) and more. Since in simulations the PML performs much better, there is the desire to find a better error bound. We will come back to that issue in chapter 5.

4 Numerical tests

Content of this chapter To investigate the properties of the PML in numerical calculations, we tested it with M++, a library programmed and used at our institute. The main focus in this chapter is on an exact solution to the PML system, which has a sharp angle of incidence into the layer. One advantage is the exact knowledge of the angle dependent damping behaviour of the layer. Another one is the existence of real non-reflecting boundary conditions for that sharp angle of incidence. That way, we can choose the parameters of the layer appropriately, so that the errors from the layer are of the same order as the overall discretization errors. Our exact solution will be presented in Section 4.1 and for the rest of this chapter we use it as a numerical test setting.

Origin of this chapter Except for the idea, the presented exact solution to the system with a layer is my contribution. The C++ code to simulate the problem without a layer was mainly written by Ekkachai Thawinan and Christian Wieners. My contribution was the addition of the layer.

4.1 An exact solution in a half-space with a PML

In Section 3.1.3 we already introduced initial values in form of a hat function, that we use for testing. The hat function spreads in every direction and therefore does not hit the layer in a sharp angle. Since the damping of the layer depends on the angle of incidence, we like to introduce another test setting, where we only have one direction of propagation and knowledge of the exact solution. This exact solution will be defined on the half-space $\Omega_{\mathrm{HS}} = \overline{\Omega}^c_{\mathrm{HS}} \cup \Omega^{\mathrm{PML}}_{\mathrm{HS}}$, where the corresponding subsets are definded by $\Omega^c_{\mathrm{HS}} = \{\mathbf{x} \in \mathbb{R}^3 : \ x_1 < a_1\}$ and $\Omega^{\mathrm{PML}}_{\mathrm{HS}} = \{\mathbf{x} \in \mathbb{R}^3 : \ a_1 < x_1 < a_1 + d\}$. The space-time cylinder will be denoted by $\Omega_{\mathrm{HS},\infty} = \Omega_{\mathrm{HS}} \times (0, \infty)$.

The setting is supposed to be two-dimensional, i.e. the fields are homogeneous in x_3-direction. On the boundary behind the layer we use PEC- as well as impedance boundary conditions.

Exact solution with PEC boundary In our test setting we set $\varepsilon = \mu = 1$, $\sigma = 0$, and $\mathbf{f}_E = \mathbf{f}_H = \mathbf{0}$. The parameter function $\theta(x_1)$ is a bounded, non-negative, piecewise continuous function of x_1, that vanishes outside of the layer, i.e.

$$0 \leq \theta(x_1) \leq \theta_{\sup}, \qquad \theta(x_1) \text{ piecewise continuous}, \qquad \theta|_{\Omega_{\text{HS}}^c} = 0.$$

We seek a solution to the system

$$\begin{aligned}
\partial_t \mathbf{E} - \nabla \times \mathbf{H} + (2\Theta - \theta\mathbb{1})\mathbf{E} + (\Theta - \theta\mathbb{1})\boldsymbol{\xi}_E &= \mathbf{0} && \text{in } \Omega_{\text{HS},\infty}, && (4.1a)\\
\partial_t \mathbf{H} + \nabla \times \mathbf{E} + (2\Theta - \theta\mathbb{1})\mathbf{H} + (\Theta - \theta\mathbb{1})\boldsymbol{\xi}_H &= \mathbf{0} && \text{in } \Omega_{\text{HS},\infty}, && (4.1b)\\
\partial_t \boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\mathbf{E} &= \mathbf{0} && \text{in } \Omega_{\text{HS},\infty}, && (4.1c)\\
\partial_t \boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\mathbf{H} &= \mathbf{0} && \text{in } \Omega_{\text{HS},\infty}, && (4.1d)\\
\mathbf{n} \times \mathbf{E} &= \mathbf{0} && \text{on } (\partial\Omega_{\text{HS}})_\infty, && (4.1e)\\
\mathbf{E}(\cdot, 0) &= \mathbf{E}_0 && \text{in } \Omega_{\text{HS}}, && (4.1f)\\
\mathbf{H}(\cdot, 0) &= \mathbf{H}_0 && \text{in } \Omega_{\text{HS}}, && (4.1g)\\
\boldsymbol{\xi}_E(\cdot, 0) &= \boldsymbol{\xi}_{E,0} && \text{in } \Omega_{\text{HS}}, && (4.1h)\\
\boldsymbol{\xi}_H(\cdot, 0) &= \boldsymbol{\xi}_{H,0} && \text{in } \Omega_{\text{HS}}. && (4.1i)
\end{aligned}$$

To specify the initial values we decompose Ω_{HS} into seven parts (see Fig. 4.1). Therefor we fix a point $\mathbf{x}_0 \in \partial\Omega_{\text{HS}}$ and choose the direction of propagation

$$\mathbf{e}_\mathbf{k} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \end{pmatrix}$$

for the incoming wavefront, where $\varphi \in (-\frac{1}{2}\pi, 0) \cup (0, \frac{1}{2}\pi)$. The reflected wave will propagate in the direction

$$\mathbf{e}_\mathbf{k}^* = \begin{pmatrix} -\cos\varphi & \sin\varphi & 0 \end{pmatrix}.$$

The thickness of the wave will be denoted by $d_s > 2d$. We define our decomposition of Ω_{HS} as follows

$$\begin{aligned}
\Omega_0 = \{&\mathbf{x} \in \Omega_{\text{HS}}^c : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} > 0\}\\
&\cup \{\mathbf{x} \in \Omega_{\text{HS}}^c : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} < -d_s, \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* > 0\}\\
&\cup \{\mathbf{x} \in \Omega_{\text{HS}}^c : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* < -d_s\}\\
&\cup \{\mathbf{x} \in \Omega_{\text{HS}}^{\text{PML}} : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} > 0\},
\end{aligned} \qquad (4.2a)$$

$$\begin{aligned}
\Omega_1 &= \{\mathbf{x} \in \Omega_{\text{HS}}^c : \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} < 0, \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* > 0\}, && (4.2b)\\
\Omega_2 &= \{\mathbf{x} \in \Omega_{\text{HS}}^c : \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} < 0, \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* < 0\}, && (4.2c)\\
\Omega_3 &= \{\mathbf{x} \in \Omega_{\text{HS}}^c : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} < -d_s, \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* < 0\}, && (4.2d)\\
\Omega_4 &= \{\mathbf{x} \in \Omega_{\text{HS}}^{\text{PML}} : \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} < 0, \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* > 0\}, && (4.2e)\\
\Omega_5 &= \{\mathbf{x} \in \Omega_{\text{HS}}^{\text{PML}} : \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} < 0, \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* < 0\}, && (4.2f)\\
\Omega_6 &= \{\mathbf{x} \in \Omega_{\text{HS}}^{\text{PML}} : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k} < -d_s, \ -d_s < (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* < 0\}, && (4.2g)\\
\Omega_7 &= \{\mathbf{x} \in \Omega_{\text{HS}}^{\text{PML}} : \ (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}_\mathbf{k}^* < -d_s\}. && (4.2h)
\end{aligned}$$

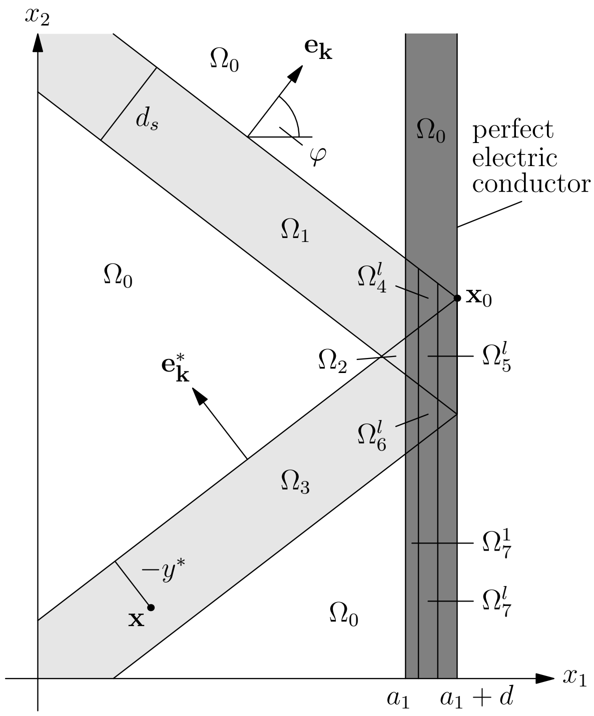


Figure 4.1: Decomposition of Ω_{HS} to specify initial values.

The lower bound $d_s > 2d$ ensures Ω_2 to be non-empty. Since our PML-parameter only is piecewise continuous, we decompose the sets Ω_4 to Ω_7 again into a finite number of sets Ω_j^l, $j = 4, \dots, 7$ in a way, that $\theta(x_1)|_{\Omega_j^l}$ is continuos. To define the initial values for our test setting, we first specify - up to some factor - the electromagnetic field of the incoming wave in Ω_1. Let $\mathbf{E_k}$ and $\mathbf{H_k}$ be defined by

$$\mathbf{E_k} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{H_k} = \begin{pmatrix} \sin\varphi & -\cos\varphi & 0 \end{pmatrix}. \tag{4.3}$$

So $\mathbf{e_k}$, $\mathbf{E_k}$ and $\mathbf{H_k}$ form a right-handed orthonormal system. Also note, that $\mathbf{E_k}$ and $\mathbf{H_k}$ form an eigenvector (3.24) of the flux operator (3.23) with eigenvalue $\lambda_{11}^B = \lambda_{12}^B = 1$

$$\begin{pmatrix} \mathbf{E_k} & \mathbf{H_k} & \mathbf{0} & \mathbf{0} \end{pmatrix} \in \operatorname{span}\left(\mathbf{w}_{11}^B, \mathbf{w}_{12}^B\right),$$

if we take $\mathbf{n} = \mathbf{e_k}$ to be the normal vector. For the reflected wave we define

$$\mathbf{E}^*_\mathbf{k} = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix} = -\mathbf{E_k}, \quad \mathbf{H}^*_\mathbf{k} = \begin{pmatrix} -\sin\varphi & -\cos\varphi & 0 \end{pmatrix} = \begin{pmatrix} -H_{k,1} & H_{k,2} & H_{k,3} \end{pmatrix},$$

so that $\mathbf{e}^*_\mathbf{k}$, $\mathbf{E}^*_\mathbf{k}$ and $\mathbf{H}^*_\mathbf{k}$ also form a right-handed orthonormal system. The damping factors induced by the layer will be expressed by the functions

$$D(x) = \exp\left(-\int_{a_1}^{x} \theta(x_1)\,\mathrm{dx}_1 \cos\varphi\right), \qquad D^*(x) = D(a_1 + d)^2 D(x)^{-1}$$

for the incident and the reflected wave, respectively. A scalar function

$$s \in \mathrm{C}^1([-d_s, 0])$$

will describe the shape of the incident wave. To shorten notation we define the integral

$$S(x) = \int_0^x s(y)\,\mathrm{d}y$$

and the coordinates

$$y(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e_k}, \qquad y^*(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{e}^*_\mathbf{k}$$

and now are able to define our initial values $\mathbf{u}_0 = (\mathbf{E}_0, \mathbf{H}_0)$ and $\boldsymbol{\xi}_0 = (\boldsymbol{\xi}_{E,0}, \boldsymbol{\xi}_{H,0})$.

Before doing so, we like to explain the idea behind our test setting in more detail. The system in (4.1) outside of the layer coincides with the one we discussed in the context of the Riemann problem in Section 2.3.4. In the vacuum region we therefore know about the travelling behaviour of a discontinuity in one direction with homogeneity in the other two directions. Instead of a discontinuity we can choose any shape s to travel through Ω^c_{HS}. This shape will be placed in Ω_1 and form the incoming wave. A cross section of Ω_1 is shown in Fig. 4.2. The shape of the

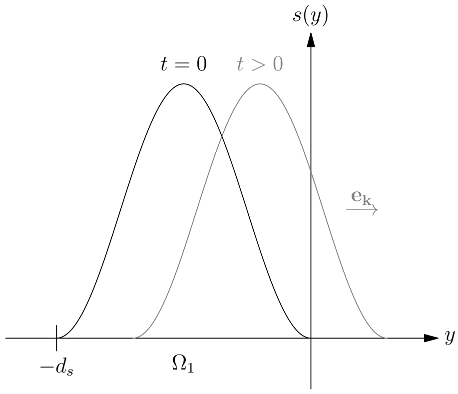


Figure 4.2: In the domain Ω_1 we place initial values with an amplitude in the shape of s. As time evolves, these initial values will travel in the direction of $\mathbf{e_k}$.

incoming wave is supposed to be homogeneous in the directions orthogonal to $\mathbf{e_k}$. Our choice of $\mathbf{E_k}$ and $\mathbf{H_k}$ in (4.3) will ensure, that the incoming wave travels in the direction of $\mathbf{e_k}$. Since we prescribe the angle of incidence φ, we also know the effect of the layer on that incoming wave. It will induce a damping with the factor $D(x_1)$. The reflections on the boundary behind the layer were already discussed in Chapter 1, page 8.

Finally, we state our initial values as follows

$$(\mathbf{u}_0+\boldsymbol{\xi}_0)(\mathbf{x}) = \begin{cases} \mathbf{0} & \text{in } \Omega_0, \\ s\big(y(\mathbf{x})\big)\begin{pmatrix}\mathbf{E}_\mathbf{k}\\ \mathbf{H}_\mathbf{k}\end{pmatrix} & \text{in } \Omega_1, \\ s\big(y(\mathbf{x})\big)\begin{pmatrix}\mathbf{E}_\mathbf{k}\\ \mathbf{H}_\mathbf{k}\end{pmatrix} + s\big(y^*(\mathbf{x})\big)D(a_1+d)^2\begin{pmatrix}\mathbf{E}^*_\mathbf{k}\\ \mathbf{H}^*_\mathbf{k}\end{pmatrix} & \text{in } \Omega_2, \\ s\big(y^*(\mathbf{x})\big)D(a_1+d)^2\begin{pmatrix}\mathbf{E}^*_\mathbf{k}\\ \mathbf{H}^*_\mathbf{k}\end{pmatrix} & \text{in } \Omega_3, \\ s\big(y(\mathbf{x})\big)D(x_1)\begin{pmatrix}\mathbf{E}_\mathbf{k}\\ \mathbf{H}_\mathbf{k}\end{pmatrix} & \text{in } \Omega_4, \\ s\big(y(\mathbf{x})\big)D(x_1)\begin{pmatrix}\mathbf{E}_\mathbf{k}\\ \mathbf{H}_\mathbf{k}\end{pmatrix} + s\big(y^*(\mathbf{x})\big)D^*(x_1)\begin{pmatrix}\mathbf{E}^*_\mathbf{k}\\ \mathbf{H}^*_\mathbf{k}\end{pmatrix} & \text{in } \Omega_5, \\ s\big(y^*(\mathbf{x})\big)D^*(x_1)\begin{pmatrix}\mathbf{E}^*_\mathbf{k}\\ \mathbf{H}^*_\mathbf{k}\end{pmatrix} & \text{in } \Omega_6, \\ \mathbf{0} & \text{in } \Omega_7. \end{cases} \tag{4.4a}$$

The corresponding auxiliary function $\boldsymbol{\xi}$ can be calculated with (4.1c) and (4.1d)

$$\boldsymbol{\xi}_0(\mathbf{x}) = \begin{cases} \mathbf{0} & \text{in } \Omega_0, \\ \mathbf{0} & \text{in } \Omega_1, \\ \mathbf{0} & \text{in } \Omega_2, \\ \mathbf{0} & \text{in } \Omega_3, \\ \theta(x_1)D(x_1)H_{k,1}S\big(y(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_4^l, \\ \theta(x_1)\big[D(x_1)H_{k,1}S\big(y(\mathbf{x})\big)+D^*(x_1)H^*_{k,1}S\big(y^*(\mathbf{x})\big)\big]\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_5^l, \\ \theta(x_1)\big[D(x_1)H_{k,1}S(-d_s)+D^*(x_1)H^*_{k,1}S\big(y^*(\mathbf{x})\big)\big]\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_6^l, \\ \theta(x_1)\big[D(x_1)H_{k,1}S(-d_s)+D^*(x_1)H^*_{k,1}S(-d_s)\big]\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_7^l. \end{cases} \tag{4.4b}$$

Since θ is allowed to be discontinuous, we obtain discontinuities in the auxiliary function $\boldsymbol{\xi}_0$. Other discontinuities may appear on the edges of the incoming and reflected wave, when the shape function s does not connect continuously to zero. For simulations, though, it is not advisable to have discontinuities crossing some cell K. In Ω_3 we can identify the damping factor $D_{\text{tot}}^{\text{PEC}}$ for the electromagnetic field by

$$D_{\text{tot}}^{\text{PEC}} = D(a_1+d)^2 = \exp\left(-2\int_{a_1}^{a_1+d}\theta(x_1)\,\mathrm{dx}_1\cos\varphi\right). \tag{4.5}$$

As time evolves these initial values will travel with a speed $\hat{v}_2 = \frac{1}{\sin\varphi}$ in x_2-direction, though this is not to be confused with the physical travelling direction of the wave. Physically, the stripe Ω_1 travels in the direction of $\mathbf{e_k}$ and Ω_3 in the direction of $\mathbf{e}^*_\mathbf{k}$, both with a velocity of $\lambda^B_{11} = \lambda^B_{12} = 1$.

Lemma 4.1.1:
With the initial values in (4.4) *a weak solution to the system* (4.1) *is given by*

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x} - \hat{v}_2 t\, \mathbf{e}_2). \tag{4.6}$$

To refer to this exact solution, we use the notation $\mathbf{v}^{\mathrm{PEC}}_{\mathrm{exact}}(\mathbf{x}, t)$. Though we already explained the construction of our exact solution, we do detailed calculations in the proof.

Proof: To transform (4.1) into a weak formulation, we multiply (4.1a) to (4.1d) with a test function $\boldsymbol{\phi} = (\boldsymbol{\phi}_u, \boldsymbol{\phi}_\xi) = (\boldsymbol{\phi}_E, \boldsymbol{\phi}_H, \boldsymbol{\phi}_{\xi_E}, \boldsymbol{\phi}_{\xi_H}) \in \mathrm{C}^\infty_0(\overline{\Omega}_{\mathrm{HS}} \times [0, \infty))^{12}$ with constraint $\mathbf{n} \times \boldsymbol{\phi}_E = \mathbf{0}$ on $(\partial\Omega_{\mathrm{HS}})_\infty$, integrate over space and time and integrate by parts to obtain

$$-\int_0^\infty \int_{\Omega_{\mathrm{HS}}} \mathbf{v} \cdot \partial_t \boldsymbol{\phi} \,\mathrm{dx}\,\mathrm{dt} - \int_{\Omega_{\mathrm{HS}}} \mathbf{v}_0 \cdot \boldsymbol{\phi}(\mathbf{x}, 0) \,\mathrm{dx} + \int_0^\infty \int_{\Omega_{\mathrm{HS}}} \mathbf{v} \cdot B^* \boldsymbol{\phi} \,\mathrm{dx}\,\mathrm{dt} = 0. \tag{4.7}$$

The operator B^* is defined by

$$B^*\mathbf{v} = \begin{pmatrix} (2\boldsymbol{\Theta} - \theta\mathbb{1})\mathbf{u} + (\theta\mathbb{1} - \boldsymbol{\Theta})\boldsymbol{\xi} - A\mathbf{u} \\ (\theta\mathbb{1} - \boldsymbol{\Theta})(\boldsymbol{\xi} - \mathbf{u}) \end{pmatrix}.$$

We seek solutions $\mathbf{v} \in \mathrm{L}^1_{loc}(\overline{\Omega}_{\mathrm{HS}} \times [0, \infty))^{12}$ and intend to show that $\mathbf{v}^{\mathrm{PEC}}_{\mathrm{exact}}(\mathbf{x}, t)$, as given in (4.6) is such a solution. Let us define a decomposition of $\Omega_{\mathrm{HS},\infty}$ by

$$\Omega_{j,\infty} = \{(\mathbf{x}, t) \in \Omega_{\mathrm{HS},\infty} : \ \mathbf{x} - \hat{v}_2 t\, \mathbf{e}_2 \in \Omega_j\}, \quad j = 0, \ldots, 7.$$

The sets $\Omega^l_{j,\infty}$, $j = 4, \ldots, 7$ are defined in the same way. Note, that the index ∞ in this context does describe a sheared space-time cylinder. Now we calculate the pointwise derivatives of $\mathbf{v}(\mathbf{x}, t)$ inside these $\Omega_{j,\infty}$ and $\Omega^l_{j,\infty}$. Therefor we define the shifted coordinates

$$y_t(\mathbf{x}) = y(\mathbf{x} - \hat{v}_2 t\mathbf{e}_2), \qquad y^*_t(\mathbf{x}) = y^*(\mathbf{x} - \hat{v}_2 t\mathbf{e}_2)$$

and with the derivative s' of s obtain the time derivatives

$$\begin{aligned}\partial_t \mathbf{u}(\mathbf{x},t) \\ + \partial_t \boldsymbol{\xi}(\mathbf{x},t)\end{aligned} = \begin{cases} \mathbf{0} & \text{in } \Omega_{0,\infty}, \\ -s'\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} & \text{in } \Omega_{1,\infty}, \\ -s'\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} - D(a_1+d)^2 s'\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_{2,\infty}, \\ -D(a_1+d)^2 s'\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_{3,\infty}, \\ -D(x_1)s'\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} & \text{in } \Omega_{4,\infty}, \\ -D(x_1)s'\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} - D^*(x_1)s'\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_{5,\infty}, \\ -D^*(x_1)s'\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_{6,\infty}, \\ \mathbf{0} & \text{in } \Omega_{7,\infty}, \end{cases}$$

$$\partial_t \boldsymbol{\xi}(\mathbf{x},t) = \begin{cases} \mathbf{0} & \text{in } \Omega_{0,\infty}, \\ \mathbf{0} & \text{in } \Omega_{1,\infty}, \\ \mathbf{0} & \text{in } \Omega_{2,\infty}, \\ \mathbf{0} & \text{in } \Omega_{3,\infty}, \\ -\theta(x_1)D(x_1)H_{k,1}s\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_{4,\infty}^l, \\ -\theta(x_1)\big[D(x_1)H_{k,1}s\big(y_t(\mathbf{x})\big) + D^*(x_1)H_{k,1}^*s\big(y_t^*(\mathbf{x})\big)\big]\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_{5,\infty}^l, \\ -\theta(x_1)D^*(x_1)H_{k,1}^*s\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_{6,\infty}^l, \\ \mathbf{0} & \text{in } \Omega_{7,\infty}^l \end{cases}$$

and the curl of the electric field

$$\nabla\times\mathbf{E}(\mathbf{x},t) = \begin{cases} \mathbf{0} & \text{in } \Omega_{0,\infty}, \\ s'\big(y_t(\mathbf{x})\big)\mathbf{e_k}\times\mathbf{E_k} & \text{in } \Omega_{1,\infty}, \\ s'\big(y_t(\mathbf{x})\big)\mathbf{e_k}\times\mathbf{E_k} + s'\big(y_t^*(\mathbf{x})\big)D(a_1+d)^2\mathbf{e_k^*}\times\mathbf{E_k^*} & \text{in } \Omega_{2,\infty}, \\ s'\big(y_t^*(\mathbf{x})\big)D(a_1+d)^2\mathbf{e_k^*}\times\mathbf{E_k^*} & \text{in } \Omega_{3,\infty}, \\ s'\big(y_t(\mathbf{x})\big)D(x_1)\mathbf{e_k}\times\mathbf{E_k} - s\big(y_t(\mathbf{x})\big)\theta(x_1)\cos\varphi\, D(x_1)\mathbf{e}_1\times\mathbf{E_k} & \text{in } \Omega_{4,\infty}^l, \\ \begin{aligned} & s'\big(y_t(\mathbf{x})\big)D(x_1)\mathbf{e_k}\times\mathbf{E_k} - s\big(y_t(\mathbf{x})\big)\theta(x_1)\cos\varphi\, D(x_1)\mathbf{e}_1\times\mathbf{E_k} \\ & + s'\big(y_t^*(\mathbf{x})\big)D^*(x_1)\mathbf{e_k^*}\times\mathbf{E_k^*} + s\big(y_t^*(\mathbf{x})\big)\theta(x_1)\cos\varphi\, D^*(x_1)\mathbf{e}_1\times\mathbf{E_k^*}\end{aligned} & \text{in } \Omega_{5,\infty}^l, \\ s'\big(y_t^*(\mathbf{x})\big)D^*(x_1)\mathbf{e_k^*}\times\mathbf{E_k^*} + s\big(y_t^*(\mathbf{x})\big)\theta(x_1)\cos\varphi\, D^*(x_1)\mathbf{e}_1\times\mathbf{E_k^*} & \text{in } \Omega_{6,\infty}^l, \\ \mathbf{0} & \text{in } \Omega_{7,\infty}. \end{cases}$$

Except for the additional auxiliary function, we obtain the same curl for the magnetic field, but have to exchange $\mathbf{E}$ by $\mathbf{H}$

$$\nabla \times \big(\mathbf{H}(\mathbf{x},t) + \boldsymbol{\xi}_H(\mathbf{x},t)\big) = \nabla \times \mathbf{E}(\mathbf{x},t)\big|_{\mathbf{E_k}=\mathbf{H_k},\ \mathbf{E}^*_\mathbf{k}=\mathbf{H}^*_\mathbf{k}},$$

where the curl of the auxiliary function is given by

$$\nabla \times \boldsymbol{\xi}_H(\mathbf{x},t) = \begin{cases} \mathbf{0} & \text{in } \Omega_{0,\infty}, \\ \mathbf{0} & \text{in } \Omega_{1,\infty}, \\ \mathbf{0} & \text{in } \Omega_{2,\infty}, \\ \mathbf{0} & \text{in } \Omega_{3,\infty}, \\ \theta(x_1) H_{k,1} D(x_1) s\big(y_t(\mathbf{x})\big)\mathbf{e_k} \times \mathbf{e}_1 & \text{in } \Omega^l_{4,\infty}, \\ \theta(x_1)\big[H_{k,1} D(x_1) s\big(y_t(\mathbf{x})\big)\mathbf{e_k} + H^*_{k,1} D^*(x_1) s\big(y^*_t(\mathbf{x})\big)\mathbf{e}^*_\mathbf{k}\big] \times \mathbf{e}_1 & \text{in } \Omega^l_{5,\infty}, \\ \theta(x_1) H^*_{k,1} D^*(x_1) s\big(y^*_t(\mathbf{x})\big)\mathbf{e}^*_\mathbf{k} \times \mathbf{e}_1 & \text{in } \Omega^l_{6,\infty}, \\ \mathbf{0} & \text{in } \Omega^l_{7,\infty}. \end{cases}$$

We also specify the terms of zeroth-order

$$\begin{aligned}(2\boldsymbol{\Theta} - \theta\mathbb{1})\mathbf{u}(\mathbf{x},t) \\ + (\boldsymbol{\Theta} - \theta\mathbb{1})\boldsymbol{\xi}(\mathbf{x},t)\end{aligned} = \begin{cases} \mathbf{0} & \text{in } \Omega_{0,\infty}, \\ \mathbf{0} & \text{in } \Omega_{1,\infty}, \\ \mathbf{0} & \text{in } \Omega_{2,\infty}, \\ \mathbf{0} & \text{in } \Omega_{3,\infty}, \\ \theta(x_1) D(x_1) s\big(y_t(\mathbf{x})\big) \begin{pmatrix} \mathbf{E_k} \\ \mathbf{H}^*_\mathbf{k} \end{pmatrix} & \text{in } \Omega^l_{4,\infty}, \\ \theta(x_1) \left[D(x_1) s\big(y_t(\mathbf{x})\big) \begin{pmatrix} \mathbf{E_k} \\ \mathbf{H}^*_\mathbf{k} \end{pmatrix} + D^*(x_1) s\big(y^*_t(\mathbf{x})\big) \begin{pmatrix} \mathbf{E}^*_\mathbf{k} \\ \mathbf{H_k} \end{pmatrix} \right] & \text{in } \Omega^l_{5,\infty}, \\ \theta(x_1) D^*(x_1) s\big(y^*_t(\mathbf{x})\big) \begin{pmatrix} \mathbf{E}^*_\mathbf{k} \\ \mathbf{H_k} \end{pmatrix} & \text{in } \Omega^l_{6,\infty}, \\ \mathbf{0} & \text{in } \Omega^l_{7,\infty}, \end{cases}$$

$$\begin{aligned}(\theta\mathbb{1} - \boldsymbol{\Theta})\mathbf{u}(\mathbf{x},t) \\ + (\theta\mathbb{1} - \boldsymbol{\Theta})\boldsymbol{\xi}(\mathbf{x},t)\end{aligned} = \begin{cases} \mathbf{0} & \text{in } \Omega_{0,\infty}, \\ \mathbf{0} & \text{in } \Omega_{1,\infty}, \\ \mathbf{0} & \text{in } \Omega_{2,\infty}, \\ \mathbf{0} & \text{in } \Omega_{3,\infty}, \\ \theta(x_1) D(x_1) s\big(y_t(\mathbf{x})\big) \sin\varphi \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{pmatrix} & \text{in } \Omega^l_{4,\infty}, \\ \theta(x_1) \sin\varphi \big[D(x_1) s\big(y_t(\mathbf{x})\big) - D^*(x_1) s\big(y^*_t(\mathbf{x})\big)\big] \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{pmatrix} & \text{in } \Omega^l_{5,\infty}, \\ -\theta(x_1) D^*(x_1) s\big(y^*_t(\mathbf{x})\big) \sin\varphi \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{pmatrix} & \text{in } \Omega^l_{6,\infty}, \\ \mathbf{0} & \text{in } \Omega^l_{7,\infty} \end{cases}$$

and use the relations

$$\begin{aligned}
&\mathbf{e}_1 \times \mathbf{E_k} = -\mathbf{e}_2, \quad \mathbf{e}_1 \times \mathbf{E_k^*} = \mathbf{e}_2, \quad \mathbf{e}_1 \times \mathbf{H_k} = -\cos\varphi\, \mathbf{e}_3, \quad \mathbf{e}_1 \times \mathbf{H_k^*} = -\cos\varphi\, \mathbf{e}_3,\\
&\mathbf{e_k} \times \mathbf{E_k} = \mathbf{H_k}, \quad \mathbf{e_k} \times \mathbf{H_k} = -\mathbf{E_k}, \quad \mathbf{e_k} \times \mathbf{e}_1 = -\sin\varphi\, \mathbf{e}_3,\\
&\mathbf{e_k^*} \times \mathbf{E_k^*} = \mathbf{H_k^*}, \quad \mathbf{e_k^*} \times \mathbf{H_k^*} = -\mathbf{E_k^*}, \quad \mathbf{e_k^*} \times \mathbf{e}_1 = -\sin\varphi\, \mathbf{e}_3
\end{aligned}$$

to confirm, that equations (4.1a) to (4.1d) are fulfilled pointwise in each $\Omega_{j,\infty}$ and $\Omega^l_{j,\infty}$, respectively. Next, we calculate the jumps on the interfaces to confirm Rankine-Hugoniot's jump conditions (see also (2.17)). On each interface we have to confirm the jump $[\mathbf{v}]$ to be an eigenvector of the flux operator $F^B_{K,f}(\cdot)$ (see (3.24) for the eigendecomposition). The corresponding eigenvalue has to be the travelling speed of the discontinuity, e.g. we have to confirm

$$F^B_{K,f}[\mathbf{v}] = \mathbf{0}$$

on $\partial\Omega_{4,\infty} \cap \partial\Omega_{1,\infty}$. To calculate the jumps on the interfaces, we need an orientation. Here, with $k > j$ we use the convention that $[\mathbf{v}](\mathbf{x},t) = \mathbf{v}|_{\Omega_{k,\infty}}(\mathbf{x},t) - \mathbf{v}|_{\Omega_{j,\infty}}(\mathbf{x},t)$ on $\partial\Omega_{k,\infty} \cap \partial\Omega_{j,\infty}$ and as well $[\mathbf{v}](\mathbf{x},t) = \mathbf{v}|_{\Omega^{l+1}_{j,\infty}}(\mathbf{x},t) - \mathbf{v}|_{\Omega^l_{j,\infty}}(\mathbf{x},t)$ on $\partial\Omega^{l+1}_{j,\infty} \cap \partial\Omega^l_{j,\infty}$. By $\theta(x_1+)$ and $\theta(x_1-)$ we denote the right- and left-hand side limit of θ at x_1, respectively, as well as the jump by $[\theta](x_1) = \theta(x_1+) - \theta(x_1-)$. For $\mathbf{v}$ we obtain the jumps

$$[\mathbf{u}](\mathbf{x},t) = \begin{cases}
s\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} & \text{on } \partial\Omega_{1,\infty} \cap \partial\Omega_{0,\infty},\\
s(0)D(a_1+d)^2\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{on } \partial\Omega_{2,\infty} \cap \partial\Omega_{1,\infty},\\
s\big(y_t^*(\mathbf{x})\big)D(a_1+d)^2\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{on } \partial\Omega_{3,\infty} \cap \partial\Omega_{0,\infty},\\
-\,s(-d_s)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} & \text{on } \partial\Omega_{3,\infty} \cap \partial\Omega_{2,\infty},\\
s(0)D(x_1)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} & \text{on } \partial\Omega^l_{4,\infty} \cap \partial\Omega_{0,\infty},\\
-\,\theta(x_1+)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{4,\infty} \cap \partial\Omega_{1,\infty},\\
-\,([\theta](x_1)D(x_1)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega^{l+1}_{4,\infty} \cap \partial\Omega^l_{4,\infty},\\
-\,\theta(x_1+)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\
\quad -\,\theta(x_1+)D(a_1+d)^2H^*_{k,1}S\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{5,\infty} \cap \partial\Omega_{2,\infty},
\end{cases}$$

$$
[\mathbf{u}](\mathbf{x},t) = \begin{cases}
s(0)D^*(x_1)\begin{pmatrix}\mathbf{E}_\mathbf{k}^*\\ \mathbf{H}_\mathbf{k}^*\end{pmatrix} & \text{on } \partial\Omega_{5,\infty}^l \cap \partial\Omega_{4,\infty}^l,\\
-[\theta](x_1)D(x_1)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\
-[\theta](x_1)D^*(x_1)H_{k,1}^*S\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{5,\infty}^{l+1} \cap \partial\Omega_{5,\infty}^l,\\
-\theta(x_1+)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\
-\theta(x_1+)D(a_1+d)^2H_{k,1}^*S\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{6,\infty} \cap \partial\Omega_{3,\infty},\\
-s(-d_s)D(x_1)\begin{pmatrix}\mathbf{E}_\mathbf{k}\\ \mathbf{H}_\mathbf{k}\end{pmatrix} & \text{on } \partial\Omega_{6,\infty}^l \cap \partial\Omega_{5,\infty}^l,\\
-[\theta](x_1)D(x_1)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\
-[\theta](x_1)D^*(x_1)H_{k,1}^*S\big(y_t^*(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{6,\infty}^{l+1} \cap \partial\Omega_{6,\infty}^l,\\
-\theta(x_1+)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\
-\theta(x_1+)D(a_1+d)^2H_{k,1}^*S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{7,\infty} \cap \partial\Omega_{0,\infty},\\
-s(-d_s)D^*(x_1)\begin{pmatrix}\mathbf{E}_\mathbf{k}^*\\ \mathbf{H}_\mathbf{k}^*\end{pmatrix} & \text{on } \partial\Omega_{7,\infty}^l \cap \partial\Omega_{6,\infty}^l,\\
-[\theta](x_1)D(x_1)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\
-[\theta](x_1)D^*(x_1)H_{k,1}^*S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{7,\infty}^{l+1} \cap \partial\Omega_{7,\infty}^l,
\end{cases}
$$

$$
[\boldsymbol{\xi}](\mathbf{x},t) = \begin{cases}
\mathbf{0} & \text{on } \partial\Omega_{1,\infty} \cap \partial\Omega_{0,\infty},\\
\mathbf{0} & \text{on } \partial\Omega_{2,\infty} \cap \partial\Omega_{1,\infty},\\
\mathbf{0} & \text{on } \partial\Omega_{3,\infty} \cap \partial\Omega_{0,\infty},\\
\mathbf{0} & \text{on } \partial\Omega_{3,\infty} \cap \partial\Omega_{2,\infty},\\
\mathbf{0} & \text{on } \partial\Omega_{4,\infty}^l \cap \partial\Omega_{0,\infty},\\
\theta(x_1+)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{4,\infty} \cap \partial\Omega_{1,\infty},\\
[\theta](x_1)D(x_1)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{4,\infty}^{l+1} \cap \partial\Omega_{4,\infty}^l,
\end{cases}
$$

$$[\boldsymbol{\xi}](\mathbf{x},t) = \begin{cases} \theta(x_1+)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\ \quad + \theta(x_1+)D(a_1+d)^2 H^*_{k,1}S\big(y^*_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{5,\infty}\cap\partial\Omega_{2,\infty}, \\ \mathbf{0} & \text{on } \partial\Omega^l_{5,\infty}\cap\partial\Omega^l_{4,\infty}, \\ [\theta](x_1)D(x_1)H_{k,1}S\big(y_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\ \quad + [\theta](x_1)D^*(x_1)H^*_{k,1}S\big(y^*_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega^{l+1}_{5,\infty}\cap\partial\Omega^l_{5,\infty}, \\ \theta(x_1+)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\ \quad + \theta(x_1+)D(a_1+d)^2 H^*_{k,1}S\big(y^*_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{6,\infty}\cap\partial\Omega_{3,\infty}, \\ \mathbf{0} & \text{on } \partial\Omega^l_{6,\infty}\cap\partial\Omega^l_{5,\infty}, \\ [\theta](x_1)D(x_1)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\ \quad + [\theta](x_1)D^*(x_1)H^*_{k,1}S\big(y^*_t(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega^{l+1}_{6,\infty}\cap\partial\Omega^l_{6,\infty}, \\ \theta(x_1+)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\ \quad + \theta(x_1+)D(a_1+d)^2 H^*_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega_{7,\infty}\cap\partial\Omega_{0,\infty}, \\ \mathbf{0} & \text{on } \partial\Omega^l_{7,\infty}\cap\partial\Omega^l_{6,\infty}, \\ [\theta](x_1)D(x_1)H_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \\ \quad + [\theta](x_1)D^*(x_1)H^*_{k,1}S(-d_s)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{on } \partial\Omega^{l+1}_{7,\infty}\cap\partial\Omega^l_{7,\infty}. \end{cases}$$

Now we can check the jump conditions on every interface and see, that they are fulfilled. To verify the weak formulation (4.7), we use the following integration by parts formula (the index l is optional)

$$\int_{\Omega^l_{j,\infty}} \mathbf{v}\cdot B^*\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} = \int_{\Omega^l_{j,\infty}} B\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} - \int_{\partial\Omega^l_{j,\infty}} \frac{1}{\sqrt{1+\lambda^2}} F^B_{K,f}\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{da}\,.$$

Considering the space-time boundary $\partial(\Omega_{j,\infty})$ or $\partial(\Omega^l_{j,\infty})$ as a travelling boundary only in space, i.e. e.g. the time-dependent set $\Gamma_j(t) = \{\mathbf{x}\in\overline{\Omega}_{\mathrm{HS}} :\ (\mathbf{x},t)\in\partial(\Omega_{j,\infty})\}$, λ denotes the local traveling speed of that boundary in normal direction. We have to set $\lambda=\infty$ on $\partial(\Omega_{j,\infty})\cap\{(\mathbf{x},t)\in\mathbb{R}^4 :\ t=0\}$ for the formula to be correct. Note, that the normal vector in the definition of $F^B_{K,f}(\mathbf{u})$ is only the spatial normal vector. The space-time normal vector again is denoted by

$$\mathbf{n}_T = \frac{1}{\sqrt{1+\lambda^2}}\begin{pmatrix}\mathbf{n} & -\lambda\end{pmatrix},$$

with the special case $\mathbf{n}_T = (\mathbf{0}, -1)$ for $\lambda = \infty$. We start with the left-hand side of (4.7)

$$\begin{aligned}
-\int_{\Omega_{\mathrm{HS},\infty}} &\mathbf{v}\cdot\partial_t\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} - \int_{\Omega_{\mathrm{HS}}} \mathbf{v}_0\cdot\boldsymbol{\phi}(\mathbf{x},0)\,\mathrm{dx} + \int_{\Omega_{\mathrm{HS},\infty}} \mathbf{v}\cdot B^*\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} \\
&= -\sum_{j=0}^{7}\sum_{l}\left[\int_{\Omega^l_{j,\infty}} \mathbf{v}\cdot\partial_t\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} + \int_{\Omega^l_j}\mathbf{v}_0\cdot\boldsymbol{\phi}(\mathbf{x},0)\,\mathrm{dx} - \int_{\Omega^l_{j,\infty}}\mathbf{v}\cdot B^*\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt}\right] \\
&= \sum_{j=0}^{7}\sum_{l}\left[\int_{\Omega^l_{j,\infty}} \partial_t\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} - \int_{\partial\Omega^l_{j,\infty}} n_{T,4}\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{da} - \int_{\Omega^l_j}\mathbf{v}_0\cdot\boldsymbol{\phi}(\mathbf{x},0)\,\mathrm{dx}\right. \\
&\qquad\qquad \left. + \int_{\Omega^l_{j,\infty}} B\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{dx}\,\mathrm{dt} - \int_{\partial\Omega^l_{j,\infty}} \frac{1}{\sqrt{1+\lambda^2}} F^B_{K,f}\mathbf{v}\cdot\boldsymbol{\phi}\,\mathrm{da}\right] \\
&\overset{!}{=} 0.
\end{aligned}$$

Regarding $\partial_t\mathbf{v} + B\mathbf{v} = \mathbf{0}$ in each $\Omega_{j,\infty}$ and $\Omega^l_{j,\infty}$, the Rankine-Hugoniot jump conditions, and the initial and boundary values, we see that the weak formulation (4.7) is fulfilled. □

Exact solution with impedance boundary Again, we set $\varepsilon = \mu = 1$, $\sigma = 0$, $\mathbf{f}_E = \mathbf{f}_H = \mathbf{0}$ and use a bounded, non-negative, piecewise continuous parameter function $\theta(x_1)$, that vanishes outside of the layer. We seek a solution to the system

$$\begin{aligned}
\partial_t\mathbf{E} - \nabla\times\mathbf{H} + (2\Theta - \theta\mathbb{1})\mathbf{E} + (\Theta - \theta\mathbb{1})\boldsymbol{\xi}_E &= \mathbf{0} && \text{in } \Omega_{\mathrm{HS},\infty}, && \text{(4.8a)}\\
\partial_t\mathbf{H} + \nabla\times\mathbf{E} + (2\Theta - \theta\mathbb{1})\mathbf{H} + (\Theta - \theta\mathbb{1})\boldsymbol{\xi}_H &= \mathbf{0} && \text{in } \Omega_{\mathrm{HS},\infty}, && \text{(4.8b)}\\
\partial_t\boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_E + (\theta\mathbb{1} - \Theta)\mathbf{E} &= \mathbf{0} && \text{in } \Omega_{\mathrm{HS},\infty}, && \text{(4.8c)}\\
\partial_t\boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\boldsymbol{\xi}_H + (\theta\mathbb{1} - \Theta)\mathbf{H} &= \mathbf{0} && \text{in } \Omega_{\mathrm{HS},\infty}, && \text{(4.8d)}\\
\mathbf{n}\times\mathbf{H} - \mathbf{n}\times(\mathbf{n}\times\mathbf{E}) &= \mathbf{0} && \text{on } (\partial\Omega_{\mathrm{HS}})_\infty, && \text{(4.8e)}\\
\mathbf{E}(\cdot,0) &= \mathbf{E}_0 && \text{in } \Omega_{\mathrm{HS}}, && \text{(4.8f)}\\
\mathbf{H}(\cdot,0) &= \mathbf{H}_0 && \text{in } \Omega_{\mathrm{HS}}, && \text{(4.8g)}\\
\boldsymbol{\xi}_E(\cdot,0) &= \boldsymbol{\xi}_{E,0} && \text{in } \Omega_{\mathrm{HS}}, && \text{(4.8h)}\\
\boldsymbol{\xi}_H(\cdot,0) &= \boldsymbol{\xi}_{H,0} && \text{in } \Omega_{\mathrm{HS}}. && \text{(4.8i)}
\end{aligned}$$

Compared to system (4.1) we only changed the boundary condition (4.1e). But we will use altered initial values as well. The reflected wave obtains an additional factor

$\alpha(\varphi)$ defined in (4.10). With the same decomposition of Ω_{HS} (see (4.2) and Fig. 4.1), we can formulate our initial values for the electromagnetic field

$$(\mathbf{u}_0+\boldsymbol{\xi}_0)(\mathbf{x}) = \begin{cases} \mathbf{0} & \text{in } \Omega_0, \\ s\big(y(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} & \text{in } \Omega_1, \\ s\big(y(\mathbf{x})\big)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} + \alpha(\varphi)s\big(y^*(\mathbf{x})\big)D(a_1+d)^2\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_2, \\ \alpha(\varphi)s\big(y^*(\mathbf{x})\big)D(a_1+d)^2\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_3, \\ s\big(y(\mathbf{x})\big)D(x_1)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} & \text{in } \Omega_4, \\ s\big(y(\mathbf{x})\big)D(x_1)\begin{pmatrix}\mathbf{E_k}\\ \mathbf{H_k}\end{pmatrix} + \alpha(\varphi)s\big(y^*(\mathbf{x})\big)D^*(x_1)\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_5, \\ \alpha(\varphi)s\big(y^*(\mathbf{x})\big)D^*(x_1)\begin{pmatrix}\mathbf{E_k^*}\\ \mathbf{H_k^*}\end{pmatrix} & \text{in } \Omega_6, \\ \mathbf{0} & \text{in } \Omega_7 \end{cases} \tag{4.9a}$$

and for the auxiliary function

$$\boldsymbol{\xi}_0(\mathbf{x}) = \begin{cases} \mathbf{0} & \text{in } \Omega_0, \\ \mathbf{0} & \text{in } \Omega_1, \\ \mathbf{0} & \text{in } \Omega_2, \\ \mathbf{0} & \text{in } \Omega_3, \\ \theta(x_1)D(x_1)H_{k,1}S\big(y(\mathbf{x})\big)\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_4^l, \\ \theta(x_1)\big[D(x_1)H_{k,1}S\big(y(\mathbf{x})\big) + \alpha(\varphi)D^*(x_1)H_{k,1}^*S\big(y^*(\mathbf{x})\big)\big]\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_5^l, \\ \theta(x_1)\big[D(x_1)H_{k,1}S(-d_s) + \alpha(\varphi)D^*(x_1)H_{k,1}^*S\big(y^*(\mathbf{x})\big)\big]\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_6^l, \\ \theta(x_1)S(-d_s)[D(x_1)H_{k,1} + \alpha(\varphi)D^*(x_1)H_{k,1}^*]\begin{pmatrix}\mathbf{0}\\ \mathbf{e}_1\end{pmatrix} & \text{in } \Omega_7^l. \end{cases} \tag{4.9b}$$

The function $\alpha(\varphi)$ is given by (see (1.9))

$$\alpha(\varphi) = \frac{1-\cos\varphi}{1+\cos\varphi} \tag{4.10}$$

and plotted in Figure 4.3. Again, in Ω_3 we can identify the damping factor for the electromagnetic field. This time it also contains the function $\alpha(\varphi)$. The total

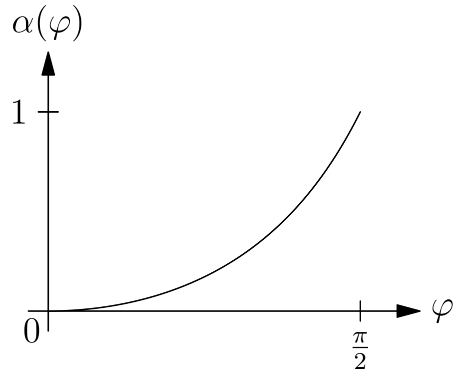


Figure 4.3: The damping factor $\alpha(\varphi)$ shows the damping by the impedance boundary and depends on the angle of incidence.

damping factor now is given by

$$D_{\text{tot}}^{\text{Impedance}} = \alpha(\varphi) D(a_1 + d)^2 = \frac{1 - \cos\varphi}{1 + \cos\varphi} \exp\left(-2 \int_{a_1}^{a_1+d} \theta(x_1)\,\mathrm{dx}_1 \cos\varphi\right). \quad (4.11)$$

In regard of a weak solution to the system (4.8), we can state the same lemma as before.

Lemma 4.1.2:
With the initial values in (4.9) *a weak solution to the system* (4.8) *is given by*

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x} - \hat{v}_2 t\, \mathbf{e}_2). \quad (4.12)$$

Here again, the initial values travel with a speed $\hat{v}_2 = \frac{1}{\sin\varphi}$ *in* x_2*-direction.*

To refer to this exact solution, we use the notation $\mathbf{v}_{\text{exact}}^{\text{Impedance}}(\mathbf{x}, t)$.

Proof: The proof is analogous to the one of Lemma 4.1.1, so we skip the details. □

The exact solution we use here can also be adjusted to have a fully absorbing boundary (FAB) without any layer. This can be done with an impedance boundary and a choice of $Z = \cos(\varphi)$ (see (1.9)). We call the corresponding exact solution $\mathbf{v}_{\text{exact}}^{\text{FAB}}(\mathbf{x}, t)$.

4.2 Numerical test setting

In our numerical tests, the PML parameter $\theta(x_1)$ is supposed to be piecewise constant, in especially constant on every cell K. To perform these tests, we restrict the exact solutions (4.6) and (4.12) to a rectangle $\Omega_c = (0, 1-d) \times (0, 0.5)$, $\Omega_{\mathrm{PML}} = (1-d, 1) \times (0, 0.5)$ in the x_1x_2-plane (see Fig. 4.4). On the outer boundary

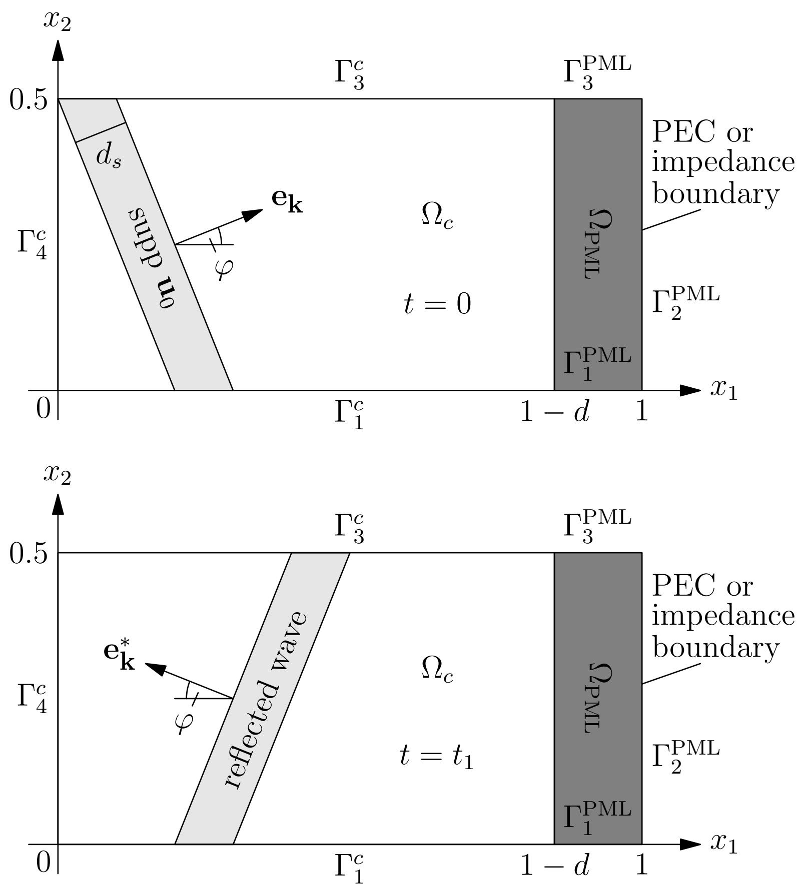


Figure 4.4: To perform numerical tests on the layer, we use an incoming initial wave with precise angle of incidence φ. The initial wave will travel towards the layer, where it will be damped and reflected at the boundary. To measure the reflections of the layer, we stop the test, when the reflected wave is still contained in Ω_c.

of the PML, we use either homogeneous PEC or impedance boundary conditions, i.e.

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \qquad \text{or} \qquad \mathbf{n} \times \mathbf{H} - Z\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = \mathbf{0} \qquad \text{on } \Gamma_2^{\mathrm{PML}}.$$

We use $Z = 1$ to test the PML. In case we want to test an optimal absorbing boundary (that only works for our special test setting), we use $Z = \cos(\varphi)$. On all the other parts of the boundary, we use the exact inhomogeneous PEC boundary condition with values taken from the exact solutions (4.6) and (4.12), i.e.

$$\mathbf{n} \times \mathbf{E} = \mathbf{n} \times \mathbf{E}^j_{\text{exact}} \qquad \text{on } \Gamma^c_1 \cup \Gamma^{\text{PML}}_1 \cup \Gamma^{\text{PML}}_3 \cup \Gamma^c_3 \cup \Gamma^c_4,$$

$j \in \{\text{PEC}, \text{Impedance}, \text{FAB}\}$. Here we only use a PEC boundary, because the impedance boundary absorbs energy and therefore as well reflections from the layer. We also like to mention, that in this context the expression 'exact' is not fully correct, since in our code we use quadrature rules to evaluate integrals like in the definition of $\mathbf{G}_h$ in (2.29). So instead of the exact values we use polynomial interpolations of the exact values. The same holds true for the initial values. Instead of the $L_2(\Omega)$-projection of the continuous initial values, we use as well an interpolation, but we will not go into any more details on that topic. We take our initial values from the exact solutions and choose the parameter $\mathbf{x}_0$, which shifts the initial values in x_2-direction, in a way that we only have an incoming initial wave. In terms of the second component of $\mathbf{x}_0$, we use

$$x_{0,2} = \frac{1}{2} - \frac{\cos\varphi - d_s}{\sin\varphi}.$$

The simulation stops, when the reflected wave is still contained in Ω_c, though the corresponding time t_1 depends on the angle of incidence φ. Here, we set $t_1(15°) = 1.2$, $t_1(30°) = 1.1$, $t_1(45°) = 0.75$ and are interested in the values of the relative reflections of layer and boundary in the L_2-norm

$$R_{\text{rel}} = \frac{\|\mathbf{u}_h\big(t_1(\varphi)\big)\|_{2,\Omega_c}}{\|\mathbf{u}_h(0)\|_{2,\Omega_c}}. \tag{4.13}$$

With a perfect absorbing layer and without discretization errors the reflections R_{rel} would vanish, whereas for a fully reflecting boundary we would obtain $R_{\text{rel}} = 1$. As shape s of the initial wave, we use a squared sine profile, that connects continuously to zero

$$s(y) = \sin^2\left(\frac{\pi y}{d_s}\right),$$

with $d_s = 0.3$ the width of the initial wave. Our triangular mesh is illustrated in the non-refined state (black) and after one step of refinement (grey) in Fig. 4.5. In each step of refinement the faces of a cell K are bisected and the middle points are connected. So we have a mesh parameter of $h_l = 2^{-l-\frac{1}{2}} = 2^{-l}h_0$, where l is the level of refinement.

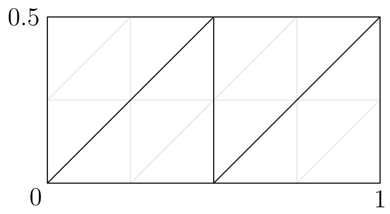
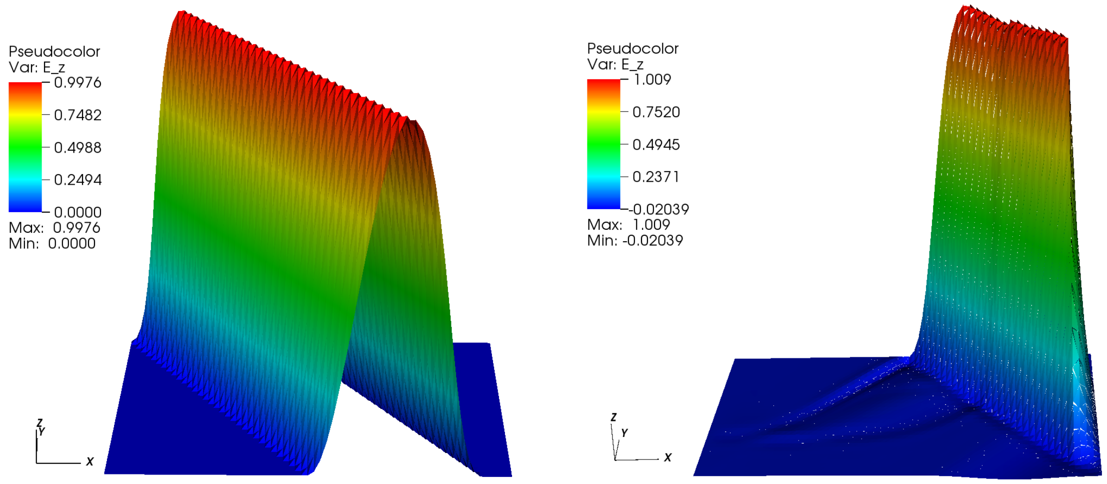


Figure 4.5: We use a triangular mesh, that is shown in the coarsest version in black and after one step of refinement in grey.

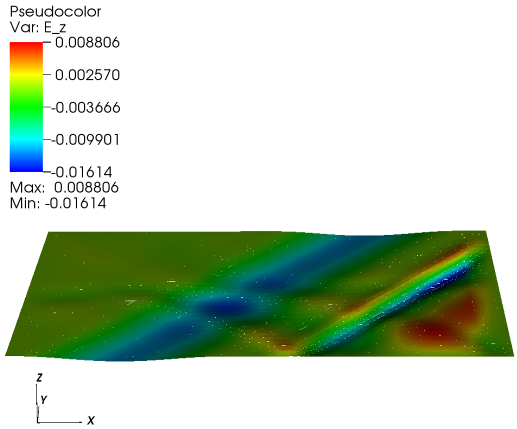
As long as not mentioned otherwise, the time integration will be done with explicit Runge-Kutta methods with time step size τ and of order r_k, e.g. $r_k = 1$ denotes the explicit Euler method. In general our choice of the order in time will be $r_k = p + 1$ and we assume, that we can neglect the error from time discretization over the error from space discretization. The number of layer cells in x_1-direction is denoted by n_c, so that $d = 2^{-l-2} n_c$. An illustration of our exact PML solution can be found in Fig. 4.6, 4.7, and 4.8.



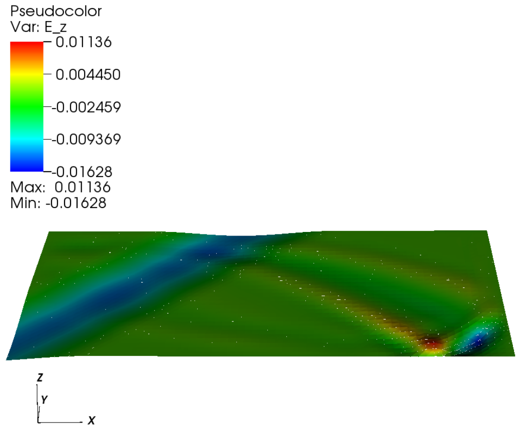
(a) The picture shows the $\sin^2$-profile of E_3 at time $t = 0$.

(b) At time $t = 0.35$ the initial wave has already reached the layer. On the right-hand side we see the exponential damping behaviour. On the left-hand side of the wave, we see errors with origin at the boundary.

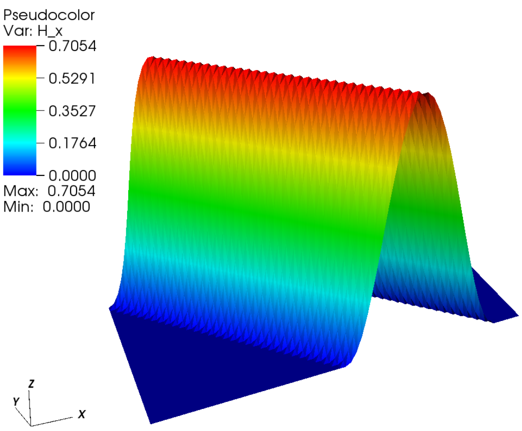
Figure 4.6: The pictures show the E_3-field at several times t as it behaves, when it hits a layer with parameter $\theta = 50$ over a thickness of eight cells. We used a mesh of level $l = 5$, a polynomial degree $p = 1$, an angle of incidence $\varphi = 45°$, and a PEC boundary.



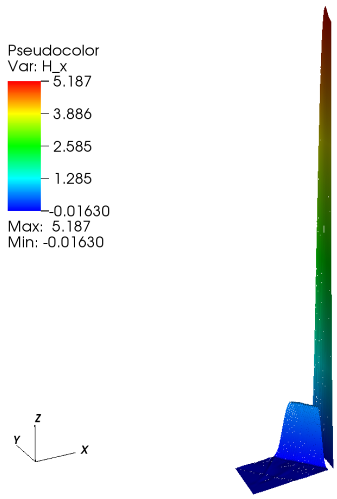
(a) After a time $t = 0.75$, we see the left travelling reflected wave on the left-hand side. The disturbances on the right-hand side are leftovers from the boundary errors.



(b) At time $t = 0.95$ the boundary errors almost dissappeared in the layer, but the reflected wave also starts to leave the domain.

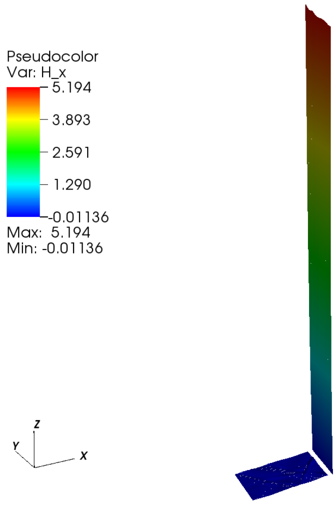


(c) The profile of H_1 at time $t = 0$ is similar to E_3, but with a factor $2^{-\frac{1}{2}}$.

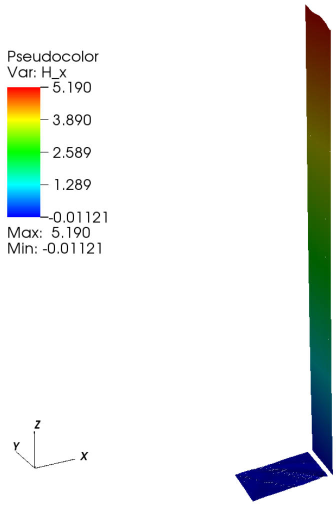


(d) Since the auxiliary function is included in H_1, at time $t = 0.35$ we can compare the magnitudes of auxiliary function and electromagnetic field.

Figure 4.7: The first two illustrations show the E_3-field, the latter two the H_1-field at several times t.



(a) As soon as the initial wave has fully crossed the layer, the auxiliary function $\xi_{H,1}$ only depends on x_1. Here, we see the time $t = 0.75$.



(b) Until the time $t = 0.95$ and later on the auxiliary function will not change any more.

Figure 4.8: We see the H_1-field for larger times t, which is dominated by the auxiliary function.

4.3 Numerical results

4.3.1 Goals of the numerical tests

We test the behaviour of the PML for several values of θ, several angles of incident, several mesh sizes and up to a third order spatial approximation. We compare the behaviour of the impedance boundary to the often used PEC boundary, have a look at the improvement achieved by a non-constant parameter $\theta(x_1)$ compared to a constant one and test the implicit midpoint rule as a representative for implicit time integration schemes. We do not test any order of convergence. Our goal is a basic understanding of the layers behaviour.

4.3.2 First order approximation in space

We compare the reflections (4.13) for several different parameter settings. In Table 4.1 we tested with a first order approximation in space for several one-valued layer parameters θ and several angles of incident φ. Looking at the first angle $\varphi = 15°$,

$R_{\text{rel}} \cdot 10^2$	$\theta = 50$	75	100	125	150	175	200	225
$\varphi = 15°$	22.17	10.44	4.964	2.511	1.632	**1.520**	1.650	1.822
$\varphi = 30°$	25.95	13.21	6.766	3.589	2.208	**1.846**	1.922	2.115
$\varphi = 45°$	33.27	19.17	11.09	6.511	4.035	2.909	**2.635**	2.785

Table 4.1: Parameters: $l = 5$, $p = 1$, $r_k = 2$, $\tau = 0.001$, $n_c = 2$, $d = 64^{-1}$, PEC.

we see decreasing reflections as the layer parameter increases up to $\theta = 175$, since the analytical damping behaviour of the layer is improved. With further increasing layer parameter, the reflections will increase again, due to increasing discretization errors. Here are two kinds of reflections involved. The first kind already exists in the continuous model. The layer damps incoming waves, but does not absorb them completely. For a reduction of these reflections we desire a large parameter θ. The values for the analytical reflections can be calculated with (4.5) and (4.11) and are presented in Table 4.2. These are the values of R_{rel} in (4.13), if the approximate solution $\mathbf{u}_h$ is replaced by the analytical solution $\mathbf{u}$. The second kind of reflection

$D_{\text{tot}}^{\text{PEC}} \cdot 10^2$	$\theta = 50$	75	100	125	150	175	200	225
$\varphi = 15°$	22.11	10.39	4.887	2.298	1.080	0.5080	0.2389	0.1123
$\varphi = 30°$	25.84	13.14	6.678	3.395	1.726	0.8773	0.4460	0.2267
$\varphi = 45°$	33.13	19.07	10.97	6.316	3.635	2.092	1.204	0.6930

Table 4.2: Parameters: $d = 64^{-1}$, PEC.

is due to the numerical discretization and expected to increase with increasing θ. Therefore the value of θ has to be choosen carefully to get an equal effect of both kinds of reflections. In order to make a good choice, we have to consider e.g. the order of discretization, the level of mesh refinement, and the thickness of the layer. As mentioned on page 88, in our testsetting we can use non-reflecting boundary conditions to obtain values for the discretization error that is not related to the layer. Here, for a first order approximation with mesh level $l = 5$ we obtain

$$R_{\text{rel}}^{\text{FAB}}(15°) = 0.2277 \cdot 10^{-2}, \quad R_{\text{rel}}^{\text{FAB}}(30°) = 0.2972 \cdot 10^{-2}, \quad R_{\text{rel}}^{\text{FAB}}(45°) = 0.7552 \cdot 10^{-2}.$$

The latter value is larger than the other two, since at time $t = 0.75$ the errors arising at the boundary Γ_3^c are still contained in Ω_c (see Fig. 4.7(a)). To compare the PEC- with the impedance boundary, we did the same calculations as before in Table 4.3. Here, we see a more significant dependence of the optimal parameter θ from the angle of incidence φ. The smaller the angle φ, the smaller the optimal θ, since

$R_{\text{rel}} \cdot 10^2$	$\theta = 50$	75	100	125	150	175	200	225
$\varphi = 15°$	0.4693	**0.4186**	0.5579	0.7597	0.9766	1.191	1.395	1.586
$\varphi = 30°$	1.905	1.054	**0.7767**	0.8549	1.077	1.347	1.629	1.904
$\varphi = 45°$	5.772	3.408	2.135	1.571	**1.485**	1.674	2.002	2.399

Table 4.3: Parameters: $l = 5$, $p = 1$, $r_k = 2$, $\tau = 0.001$, $n_c = 2$, $d = 64^{-1}$, Impedance.

the analytical reflection is more and more reduced by the impedance boundary (see Fig. 4.3 for the corresponding extra damping factor). In applications there is no reason to throw away the absorbing effect of the impedance boundary, since there is no additional computational cost in comparison with a PEC boundary. Of course, for applicational purposes there is no use to distinguish between different angles of incidence, since the distribution in φ is in general not known. But depending on the geometry of the application, angles around $\varphi \approx 90°$ are unlikely to appear and waves of normal incidence are completely absorbed by the impedance boundary, so one might focus to optimize θ for an angle around $\varphi \approx 45°$. In case of a larger layer we obtain the values shown in Table 4.4. Compared to Table 4.1, the optimal values

$R_{\text{rel}} \cdot 10^2$	$\theta = 25$	50	75	100	125	150	175	200
$\varphi = 15°$	4.913	0.3447	**0.3438**	0.5259	0.7575	1.006	1.252	1.487
$\varphi = 30°$	6.719	0.5427	**0.3735**	0.5487	0.8015	1.092	1.392	1.684
$\varphi = 45°$	11.05	1.396	**0.7166**	0.7888	0.9699	1.255	1.627	2.053

Table 4.4: Parameters: $l = 5$, $p = 1$, $r_k = 2$, $\tau = 0.001$, $n_c = 8$, $d = 16^{-1}$, PEC.

for θ decreased. This is quite expected, since the analytical reflections decrease, whereas the numerical reflections stay almost the same for a fixed θ. Inside the layer we have an exponential decay in the fields, so the numerical reflections basically arise from the vacuum layer interface and the first layer cells.

4.3.3 h-dependence

There is one interesting effect to mention, when looking at different mesh sizes. Comparing Table 4.1 with Tables 4.5 and 4.6 for finer meshes, we almost see the same values in case we double the parameter θ in each mesh refinement. For small θ we expect that behaviour, since we have almost only analytical reflections and for

$R_{\text{rel}} \cdot 10^2$	$\theta = 100$	150	200	250	300	350	400	450
$\varphi = 15°$	22.12	10.41	4.936	2.493	1.623	**1.513**	1.640	1.813
$\varphi = 30°$	25.87	13.16	6.726	3.560	2.193	**1.845**	1.931	2.129
$\varphi = 45°$	33.16	19.09	11.01	6.418	3.918	2.767	**2.495**	2.664

Table 4.5: Parameters: $l = 6$, $p = 1$, $r_k = 2$, $\tau = 0.0005$, $n_c = 2$, $d = 128^{-1}$, PEC.

$R_{\text{rel}} \cdot 10^2$	$\theta = 200$	300	400	500	600	700	800	900
$\varphi = 15°$	22.10	10.39	4.926	2.488	1.625	**1.518**	1.644	1.816
$\varphi = 30°$	25.84	13.14	6.713	3.552	2.194	**1.854**	1.945	2.144
$\varphi = 45°$	33.13	19.06	10.99	6.396	3.892	2.740	**2.472**	2.649

Table 4.6: Parameters: $l = 7$, $p = 1$, $r_k = 2$, $\tau = 0.00025$, $n_c = 2$, $d = 256^{-1}$, PEC.

half the thickness of the layer, we need to double the value of θ to obtain the same damping. But also the numerical reflections seem to only depend on the product θh. In our calculations this effect was noticed for different orders of approximation in space, though we also have seen some deviation. A possible explanation is, that in our values R_{rel} the discretization error from the vacuum region Ω_c is still included. It can be removed by substraction of the approximate solution with non-reflecting boundary in the numerator of (4.13). The θh-dependence has already been noticed for a different kind of discretization. In [CM98] (we relied on the interpretation in [Jol12, Sec. 2.2]) a $\theta^2 h^2$-dependence of the numerical reflections was calculated for some finite difference scheme.

4.3.4 Approximation with different orders in space

To investigate the PML under different spatial orders of approximation, we varied the polynomial degree from zero to three. Starting with a finite volume discretization, we have a look at Tables 4.7 and 4.8. Again, we notice the optimal parameter θ for the finer mesh to be twice as large as for the coarser mesh. The values for the non-reflecting boundary are as follows

$$R_{\text{rel}}^{\text{FAB}}\Big|_{l=6,\varphi=30°} = 1.941 \cdot 10^{-2}, \qquad R_{\text{rel}}^{\text{FAB}}\Big|_{l=7,\varphi=30°} = 1.143 \cdot 10^{-2},$$
$$R_{\text{rel}}^{\text{FAB}}\Big|_{l=6,\varphi=45°} = 6.861 \cdot 10^{-2}, \qquad R_{\text{rel}}^{\text{FAB}}\Big|_{l=7,\varphi=45°} = 4.266 \cdot 10^{-2},$$

so even with a small layer extended over two cells, we obtain a reflection behaviour, that is quite close to the fully absorbing boundary. One reason for that is the dissi-

$R_{\mathrm{rel}} \cdot 10^2$	$\theta = 50$	100	150	200	250	300	350	400
$\varphi = 30°$	4.001	**3.038**	3.862	5.109	6.243	7.241	8.105	8.807
$\varphi = 45°$	11.69	8.867	**8.030**	8.417	9.289	10.12	10.75	11.18

Table 4.7: Parameters: $l = 6$, $p = 0$, $r_k = 1$, $\tau = 0.0005$, $n_c = 2$, $d = 128^{-1}$, Impedance.

$R_{\mathrm{rel}} \cdot 10^2$	$\theta = 100$	200	300	400	500	600	700	800
$\varphi = 30°$	3.852	**2.907**	3.833	5.146	6.342	7.422	8.380	9.178
$\varphi = 45°$	10.66	7.425	**6.553**	7.314	8.626	9.820	10.74	11.42

Table 4.8: Parameters: $l = 7$, $p = 0$, $r_k = 1$, $\tau = 0.00025$, $n_c = 2$, $d = 256^{-1}$, Impedance.

pative behaviour of finite volume discretizations. Since we use the exact boundary values for the incoming wave, they do not fit the discretized wave very well and therefore produce relatively large discretization errors.

We tested as well with a third order spatial approximation in Tables 4.9 and 4.10.

$R_{\mathrm{rel}} \cdot 10^2$	$\theta = 50$	100	150	200	250	300	350
$\varphi = 45°$	1.884	0.2177	**0.1064**	0.2311	0.4881	0.8649	1.324

Table 4.9: Parameters: $l = 4$, $p = 3$, $r_k = 4$, $\tau = 0.002$, $n_c = 2$, $d = 32^{-1}$, Impedance.

$R_{\mathrm{rel}} \cdot 10^2$	$\theta = 100$	200	300	400	500	600	700
$\varphi = 45°$	1.883	0.2086	**0.08672**	0.2123	0.4551	0.8210	1.280

Table 4.10: Parameters: $l = 5$, $p = 3$, $r_k = 4$, $\tau = 0.001$, $n_c = 2$, $d = 64^{-1}$, Impedance.

With a non-reflecting boundary, we obtain the values

$$R_{\mathrm{rel}}^{\mathrm{FAB}}\big|_{l=4} = 0.06854 \cdot 10^{-2}, \qquad R_{\mathrm{rel}}^{\mathrm{FAB}}\big|_{l=5} = 0.01517 \cdot 10^{-2}. \tag{4.14}$$

Here, the layer parameter has to be doubled again for the finer mesh. Comparing Table 4.10 with Table 4.3 we also note, that the optimal θ for the third order approximation is larger than the first order one. The reason is the interplay of analytical and numerical reflections. If θ is fixed and the order of discretization is increased, the numerical reflections will decrease, whereas the analytical reflections will remain unchanged. An increase in θ can compensate for that smaller discretization error.

4.3.5 Non-constant layer parameter

For an optimal use of the PML, it is not recommended to use a constant parameter θ, but to use an increasing parameter towards the outer boundary. An explanation for this can be found in [Jol12, Sec. 2.2]. For a constant θ, the numerical reflections mainly originate in the layer vacuum interface and in the inner layer cells. When θ increases inside the layer, more of the numerical reflections rise inside the layer and therefore suffer the damping effect of the inner layer cells. Of course, this will harden the task to find the optimal layer parameters. Here, we show with two examples the improvement, that can be achieved with a varying layer parameter. At first, we use a second order approximation in space and a PEC boundary in Table 4.11. With a

$R_{\text{rel}} \cdot 10^2$	$\theta = 50$	75	100	125	150	175	200	225
$\varphi = 45°$	10.97	3.636	1.207	0.4096	**0.1829**	0.1915	0.2744	0.3910

$R_{\text{rel}} \cdot 10^2$	$\theta_2 = 260$	270	280
$\theta_1 = 100$	0.07206	0.07089	0.07118
$\theta_1 = 110$	0.06910	**0.06853**	0.06896
$\theta_1 = 120$	0.07031	0.07020	0.07076

Table 4.11: Parameters: $l = 5$, $p = 2$, $r_k = 3$, $\tau = 0.001$, $n_c = 4$, $d = 32^{-1}$, PEC.

two-valued θ we obtained optimal values $\theta_1 = 110$ and $\theta_2 = 270$. Here, θ_1 denotes the value closer to Ω_c. We have an optimal absorbtion value of

$$R_{\text{rel}}^{\text{FAB}} = 0.04714 \cdot 10^{-2}.$$

The second example is a third order spatial approximation with impedance boundary in Table 4.12. The optimal absorption for this case was already stated in (4.14). In this second example, the improvement of a varying θ is not as large as in the first one, but since we are much closer to the optimal absorbing boundary, this does not surprise us.

$R_{\text{rel}} \cdot 10^2$	$\theta = 100$	150	200	250	300	350
$\varphi = 45°$	0.2071	0.02747	**0.02350**	0.04516	0.08244	0.1378

$R_{\text{rel}} \cdot 10^2$	$\theta_2 = 270$	280	290
$\theta_1 = 120$	0.01489	0.01486	0.01489
$\theta_1 = 130$	0.01485	**0.01483**	0.01485
$\theta_1 = 140$	0.01501	0.01500	0.01502

Table 4.12: Parameters: $l = 5$, $p = 3$, $r_k = 4$, $\tau = 0.001$, $n_c = 4$, $d = 32^{-1}$, Impedance.

4.3.6 Long-time error

Since we are also interested in the long-time error evolution, we performed tests over a time $T = 31.5$ in Table 4.13, 4.14, and 4.15. In our tests the reflections stay small

$R_{\text{rel}} \cdot 10^2$	$t_1 = 4.5$	9	13.5	18	22.5	27	31.5
$\theta = \mathbf{150}$	0.1353	0.09559	0.09518	0.09518	0.09518	0.09518	0.09518
$\theta = 400$	11.77	15.23	22.48	33.58	51.25	80.56	127.6

Table 4.13: Parameters: $l = 6$, $p = 0$, $r_k = 1$, $\tau = 0.0005$, $n_c = 2$, $d = 128^{-1}$, $\varphi = 45°$, Impedance.

$R_{\text{rel}} \cdot 10^2$	$t_1 = 4.5$	9	13.5	18	22.5	27	31.5
$\theta = \mathbf{200}$	0.5345	0.2721	0.2255	0.2106	0.2049	0.2029	0.2020
$\theta = 400$	3.307	3.035	2.876	2.922	3.038	3.165	3.329
$\theta = 800$	14.69	29.67	63.39	143.9	329.9	762.3	1773

Table 4.14: Parameters: $l = 5$, $p = 1$, $r_k = 2$, $\tau = 0.001$, $n_c = 2$, $d = 64^{-1}$, $\varphi = 45°$, PEC.

up to a time $t_1 = 31.5$, as long as we choose the paramter θ around the optimal value. For a larger θ the layer may emit to much energy to produce useful results. In case of the first order approximation and $\theta = 400$, we already see slightly increasing reflections towards larger times. For $\theta = 800$ the layer is already useless in the long run.

$R_{\text{rel}} \cdot 10^2$	$t_1 = 4.5$	9	13.5	18	22.5	27	31.5
$\boldsymbol{\theta_1 = 110}$ $\boldsymbol{\theta_2 = 270}$	0.03091	0.02575	0.01573	0.01499	0.01998	0.01639	0.01282
$\theta_1 = 220$ $\theta_2 = 540$	0.04231	0.03468	0.02655	0.01930	0.02022	0.01397	0.01661
$\theta_1 = 440$ $\theta_2 = 1080$	7182000 at $t_1 = 0.75$			calculation stops at $t = 0.796$			

Table 4.15: Parameters: $l = 5$, $p = 2$, $r_k = 3$, $\tau = 0.001$, $n_c = 4$, $d = 32^{-1}$, $\varphi = 45°$, PEC.

4.3.7 Implicit midpoint rule

As we already mentioned, time integration is not the focus of this work. Nevertheless, we wanted to test, whether or not we can expect difficulties with implicit time integration. So we decided to try the implicit midpoint rule in combination with the trapezoidal rule. As the most important advantage of an implicit time integrator, there is no bound on the time step size τ. We can still run calculations for large time step sizes without any blow up in the solution. Of course a large time step size has its negative influences on the discretization errors.

For the nodes t_n and $t_{n+1} = t_n + \tau$ in time, we use the notation $\mathbf{u}_h^n = \mathbf{u}_h(t_n) \in \mathrm{V}_h^p$ and approximate the semi-discrete system (3.28a) by

$$\frac{1}{\tau}\left(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\right) + \frac{1}{2}B_{uu}^h\left(\mathbf{u}_h^{n+1} + \mathbf{u}_h^n\right) + \frac{1}{2}B_{u\xi}^h\left(\boldsymbol{\xi}_h^{n+1} + \boldsymbol{\xi}_h^n\right) = \mathbf{f}_h^{n+\frac{1}{2}} - \mathbf{G}_h^{n+\frac{1}{2}}, \quad (4.15a)$$

$$\frac{1}{\tau}\left(\boldsymbol{\xi}_h^{n+1} - \boldsymbol{\xi}_h^n\right) + \frac{1}{2}B_{\xi u}^h\left(\mathbf{u}_h^{n+1} + \mathbf{u}_h^n\right) + \frac{1}{2}B_{\xi\xi}^h\left(\boldsymbol{\xi}_h^{n+1} + \boldsymbol{\xi}_h^n\right) = \mathbf{0}. \quad (4.15b)$$

The four operators correlated to B are

$$B_{uu}^h\mathbf{u}_h^n = (2\boldsymbol{\Theta} - \theta\mathbb{1})\mathbf{u}_h^n + A_h\mathbf{u}_h^n, \qquad B_{u\xi}^h\boldsymbol{\xi}_h^n = (\boldsymbol{\Theta} - \theta\mathbb{1})\boldsymbol{\xi}_h^n,$$
$$B_{\xi u}^h\mathbf{u}_h^n = (\theta\mathbb{1} - \boldsymbol{\Theta})\mathbf{u}_h^n, \qquad B_{\xi\xi}^h\boldsymbol{\xi}_h^n = (\theta\mathbb{1} - \boldsymbol{\Theta})\boldsymbol{\xi}_h^n.$$

We choose a basis $\mathcal{B}_h^p = (\mathbf{b}_l)_{l=1}^{\dim \mathrm{V}_h^p}$ of V_h^p, where every basis vector $\mathbf{b}_l$ is located on some cell K, i.e. $\operatorname{supp}\mathbf{b}_l = K$, and in one component j, i.e. $(\mathbf{b}_l)_m|_K = 0$ for $m \neq j$. The coordinates of a vector $\mathbf{u}_h^n$ in that basis are denoted with an underline

$$\mathbf{u}_h^n = \sum_{l=1}^{\dim \mathrm{V}_h^p} (\underline{u}_h^n)_l \mathbf{b}_l.$$

After a decomposition in the basis $\mathcal{B}_h^p$ and a multiplication with the basis vector $\mathbf{b}_j$, equations (4.15) read

$$\begin{aligned}\frac{1}{\tau}\underline{\underline{M}}\left(\underline{u}_h^{n+1}-\underline{u}_h^n\right)+\frac{1}{2}\underline{\underline{B}}_{uu}^h\left(\underline{u}_h^{n+1}+\underline{u}_h^n\right)+\frac{1}{2}\underline{\underline{B}}_{u\xi}^h\left(\underline{\xi}_h^{n+1}+\underline{\xi}_h^n\right)& \\ =\underline{\underline{M}}(\underline{f}_h^{n+\frac{1}{2}}-\underline{G}_h^{n+\frac{1}{2}}),& \qquad (4.16a)\\ \frac{1}{\tau}\underline{\underline{M}}\left(\underline{\xi}_h^{n+1}-\underline{\xi}_h^n\right)+\frac{1}{2}\underline{\underline{B}}_{\xi u}^h\left(\underline{u}_h^{n+1}+\underline{u}_h^n\right)+\frac{1}{2}\underline{\underline{B}}_{\xi\xi}^h\left(\underline{\xi}_h^{n+1}+\underline{\xi}_h^n\right)=\underline{0},& \qquad (4.16b)\end{aligned}$$

with the matrices

$$(\underline{\underline{M}})_{j,l}=(\mathbf{b}_j,\mathbf{b}_l)_{\mathrm{V}}\quad\text{(mass matrix)},\qquad\qquad (\underline{\underline{B}}^h_{..})_{j,l}=(\mathbf{b}_j,B^h_{..}\mathbf{b}_l)_{\mathrm{V}}.$$

We can multiply (4.16b) by $\underline{\underline{M}}^{-1}$, so that there are only diagonal matrices left in that equation. Then we can solve for $\underline{\xi}_h^{n+1}$ and obtain

$$\underline{\xi}_h^{n+1}-\underline{\xi}_h^n=\frac{1}{2}\underline{\underline{K}}(\underline{u}_h^{n+1}+\underline{u}_h^n)+\underline{\underline{K}}\,\underline{\xi}_h^n. \qquad (4.17)$$

Here, the matrix $\underline{\underline{K}}$ is a diagonal matrix with diagonal entries

$$\left(\underline{\underline{K}}\right)_{l,l}=\begin{cases}\left.\dfrac{2\tau\theta}{2+\tau\theta}\right|_{\operatorname{supp}\mathbf{b}_l}, & \text{if } (\mathbf{b}_l)_m=0 \text{ for } m\in\{2,3,5,6\},\\ 0, & \text{otherwise.}\end{cases}$$

Using (4.17) in (4.16a) and solving for $\underline{u}_h^{n+1}$, we obtain

$$\underline{u}_h^{n+1}-\underline{u}_h^n=\underline{\underline{E}}^{-1}\left(2(\underline{\underline{M}}-\underline{\underline{E}})\underline{u}_h^n-\underline{\underline{M}}\,\underline{\underline{K}}\,\underline{\xi}_h^n+\tau\underline{\underline{M}}(\underline{f}_h^{n+\frac{1}{2}}-\underline{G}_h^{n+\frac{1}{2}})\right). \qquad (4.18)$$

The matrix $\underline{\underline{E}}$ is given by

$$\begin{aligned}\underline{\underline{E}}_{j,l}&=\sum_{K\in\mathcal{T}_h}\left(\mathbf{b}_j,\left(E_K+\frac{\tau}{2}A_h\right)\mathbf{b}_l\right)_{\mathrm{V},K},\\ E_K&=\operatorname{diag}\left(\frac{2}{2+\tau\theta},\frac{2+\tau\theta}{2},\frac{2+\tau\theta}{2},\frac{2}{2+\tau\theta},\frac{2+\tau\theta}{2},\frac{2+\tau\theta}{2}\right)\end{aligned}$$

and invertible, since A_h is positive semi-definite (see Lemma 2.4.4). We can use equation (4.18) to calculate $\underline{u}_h^{n+1}$ out of $\underline{u}_h^n$ and $\underline{\xi}_h^n$ and afterwards equation (4.17) to calculate $\underline{\xi}_h^{n+1}$.

We tested the preceeding implicit midpoint rule for a level $l=6$ mesh, where we took an eight cell layer with impedance boundary and a first order spatial approximation, together with an explicit second order time integrator for comparison. Still with

$R_{\mathrm{rel}} \cdot 10^2$	$\theta = 50$	100	150	200	250	300	350	400
$\varphi = 45°$	1.896	0.2883	**0.2683**	0.4303	0.6812	1.013	1.415	1.867

Table 4.16: Parameters: $l = 6$, $p = 1$, $r_k = 2$, $\tau = 0.0005$, $n_c = 8$, $d = 32^{-1}$, Impedance.

explicit time integration, we determined the optimal layer parameter for this setting in Table 4.16. For the value $\theta = 150$ we calculated again with a larger time step $\tau_1 = 0.001$, since the former choice for τ was not optimal. In case of a larger $\tau_2 = 0.0015$ the calculation stops at $t = 0.078$, because of a CFL-condition. With the step size τ_1 the calculation up to a time $T = 1.2$ took 1:28 minutes on 24 CPUs, with a reflection value $R_{\mathrm{rel}}^{\mathrm{explicit}} = 0.2669$ almost as in Table 4.16. Time integration with the implicit midpoint rule on the other hand took us 1:18 minutes with a comparable value for the reflections of $R_{\mathrm{rel}}^{\mathrm{implicit}} = 0.2704$. Here, we chose the time step size $\tau_3 = 0.00125$ quite close to the explicit one. For the calculations we changed some of the parameters in the code. For smoother and preconditioner we used Gauss-Seidel and linear epsilon and reduction were set to 10^{-9}.

4.3.8 Conclusions

There are two kinds of reflections to observe from the layer - an analytical one, that already exists in the continuous model and can be reduced by an increase of the paramter θ, and a numerical one, which arises from discretization errors and therefore demands a small parameter θ. In Fig. 4.9 the qualitative behaviour is shown. In order to obtain a usefull behaviour of the layer, θ has to be chosen wisely, so that neither of the reflections dominates the other one. A too large θ can even render the layer useless, as shown in Section 4.3.6. The layer can be improved by an impedance boundary behind it and by a multivalued θ (see Section 4.3.5), that increases towards the outer boundary. To control the choice in θ one may perform a-priori calculations in a simple setting, e.g. the one we presented in this chapter, before starting a costly calculation. As pointed out in Section 4.3.3, these calculations can be done on a coarse grid and the optimal parameter can be estimated by $\theta_{\mathrm{fine}} h_{\mathrm{fine}} = \theta_{\mathrm{coarse}} h_{\mathrm{coarse}}$. The layer also works fine with implicit time integration, as tested in Section 4.3.7.

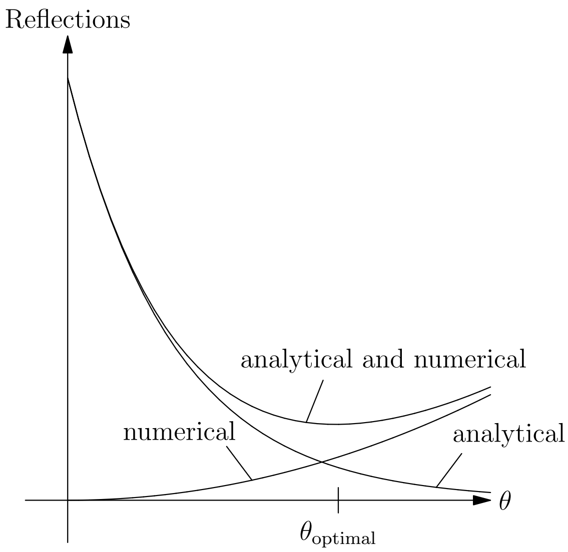


Figure 4.9: The layer produces two kinds of reflections. One of them already exists in the analytical model, since the damping of the layer is not perfect, the other one arises from discretization errors. The task is to find the optimal parameter θ, that minimizes the total reflections.

5 On the long-time behaviour of the PML

Content of this chapter In this chapter we have a closer look at the time dependence of the solution to Maxwell's equations with a PML in one direction. The first section is devoted to the expected long-time behaviour, though we were not able to prove one important aspect, namely the dissipativity in (5.3), in the discussion. To overcome this problem, in the second section we use an energy estimate, that is already available in the literature and so far works in two dimensions for a constant parameter $\theta \neq \theta(x_1)$. Based on that energy estimate, we derive an error estimate for the spatially discretized PDE of the kind

$$\|\mathbf{v} - \mathbf{v}_h\|_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})} \leq Ch^{s-2},$$

where the constant C does not explicitly show any exponential behaviour in time. The detailed constant and the assumptions on the regularity of the solution $\mathbf{v}$ can be found in Theorem 5.2.12.

Origin of this chapter The presented proof in Section 5.2 is a major modification of the proof of Theorem 2.4.1 and my contribution.

5.1 The expected long-time behaviour of the PML

At the moment, the long-time behaviour of the PML system is, to the best of our knowledge, not fully understood. Regarding the application of Lumer-Phillips' theorem in Section 3.2, we have at most an exponential growth in time

$$\|\mathbf{v}(t)\|_{\mathrm{V}_{\mathrm{PML}}} \leq \exp(\theta_{\sup} t)\|\mathbf{v}_0\|_{\mathrm{V}_{\mathrm{PML}}}$$

of the solution $\mathbf{v}$ to the PML system, as long as we have no outer forces, i.e. $\mathbf{f} = \mathbf{0}$ and no boundary input, i.e. $\mathbf{g}_j = \mathbf{0}$. Though, by construction of the layer in Section 3.1, we expect a damping and therefore dissipative behaviour in the variable $\mathbf{u} + \boldsymbol{\xi}$.

In that construction, we considered plane waves with $k_1 > 0$. In case of a negative k_1, the exponential damping behaviour becomes an exponential amplification. So in the geometry of Figure 3.2 the question is, if any part of the initial values $\mathbf{u}_0$ with negative k_1 can reach the layer. Since the sign of k_1 determines the x_1-direction of propagation of a plane wave, one may assume that such a wave will never reach the layer. But that is not true for every material. In [BFJ03] Bécache, Fauqueux, and Joly described an exponential blow up of the analytical solution to a PML system in an anisotropic medium. They spoke of backward propagating waves, where group velocity and phase velocity are in opposite directions.

We did not focus on that phenomenon, since we are interested in inhomogeneous, but isotropic media. So we assume that these backward propagating waves do not occur in our system (3.13). There also is another reason to assume, an exponential growth is not possible in this system. Regarding the second assumption in the proof of Lemma 3.2.1, we see where the possibility of an exponential growth originates. We have a look at the ODE

$$\partial_t u - \vartheta u = 0, \qquad\qquad u(0) = u_0 \tag{5.1}$$

for $\vartheta \in \mathbb{R}$, $\vartheta > 0$, and $u : [0, \infty) \to \mathbb{R}$. In the first equations of (3.13a) and (3.13b), our PDE system contains such a structure. The solution of (5.1) is

$$u(t) = \exp(\vartheta t) u_0$$

and therefore exponentially growing in time. Including the contribution of the auxiliary function $\boldsymbol{\xi}$ in (3.13) to the ODE, we obtain the system

$$\partial_t u - \vartheta u - \vartheta \xi = 0, \qquad\qquad u(0) = u_0, \tag{5.2a}$$
$$\partial_t \xi + \vartheta u + \vartheta \xi = 0, \qquad\qquad \xi(0) = \xi_0, \tag{5.2b}$$

for $u, \xi : [0, \infty) \to \mathbb{R}$. Here, we have the solution

$$u(t) = u_0 + t\vartheta(u_0 + \xi_0),$$
$$\xi(t) = \xi_0 - t\vartheta(u_0 + \xi_0)$$

and therefore a linear growth in time of $u(t)$ and $\xi(t)$ and inespecially a sum $u(t)+\xi(t)$ independent of t, which is related to the expected dissipative behaviour of $\mathbf{u}+\boldsymbol{\xi}$. A similar discussion can be found in [AGH02, Section 4.1]. The remaining problem in that context is the effect of the differential operator A. So one may try to calculate the dissipative behaviour in $\mathbf{u} + \boldsymbol{\xi}$ directly. Therefor we consider everything to be as simple as possible, i.e. $\varepsilon = \mu = 1$, $\sigma = 0$, $\mathbf{f}_E = \mathbf{f}_H = \mathbf{0}$, $\mathbf{g}_E = \mathbf{g}_H = \mathbf{g}_I = \mathbf{0}$, and the initial wave has not yet reached the boundary. In this setting, we have a look at the time derivative of the $L_2(\Omega)$-energy

$$\partial_t \frac{1}{2}\|\mathbf{u}+\boldsymbol{\xi}\|_{0,\Omega}^2 = \big(\mathbf{u}+\boldsymbol{\xi}, \partial_t(\mathbf{u}+\boldsymbol{\xi})\big)_{0,\Omega} = -(\mathbf{u}+\boldsymbol{\xi}, A\mathbf{u}+\boldsymbol{\Theta}\mathbf{u})_{0,\Omega}$$
$$= -\underbrace{(\mathbf{u}, A\mathbf{u})_{0,\Omega}}_{=0} - \underbrace{(\mathbf{u}, \boldsymbol{\Theta}\mathbf{u})_{0,\Omega}}_{\geq 0} - (\boldsymbol{\xi}, A\mathbf{u})_{0,\Omega} - \underbrace{(\boldsymbol{\xi}, \boldsymbol{\Theta}\mathbf{u})_{0,\Omega}}_{=0}.$$

Here we see, that in case of arbitrary initial conditions on $\mathbf{u}$ and $\boldsymbol{\xi}$, we do not have dissipative behaviour. Since in application we choose $\mathbf{u}_0 = \boldsymbol{\xi}_0 = \mathbf{0}$ inside the layer, the initial conditions are not that arbitrary and the question remains, whether or not we have an energy decay. In this work, we will not answer this question, but for a moment let us assume, we have this decay, solenoidal initial values, i.e.

$$\partial_t \frac{1}{2}\|\mathbf{u}+\boldsymbol{\xi}\|_{0,\Omega}^2 \leq 0, \qquad \nabla\cdot\mathbf{E}_0 = \nabla\cdot\mathbf{H}_0 = 0, \tag{5.3}$$

and the vacuum region is simply connected. Now, we have a look at what this assumption implies in terms of long-time behaviour. First we point out, that we can apply a time derivative on system (3.13) to see that $\partial_t\mathbf{v}$ solves the same system and therefore shows the same behaviour as $\mathbf{v}$. In other words, we have boundedness in $L_2(\Omega)$ of $\mathbf{u}+\boldsymbol{\xi}$ and $\partial_t\mathbf{u}+\partial_t\boldsymbol{\xi}$. The boundedness of $\partial_t\boldsymbol{\xi}$ follows from (3.13c) and (3.13d) and leads to boundedness of $\partial_t\mathbf{u}$. So we have at most a linear growth in our field $\mathbf{v}$.

Due to the solenoidal condition (5.3), we can find vector potentials

$$\mathbf{E}_0 = \nabla\times\tilde{\mathbf{H}}_0, \qquad \mathbf{H}_0 = -\nabla\times\tilde{\mathbf{E}}_0,$$

which also vanish inside the layer (see [Mon03, Sec. 3.7]). So for the solution $\tilde{\mathbf{v}}$ to (3.13) with initial values $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{H}}_0$ we have boundedness of $\partial_t\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi}$ and $\partial_t\tilde{\mathbf{u}} = \mathbf{u}$. Once we have boundedness of the solution $\mathbf{v}$, the goal is to use it for an error estimate without exponential terms. Since we were not able to show the energy decay in (5.3), we will base our error estimate in the next section on a result, which at least works in a limited geometry.

Remark 5.1.1:
In the literature [BPG04], [AGH02] there are several ideas to avoid a long-time linear growth of the solution to a PML system. At the moment we do not see a necessity in this, since we expect boundedness of the solution, as we already explained in the preceding discussion. To say it in non-strict words: A wave that enters the layer will cause a growth in the auxiliary function, but since the wave will be damped, the growth will not last forever.

5.2 A restricted error estimate for the DG-discretization with a PML

As already described, there are still some unanswered theoretical questions about the long-time behaviour of the PML. Nevertheless, in a special case there is an answer to that questions, which can be found in a work by Bécache and Joly [BJ02, Sec. 2.3]. They state an energy-like functional, probably non-physical, for the PML and give a proof, that it decays over time. This decay only works in two dimensions and only for a constant parameter $\theta = \text{const.} > 0$. So in this section, we consider a polygon Ω completely belonging to the layer and with a homogeneous PEC boundary condition, but non-homogeneous initial values. In contrast to the situation of Figure 3.2, where the initial values are supported outside of the layer and the auxiliary function $\boldsymbol{\xi}$ evolves, as the wave enters the layer, here the initial values of the electromagnetic field $\mathbf{u}$ and the auxiliary function $\boldsymbol{\xi}$ are independent. The interface in between layer and vacuum will not be covered in this chapter. Further on, we have no conductivity, i.e. $\sigma = 0$ and no outer force, i.e. $\mathbf{f} = \mathbf{0}$. The medium $\varepsilon(\mathbf{x}), \mu(\mathbf{x})$ is supposed to be piecewise homogeneous and as well homogeneous in every cell K. Our calculations in this chapter work for piecewise constant materials, though the solution to the PDE (5.4) may not fulfill the desired regularity assumptions (see Remark 2.4.2) and the material distribution may not be useful in applications (see Remark 3.1.2). Nevertheless, we will not address the regularity topic here.

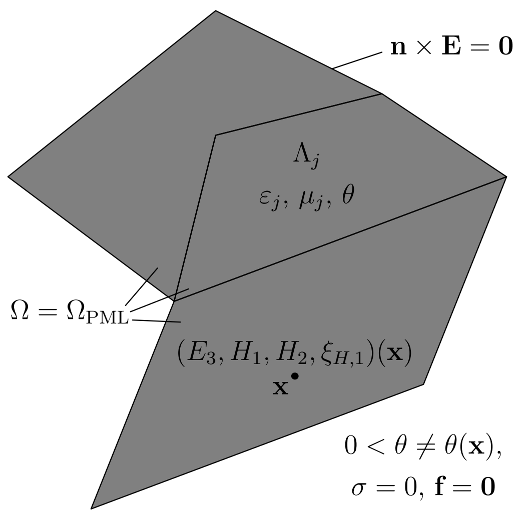


Figure 5.1: In this section we use a different geometrical setting.

Since our code works in TE-mode, we will restrict our considerations to the electromagnetic variables E_3, H_1, and H_2 and still call them $\mathbf{u}$. Together with the auxiliary function $\xi_{H,1}$ we call the fields $\mathbf{v}$. So we investigate the equations

$$\begin{aligned}
\partial_t E_3 - \varepsilon^{-1}\partial_{x_1} H_2 + \varepsilon^{-1}\partial_{x_2} H_1 + \theta E_3 &= 0 && \text{in } \Omega_\infty, && (5.4a)\\
\partial_t H_1 + \mu^{-1}\partial_{x_2} E_3 - \theta H_1 - \theta\xi_{H,1} &= 0 && \text{in } \Omega_\infty, && (5.4b)\\
\partial_t H_2 - \mu^{-1}\partial_{x_1} E_3 + \theta H_2 &= 0 && \text{in } \Omega_\infty, && (5.4c)\\
\partial_t \xi_{H,1} + \theta H_1 + \theta\xi_{H,1} &= 0 && \text{in } \Omega_\infty, && (5.4d)\\
E_3 &= 0 && \text{on } (\partial\Omega)_\infty, && (5.4e)\\
(E_3, H_1, H_2, \xi_{H,1})(\cdot, 0) &= (E_{3,0}, H_{1,0}, H_{2,0}, \xi_{H,1,0}) && \text{in } \Omega. && (5.4f)
\end{aligned}$$

Again, $\Omega_\infty = \Omega \times [0,\infty)$ denotes the space-time cylinder and $(\partial\Omega)_\infty = \partial\Omega \times [0,\infty)$ the corresponding spatial boundary. The differential operator A in this two dimensional case reads

$$A\mathbf{u} = \begin{pmatrix} (A\mathbf{u})_3 \\ (A\mathbf{u})_4 \\ (A\mathbf{u})_5 \end{pmatrix} = \begin{pmatrix} -\varepsilon^{-1}\partial_{x_1} H_2 + \varepsilon^{-1}\partial_{x_2} H_1 \\ \mu^{-1}\partial_{x_2} E_3 \\ -\mu^{-1}\partial_{x_1} E_3 \end{pmatrix}. \tag{5.5}$$

The functional introduced by Bécache and Joly is given by

$$\begin{aligned}
\mathcal{B}_{\text{PML}}(\mathbf{u}, \hat{\mathbf{u}}) &= \Big((\partial_t H_1, \partial_t \hat{H}_1)_\mu + (\partial_t H_2, \partial_t \hat{H}_2)_\mu + (\theta H_2, \theta \hat{H}_2)_\mu \\
&\qquad + \big((\partial_t + \theta)E_3, (\partial_t + \theta)\hat{E}_3\big)_\varepsilon \Big), \\
\mathcal{E}_{\text{PML}}(\mathbf{u}) &= \frac{1}{2}\mathcal{B}_{\text{PML}}(\mathbf{u}, \mathbf{u}).
\end{aligned}$$

We recall that the indices ε and μ denote weighted inner products in $\mathrm{L}_2(\Omega)$. The proof of the functionals decay is a straight forward calculation. Since θ will be exchanged with spatial derivatives, the parameter needs to be constant in space.

Lemma 5.2.1 ([BJ02], Lemma 2.2)**:**
For a solution $\mathbf{v}$ *of* (5.4) *with additional regularity of the electromagnetic field* $\mathbf{u} \in \mathrm{C}^2([0,\infty), \mathrm{L}_2(\Omega)^3) \cap \mathrm{C}^1([0,\infty), \mathrm{H}(\mathrm{curl}, \Omega)^2)$*, we have an energy-like decay*

$$\partial_t \mathcal{E}_{\text{PML}}(\mathbf{u}) = -2\theta \|\partial_t H_2\|_\mu^2 \leq 0.$$

Proof: For the single terms of the functional we calculate the time derivatives

$$\partial_t \frac{1}{2}\|\partial_t H_1\|_\mu^2 = (\partial_t H_1, \partial_t^2 H_1)_\mu = -(\partial_t H_1, (\partial_t + \theta)\partial_{x_2} E_3)_{0,\Omega},$$

$$\partial_t \frac{1}{2}(\|\partial_t H_2\|_\mu^2 + \|\theta H_2\|_\mu^2) = (\partial_t H_2, \partial_t^2 H_2)_\mu + (\theta H_2, \theta \partial_t H_2)_\mu$$

$$
\begin{aligned}
&= (\partial_t H_2, \partial_t \partial_{x_1} E_3)_{0,\Omega} - (\partial_t H_2, \theta \partial_t H_2)_\mu \\
&\quad + (\partial_{x_1} E_3, \theta \partial_t H_2)_{0,\Omega} - (\partial_t H_2, \theta \partial_t H_2)_\mu \\
&= -2(\partial_t H_2, \theta \partial_t H_2)_\mu + \big(\partial_t H_2, (\partial_t + \theta)\partial_{x_1} E_3\big)_{0,\Omega},
\end{aligned}
$$

$$
\begin{aligned}
\partial_t \frac{1}{2} \|(\partial_t + \theta) E_3\|_\varepsilon^2 &= \big((\partial_t + \theta)E_3, \partial_t(\partial_t + \theta)E_3\big)_\varepsilon \\
&= \big((\partial_t + \theta)E_3, \partial_t(\partial_{x_1} H_2 - \partial_{x_2} H_1)\big)_{0,\Omega}.
\end{aligned}
$$

With (5.5), an integration by parts, and a PEC boundary, we see that

$$
\begin{aligned}
\partial_t \mathcal{E}_{\mathrm{PML}}(\mathbf{u}) &= -(\partial_t \mathbf{u}, A \partial_t \mathbf{u})_{\mathrm{V}} - 2(\partial_t H_2, \theta \partial_t H_2)_\mu \\
&\quad - \left(\partial_t \begin{pmatrix} \mathbf{0} \\ \mathbf{H} \end{pmatrix}, A\theta \mathbf{u}\right)_{\mathrm{V}} - \left(\theta \begin{pmatrix} \mathbf{E} \\ \mathbf{0} \end{pmatrix}, A \partial_t \mathbf{u}\right)_{\mathrm{V}} \\
&= -2(\partial_t H_2, \theta \partial_t H_2)_\mu.
\end{aligned}
$$

□

The preceeding energy-like decay result shows boundedness in time of $\partial_t H_1$, H_2 and E_3. The latter one is not obvious, but was mentioned in [BPG04, Sec. II.A.] and can be seen later on with Lemma 5.2.10. The first one is in accordance with the earlier considerations about the ODE system (5.2). Now, we have a look at the discrete system

$$
\begin{aligned}
\partial_t E_{h,3} + (A_h \mathbf{u}_h)_3 + \theta E_{h,3} &= 0 && \text{in } \Omega_\infty, && (5.6a) \\
\partial_t H_{h,1} + (A_h \mathbf{u}_h)_4 - \theta H_{h,1} - \theta \xi_{H,h,1} &= 0 && \text{in } \Omega_\infty, && (5.6b) \\
\partial_t H_{h,2} + (A_h \mathbf{u}_h)_5 + \theta H_{h,2} &= 0 && \text{in } \Omega_\infty, && (5.6c) \\
\partial_t \xi_{H,h,1} + \theta H_{h,1} + \theta \xi_{H,h,1} &= 0 && \text{in } \Omega_\infty, && (5.6d) \\
(E_{h,3}, H_{h,1}, H_{h,2}, \xi_{H,h,1})(\cdot, 0) &= \Pi_h^p(E_{3,0}, H_{1,0}, H_{2,0}, \xi_{H,1,0}) && \text{in } \Omega, && (5.6e)
\end{aligned}
$$

where the operator A_h is the two dimensional analogon of (2.28). The goal of this chapter is to obtain an error estimate for the discretized solution. To a certain extent our approach is analogous to the procedure in Section 2.4. We start with calculations similar to the ones in Lemma 2.4.3.

Lemma 5.2.2:
For $\mathbf{u}, \mathbf{u}_h \in \mathrm{C}^2([0,T], \mathrm{L}_2(\Omega)^3)$ the error in the functional $\mathcal{E}_{\mathrm{PML}}$ can be reformulated in the following expression

$$
\begin{aligned}
\int_0^T \mathcal{E}_{\mathrm{PML}}(\mathbf{u} - \mathbf{u}_h)\,\mathrm{dt} &= \int_0^T \partial_t\big(\mathcal{E}_{\mathrm{PML}}(\mathbf{u}) + \mathcal{E}_{\mathrm{PML}}(\mathbf{u}_h) - \mathcal{B}_{\mathrm{PML}}(\mathbf{u}, \mathbf{u}_h)\big)\eta_T(t)\,\mathrm{dt} \\
&\quad + T\mathcal{E}_{\mathrm{PML}}(\mathbf{u} - \mathbf{u}_h)(0).
\end{aligned} \tag{5.7}
$$

The function $\eta_T(t) = T - t$ was already defined in Lemma 2.4.3.

Proof:

$$
\begin{aligned}
&\int_0^T \mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)\,\mathrm{dt} = -\int_0^T \mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)\partial_t\eta_T(t)\,\mathrm{dt}\\
&= \int_0^T \partial_t\big(\mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)\big)\eta_T(t)\,\mathrm{dt} + T\mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)(0)\\
&= \int_0^T \partial_t\big(\mathcal{E}_{\mathrm{PML}}(\mathbf{u}) + \mathcal{E}_{\mathrm{PML}}(\mathbf{u}_h) - \mathcal{B}_{\mathrm{PML}}(\mathbf{u},\mathbf{u}_h)\big)\eta_T(t)\,\mathrm{dt} + T\mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)(0).
\end{aligned}
$$

□

Next, we have a look at the discrete functional and formulate a result similar to Lemma 5.2.1.

Lemma 5.2.3:
For the discrete solution $\mathbf{v}_h$ of (5.6), the discrete functional follows the equation

$$
\begin{aligned}
\partial_t\mathcal{E}_{\mathrm{PML}}(\mathbf{u}_h) &= -(\partial_t\mathbf{u}_h, A_h\partial_t\mathbf{u}_h)_{\mathrm{V}} - 2(\partial_t H_{h,2}, \theta\partial_t H_{h,2})_\mu\\
&\quad - \left(\partial_t\begin{pmatrix}\mathbf{0}\\ \mathbf{H}_h\end{pmatrix}, A_h\theta\mathbf{u}_h\right)_{\mathrm{V}} - \left(\theta\begin{pmatrix}\mathbf{E}_h\\ \mathbf{0}\end{pmatrix}, A_h\partial_t\mathbf{u}_h\right)_{\mathrm{V}}.
\end{aligned}
\tag{5.8}
$$

Proof: We do the same procedure as in the continuous case

$$
\begin{aligned}
\partial_t\frac{1}{2}\|\partial_t H_{h,1}\|_\mu^2 &= (\partial_t H_{h,1}, \partial_t^2 H_{h,1})_\mu = -(\partial_t H_{h,1}, (\partial_t+\theta)(A_h\mathbf{u}_h)_4)_\mu,\\
\partial_t\frac{1}{2}(\|\partial_t H_{h,2}\|_\mu^2 + \|\theta H_{h,2}\|_\mu^2) &= (\partial_t H_{h,2}, \partial_t^2 H_{h,2})_\mu + (\theta H_{h,2}, \theta\partial_t H_{h,2})_\mu\\
&= -\big(\partial_t H_{h,2}, \partial_t(A_h\mathbf{u}_h)_5\big)_\mu - (\partial_t H_{h,2}, \theta\partial_t H_{h,2})_\mu\\
&\quad - \big((A_h\mathbf{u}_h)_5, \theta\partial_t H_{h,2}\big)_\mu - (\partial_t H_{h,2}, \theta\partial_t H_{h,2})_\mu\\
&= -2(\partial_t H_{h,2}, \theta\partial_t H_{h,2})_\mu - \big(\partial_t H_{h,2}, (\partial_t+\theta)(A_h\mathbf{u}_h)_5\big)_\mu,\\
\partial_t\frac{1}{2}\|(\partial_t+\theta)E_{h,3}\|_\varepsilon^2 &= \big((\partial_t+\theta)E_{h,3}, \partial_t(\partial_t+\theta)E_{h,3}\big)_\varepsilon\\
&= -\big((\partial_t+\theta)E_{h,3}, \partial_t(A_h\mathbf{u}_h)_3\big)_\varepsilon
\end{aligned}
$$

to obtain a similar result. In contrast to the continuous case, none of the terms will vanish

$$\partial_t \mathcal{E}_{\mathrm{PML}}(\mathbf{u}_h) = -(\partial_t \mathbf{u}_h, A_h \partial_t \mathbf{u}_h)_{\mathrm{V}} - 2(\partial_t H_{h,2}, \theta \partial_t H_{h,2})_\mu \\ - \left(\partial_t \begin{pmatrix} \mathbf{0} \\ \mathbf{H}_h \end{pmatrix}, A_h \theta \mathbf{u}_h \right)_{\mathrm{V}} - \left(\theta \begin{pmatrix} \mathbf{E}_h \\ \mathbf{0} \end{pmatrix}, A_h \partial_t \mathbf{u}_h \right)_{\mathrm{V}}.$$

□

The first summand on the right-hand side is handled in Lemma 2.4.4, though here we have no conductivity and only a PEC boundary. The second summand is similar to the continuous case and will be unproblematic in the proof of Lemma 5.2.6 later on. So we will have a look at the two remaining summands.

Lemma 5.2.4:
For a discrete function $\mathbf{u}_h \in \mathrm{C}^1([0,T], \mathrm{V}_h^p)$ and the latter two summands of (5.8), *we have the result*

$$- \int_0^T \left(\left(\partial_t \begin{pmatrix} \mathbf{0} \\ \mathbf{H}_h \end{pmatrix}, A_h \theta \mathbf{u}_h \right)_{\mathrm{V}} + \left(\theta \begin{pmatrix} \mathbf{E}_h \\ \mathbf{0} \end{pmatrix}, A_h \partial_t \mathbf{u}_h \right)_{\mathrm{V}} \right) \eta_T(t)\, \mathrm{dt} \\ = -\theta \int_0^T \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} \frac{1}{4} \delta_f \| \mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f} \|_{0,f}^2 + \frac{1}{4} \gamma_f \| \mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f} \|_{0,f}^2 \, \mathrm{dt} \\ + \theta T \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} \frac{1}{4} \delta_f \| \mathbf{n}_{K,f} \times [\mathbf{H}_{h,0}]_{K,f} \|_{0,f}^2 + \frac{1}{4} \gamma_f \| \mathbf{n}_{K,f} \times [\mathbf{E}_{h,0}]_{K,f} \|_{0,f}^2.$$

We obtain an initial term, as well as a non-positive term, which contains the jumps of the discrete electromagnetic field on the faces.

Proof: First, we like to mention that for a sum over inner faces of a cell and face dependent function $g_{K,f}$, we can switch the cell to the next neighbour cell

$$\sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^\circ} g_{K,f} = \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K^\circ} g_{K_f,f},$$

since every face f appears twice in the sum. On boundary faces, we usually can do the same for the terms to appear in our calculations, but we have to justify it with the virtual definitions in Lemma 2.3.6. Using integration by parts and (2.23), the following term vanishes

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} (\nabla\times\mathbf{E}_h, \partial_t\mathbf{H}_h)_{0,K} - (\nabla\times\partial_t\mathbf{H}_h, \mathbf{E}_h)_{0,K} \\
&+ \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \beta_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}, \partial_t\mathbf{H}_K)_{0,f} - \alpha_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}, \mathbf{E}_K)_{0,f} \\
&\qquad = \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \beta_{K,f}(\mathbf{n}_{K,f}\times\mathbf{E}_{K_f}, \partial_t\mathbf{H}_K)_{0,f} - \alpha_{K,f}(\mathbf{n}_{K,f}\times\partial_t\mathbf{H}_{K_f}, \mathbf{E}_K)_{0,f} \\
&\qquad = 0.
\end{aligned}
$$

With the definition of the discrete operator A_h in (2.28), we obtain

$$
\begin{aligned}
&-\left(\partial_t\begin{pmatrix}\mathbf{0}\\ \mathbf{H}_h\end{pmatrix}, A_h\theta\mathbf{u}_h\right)_{\mathrm{V}} - \left(\theta\begin{pmatrix}\mathbf{E}_h\\ \mathbf{0}\end{pmatrix}, A_h\partial_t\mathbf{u}_h\right)_{\mathrm{V}} \\
&\quad = -\theta\Bigg[\sum_{K\in\mathcal{T}_h} (\nabla\times\mathbf{E}_h, \partial_t\mathbf{H}_h)_{0,K} - (\nabla\times\partial_t\mathbf{H}_h, \mathbf{E}_h)_{0,K} \\
&\qquad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \beta_{K,f}(\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}, \partial_t\mathbf{H}_K)_{0,f} - \alpha_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}, \mathbf{E}_K)_{0,f} \\
&\qquad\qquad + \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}), \partial_t\mathbf{H}_K\big)_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}), \mathbf{E}_K\big)_{0,f}\Bigg] \\
&\quad = -\theta\sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}), \partial_t\mathbf{H}_K\big)_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}), \mathbf{E}_K\big)_{0,f}.
\end{aligned}
$$

Next, we integrate this term over time against the function $\eta_T(t)$. An integration by parts concludes the proof

$$
\begin{aligned}
&-\theta\int_0^T \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \Big(\delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}), \partial_t\mathbf{H}_K\big)_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}), \mathbf{E}_K\big)_{0,f}\Big)\eta_T(t)\,\mathrm{dt} \\
&\quad = -\theta\int_0^T \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \Big(\frac{\delta_f}{2}(\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}, \mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f})_{0,f} \\
&\qquad\qquad + \frac{\gamma_f}{2}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}, \mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f})_{0,f}\Big)\eta_T(t)\,\mathrm{dt} \\
&\quad = -\theta\int_0^T \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \Big(\frac{\delta_f}{2}\partial_t\frac{1}{2}\|\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}\|_{0,f}^2
\end{aligned}
$$

$$+ \frac{\gamma_f}{2}\partial_t \frac{1}{2}\|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2\Big)\eta_T(t)\,\mathrm{dt}$$

$$= -\theta \int_0^T \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \frac{\delta_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2 + \frac{\gamma_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2\,\mathrm{dt}$$

$$+ \theta T \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \frac{\delta_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{H}_{h,0}]_{K,f}\|_{0,f}^2 + \frac{\gamma_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{E}_{h,0}]_{K,f}\|_{0,f}^2.$$

□

Now, we have a look at the term $-\partial_t \mathcal{B}_{\mathrm{PML}}(\mathbf{u}, \mathbf{u}_h)$.

Lemma 5.2.5:
For a solution $\mathbf{v}$ *of the continuous problem* (5.4) *with additional regularity of the electromagnetic field* $\mathbf{u} \in \mathrm{C}^2([0,\infty), \mathrm{L}_2(\Omega)^3) \cap \mathrm{C}^1([0,\infty), \mathrm{H}(\mathrm{curl},\Omega)^2)$, *as well as* $\mathbf{H} \in \mathrm{C}^1([0,\infty), \mathrm{H}^{\frac{1}{2}}(\Lambda_j)^2)$, $\mathbf{E} \in \mathrm{C}([0,\infty), \mathrm{H}^{\frac{1}{2}}(\Lambda_j))$ *for every* j *and a solution* $\mathbf{v}_h$ *of the discrete problem* (5.6), *the bilinear form* $\mathcal{B}_{\mathrm{PML}}(\mathbf{u}, \mathbf{u}_h)$ *fulfills*

$$\begin{aligned}
-\partial_t \mathcal{B}_{\mathrm{PML}}(\mathbf{u}, \mathbf{u}_h) &= (\partial_t \mathbf{u}, A_h \partial_t \mathbf{u}_h)_{\mathrm{V}} + (\partial_t \mathbf{u}_h, A \partial_t \mathbf{u})_{\mathrm{V}} + 4(\partial_t H_2, \theta \partial_t H_{h,2})_\mu \\
&\quad + \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} -\alpha_{K,f}(\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}, \Pi_K^p \theta \mathbf{E} - \theta \mathbf{E})_{0,f} \\
&\qquad + \beta_{K,f}(\mathbf{n}_{K,f} \times [\theta \mathbf{E}_h]_{K,f}, \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H})_{0,f} \\
&\qquad + \gamma_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}), \Pi_K^p \theta \mathbf{E} - \theta \mathbf{E}\big)_{0,f} \\
&\qquad + \delta_f\big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\theta \mathbf{H}_h]_{K,f}), \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\big)_{0,f}.
\end{aligned}$$

The first and second summand on the right-hand side can be treated as in Lemma 2.4.5.

Proof: Again, we calculate the single terms of $-\partial_t \mathcal{B}_{\mathrm{PML}}(\mathbf{u}, \mathbf{u}_h)$

$$\begin{aligned}
&- \partial_t(\partial_t H_1, \partial_t H_{h,1})_\mu \\
&= -(\partial_t^2 H_1, \partial_t H_{h,1})_\mu - (\partial_t H_1, \partial_t^2 H_{h,1})_\mu \\
&= \big((\partial_t + \theta)\partial_{x_2} E_3, \partial_t H_{h,1}\big)_{0,\Omega} + \big(\partial_t H_1, (\partial_t + \theta)(A_h \mathbf{u}_h)_4\big)_\mu,
\end{aligned}$$

$$\begin{aligned}
&- \partial_t\big((\partial_t H_2, \partial_t H_{h,2})_\mu + (\theta H_2, \theta H_{h,2})_\mu\big) \\
&= -(\partial_t \partial_t H_2, \partial_t H_{h,2})_\mu - (\partial_t H_2, \partial_t^2 H_{h,2})_\mu - (\theta \partial_t H_2, \theta H_{h,2})_\mu - (\theta H_2, \theta \partial_t H_{h,2})_\mu \\
&= -(\partial_t \partial_{x_1} E_3, \partial_t H_{h,2})_{0,\Omega} + (\theta \partial_t H_2, \partial_t H_{h,2})_\mu + \big(\partial_t H_2, \partial_t (A_h \mathbf{u}_h)_5\big)_\mu + (\partial_t H_2, \theta \partial_t H_{h,2})_\mu
\end{aligned}$$

$$
\begin{aligned}
&+ \big(\theta\partial_t H_2, (A_h\mathbf{u}_h)_5\big)_\mu + (\theta\partial_t H_2, \partial_t H_{h,2})_\mu - (\partial_{x_1} E_3, \theta\partial_t H_{h,2})_{0,\Omega} + (\partial_t H_2, \theta\partial_t H_{h,2})_\mu \\
&= 4(\partial_t H_2, \theta\partial_t H_{h,2})_\mu + \big(\partial_t H_2, (\partial_t+\theta)(A_h\mathbf{u}_h)_5\big)_\mu - \big((\partial_t+\theta)\partial_{x_1} E_3, \partial_t H_{h,2}\big)_{0,\Omega},
\end{aligned}
$$

$$
\begin{aligned}
&- \partial_t\big((\partial_t+\theta)E_3, (\partial_t+\theta)E_{h,3}\big)_\varepsilon \\
&= -\big(\partial_t(\partial_t+\theta)E_3, (\partial_t+\theta)E_{h,3}\big)_\varepsilon - \big((\partial_t+\theta)E_3, \partial_t(\partial_t+\theta)E_{h,3}\big)_\varepsilon \\
&= -\big(\partial_t(\partial_{x_1} H_2 - \partial_{x_2} H_1), (\partial_t+\theta)E_{h,3}\big)_{0,\Omega} + \big((\partial_t+\theta)E_3, \partial_t(A_h\mathbf{u}_h)_3\big)_\varepsilon.
\end{aligned}
$$

Adding up the three terms, we obtain the expression

$$
\begin{aligned}
&-\partial_t \mathcal{B}_{\mathrm{PML}}(\mathbf{u},\mathbf{u}_h) \\
&\quad= \left(\partial_t \begin{pmatrix}\mathbf{0}\\ \mathbf{H}_h\end{pmatrix}, A\theta\mathbf{u}\right)_{\mathrm{V}} + \left(\theta \begin{pmatrix}\mathbf{E}_h\\ \mathbf{0}\end{pmatrix}, A\partial_t\mathbf{u}\right)_{\mathrm{V}} + \left(\partial_t \begin{pmatrix}\mathbf{0}\\ \mathbf{H}\end{pmatrix}, A_h\theta\mathbf{u}_h\right)_{\mathrm{V}} \\
&\qquad + \left(\theta \begin{pmatrix}\mathbf{E}\\ \mathbf{0}\end{pmatrix}, A_h\partial_t\mathbf{u}_h\right)_{\mathrm{V}} + (\partial_t\mathbf{u}, A_h\partial_t\mathbf{u}_h)_{\mathrm{V}} + (\partial_t\mathbf{u}_h, A\partial_t\mathbf{u})_{\mathrm{V}} + 4(\partial_t H_2, \theta\partial_t H_{h,2})_\mu.
\end{aligned}
$$

We have a look at the first four summands. With an integration by parts, (2.23), and since Π_h^p is an orthogonal projector, we obtain the identity

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} (\partial_t\mathbf{H}_h, \theta\nabla\times\mathbf{E})_{0,K} - (\nabla\times\partial_t\mathbf{H}, \theta\mathbf{E}_h)_{0,K} \\
&\qquad + (\theta\nabla\times\mathbf{E}_h, \Pi_K^p\partial_t\mathbf{H})_{0,K} - (\nabla\times\partial_t\mathbf{H}_h, \Pi_K^p\theta\mathbf{E})_{0,K} \\
&\qquad\quad = \sum_{K\in\mathcal{T}_h} (\nabla\times\partial_t\mathbf{H}_h, \theta\mathbf{E})_{0,K} - (\partial_t\mathbf{H}, \theta\nabla\times\mathbf{E}_h)_{0,K} \\
&\qquad\qquad\quad + (\theta\nabla\times\mathbf{E}_h, \Pi_K^p\partial_t\mathbf{H})_{0,K} - (\nabla\times\partial_t\mathbf{H}_h, \Pi_K^p\theta\mathbf{E})_{0,K} \\
&\qquad\qquad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} (\partial_t\mathbf{H}_K, \theta\mathbf{n}_{K,f}\times\mathbf{E})_{0,f} - (\mathbf{n}_{K,f}\times\partial_t\mathbf{H}, \theta\mathbf{E}_K)_{0,f} \\
&\qquad\quad = \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \alpha_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}, \theta\mathbf{E})_{0,f} - \beta_{K,f}(\mathbf{n}_{K,f}\times[\theta\mathbf{E}_h]_{K,f}, \partial_t\mathbf{H})_{0,f}
\end{aligned}
$$

and use the definition of A_h in (2.28) to conclude the proof

$$
\begin{aligned}
&\left(\partial_t \begin{pmatrix}\mathbf{0}\\ \mathbf{H}_h\end{pmatrix}, A\theta\mathbf{u}\right)_{\mathrm{V}} + \left(\theta \begin{pmatrix}\mathbf{E}_h\\ \mathbf{0}\end{pmatrix}, A\partial_t\mathbf{u}\right)_{\mathrm{V}} + \left(\partial_t \begin{pmatrix}\mathbf{0}\\ \mathbf{H}\end{pmatrix}, A_h\theta\mathbf{u}_h\right)_{\mathrm{V}} \\
&+ \left(\theta \begin{pmatrix}\mathbf{E}\\ \mathbf{0}\end{pmatrix}, A_h\partial_t\mathbf{u}_h\right)_{\mathrm{V}} \\
&= \sum_{K\in\mathcal{T}_h} (\partial_t\mathbf{H}_h, \theta\nabla\times\mathbf{E})_{0,K} - (\nabla\times\partial_t\mathbf{H}, \theta\mathbf{E}_h)_{0,K} \\
&\qquad + (\theta\nabla\times\mathbf{E}_h, \Pi_K^p\partial_t\mathbf{H})_{0,K} - (\nabla\times\partial_t\mathbf{H}_h, \Pi_K^p\theta\mathbf{E})_{0,K} \\
&\quad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \beta_{K,f}(\mathbf{n}_{K,f}\times[\theta\mathbf{E}_h]_{K,f}, \Pi_K^p\partial_t\mathbf{H})_{0,f} - \alpha_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}, \Pi_K^p\theta\mathbf{E})_{0,f}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + \delta_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\theta \mathbf{H}_h]_{K,f}), \Pi_K^p \partial_t \mathbf{H}\big)_{0,f} \\
&\qquad + \gamma_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}), \Pi_K^p \theta \mathbf{E}\big)_{0,f} \\
&= \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} \beta_{K,f} (\mathbf{n}_{K,f} \times [\theta \mathbf{E}_h]_{K,f}, \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H})_{0,f} \\
&\qquad - \alpha_{K,f} (\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}, \Pi_K^p \theta \mathbf{E} - \theta \mathbf{E})_{0,f} \\
&\qquad + \delta_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\theta \mathbf{H}_h]_{K,f}), \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\big)_{0,f} \\
&\qquad + \gamma_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}), \Pi_K^p \theta \mathbf{E} - \theta \mathbf{E}\big)_{0,f}.
\end{aligned}
$$

□

With the preceding lemmata, it is straight forward to estimate the time integral on the right-hand side of (5.7).

Lemma 5.2.6:
For a solution $\mathbf{v}$ *of the continuous problem* (5.4) *with additional regularity of the electromagnetic field* $\mathbf{u} \in \mathrm{C}^2([0,\infty), \mathrm{L}_2(\Omega)^3) \cap \mathrm{C}^1([0,\infty), \mathrm{H}(\mathrm{curl}, \Omega)^2)$*, as well as* $\mathbf{u} \in \mathrm{C}^1([0,\infty), \mathrm{H}^{\frac{1}{2}}(\Lambda_j)^3)$ *for every* j *and a solution* $\mathbf{v}_h$ *of the discrete problem* (5.6)*, we have the following estimate*

$$
\begin{aligned}
&\int_0^T \partial_t \big(\mathcal{E}_{\mathrm{PML}}(\mathbf{u}) + \mathcal{E}_{\mathrm{PML}}(\mathbf{u}_h) - \mathcal{B}_{\mathrm{PML}}(\mathbf{u}, \mathbf{u}_h)\big) \eta_T(t)\, \mathrm{dt} \\
&\qquad \le \int_0^T \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} \Bigg[\frac{3}{2} \frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}} \|\Pi_K^p \partial_t \mathbf{E} - \partial_t \mathbf{E}\|_{0,f}^2 \\
&\qquad\qquad + \left(\frac{3}{2} + \theta \eta_T(t)\right) \frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}} \|\Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\|_{0,f}^2 \\
&\qquad\qquad + \frac{3}{2} \frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}} \theta^2 \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2 \Bigg] \eta_T(t)\, \mathrm{dt} \\
&\qquad + \theta T \sum_{K \in \mathcal{T}_h} \sum_{f \in \mathcal{F}_K} \frac{1}{4} \delta_f \|\mathbf{n}_{K,f} \times [\mathbf{H}_{h,0}]_{K,f}\|_{0,f}^2 + \frac{1}{4} \gamma_f \|\mathbf{n}_{K,f} \times [\mathbf{E}_{h,0}]_{K,f}\|_{0,f}^2.
\end{aligned} \tag{5.9}
$$

The projection errors on the faces can be treated with Lemma 2.3.2. How to treat the initial term will be shown in Lemma 5.2.8 later on.

Proof: We use Lemma 2.4.4, 2.4.5, 5.2.1, 5.2.3, 5.2.4, and 5.2.5, (2.33), and Cauchy-Schwarz' inequality. This yields

$$\begin{aligned}
&\int_0^T \partial_t\big(\mathcal{E}_{\mathrm{PML}}(\mathbf{u}) + \mathcal{E}_{\mathrm{PML}}(\mathbf{u}_h) - \mathcal{B}_{\mathrm{PML}}(\mathbf{u},\mathbf{u}_h)\big)\eta_T(t)\,\mathrm{dt} \\
&\quad = \int_0^T \Bigg(-2(\partial_t H_2, \theta\partial_t H_2)_\mu - 2(\partial_t H_{h,2}, \theta\partial_t H_{h,2})_\mu + 4(\partial_t H_2, \theta\partial_t H_{h,2})_\mu \\
&\qquad - (\partial_t \mathbf{u}_h, A_h\partial_t\mathbf{u}_h)_{\mathrm{V}} + (\partial_t \mathbf{u}, A_h\partial_t\mathbf{u}_h)_{\mathrm{V}} + (\partial_t \mathbf{u}_h, A\partial_t\mathbf{u})_{\mathrm{V}} \\
&\qquad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} -\alpha_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}, \Pi_K^p\theta\mathbf{E} - \theta\mathbf{E})_{0,f} \\
&\qquad\qquad + \beta_{K,f}(\mathbf{n}_{K,f}\times[\theta\mathbf{E}_h]_{K,f}, \Pi_K^p\partial_t\mathbf{H} - \partial_t\mathbf{H})_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}), \Pi_K^p\theta\mathbf{E} - \theta\mathbf{E})_{0,f} \\
&\qquad\qquad + \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\theta\mathbf{H}_h]_{K,f}), \Pi_K^p\partial_t\mathbf{H} - \partial_t\mathbf{H}\big)_{0,f}\Bigg)\eta_T(t)\,\mathrm{dt} \\
&\quad - \theta\int_0^T \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \frac{1}{4}\delta_f\|\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}\|_{0,f}^2 + \frac{1}{4}\gamma_f\|\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}\|_{0,f}^2\,\mathrm{dt} \\
&\quad + \theta T\sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \frac{1}{4}\delta_f\|\mathbf{n}_{K,f}\times[\mathbf{H}_{h,0}]_{K,f}\|_{0,f}^2 + \frac{1}{4}\gamma_f\|\mathbf{n}_{K,f}\times[\mathbf{E}_{h,0}]_{K,f}\|_{0,f}^2 \\
&\quad = \int_0^T \Bigg(-2\theta(\partial_t H_2 - \partial_t H_{h,2}, \partial_t H_2 - \partial_t H_{h,2})_\mu \\
&\qquad - \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \frac{\gamma_f}{2}\|\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{\delta_f}{2}\|\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}\|_{0,f}^2 \\
&\qquad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} -\alpha_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}, \Pi_K^p\partial_t\mathbf{E} - \partial_t\mathbf{E})_{0,f} \\
&\qquad\qquad + \beta_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}, \Pi_K^p\partial_t\mathbf{H} - \partial_t\mathbf{H})_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}), \Pi_K^p\partial_t\mathbf{E} - \partial_t\mathbf{E}\big)_{0,f} \\
&\qquad\qquad + \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}), \Pi_K^p\partial_t\mathbf{H} - \partial_t\mathbf{H}\big)_{0,f} \\
&\qquad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} -\alpha_{K,f}(\mathbf{n}_{K,f}\times[\partial_t\mathbf{H}_h]_{K,f}, \Pi_K^p\theta\mathbf{E} - \theta\mathbf{E})_{0,f} \\
&\qquad\qquad + \beta_{K,f}(\mathbf{n}_{K,f}\times[\theta\mathbf{E}_h]_{K,f}, \Pi_K^p\partial_t\mathbf{H} - \partial_t\mathbf{H})_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\partial_t\mathbf{E}_h]_{K,f}), \Pi_K^p\theta\mathbf{E} - \theta\mathbf{E}\big)_{0,f} \\
&\qquad\qquad + \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\theta\mathbf{H}_h]_{K,f}), \Pi_K^p\partial_t\mathbf{H} - \partial_t\mathbf{H}\big)_{0,f}\Bigg)\eta_T(t)\,\mathrm{dt} \\
&\quad - \theta\int_0^T \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \frac{1}{4}\delta_f\|\mathbf{n}_{K,f}\times[\mathbf{H}_h]_{K,f}\|_{0,f}^2 + \frac{1}{4}\gamma_f\|\mathbf{n}_{K,f}\times[\mathbf{E}_h]_{K,f}\|_{0,f}^2\,\mathrm{dt}
\end{aligned}$$

$$+ \theta T \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \frac{1}{4}\delta_f \|\mathbf{n}_{K,f} \times [\mathbf{H}_{h,0}]_{K,f}\|_{0,f}^2 + \frac{1}{4}\gamma_f \|\mathbf{n}_{K,f} \times [\mathbf{E}_{h,0}]_{K,f}\|_{0,f}^2.$$

For a face f of a cell K at time $t \in (0,T)$ we can utilize

$$\begin{aligned}
&-\alpha_{K,f}(\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}, \Pi_K^p \partial_t \mathbf{E} - \partial_t \mathbf{E})_{0,f} \\
&\qquad \le \frac{\delta_f}{6} \|\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}\|_{0,f}^2 + \frac{3\alpha_{K,f}^2}{2\delta_f} \|\Pi_K^p \partial_t \mathbf{E} - \partial_t \mathbf{E}\|_{0,f}^2, \\
&\beta_{K,f}(\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}, \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H})_{0,f} \\
&\qquad \le \frac{\gamma_f}{6} \|\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{3\beta_{K,f}^2}{2\gamma_f} \|\Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\|_{0,f}^2, \\
&\gamma_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}), \Pi_K^p \partial_t \mathbf{E} - \partial_t \mathbf{E}\big)_{0,f} \\
&\qquad \le \frac{\gamma_f}{6} \|\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{3\gamma_f}{2} \|\Pi_K^p \partial_t \mathbf{E} - \partial_t \mathbf{E}\|_{0,f}^2, \\
&\delta_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}), \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\big)_{0,f} \\
&\qquad \le \frac{\delta_f}{6} \|\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}\|_{0,f}^2 + \frac{3\delta_f}{2} \|\Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\|_{0,f}^2, \\
&-\alpha_{K,f}(\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}, \Pi_K^p \theta \mathbf{E} - \theta \mathbf{E})_{0,f} \\
&\qquad \le \frac{\delta_f}{6} \|\mathbf{n}_{K,f} \times [\partial_t \mathbf{H}_h]_{K,f}\|_{0,f}^2 + \frac{3\alpha_{K,f}^2\theta^2}{2\delta_f} \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2, \\
&\beta_{K,f}(\mathbf{n}_{K,f} \times [\theta \mathbf{E}_h]_{K,f}, \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H})_{0,f} \\
&\qquad \le \frac{\gamma_f\theta}{4\eta_T(t)} \|\mathbf{n}_{K,f} \times [\mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{\beta_{K,f}^2\theta\eta_T(t)}{\gamma_f} \|\Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\|_{0,f}^2, \\
&\gamma_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}), \Pi_K^p \theta \mathbf{E} - \theta \mathbf{E}\big)_{0,f} \\
&\qquad \le \frac{\gamma_f}{6} \|\mathbf{n}_{K,f} \times [\partial_t \mathbf{E}_h]_{K,f}\|_{0,f}^2 + \frac{3\gamma_f\theta^2}{2} \|\Pi_K^p \mathbf{E} - \mathbf{E}\|_{0,f}^2, \\
&\delta_f \big(\mathbf{n}_{K,f} \times (\mathbf{n}_{K,f} \times [\theta \mathbf{H}_h]_{K,f}), \Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\big)_{0,f} \\
&\qquad \le \frac{\delta_f\theta}{4\eta_T(t)} \|\mathbf{n}_{K,f} \times [\mathbf{H}_h]_{K,f}\|_{0,f}^2 + \delta_f\theta\eta_T(t) \|\Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\|_{0,f}^2
\end{aligned}$$

to obtain the desired result

$$\begin{aligned}
&\int_0^T \partial_t \big(\mathcal{E}_{\mathrm{PML}}(\mathbf{u}) + \mathcal{E}_{\mathrm{PML}}(\mathbf{u}_h) - \mathcal{B}_{\mathrm{PML}}(\mathbf{u}, \mathbf{u}_h)\big) \eta_T(t) \,\mathrm{dt} \\
&\quad \le \int_0^T \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \Bigg[\left(\frac{3\alpha_{K,f}^2}{2\delta_f} + \frac{3\gamma_f}{2} \right) \|\Pi_K^p \partial_t \mathbf{E} - \partial_t \mathbf{E}\|_{0,f}^2 \\
&\qquad + \left(\frac{3\beta_{K,f}^2}{2\gamma_f} + \frac{3\delta_f}{2} + \frac{\beta_{K,f}^2 \eta_T(t)\theta}{\gamma_f} + \eta_T(t)\delta_f\theta \right) \|\Pi_K^p \partial_t \mathbf{H} - \partial_t \mathbf{H}\|_{0,f}^2
\end{aligned}$$

$$+\left(\frac{3\alpha_{K,f}^2\theta^2}{2\delta_f}+\frac{3\gamma_f\theta^2}{2}\right)\|\Pi_K^p\mathbf{E}-\mathbf{E}\|_{0,f}^2\Bigg]\eta_T(t)\,\mathrm{dt}$$

$$+\theta T\sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K}\frac{1}{4}\delta_f\|\mathbf{n}_{K,f}\times[\mathbf{H}_{h,0}]_{K,f}\|_{0,f}^2+\frac{1}{4}\gamma_f\|\mathbf{n}_{K,f}\times[\mathbf{E}_{h,0}]_{K,f}\|_{0,f}^2.$$

The arising material coefficients can be simplified to

$$\frac{\alpha_{K,f}^2}{\delta_f}+\gamma_f=\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}},\qquad\qquad \frac{\beta_{K,f}^2}{\gamma_f}+\delta_f=\frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}}$$

and already appeared in the error estimate without PML in Lemma 2.4.6. □

To conclude the estimate, we need to have a look at the initial terms. There is one initial term on the right-hand side of (5.7) and another one on the right-hand side of (5.9). To handle them, we need an estimate on the term $\|A_h\Pi_h^p\mathbf{u}-\Pi_h^pA\mathbf{u}\|_{\mathrm{V}}$.

Lemma 5.2.7:
Let $\mathbf{u}\in\mathrm{H}(\mathrm{curl},\Omega)^2$ with regularity $\mathbf{u}\in\mathrm{H}^{\frac{1}{2}}(\Lambda_j)^3$ for all j and boundary values $\mathbf{n}\times\mathbf{E}|_{\partial\Omega}=\mathbf{0}$. Then we obtain the following commutation error estimate for A and Π_h^p

$$\|A_h\Pi_h^p\mathbf{u}-\Pi_h^pA\mathbf{u}\|_{\mathrm{V}}^2$$
$$\leq\sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K}4c_{\mathrm{trace}}^2\left(\frac{\alpha_{K_f,f}^2}{h_K\varepsilon_K}+\frac{\alpha_{K_f,f}^2}{h_{K_f}\varepsilon_{K_f}}+\frac{\delta_f^2}{h_K\mu_K}+\frac{\delta_f^2}{h_{K_f}\mu_{K_f}}\right)\|\Pi_K^p\mathbf{H}-\mathbf{H}\|_{0,f}^2$$
$$+4c_{\mathrm{trace}}^2\left(\frac{\beta_{K_f,f}^2}{h_K\mu_K}+\frac{\beta_{K_f,f}^2}{h_{K_f}\mu_{K_f}}+\frac{\gamma_f^2}{h_K\varepsilon_K}+\frac{\gamma_f^2}{h_{K_f}\varepsilon_{K_f}}\right)\|\Pi_K^p\mathbf{E}-\mathbf{E}\|_{0,f}^2.$$

Proof: For $(\boldsymbol{\psi}_h,\boldsymbol{\varphi}_h)\in\mathrm{V}_h^p$ we use integration by parts

$$\sum_{K\in\mathcal{T}_h}-\big(\nabla\times(\Pi_K^p\mathbf{H}-\mathbf{H}),\boldsymbol{\psi}_h\big)_{0,K}+\big(\nabla\times(\Pi_K^p\mathbf{E}-\mathbf{E}),\boldsymbol{\varphi}_h\big)_{0,K}$$
$$=\sum_{K\in\mathcal{T}_h}-\big((\Pi_K^p\mathbf{H}-\mathbf{H}),\nabla\times\boldsymbol{\psi}_h\big)_{0,K}+\big((\Pi_K^p\mathbf{E}-\mathbf{E}),\nabla\times\boldsymbol{\varphi}_h\big)_{0,K}$$
$$+\sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K}-\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{H}-\mathbf{H}),\boldsymbol{\psi}_K\big)_{0,f}+\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{E}-\mathbf{E}),\boldsymbol{\varphi}_K\big)_{0,f}$$
$$=\sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K}-\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{H}-\mathbf{H}),\boldsymbol{\psi}_K\big)_{0,f}+\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{E}-\mathbf{E}),\boldsymbol{\varphi}_K\big)_{0,f}$$

and also (2.23), (2.33), and Cauchy-Schwarz' inequality to calculate

$$
\begin{aligned}
&\left(A_h\Pi_h^p\mathbf{u} - \Pi_h^p A\mathbf{u}, \begin{pmatrix}\boldsymbol{\psi}_h\\ \boldsymbol{\varphi}_h\end{pmatrix}\right)_{\mathrm{V}} \\
&\quad = \sum_{K\in\mathcal{T}_h} -\big(\nabla\times(\Pi_K^p\mathbf{H}-\mathbf{H}),\boldsymbol{\psi}_h\big)_{0,K} + \big(\nabla\times(\Pi_K^p\mathbf{E}-\mathbf{E}),\boldsymbol{\varphi}_h\big)_{0,K} \\
&\qquad + \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} -\alpha_{K,f}(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{H}]_{K,f},\boldsymbol{\psi}_K)_{0,f} + \beta_{K,f}(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{E}]_{K,f},\boldsymbol{\varphi}_K)_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{E}]_{K,f}),\boldsymbol{\psi}_K\big)_{0,f} \\
&\qquad\qquad + \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{H}]_{K,f}),\boldsymbol{\varphi}_K\big)_{0,f} \\
&\quad = \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} -\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{H}-\mathbf{H}),\boldsymbol{\psi}_K\big)_{0,f} + \big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{E}-\mathbf{E}),\boldsymbol{\varphi}_K\big)_{0,f} \\
&\qquad\qquad - \alpha_{K,f}(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{H}-\mathbf{H}]_{K,f},\boldsymbol{\psi}_K)_{0,f} \\
&\qquad\qquad + \beta_{K,f}(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{E}-\mathbf{E}]_{K,f},\boldsymbol{\varphi}_K)_{0,f} \\
&\qquad\qquad + \gamma_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{E}-\mathbf{E}]_{K,f}),\boldsymbol{\psi}_K\big)_{0,f} \\
&\qquad\qquad + \delta_f\big(\mathbf{n}_{K,f}\times(\mathbf{n}_{K,f}\times[\Pi_h^p\mathbf{H}-\mathbf{H}]_{K,f}),\boldsymbol{\varphi}_K\big)_{0,f} \\
&\quad = \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} -\alpha_{K_f,f}\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{H}-\mathbf{H}),\boldsymbol{\psi}_K\big)_{0,f} \\
&\qquad\qquad + \beta_{K_f,f}\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{E}-\mathbf{E}),\boldsymbol{\varphi}_K\big)_{0,f} \\
&\qquad\qquad - \alpha_{K,f}\big(\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{H}-\mathbf{H}),\boldsymbol{\psi}_K\big)_{0,f} \\
&\qquad\qquad + \beta_{K,f}\big(\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{E}-\mathbf{E}),\boldsymbol{\varphi}_K\big)_{0,f} \\
&\qquad\qquad + \gamma_f\Big(\mathbf{n}_{K,f}\times\big(\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{E}-\mathbf{E})\big),\boldsymbol{\psi}_K\Big)_{0,f} \\
&\qquad\qquad - \gamma_f\Big(\mathbf{n}_{K,f}\times\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{E}-\mathbf{E})\big),\boldsymbol{\psi}_K\Big)_{0,f} \\
&\qquad\qquad + \delta_f\Big(\mathbf{n}_{K,f}\times\big(\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{H}-\mathbf{H})\big),\boldsymbol{\varphi}_K\Big)_{0,f} \\
&\qquad\qquad - \delta_f\Big(\mathbf{n}_{K,f}\times\big(\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{H}-\mathbf{H})\big),\boldsymbol{\varphi}_K\Big)_{0,f} \\
&\quad \le \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} \frac{2c_{\mathrm{trace}}^2\alpha_{K_f,f}^2}{h_K\varepsilon_K}\|\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{H}-\mathbf{H})\|_{0,f}^2 + \frac{h_K\varepsilon_K}{8c_{\mathrm{trace}}^2}\|\boldsymbol{\psi}_K\|_{0,f}^2 \\
&\qquad\qquad + \frac{2c_{\mathrm{trace}}^2\beta_{K_f,f}^2}{h_K\mu_K}\|\mathbf{n}_{K,f}\times(\Pi_K^p\mathbf{E}-\mathbf{E})\|_{0,f}^2 + \frac{h_K\mu_K}{8c_{\mathrm{trace}}^2}\|\boldsymbol{\varphi}_K\|_{0,f}^2 \\
&\qquad\qquad + \frac{2c_{\mathrm{trace}}^2\alpha_{K,f}^2}{h_K\varepsilon_K}\|\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{H}-\mathbf{H})\|_{0,f}^2 + \frac{h_K\varepsilon_K}{8c_{\mathrm{trace}}^2}\|\boldsymbol{\psi}_K\|_{0,f}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2c_{\text{trace}}^2\beta_{K,f}^2}{h_K\mu_K}\|\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{E}-\mathbf{E})\|_{0,f}^2 + \frac{h_K\mu_K}{8c_{\text{trace}}^2}\|\boldsymbol{\varphi}_K\|_{0,f}^2\\
&+ \frac{2c_{\text{trace}}^2\gamma_f^2}{h_K\varepsilon_K}\|\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{E}-\mathbf{E})\|_{0,f}^2 + \frac{h_K\varepsilon_K}{8c_{\text{trace}}^2}\|\boldsymbol{\psi}_K\|_{0,f}^2\\
&+ \frac{2c_{\text{trace}}^2\gamma_f^2}{h_K\varepsilon_K}\|\mathbf{n}_{K,f}\times(\Pi_{K}^p\mathbf{E}-\mathbf{E})\|_{0,f}^2 + \frac{h_K\varepsilon_K}{8c_{\text{trace}}^2}\|\boldsymbol{\psi}_K\|_{0,f}^2\\
&+ \frac{2c_{\text{trace}}^2\delta_f^2}{h_K\mu_K}\|\mathbf{n}_{K,f}\times(\Pi_{K_f}^p\mathbf{H}-\mathbf{H})\|_{0,f}^2 + \frac{h_K\mu_K}{8c_{\text{trace}}^2}\|\boldsymbol{\varphi}_K\|_{0,f}^2\\
&+ \frac{2c_{\text{trace}}^2\delta_f^2}{h_K\mu_K}\|\mathbf{n}_{K,f}\times(\Pi_{K}^p\mathbf{H}-\mathbf{H})\|_{0,f}^2 + \frac{h_K\mu_K}{8c_{\text{trace}}^2}\|\boldsymbol{\varphi}_K\|_{0,f}^2.
\end{aligned}
$$

Now, we proceed with Lemma 2.3.3

$$
\begin{aligned}
&\left(A_h\Pi_h^p\mathbf{u}-\Pi_h^pA\mathbf{u},\begin{pmatrix}\boldsymbol{\psi}_h\\ \boldsymbol{\varphi}_h\end{pmatrix}\right)_{\text{V}}\\
&\quad\le \sum_{K\in\mathcal{T}_h}\sum_{f\in\mathcal{F}_K} 2c_{\text{trace}}^2\left(\frac{\alpha_{K_f,f}^2}{h_K\varepsilon_K}+\frac{\alpha_{K_f,f}^2}{h_{K_f}\varepsilon_{K_f}}+\frac{\delta_f^2}{h_K\mu_K}+\frac{\delta_f^2}{h_{K_f}\mu_{K_f}}\right)\|\Pi_K^p\mathbf{H}-\mathbf{H}\|_{0,f}^2\\
&\qquad\qquad + 2c_{\text{trace}}^2\left(\frac{\beta_{K_f,f}^2}{h_K\mu_K}+\frac{\beta_{K_f,f}^2}{h_{K_f}\mu_{K_f}}+\frac{\gamma_f^2}{h_K\varepsilon_K}+\frac{\gamma_f^2}{h_{K_f}\varepsilon_{K_f}}\right)\|\Pi_K^p\mathbf{E}-\mathbf{E}\|_{0,f}^2\\
&\quad + \frac{1}{2}\|\boldsymbol{\psi}_h\|_\varepsilon^2+\frac{1}{2}\|\boldsymbol{\varphi}_h\|_\mu^2.
\end{aligned}
$$

Here, we used $\mathbf{n}_{K,f}\times\mathbf{E}|_{K_f}$ and $\mathbf{n}_{K,f}\times\mathbf{H}|_{K_f}$ on boundary faces f. This is defined via the usual virtual definitions stated in Lemma 2.3.6, where we also mentioned that $h_{K_f}=h_K$ for boundary faces. With the choice $(\boldsymbol{\psi}_h,\boldsymbol{\varphi}_h)=A_h\Pi_h^p\mathbf{u}-\Pi_h^pA\mathbf{u}$, the desired result is finally obtained. □

It is mentionable, that terms like $\frac{1}{h_{K_f}}\|\Pi_K^p\mathbf{H}-\mathbf{H}\|_{0,f}^2$ appear in the preceding Lemma, which contain values of both cells belonging to a face f. The projection error can be estimated with Lemma 2.3.2 to obtain powers of h_K. For cells of equal size, one power cancels down, but in case of $h_{K_f} << h_K$ we have to expect a negative effect on the discrete solution. This may happen if e.g. two meshes of different fineness are matched together with hanging nodes. In the error calculations without layer in Chapter 2, we did not come across such terms.

Now, the initial terms can be estimated by projection errors.

Lemma 5.2.8:
Let $\mathbf{u} \in \mathrm{C}^1\big([0,T], \mathrm{H}(\mathrm{curl},\Omega)^2\big)$, *with* $\mathbf{u} \in \mathrm{C}^1\big([0,T], \mathrm{H}^{\frac{1}{2}}(\Lambda_j)^3\big)$ *for all* j *and let* $\mathbf{u}_h : [0,T] \to \mathrm{V}_h^p$ *be solutions to PDE* (5.4) *and ODE* (5.6), *respectively. Let also* $\xi_{H,1,0} \in \mathrm{L}_2(\Omega)$. *Then the initial terms can be estimated as follows*

$$
\begin{aligned}
&\theta T \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \frac{\delta_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{H}_{h,0}]_{K,f}\|_{0,f}^2 + \frac{\gamma_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{E}_{h,0}]_{K,f}\|_{0,f}^2 \\
&\quad\le \theta T \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \gamma_f \|\Pi_K^p \mathbf{E}_0 - \mathbf{E}_0\|_{0,f}^2 + \delta_f \|\Pi_K^p \mathbf{H}_0 - \mathbf{H}_0\|_{0,f}^2, \\
&T\mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)(0) \\
&\quad\le T \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} 4c_{\mathrm{trace}}^2 \left(\frac{\alpha_{K_f,f}^2}{h_K\varepsilon_K} + \frac{\alpha_{K_f,f}^2}{h_{K_f}\varepsilon_{K_f}} + \frac{\delta_f^2}{h_K\mu_K} + \frac{\delta_f^2}{h_{K_f}\mu_{K_f}} \right) \|\Pi_K^p \mathbf{H}_0 - \mathbf{H}_0\|_{0,f}^2 \\
&\qquad\qquad + 4c_{\mathrm{trace}}^2 \left(\frac{\beta_{K_f,f}^2}{h_K\mu_K} + \frac{\beta_{K_f,f}^2}{h_{K_f}\mu_{K_f}} + \frac{\gamma_f^2}{h_K\varepsilon_K} + \frac{\gamma_f^2}{h_{K_f}\varepsilon_{K_f}} \right) \|\Pi_K^p \mathbf{E}_0 - \mathbf{E}_0\|_{0,f}^2 \\
&\qquad + 3T\|A\mathbf{u}_0 - \Pi_h^p A\mathbf{u}_0\|_{\mathrm{V}}^2 + \frac{5}{2}T\theta^2\|H_{2,0} - \Pi_h^p H_{2,0}\|_\mu^2 \\
&\qquad + 3T\theta^2\big\|H_{1,0} - \Pi_h^p H_{1,0}\big\|_\mu^2 + 3T\theta^2\big\|\xi_{H,1,0} - \Pi_h^p \xi_{H,1,0}\big\|_\mu^2.
\end{aligned}
$$

Proof: We start with the first term and use $(a+b)^2 \le 2(a^2+b^2)$ and later on also $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ for $a,b,c \in \mathbb{R}$

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \frac{\delta_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{H}_{h,0}]_{K,f}\|_{0,f}^2 + \frac{\gamma_f}{4}\|\mathbf{n}_{K,f} \times [\mathbf{E}_{h,0}]_{K,f}\|_{0,f}^2 \\
&\quad= \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \frac{\delta_f}{4}\|\mathbf{n}_{K,f} \times [\Pi_h^p\mathbf{H}_0 - \mathbf{H}_0]_{K,f}\|_{0,f}^2 + \frac{\gamma_f}{4}\|\mathbf{n}_{K,f} \times [\Pi_h^p\mathbf{E}_0 - \mathbf{E}_0]_{K,f}\|_{0,f}^2 \\
&\quad\le \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \delta_f\|\Pi_K^p\mathbf{H}_0 - \mathbf{H}_0\|_{0,f}^2 + \gamma_f\|\Pi_K^p\mathbf{E}_0 - \mathbf{E}_0\|_{0,f}^2.
\end{aligned}
$$

Next, we have a look at the energy-like term

$$
\begin{aligned}
\mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)(0)& \\
&= \frac{1}{2}\|\partial_t(H_1 - H_{h,1})(0)\|_\mu^2 + \frac{1}{2}\|\partial_t(H_2 - H_{h,2})(0)\|_\mu^2 \\
&\quad + \frac{1}{2}\|\theta(H_{2,0} - H_{h,2,0})\|_\mu^2 + \frac{1}{2}\|(\partial_t + \theta)(E_3 - E_{h,3})(0)\|_\varepsilon^2 \\
&= \frac{1}{2}\big\| - \big((A\mathbf{u})_{4,0} - (A_h\mathbf{u}_h)_{4,0}\big) + \theta(H_{1,0} - H_{h,1,0}) + \theta(\xi_{H,1,0} - \xi_{H,h,1,0})\big\|_\mu^2 \\
&\quad + \frac{1}{2}\big\| - \big((A\mathbf{u})_{5,0} - (A_h\mathbf{u}_h)_{5,0}\big) - \theta(H_{2,0} - H_{h,2,0})\big\|_\mu^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2}\|\theta(H_{2,0} - H_{h,2,0})\|_\mu^2 + \frac{1}{2}\big\|\big((A\mathbf{u})_{3,0} - (A_h\mathbf{u}_h)_{3,0}\big)\big\|_\varepsilon^2 \\
\leq{}& \|\Pi_h^p A\mathbf{u}_0 - A_h\Pi_h^p\mathbf{u}_0\|_{\mathrm{V}}^2 + 3\|A\mathbf{u}_0 - \Pi_h^p A\mathbf{u}_0\|_{\mathrm{V}}^2 \\
&+ \frac{5}{2}\|\theta(H_{2,0} - \Pi_h^p H_{2,0})\|_\mu^2 + 3\big\|\theta(H_{1,0} - \Pi_h^p H_{1,0})\big\|_\mu^2 \\
&+ 3\big\|\theta(\xi_{H,1,0} - \Pi_h^p \xi_{H,1,0})\big\|_\mu^2.
\end{aligned}
$$

Lemma 5.2.7 concludes the proof. □

The functional $\mathcal{E}_{\mathrm{PML}}$ contains time derivatives. To obtain an error estimate in the weighted $\mathrm{L}_2(\Omega)$-norm, we utilize Jensen's inequality and another lemma, which leads to the shifted time derivative $\partial_t + \theta$ contained in the functional $\mathcal{E}_{\mathrm{PML}}(\mathbf{u})$.

Lemma 5.2.9 (Jensen's inequality, [Wal04, Section 11.18])**:**
Let $y : [a,b] \subset \mathbb{R} \to [c,d] \subset \mathbb{R} \in \mathrm{L}^1(a,b)$ and let $f \in \mathrm{C}([c,d],\mathbb{R})$ be a convex function, with $f \circ y \in \mathrm{L}^1(a,b)$. Then f commutes with the integral mean as follows

$$
f\left(\frac{1}{b-a}\int_a^b y(x)\,\mathrm{d}x\right) \leq \frac{1}{b-a}\int_a^b f\big(y(x)\big)\,\mathrm{d}x\,.
$$

Lemma 5.2.10:
Let $u \in \mathrm{C}^1([0,t],\mathrm{L}_2(\Omega))$ be a scalar function and $\vartheta \in \mathbb{R}$. Then u can be rewritten as

$$
u(\mathbf{x},t) = \exp(-\vartheta t)u(\mathbf{x},0) + \int_0^t \exp\big(\vartheta(\tau - t)\big)(\partial_\tau + \vartheta)u(\mathbf{x},\tau)\,\mathrm{d}\tau\,.
$$

Proof: We start with one term on the right-hand side

$$
\begin{aligned}
\int_0^t \exp\big(\vartheta(\tau - t)\big)\vartheta u(\mathbf{x},\tau)\,\mathrm{d}\tau &= \int_0^t \partial_\tau\Big(\exp\big(\vartheta(\tau - t)\big)\Big)u(\mathbf{x},\tau)\,\mathrm{d}\tau \\
&= -\int_0^t \exp\big(\vartheta(\tau - t)\big)\partial_\tau u(\mathbf{x},\tau)\,\mathrm{d}\tau \\
&\quad + u(\mathbf{x},t) - \exp(-\vartheta t)u(\mathbf{x},0).
\end{aligned}
$$

A rearrangement of the terms concludes the proof. □

Lemma 5.2.11:
Let $\mathbf{v} \in \mathrm{C}^1([0,T], \mathrm{V}_{\mathrm{PML}}) \cap \mathrm{C}([0,T], \mathrm{H}(\mathrm{curl},\Omega)^2 \times \mathrm{V})$ *and* $\mathbf{v}_h : [0,T] \to \mathrm{V}^p_{h,\mathrm{PML}}$ *be solutions to PDE* (5.4) *and ODE* (5.6), *respectively. Then the error in the weighted* $\mathrm{L}_2(\Omega)$*-norm can be estimated as follows*

$$\begin{aligned}
&\int_0^T \|H_1 - H_{h,1}\|_\mu^2 + \|H_2 - H_{h,2}\|_\mu^2 + \|E_3 - E_{h,3}\|_\varepsilon^2 + \|\xi_{H,1} - \xi_{H,h,1}\|_\mu^2 \,\mathrm{dt} \\
&\quad \le 8T\|\xi_{H,1,0} - \Pi_h^p \xi_{H,1,0}\|_\mu^2 + 14T\|H_{1,0} - \Pi_h^p H_{1,0}\|_\mu^2 + 2T\|H_{2,0} - \Pi_h^p H_{2,0}\|_\mu^2 \\
&\qquad + 2T\|E_{3,0} - \Pi_h^p E_{3,0}\|_\varepsilon^2 + 10\int_0^T t\int_0^t \|\partial_\tau(H_1 - H_{h,1})\|_\mu^2 \,\mathrm{d}\tau\,\mathrm{dt} \\
&\qquad + 2\int_0^T t\int_0^t \|\partial_\tau(H_2 - H_{h,2})\|_\mu^2 \,\mathrm{d}\tau\,\mathrm{dt} + 2\int_0^T t\int_0^t \|(\partial_\tau + \theta)(E_3 - E_{h,3})\|_\varepsilon^2 \,\mathrm{d}\tau\,\mathrm{dt}\,.
\end{aligned}$$

Proof: We start with the first term and use Jensen's inequality

$$\begin{aligned}
\int_0^T \|H_1 - H_{h,1}\|_\mu^2 \,\mathrm{dt} &= \int_0^T \left\| H_{1,0} - H_{h,1,0} + \int_0^t \partial_\tau(H_1 - H_{h,1})\,\mathrm{d}\tau \right\|_\mu^2 \mathrm{dt} \\
&\le 2T\|H_{1,0} - H_{h,1,0}\|_\mu^2 + 2\int_0^T \left(\int_0^t \|\partial_\tau(H_1 - H_{h,1})\|_\mu \,\mathrm{d}\tau\right)^2 \mathrm{dt} \\
&\le 2T\|H_{1,0} - \Pi_h^p H_{1,0}\|_\mu^2 + 2\int_0^T t\int_0^t \|\partial_\tau(H_1 - H_{h,1})\|_\mu^2 \,\mathrm{d}\tau\,\mathrm{dt}\,.
\end{aligned}$$

The same holds true for H_2. The third term has to be treated differently. We use Lemma 5.2.10

$$\begin{aligned}
&\int_0^T \|E_3 - E_{h,3}\|_\varepsilon^2 \,\mathrm{dt} \\
&\quad = \int_0^T \left\| \exp(-\theta t)(E_{3,0} - E_{h,3,0}) + \int_0^t \exp\big(\theta(\tau - t)\big)(\partial_\tau + \theta)(E_3 - E_{h,3})\,\mathrm{d}\tau \right\|_\varepsilon^2 \mathrm{dt} \\
&\quad \le 2\int_0^T \exp(-2\theta t)\,\mathrm{dt}\,\|E_{3,0} - E_{h,3,0}\|_\varepsilon^2 \\
&\qquad + 2\int_0^T \left(\int_0^t \exp\big(\theta(\tau - t)\big)\|(\partial_\tau + \theta)(E_3 - E_{h,3})\|_\varepsilon \,\mathrm{d}\tau\right)^2 \mathrm{dt} \\
&\quad \le 2T\|E_{3,0} - \Pi_h^p E_{3,0}\|_\varepsilon^2 + 2\int_0^T t\int_0^t \exp\big(2\theta(\tau - t)\big)\|(\partial_\tau + \theta)(E_3 - E_{h,3})\|_\varepsilon^2 \,\mathrm{d}\tau\,\mathrm{dt} \\
&\quad \le 2T\|E_{3,0} - \Pi_h^p E_{3,0}\|_\varepsilon^2 + 2\int_0^T t\int_0^t \|(\partial_\tau + \theta)(E_3 - E_{h,3})\|_\varepsilon^2 \,\mathrm{d}\tau\,\mathrm{dt}\,.
\end{aligned}$$

For the last term we start similar to the third term, but use equations (5.4d) and (5.6d)

$$
\begin{aligned}
&\int_0^T \|\xi_{H,1} - \xi_{H,h,1}\|_\mu^2 \,\mathrm{dt} \\
&\quad= \int_0^T \Big\| \exp(-\theta t)(\xi_{H,1,0} - \xi_{H,h,1,0}) - \int_0^t \exp\big(\theta(\tau - t)\big)\theta(H_1 - H_{h,1})\,\mathrm{d}\tau \Big\|_\mu^2 \,\mathrm{dt} \\
&\quad= \int_0^T \Big\| \exp(-\theta t)(\xi_{H,1,0} - \xi_{H,h,1,0}) - \int_0^t \partial_\tau\Big(\exp\big(\theta(\tau - t)\big)\Big)(H_1 - H_{h,1})\,\mathrm{d}\tau \Big\|_\mu^2 \,\mathrm{dt} \\
&\quad= \int_0^T \Big\| \exp(-\theta t)(\xi_{H,1,0} - \xi_{H,h,1,0}) + \int_0^t \exp\big(\theta(\tau - t)\big)\partial_\tau(H_1 - H_{h,1})\,\mathrm{d}\tau \\
&\qquad\qquad - (H_1 - H_{h,1}) + \exp(-\theta t)(H_{1,0} - H_{h,1,0}) \Big\|_\mu^2 \,\mathrm{dt} \\
&\quad\leq \int_0^T \Big(8\exp(-2\theta t)\|\xi_{H,1,0} - \xi_{H,h,1,0}\|_\mu^2 + 8\exp(-2\theta t)\|H_{1,0} - H_{h,1,0}\|_\mu^2 \\
&\qquad\qquad + 4\Big(\int_0^t \exp\big(\theta(\tau - t)\big)\|\partial_\tau(H_1 - H_{h,1})\|_\mu \,\mathrm{d}\tau\Big)^2 + 2\|H_1 - H_{h,1}\|_\mu^2 \Big)\,\mathrm{dt} \\
&\quad\leq 8T\|\xi_{H,1,0} - \Pi_h^p \xi_{H,1,0}\|_\mu^2 + 8T\|H_{1,0} - \Pi_h^p H_{1,0}\|_\mu^2 \\
&\qquad + 4\int_0^T t\int_0^t \|\partial_\tau(H_1 - H_{h,1})\|_\mu^2 \,\mathrm{d}\tau\,\mathrm{dt} + 2\int_0^T \|H_1 - H_{h,1}\|_\mu^2 \,\mathrm{dt}\,.
\end{aligned}
$$

This concludes the proof. □

Finally, we can state the desired error estimate. Because we have a loss of space regularity in $A\mathbf{u}_0$ and $\partial_t\mathbf{u}$, due to the space derivatives in A, we decreased the polynomial degree p by one compared to Theorem 2.4.1 and only estimate by semi-norms $|\cdot|_{s-1}$.

Theorem 5.2.12:
Let the relative diameter of next neighbour cells be bounded as in (2.11)*. Further on let* $\mathbf{v}$ *be the solution to Maxwell's system* (5.4) *with additional regularity of the electromagnetic field* $\mathbf{u} \in \mathrm{C}^1([0,T], \mathrm{H}(\mathrm{curl},\Omega)^2) \cap \mathrm{C}^2([0,T], \mathrm{V})$, $\mathbf{u} \in \mathrm{C}([0,T], \mathrm{H}^s(\Lambda_j)^3)$ *for every* j *and a* $s > 2$ *and with regularity of the auxiliary function* $\boldsymbol{\xi} \in \mathrm{C}^1([0,T], \mathrm{V})$, $\boldsymbol{\xi} \in \mathrm{C}\big([0,T], \mathrm{H}^{s-1}(\Lambda_j)\big)$ *for every* j*, let* $p = \bar{s} - 2$*, with* $\bar{s}$ *the rounded up integer part of* s*, and let* $\mathbf{v}_h$ *be the solution of the ODE* (5.6)*. Then the error between continuous and discrete solution can be estimated as follows*

$$\begin{aligned}
\|\mathbf{v}-\mathbf{v}_h\|^2_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})} \leq\ & 7T^3\theta c_{\gamma,\delta} c^2_{\mathrm{boundary}} h^{2s-3} |\mathbf{u}_0|^2_{s-1,\mathrm{V},\Lambda} \\
&+ 27T^3 c^2_{\mathrm{trace}} c_{\varepsilon,\mu,c_{\mathrm{h}}} c^2_{\mathrm{boundary}} h^{2s-4} |\mathbf{u}_0|^2_{s-1,\mathrm{V},\Lambda} \\
&+ 20T^3 c^2_{\mathrm{cell}} h^{2s-2} |A\mathbf{u}_0|^2_{s-1,\mathrm{V},\Lambda} \\
&+ (14T + 20T^3\theta^2) c^2_{\mathrm{cell}} h^{2s-2} |\mathbf{v}_0|^2_{s-1,\mathrm{V}_{\mathrm{PML}},\Lambda} \\
&+ 4T^4 c_{\mathrm{light}} c^2_{\mathrm{boundary}} h^{2s-3} \max_{t\in[0,T]} |\partial_t \mathbf{u}|^2_{s-1,\mathrm{V},\Lambda} \\
&+ 4T^4\theta^2 c_{\mathrm{light}} c^2_{\mathrm{boundary}} h^{2s-3} \max_{t\in[0,T]} |\mathbf{E}|^2_{s-1,\varepsilon,\Lambda} \\
&+ 2T^5\theta c_{\mathrm{light}} c^2_{\mathrm{boundary}} h^{2s-3} \max_{t\in[0,T]} |\partial_t \mathbf{H}|^2_{s-1,\mu,\Lambda} .
\end{aligned}$$

The constant c_{light} was defined in (2.36). Further on, we used the constants

$$\begin{aligned}
c_{\gamma,\delta} &= \max\left\{ \max_{K\in\mathcal{T}_h, f\in\mathcal{F}_K} \frac{\gamma_f}{\varepsilon_K}, \max_{K\in\mathcal{T}_h, f\in\mathcal{F}_K} \frac{\delta_f}{\mu_K} \right\}, \\
c_{\varepsilon,\mu,c_{\mathrm{h}}} &= \max\left\{ \max_{K\in\mathcal{T}_h, f\in\mathcal{F}_K} \frac{\alpha^2_{K_f,f}}{\varepsilon_K\mu_K} + \frac{\delta_f^2}{\mu_K^2} + c_{\mathrm{h}} \left(\frac{\alpha^2_{K_f,f}}{\varepsilon_{K_f}\mu_K} + \frac{\delta_f^2}{\mu_{K_f}\mu_K} \right), \right. \\
&\qquad \left. \max_{K\in\mathcal{T}_h, f\in\mathcal{F}_K} \frac{\beta^2_{K_f,f}}{\mu_K\varepsilon_K} + \frac{\gamma_f^2}{\varepsilon_K^2} + c_{\mathrm{h}} \left(\frac{\beta^2_{K_f,f}}{\mu_{K_f}\varepsilon_K} + \frac{\gamma_f^2}{\varepsilon_{K_f}\varepsilon_K} \right) \right\}.
\end{aligned}$$

Proof: We start with a utilization of Lemma 5.2.11

$$\begin{aligned}
\|\mathbf{v}-\mathbf{v}_h\|^2_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})} &= \int_0^T \|\mathbf{v}-\mathbf{v}_h\|^2_{\mathrm{V}_{\mathrm{PML}}}\,\mathrm{dt} \\
&\leq 14T\|\mathbf{v}_0 - \Pi_h^p \mathbf{v}_0\|^2_{\mathrm{V}_{\mathrm{PML}}} + 20\int_0^T t \int_0^t \mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\tau\,\mathrm{dt}\,.
\end{aligned}$$

The latter summand can be estimated with Lemma 5.2.2, 5.2.6, and 5.2.8

$$\begin{aligned}
&\int_0^t \mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\tau \\
&\leq \int_0^t \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \left[\frac{3}{2}\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}} \|\partial_\tau \mathbf{E} - \Pi_K^p \partial_\tau \mathbf{E}\|^2_{0,f} \right. \\
&\qquad + \left(\frac{3}{2} + \theta\eta_t(\tau)\right) \frac{\sqrt{\mu_K}}{\sqrt{\varepsilon_K}} \|\partial_\tau \mathbf{H} - \Pi_K^p \partial_\tau \mathbf{H}\|^2_{0,f} \\
&\qquad \left. + \frac{3}{2}\frac{\sqrt{\varepsilon_K}}{\sqrt{\mu_K}} \theta^2 \|\mathbf{E} - \Pi_K^p \mathbf{E}\|^2_{0,f} \right] \eta_t(\tau)\,\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \theta t \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} \gamma_f \|\mathbf{E}_0 - \Pi_K^p \mathbf{E}_0\|_{0,f}^2 + \delta_f \|\mathbf{H}_0 - \Pi_K^p \mathbf{H}_0\|_{0,f}^2 \\
&+ t \sum_{K\in\mathcal{T}_h} \sum_{f\in\mathcal{F}_K} 4c_{\text{trace}}^2 \left(\frac{\alpha_{K_f,f}^2}{h_K \varepsilon_K} + \frac{\alpha_{K_f,f}^2}{h_{K_f}\varepsilon_{K_f}} + \frac{\delta_f^2}{h_K \mu_K} + \frac{\delta_f^2}{h_{K_f}\mu_{K_f}} \right) \|\mathbf{H}_0 - \Pi_K^p \mathbf{H}_0\|_{0,f}^2 \\
&\qquad + 4c_{\text{trace}}^2 \left(\frac{\beta_{K_f,f}^2}{h_K \mu_K} + \frac{\beta_{K_f,f}^2}{h_{K_f}\mu_{K_f}} + \frac{\gamma_f^2}{h_K \varepsilon_K} + \frac{\gamma_f^2}{h_{K_f}\varepsilon_{K_f}} \right) \|\mathbf{E}_0 - \Pi_K^p \mathbf{E}_0\|_{0,f}^2 \\
&+ 3t\|A\mathbf{u}_0 - \Pi_h^p A\mathbf{u}_0\|_{\mathrm{V}}^2 + 3t\theta^2 \|\mathbf{v}_0 - \Pi_h^p \mathbf{v}_0\|_{\mathrm{V}_{\mathrm{PML}}}^2.
\end{aligned}
$$

The projection errors can be estimated with Lemma 2.3.1 and 2.3.2

$$
\begin{aligned}
\int_0^t \mathcal{E}_{\mathrm{PML}}(\mathbf{u}-\mathbf{u}_h)\,\mathrm{d}\tau \le \int_0^t \Bigg[&\frac{3}{2} c_{\text{light}} c_{\text{boundary}}^2 h^{2s-3} |\partial_\tau \mathbf{E}|_{s-1,\varepsilon,\Lambda}^2 \\
&+ \left(\frac{3}{2} + \theta \eta_t(\tau) \right) c_{\text{light}} c_{\text{boundary}}^2 h^{2s-3} |\partial_\tau \mathbf{H}|_{s-1,\mu,\Lambda}^2 \\
&+ \frac{3}{2}\theta^2 c_{\text{light}} c_{\text{boundary}}^2 h^{2s-3} |\mathbf{E}|_{s-1,\varepsilon,\Lambda}^2 \Bigg] \eta_t(\tau)\,\mathrm{d}\tau \\
&+ t\theta c_{\gamma,\delta} c_{\text{boundary}}^2 h^{2s-3} |\mathbf{u}_0|_{s-1,\mathrm{V},\Lambda}^2 \\
&+ 4t c_{\text{trace}}^2 c_{\varepsilon,\mu,c_{\mathrm{h}}} c_{\text{boundary}}^2 h^{2s-4} |\mathbf{u}_0|_{s-1,\mathrm{V},\Lambda}^2 \\
&+ 3t c_{\text{cell}}^2 h^{2s-2} |A\mathbf{u}_0|_{s-1,\mathrm{V},\Lambda}^2 \\
&+ 3t\theta^2 c_{\text{cell}}^2 h^{2s-2} |\mathbf{v}_0|_{s-1,\mathrm{V}_{\mathrm{PML}},\Lambda}^2.
\end{aligned}
$$

In total, we obtain the error estimate

$$
\begin{aligned}
\|\mathbf{v}-\mathbf{v}_h\|_{\mathrm{L}_2((0,T),\mathrm{V}_{\mathrm{PML}})}^2 \le\; &7T^3 \theta c_{\gamma,\delta} c_{\text{boundary}}^2 h^{2s-3} |\mathbf{u}_0|_{s-1,\mathrm{V},\Lambda}^2 \\
&+ 27T^3 c_{\text{trace}}^2 c_{\varepsilon,\mu,c_{\mathrm{h}}} c_{\text{boundary}}^2 h^{2s-4} |\mathbf{u}_0|_{s-1,\mathrm{V},\Lambda}^2 \\
&+ 20T^3 c_{\text{cell}}^2 h^{2s-2} |A\mathbf{u}_0|_{s-1,\mathrm{V},\Lambda}^2 \\
&+ (14T + 20T^3\theta^2) c_{\text{cell}}^2 h^{2s-2} |\mathbf{v}_0|_{s-1,\mathrm{V}_{\mathrm{PML}},\Lambda}^2 \\
&+ 4T^4 c_{\text{light}} c_{\text{boundary}}^2 h^{2s-3} \max_{t\in[0,T]} |\partial_t \mathbf{u}|_{s-1,\mathrm{V},\Lambda}^2 \\
&+ 4T^4 \theta^2 c_{\text{light}} c_{\text{boundary}}^2 h^{2s-3} \max_{t\in[0,T]} |\mathbf{E}|_{s-1,\varepsilon,\Lambda}^2 \\
&+ 2T^5 \theta c_{\text{light}} c_{\text{boundary}}^2 h^{2s-3} \max_{t\in[0,T]} |\partial_t \mathbf{H}|_{s-1,\mu,\Lambda}^2.
\end{aligned}
$$

□

Remark 5.2.13 (Comparison of the error analysis with and without PML)**:**
Since the approach to the presented error estimate is in several parts analogous to the error estimate without PML in Chapter 2, we would like to point out the main differences. The most apparent difference is the utilization of a dissipative functional $\mathcal{E}_{\mathrm{PML}}(\mathbf{u})$, which contains time derivatives. Since we are interested in an error estimate in a weighted $\mathrm{L}_2(\Omega)$-norm, we had to fill the gap with Lemma 5.2.11. Though some of the terms are similar in both estimates and only differ in the absence or presence of time derivatives, we had to find a non-positive term analogous to $-(\partial_t \mathbf{u}_h, A_h \partial_t \mathbf{u}_h)_{\mathrm{V}}$, but without time derivatives. This was done in Lemma 5.2.4. Another difference we like to mention is the bound on $\frac{h_K}{h_{K_f}}$, which was used in Theorem 5.2.12. Generating a mesh, one usually avoids large jumps in the cellsize. In the error analysis with PML we actually see the effect of those jumps, in the analysis without PML this problem did not occur. Regarding the increased regularity assumption $s > 2$ in Theorem 5.2.12, we like to point out, that we expect an assumption $s > \frac{3}{2}$, since the time derivative $\partial_t \mathbf{u}$ appears in the error bound. The stronger assumption $s > 2$ is only needed in one term of the bound and may be improved by a closer look at the details.

Remark 5.2.14 (What is missing/How to proceed)**:**
At the moment the most important question is about the long-time behaviour of the analytic PML solution. The task is to find an analog of the functional $\mathcal{E}_{\mathrm{PML}}(\mathbf{u})$, that is dissipative in three dimensions and also with jumps in the parameter θ. With that functional one may check, if any additional difficulties in the error estimate occur. We assume that everything will work quite similar to the error estimate we presented here. Another approach to the problem can be to show the dissipative behaviour in the $\mathrm{L}_2(\Omega)$-energy $\frac{1}{2}\|\mathbf{u}+\boldsymbol{\xi}\|_{0,\Omega}^2$ in case of initial values supported outside of the layer. Since in that approach we have some restriction on the initial values, we expect more difficulties in the error estimate.

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